

# Complexes from complexes

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1 de Rham complexes

2 Complexes from complexes

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# Motivation

decomposition of flows:  $\Omega \subset \mathbb{R}^3$ , for any  $u \in [L^2(\Omega)]^3$ :

$$u = \operatorname{grad} \phi + \operatorname{curl} \psi + w$$

- any flow = potential part (rotation-free)  $\oplus_{\perp}$  rotational part (source-free)  $\oplus_{\perp}$  harmonic,
- $w$ : both rotation-free and source-free (*harmonic forms*),  
dim of such functions = Betti number, reflecting topology of  $\Omega$ ,
- rough idea of cohomology theories: using  $w$  to study topology.

Key of these results: algebraic structures, complexes!

# de Rham complex

de Rham complex (3D version)

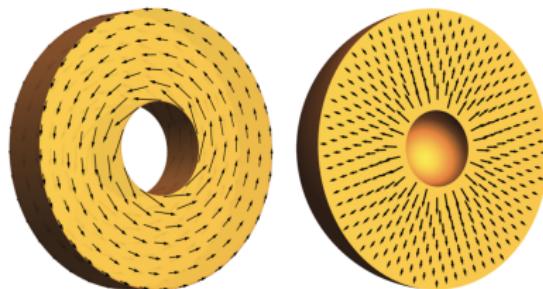
$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}.$$

- complex property:  $d^k \circ d^{k-1} = 0, \Rightarrow \mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k),$   
 $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \mathcal{N}(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \mathcal{N}(\text{div})$
- cohomology:  $\mathcal{H}^k := \mathcal{N}(d^k)/\mathcal{R}(d^{k-1}),$   
 $\mathcal{H}^0 := \mathcal{N}(\text{grad}), \quad \mathcal{H}^1 := \mathcal{N}(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathcal{H}^2 := \mathcal{N}(\text{div})/\mathcal{R}(\text{curl})$
- exactness (contractible domains):  $\mathcal{N}(d^k) = \mathcal{R}(d^{k-1}),$  i.e.,  $d^k u = 0 \Rightarrow u = d^{k-1} v$   
 $\text{curl } u = 0 \Rightarrow u = \text{grad } \phi, \quad \text{div } v = 0 \Rightarrow v = \text{curl } \psi.$

## de Rham complex and topology:

dimension of  $\mathcal{H}^k$  = number of “ $k$ -dimensional holes” (c.f. de Rham theorem)



Examples where  $\dim \mathcal{H}^1 = 1$  and  $\dim \mathcal{H}^2 = 1$ , respectively.

Left: curl-free field which is not grad, Right: div-free field with is not curl.

(figure from *Finite element exterior calculus*, D.N.Arnold, SIAM 2008. )

## From complexes to PDEs

Formal adjoint of operators:

$$\text{grad}^* = -\text{div}, \quad \text{curl}^* = \text{curl}, \quad \text{div}^* = -\text{grad}.$$

$$\int_{\Omega} \text{grad } u \cdot v = - \int_{\Omega} u \cdot \text{div } v + \text{bound. term}, \quad \int_{\Omega} \text{curl } u \cdot v = \int_{\Omega} u \cdot \text{curl } v + \text{bound. term}$$

$$(\text{grad } u, v) = (u, -\text{div } v), \quad (\text{curl } u, v) = (u, \text{curl } v)$$

Formal adjoint of de Rham complex:

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{-\text{div}} C^\infty(\Omega; \mathbb{R}^3) \xleftarrow{\text{curl}} C^\infty(\Omega; \mathbb{R}^3) \xleftarrow{-\text{grad}} C^\infty(\Omega) \longleftarrow 0.$$

$$d_2^* := -\text{div}, \quad d_1^* := \text{curl}, \quad d_0^* := -\text{grad}.$$

connections to PDEs: Hodge–Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^*d^k)u = f.$$

connections to PDEs: Hodge–Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^*d^k)u = f.$$

$$0 \xleftarrow{\quad} C^\infty(\Omega) \xrightleftharpoons[-\operatorname{div}]{} C^\infty(\Omega; \mathbb{R}^3) \qquad \qquad C^\infty(\Omega; \mathbb{R}^3) \qquad \qquad C^\infty(\Omega) \qquad 0.$$

Hodge–Laplacian problem:

$$-\operatorname{div} \operatorname{grad} u = f.$$

Poisson equation.

connections to PDEs: Hodge–Laplacian problems.

$$(d^{k-1} d_{k-1}^* + d_k^* d^k) u = f.$$

$$0 \quad C^\infty(\Omega) \xrightleftharpoons[-\operatorname{div}]{\operatorname{grad}} \textcolor{red}{C^\infty(\Omega; \mathbb{R}^3)} \xrightleftharpoons[\operatorname{curl}]{\operatorname{curl}} C^\infty(\Omega; \mathbb{R}^3) \quad C^\infty(\Omega) \quad 0.$$

Hodge–Laplacian problem:

$$-\operatorname{grad} \operatorname{div} v + \operatorname{curl} \operatorname{curl} v = f.$$

Maxwell equations.

connections to PDEs: Hodge–Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^*d^k)u = f.$$

$$0 \quad C^\infty(\Omega) \quad C^\infty(\Omega; \mathbb{R}^3) \xrightleftharpoons[\text{curl}]{\text{curl}} \textcolor{red}{C^\infty(\Omega; \mathbb{R}^3)} \xrightleftharpoons[-\text{grad}]{\text{div}} C^\infty(\Omega) \quad 0.$$

Hodge–Laplacian problem:

$$\text{curl curl } v - \text{grad div } v = f.$$

Maxwell equations.

connections to PDEs: Hodge–Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^*d^k)u = f.$$

$$0 \quad C^\infty(\Omega) \quad C^\infty(\Omega; \mathbb{R}^3) \quad C^\infty(\Omega; \mathbb{R}^3) \xrightarrow[\text{grad}]{\text{div}} \textcolor{red}{C^\infty(\Omega)} \xleftarrow{\quad} 0.$$

Hodge–Laplacian problem:

$$-\operatorname{div} \operatorname{grad} u = f.$$

Poisson equation.

## Analytic properties

$$0 \longrightarrow H(\text{grad}) \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0$$

$$H(d) := \{u \in L^2 : du \in L^2\}.$$

- Hodge decomposition:  $L^2 = \mathcal{R}(d^{k-1}) \oplus \mathcal{R}(d_k^*) \oplus \mathcal{H}^k$ ,

$$L^2 = \mathcal{R}(\text{div}) \oplus \mathcal{G},$$

$$[L^2]^3 = \mathcal{R}(\text{grad}) \oplus_{\perp} \mathcal{R}(\text{div}) \oplus \mathcal{H}.$$

- Poincaré inequalities:  $\|u\| \leq C \|d^k u\|, \forall u \perp \mathcal{N}(d^k)$ ,

$$\|u\| \leq \|\text{grad } u\|, \quad u \perp \mathcal{N}(\text{grad}),$$

$$\|u\| \leq \|\text{curl } u\|, \quad u \perp \mathcal{N}(\text{curl}),$$

$$\|u\| \leq \|\text{div } u\|, \quad u \perp \mathcal{N}(\text{div}).$$

- other properties: regular decomposition, existence of regular potentials, compactness properties, div-curl lemma, etc.

## Why are complexes useful for PDEs?

- well-posedness: analytic properties + standard variational argument,
- discretization: structure-preservation!

Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.

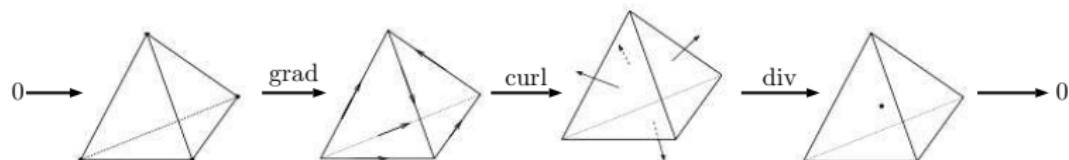
– James Clerk Maxwell, 1873

discrete differential forms (Bossavit 1988, Hiptmair 1999, ...),  
Finite Element Exterior Calculus (Arnold, Falk, Winther 2006, ...)

discrete spaces fit into de Rham complexes.

# Discretization

finite element de Rham complex in 3D



$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\text{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0.$$

Raviart-Thomas (1977), Nédélec (1980) in numerical analysis, Bossavit (1988), Hiptmair (1999) for differential forms, Whitney (1957) for studying topology.

# Periodic Table of the Finite Elements



Arnold, Logg 2014, SIAM news

1 de Rham complexes

2 Complexes from complexes

## Elasticity: deformation and mechanics of solids

elasticity equation:

$$-\operatorname{div}(A \operatorname{def} u) = f.$$

$u$

displacement (vector),

$e := \operatorname{def} u := 1/2(\nabla u + \nabla u^T)$

strain (linearized deformation),

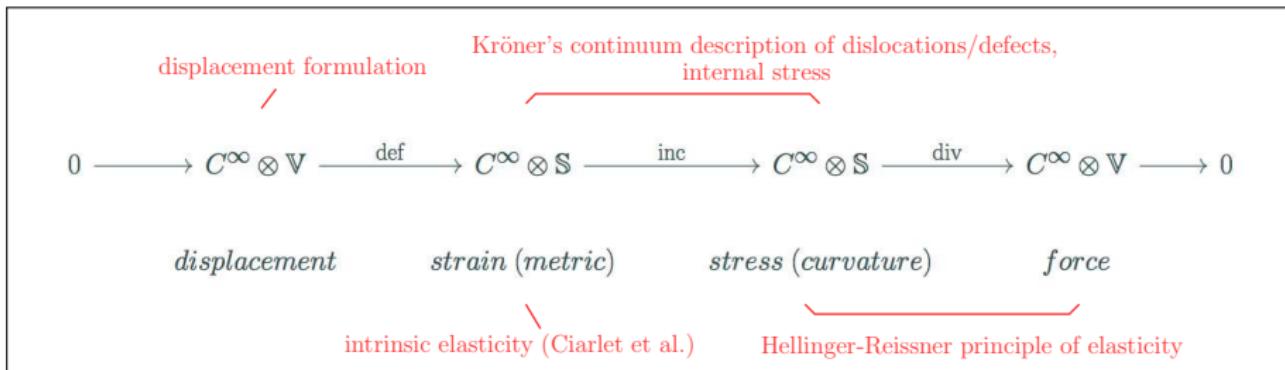
$\sigma := A \operatorname{def} u$

stress.

analogy to Poisson equation:

$$-\operatorname{div}(A \operatorname{grad} v) = g.$$

## A cohomological approach: elasticity complex



$\mathbb{V} := \mathbb{R}^3$  vectors,     $\mathbb{S} := \mathbb{R}_{\text{sym}}^{3 \times 3}$  symmetric matrices

$$\text{def } u := 1/2(\nabla u + \nabla u^T), \quad (\text{def } u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\text{inc } g := \nabla \times g \times \nabla, \quad (\text{inc } g)^{ij} = \epsilon^{ikl} \epsilon^{jst} \partial_k \partial_s g_{lt}.$$

$$\text{div } v := \nabla \times v, \quad (\text{div } v)_i = \partial^j v_{ij}.$$

$g$  metric  $\Rightarrow$  inc  $g$  linearized Einstein tensor ( $\simeq$  Riem  $\simeq$  Ric in 3D)

inc  $\circ$  def = 0: Saint-Venant compatibility  
 div  $\circ$  inc = 0: Bianchi identity

systematic study still missing, until  
*Complexes from complexes, Arnold, Hu 2020.*

Algebraic and analytic construction (Arnold, Hu 2020): derive elasticity from deRham

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^s \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{s-1} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & H^{s-2} \otimes \mathbb{M} \\
 & & S^0 := \text{mskw} & \nearrow & S^1 & \nearrow & S^2 := \text{vskw} \\
 0 & \longrightarrow & \cancel{H^{s-1} \otimes \mathbb{V}} & \xrightarrow{\text{grad}} & H^{s-2} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & H^{s-3} \otimes \mathbb{M} \\
 & & & & & & \xrightarrow{\text{div}} H^{s-4} \otimes \mathbb{V} \longrightarrow 0.
 \end{array}$$

$$S^1 u := u^T - \text{tr}(u)I.$$

key: Sobolev complexes ( $\forall s \in \mathbb{R}$ ), match indices, commuting diagrams, injectivity & surjective.

output: elasticity complex

$$0 \longrightarrow H^s \otimes \mathbb{V} \xrightarrow{\text{def}} H^{s-1} \otimes \mathbb{S} \xrightarrow{\text{curl}} \\ \xleftarrow{\text{curl}} H^{s-3} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{s-4} \otimes \mathbb{V} \longrightarrow 0. \quad (1)$$

inspired by Bernstein-Gelfand-Gelfand (BGG) resolution

(c.f., Eastwood 2000, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006).

## Theorem

Cohomology of (1) is isomorphic to the smooth de Rham cohomology:

$$\mathcal{N}(\mathcal{D}^i) = \mathcal{R}(\mathcal{D}^{i-1}) \oplus \mathcal{H}_\infty^i, \quad \mathcal{H}_\infty^i \simeq \mathcal{H}_{\text{deRham}}^i \otimes (\mathbb{V} \times \mathbb{V})$$

Proof: Homological algebra + results for de Rham by Costabel & McIntosh.

Corollary: finite dimensional cohomology  $\implies$  operators have closed range.

Consequences:

- analytic results (Poincaré inequality, Hodge decomposition, etc.)  
e.g., Korn inequality  $\|u\|_1 \leq C \|\operatorname{def} u\|$ ,  $u \perp \mathcal{N}(\operatorname{def})$ .
- explicit representatives of elasticity cohomology.

## Systematic construction: generating new complexes from existing ones

- input:  $(Z^\bullet, D^\bullet)$ ,  $(\tilde{Z}^\bullet, \tilde{D}^\bullet)$ , connecting maps  $S^i : \tilde{Z}^i \rightarrow Z^{i+1}$ , satisfying some conditions,

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z^0 & \xrightarrow{D^0} & Z^1 & \xrightarrow{D^1} & \cdots \xrightarrow{D^{n-1}} Z^n \longrightarrow 0 \\ & & S^0 \nearrow & & S^1 \nearrow & & S^{n-1} \nearrow \\ 0 & \longrightarrow & \tilde{Z}^0 & \xrightarrow{\tilde{D}^0} & \tilde{Z}^1 & \xrightarrow{\tilde{D}^1} & \cdots \xrightarrow{\tilde{D}^{n-1}} \tilde{Z}^n \longrightarrow 0 \end{array}$$

- output:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \text{coker}(S^{J-2}) & \xrightarrow{D^{J-1}} & \text{coker}(S^{J-1}) & \xrightarrow{D^J} & & \\ & & & & (S^J)^{-1} \searrow & & & \\ & & & & & & & \\ & & & & \swarrow \tilde{D}^J & & \xrightarrow{\tilde{D}^{J+1}} & \mathcal{N}(S^{J+2}) \xrightarrow{\tilde{D}^{J+2}} \cdots \end{array}$$

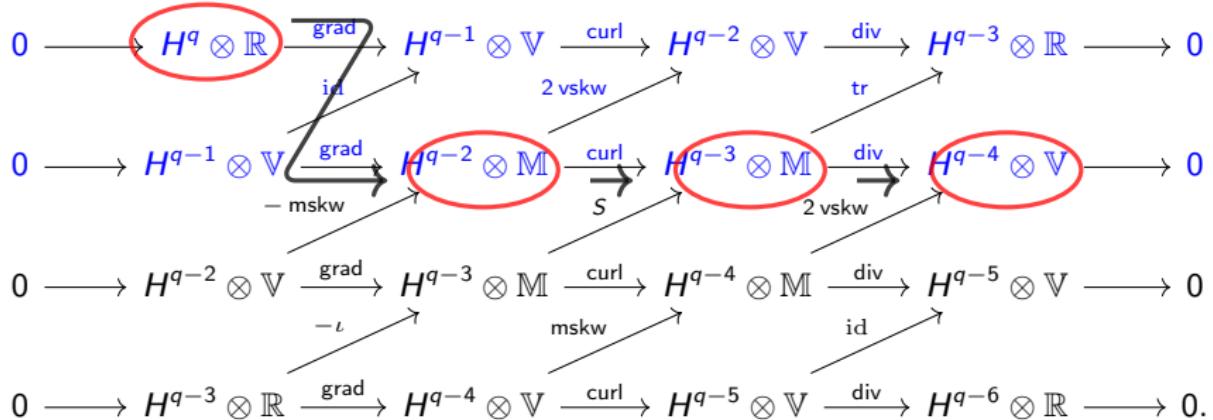
- conclusion:

$$\dim \mathcal{H}^i(\Upsilon^\bullet, \mathcal{D}^\bullet) \leq \dim \mathcal{H}^i(Z^\bullet, D^\bullet) + \dim \mathcal{H}^i(\tilde{Z}^\bullet, \tilde{D}^\bullet), \quad \forall i = 0, 1, \dots, n$$

Equality holds if and only if  $S^i$  induces the zero maps on cohomology, i.e.,  $S^i \mathcal{N}(\tilde{D}^i) \subset \mathcal{R}(D^i)$ .

More 3D examples:

(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)



Hessian complex:

$$0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\text{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\text{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

More 3D examples:

(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^q \otimes \mathbb{R} & \xrightarrow{\text{grad}} & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{curl}} & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{div}} & H^{q-3} \otimes \mathbb{R} & \longrightarrow 0 \\
 & & & \nearrow \text{id} & & \nearrow 2 \text{ vskw} & & \nearrow \text{tr} & & \\
 0 & \longrightarrow & \textcolor{red}{H^{q-1} \otimes \mathbb{V}} & \xrightarrow{\text{grad}} & \textcolor{blue}{H^{q-2} \otimes \mathbb{M}} & \xrightarrow{\text{curl}} & \textcolor{black}{H^{q-3} \otimes \mathbb{M}} & \xrightarrow{\text{div}} & \textcolor{blue}{H^{q-4} \otimes \mathbb{V}} & \longrightarrow 0 \\
 & & \searrow -\text{mskw} & & \nearrow S & & \nearrow 2 \text{ vskw} & & & \\
 0 & \longrightarrow & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & \textcolor{red}{H^{q-4} \otimes \mathbb{M}} & \xrightarrow{\text{div}} & \textcolor{red}{H^{q-5} \otimes \mathbb{V}} & \longrightarrow 0 \\
 & & \nearrow -\iota & & \nearrow \text{mskw} & & & \nearrow \text{id} & & \\
 0 & \longrightarrow & H^{q-3} \otimes \mathbb{R} & \xrightarrow{\text{grad}} & H^{q-4} \otimes \mathbb{V} & \xrightarrow{\text{curl}} & H^{q-5} \otimes \mathbb{V} & \xrightarrow{\text{div}} & H^{q-6} \otimes \mathbb{R} & \longrightarrow 0.
 \end{array}$$

elasticity complex:

$$0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.$$

elasticity, defects, metric, curvature

More 3D examples:

(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^q \otimes \mathbb{R} & \xrightarrow{\text{grad}} & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{curl}} & H^{q-2} \otimes \mathbb{V} & \xrightarrow{\text{div}} & H^{q-3} \otimes \mathbb{R} & \longrightarrow 0 \\
 & & \text{id} \nearrow & & 2 \text{ vskw} \nearrow & & \text{tr} \nearrow & & & \\
 0 & \longrightarrow & H^{q-1} \otimes \mathbb{V} & \xrightarrow{\text{grad}} & H^{q-2} \otimes \mathbb{M} & \xrightarrow{\text{curl}} & H^{q-3} \otimes \mathbb{M} & \xrightarrow{\text{div}} & H^{q-4} \otimes \mathbb{V} & \longrightarrow 0 \\
 & & -\text{mskw} \nearrow & & S \nearrow & & 2 \text{ vskw} \nearrow & & & \\
 0 & \longrightarrow & \color{red}{H^{q-2} \otimes \mathbb{V}} & \xrightarrow{\text{grad}} & \color{red}{H^{q-3} \otimes \mathbb{M}} & \xrightarrow{\text{curl}} & \color{red}{H^{q-4} \otimes \mathbb{M}} & \xrightarrow{\text{div}} & \color{blue}{H^{q-5} \otimes \mathbb{V}} & \longrightarrow 0 \\
 & & -\iota \nearrow & & \text{mskw} \nearrow & & \text{id} \nearrow & & & \\
 0 & \longrightarrow & \color{blue}{H^{q-3} \otimes \mathbb{R}} & \xrightarrow{\text{grad}} & \color{blue}{H^{q-4} \otimes \mathbb{V}} & \xrightarrow{\text{curl}} & \color{blue}{H^{q-5} \otimes \mathbb{V}} & \xrightarrow{\text{div}} & \color{red}{H^{q-6} \otimes \mathbb{R}} & \longrightarrow 0.
 \end{array}$$

divdiv complex:

$$0 \longrightarrow H^{q-2} \otimes \mathbb{V} \xrightarrow{\text{dev grad}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{sym curl}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\text{div div}} H^{q-6} \otimes \mathbb{R} \longrightarrow 0.$$

plate theory, elasticity

## Another example

“conformal complex”

ker of dev def: conformal Killing v.f. Cotton-York: flatness in conformal geometry

$$0 \longrightarrow H^q(\Omega) \otimes \mathbb{V} \xrightarrow{\text{dev def}} H^{q-1}(\Omega) \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{cott}} H^{q-4}(\Omega) \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div}} H^{q-5}(\Omega) \otimes \mathbb{V} \longrightarrow 0$$

gravitational wave variable: transverse-traceless (TT) gauge  
stress like variable def  $u$  in NS  
(= symmetric, trace-free, div-free) (Gopalakrishnan, Lederer, Schöberl, 2019)

Hodge decomp.: York split, Einstein constraint eqns.

trace-free Korn inequality:

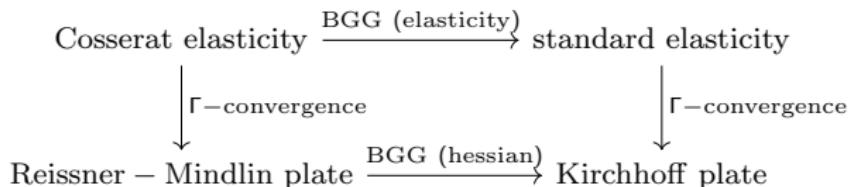
$$\|u\|_1 \leq C \|\text{dev def } u\|, \quad \forall u \in \mathcal{N}(\text{dev def}).$$

$\mathcal{N}(\text{dev def})$ : conformal Killing fields

## (potential) applications of complexes

- continuum with microstructure (Čap, Hu, in preparation):  
towards an Erlangen programme for continuum mechanics  
Cosserat continuum (Cosserat brothers 1909): introducing rotational degrees of freedom.

c.f. *E. Cartan's attempt at bridge-building between Einstein and the Cosserat - or how translational curvature became to be known as torsion*, Scholz 2019.



$\Gamma$ -convergence: Ciarlet 1997; Neff, Hong, Jeong 2010 etc.

- analysis: inequalities, compensated compactness etc. by homological algebra.
- numerical methods: fitting finite element (discrete) spaces in the BGG diagrams.
- structure-preserving numerical relativity: solving the Einstein equations.
- discrete differential geometry: discrete curvature and Regge calculus (Christiansen 2011)
- graph theory and data analysis: Hodge-Laplacian (c.f., Lim 2020), graph curvature.
- discrete differential forms in biology, discrete physics (lattice models).
- ...

## References

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