In the last episode of M1F, we proved

Theorem 5.7. Say $n, d \in \mathbb{Z}_{\geq 0}$ and $r_1, r_2, \ldots, r_d \in \mathbb{Z}_{\geq 0}$ satisfy $\sum_{i=1}^{d} r_i = n$. Then $\binom{n}{r_1, r_2, \ldots, r_d} = \frac{n!}{r_1! r_2! \cdots r_d!}$

Based on a conversation with one of you in the lecture, here's another approach to proving this.

By definition, $\binom{n}{r_1, r_2, \ldots, r_d}$ is the number of ways we can put n different objects into d different boxes, with r_1 in box 1, r_2 in box 2,… r_d in box d.

One way of counting this is as follows. In step 1 we choose r_1 objects for the first box. In step 2 we choose r_2 objects from the remaining $n - r_1$ objects, and put them in the second box. And so on.

By the multiplication principle, we deduce $\binom{n}{r_1, r_2, \ldots, r_d} = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3} \cdots \binom{n-r_1-r_2-\cdots-r_{d-1}}{r_d}$. Now use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and a bunch of stuff cancels, giving another proof of Theorem 5.7. Exercise – fill in the details!

We also proved

Theorem 5.8. [the multinomial theorem]
$(a_1 + a_2 + \cdots a_d)^n = \sum \binom{n}{r_1, r_2, \ldots, r_d} a_1^{r_1} a_2^{r_2} \cdots a_d^{r_d}$, where the sum is over the ordered d-tuples $(r_1, r_2, \ldots, r_d)$ of non-negative integers such that $\sum_{i=1}^{d} r_i = n$.

Today let's do a couple of examples.

Example 1. What's the coefficient of $x_1^2 x_2 x_3$ in $(x_1 + x_2 + x_3)^4$?

Solution. This is the students in the hotel again. Picking out x_i from the jth bracket ($1 \leq j \leq 4$) is like putting the j'th student into room i. The answer is $\binom{4}{2,1,1} = \frac{4!}{2!1!1!} = 12$.

Example 2. What's the coefficient of x^3 in $(1 - x^{-3} + 2x^2)^5$?

Proof. Well, $1^{r_1}(-x^{-3})^{r_2}(2x^2)^{r_3} = (-1)^{r_2}2^{r_3}x^{2r_3-3r_2}$ so before we can apply the multinomial theorem we need to figure out the solutions to $r_1 + r_2 + r_3 = 5$ and $2r_3 - 3r_2 = 3$ in non-negative integers r_1, r_2, r_3, so we can figure out exactly when x^3 shows up in the multinomial expansion of $(1 - x^{-3} + 2x^2)^5$.

Well certainly $r_1, r_2, r_3 \leq 5$ because their sum is 5, and $2r_3 = 3 + 3r_2$. To make the right hand even we need r_2 odd, and $r_2 = 1, 3, 5$ gives $r_3 = 3, 6, 9$. The only solution which gives $r_1 = 5 - r_2 - r_3 \geq 0$ is $r_2 = 1$ and $r_3 = 3$, so we must have $r_1 = 1$ and hence the only term in the expansion which has degree 3 is $\binom{5}{1,1,3}(1)^1(-x^{-3})^1(2x^2)^3 = -\frac{5!}{1!1!3!}2^3x^3 = -160x^3$, so the answer is -160.

Chapter 6. Completeness of the real numbers.
What *are* the real numbers? They have a technical definition. Assuming you have defined the rational numbers, the real numbers are "Cauchy sequences of rationals, modulo null sequences". We will not talk about this definition here. But as a consequence of this definition, it is possible to prove a certain key fact about the real numbers, called the "completeness axiom".
Completeness axiom for the real numbers: "Any non-empty set of real numbers which is bounded above, has a least upper bound".

We will now spend the rest of the lecture explaining what these words mean.

Definition 6.1 Let $S \subseteq \mathbb{R}$ be a set of real numbers.

- We say a real number $b \in \mathbb{R}$ is an *upper bound* for S if $\forall s \in S, s \leq b$. **Note that b does not have to be in S. But it can be.**
- We say S is *bounded above* if there exists a real number $b \in \mathbb{R}$ which is an upper bound for S.

Let's work out some examples.

Let $S \subseteq \mathbb{R}$ be the set $\{1, 2, 3\}$. Is S bounded above? Can you write down five upper bounds for S? Can you write down a real number which is not an upper bound for S?

- We say a real number $b \in \mathbb{R}$ is an *upper bound* for S if $\forall s \in S, s \leq b$. **Note that b does not have to be in S. But it can be.**
- We say S is *bounded above* if there exists a real number $b \in \mathbb{R}$ which is an upper bound for S.

Example: $S = \{1, 2, 3\}$.

Yes, S is bounded above. An example of an upper bound for S: $b = 59$. Because $1 \leq 59$, $2 \leq 59$ and $3 \leq 59$, hence $\forall s \in S, s \leq 59$.

Another example: 9999999999 is an upper bound for S.

Another example: 3 is an upper bound for S, because $1 \leq 3$, $2 \leq 3$ and $3 \leq 3$, hence $\forall s \in S, s \leq 3$.

Non-example: $2\frac{1}{2}$ is not an upper bound for S, because $3 \in S$ and $2\frac{1}{2} < 3$. The negation of $\forall s \in S, s \leq b$ is $\exists s \in S, b < s$, which is what we just showed.

What's the “best” upper bound for S? It feels like the “best” upper bound should be 3.

We've seen this before:

We say a real number $b \in \mathbb{R}$ is an *upper bound* for S if $\forall s \in S, s \leq b$. **Note that b does not have to be in S. But it can be.**

Here's a new definition.

Definition 6.2. Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that $\ell \in \mathbb{R}$ is a *least upper bound* for S if it satisfies both of the following two properties:

- ℓ is an upper bound for S;
- If $b \in \mathbb{R}$ is any upper bound for S then $\ell \leq b$.

Note that ℓ does not have to be in S. But it can be. Informally, ℓ is the least of the upper bounds for S.

Let S be the set $\{1, 2, 3\}$. We have seen that S has lots of upper bounds. Maybe 3 is a *least* upper bound? We just checked that 3 was indeed an upper bound. What else do we need to do? Can you prove that 3 is a least upper bound for S?

We say that $\ell \in \mathbb{R}$ is a *least upper bound* for S if it satisfies both of the following two properties:

- ℓ is an upper bound for S;
- If $b \in \mathbb{R}$ is any upper bound for S then $\ell \leq b$.

Set $S = \{1, 2, 3\}$ and $\ell = 3$. We know ℓ is an upper bound for S.

Now say $b \in \mathbb{R}$ is any upper bound for S. Then, by definition, $\forall s \in S, s \leq b$. And $3 \in S$, so this means $3 \leq b$. Hence $\ell \leq b$ and indeed $\ell = 3$ is a least upper bound for S.

What is the set of all upper bounds of the set $\{1, 2, 3\}$?
If b is an upper bound for S, then certainly $3 \leq b$, because $3 \in S$. Conversely, if $3 \leq b$ then $\forall s \in S, s \leq b$ because $1 \leq 3 \leq b$ and $2 \leq 3 \leq b$ and $3 \leq b$. So the set of upper bounds for S is $[3, \infty)$.

Next let's think about a harder case. Set $S = (0, 1)$, that is, $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$. In M1F Sheet 2 Q2 we proved that this set had no largest element! Let's investigate its upper bounds.

We say a real number $b \in \mathbb{R}$ is an *upper bound* for S if $\forall s \in S, s \leq b$. **Note that b does not have to be in S. But it can be.**

We say that $\ell \in \mathbb{R}$ is a *least upper bound* for S if it satisfies both of the following two properties:

- ℓ is an upper bound for S;
- If $b \in \mathbb{R}$ is any upper bound for S then $\ell \leq b$.

Note that ℓ does not have to be in S. But it can be.

Set $S = (0, 1)$. Write down three upper bounds for S, and one real number which is not an upper bound for S.

Again $b = 59$ works as an upper bound, because if $s \in S$ then $s < 1$ and $1 \leq 59$, hence $s \leq 59$, and this is what we need to check if we want to show that 59 is an upper bound.

Note that -1000000000 is not an upper bound, because $\frac{1}{2} \in S$, and it is not true that $\frac{1}{2} \leq -1000000000$.

What is the "best" upper bound here? Let's set $\ell = 1$. Is it true that ℓ is a least upper bound for S? Let's try to check!

- We say that $\ell \in \mathbb{R}$ is a *least upper bound* for S if it satisfies both of the following two properties:
  - ℓ is an upper bound for S;
  - If $b \in \mathbb{R}$ is any upper bound for S then $\ell \leq b$.

  Note that ℓ does not have to be in S. But it can be.

Set $S = (0, 1)$ and $\ell = 1$. Is it true that ℓ is an upper bound for S? Yes it is, because if $s \in S$ then $s < 1$ and hence $s \leq 1$.

Now let's do the tricky part. Say $b \in \mathbb{R}$ is any upper bound for S. Remember that this means that $\forall s \in S,\ s \leq b$. Is it true that $1 \leq b$?

Say b is an upper bound for S, i.e., $\forall s \in (0,1), s \leq b$, and let's assume for a contradiction that $b < 1$. Well, $\frac{1}{2} \in S$, so certainly $\frac{1}{2} \leq b$.

Now let s be the average of b and 1, so $s = \frac{b+1}{2}$. Then

$$0 < \frac{1}{2} \leq b < \frac{b+1}{2} = s < 1.$$

So certainly $s \in S$, because we can see $0 < s < 1$. Because b is an upper bound for S, we're supposed to have $s \leq b$. But $b < s$! What has gone wrong here?

We have a contradiction here – so our assumption that $b < 1$ is incorrect. Hence $1 \leq b$. Hence…what? 1 is indeed a least upper bound.