Mathematical Methods (M1M1) **Autumn 2017**

Chapter 6: $\mathbb{C}$omplex $\mathbb{N}$umbers

Reference G. Stephenson 'Mathematical Methods for Science Students' chapter 7.

Complex Numbers

A complex number z is specified by a pair of real numbers x and y and written

$$z = x + iy$$

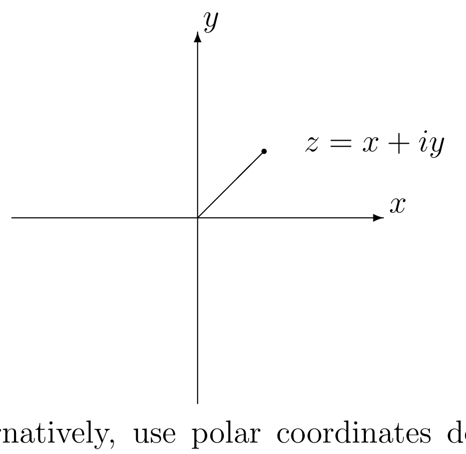
where i is the imaginary unit. i has the basic property

$$i^2 = -1.$$

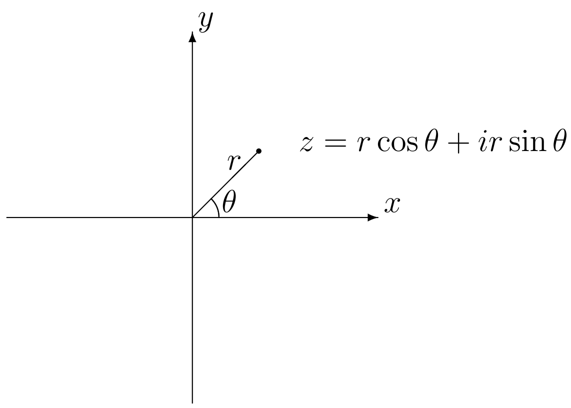
Use the abbreviation x for $x + i0$, iy for $0 + iy$ and 0 for $0 + i0$. The set of complex numbers is denoted $\mathbb{C}$. Two elements $x + iy$ and $u + iv$ are equal if and only if $x = u$ and $y = v$. Given $z = x + iy$ in $\mathbb{C}$; x is the real part of z and y is the imaginary part

$$x = \text{Re }(z), \qquad y = \text{Im }(z).$$

It is convenient to represent complex numbers geometrically as points of a plane (complex plane) known as the Argand diagram.



Alternatively, use polar coordinates defined through $x = r\cos\theta$, $y = r\sin\theta$.



Modulus and Argument

The modulus $|z|$ of $z = x + iy$ is defined to be

$$|z| = \sqrt{x^2 + y^2} = r,$$

which is the distance from z to the origin in the complex plane.

The angle θ in the polar form $z = r\cos\theta + ir\sin\theta$ is an argument of z. The angle θ is not unique; replacing θ with $\theta + 2\pi$ yields the same z. For example $z = 0 + i1$ which is normally written i has modulus 1 and argument $\pi/2$. $\arg(i) = \pi/2$ or $\arg(i) = \pi/2 + 2\pi n$ with n integer. This ambiguity can be removed by restricting the range of θ to $-\pi < \theta \leq \pi$ (or another interval of length 2π). The principal value of the argument of z, denoted Arg (z) is the argument θ for which $-\pi < \theta \leq \pi$.

Algebra in the Complex Plane

The rules of addition are the same as for reals:

$$(x + iy) + (u + iv) = (x + u) + i(y + v).$$

Multiplication rules are the same as for reals but with the additional condition $i^2 = -1$

$$(x + iy) \cdot (u + iv) = xu + iyu + xvi + i^2yv = (xu - yv) + i(xv + yu).$$

Note that $|z_1z_2| = |z_1||z_2|$ for any z_1, z_2 in $\mathbb{C}$. As for reals

$$\left.\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1z_2 &= z_2z_1 \end{aligned}\right\} \quad \text{commutative laws}$$

$$\left.\begin{aligned} z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 \\ (z_1z_2)z_3 &= z_1(z_2z_3) \end{aligned}\right\} \quad \text{associative laws}$$

for any z_1, z_2, z_3 in $\mathbb{C}$.

For $z = x+iy$ in $\mathbb{C}$ $-z$ is defined to be $(-x)+i(-y)$. Clearly $z+(-z) = 0$ (recall $0 = 0 + i0$).

Complex Reciprocal

For $z = x + iy \neq 0$, $1/z$ is defined through

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

and is also written z^{-1}. As for real numbers the reciprocal has the property

$$z \cdot \frac{1}{z} = 1.$$

Note that

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}.$$

In polar form $z = r\cos\theta + ir\sin\theta$, the reciprocal is obtained by replacing r with $1/r$ and θ with $-\theta$:

$$\frac{1}{z} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}.$$

Example

The reciprocal of $z = \sqrt{3} + i$ is

$$\frac{1}{z} = \frac{\sqrt{3}}{3+1} - \frac{i}{3+1} = \frac{\sqrt{3} - i}{4}.$$

Powers of Complex Numbers

As with reals, powers of complex numbers are defined through multiplication

$$z^2 = z \cdot z, \qquad z^3 = z^2 \cdot z \qquad \text{etc.}$$

For example, if $z = x + iy$

$$z^2 = (x+iy)(x+iy) = x^2 + i^2y^2 + 2ixy = x^2 - y^2 + 2ixy.$$

Negative powers can be defined via the reciprocal

$$z^{-2} = \frac{1}{z} \cdot \frac{1}{z}, \qquad z^{-3} = \frac{1}{z} \cdot \frac{1}{z} \cdot \frac{1}{z}, \quad \text{etc.}$$

Complex Power Series

Complex power series can be defined in much the same way as for reals:

$$\sum_{m=0}^{\infty} c_m z^m$$

where c_0, c_1, c_2, ... is an infinite list of complex numbers. The notion of absolute convergence and the ratio and root test carry through - here absolute values are replaced with the complex modulus. Such a series converges (absolutely) for $|z| < R$ and diverges for $|z| > R$ where R is a positive constant. That is the series converges for z in a disc of radius R in the complex plane.

Recall the Maclaurin series for the exponential

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

which is valid for all real x, ie. $R = \infty$. *Define* the exponential of a complex number z via the complex power series

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

This converges for any complex z, ie. $R = \infty$. As for reals this has the property that

$$\exp(z + w) = \exp(z)\exp(w),$$

for any complex numbers z and w. As for reals we will use $\exp(z)$ and e^z interchangeably.

Euler's Formula

This important formula links exponentials to trigonometric functions

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

or

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

To understand where this striking formula comes from insert $z = i\theta$ into the definition

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

This gives

$$\exp(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + ...$$

Note that odd powers are imaginary while even powers are real. Separating odd and even powers

$$\exp(i\theta) = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right)$$

$$= \cos\theta + i\sin\theta,$$

since the real part is just the Maclaurin series for $\cos\theta$ and the imaginary part is the Maclaurin series for $\sin\theta$. We have already considered the polar form $z = r\cos\theta + ir\sin\theta$. Using Euler's formula this can be shortened to

$$z = re^{i\theta}.$$

An amusing example is $z = -1$, for which $r = 1$ and $\theta = \pi$ so that $-1 = e^{i\pi}$ or

$$1 + e^{i\pi} = 0,$$

a formula involving 5 constants 0, 1, e, π and i!

Another example: $\sqrt{3} + i$ can be written in standard polar form as $2(\cos(\pi/6) + i\sin(\pi/6))$ or just $\sqrt{3} + i = 2e^{i\pi/6}$.

Euler's formula is very powerful and can be used to derive trigonometric identities with ease. Recall the addition formulas

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

To derive these formulas use (the much simpler) addition formula for the exponential $e^{z+w} = e^z e^w$; setting $z = i\alpha$ and $w = i\beta$ gives

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}.$$

Now apply Euler's formula to each of the three exponentials

$$\cos(\alpha + \beta) + i\sin(\alpha + \beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta).$$

The real part of this equation is the addition formula for the cosine while the imaginary part is the addition formula for the sine function.

De Moivre's theorem, which is the result

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

follows from Euler's formula - just use $(e^{i\theta})^n = e^{in\theta}$. For $n = 2$ de Moivre's theorem gives the double angle formulas for the sine and cosine.

Euler's formula expresses an exponential in terms of trigonometric functions - one can do the reverse

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

To derive these use $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$.

Radius of Convergence of a Complex Power Series

A complex power series $\sum_m c_m z^m$ converges (absolutely) for $|z| < R$ and diverges for $|z| > R$ where R is the radius of convergence of the power series. It is now clear why R is called a 'radius' as it is the radius of a disc in the complex plane within which the power series is meaningful. R can be evaluated in much the same way as for real power series via the application of standard tests for absolute convergence (the ratio test being the most useful).

We note that there is a very powerful method for computing R which is based on some results from year 2 Complex Analysis. The theory will not be developed here but the method itself is very simple and very useful:

There is a theorem [1] that states that if a complex function is *holomorphic* in a disc of radius r centred at $w \in \mathbb{C}$ then $f(z)$ can be expressed in the form

$$f(z) = \sum_{m=0} c_m (z-w)^m,$$

AND this converges absolutely for $|z - w| < r$.

What does holomorphic mean? It is usually understood to mean that f is *complex differentiable* an idea which we don't have time to develop. However, there is an equivalent definition we can use:

$f(z)$ is holomorphic at $w \in \mathbb{C}$ if f can be expanded as a complex Taylor series about $z = w$ with a finite radius of convergence.

We can restate the theorem as follows: If $f(z)$ is Taylor-expandable at every point in a disc then the Taylor expansion about the centre of the disc converges everywhere in the disc.

To exploit this to compute R we simply need to find the 'singular' points about which $f(z)$ cannot be expanded as a power series and from there work out the largest disc that does not include such singularities. In simple terms to compute R we just need to find the points where $f(z)$ 'blows up'.

[1] In Complex Analysis this result would be called 'Taylor's Theorem'. See for example chapter 14 of 'Complex Analysis' by H. A. Priestley (2nd edition).

For example consider the complex function

$$f(z) = \frac{1}{e^z + 1}.$$

This can be expanded as a power series of the form $\sum_m c_m z^m$. We can compute the first few coefficients c_0, c_1, c_2 etc using the methods of previous chapters. This is not going to give us R since to compute R using the ratio or root test we would require knowledge of c_m for large m. However, it is clear that $f(z)$ is well behaved unless $e^z = -1$ - so the function 'blows up' at $z = in\pi$ where n is any odd integer. A disc centred at the origin will not include singularities provided the radius of the disc is less than π. Therefore the radius of convergence is π. The radius of convergence about any point $w \in \mathbb{C}$ is the distance between w and the nearest singularity.

Complex Conjugation

Given $z = x+iy$ in $\mathbb{C}$ the complex conjugate of z is defined to be $\bar{z} = x-iy$ or in polar form $\bar{z} = re^{-i\theta}$ if $z = re^{i\theta}$. In the Argand diagram $\bar{z}$ is the reflection of z about the x (or real) axis.

The following hold if z, z_1 and z_2 are complex numbers

i) $\bar{\bar{z}} = z$
ii) Re $(z) = \frac{z+\bar{z}}{2}$ and Im $(z) = \frac{z-\bar{z}}{2i}$
iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
iv) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
v) $|\bar{z}| = |z|$

Complex Polynomials

A complex polynomial (of degree n) is an expression of the form

$$P(z) = c_0 + c_1 z + c_2 z^2 + ... + c_n z^n.$$

Here c_0, c_1, c_2, ...,c_n ($c_n \neq 0$) are complex constants, and $z = x + iy$ is a *complex variable.*

The Fundamental Theorem of Algebra asserts that every polynomial $P(z)$ has at least one root. In general, a polynomial of degree n will have n roots (in some cases these are repeated roots so that the number of solutions may be less than n). Moreover, *any* polynomial can be factorised:

$$P(z) = c_n(z - a_1)(z - a_2)...(z - a_n)$$

where a_1, a_2,...,a_n are the (complex) roots. If the coefficients c_0, c_1,..., c_n are real the roots a_1, a_2,..., a_n can still be complex. However, in this case they occur in complex conjugate pairs, ie. if a_1 is complex $\bar{a}_1$ will also be a root.

A simple example is $P(z) = z^2 + 1 = (z+i)(z-i)$ where the two roots i and $-i$ are complex conjugates of each other.

It is easy to prove that the roots of complex polynomials with real coefficients occur in complex conjugate pairs. Suppose that a is a root of $P(z) = c_0 + c_1 z + c_2 z^2 + ... + c_n z^n$ where the c_i are real. Then

$$c_0 + c_1 a + c_2 a^2 + ... + c_n a^n = 0.$$

Now take the complex conjugate of this equation

$$\bar{c}_0 + \bar{c}_1 \bar{a} + \bar{c}_2 \bar{a}^2 + ... + \bar{c}_n \bar{a}^n = 0.$$

But the coefficients c_i are assumed to be real, giving $\bar{c}_0 = c_0$, $c_1 = \bar{c}_1$, etc. Therefore

$$c_0 + c_1 \bar{a} + c_2 \bar{a}^2 + ... + c_n \bar{a}^n = 0,$$

or $P(\bar{a}) = 0$ so that $\bar{a}$ is a root of P. This means that if the coefficients of $P(z)$ are real, then any root of P is real or one of a complex conjugate pair of roots.

Example Factorise the complex polynomial $P(z) = z^4 + z^2 + 1$. Here $P(z) = 0$ can be recast as a quadratic equation for z^2

$$(z^2)^2 + z^2 + 1 = 0.$$

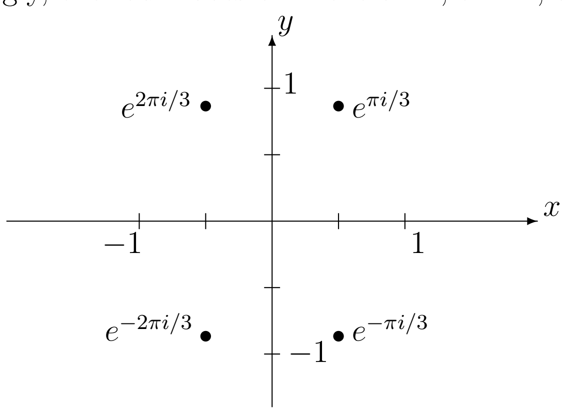
This can be solved via the standard formula for the roots of a quadratic

$$z^2 = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = e^{\pm 2\pi i/3}.$$

Now solve $z^2 = e^{2\pi i/3}$. An obvious solution is $z = e^{i\pi/3}$. Another solution is $z = e^{-2\pi i/3}$. To see this write $z^2 = e^{2\pi i/3} \cdot 1 = e^{2\pi i/3} e^{2\pi i} = e^{8\pi i/3}$ from which $z = e^{4\pi i/3} = e^{-2\pi i/3}$ is a solution.

Similarly, $z = e^{-i\pi/3}$ and $z = e^{2\pi i/3}$ are solutions of $z^2 = e^{-2\pi i/3}$.

Accordingly, the four roots of P are $e^{i\pi/3}$, $e^{-i\pi/3}$, $e^{2\pi i/3}$, $e^{-2\pi i/3}$.



$P(z)$ has the factorised form

$$P(z) = \left(z - e^{\pi i/3}\right)\left(z - e^{-i\pi/3}\right)\left(z - e^{2\pi i/3}\right)\left(z - e^{-2\pi i/3}\right),$$

or

$$P(z) = \left(z - \frac{1 + i\sqrt{3}}{2}\right)\left(z - \frac{1 - i\sqrt{3}}{2}\right)\left(z - \frac{-1 + i\sqrt{3}}{2}\right)\left(z - \frac{-1 - i\sqrt{3}}{2}\right).$$

where the complex exponentials have been converted into cartesian form.

Real Polynomials

Consider a real polynomial

$$P(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n,$$

where the coefficients c_0, c_1, ..., c_n are real and x is a real variable. Even though this is a real polynomial we can still write

$$P(x) = c_n(x - a_1)(x - a_2)...(x - a_n)$$

where a_1, a_2,...,a_n are the roots of the complex polynomial $c_0 + c_1 z + c_2 z^2 + ... + c_n z^n$. Now the a_i may be complex - but in this case they come in complex conjugate pairs - if a_1 is complex then another root, say a_2, is equal to $\bar{a}_1$. Here $(x - a_1)(x - a_2) = (x - a_1)(x - \bar{a}_1)$ is a real quadratic. For example

$$x^4 + x^2 + 1 = \left(x - e^{\pi i/3}\right)\left(x - e^{-i\pi/3}\right)\left(x - e^{2\pi i/3}\right)\left(x - e^{-2\pi i/3}\right)$$

The first two factors are complex conjugates of each other so their product is a real quadratic

$$\left(x - e^{\pi i/3}\right)\left(x - e^{-i\pi/3}\right) = x^2 - x(e^{\pi i/3} + e^{-\pi i/3}) + 1 = x^2 - 2\cos(\pi/3)x + 1 = x^2 - x + 1.$$

Similarly, $\left(x - e^{2\pi i/3}\right)\left(x - e^{-2\pi i/3}\right) = x^2 + x + 1$. Therefore

$$x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1).$$

In general, any real polynomial can be factorised into a product of real linear and real quadratic terms. The linear factors are associated with real roots and the quadratic factors with complex conjugate pairs of roots.

Complex Functions

For real variables a function f is a rule assigning a real number $f(x)$ to an element x of the domain of f. A function of a complex variable f (or just a complex function) assigns a complex number $f(z)$ to a complex number z.

Examples

i) Already considered complex polynomials.

ii) A complex power series

$$f(z) = \sum_{m=0}^{\infty} c_m z^m$$

defines a complex function for $|z| < R$ where R is the radius of convergence of the complex power series.

iii) Complex exponential. $f(z) = \exp(z)$ defines a complex function. Writing $z = x + iy$ gives

$$f(z) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

using Euler's formula.

iv) Complex sine and cosine. Recall that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Here θ is real - this can be extended to any complex number z

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The complex sine and cosine have the complex power series representations

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

which hold for all $z \in \mathbb{C}$.

Other trigonometric functions can be defined in terms of the complex sine and cosine

$$\sec z = \frac{1}{\cos z}, \qquad \tan z = \frac{\sin z}{\cos z}, \quad \text{etc.}$$

v) Hyperbolic functions. Complex versions of hyperbolic functions are defined through

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

$$\text{sech } z = \frac{1}{\cosh z}, \quad \tanh z = \frac{\sinh z}{\cosh z}, \quad \text{etc.}$$

The trigonometric functions can be expressed in terms of hyperbolic functions:

$$\cos z = \cosh iz, \qquad i \sin z = \sinh iz.$$

vi) Complex Logarithm.

For real numbers the logarithm is the inverse function to the exponential so that

$$\exp(\log x) = x.$$

Similarly, the complex logarithm is defined through

$$\exp(\log z) = z.$$

The polar form $z = re^{i\theta}$ can be rewritten as

$$z = \exp(\log r + i\theta)$$

so that

$$\log z = \log r + i\theta.$$

As θ is ambiguous (adding 2π to θ does not change $z = re^{i\theta}$) so is $\log z$. The complex logarithm can also be written in the form

$$\log z = \log |z| + i \arg(z),$$

where the ambiguity in the imaginary part is manifest. As with $\arg(z)$ the ambiguity may be removed by restricting the range of θ. One can also define the principal value of the logarithm through

$$\text{Log } z = \log |z| + i\text{Arg } (z)$$

(recall that $-\pi < \text{Arg } (z) \leq \pi$). Note that Log z is discontinuous on the negative real axis. Such discontinuities are called *branch cuts.*

vii) Powers.

If $z = re^{i\theta}$ we have $\log z = \log r + i\theta$ which is defined up to integer multiples of $2\pi i$. Therefore

$$z^n = r^n e^{in\theta} = e^{n(\log r + i\theta)} = e^{n \log z}$$

For *integer* n the ambiguity in $\log z$ drops out - one can replace $\log z$ with $\log z + 2\pi i$ but $e^{n(\log z + 2\pi i)} = e^{2\pi i n} e^{n \log z} = z^n$ since $e^{2\pi i} = 1$. Suppose p is not an integer. *Define*

$$z^p = e^{p \log z},$$

but like $\log z$ itself this is ambiguous. The ambiguity in z^p can be removed by fixing the ambiguity in $\log z$. For example, choose

$$z^p = e^{p\mathrm{Log}\ z}.$$

This, like Log z, is discontinuous on the negative x-axis.

viii) Inverse Tangent.

The (real) inverse tangent function has the power series representation

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - ...$$

which has radius of convergence 1. Is there a complex version of this function with the property $\tan(\tan^{-1} z) = z$ valid for all $z \in \mathbb{C}$? The answer is no as one cannot define an inverse tangent of i or $-i$. This is why the power series has $R = 1$ (the distance between the origin and the singularities). One can write an explicit formula for the inverse tangent

$$\tan^{-1} z = \frac{1}{2i}\mathrm{Log}\ \frac{1+iz}{1-iz},$$

which can be viewed as a 90 degree rotation (in the complex plane) of the real formula

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}.$$

The complex function $\tan^{-1} z$ has branch cuts ending at the singularities at $\pm i$.