Final Project

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1 Problem Description and Notations

We would like to buy a contract paying at maturity T the amount in USD:

$$\Pi_T = \max\left[0, \left(\frac{S(T)}{S(0)} - k\right) \cdot \left(k' - \frac{L(T-\Delta, T-\Delta, T)}{L(0, T-\Delta, T)}\right)\right] \tag{1}$$

where

- $S(t)$ the Nikkei-225 spot price quantoed from JPY into USD
- $L(t, T-\Delta, T)$ is the 3-month USD LIBOR rate between $T-\Delta$ and T
- Δ is a period of 3-month (0.25 years)
- T the expiration date (e.g. 3 year)
- $k,\ k'$ given relative strike prices (e.g. both could be 1 or ...)

Provide a pricing routine (e.g. Python script) calculating the price of this contract, taking as inputs the the deal terms ($T,\ \Delta,\ k,\ k'$) and the relevant market data (rates, volatilities, spot prices ...).

Here are the additional notations used for the project:

- r_t^d and r^f are the domestic short rate (USD) and the foreign short rate (JPY);
- B_d and B_f are the domestic and foreign riskless asset prices;
- S_t^f and $S(t)$ are the foreign and quantoed prices of Nikkei-225;
- X_t is the foreign exchange rate (units of USD/unit of JPY);
- σ_r, σ_f, σ_X are the volatilities of r_t^d, S_t^f and X_t;
- ρ_{sx} is the correlation coefficient between S_t^f and X_t. ρ_{sr} is the correlation coefficient between S_t^f and r_t^d.

2 Assumptions

In this section, we will make some assumptions about dynamics of the short rates and the stock price and illustrate the considerations behind these assumptions.

In this project, there are two currencies: the domestic currency (USD) and the foreign currency (JPY). Since we are working on a relatively short time scale, the fluctuation of the foreign short rate r_f of JPY will have little impact on the quantoed stock price $S(t)$[1]. In the meanwhile, to capture the change of USD LIBOR, we have to model the domestic short rate r_t^d as a stochastic process. We further assume that the dynamics of the domestic short rate r_t^d follows the Hull-White model.

Assumption 1. We assume that the the foreign short rate r^f is a deterministic constant and the domestic short rate r_t^d is a stochastic process. The dynamics of r_t^d under the domestic martingale measure Q^d is

$$dr_t^d = [\Theta(t) - ar_t^d]dt + \sigma_r(t)dZ_t^{Q^d} \quad \text{with} \quad a > 0, \tag{2}$$

and the volatility term is assumed to be constant, i.e., $\sigma_r(t) = \sigma_r$.

We choose the Hull-White model because the $\Theta(t)$ term gives us sufficient degrees of freedom to model the dynamics of r_t^d, and also taking the volatility term as a constant increases the tractability of the problem.

In addition, we assume that the exchange rate (USD/JPY) X_t and the foreign stock price S_t^f of Nikkei-225 are modelled by geometric Brownian motions. However, it is not necessary to model X_t explicitly since we only needs its volatility σ_X and its correlation with the foreign stock price S_t^f noted as ρ_{sx}.

Assumption 2. The P-dynamics of the entire economy is as follows :

$$dS_t^f = S_t^f \alpha_f dt + S_t^f \sigma_f dW_t^P \tag{3}$$
$$dD_t^f = q^f S_t^f dt \tag{4}$$
$$dB_t^d = r_t^d B_t^d dt \tag{5}$$
$$dB_t^f = r^f B_t^f dt \tag{6}$$

where q^f is a deterministic constant for the dividend yield of Nikkei-225 and σ_r is a constant.

3 Methodology

3.1 Q^d-dynamics of the Quantoed Stock Price $S(t)$

To simplify the question, we want to represent all related processes w.r.t. the Wiener process under martingale measure Q^d with B_t^d being the numeraire.

Consider the cumulative continuous dividend D_t^f with dynamics in (4). $S_t^f + D_t^f$ is actually a nondividend paying asset, so

$$G_t^Y = \frac{S_t^f + D_t^f}{B_t^f} = \frac{S_t^f}{B_t^f} + \int_0^t \frac{1}{B_u^f} dD_u^f$$

[1]In section 4, we see that r^f is regulated by Bank of Japan and behaves like a constant in reality.

is a martingale under Q^f with B_t^f being the numeraire. Set $Y_t = \frac{S_t^f}{B_t^f}$, then we know

$$dG_t^Y = d\frac{S_t^f}{B_t^f} + \frac{1}{B_t^f} dD_t^f = Y_t(\alpha_f - r^f + q^f)dt + Y_t \sigma_f dW_t^P$$

Suppose φ_t^f is the Girsanov kernel of the measure transformation $dW_t^P = \varphi_t^f dt + dW_t^{Qf}$, then we have

$$dG_t^Y = Y_t(\alpha_f - r^f + q^f + \sigma_f \varphi_t^f)dt + Y_t \sigma_f dW_t^{Qf}.$$

Since G_t^Y is a martingale under Q^f, we must have

$$\sigma_f \varphi_t^1 = -\alpha_f + r^f - q^f \tag{7}$$

Performing the same measure transformation from P to Q^f for S_t^f and plugging (7) into (3), we obtain

$$dS_t^f = (r^f - q^f)S_t^f dt + S_t^f \sigma_f dW_t^{Qf}$$

Now we want to get the dynamics of S_t^f under the domestic martingale measure Q^d. Suppose φ_t is the Girsanov kernel of the measure transformation $dW_t^{Q^f} = \varphi_t dt + dW_t^{Q^d}$. We can show that $\varphi_t = -\sigma_X$ (see proof in Section 6.1). Then we obtain

$$dS_t^f = (r^f - q^f - \rho_{sx}\sigma_f \sigma_X)S_t^f dt + S_t^f \sigma_f dW_t^{Q^d} \tag{8}$$

Suppose the fixed exchange rate for this quanto product is X_0. Then we have $S(t) = S_t^f \cdot X_0$, and the dynamics for $S(t)$ under Q^d is

$$dS(t) = (r^f - q^f - \rho_{sx}\sigma_f \sigma_X)S(t)dt + S(t)\sigma_f dW_t^{Q^d}. \tag{9}$$

3.2 Q^d-dynamics of r_t^d and the Hull-White Term Structure

Now, by Assumption 1, the dynamics of the domestic short rate r_t^d follows the Hull-White model from (2)

$$dr_t^d = [\Theta(t) - ar_t^d]dt + \sigma_r dZ_t^{Q^d},$$

where $a > 0$ and σ are constants while Θ is a deterministic function of time. The dynamics above of r_t^d is directly under the martingale measure Q^d.

The Hull-White term structure is given by:

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\left\{ B(t,T)f^*(0,t) - \frac{\sigma_r^2}{4a} B^2(t,T)(1 - e^{-2at}) - B(t,T)r_t^d \right\} \tag{10}$$

where $B(t,T)$ is

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]. \tag{11}$$

The Θ function can be fitted as

$$\Theta(T) = f_T^*(0,T) + \dot{g}(T) + a\left[f^*(0,T) + g(T)\right], \tag{12}$$

where

$$g(t) = \frac{\sigma_r^2}{2a^2}\left(1 - e^{-at}\right)^2. \tag{13}$$

The fitting of the Hull-White model will be detailed in section 4.4.

3.3 Price of the Contract

The payoff (in USD) of the contract at maturity T is:

$$\Pi_T = \max\left[0, \left(\frac{S(T)}{S(0)} - k\right) \cdot \left(k' - \frac{L(T-\Delta, T-\Delta, T)}{L(0, T-\Delta, T)}\right)\right].$$

By risk neutral valuation, the price of this contract at time $t = 0$ is

$$\Pi_0 = \mathbb{E}_0^{Qf}\left[e^{-\int_0^T r_s^d ds}\Pi_T\right] \tag{14}$$

The simple forward rate for $[S,T]$ contracted at t, henceforth referred to as the LIBOR forward rate, is defined as

$$L(t,S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)}. \tag{15}$$

By (15), the LIBOR quotient involved in the contract payoff (1) can be reduced to

$$\frac{L(T-\Delta, T-\Delta, T)}{L(0, T-\Delta, T)} = \frac{p^*(0,T)}{p^*(0,T-\Delta) - p^*(0,T)}\left(\frac{1}{p(T-\Delta,T)} - 1\right),$$

and we can use (10) to calculate $p(T-\Delta,T)$.

3.4 Monte-Carlo Simulation

We take $\Delta t = \frac{T}{N}$ as the stepsize of the Euler-Maruyama method with N being a positive integer. The discrete timestamps are noted as $t_i = i\Delta t$ with $t_0 = 0$ and $t_N = T$.

Assume that we simulate in total M paths. Then, for path j, we have

$$S^j(t_{i+1}) = S^j(t_i) + (r^f - q^f - \rho_{sx}\sigma_f\sigma_X)S^j(t_i)\Delta t + S^j(t_i)\sigma_f\sqrt{\Delta t}\cdot\epsilon_1$$
$$r^{d,j}(t_{i+1}) = r^{d,j}(t_i) + [\Theta(t) - ar^{d,j}(t_i)]\Delta t + \sigma_r\sqrt{\Delta t}\cdot\epsilon_2,$$

where $\epsilon_1, \epsilon_2 \sim \mathcal{N}(0,1)$ and the correlation coefficient between them is $\rho(\epsilon_1,\epsilon_2) = \rho_{sr}$. Then, the payoff for path j is

$$\Pi_T^j = \max\left[0, \left(\frac{S^j(t_N)}{S^j(t_0)} - k\right)\cdot\left(k' - \frac{L^j(T-\Delta, T-\Delta, T)}{L^j(0, T-\Delta, T)}\right)\right]$$

where

$$\frac{L^j(T-\Delta, T-\Delta, T)}{L^j(0, T-\Delta, T)} = \frac{p^*(0,T)}{p^*(0,T-\Delta) - p^*(0,T)}\left(\frac{1}{p^j(T-\Delta,T)} - 1\right).$$

The price of this contract for path j at $t = 0$ is thus

$$\Pi_0^j = \Pi_T^j \sum_{i=0}^{N} r^{d,j}(t_i)\Delta t,$$

and an estimate of Π_0 is

$$\hat{\Pi}_0 = \frac{1}{M}\sum_{j=1}^{M}\Pi_0^j.$$

4 Data Inputs and Model Fitting

4.1 Estimated Value for Parameters and Data Sources

We have summarized all the parameter values in Tab. 1.

Parameter	Value	Source
σ_f [2]	24.78%	`https://svc.qri.jp/jpx/english/nkopm/`
σ_X [3]	11.4%	`https://www.investing.com/currencies/usd-jpy-options`
σ_r	1.5%	Hypothetical input. Calculation method see section 4.3
q^f	2.35%	`https://indexes.nikkei.co.jp/en/nkave/factsheet?idx=nk225hdy`
r^f	-0.1% [4]	`https://www.boj.or.jp/en/index.htm/`
a	3%	Common value used by practitioners
ρ_{sx}	-0.242	See section 4.2
ρ_{sr}	0.30	Hypothetical input

Table 1: Parameter values

[2]Implied volatility (at the money) of the call option on Nikkei-225.
[3]Implied volatility (at the money) of the call option on the exchange rate.
[4]The short-term interest rate of JPY has been set to -0.1% by the Bank of Japan since 2016. This fact also substantiates our assumption that r^f can be seen as a constant.

4.2 Estimation of ρ_{sx}

We have assumed that the P-dynamics of S_t^f and X_t are given as follows

$$
\begin{aligned}
dS_t^f &= S_t^f \alpha_f dt + S_t^f \sigma_f dW_t^P \\
dX_t &= X_t \alpha_X dt + X_t \sigma_X dY_t^P,
\end{aligned}
$$

where W_t^P and Y_t^P are two brownian motions with a correlation coefficient ρ_{sx}. We observe the S_t^f and X_t at time $t_i = i\Delta t$ and have

$$
\begin{aligned}
\frac{S(t_{i+1}) - S(t_i)}{S(t_i)} &= \alpha_f \Delta t + \sigma_f \sqrt{\Delta t} \cdot w \\
\frac{X(t_{i+1}) - X(t_i)}{X(t_i)} &= \alpha_X \Delta t + \sigma_X \sqrt{\Delta t} \cdot y
\end{aligned}
$$

where $w, y \sim \mathcal{N}(0,1)$ and the correlation coefficient between them is $\rho(w,y) = \rho_{sx}$. Therefore, given two series $\{a_i\} = \left\{\frac{S(t_{i+1})-S(t_i)}{S(t_i)}\right\}_i$ and $\{b_i\} = \left\{\frac{X(t_{i+1})-X(t_i)}{X(t_i)}\right\}_i$, we have an estimate of ρ_{sx} as

$$
\hat{\rho}_{sx} = \frac{\sum(a_i - \bar{a})(b_i - \bar{b})}{\sqrt{\sum(a_i - \bar{a})^2 \sum(b_i - \bar{b})^2}}.
$$

We use the historical data of Nikkei-225 and USD/JPY exchange rate during the past 6 months[5], and get $\hat{\rho}_{sx} = -0.242$.

4.3 Calculation of σ_r

Due to lack of data access, we here present a calculation method for σ_r and use a hypothetical value in the simulation.

We know that a floorlet is equivalent to a call option on a zero-coupon bond. Suppose we have access to the implied volatility of the floorlet, which is noted as σ_p. Then, by results on the Hull-White model[6], we know the relationship between σ_p and σ_r is

$$
\sigma_p = \frac{1}{a}\left[1 - e^{-a(S-T)}\right]\sqrt{\frac{\sigma_r^2}{2a}\left[1 - e^{-2a(T-t)}\right]}.
$$

Note that t, S and T needs to be set corresponding to the specifications of the floorlet. If we have access to the value of σ_p, we can solve for σ_r using the above equation.

[5]Nikkei-225 historical data from https://finance.yahoo.com/quote/%5EN225/history/. USD/JPY exchange rate historical data from https://finance.yahoo.com/quote/JPYUSD%3DX/history?p=JPYUSD%3DX.

[6]See Proposition 24.9 in *Arbitrage Theory in Continuous Time* by Tomas Björk

4.4 Fitting the Hull-White Model

We start from the yield curve for US treasury bonds observed from the market for today. Then, with yields data in hand, we can use the formula

$$p(0,T) = e^{-yT}$$

to get the bond prices for all different maturities for today.

Now, the bond data we have are really just a few data points, so we use the cubic spline technique to fit the bond prices and get the curve for bond prices, denoted by $p^*(0,T)$.

Since we have

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{1}{p(t,T)}\frac{\partial p(t,T)}{\partial T}$$

We can then use the formula above to find the curve for instantaneous forward rate $f^*(0,T)$.

Lastly, for $f^*_T(0,T)$ curve, we can just take the derivative of $f^*(0,T)$ w.r.t. T to obtain it.

Now, we have all the curves required to fit the Hull-White model. We shall use the data for $f^*(0,T)$ and $f^*_T(0,T)$ on $\Theta(t)$ (12) to conduct the Monte Carlo simulations on r_d and S, and use the data for $p^*(0,T)$ and $f^*(0,T)$ to price the zero coupon bond $p(T-\Delta,T)$ via equation (10)

5 Simulation Results

Plug in all the data we've obtained from the market and run the Monte Carlo Simulations, we've got the following results:

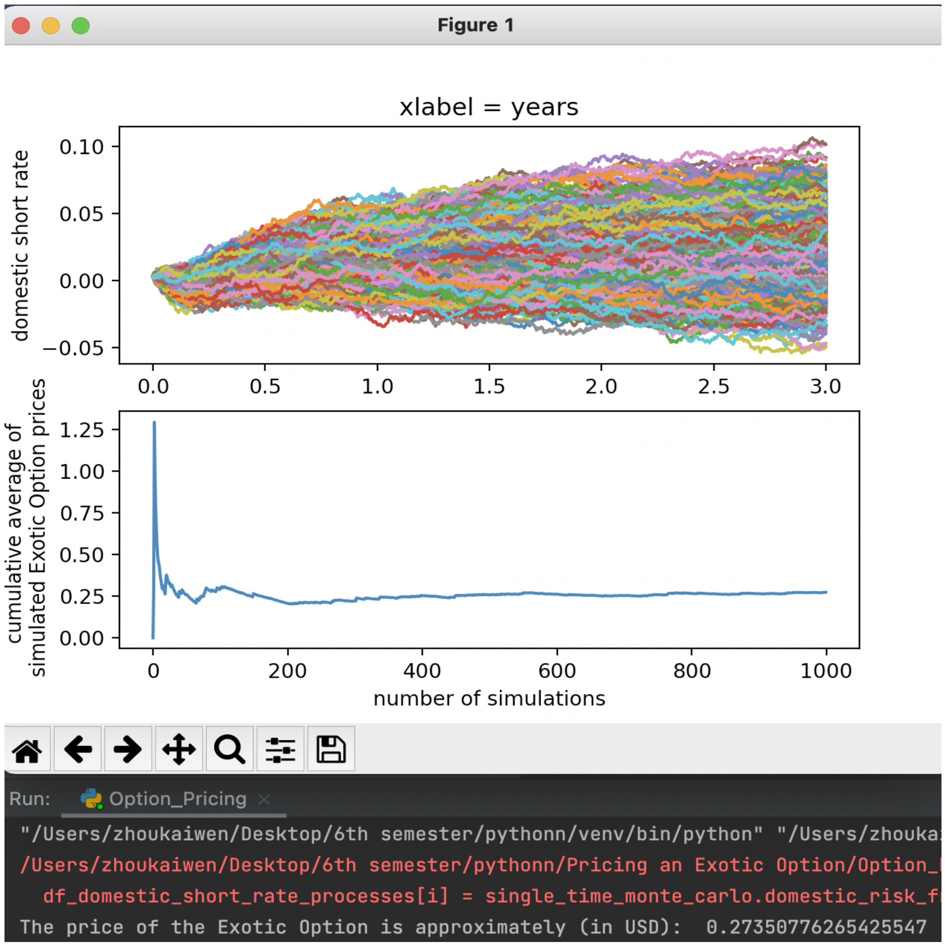

Figure 1: Monte Carlo Simulation Results

We can conclude that the price of the Exotic Option is approximately (in USD):

$$\hat{\Pi}_0 = 0.27350776265425547$$

wtih standard deviation $\sigma_\Pi = 0.02984458917087303$. (Calculated using several approximations of Π_0.)

6 Appendix

6.1 Relation between Q^f and Q^d

The domestic and foreign martingale measures are denoted by Q^d and Q^f respectively, and we define on $\mathcal{F}_t$ the following likelihood process

$$L_t = \frac{dQ^f}{dQ^d}.$$

The corresponding Wiener processes under Q^d and Q^f are denoted by W^d and W^f. The Girsanov kernel corresponding to L is denoted by φ, so the likelihood processes have the dynamics

$$dL_t = L_t \varphi_t^d W_t^{Q_d}.$$

Fix an arbitrary foreign T-claim Z. Compute foreign price and change to domestic currency. The price at $t = 0$ will be

$$\Pi_0[Z] = X_0 \mathbb{E}_0^{Q^f} \left[e^{-\int_0^T r_s^f ds} Z \right].$$

This can be written as

$$\Pi_0[Z] = X_0 \mathbb{E}_0^{Q^f} \left[L_T e^{-\int_0^T r_s^f ds} Z \right].$$

Change into domestic currency at T and then compute arbitrage free price. This gives us

$$\Pi_0[Z] = \mathbb{E}_0^{Q^f} \left[e^{-\int_0^T r_s^d ds} X_T Z \right].$$

These expressions must be equal for all choices of $Z \in \mathcal{F}_T$. We thus obtain

$$\mathbb{E}_0^{Q^f} \left[e^{-\int_0^T r_s^d ds} X_T Z \right] = X_0 \mathbb{E}_0^{Q^f} \left[L_T e^{-\int_0^T r_s^f ds} Z \right]$$

for all T-claim Z. This implies that

$$X_t = X_0 e^{\int_0^T (r_s^d - r_s^f) ds} L_t.$$

The Q^d-dynamics of X thus follows as

$$dX_t = (r_t^d - r_t^f) X_t dt + X_t \varphi_t dW_t^d.$$

Since the X_t is assumed to be a geometric brownian motion under P, we can deduce that $\varphi_t = \sigma_X$, i.e.,

$$dW^{Q_d} = \sigma_X dt + dW^{Q_f}$$