$$f(x) = \frac{P(x)}{Q(x)} \quad \text{Rational}$$

Suppose order of $P \geq$ order of Q.

Write $P(x) = \underbrace{A(x)}_{\text{polynomial}} Q(x) + \underbrace{R(x)}_{\text{polynomial}}$

$$f(x) = A(x) + \frac{R(x)}{Q(x)}$$

where order of $R <$ order of Q.

Example $f(x) = \dfrac{x^3}{1+x^2}$

$$x^3 = x(1+x^2) - x$$

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$$

$$\int \frac{x^3}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\log(1+x^2) + c$$

Complex Integration

Consider $\displaystyle\int_a^b f(x)\,dx$

Can take f to be a complex function of a real variable

x still real

$f(x)$ can be complex

Can define

$$\int_a^b f(x)\,dx = \int_a^b \operatorname{Re} f(x)\,dx$$

$$+ \; i \int_a^b \operatorname{Im} f(x)\,dx$$

Examples

(a) $\displaystyle\int_0^1 \frac{dx}{x+i}$ (b) $\displaystyle\int_{-\pi}^{\pi} e^{ix}\,dx$

$$\frac{1}{x+i} = \frac{1}{x+i} \cdot \frac{x-i}{x-i} = \frac{x-i}{1+x^2}$$

$$\int_0^1 \cancel{dxx} \frac{1}{x+i}\,dx = \int_0^1 \frac{x}{1+x^2}\,dx - i\int_0^1 \frac{dx}{1+x^2}$$

$$= \frac{1}{2}\log(1+x^2)\Big|_0^1 - i\,\frac{\pi}{4}$$

$$= \frac{1}{2}\log 2 - i\frac{\pi}{4}$$

<u>or</u>

$$\int_0^1 \frac{dx}{x+i} = \log(x+i)\Big|_0^1$$

$$= \log(1+i) - \log(i)$$

$$= \log\sqrt{2} + i\frac{\pi}{4} - i\frac{\pi}{2}$$

$$= \frac{1}{2}\log 2 - i\frac{\pi}{4}$$

$$\int_{-\pi}^{\pi} e^{ix}\, dx$$

$$= \int_{-\pi}^{\pi} (\cos x + i \sin x)\, dx$$

$$= 0 + i0 \quad = 0$$

or $$\int_{-\pi}^{\pi} e^{ix} dx = \frac{e^{ix}}{i}\Big|_{-\pi}^{\pi}$$

$$= \frac{e^{i\pi}}{i} - \frac{e^{-i\pi}}{i} = \frac{-1}{i} - \frac{-1}{i} = 0$$

Can use complex integration for real integrals

$$I = \int_{-\pi}^{\pi} \cos^8 x \, dx$$

use $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$$I = \frac{1}{2^8} \int_{-\pi}^{\pi} \left(e^{ix} + e^{-ix} \right)^8 dx$$

$$= \frac{1}{2^8} \int_{-\pi}^{\pi} \left[e^{8ix} + \binom{8}{1} e^{6ix} + \binom{8}{2} e^{4ix} \right.$$

$$\left. + \binom{8}{3} e^{2ix} + \binom{8}{4} 1 + \cdots \right] dx$$

But $\displaystyle\int_{-\pi}^{\pi} e^{inx} dx = 0 \quad n \in \mathbb{Z}$ and $n \neq 0$

$$I = \frac{1}{2^8} \binom{8}{4} 2\pi$$

Similarly

$$\int_{-\pi}^{\pi} \cos^{2p} x \, dx \qquad p \in \mathbb{Z}, \; p > 0$$

$$= \frac{1}{2^{2p}} \binom{2p}{p} 2\pi$$

$$= \frac{1}{4^p} \frac{(2p)!}{(p!)^2} 2\pi$$

Differentiating under the integral (see problems class next week!)

Bar Notation

$F(x)\Big|_a^b$ understood to be

$F(b) - F(a)$

eg $\int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0$

$= e - 1$

Improper Integrals

These are definite integrals with infinite limits or unbounded integrands

Examples

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

$$\int_{-\infty}^\infty e^{-x^2}\,dx$$

$$\int_0^1 x^{-\frac{1}{2}}\,dx$$

$$\int_0^{\frac{\pi}{2}} x\tan x\,dx$$

Riemann's definition does not work for improper integrals (Upper Riemann sum or Lower Riemann sum or both undefined)

Can define improper integrals as particular limits of Riemann integrals

$$\int_a^\infty f(x)\,dx \quad \text{interpreted}$$

as the limit

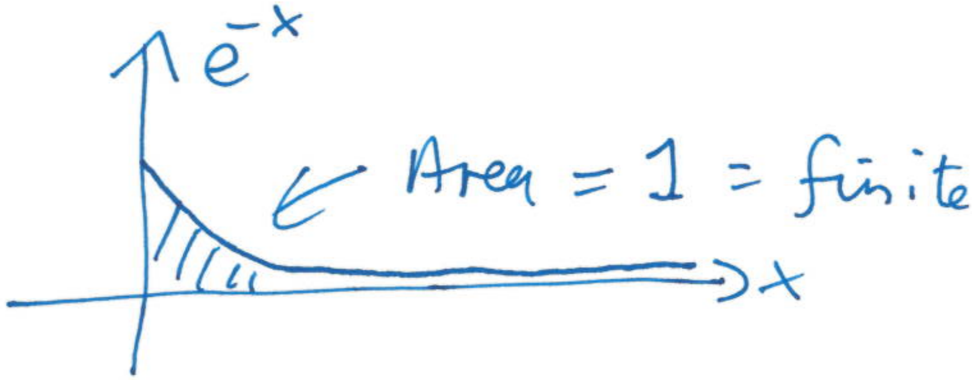
$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

Example $\quad f(x) = e^{-x}$

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left. -e^{-x} \right|_0^b$$

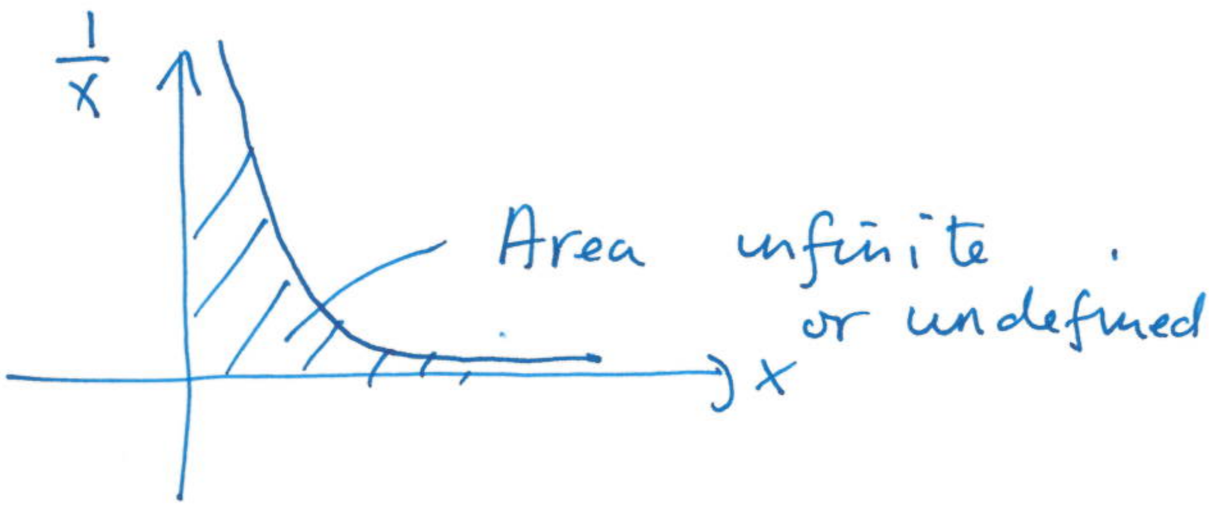
$$= \lim_{b\to\infty} \left(1 - e^{-b}\right) = 1$$



$$\int_1^{\infty} \frac{dx}{x} \stackrel{?}{=} \lim_{b \to \infty} \int_1^{b} \frac{dx}{x}$$

$$= \lim_{b \to \infty} \log x \Big|_1^b$$

$$= \lim_{b \to \infty} \log b \qquad \text{undefined}$$



Similarly $\displaystyle\int_{-\infty}^{b} f(x)\,dx$

can be interpreted a

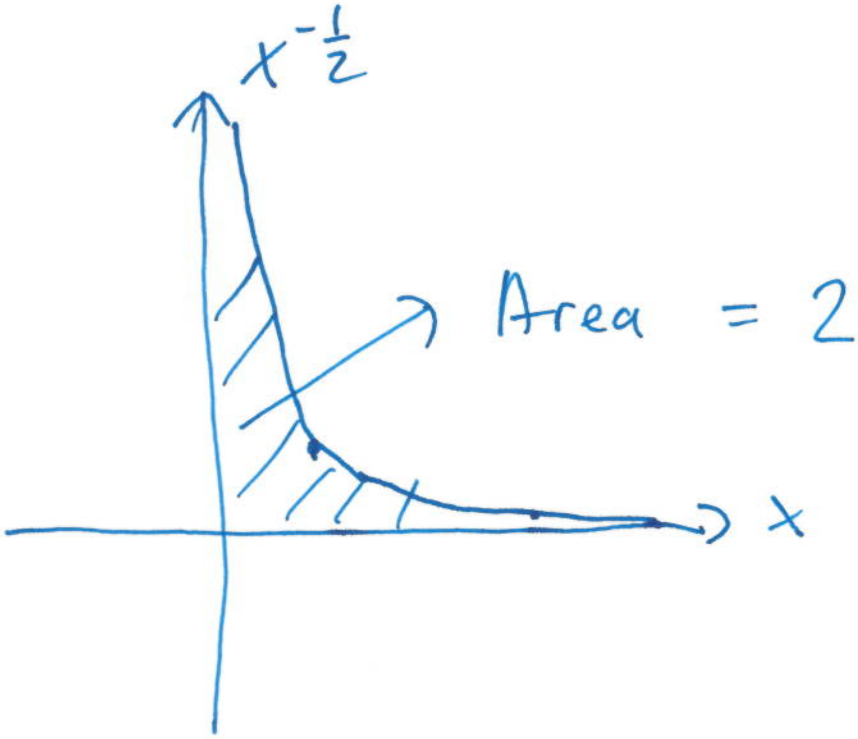
$$\lim_{a \to -\infty} \int_a^b f(x)\,dx$$

If f is unbounded (but range of integration finite) can also interpret integrals as limits.

$$\int_0^1 x^{-\frac{1}{2}}\,dx = \lim_{a \to 0^+} \int_a^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{a \to 0^+} +2x^{\frac{1}{2}} \Big|_a^1$$

$$= \lim_{a \to 0^+} \left(2 - 2a^{\frac{1}{2}}\right) = 2$$



$$\int_0^1 \frac{dx}{x} \quad \text{undefined}$$

$$\int_0^1 x^{-\frac{3}{2}}\,dx \quad \text{undefined}$$

$$\int_0^1 x^{-\frac{3}{2}}\,dx = \lim_{a \to 0^+} \int_a^1 x^{-\frac{3}{2}}\,dx$$

$$= \lim_{a \to 0^+} -2x^{-\frac{1}{2}} \Big|_a^1$$

$$= \lim_{a \to 0^+} = \left(-2 + 2a^{-\frac{1}{2}}\right)$$

undefined