Looking at the complex numbers with human eyes, and drawing from our human experience, we know that the following definitions are important.

Definition.

Formal: the *complex conjugate* of a complex number (x, y) is the complex number $(x, -y)$.

Informal: the *complex conjugate* of a complex number is what you get if you reflect the complex number about the real axis.

Happy medium: the *complex conjugate* of $x + yi$ is $x - yi$.

Notation: we write $\overline{z}$ for the complex conjugate of z.

Definition.

Formal: The *norm*, or the *modulus*, of a complex number (x, y) is the real number $\sqrt{x^2 + y^2}$ (just to be clear – I mean the non-negative square root).

Informal: the norm of a complex number z is the distance, in the complex plane, from zero to z.

Notation: $|z|$ is the norm of z.

Example: $|3 + 4i| = 5$, because $3^2 + 4^2 = 5^2$.

Question: Do you think my unimaginative friend from last time cares about examples?

Theorem 3.15. If z is a non-zero complex number, then $|z|$ is a positive (and hence non-zero) real number.

Informal proof. This is obvious.

Formal proof. Say $z = (x, y)$. Then $z = 0$ *means* $z = (0, 0)$ which *means* $(x, y) = (0, 0)$ which *means* $x = 0$ and $y = 0$.

So $z \neq 0$ *means* either $x \neq 0$ or $y \neq 0$.

So definitely $x^2 \geq 0$ and $y^2 \geq 0$ (because we proved that the square of any number was non-negative), and at least one of them is > 0 (because we proved that the square of any non-zero real number was positive).

So their sum is positive. $\square$

Theorem 3.16. If $z \in \mathbb{C}$ then $z\overline{z} = |z|^2$ (where as usal we are identifying the real number $|z|^2$ with the corresponding complex number $(|z|^2, 0)$).

Proof. Set $z = x + iy$. Then $z\overline{z} = (x + yi)(x - yi) = x^2 + y^2$. And $|z|^2 = \sqrt{x^2 + y^2}^2 = x^2 + y^2$, so we are done.

$\square$

Corollary 3.17. If $z \in \mathbb{C}$ is non-zero then there exists $w \in \mathbb{C}$ such that $zw = 1$.

Proof. Set $w = \overline{z}/|z|^2$ and use Theorem 3.16. **Happy?** Except that my unimaginative friend is objecting. Why? We can divide by $|z|^2$ because $|z|$ is non-zero by Theorem 3.15. But they are still objecting, because we haven't defined division of complex numbers! *sigh*

Let's talk a bit about multiplying a complex number by a *real* number.

If $z = (x, y)$ is a complex number and r is a real number, then let's just check that the product is what we think it is.

$rz = (r, 0) \times (x, y) = (rx - 0y, ry + 0x) = (rx, ry)$.

Geometrically, what does multiplication by a real number correspond to on the complex plane?

Multiplication by r geometrically corresponds to *scaling* by a factor of r.

Examples: multiplication by 2 sends $3+4i$ to $6+8i$ and $x+yi$ to $(2x)+(2y)i$.

Multiplication by $\frac{1}{10}$ sends $3+4i$ to $\frac{3}{10}+\frac{4}{10}i$. It is shrinking by a factor of 10.

Multiplication by -1 is the same as "reflecting about the origin". But we usually describe this as "rotating by angle π".

Multiplication by -2 is the same as rotating by angle π and then scaling by a factor of 2.

Corollary 3.17. If $z \in \mathbb{C}$ is non-zero then there exists $w \in \mathbb{C}$ such that $zw = 1$ ("complex reciprocals exist").

More careful proof. Say $z = x + yi$. Recall from Theorem 3.15 that $|z| > 0$, and so $|z|^2 > 0$ and in particular $|z|^2 \neq 0$. Define $w = \frac{1}{|z|^2}\overline{z} = (x/|z|^2) - (y/|z|^2)i$. This is multiplication by the real number $\frac{1}{|z|^2}$, which we can do – we can take reciprocals of *real* numbers! Now multiplying everything out we see $zw = \frac{1}{|z|^2} z\overline{z} = 1$.

□

Corollary 3.18. It makes sense to divide a complex number by a non-zero complex number.

Proof. We can divide by $z \neq 0$, simply by multiplying by $\overline{z}/|z|^2$ instead. □

Technical remark: Now we have $+$, $-$, $\times$, and "divide-except-by-zero", and they obey all the usual rules, so we have just proved that the complex numbers are a. . . *field*.

Definition.

Informal: the *argument* of a complex number z is the angle which the line from 0 to z makes with the positive real axis (draw a picture to explain this).

Formal: The *argument* of a non-zero complex number $z = (x, y)$ is the unique angle θ such that $\sin(\theta) = y/|z|$, $\cos(\theta) = x/|z|$.

How do we know that such an angle θ exists? Well, $(x/|z|)^2 + (y/|z|)^2 = \frac{x^2+y^2}{x^2+y^2} = 1$, and it's a standard fact about real numbers that if we have two real numbers c and s such that $s^2 + c^2 = 1$ then there's a unique angle θ such that $s = \sin(\theta)$ and $c = \cos(\theta)$.

Happy?

Wait! says our pedantic friend. If $z = (1, 0)$ then we need to solve $\sin(\theta) = 0$ and $\cos(\theta) = 1$, and $\theta = 0$ and $\theta = 2\pi$ are two different solutions!

What should we tell our friend?

The M1M1 way to deal with this sort of thing is to say that an angle by definition is a real number in $[0, 2\pi)$, and in this range there is a unique solution.

There is however another way to think about it, which we will come back to later on. We could say that an angle was not a real number at all, it was an infinite set of real numbers, whose differences were all integer multiples of 2π. For example we could say that the solution to $\sin(\theta) = 0$ and $\cos(\theta) = 1$ is the angle $\{\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots\}$.

This does look like a weird way to think about angles, but actually it has its advantages.

We will come back to this way of looking at angles later on, when we do equivalence relations.

It is true by definition that a complex number is determined by its real and imaginary part.

It is intuitively true that a non-zero complex number is also determined by its modulus and its argument, and this is not hard to check formally.

Indeed, if $r = |z|$ and $\theta = \arg(z)$, then by definition $z = x + iy$ with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Restating this:

Fundamental fact. If $r = |z|$ and $\theta = \arg(z)$ then $z = r(\cos(\theta) + i\sin(\theta))$.

Question for the audience: what is the argument of $z = 0$? [I don't know]

Whatever it is, we know $|0| = 0$, so in all cases we have $z = r(\cos(\theta) + i\sin(\theta))$.

Technical (nonexaminable!) point: we now have a new way of representing complex numbers z using only the real numbers r and θ.

So in fact we could have *defined* complex numbers this way.

We could have *defined* a complex number to be an ordered pair (r, θ) of real numbers, such that $r \geq 0$, with the caveat that we have to identify (r, θ) and (s, ψ) if either $r = s = 0$ or $r = s > 0$ and $\theta - \psi$ is an integer multiple of 2π.

We could have defined the real part of z to be $r\cos(\theta)$ and so on and so on.

This would have been a different way of formalising our intuition about the complex plane.

It doesn't make any difference which way we do it though. The same theorems are true for both definitions :-)

This “$\cos(\theta) + i\sin(\theta)$” complex number is an important construction in the complex numbers. Let’s draw a picture and give it a name.

Let's define a function from the real numbers to the complex numbers.

Let's call the function `e-to-the-i`. (Note added in proof: I'm going to change this to cis in the next lecture because it's less weird.) It's defined like this.

If θ is a real number, then $\mathtt{e-to-the-i}(\theta)$ is defined to be $\cos(\theta) + i \sin(\theta)$.

Theorem 3.19 [de Moivre's theorem] If θ and ψ are real numbers, then

$\mathtt{e\text{-}to\text{-}the\text{-}i}(\theta + \psi) = \mathtt{e\text{-}to\text{-}the\text{-}i}(\theta) \times \mathtt{e\text{-}to\text{-}the\text{-}i}(\psi)$.

Proof. By definition of the function `e-to-the-i`, the theorem is equivalent to the following claim:

$$\cos(\theta + \psi) + i \sin(\theta + \psi) = (\cos(\theta) + i \sin(\theta))(\cos(\psi) + i \sin(\psi)).$$

$$\cos(\theta+\psi)+i\sin(\theta+\psi)=(\cos(\theta)+i\sin(\theta))(\cos(\psi)+i\sin(\psi)).$$

Multiplying out, and equating real and imaginary parts, we see that the theorem is equivalent to the following two claims:

$$\cos(\theta+\psi)=\cos(\theta)\cos(\psi)-\sin(\theta)\sin(\psi)$$

and

$$\sin(\theta+\psi)=\cos(\theta)\sin(\psi)+\sin(\theta)\cos(\psi).$$

But these are both true – indeed, they are standard facts about the real numbers.

$\square$

Corollary 3.20. Multiplication by $\texttt{e-to-the-i}(\theta)$ sends $r\texttt{e-to-the-i}(\psi)$ to $r\texttt{e-to-the-i}(\theta+\psi)$.

Geometric interpretation: multiplication by $(\cos(\theta) + i\sin(\theta))$ geometrically corresponds to an anticlockwise rotation by angle θ.

Corollary 3.21 [geometric interpretation of multiplication]. If $z = r\texttt{e-to-the-i}(\theta)$ is a complex number, then multiplication by z is equivalent to scaling by a factor of r, and then rotating anticlockwise by a factor of θ.

Easy exercise: what does multiplication by i correspond to in this geometric language?