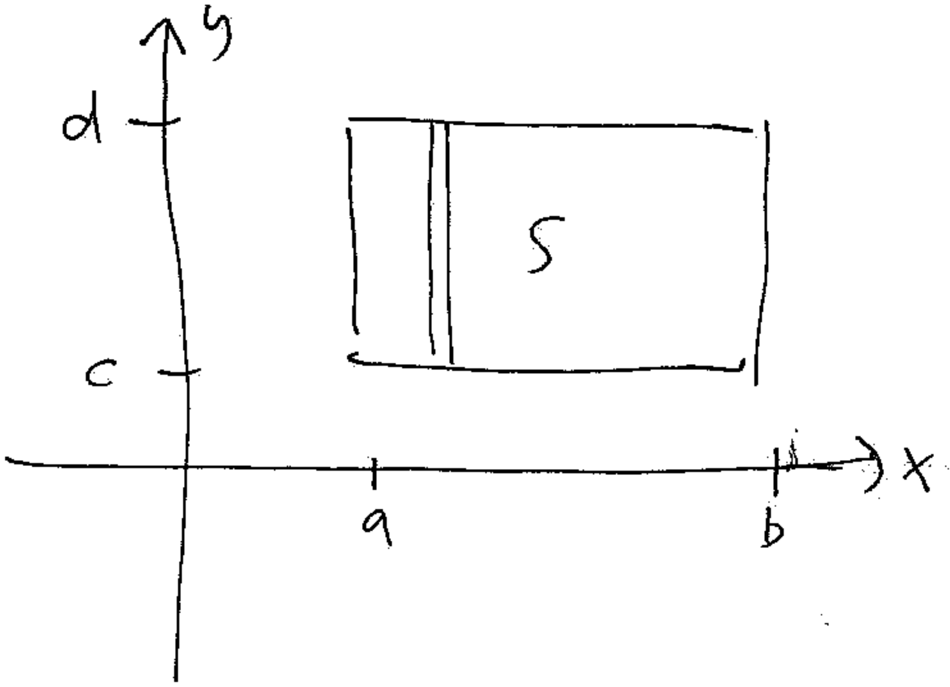
Iterated Integrals

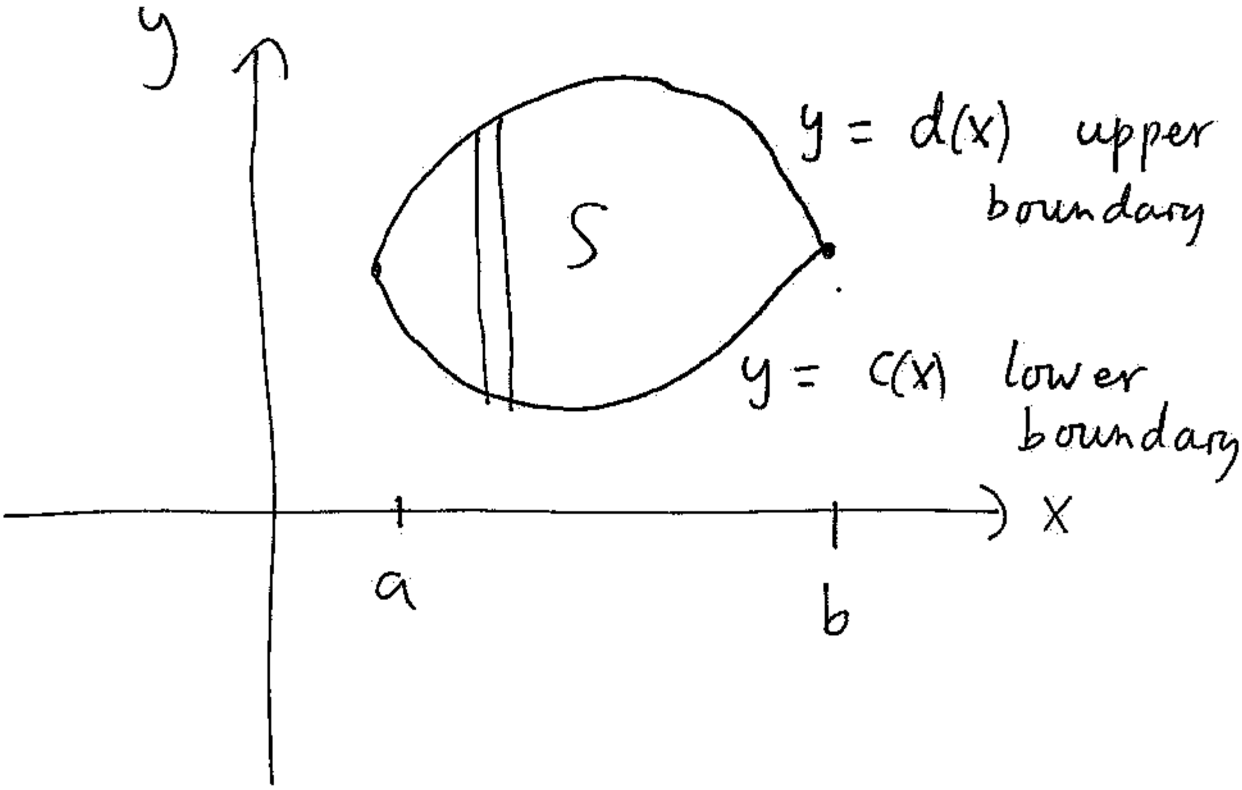


$$\iint_S f(x,y)\,dx\,dy$$

$$\int_a^b dx \int_c^d dy\, f(x,y)$$

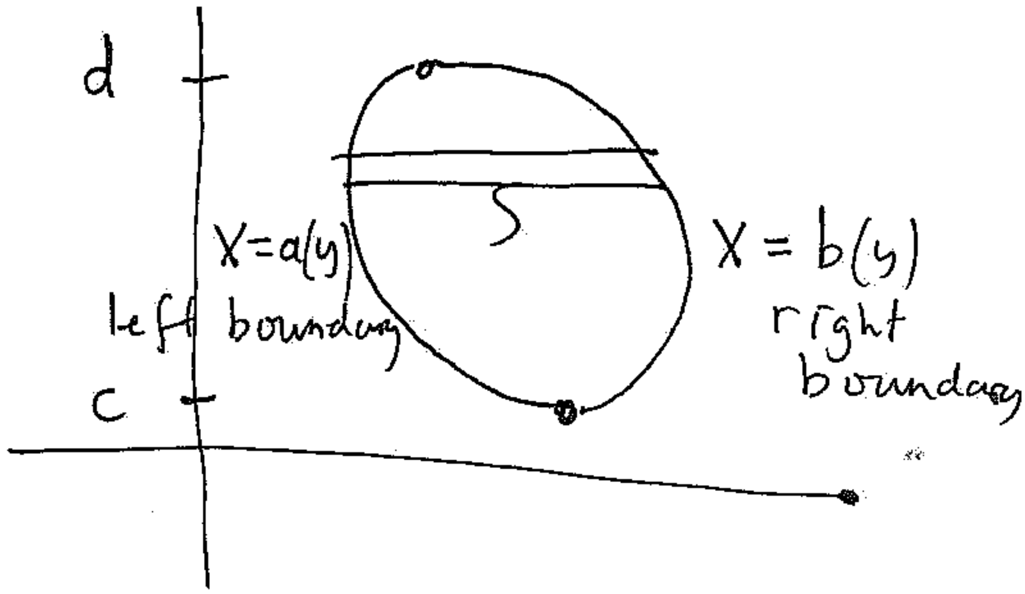
or $$\int_c^d dy \int_a^b dx\, f(x,y)$$

Non-rectangular regions



$$\int_a^b dx \int_{c(x)}^{d(x)} dy \; f(x,y)$$

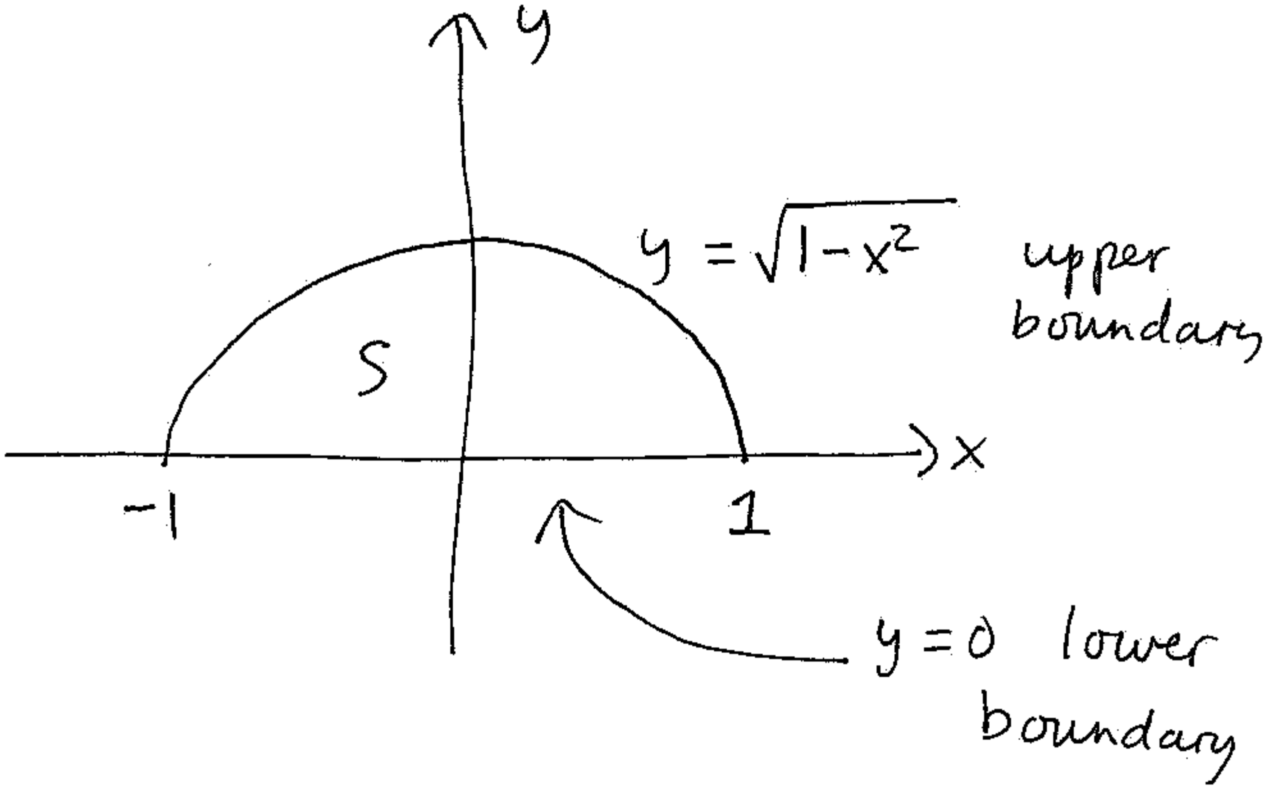
OR



$$\int_c^d dy \int_{a(y)}^{b(y)} dx \; f(x,y)$$

Example

semi-disc unit radius



$$I = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \, f(x,y)$$

$$f(x,y) = 1$$

$$\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} dy \ 1$$

$$\int_{-1}^{1} dx \quad y \Big|_{y=0}^{y=\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} dx \ \sqrt{1-x^2}$$ usual area under graph

$$= \frac{\pi}{2}$$

<u>Centroids</u>

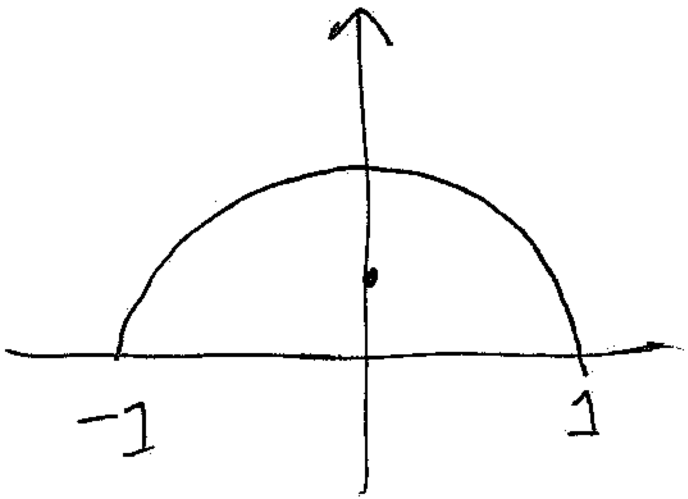
The centroid of a region $S \subset \mathbb{R}^2$ is $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{A} \iint_S x \, dx \, dy$$

$$\bar{y} = \frac{1}{A} \iint_S y \, dx \, dy$$

(centre of mass assuming constant density)

For disc example



centroid on y axis

$\bar{x} = 0$

$$\bar{y} = \frac{1}{A} \iint_S y \, dx \, dy$$

$$= \frac{1}{A} \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \, y$$

$$= \frac{1}{A} \int_{-1}^{1} dx \, \frac{y^2}{2} \Bigg|_{y=0}^{y=\sqrt{1-x^2}}$$

$$= \frac{1}{A} \int_{-1}^{1} dx \, \frac{1-x^2}{2}$$

$$= \frac{1}{A} \cancel{\int_{-1}^{1} dx} \left(\frac{x}{2} - \frac{x^3}{6} \right) \Bigg|_{x=-1}^{x=1}$$

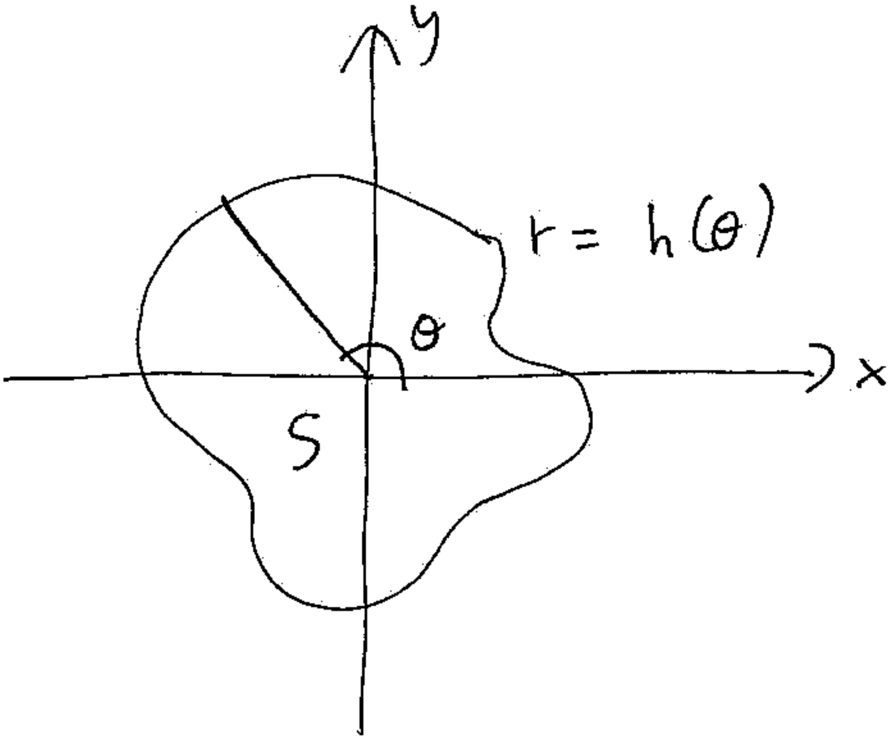
$$= \frac{1}{A} \left(1 - \frac{1}{3} \right) = \frac{\frac{2}{3}}{\frac{\pi}{2}} = \frac{4}{3\pi}$$

Polar Coordinates

As in 1d can make a change of variables or substitution (more details next term). An important ~~expam~~ example is polar coordinates

$$I = \iint_S f(x, y)\, dx\, dy$$

can convert to polar coordinates



S region enclosed

by curve $r = h(\theta)$

$0 \le \theta \le 2\pi$

Suppose $f(x,y) = g(r,\theta)$

where $x = r\cos\theta$, $y = r\sin\theta$

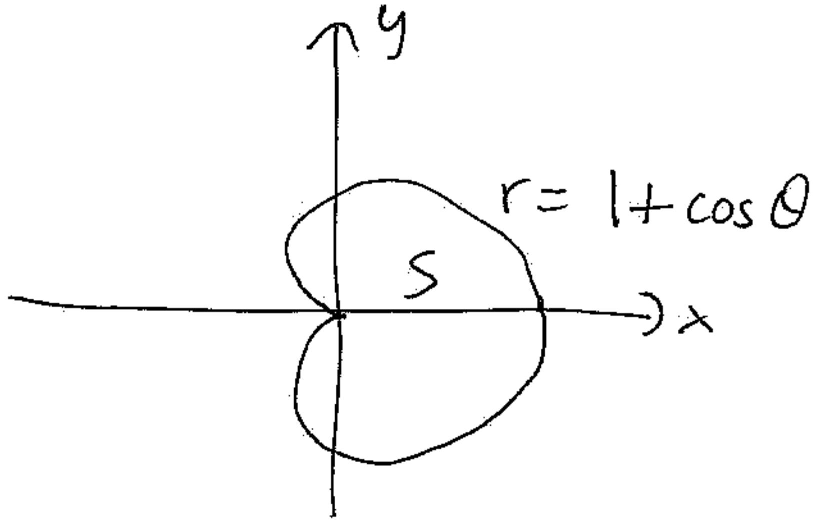
$$\iint_S f(x,y)\,dx\,dy = \int_0^{2\pi} d\theta \int_0^{h(\theta)} dr\, \underset{\substack{\uparrow \\ \text{extra factor}}}{r}\, g(r,\theta)$$

Extra factor of r

is a Jacobian (more later)

Example

Cardioid $r = 1 + \cos\theta$



Area enclosed

$$A = \iint_S 1 \, dx\, dy$$

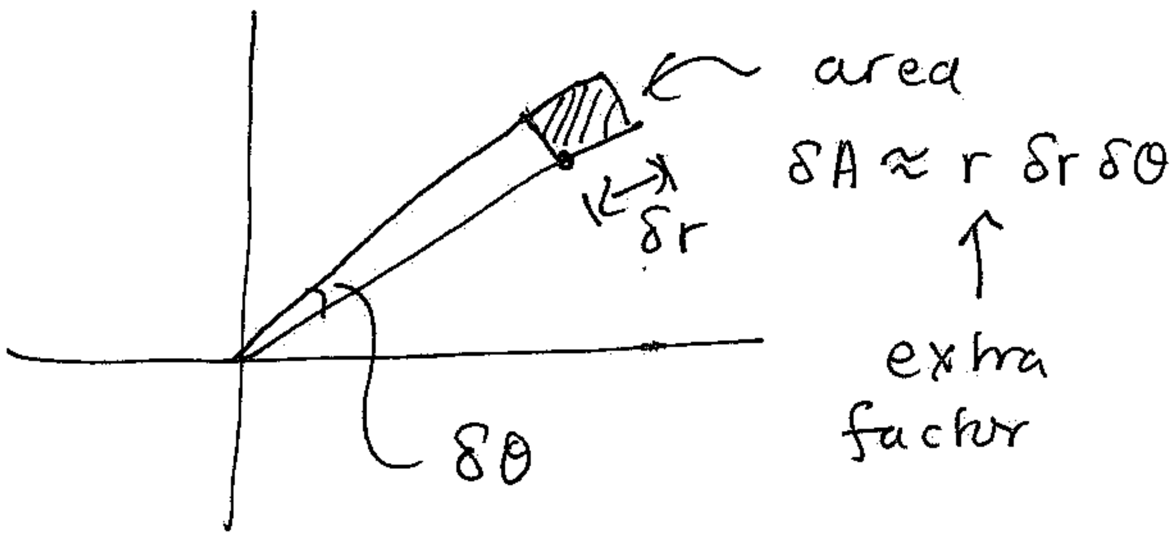
$$= \int_0^{2\pi} d\theta \int_0^{1+\cos\theta} dr \, r \, 1$$

see problems

Another nice problem

Compute centroid $(\bar{x}, \bar{y})$ of S for cardioid case.

Extra factor r can be understood by looking at element of area in polar coordinates



In cartesians $\delta A \approx \delta x\, \delta y$

The Gaussian Integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Can derive this using polar coordinates!

Trick: write I^2 as a double integral

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$$

switch to polar coordinates

to get answer (π)

see problems.

Can rescale

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

here $a > 0$

Another useful integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \, e^{\frac{b^2}{4a}}$$

(b can be complex)

Stirling's Formula

An approximate formula for $n! = \Gamma(n+1)$

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1$$

or

$$\log(n!) = n \log\left(\frac{n}{e}\right) + \frac{1}{2}\log(2\pi n)$$

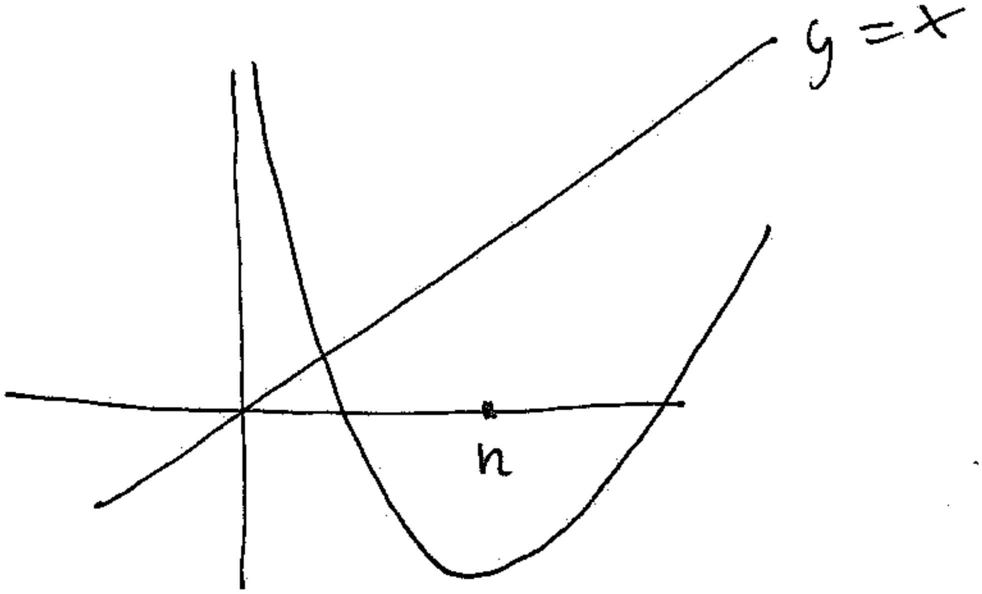
$$= n\left(\log n - 1\right) + \frac{1}{2}\log(2\pi n)$$

$$n! = \Gamma(1+n) = \int_0^\infty x^n e^{-x}\, dx$$

$$= \int_0^\infty e^{-x + n\log x}\, dx$$

$$= \int_0^\infty e^{-f(x)}\, dx$$

$$f(x) = x - n \log x$$



$$f'(x) = 1 - \frac{n}{x} = 0 \quad \text{for}$$
$$f''(x) = +\frac{n}{x^2} \qquad x = n \ (\text{min})$$

Now consider a Taylor expansion about the minimum $x = n$

$$f(x) = f(n) + \underbrace{f'(n)}_{=0}(x-n)$$
$$+ \frac{1}{2} f''(n)(x-n)^2 + \cdots$$

$$= n - n\log n + \frac{1}{2}\frac{1}{n}(x-n)^2$$
$$+ \cdots$$

Now ignore all higher order terms!

$$n! \approx \int_0^\infty e^{-\left[n - n\log n + \frac{1}{2n}(x-n)^2\right]}$$

$$= e^{n\log n - n} \int_0^\infty e^{-\frac{1}{2n}(x-n)^2} dx$$

2nd approximation

replace lower limit 0

with $-\infty$!!

$$n! \approx \left(\frac{n}{e}\right)^n \int_{-\infty}^\infty e^{-\frac{1}{2n}(x-n)^2} dx$$

$$= \left(\frac{n}{e}\right)^n \int_{-\infty}^\infty e^{-\frac{1}{2n}x^2} dx$$

$$= \left(\frac{n}{e}\right)^n \sqrt{\frac{\pi}{\frac{1}{2n}}}$$

$$= \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

9 Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation involving a function y and one or more of its derivatives. The order of an ODE is the highest derivative present

$$h\left(y(x), y'(x), y''(x), \ldots, y^{(n)}(x); x\right) = 0$$

general form of an nth order ODE.

A linear ODE has form

$$p_0(x)\, y(x) + p_1(x)\, y'(x) + p_2(x)\, y''(x) + \cdots\; p_n(x)\, y^{(n)}(x) = q(x)$$

where $p_0, p_1, p_2, \ldots, p_n, q$ are abitrary functions of x.

(can set $p_0(x) = 1$). Should be $p_n(x) = 1$

Examples

(a) $\dfrac{dy}{dx} = \tan y$ non-linear 1st order

(b) $\dfrac{d^2y}{dx^2} + y = \tan x$ linear 2nd order

(c) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \cot x$ linear 2nd order

(d) $\underline{y \dfrac{dy}{dx}} + x = y$ non-linear first order

(c) first order ODE in disguise. Write $w(x) = \dfrac{dy}{dx}$

$w' + w = \cot x$

First Order ODEs

Simplest case

$$\frac{dy}{dx} = f(x)$$

solution

$$y(x) = \int f(x)\,dx$$ ← Must include constant of integration to get general solution

$$\left[\frac{d}{dx} \int f(x)\,dx = f(x) \quad \text{by definition} \right]$$

e.g. $$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y(x) = \tan^{-1} x + c$$

c arbitrary constant

(general solution of ODE)

General linear ODE (1st order)

$$\frac{dy(x)}{dx} + p(x)\, y(x) = q(x)$$

p, q arbitrary functions of x.

Integrate equation

$$y(x) + \int p(x)\, y(x)\, dx = \int q(x)\, dx$$

An integral equation
(not so useful!)

Trick: multiply ODE by the integrating factor

$$e^{\int p(x)\,dx}$$

$$\left(\frac{dy}{dx} + py\right) e^{\int p\,dx} = q\, e^{\int p\,dx}$$

$$\frac{d}{dx}\left(y\, e^{\int p\,dx}\right) = q\, e^{\int p\,dx}$$

which integrates to

$$y\, e^{\int p\,dx} = \int q\, e^{\int p\,dx}\, dx$$

Example

$$y' + 2xy = 2x$$

Integrating factor $e^{\int 2x\,dx} = e^{x^2}$

$$(y' + 2xy)e^{x^2} = 2xe^{x^2}$$

$$\frac{d}{dx}\left(ye^{x^2}\right) = 2xe^{x^2}$$

integrate both sides

$$ye^{x^2} = \int 2xe^{x^2}\,dx$$

$$= e^{x^2} + c$$

$y = 1 + ce^{-x^2}$ $\quad c$ arbitrary constant

Separation of Variables

1st order equation of form

$$\frac{dy(x)}{dx} = f(x)\,g(y)$$

is called separable

(can be linear or non-linear)

ODE can be separated

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

Examples

(a) $y' = e^x y$ linear

(b) $y' = e^x y^2$ non-linear

(a) $$\int \frac{dy}{y} = \int e^x \, dx$$

$$\log y = e^x + c$$

or $y = Ae^{e^x}$ A arbitrary constant

(b) $$\int \frac{dy}{y^2} = \int e^x \, dx$$

$$-\frac{1}{y} = e^x + c$$

$$y = \frac{-1}{e^x + c}$$

See problems for more examples

Homogeneous ODE (1st order)

An ODE of form

$$\frac{dy(x)}{dx} = f\left(\frac{y}{x}\right)$$

called homogeneous

eg. $$\frac{dy}{dx} = \frac{y}{x} - 1 - \left(\frac{y}{x}\right)^2$$

trick write

$y(x) = x\, v(x)$ $\qquad v = \frac{y}{x}$

ODE for v is separable

$y' = x v' + v$

ODE

$x v' + v = f(v)$

or $x v' = f(v) - v$

which is separable