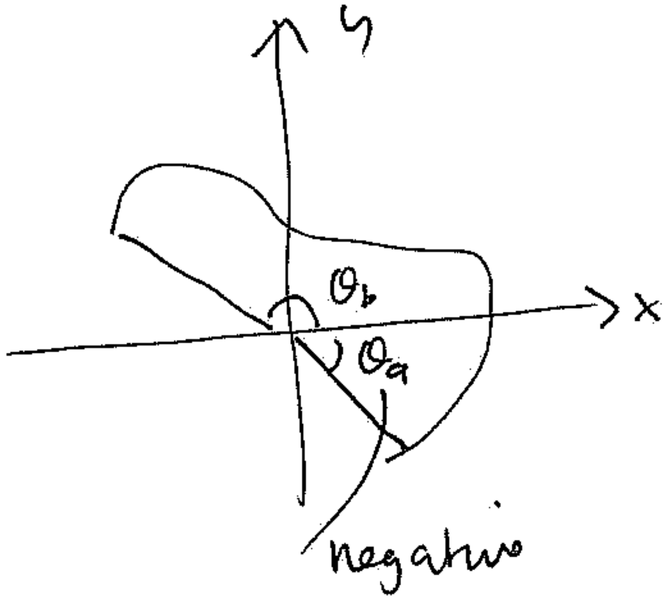
Using Polar coordinates

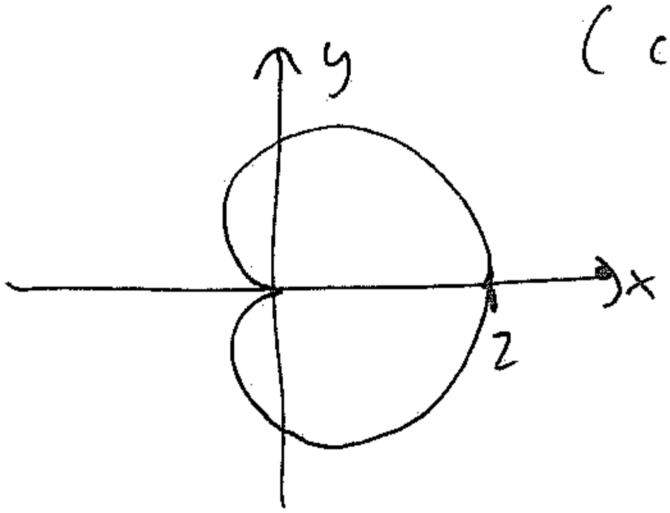
length of curve

$$r = r(\theta) \qquad \theta_a \le \theta \le \theta_b$$



<u>Example</u> $\quad r = 1 + \cos\theta$

(cardioid)

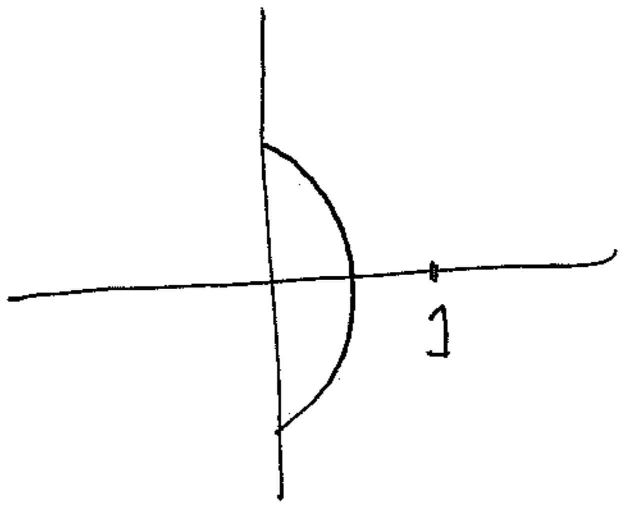


polar length formula

$$L = \int_{\theta_a}^{\theta_b} \sqrt{r^2(\theta) + (r'(\theta))^2}\, d\theta$$

see problems for cardioid case

$$r = \frac{1}{1+\cos\theta} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$



$$r'(\theta) = \frac{\sin\theta}{(1+\cos\theta)^2}$$

$$r^2 + r'^2 = \frac{1}{(1+\cos\theta)^2} + \frac{\sin^2\theta}{(1+\cos\theta)^4}$$

$$= \frac{(1+\cos\theta)^2 + \sin^2\theta}{(1+\cos\theta)^4}$$

$$= \frac{(1+2\cos\theta+\cos^2\theta) + \sin^2\theta}{(1+\cos\theta)^4}$$

$$= \frac{2}{(1+\cos\theta)^3}$$

$$L = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{(1+\cos\theta)^{\frac{3}{2}}}$$

could try

$$1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$$

or try $t = \tan\frac{\theta}{2}$

substitution

$$d\theta = \frac{2\,dt}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2}$$

$$\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{(1+\cos\theta)^{\frac{3}{2}}}$$

$$= \sqrt{2}\int_{-1}^{1} \frac{\frac{2\,dt}{1+t^2}}{\left(\frac{(1+t^2)+(1-t^2)}{1+t^2}\right)^{\frac{3}{2}}}$$

$$= \int_{-1}^{1} \sqrt{1+t^2}\,dt \qquad t = \sinh u$$

$\ldots$

Multiple Integrals

A function f of 2 variables is a rule assigning a real number $f(x, y)$ to ~~an~~ every ordered pair (x, y) in the domain of f.

$Dom(f)$ is $\mathbb{R}^2$ or a subset of $\mathbb{R}^2$:

(x, y) input

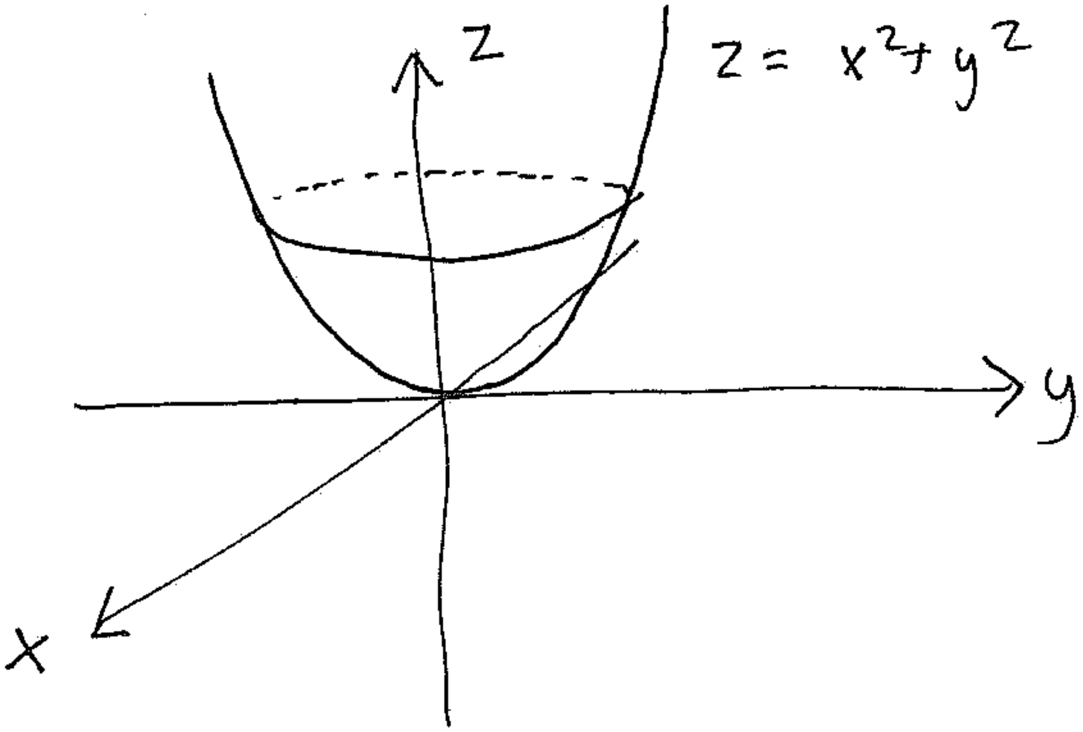
$f(x, y)$ output

Example

$f(x,y) = x^2 + y^2$ defines a function of 2 variables

The graph of a function of 2 variables is the surface $z = f(x,y)$ in 3d space

For $f(x,y) = x^2 + y^2$ surface is a paraboloid



For a function of one variable $\int_a^b f(x)\,dx$

represents area below graph $y = f(x)$ between $x = a$ and $x = b$

The double integral

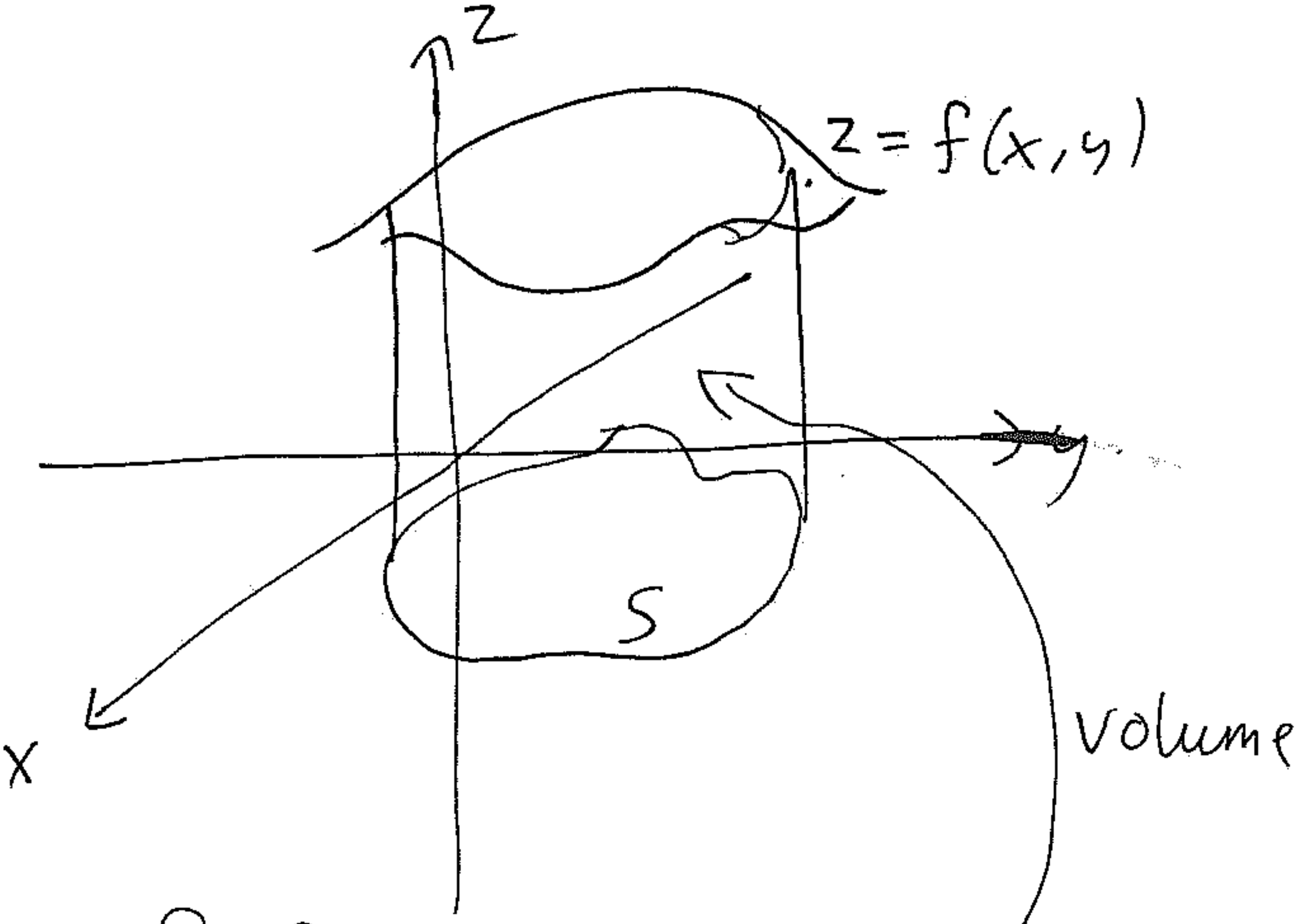
$$\iint_S f(x,y)\,dx\,dy$$

(Here S is $\mathbb{R}^2$ or a subset of $\mathbb{R}^2$)

is the volume under the surface

$z = f(x,y)$

above the set S in xy plane ($z=0$ plane)



$$\int\int_S f(x,y)\,dx\,dy$$

If surface dips below $z=0$ plane volume counts negatively

$$I = \iint_S \sqrt{1 - x^2 - y^2} \, dx\,dy$$

S unit disc

$$\{(x, y) \mid x^2 + y^2 \leq 1\}$$

I obviously $\frac{2\pi}{3}$

Since $z = \sqrt{1 - x^2 - y^2}$

is a hemisphere (unit radius). I is $\frac{1}{2}$ volume of the unit sphere.

$$\iint_S 1 \, dx \, dy$$

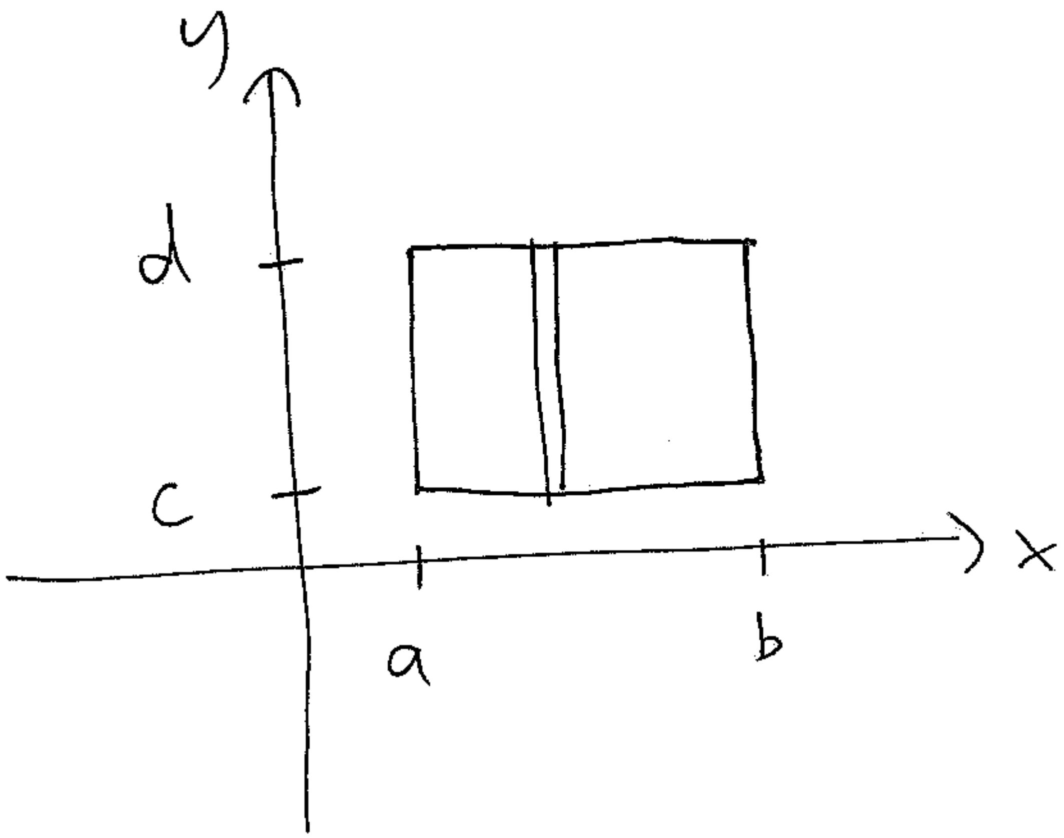
is the area of the region of integration S

Generalises

$$\int_a^b 1 \, dx = b-a$$

$=$ length of interval $[a,b]$

Integrating over a rectangle



$$\iint_S f(x,y)\,dx\,dy$$

Can be written

$$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Can do it the other way round

$$\int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

These two integrals are the same (Fubini's theorem)

These are called iterated integrals

Avoiding brackets

Write 1d integrals

$\int_a^b f(x)\,dx$. Can write

$\int_a^b dx\ f(x)$ instead

Using this convention

iterated integrals

$$\int_a^b dx \int_c^d dy \; f(x,y)$$

or

$$\int_a^b dy \int_a^b dx \; f(x,y)$$