Power Series (aka Maclaurin series) worth memorizing:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

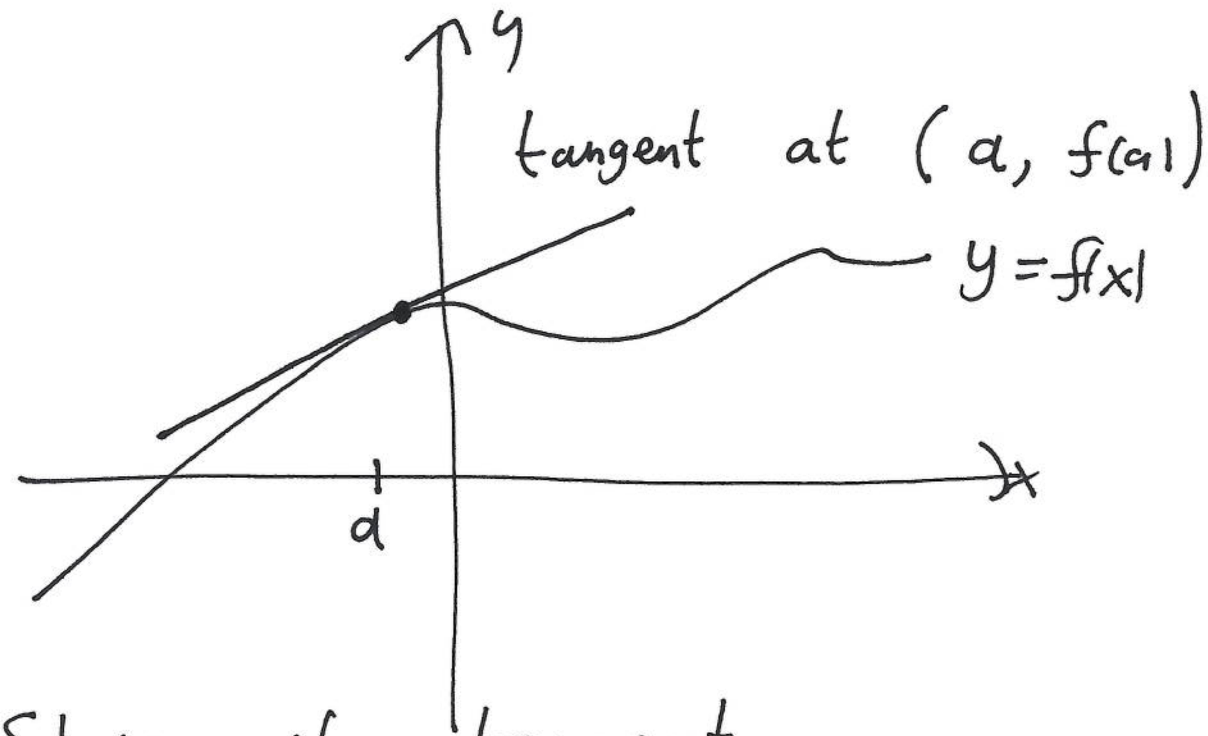
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$\tan x$, $\sin^{-1} x$ PS not so simple!

3. Differentiation

The derivative of a function f is the slope of the tangent to graph of function



Slope of tangent is $f'(a)$ or $\frac{df(a)}{da}$

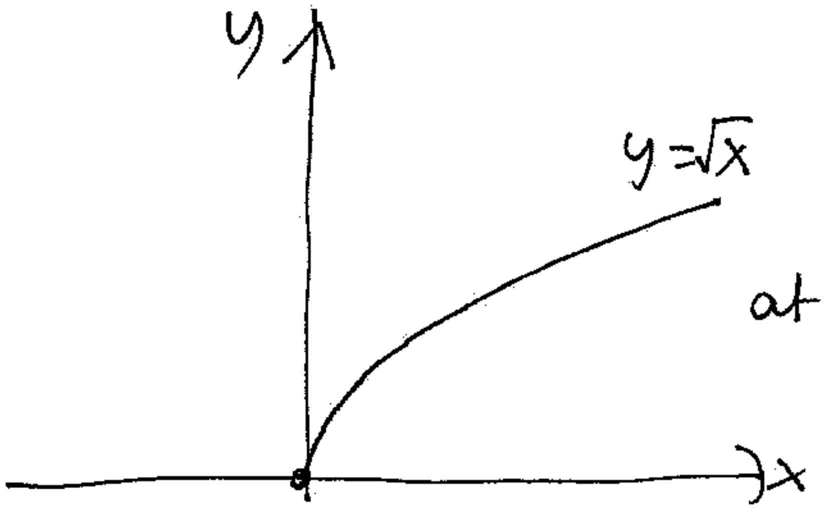
or $\left.\dfrac{df}{dx}\right|_{x=a}$

$f'(x)$ or $\dfrac{df}{dx}$ is called the derivative of f. f' is a function $\quad \mathrm{dom}(f') \subseteq \mathrm{dom}(f)$

<u>Example</u>

$f(x) = \sqrt{x} \qquad x \geq 0 \qquad \mathrm{dom}(f) = [0,\infty)$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \qquad x > 0 \qquad \mathrm{dom}(f') = (0,\infty)$



tangent vertical at origin

Can define derivative as a limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation from first Principles - use limit formula to compute derivatives

<u>Examples</u>

(i) $f(x) = x^3$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}$$

$$= \frac{\cancel{x^3} + 3hx^2 + 3h^2x + h^3 - \cancel{x^3}}{h}$$

$$= 3x^2 + 3hx + h^2 \to 3x^2$$

as $h \to 0$ (in this calculation x treated as a constant).

(ii) $f(x) = \cos x$, $f'(x) = -\sin x$

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h}$$

$$\cos A - \cos B = -2 \sin \frac{A-B}{2} \sin \frac{A+B}{2}$$

derive using addition formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\alpha = \frac{A-B}{2} \qquad \beta = \frac{A+B}{2}$$

$$\frac{f(x+h)-f(x)}{h} = -\frac{2}{h}\sin\frac{h}{2}\cdot\sin\left(x+\frac{h}{2}\right)$$

$(A = x+h, \quad B = x)$ use product rule

$$\rightarrow -1 \cdot \sin x \quad \text{as } h \to 0$$

$$-\frac{2\sin\left(\frac{1}{2}h\right)}{h} \rightarrow -1 \quad \text{as } h \to 0$$

set $k = \frac{1}{2}h$ $\left(= -\frac{\sin k}{k} \to -1 \text{ as } k \to 0 \right)$

First principles - not very practical! Instead use basic derivatives together with differentiation rules.

Basic Derivatives

$f(x)$	$f'(x)$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
x^n	$n x^{n-1}$

$f(x)$	$f'(x)$
$\log x$	$\frac{1}{x}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

have $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

or $\cos^{-1} x = \underbrace{\frac{\pi}{2}}_{\text{even}} - \underbrace{\sin^{-1} x}_{\text{odd}}$

Differentiation Rules

(i) $$\frac{d}{dx}\left(\alpha\, u(x) + \beta\, v(x)\right) = \alpha\, u'(x) + \beta\, v'(x)$$

α, β constants

symbol $\frac{d}{dx}$ means

differentiate everything to right of symbol

(Linearity of differentiation)

(ii) Product rule

$$\frac{d}{dx}\, u(x)\, v(x) = u'(x)v(x) + u(x)v'(x)$$

(iii) Quotient rule

$$\frac{d}{dx} \frac{u(x)}{v(x)} = \frac{u'(x)\, v(x) - u(x) v'(x)}{(v(x))^2}$$

for $v(x) \neq 0$

(iv) Chain rule - for differentiating composite functions

$$\frac{d}{dx} f(u(x)) = f'(u(x))\, u'(x)$$

$$\left[\text{Can be written} \quad \frac{df}{dx} = \frac{df}{du} \frac{du}{dx} \right]$$

Example

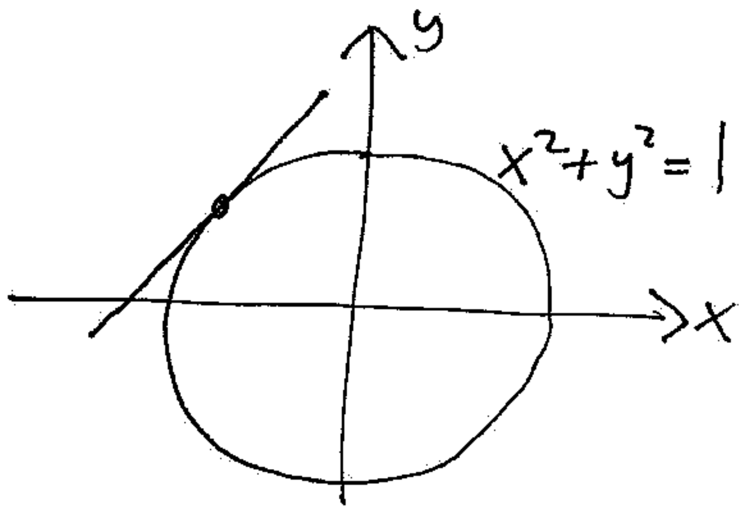
$$\frac{d}{dx} \exp(\sin x)$$

$$= \exp(\sin x) \cdot \cos x$$

Implicit Differentiation

An application of the chain rule. For example consider unit circle

$x^2 + y^2 = 1$



Using explicit differentiation "solve" equation to find y as a function of x and differentiate

$y^2 = 1 - x^2 \qquad y(x) = \pm\sqrt{1-x^2}$

$$\frac{dy}{dx} = \mp \frac{x}{\sqrt{1-x^2}}$$

Implicit differentiation –
y^2 is a composite function of x. Differentiate equation

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Further examples

(i) $(x^2+y^2-1)^3 = x^2y^3$ see problems

(ii) $y^3 - y = x^2$ Find slope at $(\sqrt{6}, 2)$

Implicit differentiation

$$3y^2y' - y' = 2x$$

$$y' = \frac{2x}{3y^2-1} = \frac{2\sqrt{6}}{11}$$

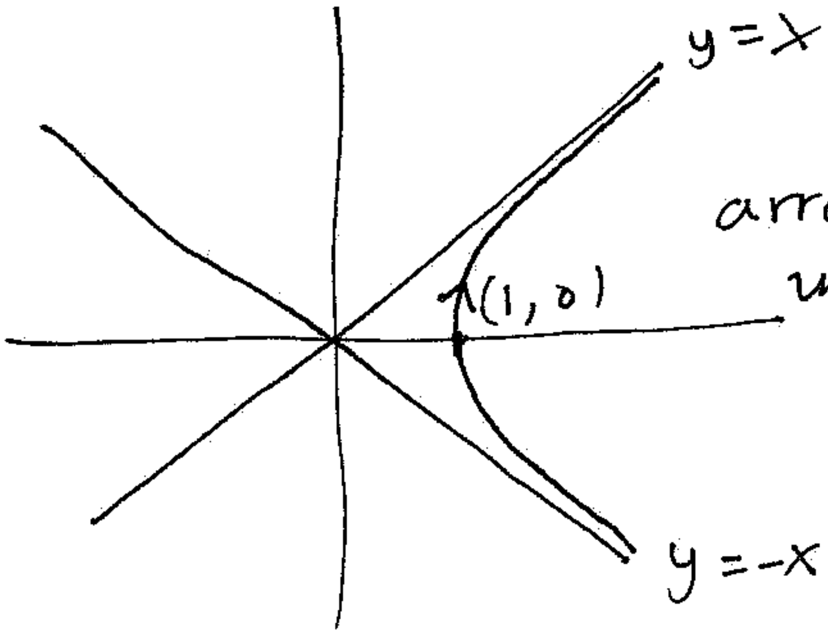
Parametric Differentiation

Can represent curves in xy plane (or in 3d space) parametrically

For example

$$x(t) = \cosh t$$
$$y(t) = \sinh t$$

t is a <u>parameter</u>



arrow indicates increasing t

Slope of tangent to curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$$

where $\bullet$ = differentiation w.r.t. t

$$\frac{dy}{dx} = \frac{\cosh t}{\sinh t} = \coth t$$

at $t = 0$ undefined

(tangent is vertical at $(1, 0)$ where $t = 0$)

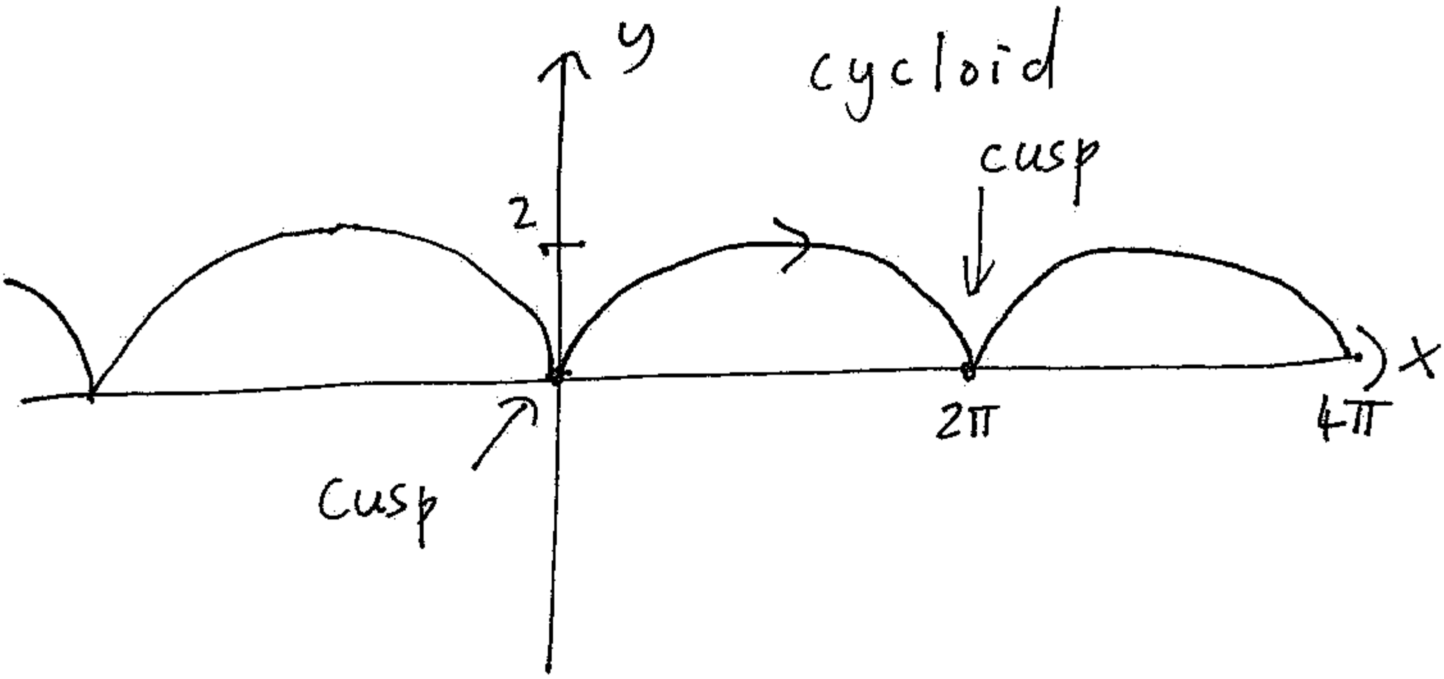
$$\frac{dy}{dx} \to 1 \quad \text{as} \quad t \to +\infty$$

$$\frac{dy}{dx} \to -1 \quad \text{at} \quad t \to -\infty$$

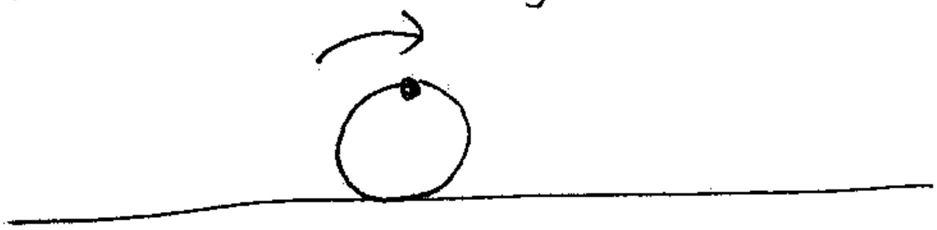
Another example

$$x(t) = t - \sin t$$

$$y(t) = 1 - \cos t$$



cycloid - path traced by a point on rolling wheel



$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{+\sin t}{1-\cos t}$$

Q: area under one arch?

$$= \frac{2\sin(\tfrac{1}{2}t)\cos(\tfrac{1}{2}t)}{2\sin^2(\tfrac{1}{2}t)}$$

$$= \cot(\tfrac{1}{2}t)$$

Higher Derivatives

Suppose f is differentiable.

Consider limit

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}.$$

If this limit exists

f is called twice differentiable

Limit is called the second derivative written

as $f''(x)$, $\frac{d^2 f(x)}{dx^2}$ or $f^{(2)}(x)$

Can repeat process to define third and higher derivatives nth derivative

$$\frac{d^n f(x)}{dx^n} \quad \text{or} \quad f^{(n)}(x)$$

<u>Example</u> $f(x) = \log x$

$$f^{(1)}(x) = \frac{1}{x}, \quad f^{(2)} = -\frac{1}{x^2}$$

$$f^{(3)}(x) = \frac{2}{x^3} \quad f^{(4)}(x) = -\frac{2 \cdot 3}{x^4}$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Product Rule

$$\frac{d}{dx} u(x)\,v(x) = u'(x)\,v(x) + u(x)\,v'(x)$$

differentiate again

$$\frac{d^2}{dx^2} u(x)\,v(x) = \left(u''(x)\,v(x) + u'(x)\,v'(x) \right) + \left(u'(x)\,v'(x) + u(x)\,v''(x) \right)$$

$$= u''v + 2u'v' + uv''$$

and again

$$\frac{d^3}{dx^3} uv = \left(u'''v + u''v' \right) + 2\left(u''v' + u'v'' \right) + \left(u'v'' + uv''' \right)$$

$$= u'''v + 3u''v' + 3u'v'' + uv'''$$

See the pattern?

The numbers are Binomial coefficients. In general

$$\frac{d^n}{dx^n} u(x)\, v(x) = \sum_{p=0}^{n} \binom{n}{p} u^{(n-p)}(x)\, v^{(p)}(x)$$

Leibniz' rule

where $\binom{n}{p} = \dfrac{n!}{p!\,(n-p)!}$

= # of ways of choosing p objects from n irrespective of order.

Formula useful if u or v is polynomial

<u>Example</u> $f(x) = x^2 e^{2x}$

$u(x) = e^{2x}$, $v(x) = x^2$

$$v^{(1)} = 2x, \quad v^{(2)} = 2, \quad 0 = v^{(3)} = v^{(4)} = \dots$$

$$u^{(n)} = 2^n e^{2x}$$

$$\frac{d^n}{dx^n} f(x) = \binom{n}{0} u^{(n)} x^2 + \binom{n}{1} u^{(n-1)} \cdot 2x$$

$$+ \binom{n}{2} u^{(n-2)} \cdot 2 + 0$$

$$= 2^n e^{2x} x^2 + n\, 2^{n-1} e^{2x}\, 2x$$

$$+ \tfrac{1}{2} n(n-1)\, 2^{n-2} e^{2x} \cdot 2$$

$$= e^{2x} \left[2^n x^2 + 2^n x n + n(n-1) 2^{n-2} \right]$$

Another example

$f(x) = \sin^{-1} x$ (not a product!)

$$f^{(1)}(x) = \frac{1}{\sqrt{1-x^2}} \qquad f^{(2)} = \frac{x}{(1-x^2)^{\frac{3}{2}}}$$

$$f^{(2)}(x) = \frac{x}{1-x^2} f^{(1)}(x)$$

or $(1-x^2) f^{(2)}(x) = x f^{(1)}(x)$

Differentiate n times using Leibniz

LHS ($f^{(2)} = u$)

$$\binom{n}{0} f^{(2+n)}(1-x^2) + \binom{n}{1} f^{(1+n)} \cdot -2x$$

$$+ \binom{n}{2} f^{(n)} \cdot -2 \qquad \binom{n}{2} = \tfrac{1}{2} n(n-1)$$

RHS $\binom{n}{0} f^{(1+n)} x + \binom{n}{1} f^{(n)} \cdot 1$

$$(1-x^2) f^{(n+2)} - 2nx\, f^{(1+n)} - n(n-1)\, f^{(n)}$$

$$= x\, f^{(1+n)} + n\, f^{(n)}$$

Consider case $x = 0$

$$f^{(n+2)}(0) - n(n-1)\, f^{(n)}(0) = n\, f^{(n)}(0)$$

$$f^{(n+2)}(0) = n^2 f^{(n)}(0)$$

$$f^{(1)}(0) = 1 \qquad f^{(2)}(0) = 0$$

$$f^{(3)}(0) = f^{(1+2)} = 1^2 f^{(1)}(0) = 1$$

$$f^{(5)}(0) = 3^2 f^{(3)}(0) = 3^2$$

$$f^{(7)}(0) = 3^2\, 5^2$$

$$f^{(9)}(0) = 3^2\, 5^2\, 7^2 \quad \text{etc}$$

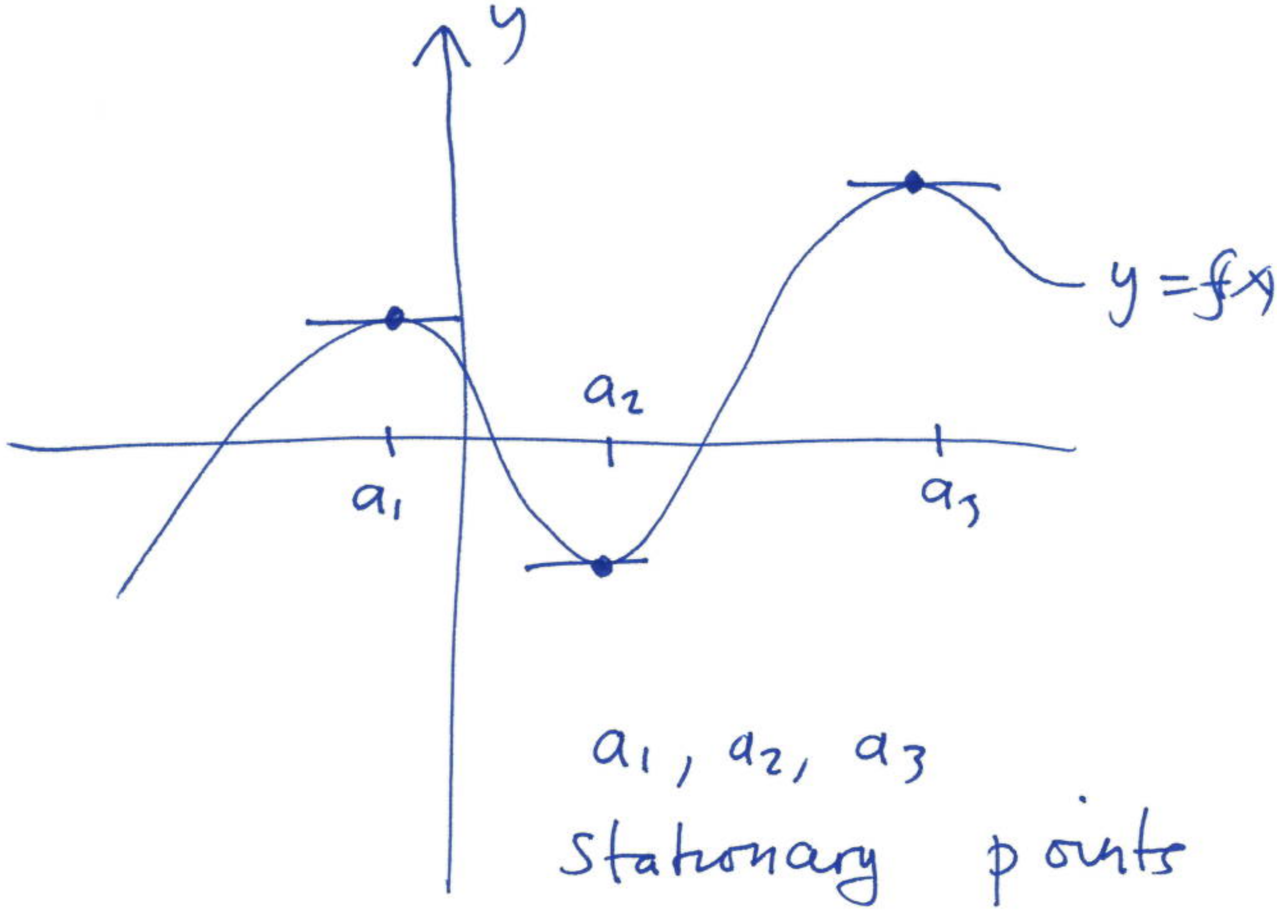
$0 = f^{(4)}(0) = f^{(6)}(0) = f^{(8)}(0) =$

(also follows as f is odd)

4 Graphs

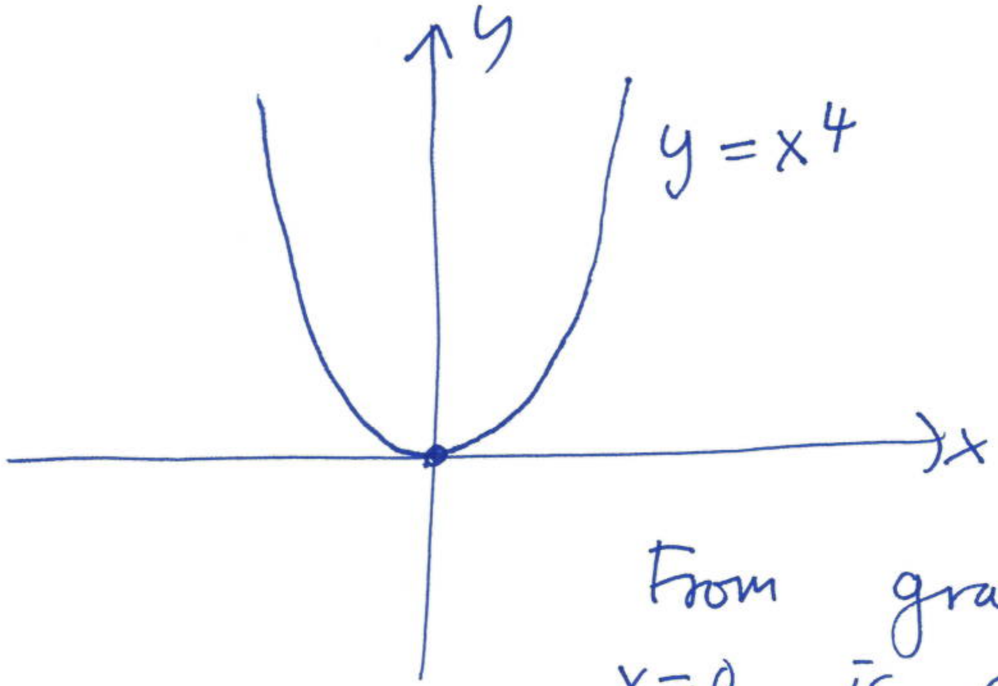
The graph of a function f is the curve defined by $y = f(x)$

$a \in dom(f)$ is called a stationary point of f if $f'(a)$ (tangent horizontal)



a_1, a_2, a_3
Stationary points

Stationary points can be local maxima (eg. a_1 and a_3 above), local minima (eg. a_2 above) or points of inflection with horizontal tangent



From graph
$x=0$ is a local
(and global) ~~max~~ minimum

$f'(x) = 4x^3$ $\quad$ $f''(x) = 12x^2$

$f'(0) = 0$ $\quad$ $x=0$ stationary point

$f''(0) = 0$ $\quad$ no information

2nd Derivative Test

Suppose a is a stationary point of f. Consider $f''(a)$.

(i) if $f''(a) > 0$ a is a local minimum

(ii) if $f''(a) < 0$ a is a local maximum

(iii) if $f''(a) = 0$ no information

<u>Example</u> $f(x) = x^4$