## (c) Fundamental Theorem of Calculus (FToC)

Suppose $F$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $F'(x) = f(x)$ on $(a, b)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}$$

here $F(x) = \dfrac{x^3}{3}$

$F'(x) = x^2$

$F$ is called an anti-derivative (or primitive) for $f$

Approach to proof

Work backwards

Take any partition of $[a, b]$

$$\underset{\substack{\| \\ x_0}}{a} < x_1 < x_2 < \cdots < x_{N-1} < \underset{\substack{\| \\ x_N}}{b}$$

$$F(b) - F(a)$$

$$= F(x_N) - F(x_0)$$

$$= F(x_N) - F(x_{N-1})$$

$$+ F(x_{N-1}) - F(x_{N-2})$$

$$+ F(x_{N-2}) - F(x_{N-3})$$

$$\vdots$$

$$F(x_1) - F(x_0)$$

$$= \sum_{i=1}^{N} F(x_i) - F(x_{i-1})$$

$$= \sum_{i=1}^{N} \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \cdot (x_i - x_{i-1})$$

Use MVT on each sub-interval

$$= \sum_{i=1}^{N} f(c_i)\,(x_i - x_{i-1})$$

where $c_i$ is between $x_i$ and $x_{i-1}$ $\left( F'(x) = f(x) \right)$

Can see that

$$L(f,P) \le F(b) - F(a) \le U(f,P)$$

for any partition $P$

If $f$ R-integrable

can show that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

see later Analysis modules

An indefinite integral is another notation for anti derivative (or primitive

$\int f(x)\,dx$ has property

$$\frac{d}{dx} \int f(x)\,dx = f(x)$$

$$\int x^2\,dx = \frac{1}{3} x^3$$

If $F(x)$ is an anti-derivative so is $F(x) + c$

where $c$ is a constant.

Customary to include this constant in tables of indefinite integrals

$$\int x^2\,dx = \frac{1}{3}x^3 + c$$

Some basic indefinite integrals

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $x^n$ | $\frac{x^{n+1}}{n+1} + c \quad n \neq -1$ |
| $\frac{1}{x}$ | $\log x + c$ |
| $\sin x$ | $-\cos x + c$ |
| $\cos x$ | $\sin x + c$ |
| $e^x$ | $e^x + c$ |
| $\frac{1}{1+x^2}$ | $\tan^{-1} x + c$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $\sin^{-1} x + c$ |

Unlike differentiation integration using FToC can be difficult

$\int x \sin x \, dx$ not so difficult

$\int \sqrt{\tan x} \, dx$ harder

$\int x \tan x \, dx$ harder still

# Integration Techniques

(a) Inspection

(b) Integration by Parts

(c) Substitution

general methods

(d) partial fractions

(e) complex numbers

(f) differentiation

adapted to specific functions

(a) Inspection

Sometimes form of anti-derivative is 'obvious'. Examples

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + c$$

$$\frac{d}{dx} e^{x^3} = 3x^2 e^{x^3}$$

$$\int \frac{dx}{1+36x^2} = \frac{1}{6}\tan^{-1}(6x) + c$$

$$\frac{d}{dx}\tan^{-1}(6x) = \frac{6}{1+(6x)^2}$$

$$\int \frac{dx}{\cosh x} = \int \frac{\cosh x \, dx}{\cosh^2 x}$$

$$= \int \frac{\cosh x}{1 + \sinh^2 x} \, dx$$

$$= \tan^{-1} (\sinh x) + c$$

Note if $f(x) = \dfrac{g'(x)}{g(x)}$

$$= \frac{d}{dx} \log g(x)$$

$$\int \frac{g'(x)}{g(x)} dx = \log g(x) + c$$

$$\int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} \, dx$$

$$= - \int \frac{(-\sin x)}{\cos x} \, dx$$

$$= - \log(\cos x) + c$$

$$= \log(\sec x) + c$$

## Integration by Parts

Idea use product rule

for differentiation to

obtain anti-derivatives

Product rule

$$\frac{d}{dx} u(x)\, v(x) = u'(x)\, v(x) + u(x)\, v'(x)$$

as an integral formula

$$\int \left[ u'(x)\, v(x) + u(x)\, v'(x) \right] dx = u(x)\, v(x) + c$$

or

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

Formula useful if second integral is 'easier' that first

( trying to integrate

$$f(x) = u(x)\, v'(x)$$

$$\int \underbrace{x}_{u} \; \underbrace{\sin x}_{v'} \, dx$$

$$u' = 1 \qquad v = -\cos x$$