I think cis is a better name than `e-to-the-i`, so let's go with that today.

Definition. If θ is a real number, then cis(θ) $:= \cos(\theta) + i\sin(\theta)$.

Important note: mathematicians usually write $e^{i\theta}$ for cis(θ). But this notation is confusing *at this point in the development of the complex numbers*, because it makes things like $(e^{i\theta})^n = e^{in\theta}$ look *obvious*, and this is not obvious – it is a theorem (which we'll prove later in this lecture).

Recall
Theorem 3.19 [de Moivre's theorem] If θ and ψ are real numbers, then $\mathrm{cis}(\theta + \psi) = \mathrm{cis}(\theta)\,\mathrm{cis}(\psi)$.

Today we'll see some cool consequences of this.

Technical note: someone asked me at the end why I didn't define $e^z := \sum_{n \geq 0} \frac{z^n}{n!}$ and proceed in that way.

I could have done it like this – and indeed in M1P1 I suspect that this is how you will do it.

The issue about doing it that way is that then you have to *define* $\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ and then the fact htat $\sin(\theta)$ is opposite over hypotenuse becomes a *theorem*, not a definition.

Defining $\sin(\theta)$ to be $\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ is a perfectly fine way to do it though. It's another example of a phenomenon we saw last time – you need to choose your definitions, and then work to prove your theorems.

The problem with defining $\sin(\theta)$ to be the infinite sum is that you have to define what you *mean* by an infinite sum, i.e. you have to *rigorously define* the $\cdots$.

This is what you will learn about in M1P1.

Corollary 3.22. If $n \in \mathbb{Z}_{\geq 1}$ then

$$\mathrm{cis}(n\theta) = (\mathrm{cis}(\theta))^n.$$

Note that in the usual notation, $e^{in\theta} = (e^{i\theta})^n$.

The usual notation makes the corollary seem *obvious* – it "obviously" follows from $a^{bc} = (a^b)^c$.

But you have probably only seen the proof of this "obvious" fact for b and c integers – or maybe real numbers. If you want to use this "obvious" fact here you are going to have to *define* a^b for b a general complex number, which needs a whole bunch of $\cdots$'s.

Corollary 3.22. (corollary of de Moivre) If $n \in \mathbb{Z}_{\geq 1}$ then

$$\mathrm{cis}(n\theta) = (\mathrm{cis}(\theta))^n.$$

We are going to prove this by...induction.

We're going to do induction in the next chapter, but how much time I spend on it depends on how many people in the room don't know what induction is.

Test! Assuming $e^{i(\theta+\psi)} = e^{i\theta}e^{i\psi}$, and remembering that e^{ix} is just *notation* for $\cos(x) + i\sin(x)$ and *doesn't mean anything to the power anything*, see if you can prove Corollary 3.22 yourselves, by induction.

Corollary 3.22 If $n \in \mathbb{Z}_{\geq 1}$ then $e^{i(n\theta)} = (e^{i\theta})^n$.

Proof. The corollary is claiming that infinitely many propositions $P(1), P(2), P(3), \ldots, P(n), \ldots$, are all true.
Here, for $d \in \mathbb{Z}_{\geq 1}$ a fixed integer, $P(d)$ is the statement that $\operatorname{cis}(d\theta) = (\operatorname{cis}(\theta))^d$.

In fancy language, we are asked to prove $\forall n \in \mathbb{Z}_{\geq 1}, P(n)$.

Well, $P(1)$ is the statement that $\operatorname{cis}(\theta) = \operatorname{cis}(\theta)^1$, which is certainly true.

And $P(2)$ is the statement that $\operatorname{cis}(2\theta) = \operatorname{cis}(\theta)^2$, which we can rewrite as

$$\operatorname{cis}(\theta + \theta) = \operatorname{cis}(\theta)\operatorname{cis}(\theta).$$

This follows from de Moivre. So we're off to a good start!

Now let's say $d \in \mathbb{Z}_{\geq 1}$ is *fixed*, and let's assume that *we've already proved that* $\text{cis}(d\theta) = \text{cis}(\theta)^d$. In other words, let's assume that $P(d)$ is true. Can we prove $P(d+1)$?

Well $P(d+1)$ says

$$\text{cis}((d+1)\theta) = \text{cis}(\theta)^{d+1}.$$

Rewriting this, it says

$$\text{cis}(d\theta + \theta) = \text{cis}(\theta)^d \,\text{cis}(\theta).$$

If we're *assuming* $P(d)$ is true, then we can assume $\text{cis}(\theta)^d = \text{cis}(d\theta)$.
So we can rewrite our goal as

$$\text{cis}(d\theta + \theta) = \text{cis}(d\theta)\,\text{cis}(\theta).$$

And this is true by de Moivre :-)

So we just proved that if $d \in \mathbb{Z}_{\geq 1}$ is fixed, then $P(d) \implies P(d+1)$.

And we also checked that $P(1)$ is true.

So $P(1) \implies P(2)$, and $P(2)$ is true, and $P(2) \implies P(3)$, so $P(3)$ is true, and $P(3) \implies P(4)$, so $P(4)$ is true, and so on.

Formally – by the *principle of mathematical induction*, which I'll talk about next lecture, we can deduce $P(n)$ for all n.

So corollary 3.22 is proved.

$\square$

Remark. In fact, if n is any *integer*, then $\text{cis}(n\theta) = \text{cis}(\theta)^n$.

Why don't you try checking this now for $n = -1$?

I am claiming that $\text{cis}(-\theta) = \text{cis}(\theta)^{-1}$. Substituting in the definition of cis, I am claiming that

$$\cos(-\theta) + i\sin(-\theta) = (\cos(\theta) + i\sin(\theta))^{-1}.$$

In other words, I am claiming that

$$(\cos(-\theta) + i\sin(-\theta))(\cos(\theta) + i\sin(\theta)) = 1.$$

(that is the *definition* of x^{-1})

Goal:

$$(\cos(-\theta) + i\sin(-\theta))(\cos(\theta) + i\sin(\theta)) = 1.$$

Because $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, multiplying everything out we see that the claim is equivalent to the assertion that

$$\cos(\theta)^2 + \sin(\theta)^2 = 1.$$

And this is a standard fact about real numbers (in fact it's Pythagoras' theorem).
The case of general negative n now follows (exercise!)

Exercise: prove $\overline{cis(\theta)} = \mathrm{cis}(-\theta)$.

Applications of de Moivre.

Q) Write $\cos(3\theta)$ as a polynomial in $\cos(\theta)$.

A) $\cos(3\theta)$ is the real part of $\operatorname{cis}(3\theta)$.

So by de Moivre (corollary) it's the real part of $\operatorname{cis}(\theta)^3$.

So if $c = \cos(\theta)$ and $s = \sin(\theta)$, it's the real part of $(c + is)^3 = c^3 + 3ic^2s - 3cs^2 - is^3$.

So it's $c^3 - 3cs^2$, which is $c^3 - 3c(1 - c^2) = 4c^3 - 3c$.

I think this is more efficient than $\cos(3\theta) = \cos(2\theta + \theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) = \cdots$.

Exercise for later: try writing $\cos(5\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$ using both methods and see which you like best.

Q) What is $(1+i)^{10}$?

Bad idea: multiply it out and use the binomial theorem.

We get

$$\left(\binom{10}{0}-\binom{10}{2}+\binom{10}{4}+\cdots\right)+i\left(\binom{10}{1}-\binom{10}{3}+\binom{10}{5}-\cdots\right).$$

Ouch.

Trick: draw a picture!

$1+i=\sqrt{2}\,\mathrm{cis}(\pi/4)$.

But $(r\,\mathrm{cis}(\theta))^n = r^n\,\mathrm{cis}(\theta)^n = r^n\,\mathrm{cis}(n\theta)$ by de Moivre.

So $(1+i)^{10} = \sqrt{2}^{10}\,\mathrm{cis}(10\pi/4)$.

Which is $2^5(\cos(5\pi/2)+i\sin(5\pi/2))$, i.e. $32i$.

Q) If z and w are two complex numbers, then show $|zw| = |z||w|$.

A) We can write $z = a\,\text{cis}(\theta)$ and $w = b\,\text{cis}(\psi)$, with $a = |z|$ and $b = |w|$.

Then $zw = ab\,\text{cis}(\theta)\,\text{cis}(\psi) = ab\,\text{cis}(\theta + \psi)$ by de Moivre.

Writing $c = \cos(\theta + \psi)$ and $s = \sin(\theta + \psi)$ we see $zw = abc + iabs$. So $|zw| = \sqrt{(abc)^2 + (abs)^2} = ab\sqrt{c^2 + s^2}$.

And $c^2 + s^2 = 1$ so we're done.

As a corollary, $|z^n| = |z|^n$. The proof is…induction on n.

Theorem 3.23. If $n \in \mathbb{Z}_{\geq 1}$ then there exactly n complex numbers ζ satisfying $\zeta^n = 1$.

Proof. If $\zeta \in \mathbb{C}$ and $\zeta^n = 1$ then $|\zeta|^n = |1|$ so $|\zeta| = 1$ which means that we must have $\zeta = \mathrm{cis}(\theta)$ for some θ.

So we need to solve $\mathrm{cis}(\theta)^n = 1$. By the corollary to de Moivre, this is equivalent to $\mathrm{cis}(n\theta) = 1$.

So we need to solve $\cos(n\theta) = 1$ and $\sin(n\theta) = 0$. By a well-known fact about cos, we see that we must have $n\theta = 2k\pi$ for some integer k.

But k can be any integer, so it looks like we're going to get infinitely many real number solutions for θ, for example $\theta_0 = 0$, $\theta_1 = \frac{2\pi}{n}$, $\theta_2 = \frac{4\pi}{n}$, $\theta_3 = \frac{6\pi}{n}$ and so on. In general we could define $\theta_k = \frac{2k\pi}{n}$ for any integer k, and hence get infinitely many solutions for θ. What's going on?

The point is that *different* real numbers can give rise to *the same angles.*

If $k \in \mathbb{Z}$ and $\theta_k = \frac{2k\pi}{n}$ then $\zeta_k = \text{cis}(\theta_k)$ satisfies $\zeta_k^n = 1$. Infinitely many solutions?

No, because even though the real numbers θ_k are different, the *angles* they represent might sometimes be the same.

Indeed, if $i, j \in \mathbb{Z}$ then $\theta_i - \theta_j = \frac{2(i-j)\pi}{n} = \frac{i-j}{n}2\pi$, and if $i - j$ is a multiple of n then this difference is an integer multiple of 2π and hence θ_i and θ_j represent *the same angle*.

For every integer i, there's some integer j with $0 \leq j \leq n - 1$ such that $i - j$ is a multiple of n. So every *angle* θ_i is one of θ_0, θ_1, ..., θ_{n-1}, and these n real numbers are easily checked to represent n different angles.

Conclusion so far: if $\zeta^n = 1$ then $\zeta = \mathrm{cis}(\theta_k)$ for some $0 \le k \le n-1$.

Note that $\theta_1 = \frac{2\pi}{n}$, and a general $\theta_j = j\theta_1$.

Conclusion: If $\zeta^n = 1$ then $\zeta = \mathrm{cis}(2\pi/n)^j$ for some $0 \le j \le n-1$, and these are the n roots of the equation.

I think a picture makes this much easier to understand.
[I then drew a picture of the five 5th roots of unity, forming a pentagon with vertices on the unit circle]