Chapter 3: in which Kevin explains to a really unimaginative person what the complex numbers are.

Remember the game: we are trying to teach a really unimaginative person what the complex numbers are. This unimaginative person is a world expert on the real numbers, they know everything about them – but they don't even know the *definition* of the complex numbers.

What is even worse – this person will *never* go "Oh – I get it! I can sort of guess how the rest of the story goes." They have to be shown *everything*. But they understand mathematical notation, and if you write something which makes sense, then they will learn it.

We will use the picture of complex numbers which we have in our mind, to guide us to the formal definition which we need to give to our unimaginative friend.

Definition. A complex number is an *ordered pair* of real numbers.

What's an ordered pair? Well, $(1, 2) = (1, 2)$, but $(1, 2) \neq (2, 1)$. In fact $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. That's the defining property of an ordered pair.

Notation: $\mathbb{C}$ denotes the set of all complex numbers.

Our *temporary* notation for a complex number: $z = (x, y)$.

We call x the *real part* of z and y the *imaginary part*.

We have $\mathbb{C} = \{(x, y) \mid x \in \mathbb{R} \wedge y \in \mathbb{R}\}$.

In our thoughts (if we know complex numbers already):
"$z = x + yi$ with $i^2 = -1$ and it all works"

But if we said this to our unimaginative friend they would say "I don't understand "$+$" and I don't understand "i" and I don't understand the "$\times$" hidden in yi either, and I don't understand how you are identifying real numbers with complex numbers."

Definition 3.2. If we have a real number r and we start talking about it as if it were a complex number, we mean the complex number $(r, 0)$.

Definition 3.3. We define the complex number i to be $(0, 1)$.

Definition 3.4. If $z = (u, v)$ and $w = (x, y)$ are complex numbers, we define their sum $z + w$ to be $(u + x, v + y)$. **Why?**

Because we are *thinking* $(u + vi) + (x + yi) = (u + x) + (v + y)i$, with real part $u + x$ and imaginary part $v + y$.

Lemma 3.5. [commutativity of addition.] If z and w are complex numbers, then $z + w = w + z$.

Proof. Set $z = (u, v)$ and $w = (x, y)$. Then $z + w = (u + x, v + y)$ and $w + z = (x + u, y + v)$. These complex numbers are equal, because their real parts are equal real numbers, and their imaginary parts are equal real numbers. $\square$

How to remember this proof: “commutativity of addition on the real numbers trivially implies commutativity of addition on the complex numbers”.

Lemma 3.6. [assocativity of addition.] If a, b and c are complex numbers, then $a + (b + c) = (a + b) + c$.

Proof. Trivially follows from associativity of addition on the real numbers. $\square$

While we're here, let's define negatives and subtraction: if $z = (x, y)$ then $-z$ is defined to be $(-x, -y)$ and $z - w$ is defined to be $z + (-w)$.

Exercise. Take any standard fact about subtraction, for example $a - (b - c) = (a - b) + c$. Assume it's true for the real numbers. Deduce it for the complex numbers.

Now let's do multiplication.

Definition 3.7 [multiplication of complex numbers.] Say $z = (x, y)$ and $w = (u, v)$ are complex numbers. Let's define the product $z \times w = zw$ to be...what?

What we think: $(x + yi)(u + vi) = (xu - yv) + (xv + yu)i$.

What we tell our friend: $(x, y) \times (u, v) = (xu - yv, xv + yu)$.

Lemma 3.8 [assocativity of multiplication.] If $t = (r, s)$, $w = (u, v)$ and $z = (x, y)$, then $(tw)z = t(wz)$.

Proof. Multiply both sides out and observe that they are the same.

$\square$

Remark. Associativity of addition (the fact that it doesn't matter which order you add three things up) on the complexes followed easily from associativity of addition on the real numbers.

But associativity of multiplication on the complexes does not follow trivially from associativity of multiplication on the reals. $((r+si)(u+vi))(x+yi) = (rux - rvy - suy - rvy) + (ruy + rvx + sux - svy)i = (r+si)((u+vi)(x+yi))$, and in some sense we're lucky that it all works out.

Remark. Did you ever check associativity of multiplication of complex numbers before? And did you ever use it?

Lemma 3.9 [distributivity of multiplication over addition.] If a, b, c are complex numbers, then $a(b+c) = ab + ac$.

Proof. Multiply both sides out and observe that they are the same. [details omitted]

$\square$

Slogan: "expanding out the brackets works with complex numbers."

Lemma 3.10 If 1 is the complex number $(1, 0)$, then for all complex numbers z we have $1 \times z = z$.

Proof. Multiply both sides out and observe that they are the same. Let's do it. Say $z = (x, y)$. Then $1 \times z = (1, 0)(x, y) = (1x - 0y, 1y + 0x)$ which does indeed equal $(x, y) = z$. $\square$

The complex numbers obey all the normal rules like associativity of addition and multiplication, and distributivity of multiplication over addition – *but these things need to be checked!* However they *only need to be checked once.*

Maybe some of you have heard of *groups*. A *group* is some abstract mathematical object with some structure (multiplication, identity, inverses), satisfying some axioms (associativity etc). You will learn about groups next term.

There are lots more words like this, like *commutative ring* and *field* and *perfectoid space* and so on, and those of you who specialise in pure maths will learn a whole bunch more before you graduate.

The lemmas above show that the complex numbers are a *commutative ring*.

Just before we go on – remember when we did the "real numbers" – they had $+ - */$ and satisfied all the usual things, and they had $<$ and satisfied axioms A1 to A4?

The axioms said stuff like $a < b$ and $b < c$ implies $a < c$.

When you were doing those questions, your mental model of what was going on was that a and b and c were elements of the "real number line".

What if I told you that I was lying to you all along, and that chapter 2 was *actually* all about the rational numbers?

Or what if I told you that it was actually all about the set of real numbers of the form $x + y\sqrt{2}$ with x and y both rational numbers?

Or what if I told you that it was actually all about the hyperreal numbers?

All of the proofs still work. In fact they work for any ordered field.

Lemma 3.11 $i^2 = -1$.

Proof. Multiply both sides out and observe that they are the same.
full proof:
$(0, 1) \times (0, 1) = (0 \times 0 - 1 \times 1, 0 \times 1 + 1 \times 0) = (-1, 0)]$

$\square$

Theorem 3.12. If x and y are real numbers, identified with complex numbers in the usual way, then $x + yi = (x, y)$.

Remark. This lemma finally implies that we can use our usual notation for complex numbers.

Proof. Multiply both sides out and observe that they are the same. Indeed
$x+yi = (x, 0)+(y, 0)(0, 1)= (x, 0)+(y\times 0-0\times 1, y\times 1+0\times 0)= (x, 0) + (0, y)= (x + 0, 0 + y)= (x, y)$.

Here's a theorem of a different nature. Can we *order* the complex numbers? How should we define $z < w$ for z and w complex numbers?

Lemma 3.13. There is no way of ordering the complex numbers in such a way that they satisfy axioms A1 to A4 of the previous chapter.

Proof. From axioms A1 to A4 we deduced that all non-zero squares were positive. In particular we deduced that $1 > 0$, and hence $-1 < 0$. But -1 is a square and hence $0 < -1$. This contradicts axiom A3, which said that these things can't both happen at the same time.

□

Corollary 3.14 If you write $z < w$ for z and w complex numbers, *you will lose marks.*
Proof. By inspection.

Let's have a look at the complex numbers with human eyes, and see what we can see that might be useful. [cut to pen and paper]
[we talked about the modulus and argument of a complex number, and complex conjugation]