Structure of the rest of M1F : around three lectures on binary relations and equivalence relations (chapter 8), around four lectures on functions (injective, surjective, bijective, plus countability / uncountability) (chapter 9), finishing on Thursday 13th Dec. All examinable. Then Friday 14th Dec one non-examinable lecture on $F - E + V = 2$.

Chapter 8 : Equivalence relations

Before I can tell you about equivalence relations, I need to tell you about binary relations.

Let S be a set.

Recall that an *ordered pair* of elements of S is a pair of elements (s, t) with $s \in S$ and $t \in S$, and where order matters: (s, t) is not the same as (t, s) if $s \neq t$. For example $\mathbb{R}^2$ is the set of ordered pairs of elements of $\mathbb{R}$.

Definition 7.1. A *binary relation* on S is a function, which takes as input an ordered pair of elements of S, and outputs either "true" or "false".

Formal definition. A *binary relation* on a set S is a function, which takes as input an ordered pair of elements of S, and spits out either “true” or “false”.

Example. Let S be the real numbers. The function “less than” is a binary relation. Why? Because if x and y are two real numbers, then $x < y$ is a true/false statement.

For example, we can feed in 3 and 7 into the “less than” function, and we get out the true/false statement $3 < 7$, which is true. Or we can feed in 7 and 3, and we get out the statement $7 < 3$, which is false.

Example. Let S be the set $\{1, 4, 8\}$ and let’s define a binary relation $\sim$ in the following way: $4 \sim 8$ is true, $4 \sim 4$ is true, and $x \sim y$ is false for all other choices of x and y in S. For example $8 \sim 1$ is false, and $8 \sim 4$ is false. And $8 \sim 2$? This *doesn’t even make sense*, because $2 \notin S$. There is no rhyme or reason to this binary relation, but it’s a binary relation.

A *binary relation* on a set S is a function from $S \times S$ to $\{\texttt{True}, \texttt{False}\}$.

Example. Let S be any set. Then "equality" is a binary relation on S. If $s, t \in S$ then $s = t$ is a true-false statement.

Example. Let S be any set. Then $\neq$ is a binary relation on S. If $s, t \in S$ then $s \neq t$ is a true-false statement.

Non-example. "Addition" is not a binary relation on the natural numbers. Why not? We can put two numbers, say 3 and 6, into the addition function, and the function spits out 9 which is not a true-false statement – it's a natural number.

Notation. Binary relations are often written using the notation we saw examples of above, where the relation on S is written between the two elements of S that we are relating, like $x < y$ or $s \sim t$. We do not have to use this notation.

Example. Let S be the set $\{1, 2\}$ and define a binary relation $f(x, y)$ on S by letting $f(1, 1)$, $f(1, 2)$ and $f(2, 1)$ be true, and letting $f(2, 2)$ be false. This is a binary relation, even though we are using traditional function notation here and not the cool "write the symbol in the middle" notation which we traditionally use when using $<$ and $=$.

Example. How many binary relations are there on the set $\{1, 2\}$? To give a binary relation $\odot$ on this set is nothing more and nothing less than deciding whether $1 \odot 1$ is true or false, whether $1 \odot 2$ is true or false, whether $2 \odot 1$ is true or false, and whether $2 \odot 2$ is true or false.
Hence we have to make four decisions, and we have two choices for each decision. So by the multiplication principle the number of binary relations on $S = \{1, 2\}$ is 2^4.

How many binary relations are there on the set $\{3, 4\}$? There are 16 again, because again we have to make four true/false decisions. What about on the set $\{10, 20, 30\}$? Work it out yourself! This time we have to make nine T/F decisions because if $S = \{10, 20, 30\}$ then $S \times S$ has size 9. So the answer is 2^9, which happens to be 512.

What's the formula for the number of binary relations on a finite set with n elements? It's 2^{n^2}. Does this formula work for small n? How many binary relations $\star$ are there on the set $S = \{37\}$? There are two – we have to decide whether $37 \star 37$ is true or false. How many binary relations are there on the empty set? The empty relation is the only one, so there is one relation – the function which says "if you give me any two elements of the empty set, I'll tell you either true or false" and then just sits there, confident that it will never be called.

Here's a binary relation we'll see later. Say S is a set containing 20 plastic triangles and squares, some red, some yellow, some green and some blue. I actually have this set in my office. Let's define a binary relation $\sim$ on S using the following rule: if $x, y \in S$ then let's say $x \sim y$ is true if and only if x and y are the same colour. I guess we could even drop the "is true" stuff and just write "$x \sim y \iff x$ and y are the same colour".

This is a binary relation on a set which has nothing to do with numbers.

Those of you doing M1GLA might have learnt that there is more than one way of thinking about a matrix – a matrix can be thought of as a bunch of numbers in a box, but it can also be thought of as a *linear map*. Having more than one way of thinking about an object is a really helpful way of getting good at doing questions which involve that object.

Let's look at another way of thinking about binary relations.

When are two binary relations equal? Well, when are two functions equal? They are equal precisely when they give the same values on all inputs. This is why maths is better than computer science – we don't have to worry about running times of our functions!

So let S be a set and let $\sim$ be a binary relation on S. The binary relation is hence *completely determined* by its values on $S \times S$, the set of all ordered pairs (s, t) of elements of S.

If $s, t \in S$ then there are only two possibilities for $s \sim t$ – it can either be true or false. So here is another way of thinking about binary relations – we could just give a list of all the pairs (s, t) for which $s \sim t$ is true.

To completely determine a binary relation on a set S, we could just give the subset of $S \times S$ consisting of the pairs (s, t) for which the relation is true. The relation will be false when evaluated at all the other elements of $S \times S$. Let's look at an example.

Example. Let $S = \{1, 2, 3\}$ and let's consider the binary relation $<$. Then $1 < 2$ and $1 < 3$ and $2 < 3$ are all true, and any other possibility like $1 < 1$ or $3 < 1$ is false. So another way of thinking about the binary relation $<$ on S is to say that it corresponds to the subset $R := \{(1, 2), (1, 3), (2, 3)\}$ of $S \times S$.

Let's think about this a little more. Given two elements $x, y \in S$, we can ask whether $x < y$ is true or false. One way to work this out is to look for the pair (x, y) in the set R. If it's in there, then $x < y$ is true, and if it's not then $x < y$ is false. Given the binary relation we worked out the subset R, and conversely given only the subset $R \subseteq S \times S$ we can work out the binary relation. So *R completely determines* the binary relation.

Let S be the set $\{10, 20, 30\}$ and let R be the subset of $S \times S$ defined as $R = \{(10, 10), (20, 20), (30, 30)\}$. Then R defines a binary relation on S, by $x \sim y$ if and only if $(x, y) \in R$. What is another name for this binary relation? This is just the subset of $S \times S$ corresponding to the binary relation $=$ on S.

In M1M1 you are sometimes asked to sketch graphs. For example you might be asked to sketch the graph of $y = x^3 - x$ or $x^2 + y^2 = 1$. What *is* a graph? A graph is just what you get if you colour in a subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. So a graph like $x^2 + y^2 = 1$ must somehow give rise to a binary relation on $\mathbb{R}$.

What is the binary relation $\sim$ on $\mathbb{R}$ corresponding to the graph of $x^2 + y^2 = 1$? Try and write it down. It is $x \sim y \iff x^2 + y^2 = 1$. See that $x^2 + y^2 = 1$ is a true/false statement, if x, y are fixed real numbers, and when in M1M1 you draw graphs of "functions" (is $x^2 + y^2 = 1$ even a function?) you are really drawing the subset of $S \times S$ corresponding to the binary relation, in the case $S = \mathbb{R}$.

Here is a cool binary relation on $\mathbb{Z}$. It depends on a fixed positive integer $m \in \mathbb{Z}_{\geq 1}$. We say $a, b \in \mathbb{Z}$ are related if $a \equiv b \bmod m$. Now given two integers a and b, $a \equiv b \bmod m$ is a true/false statement. So for fixed m, congruence mod m is a binary relation on $\mathbb{Z}$.

I talked about congruence mod m being "similar" to equals – here is one way that they are similar – they are both binary relations!

Non-examinable comment: In Lean, a binary relation on a type S looks like this:

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R : S → S → Prop.
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Can any Leaners see why?

We saw a random binary relation on a random set $S = \{1, 4, 8\}$ earlier. Mathematicians are often not interested in such random relations. Mathematicians prefer binary relations which have some extra structure attached to them.

Here are some nice properties that a binary relation may or may not have.

Definition 7.2. Let S be a set and let $\sim$ be a binary relation on S.

- We say $\sim$ is *reflexive* if $\forall s \in S, s \sim s$.
- We say $\sim$ is *symmetric* if $\forall s, t \in S, s \sim t \implies t \sim s$.
- We say $\sim$ is *transitive* if $\forall s, t, u \in S$, if $s \sim t$ and $t \sim u$, then $s \sim u$.
- We say a binary relation $\sim$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

A "random" relation is unlikely to be any of these things. But sensible relations which show up in mathematics might have one or more of these properties.

Let $\sim$ be a binary relation on S.

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- We say a binary relation $\sim$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

Example. Let's consider the relation $<$ on the real numbers. Is it reflexive? No it is not! The statement that $<$ is reflexive is the statement that $\forall s \in \mathbb{R}, s < s$, and this statement looks suspiciously false. But what is the *proof* that $<$ is not reflexive? The negation of reflexivity is the statement that $\exists s \in \mathbb{R}, s \not< s$. So to give a complete proof that $<$ is not reflexive, we must produce a real number! For example 59. We know 59 is not less than 59, and hence $<$ is not reflexive on $\mathbb{R}$.

Let $\sim$ be a binary relation on S.

- We say $\sim$ is *reflexive* if $\forall s \in S, s \sim s$.
- We say $\sim$ is *symmetric* if $\forall s, t \in S, s \sim t \implies t \sim s$.
- We say $\sim$ is *transitive* if $\forall s, t, u \in S$, if $s \sim t$ and $t \sim u$, then $s \sim u$.
- We say a binary relation $\sim$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

Is $<$ symmetric on $\mathbb{R}$? It is not! What's the proof? Now we need to produce two real numbers. For example, $1 < 59$ is true, but $59 < 1$ is false. Hence the binary relation $<$ on $\mathbb{R}$ is not symmetric.

Is $<$ transitive? In other words, is it true that $\forall s, t, u \in \mathbb{R}$, $s < t \wedge t < u \implies s < u$. It is! I considered this fact so important back in lecture 6 that I even gave it a name – "assumption A2".

Is $<$ an equivalence relation? It is not! What's the proof? We know $<$ is not reflexive, and hence $<$ is not an equivalence relation.

Let $\sim$ be a binary relation on S.

- We say $\sim$ is *reflexive* if $\forall s \in S, s \sim s$.
- We say $\sim$ is *symmetric* if $\forall s, t \in S, s \sim t \implies t \sim s$.
- We say $\sim$ is *transitive* if $\forall s, t, u \in S$, if $s \sim t$ and $t \sim u$, then $s \sim u$.
- We say a binary relation $\sim$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

Let S be any set, and consider the binary relation of equality on S. In other words, for $s, t \in S$ define $s \sim t \iff s = t$. Is this reflexive? Yes! Because $\forall s \in S, s = s$. Is this binary relation symmetric? Yes! Because if $s = t$ then $t = s$. And is equality transitive? Yes it is! Becuase if $s = t$ and $t = u$ then, as Euclid observed 2000 years ago, $s = u$. So is equality an equivalence relation? Yes it is! It is reflexive, symmetric and transitive, so it is an equivalence relation.

Let $\sim$ be a binary relation on S.

- We say $\sim$ is *reflexive* if $\forall s \in S, s \sim s$.
- We say $\sim$ is *symmetric* if $\forall s, t \in S, s \sim t \implies t \sim s$.
- We say $\sim$ is *transitive* if $\forall s, t, u \in S$, if $s \sim t$ and $t \sim u$, then $s \sim u$.
- We say a binary relation $\sim$ is an *equivalence relation* if it is reflexive, symmetric and transitive.

Consider the binary relation on $\mathbb{Z}$ defined by $a \sim b \iff a \equiv b \bmod 37$. Is this an equivalence relation?

Remember this?
Theorem 7.17. Say $m \in \mathbb{Z}_{\geq 1}$.
1) $\forall a \in \mathbb{Z},\ a \equiv a \bmod m$;
2) $\forall a, b \in \mathbb{Z}$, if $a \equiv b \bmod m$ then $b \equiv a \bmod m$;
3) $\forall a, b, c \in \mathbb{Z}$, if $a \equiv b \bmod m$ and $b \equiv c \bmod m$ then $a \equiv c \bmod m$.

“congruence mod m behaves in a similar way to equality”. **In other words, congruence mod m is an equivalence relation.**

Exercises!

1) The set of plastic squares and triangles, with the relationship defined by $s \sim t \iff s$ and t are the same colour – this idea – "equality of colour" – is it an equivalence relation?

2) Click on this link to go to an interactive Lean session where you can prove for yourself that congruence mod 37 is an equivalence relation.