Further Examples

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

(aka small angle formula
$\sin x \approx x$ for x small)

Using power series

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$\to 1 \quad \text{as } x \to 0$$

Similarly $\lim_{x \to 0} \frac{\tan x}{x} = 1$

$$\frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \cdots$$

$$\to 1 \quad \text{as} \quad x \to 0$$

On intro problem sheet

Q1 sketch

(i) $y = \dfrac{x}{e^x - 1}$ $\quad x \neq 0$

(ii) $y = \dfrac{\cos(\frac{1}{2}\pi x)}{1 - x^2}$ $\quad x \neq \pm 1$

Consider limit

$$\lim_{x \to 1} \frac{\cos\left(\frac{1}{2}\pi x\right)}{1 - x^2}$$

$$\frac{\cos\left(\frac{1}{2}\pi x\right)}{1-x^2} = \frac{\cos\left[\frac{1}{2}\pi(x-1) + \frac{1}{2}\pi\right]}{(1-x)(1+x)}$$

$$= -\frac{\sin\left[\frac{1}{2}\pi(x-1)\right]}{(1-x)(1+x)}$$

$$\left(\text{Let} \quad s = x - 1 \right)$$

$$= \frac{+\sin\frac{1}{2}\pi s}{+s(2+s)}$$

(i) $$y = \frac{x}{e^x - 1} \qquad x \neq 0$$

$$= \frac{x}{\left(\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \cancel{1}}$$

$$= \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots}$$

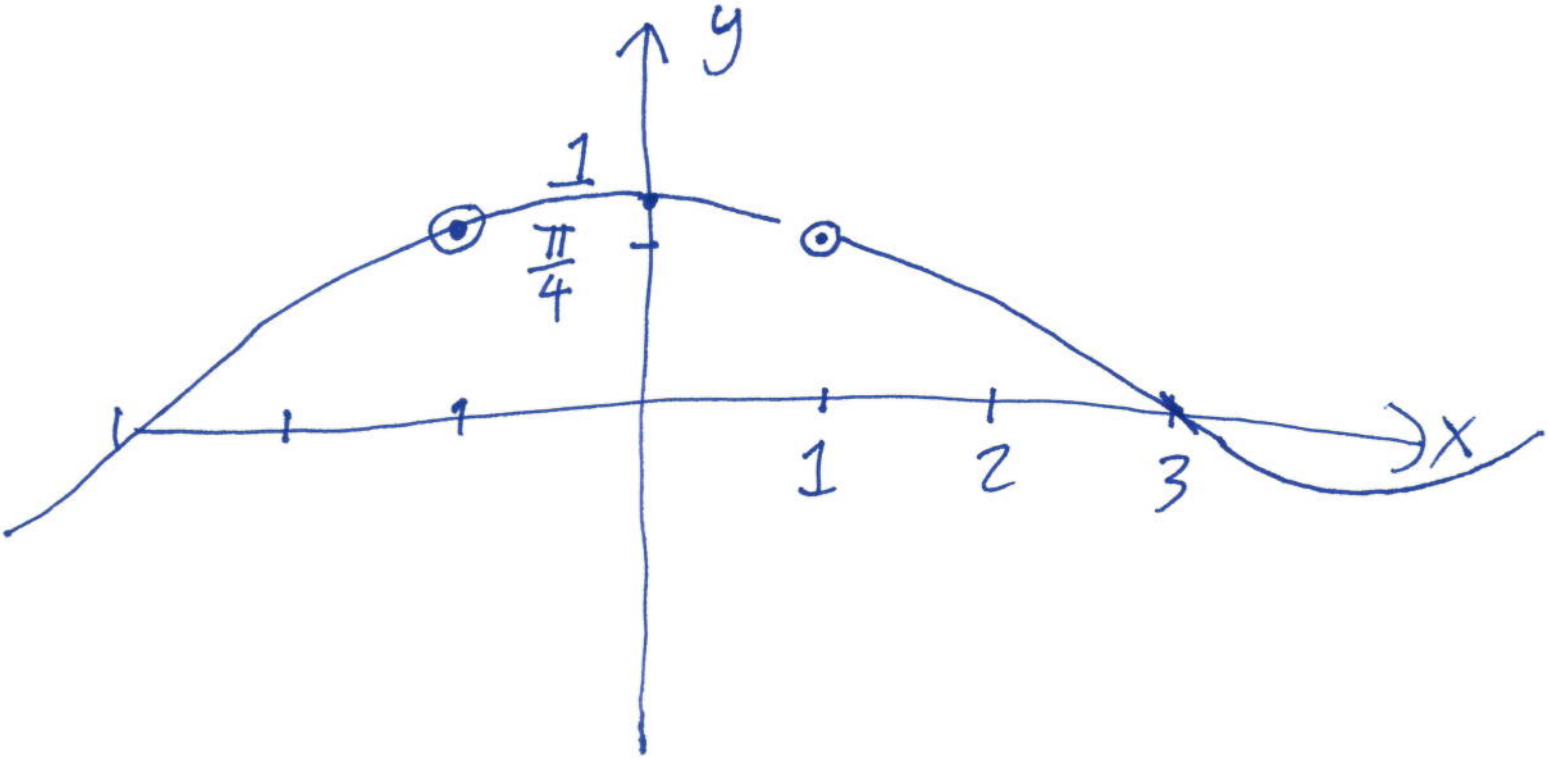
$$\to 1 \quad \text{as} \quad x \to 0$$

(ii) $$y = \frac{\cos\left(\frac{1}{2}\pi x\right)}{1 - x^2}$$

$$= \frac{\frac{1}{2}\pi s - \frac{1}{3!}\left(\frac{1}{2}\pi s\right)^3 + \cdots}{s\,(2+s)}$$

$$= \frac{\frac{1}{2}\pi - \cdots\cdots}{2+s}$$

$$\to \frac{\pi}{4} \quad \text{as} \quad s \to 0$$



$$\lim_{x \to \infty} \left[x^{\frac{1}{4}}(2+x)^{\frac{3}{4}} - x^{\frac{3}{4}}(2+x)^{\frac{1}{4}} \right]$$

of form $\infty - \infty$

$$(2+x)^{\frac{3}{4}} = x^{\frac{3}{4}}\left(1 + \frac{2}{x}\right)^{\frac{3}{4}}$$

$$= x^{\frac{3}{4}}\left[1 + \frac{3}{4}\cdot\frac{2}{x} + \frac{\frac{3}{4}\left(\frac{3}{4}-1\right)}{2!}\left(\frac{2}{x}\right)^2 + \cdots\right]$$

$\frac{2}{x}$ "small" for x large and positive

$$x^{\frac{1}{4}}(2+x)^{\frac{3}{4}}$$

$$= x^{\frac{1}{4}}\, x^{\frac{3}{4}}\left(1 + \frac{3}{2}\frac{1}{x} + \cdots\right)$$

$$= x + \frac{3}{2} + \cdots$$

$$x^{\frac{3}{4}}(2+x)^{\frac{1}{4}}$$

$$= x\left(1+\frac{2}{x}\right)^{\frac{1}{4}}$$

$$= x\left(1+\frac{1}{2}\frac{1}{x}+\cdots\right)$$

$$= x+\frac{1}{2}+\cdots$$

limit

$$[\ldots\ldots] = \left(\cancel{x}+\frac{3}{2}+\ldots\right) - \left(\cancel{x}+\frac{1}{2}+\ldots\right)$$

$$\to 1 \quad \text{as} \quad x \to \infty$$

$$\lim_{x \to \infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right)$$

limit of form $\infty \cdot 0$

from problem sheet 1

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ if $x > 0$

$$x \left(\frac{\pi}{2} - \tan^{-1} x \right) = x \cot^{-1} x$$
$$= x \tan^{-1}\left(\frac{1}{x}\right)$$

In desired limit $\frac{1}{x}$ "small"

$$= x \left(\frac{1}{x} - \frac{1}{3}\left(\frac{1}{x}\right)^3 + \frac{1}{5}\left(\frac{1}{x}\right)^5 \right)$$
$$= 1 - \frac{1}{3}\frac{1}{x^2} + \frac{1}{5}\frac{1}{x^4}$$

$\to 1 \quad \text{as} \quad x \to \infty$

$$\lim_{x \to \infty} (\tanh x - 1) = 0$$

obvious

$$\lim_{x \to \infty} x(\tanh x - 1) \quad ?$$

$$\lim_{x \to \infty} e^{2x}(\tanh x - 1) \quad ?$$

Another important limit

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

In particular

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

$$e \approx \left(1 + \frac{1}{100}\right)^{100}$$

Derivation?

$$\left(1 + \frac{a}{x}\right)^x = \exp\left[\log\left(1 + \frac{a}{x}\right)^x\right]$$

$$= \exp\left[x \log\left(1 + \frac{a}{x}\right) \right]$$

Use

$$\log(1+s) = s - \frac{s^2}{2} + \frac{s^3}{3} + \cdots$$

$$x \log\left(1 + \frac{a}{x}\right)$$

$$= x \left[\frac{a}{x} - \frac{1}{2}\left(\frac{a}{x}\right)^2 + \frac{1}{3}\left(\frac{a}{x}\right)^3 + \cdots \right]$$

$$= a - \frac{1}{2}\frac{a^2}{x} + \frac{1}{3}\frac{a^3}{x^2} - \cdots$$

$\rightarrow a$ as $x \rightarrow \infty$

$$\exp\left[x \log\left(1 + \frac{a}{x}\right) \right] \rightarrow e^a$$

as $x \to \infty$

Have shown

$[\quad] \to a$ as $x \to +\infty$

Is it true that

$e^{[\quad]} \to e^{a}$ as $x \to \infty$

Yes since exp is a continuous function

In general $f(x)$ continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$