Year 1 — Foundation of Analysis

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First Term 2018

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

M1F

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0 Introduction

In this course...

1 Propositions, Sets and Numbers

The propositions are like easy logic, and then a few sets and number concept will be discussed.

1.1 Propositions

Definition (Proposition)**.** A *proposition* is a **True** or **False** statement.

Example.

- $2 + 2 = 4$
- $2 + 2 = 100000000$
- Fermat's Last Theorm
- Riemann Hypothesis

There are some propositions that we don't know they are true or false, like Riemann hypothesis. However, in *classical mathematics*, mathematics of M1F, **every** proposition is either true or not. We are just not sure about some of them.

There are also some examples of things which are **not** propositions:

Example.

- $2 + 2$
- $2 = 2 = 4$

The first example is a number, but not proposition. It is not 'true' or 'false', it is 4. The second example doesn't even make sense. It is not a mathematical object.

1.2 Notation of proposition

There are few connectives between propositions, they are **and**, **or**, **not**, **implies**, **if and only if**

Definition (And)**.** If P and Q are propositions, "P *and* Q" is a proposition and can be written as $P \wedge Q$. $P \wedge Q$ are true when *both* P and Q are true.

We can see the relation of $P \wedge Q$, P, and Q by the truth table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example. $(2 + 2 = 4) \wedge (2 + 2 = 5)$ is false, since $2 + 2 = 5$ is false.

Definition (Or)**.** If P and Q are propositions, "P *or* Q" is a proposition and can be written as $P \vee Q$. $P \vee Q$ are true when *either* P, Q or *both* are true.

We can see the relation of $P \vee Q$, P, and Q by the truth table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example. $(2+2=4) \vee (2+2=5)$ is false, since $2+2=4$ is true.

Definition (Not)**.** If P is proposition, "not P" is a proposition and can be written as $\neg P$. $\neg P$ is the proposition which is "the opposite of P". If P is true then $\neg P$ is false, and if P is false then $\neg P$ is true.

We can see the relation of $\neg P$ and P by the truth table.

P	$\neg P$
T	F
F	T

Example. Let P be the Riemann hypothesis, then $P \vee \neg P$ is true, because in classical mathematics, the Riemann hypothesis is either true or false.

Definition (Implies)**.** If P and Q are propositions, "P *implies* Q" is a proposition and can be written as $P \implies Q$. $P \implies Q$ means if P is true, then Q is true as well.

We can see the relation of $P \implies Q$, P, and Q by the truth table.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

The only time that $P \implies Q$ is false is when P is true and Q is false.

Example. $(2+2=4) \implies (2+2=5)$ is false, but $(2+2=5) \implies (2+2=4)$ is true.

Notation. $Q \impliedby P$ is defined to be $P \implies Q$.

Definition (if and only if)**.** If P and Q are propositions, "P *if and only if* Q" is a proposition and can be written as $P \iff Q$. $P \iff Q$ is true when P and Q have the *same* truth value.

We can see the relation of $P \implies Q$, P, and Q by the truth table.

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

$\iff$ is the proposition version of $=$ for numbers. If x and y are equal numbers, we write $x = y$, but if P and Q are propositions with the same truth value, we write $P \iff Q$.

Example.

- $(P \implies Q) \iff (Q \impliedby P)$ is always true.
- $P \iff (\neg P)$ is always false.

1.3 Theorm of propositions

Theorem (Relation of *not, in* and *or*)**.** Let P and Q be propositions,

$$(\neg P) \vee (\neg Q) \iff \neg(P \wedge Q)$$

Proof. Consider the truth table,

P	Q	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q)$
T	T	$F \vee F \iff F$	$\neg(T) \iff F$
T	F	$F \vee T \iff T$	$\neg(F) \iff T$
F	T	$T \vee F \iff T$	$\neg(F) \iff T$
F	F	$T \vee T \iff T$	$\neg(F) \iff T$

We can see that the truth values of proposition $(\neg P) \vee (\neg Q)$ and $\neg(P \wedge Q)$ are always the same.

$$\therefore (\neg P) \vee (\neg Q) \iff \neg(P \wedge Q)$$

□

1.4 Sets

Definition (Set)**.** A *set* is a collection of stuff. The things in a set X are called the *elements* of X.

Note that there is a more rigorous definition of a set. The more rigorous one depends on which axiomatic foundation using for mathematics. If set theory is the foundation, the definition of a set will be "**Everything is a set.**".

1.5 Basic notation for sets.

Notation. We use { and } to denote sets.

Example.

- $\{1, 2, 3\}$ is a set.
- { me, you, the desk in my office } is a set.
- $\{\}$ is a set. It exists, but it has no elements.
- $\{1, 2, 3, 2\}$ is a set.
- $\{1, 2, 3, 4, 5, ...\}$ is a set, and it is an infinite set.

We use the symbol $\in$ to denote set membership. If a is a thing (e.g. a number) and X is a set, then $a \in X$ is a proposition. The proposition $a \in X$ is true exactly when a is in set X.

Example.

- $2 \in \{1, 2, 3\}$. This means 2 is an element of set $\{1, 2, 3\}$.
- $x \in \{\}$ makes mathematical sense, but it is a false statement.

Notation. $\{\}$ has no elements, which is called the *empty set*. We use $\varnothing$ to notate an empty set.

1.6 Fundamental fact about equality of sets

Definition (Equality of sets)**.**

$$X = Y \iff (\forall a \in \Omega, a \in X \iff a \in Y)$$

It means two sets are equal if and only if they have the same elements.

Example. $\{1, 2, 3\}$ and $\{1, 2, 3, 2\}$ are equal.

Fundamental fact above is the rule for sets. If we need to count things, we can use other things, like multisets, lists, or sequences, instead of sets.

1.7 Notation of sets

1.7.1 Subsets

Notation. We use $\subseteq$ to denote subsets. $X \subseteq Y$ is a proposition saying that X is a subset of Y.

Definition (Subset)**.**

$$X \subseteq Y \iff (\forall a \in \Omega, a \in X \implies a \in Y)$$

It means X is a subset of Y when every elements of X is also an element of Y.

Example.

- $\{1, 2\} \subseteq \{1, 2, 3\}$, since elements of set $\{1, 2\}$, 1 and 2 are both inthe set $\{1, 2, 3\}$.
- If a is my left shoe, b is my right hand, and c is my mother, then $\{a, b\} \subseteq \{a, b, c\}$

Notation. $X \supseteq Y$ means $X \subseteq Y$.

Theorem (Equality and subsets)**.** If X and Y are sets, then

$$X = Y \iff (X \subseteq Y \land Y \subseteq X)$$

Proof. From $X \subseteq Y$, we can deduce

$$a \in X \implies a \in Y \tag{1}$$

And from $Y \subseteq X$, we can deduce

$$a \in Y \implies a \in X \tag{2}$$

From (1) and (2), we can deduce that

$$a \in Y \iff a \in X$$

which is definition of $X = Y$

$$\therefore (X \subseteq Y \land Y \subseteq X) \implies X = Y \tag{a}$$

Similarly, From $X = Y$, we can deduce

$$a \in Y \iff a \in X$$

And it is equivalent to

$$a \in Y \implies a \in X$$
$$a \in X \implies a \in Y$$

which are definition of $Y \subseteq X$ and $X \subseteq Y$.

$$\therefore X = Y \implies (X \subseteq Y \land Y \subseteq X) \tag{b}$$

With (a) and (b), we can conclude that,

$$X = Y \iff (X \subseteq Y \land Y \subseteq X)$$

$\square$

1.8 Important sets

Example.

- $\mathbb{Z}$ Integers
- $\mathbb{Q}$ Rational numbers
- $\mathbb{R}$ Real numbers
- $\mathbb{C}$ Complex numbers

Definition (Integers $\mathbb{Z}$)**.**

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

There is a problem of unconsistent of natural numbers $\mathbb{N}$. Someone defined it as

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$

Someone defined it as

$$\mathbb{N} = \{1, 2, 3, ...\}$$

In M1F, we will not use $\mathbb{N}$. Instead, we will use the following notations.

Notation.

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, ...\}$$
$$\mathbb{Z}_{\geq 1} = \{1, 2, 3, ...\}$$

For set $\mathbb{R}$, there are some special notations.

Notation. Let a and b be real numbers,

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \wedge x \leq b\}$$
$$(a, b) = \{x \in \mathbb{R} \mid a < x \wedge x < b\}$$
$$[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$$

1.8.1 Universes

Notation. We use *universe* Ω to denote the set consisting of all the stuff we are interested in.

Universe means the set we are considering. For example, Ω could be a set of real numbers, or complex numbers. It depends on what we are considering.

1.8.2 For all

Notation. We use $\forall$ to say for all in mathematics.

Example. $\forall a \in \mathbb{Z}, 2a$ is even.
This means "For all integers a, $2a$ is an even number".

1.8.3 There exists

Notation. We use $\exists$ to say there exists inn mathematics.

Example. $\exists a \in \mathbb{Z}, a$ is even.
This means "There exists an integer a, which is an even number".

1.8.4 Union

Definition (Unions)**.**

$$\forall a \in \Omega, a \in X \cup Y \iff a \in X \vee a \in Y$$

It means the *union* of X and Y, $X \cup Y$ is all the stuff in either X, *or* Y, *or both*.

Example. Let $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$,
then $X \cup Y = \{1, 2, 3, 4, 5\}$.

We have some notations for intersection of large numbers of sets.
Let us define $I = \mathbb{Z}_{\geq 1} = \{1, 2, 3, ...\}$. For every $i \in I$, we have a set of real numbers $X_i \subseteq \mathbb{R}$.

Notation.

$$\bigcup_{i=1}^{\infty} X_i = \{a \in \Omega \mid \exists i \in \mathbb{Z}_{\geq 1}, a \in X_i\}$$
$$\bigcup_{i \in I} X_i = \{a \in \Omega \mid \exists i \in I, a \in X_i\}$$

Example. Let $I = \mathbb{R}$. If $i \in I$, and let $X_i = \{i\}$. What is $\bigcup_{i \in I} X_i$?

$$\bigcup_{i \in I} X_i = \{a \in \mathbb{R} \mid \exists i \in I, a \in X_i\}$$

$$\therefore \bigcup_{i \in I} X_i \subseteq \mathbb{R} \tag{1}$$

Let $a \in \mathbb{R}$,

$$a \in X_a = \{a\} \qquad \text{(by definition)}$$

$\therefore \exists i \in I = \mathbb{R}$ such that $a \in X_i = \{i\}$ when $i = a$.

$$\therefore \mathbb{R} \subseteq \bigcup_{i \in I} X_i \tag{2}$$

$$\bigcup_{i \in I} X_i = \mathbb{R}$$

1.8.5 Intersection

Definition (Intersection)**.**

$$\forall a \in \Omega, a \in X \cap Y \iff a \in X \land a \in Y$$

It means the *intersection* of X and Y, $X \cap Y$ is all the stuff in *both* X, *and* Y.

Example. Let $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$,
then $X \cap Y = \{3\}$.

We have some notations for intersection of large numbers of sets.
Let us define $I = \mathbb{Z}_{\geq 1} = \{1, 2, 3, ...\}$. For every $i \in I$, we have a set of real numbers $X_i \subseteq \mathbb{R}$.

Notation.

$$\bigcap_{i=1}^{\infty} X_i = \{a \in \Omega \mid \forall i \in \mathbb{Z}_{\geq 1}, a \in X_i\}$$

$$\bigcap_{i \in I} X_i = \{a \in \Omega \mid \forall i \in I, a \in X_i\}$$

Example. What is $\bigcap_{i=1}^{\infty} X_i$, where $X_i = [-i, i]$?
$\because X_1 \subseteq X_2 \subseteq X_3 \subseteq ...$, real numbers in all the X_i are the real numbers in X_1.
$\therefore \bigcap_{i=1}^{\infty} X_i = X_1$

1.8.6 Complements

Definition (Complements)**.**

$$\forall a \in \Omega, a \in X^c \iff \neg(a \in X)$$

It means if X is a subset of Ω, then its *complement* X^c is the set whose elements are all the things in Ω which are not in X.

Example. If our universe Ω is $\mathbb{Z}$, the integers, and if X is the set of even integers, then its *complement* X^c is the set of odd numbers.

Notation. $a \notin X$ is defined to be $\neg(a \in X)$, since a is not an element of X is also a proposition.

1.9 Notation of sets with certain property

Let X be the set of *integers*, and we want to consider the subset of X consisting of positive integers. We can write the subset as:

$$\{a \in X \mid a > 0\}$$

The line in the middle is pronounced "'such that". So the full statement can be read as "the elements a of X such that $a > 0$".

1.10 Theorm of sets

Theorem (A theorm of complement)**.** Let X and Y be sets.
If $X, Y \subseteq \Omega$,

$$(X \cup Y = \Omega) \wedge (X \cap Y = \varnothing) \implies X = Y^c$$

Proof. Let $a \in \Omega$, P be proposition $a \in X$, Q be proposition $a \in Y$,

$$\begin{aligned}
& a \in X \cup Y \iff (a \in X) \vee (a \in Y) && \text{(Union definition)} \\
\therefore\quad & a \in X \cup Y \iff P \vee Q \\
\because\quad & X \cup Y = \Omega \\
\therefore\quad & a \in X \cup Y \iff \top \\
& P \vee Q \iff \top \\
& a \in X \cap Y \iff (a \in X) \wedge (a \in Y) && \text{(Intersection definition)} \\
\therefore\quad & a \in X \cup Y \iff P \wedge Q \\
\because\quad & X \cup Y = \varnothing \\
\therefore\quad & a \in X \cup Y \iff \bot \\
& P \wedge Q \iff \bot \\
& \neg(P \wedge Q) \iff \top \\
\therefore\quad & P \vee Q \iff \neg(P \wedge Q) \\
\therefore\quad & P \iff \neg Q \\
& a \in X \iff \neg(a \in Y) \\
& a \in X \iff a \in Y^c && \text{(Complement definition)} \\
\therefore\quad & X = Y^c
\end{aligned}$$

$$(X \cup Y = \Omega) \wedge (X \cap Y = \varnothing) \implies X = Y^c$$

□

Let $S = \{a \in \mathbb{R} \mid a > 0\}$

Proposition (S has a smallest element)**.**

$$P := \exists s \in S, \forall t \in S, s \leq t$$

Proof. Consider $\neg P$,

$$\neg P = \forall s \in S, \exists t \in S, s > t$$

Let $s \in S$,
$\frac{s}{2}$ will also be a real number, and it is smaller than s.
$\therefore \neg P$ is true, and so P is a false proposition.
Hence, S does not have a smallest statement. □

1.11 Some Proof Examples

Lemma 1.1. If x is an integer, and x^2 is even, then x is even.

Proof. Assume x is an integer and x^2 is even.
Assume for contradiction that x is odd.
Then, $x = 2t + 1$, so $x^2 = 4t^2 + 4t + 1$.
$x^2 = 2(2t^2 + 2t) + 1$, which is an odd number.
However, we assumed that x^2 is even at the beginning, so contradiction occurs. ($\Rightarrow\Leftarrow$)
Hence, the assumption that x is odd must be wrong, so x should be even. $\square$

Lemma 1.2. $\sqrt{2}$ is irrational.

Proof. Assume for a conntradiction that $\sqrt{2}$ is rational.
Write $\sqrt{2} = \frac{a}{b}$, with $a, b \in \mathbb{Z}_{\geq 1}$, and at least one of them is odd.
By squaring both sides, we can deduce

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

It shows that a^2 is an even number. By Lemma 1.1, a will be even.
Write $a = 2c$, with $c \in \mathbb{Z}_{\geq 1}$, we can deduce

$$2b^2 = (2c)^2$$

$$2b^2 = 4c^2$$

$$b^2 = 2c^2$$

Similarly, by Lemma 1.1, b will be even.
However, we assumed that one of a, b is odd, so contradiction occurs.($\Rightarrow\Leftarrow$)
Hence, the assumption that $\sqrt{2}$ is rational is wrong, so $\sqrt{2}$ is irrational. $\square$

Lemma 1.3.

$$a, b \in \mathbb{Q} \implies a + b, a - b, ab \in \mathbb{Q}$$

Proof. Write $a = \frac{m}{n}$, $b = \frac{r}{s}$, with $m, n, r, s \in \mathbb{Z}$ annd $n, s \neq 0$
We can deduce,

$$a \pm b = \frac{ms \pm rn}{ns}$$

Since $ms \pm rn \in \mathbb{Z}$ and $ns \neq 0$, therefore $a \pm b \in \mathbb{Q}$
We cann also deduce,

$$ab = \frac{mr}{ns}$$

Since $mr \in \mathbb{Z}$ and $ns \neq 0$, therefore $ab \in \mathbb{Q}$

$\square$

Corollary 1.4.

$$a \in \mathbb{Q}, b \notin \mathbb{Q} \implies a + b \notin \mathbb{Q}$$

Proof. Assume $a \in \mathbb{Q}, b \notin \mathbb{Q}$. And we also assume, $a + b \in \mathbb{Q}$ for contradiction.
We know $b = (a + b) - a$, and $a + b, a \in \mathbb{Q}$ by assumption.
By Lemma 1.3, $b \in \mathbb{Q}$.
However, we assumed $b \in \mathbb{Q}$ at the beginning. ($\Rightarrow\Leftarrow$)
Hence, assumption $a + b \in \mathbb{Q}$ is false, so Corollary 1.4 is proved. $\square$

Corollary 1.5. There are infinitely many irrational numbers.

Proof. There are infinitely many integers. Consider $n \in \mathbb{Z}$,

$$a = n + \sqrt{2}$$

$\sqrt{2}$ is irrational by Lemma 1.2, and $a \notin \mathbb{Q}$ by Corollary 1.4.
Thus there are infinitely many a.
Therefore, there are infinitely many irrational numbers. $\square$