Times Series Foundamentals

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Contents

1 Topic: Stationary Process

Problem 1.1. *A stationary process $\{Y_t\}$ is given, show that:*

(a) $Z_t = (Y_t - Y_{t-1})$ *is a stationary process.*

(b) $Z'_t = (Y_t - Y_{t-d})$ *is a stationary process*

Solution:

(a) Since $\{Y_t\}$ is stationary, we have

$$\mathbb{E}[Y_t^2] < \infty; \quad \mathbb{E}[Y_t] = m, m \in \mathbb{R}, \forall t; \quad Cov[Y_t, Y_s] = \gamma_Y(r,s) = \gamma_Y(r+t, s+t) \equiv \gamma_Y(r-s), \forall t.$$

By the definition of the stationary process, we need to verify the three criterions. We have

(i) **Square-Integrable:**

$$\begin{aligned}
\mathbb{E}[Z_t^2] &= \mathbb{E}[(Y_t - Y_{t-1})^2] \\
&= \mathbb{E}[Y_t^2] + \mathbb{E}[Y_{t-1}^2] - 2\mathbb{E}[Y_t Y_{t-1}] \\
&\le \mathbb{E}[Y_t^2] + \mathbb{E}[Y_{t-1}^2] + 2|\mathbb{E}[Y_t Y_{t-1}]| \\
\text{AM-GM} \implies &\le 2\mathbb{E}[Y_t^2] + 2\mathbb{E}[Y_{t-1}^2] \\
&< \infty + \infty = \infty
\end{aligned}$$

(ii) **Constant Mean:**

$$\mathbb{E}[Z_t] = \mathbb{E}[Y_t - Y_{t-1}] = \mathbb{E}[Y_t] - \mathbb{E}[Y_{t-1}] = m - m = 0$$

(iii) **Time independent autocovariance:**

$$\begin{aligned}
\gamma_Y(r,s) &= Cov[Z_r, Z_s] \\
&= Cov[Y_r - Y_{r-1}, Y_s - Y_{s-1}] \\
&= \mathbb{E}[(Y_r - Y_{r-1} + \mathbb{E}[Y_r - Y_{r-1}])\,(Y_s - Y_{s-1} + \mathbb{E}[Y_s - Y_{s-1}])] \\
&= \mathbb{E}[((Y_r - \mathbb{E}[Y_r]) + (Y_{r-1} - \mathbb{E}[Y_{r-1}]))\,((Y_s - \mathbb{E}[Y_s]) + (Y_{s-1} - \mathbb{E}[Y_{s-1}]))] \\
&= \mathbb{E}[(Y_r - \mathbb{E}[Y_r])(Y_s - \mathbb{E}[Y_s])] + \mathbb{E}[(Y_{r-1} - \mathbb{E}[Y_{r-1}])(Y_{s-1} - \mathbb{E}[Y_{s-1}])] \\
&\quad + \mathbb{E}[(Y_r - \mathbb{E}[Y_r])(Y_{s-1} - \mathbb{E}[Y_{s-1}])] + \mathbb{E}[(Y_{r-1} - \mathbb{E}[Y_{r-1}])(Y_s - \mathbb{E}[Y_s])] \\
&= Cov[Y_r, Y_s] + Cov[Y_{r-1}, Y_{s-1}] + Cov[Y_r, Y_{s-1}] + Cov[Y_{r-1}, Y_s] \\
&= \gamma_Y(r,s) + \gamma_Y(r-1, s-1) + \gamma_Y(r, s-1) + \gamma_Y(r-1, s) \\
&= \gamma_Y(r-s) + \gamma_Y(r-s) + \gamma_Y(r-s+1) + \gamma_Y(r-s-1) \\
&= 2\gamma_Y(r-s) + \gamma_Y(r-s+1) + \gamma_Y(r-s-1) \text{ independent of the value of } t
\end{aligned}$$

Therefore, by the definition of the stationary process, we can conclude that $Z_t = (Y_t - Y_{t-1})$ is a stationary process.

(b) By the definition of the stationary process, we need to verify the three criterions. We have

(i) **Square-integrable:**

$$\begin{aligned}
\mathbb{E}[Z_t^2] &= \mathbb{E}[(Y_t - Y_{t-d})^2] \\
&= \mathbb{E}[Y_t^2] + \mathbb{E}[Y_{t-d}^2] - 2\mathbb{E}[Y_t Y_{t-d}] \\
&\le \mathbb{E}[Y_t^2] + \mathbb{E}[Y_{t-d}^2] + 2|\mathbb{E}[Y_t Y_{t-d}]| \\
\text{AM-GM} \implies &\le 2\mathbb{E}[Y_t^2] + 2\mathbb{E}[Y_{t-d}^2] \\
&< \infty + \infty = \infty
\end{aligned}$$

(ii) **Constant Mean:**

$$\mathbb{E}[Z_t] = \mathbb{E}[Y_t - Y_{t-d}] = \mathbb{E}[Y_t] - \mathbb{E}[Y_{t-d}] = m - m = 0.$$

(iii) **Time independent autocovariance:**

$$\begin{aligned}
\gamma_Y(r,s) &= Cov[Z_r, Z_s] \\
&= Cov[Y_r - Y_{r-d}, Y_s - Y_{s-d}] \\
&= \mathbb{E}[(Y_r - Y_{r-d} + \mathbb{E}[Y_r - Y_{r-d}])\,(Y_s - Y_{s-d} + \mathbb{E}[Y_s - Y_{s-d}])] \\
&= \mathbb{E}[((Y_r - \mathbb{E}[Y_r]) + (Y_{r-d} - \mathbb{E}[Y_{r-d}]))\,((Y_s - \mathbb{E}[Y_s]) + (Y_{s-d} - \mathbb{E}[Y_{s-d}]))] \\
&= \mathbb{E}[(Y_r - \mathbb{E}[Y_r])(Y_s - \mathbb{E}[Y_s])] + \mathbb{E}[(Y_{r-d} - \mathbb{E}[Y_{r-d}])(Y_{s-d} - \mathbb{E}[Y_{s-d}])] \\
&\quad + \mathbb{E}[(Y_r - \mathbb{E}[Y_r])(Y_{s-d} - \mathbb{E}[Y_{s-d}])] + \mathbb{E}[(Y_{r-d} - \mathbb{E}[Y_{r-d}])(Y_s - \mathbb{E}[Y_s])] \\
&= Cov[Y_r, Y_s] + Cov[Y_{r-d}, Y_{s-d}] + Cov[Y_r, Y_{s-d}] + Cov[Y_{r-d}, Y_s] \\
&= \gamma_Y(r,s) + \gamma_Y(r-d, s-d) + \gamma_Y(r, s-d) + \gamma_Y(r-d, s) \\
&= \gamma_Y(r-s) + \gamma_Y(r-s) + \gamma_Y(r-s) + \gamma_Y(r-s) \\
&= 2\gamma_Y(r-s) + \gamma_Y(r-s+d) + \gamma_Y(r-s-d) \text{ independent of the value of } t
\end{aligned}$$

Therefore, by the definition of the stationary process, we can conclude that $Z_t = (Y_t - Y_{t-d})$ is a stationary process.

2 Topic: Moving Average, White Noise

Problem 2.1. *Write the autocovariance function for the $MA(1)$ process and show that $MA(1)$ is a stationary process.*

Solution: By definition, the $MA(1)$ process is defined to be $Y_t = Z_t + \theta Z_{t-1}$ where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta \in \mathbb{R}$ is a constant.

By the definition of the stationary process, we need to verify the three criterions. We have

(i) **Square-integrable:**

$$\mathbb{E}[Y_t^2] = \mathbb{E}[(Z_t + \theta Z_{t-1})^2] = \mathbb{E}[Z_t^2] + \theta^2\mathbb{E}[Z_{t-1}^2] + 2\theta\mathbb{E}[Z_t Z_{t-1}] \xlongequal{Z\sim WN(0,\sigma^2)} \mathbb{E}[Z_t^2] + \theta^2\mathbb{E}[Z_{t-1}^2] = \sigma^2 + \theta^2\sigma^2 = (1+\theta^2)\sigma^2 < \infty$$

(ii) **Constant Mean:**

$$\mathbb{E}[Y_t] = \mathbb{E}[Z_t + \theta Z_{t-1}] = \mathbb{E}[Z_t] + \theta\mathbb{E}[Z_{t-1}] \xlongequal{Z\sim WN(0,\sigma^2)} 0$$

(iii) **Time independent autocovariance:**

$$\begin{aligned}
\gamma_Y(r,s) &= Cov[Y_r, Y_s] \\
&= Cov[Z_r + \theta Z_{r-1}, Z_s + \theta Z_{s-1}] \\
&= \mathbb{E}[(Z_r + \theta Z_{r-1} + \mathbb{E}[Z_r + \theta Z_{r-1}])\,(Z_s + \theta Z_{s-1} + \mathbb{E}[Z_s + \theta Z_{s-1}])] \\
&= \mathbb{E}[((Z_r - \mathbb{E}[Z_r]) + \theta(Z_{r-1} - \mathbb{E}[Z_{r-1}]))\,((Z_s - \mathbb{E}[Z_s]) + \theta(Z_{s-1} - \mathbb{E}[Z_{s-1}]))] \\
&= \mathbb{E}[(Z_r - \mathbb{E}[Z_r])(Z_s - \mathbb{E}[Z_s])] + \theta^2\mathbb{E}[(Z_{r-1} + \mathbb{E}[Z_{r-1}])(Z_{s-1} - \mathbb{E}[Z_{s-1}])] \\
&\quad + \theta\mathbb{E}[(Z_r - \mathbb{E}[Z_r])(Z_{s-1} - \mathbb{E}[Z_{s-1}])] + \theta\mathbb{E}[(Z_{r-1} - \mathbb{E}[Z_{r-1}])(Z_s - \mathbb{E}[Z_s])] \\
&= Cov[Z_r, Z_s] + \theta^2 Cov[Z_{r-1}, Z_{s-1}] + \theta Cov[Z_r, Z_{s-1}] + \theta Cov[Z_{r-1}, Z_s] \\
&= \gamma_Z(r,s) + \theta^2\gamma_Z(r-1, s-1) + \theta\gamma_Z(r, s-1) + \theta\gamma_Z(r-1, s) \\
&= \gamma_Z(r-s) + \theta^2\gamma_Z(r-s) + \theta\gamma_Z(r-s+1) + \theta\gamma_Z(r-s-1) \\
&= \begin{cases} (1+\theta^2)\sigma^2 & \text{if } r = s \\ \theta\sigma^2 & \text{if } r = s-1 \\ \theta\sigma^2 & \text{if } r = s+1 \\ 0 & \text{otherwise} \end{cases} \longrightarrow \text{ independent of the value of } t
\end{aligned}$$

Therefore, by the definition of the stationary process, we can conclude that the $MA(1)$ process $Y_t = Z_t + \theta Z_{t-1}$ is a stationary process, and its autocovariance is

$$\gamma_Y(r,s) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } r = s \\ \theta\sigma^2 & \text{if } r = s-1 \\ \theta\sigma^2 & \text{if } r = s+1 \\ 0 & \text{otherwise} \end{cases}$$

3 Topic: Classical Decomposition

Problem 3.1. *Load the file "Data_For_HW1.xls" file from the course website.*

(a) Plot the time series.

(b) Obtain the equation for its linear trend, de-trend the original time series, and plot it.

(c) Plot the sample autocorrelation function for the de-trended time series. What process do you recommend to model the de-trended time series?

(d) Plot the first difference of the original time series.

(e) Plot the sample autocorrelation function for the differenced time series. What process model do you recommend for the differenced time series?

Solution: For the plots and code see the attached jupyter notebook, `Time Series and Statistical Arbitrage HW1.ipynb`.

4 Topic: Generalized Linear Process

Problem 4.1. *Let $\{Y_t\}$ be a stationary time series with mean 0 and covariance function γ_Y. If $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty$ then show that the time series*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}$$

is stationary with mean 0 and autocovariance function

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j)$$

Solution: By the definition of the stationary process, we need to verify the three criterions. We have

(i) **Square-integrable:**

$$\begin{aligned}
\mathbb{E}[X_t^2] &= \mathbb{E}[(\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j})^2] \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \mathbb{E}[Y_{t-j}Y_{t-k}] \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \left(Cov(Y_{t-j}, Y_{t-k}) + \mathbb{E}[Y_{t-j}]\mathbb{E}[Y_{t-k}]\right) \\
\mathbb{E}[Y_t] = 0, \forall t \in \mathbb{Z} \implies &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(k-j) \\
&\leq \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\psi_j||\psi_k||\gamma_Y(k-j)| \\
\gamma_Y(0) \geq |\gamma_Y(f)|, \forall h \in \mathbb{Z} \implies &\leq \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\psi_j||\psi_k|\gamma_Y(0) \\
&= \left(\sum_{j=-\infty}^{\infty} |\psi_j|\right)^2 \gamma_Y(0) \\
\sum_{j=-\infty}^{\infty} |\psi_j| < \infty \implies &< \infty
\end{aligned}$$

(ii) **Constant Mean:**

$$\mathbb{E}[X_t] = \mathbb{E}[\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}] = \sum_{j=-\infty}^{\infty} \psi_j \mathbb{E}[Y_{t-j}] \xlongequal{\mathbb{E}[Y_t]=0, \forall t \in \mathbb{Z}} \sum_{j=-\infty}^{\infty} \psi_j \cdot 0 = 0$$

(iii) **Time independent autocovariance:**

$$\begin{aligned}
\gamma_X(t+h, t) &= Cov[X_{t+h}, X_t] \\
&= \mathbb{E}[X_{t+h}X_t] - \mathbb{E}[X_{t+h}]\mathbb{E}[X_t] \\
\mathbb{E}[X_t] = 0, \forall t \in \mathbb{Z} \implies &= \mathbb{E}[X_{t+h}X_t] \\
&= \mathbb{E}\left[\left(\sum_{j=-\infty}^{\infty} \psi_j Y_{t+h-j}\right)\left(\sum_{k=-\infty}^{\infty} \psi_k Y_{t-k}\right)\right] \\
&= \mathbb{E}\left[\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k Y_{t+h-j} Y_{t-k}\right] \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \mathbb{E}\left[Y_{t+h-j} Y_{t-k}\right] \\
\mathbb{E}[Y_t] = 0, \forall t \in \mathbb{Z} \implies &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k Cov[Y_{t+h-j} Y_{t-k}] \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(t+h-j, t-k) \\
\{Y_t\} \text{ is stationary } \implies &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j) \text{ independent of the value of } t
\end{aligned}$$

Therefore, by the definition of the stationary process, we can conclude that the generalized linear process defined by $X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}$ is stationary with mean 0 provided $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, and its autocovariance is

$$\gamma_X(t+h,t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j) \tag{1}$$

5 Topic: Linear Process

Problem 5.1. *In Problem 4, when $\{X_t\}$ is a linear process (i.e. $\{Y_t\} = \{Z_t\}$), show that:*

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2$$

Solution: With the result proved in Problem 4.1, we just need to substitute $\gamma_Y(h+k-j)$ in Equation (1) with $\gamma_Z(h+k-j)$, and we obtain

$$\gamma_X(t+h,t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Z(h+k-j) \xlongequal[\gamma_Z(h+k-j)=0 \text{ if } h+k-j\neq 0]{Z\sim WN(0,\sigma^2)} \sum_{j=-\infty}^{\infty} \psi_j \psi_{j-h} \gamma_Z(0) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j-h} \sigma^2 \xlongequal{j=j-h} \sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j \sigma^2$$

which is exactly what we desired.