Today:

* Review & talk around what happened last time (eg $0<1$)

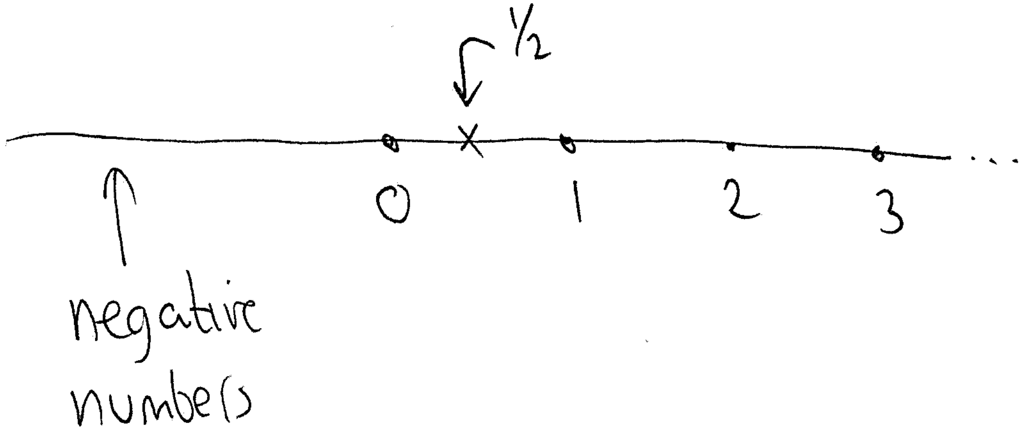
* Solving inequalities]
* Decimal expansions.]

(added later)
[note: I'll do these next time]

What was going on in the last lecture?

What we saw was how mathematicians <u>rigorously</u> <u>formalize</u> stuff that we think is "intuitively true" about the <u>ideas</u> which maths tries to represent.

Idea: The real number line.



Problem

Is $0 \cdot 9999\ldots = 1$?

Is $dx = \lim_{\delta x \to 0} \delta x$ - is it zero? or "a little bit bigger?"

Solution

Write down ~~a~~, once & for all, the definition of $\mathbb{R}$ as a set

Define $+ \times \div -$ as functions eg $+$ eats 2 real numbers & spits out ~~one~~ another (the sum).

Mathematical defn of $\mathbb{R}$

Assume we already have a mathematical ~~of~~ definition of $\mathbb{Q}$, the rationals.

$$\left(\frac{a}{b} + \frac{c}{d} := \frac{ad+bc}{bd} \quad \text{etc etc}\right)$$

Build $\mathbb{R}$ like this:

Idea:

$\pi = 3.14159\ldots$

- not rational
- but it should be the <u>limit</u> of the <u>sequence</u>

$a_0 = 3$
$a_1 = 3.1$
$a_2 = 3.14$
$a_3 = 3.141$
$a_4 = 3.1415$
$\vdots$

1st attempt:

Let $\mathbb{R}$ be the set of sequences of rational numbers!

NO GOOD

- too many

  (eg $1, -2, 3, -4, 5, -6, 7, -8, \ldots$)

- fix: Only allow sequences which "look like they converge"

- Cauchy sequences

2nd attempt:

$\mathbb{R}$ = set of Cauchy sequences of rationals.

NO GOOD

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

* Cauchy sequence
* tends to 0

$\therefore$ represents 0.

$0, 0, 0, 0, 0, \ldots$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots$

All Cauchy sequences!

3, 3·1, 3·14, 3·141, ...

Cauchy sequence representing π

4, 3·2, 3·15, 3·142, ...

↑

another Cauchy sequence representing π.

3rd attempt: π should be the set of all Cauchy sequences whose limit is π.

Turns out there is a way of saying 2 Cauchy sequences "tend to the same limit" - ie their difference tends to 0.

Final def^{n} of $\mathbb{R}$:

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an element of $\mathbb{R}$ is an infinite set of Cauchy sequences of rationals, such that any 2 elements in this set have the property that their difference tends to 0.

NOTE All Cauchy sequences with the same limit are in this set!

Intuition

$\mathbb{R}$ = number line &

$2+3$ means

"go 2 along, then go 3 along"

Actual defⁿ A real number is an infinitely big & really rather complicated set.

WHY spoil maths?

Why make $\mathbb{R}$ such an ugly & complicated object?

History tells us that you need definitions.

Problem: once you *have* a definition, you then need to prove that it coincides with your intuition.

For example,

* you have to prove that

$$0 < 1$$

* you have to define addition

* you have to prove

$$x + y = y + x.$$

Last lecture:

I zoomed in on this.

I assumed we had

* definitions of $\mathbb{R}$, $+$, $-$, $\times$, $\div$

* All standard theorems about these eg $x + y = y + x$

* Definition of $x < y$

* HARDLY ANY THEOREMS about $x < y$.

I proved lemma after lemma, & each lemma showed us that our formalist definiton of $\mathbb{R}$ agreed, in some small way, with our intuitive feeling for the number line.