Office Hours

Tue 2-3 657 Huxley

Thu 9-10

General Binomial

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

$(R=1)$

Can also write this in the form

$$(1+x)^p = \sum_{m=0}^{\infty} \binom{p}{m} x^m$$

where $\binom{p}{m} = \frac{p!}{m!\,(p-m)!}$ unless p is a positive integer

have factorials of non-integers ~~or~~ or negative integers.
Can define these using Gamma function (more later...)

L'Hôpital's Rule

Suppose $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(if both limits exist)

Examples (i) $$\lim_{x \to 1} \frac{\cos\left(\frac{1}{2}\pi x\right)}{1 - x^2}$$

of form $\frac{0}{0}$

$$= \lim_{x \to 1} \frac{-\frac{1}{2}\pi \sin(\frac{1}{2}\pi x)}{-2x} = \frac{\pi}{4}$$

(ii) $$\lim_{x \to 0} \frac{\sinh x - \sin x}{x^3}$$

quickest way is power series

$$= \lim_{x \to 0} \frac{\cosh x - \cos x}{3x^2}$$

still undeterminate

use L'Hôpital again

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{6x}$$

and again

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{6} = \frac{1+1}{6}$$

$$= \frac{1}{3}$$

Can understand how L'Hôpital 'works' by expanding $f(x)$ and $g(x)$ as infinite Taylor series:

$$\frac{f(x)}{g(x)} = \frac{f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots}{g(a) + g'(a)(x-a) + \frac{1}{2} g''(a)(x-a)^2 + \dots}$$

If $g(a) \neq 0$ clear that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$$

Now suppose $g(a) = f(a) = 0$

$$= \frac{f'(a) + \frac{1}{2} f''(a)(x-a) + \cdots}{g'(a) + \frac{1}{2} g''(a)(x-a) + \cdots}$$

$x \neq a$

If $g'(a) \neq 0$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

If $g'(a) = f'(a) = 0$

$$\frac{f(x)}{g(x)} = \frac{\frac{1}{2} f''(a) + \cdots}{\frac{1}{2} g''(a) + \cdots}$$

$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}$ if $g''(a) \neq 0$

and so on

A few questions on Series

① For each of the following series decide whether it is absolutely convergent, conditionally convergent or divergent

(i) $\displaystyle\sum_{m=0}^{\infty} \frac{2^m}{2^m+3^m}$ (ii) $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{\cosh m}$

(iii) $\displaystyle\sum_{m=0}^{\infty} \left(\tan^{-1} m - \frac{\pi}{2}\right)$ (iv) $\displaystyle\sum_{m=2}^{\infty} \frac{\log m}{m^2}$

(2) Determine the radius of convergence of the following power series

(i) $\displaystyle\sum_{m=0}^{\infty} \frac{2^m x^{3m}}{1+m^2}$ (ii) $\displaystyle\sum_{m=0}^{\infty} e^{\sqrt{m}} x^m$

(iii) $\displaystyle\sum_{m=0}^{\infty} (1-\tanh m)\, x^m$

(iv) $x + x^2 + 2x^3 + 3x^4 + 5x^5$

$+ 8x^6 + 13x^7 + \cdots$

Fibonacci numbers!

6 Complex Numbers

Numbers of form

$$z = x + iy$$

are called complex numbers.

Here x and y are real numbers and i is the imaginary unit with basic property

$$i^2 = -1.$$

$\mathbb{C}$ is the set of all complex numbers

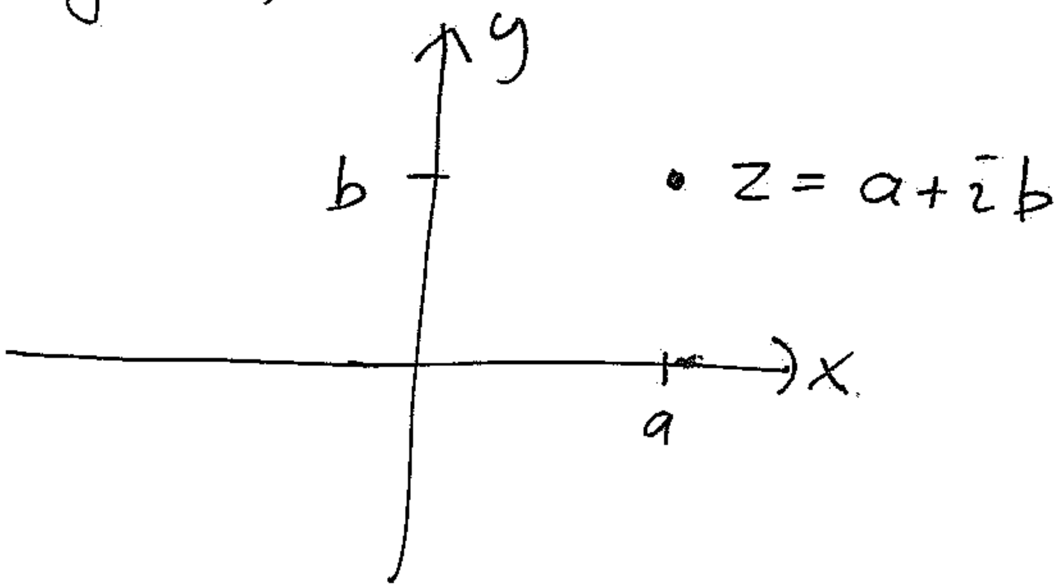
Can write

$x = \mathrm{Re}(z)$ $\qquad$ $y = \mathrm{Im}(z)$

$\uparrow$ $\qquad$ $\uparrow$

real part $\qquad$ Imaginary part

Can represent complex numbers as points in xy plane (or Argand diagram)



Using polar coordinates (r, θ) can write

$$z = r\cos\theta + i r \sin\theta$$

$x = r\cos\theta \quad , \quad y = r\sin\theta$

Modulus and Argument

The modulus of a complex number

$$z = x + iy = r\cos\theta + ir\sin\theta$$

is defined through

$$|z| = \sqrt{x^2 + y^2} = r$$

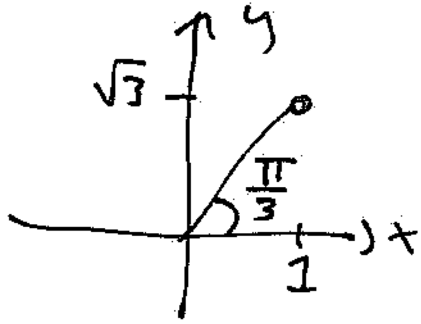
$(r \geq 0)$

The polar angle θ is an argument for the complex number z

$$\arg(z) = \theta$$

θ is not unique-
replacing θ with $\theta + 2\pi$
has no effect.

<u>Example</u> $z = 1 + i\sqrt{3}$



$|z| = 2$

$\arg(z) = \frac{\pi}{3}$

or $\arg(z) = \frac{7\pi}{3}$ also true

Can fix ambiguity by
restricting θ to interval

$$-\pi < \theta \leq \pi$$

This is called the principal value of the argument

$$\operatorname{Arg}(z) = \theta \quad \text{where}$$

$$-\pi < \theta \leq \pi$$

Eg. $\operatorname{Arg}(1 + i\sqrt{3}) = \frac{\pi}{3}$

Rules of addition and multiplication same as for reals augmented with rule $i^2 = -1$

(see MIF)

The reciprocal of a complex number $z = x + iy = r\cos\theta + i r\sin\theta \neq 0$

defined as

$$\frac{1}{z} = \frac{x - iy}{x^2+y^2} = \frac{\cos\theta - i\sin\theta}{r}$$

also written as z^{-1}

has property $z \cdot \frac{1}{z} = 1$

eg. $(1+i\sqrt{3})^{-1} = \dfrac{1-i\sqrt{3}}{4}$

<u>Powers</u> of complex numbers defined by multiplication

$$z^2 = z \cdot z \qquad z^3 = z \cdot z^2 \quad \text{etc}$$

eg. if $z = x+iy$

$$z^2 = (x+iy)^2 = x^2 + i^2 y^2 + 2ixy$$
$$= x^2 - y^2 + 2ixy$$

Negative powers defined through reciprocal

$$z^{-2} = \frac{1}{z} \cdot \frac{1}{z}$$

$$z^{-3} = \frac{1}{z} \cdot \frac{1}{z} \cdot \frac{1}{z} \quad \text{etc.}$$

Complex Power Series

A complex power series has form

$$\sum_{m=0}^{\infty} c_m z^m$$

$c_0, c_1, c_2, \ldots$ infinite list of complex numbers

Much as for reals a complex power series converges absolutely for $|z| < R$ and diverges for $|z| > R$ where $R =$ radius of convergence of complex power series

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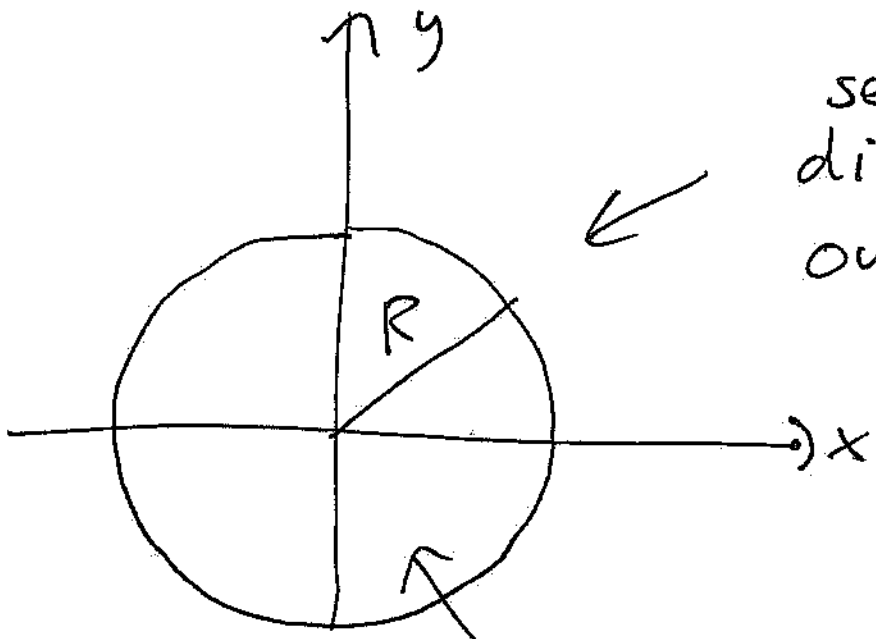


series diverges outside circle

series converges absolutely inside circle

On boundary circle series may converge or diverge.

a complex series $\sum_{m=0}^{\infty} a_m$

$a_0, a_1, a_2, \ldots$ complex

is called absolutely

convergent if $\sum_{m=0}^{\infty} |a_m|$

is convergent]

Ratio and root test apply much as for reals.

<u>Important Example</u>

The real exponential series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

valid for all real x

(or $R = \infty$)

Can define complex exponential through

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

converges absolutely for any complex z (ie $R=\infty$)

Addition formula

$$\exp(z+w) = \exp(z) \cdot \exp(w)$$

holds for any complex z and w. As with reals use notation $\exp(z)$ and e^z interchangeably.

Euler's Formula

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

with θ real. Relates a complex exponential to real trig functions.

Insert $z = i\theta$ into power series for $\exp(z)$

$$\exp(i\theta) = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \cdots$$

$$+ i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$= \cos\theta + i \sin\theta$

Euler can be written

as $e^{i\theta} = \cos\theta + i\sin\theta$

Polar form $z = r\cos\theta + i r \sin\theta$

can be written in the compact form $z = re^{i\theta}$

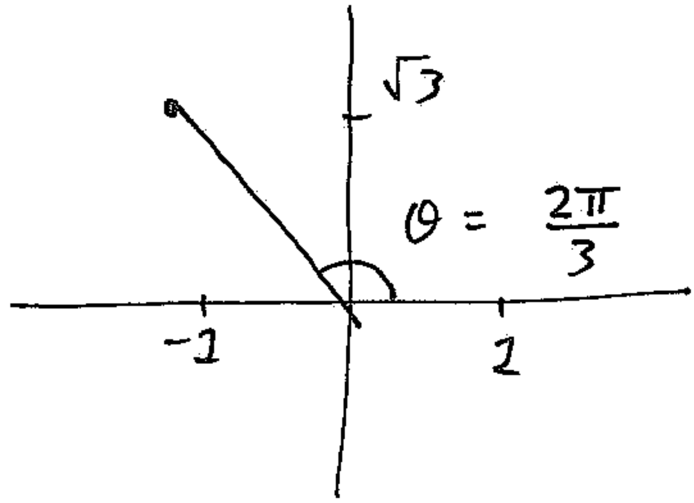
Examples

(i) $e^{i\pi} + \underline{1} = 0$

$\left(e^{i\pi} = -1, \quad \cos\pi = -1, \sin\pi = 0 \right)$

(ii) $-1 + i\sqrt{3}$ Here $|-1+i\sqrt{3}| = 2$

an argument is



$$-1 + i\sqrt{3} = 2e^{2\pi i/3}$$

Addition formulas for sine and cosine:

$$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

Follows from

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$$

Apply Euler to three exponentials

$\cos(\alpha+\beta) + i \sin(\alpha+\beta)$ LHS

$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$ RHS

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$+ i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

Real part $\rightarrow$ addition formula for cosine

Imaginary part $\rightarrow$ addition formula for sine

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Follows from

$$(e^{i\theta})^n = e^{in\theta}$$

and Euler's formula

Can recast Euler's formula as two formulas for $\sin\theta$ and $\cos\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Derive using

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$