Last time:

FACT (the "completeness axiom" for the real numbers): If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then S has a least upper bound.

(…and hence exactly one least upper bound, by Theorem 6.3.)

Last time we used this fact to prove the intuitively clear result that the integers were unbounded. I wish I'd given that example a number, so let's do that now.

Theorem 6.4. (Archimedean property of the reals). If $x \in \mathbb{R}$ then there exists $n \in \mathbb{Z}$ with $x < n$.
Proof. We did this last time.

FACT (the "completeness axiom" for the real numbers): If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then S has a least upper bound.

Today we will prove several more intuitively clear results. Note that just because a result is "obvious for the number line" does *not* make it true! To some people, it's "obvious" that

$0.9999999\ldots = 1$ and to others it's "obvious" that $0.9999999\ldots < 1$. *Use of intuition leads us into trouble. We need proven facts.*

Way back in Chapter 2 I had to assume the "floor" function existed (assumption A5, used only for decimal expansions). We wanted $\lfloor x \rfloor$ to be "the largest integer which is at most x". The key property we needed was that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

Let's prove that this function actually exists.

Theorem 6.5 (former "assumption A5"). The "floor function" $\lfloor x \rfloor$ exists. In other words, if x is any real number, then there is a unique integer n with $n \leq x < (n+1)$.

Notation: we write $n = \lfloor x \rfloor$.

Proof. Existence: By Theorem 6.4 there exist integers A and B with $x < B$ and $-x < A$. Hence $-A < x < B$. Now look at the finite set of integers i with $-A \leq i$ and $i \leq x$. This set is finite, and definitely $-A$ is in it, and B is not, by construction of A and B. So there is some largest integer n in this set. Because n is in and $n+1$ is not, we must have $n \leq x < n+1$. This does existence.

Uniqueness: Say $n \leq x < n+1$ and $m \leq x < m+1$. If $n \neq m$ then either $n < m$ or $m < n$. If $n < m$ then $n+1 \leq m$, giving

$$x < n+1 \leq m \leq x,$$

a contradiction. By a symmetric argument, $m < n$ can also be ruled out. Hence $m = n$.

$\square$

Theorem 6.6 ("density of the rationals in the reals.") If $x < y$ are real numbers, then there is a rational number q with $x < q < y$.

Remark. This result is called "density of the rationals in the reals". It is yet another example of something which is "intuitively true" but needs rigorous justification.

Proof. If $x < y$ then $y - x > 0$, so $\frac{2}{(y-x)} > 0$. Choose a large positive integer $D \in \mathbb{Z}_{\geq 1}$ with $\frac{2}{y-x} < D$. Then $\frac{2}{D} < y - x$.

Now set $N = \lfloor Dy \rfloor - 1$, so $N < Dy$ and hence $\frac{N}{D} < y$. Moreover $N + 2 = \lfloor Dy \rfloor + 1 > Dy$, so $Dy - 2 < N$.
We deduce $x = y - (y - x) < y - \frac{2}{D} = \frac{Dy-2}{D} < \frac{N}{D}$.

So $q = \frac{N}{D}$ has the properties we require.

Here's yet *another* example of something which is intuitively true but now we can actually *prove*, using the completeness axiom.

Theorem 6.7. There is a positive real number whose square is 2.

We are not quite ready to prove this theorem yet! We need to prove two boring "helper lemmas" first.

Boring lemma 6.8. Say $\ell \in \mathbb{R}_{\geq 1}$ and $\ell^2 > 2$. Then there exists some positive real number $\epsilon > 0$ such that $\ell - \epsilon > 0$ and $(\ell - \epsilon)^2 > 2$.

Proof. Define $\delta = \frac{\ell^2 - 2}{2} > 0$, so $\ell^2 > \ell^2 - \delta > 2$. Now set $\epsilon = \min\{\frac{\delta}{2\ell}, \frac{\ell}{2}\} > 0$. In words, "$\epsilon$ is smaller than everything we need it to be smaller than to make this proof work".

Firstly $\epsilon \leq \frac{\ell}{2} < \ell$ so $\ell - \epsilon > 0$.

And finally,

$$\begin{aligned} (\ell - \epsilon)^2 &= \ell^2 - 2\ell\epsilon + \epsilon^2 \\ &\geq \ell^2 - 2\ell\epsilon \\ &\geq \ell^2 - \delta > 2. \end{aligned}$$

$\square$

Even more boring lemma 6.9. Say $\ell \in \mathbb{R}_{\geq 1}$ and $\ell^2 < 2$. Then there exists some positive real number $\epsilon > 0$ such that $(\ell + \epsilon)^2 < 2$.

Proof. Define $\delta = \frac{2-\ell^2}{2} > 0$, so $\ell^2 + \delta < 2$. Now set $\epsilon = \min\{\frac{\delta}{2}, 1, \frac{\delta}{4\ell}\} > 0$. In words, ϵ is “positive, and smaller than everything we need it to be smaller than”.

Then $\epsilon^2 \leq \epsilon$ (because $0 < \epsilon \leq 1$) and $\epsilon \leq \frac{\delta}{2}$, hence $\epsilon^2 \leq \frac{\delta}{2}$.

Also $2\ell\epsilon \leq \frac{\delta}{2}$, so $2\ell\epsilon + \epsilon^2 \leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta$.
Hence

$$\begin{aligned}(\ell + \epsilon)^2 &= \ell^2 + 2\ell\epsilon + \epsilon^2 \\ &\leq \ell^2 + \delta \\ &< 2.\end{aligned}$$

$\square$

We just proved

Lemma 6.8 Say $\ell \in \mathbb{R}_{\geq 1}$ and $\ell^2 > 2$. Then there exists some positive real number $\epsilon > 0$ with $\ell - \epsilon > 0$ and $(\ell - \epsilon)^2 > 2$.
Lemma 6.9 Say $\ell \in \mathbb{R}_{\geq 1}$ and $\ell^2 < 2$. Then there exists some positive real number $\epsilon > 0$ such that $(\ell + \epsilon)^2 < 2$.

We are actually doing a special case of a much more general thing. Does anyone know what is going on here? We are proving that the squaring function $y = x^2$ is a *continuous function* near $x = \sqrt{2}$. You will learn about continuous functions in M1P1 – the same course where you learn the rigorous definition of limits of sequences, infinite sums and so on.

We will now prove
Theorem 6.7 There is a positive real number whose square is 2.

Lemma 6.8 Say $\ell \in \mathbb{R}_{\geq 1}$ and $\ell^2 > 2$. Then there exists some positive real number $\epsilon > 0$ with $\ell - \epsilon > 0$ and $(\ell - \epsilon)^2 > 2$.
Lemma 6.9 Say $\ell \in \mathbb{R}_{\geq 1}$ and $\ell^2 < 2$. Then there exists some positive real number $\epsilon > 0$ such that $(\ell + \epsilon)^2 < 2$.
Theorem 6.7 There is a positive real number whose square is 2.

Proof. Set $S = \{ x \in \mathbb{R}_{\geq 0} \mid x^2 < 2 \}$. In words, S is the non-negative real numbers whose square is less than 2. Note: there is no mention of "the square root of 2" in the definition of S! Only things we are allowed, like $\times$ (because x^2 means $x \times x$) and $<$.

Claim: S is non-empty. Indeed $1 \in S$ because $1 \geq 0$ and $1^2 = 1 < 2$. Claim: S is bounded above. Indeed, I claim that 10 is an upper bound for S. For if $x > 10$ then $x^2 > 10^2 = 100 > 2$ so $x \notin S$. Hence if $x \in S$ then $x \leq 10$, by the contrapositive.

So S is non-empty and bounded above, and hence, by completeness of the real numbers, S has a least upper bound ℓ. Because $1 \in S$ we know $\ell \geq 1$. Claim: $\ell^2 = 2$.

$S = \{ x \in \mathbb{R}_{\geq 0} \mid x^2 < 2 \}$ is non-empty and bounded above; let ℓ be its least upper bound, which exists by the completeness axiom. Claim: $\ell^2 = 2$.

One can prove this claim by firstly showing that $\ell^2 < 2$ is impossible, and secondly by showing $\ell^2 > 2$ is impossible. Let's rule out both of these possibilities using a proof by contradiction.

So suppose for a contradiction that $\ell^2 < 2$. Then by Lemma 6.9 one can find an element $x = \ell + \epsilon$ slightly greater than ℓ such that $x^2 < 2$, meaning that $x \in S$ and $\ell < x$. This contradicts the fact that ℓ is an upper bound! So $\ell^2 < 2$ cannot be correct.

Now suppose for a contradiction that $\ell^2 > 2$. Then by Lemma 6.8 one can find an element $b = \ell - \epsilon$, positive and slightly less than ℓ, such that $b^2 > 2$. Such an element b must be an upper bound for S! Indeed, if $x \in \mathbb{R}$ with $x > b$ then $x^2 > b^2 > 2$, so $x \notin S$. Hence $x \in S \implies x \leq b$. This means that b is an upper bound for S and hence that ℓ is not a *least* upper bound. Contradiction! So $\ell^2 > 2$ cannot be correct either.

Either way, we reach a contradiction, and this finishes the proof. $\square$

The same techniques can be used to prove
Theorem. If $r \in \mathbb{R}$ with $r > 0$ and $n \in \mathbb{Z}_{\geq 1}$, then there exists a positive real number s with $s^n = r$.

The full proof uses two generalised boring lemmas, which I am not going to prove, because there is *a better way*.

The better way is to prove that the function $\lambda x, x^n$ is a continuous function (oh – that's computer science notation – I just mean $f(x) = x^n$) and then to deduce the theorem from the *intermediate value theorem*. You will see this more powerful strategy later on in your careeer.