# Conics

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

Degenerate cases

nothing
point
line
2 lines

Non-degenerate cases

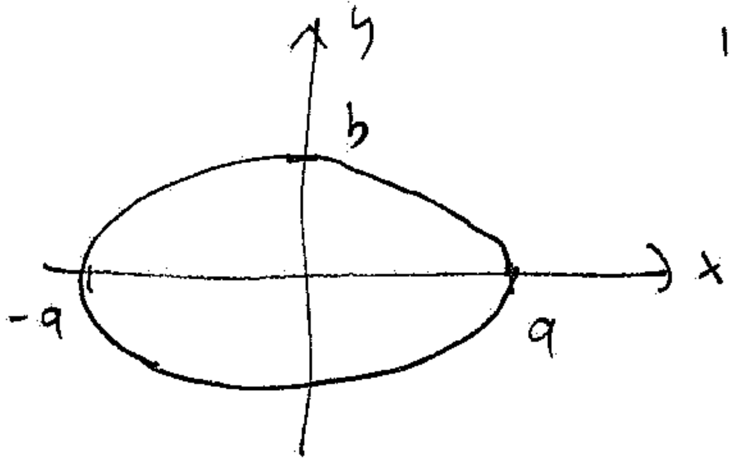
ellipse

parabola

hyperbola

# Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a, b \neq 0$$



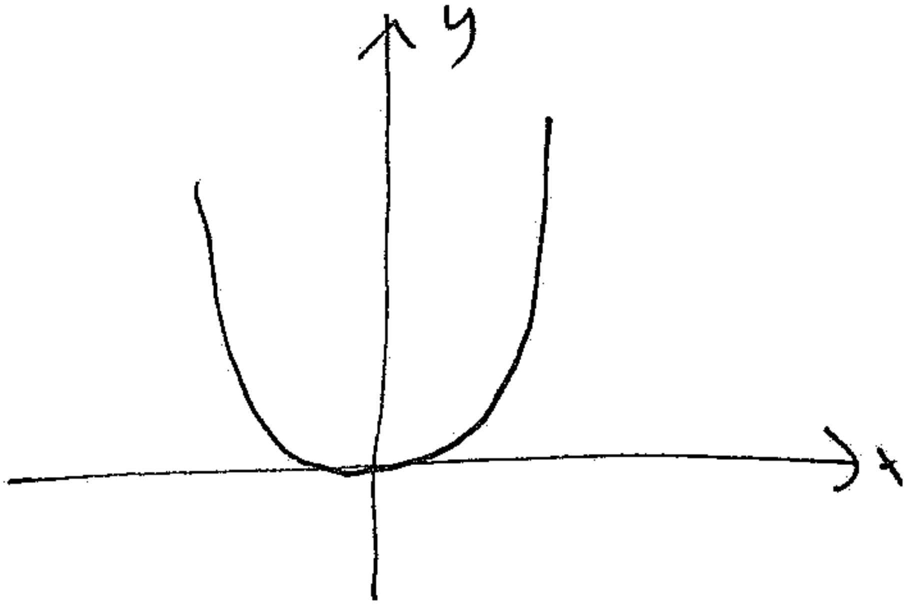

if $a = b$ reduces to a circle

defines an ellipse. Any graph which can be obtained from this ~~ell~~ standard ellipse by rotation or translation is also an ellipse

# Parabola

$$y = c x^2 \qquad c \neq 0$$

defines a parabola



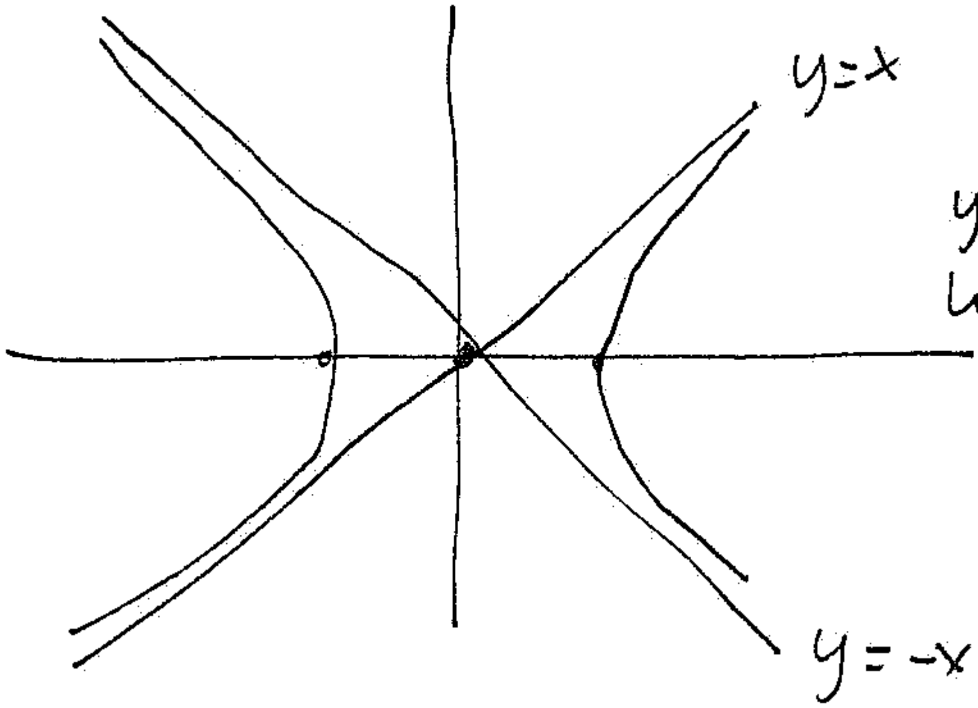

Again translating or rotating this basic parabola also gives a parabola

# Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ $a, b$ non zero

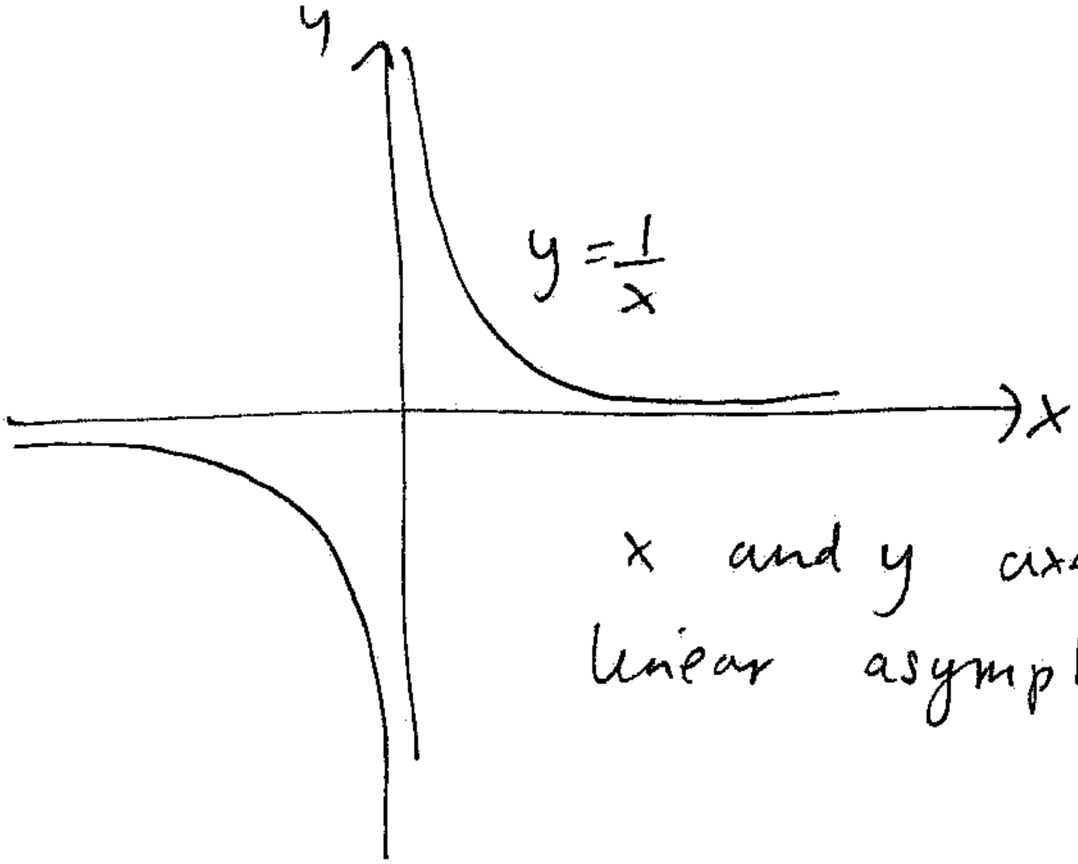
defines a hyperbola

eg. $a = b = 1$




$y = \pm x$ linear asymptotes

$y = \frac{1}{x}$ is also a hyperbola




x and y axes
linear asymptotes

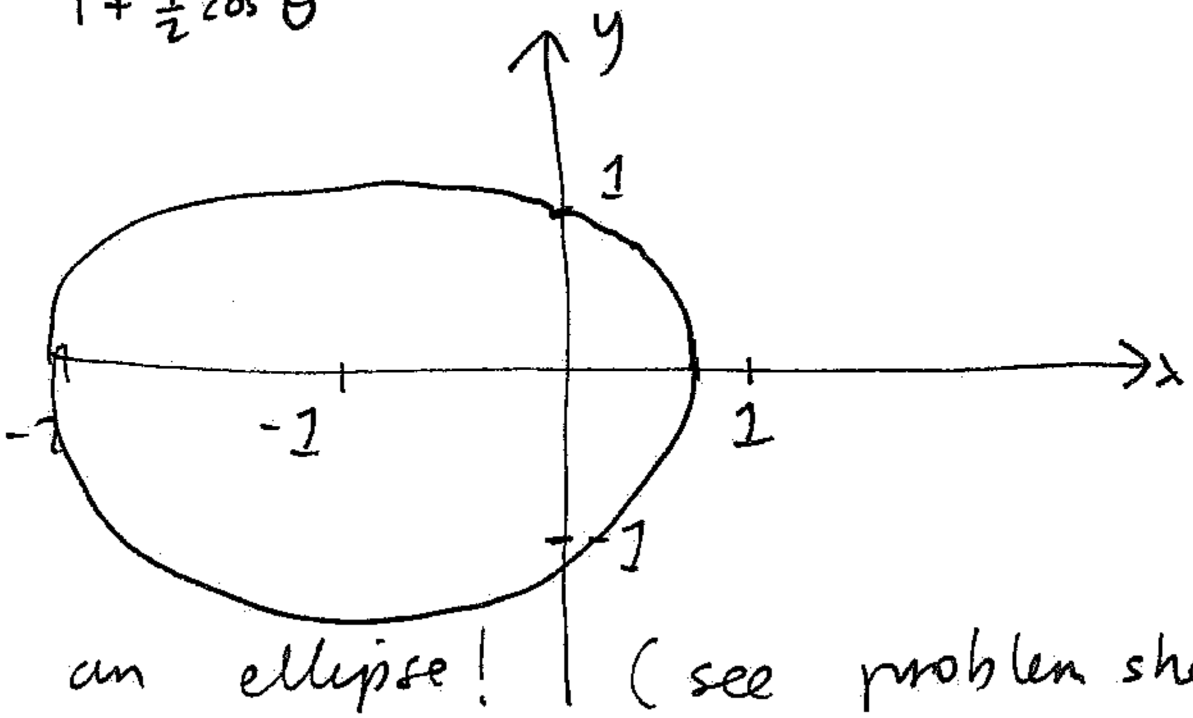
Returning to polar coordinates

$$r = \frac{L}{1 + e\cos\theta}$$

defines a conic

<u>Example</u> $L = 1, \quad e = \frac{1}{2}$

$$r = \frac{1}{1 + \frac{1}{2}\cos\theta}$$



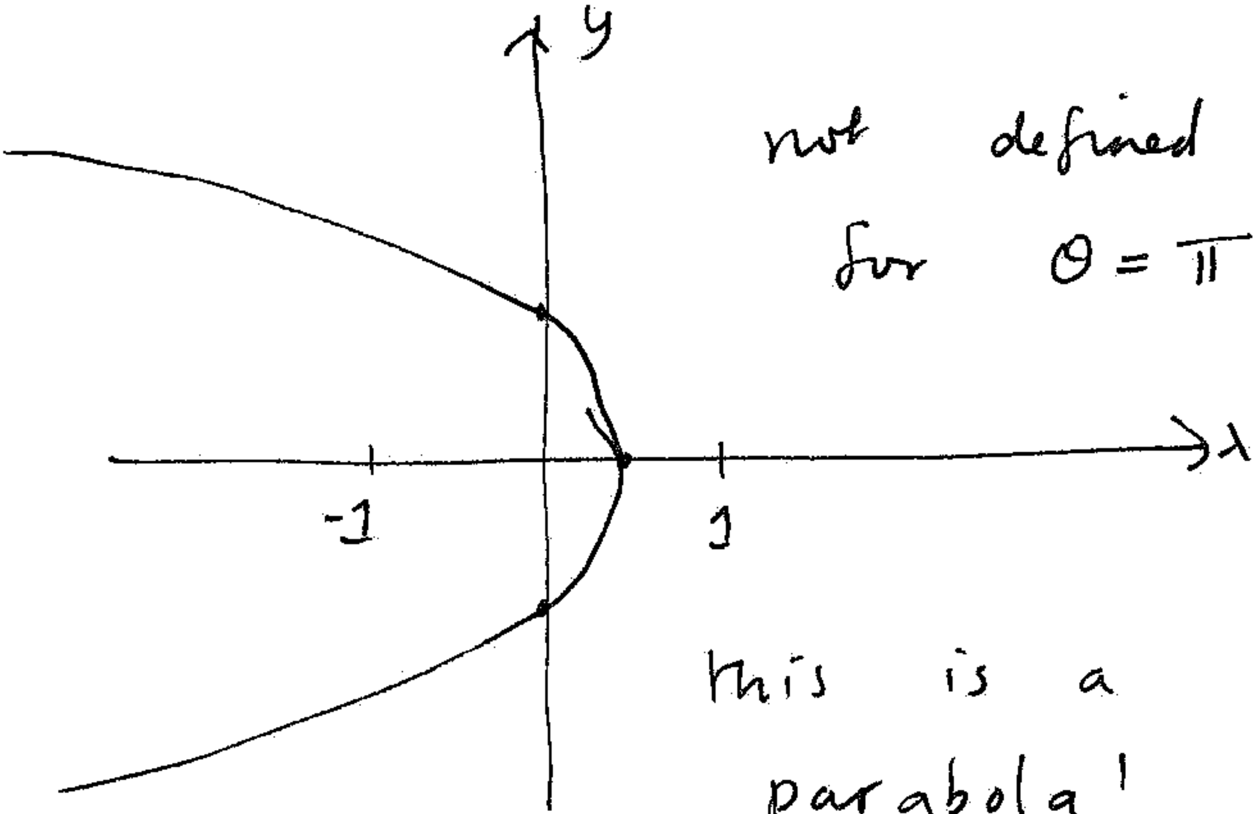

an ellipse! (see problem sheet 4)
centre of ellipse not at origin. Origin is one ~~fou~~ focus of the ellipse

Claim $r = \dfrac{L}{1 + e\cos\theta}$

is an ellipse if $0 \leq e < 1$

as $e$ increases ellipse is stretched

$\underline{e=1}$ $$r = \frac{1}{1+\cos\theta}$$



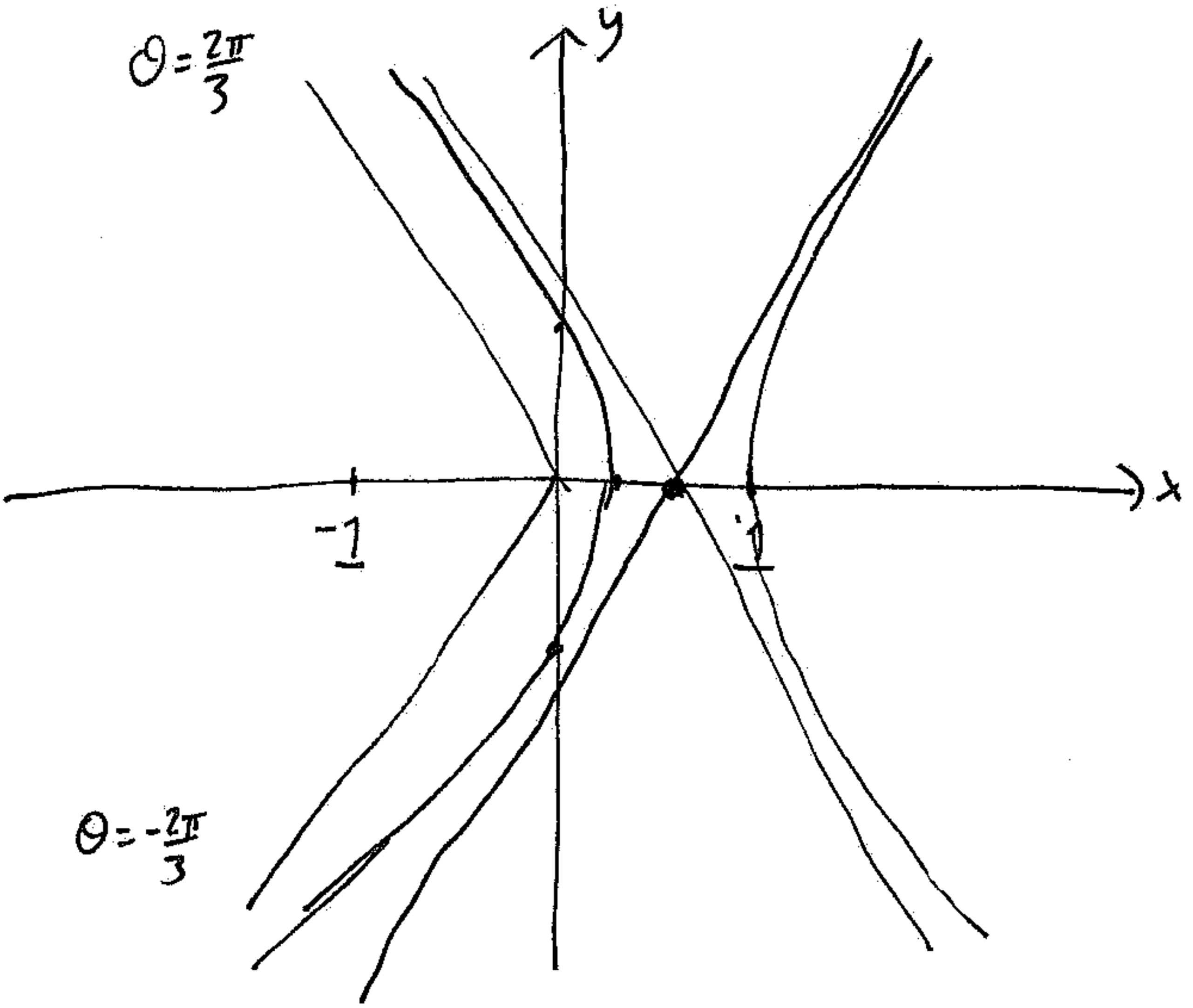

not defined for $\theta = \pi$

this is a parabola!

for $e>1$ polar curve is a hyperbola (or a branch thereof)

$e=2$

$$r = \frac{1}{1 + 2\cos\theta}$$




$r$ undefined when $\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$ or $-\frac{2\pi}{3}$

linear asymptote

$x = r\cos\theta$

$y = r\sin\theta$

One prescription

is to identify $(-r, \theta)$ with $(r, \theta + \pi)$

Another prescription is to <u>discard</u> points for which $r$ is negative

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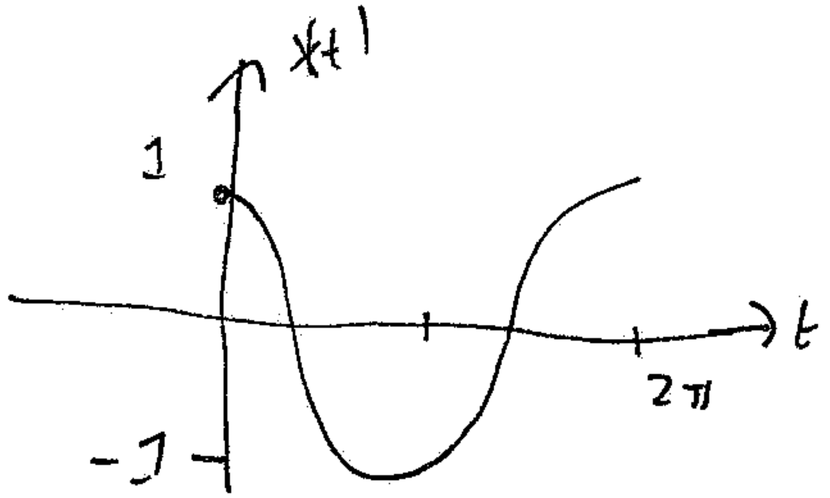
## <u>Parametric Plots</u>

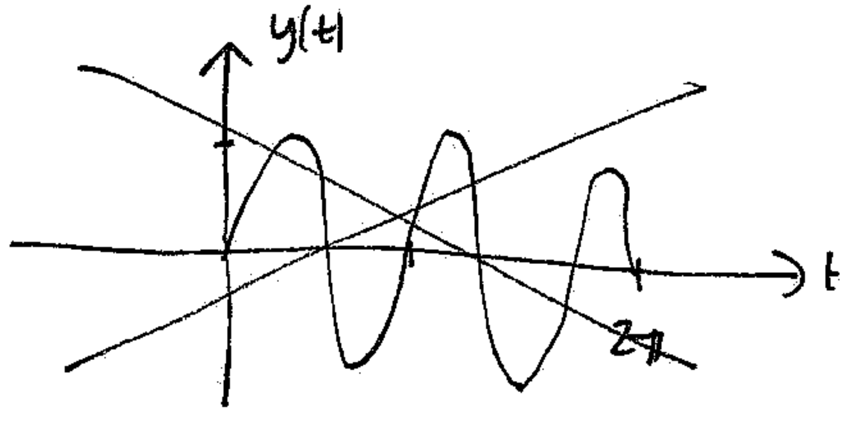
Write $x$ and $y$ as a function of a parameter $t$

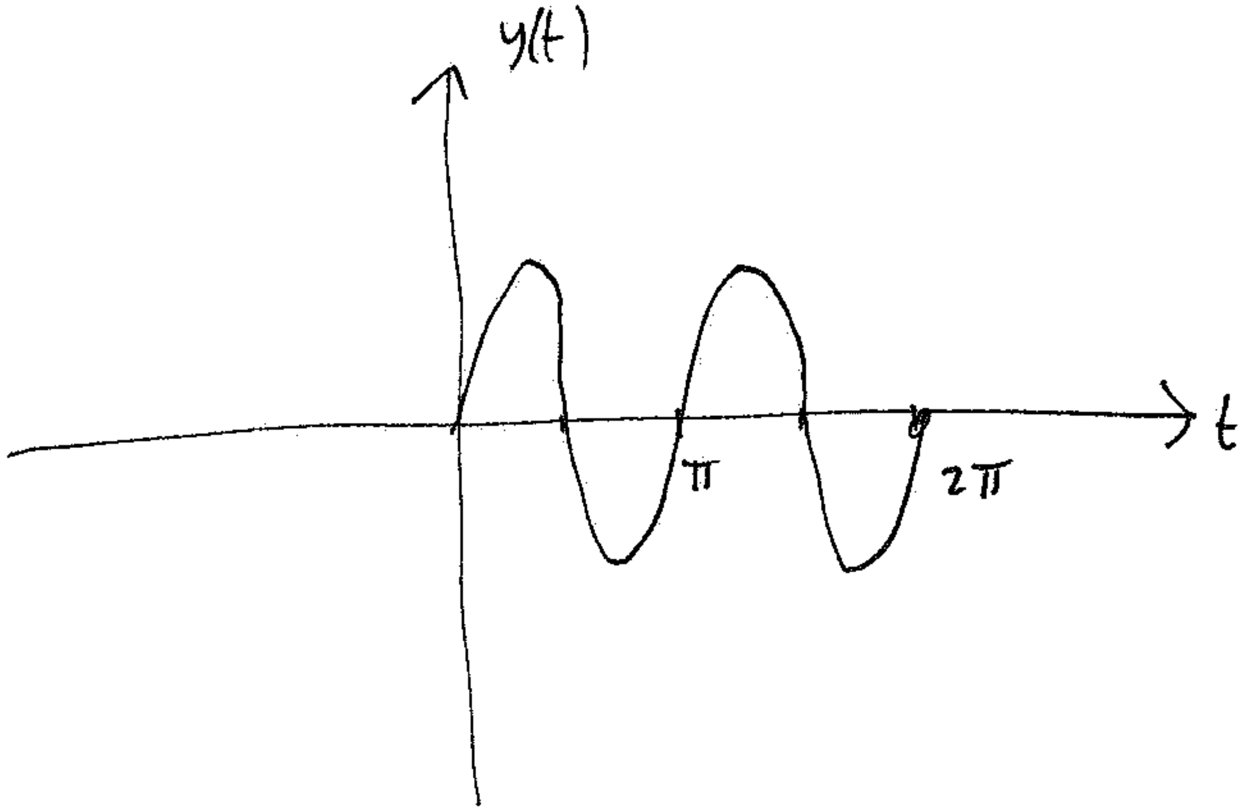
( eg. previous cycloid example )

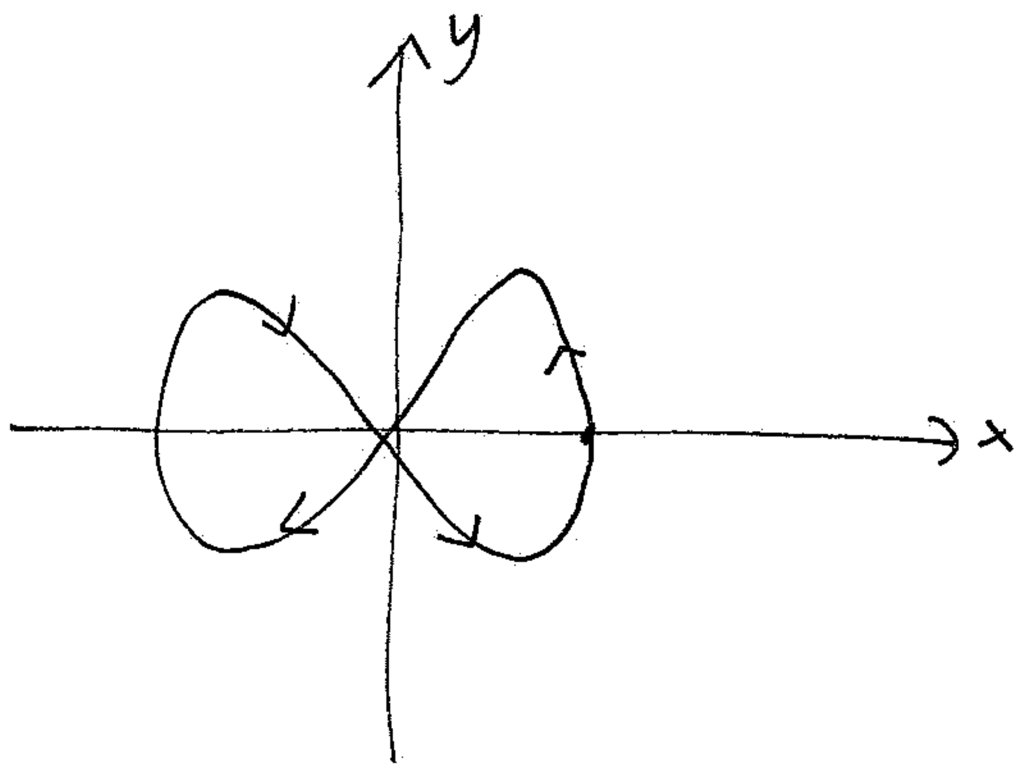
To sketch these graphs sometimes useful to make a separate plot of x and y as a function of t. For example

$$x = \cos t \qquad 0 \le t \le 2\pi$$
$$y = \sin(2t)$$




y(t)
t
π
2π


y
x

To compute slope of tangent at $(1, b)$ and $(1, 0)$ use implicit differentiation

tricky at $(1, 0)$

What is slope of tangent to curve $y^3 = x^3$ at origin. Obviously slope $= 1$ as curve is line $y = x$

But implicit differentiation problematic.

## Another example

$$x = \cos(11t)$$

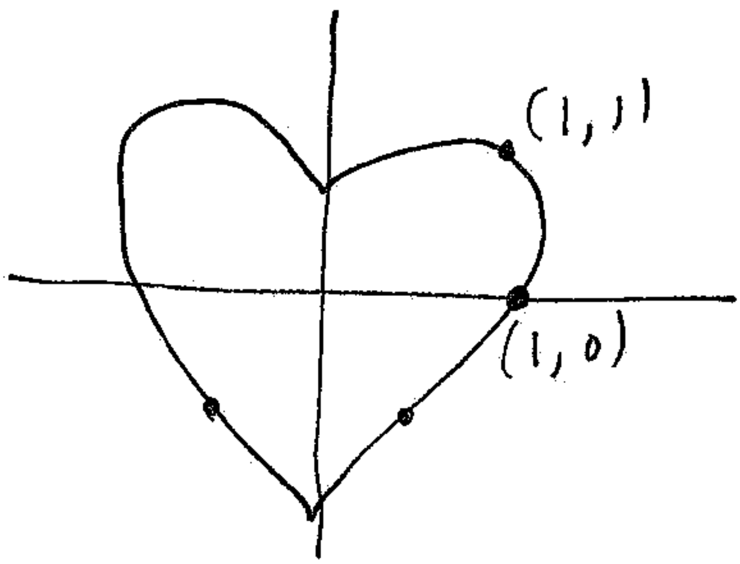
$$y = \sin(13t)$$

use a computer!

## From Problem sheet 3

$$(x^2+y^2-1)^3 = x^2y^3$$

use a computer




Q what points on graph closest to origin?

# 5. Series

A sequence denoted $a_m$ is an infinte list of numbers

$$a_0, a_1, a_2, a_3, a_4, \ldots$$

[ More formally a sequence is a function with domain $\mathbb{N}$. ]

<u>Example</u> $a_m = m^2$

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \ldots$

defines a sequence

[ $a : \mathbb{N} \to \mathbb{R}$ denote outputs as $a_m$ rather than $a(m)$ ]

An infinte series $\sum_{m=0}^{\infty} a_m$ is the result of adding up all elements of the sequence $a_m$. This is actually a <u>limit</u>

$$\sum_{m=0}^{\infty} a_m = \lim_{n\to\infty} \sum_{m=0}^{n} a_m$$

If this limit exists the series $\sum_{m=0}^{\infty} a_m$ is said to be <u>convergent</u>. Otherwise $\sum_{m=0}^{\infty} a_m$ is called <u>divergent</u>

## Examples

(i) $a_m = m^2$

$$\sum_{m=0}^{\infty} m^2 = 0+1+4+9+16+\dots$$

is divergent

(ii) $a_m = \frac{1}{m^2} \quad (m \geq 1)$

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

converges ( actually $= \zeta(2) = \frac{\pi^2}{6}$ )

(iii) $a_m = \frac{1}{m} \quad (m \geq 1)$

$$\sum_{m=1}^{\infty} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

(harmonic series)

is divergent

(iv) $a_m = (-1)^m$

$$\sum_{m=0}^{\infty} (-1)^m = 1 + (-1) + 1 + (-1) + 1 + (-1)$$

$$\sum_{m=0}^{n} (-1)^m = \frac{(-1)^n + 1}{2}$$ has no $n \to \infty$ limit

series is divergent

How to tell if a series is convergent?

(i) Evaluate series! Can be difficult

(ii) Apply convergence tests.

# Convergence Tests

(a) Preliminary Test 'easy test'

If $a_m \not\to 0$ as $m \to \infty$

then $\sum_{m=0}^{\infty} a_m$ is divergent

This test cannot be used to establish convergence.

**Examples** $a_m = m^2$

$\sum a_m$ diverges as $a_m \not\to 0$ as $m \to \infty$

$a_m = \frac{1}{m^2}$ here $a_m \to 0$ as $m \to \infty$

Here $\sum a_m$ is convergent but we cannot deduce this ~~from~~ from preliminary test

$a_m = \frac{1}{m}$ $\sum a_m$ diverges

but $a_m \to 0$ as $m \to \infty$

preliminary test is again inconclusive

$a_m = \frac{1}{3} + \frac{1}{m^3}$ here $\sum_{m=1}^{\infty} a_m$

diverges by preliminary test

as $a_m \to \frac{1}{3}$ as $m \to \infty$

(b) Alternating Series Test

Suppose $a_m$ is

1. Alternating. $a_{m+1}$ has opposite sign to $a_m$ for all $m$

2. $|a_m|$ is decreasing

$|a_{m+1}| < |a_m|$ for all $m$

( or for all $m \geq N$ )

3. $a_m \to 0$ as $m \to \infty$

Then $\sum_m a_m$ is <u>convergent</u>

<u>Examples</u> $a_m = \frac{(-1)^{m+1}}{m}$ $(m \geq 1)$

then $\sum_m a_m$ converges by alternating series test

$$\sum_m a_m = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

is finite ( actually $= \log(2)$ )

A similar example

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$$

is a finite number by alternating series test

$a_m = \dfrac{(-1)^{m+1}}{\sqrt{m}}$ $(m \geq 1)$ is alternating

$|a_m| = \dfrac{1}{\sqrt{m}}$ is decreasing

and $a_m \to 0$ as $m \to \infty$

<u>Question</u> Find a divergent series $\sum_m a_m$ such that

$a_m$ is alternating and satisfies $a_m \to 0$ as $m \to \infty$