

# Year 1 — Foundation of Analysis

Based on lectures by Kevin Buzzard

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

**M1F**

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## **0 Introduction**

In this course...

# 1 Propositions, Sets and Numbers

The propositions are like easy logic, and then a few sets and number concept will be discussed.

## 1.1 Propositions

**Definition** (Proposition). A *proposition* is a **True** or **False** statement.

**Example.**

- $2 + 2 = 4$
- $2 + 2 = 100000000$
- Fermat's Last Theorem
- Riemann Hypothesis

There are some propositions that we don't know they are true or false, like Riemann hypothesis. However, in *classical mathematics*, mathematics of M1F, **every** proposition is either true or not. We are just not sure about some of them.

There are also some examples of things which are **not** propositions:

**Example.**

- $2 + 2$
- $2 = 2 = 4$

The first example is a number, but not proposition. It is not 'true' or 'false', it is 4. The second example doesn't even make sense. It is not a mathematical object.

## 1.2 Notation of proposition

There are few connectives between propositions, they are **and**, **or**, **not**, **implies**, **if and only if**

**Definition** (And). If  $P$  and  $Q$  are propositions, " $P$  and  $Q$ " is a proposition and can be written as  $P \wedge Q$ .  $P \wedge Q$  are true when *both*  $P$  and  $Q$  are true.

We can see the relation of  $P \wedge Q$ ,  $P$ , and  $Q$  by the truth table.

$P$	$Q$	$P \wedge Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$F$

**Example.**  $(2 + 2 = 4) \wedge (2 + 2 = 5)$  is false, since  $2 + 2 = 5$  is false.

**Definition** (Or). If  $P$  and  $Q$  are propositions, " $P$  or  $Q$ " is a proposition and can be written as  $P \vee Q$ .  $P \vee Q$  are true when *either*  $P$ ,  $Q$  or *both* are true.

We can see the relation of  $P \vee Q$ ,  $P$ , and  $Q$  by the truth table.

$P$	$Q$	$P \vee Q$
$T$	$T$	$T$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

**Example.**  $(2 + 2 = 4) \vee (2 + 2 = 5)$  is false, since  $2 + 2 = 4$  is true.

**Definition (Not).** If  $P$  is proposition, "not  $P$ " is a proposition and can be written as  $\neg P$ .  $\neg P$  is the proposition which is "the opposite of  $P$ ". If  $P$  is true then  $\neg P$  is false, and if  $P$  is false then  $\neg P$  is true.

We can see the relation of  $\neg P$  and  $P$  by the truth table.

$P$	$\neg P$
$T$	$F$
$F$	$T$

**Example.** Let  $P$  be the Riemann hypothesis, then  $P \vee \neg P$  is true, because in classical mathematics, the Riemann hypothesis is either true or false.

**Definition (Implies).** If  $P$  and  $Q$  are propositions, " $P$  implies  $Q$ " is a proposition and can be written as  $P \implies Q$ .  $P \implies Q$  means if  $P$  is true, then  $Q$  is true as well.

We can see the relation of  $P \implies Q$ ,  $P$ , and  $Q$  by the truth table.

$P$	$Q$	$P \implies Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

The only time that  $P \implies Q$  is false is when  $P$  is true and  $Q$  is false.

**Example.**  $(2 + 2 = 4) \implies (2 + 2 = 5)$  is false, but  $(2 + 2 = 5) \implies (2 + 2 = 4)$  is true.

**Notation.**  $Q \iff P$  is defined to be  $P \implies Q$ .

**Definition (if and only if).** If  $P$  and  $Q$  are propositions, " $P$  if and only if  $Q$ " is a proposition and can be written as  $P \iff Q$ .  $P \iff Q$  is true when  $P$  and  $Q$  have the *same* truth value.

We can see the relation of  $P \iff Q$ ,  $P$ , and  $Q$  by the truth table.

$P$	$Q$	$P \iff Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$T$

$\iff$  is the proposition version of  $=$  for numbers. If  $x$  and  $y$  are equal numbers, we write  $x = y$ , but if  $P$  and  $Q$  are propositions with the same truth value, we write  $P \iff Q$ .

**Example.**

- $(P \implies Q) \iff (Q \impliedby P)$  is always true.
- $P \iff (\neg P)$  is always false.

### 1.3 Theorem of propositions

**Theorem** (Relation of *not*, *in* and *or*). Let  $P$  and  $Q$  be propositions,

$$(\neg P) \vee (\neg Q) \iff \neg(P \wedge Q)$$

*Proof.* Consider the truth table,

$P$	$Q$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q)$
$T$	$T$	$F \vee F \iff F$	$\neg(T) \iff F$
$T$	$F$	$F \vee T \iff T$	$\neg(F) \iff T$
$F$	$T$	$T \vee F \iff T$	$\neg(F) \iff T$
$F$	$F$	$T \vee T \iff T$	$\neg(F) \iff T$

We can see that the truth values of proposition  $(\neg P) \vee (\neg Q)$  and  $\neg(P \wedge Q)$  are always the same.

$$\therefore (\neg P) \vee (\neg Q) \iff \neg(P \wedge Q)$$

□

### 1.4 Sets

**Definition** (Set). A *set* is a collection of stuff. The things in a set  $X$  are called the *elements* of  $X$ .

Note that there is a more rigorous definition of a set. The more rigorous one depends on which axiomatic foundation using for mathematics. If set theory is the foundation, the definition of a set will be "**Everything is a set.**".

### 1.5 Basic notation for sets.

**Notation.** We use  $\{$  and  $\}$  to denote sets.

**Example.**

- $\{1, 2, 3\}$  is a set.
- $\{ \text{me, you, the desk in my office} \}$  is a set.
- $\{\}$  is a set. It exists, but it has no elements.
- $\{1, 2, 3, 2\}$  is a set.
- $\{1, 2, 3, 4, 5, \dots\}$  is a set, and it is an infinite set.

We use the symbol  $\in$  to denote set membership. If  $a$  is a thing (e.g. a number) and  $X$  is a set, then  $a \in X$  is a proposition. The proposition  $a \in X$  is true exactly when  $a$  is in set  $X$ .

**Example.**

- $2 \in \{1, 2, 3\}$ . This means 2 is an element of set  $\{1, 2, 3\}$ .
- $x \in \{\}$  makes mathematical sense, but it is a false statement.

**Notation.**  $\{\}$  has no elements, which is called the *empty set*. We use  $\emptyset$  to notate an empty set.

## 1.6 Fundamental fact about equality of sets

**Definition** (Equality of sets).

$$X = Y \iff (\forall a \in \Omega, a \in X \iff a \in Y)$$

It means two sets are equal if and only if they have the same elements.

**Example.**  $\{1, 2, 3\}$  and  $\{1, 2, 3, 2\}$  are equal.

Fundamental fact above is the rule for sets. If we need to count things, we can use other things, like multisets, lists, or sequences, instead of sets.

## 1.7 Notation of sets

### 1.7.1 Subsets

**Notation.** We use  $\subseteq$  to denote subsets.  $X \subseteq Y$  is a proposition saying that  $X$  is a subset of  $Y$ .

**Definition** (Subset).

$$X \subseteq Y \iff (\forall a \in \Omega, a \in X \implies a \in Y)$$

It means  $X$  is a subset of  $Y$  when every elements of  $X$  is also an element of  $Y$ .

**Example.**

- $\{1, 2\} \subseteq \{1, 2, 3\}$ , since elements of set  $\{1, 2\}$ , 1 and 2 are both in the set  $\{1, 2, 3\}$ .
- If  $a$  is my left shoe,  $b$  is my right hand, and  $c$  is my mother, then  $\{a, b\} \subseteq \{a, b, c\}$

**Notation.**  $X \supseteq Y$  means  $X \subseteq Y$ .

**Theorem** (Equality and subsets). If  $X$  and  $Y$  are sets, then

$$X = Y \iff (X \subseteq Y \wedge Y \subseteq X)$$

*Proof.* From  $X \subseteq Y$ , we can deduce

$$a \in X \implies a \in Y \quad (1)$$

And from  $Y \subseteq X$ , we can deduce

$$a \in Y \implies a \in X \quad (2)$$

From (1) and (2), we can deduce that

$$a \in Y \iff a \in X$$

which is definition of  $X = Y$

$$\therefore (X \subseteq Y \wedge Y \subseteq X) \implies X = Y \quad (a)$$

Similarly, From  $X = Y$ , we can deduce

$$a \in Y \iff a \in X$$

And it is equivalent to

$$\begin{aligned} a \in Y &\implies a \in X \\ a \in X &\implies a \in Y \end{aligned}$$

which are definition of  $Y \subseteq X$  and  $X \subseteq Y$ .

$$\therefore X = Y \implies (X \subseteq Y \wedge Y \subseteq X) \quad (b)$$

With (a) and (b), we can conclude that,

$$X = Y \iff (X \subseteq Y \wedge Y \subseteq X)$$

□

## 1.8 Important sets

**Example.**

- $\mathbb{Z}$  Integers
- $\mathbb{Q}$  Rational numbers
- $\mathbb{R}$  Real numbers
- $\mathbb{C}$  Complex numbers

**Definition** (Integers  $\mathbb{Z}$ ).

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

There is a problem of inconsistent of natural numbers  $\mathbb{N}$ . Someone defined it as

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Someone defined it as

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

In M1F, we will not use  $\mathbb{N}$ . Instead, we will use the following notations.



**Notation.**

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$$

$$\mathbb{Z}_{\geq 1} = \{1, 2, 3, \dots\}$$

For set  $\mathbb{R}$ , there are some special notations.

**Notation.** Let  $a$  and  $b$  be real numbers,

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \wedge x \leq b\}$$

$$(a, b) = \{x \in \mathbb{R} \mid a < x \wedge x < b\}$$

$$[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$$

### 1.8.1 Universes

**Notation.** We use *universe*  $\Omega$  to denote the set consisting of all the stuff we are interested in.

*Universe* means the set we are considering. For example,  $\Omega$  could be a set of real numbers, or complex numbers. It depends on what we are considering.

### 1.8.2 For all

**Notation.** We use  $\forall$  to say for all in mathematics.

**Example.**  $\forall a \in \mathbb{Z}, 2a$  is even.

This means “For all integers  $a$ ,  $2a$  is an even number”.

### 1.8.3 There exists

**Notation.** We use  $\exists$  to say there exists in mathematics.

**Example.**  $\exists a \in \mathbb{Z}, a$  is even.

This means “There exists an integer  $a$ , which is an even number”.

### 1.8.4 Union

**Definition** (Unions).

$$\forall a \in \Omega, a \in X \cup Y \iff a \in X \vee a \in Y$$

It means the *union* of  $X$  and  $Y$ ,  $X \cup Y$  is all the stuff in either  $X$ , or  $Y$ , or both.

**Example.** Let  $X = \{1, 2, 3\}$  and  $Y = \{3, 4, 5\}$ , then  $X \cup Y = \{1, 2, 3, 4, 5\}$ .

We have some notations for intersection of large numbers of sets. Let us define  $I = \mathbb{Z}_{\geq 1} = \{1, 2, 3, \dots\}$ . For every  $i \in I$ , we have a set of real numbers  $X_i \subseteq \mathbb{R}$ .

**Notation.**

$$\bigcup_{i=1}^{\infty} X_i = \{a \in \Omega \mid \exists i \in \mathbb{Z}_{\geq 1}, a \in X_i\}$$

$$\bigcup_{i \in I} X_i = \{a \in \Omega \mid \exists i \in I, a \in X_i\}$$

**Example.** Let  $I = \mathbb{R}$ . If  $i \in I$ , and let  $X_i = \{i\}$ . What is  $\bigcup_{i \in I} X_i$ ?

$$\begin{aligned} \bigcup_{i \in I} X_i &= \{a \in \mathbb{R} \mid \exists i \in I, a \in X_i\} \\ \therefore \bigcup_{i \in I} X_i &\subseteq \mathbb{R} \end{aligned} \tag{1}$$

Let  $a \in \mathbb{R}$ ,

$$a \in X_a = \{a\} \quad (\text{by definition})$$

$\therefore \exists i \in I = \mathbb{R}$  such that  $a \in X_i = \{i\}$  when  $i = a$ .

$$\begin{aligned} \therefore \mathbb{R} &\subseteq \bigcup_{i \in I} X_i \\ \bigcup_{i \in I} X_i &= \mathbb{R} \end{aligned} \tag{2}$$

### 1.8.5 Intersection

**Definition** (Intersection).

$$\forall a \in \Omega, a \in X \cap Y \iff a \in X \wedge a \in Y$$

It means the *intersection* of  $X$  and  $Y$ ,  $X \cap Y$  is all the stuff in *both*  $X$ , *and*  $Y$ .

**Example.** Let  $X = \{1, 2, 3\}$  and  $Y = \{3, 4, 5\}$ , then  $X \cap Y = \{3\}$ .

We have some notations for intersection of large numbers of sets. Let us define  $I = \mathbb{Z}_{\geq 1} = \{1, 2, 3, \dots\}$ . For every  $i \in I$ , we have a set of real numbers  $X_i \subseteq \mathbb{R}$ .

**Notation.**

$$\begin{aligned} \bigcap_{i=1}^{\infty} X_i &= \{a \in \Omega \mid \forall i \in \mathbb{Z}_{\geq 1}, a \in X_i\} \\ \bigcap_{i \in I} X_i &= \{a \in \Omega \mid \forall i \in I, a \in X_i\} \end{aligned}$$

**Example.** What is  $\bigcap_{i=1}^{\infty} X_i$ , where  $X_i = [-i, i]$ ?

$\therefore X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ , real numbers in all the  $X_i$  are the real numbers in  $X_1$ .  
 $\therefore \bigcap_{i=1}^{\infty} X_i = X_1$

### 1.8.6 Complements

**Definition** (Complements).

$$\forall a \in \Omega, a \in X^c \iff \neg(a \in X)$$

It means if  $X$  is a subset of  $\Omega$ , then its *complement*  $X^c$  is the set whose elements are all the things in  $\Omega$  which are not in  $X$ .

**Example.** If our universe  $\Omega$  is  $\mathbb{Z}$ , the integers, and if  $X$  is the set of even integers, then its *complement*  $X^c$  is the set of odd numbers.

**Notation.**  $a \notin X$  is defined to be  $\neg(a \in X)$ , since  $a$  is not an element of  $X$  is also a proposition.

### 1.9 Notation of sets with certain property

Let  $X$  be the set of *integers*, and we want to consider the subset of  $X$  consisting of positive integers. We can write the subset as:

$$\{a \in X \mid a > 0\}$$

The line in the middle is pronounced “such that”. So the full statement can be read as “the elements  $a$  of  $X$  such that  $a > 0$ ”.

### 1.10 Theorem of sets

**Theorem** (A theorem of complement). Let  $X$  and  $Y$  be sets.

If  $X, Y \subseteq \Omega$ ,

$$(X \cup Y = \Omega) \wedge (X \cap Y = \emptyset) \implies X = Y^c$$

*Proof.* Let  $a \in \Omega$ ,  $P$  be proposition  $a \in X$ ,  $Q$  be proposition  $a \in Y$ ,

$$\begin{aligned} a \in X \cup Y &\iff (a \in X) \vee (a \in Y) && \text{(Union definition)} \\ \therefore a \in X \cup Y &\iff P \vee Q \\ \therefore X \cup Y = \Omega & \\ \therefore a \in X \cup Y &\iff \top \\ &P \vee Q \iff \top \\ a \in X \cap Y &\iff (a \in X) \wedge (a \in Y) && \text{(Intersection definition)} \\ \therefore a \in X \cap Y &\iff P \wedge Q \\ \therefore X \cap Y = \emptyset & \\ \therefore a \in X \cap Y &\iff \perp \\ &P \wedge Q \iff \perp \\ \neg(P \wedge Q) &\iff \top \\ \therefore P \vee Q &\iff \neg(P \wedge Q) \\ \therefore P &\iff \neg Q \\ a \in X &\iff \neg(a \in Y) \\ a \in X &\iff a \in Y^c && \text{(Complement definition)} \\ \therefore X &= Y^c \\ (X \cup Y = \Omega) \wedge (X \cap Y = \emptyset) &\implies X = Y^c \end{aligned}$$

□

Let  $S = \{a \in \mathbb{R} \mid a > 0\}$

**Proposition** ( $S$  has a smallest element).

$$P := \exists s \in S, \forall t \in S, s \leq t$$

*Proof.* Consider  $\neg P$ ,

$$\neg P = \forall s \in S, \exists t \in S, s > t$$

Let  $s \in S$ ,

$\frac{s}{2}$  will also be a real number, and it is smaller than  $s$ .

$\therefore \neg P$  is true, and so  $P$  is a false proposition.

Hence,  $S$  does not have a smallest statement.

□

### 1.11 Some Proof Examples

**Lemma 1.1.** If  $x$  is an integer, and  $x^2$  is even, then  $x$  is even.

*Proof.* Assume  $x$  is an integer and  $x^2$  is even.

Assume for contradiction that  $x$  is odd.

Then,  $x = 2t + 1$ , so  $x^2 = 4t^2 + 4t + 1$ .

$x^2 = 2(2t^2 + 2t) + 1$ , which is an odd number.

However, we assumed that  $x^2$  is even at the beginning, so contradiction occurs.

( $\Rightarrow \Leftarrow$ )

Hence, the assumption that  $x$  is odd must be wrong, so  $x$  should be even.  $\square$

**Lemma 1.2.**  $\sqrt{2}$  is irrational.

*Proof.* Assume for a contradiction that  $\sqrt{2}$  is rational.

Write  $\sqrt{2} = \frac{a}{b}$ , with  $a, b \in \mathbb{Z}_{\geq 1}$ , and at least one of them is odd.

By squaring both sides, we can deduce

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

It shows that  $a^2$  is an even number. By *square is even*

$\square$