Remember that 2 lectures ago, we started off with four very basic assumptions about $<$ on $\mathbb{R}$, and then went on to prove about ten other things about $<$?
Here are some lemmas which I did not prove, but which are easy.

Lemma 2.9. The product of a positive and a negative number is negative (see example sheet).
Lemma 2.10. The product of two negative numbers is positive (see example sheet).
Lemma 2.11. If $x > 0$ then $1/x > 0$, and if $x < 0$ then $1/x < 0$. (use the previous lemmas, and the fact that $1 > 0$).

So now we know that the product of two positives is positive, the product of two negatives is positive, and the product of a positive and a negative is negative.
From this it is not hard to check that

$+ * + * + = +$,

$+ * + * - = -$,

$+ * - * - = +$,

$- * - * - = -$.

More generally, if you know about induction, you could even prove that if you have *n* non-zero real numbers, and *k* of them are negative, then the product is positive if *k* is even and negative if *k* is odd.
All from those four axioms.
Now let's do some normal maths.

Example. For which $x \in \mathbb{R}$, $x \neq 1$, is $x < 6/(x-1)$?

Have a go yourselves.
One solution. Subtracting x from both sides, the question is equivalent to finding which $x \neq 1$ satisfy

$$0 < \frac{6}{x-1} - x$$

i.e.

$$0 < \frac{6}{x-1} - \frac{x^2 - x}{x-1} = -\frac{x^2 - x - 6}{x-1}$$

i.e.

$$0 > (x-3)(x+2)(x-1)^{-1}.$$

So the question asks us to find all $x \neq 1$ such that $(x-3)(x+2)(x-1)^{-1} < 0$.

One of these factors changes sign at $x = 3$, $x = -2$ and $x = 1$.

Note also that none of these values of x work, because $x = 3$ and $x = -2$ give zero, and $x = 1$ is not allowed in the question.

So we need to consider the four regions $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$.

In these four regions, the signs are $- - -$, $- + -$, $- + +$ and $+ + +$ respectively.

So the product is negative in the first and third case, i.e., $x \in (-\infty, -2) \cup (1, 3)$.

So have we got a fully working real numbers yet? No, not really. One thing we don't have is decimal expansions. And to get these, we have to add a new assumption!

Assumption A5: If x is any real number, then there exists a unique integer n such that $n \leq x < (n+1)$ (we write $n = \lfloor x \rfloor$).

Now we can define decimal expansions!

Definition. The *decimal expansion* of a non-negative real number x is an infinite sequence $a_0, a_1, a_2, a_3, \ldots$ of non-negative integers, with the property that $0 \leq a_i \leq 9$ for every $i \geq 1$ (note that a_0 can be any non-negative integer). We will define this sequence below.
Notation: we write $x = a_0.a_1 a_2 a_3 \ldots$. Take a few moments to think about how to define this sequence.

Example: $\pi + 10 = 13.1415926\cdots$.
The way I set it up with the sequence language, $a_0 = 13$, $a_1 = 1$, $a_2 = 4$, $a_3 = 1$ and so on.
What if $x < 0$? Just define the decimal expansion to be $-$(decimal expansion of $-x$).

Construction of the sequence.

Let x be a positive real number. Define a_0 to be the integer part $\lfloor x \rfloor$ of x.
Define r_0, (the *remainder*), to be $x - a_0$, and note that by assumption A5 we have $0 \leq r_0 < 1$.

We have $x = a_0 + r_0$ with a_0 an integer, and $0 \leq r_0 < 1$.

Hence $0 \leq 10r_0 < 10$. Set $a_1 = \lfloor 10r_0 \rfloor$ and $r_1 = 10r_0 - a_1$.

It's not hard to check that $0 \leq a_1 \leq 9$ and $0 \leq r_1 < 1$.

It's also not hard to check (using A5) that $x = a_0 + \frac{a_1}{10} + \frac{r_1}{10}$.

Now repeat: set $a_2 = \lfloor 10r_1 \rfloor$ and $r_2 = 10r_1 - a_2$.

Then $0 \leq a_2 \leq 9$, $0 \leq r_2 < 1$, and $x = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \frac{r_2}{100}$.

Repeating this forever give us our sequence.

At the Nth step, $x = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_N}{10^N} + \frac{r_N}{10^N}$, and $0 \leq r_N < 1$.

Let's do an example. Set $x = 1/6$. What is the decimal expansion?

Well, $a_0 = \lfloor x \rfloor = 0$, and $r_0 = 1/6$.

So $1/6 = 0$.[something].

Now $10r_0 = \frac{10}{6} = 1\frac{2}{3}$, so $a_1 = 1$ and $r_1 = \frac{2}{3}$.

So $1/6 = 0.1$[something].

Now $10r_1 = \frac{20}{3} = 6\frac{2}{3}$ so $a_2 = 6$ and $r_2 = \frac{2}{3}$ again.

So $1/6 = 0.16$[something].

And $10r_2 = \frac{20}{3} = 6\frac{2}{3}$ so $a_3 = 6$ and $r_3 = \frac{2}{3}$...we are in a loop.

So $\frac{1}{6} = 0.16666\ldots$.

Are we all happy that we have defined the decimal expansion of a positive real number here?

Theorem 2.12. There is *no real number x* whose decimal expansion is $0.99999\ldots$.

Corollary. Anyone who has an opinion on whether $0.99999\ldots = 1$ had better know exactly what they *mean* when they talk about $0.99999\ldots$. They had better know a *definition* of $0.99999\ldots$!

Proof of 2.12. Let's assume, for a contradiction, that such a real number x exists with decimal expansion $0.99999\ldots$.

Because $\lfloor x \rfloor = a_0 = 0$ we must have $0 \leq x < 1$.

Because $x < 1$ we can write $x = 1 - \varepsilon$ with $\varepsilon > 0$.

One can check, using assumption $A5$, that there exists some large integer N with $1/\varepsilon < 10^N$.

Hence $\frac{1}{10^N} < \varepsilon$.

$x = 1 - \varepsilon$, $\frac{1}{10^N} < \varepsilon$.
By definition of the decimal expansion, we have

$$
\begin{aligned}
1 - \varepsilon &= x \\
&= a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_N}{10^N} + \frac{r_N}{10^N} \\
&= \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9}{10^N} + \frac{r_N}{10^N} \\
&\geq \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9}{10^N} \\
&= 1 - \frac{1}{10^N} \\
&> 1 - \varepsilon
\end{aligned}
$$

and this is a contradiction! $\square$

Chapter 3 : Complex numbers.

In chapter 2, we saw a part of the development of the real numbers – we assumed we had proved a few basic facts about $<$ and then we proved a whole lot more.

In this chapter, we will assume *everything you learnt at school about the real numbers*. We will assume that addition and $\sin(x)$ and $\sqrt{2}$ and π and other stuff like that all exist and all satisfy everything you already know, like $x + y = y + x$ and $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$ and so on.

We will assume *nothing* about the complex numbers.

We will start by *defining* the complex numbers!

I will then give you some sort of a feeling as to what needs to be done next, and how to do it.

Definition 3.1. A *complex number* is an ordered pair (x, y) of real numbers.

That's it.

Definition 3.1. A *complex number* is an ordered pair (x, y) of real numbers.

Notation: If $z = (x, y)$ is a complex number, we call x the *real part* of z and we call y the *imaginary part* of z.

I am a bit reluctant to write $x + yi$ instead of (x, y) at this point. Why?

Because we have no definition of $+$ and no definition of i yet!

Let's fix that now.

Definition 3.2 [the natural map from the reals to the complexes.] We identify the real number r with the complex number $(r, 0)$. Using this identification, we can think of real numbers as being complex numbers with no imaginary part.

Definition 3.3. We define the complex number i to be $(0, 1)$.

Definition 3.4 [addition of complex numbers.] If $z = (u, v)$ and $w = (x, y)$ are complex numbers, we define their sum $z + w$ to be… $(u + x, v + y)$.