ELEN90054 Probability and Random Models

MATLAB Workshop 3 on Gaussian Noise Channel Simulation and Symbol Detection;

week 7 = week of 16 April 2018

Prepare this workshop individually, in particular questions 1,2,3,4 and 7 and read through all questions before you come to your workshop session.

This workshop is worth 5% of the overall subject assessment and should be completed in pairs. Allocation will be random by your demonstrator, so you should not choose your project partner yourself. The next workshop will have a new allocation. Be aware that seeking or providing detailed assistance from/to people other than your workshop partner is collusion - see
http://academichonesty.unimelb.edu.au/plagiarism.html.

Each group is expected to upload two files:

1. A pdf (scanned/typed) file containing their worked solutions;
   a. Only one member of the group needs to upload the pdf file.
   b. The naming convention of the file should include workshop number, day, time and the assigned group number, e.g. Workshop_2_Mon09_Gp5.pdf for group 5 in the Monday 0900 hrs workshop slot.
   c. You also need to **read and attach** the Engineering cover-sheet (see the ELEN90054 LMS site Workshops) signed by both of you.
2. A single zip file, containing all the required functions and **a single main script** which calls these functions to generate the required outputs as outlined in the workshop questions.
   a. Only one member of the group needs to upload the zip file.
   b. The following naming convention should strictly be followed: Workshop_2_Mon12_Gp5_Matlab.zip for group 5 in the Monday 1200 hrs workshop slot.

The workshop times are: Mon09, Mon12, Tue10, Tue17, Wed13, Wed17, Wed18, Wed19, Thu11, Thu19, Fri10, Fri13. The group numbers are randomly assigned by your demonstrator at the start of the workshop.

Both submissions should be made before the start of your workshop time in week 8. **This is a strict deadline**.

In special circumstances you can email the pdf/code to your demonstrator. For demonstrator email address information, see "staff info" on the ELEN90054 LMS site.

Note: The main purpose of this workshop is to get you to experiment with continuous channel simulation and symbol detector performance. You will get to use random variable transformations as well as random vector transformations and will also get a taste of the workings of a symbol detector, as used in digital communications.

Gaussian Noisy Communication Channel and Symbol Detection

(total=36 marks + 4 on-time attendance marks)
Important:

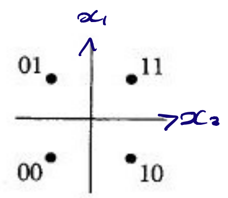
- Do not use the built-in exprnd (or any similar) command in MATLAB to generate exponential random variables in this workshop
- Make sure that all your MATLAB coding has clear comments, so that the marker (and yourself) can understand what is going on.

Communication channels are often modelled using complex numbers. Here we consider a noisy communication channel described by

$$Y = x + N,$$

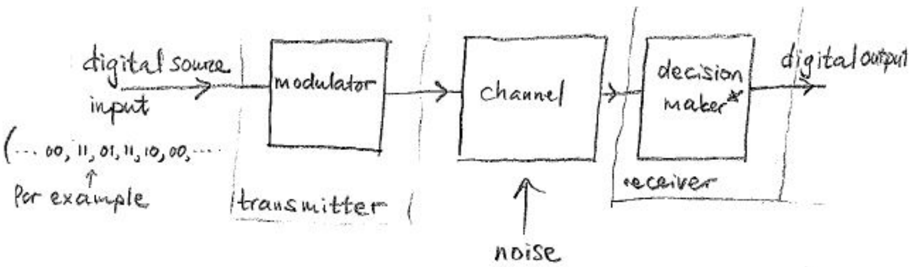
where the input $x = x_1 + jx_2$ is a **complex-valued symbol**[1]. Here $Y = Y_1 + jY_2$ is the received symbol and

- x_1 and x_2 are equal to either 1 or -1. Thus there are 4 possible combinations, corresponding to information bit pairs 11,10,00 and 01:



$y_1 + jy_2 = (x_1+n_1) + j(x_2+n_2)$

- $N = N_1 + jN_2$ is a **complex random variable**, modelling noise which corrupts the input signal.

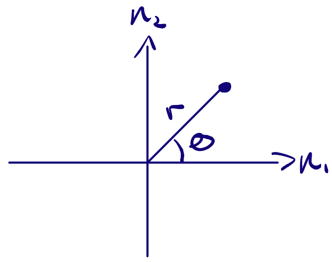


* is called "detector" in comms theory

The **detector's decision rule**[2] is as follows: it decodes x_1 as 1 if the channel output $Y_1 \geq 0$, and as -1 otherwise; similarly for x_2 and Y_2.

[1]This is called a *lowpass representation*, and is a useful, shorthand way of representing a *real* bandpass signal $s(t) = x_1 \cos(\omega_0 t) - x_2 \sin(\omega_0 t)$. The real part x_1 *modulates* a cosine wave at some frequency and is called the *in-phase component*, and the imaginary part x_2 modulates a -sine wave at the same frequency, and is called the *quadrature* component.

[2]More about this in follow-on subjects ELEN90057 Communication Systems and ELEN90051 Advanced Communication Systems



We specify the noise in terms of two other random variables:

$$N = N_1 + jN_2 = Re^{j\Theta} = R\cos\Theta + jR\sin\Theta,$$

where $R^2 \sim \exp(\lambda)$ and $\Theta \sim U(0, 2\pi)$, and R and Θ are independent. That is, the square magnitude of the noise is assumed to be exponentially distributed with parameter λ, and the phase is assumed to be uniformly distributed from 0 to 2π.

Questions

1. (1 mark) Visualize the detector's decision rule in the complex plane.
2. (3 marks) Find the CDF of R, that is, $F_R(r) = \Pr[R \leq r]$. Hence find the PDF of R, that is $f_R(r)$, and the joint PDF of R and Θ, that is $f_{R,\Theta}(r, \theta)$.
3. (3 marks) Compute $E[N_1]$, $E[N_1^2]$ and $E[N_1 N_2]$. The identities $\cos^2(\alpha) = \frac{1}{2}(1+\cos(2\alpha))$ and $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ may be useful.
4. (3 marks) Using the CDF of R^2, express R^2 in terms of a uniformally distributed random variable U. That is, find a function f such that $f(U)$ has the same distribution as R^2. Implement the function in MATLAB and call it 30,000 times with a $U(0, 1)$ argument (generated by the **rand** command) and a λ argument. Plot the sample CDF for $\lambda = 1$, using **cdfplot** on the same axes as the exact CDF (the command **hold on** may be useful).
5. Assume that $\lambda = 1$. Write a function that calls the function from the previous question, to obtain a value for R^2, calls **rand** to obtain a value for Θ, then computes and returns the resulting values of N_1, N_2 and R. Call this function 30,000 times and build up three arrays for the corresponding values. Using these:

   (a) (2 marks) Produce a scatterplot of the first 1,000 samples of N_1 and N_2 (for example if n1 and n2 are the arrays, use **scatter(n1(1:1000), n2(1:1000)**,'.'))

   (b) (2 marks) Suppose that the following assumption holds:

   Assumption 1: the input is deterministic and equals $x = -1 + j$ on every use of the channel

   Compute empirical values for the following probabilities:

   i. $P[x_1 \text{ is decoded correctly}]$
   ii. $P[x_2 \text{ is decoded correctly}]$
   iii. $P[x_1 \text{ and } x_2 \text{ are both decoded correctly}]$

6. (2 marks) Again assume that $\lambda = 1$. From your results in answering part (b) of the previous question, can you confidently conclude anything about whether N_1 and N_2 are independent?
7. (4 marks) Again assume that $\lambda = 1$. It can be shown that then $N_1, N_2 \sim \mathcal{N}(0, 0.5)$ and that N_1 and N_2 are independent[3]. Using this result (or otherwise), theoretically derive the probabilities from Q5b and compare them to your simulations (the **normcdf** command in MATLAB may be useful).

Questions continue on the next page

[3]in fact, it may sound familiar to you that $N_1^2 + N_2^2 \sim \exp(0.5)$, where N_1 and N_2 are independent $\mathcal{N}(0, 1)$ random variables, see lectures.

$\lambda = \frac{1}{2\sigma^2}$

$\sigma^2 = \frac{1}{2\lambda}$

$P[x_1 \cap x_2 \text{ decodeded correctly}]$
$= P[x_1 \text{ decoded correctly}] \cdot P[x_2 \text{ decoded correctly}]$

8. (a) (4 marks) Again assume that $\lambda = 1$. Replace Assumption 1 by an assumption that is more relevant to digital communications, namely

   > **Assumption 2**: the input X is a discrete random variable with outcomes $x \in \{1+j, 1-j, -1-j, -1+j\}$ with equal probabilities; the inputs are independent for different channel uses.

   Simulate a sequence of 10,000 inputs, corrupted by noise N, resulting in a sequence of 10,000 outputs Y. Visualize your simulation via a scatterplot of the y-values, where each point y has one of 4 different colours **depending on the value of x that it originates from.**

   (b) (2 marks) Plot your scatterplot from a) again (keep all colours), but this time without all correctly detected y points.

   (c) (2 marks) Compute an empirical value of the detector error probability which is given by

$$1 - P[x_1 \text{ and } x_2 \text{ are both decoded correctly}].$$

   (d) (3 marks) Repeat parts a), b) and c) for the values $\lambda = 0.5$ and $\lambda = 5$. Discuss your results, in particular compare the detector error probabilities of the three cases $\lambda = 1$, $\lambda = 0.5$ and $\lambda = 5$ (include the theoretical derivation for $\lambda = 0.5$ and also for $\lambda = 5$); interpret the differences between these three cases.

9. (3 marks) Repeat Q8 but this time with Assumption 2 replaced by

   > **Assumption 3**: the input X is a discrete random variable with outcomes $x \in \{1+j, 1-j, -1-j, -1+j\}$ with probabilities $\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$, respectively; the inputs are independent for different channel uses.

   Discuss your results, in particular compare the detector error probability under Assumption 3 with the detector error probability under Assumption 2. Can you think of ways to change the decision rule to make the detector error probability under Assumption 3 smaller?

10. (2 marks) Repeat Q9 but this time with Assumption 3 replaced by

   > **Assumption 4**: the input X is a discrete random variable with outcomes $x \in \{1+j, 1-j, -1-j, -1+j\}$ with probabilities $1, 0, 0, 0$, respectively.

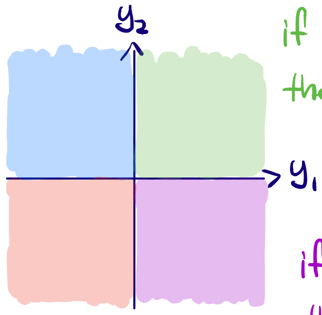
   Can you think of a decision rule that yields zero detector error probability under Assumption 4?

End of MATLAB Workshop 3 Questions

1. (1 mark) Visualize the detector's decision rule in the complex plane.

if $y = -(n_1+x_1) + j(n_2+x_2)$
then x should be $x = -1+j$

if $y = (n_1+x_1) + j(n_2+x_2)$
then x should be $x = 1+j$



if $y = -(n_1+x_1) - j(n_2+x_2)$
then x should be $x = -1-j$

if $y = (n_1+x_1) - j(n_2+x_2)$
then x should be $x = 1-j$

2. (3 marks) Find the CDF of R, that is, $F_R(r) = \Pr[R \le r]$. Hence find the PDF of R, that is $f_R(r)$, and the joint PDF of R and Θ, that is $f_{R,\Theta}(r,\theta)$.

$$F_{R^2}(r) = 1 - e^{-\lambda r}$$

$$\begin{aligned} F_R(r) &= \Pr[R \le r] \\ &= \Pr[\sqrt{R^2} \le r] \\ &= \Pr[R^2 \le r^2] \\ &= F_{R^2}(r^2) \\ &= 1 - e^{-\lambda r^2} \end{aligned}$$

$$\therefore F_R(r) = \begin{cases} 1 - e^{-\lambda r^2} & \text{for } r \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} f_R(r) &= \frac{d}{dr}[F_R(r)] \\ &= 2\lambda r e^{-\lambda r^2} \end{aligned}$$

$$\therefore f_R(r) = \begin{cases} 2\lambda r e^{-\lambda r^2} & \text{for } r \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$g(f) = 1 - e^{f}$$
$$f(r) = -\lambda r^2$$
$$g(f(r)) = 1 - e^{-\lambda r^2}$$
$$\frac{dg}{dr} = \frac{dg}{df}\frac{df}{dr}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \le \theta < 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

R and Θ are independent,

$$\begin{aligned} \therefore f_{R,\Theta}(r,\theta) &= f_R(r) \cdot f_\Theta(\theta) \\ &= \frac{\lambda r}{\pi} e^{-\lambda r^2} \end{aligned}$$

$$\therefore f_{R,\Theta}(r,\theta) = \begin{cases} \frac{\lambda r}{\pi} e^{-\lambda r^2} & \text{for } r \ge 0,\ 0 \le \theta < 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

3. (3 marks) Compute $E[N_1]$, $E[N_1^2]$ and $E[N_1 N_2]$. The identities $\cos^2(\alpha) = \frac{1}{2}(1+\cos(2\alpha))$ and $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ may be useful.

$$E[N_1] = E\left[R\cos(\Theta)\right]$$
$$= \int_0^\infty \int_0^{2\pi} r\cos(\theta)\,\frac{\lambda r}{\pi} e^{-\lambda r^2}\,d\theta\,dr$$
$$= \int_0^\infty \frac{\lambda r^2}{\pi}\sin(\theta)\, e^{-\lambda r^2}\Big|_0^{2\pi} dr$$
$$= \int_0^\infty 0\,dr$$
$$= 0$$

$$E[N_1 N_2] = E\left[R^2\cos(\Theta)\sin(\Theta)\right]$$
$$= E\left[\frac{R^2}{2}\, 2\cos(\Theta)\sin(\Theta)\right]$$
$$= E\left[\frac{R^2}{2}\sin(2\Theta)\right]$$
$$= \int_0^\infty \int_0^{2\pi} \frac{r^2}{2}\sin(2\theta)\,\frac{\lambda r}{\pi} e^{-\lambda r^2}\,d\theta\,dr$$
$$= \int_0^\infty \frac{-\lambda r^3}{4\pi}\cos(2\theta)\, e^{-\lambda r^2}\Big|_0^{2\pi} dr$$
$$= \frac{\lambda}{4\pi}\int_0^\infty -r^3 e^{-\lambda r^2} + r^3 e^{-\lambda r^2}\,dr$$
$$= \frac{\lambda}{4\pi}\int_0^\infty 0\,dr$$
$$= 0$$

$$E[N_1^2] = E\left[R^2\cos^2(\Theta)\right]$$
$$= \int_0^\infty \int_0^{2\pi} \underbrace{r^2\cos^2(\theta)\,\frac{\lambda r}{\pi} e^{-\lambda r^2}}_{\left[\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right]\frac{\lambda r^3}{\pi} e^{-\lambda r^2}}\,d\theta\,dr$$
$$= \int_0^\infty \int_0^{2\pi} \frac{\lambda r^3}{2\pi} e^{-\lambda r^2} + \cos(2\theta)\,\frac{\lambda r^3}{2\pi} e^{-\lambda r^2}\,d\theta\,dr$$
$$= \int_0^\infty \frac{\lambda r^3\theta}{2\pi} e^{-\lambda r^2} + \sin(2\theta)\,\frac{\lambda r^3}{2\pi} e^{-\lambda r^2}\Big|_0^{2\pi} dr$$
$$= \int_0^\infty \lambda r^3 e^{-\lambda r^2}\,dr \qquad u = r^2 \quad du = 2r\,dr \qquad u(\infty) = \infty \quad u(0) = 0$$
$$= \frac{\lambda}{2}\int_0^\infty r^2 e^{-\lambda r^2}\, 2r\,dr$$
$$= \frac{\lambda}{2}\int_0^\infty u\, e^{-\lambda u}\,du$$

$\frac{-1}{\lambda}e^{-\lambda u}$ $\quad u$

$e^{-\lambda u}$ $\quad 1$

$$= \frac{\lambda}{2}\left[\frac{-u}{\lambda}e^{-\lambda u} + \int_0^\infty \frac{1}{\lambda}e^{-\lambda u}\,dr\right]$$
$$= \frac{\lambda}{2}\left[\frac{-u}{\lambda}e^{-\lambda u} - \frac{1}{\lambda^2}e^{-\lambda u}\right]_0^\infty$$
$$= \frac{\lambda}{2}\left(\frac{1}{\lambda^2}\right)$$
$$= \frac{1}{2\lambda}$$

4. (3 marks) Using the CDF of R^2, express R^2 in terms of a uniformally distributed random variable U. That is, find a function f such that $f(U)$ has the same distribution as R^2. Implement the function in MATLAB and call it 30,000 times with a $U(0,1)$ argument (generated by the **rand** command) and a λ argument. Plot the sample CDF for $\lambda = 1$, using **cdfplot** on the same axes as the exact CDF (the command **hold on** may be useful).

$R^2 \sim$ exponential(λ)

$$F_{R^2}(r^2) = \begin{cases} 0 & r<0 \\ 1-e^{-\lambda r^2} & r \geq 0 \end{cases}$$

$U \sim$ Uniform$(0,1)$

$$F_U(u) = \begin{cases} 0 & u<0 \\ u & 0 \leq u < 1 \\ 1 & u \geq 1 \end{cases}$$

Transform U to R^2

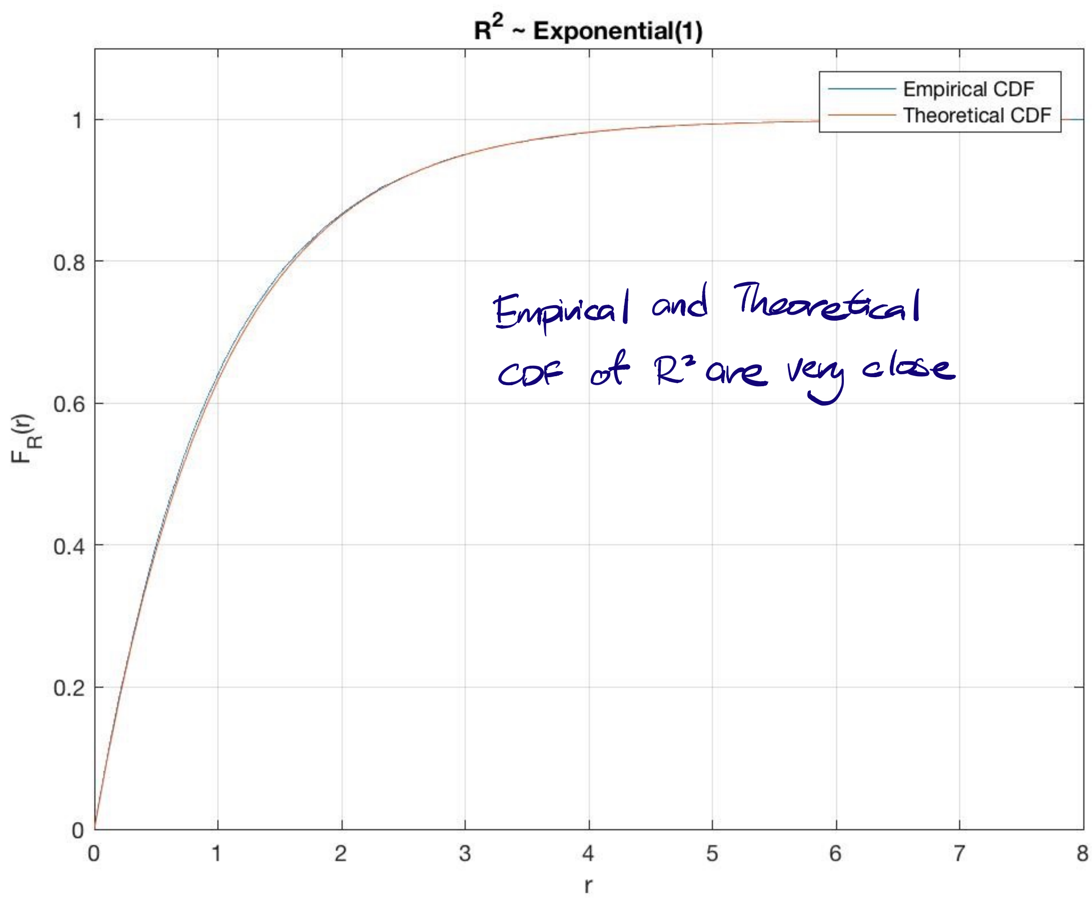
$$u = 1-e^{-\lambda r^2}$$
$$e^{-\lambda r^2} = 1-u$$
$$-\lambda r^2 = \ln(1-u)$$
$$r^2 = \frac{-1}{\lambda}\ln(1-u)$$
$$f(u) = \frac{-1}{\lambda}\ln(1-u)$$

Express CDF of R^2 in terms of U,
for $0 < u \leq 1$

$$\begin{aligned} F_U(u) &= P[U \leq u] \\ &= P[1-e^{-\lambda R^2} \leq u] \\ &= P[e^{-\lambda R^2} \geq 1-u] \\ &= P[-\lambda R^2 \geq \ln(1-u)] \\ &= P[R^2 \leq -\tfrac{1}{\lambda}\ln(1-u)] \\ &= F_{R^2}(-\tfrac{1}{\lambda}\ln(1-u)) \end{aligned}$$

$$F_U(u) = F_{R_2}(f(u))$$

$$F_{R_2}(f(u)) = \begin{cases} 0 & u<0 \\ \frac{-1}{\lambda}\ln(1-u) & 0 \leq u < 1 \\ \text{undefined} & u \geq 1 \end{cases}$$



Empirical and Theoretical CDF of R^2 are very close

5. Assume that $\lambda = 1$. Write a function that calls the function from the previous question, to obtain a value for R^2, calls **rand** to obtain a value for Θ, then computes and returns the resulting values of N_1, N_2 and R. Call this function 30,000 times and build up three arrays for the corresponding values. Using these:

   (a) (2 marks) Produce a scatterplot of the first 1,000 samples of N_1 and N_2 (for example if n1 and n2 are the arrays, use **scatter(n1(1:1000), n2(1:1000)**,'.'))

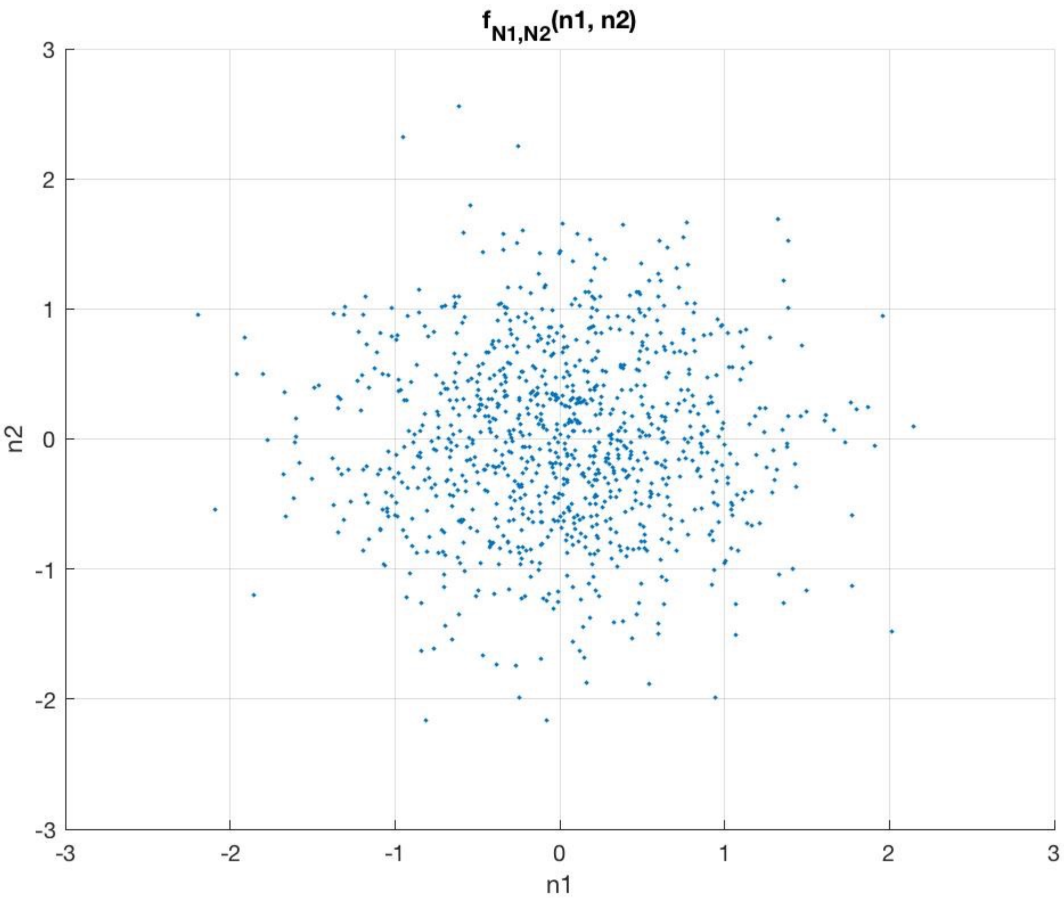
   (b) (2 marks) Suppose that the following assumption holds:

   Assumption 1: the input is deterministic and equals $x = -1 + j$ on every use of the channel

   Compute empirical values for the following probabilities:

   i. $P[x_1$ is decoded correctly$]$
   ii. $P[x_2$ is decoded correctly$]$
   iii. $P[x_1$ and x_2 are both decoded correctly$]$

5a



5bi $P[x_1$ is decoded correctly$] \approx 0.92077$

5bii $P[x_2$ is decoded correctly$] \approx 0.92433$

5biii $P[x_1$ and x_2 is decoded correctly$] \approx 0.8481$

6. (2 marks) Again assume that $\lambda = 1$. From your results in answering part (b) of the previous question, can you confidently conclude anything about whether N_1 and N_2 are independent?

P[x_1 is decoded correctly] · P[x_2 is decoded correctly]

$\approx$ P[x_1 and x_2 is decoded correctly]

$0.92 \cdot 0.92 \approx 0.847$

$\therefore$ The event that x_1 is decoded correctly is indepedent to the event that x_2 is decoded correctly.

Since x_1 and x_2 are deterministic, the added noise n_1 and n_2 are therefore independent to each other.

7. (4 marks) Again assume that $\lambda = 1$. It can be shown that then $N_1, N_2 \sim \mathcal{N}(0, 0.5)$ and that N_1 and N_2 are independent[3]. Using this result (or otherwise), theoretically derive the probabilities from Q5b and compare them to your simulations (the **normcdf** command in MATLAB may be useful).

$P[Y_1 < 0 \mid X_1 = -1]$

$= P[Y_1 < 0]$

$= P[-1 + N_1 < 0]$

$= P[N_1 < 1]$

$= F_{N_1}(1)$

$= 0.921$

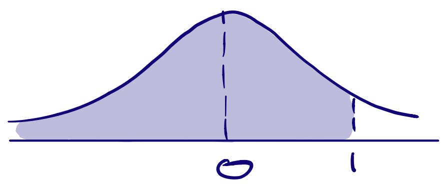
X is deterministic

$\therefore P[X_1 = -1] = 1$

$x_1 = -1$

$\therefore Y_1 = -1 + N_1$

where $N_1 \sim N(0, 0.5)$



$P[Y_2 \geq 0 \mid X_2 = 1]$

$= P[Y_2 \geq 0]$

$= P[1 + N_2 \geq 0]$

$= P[N_2 \geq -1]$

$= 1 - P[N_2 < -1]$

$= 1 - F_{N_2}(-1)$

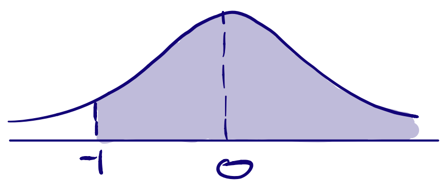
$= 0.921$

$\therefore P[X_2 = 1] = 1$

$x_2 = 1$

$\therefore Y_2 = 1 + N_2$

where $N_2 \sim N(0, 0.5)$

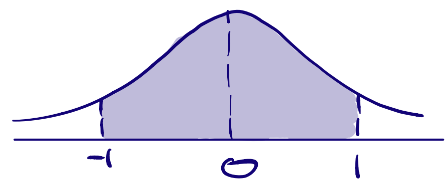


$$P[(Y_1<0)\cap(Y_2\geq 0)\,|\,(x_1=-1)\cap(x_2=1)]$$

$$=P[Y_1<0\cap Y_2\geq 0]$$

$$=P[N_1<1\cap N_2\geq -1]$$

N_1 and N_2 are both $N\sim(0,0.5)$



$$=F_N(1)-F_N(-1)$$

$$=0.8427$$

	Empirical Probability	Theoretical Probability
$P[x_1$ decoded correctly$]$	0.92077	0.92135
$P[x_2$ decoded correctly$]$	0.92433	0.92135
$P[x_1$ and x_2 decoded correctly$]$	0.8481	0.8427

8. (a) (4 marks) Again assume that $\lambda = 1$. Replace Assumption 1 by an assumption that is more relevant to digital communications, namely

> **Assumption 2**: the input X is a discrete random variable with outcomes $x \in \{1+j, 1-j, -1-j, -1+j\}$ with equal probabilities; the inputs are independent for different channel uses.

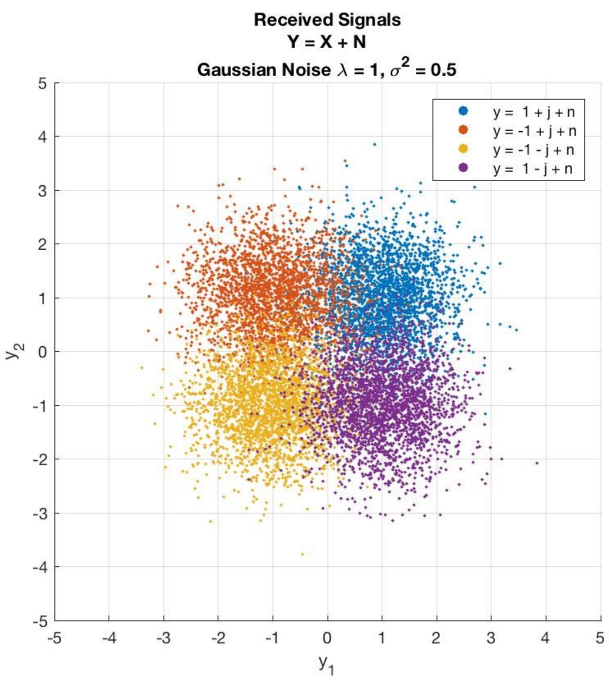
Simulate a sequence of 10,000 inputs, corrupted by noise N, resulting in a sequence of 10,000 outputs Y. Visualize your simulation via a scatterplot of the y-values, where each point y has one of 4 different colours **depending on the value of x that it originates from**.

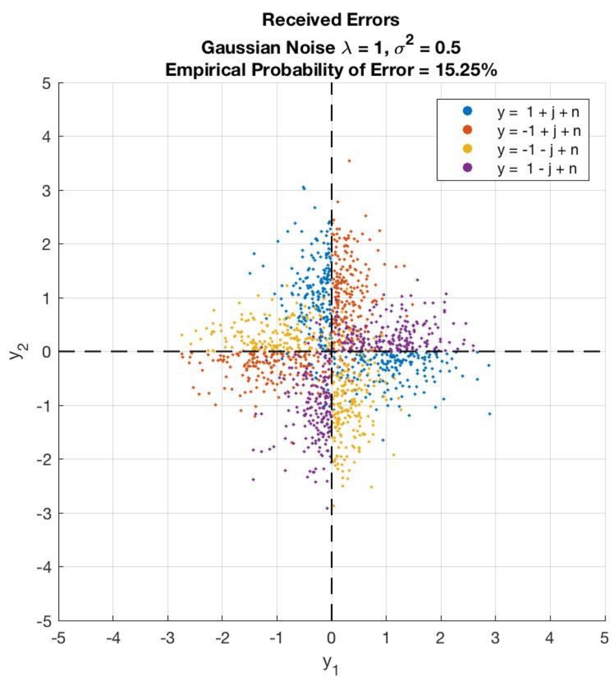
(b) (2 marks) Plot your scatterplot from a) again (keep all colours), but this time without all correctly detected y points.

(c) (2 marks) Compute an empirical value of the detector error probability which is given by

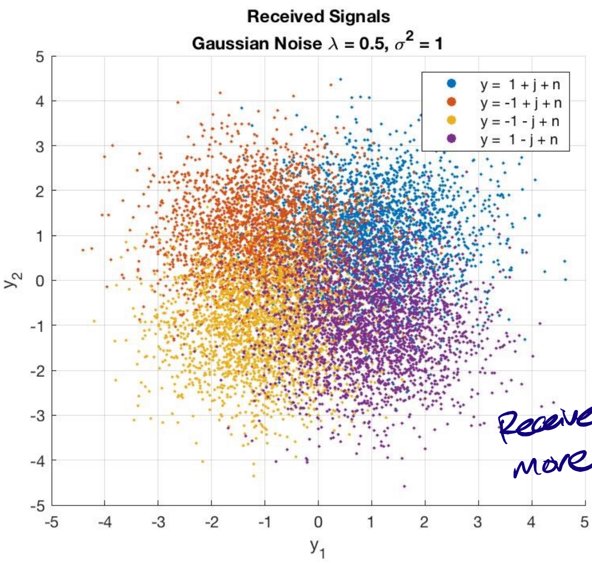
$$1 - P[x_1 \text{ and } x_2 \text{ are both decoded correctly}].$$

(d) (3 marks) Repeat parts a), b) and c) for the values $\lambda = 0.5$ and $\lambda = 5$. Discuss your results, in particular compare the detector error probabilities of the three cases $\lambda = 1$, $\lambda = 0.5$ and $\lambda = 5$ (include the theoretical derivation for $\lambda = 0.5$ and also for $\lambda = 5$); interpret the differences between these three cases.

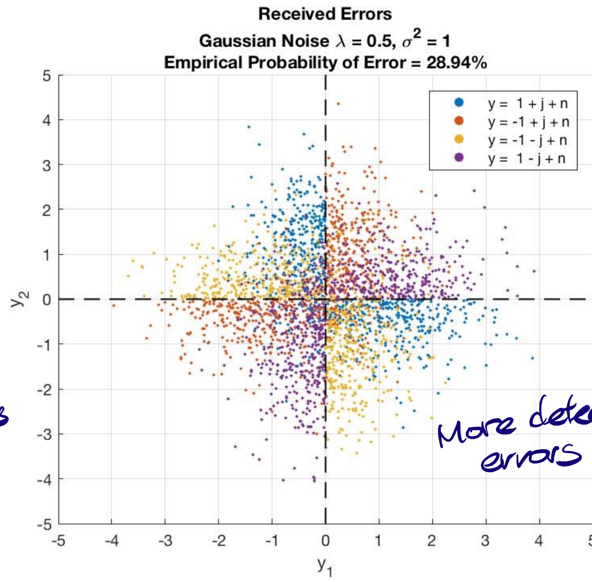




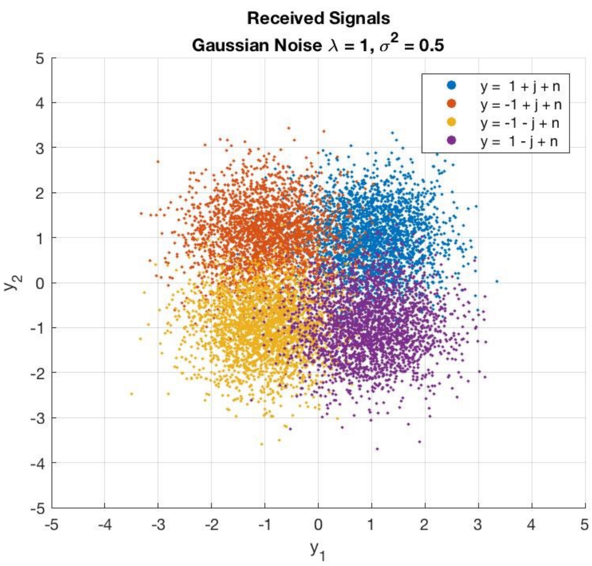
8d

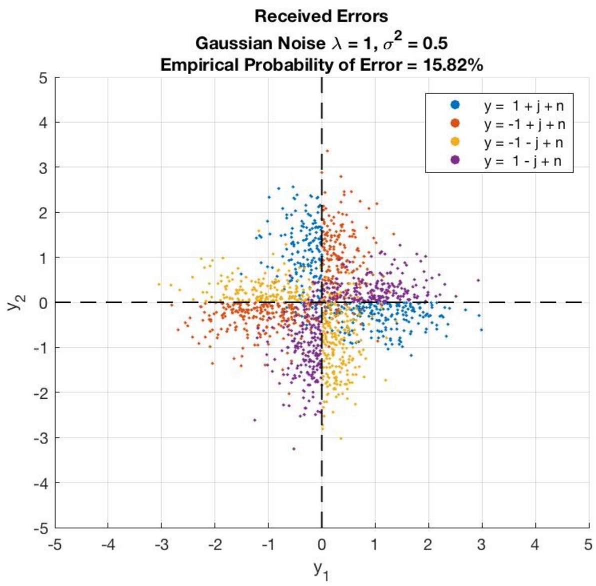


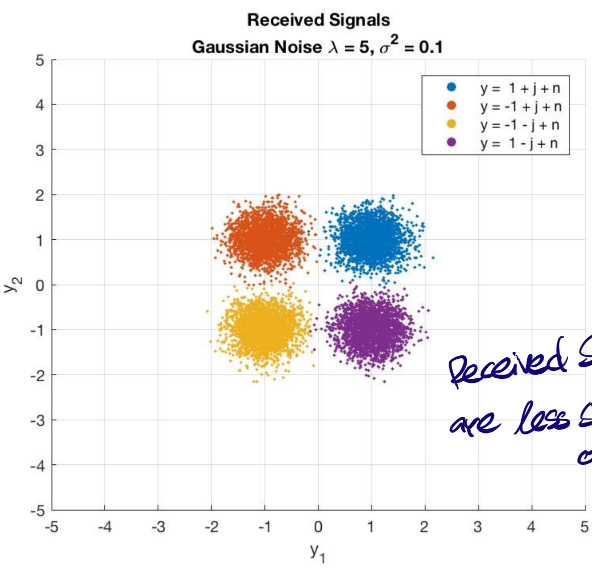
Received Signals more spread out



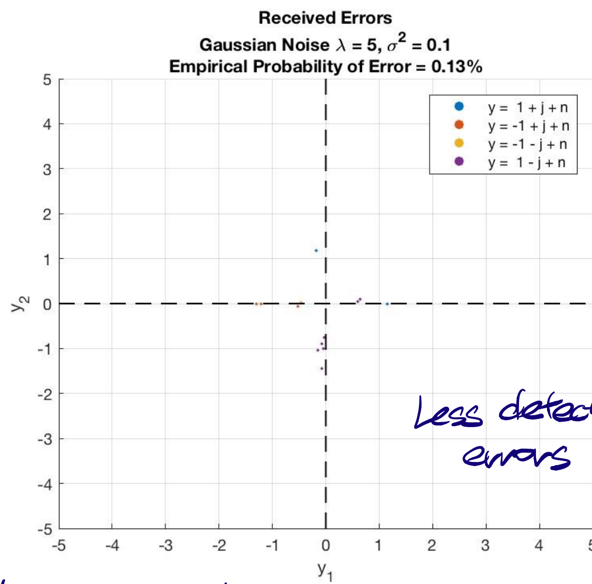
More detector errors







Received Signals are less spread out



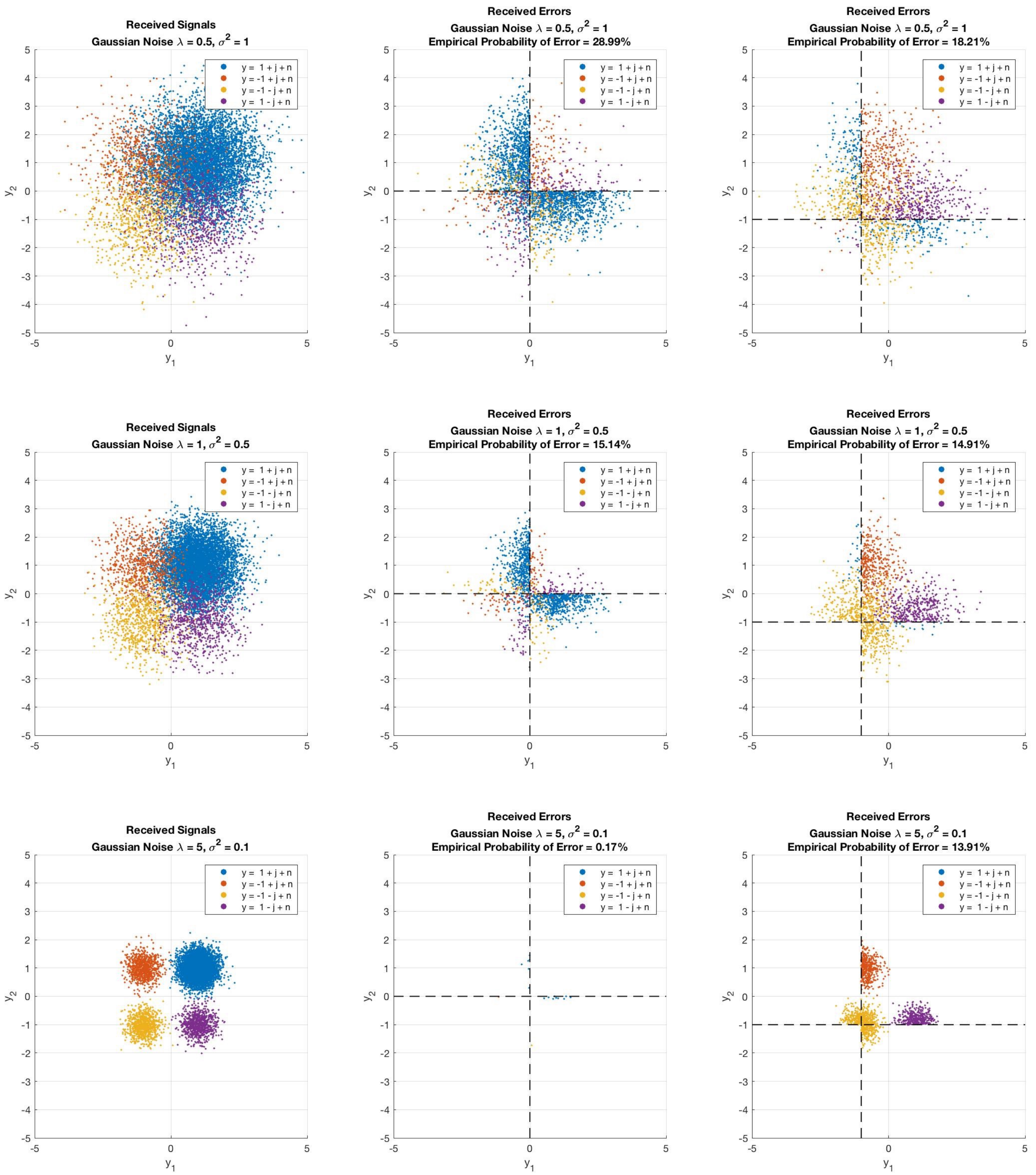
Less detector errors

Please see Matlab code under "observations" for discussions and Matlab output for tabulated results.

9. (3 marks) Repeat Q8 but this time with Assumption 2 replaced by

Assumption 3: the input X is a discrete random variable with outcomes $x \in \{1+j, 1-j, -1-j, -1+j\}$ with probabilities $\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$, respectively; the inputs are independent for different channel uses.

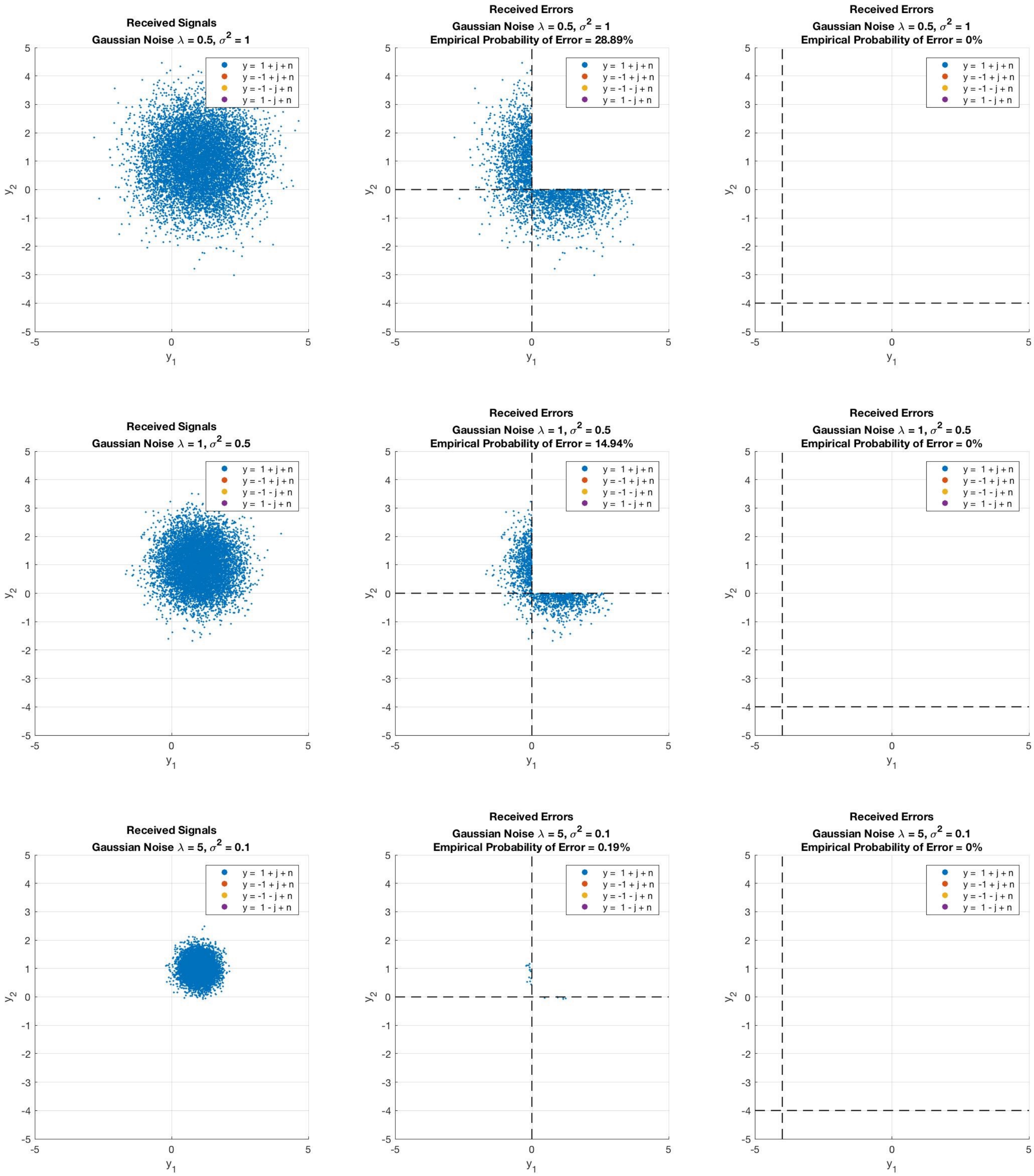
Discuss your results, in particular compare the detector error probability under Assumption 3 with the detector error probability under Assumption 2. Can you think of ways to change the decision rule to make the detector error probability under Assumption 3 smaller?



Please see Matlab code under "observations" for discussions and Matlab output for tabulated results.

10. (2 marks) Repeat Q9 but this time with Assumption 3 replaced by

Assumption 4: the input X is a discrete random variable with outcomes $x \in \{1+j, 1-j, -1-j, -1+j\}$ with probabilities $1, 0, 0, 0$, respectively.



Using the follow decision rule will ensure zero detector errors under assumption 4,

↳ Decodes x_1 as 1 if channel output $Y_1 \geq -4$

↳ Decodes x_1 as -1 if channel output $Y_1 < -4$

↳ Decodes x_2 as 1 if channel output $Y_2 \geq -4$

↳ Decodes x_2 as -1 if channel output $Y_2 < -4$