Office Hours

Tue 2-3
Thu 9-10 room 657

If a series $\sum_m a_m$ is absolutely convergent the sum is independent of the order of summation

---

A power series is an infinite sum of the form

$$\sum_{m=0}^{\infty} c_m x^m$$

where $c_0, c_1, c_2, \ldots$

an infinite list of numbers.

A power series is also a <u>function</u> of the variable x. A power series may converge for all x or a specific range of x values.

<u>Examples</u>

(i) Exponential series

$$\exp(x) = \sum_{m=0}^{\infty} \frac{1}{m!} x^m$$

$$\left(c_m = \frac{1}{m!} \, , \quad c_0 = 1 \, , \quad 0! = 1 \right)$$

this series converges for all x

(ii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

does not converge for all x even though $\tan^{-1} x$ defined for all x

Apply ratio test to example (i) write $\sum_{m=0}^{\infty} \frac{x^m}{m!} = \sum_{m=0}^{\infty} a_m$

$a_m = \frac{x^m}{m!}$. Now treat x as a constant

$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{\frac{x^{m+1}}{(m+1)!}}{\frac{x^m}{m!}}\right| = \frac{|x|}{m+1}$$

$\frac{m!}{(m+1)!} = \frac{1}{m+1}$ $\to 0$ as $m \to \infty$ (for fixed x)

$L = 0 < 1$ series is absolutely convergent for any x.

Second example

$$\sum_{m=0}^{\infty} a_m \qquad a_m = \frac{(-1)^{m}}{2m+1} x^{2m+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

$$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{\frac{x^{2m+3}}{2m+3}}{\frac{x^{2m+1}}{2m+1}} \right|$$

$$= \frac{2m+1}{2m+3} |x|^2$$

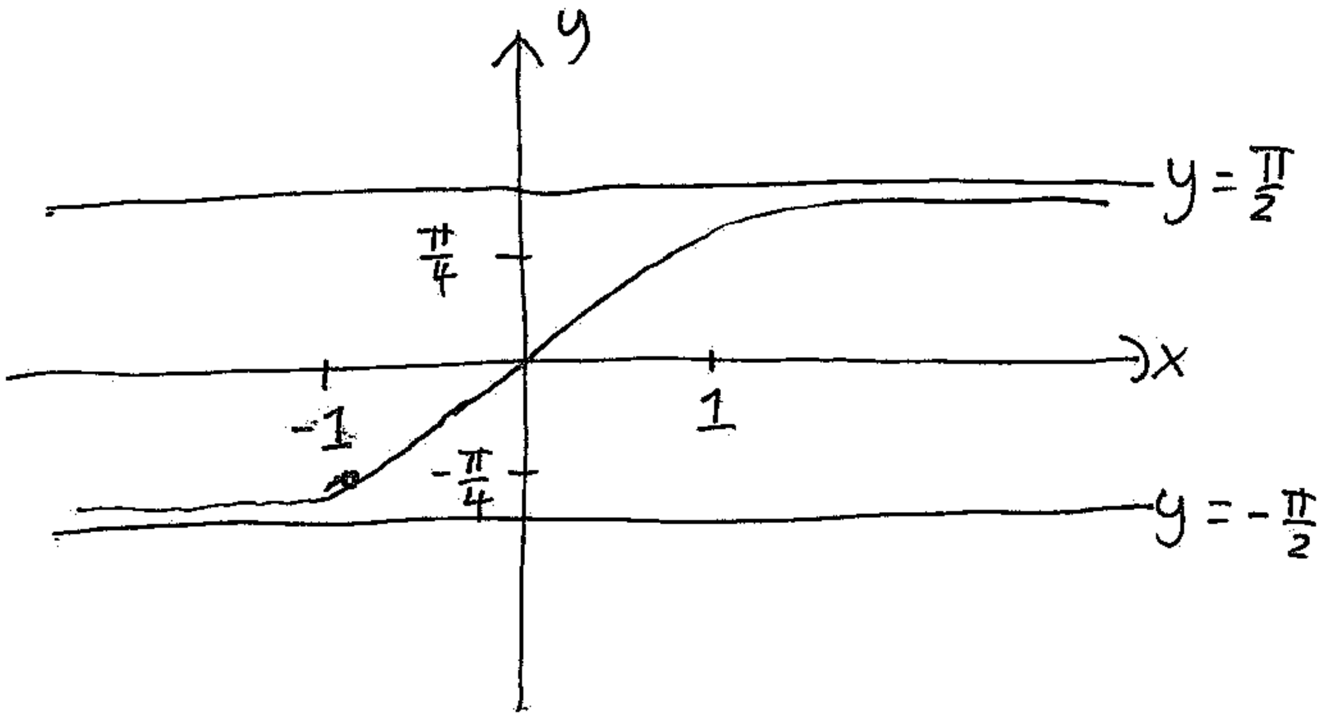
$$\rightarrow |x|^2 \quad \text{as} \quad m \rightarrow \infty$$

$L = |x|^2$ by ratio test

series absoutely convergent

for $|x|^2 < 1$ or $-1 < x < 1$

series diverges if $|x| > 1$
(here $L > 1$)



formula $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$
valid for $|x| < 1$

<u>Claim</u> Any power series $\sum_m c_m x^m$ has one of three properties

(i) converges for all x (entire function)

(ii) converge absolutely for $|x| < R$ and diverges for $|x| > R$ where R is a constant. R is called radius of convergence of power series

(iii) power series diverges for all $x \neq 0$

for case (i) can write $R = \infty$

for case (iii) can write $R = 0$

Example of case (iii)

$$\sum_{m=0}^{\infty} m!\, x^m$$ diverges for all $x \neq 0$

Here $a_m = m!\, x^m$

$$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{(m+1)!\, x^{m+1}}{m!\, x^m} \right|$$

$$= (m+1)\, |x|$$

so limit $m \to \infty$ does not exist (unless $x = 0$)

A power series may or may not converge (absolutely or conditionally) for $x = \pm R$

<u>Examples</u>

$\tan^{-1} x$ series does converge for $x = \pm 1$ (by alternating series test)

$\underline{x=1}$ $\quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$

$\underline{x=-1}$ $\quad -1 + \frac{1}{3} - \frac{1}{5} + \cdots = -\frac{\pi}{4}$

series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

has $R = 1$

this converges (conditionally)

for $x=1$ to $\log 2$

diverges for $x=-1$

Examples

(i) $c_m = \sqrt{m}\, 2^{-m}$

$$\sum_{m=0}^{\infty} \sqrt{m}\, 2^{-m} x^m \qquad \text{Here} \quad a_m = \sqrt{m}\, 2^{-m} x^m$$

$$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{\sqrt{m+1}\, 2^{-(m+1)} x^{m+1}}{\sqrt{m}\, 2^{-m} x^m} \right|$$

$$= \sqrt{\frac{m+1}{m}}\, \frac{1}{2} |x| \to \frac{1}{2}|x|$$

as $m \to \infty$

series converges absolutely $\frac{1}{2}|x| < 1$
diverges $\frac{1}{2}|x| > 1$

$R = 2$ (abs conv for $|x|<2$
div for $|x|>2$)

behaviour at $x=2$ and $x=-2$?

(ii) $x^3 + 2x^6 + 3x^9 + \dots$

$a_m = m x^{3m}$ $m \geq 1$

$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{(m+1)\, x^{3(m+1)}}{m\, x^{3m}}\right|$$

$$= \frac{m+1}{m} |x|^3 \to |x|^3$$

as $m \to \infty$

$R = 1$

(iii) $\sum_{p \text{ prime}} x^p = x^2 + x^3 + x^5 + x^7 + x^{11} + x^{13} + \cdots$

What is R for this power series?

Further Claim

A power series can be differentiated term by term within its inverval of convergence.

Suppose the power series $f(x) = \sum_{m=0}^{\infty} c_m x^m$ has radius of convergence R

then f is differentiable for $|x| < R$ and

$$f'(x) = \sum_{m=1}^{\infty} m c_m x^{m-1}$$

The power series for $f(x)$ and $f'(x)$ have <u>same</u> radius of convergence.

<u>Example</u>

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

valid for $|x|<1$

that is $R = 1$

Differentiating

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \dots$$

valid for $|x| < 1$

that is $R = 1$

Finite Power Series

Have considered infinite power series of form $\sum_{m=0}^{\infty} c_m x^m$. This can be viewed as a polynomial of infinite degree. A polynomial of degree n can be written as

$$\sum_{m=0}^{n} c_m x^m$$ degree n if $c_n \neq 0$

$c_0, c_1, c_2, \ldots, c_n$ are coefficients of the polynomial

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

Set $x = 0$ $\quad f(0) = c_0$

Differentiate formula

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1}$$

Set $x=0$ $\quad f'(0) = c_1$

Repeating process (differentiate and set $x=0$)

$f''(0) = 2c_2$

Continuing $f^{(m)}(0) = m!\, c_m$

or $$\boxed{c_m = \frac{f^{(m)}(0)}{m!}}$$

Therefore a polynomial of degree n satisfies the 'reconstruction formula'

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(0)}{m!} x^m$$

Formula is false if f is not a polynomial of degree n (or lower)

If f not a polynomial of degree n or lower

then

$$S(x) = \sum_{m=0}^{n} \frac{f^{(m)}(0)}{m!} x^m$$

is a polynomial approximation to $f(x)$. Approximation 'works' in neighbourhood of $x = 0$

$f(x)$, $S(x)$ and ~~its~~ their first n derivatives agree at $x = 0$

$f(0) = S(0)$, $f^{(1)}(0) = S^{(1)}(0)$, $\ldots$ $f^{(n)}(0) = S^{(n)}(0)$.

$S(x)$ called a Maclaurin series. Can shift point at which approximation works from $x=0$ to another point $x=a$:

$$S(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m$$

This is called a Taylor Series.

<u>Example</u> $f(x) = \sin x$, $a=0$ (Maclaurin series) $n=4$

$$f^{(1)}(x) = \cos x, \quad f^{(2)}(x) = -\sin x$$

$$f^{(3)}(x) = -\cos x, \quad f^{(4)} = \sin x$$

$$S(x) = \sum_{m=0}^{4} \frac{f^{(m)}(0)}{m!} x^m$$

$$= 0 + 1x + 0 - \frac{1}{3!}x^3 + 0$$

$$= x - \frac{x^3}{3!}$$

This approximatos $\sin x$ near $x = 0$. How good is such an approximation?

Returning to general case

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m + R_m(x)$$

where $R_n(x)$ is the error or remainder term.

There are a number of formulas for the remainder term $R_n(x)$. Here are three!

~~(i) Lagrange form~~

$$\cancel{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$ ~~where c is between a and x~~

(i) Lagrange form

$$R_n(x) = \frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!}$$

c between a and x

(ii) Cauchy form

$$R_n(x) = \frac{f^{(n+1)}(c)\,(x-c)^{n}\,(x-a)}{n!}$$

c between a and x

(iii) Integral form

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt$$

Application of Lagrange form of error.

Returning to previous example

$f(x) = \sin x$, $a = 0$ (Maclaurin)

$n = 4$

$$\sin x = x - \frac{x^3}{3!} + R_4(x)$$

Can write

$$R_4(x) = \frac{f^{(5)}(c)}{5!}(x-0)^5$$

c is between 0 and x

Here $f^{(5)}(x) = \cos x$

$$R_4(x) = \frac{\cos(c)}{5!} x^5$$

As $|\cos(c)| \leq 1$

$$|R_4(x)| \leq \frac{|x|^5}{5!}$$

can be written as

$$\left| \sin x - x + \frac{x^3}{6} \right| \leq \frac{|x|^5}{5!}$$

Indicates approximation is good for moderate angles (say ≈ 1 radian) as well as small angles

Another Example

$f(x) = e^x, \quad a = 0, \quad n = 3$

$f^{(n)}(x) = e^x$

approximation

$$S(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^x = S(x) + R_3(x)$$

Lagrange $\quad R_3(x) = \dfrac{f^{(4)}(c)}{4!} x^4$

$f^{(4)}(x) = e^x, \quad = \dfrac{e^c}{4!} x^4$

c between 0 and x

$$\left| e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} \right| = \frac{e^c}{4!} x^4$$

If x negative c is negative $e^c < 1$

For negative x

$$\left| e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} \right| < \frac{x^4}{4!}$$