## Chapter 9 [last chapter!]. Functions and countability.

Let $X$ and $Y$ be sets. A *function* from $X$ to $Y$ is an "algorithm", or a "method", or a "rule", or a "process", or a "machine", or a "black box", which takes as input an element of $X$, and outputs an element of $Y$.

Example: cos, the cosine function, is a function from the real numbers to the real numbers; given a real number $\theta$ as input, we get an output $\cos(\theta)$, which is again a real number.

Notation: $f : X \to Y$ means that $f$ is a function from $X$ to $Y$. If we feed in $x \in X$ to $f$, the output is denoted $f(x)$ (in computer science it is sometimes denoted `f x` without the brackets).

The set $X$ is called the *domain* of the function, and the set $Y$ is called the *codomain*. Some people call $Y$ the *range* of the function, however other people use "range" to mean what some people call the "image" of the function. I'll talk about images later on, but I will avoid the word "range", for the same reason I avoid $\mathbb{N}$ – it seems to me to be ambiguous.

Let $X$ and $Y$ be sets. A *function* from $X$ to $Y$ is a process which takes as input an element of $X$, and outputs an element of $Y$.

**Properties that a "process" *must* satisfy in order to be a function from $X$ to $Y$.**

1) There can be no randomness involved! If I have a function from $X$ to $Y$, and I feed in the same element of $X$ twice, I *must* get the same element of $Y$ out twice.

2) Related: the function must output *exactly one* element of $Y$. The "function" $f(x) = \pm\sqrt{x}$ is not a function from the positive reals to the reals, because it cannot seem to make up its mind about whether to output $+2$ or $-2$ given an input of 4.

3) The function must be defined on *all input values*, that is, on all elements of $X$. For example the "function" $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x^2+3}{x^2-9}$ is not a function, because it is undefined at $x = \pm 3$ (and $\infty$ is not a real number). However $f(x) = \frac{x^2+3}{x^2-9}$ *is* a well-defined function from $\mathbb{R} \setminus \{3, -3\}$ to $\mathbb{R}$ (the notation means "start with $\mathbb{R}$ and then remove 3 and $-3$").

[Draw some pictures of functions and non-functions]

Other words for a function: a “mapping” or a “map”.

Sometimes you want to talk about a function without giving it a name. For example, let’s say I want to talk about the function from the reals to the reals which sends $x \in \mathbb{R}$ to $4x + 3$. I could say “define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 4x + 3$”, but then I just gave this function the name $f$. Can I talk about the function without giving it a name?

Mathematicians have some notation for this, which I used to think was pretty cool.

$$\begin{aligned} \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 4x + 3 \end{aligned}$$

That’s `\mapsto` in LaTeX.

But it turns out that computer scientists have an even cooler name for it: `λ x, 4 * x + 3`.

But actually $\mapsto$ does seem to have one advantage over $\lambda$ – if we're talking about the function $x \mapsto 4x + 3$ then we can write $2 \mapsto 11$ to mean "2 maps to 11".

There's now *loads* of definitions we need to get through.

Say $X$ and $Y$ are sets, and $f : X \to Y$ is a function from $X$ to $Y$.

If $S \subseteq X$ is a subset of $X$, then $f$ also gives us a rule sending elements of $S$ to elements of $Y$, because if $a \in S$ then $a \in X$ so $f(a)$ makes sense. We call this new function from $S$ to $Y$ the *restriction* of $f$ to $S$.

For example we could restrict the cosine function to $\mathbb{Z}$ and get a new function $\mathbb{Z} \to \mathbb{R}$ sending $n \in \mathbb{Z}$ to $\cos(n)$. People still call this function cos of course.

The *image* of $f$ (note that some, but not all, people call this the *range* of $f$) is the subset $\{\, y \in Y \mid \exists x \in X, f(x) = y \,\}$. Alternatively it's $\{\, f(x) \mid x \in X \,\}$. It's *much* more easily explained in words!

In words, the *image* of $f$ is the subset of $Y$ consisting of elements that $f$ can actually output, as its inputs run through all the elements of $X$. For example if $f : \mathbb{R} \to \mathbb{R}$ is the squaring function $f(x) = x^2$, then $f$ only outputs squares, and squares are non-negative. So whatever the value of $x$, we have $f(x) \geq 0$. And conversely, if $y \geq 0$ and if we believe in the existence of square roots, then setting $x = \sqrt{y}$ we have $f(x) = x^2 = y$. So a real number $y$ is an output of the function $f$ if and only if $y \geq 0$. So in this case, the image of $f$ is $\mathbb{R}_{\geq 0}$, the non-negative real numbers.

Notation: we sometimes write $f(X)$ for the image of $X$. This is *abuse of notation*; $f$ is supposed to take an element and we have given it a whole bunch of elements at once.

More abuse of notation: if $S \subseteq X$ then $f(S)$ is defined to be $\{ y \in Y \mid \exists x \in S,\ f(x) = y \}$. Alternatively, $f(S) = \{ f(a) \mid a \in S \}$. In words, $f(S)$ is the *subset* of $Y$ consisting of elements of the form $f(a)$ as $a$ ranges through all of $S$.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x$, then $f(6) = 12$ and $f([1, 2]) = [2, 4]$. (note to self: remind students that they should use $(3, 4]$ notation and not $]3, 4]$ or $(3; 4]$ etc).

Now say $f : X \to Y$ is a function, and $T$ is a subset of $Y$ such that the image $f(X)$ of $f$ satisfies $f(X) \subseteq T$. In words – whenever we feed $x \in X$ into $f$, our output is guaranteed to land in $T$. Then $f$ gives rise to a function from $X$ to $T$. Mathematicians just call this function $f$ again :-/ .

For example $\cos : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to [-1, 1]$ are, to a mathematician, "the same function", even though they have different codomains. Mathematicians are just sloppy like that, but it usually doesn't cause confusion.

Now say $X$, $Y$ and $Z$ are all sets, and $f : X \to Y$ and $g : Y \to Z$ are functions.

Given an element $x \in X$, how can we make an element of $Z$? Think about it.

First we apply $f$, to get an element $f(x) \in Y$. And then we can apply $g$ to get an element $g(f(x)) \in Z$.

So we have a process which, given an element of $X$, spits out an element of $Z$. So we have a new function from $X$ to $Z$, built from a function $f : X \to Y$ and a function $g : Y \to Z$. This is called *composition of functions*. We just saw that this new function from $X$ to $Z$ sends $x$ to $g(f(x))$. There is notation for this function; this function from $X$ to $Z$ is called $g \circ f$.

It's easiest to understand this concept with a picture [draw a commuting triangle].

$f : X \to Y$ and $g : Y \to Z$. The composition of these functions is $g \circ f : X \to Z$. It's defined by $(g \circ f)(x) = g(f(x))$.

**Warning.** It looks like $g \circ f$ means "do $g$, then $f$" because $g$ comes before $f$ in the notation. But as we saw above, it means "do $f$, then $g$". In a parallel universe, humans decided that the notation for functions $f$ applied to elements $x$ should be $(x)f$ instead of $f(x)$. "Maps go on the right". In this parallel universe, doing $f$ then $g$ came out as $((x)f)g$ with $f$ before $g$ when you write it down, and they call function composition $f \circ g$, meaning "do $f$ then $g$". But we don't live in this parallel universe, we're stuck in this universe, so we're stuck with "do $f$, then $g$" being written as $g \circ f$.

*Example.*
Say $X = Y = Z = \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = y + 1$. What is the function $f \circ g$? I've made it extra-hard, with $X = Y = Z$, to maximise the chance that you go wrong.

Well, $(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2$. So $f \circ g$ is $x \mapsto (x + 1)^2$. What is $g \circ f$? That's what you get when you do $f$ then $g$, so it sends $x$ to $x^2 + 1$ (which is not the same as $f \circ g$ – function composition is not commutative. What other multiplication is not commutative? Matrix multiplication is not commutative – but matrices can be regarded as functions so perhaps this is not surprising).

Given any set $S$, there is always one natural (and rather boring) function from $S$ to $S$. Can anyone see what the definition of that function could be?

The *identity function*, usually called id by mathematicians, and $\lambda x, x$ by computer scientists, is the process which takes an element of $S$ and then gives it straight back :-) It satisfies $\mathrm{id}(x) = x$ for all $x \in S$. It even works when $S$ is the empty set. Note that there are lots of different identity functions – one for each set!

Say $\mathrm{id} : S \to S$ is the identity function. What is $\mathrm{id} \circ \mathrm{id}$? Well, $\mathrm{id}(\mathrm{id}(s)) = \mathrm{id}(s) = s$, so $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$. Actually, is this definitely right? What does it *mean* for two functions to be equal? To a mathematician it *means* that these functions give the same outputs on all inputs. We proved that $\mathrm{id} \circ \mathrm{id}$ and id give the same outputs on all inputs, so they are equal as functions.

Say $f : X \to Y$ is a function. Here are some *really important definitions*.

**Definition 9.1.**

(i) We say $f$ is *injective*, or *an injection*, if $\forall a, b \in X, f(a) = f(b) \implies a = b$.

(ii) We say $f$ is *surjective*, or *a surjection*, if $f(X) = Y$, or equivalently if $\forall y \in Y, \exists x \in X, f(x) = y$.

(iii) We say $f$ is *bijective*, or *a bijection*, if it is both injective and surjective.

What does all that mean? Let's go through them one by one. Remember that a true/false statement $P \implies Q$ is logically equivalent to its contrapositive $\neg Q \implies \neg P$. The true/false statement in the definition of *injective* is $f(a) = f(b) \implies a = b$. Its contrapositive is $a \neq b \implies f(a) \neq f(b)$. "Distinct elements of $X$ map to distinct elements of $Y$".

A function $f : X \to Y$ is injective if *distinct elements of $X$ map to distinct elements of $Y$*. That's what you should *think*. But if I ask you in an exam, write down a formal statement.

Another word for injective – some people say "$f$ is one-to-one".

**Definition 9.1.**
(i) We say $f$ is *injective*, or *an injection*, or *one-to-one*, if $\forall a, b \in X, f(a) = f(b) \implies a = b$.
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(iii) We say $f$ is *bijective*, or *a bijection*, if it is both injective and surjective.

We've talked about injective – what does surjective mean? It means that the image $f(X)$ is all of $Y$, or equivalently that every element of $Y$ is "hit" by an element of $X$.

Another word for surjective is "onto".

Looking at some of your weekly tests, I get the impression that some of you might not know the difference between $\forall y \in Y, \exists x \in X, f(x) = y$ and $\exists x \in X, \forall y \in Y, f(x) = y$. If you are a maths student, this will become very problematic when you do M1P1 next term. So let's go through this carefully.

Let $X = Y = \mathbb{Z}$ and let $f$ be the *identity function*.

"Every element of $Y$ is hit by an element of $X$" is an ambiguous statement. It might mean either of the below. Which of the below statements is true?

**Quiz!**

1) True or false? $\forall y \in Y, \exists x \in X, f(x) = y$.

2) True or false? $\exists x \in X, \forall y \in Y, f(x) = y$.

Answers:
1) is true – you can let $x = y$. But 2) is false, if you choose $x$, then you can let $y$ be anything, so you can let $y$ be $x + 1$, and then it's not true that $f(x) = y$.

Injective means that distinct elements of $X$ must map to distinct elements of $Y$. Surjective means that every element of $Y$ gets hit by some element of $X$. *Don't write this if I ask you what these words mean. Write what I wrote in definition 9.1.*

Example: the squaring function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$. Is it injective? No, because $-2 \neq 2$ but $f(-2) = 4 = f(2)$. So different elements of the domain get mapped to the same element of the codomain. Is the squaring function from $\mathbb{R}$ to $\mathbb{R}$ surjective? No, because if $y = -1 \in \mathbb{R}$ then there is no $x \in \mathbb{R}$ such that $f(x) = -1$, as $x^2 = -1$ has no real solutions.

Example: the cosine function. Is it injective? No, because $\cos(0) = \cos(2\pi)$. Is it surjective? This question is not well-defined. $\cos : \mathbb{R} \to \mathbb{R}$ is not surjective, but $\cos : \mathbb{R} \to [-1, 1]$ is surjective. Actually $\cos : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is injective! These notions of injective and surjective *heavily depend* on the domain and the codomain, and because mathematicians are often sloppy about what the domain and the codomain are, you need to be careful.