Last time: a *binary relation* on a set S is a function from $S \times S$ to $\{\text{true}, \text{false}\}$.

Let $\sim$ be a binary relation on S.

- We say $\sim$ is *reflexive* if $\forall s \in S, s \sim s$.
- We say $\sim$ is *symmetric* if $\forall s, t \in S, s \sim t \implies t \sim s$.
- We say $\sim$ is *transitive* if $\forall s, t, u \in S$, if $s \sim t$ and $t \sim u$, then $s \sim u$.
- We say a binary relation $\sim$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

Last time I left you with:

1) The set of plastic squares and triangles, with the relationship defined by $s \sim t \iff s$ and t are the same colour – this idea – "equality of colour" – is it an equivalence relation?

2) Click on this link to go to an interactive Lean session where you can prove for yourself that congruence mod 37 is an equivalence relation.

Let S be a set of plastic squares and triangles, some red, some green, some yellow and some blue. Define a binary relation on these plastic shapes – two pieces are related if (and only if) they are the same colour.

Let's first check that this relation is an equivalence relation. Recall the definitions. We say a binary relation $\sim$ on a set S is *reflexive* if $\forall a \in S, a \sim a$.

So what does it *mean* to say that our relation is reflexive? It means that if s is a plastic shape, then s is the same colour as itself. This is true.

S is plastic shapes. If $a, b \in S$ then $a \sim b$ iff a and b are the same colour.

A binary relation is *symmetric* if $\forall a, b \in S, a \sim b \implies b \sim a$.

What does it *mean* to say that our binary relation is symmetric? It *means* that if a and b are plastic shapes and a is the same colour as b, then b is the same colour as a. This is true!

A binary relation is *transitive* if $\forall a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Is our binary relation transitive? What does this question *mean*? It means that if a, b, c are three pieces, and a is the same colour as b, and if b is the same colour as c, then a is the same colour as c. This is true!

The relation on the pieces – “we are the same colour” – is hence an equivalence relation.

Let's generalise this. Let's let S be a general set. Let's say we have partitioned S up into non-empty subsets C_1, C_2, C_3, ... What does it mean to *partition* S into non-empty subsets? It means that we have an *index set* I (for example $I = \{1, 2, 3, \ldots\}$ or $I = \{\text{red}, \text{blue}, \text{yellow}, \text{green}\}$) and then for each $i \in I$ we have a non-empty subset C_i of S such that every element of S is in *exactly one* of the C_i.

Informally, I'm saying that we've broken S up into non-empty subsets C_i, and the C_i don't overlap, and their union is S.

For example, S could be my plastic shapes, I could be $\{\text{red}, \text{blue}, \text{yellow}, \text{green}\}$ and we could partition S up into four subsets C_{red} (the red pieces), C_{green} (the green pieces), C_{yellow} (the yellow pieces) and C_{blue} (the blue pieces).

Or another example, S could be the integers, and I could be $\{0, 1\}$ and we could let C_0 be all the even integers and C_1 be all the odd integers.

Or an extreme example: S could be the real numbers, and we could partition it into infinitely many sets all of size 1: we could set $I = \mathbb{R}$ and we could define $C_1 = \{1\}$ and $C_{-23} = \{-23\}$ and $C_\pi = \{\pi\}$ and $C_{\sqrt{2}} = \{\sqrt{2}\}$ and so on. This is a partition of the real numbers into (uncountably) infinitely many subsets of size 1.

Just remember: the C_i are non-empty, different C_i's don't overlap, and every element of S is in exactly one C_i.
[draw a picture]

Here's a formal definition.

Definition 7.3. A *partition* of a set S is: an index set I, and non-empty subsets C_i of S for every $i \in I$, with the property that $i \neq j \implies C_i \cap C_j = \emptyset$, and $\bigcup_{i \in I} C_i = S$.

The C_i are called the *parts* of the partition (or the *blocks* or the *cells*).

What you need to know: every element of S is in exactly one of the C_i. For example every integer is exactly one of odd or even. Or every plastic shape is either red, blue, yellow or green.

Now let's generalise the "colours of plastic shapes" binary relation. Let S be a set and let $C_i, i \in I$ be a partition of S into non-empty subsets. Let's define a binary relation $\sim$ on S in the following way. Say $a, b \in S$. By definition of a partition, we know there exists $i \in I$ such that $a \in C_i$, and there exists $j \in I$ such that $b \in C_j$. "The colour of a is i and the colour of b is j". We say $a \sim b$ is true if $i = j$, and $a \sim b$ is false otherwise.

S a set, and $C_i, i \in I$ a partition of S. Say $a, b \in S$. Say $a \in C_i$ and $b \in C_j$. Define $a \sim b$ if $i = j$ and $a \not\sim b$ if $i \neq j$.

Another way to think about it: S is the union of lots of parts, the C_i, and $a \sim b$ if and only if they're both in the same part.
This is certainly a binary relation. Is it an equivalence relation?

Well, if $a \in C_i$ then a is in the same part as itself, so $a \sim a$.
Hence $\sim$ is reflexive.

Now say $a, b \in S$ and $a \sim b$. Then $a \in C_i$ and $b \in C_i$, so a and b are in the same part of the partition, so b and a are in the same part of the partition, so $b \sim a$. Hence $\sim$ is symmetric.

Finally say $a, b, c \in S$ with $a \sim b$ and $b \sim c$. Then a and b are both in C_i for some i, and b and c are both in C_j for some j, but now $b \in C_i$ and $b \in C_j$, so because the C's partition S we must have $i = j$. Hence $a, c \in C_i$, so $\sim$ is transitive.

Conclusion: if we partition S up into parts, we can define a binary relation on S, by $a \sim b$ iff a and b are in the same part, and this binary relation is an equivalence relation.

If you struggled to follow the previous argument, think about the case of the plastic shapes, and the relation defined by a and b are related if (and only if) they are the same colour. The general case is just the same idea, but put naturally in a more abstract form.

Let's do some basic questions about equivalence relations in Lean.

We have seen how given a partition of S into non-empty subsets, we can construct an equivalence relation on S. Here's an interesting question. Given an equivalence relation on S, can we construct a partition of S into non-empty subsets?
[do example with plastic shapes]

Our investigations lead us to the following definition.

Definition 7.4. Let S be a set and let $\sim$ be an equivalence relation on S. Let $a \in S$ be an element. The *equivalence class* $Cl(a)$ of a is the following subset of S:

$$Cl(a) := \{\, b \in S \mid a \sim b \,\}.$$

Informally, the equivalence class of a is all the elements of S that a is related to.

For example, if S is a set of plastic shapes, and $a \sim b$ iff a and b are the same colour, and if $x \in S$ is a yellow square, then $Cl(x)$ is all the pieces which are the same colour as x, that is, all the yellow pieces.

We will talk a lot more about equivalence classes in the next lecture.