Taylor with error term

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} (x-a)^m + R_n(x)$$

error or remainder

Formulas for $R_n(x)$

(Lagrange)

(i) $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

(Cauchy)

(ii) $$R_n(x) = \frac{f^{(n+1)}(c)}{n!} (x-c)^n (x-a)$$

(iii) $$R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t) (x-t)^n dt$$

(Integral)

Consider $n = 0$

$$f(x) = f(a) + R_0(x)$$

(i) $R_0(x) = f^{(1)}(c)\,(x-a)$

(ii) $R_0(x) = f'(c)\,(x-a)$

same as Lagrange

(iii) $R_0(x) = \int_a^x f'(t)\,dt$

$= f(x) - f(a)$ correct ✓

FToC (Fundamental Theorem of Calculus)

Can prove general case by induction (see later problems on integration)

For $n=0$ Lagrange and Cauchy give

$$f(x) = f(a) + f'(c)(x-a)$$

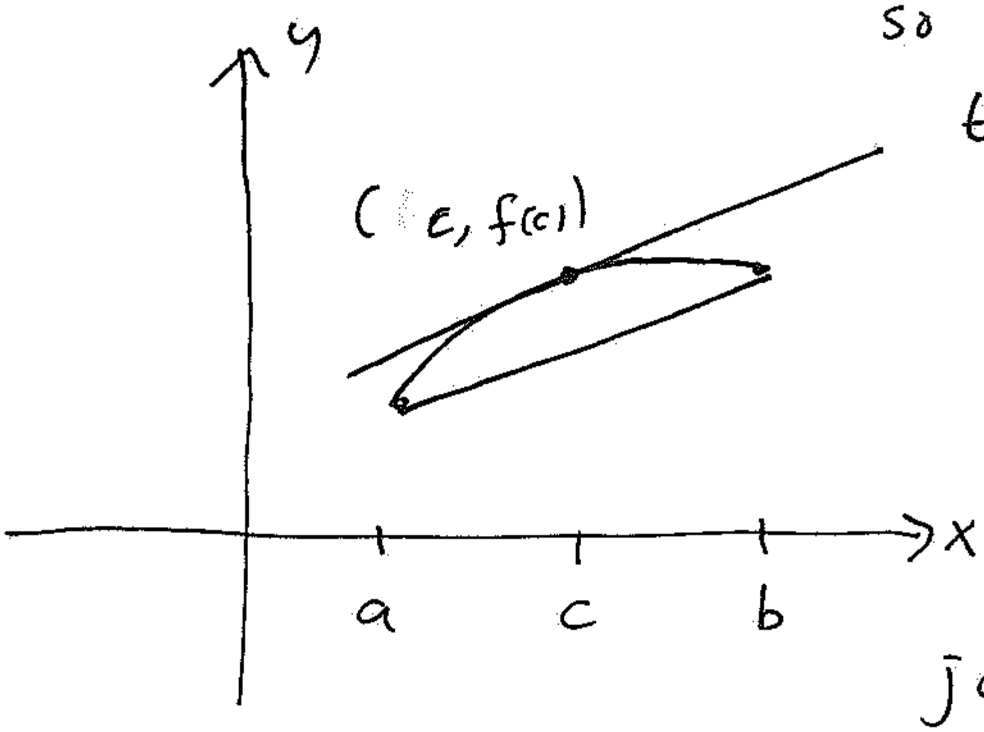
or
$$\frac{f(x)-f(a)}{x-a} = f'(c)$$

This is the Mean Value Theorem (MVT):

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Graphically $\exists c \in (a,b)$ so that the tangent at $(c, f(c))$ is parallel to the chord joining $(a, f(a))$ and $(b, f(b))$



See your analysis modules!

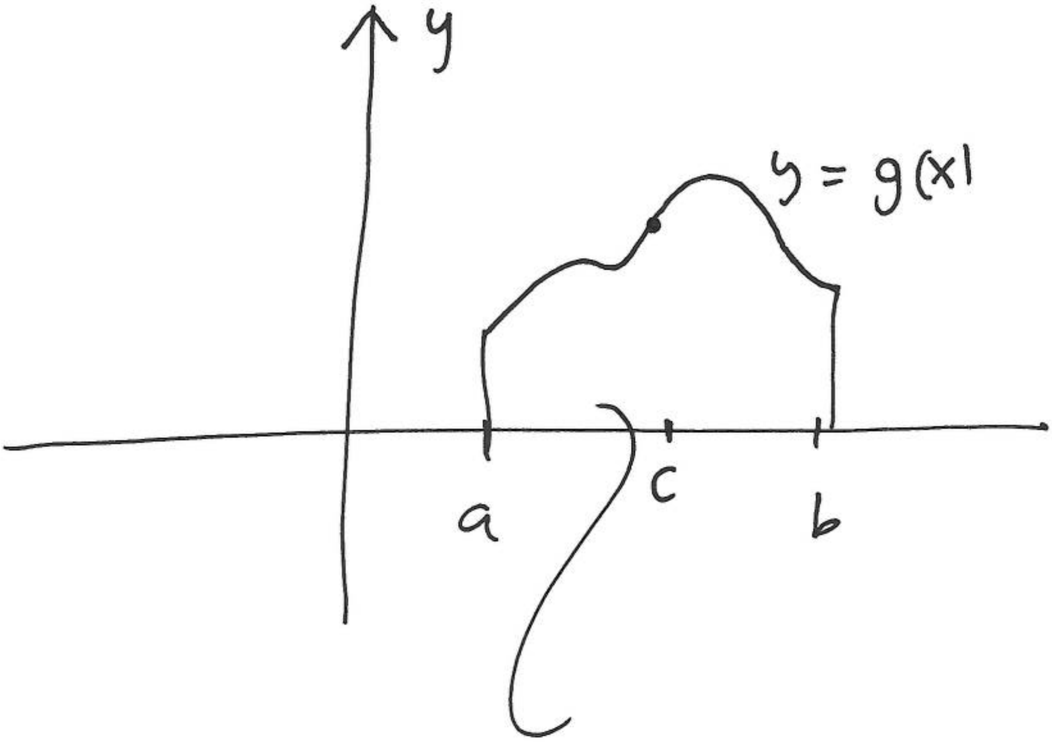
Cauchy form 'easier' to derive than Lagrange form (see problems)

Can derive Cauchy form from the Mean Value Theorem for integrals!

Suppose g is continuous on $[a,b]$

$$\int_a^b g(x)\,dx = g(c)\,(b-a)$$

for some $c \in (a,b)$



$$\text{Area} = (b-a)\ g(c)$$

where c is between a and b

$$\int_a^x g(t)\,dt = g(c)\,(x-a)$$

for some $c \in \cancel{(a,b)}$ (a,x)

Take $g(t) = \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n$

This gives Cauchy form of

remainder (assuming integral form is true).

What happens as $n \to \infty$

If $R_n(x) \to 0$ as $n \to \infty$ (for some or all x - x kept fixed as limit taken)

conclude that

$$f(x) = \lim_{n \to \infty} \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} (x-a)^m$$

or $$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!} (x-a)^m$$

This is the infinite form

$$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!} (x-a)^m$$

which is valid if f is a polynomial of degree n (or less)

<u>Example</u> $f(x) = e^x$, $a=0$ Maclaurin

$f^{(m)}(x) = e^x$, $f^{(m)}(0) = 1$

$$f(x) = \sum_{m=0}^{n} \frac{x^m}{m!} + R_n(x)$$

Use Lagrange form

$$R_n(x) = \frac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!}$$

$$= \frac{e^c x^{n+1}}{(n+1)!}$$ c between 0 and x

$$|R_n(x)| \leq \frac{e^{|x|}\,|x|^{n+1}}{(n+1)!}$$

$\to 0$ as $n \to \infty$

factorial wins!

Can write

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$ for any x

However computing limit

$\lim_{n\to\infty} R_n(x)$ can be difficult

Have formula

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!} (x-a)^m$$

which holds if $R_n(x) \to 0$ as $n \to \infty$.

The RHS of formula is a power series [in $(x-a)$ rather than x]

This has a radius of convergence R. RHS is meaningful for $-R < x-a < R$

Claim If $|x-a| < R$

can show (using Complex analysis) that $R_n(x) \to 0$ as $n \to \infty$.

That is if infinite Taylor series $\displaystyle\sum_m \frac{f^{(m)}(a)}{m!}(x-a)^m$

is absolutely convergent then it equals $f(x)$

~~Also~~ Example

General Binomial Expansion

$$f(x) = (1+x)^p \qquad p \text{ constant}$$

$$f'(x) = p(1+x)^{p-1}$$

$$f''(x) = p(p-1)(1+x)^{p-2}$$

$$\vdots$$

$$f^{(m)}(x) = p(p-1)\cdots(p-m+1)(1+x)^{p-m}$$

Infinite Maclaurin series

$$(1+x)^p = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m$$

$$f^{(m)}(0) = p(p-1)\cdots(p-m+1)$$

$$(1+x)^p = \sum_{m=0}^{\infty} c_m x^m$$

$$c_m = \frac{p(p-1)\cdots(p-m+1)}{m!}$$

Formula valid for $|x| < R$

R = radius of convergence

of power series.

Apply ratio test

to $a_m = \dfrac{p(p-1)\cdots(p-m+1)}{m!} x^m$

compute limit

$$\lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right|$$