A couple more proofs...

I couldn't face finishing the introductory chapter by doing a couple more quick proofs by contradiction.

Lemma 1.1. If x is an integer and x^2 is even, then x is even.

Proof. By contradiction. Assume x is an integer and x^2 is even.

Now assume for a contradiction that x is odd.

Then $x = 2t + 1$, so $x^2 = 4t^2 + 4t + 1$.

So $x^2 = 2(2t^2 + 2t) + 1$ is odd as well as being even – a contradiction.

Our extra assumption must be wrong – so x is even.

$\square$

Theorem 1.2. $\sqrt{2}$ is irrational.

Proof. By contradiction.

Assume for a contradiction that $\sqrt{2}$ is rational.

Write $\sqrt{2} = \frac{a}{b}$ with $a, b \in \mathbb{Z}_{\geq 1}$ and at least one of a and b odd (just cancel 2's to ensure this happens).

Square both sides and multiply up, and deduce

$$2b^2 = a^2.$$

$2b^2 = a^2$ (and at least one of a and b odd).

Now $a^2 = 2b^2$, so a^2 is even, so $a = 2c$ is even.

Hence $4c^2 = 2b^2$ giving $b^2 = 2c^2$ is even, so b is also even.

This is a contradiction! Our assumption $\sqrt{2} \in \mathbb{Q}$ is hence incorrect, and the theorem is proved.

$\square$

Lemma 1.3. $a, b \in \mathbb{Q} \implies a + b, a - b, ab \in \mathbb{Q}$.

Proof. Write $a = \frac{m}{n}$ and $b = \frac{r}{s}$ with $m, n, r, s \in \mathbb{Z}$ and $n, s \neq 0$.

Now observe that $a \pm b = \frac{ms \pm nr}{ns}$ and $ab = \frac{mr}{ns}$, so we're done …except because this is M1F it's probably also worth remarking that $ns \neq 0$.

Corollary 1.4. $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$ implies $a + b \notin \mathbb{Q}$.

Proof. By contradiction.

Let's assume our hypotheses $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$.

Let's also assume for a contradiction that the conclusion is wrong, that is, that $a + b \in \mathbb{Q}$.

Then $b = (a + b) - a$ is the difference of two rationals, and hence rational by Lemma 1.3 (the last lemma).

This contradicts our hypotheses and the proof is complete. $\square$

Corollary 1.5. There are infinitely many irrational numbers.

Proof. The real numbers $n + \sqrt{2}$ are irrational for every integer n, by the previous results. $\square$

Chapter 2: the real numbers.
I want to show you a fragment of how mathematics is built.

Let's take a look at the real numbers, and develop the theory of inequalities. Let us *assume* that we have constructed the real numbers complete with addition, subtraction, multiplication and division, and we have proved all the basic facts about these objects such as if a, b, c are real numbers then $a + b = b + a$ and $a(b + c) = ab + ac$ and $1 \times a = a$ and $0 \neq 1$ and so on.

Example: we can definitely assume $a + b + c + (d + e) = (c + e) + (d + b) + a$ for real numbers a, b, c, d, e.

Let's also assume that we have defined $<$ on the real numbers (that is, if $a, b \in \mathbb{R}$ then we have a proposition $a < b$) and let's *assume* that these propositions satisfy four fundamental properties:

A1) $\forall a, b, t \in \mathbb{R},\ a < b \implies a + t < b + t$.

A2) $\forall a, b, c \in \mathbb{R},\ a < b \wedge b < c \implies a < c$.

A3) For all $a \in \mathbb{R}$, exactly one of $a < 0$, $a = 0$ or $0 < a$ is true.

A4) $\forall a, b \in \mathbb{R}$, $0 < a \wedge 0 < b \implies 0 < ab$.

And that's all.

Define $a > b$ to mean $b < a$.
[NB to get an idea about how to prove these assumptions, try [this link]]

A1) $\forall a, b, t \in \mathbb{R},\ a < b \implies a + t < b + t$.
A2) $\forall a, b, c \in \mathbb{R},\ a < b \wedge b < c \implies a < c$.
A3) For all $a \in \mathbb{R}$, exactly one of $a < 0$, $a = 0$ or $0 < a$ is true.
A4) $\forall a, b \in \mathbb{R}$, $0 < a \wedge 0 < b \implies 0 < ab$.

Lemma 2.1. For all real numbers a and b, $a < b$ implies $-b < -a$.

Proof. Apply assumption A1 with $t = -a - b$. □

Lemma 2.2. $x < 0$ implies $-x > 0$.

Proof. Apply Lemma 2.1 with $a = x$ and $b = 0$.

A1) $\forall a, b, t \in \mathbb{R},\ a < b \implies a + t < b + t$.
A2) $\forall a, b, c \in \mathbb{R},\ a < b \wedge b < c \implies a < c$.
A3) For all $a \in \mathbb{R}$, exactly one of $a < 0$, $a = 0$ or $0 < a$ is true.
A4) $\forall a, b \in \mathbb{R}$, $0 < a \wedge 0 < b \implies 0 < ab$.

Lemma 2.2. $x < 0$ implies $-x > 0$. (proved already.)

Lemma 2.3. If $a \neq 0$ then $a^2 > 0$.

Proof. By assumption A3, either $a > 0$ or $a < 0$.

The case $a > 0$ is easy: if $a > 0$ then $a^2 = a \times a$ and we know $0 < a \times a$ by assumption A4.

The other case : If $a < 0$ then $(-a) > 0$ by Lemma 2.2.

And so $a^2 = (-a) \times (-a) > 0$ by assumption A4.

$\square$

Definition. If $a, b \in \mathbb{R}$ then we define $a \leq b$ to mean either $a < b$ or $a = b$ and we define $a \geq b$ to mean $b \leq a$.

Corollary 2.4. If $x \in \mathbb{R}$ then $x^2 \geq 0$ with equality iff $x = 0$.
Proof. If $x = 0$ then $x^2 = 0$ so $x^2 \geq 0$.

If on the other hand $x \neq 0$ then $x^2 > 0$ by Lemma 2.3, and hence $x^2 \geq 0$. $\square$

A1) $\forall a, b, t \in \mathbb{R},\ a < b \implies a + t < b + t$.
A2) $\forall a, b, c \in \mathbb{R},\ a < b \wedge b < c \implies a < c$.
A3) For all $a \in \mathbb{R}$, exactly one of $a < 0$, $a = 0$ or $0 < a$ is true.
A4) $\forall a, b \in \mathbb{R}$, $0 < a \wedge 0 < b \implies 0 < ab$.

Lemma 2.5. If $x < y$ and $c > 0$ then $cx < cy$.

Proof. By assumption A1 with $t = -x$ we have $0 < y - x$.

By assumption A4 we have $0 < c(y - x) = cy - cx$.

By assumption A1 we have $cx < cy$.

$\square$

Lemma 2.6. If $0 < a < b$ and $0 < c < d$ then $ac < bd$.
Proof. We have $ac < bc$ by Lemma 2.5.

We have $b > 0$ by Assumption A2.

So we have $bc < bd$ by Lemma 2.5.

So we have $ac < bd$ by assumption A2.

$\square$

Corollary 2.7 If $x > y > 0$ then $x^2 > y^2$.

Proof. Immediate from Lemma 2.6. $\square$

Remark. More generally one can prove $x^n > y^n$ for all $n \geq 1$ by induction.

Corollary 2.8. $1 > 0$.

Proof. We know $1 \neq 0$.

So by Lemma 2.3, $1^2 > 0$.

So $1 > 0$.

$\square$

Note added after lecture.
This is where the lecture ended (I had one more lemma – if $x > 0$ then $1/x > 0$ – but I skipped it for time reasons). But during the lecture someone said to me that it basically it was hard to see why I was proving $1 > 0$, because $1 > 0$ "was obvious".
To this person – and to the others who think this way – I think you have missed the point.
We have an intuitive geometric view of the real numbers, where most of the things we proved in this lecture are of course completely obvious. The point of this lecture is to show you that the formalist's view of mathematics is not like this at all. The

point is that pure mathematics *does not allow you* to identify a formal rigorous definition of the abstract set which a pure mathematician calls "the set of real numbers", with the fluid geometric intuitive idea you have of what the real numbers actually is "in practice". The whole point of formalising the definition is to *remove all debate*.

The problem with the fluid intuitive geometric idea of the real numbers is that it leads to problems, when one person says "this fact about the real numbers is true because it is geometrically obvious", and then someone else says "my intuition does not quite agree with yours, and I think you need to prove this – or even "my intuition says it's false". This is not philosophy – we are doing mathematics here. I don't care whether your intuition says $0.99999\ldots < 1$ or $0.9999\ldots = 1$. I can *prove* that it is 1, and so that is the end of the matter.

Working with mathematics on an intuitive level is fine at school (especially when your teacher is guiding your intuition), and in courses like M1M1 – but when you start doing harder pure mathematics, people having different opinions about what is true and what needs a proof is *simply not acceptable*. Pure mathematics is black and white – there cannot be a debate. The formalist's view is what happens when you take this to extremes; I want to argue that the real numbers is an abstract mathematical object with a precise definition (which I've not given you), and *nothing* about this object can be regarded as intuitively obvious – nothing at all. The point of the lecture is to demonstrate the consequences of this viewpoint – where *everything* needs proving – absolutely everything. Even our assumptions A1 to A4 need proving – I just had to start somewhere so I assumed that we had already proved them. To build the real numbers properly from the axioms of maths would fill up the entire course.

The point of the lecture is to show you that it can be done this way, and that everything which you geometric intuitive people think is "obvious" *can be proved* – if your intuition is correct. And if you think something is obvious but I can prove it's false, then it is false and the problem is not with pure mathematics but with your human intuition.