Summary of last lecture. In the last lecture, we saw that if you have a set of plastic shapes, and you put a binary relation on this set defined by a is related to b if (and only if) they are the same colour, then this is an equivalence relation.

We then generalised this – if you have a set S, and then you *partition* it into non-empty subsets C_i (e.g. "the red ones", "the blue ones" etc), and then you define a binary relation on S by saying a is related to b if and only if they're in the same part, i.e. if $a \in C_i$ then $b \in C_i$, then this is an equivalence relation.

Example: if we partition the integers up into C_{even}, the even integers, and C_{odd}, the odd integers, then this is a partition (because every integer is exactly one of even and odd), and the associated binary relation is just that a and b are related if (and only if) $a \equiv b \bmod 2$.

Recall that a partition of S is just a decomposition of S into disjoint non-empty subsets C_i. We could think of a partition of S in the following way. The elements of S might have certain properties. For example if S is my set of plastic pieces, then each pieces has a colour (red, yellow, green, blue), and each piece has a shape (square or triangle). We could just decide to focus on one of these properties, for example the colour, and we could partition S up into subsets, each of which has the same colour. The associated binary relation is that a is related to b if (and only if) their colours are equal. Equality is a *fundamental example* of an equivalence relation, and this is the reason why equality of colour is an equivalence relation.

Here's another example. If S is the positive integers, then for an element $a \in S$ we could just decide to focus on its last digit, which will be a number between 0 and 9. We could break S up into ten subsets C_0, C_1, C_2, ..., C_9, with C_i containing all the positive integers whose last digit is i. For example $C_3 = \{3, 13, 23, 33, \ldots\}$. Associated to this partition is an equivalence relation – a is related to b if and only if a and b end in the same digit. This is an equivalence relation – and indeed the relation is that a and b are related if and only if $a \equiv b$ mod 10, and we saw in the last chapter that this was an equivalence relation.

Upshot so far: given a partition of S (that is, a decomposition of S into disjoint non-empty subsets C_i), we can construct an equivalence relation on S.

Today, we're going to go the other way. Given an equivalence relation on S, we're going to construct a partition.

At the end of the last lecture, I showed that if you have a yellow shape a, and we have the standard equivalence relation – being the same colour – on our shapes, then we can find all the yellow shapes because they are just the shapes which are related to a. Let's generalise this idea.

Let S be a set and let $\sim$ be an equivalence relation on S. If $a \in S$ then let's define the *equivalence class* of a to be

$$\mathrm{cl}(a) := \{\, b \in S \mid a \sim b \,\}.$$

In words, $\mathrm{cl}(a)$ is the set of things related to a.
Alternative notation: $[a]_\sim = \mathrm{cl}(a)$.
[NB : the advantage of the alternative notation is that it actually mentions the equivalence relation that we're making the equivalence class for; this can be handy when there is more than one equivalence relation around].

Note that $\sim$ is an equivalence relation, and hence reflexive. So one thing we know about $\mathrm{cl}(a)$ is that $a \in \mathrm{cl}(a)$, because $a \sim a$. In particular, $\mathrm{cl}(a) \neq \emptyset$. Who can guess where this is going?

If $\sim$ is an equivalence relation on S, define

$$[a]_\sim := \mathrm{cl}(a) := \{\, b \in S \mid a \sim b \,\}.$$

Lemma 7.5. Say $\sim$ is an equivalence relation on S, and $a, b \in S$. If $a \sim b$ then $\mathrm{cl}(b) \subseteq \mathrm{cl}(a)$.

Proof. How do we prove that a set is a subset of another set? What does this *mean*? By *definition* of $\subseteq$, we need to prove that

$$\forall x \in S, x \in \mathrm{cl}(b) \implies x \in \mathrm{cl}(a).$$

That's what it *means*. Now what does $x \in \mathrm{cl}(a)$ *mean*? By definition of $\mathrm{cl}(a)$, $x \in \mathrm{cl}(a)$ *means* $a \sim x$. Similarly $x \in \mathrm{cl}(b)$ *means* $b \sim x$.
So my claim that if $\sim$ is an equivalence relation then $a \sim b \implies \mathrm{cl}(b) \subseteq \mathrm{cl}(a)$ is, when you unravel the definition, is no more and no less than the claim that if $a \sim b$ and if $x \in S$ with $b \sim x$, then $a \sim x$.
And indeed $a \sim b$ and $b \sim x$ implies $a \sim x$, by transitivity.

$\square$

Lemma 7.5. Say $\sim$ is an equivalence relation on S, and $a, b \in S$. If $a \sim b$ then $\mathrm{cl}(b) \subseteq \mathrm{cl}(a)$.
[we just proved that]

Corollary 7.6. Say $\sim$ is an equivalence relation on S, and $a, b \in S$. If $a \sim b$ then $\mathrm{cl}(a) = \mathrm{cl}(b)$.

Proof. How do we prove that two subsets of S are equal? What does it *mean* to prove that two subsets of S are equal? It means that for all $s \in S$, $s \in \mathrm{cl}(a) \iff s \in \mathrm{cl}(b)$. So it means $s \in \mathrm{cl}(a) \implies s \in \mathrm{cl}(b)$ and $s \in \mathrm{cl}(b) \implies s \in \mathrm{cl}(a)$. So it means $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$ and $\mathrm{cl}(b) \subseteq \mathrm{cl}(a)$. In other words, two sets X and Y are equal if and only if $X \subseteq Y$ and $Y \subseteq X$.

Well, say $a, b \in S$ and $a \sim b$. Then $\mathrm{cl}(b) \subseteq \mathrm{cl}(a)$, by Lemma 7.5. However, by symmetry of $\sim$ we know that $b \sim a$. Hence by 7.5 again $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$. Hence $\mathrm{cl}(a) = \mathrm{cl}(b)$, by the above remarks.

$\square$

Corollary 7.6. Say $\sim$ is an equivalence relation on S, and $a, b \in S$. If $a \sim b$ then $\mathrm{cl}(a) = \mathrm{cl}(b)$.
[we just proved that]

Proposition 7.7. If $a, b \in S$ then either $\mathrm{cl}(a) = \mathrm{cl}(b)$, or $\mathrm{cl}(a) \cap \mathrm{cl}(b) = \emptyset$.

In words – distinct equivalence classes are disjoint.

Proof. Say $a, b \in S$. If $\mathrm{cl}(a) \cap \mathrm{cl}(b)$ is empty, we're done. So let's assume it's not empty and hence contains some element c of S. Then $c \in \mathrm{cl}(a) \cap \mathrm{cl}(b)$, so $c \in \mathrm{cl}(a)$ and $c \in \mathrm{cl}(b)$. Hence $a \sim c$ and $b \sim c$. Well, $b \sim c$, so by symmetry, $c \sim b$. Hence $a \sim c$ and $c \sim b$, so by transitivity $a \sim b$. By Corollary 7.6 we have $\mathrm{cl}(a) = \mathrm{cl}(b)$. $\square$

Click here to try proving 7.7 in Lean! Come along to Xena on Thursdays if you need some help, or just ask in the secret Imperial chat on Zulip.

So now let S be a set equipped with an equivalence relation $\sim$, and let's think about what we've proved about the equivalence classes $\mathrm{cl}(a)$ as a runs through S.

- We noted that $\mathrm{cl}(a)$ was non-empty – indeed $a \in \mathrm{cl}(a)$ by reflexivity.
- We proved in 7.7 that distinct equivalence classes were disjoint.

What can we conclude about this set of sets – the set of equivalence classes for $\sim$?

Recall: a *partition* is a set of non-empty subsets of S, with the property that every element of S is in exactly one of the subsets.

So let's look at the set of equivalence classes for $\sim$.

$\sim$ is an equivalence relation on S. If $a \in S$ then $a \in \mathrm{cl}(a)$, and distinct equivalence classes are disjoint.

Is the set of all equivalence classes for $\sim$ a partition of S?

Every equivalence class is non-empty (as $a \in \mathrm{cl}(a)$). Every element of S is in at least one equivalence class (as $a \in \mathrm{cl}(a)$). And if $x \in \mathrm{cl}(a)$ and $x \in \mathrm{cl}(b)$ then $\mathrm{cl}(a)$ and $\mathrm{cl}(b)$ are not disjoint, so they are the same by 7.7. So the equivalence classes form a partition of S.

[Formally we could let I be the set of all equivalence classes for $\sim$, and if $i \in I$ we could define $C_i = i$.]

So this is interesting! In the last lecture, given a partition of S we constructed an equivalence relation on S; in this lecture, given an equivalence relation on S, we constructed a partition of S.

But this is *not a big deal*. For example if r is a real number I can make a complex number z by $z = ir^2 + 5$, and given a complex number z I can make a real number r by $r = |z| - 23$. All I'm saying is that I can think of a map from real numbers to complex numbers, and a map from complex numbers to real numbers. It doesn't tell us anything.

Why is the map from partitions to equivalence relations and the map from equivalence relations to partitions different, and better, than those silly maps above?

The reason our constructions are so much better than those random maps above, are that if we start with a partition P, and then we make a equivalence relation via last lecture's construction, and then we make a new partition from the equivalence relation via this lecture's construction, then the new partition is *equal to* P. And similarly, if we start with an equivalence relation, we make the partition, and then we make the equivalence relation corresponding to that partition, it's the equivalence relation we started with. The two constructions are the *inverses* of one another. We will talk about this more in the next (and final) chapter, when we talk about *bijections*.

Theorem 7.8 Let S be a set.
(i) If $\sim$ is an equivalence relation on S, and P is the partition of S corresponding to equivalence classes for $\sim$, then the equivalence relation associated to P is the same as $\sim$.
(ii) If P is a partition of S, and $\sim$ is the associated equivalence relation, then the equivalence classes for $\sim$ are equal to the parts of P.

In words – there is a bijective (that is, one-to-one and onto – see next chapter) correspondence between the partitions of S and the equivalence relations on S.

(i) If $\sim$ is an equivalence relation on S, and P is the partition of S corresponding to equivalence classes for $\sim$, then the equivalence relation associated to P is the same as $\sim$.

Proof of (i).

Say $\sim$ is an equivalence relation, and let $P = \{C_1, C_2, \ldots\}$ be the equivalence classes for $\sim$. Now let's define a new equivalence relation $\odot$ on S by $a \odot b$ is true if (and only if) a and b are in the same C_i. In other words, $a \odot b$ is true if and only if there exists $x \in S$ such that $a \in \mathrm{cl}(x)$ and $b \in \mathrm{cl}(x)$ (actually it might be better to write $a \in [x]_\sim$ and $b \in [x]_\sim$ – do you see why?). Our goal is to prove that $\odot = \sim$. In other words, our goal is to prove that for all $a, b \in S$, $a \odot b \iff a \sim b$.

Well, if $a \sim b$ then $b \in \mathrm{cl}(a)$ (by definition), and $a \in \mathrm{cl}(a)$ (by reflexivity), and hence (setting $x = a$) we have $a \odot b$. So $a \sim b \implies a \odot b$.

Conversely, if $a \odot b$ then there exists x such that $a \in \mathrm{cl}(x)$ and $b \in \mathrm{cl}(x)$, hence $x \sim a$ and $x \sim b$, so by symmetry $a \sim x$ and now by transitivity $a \sim b$. Hence $a \odot b \implies a \sim b$.

We just showed $a \sim b \implies a \odot b$, and $a \odot b \implies a \sim b$. Hence $a \odot b \iff a \sim b$, so $\odot$ and $\sim$ agree on all inputs and are hence the same equivalence relation.

[end of proof of (i)]

(ii) If P is a partition of S, and $\sim$ is the associated equivalence relation, then the equivalence classes for $\sim$ are equal to the parts of P.

Proof of (ii).

Say P is a partition, and let $\sim$ be the corresponding equivalence relation. Let C be one of the parts of P. By definition $C \neq \emptyset$ so pick $c \in C$. By definition of $\sim$, two things are related if and only if they are in the same part. Because $c \in C$, which is a part for P, we conclude that for $d \in S$ we have $c \sim d$ if and only if $d \in C$.

But for $d \in S$, we have $c \sim d \iff d \in \mathrm{cl}(c)$! So $C = \mathrm{cl}(c)$. So every part is an equivalence class. Conversely, if $\mathrm{cl}(x)$ is an equivalence class, then $x \in C$ for some part C of P by definition of a partition, so and then by the argument above we have $C = \mathrm{cl}(X)$. So every equivalence class is a part. Hence the parts of P are just the equivalence classes for $\sim$, and this is what we wanted to prove. $\square$

The set of equivalence classes.

I will not do much with this notion, but I need to mention it. I think it is one of the hardest conceptual notions that you meet as a first year.

Everyone is happy with the idea of a subset of a set. For example the prime numbers are a subset of the positive integers, and the integers are a subset of the real numbers. Notation for this: $\mathbb{Z} \subseteq \mathbb{R}$.

There is a "dual notion" to this, that of a *quotient*. Let us talk about an example, and we'll maybe talk about the general case in the next chapter.

Here's the example. The "mod 10 world" is a consistent number system, with addition, subtraction and multiplication. The mod 10 world only has ten numbers in, namely $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If you "overflow" off the top, then you divide by 10 and take the remainder. For example $7 + 6 = 3$, and $9 \times 8 = 2$, and $3 - 9 = 4$.

The mod 10 world has a cool maths name, namely $\mathbb{Z}/10\mathbb{Z}$. When I teach schoolchildren about the mod 10 world, I *implicitly* suggest that the mod 10 world is a *subset* of the integers. It is the subset $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \subseteq \mathbb{Z}$.

This is not how I think of the mod 10 world. Why not? Hint: addition.

In the mod 10 world, we want $7 + 6 = 3$. However $7 + 6$ *is not actually 3*. So if we think of the 7 in the mod 10 world as being *equal* to the integer 7, and if we think of 6 as being *equal* to the integer 6, then we have two *different* interpretations of $7 + 6$; it might be 13, or it might be 3, depending on how we are "understanding" or "interpreting" 7.

Here's how I think of the mod 10 world. Let's define an equivalence relation on $\mathbb{Z}$ by $a \sim b$ iff $a \equiv b$ mod 10. What is cl(7), the equivalence class of 7, for this equivalence relation? By definition, it is the set of all $n \in \mathbb{Z}$ such that $7 \sim n$. So it's the set of all $n \in \mathbb{Z}$ such that $7 \equiv n$ mod 10.

For example $27 \in$ cl(7) and $1000000000000007 \in$ cl(7) and $-13 \in$ cl(7). We have cl(7) $= \{\dots, -23, -13, -3, 7, 17, 27, \dots\}$. All of those numbers in cl(7) are *the same* in the mod 10 world. More generally, every integer has a "representative" in the mod 10 world, namely its remainder after division by 10.

I don't think there is a natural map from $\mathbb{Z}/10\mathbb{Z}$ to $\mathbb{Z}$. I think that the natural map is from $\mathbb{Z}$ to $\mathbb{Z}/10\mathbb{Z}$, and this map sends 7 to 7 and it also sends 17 to 7.

When I talk about 7 in the mod 10 world, I mean cl(7).

I think that $\mathbb{Z}/10\mathbb{Z}$ should be defined to be the set of equivalence classes for the equivalence relation $a \equiv b$ mod 10. So $\mathbb{Z}/10\mathbb{Z} = \{\mathrm{cl}(0), \mathrm{cl}(1), \mathrm{cl}(2), \mathrm{cl}(3), \mathrm{cl}(4), \mathrm{cl}(5), \mathrm{cl}(6), \mathrm{cl}(7), \mathrm{cl}(8), \mathrm{cl}(9)\}$.

And I think that the natural map from $\mathbb{Z}$ to $\mathbb{Z}/10\mathbb{Z}$ is *not* "reduce mod 10 and take the remainder". I think the natural map is cl. This is what separates me from an applied mathematician. [NB I finished the lecture here, but I had prepared 1.5 more slides; here they are]

And now we don't have to break addition, because

$$\mathrm{cl}(8) + \mathrm{cl}(9) = \{\, x + y \mid x \in \mathrm{cl}(8) \wedge y \in \mathrm{cl}(9) \,\} = \mathrm{cl}(7)$$

where $+$ is here normal integer addition.

Here's another example of how equivalence classes can be used to do thing more naturally. For sure 0 and 2π are different real numbers, but they represent the same angle. How about this for a fix? Let's define a binary relation on $\mathbb{R}$ by $a \sim b$ if (and only if) $a - b = 2\pi n$ for some integer n.

The equivalence class of a number like $\pi/6$ is

$$\{\ldots, -4\pi + \pi/6, -2\pi + \pi/6, \pi/6, \pi/6 + 2\pi, \pi/6 + 4\pi, \ldots\}.$$

We could define an angle to *be* an equivalence class, and the set of all angles to be the set of all equivalence classes. One checks that if θ is an angle, then $\sin(\theta)$ is well-defined, because choosing any real number $r \in \theta$ and computing $\sin(r)$ gives us the same answer independent of our choice of r.

One can also use this equivalence class trick to define the integers and the rational numbers, given only a definition of the natural numbers – see this week's room 342.