Basic notation for sets.

KMB, 12/10/18

1 Sets again

Recall that a *set* is a collection of stuff. In the handout for the last lecture we talked about subsets. A reminder: if X and Y are sets, we say $X \subseteq Y$ if every element of X is an element of Y.

Sometimes we need to talk about subsets of X with a certain property, and there is some cool notation to write this. Let's say X is the set of integers, and we want to consider the subset of X consisting of positive integers. We can write this subset like this:

$$\{a \in X \mid a > 0\}.$$

That line in the middle is pronounced "such that". So you can read this as "the elements a of X such that $a > 0$.

Warning: some people use colons instead. They would write $\{a \in X : a > 0\}$. I will use the straight line notation $|$.

Exercise. Why do you think I prefer $|$ to $:$?

2 Universes.

It is a bit inconvenient having to talk about "stuff" because this word is not very mathematical. So why don't we fix once and for all a *universe* Ω consisting of all the stuff we're interested in (for example Ω could be the real numbers, and then we will be talking about subsets of the real numbers).

Exercise (straight maths students only) – which other course have you seen this idea in? What was Ω called there?

3 Even more notation I'm afraid

Let X and Y be subsets of Ω. You can think of X and Y as just general sets. I want to talk about unions and intersections.

3.1 "For all" is an upside down A

One of my most favourite notations in mathematics is the symbol for "for all". It is written like this: $\forall$. It looks cool and complicated, it confuses your friends who did maths at school but are not doing maths at uni, but the best part is that *it's really easy to understand*. Here's an example. Oh – by the way – by $\mathbb{Z}$ below I mean the set of integers, i.e. the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

Example. $\forall a \in \mathbb{Z}, 2a$ is even.

This just means "for all integers a, $2a$ is an even number".

3.2 Union of two sets.

The *union* of two sets X and Y, written $X \cup Y$, is all the stuff which is in *either* X, *or* Y, *or both*. For example, if Ω is the real numbers, and $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$, then $X \cup Y = \{1, 2, 3, 4, 5\}$.

Exercise: Why did I not write $X \cup Y = \{1, 2, 3, 3, 4, 5\}$? Could I have written $X \cup Y = \{5, 4, 3, 2, 1\}$?

If a is an arbitrary element of our universe Ω, then we see that $a \in X \cup Y$ if and only if $a \in X$ or $a \in Y$. Let's write this using the cool $\forall$ notation.

Fundamental fact about unions: $\forall a \in \Omega, a \in X \cup Y \iff a \in X \vee a \in Y$.

3.3 Intersection of two sets.

The *intersection* of X and Y, written $X \cap Y$, is all the stuff which is in both X *and* Y. For example, if Ω is the real numbers, and $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$, then $X \cap Y = \{3\}$.

If a is an arbitrary element of our universe Ω, then we see that $a \in X \cap Y$ if and only if $a \in X$ and $a \in Y$. Let's write this using the cool $\forall$ notation.

Fundamental fact about intersections: $\forall a \in \Omega, a \in X \cap Y \iff a \in X \wedge a \in Y$.

3.4 Complements.

(Important technical note: this only works if we have fixed our universe Ω).

If X is a subset of our universe Ω, then its *complement* X^c is the set whose elements are all the things in Ω which are not in X.

For example, if our universe Ω is $\mathbb{Z}$, the integers, and if X is the even integers, then its complement X^c is the odd integers.

To talk about complements, it is convenient to introduce the notation $\notin$. If X is a set (regarded, as usual, as a subset of our universe Ω) and if $a \in \Omega$, then there are two possibilities. Either $a \in X$, that is, a is an element of X, or a is not an element of X, in which case we write $a \notin X$. Note that for general X and a, both $a \in X$ and $a \notin X$ are propositions – they are true/false statements. Furthermore if $a \in X$ is true then $a \notin X$ is false, and if $a \in X$ is false then $a \notin X$ is true! We can write

$$\forall a \in \Omega,\ a \notin X \iff \neg(a \in X).$$

Using our complement notation, another way of writing this is

$$\forall a \in \Omega,\ a \in X^c \iff \neg(a \in X).$$