Office Hours 657
Tuesday 2-3, Thu 9-10

$$\int \underset{\overset{\parallel}{u}}{x} \underset{\overset{\parallel}{v'}}{\sin x}\, dx = -x\cos x - \int 1(-\cos x)\,dx$$

$$(u' = 1 \qquad v = -\cos x)$$

$$= -x\cos x + \sin x + c$$

A related example

$$\int \underset{u}{x^2} \underset{v'}{\cos x}\, dx = x^2 \sin x$$

$u' = 2x, \ v = \sin x$
$$- \int (2x)\sin x\, dx$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + c$$

using previous result

$\int x^3 \sin x \, dx$ use parts 3 times

$\int x^n e^x \, dx$ n times

Choice of u and v' not always obvious, e.g.

$\int \log x \, dx$ $\int \tan^{-1} x \, dx$

or $\int \sin^{-1} x \, dx$ integrand not a product!

trick $\int \log x \, dx = \int \underset{\overset{\shortparallel}{u}}{\log x} \cdot \underset{v'}{1} \, dx$

see problems

Another example

$$\int \underset{v'}{x} \; \underset{u}{\tan^{-1} x} \, dx$$

$$\left(u' = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2} \right)$$

$$= \quad u v - \int u' \, v \, dx$$

$$= \quad \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{1+x^2} \, \frac{x^2}{2} \, dx$$

$$= \quad \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{(1+x^2)-1}{1+x^2} \, dx$$

$$= \quad \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

~~Sub~~

$$\int e^x \sin x \, dx \quad ?$$

Use parts

$$I = \int \underset{u}{e^x} \, \underset{v'}{\sin x} \, dx$$

$$(u' = e^x \qquad v = -\cos x$$

$$I = -e^x \cos x + \int e^x \cos x \, dx$$

$$= -e^x \cos x + \Big(e^x \sin x - \underbrace{\int e^x \sin x dx}_{I} \Big)$$

Alternative : no parts

use complex numbers

$$e^x \sin x = \operatorname{Im} e^x e^{ix}$$
$$= \operatorname{Im} e^{x(1+i)}$$

$$\int e^x \sin x \, dx = \operatorname{Im} \int e^{x(1+i)} dx$$

$$= \operatorname{Im} \frac{e^{x(1+i)}}{1+i} + c$$

...

Substitution

An application of chain rule to integration problems.

Have already seen examples

$$\int \cos x \, e^{\sin x} \, dx = e^{\sin x} + c$$

since $\frac{d}{dx} e^{\sin x} = \cos x \, e^{\sin x}$

Substitution formula:

Wish to compute $\int_a^b f(x)\,dx$

idea replace x with
a function of a new
variable u
$(x \to h(u)$, or $x = x(u))$

$$\int_c^d f(h(u))\, h'(u)\, du = \int_{h(c)}^{h(d)} f(x)\,dx$$

Proof apply FToC to

$F(h(u))$ where F is an anti-derivative for f.

In example

$f(x) = e^x$

$h(u) = \sin u$

$$\int_c^d e^{\sin u} \cos u = \int_{\sin c}^{\sin d} e^x \, dx$$

$$= e^{\sin d} - e^{\sin c}$$

not very efficient - easier to use chain rule directly

Formula useful in other direction

suppose h is invertible

let $b = h(d)$, $a = h(c)$

$d = h^{-1}(b)$ $c = h^{-1}(a)$

$$\int_a^b f(x)\,dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u))\,h'(u)\,du$$

also written

$$\int_a^b f(x)\,dx = \int_{u(a)}^{u(b)} f(h(u))\,h'(u)\,du$$

Not worth memorizing!

To compute

$$\int f(x)\, dx = \int f(x(u))\, x'(u)\, du$$

replace x with a function of u $\quad x \to x(u)$

replace dx with $\frac{dx(u)}{du}\, du$

RHS function of u

rewrite in terms of x

Examples

$$\int \frac{dx}{1+x^2}$$

Use substitution

$$x = \tan u$$

$$\frac{dx}{du} = \sec^2 u$$

$$\int \frac{dx}{1+x^2} = \int \frac{1}{1+\tan^2 u} \sec^2 u \, du$$

$$= \int 1 \, du = u + c$$

$$= \tan^{-1} x + c$$

A related example

$$\int \frac{dx}{(1+x^2)^2} \qquad \begin{aligned} x &= \tan u \\ dx &= \sec^2 u \, du \end{aligned}$$

$$= \int \frac{\sec^2 u \, du}{\sec^4 u}$$

$$= \int \cos^2 u \, du = \int \frac{1 + \cos 2u}{2} \, du$$

$$\frac{u}{2} + \frac{1}{4}\sin 2u + c$$

$$= \frac{u}{2} + \frac{1}{2}\sin u \cos u + c$$

$$= \frac{u}{2} + \frac{1}{2}\tan u \cos^2 u + c$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2} x \frac{1}{1+x^2} + c$$

Other substitutions

$\int \sqrt{1 - x^2}\, dx$ use $x = \sin u$

$\int \sqrt{1 + x^2}\, dx$ use $x = \sinh u$

$$\int \sqrt{x^2-1}\,dx \qquad x = \cosh u$$

$$dx = \sinh u\,du$$

$$= \cancel{\int \sinh u \cosh u\,du} \qquad \cosh^2 u - 1$$

$$= \sinh^2 u$$

$$\cancel{= \frac{1}{2}\int \sinh(2u)\,du}$$

$$= \int \sinh^2 u\,du$$

$$= \frac{1}{2}\int \cancel{\frac{1 - \cosh 2u}{2}}\,du$$

$$= \cancel{\frac{1}{4}}\int \frac{(\cosh 2u - 1)}{2}\,du$$

If integrand rational in $\sin x$ or $\cos x$ use

$$x = 2\tan^{-1} u$$

$$\left(\text{or} \quad u = \tan\frac{x}{2}\right)$$

Express $\sin x$ and $\cos x$ in terms of u

$$\sin x = \frac{2u}{1+u^2}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

and $$dx = \frac{2}{1+u^2}\,du$$

Substitution converts integrand into a rational function of u

$$\int \frac{dx}{\sin x} \qquad \text{use} \quad u = \tan\frac{x}{2}$$

$$= \int \frac{\frac{2\,du}{1+u^2}}{\frac{2u}{1+u^2}} = \int \frac{du}{u}$$

$$= \log u + c = \underline{\log \tan\frac{x}{2} + c}$$

$$\int_0^\pi \sin x \, dx$$

$u = \sin x$

$x = \sin^{-1} u$??

$dx = \frac{du}{\sqrt{1-u^2}}$

$$\stackrel{?}{=} \int_0^0 \frac{u}{\sqrt{1-u^2}}\, du$$

Partial Fractions

Useful for integration rational functions

$f(x) = \dfrac{P(x)}{Q(x)}$ P, Q polynomial

(P may be constant)

Simple Case: degree of P less than degree of Q and roots of Q distinct

$Q(x) = c(x-a_1)(x-a_2)\cdots(x-a_n)$

$a_1, a_2, \ldots, a_n$ roots of Q

which may be complex
- but distinct.

Claim : can always write

$$f(x) = \frac{P(x)}{Q(x)}$$

$$= \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \frac{c_3}{x-a_3} + \cdots \frac{c_n}{x-a_n}$$

where $c_1, c_2, \ldots, c_n$
are constants

Integration of $f(x)$ straightforward

$$\int f(x)\,dx = c_1 \log(x-a_1) + c_2 \log(x-a_2) + \cdots + c_n \log(x-a_n) + c$$

Examples

$$f(x) = \frac{1}{x^2+5x+6}$$

$$= \frac{1}{(x+2)(x+3)}$$

$$= \frac{1}{x+2} + \frac{-1}{x+3}$$

$$g(x) = \frac{1}{x(x+1)(x+2)}$$

$$= \frac{\frac{1}{1\cdot 2}}{x} + \frac{\frac{1}{-1(-1+2)}}{x+1} + \frac{\frac{1}{-2(-2+1)}}{x+2}$$

use cover up rule

$$= \frac{\frac{1}{2}}{x} - \frac{1}{x+1} + \frac{\frac{1}{2}}{x+2}$$

$$\int \frac{dx}{x(x+1)(x+2)}$$

$$= \frac{1}{2} \log x - \log(x+1)$$

$$+ \frac{1}{2} \log(x+2) + c$$

$$= \frac{1}{2} \log \frac{x(x+2)}{(x+1)^2} + c$$

An example with complex roots

$$f(x) = \frac{1}{1+x^2} = \frac{1}{(x+i)(x-i)}$$

$$= \frac{-\frac{1}{2i}}{x+i} + \frac{\frac{1}{2i}}{x-i}$$

$$\int f(x)\,dx = \frac{1}{2i}\Big[\log(x-i) - \log(x+i)\Big] + c$$

$$= \frac{1}{2i}\log\frac{(x-i)}{x+i} + c$$

$$= \frac{1}{2i}\log\frac{1+ix}{-(1-ix)} + c$$

$$= \frac{1}{2i}\log\frac{1+ix}{1-ix} + c'$$

($\log(-1)$ is a constant)

$\frac{1}{2i} \log \frac{1+ix}{1-ix}$ is a

valid formula for inverse

tangent - recall formula

for inverse hyperbolic

tangent

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$$

for $-1 < x < 1$

$$\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C$$

Can avoid complex

roots by using a

different but real PF expansion. With a c.c pair of complex roots PF expansion will include

$$\frac{c}{x-a} + \frac{\bar{c}}{x-\bar{a}}$$

$$= \frac{c(x-\bar{a}) + \bar{c}(x-a)}{(x-a)(x-\bar{a})}$$

$$= \frac{(c+\bar{c})x - c\bar{a} - \bar{c}a}{x^2 - (a+\bar{a})x + a\bar{a}}$$

Numerator linear in x
Denominator quadratic

Example

$$\frac{1}{x(x^2+1)}$$

Complex roots
$0, \; i, \; -i$

or use real form

$$\frac{1}{x(x^2+1)} = \frac{C}{x} + \frac{Ax+B}{x^2+1}$$

Then determine A, B, C

(can use cover up rule
to determine C not
A and B)

What if roots Not Distinct, eg.

$$f(x) = \frac{1}{x(x+1)^2}$$

PF decomposition has form

$$f(x) = \frac{\circ}{x} + \frac{\circ}{x+1} + \frac{\circ}{(x+1)^2}$$

Can do a 2 step calculation

$$f(x) = \frac{1}{x+1} \circ \frac{1}{x(x+1)}$$

$$= \frac{1}{x+1} \circ \left(\frac{1}{x} - \frac{1}{x+1} \right)$$

$$= -\frac{1}{(x+1)^2} + \frac{1}{x(x+1)}$$