Finite element exterior calculus

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Michaelmas Term

What is the *finite element exterior calculus (FEEC)*?

An algebraic and topological/geometric approach for studying numerical PDEs.

Plan of lectures

1. Introduction
2. Continuous level: PDEs, analysis v.s. topology, geometry, algebra
3. Discrete level: numerical schemes, finite elements
4. Applications: fluid and solid mechanics, electromagnetism etc.

References

- Arnold, D.N.: Finite Element Exterior Calculus. SIAM (2018)
- Arnold, D.N., Falk, R.S., Winther, R.: Finite element exterior calculus, homological techniques, and applications. Acta Numerica 15, 1 (2006)
- Arnold, D.N., Falk, R.S., Winther, R.: Finite element exterior calculus: from Hodge theory to numerical stability. Bulletin of the American Mathematical Society 47(2), 281354 (2010)
- Arnold, D.N., Hu, K.: Complexes from complexes. Foundations of Computational Mathematics (2021)

+ research papers (for applications)

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slides: https://kaibohu.github.io/homepage/feec_slides.pdf
any comment/suggestion welcome!

Finite element methods: Poisson equation

$$
\begin{aligned}
-\Delta u &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial\Omega.
\end{aligned}
$$

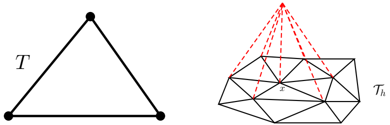
Weak form: find $u \in H_0^1(\Omega)$, such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega).$$

Galerkin methods: find $u_h \in V_h \subset H_0^1$, such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h, \quad \forall v_h \in V_h.$$

Lagrange finite element: $V_h := \{u_h \in C^0(\mathcal{T}_h) : u_h|_T \in \mathscr{P}_1(T)\} \cap H_0^1(\Omega)$.



Notation:

- 2D: grad / curl: scalar to vector, grad $u := (\partial_1 u, \partial_2 u)$, curl $u := (-\partial_2 u, \partial_1 u)$,
  rot / div: vector to scalar, rot $\boldsymbol{v} = -\partial_2 v_1 + \partial_1 v_2$, div $\boldsymbol{v} = \partial_1 v_1 + \partial_2 v_2$.
  $\operatorname{div} \circ \operatorname{curl} = 0$, $\operatorname{rot} \circ \operatorname{grad} = 0$
- 3D: grad $u := (\partial_1 u, \partial_2 u, \partial_3 u)$, curl $\boldsymbol{v} := (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)$,
  div $\boldsymbol{w} = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3$.
  $\operatorname{div} \circ \operatorname{curl} = 0$, $\operatorname{curl} \circ \operatorname{grad} = 0$
- $H(D) := \{u \in L^2 : Du \in L^2\}$, e.g., $H(\mathrm{curl})$, $H(\mathrm{div})$.

Example 1: vector Laplacian Ω: 2D bounded Lipschitz domain

$$\operatorname{curl} \operatorname{rot} \boldsymbol{u} - \operatorname{grad} \operatorname{div} \boldsymbol{u} = \boldsymbol{f}, \qquad \text{in } \Omega,$$
$$\boldsymbol{u} \cdot \boldsymbol{n}_{\partial\Omega} = 0, \quad \operatorname{rot} \boldsymbol{u}_{\partial\Omega} = 0, \qquad \text{on } \partial\Omega.$$

$\boldsymbol{n}_{\partial\Omega}$: (unit) normal vector, $\boldsymbol{u} \cdot \boldsymbol{n}_{\partial\Omega}$: normal component

Example: Maxwell equations

$$\boldsymbol{B}_t + \operatorname{curl} \boldsymbol{E} = 0, \quad \boldsymbol{D}_t - \operatorname{curl} \boldsymbol{H} = -\boldsymbol{j},$$
$$\operatorname{div} \boldsymbol{B} = 0, \quad \operatorname{div} \boldsymbol{D} = \rho,$$

with constitutive laws $\boldsymbol{D} = \epsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$.

Special case: stationary case ($\boldsymbol{B}_t = \boldsymbol{E}_t = 0$) with $\epsilon = \mu = 1$, and planar magnetic fields, i.e., $\boldsymbol{B} = (B_1(x,y), B_2(x,y), 0)$, curl $\boldsymbol{B} = (0, 0, \operatorname{rot}(B_1, B_2))$,

$$\operatorname{curl} \operatorname{rot} \boldsymbol{B} - \operatorname{grad} \operatorname{div} \boldsymbol{B} = \operatorname{curl} j,$$

with slight abuse of notation $\boldsymbol{B} = (B_1(x,y), B_2(x,y))$.

Identity:

$$\operatorname{curl}\operatorname{rot} - \operatorname{grad}\operatorname{div} = -\Delta.$$

Natural attempt for numerics: solving each component as a Poisson equation - find $\boldsymbol{u}_h \in \tilde{V}_h \times \tilde{V}_h$, s.t. for any $\boldsymbol{v}_h \in \tilde{V}_h$,

$$(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h).$$

$\tilde{V}_h$: V_h with different boundary conditions.

Numerical results

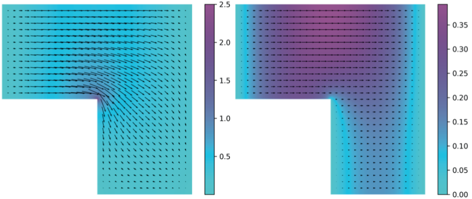


Figure 5.1. *Finite element solution to the Hodge Laplacian problem on an L-shaped domain (with $f = (1, 0)$). The left figure is calculated with a mixed method which is known to converge to the solution in L^2. The right figure is based on the primal formulation using 24,576 piecewise linear elements. The primal-based numerical solution entirely misses the dominant behavior at the reentrant corner and produces a wholly inaccurate solution.*

Arnold, *Finite element exterior calculus*, SIAM, Chapter 5

Cause of the problem: Lagrange element in H^1, cannot approximate solutions with singularity.

Example 2: nontrivial topology

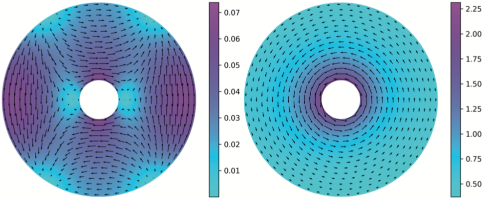


Figure 5.2. *Approximation of the Hodge Laplacian problem on an annulus ($f = (0, x)$). The true solution shown here on the left is an (accurate) approximation by a mixed method. The standard Galerkin solution using continuous piecewise linear vector fields, shown on the right, is totally different.*

Arnold, *Finite element exterior calculus*, SIAM, Chapter 5

Continuous problem does not have uniqueness ($\operatorname{curl}\operatorname{curl}\boldsymbol{u} - \operatorname{grad}\operatorname{div}\boldsymbol{u} = 0$ has nontrivial solution), but Lagrange finite element methods have a unique (spurious) solution.

Example 3: eigenvalue problem

$$\begin{aligned} \operatorname{curl} \operatorname{rot} \boldsymbol{u} &= \lambda \boldsymbol{u}, \qquad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n}_{\partial\Omega} &= 0, \qquad \text{on } \partial\Omega. \end{aligned}$$

Variational form: find $\lambda \in \mathbb{R}$, $\boldsymbol{u}_h \in \tilde{\boldsymbol{V}}_h$, such that for any $\boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h$,

$$(\operatorname{rot} \boldsymbol{u}_h, \operatorname{rot} \boldsymbol{v}_h) = \lambda(\boldsymbol{u}_h, \boldsymbol{v}_h).$$

Real eigenvectors when Ω is the unit square:

$$u(x, y) = \operatorname{curl}(\sin mx \sin ny),$$

where m, n are nonnegative integers, not both zero.
Eigenvalues: $\lambda = m^2 + n^2$, i.e., $\lambda = 1, 1, 2, 4, 4, 5, 5, 6, 7, 7, 10, 10, 13, 13...$

Numerical results

True solutions (eigenvalues): $\lambda = m^2 + n^2$, i.e., $\lambda = 1, 1, 2, 4, 4, 5, 5, 6, 7, 7, 10, 10, 13, 13...$

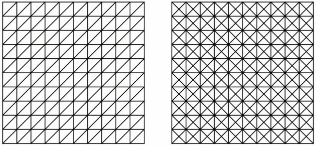


Figure 1.4. *Uniform meshes, diagonal and crisscross. The meshes used for computation were finer, with* 16 *times as many elements.*

Table 1.4. *First* 12 *Maxwell eigenvalues and Galerkin approximations of them.*

Exact	1	1	2	4	4	5	5	8	9	9	10	10
Diagonal mesh												
Lagrange	5.16	5.26	5.26	5.30	5.39	5.45	5.53	5.61	5.61	5.62	5.71	5.73
FEEC	1.00	1.00	2.00	4.00	4.00	5.00	5.00	8.01	8.98	8.99	9.99	9.99
Crisscross mesh												
Lagrange	1.00	1.00	2.00	4.00	4.00	5.00	5.00	6.00	8.01	9.01	9.01	10.02
FEEC	1.00	1.00	2.00	4.00	4.00	5.00	5.00	7.99	9.00	9.00	10.00	10.00

Arnold, *Finite element exterior calculus*, SIAM, Chapter 1

$$\cdots \longrightarrow V^{i-1} \xrightarrow{d^{i-1}} V^i \xrightarrow{d^i} V^{i+1} \longrightarrow \cdots$$

V^i: vector spaces, d^i: linear (or nonlinear) operators

- complex: $d^i V^i \subset V^{i+1}$, $d^{i+1} \circ d^i = 0$, $\forall i$, (implies $\mathcal{R}(d^{i-1}) \subset \mathcal{N}(d^i)$)
- exact: $\mathcal{N}(d^i) = \mathcal{R}(d^{i-1})$,
- cohomology (when d is linear): $\mathscr{H}^i := \mathcal{N}(d^i)/\mathcal{R}(d^{i-1})$.

Some useful observations:

- cochain map:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V^{i-1} & \xrightarrow{d^{i-1}} & V^i & \xrightarrow{d^i} & V^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow \Pi^{i-1} & & \downarrow \Pi^i & & \downarrow \Pi^{i+1} & & \\
\cdots & \longrightarrow & \tilde{V}^{i-1} & \xrightarrow{\tilde{d}^{i-1}} & \tilde{V}^i & \xrightarrow{\tilde{d}^i} & \tilde{V}^{i+1} & \longrightarrow & \cdots
\end{array}$$

  linear maps $\Pi^i : V^i \to \tilde{V}^i$ satisfying $\Pi^{i+1} d^i = \tilde{d}^i \Pi^i$ are called *cochain maps*. Cochain maps induce maps on cohomology.
- cochain isomorphism: bijective cochain maps. Cochain isomorphism induces bijections on cohomology. In this case $(V^\bullet, d^\bullet)$ and $(\tilde{V}^\bullet, \tilde{d}^\bullet)$ carry the same information.
- necessary condition for exactness when each V^i is finite dimensional:

$$\dim V^0 - \dim V^1 + \cdots + (-1)^N \dim V^N = 0.$$

  not a sufficient condition, useful for constructing finite elements.

Example: de Rham complex in $\mathbb{R}^3$

Notation: $\mathbb{V} := \mathbb{R}^n$, where n is the space dimension (in this case $n = 3$)

$$0 \longrightarrow C^\infty \xrightarrow{\text{grad}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{div}} C^\infty \longrightarrow 0.$$

Complex:

$\text{curl} \circ \text{grad} = 0$ (gradient fields have no rotation), $\text{div} \circ \text{curl} = 0$ (rotation fields have no source).

Exactness:

holds on domains with trivial topology, $\text{curl}\, \boldsymbol{u} = 0 \;\Rightarrow \boldsymbol{u} = \text{grad}\, \phi$ for some ϕ,
$\text{div}\, \boldsymbol{v} = 0 \;\Rightarrow \boldsymbol{v} = \text{curl}\, \boldsymbol{\psi}$ for some $\boldsymbol{\psi}$.

2D versions:

$$0 \longrightarrow C^\infty \xrightarrow{\text{grad}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{rot}} C^\infty \longrightarrow 0.$$

$$0 \longrightarrow C^\infty \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{div}} C^\infty \longrightarrow 0.$$

Elasticity complex

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0.$$

- spaces and operators
  $\mathbb{V} := \mathbb{R}^3$ vectors, $\mathbb{S} := \mathbb{R}^{3\times 3}_{\text{sym}}$ symmetric matrices
  $\text{def}\, u := \text{sym grad}\, u = 1/2(\nabla u + \nabla u^T)$, $(\text{def}\, u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i)$.
  $\text{inc}\, g := \text{curl} \circ \mathrm{T} \circ \text{curl}\, g = \nabla \times g \times \nabla$, $(\text{inc}\, g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k \partial_s g_{lt}$.
  $\text{div}\, v := \nabla \cdot v$, $(\text{div}\, v)_i = \partial^j u_{ij}$.
  g metric $\Rightarrow$ $\text{inc}\, g$ linearized Einstein tensor ($\backsimeq \mathrm{Riem} \backsimeq \mathrm{Ric}$ in 3D)
- complex property
  $\text{inc} \circ \text{def} = 0$: Saint-Venant compatibility
  $\text{div} \circ \text{inc} = 0$: Bianchi identity

different names: Calabi complex (works for Riemannian manifolds with constant sectional curvature), Kröner complex (modeling defects in continuum with inc), elasticity complex (most FEEC literature)

Example: conformal deformation complex

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{dev def}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{cot}} C^\infty \otimes (\mathbb{S} \cap \mathbb{T}) \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0.$$

- deviator $\operatorname{dev} u := u - \frac{1}{n}\operatorname{tr}(u)I$, where n is the space dimension. Thus dev def is symmetric and traceless. $\mathbb{T}$: traceless matrices ($u_{11} + u_{22} + u_{33} = 0$ for $u \in \mathbb{T}$).
  $\operatorname{cot} := \operatorname{curl} \circ \mathrm{T} \circ \operatorname{curl} \circ \mathrm{T} \circ \operatorname{curl}$: linearised Cotton operator (tensor), plays the role of curvature in conformal geometry.

Variables in gravitational waves: TT (transverse traceless) tensors, i.e., symmetric, traceless ($\mathbb{S} \cap \mathbb{T}$), divergence-free.
incompressible flows: velocity u, the strain-like tensor $\sigma := \operatorname{def} u$ is symmetric and traceless (since $\operatorname{tr} \operatorname{def} u = \operatorname{div} u = 0$), and $\operatorname{div} \sigma$ contributes to the force balance.

Some concrete questions related to these complexes (as a motivation for further investigation)

1. (for electromagnetic waves) how to characterise divergence-free *vector fields* $\operatorname{div} B = 0$?
2. (for elasticity) how to characterise divergence-free *symmetric matrix field* $\operatorname{div} \sigma = 0$?
3. (for gravitational waves) how to characterise *TT tensors (symmetric, trace-free, divergence-free)*?

We shall see that the answer depends on the topology, and from (1) to (3) more and more algebraic machinery is involved. Moreover, we have to answer these questions not only for smooth (C^∞) functions, but also for broader classes of function spaces, e.g., Sobolev spaces. For example, spaces in

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0, \tag{1}$$

where $H(D) := \{u \in L^2 : Du \in L^2\}$, and

$$0 \longrightarrow H^q \xrightarrow{\text{grad}} H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{curl}} H^{q-2} \otimes \mathbb{V} \xrightarrow{\text{div}} H^{q-3} \longrightarrow 0, \tag{2}$$

where q is any real number.

To further incorporate analysis into complexes, we introduce unbounded operators. This is to relate complexes to PDEs, which involves *adjoint operators* defined in this functional analysis framework.

Unbounded (linear) operator: linear operator between Hilbert spaces $T : X \to Y$, not necessarily bounded, e.g., grad : $L^2 \to L^2 \otimes \mathbb{R}^3$, curl : $L^2 \otimes \mathbb{R}^3 \to L^2 \otimes \mathbb{R}^3$.
Domain: a subspace of X where T is bounded (not unique and not necessarily closed), e.g., H^1 or H_0^1 for grad.
Adjoint: the adjoint of an unbounded operator T is defined to be an unbounded operator with the domain

$$D(T^*) := \{w \in Y : \exists c_w > 0 \ s.t. \ |(w, Tv)_Y| \leq c_w \|v\|_X, \ \forall v \in D(T)\}.$$

and for any $w \in D(T^*)$:

$$(T^* w, v)_X := (w, Tv)_Y, \quad \forall v \in D(T).$$

For unbounded operator $T : X \to Y$, kernel and range

$$\mathcal{N}(T) := \{x \in D(T) : Tx = 0\} \subset D(T) \subset X, \quad \mathcal{R}(T) := \{Tx : x \in D(T)\} \subset Y,$$

graph

$$\Gamma(T) := \{(x, Tx) : x \in D(T)\} \subset X \times Y.$$

Definition

An unbounded operator T is a closed operator if $\Gamma(T)$ is closed.

Different from “operators with closed range” ($\mathcal{R}(T)$ is closed).

Theorem

Let $T : X \to Y$ be a closed densely defined unbounded operator. Then

$$\mathcal{R}(T)^\perp = \mathcal{N}(T^*), \quad \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}, \quad \mathcal{R}(T^*)^\perp = \mathcal{N}(T), \quad \mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}.$$

Lemma

If T is a closed densely defined unbounded operator from X to Y then $T^{**} = T$.

Example: grad, curl, div: unbounded operators from L^2 to L^2.

$$\int_\Omega \operatorname{grad} u \cdot v = -\int_\Omega u \operatorname{div} v + \int_{\partial\Omega} un \cdot v,$$

- adjoint of $(\operatorname{grad}, H^1)$ is $(-\operatorname{div}, H_0(\operatorname{div}))$,
- adjoint of $(\operatorname{grad}, H_0^1)$ is $(-\operatorname{div}, H(\operatorname{div}))$.

$(\operatorname{grad}, H^1)^{**} = (-\operatorname{div}, H_0(\operatorname{div}))^* = (\operatorname{grad}, H^1)$,
$(\operatorname{grad}, H_0^1)^{**} = (-\operatorname{div}, H(\operatorname{div}))^* = (\operatorname{grad}, H_0^1)$

curl and div are similar.

Hilbert complex: sequence $(W^\bullet, d^\bullet)$ where $W^\bullet$ are Hilbert spaces with the norm $\|\cdot\|$ and $d^\bullet$ are closed densely defined linear operators, such that $\mathcal{R}(D^k) \subset \mathcal{N}(D^{k+1})$.

Let domain $V^k := D(d^k) \subset W^k$. Then V^k is a Hilbert space equipped with the norm $\|u\|_V^2 := \|u\|^2 + \|du\|^2$ (since d is closed). By the definition of $\mathcal{R}$ and $\mathcal{N}$, $(V^\bullet, d^\bullet)$ is a complex with bounded operators, called the *domain complex*.

Dual complex: complex of adjoint operators (as unbounded operators, with associated domain complex).
If $\mathcal{R}(d^k) \subset \mathcal{N}(d^{k+1})$, then $\mathcal{R}(d^*_{k+1}) \subset \mathcal{N}(d^*_k)$ (by definition).

Example: $(W^\bullet, d^\bullet)$

$$0 \longrightarrow L^2 \xrightarrow{\text{grad}} L^2 \otimes \mathbb{V} \xrightarrow{\text{curl}} L^2 \otimes \mathbb{V} \xrightarrow{\text{div}} L^2 \longrightarrow 0,$$

grad, curl, div: unbounded operators from L^2 to L^2.

Example (continued)

- Option 1: define domain to be H^1, $H(\mathrm{curl})$, $H(\mathrm{div})$, i.e., the domain complex $(V^\bullet, d^\bullet)$

$$0 \longrightarrow H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0.$$

Then the dual complex $(W^\bullet, d^*_\bullet)$ is

$$0 \longleftarrow L^2 \xleftarrow{-\mathrm{div}} L^2 \otimes \mathbb{V} \xleftarrow{\mathrm{curl}} L^2 \otimes \mathbb{V} \xleftarrow{-\mathrm{grad}} L^2 \longleftarrow 0,$$

with the domain $(V^*_\bullet, d^*_\bullet)$

$$0 \longleftarrow L^2 \xleftarrow{-\mathrm{div}} H_0(\mathrm{div}) \xleftarrow{\mathrm{curl}} H_0(\mathrm{curl}) \xleftarrow{-\mathrm{grad}} H^1_0 \longleftarrow 0.$$

- Option 2: define domain to be H_0^1, $H_0(\mathrm{curl})$, $H_0(\mathrm{div})$, i.e., the domain complex $(V^\bullet, d^\bullet)$

$$0 \longrightarrow H_0^1 \xrightarrow{\mathrm{grad}} H_0(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H_0(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0.$$

where $H_0(D) := \{u \in L^2 : Du \in L^2, \mathrm{tr}\, u = 0 \text{ on } \partial\Omega\}$, with

$$\mathrm{tr}\, u := \begin{cases} u|_{\partial\Omega} \in H^{1/2}(\partial\Omega), & \text{if } D = \mathrm{grad}, \\ u \times n \in H^{-1/2}(\partial\Omega), & \text{if } D = \mathrm{curl}, \\ u \cdot n \in H^{-1/2}(\partial\Omega), & \text{if } D = \mathrm{div}\,. \end{cases}$$

Then the dual complex $(W^\bullet, d^*_\bullet)$ is

$$0 \longleftarrow L^2 \xleftarrow{-\mathrm{div}} L^2 \otimes \mathbb{V} \xleftarrow{\mathrm{curl}} L^2 \otimes \mathbb{V} \xleftarrow{-\mathrm{grad}} L^2 \longleftarrow 0,$$

with the domain $(V^*_\bullet, d^*_\bullet)$

$$0 \longleftarrow L^2 \xleftarrow{-\mathrm{div}} H(\mathrm{div}) \xleftarrow{\mathrm{curl}} H(\mathrm{curl}) \xleftarrow{-\mathrm{grad}} H^1 \longleftarrow 0.$$

To establish further properties of Hilbert complexes, we will resort to Sobolev complexes. In the following, we consider sequences of the form

$$\cdots \to L^2 \otimes \mathbb{W}^k \xrightarrow{D^k} L^2 \otimes \mathbb{W}^{k+1} \to \cdots, \tag{3}$$

where $\mathbb{W}^k$ is a finite dimensional space (scalars, vectors, matrices etc.), D^k is a differential operator with constant (or more generally, smooth) coefficients defined in the sense of distributions and viewed as a closed unbounded operator with domain

$$H\mathbb{W}^k := \{\, u \in L^2 \otimes \mathbb{W}^k \mid D^k u \in L^2 \otimes \mathbb{W}^{k+1} \,\}.$$

This leads to the following L^2 domain complex

$$\cdots \to H\mathbb{W}^k \xrightarrow{D^k} H\mathbb{W}^{k+1} \to \cdots. \tag{4}$$

All the properties (Poincaré equalities, existence of regular potential, regular decomposition, compactness, div-curl type lemma...) will be established via the following *Sobolev complex*:

$$\cdots \to H^{q_k} \otimes \mathbb{W}^k \xrightarrow{D^k} H^{q_{k+1}} \otimes \mathbb{W}^{k+1} \to \cdots, \tag{5}$$

Remark. Hilbert complex usually refers to the L^2 complex and its domain complex. We do not view the Sobolev complex (5) as a "Hilbert complex" although each $H^{q_k} \otimes \mathbb{W}^k$ is a Hilbert space. The introduce of (5) is to establish further results of the L^2 and its domain complex, although (5) is important in its own right.

Assumption: cohomology of (5) is finite dimensional, and there exists a uniform representation of cohomology for any $q_k \in \mathbb{R}$ and $q_{k+1} - q_k = \gamma_k$ where γ_k is the order of D^k i.e.,

$$\mathcal{N}(D^{k+1}, H^{q_{k+1}} \otimes \mathbb{W}^{k+1}) = D^k(H^{q_k} \otimes \mathbb{W}^k) \oplus G_\infty^{k+1}, \quad \forall q_k \in \mathbb{R}.$$

Remark: Since $G_\infty^{k+1} \subset H^q \otimes \mathbb{W}^{k+1}$ for any q, by Sobolev imbedding,

$$G_\infty^{k+1} \subset C^\infty \otimes \mathbb{W}^{k+1}.$$

With the above assumption, the Sobolev complex is a *Hilbert scale*.
Hilbert scale: a family of complexes

$$\cdots \to Z^k_{[q]} \xrightarrow{D^k_{[q]}} Z^{k+1}_{[q]} \to \cdots \tag{6}$$

indexed by a parameter $q \in \mathbb{R}$, such that if $q' \geq q$, then

$$Z^k_{[q']} \subset Z^k_{[q]} \quad \text{and} \quad D^k_{[q']} = D^k_{[q]}|_{Z^k_{[q']}}.$$

Definition

A sequence of finite-dimensional spaces $G^k_\infty \subset L^2(\Omega) \otimes \mathbb{W}^k$ is said to *uniformly represent the cohomology* of a scale of complexes (6) if, for each $k \in \mathbb{Z}$ and each $q \in \mathbb{R}$,

$$\mathcal{N}(D^k, Z^k_{[q]}) = \mathcal{R}(D^{k-1}, Z^{k-1}_{[q]}) \oplus G^k_\infty. \tag{7}$$

Below we *assume that the Sobolev complexes are a Hilbert scale* and establish various results based on this fact. They are useful not only for studying the Hilbert complex, but also play a key role in, e.g., fluid and solid mechanics.

Since $H^{\gamma_{k-1}} \otimes \mathbb{W}^k \subset H\mathbb{W}^k \subset L^2 \otimes \mathbb{W}^k$, the cohomology of the L^2 complex (3) can be represented by the same representatives as the Sobolev complex (5). Specifically:

Theorem

Suppose that the scale of complexes

$$\cdots \to H^{q_k} \otimes \mathbb{W}^k \xrightarrow{D^k} H^{q_{k+1}} \otimes \mathbb{W}^{k+1} \to \cdots,$$

admits a uniform set of cohomology representatives G^k_∞. Then the same spaces are cohomology representatives for the domain complex

$$\cdots \to H\mathbb{W}^k \xrightarrow{D^k} H\mathbb{W}^{k+1} \to \cdots.$$

as well:

$$\mathcal{N}\left(D^k, H\mathbb{W}^k\right) = \mathcal{R}(D^{k-1}, H\mathbb{W}^{k-1}) \oplus G^k_\infty.$$

Proof.

We have

$$\begin{aligned}\mathcal{N}\left(D^k, H\mathbb{W}^k\right) = \mathcal{N}\left(D^k, L^2 \otimes \mathbb{W}^k\right) &= \mathcal{R}(D^{k-1}, H^{\gamma_{k-1}} \otimes \mathbb{W}^{k-1}) \oplus G^k_\infty \\ &\subset \mathcal{R}(D^{k-1}, H\mathbb{W}^{k-1}) \oplus G^k_\infty \subset \mathcal{N}\left(D^k, H\mathbb{W}^k\right),\end{aligned}$$

where the first equality is by definition. This implies the result.

Closed range: $D^k : H^{q_k} \otimes \mathbb{W}^k \to H^{q_{k+1}} \otimes \mathbb{W}^{k+1}$ has closed range.

Trivial cohomology: range = kernel, which is closed.
In general:

Lemma (Hömander, vol 3, Lemma 19.1.1)

Let B_1, B_2 be Banach spaces. If $T \in \mathcal{L}(B_1, B_2)$ and the range TB_1 has finite codimensions in B_2, then TB_1 is closed.

Proof. We may assume that T is injective (otherwise consider $B_1/\mathcal{N}(T)$). If N is the codimension of TB_1, consider a linear map

$$S : \mathbb{R}^N \to B_2,$$

such that $S\mathbb{R}^N$ is a supplement of TB_1, that is, the map

$$T_1 : B_1 \oplus \mathbb{R}^N \ni (x, y) \mapsto Tx + Sy \in B_2$$

is bijective. By Banach's theorem it is then a homeomorphism, which proves that $TB_1 = T_1(B_1 \oplus \{0\})$ is closed.

Hodge decomposition, harmonic forms

We have

$$L^2 \otimes \mathbb{W}^k = \mathcal{N}(D^k) \oplus \mathcal{N}(D^k)^\perp = \mathcal{N}(D^k) \oplus \mathcal{R}(D^*_{k+1}) = \mathcal{R}(D^{k-1}) \oplus \mathfrak{H}^k \oplus \mathcal{R}(D^*_{k+1}), \tag{8}$$

where

$$\mathfrak{H}^k = \{\, u \in \mathcal{N}(D^k) \mid u \perp \mathcal{R}(D^{k-1}) \,\}$$

denote the *harmonic forms* for this Hilbert complex.

Proof.

The following is by definition and duality:

$$L^2 \otimes \mathbb{W}^k = \mathcal{N}(D^k) \oplus \mathcal{N}(D^k)^\perp = \mathcal{N}(D^k) \oplus \overline{\mathcal{R}(D^*_{k+1})} = \overline{\mathcal{R}(D^{k-1})} \oplus \mathfrak{H}^k \oplus \overline{\mathcal{R}(D^*_{k+1})}. \tag{9}$$

The closed range property ensures that we can drop the closure (using the fact that T has closed range if and only if T^* does, due to the Banach closed range theorem). □

Theorem

$$\|u\|_{q_k} \leq C\|D^k u\|_{q_{k+1}}, \quad \forall u \in H^{q_k} \otimes \mathbb{W}^k,\ u \perp_{H^{q_k}} \mathcal{N}(D^k, H^q \otimes \mathbb{W}^k).$$

$$\|u\| \leq C\|D^k u\|, \quad \forall u \in H\mathbb{W}^k,\ u \perp_{L^2} \mathcal{N}(D^k, H\mathbb{W}^k).$$

Proof: Banach theorem.
e.g., (if the cohomology is trivial) de Rham complex:

$$\|u\|_1 \leq C\|\operatorname{grad} u\|, \quad \int u = 0, \quad \text{Poincaré}$$

$$\|u\|_1 \leq C\|\operatorname{curl} u\|, \quad u : \operatorname{div} u = 0, u \cdot n = 0 \text{ on } \partial\Omega,$$
$$\|u\|_1 \leq C\|\operatorname{div} u\|, \quad u : \operatorname{curl} u = 0, u \times n = 0 \text{ on } \partial\Omega.$$

elasticity complex:

$$\|u\|_1 \leq C\|\operatorname{def} u\|, \quad \int u \cdot q = 0, \forall q \in \mathcal{N}(\operatorname{def}), \quad \text{Korn}$$

$\mathcal{N}(\operatorname{def}) = \{a + b \times x : a, b \in \mathbb{R}^3\}$: infinitesimal rigid body motion

$$\|u\|_2 \leq C\|\operatorname{inc} u\|, \quad u : \operatorname{div} u = 0, \ + \text{b.c.s} \ ,$$

$$\|u\|_1 \leq C\|\operatorname{div} u\|, \quad u : \operatorname{inc} u = 0, \ + \text{b.c.s} \ .$$

Theorem (Existence of bounded regular potentials)

Let $q, r \in \mathbb{R}$, $k \in \mathbb{Z}$. There is a constant C such that for any $v \in H^q \otimes \mathbb{W}^{k+1} \cap \mathcal{R}(D^k, H^r \otimes \mathbb{W}^k)$, there exists $u \in H^{q+\gamma_k} \otimes \mathbb{W}^k$ such that $D^k u = v$ and

$$\|u\|_{q+\gamma_k} \leq C\|v\|_q. \tag{10}$$

Proof.

By the assumptions on v, it belongs to $\mathcal{N}(D^{k+1}, H^q \otimes \mathbb{W}^{k+1})$. Now, from the uniform representation of cohomology applied to the sequence

$$H^{q+\gamma_k} \otimes \mathbb{W}^k \xrightarrow{D^k} H^q \otimes \mathbb{W}^{k+1} \xrightarrow{D^{k+1}} H^{q-\gamma_{k+1}} \otimes \mathbb{W}^{k+2},$$

we have $\mathcal{N}(D^{k+1}, H^q \otimes \mathbb{W}^{k+1}) = \mathcal{R}(D^k, H^{q+\gamma_k} \otimes \mathbb{W}^k) + G_\infty^{k+1}$, so there exists $u \in H^{q+\gamma_k} \otimes \mathbb{W}^k$ and $s \in G_\infty^{k+1}$ for which $v = Du + s$. But $v \in \mathcal{R}(D^k, H^r \otimes \mathbb{W}^k)$ and $Du \in \mathcal{R}(D^k, H^{q+\gamma_k} \otimes \mathbb{W}^k)$, so $s \in G_\infty^{k+1} \cap \mathcal{R}(D^k, H^t \otimes \mathbb{W}^k)$ with $t = \min(q + \gamma_k, r)$. But the last space reduces to zero. Further, we may subtract from u its projection onto $\mathcal{N}(D^k, H^{q+\gamma_k} \otimes \mathbb{W}^k)$ without changing $D^k u$. Then u is the desired regular potential and the bound (10) is immediate from Poincaré inequality. $\square$

Theorem

The regular decomposition holds:

$$H\mathbb{W}^k = D^{k-1}\left(H^{\gamma_{k-1}} \otimes \mathbb{W}^{k-1}\right) + H^{\gamma_k} \otimes \mathbb{W}^k. \tag{11}$$

Proof.

Let $w \in H\mathbb{W}^k$. Applying Theorem 8 to $v = D^k w$, we obtain $u \in H^{\gamma_k} \otimes \mathbb{W}^k$ such that $D^k u = D^k w$. Then $w - u \in \mathcal{N}(D^k, L^2 \otimes \mathbb{W}^k)$, so, by uniform representation of cohomology, there exists $y \in H^{\gamma_{k-1}} \otimes \mathbb{W}^{k-1}$ and $s \in G^k_\infty \otimes \mathbb{W}^k$ such that $w - u = D^{k-1}y + s$. Then $w = D^{k-1}y + (u + s)$ provides a regular decomposition for u. □

decompose a field into a regular part plus something in the range, e.g.,
$H(\mathrm{curl}) = [H^1]^3 + \mathrm{grad}\, H^1$.
compared with the Hodge decomposition: lose orthogonality, gain regularity
Useful for constructing preconditioners - reducing a curl problem to several Poisson problems which are easier to solve.

c.f. *Hiptmair, R., & Xu, J. (2007). Nodal auxiliary space preconditioning in H (curl) and H (div) spaces. SIAM Journal on Numerical Analysis, 45(6), 2483-2509.*

Compactness: define $H(D^k) := \{u \in L^2 \otimes \mathbb{W}^k : D^k u \in L^2 \otimes \mathbb{W}^{k+1}\}$,
$H(D_k^*) := \{u \in L^2 \otimes \mathbb{W}^k : D_k^* u \in L^2 \otimes \mathbb{W}^{k-1}\}$

Theorem

The imbedding $H(D^k) \cap H(D_k^*) \hookrightarrow L^2 \otimes \mathbb{W}^k$ is compact.

Idea of proof: Hodge decomposition: $H(D^k) \cap H(D_k^*) = \mathcal{R}(D) + \mathcal{R}(D^*) + \mathscr{H}$,
$u = D\phi + D^*\psi + h$,

- $\mathscr{H}$ is finite dimensional,
- $\mathcal{R}(D)$ part: potential ϕ can be chosen in H^1. use Rellich compactness.
- $\mathcal{R}(D^*)$ part: similar to $\mathcal{R}(D)$ part.

Example: de Rham:

$$H(\text{curl}) \cap H_0(\text{div}) \hookrightarrow L^2,$$

elasticity/geometry:

$$H(\text{inc}) \cap H_0(\text{div}) \hookrightarrow L^2.$$

Curvature L^2, "Bianchi" L^2 $\Rightarrow$ compact imbedding.

D-D^* lemma (compensated compactness)

$$H(D^{i-1}) \xrightarrow{D^{i-1}} H(D^i) \xrightarrow{D^i} H(D^{i+1})$$

adjoint (automatically encodes b.c.s by definition)

$$H(D^*_{i-1}) \xleftarrow{D^*_i} H(D^*_i) \xleftarrow{D^*_{i+1}} H(D^*_{i+1})$$

Theorem

Let $\{x_n\} \subset H(D^i)$ and $\{y_n\} \subset H(D^*_i)$ be two bounded sequences (with the corresponding norms). Then there exist subsequences (again denoted by $\{x_n\}$, $\{y_n\}$) and $x \in H(D^i)$, $y \in H(D^*_i)$, such that

- $x_n \rightharpoonup x$ *in* $H(D^i)$,
- $y_n \rightharpoonup y$ *in* $H(D^*_i)$,
- $(x_n, y_n)_{L^2} \to (x, y)_{L^2}$.

Due to D. Pauly 2018. The form and norms can be more general.

Proof. Pick weakly convergent sequences, still denoted by $\{x_n\}$, $\{y_n\}$ (to x, y). Regular decomposition:

$$x_n = Dz_n + \tilde{x}_n,$$

with $z_n \in H^1$, $\tilde{x}_n \in H^1$, both bounded.
Rellich compactness: there exists $z \in L^2$, $z_n \to z$ in L^2, there exists $\tilde{x} \in L^2$, $\tilde{x}_n \to \tilde{x}$ in L^2.
Then

$$(x, y)_{L^2} \leftarrow (x_n, \phi)_{L^2} = (Dz_n, \phi)_{L^2} + (\tilde{x}_n, \phi)_{L^2} \to (Dz, \phi)_{L^2} + (\tilde{x}, \phi)_{L^2},$$

$$\begin{aligned}(x_n, y_n)_{L^2} &= (Dz_n, y_n)_{L^2} + (\tilde{x}_n, y_n)_{L^2} = (z_n, D^* y_n)_{L^2} + (\tilde{x}_n, y_n)_{L^2} \\ &\to (z, D^* y)_{L^2} + (\tilde{x}, y)_{L^2} = (Dz, y)_{L^2} + (\tilde{x}, y)_{L^2} = (x, y)_{L^2}.\end{aligned}$$

Example: de Rham complex: div-curl lemma. Elasticity complex: div-inc lemma ("Bianchi-Riemann lemma")

We see that all the analytic results follow from the assumption that the cohomology of the Sobolev complexes (scale) is finite dimensional. In this section, we verify this assumption for the de Rham complex.

In fact, we first introduce de Rham complexes in terms of differential forms. The grad-curl-div complex is a special case (vector proxy) of it. Then we show that the cohomology of de Rham complexes with smooth functions is determined by the topology. In the end we present Costabel and McIntosh's fundamental result which establishes that the de Rham cohomology is independent of the regularity (thus a Hilbert scale).

Topology and the machinery of differential forms are useful in their own right.

We start with algebraic alternating forms. Let V be a linear space and $\mathrm{Lin}^k V : V \times V \times \cdots \times V \to \mathbb{R}$ be the space of k-linear maps. If $\omega \in \mathrm{Lin}^k V$ and $\mu \in \mathrm{Lin}^j V$, then the *tensor product* $\omega \otimes \mu \in \mathrm{Lin}^{k+j}$ is defined by

$$(\omega \otimes \mu)(v_1, \cdots, v_{j+k}) = \omega(v_1, \cdots, v_j)\mu(v_{j+1}, \cdots, v_{j+k}), \quad v_1, \cdots, v_{j+k} \in V.$$

Alternating k-forms, denoted by $\mathrm{Alt}^k V$, are defined to be the subspace of $\mathrm{Lin}^k V$ which are skew-symmetric with the arguments, i.e.,

$$\omega(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -\omega(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

The exterior product $\wedge$ is the skew-symmetrisation of the tensor product, i.e., if $\omega \in \mathrm{Alt}^j V$, $\mu \in \mathrm{Alt}^k V$, then $\omega \wedge \mu \in \mathrm{Alt}^{j+k}$ is defined by

$$\omega \wedge \mu(v_1, \cdots, v_{j+k}) = \sum_\sigma (-1)^\sigma \omega(v_{\sigma_1}, \cdots, v_{\sigma_j})\mu(v_{\sigma_{j+1}}, \cdots, v_{\sigma_{j+k}}).$$

Canonical bases in $\mathbb{R}^n$

If $V = \mathbb{R}^n$, let $e_i = (0, \cdots, 1, \cdots, 0)$ be the i-th canonical basis. Let dx^j be the dual basis, i.e., $dx^j(e_i) = \delta_i^j$. Then $dx^i \wedge dx^j = -dx^j \wedge dx^i$, $dx^i \wedge dx^i = 0$ and any k-form vanishes in nD if $k > n$.

Contraction Let $\omega \in \mathrm{Alt}^k V$ and $v \in V$. Then contraction $i_v\omega \in \mathrm{Alt}^{k-1} V$ is defined to be

$$(i_v\omega)(v_1, \cdots, v_{k-1}) := \omega(v, v_1, \cdots, v_{k-1}).$$

Inner product If V^* is equipped with an inner product $(\cdot, \cdot)$, then we can define inner product in Alt^k by

$$(u^1 \wedge \cdots \wedge u^k, v^1 \wedge \cdots \wedge v^k) = \det[(u^i, v^j)_{i,j=1}^n].$$

Volume form In general, one can specify any non-zero element in Alt^n as the *volume form*, which measures (or, rather, defines) the volume enclosed by n vectors. In $\mathbb{R}^n$ one usually uses the wedge product of the orthonormal bases:

$$\mathsf{vol} := dx^1 \wedge \cdots \wedge dx^n.$$

Hodge star The definition of the Hodge star operator uses inner product and volume form, which therefore also relies on an inner product (metric) on V. The Hodge star $* : \mathrm{Alt}^k \to \mathrm{Alt}^{n-k}$ is defined by

$$\omega \wedge \mu = (*\omega, \mu)\mathsf{vol}, \quad \omega \in \mathrm{Alt}^k V,\ \mu \in \mathrm{Alt}^{n-k} V.$$

In terms of the canonical basis,

$$*dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \pm dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n},$$

where the sign depends on whether $\{i_1, i_2, \cdots, i_n\}$ is an even or odd permutation. For odd dimensions, $** u = u$, and for even dimensions, $** u = (-1)^k u$.

Vector proxies In $\mathbb{R}^3$, any 0-form is a function. With a basis, any 1-form can be written as $u = u_1 dx^1 + u_x dx^2 + u_3 dx^3$. Therefore u can be identified with a vector (u_1, u_2, u_3). Similarly, any 2-form $v = v_1 dx^2 \wedge dx^3 + v_2 dx^3 \wedge dx^1 + v_3 dx^1 \wedge dx^2$ can be identified with a vector (v_1, v_2, v_3). Finally, any 3-form $\omega = w dx^1 \wedge dx^2 \wedge dx^3$ is identified with a scalar function w. Operators on forms can be written in terms of vectors/matrices. We call this identification the *vector proxy*.

Table 6.1. *Exterior algebra of $\mathbb{R}^3$ in terms of scalar and vector proxy fields.*

Proxies	
$\mathrm{Alt}^0\,\mathbb{R}^3 = \mathbb{R}$	$c \leftrightarrow c$
$\mathrm{Alt}^1\,\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3$	$u_1\,dx^1 + u_2\,dx^2 + u_3\,dx^3 \leftrightarrow u$
$\mathrm{Alt}^2\,\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3$	$u_1\,dx^2 \wedge dx^3 - u_2\,dx^1 \wedge dx^3 + u_3\,dx^1 \wedge dx^2 \leftrightarrow u$
$\mathrm{Alt}^3\,\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}$	$c \leftrightarrow c\,dx^1 \wedge dx^2 \wedge dx^3$
Exterior product	
$\wedge : \mathrm{Alt}^1\,\mathbb{R}^3 \times \mathrm{Alt}^1\,\mathbb{R}^3 \to \mathrm{Alt}^2\,\mathbb{R}^3$	$\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$
$\wedge : \mathrm{Alt}^1\,\mathbb{R}^3 \times \mathrm{Alt}^2\,\mathbb{R}^3 \to \mathrm{Alt}^3\,\mathbb{R}^3$	$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$
Contraction with a vector $v \in \mathbb{R}^3$	
$\lrcorner v : \mathrm{Alt}^1\,\mathbb{R}^3 \to \mathrm{Alt}^0\,\mathbb{R}^3$	$v\cdot : \mathbb{R}^3 \to \mathbb{R}$
$\lrcorner v : \mathrm{Alt}^2\,\mathbb{R}^3 \to \mathrm{Alt}^1\,\mathbb{R}^3$	$v\times : \mathbb{R}^3 \to \mathbb{R}^3$
$\lrcorner v : \mathrm{Alt}^3\,\mathbb{R}^3 \to \mathrm{Alt}^2\,\mathbb{R}^3$	$v : \mathbb{R} \to \mathbb{R}^3 \quad (c \mapsto cv)$
Inner product	
Inner product on $\mathrm{Alt}^k\,\mathbb{R}^3$	Euclidean inner product on $\mathbb{R}$ or $\mathbb{R}^3$
Volume form	
$\mathsf{vol} = dx^1 \wedge dx^2 \wedge dx^3$	$(v_1, v_2, v_3) \mapsto \det[v_1 \vert v_2 \vert v_3]$
Hodge star	
$\star : \mathrm{Alt}^0\,\mathbb{R}^3 \to \mathrm{Alt}^3\,\mathbb{R}^3$	$\mathrm{id} : \mathbb{R} \to \mathbb{R}$
$\star : \mathrm{Alt}^1\,\mathbb{R}^3 \to \mathrm{Alt}^2\,\mathbb{R}^3$	$\mathrm{id} : \mathbb{R}^3 \to \mathbb{R}^3$
$\star : \mathrm{Alt}^2\,\mathbb{R}^3 \to \mathrm{Alt}^1\,\mathbb{R}^3$	$\mathrm{id} : \mathbb{R}^3 \to \mathbb{R}^3$
$\star : \mathrm{Alt}^3\,\mathbb{R}^3 \to \mathrm{Alt}^0\,\mathbb{R}^3$	$\mathrm{id} : \mathbb{R} \to \mathbb{R}$
Pullback by a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$	
$L^* : \mathrm{Alt}^0\,\mathbb{R}^3 \to \mathrm{Alt}^0\,\mathbb{R}^3$	$\mathrm{id} : \mathbb{R} \to \mathbb{R}$
$L^* : \mathrm{Alt}^1\,\mathbb{R}^3 \to \mathrm{Alt}^1\,\mathbb{R}^3$	$L^T : \mathbb{R}^3 \to \mathbb{R}^3$
$L^* : \mathrm{Alt}^2\,\mathbb{R}^3 \to \mathrm{Alt}^2\,\mathbb{R}^3$	$(\det L)L^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$
$L^* : \mathrm{Alt}^3\,\mathbb{R}^3 \to \mathrm{Alt}^3\,\mathbb{R}^3$	$(\det L) : \mathbb{R} \to \mathbb{R} \quad (c \mapsto c \det L)$

Arnold, *Finite element exterior calculus*, SIAM, Chapter 6

Differential forms

A differential k-form on a manifold Ω is a field of alternating k-forms, i.e., for any vector field $v_1, \cdots, v_k$, $\omega_x(v_1, \cdots, v_k)$ is a function of x on Ω. In the coordinate form, $\omega = \sum_\sigma w_\sigma(x) dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \cdots \wedge dx^{\sigma_k}$ can be written as a combination of algebraic k-forms with coefficients being functions of x.

Exterior derivatives For differential forms we can define exterior derivatives $d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$.

$$(d\omega)_x(v_0, \cdots, v_k) = \sum_{j=0}^{k} (-1)^j \partial_{v_j} \omega_x(v_0, \cdots, \hat{v}_j, \cdots, v_k).$$

In coordinates, for

$$\omega = \sum_{1 \le \sigma_1 \le \cdots \le \sigma_k \le n} f_\sigma dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k},$$

its exterior derivative

$$d^k \omega = \sum_{j=1}^{n} \sum_{1 \le \sigma_1 \le \cdots \le \sigma_k \le n} \frac{\partial f_\sigma}{\partial x^j} dx^j \wedge dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}.$$

With this definition, we can check the vector proxy in 3D. Let $u \in \Lambda^0$, then $d^0 u = \frac{\partial u}{\partial x^1} dx^1 + \frac{\partial u}{\partial x^2} dx^2 + \frac{\partial u}{\partial x^3} dx^3$. This shows that d^0 corresponds to the grad operator in the vector proxy with the canonical basis in $\mathbb{R}^3$. Similarly, for a 1-form $v = v_1 dx^1 + v_2 dx^2 + dx^3$,

$$\begin{aligned} d^1 v = {} & \frac{\partial v_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial v_1}{\partial x^3} dx^3 \wedge dx^1 + \frac{\partial v_2}{\partial x^1} dx^1 \wedge dx^2 \\ & + \frac{\partial v_2}{\partial x^3} dx^3 \wedge dx^2 + \frac{\partial v_3}{\partial x^1} dx^1 \wedge dx^3 + \frac{\partial v_3}{\partial x^2} dx^2 \wedge dx^3 \\ & = (\frac{\partial v_2}{\partial x^1} - \frac{\partial v_1}{\partial x^2}) dx^1 \wedge dx^2 + (\frac{\partial v_3}{\partial x^2} - \frac{\partial v_2}{\partial x^3}) dx^2 \wedge dx^3 + (\frac{\partial v_1}{\partial x^3} - \frac{\partial v_3}{\partial x^1}) dx^3 \wedge dx^1, \end{aligned}$$

where we have used $dx^i \wedge dx^j = -dx^j \wedge dx^i$, and in particular, $dx^i \wedge dx^i = 0$. This shows that d^1 corresponds to curl in 3D. Similarly, d^2 corresponds to div.

For $\omega \in \Lambda^k(\Omega)$, $\mu \in \Lambda^l(\Omega)$,

$$d(\omega \wedge \mu) = (d\omega) \wedge \mu + (-1)^k \omega \wedge d\mu.$$

Pullback and trace

Transforms in the finite element methods correspond to pullbacks in the language of differential forms.

Let $f : \hat{\Omega} \to \Omega$ be an isomorphism. Then the pullback f^* maps a tensor field on Ω to a tensor field on $\hat{\Omega}$. To give a precise definition, we start with the pushforward of vectors. A vector at $\hat{x} \in \hat{\Omega}$ is the tangent vector of some curve $\hat{\gamma}(t) \subset \hat{\Omega}, t \in (-1, 1)$. Then f maps $\hat{\gamma}(t)$ to $f\hat{\gamma}(t)$. The pushforward of v by f evaluated at $f(\hat{x})$, denoted by $f_* v(f(\hat{x}))$, is the tangent vector of $f\hat{\gamma}(t)$ at $f(\hat{x})$. Let v be a vector field on $\hat{\Omega}$. In coordinates, $v = v^i \frac{\partial}{\partial \hat{x}^i}$. Then the pushforward $f_* v = \frac{\partial f^j}{\partial \hat{x}^i} v^i \frac{\partial}{\partial x^j}$. The pullback $f^*\omega(v_1, \cdots, v_k) := \omega(f_* v_1, \cdots, f_* v_k)$.

Let Ω be a bounded domain. Then $\partial\Omega$ is a submanifold. The inclusion map $i : \partial\Omega \to \Omega$ induces a pullback $i^* : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$, which is defined to be the trace operator $\mathrm{tr} := i^*$.

Pullback:

$$df^* = f^* d,$$

$$di^* = i^* d.$$

trace: in $\mathbb{R}^3$

$\mathrm{tr} : \Lambda^0(\Omega) \to \Lambda^0(\partial\Omega)$ $\quad u \mapsto u$ restriction

$\mathrm{tr} : \Lambda^1(\Omega) \to \Lambda^1(\partial\Omega)$ $\quad u \mapsto u - (u \cdot n)n = n \times u \times n$ tangential components

$\mathrm{tr} : \Lambda^2(\Omega) \to \Lambda^2(\partial\Omega)$ $\quad u \mapsto u \cdot n$ normal component

$\mathrm{tr} : \Lambda^3(\Omega) \to \Lambda^3(\partial\Omega)$ $\quad u \mapsto 0$ trivial.

Remark $u \times n$ carries the same information as $n \times u \times n$ as a rotation in the tangent plane (both orthogonal to n).

vector proxies for $d\,\mathrm{tr} = \mathrm{tr}\,d$:

$d^0_{\partial\Omega}\,\mathrm{tr}^0 = \mathrm{tr}^1\,d^0$: $\quad \mathrm{grad}_{\partial\Omega}\,u|_{\partial\Omega} = n \times (\mathrm{grad}\,u) \times n$

$d^1_{\partial\Omega}\,\mathrm{tr}^1 = \mathrm{tr}^2\,d^1$: $\quad \mathrm{rot}_{\partial\Omega}(n \times u \times n) = \mathrm{curl}\,u \cdot n$,

or its rotated version $\mathrm{div}_{\partial\Omega}(u \times n) = \mathrm{curl}\,u \cdot n$

Integration

Integration is carefully defined such that the definition is coordinate independent. We will not use details of the definition (see textbooks on manifolds). It suffices to mention that a k-form can be only integrated on k-dimensional manifolds (for example, in this language one cannot integrate a function / 0-form u on $\Omega \subset \mathbb{R}^3$, but has to integrate the volume form $u\, dx^1 \wedge dx^2 \wedge dx^3$). Moreover,

$$\int_\Omega u\, dx^1 \wedge dx^2 \wedge dx^3 = -\int_\Omega u\, dx^2 \wedge dx^1 \wedge dx^3 = \ldots$$

So the orientation matters.

Stokes formula

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

Corollary: integration by parts

$$\int_\Omega d\omega \wedge \mu + (-1)^k \int_\Omega \omega \wedge d\mu = \int_{\partial\Omega} \omega \wedge \mu$$

(both sides $= \int_\Omega d(\omega \wedge \mu)$)

Vector proxies in 3D $\omega \in \Lambda^k$, $\mu \in \Lambda^{n-k-1}$.

$k = 0$: $\int_\Omega \operatorname{grad} u \cdot v + \int_\Omega u \operatorname{div} v = \int_{\partial\Omega} uv \cdot n$,
$k = 1$: $\int_\Omega \operatorname{curl} u \cdot v - \int_\Omega u \cdot \operatorname{curl} v = \int_{\partial\Omega} (u \times v) \cdot n$,
$k = 0$: $\int_\Omega \operatorname{div} uv + \int_\Omega \operatorname{grad} uv = \int_{\partial\Omega} uv \cdot n$.

Homotopy operators

How to prove local exactness? If $d^k u = 0$, there exists ϕ, such that $u = d^{k-1}\phi$.

Theorem

Let Ω be a star-shaped domain. There exists $\mathfrak{p}^k : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ such that

$$d^{k-1}\mathfrak{p}^k + \mathfrak{p}^{k+1}d^k = I.$$

The assumption that Ω is star-shaped can be relaxed by a partition of unity.

Once we have homotopy operators, for $d^k u = 0$ we have $u = (dp + pd)u = dpu$. Now set $\phi = pu$.

Proof.

Without loss of generality, assume that Ω is star-shaped with respect to $x = 0$. Define

$$\mathfrak{p}^k\omega(x)(v_1, \cdots, v_{k-1}) := \int_0^1 t^{k-1}\omega(tx)(x, v_1, \cdots, v_{k-1})\, dt.$$

$$\begin{aligned} d\mathfrak{p}\omega(x)(v_1, \cdots, v_k) &= \sum_{i=1}^{k} \partial_{v_i}(\mathfrak{p}\omega)(x)(v_1, \cdots, \widehat{v_i}, \cdots, v_k) \\ &= \sum (-1)^{i+1} \int_0^1 t^{k-1}\omega(tx)(v_i, v_1, \cdots, \widehat{v_i}, \cdots, v_{k-1})\, dt \\ &\quad + \sum (-1)^{i+1} \int_0^1 t^k(\partial_{v_i}\omega)(tx)(x, v_1, \cdots, \widehat{v_i}, \cdots, v_{k-1})\, dt. \end{aligned}$$

$$\mathfrak{p}d\omega(x)(v_1, \cdots, v_k) = \sum_{i=1}^{k} t^k(d\omega)(x)(x, v_1, \cdots, v_k)$$

$$= \int_0^1 t^k(\partial_x\omega)(tx)(v_1, \cdots, v_k)\, dt + \sum (-1)^i \int_0^1 t^k(\partial_{v_i}\omega)(tx)(x, v_1, \cdots, \widehat{v_i}, \cdots, v_{k-1})\, dt.$$

This implies that

$$\begin{aligned} (d\mathfrak{p} + \mathfrak{p}d)\omega &= \int_0^1 kt^{k-1}\omega(tx)(x, v_1, \cdots, v_k)\, dt + \int_0^1 t^k(\partial_x\omega)(tx)(v_1, \cdots, v_k)\, dt \\ &= \int_0^1 \frac{d}{dt}[t^k\omega(tx)(v_1, \cdots, v_k)]\, dt = \omega(x)(v_1, \cdots, v_k). \end{aligned}$$

Summary of the Poincaré (homotopy) operators

$$\mathfrak{p}^k\omega(x)(v_1, \cdots, v_{k-1}) := \int_0^1 t^{k-1}\omega(tx)(x, v_1, \cdots, v_{k-1})\, dt.$$

given a base point $W = 0$ and choose a path $\gamma(t) = W + t(x - W)$,

simple example: given $u : \mathrm{curl}\, u = 0$, find $\phi := \mathfrak{p}u$, s.t. $u = \mathrm{grad}\, \phi$. ($\phi = \int_\gamma u\, ds$)

3D vector proxy (with $W = 0$):

$$\mathfrak{p}_1 u = \int_0^1 u_{tx} \cdot x dt, \quad \mathfrak{p}_2 v = \int_0^1 t v_{tx} \times x dt, \quad \mathfrak{p}_3 w = \int_0^1 t^2 w_{tx} x dt.$$

Properties

- homotopy property: $d^{k-1}\mathfrak{p}^k + \mathfrak{p}^{k+1}d^k = I$. (finding potential, $du = 0 \Rightarrow u = d(\mathfrak{p}u)$)
- complex property $\mathfrak{p}^k \circ \mathfrak{p}^{k+1} = 0$,
- polynomial-preservation: if $q \in \mathcal{P}_r\Lambda^k$, then $\mathfrak{p}^k q \in \mathcal{P}_{r+1}\Lambda^{k-1}$.

The Poincaré operators will be important for constructing finite elements.

Koszul operators: Poincaré operators on homogeneous polynomials

$$\mathfrak{p}^k\omega(x)(v_1,\cdots,v_{k-1}) := \int_0^1 t^{k-1}\omega(tx)(x,v_1,\cdots,v_{k-1})\,dt.$$

If $\omega \in \mathcal{H}_r\Lambda^k$ (k-forms with homogeneous polynomial of degree r coefficients):

$$\begin{aligned}\mathfrak{p}^k\omega(x)(v_1,\cdots,v_{k-1}) := \int_0^1 t^{r+k-1}\omega(x)(x,v_1,\cdots,v_{k-1})\,dt &= \frac{1}{r+k}\omega(x)(x,v_1,\cdots,v_{k-1})\\ &= \frac{1}{r+k}i_x.\end{aligned}$$

Define $\kappa^k := i_x$. Then on $\mathcal{H}_r\Lambda^k$, $\mathfrak{p}^k = \frac{1}{r+k}\kappa^k$.

- null-homotopy identity *on homogeneous polynomials*: $(d^{k-1}\kappa^k + \kappa^{k+1}d^k)u = (r+k)u,\ u \in \mathcal{H}_r\Lambda^k$,
- polynomial-preserving property:

$$u \in \mathcal{P}_r\Lambda^k \Rightarrow \mathfrak{p}^k u \in \mathcal{P}_{r+1}\Lambda^{k-1},$$

$$u \in \mathcal{H}_r\Lambda^k \Rightarrow \mathfrak{p}^k u \in \mathcal{H}_{r+1}\Lambda^{k-1}.$$

Now we can finally present the de Rham complex in a more general form:

$$\cdots \longrightarrow C^\infty\Lambda^{k-1} \xrightarrow{d^{k-1}} C^\infty\Lambda^k \xrightarrow{d^k} C^\infty\Lambda^{k+1} \longrightarrow \cdots,$$

where $C^\infty\Lambda^k$ is the space of differential k-forms with smooth coefficients.
We also have the L^2 version with unbounded operators

$$\cdots \longrightarrow L^2\Lambda^{k-1} \xrightarrow{d^{k-1}} L^2\Lambda^k \xrightarrow{d^k} L^2\Lambda^{k+1} \longrightarrow \cdots,$$

and one possible domain complex

$$\cdots \longrightarrow H\Lambda^{k-1} \xrightarrow{d^{k-1}} H\Lambda^k \xrightarrow{d^k} H\Lambda^{k+1} \longrightarrow \cdots,$$

where $L^2\Lambda^k$ is the space of differential k-forms with coefficients in L^2, and

$$H\Lambda^k := \{u \in L^2\Lambda^k : d^k u \in L^2\Lambda^{k+1}\}.$$

The following Sobolev version will also be used

$$\cdots \longrightarrow H^{q-(k-1)}\Lambda^{k-1} \xrightarrow{d^{k-1}} H^{q-k}\Lambda^k \xrightarrow{d^k} H^{q-(k+1)}\Lambda^{k+1} \longrightarrow \cdots.$$

We establish the cohomology of the above complexes in the rest of this section. As we shall see, all these complexes have uniform smooth representatives of cohomology (Hilbert scales). The dimension is finite and depends on the topology. Therefore we first introduce essential facts on topology.

One cannot deform to another via *continuous* maps. Goal of topology: categorise these families. Idea of *homology theory*: find out circles that are not boundary of a disk (and higher dimensional generalisations). This idea can be made precise by using language of complexes.

Definition (Simplex)

Given $k+1$ points $x_0, x_1, \cdots, x_k$ in $\mathbb{R}^n$ in general position (no $m+2$ points on a common m-dimensional plane), the convex hull $\sigma := [x_0, x_1, \cdots, x_k]$ is called a k-simplex.

Example: point ($k=0$), line ($k=1$), triangle ($k=2$), tetrahedron ($k=3$)...

Definition (Simplicial complex)

A simplicial complex Σ is the union of simplices such that

1. if a simplex $\sigma \subset \Sigma$, then all subsimplices of σ belongs to Σ;
2. if two simplices $\sigma_1, \sigma_2 \subset \Sigma$, then their intersection is either empty or their common face.

Example: triangulation, counting all the vertices, edges, faces...

Now we want to define a complex

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots, \tag{12}$$

with each C_k being "the collection of k-dimensional simplices", and $\partial_k \sigma$ "the boundary of σ" for $\sigma \in C_k$.

Next we make this idea precise.

Definition (k-chain)

Given a simplical complex Σ, a k-chain is a (formal) combination of all k-simplices, i.e.,

$$\sum_i c_i \sigma_1, \quad \lambda_i \in \mathbb{Z}.$$

Definition (chain groups)

All k-chains form an Abelian group, denoted by C_k, by

$$\sum \alpha_j \sigma_j + \sum \beta_j \sigma_j = \sum (\alpha_j + \beta_j) \sigma_j.$$

Definition (boundary operator $\partial_\bullet$)

$\partial_k : C_k(\Sigma) \to C_{k-1}(\Sigma)$ is a morphism which on each simplex

$$\partial_k[v_0, v_1, \cdots, v_k] = \sum_i (-1)^i [v_0, \cdots, v_{i-1}, \widehat{v_i}, v_{i+1}, \cdots, v_k],$$

and on chains

$$\partial_k \sum_i \alpha_i \sigma_i = \sum_i \alpha_i \partial_k \sigma_i, \quad \alpha_i \in \mathbb{Z}.$$

Theorem

Boundary of boundary vanishes, i.e., $\partial_{k-1} \circ \partial_k = 0$.

Idea of proof: cancelation.

Example:

$$T = [x_0, x_1, x_2],$$

$$\partial T = [x_1, x_2] - [x_0, x_2] + [x_0, x_1],$$

$$\partial\partial T = (x_2 - x_1) - (x_2 - x_0) + (x_1 - x_0) = 0.$$

Simplicial homology $H_i := \mathcal{N}(\partial_i)/\mathcal{R}(\partial_{i+1})$

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots .$$

Recall that all k-dimensional simplices form a basis of C_k. Elements of the dual space of C_k, denoted by C^k are called *k-cochains*. That is, a k-cochain assign a number to each k-simplex. Define $\partial^k : C^k \to C^{k+1}$ to be the dual of $\partial_{k+1} : C_{k+1} \to C_k$, i.e.,

$$\langle \partial^k g, \sigma \rangle := \langle g, \partial_{k+1}\sigma \rangle, \quad \sigma \in C_{k+1}, g \in C^k.$$

By definition, $\partial^{k+1} \circ \partial^k = 0$, and

$$\cdots \longrightarrow C^{i-1} \xrightarrow{\partial^{i-1}} C^i \xrightarrow{\partial^i} C^{i+1} \longrightarrow \cdots .$$

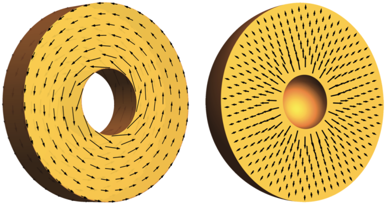
is called the simplicial cochain complex. cohomology $H^k := \mathcal{N}(\partial^k)/\mathcal{R}(\partial^{k-1})$.
Remark. In the "simplicial homology" the indices decrease and we write ∂_k rather than d^k since (12) consists of topological objects, rather than functions and differential operators. This difference between "homology" and "cohomology" is only formal and conventional.

Algebraic fact: $H^k \cong H_k$

Betti number: $b_k := \dim H^k = \dim H_k$. "dimension of k-dimensional closed loops that are not boundary" de Rham theorem The (smooth) de Rham cohomology is isomorphic to the simplicial cohomology $\mathscr{H}^k \cong H^k \cong H_k$.

de Rham complex and topology:

dimension of $\mathscr{H}^k$ = number of "k-dimensional holes" (c.f. de Rham theorem)



Examples where $\dim \mathscr{H}^1 = 1$ and $\dim \mathscr{H}^2 = 1$, respectively.
Left: curl-free field which is not grad, Right: div-free field with is not curl.

(figure from *Finite element exterior calculus*, D.N.Arnold, SIAM 2008.)

Remarks on topology

Example

- disk (solid circle) $D^2 \subset \mathbb{R}^2$: $b_0 = 1$ (boundary of any point vanishes), $b_1 = 0$, $b_2 = 0$.
- sphere S^2: $b_0 = 1$, $b_1 = 0$, $b_2 = 1$ (boundary of sphere itself vanishes).

Simply connected domains concept in homotopy theory - if any loop (or higher dimensional analogue) can continuously shrink to a point. Different concept than (co)homology, although there are connections.

- D^2 and S^2 are both simply connected, although they have different (co)homology groups
- more subtle in 3D.

The theory of FEEC is directly built on the (co)homology groups. It is possible to state results in terms of homotopy (simply connectedness) and explicit characterisation of topology. But this has to be done carefully. We say "Ω *is contractible*" or "Ω *has trivial topology*" if Ω is isomorphic to $\mathbb{R}^n$ (although $b_0 = 1$).

Nontrivial cohomology may appear even when the topology is trivial
e.g., solving PDEs with periodic boundary conditions on a square. (identifing the opposite sides of a square leads to a topological torus).

Sobolev complex

$$0 \longrightarrow H^q\Lambda^0 \xrightarrow{d^0} H^{q-1}\Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H^{q-n}\Lambda^n \longrightarrow 0. \qquad (13)$$

Theorem (Costabel, McIntosh, 2010)

The complex (13) *has uniform representation of cohomology for any real number q, i.e.,*

$$\mathcal{N}(d^k, H^{q-k}\Lambda^k) = d^{k-1}H^{q-(k-1)}\Lambda^{k-1} \oplus \mathscr{H}^k,$$

where $\mathscr{H}^k \subset C^\infty\Lambda^k$, not depending on q. Moreover, $\dim \mathscr{H}^k = \beta_k$*, the k-th Betti number.*

Idea of proof: partition of unity, on each piece construct $P^i : H^{q-i}\Lambda^i \to H^{q-(i-1)}\Lambda^{i-1}$, s.t. $d^{k-1}P^k + P^{k+1}d^k = I$ (implies local exactness immediately: $du = 0$ implies $u = (dP + Pd)u = d(Pu)$). For smooth functions: P found in manifold textbooks, as path integrals. For Sobolev functions: mollify the base points, prove they are pseudo-differential operators of order -1. Then globally, $dP + Pd = I - L$, where L comes from the partition of unity, reflecting topology.

Consequence of pseudo-differential operators: a wide range of spaces ($W^{k,p}$, Besov, Triebel-Lizorkin etc.)

Results also available for spaces with vanishing boundary conditions, e.g., H_0^q.

To investigate the cohomology of the elasticity and the conformal complex, we will derive these complexes from (several copies of) de Rham. The process is cohomology-preserving. So these complexes satisfy the assumptions for a Hilbert scale.

The machinery will be useful by itself : it provides a systematic way to generalise results for de Rham to other complexes.

A bit history: In FFEC, the investigation of these complexes started from Arnold, Falk, Winther's look at the 3D elasticity problem (Hellinger-Reissner principle). To obtain a complex involving symmetric matrices, they had communications with Eastwood (c.f., Eastwood "A complex from linear elasticity." 2000). The elasticity complex turns out to be a special case of the "Bernstein-Gelfand-Gelfand (BGG) resolution", which associates complexes to representation of Lie algebras. There are infinitely number of complexes derived in this way. This interaction between algebra, geometry, analysis (e.g., Korn type inequalities) and numerical schemes is still in progress.

For results and references before this unified treatment, c.f., Giuseppe Geymonat and Franoise Krasucki. "Hodge decomposition for symmetric matrix fields and the elasticity complex in Lipschitz domains." Communications on Pure and Applied Analysis 8.1 (2009): 295-309.

Bernstein-Gelfand-Gelfand construction: intuitive ideas

$$
\begin{tikzcd}
C^\infty \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & C^\infty \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] & C^\infty \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] & \cancel{C^\infty \otimes \mathbb{V}} \\
\cancel{C^\infty \otimes \mathbb{V}} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0 := \mathrm{mskw}"] & C^\infty \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & C^\infty \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2 := \mathrm{vskw}"] & C^\infty \otimes \mathbb{V}
\end{tikzcd}
$$

$$
0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\mathsf{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\mathsf{curl}\,\mathrm{T}\,\mathsf{curl}} C^\infty \otimes \mathbb{S} \xrightarrow{\mathsf{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0.
$$

Explicit way of doing homological algebra is needed.

Systematic construction: generating new complexes from existing ones

- input: $(Z^\bullet, D^\bullet)$, $(\tilde{Z}^\bullet, \tilde{D}^\bullet)$, connecting maps $S^i : \tilde{Z}^i \to Z^{i+1}$,
$$Z^i := V^i \otimes \mathbb{E}^i, \quad \tilde{Z}^i := V^{i+1} \otimes \tilde{\mathbb{E}}^i, \quad S^i := I \otimes s^i,$$
where $V^\bullet$ are Hilbert spaces, $\mathbb{E}^\bullet$ and $\tilde{\mathbb{E}}^\bullet$ are finite dimensional, $s^i : \tilde{\mathbb{E}}^i \to \mathbb{E}^{i+1}$ is a linear map. Assumptions:
    - (anti-)commutativity: $S^{i+1}\tilde{D}^i = -D^{i+1}S^i$,
    - injectivity/surjectivity condition: s^i injective for $i \leq J$, surjective for $i \geq J$.
$$\begin{tikzcd}
0 \arrow[r] & Z^0 \arrow[r, "D^0"] & Z^1 \arrow[r, "D^1"] & \cdots \arrow[r, "D^{n-1}"] & Z^n \arrow[r] & 0 \\
0 \arrow[r] & \tilde{Z}^0 \arrow[r, "\tilde{D}^0"] \arrow[ur, "S^0"] & \tilde{Z}^1 \arrow[r, "\tilde{D}^1"] \arrow[ur, "S^1"] & \cdots \arrow[r, "\tilde{D}^{n-1}"] \arrow[ur, "S^{n-1}"] & \tilde{Z}^n \arrow[r] & 0
\end{tikzcd}$$
- output:
$$0 \to \Upsilon^0 \xrightarrow{\mathscr{D}^0} \Upsilon^1 \xrightarrow{\mathscr{D}^1} \cdots \xrightarrow{\mathscr{D}^{J-1}} \Upsilon^J \xrightarrow{\mathscr{D}^J} \Upsilon^{J+1} \xrightarrow{\mathscr{D}^{J+1}} \cdots \xrightarrow{\mathscr{D}^{n-1}} \Upsilon^n \to 0, \quad (14)$$
where
$$\Upsilon^i := \begin{cases} V^i \otimes \mathcal{R}(s^{i-1})^\perp, & 0 \leq i \leq J, \\ V^{i+1} \otimes \mathcal{N}(s^i), & J < i \leq n, \end{cases} \qquad \mathscr{D}^i = \begin{cases} (\mathrm{id} \otimes P_{\mathcal{R}^\perp})D^i, & i < J; \\ \tilde{D}^J(S^J)^{-1}D^J, & i = J; \\ \tilde{D}^i, & i > J. \end{cases} \quad (15)$$
- conclusion:
$$\dim \mathscr{H}^i(\Upsilon^\bullet, \mathscr{D}^\bullet) \leq \dim \mathscr{H}^i(Z^\bullet, D^\bullet) + \dim \mathscr{H}^i\left(\tilde{Z}^\bullet, \tilde{D}^\bullet\right), \quad \forall i = 0, 1, \cdots, n$$
Equality holds if and only if S^i induces the zero maps on cohomology, i.e., $S^i\mathcal{N}(\tilde{D}^i) \subset \mathcal{R}(D^i)$.

A sufficient condition for checking $S^i\mathcal{N}(\tilde{D}^i) \subset \mathcal{R}(D^i)$: there exists $K^i : \tilde{Z}^i \to Z^i$, such that $S^i = D^i K^i - K^{i+1}\tilde{D}^i$.
Then $S\mathcal{N}(\tilde{D}) = (DK - K\tilde{D})\mathcal{N}(\tilde{D}) = D(K\mathcal{N}(\tilde{D}))$.
Therefore if we further assume the existence of such K operators, we precisely know the cohomology (not only an inequality).

Overview of the constructive proof:

cohomology-preserving surjective maps from the de Rham complex to BGG complexes

$$\begin{tikzcd} \cdots \arrow[r] & Y^{i-1} \arrow[r, "D^{i-1}"] \arrow[d, "\pi^{i-1}"] & Y^{i} \arrow[r, "D^{i}"] \arrow[d, "\pi^{i}"] & Y^{i+1} \arrow[r] \arrow[d, "\pi^{i+1}"] & \cdots \\ \cdots \arrow[r] & \Upsilon^{i-1} \arrow[r, "\mathscr{D}^{i-1}"] & \Upsilon^{i} \arrow[r, "\mathscr{D}^{i}"] & \Upsilon^{i+1} \arrow[r] & \cdots, \end{tikzcd}$$

where $(Y^\bullet, D^\bullet)$ is the product of de Rham complexes (two rows in the BGG diagram),

$$D^i := \begin{pmatrix} d^i & 0 \\ 0 & d^i \end{pmatrix},$$

and $(\Upsilon^\bullet, \mathscr{D}^\bullet)$ is the BGG complex (e.g., elasticity complex).

Details of construction: two steps.

- Step 1: from de Rham complex to twisted de Rham complex.

$$
\begin{tikzcd}
\cdots \arrow[r] & Y^{i-1} \arrow[r, "D^{i-1}"] \arrow[d, "\Pi^{i-1}"] & Y^{i} \arrow[r, "D^{i}"] \arrow[d, "\Pi^{i}"] & Y^{i+1} \arrow[r] \arrow[d, "\Pi^{i+1}"] & \cdots \\
\cdots \arrow[r] & Y^{i-1} \arrow[r, "\mathscr{A}^{i-1}"] & Y^{i} \arrow[r, "\mathscr{A}^{i}"] & Y^{i+1} \arrow[r] & \cdots,
\end{tikzcd}
$$

  where

$$\Pi^i := \begin{pmatrix} I & K^i \\ 0 & I \end{pmatrix}, \quad \mathscr{A}^i := \begin{pmatrix} d^i & -S^i \\ 0 & d^i \end{pmatrix}.$$

  $\mathscr{A} = d - S$: covariant derivative with flat connection.

- Step 2. twisted de Rham complex to a subcomplex by cohomology-preserving projections (a general algebraic argument).
  The BGG complex is isomorphic to the subcomplex.

Example: $\mathrm{Alt}^i = \mathrm{Alt}^i\mathbb{R}^n$: space of algebraic i-forms, $\mathrm{Alt}^{i,J} = \mathrm{Alt}^i \otimes \mathrm{Alt}^J$. Define $s^{i,J} : \mathrm{Alt}^{i,J} \to \mathrm{Alt}^{i-1,J+1}$:

$$s^{i,J}\mu(v_0,\cdots,v_i)(w_1,\cdots,w_{J-1}) := \sum_{l=0}^{i}(-1)^l\mu(v_0,\cdots,\widehat{v_l},\cdots,v_i)(v^l,w_1,\cdots,w_{J-1}),$$

$$\forall v_0,\cdots,v_i,w_1,\cdots,w_{J-1} \in \mathbb{R}^n.$$

$s^{i,J}$ is injective for $i \le J-1$, surjective for $i \ge J-1$.

$S^{i,J} = \mathrm{id} \otimes s^{i,J} : H^q \otimes \mathrm{Alt}^{i,J} \to H^q \otimes \mathrm{Alt}^{i+1,J-1}$

$$\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathrm{Alt}^{0,0} \arrow[r, "d"] & H^{q-1} \otimes \mathrm{Alt}^{1,0} \arrow[r, "d"] & \cdots \arrow[r, "d"] & H^{q-n} \otimes \mathrm{Alt}^{n,0} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathrm{Alt}^{0,1} \arrow[r, "d"] \arrow[ur, "S^{0,1}"] & H^{q-2} \otimes \mathrm{Alt}^{1,1} \arrow[r, "d"] \arrow[ur, "S^{1,1}"] & \cdots \arrow[r, "d"] \arrow[ur, "S^{n-1,1}"] & H^{q-n-1} \otimes \mathrm{Alt}^{n,1} \arrow[r] & 0 \\
 & \vdots & \vdots & & \vdots & \\
0 \arrow[r] & H^{q-n+1} \otimes \mathrm{Alt}^{0,n-1} \arrow[r, "d"] & H^{q-n} \otimes \mathrm{Alt}^{1,n-1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & H^{q-2n+1} \otimes \mathrm{Alt}^{n,n-1} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-n} \otimes \mathrm{Alt}^{0,n} \arrow[r, "d"] \arrow[ur, "S^{0,n}"] & H^{q-n-1} \otimes \mathrm{Alt}^{1,n} \arrow[r, "d"] \arrow[ur, "S^{1,n}"] & \cdots \arrow[r, "d"] \arrow[ur, "S^{n-1,n}"] & H^{q-2n} \otimes \mathrm{Alt}^{n,n} \arrow[r] & 0.
\end{tikzcd}$$

Vector proxies in 3D:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathsf{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathsf{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathsf{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathsf{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathsf{curl}"] \arrow[ur, "2\,\mathsf{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathsf{div}"] \arrow[ur, "\mathsf{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathsf{grad}"] \arrow[ur, "-\,\mathsf{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathsf{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathsf{div}"] \arrow[ur, "2\,\mathsf{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathsf{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathsf{curl}"] \arrow[ur, "\mathsf{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathsf{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$0 \longrightarrow H^q \otimes \mathbb{R} \xrightarrow{\mathsf{hess}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathsf{curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\mathsf{div}} H^{q-4} \otimes \mathbb{V} \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

Vector proxies in 3D:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$
0 \longrightarrow H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{def}} H^{q-2} \otimes \mathbb{S} \xrightarrow{\mathrm{inc}} H^{q-4} \otimes \mathbb{S} \xrightarrow{\mathrm{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.
$$

elasticity, defects, metric, curvature

Vector proxies in 3D:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^{q}\otimes\mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2}\otimes\mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3}\otimes\mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1}\otimes\mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2}\otimes\mathbb{V}; \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3}\otimes\mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4}\otimes\mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5}\otimes\mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3}\otimes\mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4}\otimes\mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5}\otimes\mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6}\otimes\mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

divdiv complex:

$$
0 \longrightarrow H^{q-2}\otimes\mathbb{V} \xrightarrow{\mathrm{dev\,grad}} H^{q-3}\otimes\mathbb{T} \xrightarrow{\mathrm{sym\,curl}} H^{q-4}\otimes\mathbb{S} \xrightarrow{\mathrm{div\,div}} H^{q-6}\otimes\mathbb{R} \longrightarrow 0.
$$

plate theory, elasticity

Vector proxies in 3D:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\text{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\text{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\text{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\text{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\text{curl}"] \arrow[ur, "2\,\text{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\text{div}"] \arrow[ur, "\text{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\text{grad}"] \arrow[ur, "-\,\text{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\text{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\text{div}"] \arrow[ur, "2\,\text{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\text{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\text{curl}"] \arrow[ur, "\text{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\text{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

grad curl complex:

$$
0 \longrightarrow H^q \xrightarrow{\text{grad}} H^{q-1} \otimes \mathbb{V} \xrightarrow{\text{grad curl}} H^{q-3} \otimes \mathbb{T} \xrightarrow{\text{curl}} H^{q-4} \otimes \mathbb{M} \xrightarrow{\text{div}} H^{q-5} \otimes \mathbb{V} \longrightarrow 0.
$$

couple stress, size effects

Vector proxies in 3D:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\text{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\text{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\text{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\text{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\text{curl}"] \arrow[ur, "2\,\text{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\text{div}"] \arrow[ur, "\text{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\text{grad}"] \arrow[ur, "-\,\text{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\text{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\text{div}"] \arrow[ur, "2\,\text{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\text{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\text{curl}"] \arrow[ur, "\text{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\text{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

curl div complex:

$$0 \to H^q \otimes \mathbb{V} \xrightarrow{\text{grad}} H^{q-1} \otimes \mathbb{M} \xrightarrow{\text{dev curl}} H^{q-2} \otimes \mathbb{T} \xrightarrow{\text{curl div}} H^{q-4} \otimes \mathbb{V} \xrightarrow{\text{div}} H^{q-5} \to 0.$$

couple stress, size effects

Vector proxies in 3D:
(diagonal maps: bijective; superdiagonal: surjective; subdiagonal: injective.)

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\,\mathrm{mskw}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-3} \otimes \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & H^{q-4} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & H^{q-6} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

grad div complex:

$$
0 \to H^q \xrightarrow{\mathrm{grad}} H^{q-1} \otimes \mathbb{V} \xrightarrow{\mathrm{curl}} H^{q-2} \otimes \mathbb{V}
$$

$$
\xrightarrow{\mathrm{grad\,div}} H^{q-4} \otimes \mathbb{V} \xrightarrow{\mathrm{curl}} H^{q-5} \otimes \mathbb{V} \xrightarrow{\mathrm{div}} H^{q-6} \to 0.
$$

K operators for deriving the precise cohomology

For the smooth version, the K operators are defined by the Koszul

$$\tilde{K}^J(\mu)(w_1, \cdots, w_{J-1}) := \mu(x, w_1, \cdots, w_{J-1}), \quad \forall w_1, \cdots, w_{J-1} \in \mathbb{R}^n.$$

(For the elasticity complex, vector proxies gives $\tilde{K}^J u := x \times u$.) Then one can check $d\tilde{K} - \tilde{K}d = S$.

Nevertheless, in the Sobolev case $\tilde{K}$ defined in this way does not map $\tilde{Z}^i$ to Z^i which has higher regularity. To fix this, define $K^i = P^{i+1}S^{i,J} + L^i\tilde{K}^{i,J}$., where P is the bounded Costabel-McIntosh homotopy operators. Then it is simple algebra to check:

$$\begin{aligned} dK - Kd = d(PS + L\tilde{K}) - (PS + L\tilde{K})d = dPS + PdS + dL\tilde{K} - L\tilde{K}d \\ = (I - L)S + Ld\tilde{K} - L\tilde{K}d = S. \end{aligned}$$

where we have used $dP + Pd = I - L$ and $dS = -Sd$.

The BGG diagrams provide a general machinery to generalise results for the de Rham complex to the BGG (e.g., elasticity) cases. The idea is to use the cohomology-projections. This approach lead to interesting results and many more to discover. To show an example, we investigate *Poincaré operators for elasticity*.

Explicit projections

$$
\begin{tikzcd}[column sep=large, row sep=large]
\begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d, "\begin{pmatrix} I & K \\ & I \end{pmatrix}"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d] & \begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[d] \\
\begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^0"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^1"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^2"'] & \begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[d, "\Pi^3"'] \\
\Gamma^0 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^1 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^2 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^3 \arrow[d, "\mathcal{I}"'] \\
C^\infty \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{inc}"] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] & C^\infty \otimes \mathbb{V}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d, "\Phi := \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d] \arrow[l, dashed, bend right, "\begin{pmatrix} \mathfrak{p} & \\ & \mathfrak{p} \end{pmatrix}"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & \\ & d \end{pmatrix}"] \arrow[d] & \begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[d] \\
\begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^0"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^1"'] & \begin{pmatrix} C^\infty \otimes \mathbb{M} \\ C^\infty \otimes \mathbb{M} \end{pmatrix} \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\Pi^2"'] & \begin{pmatrix} C^\infty \otimes \mathbb{V} \\ C^\infty \otimes \mathbb{V} \end{pmatrix} \arrow[d, "\Pi^3"'] \\
\Gamma^0 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^1 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] \arrow[uu, dashed, bend right, "\mathrm{id}"'] & \Gamma^2 \arrow[r, "\begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}"] \arrow[d, "\mathcal{I}"'] & \Gamma^3 \arrow[d, "\mathcal{I}"'] \\
C^\infty \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{inc}"] \arrow[l, bend left, "\mathscr{P}"] \arrow[u, dashed, bend right, "\mathcal{I}^{-1}"'] & C^\infty \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] & C^\infty \otimes \mathbb{V}
\end{tikzcd}
$$

Projections and lifting

Let $(W^\bullet, d^\bullet)$ be a subcomplex of $(V^\bullet, d^\bullet)$ and $\Pi^\bullet$ be cochain projections $((\Pi)^2 = \Pi$, $d\Pi = \Pi d)$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V^{i-1} & \xrightarrow{d} & V^i & \xrightarrow{d} & V^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \Pi^{i-1}} & & \downarrow{\scriptstyle \Pi^{i}} & & \downarrow{\scriptstyle \Pi^{i+1}} & & \\
\cdots & \longrightarrow & W^{i-1} & \xrightarrow{d} & W^i & \xrightarrow{d} & W^{i+1} & \longrightarrow & \cdots
\end{array}$$

Lemma

If right inverse $\Pi_\dagger$ ($\Pi\Pi_\dagger = \mathrm{id}$) commutes with d, then

$$\tilde{\mathfrak{p}}^i := \Pi^{i-1}\mathfrak{p}^i\Pi^i_\dagger$$

defines $\tilde{\mathfrak{p}}^i : W^i \mapsto W^{i-1}$ for the subcomplex $(W^\bullet, d^\bullet)$ satisfying

$$d^{i-1}\tilde{\mathfrak{p}}^i + \tilde{\mathfrak{p}}^{i+1}d^i = \mathrm{id}.$$

Result

Theorem

$$\mathscr{P}_1(\omega) := \int_0^1 \omega_{tx} \cdot x dt + \int_0^1 (1-t)x \wedge (\nabla \times \omega_{tx}) \cdot x dt,$$

$$\mathscr{P}_2 : \mu \mapsto x \wedge \left(\int_0^1 t(1-t)\mu_{tx} dt \right) \wedge x,$$

$$\mathscr{P}_3 : \mu \mapsto \operatorname{sym} \left(\int_0^1 t^2 x \otimes \mu dt - \left(\int_0^1 t^2(1-t) x \otimes \mu \wedge x dt \right) \times \nabla \right).$$

Then we have

$$\mathscr{P}_1(\operatorname{def} u) = u + \mathrm{RM}, \quad \forall u \in C^\infty \otimes \mathbb{V},$$

$$\mathscr{P}_2 \operatorname{inc} \mu + \operatorname{def} \mathscr{P}_1 \mu = \mu, \quad \forall \mu \in C^\infty \otimes \mathbb{S},$$

$$\mathscr{P}_3 \operatorname{div} \omega + \operatorname{inc} \mathscr{P}_2 \omega = \omega, \quad \forall \omega \in C^\infty \otimes \mathbb{S},$$

$$\operatorname{div} \mathscr{P}_3 v = v, \quad \forall v \in C^\infty \otimes \mathbb{V}.$$

- for $\mu \in C^\infty \otimes \mathbb{S}$ satisfying $\operatorname{inc} \mu = 0$, the Cesàro-Volterra path integral (1906, 1907)

$$\mu = \operatorname{def} (\mathscr{P}_1 \mu).$$

- complex property, polynomial-preserving property hold.

Koszul type operators

Define $\mathscr{K}_1^r : C^\infty \otimes \mathbb{S} \mapsto C^\infty \otimes \mathbb{V}$ by

$$\mathscr{K}_1^r : \omega \mapsto x \cdot \omega + \frac{1}{r+2} x \wedge (\nabla \times \omega) \cdot x, \quad \forall \omega \in C^\infty \otimes \mathbb{S},$$

and $\mathscr{K}_2^r : C^\infty \otimes \mathbb{S} \mapsto C^\infty \otimes \mathbb{S}$:

$$\mathscr{K}_2^r : u \mapsto x \wedge u \wedge x, \quad \forall u \in C^\infty \otimes \mathbb{S},$$

and define $\mathscr{K}_3^r : C^\infty \otimes \mathbb{V} \mapsto C^\infty \otimes \mathbb{S}$ by:

$$\mathscr{K}_3^r : v \mapsto \operatorname{sym}(x \otimes v) - \frac{1}{r+4} \operatorname{sym}\left((x \otimes v \wedge x) \times \nabla\right), \quad \forall v \in C^\infty \otimes \mathbb{V}.$$

- null-homotopy, polynomial preserving, Koszul type complex.
- duality:

$$\mathscr{K}_2^r u : v = u : \mathscr{K}_2^r v,$$
$$\int \mathscr{K}_1^{r+2} u : v = \int u : \mathscr{K}_3^r v.$$

Iterating the construction

Motivation: Korn inequality obtained by eliminating skew-symmetric components from grad.
"Generalised Korn inequalities":

$$\|u\|_1 \le C \| \operatorname{dev} \operatorname{sym} \operatorname{grad} u \|, \quad u \perp \mathcal{N}(\operatorname{dev} \operatorname{sym} \operatorname{grad}).$$

coming from any complex? Idea: eliminating trace component from sym grad (the elasticity complex), or eliminating the skew-symmetric components from dev grad (the divdiv complex).

$$\begin{tikzcd}
0 \arrow[r] & H^{q+1} \otimes \mathbb{R} \arrow[r, "\operatorname{grad}"] & H^{q} \otimes \mathbb{V} \arrow[r, "\operatorname{curl}"] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\operatorname{div}"] & H^{q-2} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q} \otimes \mathbb{V} \arrow[r, "\operatorname{dev} \operatorname{grad}"] \arrow[ur, "-\mathrm{id}"] & H^{q-1} \otimes \mathbb{T} \arrow[r, "\operatorname{sym} \operatorname{curl}"] \arrow[ur, "2 \operatorname{vskw}"] & H^{q-2} \otimes \mathbb{S} \arrow[r, "\operatorname{div} \operatorname{div}"] \arrow[ur, "\operatorname{tr}"] & H^{q-4} \otimes \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\operatorname{def}"] \arrow[ur, "-\operatorname{mskw}"] & H^{q-2} \otimes \mathbb{S} \arrow[r, "\operatorname{inc}"] \arrow[ur, "S"] & H^{q-4} \otimes \mathbb{S} \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{tr}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \arrow[r, "\operatorname{hess}"] \arrow[ur, "\iota"] & H^{q-4} \otimes \mathbb{S} \arrow[r, "\operatorname{curl}"] \arrow[ur, "S"] & H^{q-5} \otimes \mathbb{T} \arrow[r, "\operatorname{div}"] \arrow[ur, "2 \operatorname{vskw}"] & H^{q-6} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-4} \otimes \mathbb{R} \arrow[r, "\operatorname{grad}"] \arrow[ur, "\iota"] & H^{q-5} \otimes \mathbb{V} \arrow[r, "\operatorname{curl}"] \arrow[ur, "\operatorname{mskw}"] & H^{q-6} \otimes \mathbb{V} \arrow[r, "\operatorname{div}"] \arrow[ur, "-\mathrm{id}"] & H^{q-7} \otimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}$$

Example from iterative constructions

conformal deformation complex

ker of dev def: conformal Killing v.f. Cotton-York: flatness in conformal geometry

$$0 \longrightarrow H^q(\Omega)\otimes\mathbb{V} \xrightarrow{\operatorname{dev}\operatorname{def}} \underbrace{H^{q-1}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T}) \xrightarrow{\operatorname{cott}} H^{q-4}(\Omega)\otimes(\mathbb{S}\cap\mathbb{T})} \xrightarrow{\operatorname{div}} H^{q-5}(\Omega)\otimes\mathbb{V} \longrightarrow 0$$

gravitational wave variable: transverse-traceless (TT) gauge
(= symmetric, trace-free, div-free)

stress like variable def$u$ in NS
(Gopalakrishnan, Lederer, Schöberl, 2019)

Hodge decomp.: York split, Einstein constraint eqns.

trace-free Korn inequality:

$$\|u\|_1 \le C\|\operatorname{dev}\operatorname{def} u\|, \quad \forall u \in \mathcal{N}(\operatorname{dev}\operatorname{def}).$$

$\mathcal{N}(\operatorname{dev}\operatorname{def})$: conformal Killing fields

The trace-free Korn inequality does not hold in 2D.
open problem (Chipot 2020): minimal number of linear functionals l_i, s.t. generalized Korn inequality holds

$$\|u\|_1 \le C(\sum_{i=1}^{N}\|l_i(\nabla u)\|_{L^2} + \|u\|_{L^2}).$$

e.g., 3D Poincaré: N=9; Korn: N= 6; trace-free Korn: N=5.

Conclusion from the iterative diagram: dimensional of cohomology $\leq$ input. Nevertheless, it turns out to be hard to figure out the precise information of cohomology.

Moreover, the idea of eliminating trace component from the elasticity complex or eliminating the skew-symmetric components from the divdiv complex does not work in 2D. The following diagram

$$\begin{tikzcd} 0 \arrow[r] & H^q \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & H^{q-1} \otimes \mathbb{S} \arrow[r, "\mathrm{rot}\,\mathrm{rot}"] & H^{q-3} \arrow[r] & 0 \\ 0 \arrow[r] & H^{q-1} \arrow[r, "\mathrm{grad}\,\mathrm{grad}"] \arrow[ur, "\iota"] & H^{q-3} \otimes \mathbb{S} \arrow[r, "\mathrm{rot}"] \arrow[ur, "\mathrm{tr}"] & H^{q-4} \arrow[r] & 0, \end{tikzcd} \tag{16}$$

does not have a bijective operator. This is consistent with the fact that the conformal Korn inequality does not hold in 2D. Can we fix it?

This inspired a generalisation of the framework (Čap, Hu 2021 in preparation).

$$
\begin{tikzcd}
0 \arrow[r] & Z^{0,0} \arrow[r, "d^{0,0}"] & Z^{1,0} \arrow[r, "d^{1,0}"] & \cdots \arrow[r, "d^{n-1,0}"] & Z^{n,0} \arrow[r] & 0 \\
0 \arrow[r] & Z^{0,1} \arrow[r, "d^{0,1}"] \arrow[u] \arrow[ur, "S^{0,1}"] & Z^{1,1} \arrow[r, "d^{1,1}"] \arrow[u] \arrow[ur, "S^{1,1}"] & \cdots \arrow[r, "d^{n-1,1}"] \arrow[u] \arrow[ur, "S^{n-1,1}"] & Z^{n,1} \arrow[r] \arrow[u] & 0 \\
 & \cdots \arrow[u] \arrow[ur, "S^{0,2}"] & \cdots \arrow[u] \arrow[ur, "S^{1,2}"] & \cdots \arrow[u] \arrow[ur, "S^{n-1,2}"] & \cdots \arrow[u] & \\
0 \arrow[r] & Z^{0,N} \arrow[r, "d^{0,N}"] \arrow[u] \arrow[ur, "S^{0,N}"] & Z^{1,N} \arrow[r, "d^{1,N}"] \arrow[u] \arrow[ur, "S^{1,N}"] & \cdots \arrow[r, "d^{n-1,N}"] \arrow[u] \arrow[ur, "S^{n-1,N}"] & Z^{n,N} \arrow[r] \arrow[u] & 0
\end{tikzcd}
\tag{17}
$$

Assumptions:

$$S^{i+1,j-1} \circ S^{i,j} = 0, \quad \forall i, j \geq 0, \tag{18}$$

and there exist linear operators $K^{i,j} : Z^{i,j} \to Z^{i,j-1}$, $1 \leq i \leq n, 0 \leq j \leq N$, such that

$$S^{i,j-1} K^{i,j} = K^{i+1,j-1} S^{i,j}, \tag{19}$$

$$S^{i,j} = d^{i,j-1} K^{i,j} - K^{i+1,j} d^{i,j}. \tag{20}$$

Corollary: $dS = -Sd$. Moreover, we assume that $\mathcal{R}(S^{i,j})$ is closed for any i, j.
No injectivity/surjectivity condition is assumed! advantage for discretisation

Each sequence $(V^\bullet, S^\bullet)$ in the northeast direction $\nearrow$ is also a complex with S (Lie algebra cohomology).

$$Z^{i,j} = \mathcal{R}(S^{i-1,j+1}) \oplus \mathcal{R}(S^{i-1,j+1})^\perp = \mathcal{R}(S^{i-1,j+1}) \oplus \mathcal{N}(S^{i,j})^\perp \oplus H^{i,j}, \tag{21}$$

where $H^{i,j} := \mathcal{R}(S^{i-1,j+1})^\perp \cap \mathcal{N}(S^{i,j})$ is an analogy of the space of harmonic forms. These "algebraic harmonic forms" will be the spaces in the final "BGG sequences".

Step 1: twisted complex

Define the twisted operator $d_V^i : Y^i \to Y^{i+1}$ by

$$d_V^i := \begin{pmatrix} d^{i,0} & -S^{i,1} & 0 & 0 & \cdots & 0 \\ 0 & d^{i,1} & -S^{i,2} & 0 & \cdots & 0 \\ & & \cdots & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & d^{i,N} \end{pmatrix}.$$

From $dS = -Sd$, we have $d_V^{i+1} \circ d_V^i = 0$, $0 \le i \le n-1$. The complex $(Y^\bullet, d_V^\bullet)$, i.e.,

$$\cdots \longrightarrow Y^{i-1} \xrightarrow{d_V^{i-1}} Y^i \xrightarrow{d_V^i} Y^{i+1} \longrightarrow \cdots, \tag{22}$$

is referred to as the *twisted complex*, or the d_V-complex, as a variation of the sum $(Y^\bullet, d^\bullet)$.

Cochain isomorphism between d and d_V complexes:

$$K^i := \begin{pmatrix} 0 & K^{i,1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & K^{i,2} & 0 & \cdots & 0 \\ & & \cdots & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

define $F^i := \exp(K^i)$, the matrix exponential of K^i, i.e.,

$$\begin{aligned} F^i &:= I + K + \frac{1}{2}K^2 + \frac{1}{6}K^3 + \cdots \\ &= \begin{pmatrix} I & K^{i,1} & \frac{1}{2}K^{i,1}K^{i,2} & \frac{1}{6}K^{i,1}K^{i,2}K^{i,3} & \cdots & \frac{1}{N!}K^{i,1}K^{i,2}\cdots K^{i,N} \\ 0 & I & K^{i,2} & \frac{1}{2}K^{i,2}K^{i,3} & \cdots & \frac{1}{(N-1)!}K^{i,2}K^{i,3}\cdots K^{i,N} \\ & & \cdots & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & I \end{pmatrix}. \end{aligned}$$

$F^i : Y^i \to Y^{i+1},\ i = 0, 1, \cdots, n$ are cochain isomorphism, such that the following diagram commutes:

$$\begin{tikzcd} Y^i \arrow[r, "d^i"] \arrow[d, "F^i"] & Y^{i+1} \arrow[d, "F^{i+1}"] \\ Y^i \arrow[r, "d_V^i"] & Y^{i+1} \end{tikzcd}$$

Step 2: from twisted (d_V) complex to BGG complex

BGG complex is the subcomplex of $(Y^\bullet, d_V^\bullet)$ restricted to the algebraic harmonic spaces. E.g., elasticity complex.

deriving the conformal deformation complex

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] & H^{q-2} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] & H^{q-3} \otimes \mathbb{V} \arrow[r] & {} \\
0 \arrow[r] & H^s \otimes (\mathbb{R} \times \mathbb{V}) \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,1}"] & H^{s-1} \otimes (\mathbb{V} \times \mathbb{M}) \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^{1,1}"] & H^{s-2} \otimes (\mathbb{V} \times \mathbb{M}) \arrow[r, "\mathrm{div}"] \arrow[ur, "S^{2,1}"] & H^{s-3} \otimes (\mathbb{R} \times \mathbb{V}) \arrow[r] & {} \\
0 \arrow[r] & H^t \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,2}"] & H^{t-1} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^{1,2}"] & H^{t-2} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^{2,2}"] & H^{t-3} \otimes \mathbb{V} \arrow[r] & {}
\end{tikzcd}
\tag{23}
$$

$S^{0,1} = (\iota, -\operatorname{mskw})^T$, i.e., $S^{0,1}(w, v) = wI - \operatorname{mskw} v$. Similarly, $S^{1,1} = (-\operatorname{mskw}, S)^T$, $S^{2,1} = (I, 2\operatorname{vskw})^T$, $S^{0,2} = (I, -\operatorname{mskw})^T$, $S^{1,2} = (2\operatorname{vskw}, S)^T$, $S^{2,2} = (\operatorname{tr}, 2\operatorname{vskw})^T$. The K operators are defined by $K^{1,i} = (x\otimes, x\wedge)^T$, $K^{2,i} = (x\cdot, x\wedge)$.

Conclusion: cohomology of the conformal deformation complex is isomorphic to the input (de Rham) complexes.

A nontrivial example: 2D conformal Korn inequality

- what if the injectivity/surjectivity does not hold

$$
\begin{tikzcd}
0 \arrow[r] & H^q \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{q-1} \otimes \mathbb{M} \arrow[r, "\mathrm{rot}"] & H^{q-2} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes (\mathbb{R} \times \mathbb{R}) \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,1}"] & H^{q-2} \otimes (\mathbb{V} \times \mathbb{V}) \arrow[r, "\mathrm{rot}"] \arrow[ur, "S^{1,1}"] & H^{q-3} \otimes (\mathbb{R} \times \mathbb{R}) \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,2}"] & H^{q-3} \otimes \mathbb{M} \arrow[r, "\mathrm{rot}"] \arrow[ur, "S^{1,2}"] & H^{q-4} \otimes \mathbb{V} \arrow[r] & 0.
\end{tikzcd}
\tag{24}
$$

Here $S^{0,1} = -\perp$, $S^{1,1} = \mathrm{tr}$, $S^{0,2} = -\iota$, and $S^{1,2} = I$. These operators can be obtained by (20) with $K^{i,1}(u,v) := x \otimes u + x^{\perp} \otimes v$, and $K^{i,2}(u) := (x \cdot u, x^{\perp} \cdot u)$.
The output complex is

$$
0 \longrightarrow H^q \otimes \mathbb{V} \xrightarrow{D^0} \begin{pmatrix} H^{q-1} \otimes (\mathbb{S} \cap \mathbb{T}) \\ H^{q-3} \otimes (\mathbb{S} \cap \mathbb{T}) \end{pmatrix} \xrightarrow{D^1} H^{q-3} \longrightarrow 0, \tag{25}
$$

where

$$
D^0 := (\mathrm{dev}\,\mathrm{def}, \mathrm{hess}\,\mathrm{sskw}\,\mathrm{grad} - \frac{1}{2}\mathrm{grad} \perp \mathrm{grad}\,\mathrm{div}),
$$

and

$$
D^1 := (\mathrm{rot}[-2\,\mathrm{mskw} + \frac{1}{2}\iota]\,\mathrm{rot}\,\mathrm{rot}, \mathrm{rot})^T.
$$

We first recall inf-sup conditions and the Babuška theory. They will be used to prove the well-posedness of the continuous and discrete Hodge-Laplacian problems.

inf-sup condition: finite dimensional case

For $A \in \mathbb{R}^{n\times n}$, $u, f \in \mathbb{R}^n$, consider

$$Au = f. \tag{26}$$

Inf-sup conditions: there exists a positive constant γ such that

$$\inf_{w\in\mathbb{R}^n} \sup_{v\in\mathbb{R}^n} \frac{(Aw, v)}{\|w\|_{l^2}\|v\|_{l^2}} \equiv \inf_{w\in\mathbb{R}^n} \sup_{v\in\mathbb{R}^n} \frac{(w, A^T v)}{\|w\|_{l^2}\|v\|_{l^2}} \geq \gamma > 0, \tag{27}$$

and

$$\inf_{v\in\mathbb{R}^n} \sup_{w\in\mathbb{R}^n} \frac{(Aw, v)}{\|w\|_{l^2}\|v\|_{l^2}} \geq \gamma > 0. \tag{28}$$

(27): for any w there is $v \in \mathbb{R}^n$ such that $A^T v = w$, i.e., surjectivity of A^T (injectivity of A).
(28): surjectivity of A (injectivity of A^T).
Therefore, (27)-(28) implies the existence and uniqueness of the solution to (26).

Norm estimates

From (27),

$$\|Aw\|_{l^2} \geq \gamma\|w\|_{l^2}, \quad \forall w \in \mathbb{R}^n.$$

This implies that

$$\|w\|_{l^2} \geq \gamma\|A^{-1}w\|_{l^2} \quad \forall w \in \mathbb{R}^n,$$

and therefore

$$\|A^{-1}\|_{\mathcal{L}(l^2,l^2)} \leq \gamma^{-1}.$$

Infinite dimensional cases

Let V and Q be two Hilbert spaces and $B : V \mapsto Q'$ be a linear operator, $b(v, q) := \langle Bv, q\rangle$.
Inf-sup condition: there exist a positive constant γ such that

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \gamma > 0. \tag{29}$$

$$\inf_{v\in V} \sup_{q\in Q} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \gamma > 0, \tag{30}$$

(29): B is injective (B^* is surjective). To see this, by the Banach closed range theorem

$$\mathcal{R}(B) = Q' \quad \Leftrightarrow \quad \|B^* q\|_{V'} \geq \beta \|q\|_Q, \ \forall q \in Q \quad \Leftrightarrow \quad \inf_{q\in Q} \sup_{v\in V} \frac{(Bv, q)}{\|v\|_V \|q\|_Q} \geq \gamma > 0.$$

(30): B is surjective (B^* is injective).

Babuška theory Let X be a Banach space. Given a bilinear form $\mathscr{A}(\cdot,\cdot) : X \times X \mapsto \mathbb{R}$ and $F \in X^*$, consider the variational problem: find $\xi \in X$ such that for any $\eta \in X$,

$$\mathscr{A}(\xi,\eta) = \langle F, \eta \rangle. \tag{31}$$

Next theorem summarizes the Babuška theory.

Theorem

(Babuška theory) If the following conditions hold:

1. *boundedness: there is a positive constant C such that*

$$\mathscr{A}(\xi,\eta) \leq C\|\xi\|_X \|\eta\|_X, \tag{32}$$

2. *inf-sup conditions: there exists a positive constant γ such that*

$$\inf_{\xi \in X} \sup_{\eta \in X} \frac{\mathscr{A}(\xi,\eta)}{\|\xi\|_X \|\eta\|_X} \geq \gamma > 0, \tag{33}$$

and

$$\inf_{\eta \in X} \sup_{\xi \in X} \frac{\mathscr{A}(\xi,\eta)}{\|\xi\|_X \|\eta\|_X} \geq \gamma > 0, \tag{34}$$

then the variational problem (31) has a unique solution ξ, satisfying the stability estimate:

$$\|\xi\|_X \leq C\|F\|_{X^*}.$$

in the context of complexes: Poincaré inequality implies inf-sup condition

$$V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1}$$

$$\|v\|_V^2 = \|v\|^2 + \|dv\|^2,$$

$$\inf_{w \in \mathcal{R}(d^k)} \sup_{v \in V^k} \frac{(d^k v, w)}{\|v\|_V \|w\|_V} \geq \gamma > 0.$$

Proof.

For any $w \in \mathcal{R}(d)$, take $v : dv = w, \ v \perp \mathcal{N}(d)$,

$$\|v\|_V \leq C\|dv\|.$$

Note that $\|w\|_V = \|w\|$ since $dw = 0$. Now

$$(dv, w) = \|dv\|\|w\| \geq C^{-1}\|v\|_V\|w\|_V.$$

$\square$

Therefore in the context of complexes, the inf-sup condition holds as long as the algebra (cohomology, thus Poincaré inequalities) is done right.

The discrete theory will be established with a complex with finite dimensional spaces

$$\cdots \longrightarrow V_h^{j-1} \xrightarrow{d^{j-1}} V_h^j \xrightarrow{d^j} V_h^{j+1} \longrightarrow \cdots, \tag{35}$$

which is considered as a discretisation of

$$\cdots \longrightarrow V^{j-1} \xrightarrow{d^{j-1}} V^j \xrightarrow{d^j} V^{j+1} \longrightarrow \cdots. \tag{36}$$

Discrete L^2-adjoint operator $d_h^{*,j} : V_h^j \to V_h^{j-1}$, defined as the adjoint of $d : V_h^{j-1} \to V_h^j$, i.e.,

$$(d_h^{*,j} u_h, v_h) = (u_h, d^{j-1} v_h), \quad \forall v_h \in V_h^{j-1}.$$

In principle, in finite element methods one can invert a mass matrix to get explicit form of $d_h^{*,j}$.

The question now is, how can we carry over the properties for $(V^\bullet, d^\bullet)$ to $(V_h^\bullet, d^\bullet)$?

A key concept in FEEC is the *bounded cochain projections*, i.e., linear operators $\Pi^j : V^j \to V_h^j, j = 1, 2, \cdots$, such that

- W-boundedness:
$$\left\|\Pi^j u\right\| \leq C\|u\|,$$
or the V-boundedness:
$$\left\|\Pi^j u\right\|_V \leq C\|u\|_V,$$
$(\|u\|_V^2 := \|u\|^2 + \|du\|^2)$
- (cochain maps) the diagram commutes, i.e., $d^j \Pi^j = \Pi^{j+1} d^j, \ \forall j,$
$$\begin{array}{ccccccccc} \cdots & \longrightarrow & V^{j-1} & \xrightarrow{d^{j-1}} & V^j & \xrightarrow{d^j} & V^{j+1} & \longrightarrow & \cdots \\ & & \big\downarrow \Pi^{j-1} & & \big\downarrow \Pi^j & & \big\downarrow \Pi^{j+1} & & \\ \cdots & \longrightarrow & V_h^{j-1} & \xrightarrow{d^{j-1}} & V_h^j & \xrightarrow{d^j} & V_h^{j+1} & \longrightarrow & \cdots, \end{array}$$
- Π^j is a projection: $\Pi^j|_{V_h^j} = I$ and $(\Pi^j)^2 = \Pi^j$.

The W-boundedness implies the V-boundedness, since
$$\|d\pi u\| = \|\pi du\| \leq C\|du\|,$$
and thus
$$\|\pi u\|_V \leq C\|u\|_V.$$
For many applications, the V-boundedness is enough. For nonlinear problems and eigenvalue problems, we will need the W-boundedness.

Bounded cochain projections imply the discrete Poincaré inequalities.

Theorem (discrete Poincaré inequality)

For any $u_h \in V_h^k$, $u_h \perp \mathcal{N}(d^k, V_h^k)$, there exists a positive constant C such that

$$\|u_h\| \leq C\|d^k u_h\|. \tag{37}$$

Theorem 23 is not an immediate corollary of the continuous inequality, as $u_h \perp \mathcal{N}(d^k, V_h^k)$ does not imply $u_h \perp \mathcal{N}(d^k, V^k)$ which is required for the Poincaré inequality at the continuous level.

Proof.

There exists $z \in V^k$ satisfying $dz = du_h$ and $z \in \mathcal{N}(d^k, V^k)^{\perp}$. From the Poincaré inequality at the continuous level,

$$\|z\| \leq C\|dz\| = C\|du_h\|.$$

Consider $w_h := u_h - \Pi^k z$. By the commutativity, we have $dw_h = 0$ and therefore $(u_h, u_h - \Pi^k z) = 0$. Thus

$$\|u_h\|^2 = (u_h, \Pi^k z),$$

and

$$\|u_h\| \leq \|\Pi^k z\| \leq C\|du_h\|.$$

□

Theorem (dual discrete Poincaré inequality)

For any $v_h \in V_h^k$, $v_h \perp \mathcal{N}(d_h^*, V_h^k)$,

$$\|v_h\| \leq C^{-1}\|d_h^* v_h\|,$$

where C *is the constant in the Poincaré inequality* (37).

Proof.

The condition $v_h \perp \mathcal{N}(d_h^*, V_h^k)$ implies $v_h \in d_h^{k-1}V_h^{k-1}$. Therefore

$$\begin{aligned}
\|v_h\| &= \sup_{w_h \in d_h^{k-1}V_h^{k-1}} (v_h, w_h)/\|w_h\| = \sup_{\phi_h \perp \mathcal{N}(d^{k-1}, V_h^{k-1})} (v_h, d\phi_h)/\|d\phi_h\| \\
&= \sup_{\phi_h \perp \mathcal{N}(d^{k-1}, V_h^{k-1})} (d_h^* v_h, \phi_h)/\|d\phi_h\| \\
&\leq \sup_{\phi_h \perp \mathcal{N}(d^{k-1}, V_h^{k-1})} \|d_h^* v_h\|\|\phi_h\|/\|d\phi_h\| \leq C^{-1}\|d_h^* v_h\|.
\end{aligned}$$

□

Combining the two discrete Poincaré inequalities (for the two components in the Hodge decomposition, respectively), we have *when the cohomology is trivial*,

$$\|u_h\| \leq C(\|du_h\| + \|d_h^* u_h\|), \quad \forall u_h \in V_h^k.$$

For a Hilbert complex $(W^\bullet, d^\bullet)$, the Hodge-Laplacian operator $L^k : W^k \to W^k$ is

$$L^k := d^{k-1}d_k^* + d_{k+1}^* d^k.$$

L^k is an unbounded operator, and its domain is

$$\mathscr{D}(L^k) := \{u \in V^k \cap V_k^* : d^k u \in V_{k+1}^*, d_k^* u \in V^{k-1}\}.$$

Strong formulation
Given $f \in W^k$, the Hodge Laplacian problem is to solve $u \in \mathscr{D}(L^k),\ u \perp \mathscr{H}^k$ satisfying

$$(d^{k-1}d_k^* + d_{k+1}^* d^k)u = f - P_{\mathscr{H}^k} f. \tag{38}$$

Primal weak formulation Find $u \in V^k \cap V_k^*,\ u \perp \mathscr{H}^k$ such that for all $v \in V^k \cap V_k^*$,

$$(du, dv) + (d^* u, d^* v) = (f - P_{\mathscr{H}^k} f, v).$$

Mixed weak formulation introduce $\sigma = d^* u,\ p = P_{\mathscr{H}^k} u$.
Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathscr{H}^k$, such that for all $(\tau, v, q) \in V^{k-1} \times V^k \times \mathscr{H}^k$,

$$\begin{aligned}
(\sigma, \tau) - (u, d\tau) &= 0, \\
(d\sigma, v) + (du, dv) + (p, v) &= (f, v) \\
(u, q) &= 0.
\end{aligned}$$

The three formulations are equivalent to each other.

Example: de Rham complex

$$(d^{k-1}d_k^* + d_{k+1}^* d^k)u = f.$$

On the boundary conditions: we can choose the domain complex to be either with or without boundary conditions. Then the boundary conditions are what we get from the weak form. In the example below, we consider spaces with boundary conditions as the domain complex, i.e., in 3D

$$0 \longrightarrow H_0^1 \xrightarrow{\text{grad}} H_0(\text{curl}) \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

Example: de Rham complex

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$$0 \longrightarrow H_0^1 \xrightarrow{\text{grad}} H_0(\text{curl}) \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

$$0 \underset{}{\overset{}{\rightleftarrows}} {\color{red}L^2(\Omega)} \underset{-\,\text{div}}{\overset{\text{grad}}{\rightleftarrows}} L^2(\Omega;\mathbb{R}^3) \qquad L^2(\Omega;\mathbb{R}^3) \qquad L^2(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\,\text{div}\,\text{grad}\, u = f.$$

$$u = 0, \quad \text{on } \partial\Omega.$$

Poisson equation, Dirichlet boundary condition.

Example: de Rham complex

$$(d^{k-1}d_k^* + d_{k+1}^* d^k)u = f.$$

On the boundary conditions: we can choose the domain complex to be either with or without boundary conditions. Then the boundary conditions are what we get from the weak form. In the example below, we consider spaces with boundary conditions as the domain complex, i.e., in 3D

$$0 \longrightarrow H_0^1 \xrightarrow{\text{grad}} H_0(\text{curl}) \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

$$0 \qquad L^2(\Omega) \underset{-\operatorname{div}}{\overset{\text{grad}}{\rightleftarrows}} {\color{red} L^2(\Omega;\mathbb{R}^3)} \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} L^2(\Omega;\mathbb{R}^3) \qquad L^2(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{grad}\operatorname{div} v + \operatorname{curl}\operatorname{curl} v = f.$$

$$v \times n = 0,\ \operatorname{div} v = 0, \quad \text{on } \partial\Omega.$$

Maxwell equations, electric boundary condition.

Example: de Rham complex

$$(d^{k-1}d_k^* + d_{k+1}^* d^k)u = f.$$

On the boundary conditions: we can choose the domain complex to be either with or without boundary conditions. Then the boundary conditions are what we get from the weak form. In the example below, we consider spaces with boundary conditions as the domain complex, i.e., in 3D

$$0 \longrightarrow H_0^1 \xrightarrow{\text{grad}} H_0(\text{curl}) \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

$$0 \qquad L^2(\Omega) \qquad L^2(\Omega;\mathbb{R}^3) \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} {\color{red}L^2(\Omega;\mathbb{R}^3)} \underset{-\,\text{grad}}{\overset{\text{div}}{\rightleftarrows}} L^2(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$\text{curl}\,\text{curl}\, v - \text{grad}\,\text{div}\, v = f.$$

$$v \cdot n = 0,\ \text{curl}\, v \times n = 0, \quad \text{on } \partial\Omega.$$

Maxwell equations, magnetic boundary condition.

Example: de Rham complex

$$(d^{k-1}d_k^* + d_{k+1}^* d^k)u = f.$$

On the boundary conditions: we can choose the domain complex to be either with or without boundary conditions. Then the boundary conditions are what we get from the weak form. In the example below, we consider spaces with boundary conditions as the domain complex, i.e., in 3D

$$0 \longrightarrow H_0^1 \xrightarrow{\mathrm{grad}} H_0(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H_0(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0.$$

$$0 \qquad L^2(\Omega) \qquad L^2(\Omega;\mathbb{R}^3) \qquad L^2(\Omega;\mathbb{R}^3) \underset{-\,\mathrm{grad}}{\overset{\mathrm{div}}{\rightleftarrows}} {\color{red}L^2(\Omega)} \rightleftarrows 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div}\operatorname{grad} u = f.$$

$$\operatorname{grad} u \cdot n = 0, \quad \text{on } \partial\Omega.$$

Poisson equation, Neumann boundary condition.

To simplify the presentation, in the following we assume that the cohomology is trivial. Recall that

$$\|\phi\|_V^2 := \|\phi\|^2 + \|d\phi\|^2.$$

We define $X := V^k \times V^{k-1}$ and recast the mixed formulation as: Find $(u, \sigma) \in X$ such that for all $(v, \tau) \in X$,

$$\mathscr{A}(u, \sigma; v, \tau) := (du, dv) + (d\sigma, v) + (u, d\tau) - (\sigma, \tau) = (f, v). \tag{39}$$

As a product space, the space X is equipped with the norm

$$\|(u, \sigma)\|_X^2 := \|u\|_V^2 + \|\sigma\|_V^2.$$

Now we can prove the well-posedness of (39) with the X- norm.

Theorem

The problem (39) *has a unique solution* $(u, \sigma) \in X$, *and the following estimate holds:*

$$\|(u, \sigma)\|_X \leq C\|f\|_{(V^k)^*}.$$

Proof The boundedness of $\mathscr{A}$ is obvious. We only prove the inf-sup condition.
Given $(v,p) \in X$, we have $v = v_0 + v_\perp$, where $v_0 \in \mathcal{N}(d)$ and $v_\perp \in \mathcal{N}(d)^\perp$. From the Hodge decomposition and the Poincaré inequalities, there exists $\phi \in V^{k-1}$ s.t. $d\phi = u_0$ and $\|\phi\|_V \leq c_1\|u_0\| \leq c_1\|u\|$. On the other hand, there exists c_2 s.t. $\|u_\perp\| \leq c_2\|du\|$.
Taking $(u,\sigma) = (v + d\tau, -\tau + s\phi)$, we have

$$\begin{aligned}\mathscr{A}(u,\sigma;v,\tau) &= \|dv\|^2 + s(v_0,v) + \|d\tau\|^2 + \|\tau\|^2 + s(\phi,\tau) \\ &\geq \frac{1}{2}\|dv\|^2 + \frac{1}{2}c_2^{-2}\|v_\perp\|^2 + s\|v_0\|^2 + \|d\tau\|^2 + \|\tau\|^2 - \frac{1}{2}s^2\|\phi\|^2 - \frac{1}{2}\|\tau\|^2 \\ &\geq \frac{1}{2}\|dv\|^2 + \frac{1}{2}\left(c_2^{-2} - s^2c_1^2\right)\|v_\perp\|^2 + s\|v_0\|^2 + \|d\tau\|^2 + \frac{1}{2}\|\tau\|^2.\end{aligned}$$

Choosing $s = \frac{1}{2}c_1^{-1}c_2^{-1}$, we have

$$\mathscr{A}(u,\sigma;v,\tau) \geq \frac{1}{2}\|dv\|^2 + \frac{1}{4}c_2^{-2}\|v_\perp\|^2 + s\|v_0\|^2 + \|d\tau\|^2 + \frac{1}{2}\|\tau\|^2 \geq \|(v,\tau)\|_X^2.$$

On the other hand, we have

$$\|(u,\sigma)\|_X^2 = \|(v + d\tau, -\tau + 1/2c_1^{-1}c_2^{-1}\phi)\|^2 \lesssim \|(v,\tau)\|_X^2,$$

where the bounds only depend on the constants c_1 and c_2. Therefore the inf-sup condition

$$\inf_{(v,\tau)\in X}\sup_{(u,\sigma)\in X}\frac{\mathscr{A}(u,\sigma;v,\tau)}{\|(u,\sigma)\|_X\|(v,\tau)\|_X} \geq \gamma > 0,$$

where γ is a positive constant. Similarly we can prove another inf-sup condition:

$$\inf_{(u,\sigma)\in X}\sup_{(v,\tau)\in X}\frac{\mathscr{A}(u,\sigma;v,\tau)}{\|(u,\sigma)\|_X\|(v,\tau)\|_X} \geq \gamma > 0.$$

This proves the well-posedness of the weak formulation (39).

A natural way to get a numerical scheme is to replace the function spaces by finite dimensional spaces. Although the three (strong, primal weak and mixed weak) formulations are equivalent on the continuous level, discretisations based on these formulations usually differ from each other. Discretisations based on the mixed weak formulations are called *mixed methods*. The mixed methods use more variables, and usually enjoy better mathematical properties (esp. conservative properties). The key of a mixed method is the choice of the spaces.

It is possible to analyse a mixed method directly. Nevertheless, we follow a different approach by formally eliminate some variables. This will be convenient for more complicated problems.

Therefore, the mixed methods will be viewed as a nonconforming discretisation for the primal formulation.

Why "conforming" discretisation does not work

Primal formulation for the vector Laplacian: find $\boldsymbol{u} \in H(\mathrm{rot}) \cap H_0(\mathrm{div})$, such that for any $\boldsymbol{v} \in H(\mathrm{rot}) \cap H_0(\mathrm{div})$,

$$(\mathrm{rot}\, \boldsymbol{u}, \mathrm{rot}\, \boldsymbol{v}) + (\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}).$$

Spurious solutions for source problems

With the boundary conditions from the variational form (PDEs), $\|\nabla \boldsymbol{u}\|^2 = \|\mathrm{rot}\, \boldsymbol{u}\|^2 + \|\mathrm{div}\, \boldsymbol{u}\|^2$. Therefore $[H^1(\Omega)]^2$ is a *closed* subspace of $H(\mathrm{rot}) \cap H_0(\mathrm{div})$. If the domain is regular (convex or with $C^{1,1}$ boundary), $[H^1(\Omega)]^2 = H(\mathrm{rot}) \cap H_0(\mathrm{div})$. Otherwise (e.g., for L-shape domains) $[H^1(\Omega)]^2$ is not dense in $H(\mathrm{rot}) \cap H_0(\mathrm{div})$. Therefore Lagrange elements as a subspace of $[H^1(\Omega)]^2$ cannot approximate true solution in $H(\mathrm{rot}) \cap H_0(\mathrm{div})$.

Solution from FEEC

Seek $\boldsymbol{u} \in H_0(\mathrm{div})$ or $\boldsymbol{u} \in H(\mathrm{rot})$ instead. However, $(\mathrm{rot}\, \boldsymbol{u}, \mathrm{rot}\, \boldsymbol{v})$ does not make sense for $\boldsymbol{u} \in H_0(\mathrm{div})$ ($(\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v})$ does not make sense for $\boldsymbol{u} \in H(\mathrm{rot})$). Therefore one has to define proper discrete rot_h (div_h) and the resulting scheme can be viewed as a nonconforming method. As we shall see, if rot_h (div_h) is defined by the L^2 adjoint, then we get the mixed method.

The discrete theory will be established with a complex with finite dimensional spaces

$$\cdots \longrightarrow V_h^{j-1} \xrightarrow{d^{j-1}} V_h^j \xrightarrow{d^j} V_h^{j+1} \longrightarrow \cdots , \tag{40}$$

which is considered as a discretisation of

$$\cdots \longrightarrow V^{j-1} \xrightarrow{d^{j-1}} V^j \xrightarrow{d^j} V^{j+1} \longrightarrow \cdots . \tag{41}$$

finite dimensional subcomplex (V_h^k, d) of (V, d). The spaces V_h^k are d- conforming in the sense that $V_h^k \subset V^k = D(d)$.

Assumption: there exist bounded cochain projections. (Recall that bounded cochain projections imply discrete Poincaré inequality and discrete inf-sup conditions.)

Mixed methods

Then the Hodge-Laplacian problem is discretized as the following: find $u_h \in V_h^k$ such that

$$(dd_h^* + d_h^* d)u_h = \mathbb{P}f, \tag{42}$$

where $\mathbb{P}$ is the projection to V_h^k with respect to the W- inner product.

The above formulation is equivalent to: Find $u_h \in V_h^k$ such that

$$(du_h, dv_h) + (d_h^* u_h, d_h^* v_h) = (f, v_h), \quad \forall u_h \in V_h^k, \tag{43}$$

or,
Find $(u_h, \sigma_h) \in V_h^k \times V_h^{k-1}$ such that

$$\begin{cases} (u_h, d\tau_h) - (\sigma_h, \tau_h) = 0, \\ (d\sigma_h, v_h) + (du_h, dv_h) = (f, v_h), \end{cases} \tag{44}$$

where the first equation of (44) is equivalent to introducing $\sigma_h = d_h^* u_h$ and the second is equivalent to the operator form $d\sigma_h + d_h^* du_h = \mathbb{P}f$.

The approach we present below is to show the well-posedness of (44), i.e., Find $u_h \in V_h^k$ such that

$$(du_h, dv_h) + (d_h^* u_h, d_h^* v_h) = (f, v_h), \quad \forall u_h \in V_h^k, \tag{45}$$

We define a new norm which depends on the mesh:

$$\|u_h\|_{A_h}^2 := \|u_h\|^2 + \|du_h\|^2 + \|d_h^* u_h\|^2.$$

It is obvious that $\|\cdot\|_{A_h}$ is a norm.

Theorem

The variational problem (43) has a unique solution $u_h \in V_h^k$, which satisfies the estimates:

$$\|v_h\|_{A_h} \leq C\|f\|, \tag{46}$$

and

$$\|\sigma_h\|_V \leq C\|f\|, \tag{47}$$

where $\sigma_h = d_h^* u_h$.

Proof.

The conclusion follows from the Lax-Milgram theorem. Actually, with $\|\cdot\|_{A_h}$ norm, the boundedness of the variational form is obvious. To show the coercivity, we use the discrete Poincaré inequality:

$$\|u_h\| \le C(\|du_h\| + \|d_h^* u_h\|).$$

The estimate (46) is obvious by taking $v_h = u_h$. To show the bound (47), we take $v_h = u_h$ and (43) implies

$$\|\sigma_h\|^2 = \|d_h^* u_h\|^2 := (f, u_h) - (du_h, du_h) \le \|f\|\|u_h\| + \|du_h\|^2 \le C^2\|f\|^2,$$

and hence

$$\|\sigma_h\| \le C\|f\|.$$

Furthermore, we take $v_h = dd_h^* u_h$ in (43). This implies

$$(d\sigma_h, d\sigma_h) = (f, d\sigma_h),$$

and therefore

$$\|d\sigma_h\| \le \|f\|.$$

□

Let $Q : L^2 \to V_h^{k-1}$ be the L^2 projection.

Lemma

Let v_I be any function in V_h^k. Then

$$\|u - u_h\|_{V_h} \leq C(\|u - u_I\|_{V_h} + \|(Q - I)d^* u\| + \|d_h^* u_I - Qd^* u\|). \tag{48}$$

Moreover, if the mesh is quasi-uniform (so that the inverse estimate holds) and assume that

$$\|u_I - u\| \leq Ch^{r+\sigma}\|u\|_{r+1}, \tag{49}$$

where $r \geq 1$ and $0 < \sigma \leq 1$. Then

$$\|d_h^* u_I - Qd^* u\| \leq Ch^{r-1+\sigma}\|u\|_{r+1}.$$

Proof.

$$u - u_h = u - u_I + u_I - u_h.$$

$$\begin{aligned}\|u_I - u_h\|_{A_h}^2 &= (du_I, d(u_I - u_h)) + (d_h^* u_I, d_h^*(u_I - u_h)) - (f, u_I - u_h) \\ &= (du_I, d(u_I - u_h)) - (d^* du, u_I - u_h) + (d_h^* u_I, d_h^*(u_I - u_h)) - (dd^* u, u_I - u_h) \\ &= (d_h^* du_I - d^* du, u_I - u_h) + (dd_h^* u_I - dd^* u, u_I - u_h) =: I_1 + I_2\end{aligned}$$

$$|I_1| = |(d_h^* du_I - d^* du, u_I - u_h)| = |(d(u_I - u), d(u_I - u_h))| \leq \|d(u_I - u)\| \|d(u_I - u_h)\|,$$

$$\begin{aligned}|I_2| &= |(dd_h^* u_I - dd^* u, u_I - u_h)| = |d(d_h^* u_I - Qd^* u + Qd^* u - d^* u, u_I - u_h)| \\ &\leq |d_h^* u_I - Qd^* u, d_h^*(u_I - u_h))| + |((Q - I)d^* u, u_I - u_h)|.\end{aligned}$$

This proves (48). Moreover,

$$\|d_h^* u_I - Qd^* u\| = \sup_{\|v_h\|=1} (d_h^* u_I - Qd^* u, v_h) = \sup_{\|v_h\|=1} (u_I - u, dv_h) \leq Ch^{r-1+\sigma} \|u\|_{r+1},$$

where in the last step we used the inverse inequality and (49).

Definition of finite elements à la Ciarlet: triple (K, P, Σ)

- K: closed subset of $\mathbb{R}^n$: simplices, cubes...
- P: *local shape function space*, $\mathcal{P}_r$...
- Σ: *degrees of freedom (DoFs)*, set of linear functional on P, vertex evaluation...

Global finite element function: piecewise in P, single value on Σ

Another perspective: Finite element = local shape functions + interelement continuity

Unisolvency: (K, P, Σ) is unisolvent if any function in P is uniquely determined by Σ ($\Rightarrow \dim P = \dim \Sigma$)

Conformity: (K, P, Σ) is H^q-conforming if the global FE space is in H^q

e.g., Lagrange element is H^1-conforming (due to C^0 continuity)

In 3D: piecewise smooth function u is

- H^1 iff u is C^0,
- $H(\text{curl})$ iff u has continuous tangential components,
- $H(\text{div})$ iff u has continuous normal components.

Construction of local shape functions Let T be a simplex in $\mathbb{R}^n$ and $\mathcal{P}_r\Lambda^k(T)$ be the space of differential k-forms with polynomial coefficients of degree r in T, i.e.,

$$\mathcal{P}_r\Lambda^k(T) := \{\sum_\sigma p_\sigma dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k} : p_\sigma \in \mathcal{P}_r(T)\}.$$

Let $\mathcal{H}_r\Lambda^k$ be the space of k-forms with homogeneous polynomial coefficients of degree r. The Poinaré operators have a simpler form on homogeneous polynomials. Recall that

$$\mathfrak{p}^k\omega(x)(v_1, \cdots, v_{k-1}) := \int_0^1 t^{k-1}\omega(tx)(x, v_1, \cdots, v_{k-1})\, dt,$$

i.e.,

$$\mathfrak{p}^k\omega(x) := \int_0^1 t^{k-1} i_x\omega(tx)\, dt,$$

satisfies

- homotopy property $d^{k-1}\mathfrak{p}^k + \mathfrak{p}^{k+1}d^k = I$,
- complex property $\mathfrak{p}^{k-1} \circ \mathfrak{p}^k = 0$,
- polynomial-preservation: $\mathfrak{p}^k q \in \mathcal{P}_{r+1}\Lambda^{k-1}$ if $q \in \mathcal{P}_r\Lambda^k$.

If $\omega \in \mathcal{H}_r\Lambda^k$, then $\mathfrak{p}^k\omega = \frac{1}{r+k} i_x\omega$. Define the Koszul operator $\kappa^k : \Lambda^k \to \Lambda^{k-1}$ by $\kappa^k\omega := i_x\omega$. We have

$$d^{k-1}\kappa^k + \kappa^{k+1}d^k = \frac{1}{r+k} I.$$

The above identity is due to the fact that d increases the form degree and decreases the polynomial degree by one.

$$\cdots \rightleftarrows V^{i-1} \underset{\mathfrak{p}^i}{\overset{d^{i-1}}{\rightleftarrows}} V^i \underset{\mathfrak{p}^{i+1}}{\overset{d^i}{\rightleftarrows}} V^{i+1} \rightleftarrows \cdots$$

The following lemma is useful when we construct exact sequences.

Lemma

If a sequence $V^\bullet$ is both a sequence with $d^\bullet$ and $\mathfrak{p}^\bullet$, i.e., $d^k V^k \subset V^{k+1}$ and $\mathfrak{p}^k V^k \subset V^{k-1}$, then $(V^\bullet, d^\bullet)$ and $V^\bullet, \mathfrak{p}^\bullet$ are both exact.

Proof.

By the homotopy identity. □

The next question is how to construct such a sequence of spaces which is closed with both d and $\mathfrak{p}$. To this end, assume that

$$\cdots \xrightarrow{d} V^{j-1} \xrightarrow{d} V^{j} \xrightarrow{d} V^{j+1} \xrightarrow{d} \cdots, \tag{50}$$

is a complex, i.e. $dV^j \subset V^{j+1}$. We do not require (50) to be exact. Define

$$\tilde{V}^j := V^j + \mathfrak{p} V^{j+1}.$$

We have

$$d\tilde{V}^j := dV^j + d\mathfrak{p}V^{j+1} = dV^j + (I - \mathfrak{p}d)V^{j+1} \subset \tilde{V}^{j+1}.$$

Furthermore,

$$\mathfrak{p}\tilde{V}^{j+1} = \mathfrak{p}V_{j+1} \subset \tilde{V}^j.$$

Consequently, $\tilde{V}^\bullet$ are the spaces that we are looking for, i.e.,

$$\cdots \xrightarrow{d} V^{j-1} \xrightarrow{d} V^{j} \xrightarrow{d} V^{j+1} \xrightarrow{d} \cdots, \tag{51}$$

and

$$\cdots \xleftarrow{\mathfrak{p}} \tilde{V}^{j-1} \xleftarrow{\mathfrak{p}} \tilde{V}^{j} \xleftarrow{\mathfrak{p}} \tilde{V}^{j+1} \xleftarrow{\mathfrak{p}} \cdots, \tag{52}$$

are both exact sequences.
Therefore, now we have a general recipe to generate exact sequences from any complex.

Another consequence of the homotopy identity is that $\tilde{V}^j$ can be decomposed as the direct sum of the image of d and the image of $\mathfrak{p}$.

Theorem

We have

$$\tilde{V}^j = d\tilde{V}^{j-1} \oplus \mathfrak{p}\tilde{V}^{j+1}.$$

Proof.

We have

$$\tilde{V}^j = V^j + \mathfrak{p}V^{j+1} = d\mathfrak{p}V^j + \mathfrak{p}dV^j + \mathfrak{p}V^{j+1} = d\mathfrak{p}V^j + \mathfrak{p}\left(dV^j + V^{j+1}\right).$$

This implies the decomposition

$$\tilde{V}^j = d\tilde{V}^{j-1} + \mathfrak{p}\tilde{V}^{j+1}.$$

To verify it is a direct sum decomposition, we assume that there exists $\phi_1 \in \tilde{V}^{j-1}$ and $\phi_2 \in \tilde{V}^{j+1}$ such that $u = d\phi_1$ and $u = \mathfrak{p}\phi_2$. Therefore

$$u = d\mathfrak{p}\mathfrak{p}\phi_2 + \mathfrak{p}dd\phi_1 = 0.$$

□

Example

$$\cdots \xrightarrow{d} \mathcal{P}_r\Lambda^{k-1}(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}_r\Lambda^{k}(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}_r\Lambda^{k+1}(\mathbb{R}^n) \xrightarrow{d} \cdots, \tag{53}$$

The complex (53) is not exact. By the Poincaré type construction, we obtain

$$\cdots \xrightarrow{d} \mathcal{P}_{r+1}^{-}\Lambda^{k-1}(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}_{r+1}^{-}\Lambda^{k}(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}_{r+1}^{-}\Lambda^{k+1}(\mathbb{R}^n) \xrightarrow{d} \cdots, \tag{54}$$

where

$$\mathcal{P}_{r+1}^{-}\Lambda^{k}(\mathbb{R}^n) := \mathcal{P}_r\Lambda^{k}(\mathbb{R}^n) + \kappa^{k+1}\mathcal{P}_r\Lambda^{k+1}(\mathbb{R}^n).$$

The next example is to start from the complex

$$\cdots \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^{k-1}(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}_r\Lambda^{k}(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^{k+1}(\mathbb{R}^n) \xrightarrow{d} \cdots, \tag{55}$$

Since $\mathfrak{p}\mathcal{P}_r\Lambda^k(\mathbb{R}^n) \subset \mathcal{P}_{r+1}\Lambda^{k-1}(\mathbb{R}^n)$, the complex (55) is already invariant with Poincaré's operators (the Poincaré's construction is (55) is itself), therefore (55) is exact.

Lemma

(Lemma 3.8, [Arnold, Falk, Winther 2006 Acta Num.])

1. *The following four restrictions of d each have the same kernel:*

$$d : \mathcal{P}_r\Lambda^k \mapsto \mathcal{P}_{r-1}\Lambda^{k+1}, \quad d : \mathcal{P}_r\Lambda^k \mapsto \mathcal{P}_r^-\Lambda^{k+1},$$

$$d : \mathcal{P}_{r+1}^-\Lambda^k \mapsto \mathcal{P}_r\Lambda^{k+1}, \quad d : \mathcal{P}_{r+1}^-\Lambda^k \mapsto \mathcal{P}_{r+1}^-\Lambda^{k+1}.$$

2. *The following four restrictions of d each have the same image:*

$$d : \mathcal{P}_r\Lambda^k \mapsto \mathcal{P}_{r-1}\Lambda^{k+1}, \quad d : \mathcal{P}_r\Lambda^k \mapsto \mathcal{P}_r^-\Lambda^{k+1},$$

$$d : \mathcal{P}_r^-\Lambda^k \mapsto \mathcal{P}_{r-1}\Lambda^{k+1}, \quad d : \mathcal{P}_r^-\Lambda^k \mapsto \mathcal{P}_r^-\Lambda^{k+1}.$$

Degrees of freedom (DoFs)

- DoFs for $\mathcal{P}_r\Lambda^k(T)$:

$$\omega \mapsto \int_f (\mathrm{tr}_f\,\omega) \wedge \mu, \quad \mu \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f),\ f \in \Delta^d(T),\ d \geq k.$$

- DoFs for $\mathcal{P}^-_r\Lambda^k(T)$:

$$\omega \mapsto \int_f (\mathrm{tr}_f\,\omega) \wedge \mu, \quad \mu \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f),\ f \in \Delta^d(T),\ d \geq k.$$

Whitney forms

For $1 \leq k \leq d$, let σ be an increasing map $\{1, \ldots, k\} \mapsto \{1, \ldots, d\}$ and $x_0, x_1, \ldots, x_d$ be $d+1$ points in $\mathbb{R}^d$ such that $x_1 - x_0, x_2 - x_0, \ldots, x_d - x_0$ give a positively oriented frame.
Define

$$f_\sigma := \left[x_{\sigma(0)}, \ldots, x_{\sigma(k)}\right].$$

to be the subsimplex of dimension k which is the convex hull of $x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(k)}$. The Whitney form associated with σ is defined as

$$\phi_\sigma := \sum_{i=0}^{k} (-1)^i \lambda_{\sigma(i)} d\lambda_{\sigma(0)} \wedge \cdots \wedge \widehat{d\lambda_{\sigma(i)}} \wedge \cdots \wedge d\lambda_{\sigma(k)}.$$

The Whitney form ϕ_σ has the property that $\mathrm{tr}_g \phi_\sigma = 0, \ \forall g \in \Delta_k, g \neq f_\sigma$, since at least one factor $\lambda_{\sigma(j)}$ vanishes on g or $\mathrm{tr}\left(d\lambda_j\right) = 0$ on g. On the other hand, by straightforward calculations one has

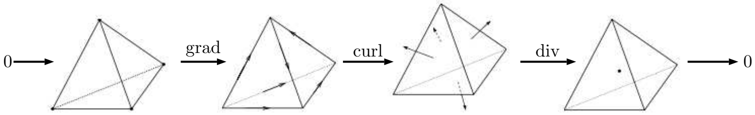
$$\int_{f_\sigma} \phi_\sigma = \pm (k!)^{-1}.$$

This implies that the Whiney k- forms form a basis of k dimensional simplicial cochains.
On the other hand, $\phi_\sigma \in \mathcal{P}_1^- \Lambda^k$. This shows that the lowest order elements in the $\mathcal{P}_r^-$ family coincide with Whitney forms.

$$0 \longrightarrow H(\mathrm{grad}) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0,$$

where $H(d) := \{u \in L^2 : du \in L^2\}$, $H(\mathrm{grad}) = H^1$.

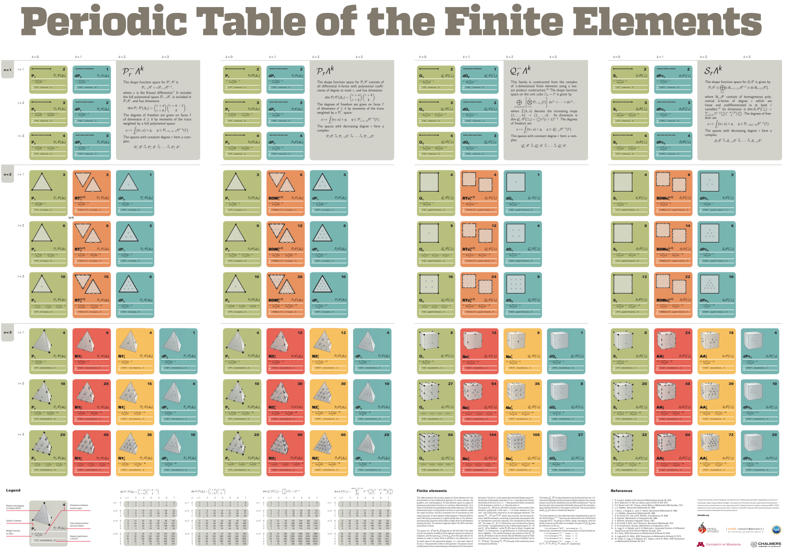
- finite element de Rham complex (Whitney, lowest order, 3D)



$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\mathrm{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\mathrm{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\mathrm{div}} \mathcal{P}_0 \longrightarrow 0.$$

Raviart-Thomas, Nédélec in numerical analysis, Whitney (earlier) for studying topology.

Finite element periodic table



Bounded cochain projections

Canonical interpolation

define $\Pi_r^k : \Lambda^k(T) \mapsto \mathcal{P}_r\Lambda^k(T)$ as the interpolation operator by DoFs, i.e., $\Pi_r^k\omega$ is the unique element in $\mathcal{P}_r\Lambda^k(T)$ that satisfies

$$\int_f \mathrm{tr}_f(\omega - \Pi_r^k\omega) \wedge q = 0, \quad \forall q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f), \forall f \in \Delta(T), \dim f \geq k.$$

Similarly, we can define $\Pi_{r-}^k\omega$ as the unique element in $\mathcal{P}_r^-\Lambda^k(T)$ that satisfies

$$\int_f \mathrm{tr}_f(\omega - \Pi_{r-}^k\omega) \wedge q = 0, \quad \forall q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f), \forall f \in \Delta(T), \dim f \geq k.$$

Incompressible Navier-Stokes equations

$$u_t + (u \cdot \nabla)u - R_e^{-1}\Delta u + \nabla p = f, \tag{56a}$$

$$\nabla \cdot u = 0, \tag{56b}$$

with nonslip boundary condition $u = 0$ on $\partial\Omega$.
Let $\omega = \nabla \times u$. We have the identity

$$(u \cdot \nabla)u = \omega \times u + \frac{1}{2}\nabla|u|^2.$$

We get the rotational form of the Navier-Stokes equation:

$$u_t + \omega \times u - R_e^{-1}\Delta u + \nabla P = f, \tag{57a}$$

$$\nabla \cdot u = 0, \tag{57b}$$

where the *total pressure* $P := p + \frac{1}{2}|u|^2$.
We can also derive the vorticity equation

$$\omega_t - \nabla \times (u \times \omega) - R_e^{-1}\Delta\omega = \nabla \times f, \tag{58a}$$

$$\nabla \cdot \omega = 0, \tag{58b}$$

Theorem (energy conservation)

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{1}{2}R_e^{-1}\|\nabla u\|^2 \leq \frac{1}{2}C^{-1}R_e\|f\|^2,$$

where C is the constant in the Poincaré inequality $\|u\| \leq C\|\nabla u\|$.

Idea of proof: testing u on both sides of equation, note that $\int (u \cdot \nabla)u \cdot u = 0$ for $\nabla \cdot u = 0$ and $u = 0$ on $\partial\Omega$ (convection does not change energy for incompressible flows).

Importance for computation: some instability of computation (in particular, for coupled systems) is caused by the violation of the energy law at the discrete level.

Theorem

The enstrophy $\int (\mathrm{rot}\, u)^2\, dx$ *is conserved by the Euler equation.*

Proof.

Taking rot on the Euler equation, we get the vorticity equation

$$\omega_t - \mathrm{rot}(u^\perp \omega) = 0,$$

where $u^\perp := (-u^2, u^1)$. By the Leibniz rule, $\mathrm{rot}(u^\perp \omega)$ has two parts: one is to apply rot to $u^\perp$, which is equal to div on u. This term vanishes for incompressible fluids. The second term in $\mathrm{rot}(u^\perp \omega)$ is $u^\perp\, \mathrm{curl}\, \omega$. Therefore

$$(\omega_t, \omega) = (\mathrm{rot}(u^\perp \omega), \omega) = (u^\perp\, \mathrm{curl}\, \omega, \omega).$$

On the other hand, by the integration by parts,

$$(\mathrm{rot}(u^\perp \omega), \omega) = -(u^\perp\, \mathrm{curl}\, \omega, \omega).$$

This implies that

$$(\mathrm{rot}(u^\perp \omega), \omega) = 0.$$

Therefore $\frac{d}{dt}\|\omega\|^2 = (\omega_t, \omega) = 0$. □

The enstropy conservation shows that the “vorticity energy” is conserved in 2D. It does not hold in 3D, as 3D vorticities have more complicated structures.

2D enstrophy-preserving algorithms have been considered in early CFD literature in the finite difference context, referred to as the *Arakawa scheme*. The “Arakawa grids” involve stagger grids. Therefore the Arakawa scheme has a flavor of mixed schemes. In 1970s it is claimed that “the Arakawa scheme is a finite element method” [Jespersen 1974, JCP].

Theorem (helicity conservation)

For the Euler equation (formally, $R_e = \infty$) with $f = 0$,

$$\frac{d}{dt}\int u \cdot \omega \, dx = 0.$$

The condition in Theorem 33 can be relaxed to any gradient field f.

Proof.

$$\frac{d}{dt}\int u\cdot\omega = (u_t,\omega)+(u,\omega_t) = 2(u,\omega_t) = 2(u,\nabla\times(u\times\omega)) = 2(\nabla\times u, u\times\omega) = 2(u,\omega\times\omega) = 0.$$

□

Stokes equations : strongly diffusive, stationary, incompressible

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p & = \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} & = 0, \end{cases}$$

Boundary condition: $\boldsymbol{u} = 0$ on $\partial\Omega$.

Variational form : find $\boldsymbol{u} \in [H_0^1(\Omega)]^n$, $p \in L_0^2(\Omega)$ s.t.,

$$\begin{cases} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) & = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in [H_0^1(\Omega)]^n, \\ (\nabla \cdot \boldsymbol{u}, q) & = 0, \quad \forall q \in L_0^2(\Omega). \end{cases}$$

Discrete variational form : find $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p \in Q_h$ s.t.,

$$\begin{cases} (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) & = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ (\nabla \cdot \boldsymbol{u}_h, q_h) & = 0, \quad \forall q_h \in Q_h. \end{cases}$$

How to choose $\boldsymbol{V}_h$ and Q_h?

- numerical stability (inf-sup condition): for any $q_h \in Q_h$, there exists $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\|\boldsymbol{v}_h\| = 1$, s.t. $(\nabla \cdot \boldsymbol{v}_h, q_h) \geq \|q_h\|_{L^2}$; $\nabla \cdot \boldsymbol{V}_h$ should be "larger" than Q_h.
- precise divergence-free constraint:

  $(\nabla \cdot \boldsymbol{v}_h, q_h) = 0, \ \forall q_h \in Q_h \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{v}_h = 0$ Q_h should be "larger" than $\nabla \cdot \boldsymbol{V}_h$.

  mass conservation, important for numerics, John et al. *SIAM Review 2017*
- balance: $\nabla \cdot \boldsymbol{V}_h = Q_h$,
- constructing finite elements satisfying $\nabla \cdot \boldsymbol{V}_h = Q_h$ turns out to be very challenging.

  Scott-Vogelius, stable Stokes pairs etc.

Pressure robustness

Let $V_h = V_h^0 \oplus V_h^\perp$, where V_h^0 is the discrete divergence-free space defined by

$$V_h^0 := \{u_h \in V_h : (\operatorname{div} u_h, q_h) = 0, \quad \forall q_h \in Q_h\},$$

and $V_h^\perp$ is the L^2-orthogonal complement of V_h^0 in V_h. On the continuous level, recall the Hodge decomposition

$$L^2 = \operatorname{grad} H_0^1 \oplus \operatorname{curl} H(\operatorname{curl}).$$

If $\operatorname{div} V_h \subset Q_h$, then $u_h \in V_h^0$ implies $\operatorname{div} u_h = 0$ and thus $u_h \in \operatorname{curl} H(\operatorname{curl})$. This implies that $u_h \perp \operatorname{grad} H_0^1$.

We get the error equation

$$\begin{aligned} a(u - u_h, v_h) + b(v_h, p - p_h) &= 0, \quad \forall v_h \in V_h^0, & (59)\\ b(v_h, q_h) &= 0, \quad \forall q_h \in Q_h. & (60) \end{aligned}$$

Combining the two equations,

$$a(u - u_h, v_h) + b(v_h, p) = 0, \quad \forall v_h \in V_h^0.$$

In general, the term $b(v_h, p)$ does not vanish and thus the estimate for $u - u_h$ involves p. If $\operatorname{div} V_h \subset Q_h$, then $u_h \perp \operatorname{grad} H_0^1$ and thus $b(v_h, p) = 0$. Therefore the complex property (rather than the single fact that v_h being divergence-free) implies pressure robustness.

Smoother de Rham complexes

many variants of the de Rham complex are available and have uniform cohomology representatives. For example, the complex consisting of H^1 differential forms with exterior derivatives in H^1, i.e.,

$$\cdots \longrightarrow H^1_d\Lambda^{k-1} \xrightarrow{d^{k-1}} H^1_d\Lambda^k \xrightarrow{d^k} H^1_d\Lambda^{k+1} \longrightarrow \cdots,$$

where $H^1_d\Lambda^k := \{u \in H^1\Lambda^k : d^k u \in H^1\Lambda^{k+1}\}$.
In particular, the following complexes play an important role for Stokes problems:

$$0 \longrightarrow H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} H^1 \otimes \mathbb{V} \xrightarrow{\text{div}} L^2 \longrightarrow 0, \tag{61}$$

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{grad}\,\text{curl}) \xrightarrow{\text{curl}} H^1 \otimes \mathbb{V} \xrightarrow{\text{div}} L^2 \longrightarrow 0, \tag{62}$$

in three space dimensions, and

$$0 \longrightarrow H^2 \xrightarrow{\text{curl}} H^1 \otimes \mathbb{V} \xrightarrow{\text{div}} L^2 \longrightarrow 0, \tag{63}$$

in two space dimensions. Here

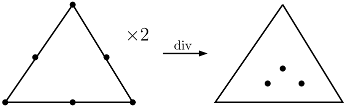
$$H^1(\text{curl}) := \{u \in H^1 \otimes \mathbb{V} : \text{curl}\, u \in H^1 \otimes \mathbb{V}\},$$

and

$$H(\text{grad}\,\text{curl}) := \{u \in L^2 \otimes \mathbb{V} : \text{grad}\,\text{curl}\, u \in L^2 \otimes \mathbb{M}\} = \{u \in L^2 \otimes \mathbb{V} : \text{curl}\, u \in H^1 \otimes \mathbb{V}\}.$$

Conservative Stokes discretization

- puzzle of Scott-Vogelius ($[C^0\mathcal{P}_r]^n$-$C^{-1}\mathcal{P}_{r-1}$) : 2D stable for $r \geq 4$, no "singular vertices"; 3D open.



- Scott-Vogelius elements are stable on certain meshes (Arnold-Qin 1992, S.Zhang),

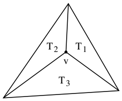
Why constructing Stokes finite elements are hard?

From Stokes complexes, a Stokes pair is closely related to a discretisation of H^2.

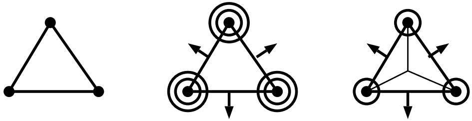
$$0 \longrightarrow H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} [H^1]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

$$0 \longrightarrow C^1 \text{ spline} \xrightarrow{\text{grad}} * \xrightarrow{\text{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0.$$

Supersmoothness: e.g., mesh consisting of three triangles, piecewise smooth C^1 function is automatically C^2 at v



Therefore H^2 (C^1) finite element on a general mesh uses second order derivatives at vertices as DoFs. One may also avoid the effect of supersmoothness by using piecewise polynomials (thus on the large triangle the shape function is not smooth). more triangles/partition around a vertex, less supersmoothness



Maximal order of supersmoothness : $C^1 \Rightarrow C^?$

$$S^r_d(\Delta) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d, T \in \Delta\},$$

Definition

We will say that $S^r_d(\Delta)$ is *degenerate* if $S^r_d(\Delta) = \mathcal{P}_d$.

Theorem

For $0 \leq r \leq \rho$, $S^r(\Delta) \in C^\rho(v)$ *if and only if* $S^r_\rho(\Delta)$ *is degenerate.*

- if: Taylor expansion up to order ρ on each piece. The piecewise expansion is a global polynomial (due to degeneracy), which implies continuity.
- only if: since $S^r_\rho(\Delta) \subset S^r(\Delta)$, every polynomial spline $s \in S^r_\rho(\Delta)$ is in $C^\rho(v)$ and therefore $s \in \mathcal{P}_\rho$.

$$\operatorname{mos} S^r(\Delta) := \max\{\rho \geq r : S^r(\Delta) \in C^\rho(v)\}.$$

Corollary

$\operatorname{mos} S^r(\Delta) = \max\{d \geq r : S^r_d(\Delta) \textit{ is degenerate}\}$.

- Floater, M.S. and Hu, K :A characterization of supersmoothness of multivariate splines." Advances in Computational Mathematics 46.5 (2020): 1-15.

Main messages

- supersmoothness is a property for general functions, not necessarily polynomials,
- boils down to the question of dimension of spline spaces (piecewise polynomials with certain continuity). algebraic geometric methods.

For some cases, dimension of spline spaces is known, e.g., 2D vertex patch.
$\dim(S_r^k(\Delta)) =$?? Strang's Conjecture.

- $k = 1$ in $\mathbb{R}^2$: Billera, algebraic geometry and homological techniques,
- in general: open.

Lemma

Suppose Δ has m triangles and suppose there are m_v different slopes among the interior edges of Δ. For $0 \le r \le d$,

$$\dim S_d^r(\Delta) = \dim \Pi_d + (m - m_v) \dim \Pi_{d-r-1} + \sum_{j=1}^{d-r} (\tau_{v,j})_+, \tag{64}$$

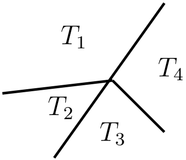
where $\tau_{v,j} := j(m_v - 1) - (r+1)$, and $(x)_+ := x$ if $x > 0$ and $(x)_+ := 0$ otherwise.

Theorem

Suppose Δ has m triangles and suppose there are m_v different slopes among the interior edges of Δ. Then for $r \ge 0$,

$$\operatorname{mos} S^r(\Delta) = \begin{cases} r + \left\lfloor \frac{r+1}{m-1} \right\rfloor, & m_v = m; \\ r, & m_v < m. \end{cases} \tag{65}$$

no supersmoothness if two lines are colinear; more triangles/partition, less supersmoothness



We have seen that from the FEEC perspective, constructing conservative Stokes pairs or complexes has the same difficulty as constructing elements with higher smoothness. In particular, the supersmoothness issue has to be resolved in some way. A simple solution is to consider finite elements or splines on cubical meshes with a tensor product structure since supersmoothness does not exist when two edges connecting a vertex are colinear (which is the case of cubical meshes). For some applications, cubical meshes are too restrictive. On simplicial meshes, there are usually two ways to handle supersmoothness: using high order derivatives as DoFs, or using macroelements. Correspondingly, most Stokes complexes have these two versions.

From the FEEC perspective, one may construct a Stokes pair in 2D by completing a scalar C^1 element by taking derivatives. The first construction in this way by Falk and Neilan in 2013 starts with the Argyris C^1 triangle.

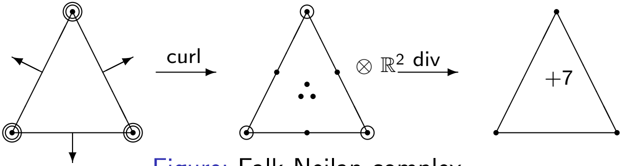


Figure: Falk-Neilan complex

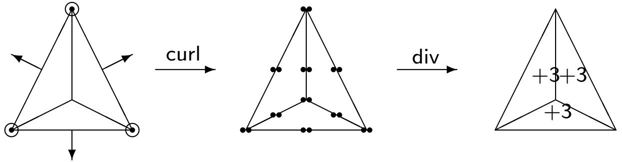


Figure: Resolution of the Clough-Tocher C^1 element. The figure shows the lowest order case: the first space is piecewise cubic, the second is piecewise quadratic (C^0 Lagrange) and the third is piecewise constant.

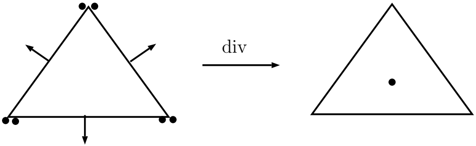
An easy check for finite element complexes: if exactness holds, the dimension of spaces (equal to the number of DoFs) should satisfy the alternating sum condition (although not a sufficient condition). We may check that this condition holds for the above complexes. Let V, E, T be the number of vertices, edges, and triangles of a mesh, respectively. For example, the Falk-Neilan sequence: $1 \to 6V + E \to 2 * (3V + E + 3T) \to 7T$. For domains isomorphic to $\mathbb{R}^n$: Euler's formula $1 = V - E + T$.

stabilising the S-V by Bernardi-Raugel bubbles

The routine of picking up a C^1 element and completing to a sequence is a natural idea (although the construction may not be easy). The resulting Stokes elements can be seen as a stabilised version of the Scott-Vogelius (S-V) pair by using either supersmoothness or subdivision. We may ask the question: what is the "minimal" space to stabilise the S-V? The Whitney sequence gives us a hint. For the lowest order Raviart-Thomas element, div is onto constants because of the integration by parts

$$\int_T \operatorname{div} uc = \int_{\partial T} cu \cdot n,$$

where the integration against constant is transformed to the normal DoFs on the boundary. In fact, what is missing in low order S-V for stability is exactly such face normal modes. This inspires the idea of Bernardi-Raugel bubbles (1985):



$\boldsymbol{V}_h := [\mathcal{P}_1(T)]^2 \oplus \mathrm{span}\{B_i\}_{i=1}^3$, where $B_i = \lambda_j \lambda_k \lambda_l n_i$,
$Q_h := \mathcal{P}_0(T)$.

Correcting the divergence

The Bernardi-Raugel pair is not part of a complex as div $\lambda_j\lambda_k\lambda_l n_i \notin \mathcal{P}_0(T)$. Next, we make a correction to B_i to make div constant and not affect the trace.

From results for the Clough-Tocher complex, we have div : $[\mathcal{P}_2(T_A) \cap H_0^1(T)]^2 \to \mathcal{P}_1(T_A) \cap L_0^2(T)$ is onto and satisfies the inf-sup condition. This means that we can use C^0 bubbles on the Alfeld split to control everything except for constants. Therefore we can choose proper $w_i \in [\mathcal{P}_2(T_A) \cap H_0^1(T)]^2$, such that the *modified Bernardi-Raugel bubbles* $\tilde{B}_i := B_i - w_i$ is a constant on T. At the same time, $\tilde{B}_i$ has the same value on any edge as B_i (by definition).

modified Bernardi-Raugel pair: $\boldsymbol{V}_h := [\mathcal{P}_1(T)]^2 \oplus \mathrm{span}\{\tilde{B}_i\}_{i=1}^3$, $Q_h := \mathcal{P}_0(T)$.
Same DoFs.

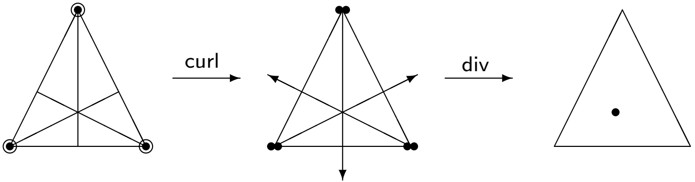
Reference: Guzmn J, Neilan M. Inf-sup stable finite elements on barycentric refinements producing divergence–free approximations in arbitrary dimensions. SIAM Journal on Numerical Analysis. 2018;56(5):2826-44.

The Bernardi-Raugel element is minimal in the sense that one needs $\mathcal{P}_1$ to get approximation (vertex dofs) and face bubbles such that div is onto constants (face normal dofs). How to construct shape functions?

Dimension count gives us a hint.

$$1 \to ? \to 2V + E \to T,$$

exactness implies $? = 3V$. To get a C^1 element with three vertex dofs, we can use the Powell-Sabin split.



To be completed beyond this point.