Absolute Convergence

A series $\sum_m a_m$ is called <u>absolutely convergent</u> if $\sum_m |a_m|$ is convergent

Every absolutely convergent series is convergent. Converse is not true

abs conv $\Rightarrow$ conv

<u>Example</u> alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \log 2$$

is convergent. It is not absolutely convergent since

$$1 + \left|-\frac{1}{2}\right| + \frac{1}{3} + \left|-\frac{1}{4}\right| + \dots$$

harmonic serie which is divergent.

Tests for Absolute Convergene

(a) Comparison test

Suppose $|a_m| \leq |b_m|$ for all m (can be weakened to $m \geq N$, N finite)

If $\sum b_m$ absolutely convergent

so is $\sum a_m$

Works in other direction

If $\sum |a_m|$ diverges

then so does $\sum |b_m|$

Example $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m^2+m}$

is convergent

$$\frac{1}{m^2+m} < \frac{1}{m^2} \quad \text{but}$$

$\sum \frac{1}{m^2}$ convergent. By comparison

test $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m^2+m}$ convergent

(b) Integral test:

Suppose $a_m = f(m) \geq 0$

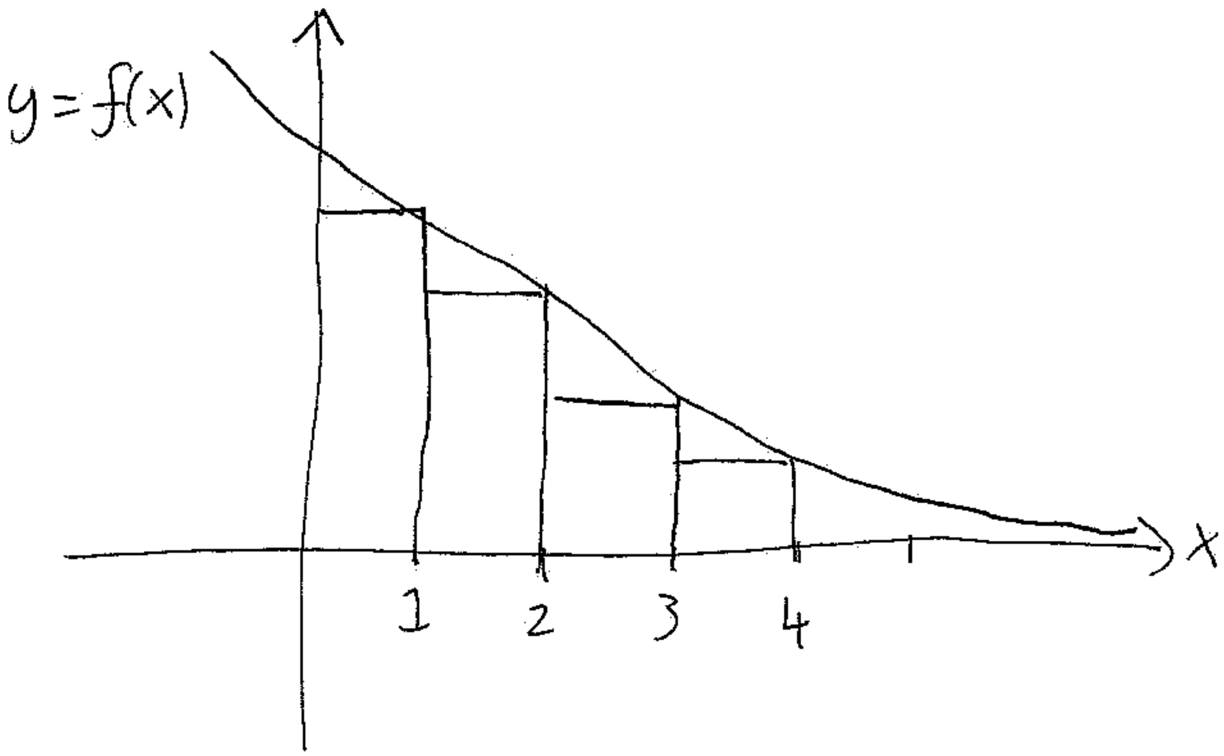
where $f(x)$ is a decreasing (non-negative) function of x

Consider the integral

$$\int_N^\infty f(x)\, dx$$

If integral undefined for any N then $\sum a_m$ is divergent

If integral finite for some N then $\sum a_m$ convergent



Area under graph from

$x = 0$ to $x = \infty$

$\geqslant$ Area of rectangles

$$= a_1 + a_2 + a_3 + \dots$$

$$= \sum_{m=1}^{\infty} a_m$$

So if area represented by integral finite so is $\sum_m a_m$

A similar argument works for divergent case (to prove a sum diverges)

<u>Examples</u> $a_m = \frac{1}{m}$

$\sum_m \frac{1}{m}$ diverges. Consider

$f(x) = \frac{1}{x}$

$$\int_N^\infty \frac{dx}{x} = \log x \Big|_{x=N}^{x=\infty}$$

$$= \log \infty - \log N$$

undefined. Harmonic series diverges by integral test

$a_m = \frac{1}{m^2}$ converges.

Justification: consider

$f(x) = \frac{1}{x^2}$

Integral $\int_N^\infty \frac{dx}{x^2}$

$$= -\frac{1}{x}\Big|_N^\infty = 0 - -\frac{1}{N} = \frac{1}{N}$$

finite $(N \neq 0)$ $\sum_m \frac{1}{m^2}$ converges

by integral test

To more test

(c) ratio test

(d) root test

Consider

ratio test the limit (if it exists)

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

root test the limit (if it exists)

$$L = \lim_{n \to \infty} |a_n|^{\frac{1}{n}}$$

If $L > 1$ then $\sum a_m$ is divergent

If $L < 1$ then $\sum a_m$ is absolutely convergent

If $L = 1$ test(s) indecisive

<u>Example</u> $a_m = m e^{-m}$

$\sum a_m$ converges. Check using ratio / root test ratio test

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(m+1)\, e^{-(m+1)}}{m\, e^{-m}}$$

$$= \frac{m+1}{m} \cdot e^{-1}$$

$$\rightarrow 1 \cdot e^{-1} = \frac{1}{e} \quad \text{as}$$

$m \to \infty$ $\qquad L = \frac{1}{e} < 1$

root test

$$|a_m|^{\frac{1}{m}} = m^{\frac{1}{m}}\, e^{-1}$$

$$\rightarrow e^{-1} \quad \text{as} \quad m \to \infty$$

since $\lim_{m \to \infty} m^{\frac{1}{m}} = 1$

$$m^{\frac{1}{m}} = e^{\frac{1}{m}\log m}$$

but $\dfrac{\log m}{m} \to 0$ as $m \to \infty$

$m^{\frac{1}{m}} \to 1$ as $m \to \infty$

$L = \frac{1}{e} < 1$

$a_m = \dfrac{1}{m}$ $\qquad$ $a_m = \dfrac{1}{m^2}$

both give $L = 1$

so ratio / root test

~~indecive~~ undecisive for these

examples

Conditional Convergence

If $\sum a_m$ is convergent but not absolutely convergent it is called <u>conditionally convergent</u>

Example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \log 2$$

$$a_m = \frac{(-1)^{m+1}}{m} \qquad m \geq 1$$

$$\lim_{n \to \infty} \sum_{m=1}^{n} a_m = \log 2$$

Claim A condionally convergnt series depends on order of summation – changing order gives a different result

In example

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$$

$$- \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \ldots$$

Reorder to obtain any number (say 8)

Take positive terms (in order given) until 8 is exceeded

Then add negative terms (in order given until sum < 8

Now resume adding positive terms until 8 exceeded

Then continue with negative terms

Result is a reordering so that limit is 8 rather than log 2