Office Hours

Tue 2-3 Thu 9-10

If singularity is inside range of integration split integral into 2 (or more) improper integrals

For example

$$\int_{-1}^{1} |x|^{-\frac{1}{2}}\,dx$$

Split into 2

$$\int_{-1}^{0} |x|^{-\frac{1}{2}}\,dx + \int_{0}^{1} |x|^{-\frac{1}{2}}\,dx$$

Both improper integrals finite. Another example

$$\int_{-1}^{1} \frac{dx}{x}$$

$$= \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x}$$

both undefined

Similarly for integrals over $\mathbb{R}$ (with both $+\infty$ and $-\infty$ limits)

Split integral into 2 or more improper integrals

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{7} e^{-x^2}\,dx + \int_{7}^{\infty} e^{-x^2}\,dx$$

How to decide if an improper integral is well defined?

(a) Compute integral!

(b) Examine integrand $f(x)$ near singularities or as $x \to \pm\infty$ – decide if singularity is integrable

Following power integrals very useful

$$\int_0^1 x^p \, dx \quad \text{finite if } p > -1$$

$$\int_1^\infty x^p \, dx \quad \text{finite if } p < -1$$

<u>Examples</u>

$$\int_0^{\frac{\pi}{2}} \frac{x}{\cos x} \, dx \qquad \int_0^{\frac{\pi}{2}} \sqrt{\frac{x}{\cos x}} \, dx$$

Approximate integrand near $x = \frac{\pi}{2}$

Near $\quad x = \frac{\pi}{2} \qquad x \approx \frac{\pi}{2}$

$$\cos x = \cos\frac{\pi}{2} - \sin\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) + \cdots$$

(Taylor expansion about $x = \frac{\pi}{2}$)

$$= 0 + \left(\frac{\pi}{2} - x\right) + \cdots$$

$$\frac{x}{\cos x} \approx \frac{\frac{\pi}{2}}{\frac{\pi}{2} - x}$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{\frac{\pi}{2} - x}\, dx \quad \text{undefined}$$

$\dfrac{1}{\frac{\pi}{2} - x}$ cannot be integrated up to $x = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \frac{x}{\cos x}\,dx$$ does not exist

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{x}{\cos x}}\,dx$$ does exist

$$\sqrt{\frac{x}{\cos x}} \approx \sqrt{\frac{\frac{\pi}{2}}{\frac{\pi}{2}-x}}$$

power $-\frac{1}{2}$

<u>Further examples</u>

$$\int_0^2 \log x\,dx$$ finite

singularity at $x=0$

integrable – better behaved

than any negative power of x

$$\int_0^2 \frac{dx}{\log x}$$

$\frac{1}{\log x}$ singular at $x=1$

$$\log x = \log 1 + \frac{1}{1}(x-1) + \cdots$$

$$\simeq x-1 \quad \text{near } x=1$$

$\frac{1}{\log x} \simeq \frac{1}{x-1}$ not integrable at $x=1$

$$-\int_{-\pi}^{\pi} \log\left(\frac{\sin x}{x}\right) dx \quad \text{finite}$$

$$\log\left(\frac{\sin x}{x}\right) \approx \log(\pi - x) - \log \pi$$

$$\sin x \approx \pi - x \quad \text{for } x \approx \pi$$

$$\int_1^{\infty} \tan^{-1}\left(\frac{1}{x}\right) dx$$

undefined

for x large and positive

$$\tan^{-1}\left(\frac{1}{x}\right) \approx \frac{1}{x}$$

not integrable at ∞

Another Example

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

is defined. Use alternating series test!?

define $\quad a_m = \displaystyle\int_{m\pi}^{(m+1)\pi} \frac{\sin x}{x}\,dx$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \sum_{m=0}^{\infty} a_m$$

Can show a_m satisfied all conditions for alternating series test

$$\int_0^\infty \sin x \, dx \qquad \int_0^\infty \sin(x^2) \, dx$$

First integral not defined

$$\int_0^b \sin x \, dx = -\cos b + 1$$

the limit $b \to \infty$ does not exist

2nd integral well defined Why?

Gamma Function

$$\int_0^\infty x^n e^{-x}\, dx = n! \qquad n = 1, 2, 3, 4, 5, \ldots$$

Proof uses induction and integration by parts

Consider

$$p! = \int_0^\infty x^p e^{-x}\, dx$$

where p is not necessarily integer (p can also be complex) eg $\quad (-\frac{1}{2})! = \sqrt{\pi}$

Gamma function

defined

~~at~~ as $\Gamma(p) = (p-1)!$

$$= \int_0^\infty x^{p-1} e^{-x}\, dx$$

has property

$$p\,\Gamma(p) = \Gamma(p+1)$$

generalises

$$(n+1)\,n! = (n+1)!$$

$\Gamma(p)$

Defined as integral

of $f(x) = x^{p-1} e^{-x}$

from $x=0$ to ∞

integrable at $x=\infty$
due to exponential e^{-x}

Near $x=0$ $f(x) \propto x^{p-1}$

requires $p>0$ (so that

$p-1 > -1$)

8 Integration (Extensions and Applications)

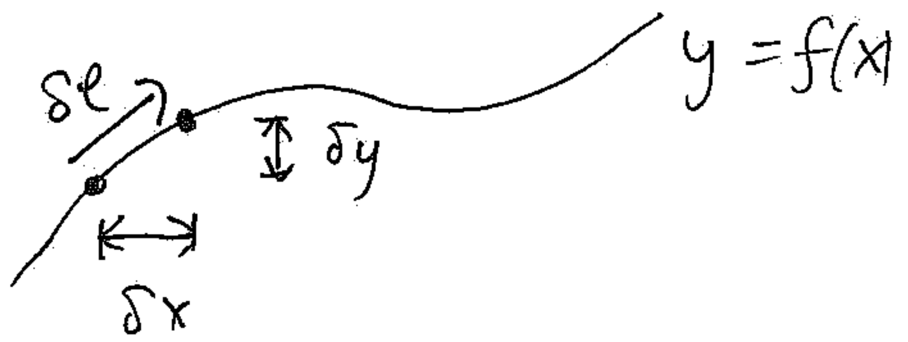
Lengths, Areas and Volumes

$\int_a^b f(x)\,dx$ represents area under graph $y = f(x)$. The length of $y = f(x)$ between $x = a$ and $x = b$ can be written as an integral

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

or

$$L = \int_a^b \sqrt{1 + (y'(x))^2}\, dx$$



Pythagoras $\delta l \approx \sqrt{(\delta x)^2 + (\delta y)^2}$

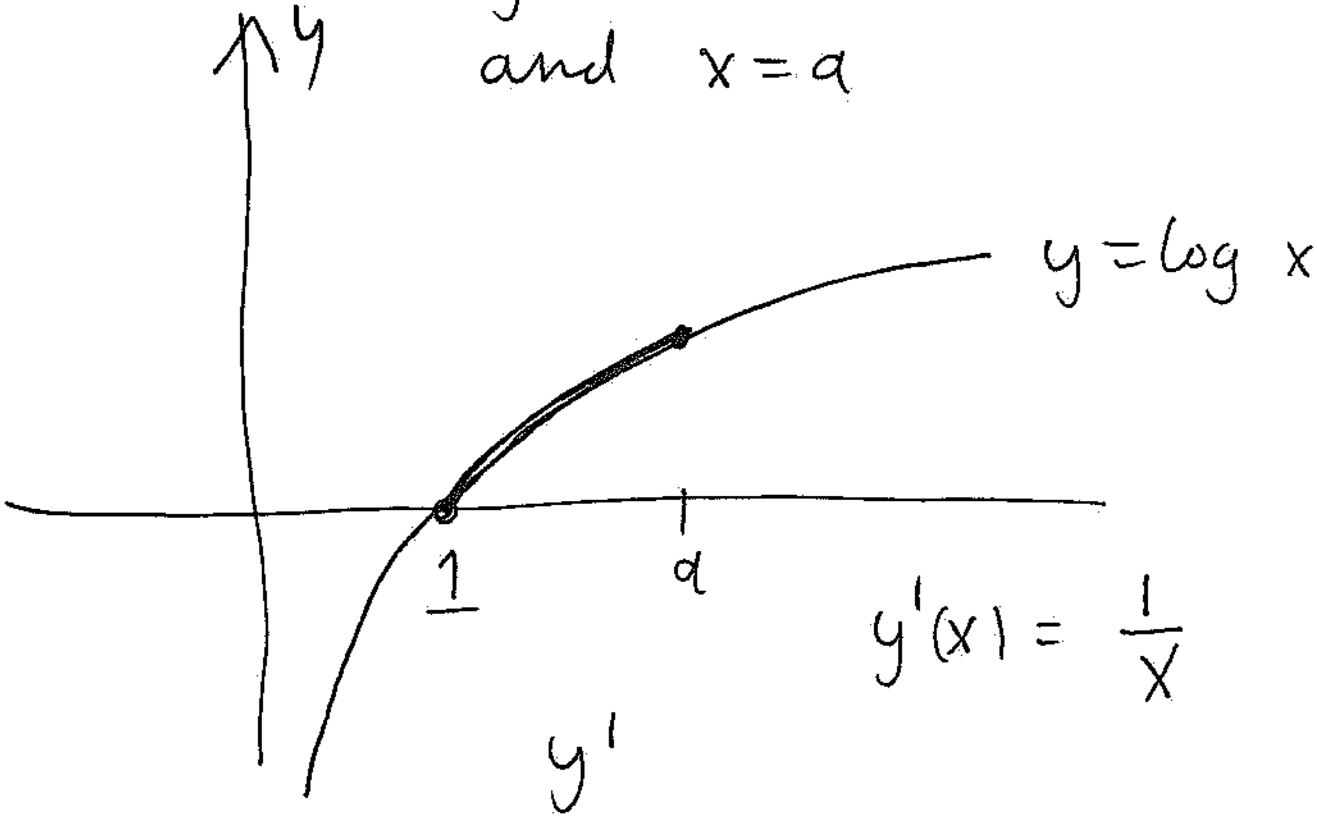
But $\delta y \approx y'(x)\, \delta x$

$\delta l \approx \sqrt{1 + (y'(x))^2}\, \delta x$

Due to square root length calculations can be tricky

<u>Example</u> log curve

length between $x = 1$ and $x = a$



$$y'(x) = \frac{1}{x}$$

y'

$$L = \int_1^a \sqrt{1 + \frac{1}{x^2}}\, dx$$

$$L = \int_1^a \frac{\sqrt{x^2+1}}{x}\, dx$$

Substitute $x = \tan u$

or $x = \sinh u$

$x = \sinh u \qquad dx = \cosh u \, du$

$$\int \frac{\sqrt{x^2+1}}{x}\, dx \qquad \cosh^2 x - \sinh^2 x = 1$$

$$= \int \frac{\sqrt{\sinh^2 u + 1}}{\sinh u} \cosh u \, du$$

$$= \int \frac{\cosh^2 u}{\sinh u}\, du$$

$$= \int \frac{\sinh^2 u + 1}{\sinh u}\, du$$

$$= \int \left(\sinh u + \frac{1}{\sinh u} \right) du$$

$$= \cosh u + \log\left(\tanh \frac{u}{2}\right) + c$$

$$\left[\int \frac{dx}{\sin x} = \log\left(\tan \frac{x}{2}\right) + c \right]$$

$$x = \sinh u , \qquad \cosh u = \sqrt{1+x^2}$$

$$\left[\sin x = \frac{2t}{1+t^2} \qquad t = \tan \frac{x}{2} \right]$$

$$x = \sinh u = \frac{2t}{1-t^2} \qquad t = \tanh \frac{u}{2}$$

$$(1-t^2)\,x = 2t$$

$$+\,x t^2 + 2t - x = 0$$

$$t = \frac{-2 \pm \sqrt{4 + 4x^2}}{2x}$$

$$= \frac{-1 \pm \sqrt{1+x^2}}{x}$$

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \sqrt{1+x^2}$$

$$+ \log\left(\frac{-1+\sqrt{1+x^2}}{x}\right) + c$$

$$\int_1^a \frac{\sqrt{1+x^2}}{x}\,dx = \sqrt{1+a^2} - \sqrt{2}$$

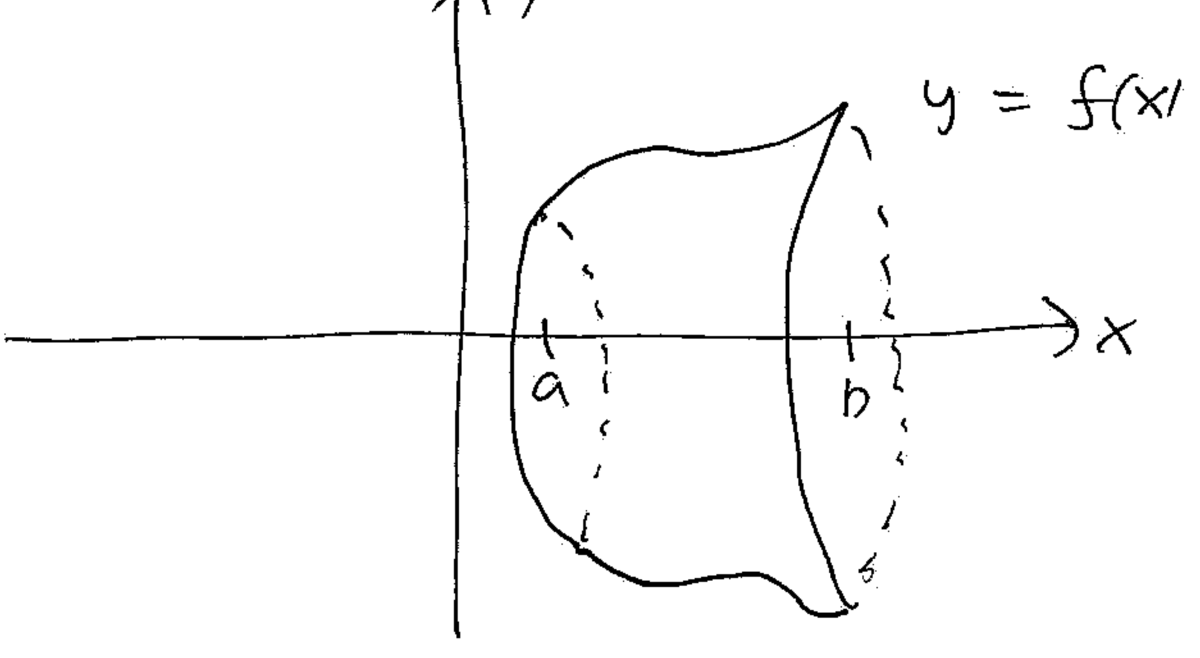
$$+ \log \left(\frac{-1 + \sqrt{1+a^2}}{a} \right)$$

$$- \log \left(-1 + \sqrt{2} \right)$$

Other Formulas

Rotate graph $y = f(x)$

about x-axis



to give a surface of revolution

Area of surface is

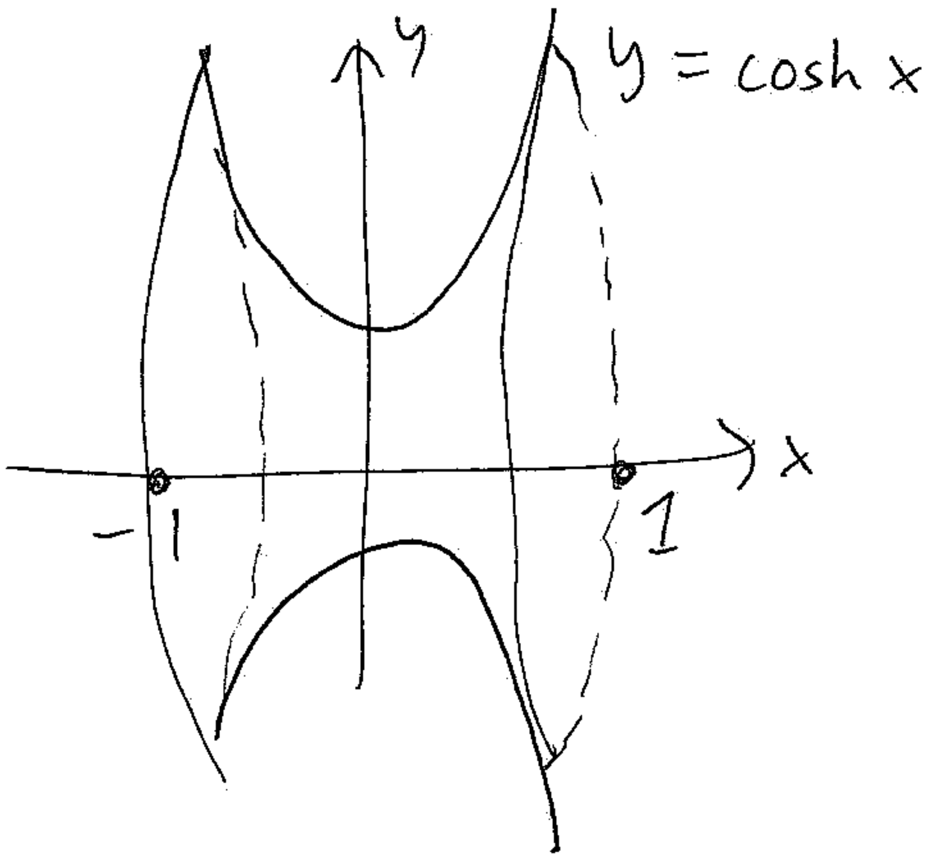
$$A = 2\pi \int_a^b y(x) \sqrt{1 + (y'(x))^2}\, dx$$

Volume enclosed between $x = a$ and $x = b$

$$V = \pi \int_a^b (y(x))^2\, dx$$

Example $y = \cosh x$

a catenary



rotate about x-axis

$\longrightarrow$ catenoid $\quad y' = \sinh x$

$$A = 2\pi \int_{-1}^{1} y \underbrace{\sqrt{1+y'^2}}_{\cosh x}\, dx$$

$$= 2\pi \int_{-1}^{1} \cosh^2 x \, dx$$

see 2016 exam (May)

Other Length Formulas

$$L = \int_a^b \sqrt{1 + (y'(x))^2}\, dx$$

Can also define curves parametrically

write $x = x(t)$, $y = y(t)$

$t_a \leq t \leq t_b$ $\quad$ t = parameter

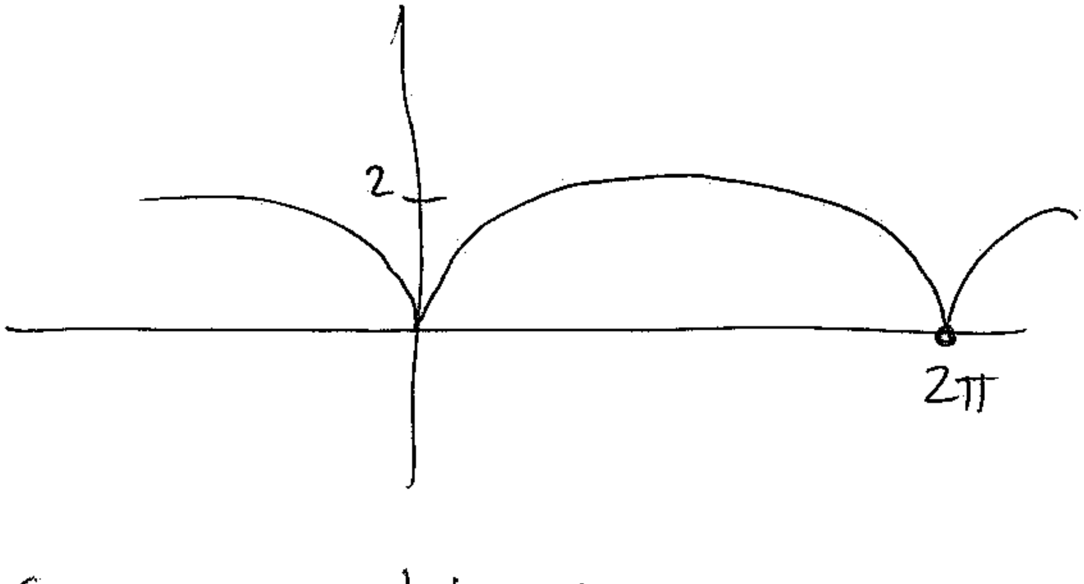
eg. $x(t) = t - \sin t$
$y(t) = 1 - \cos t$ $\quad$ cycloid

length formula

$$L = \int_{t_a}^{t_b} \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\, dt$$

$$\dot{x}(t) = \frac{dx(t)}{dt} \qquad \dot{y}(t) = \frac{dy(t)}{dt}$$

Compute length of cycloid arch $0 \le t \le 2\pi$



See problems