Stuff

- Please don’t email me with questions about the problem sheet.
- PLUS 1300-1400 today (room 340)
- M1F Problem class 1500-1600 today (rooms 340,341,342)
- Xena tonight, after...
- **Lecture 1800-1845 on “The foundations of mathematics”, in the Clore.** Optional lecture for first year students (maths and computing).

These first few lectures have been “Chapter 1: propositions, sets and numbers”.

You are supposed to learn:

- what a proposition and a set are
- a whole bunch of notation ($\implies$ etc, $\cup$ etc, $\mathbb{R}$ etc)
- Basic methods for proving things.

Today: I will talk about $\exists$, about infinite unions, and we will do a couple of simple proofs. Then it will be the end of chapter 1.

Remember $\forall$. It means “For all”.

Example: $\forall n \in \mathbb{Z}$, n is even or n is odd.

Let me show you a link between $\forall$ and intersection.

Let us define $I = \mathbb{Z}_{\geq 1} = \{1, 2, 3, \ldots\}$.

Now let us say that for every $i \in \mathbb{Z}_{\geq 1}$ we have a set of real numbers $X_i \subseteq \mathbb{R}$.

For example X_i could be the set $[-i, i] = \{x \in \mathbb{R} \mid -i \leq x \leq i\}$.

So we have sets X_1, X_2, X_3, ... of real numbers.
I want to consider their union and their intersection.

We have sets X_1, X_2, X_3, $\ldots$ of real numbers.

We defined $X_1 \cap X_2$ already – the intersection of two sets.

What do you think the infinite intersection $\bigcap_{i=1}^{\infty} X_i$ means?

Another way to write this infinite intersection (recall $I = \{1, 2, 3, \ldots\}$ is

$$\bigcap_{i \in I} X_i.$$

But what does this mean?

Recall that $X \cap Y$ means the real numbers which are in X *and* Y.

$$\bigcap_{n=1}^{\infty} X_i$$

means the real numbers which are in *all* of the X_i.

To put it in a more mathematical way:

$$\bigcap_{i=1}^{\infty} X_i = \{\, a \in \mathbb{R} \mid \forall i \in \mathbb{Z}_{\geq 1},\ a \in X_i \,\}.$$

Can you read and understand the set on the right hand side of this equation?

Definition of a countable intersection:

$$\bigcap_{i=1}^{\infty} X_i = \{\, a \in \mathbb{R} \mid \forall i \in \mathbb{Z}_{\geq 1},\, a \in X_i \,\}.$$

This line of mathematics says that the intersection of all the X_i equals the set of real numbers a such that for *every* positive integer i, we have $a \in X_i$.

Note: “for every positive integer i” means the same thing as “for all positive integers i”.
(and note for French speakers – “positive” means “ > 0” in English, so it is not the same as “positif”).

Example: if $X_i = [-i, i]$, what is $\bigcap_{i=1}^{\infty} X_i$?

We have $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$, so the real numbers in all the X_i are just the real numbers in X_1.

So $\bigcap_{i=1}^{\infty} X_i = X_1$.

Now let's talk about unions. What do you think $\bigcup_{i=1}^{\infty} X_i$ means, if the X_i are all subsets of a "universe" Ω?

When should an element $a \in \Omega$ be in the union of the X_i?

a should be in the union if it is in *at least one* of the X_i.

To put it another way: a should be in $\bigcup_{i=1}^{\infty} X_i$ if *there exists* a positive integer j such that $a \in X_j$.

There is cool notation for this: $\exists$.

If X_1, X_2, X_3,… are all subsets of a big set Ω, then

$$\bigcup_{i=1}^{\infty} X_i = \{a \in \Omega \mid \exists i \in \mathbb{Z}_{\geq 1}, a \in X_i\}.$$

This is called a “countable union”, because the *index set I* is the set of counting numbers $\{1, 2, 3, \ldots\}$.

This whole thing works with I any indexing set though! In particular it works with “sets you can’t count”, like the real numbers (we will talk more about countability and uncountability later on).

Example: Say $I = \mathbb{R}$, the real numbers.

If $i \in I$, let's define a subset $X_i \subseteq \mathbb{R}$ by $X_i = \{i\}$.

What do you think $\bigcup_{i \in I} X_i$ is?

This *uncountable union* (I will explain "uncountable" later in the course) is the whole set of real numbers!

$I = \mathbb{R}$, $X_i = \{i\}$ for $i \in I$, so we know

$$\bigcup_{i \in I} X_i = \{\, a \in \mathbb{R} \mid \exists i \in I, a \in X_i \,\}.$$

So certainly $\bigcup_{i \in I} X_i \subseteq \mathbb{R}$.

Recall that if $P \implies Q$ and $Q \implies P$ then $P \iff Q$.

The corresponding theorem for sets: if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.

We know $\bigcup_{i \in I} X_i \subseteq \mathbb{R}$, so to prove $\bigcup_{i \in I} X_i = \mathbb{R}$ it suffices to prove that $\mathbb{R} \subseteq \bigcup_{i \in I} X_i$.

Goal: $\mathbb{R} \subseteq \bigcup_{i \in I} X_i$.

So we have to prove that if a is any real number at all, then $a \in \bigcup_{i \in I} X_i$.

By definition, $\bigcup_{i \in I} X_i = \{ a \in \mathbb{R} \mid \exists i \in I, a \in X_i \}$.

So we have to prove that if a is any real number at all, then $\exists i \in I = \mathbb{R}$ such that $a \in X_i = \{i\}$.

Brilliant idea: i is allowed to depend on a, so let's set $i = a$.

Now we have to check $a \in X_a = \{a\}$. But this is true by definition.

Conclusion: $\bigcup_{i \in I} X_i = \mathbb{R}$.

Let's finish the lecture with some simple proofs.

Let's start by thinking about how $\neg$ interacts with $\forall$ and $\exists$.

Let's say S is a set.

Let P be the proposition

$$\exists a, a \in S.$$

In words, this proposition says "the set S has at least one element".

So what is the *negation* of this proposition? What is the opposite statement? How can we understand $\neg P$?

If P is the proposition $\exists a, a \in S$, then P is the assertion that S has at least one element – it is the assertion that S is non-empty.

So $\neg P$ must be the statement that S is empty.

How can we write this using $\forall$?

$\neg P$ is the statement

$$\forall a, a \notin S.$$

The rule: if P is $\forall a \in \Omega, Q(a)$ where $Q(a)$ is some proposition that depends on a, then $\neg P$ is $\exists a \in \Omega, \neg Q(a)$.

If I have a collection of propositions, then the *logical negation* of "all of the propositions in my collection are true" is "there exists a proposition in my collection which is false".

Similarly, the opposite of "there exists a proposition in my collection which is true", is… what?

It's "all the propositions in my collection are false".

Let S be the set of positive real numbers.

In mathematical notation, $S = \{ a \in \mathbb{R} \mid a > 0 \}$.

Question: Does S have a smallest element?

I would like to phrase this more mathematically.

Let P be the proposition $\exists s \in S, \forall t \in S, s \leq t$.

Do you think P says "S has a smallest element"?

S is the positive reals.

$P := \exists s \in S, \forall t \in S, s \leq t$.
P says "there exists an element of S (call it s) such that for every element t of S, we have $s \leq t$.
So P says that S has a smallest element.

Is P true or false?

Maybe $0.0000000\ldots1$ is the smallest element of S?
Is that a real number?
Is it $1 - 0.9999999999\ldots$?

Let’s try and work out what the negation $\neg P$ of P says. Remember that P is

$$\exists s \in S, \forall t \in S, s \leq t.$$

What is $\neg P$?

It’s $\forall s \in S, \neg(\forall t \in S, s \leq t)$.

So it’s $\forall s \in S, \exists t \in S, \neg(s \leq t)$.

So it’s $\forall s \in S, \exists t \in S, t < s$.

In words – the negation of P is the proposition “for every positive real s, I can find a positive real t such that $t < s$.

Is this true?

If I have a positive real number s, how can I construct a smaller positive real number?

The easiest way I know is just to divide s by 2.

So $\neg P$ is true, so P is false and there is no smallest positive real number!