Way back in Chapter 2, we developed the theory of ordered fields.

I didn't explain it like that. I said "let's assume all the usual things about $+$ and $-$ and $\times$ and $/$ on the real numbers" and I meant "let's assume we have a field".

I also said "let's assume just a few things – four axioms – about inequalities and I meant "let's assume furthermore that the field satisfies the axioms for an ordered field."

You'll learn about this more "structural" way of doing mathematics later on in your career. In fact you'll learn a bit about it in M1F, when we do equivalence relations.

But all that stuff in Chapter 2, which I said was about the real numbers, could have been about a much smaller ordered field like the rational numbers, or a much bigger ordered field like the hyperreal numbers.

You don't have to understand all of that stuff now. But what you do need to understand is that today we will finally learn about the extra property which the reals satisfy, which no other ordered field satisfies – the so-called "completeness axiom". In other words, this chapter really *is* about the real numbers.

Recall in the last lecture we learnt about the following ideas:

- What it means for a real number $b \in \mathbb{R}$ to be an *upper bound* for S.
- What it means for a set $S \subseteq \mathbb{R}$ to be *bounded above*.
- What it means for a real number $\ell \in \mathbb{R}$ to be a *least upper bound* for S.

Can you write down the definitions? We saw two examples of sets which had a least upper bound – the set $\{1, 2, 3\}$ had 3 as a least upper bound, and the set $(0, 1)$ had 1 as a least upper bound.

Note that "least upper bound" has a very precise definition – 1 is not the "max" of $(0, 1)$ – indeed it's not even an *element* of $(0, 1)$ – but it is a least upper bound.

Let's remind ourselves one last time of the definitions. Let $S \subseteq \mathbb{R}$ be a set of real numbers.

- We say $b \in \mathbb{R}$ is an *upper bound* for S if $\forall x \in S, x \leq b$.
- We say S is *bounded above* if there exists an upper bound for S.
- We say $\ell \in \mathbb{R}$ is a *least upper bound* for S if both the below conditions hold:
  - ℓ is an upper bound for S;
  - If $b \in \mathbb{R}$ is any upper bound for S, then $\ell \leq b$.

It's time we proved a basic theorem about least upper bounds.

Theorem 6.3. Say S is a set of real numbers, and ℓ_1 and ℓ_2 are both least upper bounds for S. Then $\ell_1 = \ell_2$.

Least upper bound is: (a) an upper bound, and (b) less than or equal to all upper bounds.

Theorem 6.3. Say S is a set of real numbers, and ℓ_1 and ℓ_2 are both least upper bounds for S. Then $\ell_1 = \ell_2$.

Proof. By the part (a) of definition of least upper bound applied to ℓ_1, ℓ_1 is an upper bound for S. By part (b) of the definition applied to ℓ_2, we have $\ell_2 \leq \ell_1$.

Simiarly, ℓ_2 is an upper bound for S, and hence $\ell_1 \leq \ell_2$.

Hence $\ell_1 \leq \ell_2$ and $\ell_2 \leq \ell_1$, so $\ell_1 = \ell_2$.

$\square$

Slogan: “a set of reals can have at most one least upper bound”.

This theorem enables us to stop talking about “a” least upper bound for a set, and to start talking about “the” least upper bound for a set.

(Remember from last time though – a set can have many many upper bounds.)

A fancy Latin word for "least upper bound" is *supremum*, and whilst we don't do science in Latin any more (thank goodness!), one standard abbreviation for the least upper bound of a set S is $\sup(S)$.

Let's take a quick look at an example of a set of reals with no supremum. Set $S = \mathbb{R}$. Write down a proof that S has no least upper bound.

Suppose for a contradiction that $S = \mathbb{R}$ had a least upper bound ℓ. Then by definition of least upper bound, ℓ is an upper bound for $\mathbb{R}$. However this would mean that $\forall x \in \mathbb{R}, x \leq \ell$. Whilst this definitely looks fishy, we still need to finish the job. Set $x = \ell + 1$. Then $\ell + 1 > \ell$ (because $1 > 0$ if you like), and hence $\ell + 1$ is not $\leq \ell$, a contradiction. Hence $\mathbb{R}$ has no least upper bound. In fact, we have proved slightly more – what have we proved? We've proved that $\mathbb{R}$ has *no upper bounds at all* – it is not bounded above.

So far we have analysed three sets – $\{1, 2, 3\}$, $(0, 1)$ and $\mathbb{R}$. We found least upper bounds for the first two, and the third set was not even bounded above so didn't even have an *upper bound*, let alone a *least upper bound*.

So here's an interesting question – if a set $S \subseteq \mathbb{R}$ is *bounded above*, does it always have a least upper bound?

Let's have a vote!

Let $S = \varnothing$, the empty set of reals.

My experience is that of all the sets of reals we consider in this chapter, some of you will find the empty set the hardest one.

Recall that $b \in \mathbb{R}$ is an *upper bound* for $S \subseteq \mathbb{R}$ if $\forall x \in S, x \leq b$.

Q) *Is 59 an upper bound for the empty set?* Let's have a vote!

Here are two propositions; each is the negation of the other, so exactly one of them is true.

1) $\forall x \in \varnothing, x \leq 59$.
2) $\exists x \in \varnothing, x > 59$.

The rules of logic say that exactly one of these is true. Which one is it?

Statement 2) *cannot be true*, because it implies that the empty set has an element, which it does not. So statement 1) *must be the true statement*.

Conclusion: 59 *is* an upper bound for the empty set.

Question: is -999999999999999999999 an upper bound for the empty set?
Yes it is, because no element of the empty set is bigger than that number, so all elements are at most that number.

Question: does the empty set have a *least* upper bound?

No it does not – because *every real number* is an upper bound for the empty set, and so if ℓ is an upper bound, then so is $\ell - 1 < \ell$, meaning that there is no upper bound which is less than or equal to all upper bounds.

Maybe the empty set is just weird. Does every *non-empty* bounded-above set have a least upper bound?

Does $S = \{ x \in \mathbb{R}_{\geq 0} \mid x^2 < 2 \} = [0, \sqrt{2})$ have a least upper bound?
Of course $\sqrt{2} \notin S$.
But one can prove that the least upper bound of S is $\sqrt{2}$.

What do you think? Does every non-empty set of reals which is bounded above always have a least upper bounds? Even sets like $\{ 1 - \frac{1}{p} \mid p \text{ is prime} \}$? Let's have a vote!

FACT (the “completeness axiom” for the real numbers): any non-empty bounded-above set of reals has a least upper bound.

We will *assume* this fact about the real numbers.

It is possible to prove it, if you know a *rigorous mathematical definition* of the real numbers (e.g. as Cauchy sequences, or as Dedekind cuts). We will not be concerning ourselves with these things – we will be *assuming* it. I think that maybe in 342 this Thursday they will say something about how to prove it.

Fact. If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then S has a supremum.

This fact is called the completeness axiom and it implies a whole bunch of stuff which you have believed for a long time but the only proof you had was that it was “intuitively correct”. Here is an example.

Example. If x is any real number, then there is an integer n with $n > x$.

This result is called “the archimedean property of the real numbers”. We assumed this in chapter 2 – now we can prove it, using the completeness axiom.

Completeness axiom. If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then S has a least upper bound.

Example. If x is any real number, then there is an integer n with $n > x$.

Let's prove it by contradiction. Say there was some super-huge real number B such that no integer n existed with $n > B$. Then $\forall n \in \mathbb{Z}, n \leq B$. Hence B would be an upper bound for the integers! So the integers would be non-empty (because 59 is an integer) and bounded above, so by the completeness axiom they would have a least upper bound $L = \sup(\mathbb{Z})$. Now

$L - 1 < L$ so $L - 1$ is definitely *not* an upper bound for the integers. This means that there exists an integer n with $n > L - 1$. But now $n + 1 > L$, a contradiction! Hence B cannot exist, and the integers are indeed not bounded above.