Basic notation for sets.

KMB, 11/10/18

1 Sets

A *set* is a collection of stuff. The things in a set X are called the *elements* of X.

[technical note, to be skipped at first reading: There is a more rigorous definition of a set, but let's not worry about this for now. The more rigorous definition depends on which axiomatic foundation you are using for mathematics. If you are building mathematics using set theory as your foundation, then here is the definition of a set in mathematics: everything is a set. In particular that means that π is a set! How stupid is that!]

2 Basic notation for sets.

We use the "squiggly bracket" notation for sets, also known as the "curly bracket" notation. I'm talking about these things: { and }.

Examples of sets:

- $\{1, 2, 3\}$ is a set.
- { me, you, the desk in my office } is a set.
- $\{\}$ is a set – it exists, but it has no elements.
- $\{1, 2, 3, 2\}$ is a set.
- $\{0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$ is a set – it's an infinite set.

We use the symbol $\in$ to denote set membership. If a is a thing (e.g. a number) and X is a set, then $a \in X$ is a proposition – a true/false statement. The proposition $a \in X$ is true exactly when a is one of the bits of stuff which is in the set X.

Examples:

First example: $2 \in \{1, 2, 3\}$. This means that the number 2 is an element of the set $\{1, 2, 3\}$.

Second example: if x is any number, then $x \in \{\}$ makes mathematical sense, but it is a false statement.

The set $\{\}$ has no elements at all. It has a special name – the *empty set*. Mathematicians use the notation $\varnothing$ for the empty set.

Is the set $\{1, 2, 3\}$ equal to the set $\{1, 2, 3, 2\}$? That's an interesting question! It's not a question you can "work out the answer to" though – the answer to this question is simply that we have to look at the *rules* for sets and find out what it means for two sets to be equal.

3 The fundamental fact about equality of sets.

The fundamental fact is:

Two sets are equal if, and only if, they have the same elements.

To put it another way: the sets $\{1, 2, 3\}$ and $\{1, 2, 3, 2\}$ are equal.

To put it yet another way, using the notation of the previous lecture: if X and Y are sets, then $X = Y$ if and only if for all a, $a \in X \iff a \in Y$.

You might think this convention is inconvenient! Indeed there are times in life when you *do* want to remember how many times you counted something. But the fundamental fact above *is the rule for sets*. If you need to remember how many times you have counted something – *don't use sets*! Mathematics has other things, like multisets, or lists, or sequences, which might suit you better in some use cases.[1]

4 Notation

I have already talked about the notation { and } (the brackets which "contain" the stuff in the set), the notation $\in$ (to talk about the elements of a set, so $a \in X$ is a proposition and it's true exactly when a is in X) and the notation $=$ (two sets are equal if and only if they have the same elements). Here is some more notation.

4.1 $\subseteq$

If X and Y are sets, then $X \subseteq Y$ is a proposition. The proposition $X \subseteq Y$ is true if and only if every element of X is also an element of Y. Slightly more formally, $X \subseteq Y$ means that for all things a, $a \in X \implies a \in Y$.

Example: $\{1, 2\} \subseteq \{1, 2, 3\}$, because the only elements of the set $\{1, 2\}$ are the two numbers 1 and 2, and both of these numbers are in the set $\{1, 2, 3\}$.

Example: if a is my left shoe, and b is my right hand, and c is my mother, then $\{a, b\} \subseteq \{a, b, c\}$, because the only elements of the set $\{a, b\}$ are the two things a and b, and both of these things are in the set $\{a, b, c\}$.

Variant: $X \supseteq Y$. This just means $Y \subseteq X$.

Exercise: draw a picture of two sets X and Y, with $X \subseteq Y$. Understand that this picture is *really important* for understanding sets properly. And then understand that this picture will get you *no marks* in an exam.

Important result: if X and Y are sets, then to prove $X = Y$ it suffices to prove that $X \subseteq Y$ and $Y \subseteq X$.

Why is this theorem true? Well, to prove that two sets are equal, we need to check that for every a, we have $a \in X \iff a \in Y$.

Now from $X \subseteq Y$ we can deduce that $a \in X \implies a \in Y$. And from $Y \subseteq X$ we can deduce that $a \in Y \implies a \in X$. So if we know $X \subseteq Y$ and $Y \subseteq X$ we can deduce that $a \in X \iff a \in Y$, which is exactly the statement that X and Y have the same elements. Hence the sets X and Y are equal, by the very definition of what it means for two sets to be equal.

Exercise. Clearly there is some sort of link between $\subseteq$ for sets and $\implies$ for propositions. There is also some sort of link between $=$ for sets and $\iff$ for propositions.

Which concepts in set theory correspond to $\wedge$ and $\vee$ for propositions? Write down a rigorous mathematical statement showing the link.

[1] technical note, to be omitted at first reading: in set theory, everything is a set. Can you see how this might cause problems in practice?