Complexes from complexes

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Maths Afternoon at Christ Church

15 July, 2022

1. de Rham complexes

2. Complexes from complexes

Motivation

decomposition of flows: $\Omega \subset \mathbb{R}^3$, for any $u \in [L^2(\Omega)]^3$:

$$u = \operatorname{grad} \phi + \operatorname{curl} \psi + w$$

- any flow = potential part (rotation-free) $\oplus_\perp$ rotational part (source-free) $\oplus_\perp$ harmonic,
- w: both rotation-free and source-free (*harmonic forms*),
  dim of such functions = Betti number, reflecting topology of Ω,
- rough idea of cohomology theories: using w to study topology.

Key of these results: algebraic structures, complexes!

de Rham complex

de Rham complex (3D version)

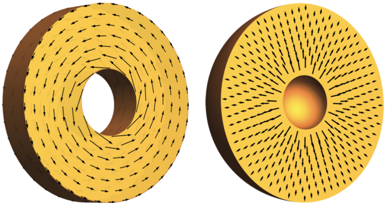
$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \;\Rightarrow\; \mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \mathcal{N}(\text{curl})$, $\quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \mathcal{N}(\text{div})$
- cohomology: $\mathscr{H}^k := \mathcal{N}(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \mathcal{N}(\text{grad})$, $\quad \mathscr{H}^1 := \mathcal{N}(\text{curl})/\mathcal{R}(\text{grad})$, $\quad \mathscr{H}^2 := \mathcal{N}(\text{div})/\mathcal{R}(\text{curl})$
- exactness (contractible domains): $\mathcal{N}(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \;\Rightarrow\; u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\, \phi$, $\quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\, \psi$.

de Rham complex and topology:

dimension of $\mathscr{H}^k$ = number of "k-dimensional holes" (c.f. de Rham theorem)



Examples where $\dim \mathscr{H}^1 = 1$ and $\dim \mathscr{H}^2 = 1$, respectively.
Left: curl-free field which is not grad, Right: div-free field with is not curl.

(figure from *Finite element exterior calculus*, D.N.Arnold, SIAM 2008.)

From complexes to PDEs

Formal adjoint of operators:

$$\operatorname{grad}^* = -\operatorname{div}, \quad \operatorname{curl}^* = \operatorname{curl}, \quad \operatorname{div}^* = -\operatorname{grad}.$$

$$\int_\Omega \operatorname{grad} u \cdot v = -\int_\Omega u \operatorname{div} v + \text{bound. term}, \quad \int_\Omega \operatorname{curl} u \cdot v = \int_\Omega u \cdot \operatorname{curl} v + \text{bound. term}$$

$$(\operatorname{grad} u, v) = (u, -\operatorname{div} v), \qquad (\operatorname{curl} u, v) = (u, \operatorname{curl} v)$$

Formal adjoint of de Rham complex:

$$0 \longleftarrow C^\infty(\Omega) \xleftarrow{-\operatorname{div}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{\operatorname{curl}} C^\infty(\Omega;\mathbb{R}^3) \xleftarrow{-\operatorname{grad}} C^\infty(\Omega) \longleftarrow 0.$$

$$d_2^* := -\operatorname{div}, \quad d_1^* := \operatorname{curl}, \quad d_0^* := -\operatorname{grad}.$$

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^* d^k)u = f.$$

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d^*_{k-1} + d^*_k d^k)u = f.$$

$$0 \rightleftarrows {\color{red}C^\infty(\Omega)} \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div}\operatorname{grad} u = f.$$

Poisson equation.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^* d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \underset{-\operatorname{div}}{\overset{\operatorname{grad}}{\rightleftarrows}} {\color{red} C^\infty(\Omega;\mathbb{R}^3)} \underset{\operatorname{curl}}{\overset{\operatorname{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \qquad C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$-\operatorname{grad}\operatorname{div} v + \operatorname{curl}\operatorname{curl} v = f.$$

Maxwell equations.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^{*} + d_k^{*} d^k)u = f.$$

$$0 \qquad C^\infty(\Omega) \qquad C^\infty(\Omega;\mathbb{R}^3) \underset{\text{curl}}{\overset{\text{curl}}{\rightleftarrows}} C^\infty(\Omega;\mathbb{R}^3) \underset{-\,\text{grad}}{\overset{\text{div}}{\rightleftarrows}} C^\infty(\Omega) \qquad 0.$$

Hodge-Laplacian problem:

$$\operatorname{curl}\operatorname{curl} v - \operatorname{grad}\operatorname{div} v = f.$$

Maxwell equations.

connections to PDEs: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^{*} + d_k^{*}d^{k})u = f.$$

$$0 \qquad C^{\infty}(\Omega) \qquad C^{\infty}(\Omega;\mathbb{R}^3) \qquad C^{\infty}(\Omega;\mathbb{R}^3) \underset{-\operatorname{grad}}{\overset{\operatorname{div}}{\rightleftarrows}} C^{\infty}(\Omega) \rightleftarrows 0.$$

Hodge-Laplacian problem:

$$-\operatorname{div}\operatorname{grad} u = f.$$

Poisson equation.

Analytic properties

$$0 \longrightarrow H(\mathrm{grad}) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0$$

$H(d) := \{u \in L^2 : du \in L^2\}.$

- Hodge decomposition: $L^2 = \mathcal{R}(d^{k-1}) \oplus \mathcal{R}(d_k^*) \oplus \mathscr{H}^k$,

$$L^2 = \mathcal{R}(\mathrm{div}) \oplus \mathscr{G},$$
$$[L^2]^3 = \mathcal{R}(\mathrm{grad}) \oplus_\perp \mathcal{R}(\mathrm{div}) \oplus \mathscr{H}.$$

- Poincaré inequalities: $\|u\| \leq C\|d^k u\|$, $\forall u \perp \mathcal{N}(d^k)$,

$$\|u\| \leq \|\mathrm{grad}\, u\|, \quad u \perp \mathcal{N}(\mathrm{grad}),$$
$$\|u\| \leq \|\mathrm{curl}\, u\|, \quad u \perp \mathcal{N}(\mathrm{curl}),$$
$$\|u\| \leq \|\mathrm{div}\, u\|, \quad u \perp \mathcal{N}(\mathrm{div}).$$

- other properties: regular decomposition, existence of regular potentials, compactness properties, div-curl lemma, etc.

Why are complexes useful for PDEs?

- well-posedness: analytic properties + standard variational argument,
- discretization: structure-preservation!

> Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.
>
> – James Clerk Maxwell, 1873

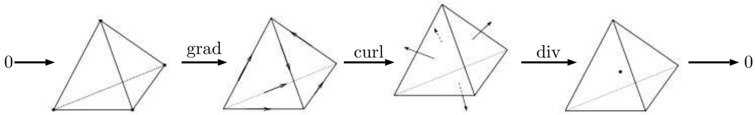
discrete differential forms (Bossavit 1988, Hiptmair 1999, ...),
Finite Element Exterior Calculus (Arnold, Falk, Winther 2006, ...)

discrete spaces fit into de Rham complexes.

Very roughly speaking, numerical methods work if discrete spaces fit in a complex (preserving the cohomological structure).

Discretization

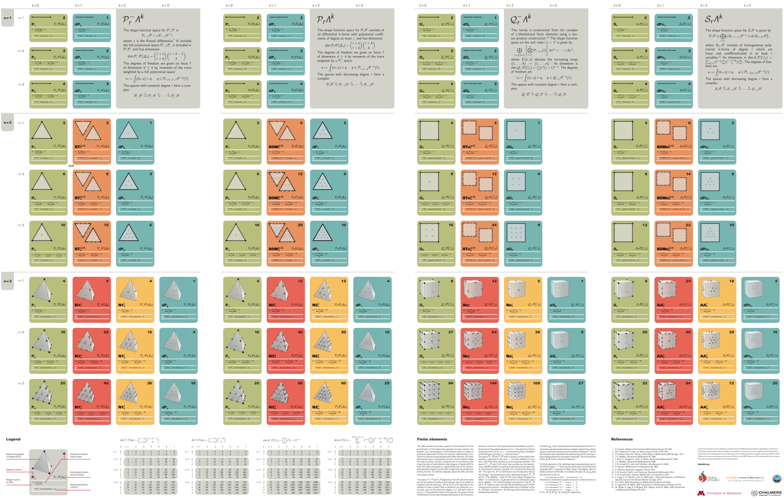
finite element de Rham complex in 3D



$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\text{grad}} [\mathcal{P}_0]^3 + [\mathcal{P}_0]^3 \times x \xrightarrow{\text{curl}} [\mathcal{P}_0]^3 + \mathcal{P}_0 \otimes x \xrightarrow{\text{div}} \mathcal{P}_0 \longrightarrow 0.$$

Raviart-Thomas (1977), Nédélec (1980) in numerical analysis, Bossavit (1988), Hiptmair (1999) for differential forms, Whitney (1957) for studying topology.

Periodic Table of the Finite Elements



Arnold, Logg 2014, SIAM news

Elasticity: deformation and mechanics of solids

elasticity equation:

$$-\operatorname{div}(A \operatorname{def} u) = f.$$

u	displacement (vector),
$e := \operatorname{def} u := 1/2(\nabla u + \nabla u^T)$	strain (linearized deformation),
$\sigma := A \operatorname{def} u$	stress.

analogy to Poisson equation:

$$-\operatorname{div}(A \operatorname{grad} v) = g.$$

A cohomological approach: elasticity complex

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.) Hellinger-Reissner principle of elasticity

$\mathbb{V} := \mathbb{R}^3$ vectors, $\mathbb{S} := \mathbb{R}^{3\times 3}_{\text{sym}}$ symmetric matrices

$$\text{def } u := 1/2(\nabla u + \nabla u^T), \quad (\text{def } u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\text{inc}\, g := \nabla \times g \times \nabla, \quad (\text{inc}\, g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k \partial_s g_{lt}.$$

$$\text{div}\, v := \nabla \times v, \quad (\text{div}\, v)_i = \partial^j u_{ij}.$$

g metric $\Rightarrow$ $\text{inc}\, g$ linearized Einstein tensor ($\backsimeq \mathrm{Riem} \backsimeq \mathrm{Ric}$ in 3D)

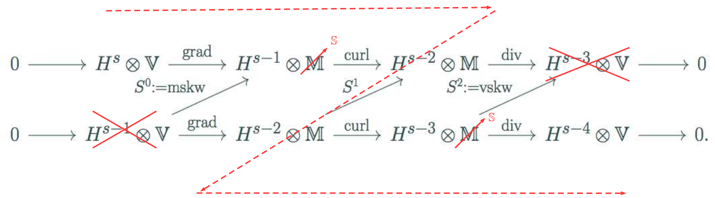
$\text{inc} \circ \text{def} = 0$: Saint-Venant compatibility

$\text{div} \circ \text{inc} = 0$: Bianchi identity

systematic study still missing, until

Complexes from complexes, Arnold, Hu 2021.

Algebraic and analytic construction (Arnold, Hu 2020): derive elasticity from deRham



$S^1 u := u^T - \mathrm{tr}(u) I.$

key: Sobolev complexes ($\forall s \in \mathbb{R}$), match indices, commuting diagrams, injectivity & surjective.

output: elasticity complex

$$\begin{tikzcd}
0 \arrow[r] & H^s \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & H^{s-1} \otimes \mathbb{S} \arrow[r, "\mathrm{curl}"] & {} \arrow[dll, "\mathrm{T}"] & & \\
 & {} \arrow[r, "\mathrm{curl}"] & H^{s-3} \otimes \mathbb{S} \arrow[r, "\mathrm{div}"] & H^{s-4} \otimes \mathbb{V} \arrow[r] & 0.
\end{tikzcd} \tag{1}$$

Theorem

Cohomology of (1) *is isomorphic to the smooth de Rham cohomology:*

$$\mathcal{N}(\mathscr{D}^i) = \mathcal{R}(\mathscr{D}^{i-1}) \oplus \mathscr{H}^i_\infty, \quad \mathscr{H}^i_\infty \backsimeq \mathcal{H}^i_{\mathrm{deRham}} \otimes (\mathbb{V} \times \mathbb{V})$$

Proof: Homological algebra + results for de Rham by Costabel & McIntosh.

Corollary: finite dimensional cohomology $\Longrightarrow$ operators have closed range.

inspired by Bernstein-Gelfand-Gelfand (BGG) resolution
(c.f., Eastwood 2000, Čap, Slovák, Souček 2001, Arnold, Falk, Winther 2006).

Consequences:

- analytic results (Poincaré inequality, Hodge decomposition, etc.)
  e.g., Korn inequality $\|u\|_1 \leq C\|\operatorname{def} u\|, \;\; u \perp \mathcal{N}(\operatorname{def})$.
- explicit representatives of elasticity cohomology.

Take home messages:

- cohomological structures play a key role in modeling, analysis, and numerics,
- (elasticity, continuum mechanics, geometry, relativity) complexes from (de Rham) complexes.

References

- *Complexes from complexes*, Douglas Arnold, Kaibo Hu; *Foundations of Computational Mathematics* (2021).

  framework, analytic results from homological algebraic structures

- *BGG sequences with weak regularity and applications*, Andreas Čap, Kaibo Hu; *arXiv:2203.01300* (2022)

  more general framework, conformal complexes, applications