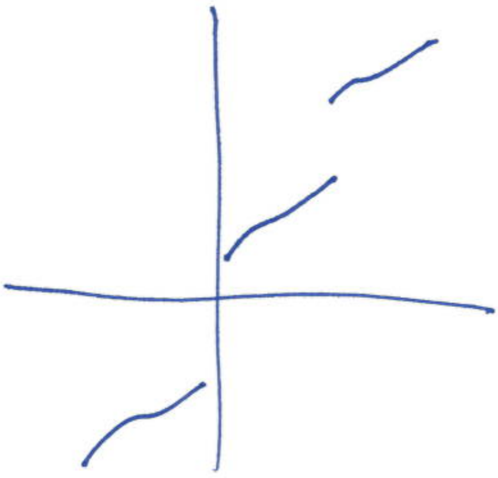
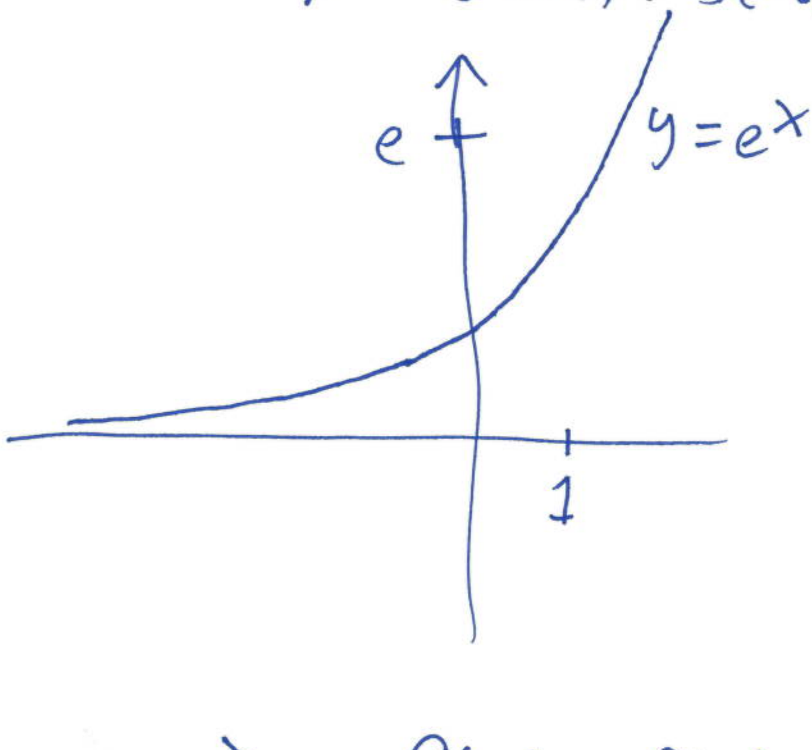
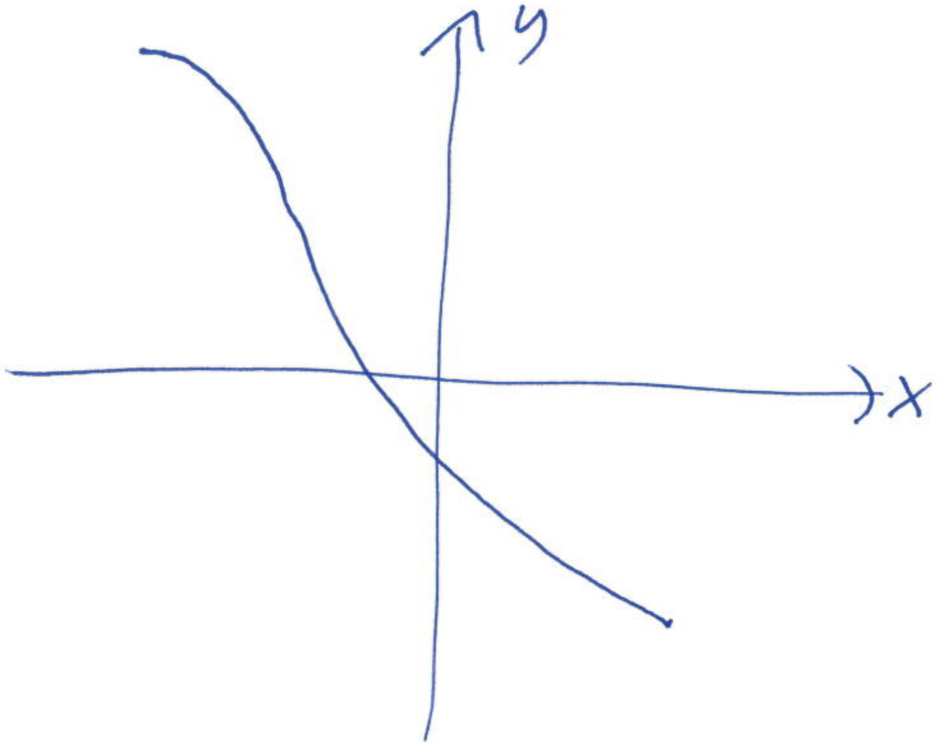
If f is strictly increasing or strictly decreasing f is injective

SI: $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$



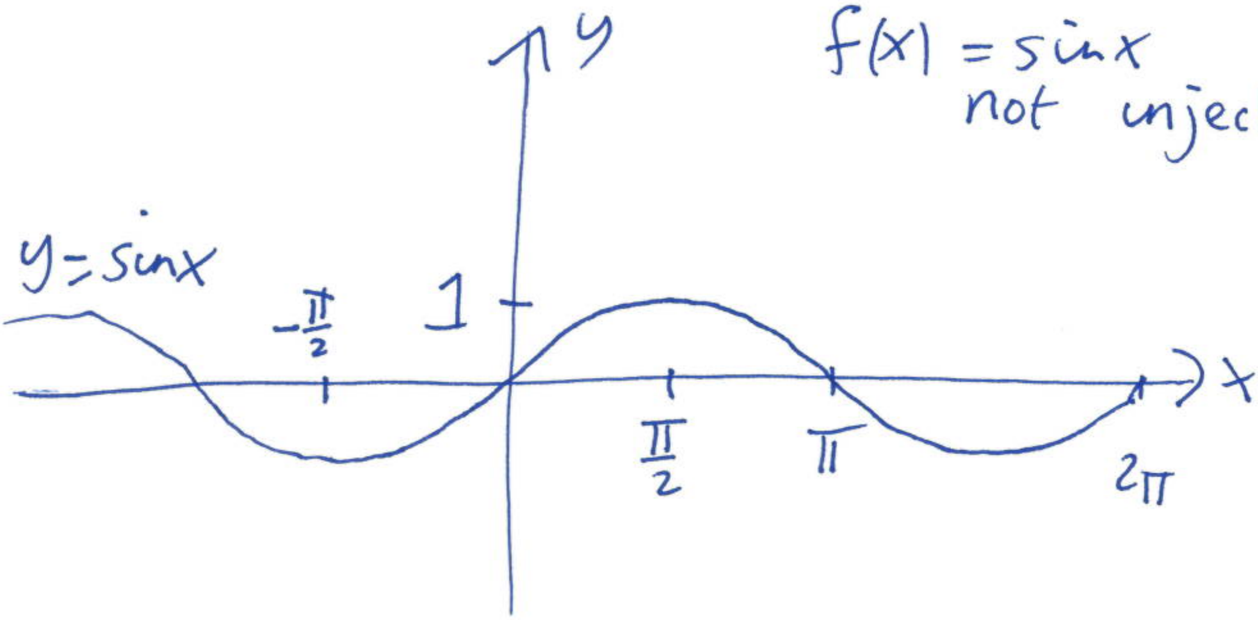


SD: $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$



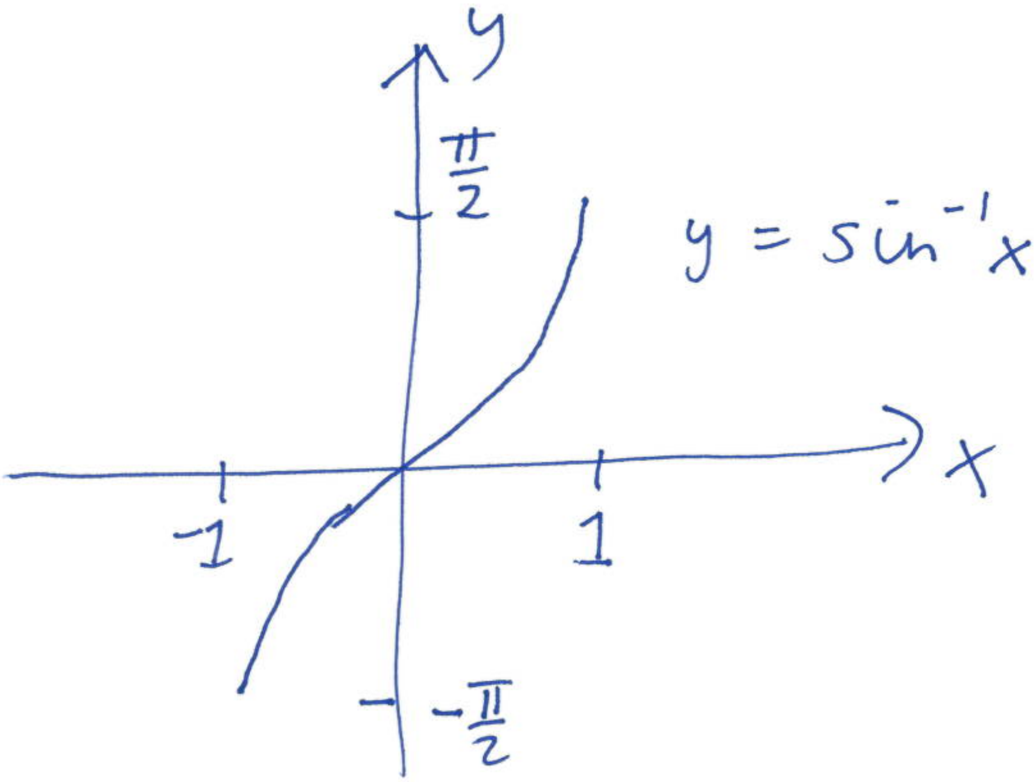
Trig Functions

$f(x) = \sin x$
not injective



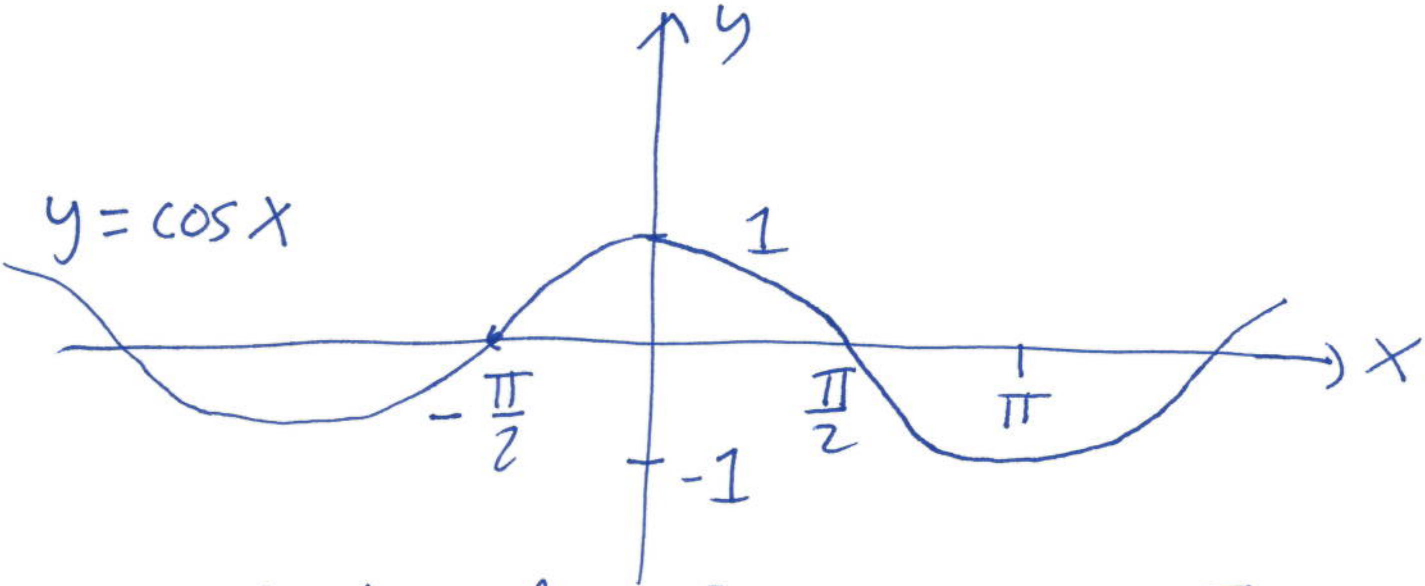
restrict domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ $\sin x$ strictly increasing



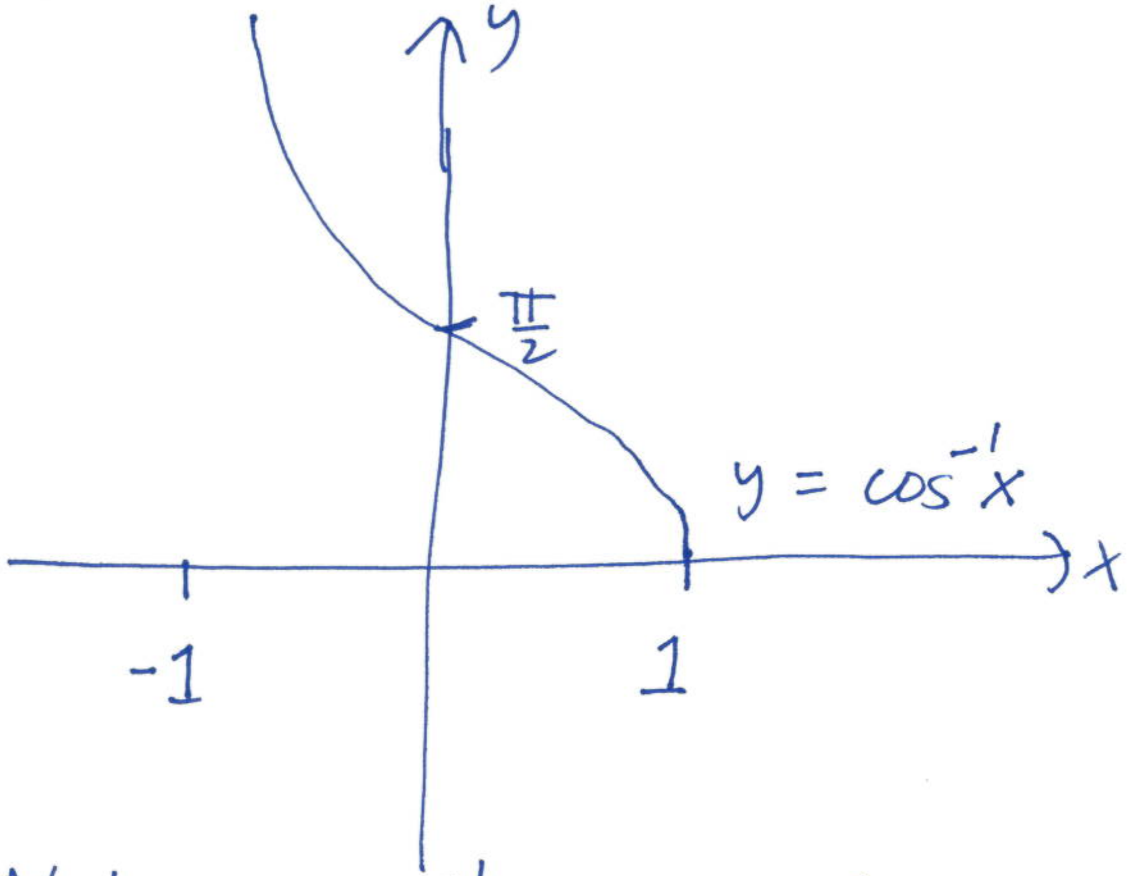
Other domains possible

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is a standard choice



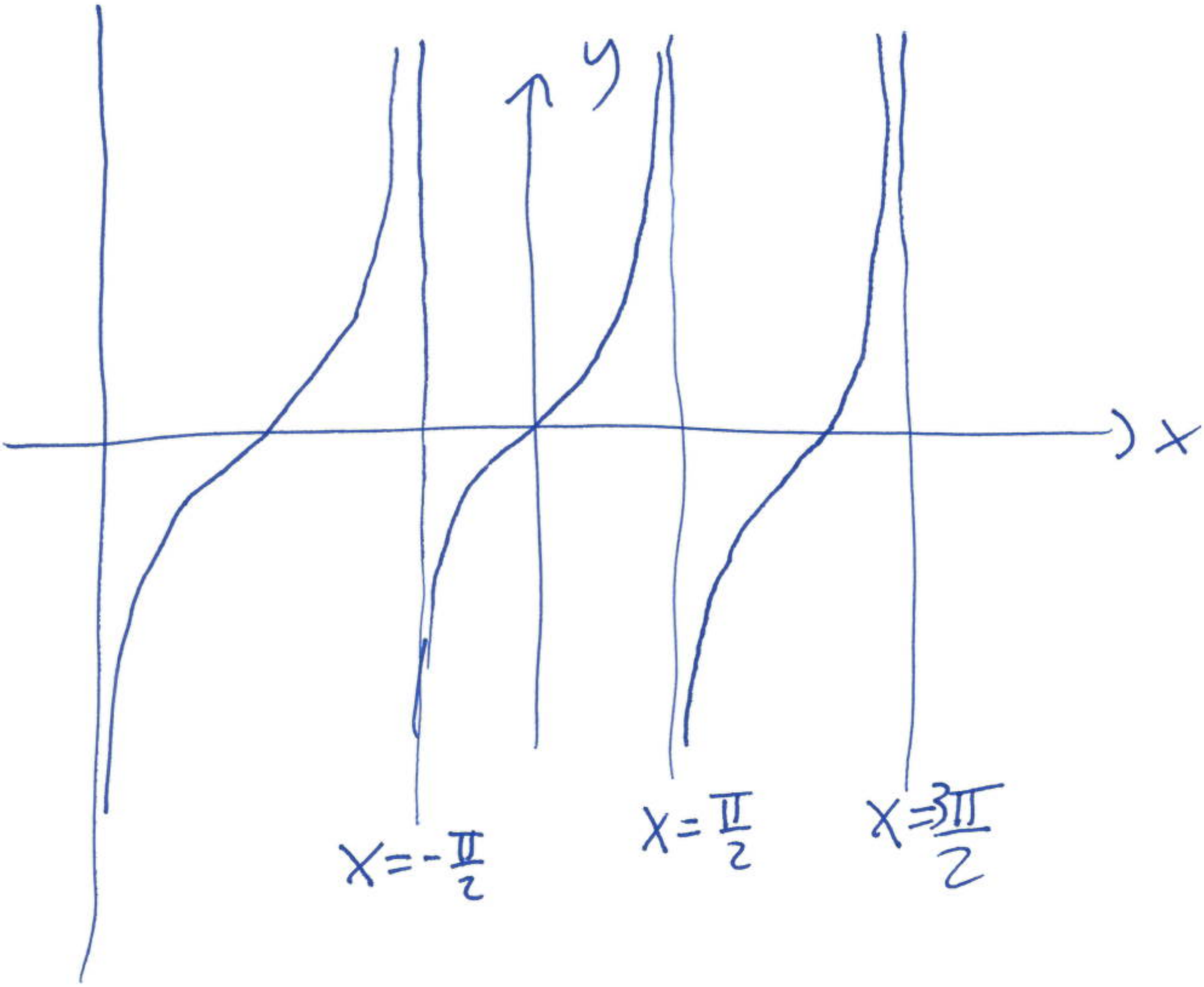
restrict domain to $[0, \pi]$

$\cos x$ strictly decreasing



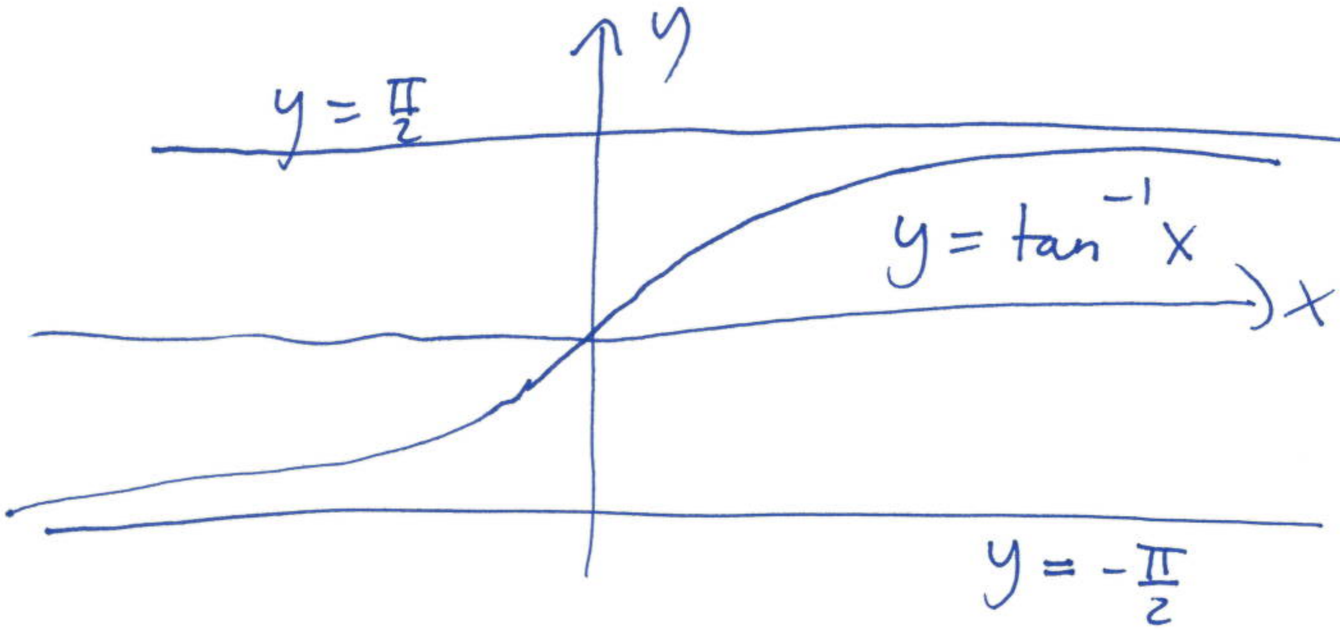
Note $\cos^{-1} x$ neither odd nor even. Decompose $\cos^{-1} x$ into odd and even parts!

tangent function



Restrict domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$



Analytical approach

Suppose f is invertible

Write $y = f(x)$ then $x = f^{-1}(y)$

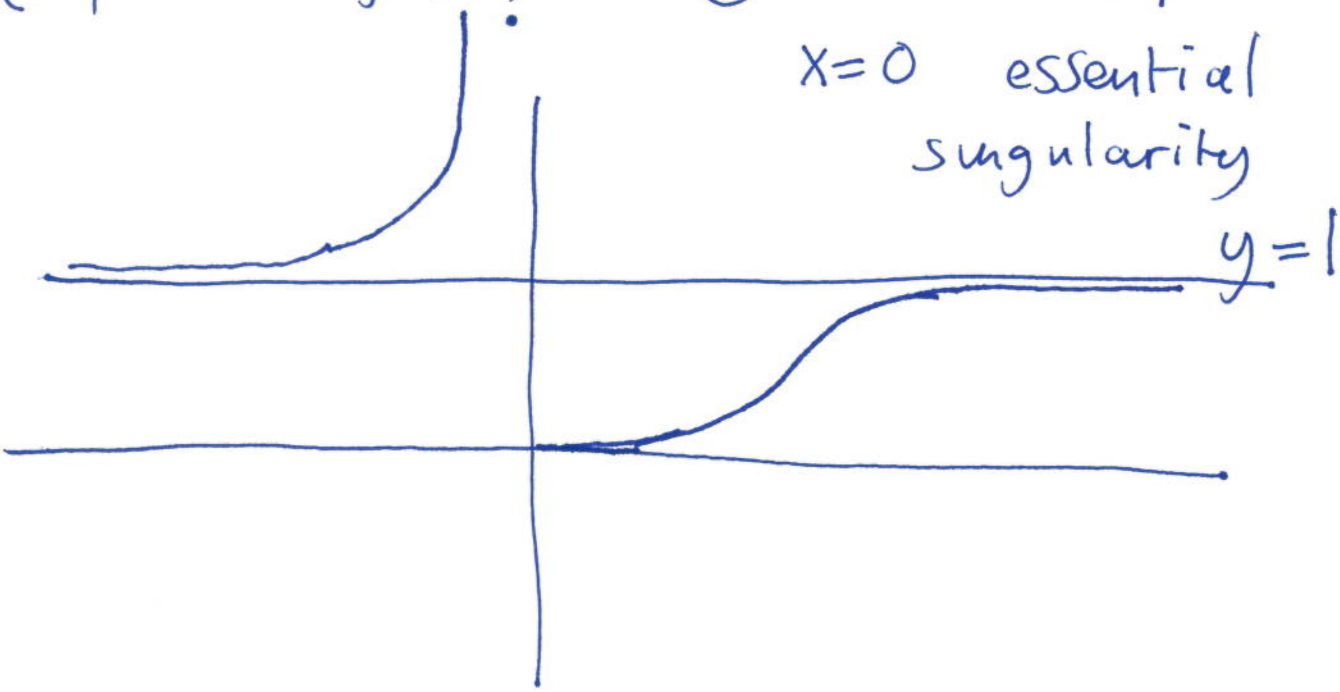
idea is to 'solve' $y = f(x)$

to express x in terms of y

Examples

(i) $f(x) = e^{-1/x}$ $x \neq 0$

$x = 0$ essential singularity



$$y = e^{-1/x}$$

$$\log y = -\frac{1}{x}$$

$$x = -\frac{1}{\log y} = f^{-1}(y)$$

can write $f^{-1}(x) = -\dfrac{1}{\log x}$

domain of f^{-1}?

$x > 0$ and $x \neq 1$

(ii) Inverse Hyperbolic Functions

$f(x) = \sinh x$ injective

$$y = \sinh x = \tfrac{1}{2}(e^{x} - e^{-x})$$

Multiply by e^{x}

$$ye^{x} = \tfrac{1}{2}e^{2x} - \tfrac{1}{2}$$

or

$$e^{2x} - 2ye^{x} - 1 = 0$$

$$(e^{x})^2 - 2ye^{x} - 1 = 0$$

quadratic equation for e^x

$$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2}$$

$$= y \pm \sqrt{y^2+1}$$

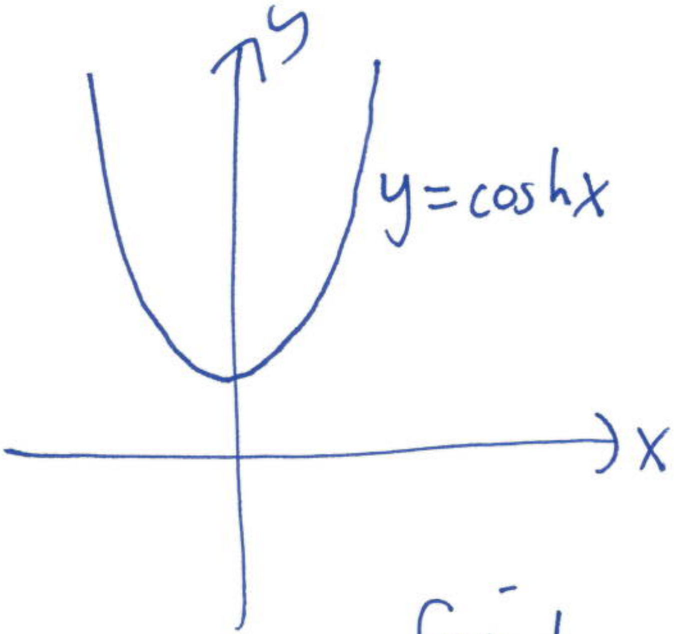
(discard − sign case as $e^x > 0$)

$$x = \log\left(y + \sqrt{y^2+1}\right)$$

$$= \sinh^{-1} y$$

or $\boxed{\sinh^{-1} x = \log\left(x + \sqrt{1+x^2}\right)}$

For cosh function



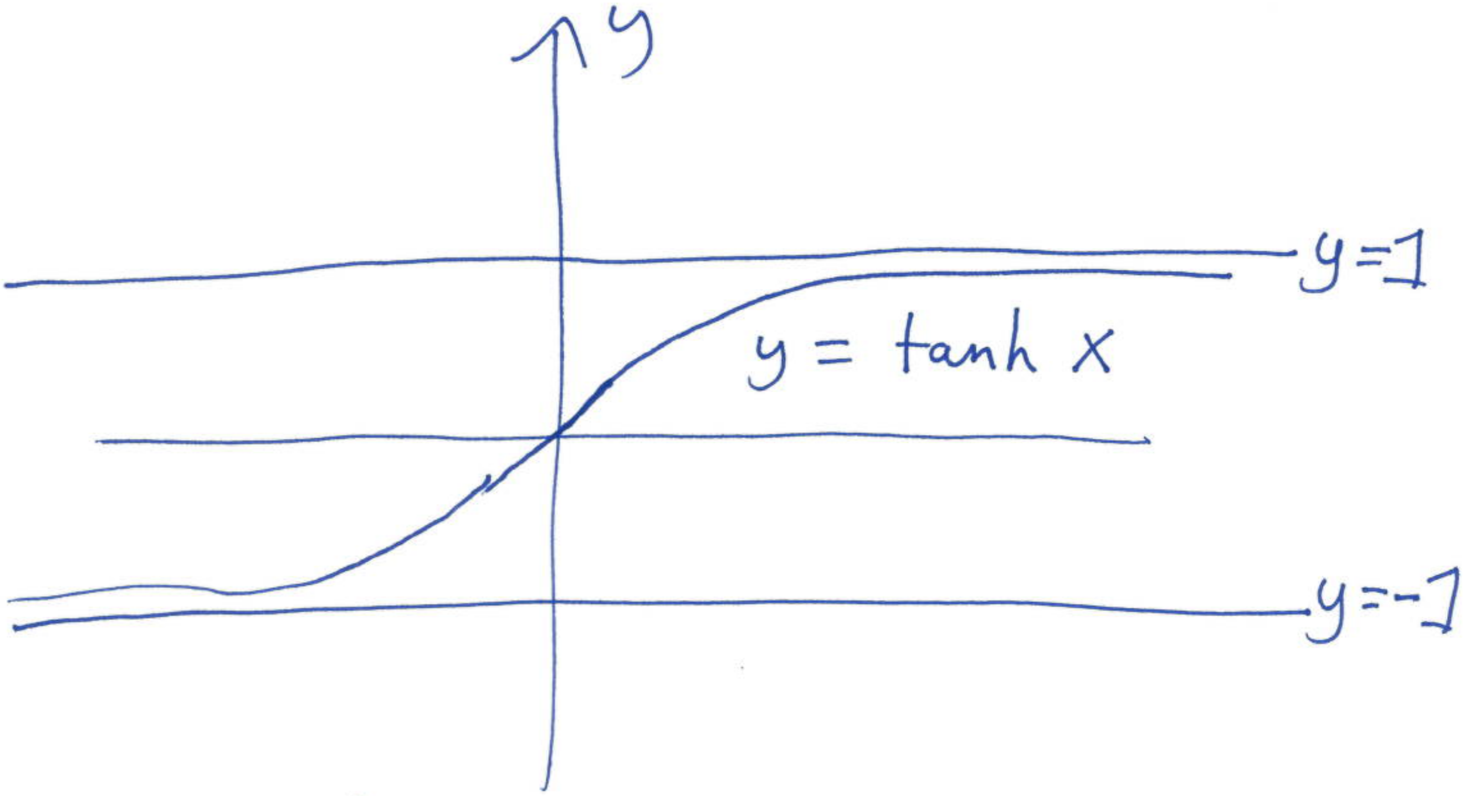
restrict domain to $[0, \infty)$

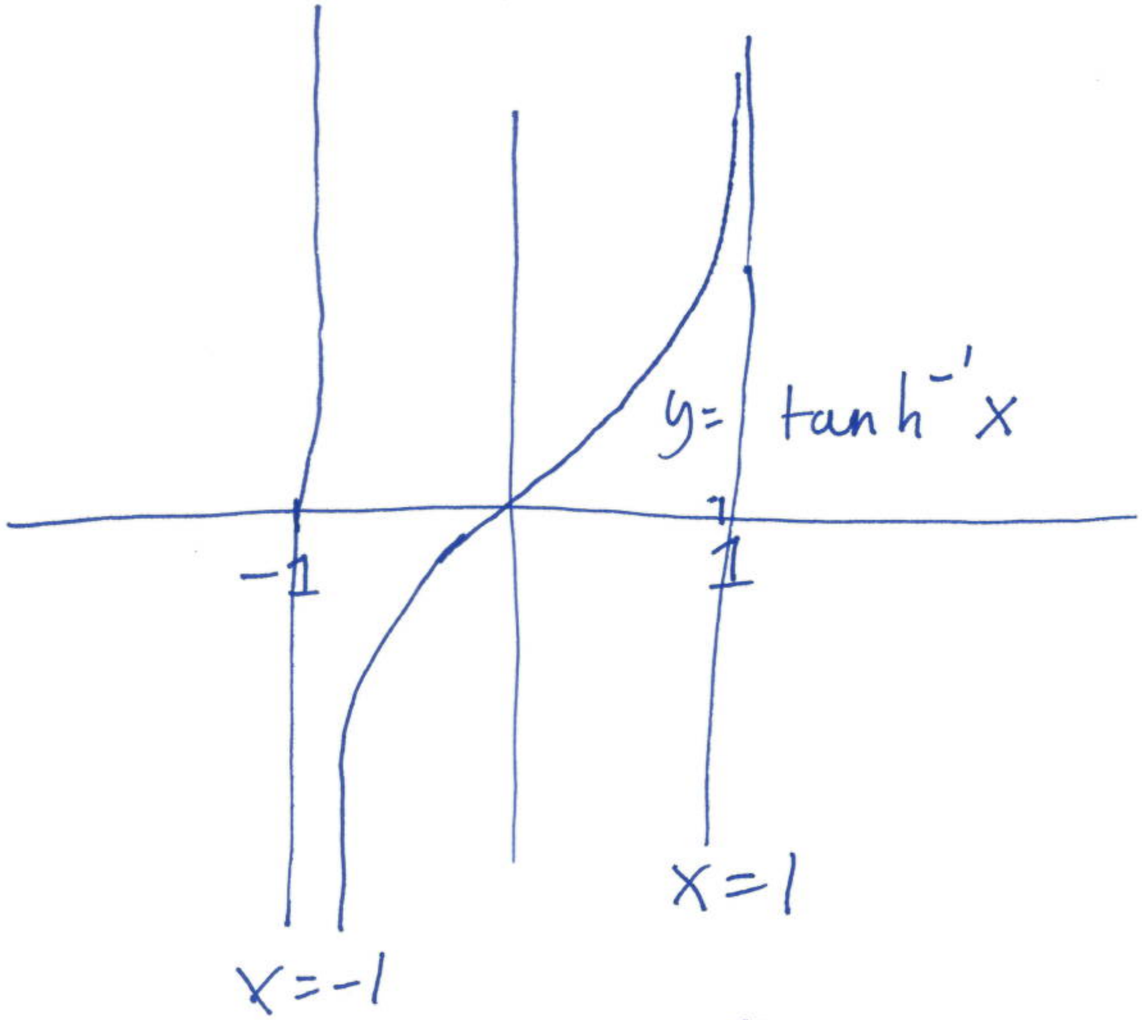
or $0 \leq x < \infty$

find a formula for $\cosh^{-1} x$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x}$$

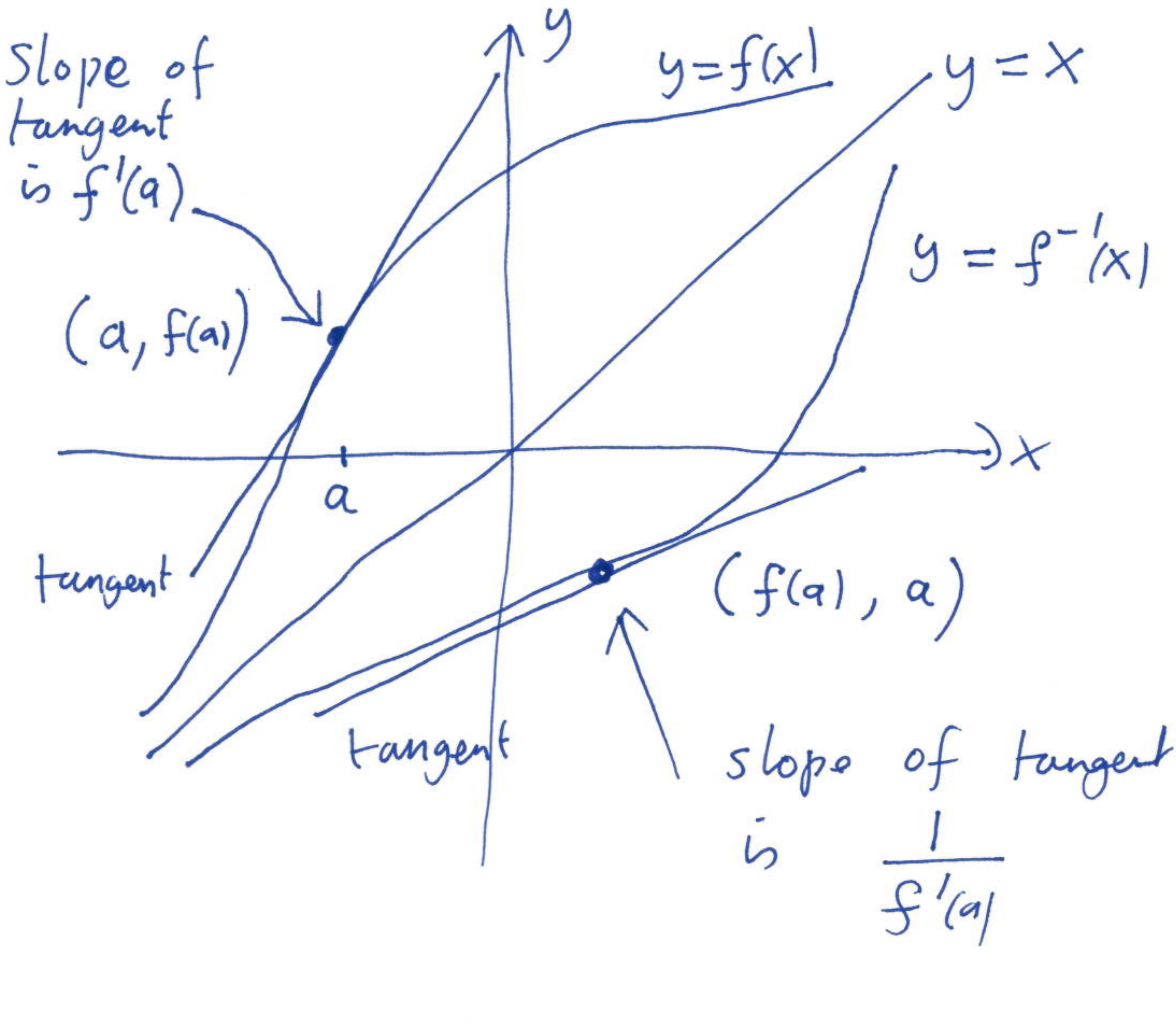


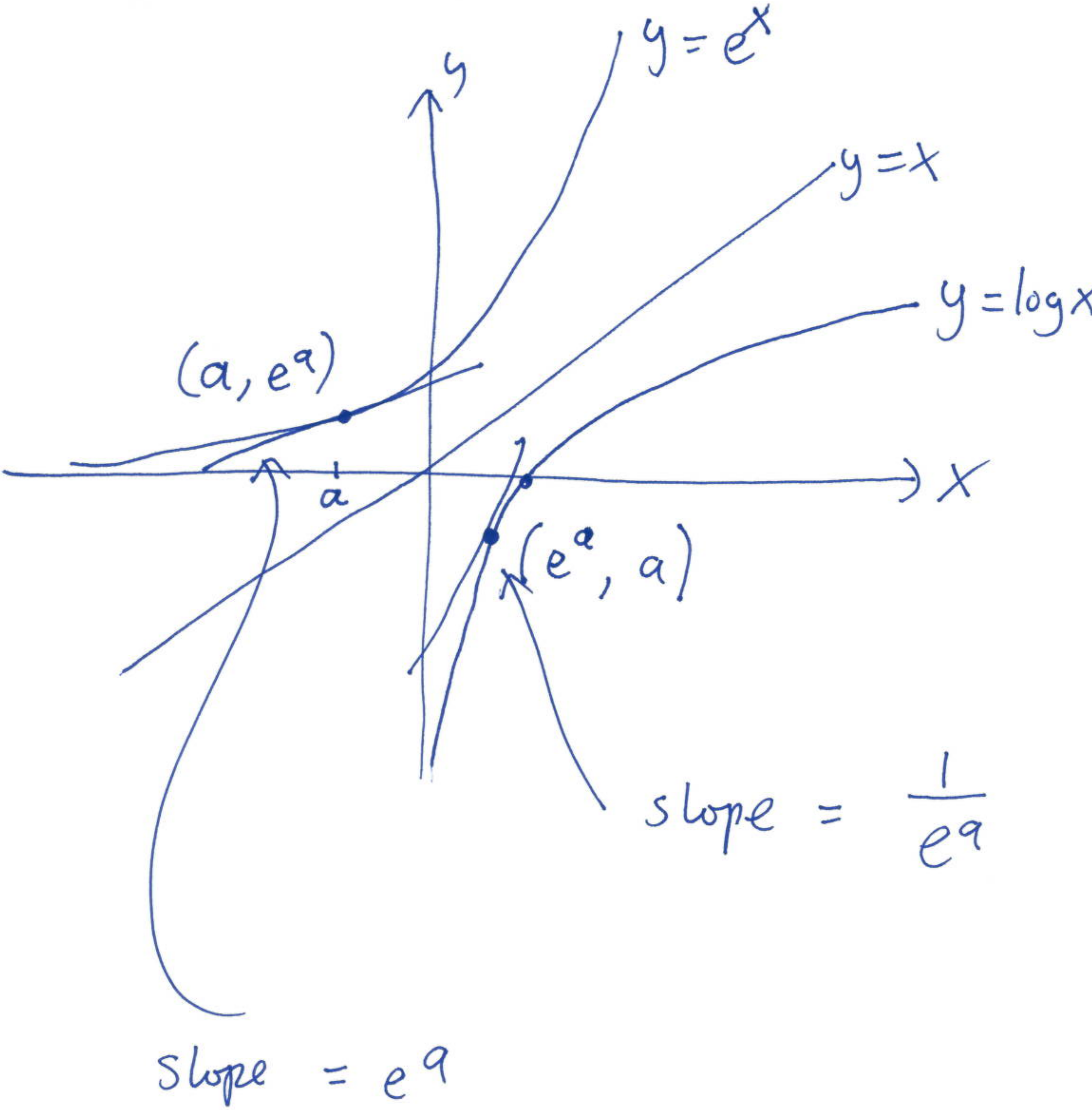


Can show that $\boxed{\tanh^{-1}x = \frac{1}{2}\log\frac{1+x}{1-x}}$

Derivatives of Inverse Functions

Graphically





If $f(x) = \log x$

$f'(x)$ at $x = e^a$ is $\frac{1}{e^a}$

$$\frac{d}{dx} \log x = \frac{1}{x}$$

Formula

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Not worth memorizing!