Towards Computational Topological (Magneto)Hydrodynamics

Long term evolution of fluids / plasma

Kaibo Hu

University of Oxford

Feng Kang visitor program
20 November 2025, Chinese Academy of Science

Feng Kang's Principle

"A fundamental principle in the study of computational methods is that the essential characteristics of the original problem should be preserved as much as possible after discretization."

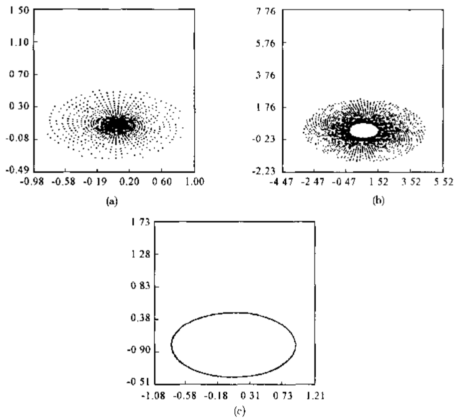
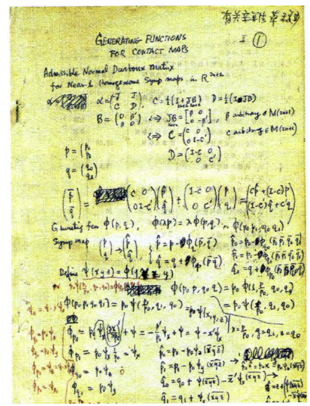


图 0.1 谐振子形成的椭圆轨道的计算

Feng Kang



Fluid and plasma systems

Various conserved quantities:

- energy,
- mass $\operatorname{div} \boldsymbol{u} = 0$ (for incompressible flows),
- charge $\operatorname{div} \boldsymbol{j} = \rho$,
- helicity $\int \boldsymbol{A} \cdot \boldsymbol{B} \, dx$,
- enstrophy $\int |\operatorname{curl} \boldsymbol{u}|^2 \, dx$,
- $\cdots$

A structure-preserving faith: preserve them as much as possible.

Fluid and plasma systems

Various conserved quantities:

- energy,
- mass $\operatorname{div} \boldsymbol{u} = 0$ (for incompressible flows),
- charge $\operatorname{div} \boldsymbol{j} = \rho$,
- helicity $\int \boldsymbol{A} \cdot \boldsymbol{B} \, dx$,
- enstrophy $\int |\operatorname{curl} \boldsymbol{u}|^2 \, dx$,
- $\cdots$

A structure-preserving faith: preserve them as much as possible.

But, what are the precise effects of (not) preserving them for numerics?

Fluid and plasma systems

Various conserved quantities:

- energy,
- mass $\operatorname{div} \boldsymbol{u} = 0$ (for incompressible flows),
- charge $\operatorname{div} \boldsymbol{j} = \rho$,
- helicity $\int \boldsymbol{A} \cdot \boldsymbol{B} \, dx$,
- enstrophy $\int |\operatorname{curl} \boldsymbol{u}|^2 \, dx$,
- $\cdots$

A structure-preserving faith: preserve them as much as possible.

But, what are the precise effects of (not) preserving them for numerics?

This talk: **long term dynamics and computation of fluids and plasma**

- relaxation : *Does plasma system evolve to a stationary state?*
  corona heating, MHD equilibrium (stellarator, tokamak) etc.
- dynamo : *Does there magnetic field exponentially grow?*
  generation of magnetic fields in earth and sun etc.

Fluid and plasma systems

Various conserved quantities:

- energy,
- mass $\operatorname{div} \boldsymbol{u} = 0$ (for incompressible flows),
- charge $\operatorname{div} \boldsymbol{j} = \rho$,
- helicity $\int \boldsymbol{A} \cdot \boldsymbol{B} \, dx$,
- enstrophy $\int |\operatorname{curl} \boldsymbol{u}|^2 \, dx$,
- $\cdots$

A structure-preserving faith: preserve them as much as possible.

But, what are the precise effects of (not) preserving them for numerics?

This talk: **long term dynamics and computation of fluids and plasma**

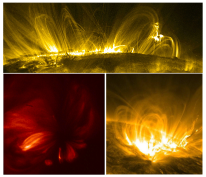
- relaxation : *Does plasma system evolve to a stationary state?*
  corona heating, MHD equilibrium (stellarator, tokamak) etc.

- dynamo : *Does there magnetic field exponentially grow?*
  generation of magnetic fields in earth and sun etc.

and limits of preserving invariants for fluids?

Motivation: structure-preserving discretisation

Fundamental question in plasma physics: given initial data, what does the system evolve to?
heating of solar corona, plasma equilibria (magnetic configurations) etc.



Magneto-friction (simplified MHD) :

$$
\begin{aligned}
\boldsymbol{B}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= 0, \\
\boldsymbol{j} &= \nabla \times \boldsymbol{B}, \\
\boldsymbol{u} &= \tau \boldsymbol{j} \times \boldsymbol{B}.
\end{aligned}
$$

Energy decay

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau \|\boldsymbol{B} \times \boldsymbol{j}\|^2.$$

Helicity conservation

$$\frac{d}{dt}\mathcal{H}_m = 0, \quad \text{with } \mathcal{H}_m := \int \boldsymbol{A}\cdot\boldsymbol{B}\, dx, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}.$$

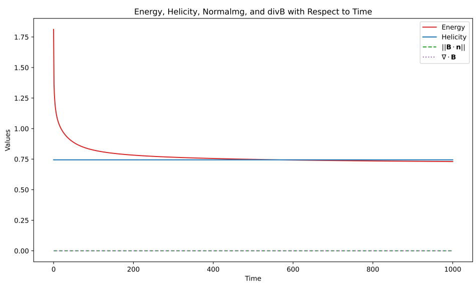


Figure. Helicity-preserving scheme

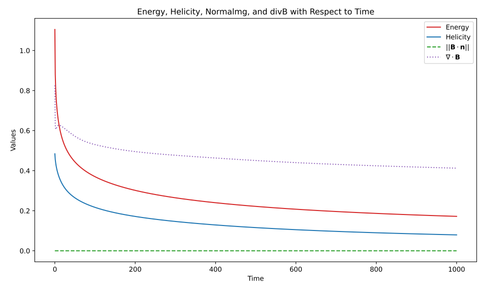
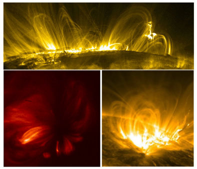
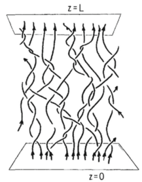
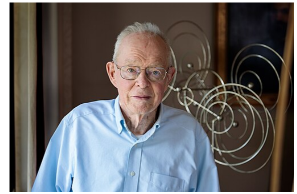


Figure. CG scheme (non-preserving)

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, KH, B. Andrews, SISC (2025).





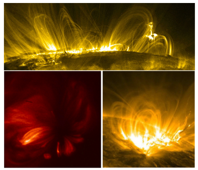


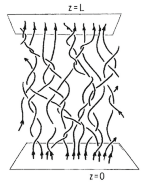
Eugene Parker

Parker hypothesis (Still Open)

For “almost any initial data”, *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.*

Ideal magnetic relaxation





Eugene Parker

Parker hypothesis (Still Open)

For “almost any initial data”, *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.*

How confident are we in what we compute?

Outline

Magnetohydrodynamics (MHD): macroscopic description of plasma, an incompressible model

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) - R_e^{-1} \Delta \boldsymbol{u} - s\boldsymbol{j} \times \boldsymbol{B} + \nabla P &= \boldsymbol{f} && \text{momentum equation,} \\
\boldsymbol{j} - \nabla \times \boldsymbol{B} &= \boldsymbol{0} && \text{Ampere's law,} \\
\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= \boldsymbol{0} && \text{Faraday's law,} \\
R_m^{-1} \boldsymbol{j} - (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) &= \boldsymbol{0} && \text{Ohm's law,} \\
\nabla \cdot \boldsymbol{B} &= 0 && \text{Gauss law,} \\
\nabla \cdot \boldsymbol{u} &= 0,
\end{aligned}
$$

initial conditions $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x})$,

boundary conditions on $\partial\Omega$: $\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}$.

Three nonlinear terms:

fluid advection $-\boldsymbol{u} \times (\nabla \times \boldsymbol{u})$ (in the vorticity form)
Lorentz force $-s\boldsymbol{j} \times \boldsymbol{B}$
magnetic advection $-\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$

For relaxation, we are interested in zero magnetic diffusion, nonzero fluid diffusion ($R_m = \infty$, $R_e < \infty$).

Energy structures of MHD

Energy dissipation or conservation:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_0^2 + \frac{S}{2}\frac{d}{dt}\|\boldsymbol{B}\|_0^2 + R_e^{-1}\|\nabla\boldsymbol{u}\|_0^2 + SR_m^{-1}\|\boldsymbol{j}\|_0^2 = (\boldsymbol{f}, \boldsymbol{u}),$$

and hence

$$\max_{0\leq t\leq T}\left(\|\boldsymbol{u}\|_0^2 + S\|\boldsymbol{B}\|_0^2\right) + R_e^{-1}\int_0^T \|\nabla\boldsymbol{u}\|_0^2 \,\mathrm{d}\tau + 2SR_m^{-1}\int_0^T \|\boldsymbol{j}\|_0^2 \,\mathrm{d}\tau$$
$$\leq \|\boldsymbol{u}_0\|_0^2 + S\|\boldsymbol{B}_0\|_0^2 + R_e\int_0^T \|\boldsymbol{f}\|_{-1}^2 \,\mathrm{d}\tau.$$

With $\boldsymbol{f} = 0$, $R_m^{-1} = 0$, total energy is non-increasing. However, some key information is not clear:

- whether the total energy decays to zero?
- how does total energy split into the fluid part $(\|\boldsymbol{u}\|^2)$ $+$ magnetic part $(S\|\boldsymbol{B}\|^2)$?

Helicity: fine structures

Magnetic helicity: for *any* potential $\boldsymbol{A}$ satisfying $\nabla \times \boldsymbol{A} = \boldsymbol{B}$,

$$\mathcal{H}_m := \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\, dx$$

Idea started from Helmholtz & Kelvin. MHD: Woltjer's invariant, ideal fluid: Moffatt (giving the name).

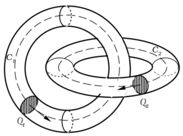


Figure: linking/knottedness of $\boldsymbol{B}$.

$\mathcal{H}_{\boldsymbol{\xi}} = 2\ell(C_1, C_2)Q_1 \cdot Q_2$. $\ell = 1$: Gauss linking number, Q_i: flux

Helicity = averaging asymptotic linking number (V.I. Arnold)

Cross helicity:

$$\mathcal{H}_c := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{B}\, dx$$

linking of vorticity and magnetic fields

A topological mechanism

Arnold inequality (V.I. Arnold 1974): helicity provides lower bound for energy

$$\left|\int \boldsymbol{A}\cdot\boldsymbol{B}\,dx\right| \leq C\int |\boldsymbol{B}|^2\,dx$$

Proof. Cauchy-Schwarz $|\int \boldsymbol{A}\cdot\boldsymbol{B}\,dx| \leq \|\boldsymbol{A}\|_{L^2}\|\boldsymbol{B}\|_{L^2}$ + Poincaré inequality $\|\boldsymbol{A}\|_{L^2} \leq C\|\nabla\times\boldsymbol{A}\|_{L^2}$.

Vladimir I. Arnold

Differential form point of view: $\boldsymbol{A}$: 1-form, $\boldsymbol{B}$: 2-form

$$\int \boldsymbol{A}\wedge\boldsymbol{B} \quad \leq C \quad \int \boldsymbol{B}\wedge *\boldsymbol{B}$$

Helicity, Topology *Energy, Geometry*

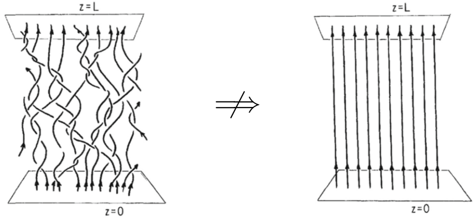


Fig: Pontin, Hornig, Living Rev. Sol. Phys. 2020.

knots are topological barriers that prevent energy from dissipation

Magnetic helicity conservation

Conservative for ideal MHD ($R_e^{-1} = R_m^{-1} = 0$):

$$\frac{d}{dt}\int \mathbf{A}\cdot\mathbf{B}\,dx = 0, \qquad \frac{d}{dt}\int \mathbf{u}\cdot\mathbf{B}\,dx = 0.$$

Proof. Magnetic field advection: $\mathbf{B}_t = \nabla \times (\mathbf{u} \times \mathbf{B})$.
Then

$$\frac{d}{dt}\int \mathbf{A}\cdot\mathbf{B}\,dx = 2\int \mathbf{A}\cdot\nabla\times(\mathbf{u}\times\mathbf{B})\,dx$$
$$\overset{\text{IBP}}{=} 2\int(\nabla\times\mathbf{A})\cdot(\mathbf{u}\times\mathbf{B})\,dx = 2\int \mathbf{u}\cdot(\mathbf{B}\times\mathbf{B})\,dx = 0.$$

(IBP: integral by parts with vanishing boundary conditions.)

Remark. Proof holds for any $\mathbf{u}$. Magnetic helicity remains conserved even when $R_e^{-1} \neq 0$.

Consequence: topological constraint

In the ideal limit $R_m = \infty$ (finite R_e), energy may decay but has a lower bound determined by magnetic helicity. Thus topologically nontrivial initial data cannot relax to a trivial field.

Magnetic helicity conservation

Conservative for ideal MHD ($R_e^{-1} = R_m^{-1} = 0$):

$$\frac{d}{dt}\int \mathbf{A}\cdot\mathbf{B}\,dx = 0, \qquad \frac{d}{dt}\int \mathbf{u}\cdot\mathbf{B}\,dx = 0.$$

Proof. Magnetic field advection: $\mathbf{B}_t = \nabla\times(\mathbf{u}\times\mathbf{B})$.
Then

$$\frac{d}{dt}\int \mathbf{A}\cdot\mathbf{B}\,dx = 2\int \mathbf{A}\cdot\nabla\times(\mathbf{u}\times\mathbf{B})\,dx$$
$$\stackrel{\text{IBP}}{=} 2\int(\nabla\times\mathbf{A})\cdot(\mathbf{u}\times\mathbf{B})\,dx = 2\int \mathbf{u}\cdot(\mathbf{B}\times\mathbf{B})\,dx = 0.$$

(IBP: integral by parts with vanishing boundary conditions.)

Remark. Proof holds for any $\mathbf{u}$. Magnetic helicity remains conserved even when $R_e^{-1} \neq 0$.

Consequence: topological constraint

In the ideal limit $R_m = \infty$ (finite R_e), energy may decay but has a lower bound determined by magnetic helicity. Thus topologically nontrivial initial data cannot relax to a trivial field.

Topological mechanism may be lost due to discretization errors, leading to **wrong solutions**!

Structure-preserving MHD: literature

Existing numerical methods for magnetic relaxation: Lagrange method, issues with mesh deformation

- *Mimetic methods for Lagrangian relaxation of magnetic fields*, S.Candelaresi, D.Pontin, G.Hornig, SIAM Journal on Scientific Computing (2014).

Structure-preserving discretization for MHD:

- energy conservation: e.g., Armero,Simo 1996 etc.
- $\nabla \cdot \boldsymbol{B} = 0$: e.g., Brackbill, Barnes 1980, Hu,Ma,Xu 2017, Hiptmair,Mao,Zheng 2018
- charge conservation: Li,Ni,Zheng 2019
- helicity conservation: less attention, Liu,Wang 2004 (axisymmetric MHD flow, finite difference methods); Kraus,Maj 2017 (DEC, variational integrator), Sullivan 2018 ('Lattice hydrodynamics').

Helicity-preserving finite element for NS:
Rebholz 2007; Zhang, Palha, Gerritsma, Rebholz 2022 (dual field approach).

Helicity-preserving finite element for MHD:
KH, Lee, Xu 2021; Gawlik, Gay-Balmaz 2022; Laakmann, KH, Farrell 2023 (Hall MHD), Zhang, Palha, Brugnoli, Toshniwal, Gerritsma 2024.

The numerics below are based on the projection approach (Rebholz 2007, KH, Lee, Xu 2021).

Why complexes matter?

Example: Gauss law in Maxwell equations.

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad \Longrightarrow \quad \partial_t(\nabla \cdot \mathbf{B}) = 0.$$

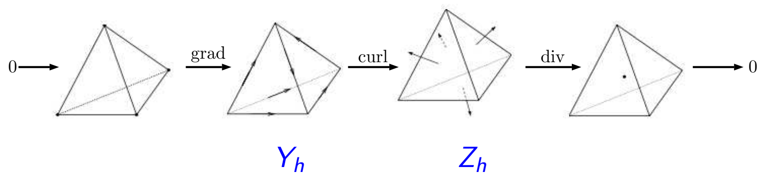
Typical Galerkin formulation: Find $\mathbf{E}_h \in Y_h$, $\mathbf{B}_h \in Z_h$ such that

$$\int \partial_t \mathbf{B}_h \cdot \mathbf{C}_h \, dx + \int (\nabla \times \mathbf{E}_h) \cdot \mathbf{C}_h \, dx = 0, \qquad \forall \mathbf{C}_h \in Z_h.$$

This implies $\partial_t \mathbf{B}_h + \mathbb{P}(\nabla \times \mathbf{E}_h) = 0 \quad \Longrightarrow \quad \partial_t(\nabla \cdot \mathbf{B}_h) = -\nabla \cdot \mathbb{P}(\nabla \times \mathbf{E}_h) \neq 0,$ (discretization errors)
$\mathbb{P} : \nabla \times Y_h \to Z_h$: L^2-projection

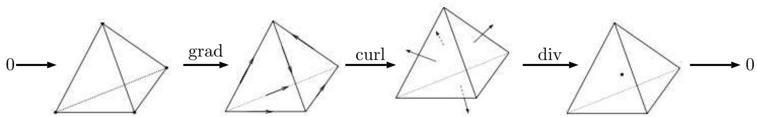
$$\mathbb{P} = I \quad \text{iff} \quad \nabla \times Y_h \subset Z_h.$$

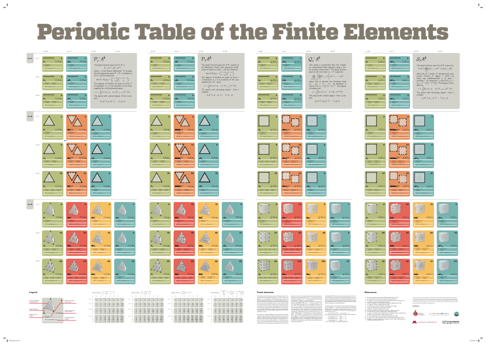
Relations such as $\nabla \times Y_h \subset Z_h$ are precisely the structure of differential complexes (homological algebra).



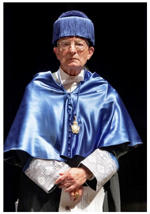
Finite elements forming a complex **preserve Gauss-type constraints**.

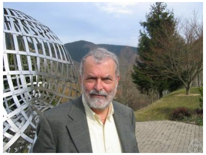
Canonical finite elements for the de Rham complex





Raviart-Thomas (1977), Nédélec (1980) in numerical analysis
Bossavit (1988): differential forms and complex
Hiptmair (1999), Arnold, Falk, Winther (2006): systematic study, “Finite Element Exterior Calculus”


Pierre-Arnaud Raviart


Jean-Claude Nédélec


Franco Brezzi


Donatella Marini


Jim Douglas

在以上数学理论的基础上，作者证明了将有限元方法用于组合弹性结构的普遍性收敛定理[4]． 正是出于后一动机，才引起作者研究组合弹性结构的数学基础乃至于更一般的组合流形上的椭圆方程的理论．看来组合流形的微分方程会有很广泛的应用．

最后指出一些有关本文主题的值得探讨的问题：

1）组合流形上的椭圆方程的解的正则性问题．G. Fichera 向作者指出，鉴于不同构件交接处在某种条件下可能引起某种奇异性，值得探讨把解空间从标准的 Sobolev 空间加以扩大．

2）组合流形上 Sobolev 空间理论的发展．

3）简单的典型性的组合流形上耦合 Laplace 方程的格林函数的解析构成．

4）与组合流形上的椭圆方程相联系的积分方程．

5）组合流形上的演化型方程理论．

6）de Rham-Hodge 调和积分理论对于组合流形的推广．

Elliptic equations on composite manifold and composite elastic structures, Feng Kang, Mathematica Numerica Sinica 1979

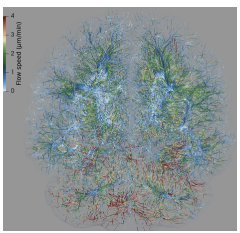


Figure: M. E. Rognes (2022), EMS Magazine

Some applications of mixed dimensional manifolds:

- brain's water scales
  M. E. Rognes (2022). *Waterscales: Mathematical and computational foundations for modelling cerebral fluid flow.* European Mathematical Society Magazine, (126), 13-26.
- combinatorial structures, contact mechanics, porous media...
  W. M. Boon, J. M. Nordbotten, J. E. Vatne (2021). *Functional analysis and exterior calculus on mixed-dimensional geometries.* Annali di Matematica Pura ed Applicata, 200(2), 757-789.
  Čech-de Rham double complex

Structure-Preserving Discretization for MHD

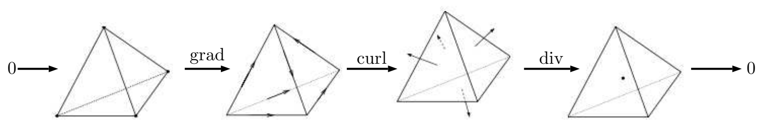
Core idea

Use finite element spaces from an exact de Rham complex

Discrete variables

- $\boldsymbol{E}_h \in H_0^h(\mathrm{curl})$, $\boldsymbol{B}_h \in H_0^h(\mathrm{div}) \implies \nabla\cdot \boldsymbol{B}_h = 0$ exactly
  Complex: $\mathrm{curl}\, H_0^h(\mathrm{curl}) \subset H_0^h(\mathrm{div})$!
- $\boldsymbol{u}_h \in H_0^h(\mathrm{curl})$
- Project nonlinear terms into the right space:

$$\boldsymbol{H}_h := \mathbb{Q}_h^{\mathrm{curl}} \boldsymbol{B}_h \quad \text{(preserves magnetic helicity)}$$
$$\boldsymbol{\omega}_h := \mathbb{Q}_h^{\mathrm{curl}} (\nabla \times \boldsymbol{u}_h) \quad \text{(preserves cross helicity)}$$



$\boldsymbol{E}_h, \boldsymbol{u}_h, \boldsymbol{H}_h$ $\boldsymbol{B}_h$

Why the projection is mandatory

Continuous identity:

$$\int (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{B} = 0$$

Naive discretization fails:

$$\int (\boldsymbol{u} \times \boldsymbol{B}_h) \cdot \mathbb{Q}_h^{\mathrm{curl}} \boldsymbol{B}_h \neq 0$$

With projection $\boldsymbol{H}_h = \mathbb{Q}_h^{\mathrm{curl}} \boldsymbol{B}_h$:

$$\int (\boldsymbol{u} \times \boldsymbol{H}_h) \cdot \boldsymbol{H}_h = 0 \quad \checkmark$$

+ Any quadratic-invariant-preserving time integrator (implicit midpoint, etc.)

Numerical scheme (magneto-friction)

Apply the same idea of choosing finite elements in a de Rham complex and adding projections :

Find $(\boldsymbol{B}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{j}, \boldsymbol{u}) \in H^h(\text{div}) \times H^h(\text{curl}) \times H^h(\text{curl}) \times H^h(\text{curl}) \times H^h(\text{div})$, such that for any $(\hat{\boldsymbol{B}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{H}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{u}})$ in the same space,

$$
\begin{aligned}
(\boldsymbol{B}_t, \hat{\boldsymbol{B}}) + (\nabla \times \boldsymbol{E}, \hat{\boldsymbol{B}}) &= 0, \\
(\boldsymbol{E}, \hat{\boldsymbol{E}}) &= -(\boldsymbol{u} \times \boldsymbol{H}, \hat{\boldsymbol{E}}), \\
(\boldsymbol{u}, \hat{\boldsymbol{v}}) &= \tau(\boldsymbol{j} \times \boldsymbol{H}, \hat{\boldsymbol{v}}), \\
(\boldsymbol{j}, \hat{\boldsymbol{j}}) &= (\boldsymbol{B}, \nabla \times \hat{\boldsymbol{j}}), \\
(\boldsymbol{H}, \hat{\boldsymbol{H}}) &= (\boldsymbol{B}, \hat{\boldsymbol{H}}).
\end{aligned}
\qquad\qquad
\begin{aligned}
\boldsymbol{B}_t + \nabla \times \boldsymbol{E} &= 0, \\
\boldsymbol{E} &= -\mathbb{P}(\boldsymbol{u} \times \boldsymbol{H}), \\
\boldsymbol{u} &= \tau\mathbb{Q}(\boldsymbol{j} \times \boldsymbol{H}), \\
\boldsymbol{j} &= \nabla_h \times \boldsymbol{B}, \\
\boldsymbol{H} &= \mathbb{P}\boldsymbol{B}.
\end{aligned}
$$

Energy law

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau\|\mathbb{Q}(\boldsymbol{H} \times \boldsymbol{j})\|^2.$$

Helicity conservation

$$\frac{d}{dt}\int \boldsymbol{A} \cdot \boldsymbol{B} = 0.$$

Numerical test: Hopf fibration

$$\boldsymbol{B}_0 = \frac{4\sqrt{a}}{\pi(1+r^2)^3}(2y(y-xz), -2(x+yz), (-1+x^2+y^2-z^2))$$

Every single field line of this field is a perfect circle, and every single field line is linked with every other one.

c.f. Smiet, C.B., Candelaresi, S. and Bouwmeester, D., 2017. Ideal relaxation of the Hopf fibration. Physics of Plasmas, 24(7).

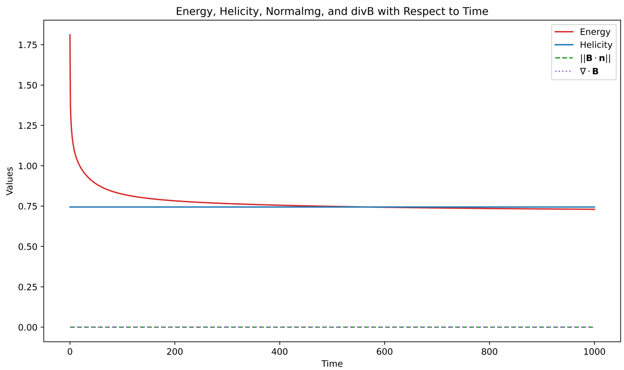


Figure. Helicity-preserving scheme

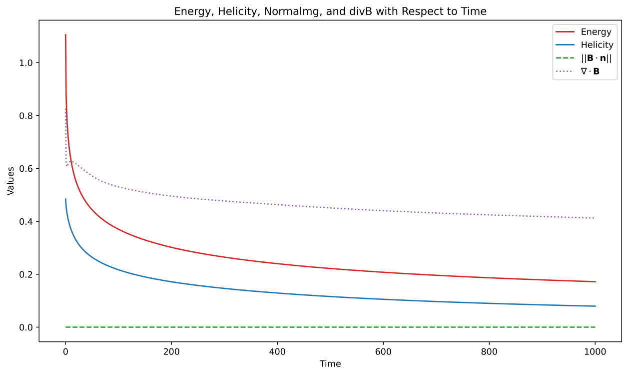
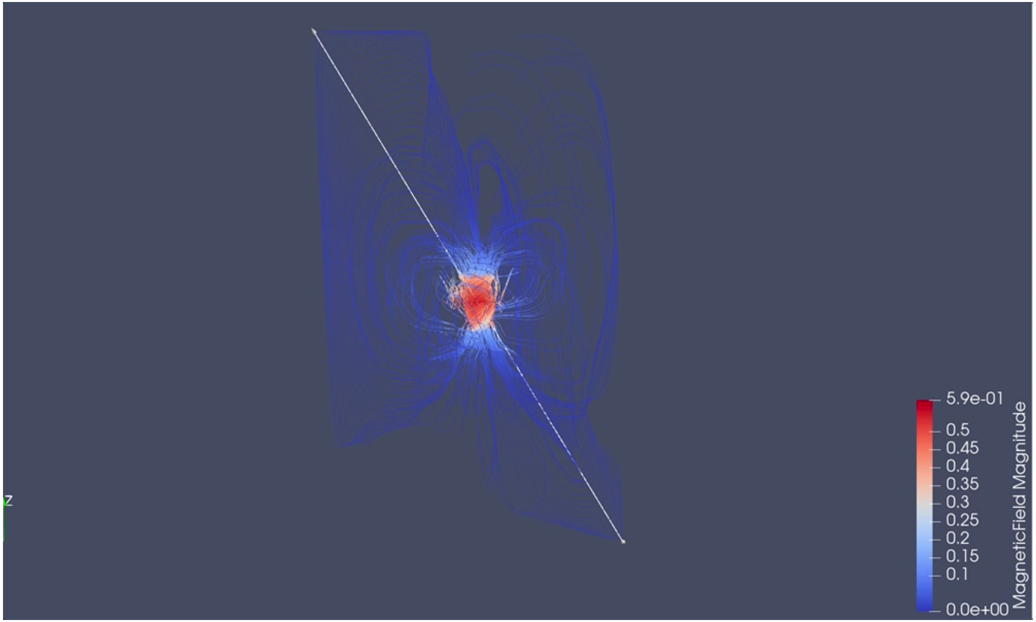
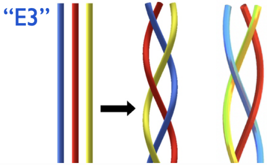


Figure. CG scheme (non-preserving)

$\tau = 10$, $dt = 1$ and $T = 1000$.



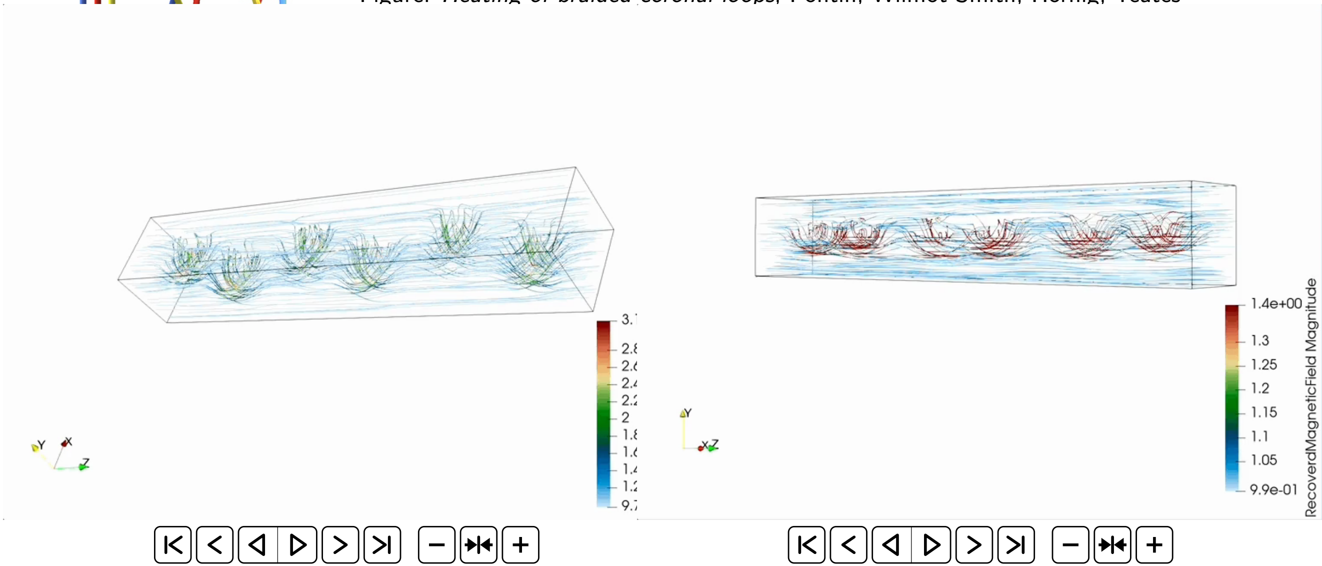
5.9e-01
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.0e+00
MagneticField Magnitude
Z



Helicity-preserving algorithms crucial even for nontrivial topology with zero helicity (twists, but no knots).

braiding, corona loops

Figure: *Heating of braided coronal loops*, Pontin, Wilmot-Smith, Hornig, Yeates



Left: helicity-preserving algorithm Right: preserving integrated $\int \boldsymbol{A} \cdot \boldsymbol{B}\, dx$ by Lagrange multiplier.

preserving *global* $\int \boldsymbol{A} \cdot \boldsymbol{B}\, dx$ not enough! local structures matter.

Full MHD (KH, Lee, Xu 2021)

Find $(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{j}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{B}, p) \in [H_0^h(\mathrm{curl}, \Omega)]^5 \times H_0^h(\mathrm{div}, \Omega) \times H_0^h(\mathrm{grad})$ such that

$$
\begin{aligned}
(D_t \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{u} \times \boldsymbol{\omega}, \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) - S(\boldsymbol{j} \times \boldsymbol{H}, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), && \text{(1a)} \\
(\boldsymbol{\omega}, \boldsymbol{\mu}) - (\nabla \times \boldsymbol{u}, \boldsymbol{\mu}) &= 0, && \text{(1b)} \\
(\boldsymbol{u}, \nabla q) &= 0, && \text{(1c)} \\
(D_t \boldsymbol{B}, \boldsymbol{C}) + (\nabla \times \boldsymbol{E}, \boldsymbol{C}) &= 0, && \text{(1d)} \\
(\boldsymbol{j}, \boldsymbol{k}) - (\boldsymbol{B}, \nabla \times \boldsymbol{k}) &= 0, && \text{(1e)} \\
(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{H}, \boldsymbol{G}) &= 0, && \text{(1f)} \\
(\boldsymbol{B}, \boldsymbol{F}) - (\boldsymbol{H}, \boldsymbol{F}) &= 0, && \text{(1g)}
\end{aligned}
$$

where $D_t \boldsymbol{u} = (\boldsymbol{u}^{new} - \boldsymbol{u}^{old})/\Delta t$, $D_t \boldsymbol{B} = (\boldsymbol{B}^{new} - \boldsymbol{B}^{old})/\Delta t$ and other variables are average of new and old values (*time stepping: implicit mid-point*).

$$
\begin{aligned}
\boldsymbol{E} &= -\mathbb{Q}_h^{\mathrm{curl}}(\boldsymbol{u} \times \boldsymbol{H}), \\
\boldsymbol{\omega} &= \mathbb{Q}_h^{\mathrm{curl}}(\nabla \times \boldsymbol{u}) \\
\boldsymbol{j} &= \nabla_h \times \boldsymbol{B}, \quad \boldsymbol{H} = \mathbb{Q}_h^{\mathrm{curl}} \boldsymbol{B}.
\end{aligned}
$$

Convergence

Algorithms converge well for *smooth true solutions*.

Theorem 1 (L. Beirão da Veiga, KH, L. Mascotto 2024)

Consider sequences $\{\mathcal{T}_h\}$ *of shape-regular, quasi-uniform meshes. Let the true solution be* **sufficiently smooth**. *Then, there exists a positive constant* C *independent of* h *such that, for all* t *in* $(0, T]$,

$$\| \mathbf{e}_\mathsf{h}^\mathbf{u}(t)\|^2 + \| \mathbf{e}_\mathsf{h}^\mathbf{B}(t)\|^2 + \int_0^t \| \operatorname{curl} \mathbf{e}_\mathsf{h}^\mathbf{u}(s)\|^2 \, \mathsf{ds} + \int_0^t \| \mathbf{e}_\mathsf{h}^\mathbf{j}(s)\|^2 \, \mathsf{ds} \leq C(\| \mathbf{e}_\mathsf{h}^\mathbf{u}(0)\|^2 + \| \mathbf{e}_\mathsf{h}^\mathbf{B}(0)\|^2 + h^{2(k+1)}).$$

The constant C *includes regularity terms of the numerical solution, the shape-regularity parameter of the mesh, and the polynomial degree* k.

Convergence

Algorithms converge well for *smooth true solutions*.

Theorem 1 (L. Beirão da Veiga, KH, L. Mascotto 2024)

Consider sequences $\{\mathcal{T}_h\}$ *of shape-regular, quasi-uniform meshes. Let the true solution be* **sufficiently smooth**. *Then, there exists a positive constant* C *independent of* h *such that, for all* t *in* $(0, T]$,

$$\|\mathbf{e}_\mathrm{h}^\mathbf{u}(t)\|^2 + \|\mathbf{e}_\mathrm{h}^\mathbf{B}(t)\|^2 + \int_0^t \|\operatorname{curl} \mathbf{e}_\mathrm{h}^\mathbf{u}(s)\|^2 \,\mathrm{ds} + \int_0^t \|\mathbf{e}_\mathrm{h}^\mathbf{j}(s)\|^2 \,\mathrm{ds} \leq C(\|\mathbf{e}_\mathrm{h}^\mathbf{u}(0)\|^2 + \|\mathbf{e}_\mathrm{h}^\mathbf{B}(0)\|^2 + h^{2(k+1)}).$$

The constant C *includes regularity terms of the numerical solution, the shape-regularity parameter of the mesh, and the polynomial degree* k.

Further question

What if the true solution is **nonsmooth**?

Onsager's conjecture; energy/helicity conservation may fail. But most FE preserves energy by definition. $O(1)$ error!

invariants-preservation for fluids has a limit!

Limit of structure-preservation

in numerical simulations. We call this an indirect approach, since it exploits the connection between singular behaviour and anomalous energy dissipation according to Onsager's conjecture. A decisive point is that this technique requires suitable discretisation schemes that remain robust in the presence of singularities and provide mechanisms of dissipation in case no viscous dissipation is present, which is a challenge in itself. Then, the idea is that observing energy-dissipating behaviour for a sequence of mesh refinement levels provides insight into the physical dissipation behaviour of the problem under investigation.

— Fehn, N., Kronbichler, M., Munch, P., & Wall, W. A. (2022). *Numerical evidence of anomalous energy dissipation in incompressible Euler flows: towards grid-converged results for the inviscid Taylor–Green problem.* Journal of Fluid Mechanics, 932, A40.

Motivation: detecting NS singularity

- direct approach: blow-up of $\boldsymbol{u}$, $\boldsymbol{\omega}$ etc. (e.g., Hou-Luo 2014)
- indirect approach: **energy dissipation for inviscid flows**

Enforcing energy conservation has to be wrong.

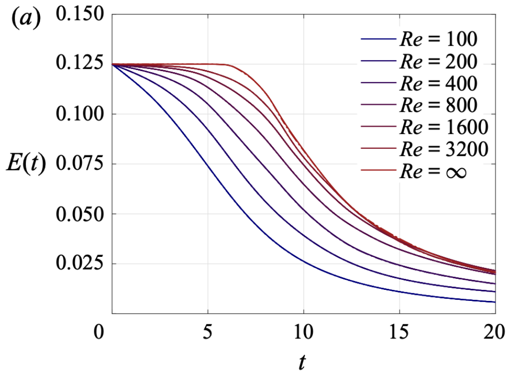
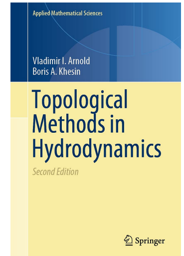


Figure 1. Temporal evolution of kinetic energy

high-order projection methods + high-order DG

Towards *Computational Topological Hydrodynamics*

A subject back to Kelvin, Helmholtz, and more recently by Arnold, Khesin, Moffatt, Sullivan...
limited applications due to lack of topology-preserving algorithms

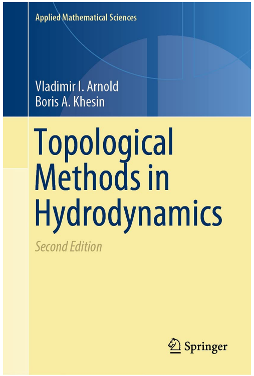


Lord Kelvin | von Helmholtz | Vladimir Arnold | Boris Khesin | Keith Moffatt | Dennis Sullivan

> *Direct computational assessment of* ***Parker's hypothesis*** *brings a number of challenges. Foremost among these is the requirement to* ***precisely maintain the magnetic topology during the simulated evolution****, i.e., precisely maintain the magnetic field line mapping between the two line-tied boundaries. . . . In the following sections, two methods are described which seek to mitigate against these difficulties. However, in all cases the* ***representation of current singularities remains problematic****. . .*

— *The Parker problem: existence of smooth force-free fields and coronal heating, Pontin, Hornig, Living Rev. Sol. Phys. 2020.*

Dynamo theory, another example:



mechanism of generation of magnetic fields in astrophysical objects (e.g., change of magnetic fields of stars and planets)

Fast dynamo: in MHD, exponential growth of magnetic field $\boldsymbol{B}$
First eigenvalue of magnetic advection-diffusion (given $\boldsymbol{u}$)

$$-\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - R_m^{-1} \nabla \times \nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}.$$

Does there exist a divergence-free field $\boldsymbol{u}$ on a manifold that is a fast kinematic dynamo?

DYNAMO

V.I.Arnold, E.I.Korkina 1983 computation: 'Galerkin methods', magnetic Reynolds number $R_m \leq 19$.

Are there spurious solutions like in Maxwell equations?

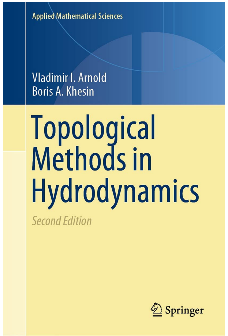
Dynamo

V.I.Arnold, E.I.Korkina 1983 computation: 'Galerkin methods', magnetic Reynolds number $R_m \leq 19$.

Are there spurious solutions like in Maxwell equations?

> *... It is* **still unknown** *whether this field (ABC flow) is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as* $R_m \to \infty$.
>
> ...
>
> *Numerically, the kinematic fast dynamo problem is the first eigenvalue problem for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers* R_m *are of order of magnitude* 10^8. **The corresponding matrices are (and will remain) beyond the reach of any computer.**

— *Topological Methods in Hydrodynamics*, V.I.Arnold, B.A.Khesin 2021.



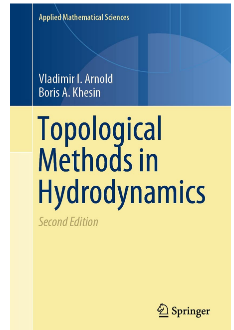
Dynamo

V.I.Arnold, E.I.Korkina 1983 computation: 'Galerkin methods', magnetic Reynolds number $R_m \leq 19$.

Are there spurious solutions like in Maxwell equations?

> *... It is* **still unknown** *whether this field (ABC flow) is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as $R_m \to \infty$.*
>
> ...
>
> *Numerically, the kinematic fast dynamo problem is the first eigenvalue problem for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8.* **The corresponding matrices are (and will remain) beyond the reach of any computer.**

— *Topological Methods in Hydrodynamics*, V.I.Arnold, B.A.Khesin 2021.



Is this true?

MHD via Differential Forms

$$\underbrace{-\nabla \times (\boldsymbol{u} \times \boldsymbol{B})}_{\substack{\text{Lie derivative} \\ \text{advection}}} \underbrace{-R_m^{-1}\nabla \times \nabla \times \boldsymbol{B}}_{\substack{\text{Hodge Laplacian} \\ \text{diffusion}}} = \lambda \boldsymbol{B}$$

- Diffusion: Hodge Laplacian . $\Delta_{\mathrm{HL}} := d\delta + \delta d$ (diffusion)
- Lie Derivative: For vector field u on manifold $\mathcal{M}$. For a k-form ω,

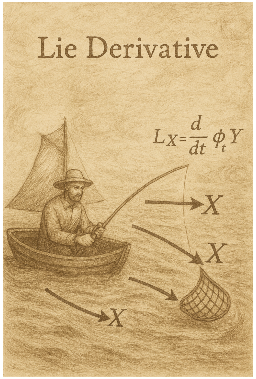
$$\mathrm{L}_u\omega = \lim_{\tau \to 0} \frac{\Phi_\tau^*\omega - \omega}{\tau}$$

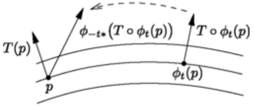
where flow $\Phi(t,x)$ satisfies $\partial_t\Phi = u(\Phi,t)$, $\Phi(0,x) = x$.

- Cartan's Magic Formula: For vector field β:

$$\mathrm{L}_\beta^k = d^{k-1}i_\beta^k + i_\beta^{k+1}d^k$$

where $i_\beta^k : \Lambda^k \to \Lambda^{k-1}$ is contraction.





' Fisherman derivative ':
sitting on boat, differentiating along the flow

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{\cdot\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\times\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\otimes\beta} C^\infty(\Omega) \longleftarrow 0.$$

$$\underbrace{\mathrm{L}_u w}_{\substack{\text{Lie derivative} \\ \text{advection}}} + \underbrace{\Delta_{HL} w}_{\substack{\text{Hodge Laplacian} \\ \text{diffusion}}}$$

Numerical application: (semi-)Lagrange methods for MHD Heumann,Hiptmair,Xu 2009

Advection-diffusion of differential forms: in coordinates

$$\mathsf{L}_\beta w + \Delta_{HL} w = f.$$

$$0 \rightleftarrows C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{-\operatorname{grad}}{\overset{\operatorname{div}}{\rightleftarrows}} C^\infty(\Omega) \rightleftarrows 0.$$

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{\cdot\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\times\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\otimes\beta} C^\infty(\Omega) \longleftarrow 0.$$

Advection-diffusion of differential forms: in coordinates

$$\mathrm{L}_\beta w + \Delta_{HL} w = f.$$

$$0 \rightleftarrows C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0$$

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{\cdot\beta} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0$$

$$(d^{k-1} i^k_\beta \quad + \quad i^{k+1}_\beta d^k) w \quad + \quad (d^{k-1} d^*_{k-1} \quad + \quad d^*_k d^k) w \quad = \quad f$$

$$\beta \cdot \nabla w \qquad - \qquad \operatorname{div}\operatorname{grad} w \quad = \quad f$$

scalar advection-diffusion.

Advection-diffusion of differential forms: in coordinates

$$\mathsf{L}_\beta w + \Delta_{HL} w = f.$$

$$0 \qquad C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0$$

$$0 \qquad C^\infty(\Omega) \xleftarrow{\cdot\beta} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\times\beta} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0$$

$$(d^{k-1} i_\beta^k \quad + \quad i_\beta^{k+1} d^k) w \quad + \quad (d^{k-1} d_{k-1}^* \quad + \quad d_k^* d^k) w \quad = \quad f$$

$$\operatorname{grad}(\boldsymbol{\beta} \cdot \boldsymbol{A}) \quad - \quad \boldsymbol{\beta} \times (\operatorname{curl} \boldsymbol{A}) \quad + \quad (-\operatorname{grad}\operatorname{div} \quad + \quad \operatorname{curl}\operatorname{curl}) \boldsymbol{A} \quad = \quad \boldsymbol{f}$$

advection-diffusion of magnetic potential.

Advection-diffusion of differential forms: in coordinates

$$\mathsf{L}_\beta w + \Delta_{HL} w = f.$$

$$\begin{array}{ccccccc}
0 & C^\infty(\Omega) & C^\infty(\Omega;\mathbb{R}^3) \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} & C^\infty(\Omega;\mathbb{R}^3) & \underset{-\,\text{grad}}{\overset{\text{div}}{\rightleftarrows}} C^\infty(\Omega) & & 0 \\
0 & C^\infty(\Omega) & C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\times\beta} & C^\infty(\Omega;\mathbb{R}^3) & \xleftarrow{\otimes\beta} C^\infty(\Omega) & & 0
\end{array}$$

$$\begin{array}{ccccccccc}
(d^{k-1} i_{\boldsymbol{B}}^k & + & i_{\boldsymbol{B}}^{k+1} d^k) w & + & (d^{k-1} d_{k-1}^* & + & d_k^* d^k) w & = & f \\
-\operatorname{curl}(\beta \times \boldsymbol{B}) & + & (\operatorname{div} \boldsymbol{B})\beta & + & (\operatorname{curl}\operatorname{curl} & - & \operatorname{grad}\operatorname{div})\boldsymbol{B} & = & \boldsymbol{f}
\end{array}$$

If imposing $\operatorname{div} \boldsymbol{B} = 0$:

$$-\operatorname{curl}(\beta \times \boldsymbol{B}) + \operatorname{curl}\operatorname{curl} \boldsymbol{B} = \boldsymbol{f}.$$

magnetic advection-diffusion.

Advection-diffusion of differential forms: in coordinates

$$\mathsf{L}_\beta w + \Delta_{HL} w = f.$$

$$
\begin{array}{ccccccccc}
0 & & C^\infty(\Omega) & & C^\infty(\Omega;\mathbb{R}^3) & & C^\infty(\Omega;\mathbb{R}^3) & \underset{-\operatorname{grad}}{\overset{\operatorname{div}}{\rightleftarrows}} & C^\infty(\Omega) & \rightleftarrows & 0 \\
0 & & C^\infty(\Omega) & & C^\infty(\Omega;\mathbb{R}^3) & & C^\infty(\Omega;\mathbb{R}^3) & \xleftarrow{\otimes\beta} & C^\infty(\Omega) & \longleftarrow & 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
(d^{k-1} i^k_{\boldsymbol{B}} & + & i^{k+1}_{\boldsymbol{B}} d^k) w & + & (d^{k-1} d^*_{k-1} & + & d^*_k d^k) w & = & f \\
\operatorname{div}(u\beta) & & & - & & & \operatorname{div}\operatorname{grad} u & = & f
\end{array}
$$

Fokker-Planck type equation (transport of density)

Convergence of Advection-Diffusion Eigenvalue Problems

Find $\boldsymbol{B} \in H(\mathrm{curl})$, $\lambda \in \mathbb{C}$:

$$R_m^{-1}(\nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) - (\boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) = \lambda(\boldsymbol{B}, \boldsymbol{C}), \quad \forall \boldsymbol{C} \in H(\mathrm{curl})$$

Bramble-Osborn Theory: Under assumptions

- *Solution operator* $T : X \to X$ is compact

  $T : \boldsymbol{f} \mapsto \boldsymbol{B}$ solves $R_m^{-1}(\nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) - (\boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) = (\boldsymbol{f}, \boldsymbol{C})$.

- $T_h : X_h \to X_h$ is compact and finite rank

$$\|T - T_h\| \to 0 \implies \text{convergence}$$

James Bramble

John Osborn

Application to MHD: Boils down to regularity of T: $V_0 := T(L^2) \hookrightarrow\hookrightarrow H(\mathrm{curl})$

Theorem [KH, Liang, Zerbinati]: For given **smooth** $\boldsymbol{u}$, $V_0 \hookrightarrow\hookrightarrow H(\mathrm{curl}) \implies$ eigenvalue convergence

Rayleigh quotient (min-max) fails due to non-self-adjoint advection, losing information (e.g., convergence of individual eigenvalues with multiplicity)

Witten transform: when wind is potential (gradient)

$$\begin{tikzcd} \cdots \arrow[r] & \Lambda^{k-1} \arrow[r, "d"] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k} \arrow[r, "d"] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k+1} \arrow[r, "d"] \arrow[d, "e^{\theta(x)}"] & \cdots \\ \cdots \arrow[r] & \Lambda^{k-1} \arrow[r, "d_\theta"] & \Lambda^{k} \arrow[r, "d_\theta"] & \Lambda^{k+1} \arrow[r, "d_\theta"] & \cdots \end{tikzcd}$$

Diagram commutes:

$$e^{\theta(\boldsymbol{x})} d(e^{-\theta(\boldsymbol{x})} w) = -\nabla\theta \wedge w + dw.$$

gauge transform. Compare to covariant derivatives $\nabla w = \partial w + \Gamma \cdot w$.

$$\begin{tikzcd} \cdots & \Lambda^{k-1} \arrow[l] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k} \arrow[l, "\delta"'] \arrow[d, "e^{\theta(x)}"] & \Lambda^{k+1} \arrow[l, "\delta"'] \arrow[d, "e^{\theta(x)}"] & \cdots \arrow[l, "\delta"'] \\ \cdots & \Lambda^{k-1} \arrow[l] & \Lambda^{k} \arrow[l, "\delta_\theta"'] & \Lambda^{k+1} \arrow[l, "\delta_\theta"'] & \cdots \arrow[l, "\delta_\theta"'] \end{tikzcd}$$

$$\delta_\theta u := e^{-\theta(\boldsymbol{x})} \delta e^{\theta(\boldsymbol{x})} u = \iota_{\pm(d\theta)^\sharp} u + \delta u.$$

$$\boxed{d\delta_{\pm\theta} + \delta_{\pm\theta} d = \Delta_{HL} + \mathrm{L}_{\nabla\theta}}$$

Hodge Laplacian on transformed coordinates $=$ advection-diffusion

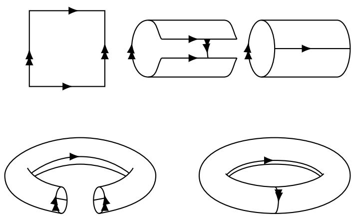
Numerical applications: stablizing numerical oscillation Brezzi,Marini,Pietra 1989 : exponential fitting, scalar problem; Wu,Xu 2018 : forms in 3D; Christiansen,Halvorsen,Sørensen 2014 : Petrov Galerkin

Edward Witten

Supersymmetry and Morse theory, Witten (1982) J. Diff. Geo.

Witten deformation
Witten complex
Witten Laplacian

Consequence 1: for potential (gradient) winds, eigenvalues are real.



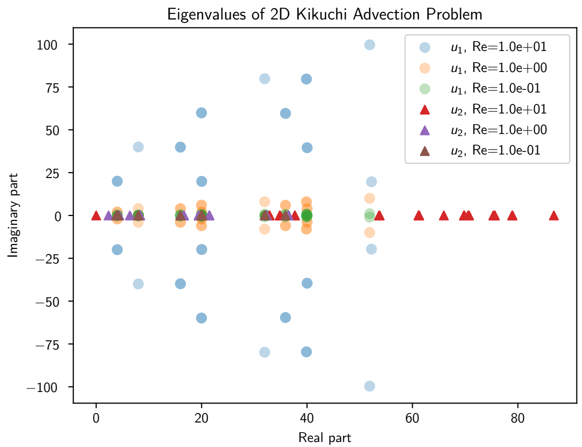


Figure. Eigenvalues for the toroidal surface with wind $\mathbf{u}_1$ (non-gradient) and $\mathbf{u}_2$ (gradient).

$$\mathbf{u}_1 = (1, 1), \quad \mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y))$$

Consequence 2: improved estimates (essentially self-adjoint)

Generalizing Hodge theory

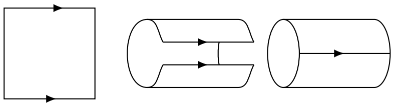
Theorem 2 (V. I. Arnold)

The number of linearly independent stationary k-forms is **not less than** *the k-th betti number of the manifold $\mathcal{M}$.*

Theorem 3 (V. I. Arnold)

If the diffusion coefficient R_m^{-1} is sufficiently large, then the number of linearly independent stationary k-forms is **equal to** *the k-th betti number of the manifold $\mathcal{M}$.*

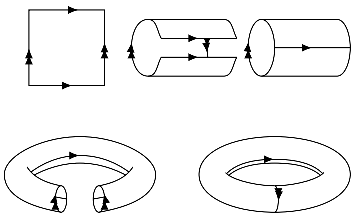
Test 1



$b_0 = 1, b_1 = 1, b_2 = 0$. $\mathbf{u}_1 = (1, 1)$

R_m	100	10	1	0.1
$\dim(\lambda_0)$	1	1	1	1

Test 2



$b_0 = 1, b_1 = 2, b_2 = 1$.
$\mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y))$

R_m	100	10	1	0.1
$\dim(\lambda_0)$	2	2	2	2

Back to the very first assumption...

> *Numerically,* **the kinematic fast dynamo problem is the first eigenvalue problem** *for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8. The corresponding matrices are (and will remain) beyond the reach of any computer.*

— *Topological Methods in Hydrodynamics*, V.I.Arnold, B.A.Khesin 2021.

Back to the very first assumption...

> *Numerically,* **the kinematic fast dynamo problem is the first eigenvalue problem** *for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8. The corresponding matrices are (and will remain) beyond the reach of any computer.*

— *Topological Methods in Hydrodynamics*, V.I.Arnold, B.A.Khesin 2021.

Eigenvalue analysis is often **misleading** for telling (in)stability for non-normal operators ($AA^* \neq A^*A$).

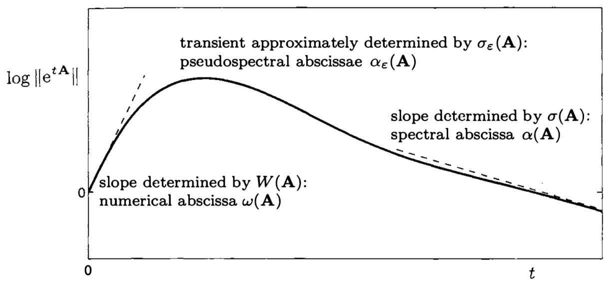
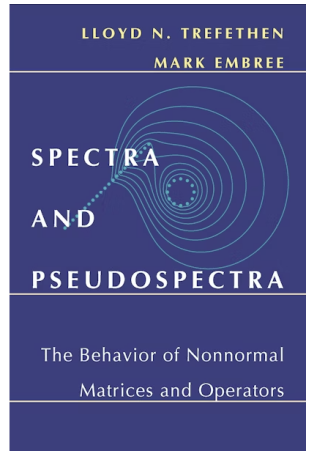


Figure 14.1: Initial, transient, and asymptotic behavior of $\|e^{t\mathbf{A}}\|$ for a nonnormal matrix or operator $\mathbf{A}$.

In transient (better described by pseudo-spectra), nonlinear effects become dominating, Asymptotics described by eigenvalues never reached .



Structure-preserving FE also computes pseudo-spectra

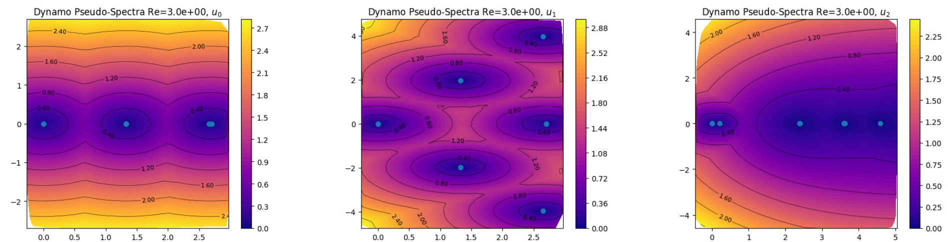
$$\sigma_\epsilon(\boldsymbol{A}) = \{z \in \mathbb{C} : \|(z - \boldsymbol{A})^{-1}\| > \epsilon^{-1}\}$$

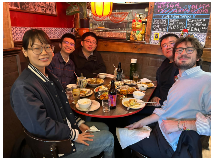
$$\sigma_\epsilon(\boldsymbol{A}) = \{z \in \mathbb{C} : s_{\min}(z - \boldsymbol{A}) < \epsilon\}$$

$s_{\min}$: minimal singular value

Theorem 4 (Zerbinati)

Finite element for pseudo-spectra converges.





Umberto Zerbinati, Mini-course Edinburgh 2025

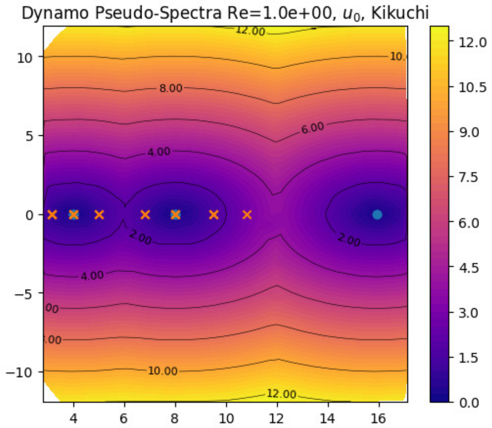
$$\mathbf{u}_0 = 0 \quad \text{Hodge Laplacian}$$

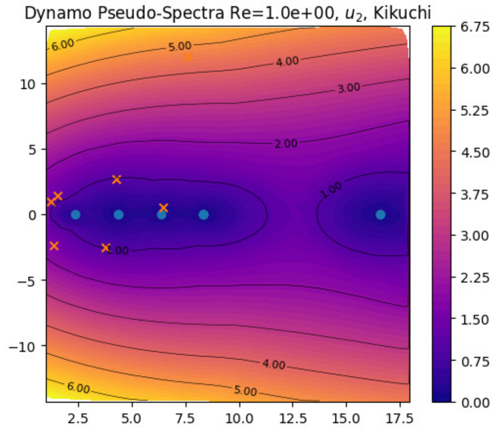
$$\mathbf{u}_1 = (1, 1) \quad \text{non potential}$$

$$\mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y)) \quad \text{potential}$$

Lagrange elements lead to spurious modes again

Lagrange elements for Maxwell lead to spurious eigenmodes. Same for Lagrange elements for MHD (Maxwell+advection).





$$\mathbf{u}_0 = 0 \qquad \text{Hodge Laplacian}$$

$$\mathbf{u}_2 = (2\cos(2x)\sin(2y), 2\sin(2x)\cos(2y)) \quad \text{potential}$$

×: from nodal elements ○: Nédélec (FEEC)

Back to Arnold&Korkina 1983 computation

Arnold, V. I., & Korkina, E. I. (1983). *The growth of a magnetic field in the three-dimensional steady flow of an incompressible fluid.* Moskovskii Universitet Vestnik Seriia Matematika Mekhanika, 43-46.

Fourier basis $e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = e^{i(k_1x_1+\cdots+k_nx_n)}$

$$0 \longrightarrow e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \xrightarrow{\text{grad}} \begin{pmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \end{pmatrix} \xrightarrow{\text{curl}} \begin{pmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \end{pmatrix} \xrightarrow{\text{div}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \longrightarrow 0.$$

"good" complex. Ongoing work with Andrea Bressan, Yuechen Zhu.

Compared to **finite element** exterior calculus, less attention paid to **spectral** basis and spectral methods.

An exciting start...

FEEC for computational topological hydrodynamics

Long term computation of fluids/plasma

Long-term computation: finite-dim vs infinite-dim

	Finite-dim Hamiltonian systems	Fluids / plasmas (infinite-dim)
Volume-preserving	phase space	infinite dim. Lie group
Structures	symplectic form ω	topology (helicity), geometry (energy, enstrophy)
How many invariants preserve?	as many Casimirs as possible	has a limit (Onsager conjecture, singularities)
Stability	KAM (Feng, Shang 1980s)	dynamo (instability)
Decay/Growth	'rigid'	decay (relaxation), growth (dynamo)
Discretization	geometric integrators	geometric integrators + **de Rham complex**

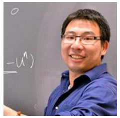
Further questions

- Long-term evolution and rough solutions.
- Fast solvers and preconditioners.
- Nonlinear eigenvalue problems, pseudo-spectra.
- (Semi-)Lagrange methods.
- Turbulence: LES, DNS.
- Flows on manifolds, relativistic fluids.
- (Pseudo)spectra beyond compactness.
- ...

Collaborators: Algorithms & Analysis

Jinchao Xu
Penn State, KAUST

Yicong Ma
Barclays



Xiaozhe Hu
Tufts

Young-Ju Lee
Texas

Patrick Farrell
Oxford

Fabian Laakman
ASML

Lourenco Beirão da Veiga
Milano

Lorenzo Mascotto
Milano

- *Stable finite element methods preserving $\nabla \cdot \boldsymbol{B} = 0$ exactly for MHD models*, K. Hu, Y. Ma, J. Xu; Numerische Mathematik, 135(2), 371-396 (2017). divergence-free preservation
- *Robust preconditioners for incompressible MHD models*, Y. Ma, K. Hu, X. Hu, J. Xu; Journal of Computational Physics, 316, 721-746 (2016). preconditioning
- *Helicity-conservative finite element discretization for incompressible MHD systems*, K. Hu, Y.-J. Lee, J. Xu; Journal of Computational Physics (2021). helicity preservation
- *Structure-preserving and helicity-conserving finite element approximations and preconditioning for the Hall MHD equations*, F. Laakmann, K. Hu, P. E. Farrell; Journal of Computational Physics (2023). Hall MHD
- *Finite element exterior calculus for multiphysics problems*, K. Hu; Peking University (2017) PhD thesis

Collaborators: Relaxation

Mingdong He
Oxford

Patrick Farrell
Oxford

Boris Andrews
Oxford

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, K. Hu, B. D. Andrews; SISC (2025)

Collaborators: Dynamo

Daniele Boffi
KAUST

Yizhou Liang
Oxford

Stefano Zampini
KAUST

Umberto Zerbinati
Oxford

Jindong Wang
KAUST, incoming Newton Fellow

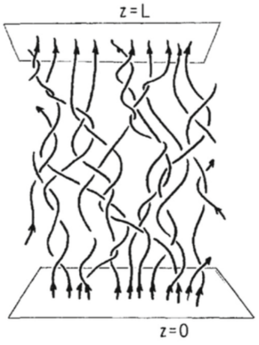
- *FEEC for dynamo*, D. Boffi, K. Hu, Y. Liang, S. Zampini, U. Zerbinati; in preparation
- *Pseudospectra of advection-diffusion of differential forms*, D. Boffi, K. Hu, U. Zerbinati; in preparation

Aknowledgement

Royal Society University Research Fellowship (URF\R1\221398)

ERC starting grant (101164551): **GeoFEM** (Geometric Finite Element Methods)

Appendix: Nontrivial cohomology & generalized helicity



Additional difficulty (torus topology):
nontrivial cohomology $\Rightarrow$ harmonic field $\boldsymbol{B}_0 = (0,0,1)$

$$\nabla \times \boldsymbol{B}_0 = 0, \quad \nabla \cdot \boldsymbol{B}_0 = 0, \quad \text{satisfies BCs}$$

- Classical helicity not well-defined
- Harmonic part $\boldsymbol{B}_H$ is time-invariant ($\partial_t \boldsymbol{B} \in \mathcal{R}(\text{curl})$) but interferes with exact part

Generalized helicity (gauge-dependent but conserved):

$$\boldsymbol{B} = \boldsymbol{B}_R + \boldsymbol{B}_H, \quad \boldsymbol{B}_R = \text{curl}\, \boldsymbol{A}$$

$$\tilde{H} := \int \boldsymbol{A} \cdot (\boldsymbol{B}_R + 2\boldsymbol{B}_H) = \int \boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{B}_H) \quad \Longrightarrow \frac{d}{dt}\tilde{H} = 0$$

Generalized Arnold inequality:

$$|\tilde{H}| \leq C(\|\boldsymbol{B}_R\|^2 + 4\|\boldsymbol{B}_2 + 2\boldsymbol{B}_H\|^2)$$

Our algorithm: exactly preserves the discrete analogue of the above.