COMPLEXES FROM COMPLEXES

Kaibo Hu

ICBS, Frontiers of Science Award talk, July 2025

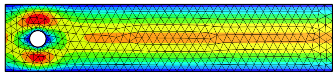
Motivation: compatible discretisation

Fundamental question: how to discretise a system with more than one variable?

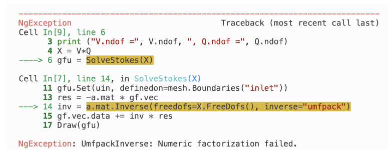
Stokes system:

$$
\begin{aligned}
\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, dx - \int_\Omega p \nabla \cdot \boldsymbol{v} \, dx &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx, & \forall \boldsymbol{v}, \\
\int_\Omega \nabla \cdot \boldsymbol{u} q \, dx &= 0, & \forall q.
\end{aligned}
$$

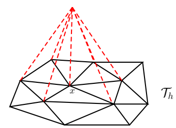
Velocity continuous $\mathcal{P}_4$, pressure discontinuous $\mathcal{P}_3$



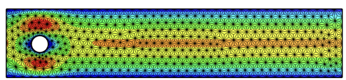
Velocity continuous $\mathcal{P}_2$, pressure discontinuous $\mathcal{P}_1$

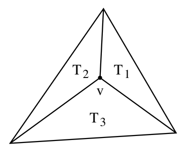


```
---------------------------------------------------------------------------
NgException                               Traceback (most recent call last)
Cell In[9], line 6
      3 print ("V.ndof =", V.ndof, ", Q.ndof =", Q.ndof)
      4 X = V*Q
----> 6 gfu = SolveStokes(X)

Cell In[7], line 14, in SolveStokes(X)
     11 gfu.Set(uin, definedon=mesh.Boundaries("inlet"))
     13 res = -a.mat * gf.vec
---> 14 inv = a.mat.Inverse(freedofs=X.FreeDofs(), inverse="umfpack")
     15 gf.vec.data += inv * res
     17 Draw(gfu)

NgException: UmfpackInverse: Numeric factorization failed.
```



Velocity continuous $\mathcal{P}_2$, pressure discontinuous $\mathcal{P}_1$, on Alfeld split

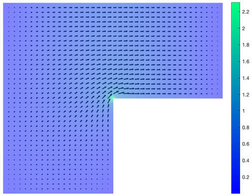


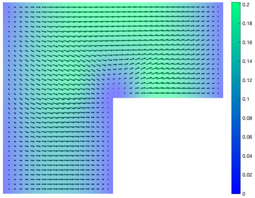


Numerical results obtained by NGSolve.

There may be no visible clues to tell spurious solutions.

$$-\operatorname{curl} \Delta \operatorname{rot} \boldsymbol{u} - \operatorname{grad} \operatorname{div} \boldsymbol{u} = \boldsymbol{f}.$$


standard finite elements


"intrinsic" FEs

model problem for generalised continua, standard finite element converges to a wrong solution.

- K. Hu et al. *Spurious solutions for high order curl problems*, IMA Journal of Numerical Analysis (2023).

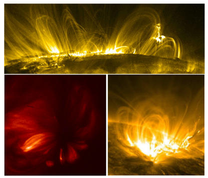
	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Intrinsic FEs	0.000000	0.593379	0.595179	1.801959	2.837796	4.458048	4.492200	5.463407
Scalar FEs	1.947637	2.579732	2.731537	3.781333	5.542562	7.373284	7.571471	7.797919

Table. Eigenvalues λ_1 to λ_8.

Many more examples available.

Motivation: structure-preserving discretisation

Fundamental question in plasma physics: given initial data, what does the system evolve to?
heating of solar corona, plasma equilibria (magnetic configurations) etc.



$$
\begin{aligned}
\boldsymbol{B}_t - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) &= 0, \\
\boldsymbol{j} &= \nabla \times \boldsymbol{B}, \\
\boldsymbol{u} &= \tau \boldsymbol{j} \times \boldsymbol{B}.
\end{aligned}
$$

Energy decay

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2 = -\tau\|\boldsymbol{B} \times \boldsymbol{j}\|^2.$$

Helicity conservation

$$\frac{d}{dt}\mathcal{H}_m = 0, \quad \text{with } \mathcal{H}_m := \int \boldsymbol{A}\cdot\boldsymbol{B}\, dx, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}.$$

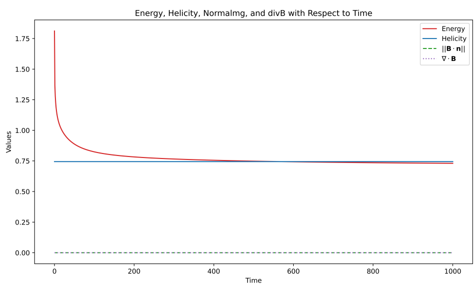


Figure. Helicity-preserving scheme

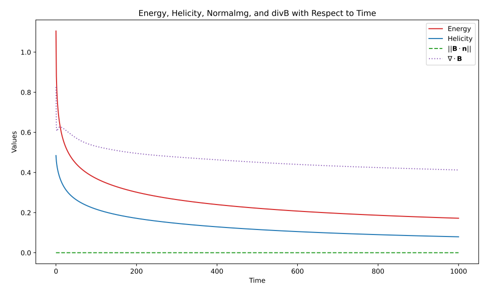


Figure. CG scheme (non-preserving)

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, KH, B. Andrews, arXiv (2025).

Reliable numerical computation

Computation is used for computing gravitational wave templates , investigating magnetic configurations for fusion devices , designing quantum computing devices etc.

How confident are we in what we compute?

Key: *differential complexes and cohomology* encode fundamental structures in mathematical models.

RELIABLE NUMERICAL COMPUTATION

Computation is used for computing gravitational wave templates , investigating magnetic configurations for fusion devices , designing quantum computing devices etc.

How confident are we in what we compute?

Key: *differential complexes and cohomology* encode fundamental structures in mathematical models.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \longrightarrow \cdots$$

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}\,.$$

- complex property: $d^k \circ d^{k-1} = 0, \;\Rightarrow\; \mathcal{R}(d^{k-1}) \subset \ker(d^k)$,
  $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \ker(\text{curl}), \quad \text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \ker(\text{div})$
- cohomology: $\mathscr{H}^k := \ker(d^k)/\mathcal{R}(d^{k-1})$,
  $\mathscr{H}^0 := \ker(\text{grad}), \quad \mathscr{H}^1 := \ker(\text{curl})/\mathcal{R}(\text{grad}), \quad \mathscr{H}^2 := \ker(\text{div})/\mathcal{R}(\text{curl})$
- exactness: $\ker(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \;\Rightarrow\; u = d^{k-1} v$
  $\text{curl}\, u = 0 \Rightarrow u = \text{grad}\,\phi, \quad \text{div}\, v = 0 \Rightarrow v = \text{curl}\,\psi.$

Outline

Stokes Pairs and Compatible Discretisation

The Stokes problem:

$$\begin{aligned}\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, dx - \int_\Omega p \nabla \cdot \boldsymbol{v}\, dx &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \quad \forall \boldsymbol{v},\\ \int_\Omega \nabla \cdot \boldsymbol{u}\, q\, dx &= 0, \qquad\qquad \forall q.\end{aligned}$$

Continuous Level (PDEs):
Well-posedness via inf-sup condition:

$$\inf_{q \in L^2/\mathbb{R}} \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1} \frac{\int \operatorname{div} \boldsymbol{v} q\, dx}{\|\boldsymbol{v}\|_{H^1} \|q\|_{L^2}} \geq \gamma > 0$$

From exact de Rham complex:

$$0 \longrightarrow V^0 \xrightarrow{\text{grad}} V^1 \xrightarrow{\text{curl}} V^2 \xrightarrow{\text{div}} V^3 \longrightarrow 0$$

velocity pressure

Discrete Level (Numerics):
Stability via discrete inf-sup:

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{\int \operatorname{div} \boldsymbol{v}_h q_h\, dx}{\|\boldsymbol{v}_h\|_{H^1} \|q_h\|_{L^2}} \geq \gamma > 0$$

Achieved by discrete de Rham complex:

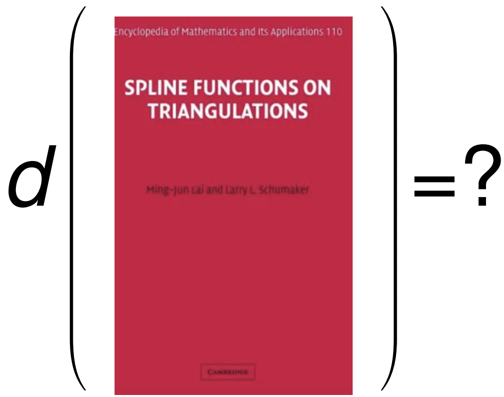
$$0 \longrightarrow \cdots \xrightarrow{\text{grad}} \cdots \xrightarrow{\text{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

velocity pressure

Constructing Finite Element Stokes Pair

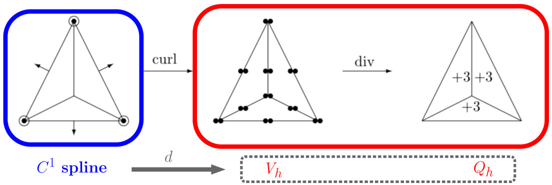
A Long-Standing Challenge

Construct velocity space $\boldsymbol{V}_h \subset \boldsymbol{H}^1$ and pressure space $Q_h \subset L^2$ such that $\operatorname{div} \boldsymbol{V}_h = Q_h$.

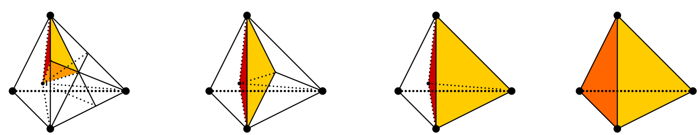


Alfeld Split: Arnold-Qin 1992, Christiansen-KH 2018

- Continuous $\mathcal{P}_2$, discontinuous $\mathcal{P}_1$
- $\mathcal{C}^1$ scalar spline on this triangulation
- Differentiating it yields the $\mathcal{P}_2$-$\mathcal{P}_1$ pair
- Ensures $\operatorname{div} \boldsymbol{V}_h = Q_h$

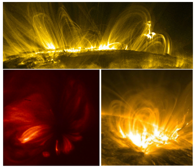


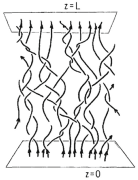
Christiansen-KH 2018: Systematic construction of Stokes complexes via scalar spline differentiation.



Christiansen, S. H., & Hu, K. (2018). Generalized finite element systems for smooth differential forms and Stokes' problem. *Numerische Mathematik*, 140, 327–371.

Ideal magnetic relaxation





Eugene Parker

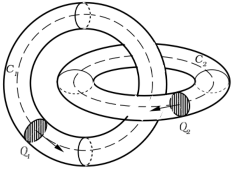
Parker hypothesis (Still Open)

For "almost any initial data", *the magnetic field develops tangential discontinuities (current sheet) during the relaxation to static equilibrium.*

Finer structure: helicity [MHD: Woltjer's invariant, ideal fluid: Moffatt (giving the name)]

$$\mathcal{H}_m := \int \boldsymbol{A} \cdot \boldsymbol{B} \, dx.$$

Describe *knots of divergence-free fields*. Conserved in ideal MHD.



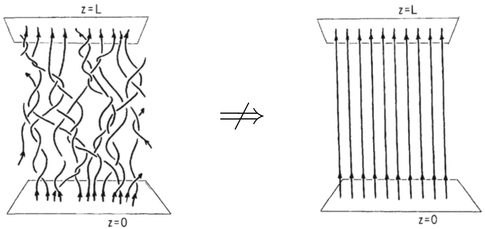
A TOPOLOGICAL MECHANISM

Arnold inequality (V.I. Arnold 1974): helicity provides lower bound for energy

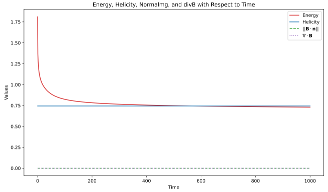
$$\left| \int \boldsymbol{A} \cdot \boldsymbol{B} \, dx \right| \leq C \int |\boldsymbol{B}|^2 \, dx$$

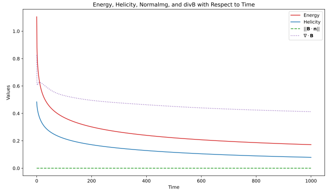
Proof. Cauchy-Schwarz $|\int \boldsymbol{A} \cdot \boldsymbol{B} \, dx| \leq \|\boldsymbol{A}\|_{L^2} \|\boldsymbol{B}\|_{L^2}$ + Poincaré inequality $\|\boldsymbol{A}\|_{L^2} \leq C \|\nabla \times \boldsymbol{A}\|_{L^2}$.

Vladimir Igorevich Arnold



Knots: topological barriers preventing energy decay. **Mechanism** lost if algorithms do not preserve helicity.





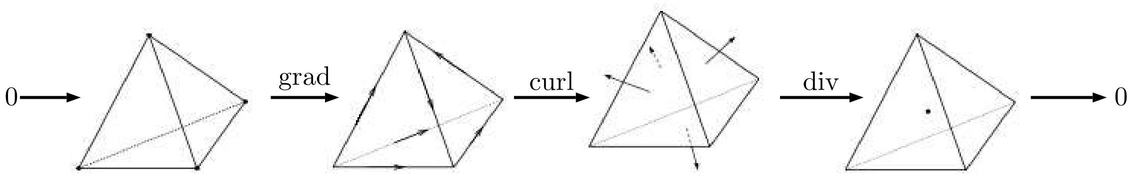
Patrick Farrell Mingdong He Boris Andrews

Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture, M. He, P. E. Farrell, KH, B. Andrews, arXiv (2025).

How to preserve helicity: discrete de Rham complex

- Raviart–Thomas (1977), Nédélec (1980): Early finite elements
- Bossavit (1988): Differential forms and complex
- Hiptmair (1999), Arnold, Falk, Winther (2006): Systematic Finite Element Exterior Calculus

Classical Whitney forms



$\boldsymbol{E}_h, \boldsymbol{H}_h$ $\boldsymbol{B}_h$

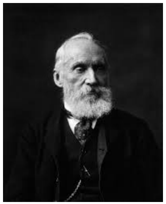
- Faraday's law $\partial_t \boldsymbol{B}_h + \nabla \times \boldsymbol{E}_h = 0$ holds exactly $\Longrightarrow \frac{d}{dt}(\nabla \cdot \boldsymbol{B}_h) = 0$.
- Introducing projection $\boldsymbol{H}_h = Q_{L^2}\boldsymbol{B}_h \quad \Longrightarrow \quad (\boldsymbol{u}_h \times \boldsymbol{H}_h, Q_{L^2}\boldsymbol{B}_h) = 0$.

First finite element method for MHD preserving $\nabla \cdot \boldsymbol{B} = 0$, energy & helicity:

KH,Hu,Ma,Xu 2016, KH,Ma,Xu 2017, KH,Lee,Xu 2021, Laakmann,KH,Farrell 2023.

Towards *Computational Topological Hydrodynamics*

A subject back to Kelvin, Helmholtz, and more recently by Arnold, Moffatt, Sullivan...
limited applications due to lack of topology-preserving algorithms



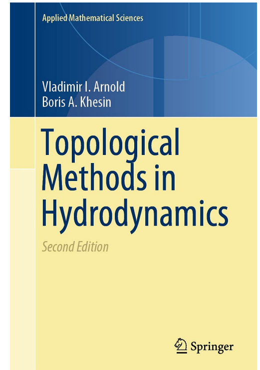
The Lord Kelvin

Hermann von Helmholtz

Vladimir Arnold

Keith Moffatt

Dennis Sullivan



> *Direct computational assessment of **Parker's hypothesis** brings a number of challenges. Foremost among these is the requirement to **precisely maintain the magnetic topology during the simulated evolution**, i.e., precisely maintain the magnetic field line mapping between the two line-tied boundaries. . . . In the following sections, two methods are described which seek to mitigate against these difficulties. However, in all cases the **representation of current singularities remains problematic**. . .*

— *The Parker problem: existence of smooth force-free fields and coronal heating*, Pontin, Hornig, Living Rev. Sol. Phys. 2020.

Solid mechanics and differential geometry: complexes from complexes

Motivation

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

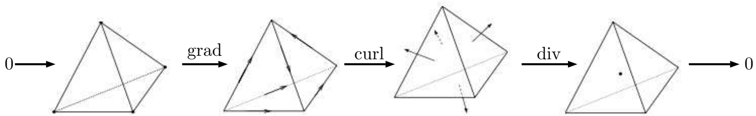
Are there discrete analogues of such tensors with symmetries and differential structures?

MOTIVATION

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

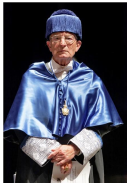
> *Are there discrete analogues of such tensors with symmetries and differential structures?*

A special case: differential forms (fully skew-symmetric tensors), exterior derivatives

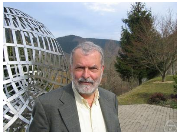


Whitney forms and higher order versions are well accepted as the canonical discretization for differential forms (skew-symmetric tensors). k-forms discretized on k-cells, unisolvent with $\mathcal{P}^-\Lambda^k$, conformity

Raviart-Thomas (1977), Nédélec (1980) in numerical analysis
Bossavit (1988): differential forms and complex
Hiptmair (1999), Arnold, Falk, Winther (2006): systematic study, "Finite Element Exterior Calculus"



Pierre-Arnaud Raviart



Jean-Claude Nédélec

Franco Brezzi

Donatella Marini

Jim Douglas

Differential structures in elasticity

Linear elasticity (Calabi, Kröner) complex

$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\mathsf{sym\,grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\mathsf{sym}} \xrightarrow{\mathsf{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\mathsf{sym}} \xrightarrow{\mathsf{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

Differential structures in elasticity

Linear elasticity (Calabi, Kröner) complex

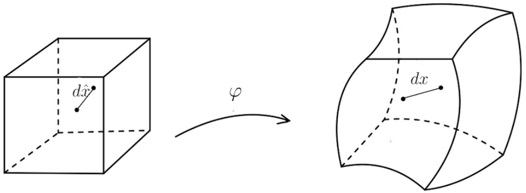
embedding $\mathbb{R}^3 \to \mathbb{R}^3$ | change of metric (strain)

$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$\varphi \longrightarrow \boldsymbol{e} = (\widehat{\nabla}\varphi)\cdot(\varphi\widehat{\nabla}) - I$$

$\boldsymbol{e} = 0$ iff φ is a rigid body motion.

Linearisation: $\boldsymbol{e} = \text{sym grad}\, \boldsymbol{u}$, in terms of displacement $\boldsymbol{u}(\widehat{\boldsymbol{x}}) = \varphi(\widehat{\boldsymbol{x}}) - \widehat{\boldsymbol{x}}$.



$dx - d\hat{x} = d\hat{x}\cdot e\cdot d\hat{x}$

Differential structures in elasticity

Linear elasticity (Calabi, Kröner) complex

metric (strain) Riemann curvature

$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

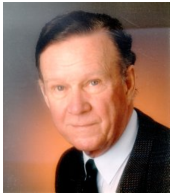
$$\boldsymbol{e} \longrightarrow \mathrm{Riem}(\boldsymbol{e})$$

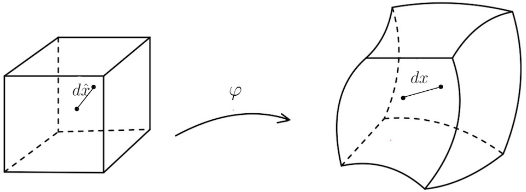
Strain tensor (change of metric) $\boldsymbol{e} = (\widehat{\nabla}\boldsymbol{\varphi}) \cdot (\boldsymbol{\varphi}\widehat{\nabla}) - \boldsymbol{I}$ satisfies $\mathrm{Riem}(\boldsymbol{e}) = 0$.

Defect theory: Kröner et al. used violation of compatibility conditions to model defects and incompatibility

Linearisation: Saint-Venant compatibility condition $\mathrm{inc}\, \boldsymbol{e} := \nabla \times \boldsymbol{e} \times \nabla = 0$.


Bernhard Riemann


Ekkehart Kröner



$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$

Differential structures in elasticity

Linear elasticity (Calabi, Kröner) complex

curvature / stress

covector / force

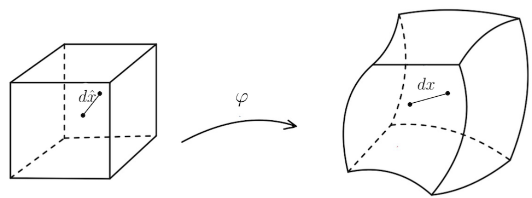
$$\mathrm{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}^{3\times 3}_{\text{sym}} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$\boldsymbol{\sigma} \longrightarrow \nabla \cdot \boldsymbol{\sigma}$$

Cauchy stress tensor $\boldsymbol{\sigma}$ balances load $\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f}$ with $\boldsymbol{\sigma} = A\boldsymbol{e}$ (Hooke's law);
incompatibility causes internal stress $\operatorname{inc} \boldsymbol{e}$.


Robert Hooke
(Christ Church PDRA room)


Augustin-Louis Cauchy



$$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$$

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 1: connect two (or more) de Rham complexes

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}
$$

$S^\bullet$: algebraic operators, connecting components of vectors/matrices

Complexes from complexes

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 2: elimination

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\operatorname{grad}"] & \mathbb{S}+\mathbb{K} \arrow[r, "\operatorname{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\operatorname{div}"] & \mathbb{R}^3 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\operatorname{grad}"'] \arrow[ur, "-\operatorname{mskw}"] & \mathbb{R}^{3\times 3} \arrow[r, "\operatorname{curl}"'] \arrow[ur, "\mathcal{S}"] & \mathbb{S}+\mathbb{K} \arrow[r, "\operatorname{div}"'] \arrow[ur, "2\operatorname{vskw}"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}
$$

$\mathbb{S}$: symmetric matrix, $\mathbb{K}$: skew-symmetric matrix

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 2: elimination

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{S}+\cancel{\mathbb{K}} \arrow[r, "\mathrm{curl}"] & \cancel{\mathbb{R}^{3\times 3}} \arrow[r, "\mathrm{div}"] & \cancel{\mathbb{R}^3} \arrow[r] & 0 \\
0 \arrow[r] & \cancel{\mathbb{R}^3} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & \cancel{\mathbb{R}^{3\times 3}} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathcal{S}"] & \mathbb{S}+\cancel{\mathbb{K}} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{R}^3 \arrow[r] & 0
\end{tikzcd}
$$

$\mathbb{S}$: symmetric matrix, $\mathbb{K}$: skew-symmetric matrix

Complexes from complexes

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 3: connect rows by zig-zag

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{sym\,grad}"] & \mathbb{S} \arrow[r, "\mathrm{curl}"] & {} \arrow[dll] & & \\
 & {} \arrow[r, "\mathrm{curl}^{\mathrm{T}}"] & \mathbb{S} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0.
\end{tikzcd}
$$

Conclusion: cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Analytic results follow: Poincaré–Korn inequalities, Hodge decomposition, compactness...

Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction (B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006, Arnold, KH 2021, Čap, KH 2023)

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 3: connect rows by zig-zag

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}^3 \arrow[r, "\mathrm{sym\,grad}"] & \mathbb{S} \arrow[r, "\mathrm{curl}"] & {} \arrow[dl] & & \\ & & {} \arrow[r, "\mathrm{curl}^{T}"] & \mathbb{S} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \arrow[r] & 0. \end{tikzcd}$$

Conclusion: cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Analytic results follow: Poincaré–Korn inequalities, Hodge decomposition, compactness...

But, is it purely mathematical?

embedding/displacement

metric/strain

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "\text{sym grad}"] & \mathbb{S} \arrow[r, "\text{curl}"] & {} \arrow[dl, "T"] & & \\
& {} \arrow[r, "\text{curl}"] & \mathbb{S} \arrow[r, "\text{div}"] & \mathbb{R}^3
\end{tikzcd}$$

curvature/stress

covector/load

Riemann, Kröner, Cauchy, Hooke

embedding/displacement

metric/strain

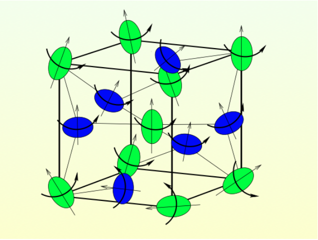
$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] & {} \arrow[dl] & \\
\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0"] & {} \arrow[r, "\mathrm{curl}^{T}"] & \mathbb{S} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3
\end{tikzcd}
$$

rotation

curvature/stress

covector/load

Cosserat brothers



Cosserat continua: microstructures (rotation, stretch etc.)

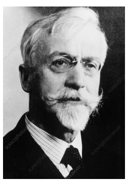
Observations: A. Čap & KH, *BGG sequences with weak regularity and applications.* FoCM (2024).

Leading to first parameter-robust scheme for Cosserat model: A.Dziubek, KH, M.Karow & M. Neunteufel, arXiv (2024).

embedding/displacement | coframes | torsion 2-form dislocation | Bianchi

$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] & \mathbb{R}^3 \\
\mathbb{R}^3 \arrow[r, "\mathrm{grad}"] \arrow[ur, "S^0"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S^1"] & \mathbb{R}^{3\times 3} \arrow[r, "\mathrm{div}"] \arrow[ur, "S^2"] & \mathbb{R}^3
\end{tikzcd}
$$

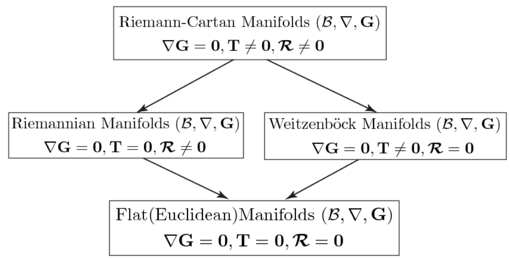
rotation | connection 1-form | curvature/stress disclination | covector/load


Élie Cartan


Arash Yavari


Alain Goriely

Cartan's bridge between Einstein and Cosserat brothers – torsion



Riemann-Cartan Geometry of Nonlinear Dislocation Mechanics,
Yavari and Goriely, ARMA (2012)

Observations: Christiansen, KH, & Lin, *Extended Regge complex for linearized Riemann-Cartan geometry and cohomology*. arXiv (2023). BGG construction is thus cohomology-preserving elimination of microstructures!

Recall: differential forms

On $\mathbb{R}^n$, a k-form is an element in $C^\infty(\Omega) \otimes \mathrm{Alt}^k$.

- A general k-form: $\omega = \sum_I f_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.
- antisymmetric: $dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad dx_i \wedge dx_i = 0$.

$$
\begin{aligned}
&f = f \Rightarrow \text{0-form}, \mathbb{R}\\
&\omega = udx + vdy + wdz \Rightarrow \text{1-form}, \mathbb{V}\\
&\eta = ady \wedge dz + bdz \wedge dx + cdx \wedge dy \Rightarrow \text{2-form}, \mathbb{V}\\
&\mu = fdx \wedge dy \wedge dz \Rightarrow \text{3-form}, \mathbb{R}
\end{aligned}
$$

- Differential operator: $d\omega = \sum_I \left(\sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

$$d = \mathrm{grad}(\text{0-form}), \mathrm{curl}(\text{1-form}), \mathrm{div}(\text{2-form}), 0(\text{3-form})$$

- Traces $\iota_F^* \omega : \sum_I (f_I|_F) \, (dx_{i_1}|_F) \ \wedge \ \cdots \wedge \ (dx_{i_k}|_F)$.

$$\iota^* = \text{value}(\text{0-form}), t(\text{1-form}), n(\text{2-form}), 0(\text{3-form})$$

Form-value Forms

On $\mathbb{R}^n$, a (k,ℓ)-form is an element in $\underbrace{(C^\infty(\Omega)\otimes\mathrm{Alt}^k)}_{k \text{ forms}}\ \underbrace{\otimes\mathrm{Alt}^\ell}_{\ell\text{-form valued}} =: C^\infty(\Omega)\otimes\mathrm{Alt}^{k,\ell}$.

$$g = g_{ij}dx^i\otimes dx^j \quad \text{(sym (1,1)-form)}, \quad R = R_{ijpq}dx^i\wedge dx^j\otimes dx^p\wedge dx^q \quad \text{(sym (2,2)-form)}.$$

In three dimensions:

	dx	dy	dz
dx	*	*	*
dy	*	*	*
dz	*	*	*

(1,1) forms

	dx	dy	dz
$dy\wedge dz$	*	*	*
$dz\wedge dx$	*	*	*
$dx\wedge dy$	*	*	*

(2,1) forms

	$dy\wedge dz$	$dz\wedge dx$	$dx\wedge dy$
dx	*	*	*
dy	*	*	*
dz	*	*	*

(1,2) forms

	$dy\wedge dz$	$dz\wedge dx$	$dx\wedge dy$
$dy\wedge dz$	*	*	*
$dz\wedge dx$	*	*	*
$dx\wedge dy$	*	*	*

(2,2) forms

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots & \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell-1} \arrow[r, "d^{k-1}"] \arrow[ur] & \mathrm{Alt}^{k,\ell-1} \arrow[r, "d^{k}"] \arrow[ur] & \mathrm{Alt}^{k+1,\ell-1} \arrow[r, "d^{k+1}"] \arrow[ur] & \mathrm{Alt}^{k+2,\ell-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell}"'] & \mathrm{Alt}^{k,\ell} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell}"'] & \mathrm{Alt}^{k+1,\ell} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell}"'] & \mathrm{Alt}^{k+2,\ell} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+1} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+1}"'] & \mathrm{Alt}^{k,\ell+1} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+1}"'] & \mathrm{Alt}^{k+1,\ell+1} \arrow[r, "d^{2}"] \arrow[ur, "S^{k+1,\ell+1}"'] & \mathrm{Alt}^{k+2,\ell+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+2} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+2}"'] & \mathrm{Alt}^{k,\ell+2} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+2}"'] & \mathrm{Alt}^{k+1,\ell+2} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell+2}"'] & \mathrm{Alt}^{k+2,\ell+2} \arrow[r] & \cdots \\
 & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

where $\mathrm{Alt}^{k,\ell} := \mathrm{Alt}^k \otimes \mathrm{Alt}^\ell$ $\quad$ ℓ-form-valued k-forms (k-forms taking value in ℓ-forms; (k, ℓ)-form)

$$
S^{i,J}\mu(v_0, \cdots, v_i)(w_1, \cdots, w_{J-1}) := \sum_{l=0}^{i} (-1)^l \mu(v_0, \cdots, \widehat{v_l}, \cdots, v_i)(v^l, w_1, \cdots, w_{J-1}),
$$

$$
S_\dagger^{i,J}\mu(v_1, \cdots, v_{i-1})(w_0, \cdots, w_J) := \sum_{l=0}^{i} (-1)^l \mu(w_l, v_1, \cdots, v_{i-1})(w_1, \cdots, \widehat{w_l}, \cdots, w_J),
$$

$$
\forall v_0, \cdots, v_i, w_1, \cdots, w_J \in \mathbb{R}^n.
$$

$$\begin{tikzcd}
 & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots & \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell-1} \arrow[r, "d^{k-1}"] \arrow[ur] & \mathrm{Alt}^{k,\ell-1} \arrow[r, "d^{k}"] \arrow[ur] & \mathrm{Alt}^{k+1,\ell-1} \arrow[r, "d^{k+1}"] \arrow[ur] & \mathrm{Alt}^{k+2,\ell-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell}"'] & \mathrm{Alt}^{k,\ell} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell}"'] & \mathrm{Alt}^{k+1,\ell} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell}"'] & \mathrm{Alt}^{k+2,\ell} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+1} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+1}"'] & \mathrm{Alt}^{k,\ell+1} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+1}"'] & \mathrm{Alt}^{k+1,\ell+1} \arrow[r, "d^{2}"] \arrow[ur, "S^{k+1,\ell+1}"'] & \mathrm{Alt}^{k+2,\ell+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+2} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+2}"'] & \mathrm{Alt}^{k,\ell+2} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+2}"'] & \mathrm{Alt}^{k+1,\ell+2} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell+2}"'] & \mathrm{Alt}^{k+2,\ell+2} \arrow[r] & \cdots \\
 & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}$$

Tensor symmetries encoded in diagrams: e.g.,

differential forms: $(k, 0)$ metric, strain: $(1, 1)$ curvature, stress: $(2, 2)$ torsion: $(2, 1)$

Example: Riemannian tensor

$\ker(S^{2,2}) \subset \mathrm{Alt}^{2,2}$: symmetry of Riemannian tensor (algebraic Bianchi identity)

$R_{ab;cd} = -R_{ba;cd} = -R_{ab;dc}$, $\quad R_{ab;cd} = R_{cd;ab}$, $\quad R_{ab;cd} + R_{bc;ad} + R_{ca;bd} = 0.$

$$\dim(\ker(S^{2,2})) = \dim \mathrm{Alt}^{2,2} - \dim \mathrm{Alt}^{3,1} = \begin{cases} 1 & \text{in 2D} \quad \text{Gauss curvature,} \\ 6 & \text{in 3D} \quad \text{Ricci/Einstein,} \\ 20 & \text{in 4D} \quad \text{Riemann} \\ \cdots & \end{cases}$$

Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction. (B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006, Arnold,KH 2021, Čap,KH 2023)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Alt}^{0,J-1} \arrow[r, "d"] & \mathrm{Alt}^{1,J-1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathrm{Alt}^{n,J-1} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Alt}^{0,J} \arrow[r, "d"] \arrow[ur, "S^{0,J}"] & \mathrm{Alt}^{1,J} \arrow[r, "d"] \arrow[ur, "S^{1,J}"] & \cdots \arrow[r, "d"] \arrow[ur, "S^{n-1,J}"] & \mathrm{Alt}^{n,J} \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{Alt}^{i,J} := \mathrm{Alt}^{i} \otimes \mathrm{Alt}^{J}$ J-form-valued i-forms (i-forms taking value in J-forms; (i, J)-form)

Output:

$$\begin{tikzcd}
0 \arrow[r] & \ker(S_{\dagger}^{-1,J}) \arrow[r] & \cdots \arrow[r] & \ker(S_{\dagger}^{J-2,J}) \arrow[r, "d"] & {} \arrow[dl, "S^{-1}"] & & & \\
& & & {} \arrow[r, "d"] & \ker(S^{J,J}) \arrow[r] & \cdots \arrow[r] & \ker(S^{n,J}) \arrow[r] & 0.
\end{tikzcd}$$

$S_{\dagger}$: adjoint of S

3D EXAMPLES

$\mathbb{R}$: scalar $\quad$ $\mathbb{V}$: vector $\quad$ $\mathbb{M}$: matrix $\quad$ $\mathbb{S}$: symmetric matrix $\quad$ $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\operatorname{grad}"] & \mathbb{V} \arrow[r, "\operatorname{curl}"] & \mathbb{V} \arrow[r, "\operatorname{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\operatorname{grad}"] \arrow[ur, "\operatorname{id}"] & \mathbb{M} \arrow[r, "\operatorname{curl}"] \arrow[ur, "2\operatorname{vskw}"] & \mathbb{M} \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{tr}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\operatorname{grad}"] \arrow[ur, "-\operatorname{mskw}"'] & \mathbb{M} \arrow[r, "\operatorname{curl}"] \arrow[ur, "S"'] & \mathbb{M} \arrow[r, "\operatorname{div}"] \arrow[ur, "2\operatorname{vskw}"'] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\operatorname{grad}"] \arrow[ur, "-\iota"] & \mathbb{V} \arrow[r, "\operatorname{curl}"] \arrow[ur, "\operatorname{mskw}"] & \mathbb{V} \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{id}"] & \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

Hessian complex:

$$0 \longrightarrow C^\infty \xrightarrow{\operatorname{hess}} C^\infty(\mathbb{S}) \xrightarrow{\operatorname{curl}} C^\infty(\mathbb{T}) \xrightarrow{\operatorname{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{tr}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"] & \mathbb{M} \arrow[r, "\mathrm{curl}"] \arrow[ur, "S"] & \mathbb{M} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\,\mathrm{vskw}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\iota"] & \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{mskw}"] & \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "\mathrm{id}"] & \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

elasticity complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\mathrm{def}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{inc}} C^\infty(\mathbb{S}) \xrightarrow{\mathrm{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

elasticity, defects, metric, curvature

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathsf{grad}"] & \mathbb{V} \arrow[r, "\mathsf{curl}"] & \mathbb{V} \arrow[r, "\mathsf{div}"] & \mathbb{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathsf{grad}"] \arrow[ur, "\mathrm{id}"] & \mathbb{M} \arrow[r, "\mathsf{curl}"] \arrow[ur, "2\,\mathsf{vskw}"] & \mathbb{M} \arrow[r, "\mathsf{div}"] \arrow[ur, "\mathsf{tr}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{V} \arrow[r, "\mathsf{grad}"] \arrow[ur, "-\,\mathsf{mskw}"] & \mathbb{M} \arrow[r, "\mathsf{curl}"] \arrow[ur, "S"] & \mathbb{M} \arrow[r, "\mathsf{div}"] \arrow[ur, "2\,\mathsf{vskw}"] & \mathbb{V} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\mathsf{grad}"] \arrow[ur, "-\iota"] & \mathbb{V} \arrow[r, "\mathsf{curl}"] \arrow[ur, "\mathsf{mskw}"] & \mathbb{V} \arrow[r, "\mathsf{div}"] \arrow[ur, "\mathrm{id}"] & \mathbb{R} \arrow[r] & 0.
\end{tikzcd}
$$

divdiv complex:

$$
0 \longrightarrow C^{\infty}(\mathbb{V}) \xrightarrow{\mathsf{dev\,grad}} C^{\infty}(\mathbb{T}) \xrightarrow{\mathsf{sym\,curl}} C^{\infty}(\mathbb{S}) \xrightarrow{\mathsf{div\,div}} C^{\infty} \longrightarrow 0.
$$

plate theory, elasticity

Discrete level: intrinsic finite elements

Discretization of complexes: finite elements and splines

- 2D stress: Arnold-Winther 2002, J.Hu-S.Zhang 2014, Christiansen-KH 2018,
- 2D strain: Chen-J.Hu-Huang 2014 (Regge/HHJ), Christiansen-KH 2018 (conforming), Chen-Huang 2020, DiPietro-Droniou 2021 (polygonal meshes), KH 2023
- 3D elasticity: various results on last part of complex, Hauret-Kuhl-Ortiz 2007 (discrete geometry/mechanics), Arnold-Awanou-Winther 2008, Christiansen 2011 (Regge), Christiansen-Gopalakrishnan-Guzmán-KH 2020, Chen-Huang 2021, J.Hu-Liang-Lin 2023, Gong-Gopalakrishnan-Guzmán-Neilan 2023
- 3D Hessian: Chen-Huang 2020, J.Hu-Liang 2021, Arf-Simeon 2021 (splines)
- 3D divdiv: Chen-Huang 2021, J.Hu-Liang-Ma 2021, Sander 2021 ($H(\mathrm{sym\,curl})$, $H(\mathrm{dev\,sym\,curl})$), J.Hu-Liang-Ma-Zhang 2022, J.Hu-Liang-Lin 2023, DiPietro-Hanot 2023
- nD: Chen-Huang 2021 (last two spaces), 2D arbitrary regularity: Chen-Huang 2022, Bonizzoni-KH-Kanschat-Sap 2023 (tensor product construction, nD, (k, ℓ)-forms)
- conformal complexes: KH-Lin-Shi 2023.

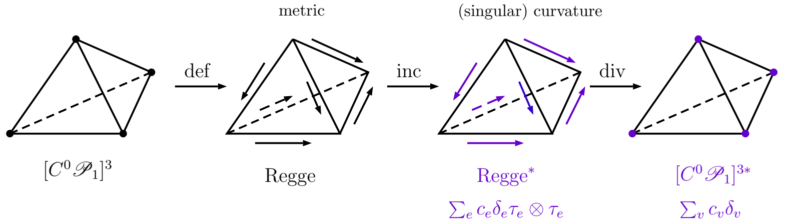
Question: Canonical finite elements, analogue of the Whitney forms?

correct cohomology, discrete topological/geometric structures...

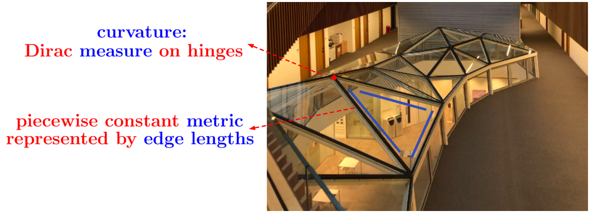
Gathering pieces to solve the puzzle

As the development of FEEC, several individual ingredients are already in the literature. Particularly, it is perhaps with little hesitation to accept Christiansen-Regge complex as the *canonical* discretization for the elasticity complex, due to simple dofs, geometric interpretations, formal self-adjointness, correct cohomology.

Christiansen 2011: Regge calculus = finite elements



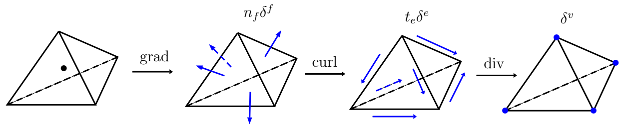
Regge finite element v.s. Regge Calculus (*'General relativity without coordinates'*, quantum gravity)



nD: Lizao Li (2018 Minnesota thesis), nonlinear curvature with Regge elements (Berchenko-Kogan,Gawlik 2022, Gopalakrishnan,Neunteufel,Schöberl,Wardetzky 2022, Gawlik,Neunteufel 2023)

Schöberl and collaborators systematically used finite elements + distributions to design numerical schemes.

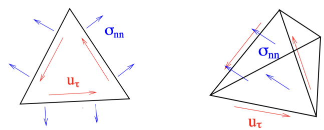
- Equilibrated residual error estimator



distributional finite elements, Braess-Schöberl 2008, Licht 2017

- Tangential Displacement Normal-Normal Stress (TDNNS) method for elasticity

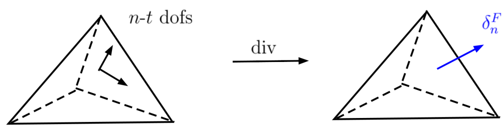
$$\sigma = -\varepsilon(\boldsymbol{u}), \quad \nabla \cdot \sigma = \boldsymbol{f}. \quad \sigma : \text{ symmetric matrix}$$



div σ: tangential distribution Schöberl-Sinwel 2007

- Mass-Conserving mixed Stress (MCS) method for Stokes equations

$$\sigma = -\nabla \boldsymbol{u}, \quad \nabla \cdot \sigma + \nabla p = \boldsymbol{f}. \quad \sigma : \text{ trace-free matrix}$$



div σ: normal distribution Gopalakrishnan-Lederer-Schöberl 2020

In this way and more broadly, we develop structure-aware and computation-friendly modelling via complexes .
microstructures, defects, dimension reduction, contact mechanics, porous media...

Our ‘BGG construction’ is much broader. e.g. A generalisation to form-valued forms (double forms à ala Cartan)

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots & \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell-1} \arrow[r, "d^{k-1}"] \arrow[ur] & \mathrm{Alt}^{k,\ell-1} \arrow[r, "d^{k}"] \arrow[ur] & \mathrm{Alt}^{k+1,\ell-1} \arrow[r, "d^{k+1}"] \arrow[ur] & \mathrm{Alt}^{k+2,\ell-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell}"] & \mathrm{Alt}^{k,\ell} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell}"] & \mathrm{Alt}^{k+1,\ell} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell}"] & \mathrm{Alt}^{k+2,\ell} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+1} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+1}"] & \mathrm{Alt}^{k,\ell+1} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+1}"] & \mathrm{Alt}^{k+1,\ell+1} \arrow[r, "d^{2}"] \arrow[ur, "S^{k+1,\ell+1}"] & \mathrm{Alt}^{k+2,\ell+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell+2} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell+2}"] & \mathrm{Alt}^{k,\ell+2} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell+2}"] & \mathrm{Alt}^{k+1,\ell+2} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell+2}"] & \mathrm{Alt}^{k+2,\ell+2} \arrow[r] & \cdots \\
 & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

$\mathrm{Alt}^{k,\ell} := \mathrm{Alt}^k \otimes \mathrm{Alt}^\ell$: ℓ-form-valued k-forms

differential forms: $(k, 0)$ metric, strain: $(1, 1)$ curvature, stress: $(2, 2)$ torsion: $(2, 1)$

In this way and more broadly, we develop structure-aware and computation-friendly modelling via complexes .
microstructures, defects, dimension reduction, contact mechanics, porous media...

Our ‘BGG construction’ is much broader. e.g. A generalisation to form-valued forms (double forms à ala Cartan)

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots & \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell-1} \arrow[r, "d^{k-1}"] \arrow[ur] & \mathrm{Alt}^{k,\ell-1} \arrow[r, "d^{k}"] \arrow[ur] & \mathrm{Alt}^{k+1,\ell-1} \arrow[r, "d^{k+1}"] \arrow[ur] & \mathrm{Alt}^{k+2,\ell-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Alt}^{k-1,\ell} \arrow[r, "d^{k-1}"] \arrow[ur, "S^{k-1,\ell}"] & \mathrm{Alt}^{k,\ell} \arrow[r, "d^{k}"] \arrow[ur, "S^{k,\ell}"] & \mathrm{Alt}^{k+1,\ell} \arrow[r, "d^{k+1}"] \arrow[ur, "S^{k+1,\ell}"] & \mathrm{Alt}^{k+2,\ell} \arrow[r] & \cdots \\
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 & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

$\mathrm{Alt}^{k,\ell} := \mathrm{Alt}^k \otimes \mathrm{Alt}^\ell$: ℓ-form-valued k-forms

differential forms: $(k, 0)$ metric, strain: $(1, 1)$ curvature, stress: $(2, 2)$ torsion: $(2, 1)$

Questions: Canonical discretisation of double forms?

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms (I): Construction.* arXiv: 2503.03243 (2025)

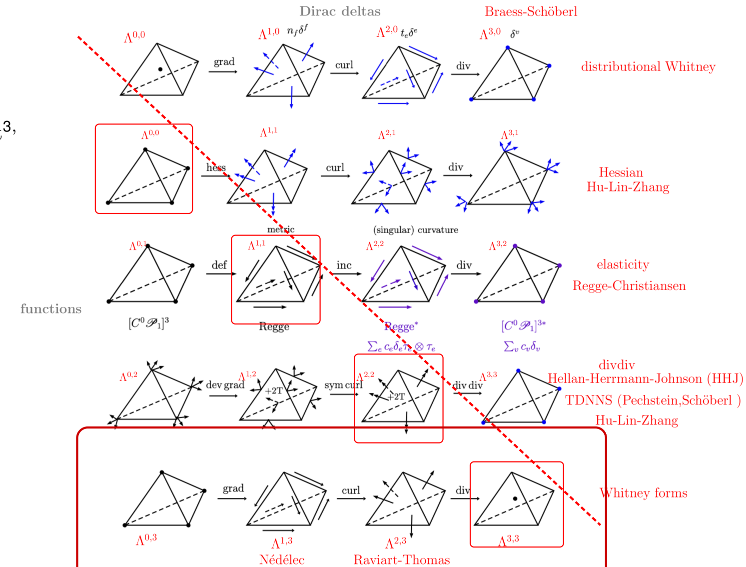
$$\mathrm{Alt}^{-1,-1} \dashrightarrow \mathrm{Alt}^{0,0} \longrightarrow \mathrm{Alt}^{1,0} \longrightarrow \mathrm{Alt}^{2,0} \longrightarrow \mathrm{Alt}^{3,}$$

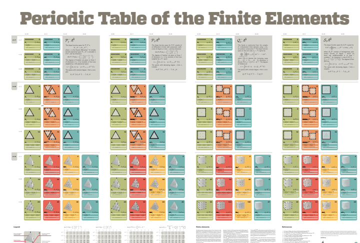
$$\mathrm{Alt}^{0,0} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \mathrm{Alt}^{2,1} \longrightarrow \mathrm{Alt}^{3,1}$$

$$\mathrm{Alt}^{0,1} \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \mathrm{Alt}^{3,2}$$

$$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{3,3}$$

$$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3} \dashrightarrow \mathrm{Alt}^{4,4}$$





classical Finite Element Exterior Calculus

Nédélec, Raviart–Thomas, Whitney, Bossavit, Hiptmair, Arnold, Falk, Winther...

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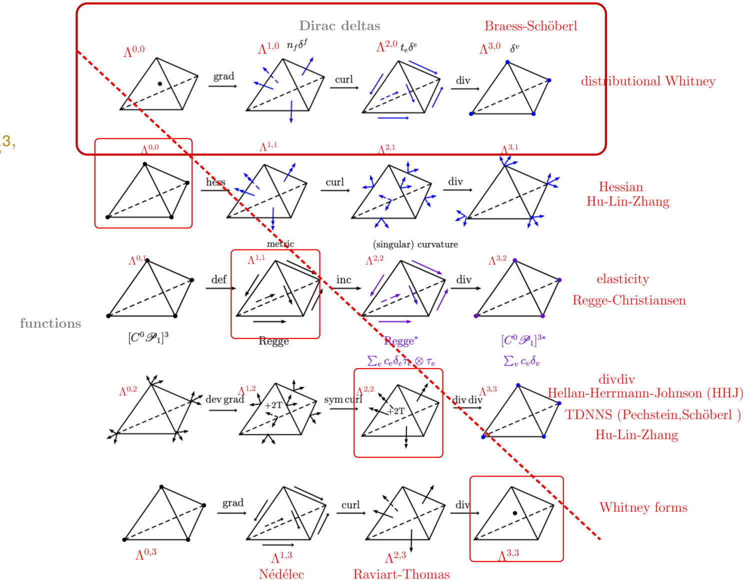

distributional de Rham complex (currents).

Braess, Schöberl 2008: equilibrated residual error estimator

Licht 2017: double complex

Dietrich Braess

Joachim Schöberl

Towards a finite element periodic table for tensors

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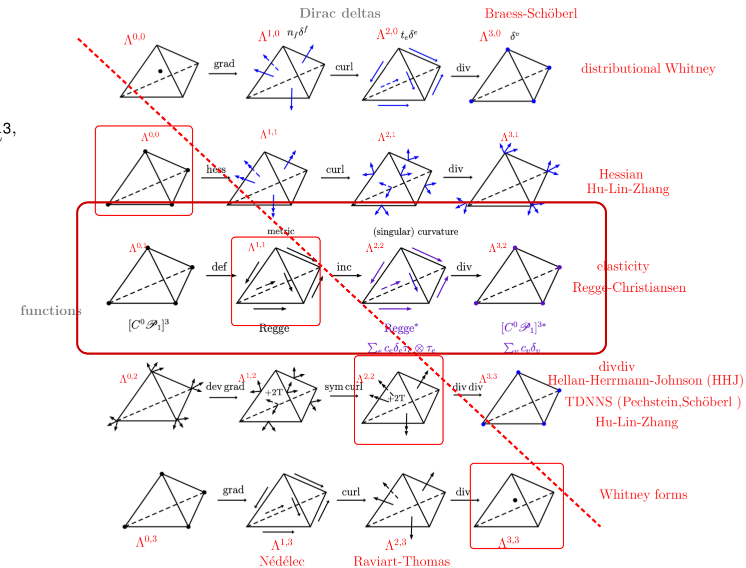

Christiansen's interpretation of Regge calculus as finite elements

Regge calculus (quantum & numerical gravity) :
edge length as metric, angle deficit as curvature

Regge finite element : piecewise constant symmetric tensor field


Tullio Regge


Snorre Christiansen

Towards a finite element periodic table for tensors

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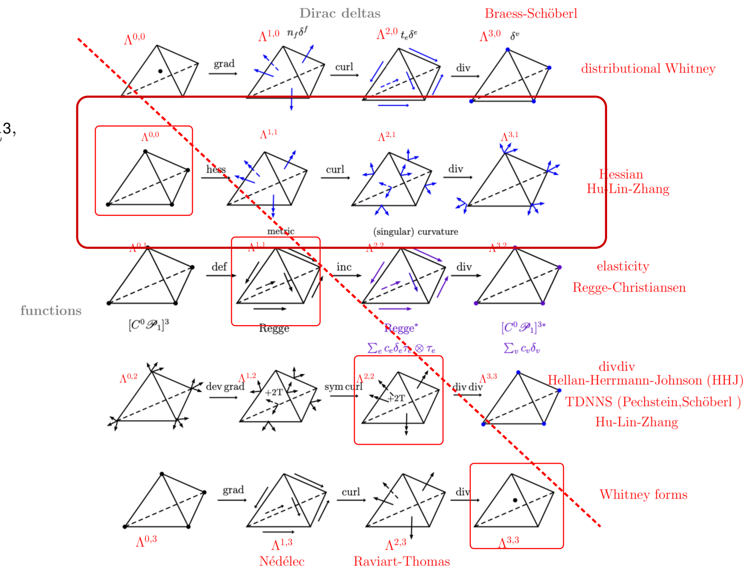
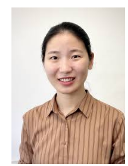


Hessian complex, unified structures identified.

Kaibo Hu, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).


Ting Lin


Qian Zhang

Towards a finite element periodic table for tensors

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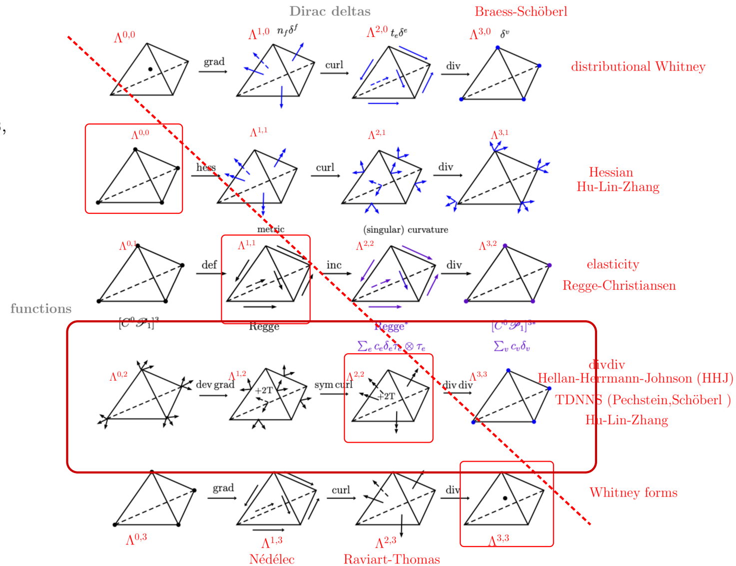

divdiv complex , dual to Hessian complex.

TDNNS for elasticity (Schöberl, Sinwel 2007), Hellan-Herrmann-Johnson (HHJ) element for plate.

Implemented by J.Schöberl in NGSolve with relativity applications

KH, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).

Astrid Pechstein

Joachim Schöberl

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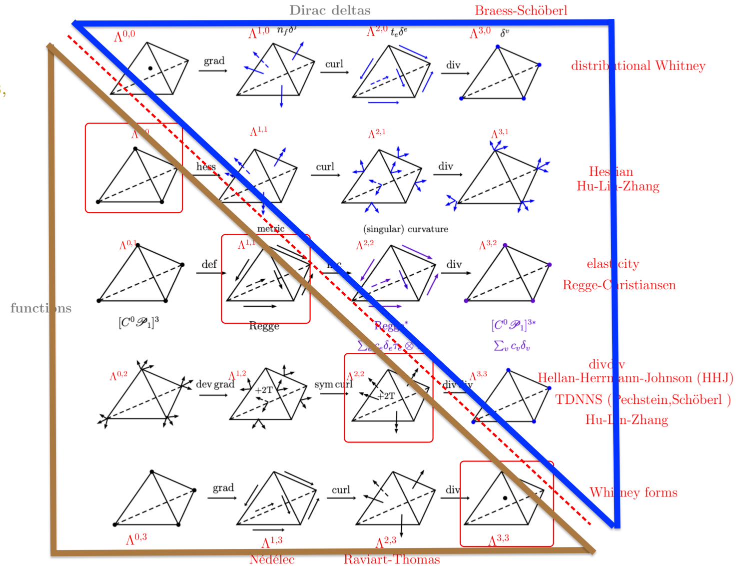


Patterns, Symmetries, Duality.

functions (classical finite elements) v.s. Dirac measures (currents). any dimension, any degree.

Georges de Rham

Classical Finite Element Periodic Table (last row) is the special case of the generalised Table where all spaces are piecewise polynomials.

Discrete differential geometry and data sciences: discrete structures v.s. discretisation

Discrete Differential Geometry v.s. PDEs

Curvature identity:

well defined: moving derivatives to test

$$u_{xx}u_{yy} - u_{xy}^2 = -\partial_x^2(\tfrac{1}{2}u_y^2) + \partial_x\partial_y(u_x u_y) - \partial_y^2(\tfrac{1}{2}u_x^2)$$

Gauss curvature, not defined if $u \notin C^1$

John Ball

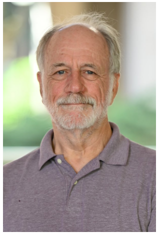
- Ball's existence proof of nonlinear elasticity (compensated compactness)
- conservation law (divergence structure)

Discrete Differential Geometry v.s. PDEs

Curvature identity:

$$\underbrace{u_{xx}u_{yy} - u_{xy}^2}_{\boxed{\det(\operatorname{hess} u)}} = \overbrace{-\partial_x^2(\tfrac{1}{2}u_y^2) + \partial_x\partial_y(u_x u_y) - \partial_y^2(\tfrac{1}{2}u_x^2)}^{\boxed{\operatorname{rot}\operatorname{rot}(\nabla u \otimes \nabla u)}}$$

John Ball



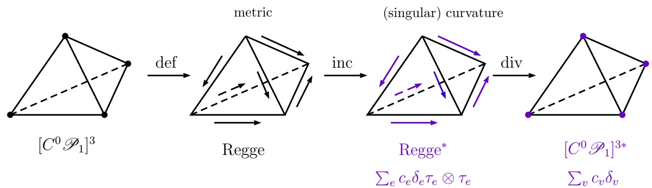
Peter Olver

- Peter Olver, *Differential hyperforms I.* (1982)

Generalising the structures in calculus of variations using representation theory, including many BGG complexes.

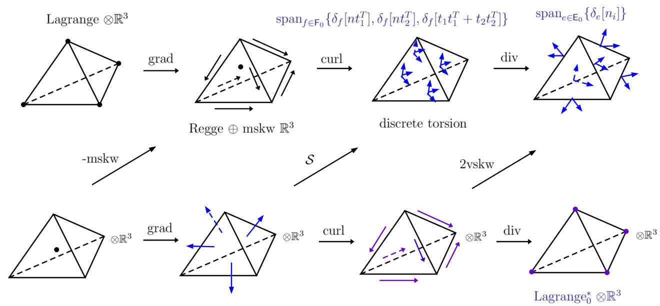
Extend Regge calculus/finite element (S.Christiansen 2011, Regge for curvature)

metric | curvature



to Riemann-Cartan geometry (S.Christiansen, KH, L.Ting 2023, extended Regge for curvature + torsion)

coframe (metric) | discrete torsion

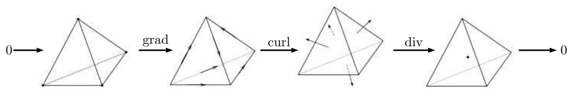


discrete connection | curvature

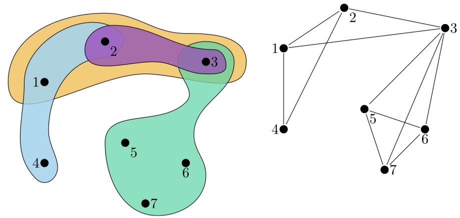
Question: the first systematic discretisation for generalised continuum with defects, as in Yavari–Goriely?

TOPOLOGICAL/GEOMETRIC DATA ANALYSIS

Canonical finite elements generalise to graphs and networks



Finite element de Rham complex (shown in 3D)



Hypergraphs. Cliques (loops) can exist in any dimension.

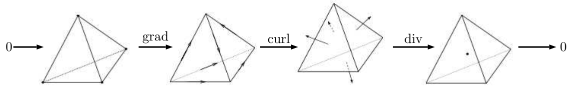
Many applications in Topological Data Analysis (persistent homology), Hodge Laplacian on graphs (ranking, data representation, geometric deep learning...), random graphs and phase transition

Hodge Laplacians on graphs. L. H. Lim, SIAM Review (2020).

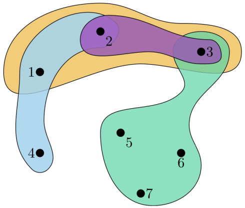
What are higher-order networks?. C. Bick, E. Gross, H.A. Harrington, & M.T. Schaub, SIAM Review (2023).

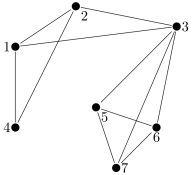
Topological/geometric data analysis

Canonical finite elements generalise to graphs and networks



Finite element de Rham complex (shown in 3D)

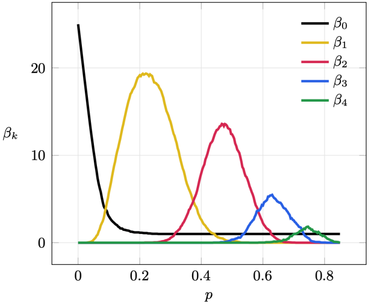




Hypergraphs. Cliques (loops) can exist in any dimension.

Many applications in Topological Data Analysis (persistent homology), Hodge Laplacian on graphs (ranking, data representation, geometric deep learning...), random graphs and phase transition

> *I predict a new subject of statistical topology. Rather than count the number of holes, Betti numbers, etc., one will be more interested in the distribution of such objects on non-compact manifolds as one goes out to infinity.*
>
> *– Isadore Singer*



Lázár Bertók, MSc thesis at University of Edinburgh, 2024
Betti number β_k changes with the probability p of a random graph

Many questions and opportunitites: e.g., revisit graph curvature, data torsion, quantum graphs...