Discretization of Hilbert complexes

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Hilbert Complexes: Analysis, Applications, and Discretizations

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Aims:

- finite dimensional spaces fitting in a complex,
- certain continuity,
  p.w. smooth fields: $H^r \Longleftrightarrow C^{r-1}$ $(r \geq 1)$, $H(\mathrm{curl}) \Longleftrightarrow C^t$, $H(\mathrm{div}) \Longleftrightarrow C^n$
- algebraic and analytic structures (exactness, bounded cochain projections).

Spectrum of continuity

splines (nonlocal FEs) $\longrightarrow$ FEs $\longleftarrow$ lattice theory (distributional FEs)

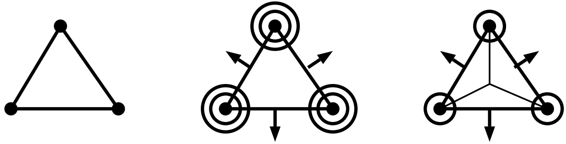
approximation theory
hard construction (supersmoothness)
cubical v.s. simplicial (single cell, subdivision)
isogeometric analysis

discrete physics/geometry
hard schemes (distributions)
triangulations, graphs
DEC, DDG

Smooth side

Splines piecewise polynomials with certain continuity

Finite elements



Lagrange, Argyris, Clough-Tocher

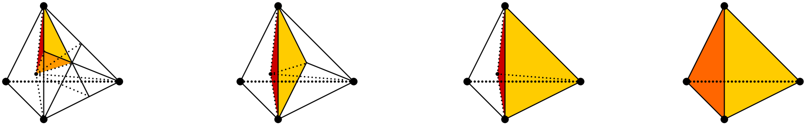
- local shape function + interelement continuity (imposed by degrees of freedom),
- finite v.s. virtual elements: explicit functions v.s. determined by solving PDEs.

- challenge raised by high continuity: supersmoothness

  Supersmoothness: p.w. smooth functions have automatic higher order continuity at vertices, edges etc. (Sorokina 2010).

  Determining supersmoothness $\Longleftrightarrow$ dimension of spline spaces (Floater-KH 2020), algebraic geometry is relevant.

  - use supersmoothness as dofs (Argyris),
  - macroelements, subdivision (Clough-Tocher).



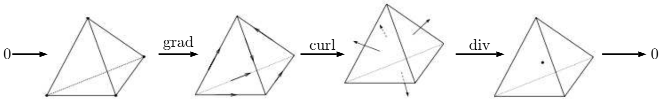
Refinements of a tetrahedron relative to one face: Worsey-Piper, Worsey-Farin, Alfeld, no split. (each 2D face: Powell-Sabin, Clough-Tocher/Alfeld, no split, no split)

Lai & Schumaker 2007, *Spline functions on triangulations*

General experience: If a construction can be done on one type of cells, then it can probably be done on others as well. Finer refinement requires less supersmoothness.

Rough side : discrete theories (discrete calculus/geometry, lattice methods)

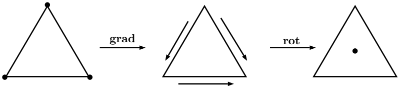
- cochains (exterior calculus) v.s. Whitney forms/finite elements (Raviart-Thomas 1975, Nédélec 1980, Bossavit 1988, Hiptmair 1999, Arnold-Falk-Winther 2006)

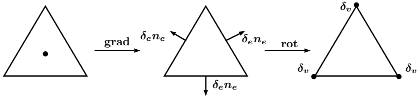


$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\text{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0.$$

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

- distributional finite elements: Braess-Schöberl 2008 (a posterior estimators), Licht 2017 (double complexes), Christiansen-Licht 2016 (Poincaré inequalities)



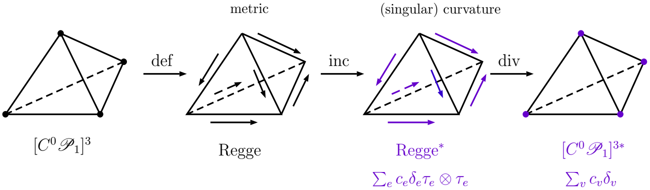


(slightly nonconforming) discretization for

$$0 \longrightarrow L^2 \xrightarrow{\text{grad}} H^{-1}(\text{rot}) \xrightarrow{\text{rot}} H^{-1} \longrightarrow 0.$$

- Regge calculus v.s. Regge finite elements (Christiansen 2011, Li 2018) diamond element, inspired by discrete mechanics (Hauret-Kuhl-Ortiz 2007)

  Regge metric: constant symmetric matrix, continuous tangential-tangential (tt) components $\int_e t_e \cdot u \cdot t_e \, ds$.



  inc: piecewise constants $\to$ tangential Dirac delta on codim 2 (cancelation due to tt-continuity), *analogy of Whitney forms for the elasticity complex*

  Increased interests in the nonlinear context Christiansen 2013 (justification of nonlinear curvature with Regge metric), Berchenko-Kogan-Gawlik 2021 (high order Regge, geodesic curvature), Schöberl (2022 EFEF talk).

  distributional element based methods (TDNNS etc., Schöberl-Sinwel 2007): evaluating Dirac delta only on continuous functions

Construction of shape functions as an exact sequence: Poincaré/Koszul operators
(manifold textbooks, Hiptmair 1999, Arnold-Falk-Winther 2006)

$$\cdots \rightleftarrows V^{i-1} \underset{\mathfrak{p}^i}{\overset{d^{i-1}}{\rightleftarrows}} V^i \underset{\mathfrak{p}^{i+1}}{\overset{d^i}{\rightleftarrows}} V^{i+1} \rightleftarrows \cdots$$

$\mathfrak{p}^k : C^\infty\Lambda^k \mapsto C^\infty\Lambda^{k-1}$, satisfying

- null-homotopy property: $d^{k-1}\mathfrak{p}^k + \mathfrak{p}^{k+1}d^k = \mathrm{id}_{C^\infty\Lambda^k}$,
- complex property: $\mathfrak{p}^{k-1} \circ \mathfrak{p}^k = 0$,
- polynomial preserving property: $u \in \mathcal{P}_r\Lambda^k \implies \mathfrak{p}^k u \in \mathcal{P}_{r+1}\Lambda^{k-1}$,

exactness: $du = 0 \Rightarrow u = (d\mathfrak{p} + \mathfrak{p}d)u = d(\mathfrak{p}u)$.

Construction of exact sequence: starting from any *complex* $(V^\bullet, d^\bullet)$, complete it with $\mathfrak{p}$ to get $(\tilde{V}^\bullet, d^\bullet)$, where $\tilde{V}^k := V^k + \mathfrak{p}^{k+1}V^{k+1}$.

- $$\cdots \longrightarrow \mathcal{P}_r\Lambda^{k-1} \xrightarrow{d^{k-1}} \mathcal{P}_r\Lambda^k \xrightarrow{d^k} \mathcal{P}_r\Lambda^{k+1} \longrightarrow \cdots$$

  leads to

  $$\cdots \longrightarrow \mathcal{P}_r^-\Lambda^{k-1} \xrightarrow{d^{k-1}} \mathcal{P}_r^-\Lambda^k \xrightarrow{d^k} \mathcal{P}_r^-\Lambda^{k+1} \longrightarrow \cdots$$

  where $\mathcal{P}_r^-\Lambda^k := \mathcal{P}_r\Lambda^k + \mathfrak{p}^k(\mathcal{P}_r\Lambda^{k+1})$.

- $$\cdots \longrightarrow \mathcal{P}_{r-(k-1)}\Lambda^{k-1} \xrightarrow{d^{k-1}} \mathcal{P}_{r-k}\Lambda^k \xrightarrow{d^k} \mathcal{P}_{r-(k+1)}\Lambda^{k+1} \longrightarrow \cdots$$

  remains invariant under completion.

Geometric decomposition: dofs of the $\mathscr{P}_r$ family are given by $\mathscr{P}_s^-$, and vice versa.
(Arnold-Falk-Winther 2006)

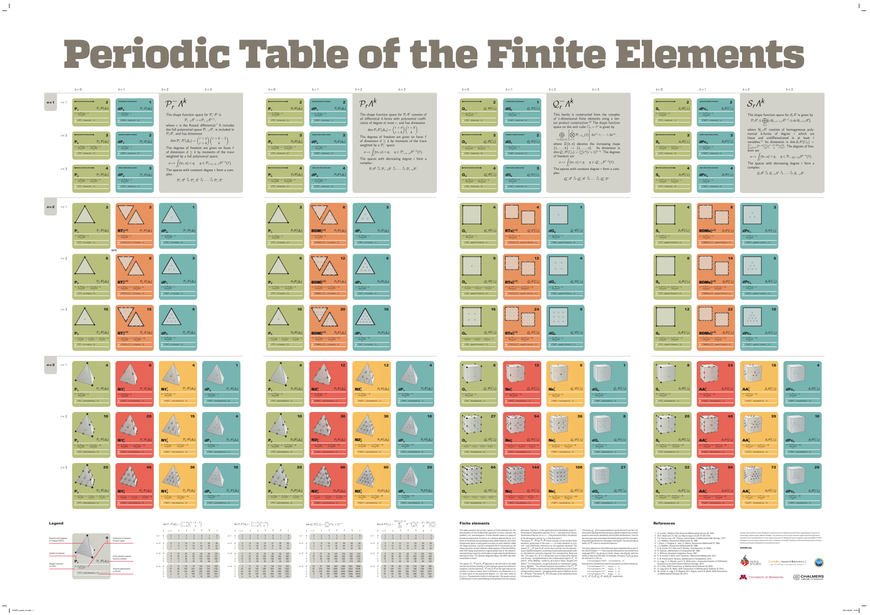
Cubical elements

- tensor product construction (a general strategy) $\mathscr{Q}_r^- \Lambda^k$ (Christiansen 2009, Arnold-Boffi-Bonizzoni 2015): approximating $u(x, y)$ by $f(x)g(y)$, and complete to a sequence, interactions between tensor product and de-Rham structures

- serendipity family $\mathcal{S}_r \Lambda^k$ (Arnold-Awanou 2014): analogy of the $\mathscr{P}_r \Lambda^k$ family

- trimmed serendipity family: completing $\mathcal{S}_r \Lambda^k$ by Poincaré/Koszul operators $\mathcal{S}_r^- \Lambda^k(\Box^n) := \mathcal{S}_r \Lambda^k(\Box^n) + \kappa \mathcal{S}_r \Lambda^{k+1}(\Box^n)$.

Finite element periodic table (Arnold-Logg 2014)

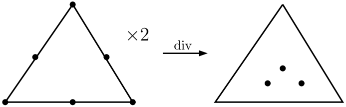


Variants (at least in 2D/3D):

- variants of Lagrange elements (Stenberg 2010, Christiansen-J.Hu-KH 2016, J.Hu-KH-Q.Zhang 2022)
- splines: Buffa-Rivas-Sangalli-Vázquez 2011, Patrizi 2021 (toroidal solid), Y.Zhang-Jain-Palha-Gerritsma (dual complex 2021)
- polyhedral/polygonal meshes: DiPietro-Droniou-Rapetti 2020, DiPietro-Droniou 2021
- virtual element: Veiga-Brezzi-Marini-Russo 2016
- distributional elements: Braess-Schöberl 2008, Licht 2017, Christiansen-Licht 2016
- discrete exterior calculus: Hirani 2003
- graphs: Lim 2020 ("cochains" on graphs)
- neuron network: Longo-Opschoor-Disch-Schwab 2022 (based on classical definitions of de-Rham finite elements)

Stokes problem: construct $V_h \subset [H^1]^n$, $Q_h \subset L^2$, such that div $V_h = Q_h$ and inf-sup condition.

- puzzle of Scott-Vogelius ($[C^0\mathcal{P}_r]^n$-$C^{-1}\mathcal{P}_{r-1}$) : 2D stable for $r \geq 4$, no "singular vertices"; 3D open.



Scott-Vogelius elements are stable on certain meshes (Arnold-Qin 1992, Qin-S.Zhang 2007, S.Zhang 2008),

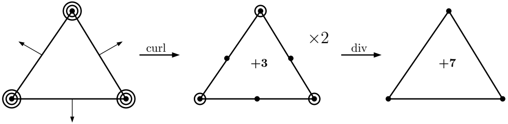
- Strang's Conjecture: $\dim(S_r^k(\Delta)) =??$
  - $k = 1$ in $\mathbb{R}^2$: Billera, algebraic geometry and homological techniques,
  - in general: open.

de-Rham complex relates Stokes problem with C^1 splines:

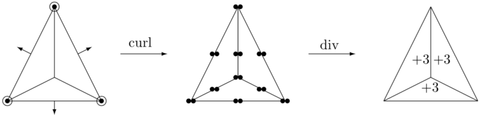
$$0 \longrightarrow C^1 \text{ spline} \xrightarrow{\text{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0.$$

Stokes complex in 2D

$$0 \longrightarrow H^2 \xrightarrow{\text{grad}} [H^1]^2 \xrightarrow{\text{rot}} L^2 \longrightarrow 0.$$



supersmoothness: Falk-Neilan 2013



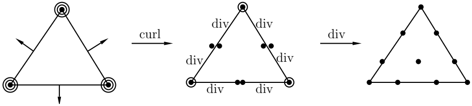
macroelements/splits: Arnold-Qin 1992 (last two spaces), Christiansen-KH 2018

Guzmán-Lischke-Neilan 2019: Powell-Sabin split (Stokes pair: S.Zhang 2008)

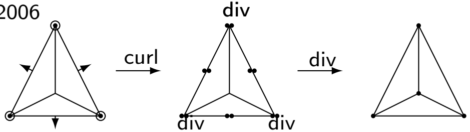
Other smoother de-Rham complexes in 2D

- $$0 \longrightarrow H^2 \xrightarrow{\text{curl}} H^1(\text{div}) \xrightarrow{\text{div}} H^1 \longrightarrow 0.$$

  $H^1(\text{div}) := \{u \in [H^1]^3 : \text{div}\, u \in H^1\}$

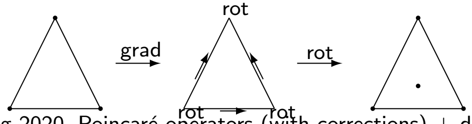
  

  Arnold-Falk-Winther 2006

  Christiansen-KH 2018
- generalized continuum models (size effects), electromagetism

  $$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{grad}\,\text{rot}) \xrightarrow{\text{rot}} H^1 \longrightarrow 0.$$

  KH-Q.Zhang-Z.Zhang 2020, Poincaré operators (with corrections) + dimension count
- arbitrarily high continuity in 2D: J.Hu-Lin-Wu 2021, Chen-Huang 2022 (Bernstein-Beziér technique, dimensional count), Chen-Huang 2022 (3D)

Stokes complex in 3D (supersmoothness)

$$0 \longrightarrow H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} [H^1]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

Neilan 2015 (starting with $C^1\mathscr{P}_9$, vertex dofs: C^4-C^3-C^2-C^1)

$$0 \longrightarrow H^1 \xrightarrow{\text{grad}} H_+(\text{curl}) \xrightarrow{\text{curl}} [H^1]^3 \xrightarrow{\text{div}} L^2 \longrightarrow 0.$$

Q.Zhang-Z.Zhang 2020 (1-forms)

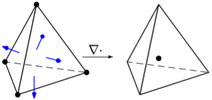
Smoother de-Rham complexes in nD (macroelements)

$$\cdots \longrightarrow H^1(d^{k-2}) \xrightarrow{d^{k-2}} H^1(d^{k-1}) \xrightarrow{d^{k-1}} H^1\Lambda^k \xrightarrow{d^k} H\Lambda^{k+1} \xrightarrow{d^{k+1}} \cdots$$

$H^1(d) := \{u \in H^1 : du \in H^1\}, \quad H\Lambda^k := \{u \in L^2 : du \in L^2\}.$

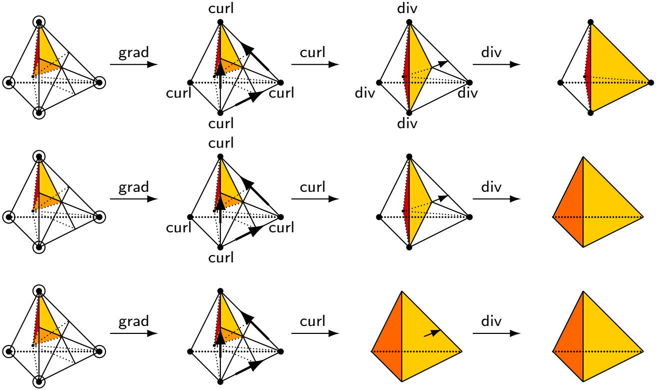
Fu-Guzmán-Neilan 2020: nD Alfeld split
Guzmán-Lischke-Neilan 2021: 3D Worsey-Farin split

Bernardi-Raugel 1985: $V_h := [\mathscr{P}_1]^3 \oplus b_{BR}$, where Bernardi-Raugel bubbles $b_{BR} := \{b_F n_F\}$, $Q_h = \mathscr{P}_0$. Not complex $\operatorname{div} V_h \notin Q_h$ ($\operatorname{div} b_{BR} \notin \mathscr{P}_0$).
"Minimal" dofs: vertex evaluation (approximation) + face normal (surjectivity of div)



low order Stokes complexes: Christiansen-KH 2018

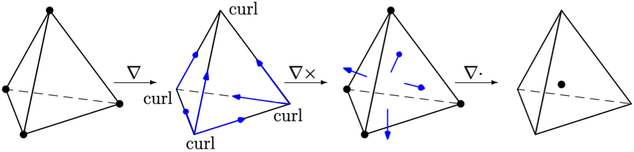
any dimension, branch into Whitney, use of Poincaré operators   See Christiansen talk



Applications: Burman-Christiansen-Hansbo 2020, elasticity, turbulence

Guzman-Neilan 2018: fix Bernardi-Raugel bubbles using functions from the Alfeld split (such that div maps onto constants)
$V_h := [\mathscr{P}_1]^3 \oplus b_{GN}$, where b_{GN} modifies b_{BR} by functions from the Alfeld split s.t. div $b_{GN} = \mathcal{P}_0$.

Complex of any degree in 3D: KH-Q.Zhang-Z.Zhang 2021



Construct 1-form by Poincaré operators (with corrections): $V_h^1 := \text{grad}\, V_h^1 \oplus \tilde{\mathscr{P}}_2 V_h^2$.

Existing finite element Stokes complexes (at least in 2D/3D)

- 2D conforming triangle: Falk-Neilan 2013, Christiansen-KH 2018 (Clough-Tocher), Guzmán-Lischke-Neilan 2020 (Powell-Sabin), Alfeld-Sorokina 2016 (Bernstein-Bézier techniques), J.Hu-Lin-Wu 2021, Chen-Huang 2022
- 3D tetrahedral element: Neilan 2015, Christiansen-KH 2018 (low order), Fu-Guzmán-Neilan 2018 (Alfeld split, any dim), Guzmán-Lischke-Neilan, 2020 (Worsey-Farin), KH-Q.Zhang-Z.Zhang 2020 (extending Guzmán-Neilan)
- nonconforming elements: Brenner 2014 (Morley, Crouzeix-Raviart), Mardal-Tai-Winther 2002, Tai-Winther 2006, Huang 2020
- virtual elements: Beirão da Veiga-Dassi-Vacca 2020 (conforming), Zhao-Zhang-Mao-Chen 2019 (nonconforming)
- quadrilateral grids: S.Zhang 2016, Neilan-Sap 2018 (macroelement), Quan-Ji-S.Zhang 2020 (nonconforming)
- polygonal meshes: Hanot 2021

Approaches

- discrete differential geometry inspired methods (distributional finite elements),
- BGG diagram chasing,
- characterizing bubbles,
- geometric decomposition,
- weakly imposed symmetry.

Additional motivation for discretizing the entire BGG diagram: twisted complexes

BGG diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & W^{k-1} \arrow[r, "d"] & W^{k} \arrow[r, "d"] & W^{k+1} \arrow[r, "d"] & \cdots \\
\cdots \arrow[r] \arrow[ur] & V^{k-1} \arrow[r, "d"] \arrow[ur, "S^{k-1}"] & V^{k} \arrow[r, "d"] \arrow[ur, "S^{k}"] & V^{k+1} \arrow[r, "d"] \arrow[ur, "S^{k+1}"] & \cdots .
\end{tikzcd}
$$

twisted complex:

$$
\cdots \longrightarrow \begin{pmatrix} W^{k-1} \\ V^{k-1} \end{pmatrix} \xrightarrow{d_V^{k-1}} \begin{pmatrix} W^{k} \\ V^{k} \end{pmatrix} \xrightarrow{d_V^{k}} \begin{pmatrix} W^{k+1} \\ V^{k+1} \end{pmatrix} \longrightarrow \cdots,
$$

$$
d_V := \begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}
$$

BGG cohomology-preserving projections: twisted complex to BGG complex

$$
\begin{tikzcd}
\text{Cosserat elasticity} \arrow[r, "\text{BGG (elasticity)}"] \arrow[d, "\Gamma-\text{convergence}"] & \text{classical elasticity} \arrow[d, "\Gamma-\text{convergence}"] \\
\text{(modified) Reissner} - \text{Mindlin plate} \arrow[r, "\text{BGG (hessian)}"] & \text{Kirchhoff plate}
\end{tikzcd}
$$

Γ-convergence: Ciarlet 1997; Neff, Hong, Jeong 2010 etc.

Quenneville-Bélair 2015, Arnold 2015 (PKU lecture), Čap-KH 2022

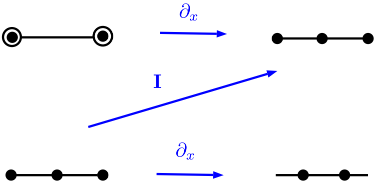
Diagram chasing: fit finite elements in diagrams

Continuous level

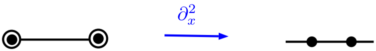
$$0 \longrightarrow H^2 \xrightarrow{\partial_x^2} L^2 \longrightarrow 0.$$

$$\begin{tikzcd}
0 \arrow[r] & H^2 \arrow[r, "\partial_x"] & H^1 \arrow[r] & 0 \\
0 \arrow[r] & H^1 \arrow[r, "\partial_x"] \arrow[ur, "I"] & L^2 \arrow[r] & 0.
\end{tikzcd}$$

Discrete level

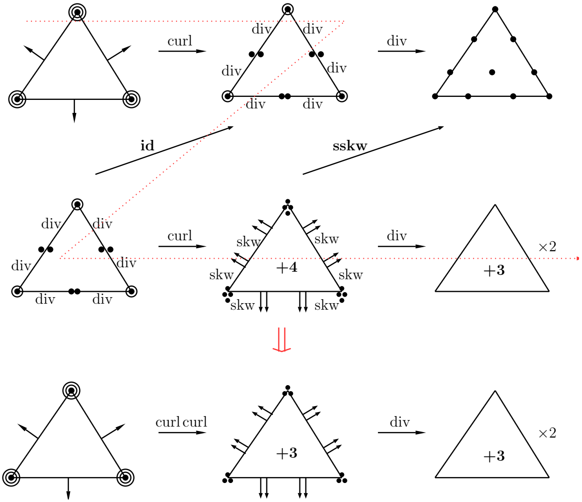


implies



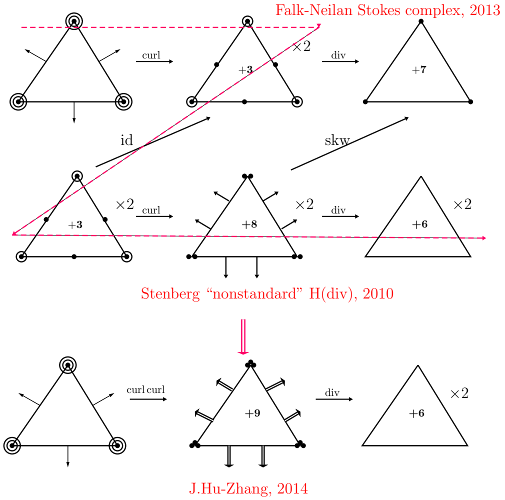
- two de-Rham complexes with different continuity,
- cohomology: $\mathcal{N}(\partial_x^2) \cong \mathcal{N}(\partial_x) \oplus \mathcal{N}(\partial_x)$, ∂_x^2 is onto.

Re-interpretation of Arnold-Winther elasticity (stress) complex: (Arnold-Falk-Winther 2006)

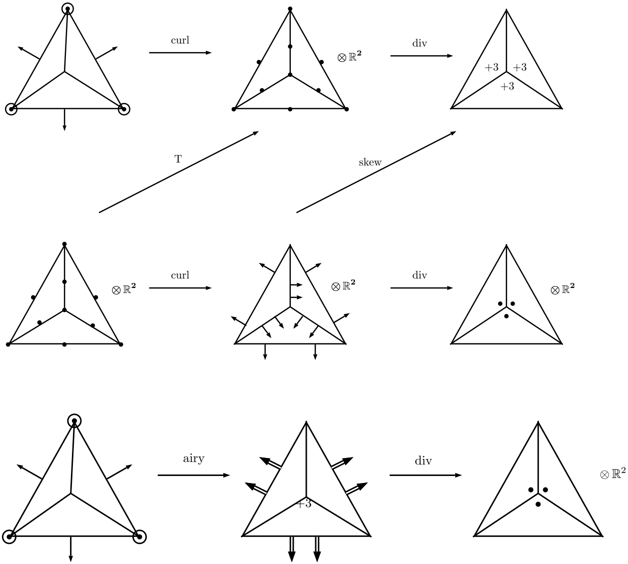


Another stress complex

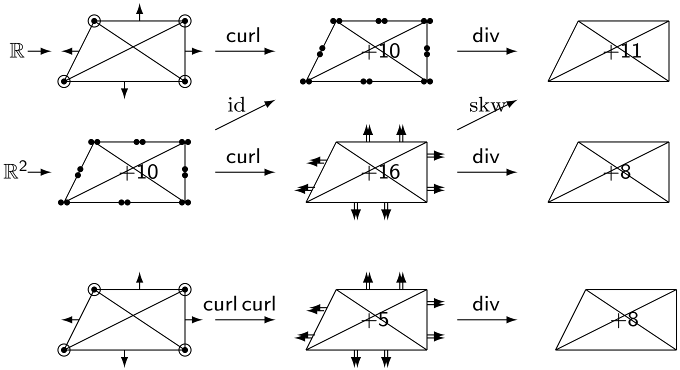
$$0 \longrightarrow H^2 \xrightarrow{\text{curl curl}} H(\text{div}, \mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{R}^2 \longrightarrow 0.$$



Christiansen-J.Hu-KH 2018



scalar: Clough-Tocher; vector: Arnold-Qin 1992 (Stokes pair);
matrix: (modified) Johnson-Mericier 1978



scalar: Lai-Schumaker (book); vector: Arnold-Qin 1992 (Stokes pair); matrix: Johnson-Mercier

To do: Powell-Sabin version?

2D strain complex: Christiansen-KH 2022
See Christiansen talk.

3D elasticity complex: Christiansen-Gopalakrishnan-Guzmán-KH 2020
See Gopalakrishnan talk.

Tensor product construction for “first order BGG”, any dimension & degree:
Bonizzoni-KH-Kanschat-Sap, in preparation
See Bonizzoni talk.

Splines and IGA: Arf-Simeon 2021, Arf-Simeon 2022 (nontrivial pullback, planar elasticity)

Weak symmetry: when rows do not match exactly

Example: elasticity $\operatorname{div}\sigma = f$, $\operatorname{vskw}\sigma = 0$. Weak form

$$(\operatorname{div}\sigma, v) = (f, v), \quad (\operatorname{vskw}\sigma, q) = 0, \quad \forall v, q.$$

Inf-sup condition: $(\operatorname{div}, \operatorname{vskw}) : \Sigma \to V \times Q$ is onto. On the continuous level, surjectivity can be verified by explicit diagram chasing or general BGG construction.

$$\begin{tikzcd}
0 \arrow[r] & W^0 \arrow[r, "\operatorname{grad}"] & W^1 \arrow[r, "\operatorname{curl}"] & W^2 \arrow[r, "\operatorname{div}"] & W^3 \arrow[r] & 0 \\
0 \arrow[r] & V^0 \arrow[r, "\operatorname{grad}"] \arrow[ur, "-\operatorname{mskw}"] & V^1 \arrow[r, "\operatorname{curl}"] \arrow[ur, "\mathcal{S}"] & V^2 \arrow[r, "\operatorname{div}"] \arrow[ur, "2\operatorname{vskw}"] & V^3 \arrow[r] & 0.
\end{tikzcd}$$

$$\begin{tikzcd}
W^0 \arrow[r, "\operatorname{grad}"] & W^1 \arrow[dr, "\operatorname{curl} S^{-1} \operatorname{curl}"] & & W^3 \\
V^0 \arrow[ur, "-\mathrm{mskw}"] & & V^2 \arrow[r, "\operatorname{div}"] \arrow[ur, "\mathrm{vskw}"] & V^3,
\end{tikzcd}$$

Discrete level

$$\begin{tikzcd}
0 \arrow[r] & W_h^0 \arrow[r, "\operatorname{grad}"] & W_h^1 \arrow[r, "\operatorname{curl}"] & W_h^2 \arrow[r, "\operatorname{div}"] & W_h^3 \arrow[r] & 0 \\
0 \arrow[r] & V_h^0 \arrow[r, "\operatorname{grad}"] \arrow[ur, "-\operatorname{mskw}_h"] & V_h^1 \arrow[r, "\operatorname{curl}"] \arrow[ur, "\mathcal{S}_h"] & V_h^2 \arrow[r, "\operatorname{div}"] \arrow[ur, "2\operatorname{vskw}_h"] & V_h^3 \arrow[r] & 0.
\end{tikzcd}$$

$S_h := \pi_h \circ S$. Key condition in Arnold-Falk-Winther 2005: choosing spaces s.t. $\mathcal{S}_h$ is onto.
Gopalakrishnan-Guzmán 2011: with certain conditions, weak symmetry implies strong symmetry.
Weakly imposed high order operators: Quenneville-Bélair 2015, Arnold 2015 (PKU lecture).

Characterizing bubbles

- finite elements = skeleton + bubbles (e.g., Raviart-Thomas, Bernadi-Raugel),
- key ideas (J.Hu-S.Zhang 2014, 2015):
  - explicit characterization of bubbles,
  - Lagrange type basis.

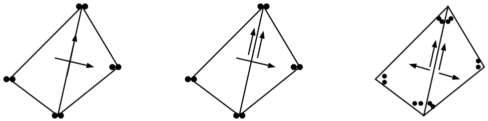
Example: div : $H(\mathrm{div}, \mathbb{S}) \to L^2(\mathbb{V})$. On a tetrahedron T, edge-based $H(\mathrm{div}, \mathbb{S})$-bubbles:

$$\Sigma_{r,b}(T) := \sum \lambda_i \lambda_j \mathscr{P}_{r-2}(T) e_{ij} \otimes e_{ij}.$$

e_{ij}: unit vector of edge connecting vertices i, j. $e_{ij} \otimes e_{ij}$: Regge basis.

- symmetric,
- normal component vanishes on all faces.

$\Sigma_h := \sum_T \Sigma_{r,b}(T) + \mathrm{Lagrange}$, $V_h = \mathscr{P}_{r-1}$.



vector $H(\mathrm{div})$ case: Lagrange (partially discontinuous), Stenberg, DG

Generalizations

J.Hu-Liang 2021 (3D Hessian complex), J.Hu-Liang-Ma 2021, J.Hu-Liang-Ma-Zhang 2022 (3D divdiv complex)

Chen-Huang:

- polynomial BGG exact sequences: symbols of differentials

$$\mathcal{RM} \underset{\pi_{RM}}{\overset{\subset}{\rightleftarrows}} \mathscr{P}_{k+1}(K;\mathbb{R}^3) \underset{\boldsymbol{\tau}\cdot\boldsymbol{x}}{\overset{\text{def}}{\rightleftarrows}} \mathscr{P}_{k}(K;\mathbb{S}) \underset{\boldsymbol{x}\times\boldsymbol{\tau}\times\boldsymbol{x}}{\overset{\text{inc}}{\rightleftarrows}} \mathscr{P}_{k-2}(K;\mathbb{S}) \underset{\text{sym}(\boldsymbol{v}\boldsymbol{x}^{\intercal})}{\overset{\text{div}}{\rightleftarrows}} \mathscr{P}_{k-3}(K;\mathbb{R}^3) \rightleftarrows 0 \ .$$

  not null-homotopy operators, but the resulting spaces form an exact sequence. (Poincaré operators: Christiansen-KH-Sande 2020, Čap-KH in preparation)

- geometric decomposition

<table>
<tr><td>$E_0 := \ker(\mathrm{d})$
$\cap \ker(\mathrm{tr}^{\mathrm{d}})$</td><td>$E_0^{\perp}$</td></tr>
<tr><td colspan="2">tr^{d}</td></tr>
</table>

  - characterization of trace dofs: Green's formula,
  - bubbles: use polynomial complex, Koszul operators and harmonic interpolations.

Chen-Huang: all examples from "first order BGG" in 2D/3D
Chen-Huang 2021: geometric decomposition, nD (n-1)-form
Chen-Huang 2022: arbitrarily high regularity in 2D

Conformal deformation complex: an open question

"three" de-Rham complexes $\Longrightarrow$ conformal (Čap-KH 2022)

$$\begin{tikzcd}
0 \arrow[r] & H^{q_0} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & H^{q_0-1} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] & H^{q_0-2} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] & H^{q_0-3} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q_1} \otimes (\mathbb{R} \oplus \mathbb{V}) \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,1}"] & H^{q_1-1} \otimes (\mathbb{V} \oplus \mathbb{M}) \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^{1,1}"] & H^{q_1-2} \otimes (\mathbb{V} \oplus \mathbb{M}) \arrow[r, "\mathrm{div}"] \arrow[ur, "S^{2,1}"] & H^{q_1-3} \otimes (\mathbb{R} \oplus \mathbb{V}) \arrow[r] & 0 \\
0 \arrow[r] & H^{q_2} \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^{0,2}"] & H^{q_2-1} \otimes \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^{1,2}"] & H^{q_2-2} \otimes \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^{2,2}"] & H^{q_2-3} \otimes \mathbb{V} \arrow[r] & 0.
\end{tikzcd}$$

Alternatively, from two more complicated complexes (Arnold-KH 2021) - either eliminate trace from symmetric matrices:

$$\begin{tikzcd}
0 \arrow[r] & H^{q} \otimes \mathbb{V} \arrow[r, "\mathrm{dev\,grad}"] & H^{q-1} \otimes \mathbb{T} \arrow[r, "\mathrm{sym\,curl}"] & H^{q-2} \otimes \mathbb{S} \arrow[r, "\mathrm{div\,div}"] & H^{q-4} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] \arrow[ur, "-\mathrm{mskw}"] & H^{q-2} \otimes \mathbb{S} \arrow[r, "\mathrm{inc}"] \arrow[ur, "\mathcal{S}"] & H^{q-4} \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}$$

or eliminate skew-symmetric part from trace-free matrices:

$$\begin{tikzcd}
0 \arrow[r] & H^{q-1} \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & H^{q-2} \otimes \mathbb{S} \arrow[r, "\mathrm{inc}"] & H^{q-4} \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] & H^{q-5} \otimes \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2} \arrow[r, "\mathrm{hess}"] \arrow[ur, "\iota"] & H^{q-4} \otimes \mathbb{S} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathcal{S}"] & H^{q-5} \otimes \mathbb{T} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & H^{q-6} \otimes \mathbb{V} \arrow[r] & 0.
\end{tikzcd}$$

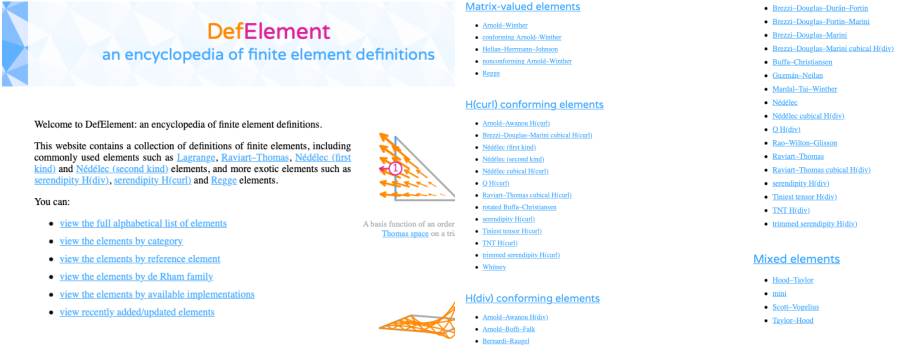
weak symmetry (with curl bubbles of Stenberg's type): Gopalakrishnan-Lederer-Schöberl 2020
BGG diagrams may provide new constructions

Discretization of complexes:

- 2D stress: Arnold-Winther 2002, J.Hu-S.Zhang 2014, Christiansen-KH 2018,
- 2D strain: Chen-J.Hu-Huang 2014 (Regge/HHJ), Christiansen-KH 2018 (conforming), Chen-Huang 2020, DiPietro-Droniou 2021 (polygonal meshes)
- 3D elasticity: various results on last part of complex, Hauret-Kuhl-Ortiz 2007 (discrete geometry/mechanics), Arnold-Awanou-Winther 2008, Christiansen 2011 (Regge), Christiansen-Gopalakrishnan-Guzmán-Hu 2020, Chen-Huang 2021
- 3D Hessian: Chen-Huang 2020, J.Hu-Liang 2021, Arf-Simeon 2021 (splines)
- 3D divdiv: Chen-Huang 2021, J.Hu-Liang-Ma 2021, Sander 2021 ($H(\operatorname{sym}\operatorname{curl})$, $H(\operatorname{dev}\operatorname{sym}\operatorname{curl})$), J.Hu-Liang-Ma-Zhang 2022
- nD: Chen-Huang 2021 (last two spaces), 2D arbitrary regularity: Chen-Huang 2022
- conformal complexes: open.

What have been implemented? (apologies for omission - please correct!)

- de-Rham: FEniCS, Firedrake, NGSolve, deal.II...
- smoother de-Rham: several variants of Scott-Vogelius, Christiansen-Hu, Guzmán-Neilan (K.Hu-Q.Zhang-Z.Zhang)... (research papers, not many in libraries)
- tensors: Arnold-Winther (Firedrake), J.Hu-S.Zhang, Regge (FEniCS, Firedrake, NGSolve), other distributional elements (NGSolve).


DefElement

an encyclopedia of finite element definitions

Welcome to DefElement: an encyclopedia of finite element definitions.

This website contains a collection of definitions of finite elements, including commonly used elements such as Lagrange, Raviart–Thomas, Nédélec (first kind) and Nédélec (second kind) elements, and more exotic elements such as serendipity H(div), serendipity H(curl) and Regge elements.

You can:

- view the full alphabetical list of elements
- view the elements by category
- view the elements by reference element
- view the elements by de Rham family
- view the elements by available implementations
- view recently added/updated elements

A basis function of an order

Thomas space on a tri

Matrix-valued elements

- Arnold–Winther
- conforming Arnold–Winther
- Hellan–Herrmann–Johnson
- nonconforming Arnold–Winther
- Regge

H(curl) conforming elements

- Arnold–Awanou H(curl)
- Brezzi–Douglas–Marini cubical H(curl)
- Nédélec (first kind)
- Nédélec (second kind)
- Nédélec cubical H(curl)
- Q H(curl)
- Raviart–Thomas cubical H(curl)
- rotated Buffa–Christiansen
- serendipity H(curl)
- Tiniest tensor H(curl)
- TNT H(curl)
- trimmed serendipity H(curl)
- Whitney

H(div) conforming elements

- Arnold–Awanou H(div)
- Arnold–Boffi–Falk
- Bernardi–Raugel
- Brezzi–Douglas–Durán–Fortin
- Brezzi–Douglas–Fortin–Marini
- Brezzi–Douglas–Marini
- Brezzi–Douglas–Marini cubical H(div)
- Buffa–Christiansen
- Guzmán–Neilan
- Mardal–Tai–Winther
- Nédélec
- Nédélec cubical H(div)
- Q H(div)
- Rao–Wilton–Glisson
- Raviart–Thomas
- Raviart–Thomas cubical H(div)
- serendipity H(div)
- Tiniest tensor H(div)
- TNT H(div)
- trimmed serendipity H(div)

Mixed elements

- Hood–Taylor
- mini
- Scott–Vogelius
- Taylor–Hood

defelement.com

Seems no "universal theory" yet, but some general ideas and tools are available: Bernstein-Bézier techniques, Poincaré/Koszul operators, dimension count, supersmoothness v.s. subdivision, tensor product construction, BGG diagrams, Finite Element System...

Further questions

- computational issues: implementation, basis, supersmoothness v.s. subdivision (simplicial and cubical spline techniques)
- general theory of distributional elements
- weak symmetry (Čap-KH 2022: BGG without inj/surj conditions)
- conformal complexes and high-order BGG
- nonlinear and curved theory
- applications: numerical relativity
- what finite elements can bring to splines and discrete calculus?

$$\boxed{\text{splines}} \longleftarrow \boxed{\text{FEs}} \longrightarrow \boxed{\text{lattice theory}}$$

  - spline dimensions: equivalent problems of Strang Conjecture using de-Rham or BGG complexes (lower regularity, vector/tensor structures)
  - FE-inspired discrete geometry (following Christiansen 2011): supplying local functions, PDE and numerical analysis, measure-valued solutions