$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

for real θ

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

extends to complex numbers

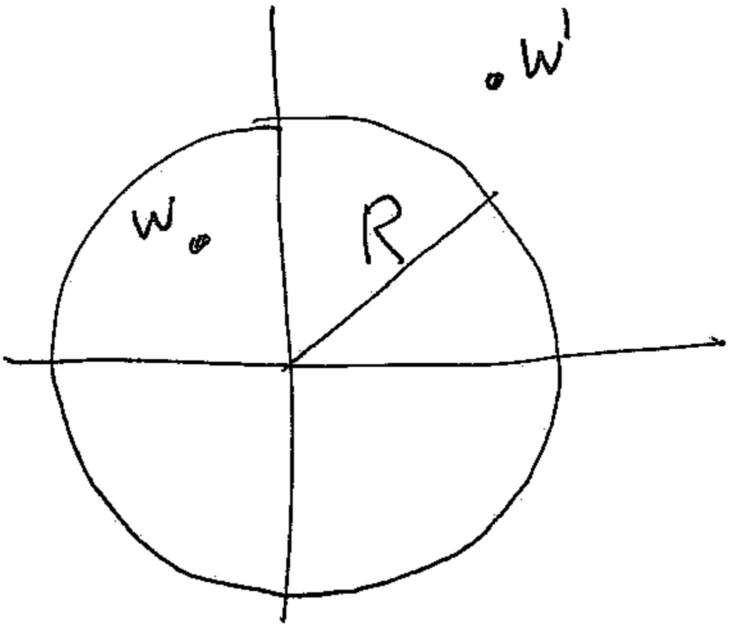
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$z \in \mathbb{C}$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Complex Power Series

$$\sum_{m=0}^{\infty} c_m z^m \qquad z \in \mathbb{C}$$



R radius of convergence

series converges absolutely for $|z| < R$

diverges for $|z| > R$

Proof (outline)

Assume series converges absolutely for ~~[illegible]~~ $z = w \in \mathbb{C}$. Show that series converges if $|z| < |w|$.

Assume series not absolutely

convergent for $z = w' \in \mathbb{C}$

show that series diverges if $|z| > |w'|$.

How to compute R?

Can use root / ratio test. Can also exploit properties of complex power series (see Complex Analysis next year)

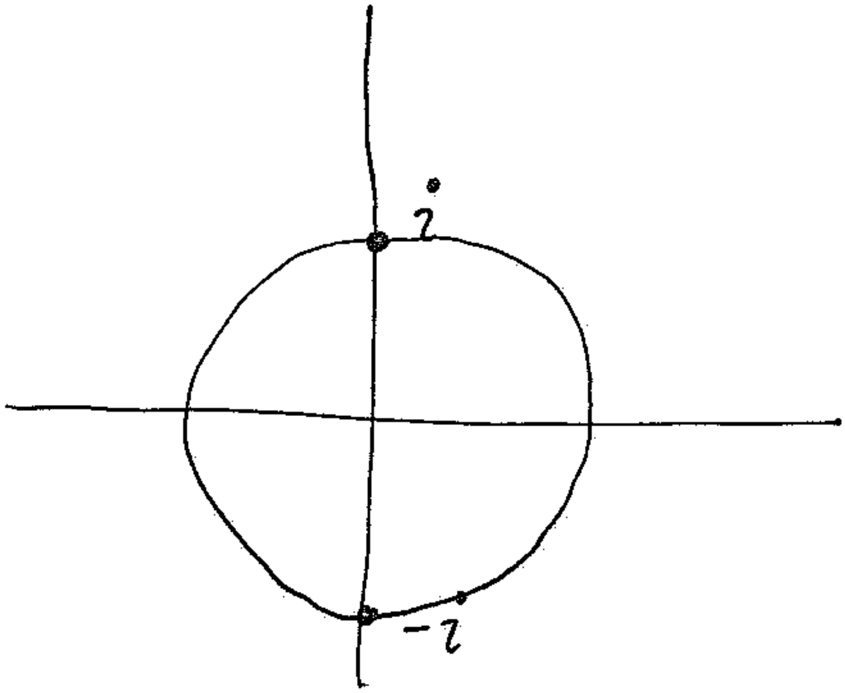
Consider power series

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

$$(R = 1)$$

LHS 'blows up' at

$z = i$ or $z = -i$



RHS = a power series

which converges in a

disc of radius R

R must be ≤ 1

since otherwise ~~singul~~ RHS

would be well defined for $z = \pm i$

Actually in this case R is equal 1. Some theory (holomorphic functions)

$R =$ distance between origin and 'nearest singularity.

Examples

(i) $\tan z$

$= z + \frac{z^3}{3} + \frac{2}{15} z^5 + \cdots$

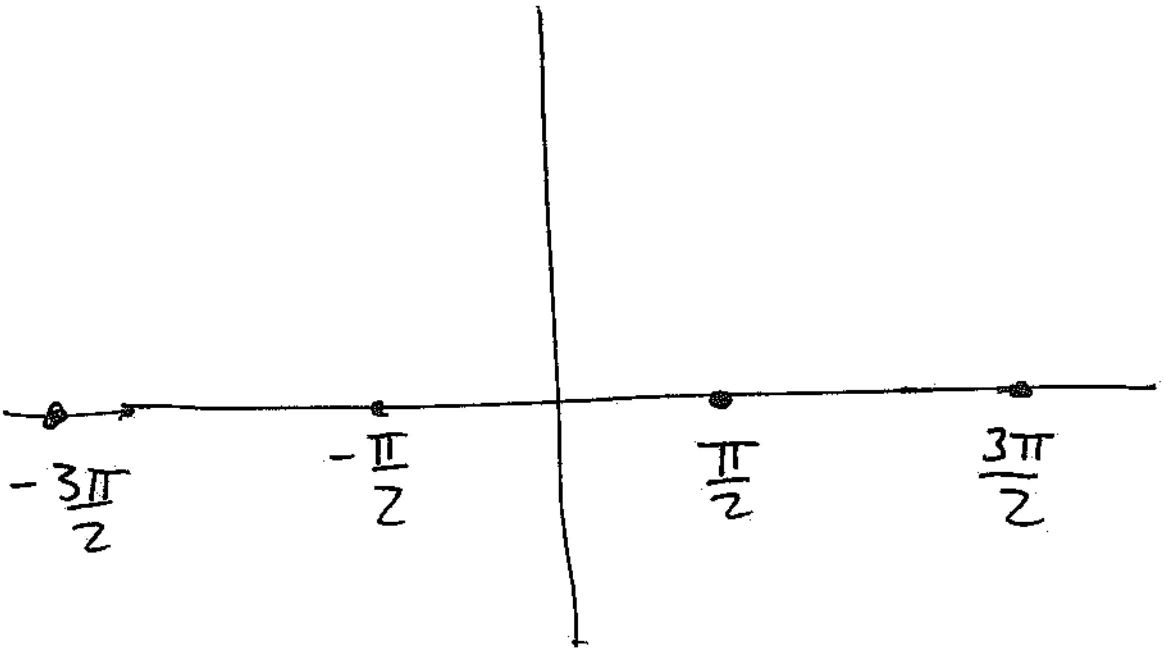
(ii) $\frac{1}{1+e^z}$

(i) $\tan z = \frac{\sin z}{\cos z}$

Singularities where $\cos z = 0$

$z = \frac{n}{2}\pi$ n odd integer

can show $\cos z$ has no other zeros



$R = \frac{\pi}{2}$ = distance between origin and singularities at $z = \frac{\pi}{2}$ and $-\frac{\pi}{2}$

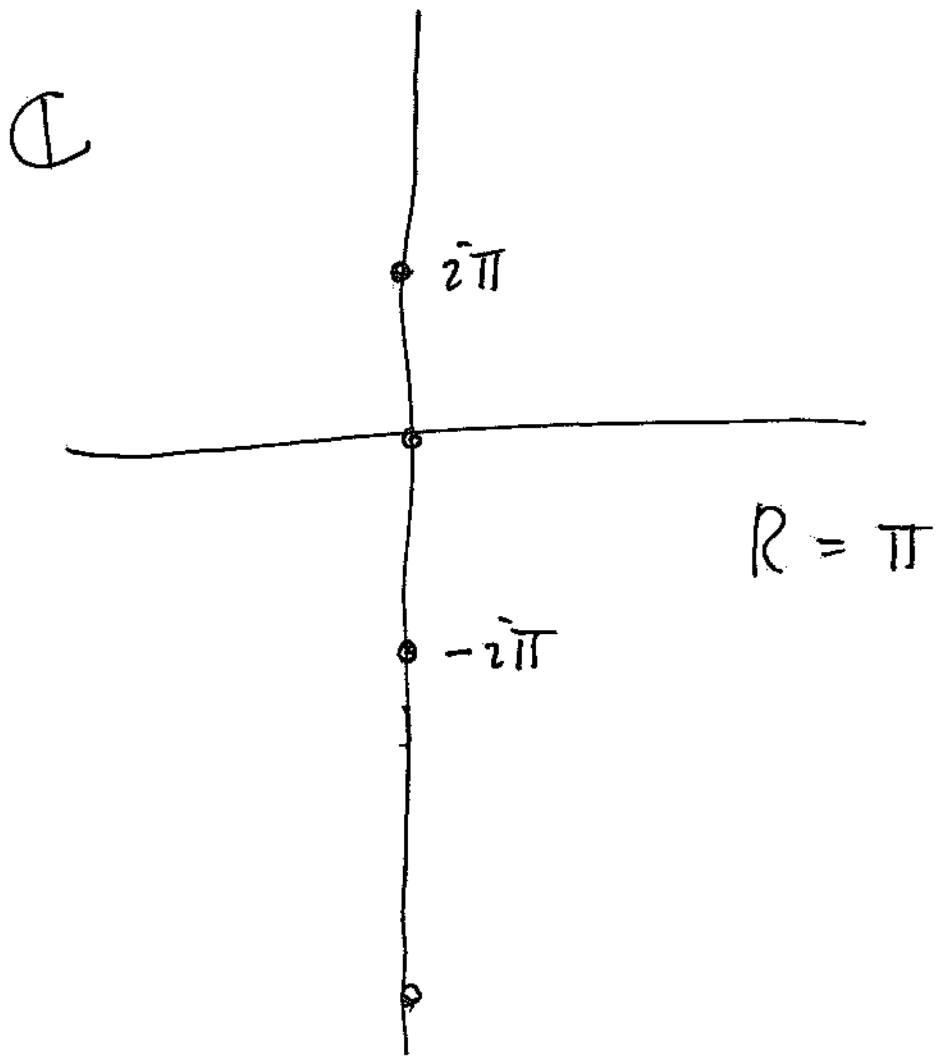
Q What is R for

$$\tanh z = z - \frac{z^3}{3} + \frac{2}{15} z^5 - \ldots$$

(ii) $$\frac{1}{e^z+1} = \frac{1}{2} + \dots$$

singular if $e^z = -1$

$z = i n \pi$ $\quad n$ odd integer



Complex Conjugation

The complex conjugate of $z = x + iy = re^{i\theta}$ is defined through

$$\bar{z} = x - iy = re^{-i\theta}$$

Properties:

$$\bar{\bar{z}} = z \qquad \overline{z^n} = (\bar{z})^n$$

$$|z|^2 = z\bar{z}$$

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$$

$$\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$$

eg. $z = e^{i\theta} = \cos\theta + i\sin\theta$

$$\mathrm{Re}\,(e^{i\theta}) = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\mathrm{Im}\,(e^{i\theta}) = \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Polynomials

A complex polynomial of degree n has form

$$c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n$$

$(c_n \neq 0)$ $\quad z$ complex

$c_0, c_1, \dots, c_n$ complex numbers (coefficients)

Fundamental Theorem of Algebra any polynomial (of degree ≥ 1) has at least one root

ie. $P(z)=0$ has at least one solution.

In general for degree n there are n roots (these can be 'repeated' so number of roots can be less than n)

A complex polynomial can be factorised

$$P(z) = c_n (z-a_1)(z-a_2)\cdots(z-a_n)$$

where $a_1, a_2, \ldots, a_n$ are the roots of P (can be repeated roots)

If coefficients $c_0, \ldots, c_n$ are real

roots $a_1, \ldots, a_n$ can still be complex –

complex roots will appear in complex conjugate pairs

If a is a root so is $\bar{a}$.

<u>Proof</u> Suppose $c_0, c_1, \ldots, c_n$ are real and a is a root of $P(z) = c_0 + c_1 z + \cdots + c_n z^n$

we have

$$P(a) = c_0 + c_1 a + c_2 a^2 + \cdots + c_n a^n$$
$$= 0$$

Take complex conjugate

$$\bar{c}_0 + \bar{c}_1 \bar{a} + \bar{c}_2 \overline{a^2} + \dots \bar{c}_n \overline{a^n}$$

$$= c_0 + c_1 \bar{a} + c_2 (\bar{a})^2 + \dots + c_n (\bar{a})^n = 0$$

so that $P(\bar{a}) = 0$

$(\bar{c}_0 = c_0,\ \bar{c}_1 = c_1,\ \text{etc.})$

Simple example

$$P(z) = 1 + z^2 = (z+i)(z-i)$$

Roots $\pm i$ a complex conjugate pair

Another example

$$P(z) = z^6 - 7z^3 - 8$$

$$= (z^3 - 8)(z^3 + 1)$$

roots

$z^3 = 8 = 8e^{2\pi i} = 8e^{4\pi i}$

$z = 2$, $z = 2e^{2\pi i/3}$, $z = 2e^{4\pi i/3} = 2e^{-2\pi i/3}$

$z^3 = -1 = e^{i\pi} = e^{3i\pi} = e^{5i\pi}$

$z = e^{i\pi/3}$, $z = e^{i\pi} = -1$, $z = e^{5i\pi/3}$
$= e^{-i\pi/3}$