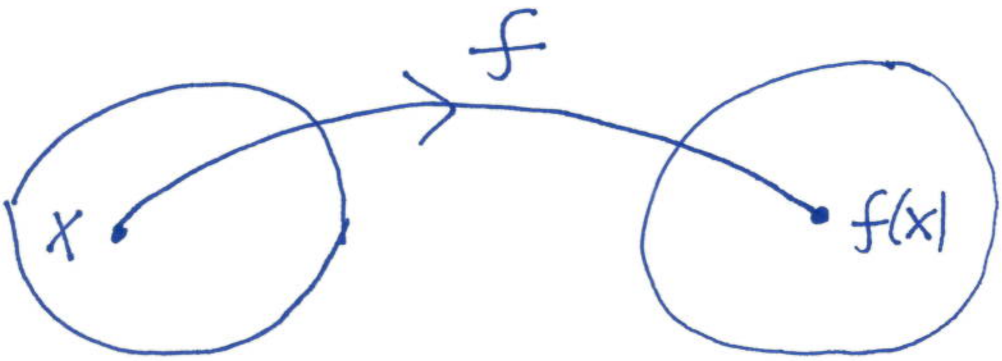
1. Functions

A function f is rule which assigns a element $f(x)$ of a set B to every element x of a set A.



A (domain) B (codomain)
dom(f)

Functions of one variable

A and B ~~are~~ is $\mathbb{R}$
(or subsets thereof)

Examples

A polynomial is a function

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$$

(if $c_n \neq 0$ has degree n)

$c_0, c_1, \dots, c_n$ constants
(or coefficients)

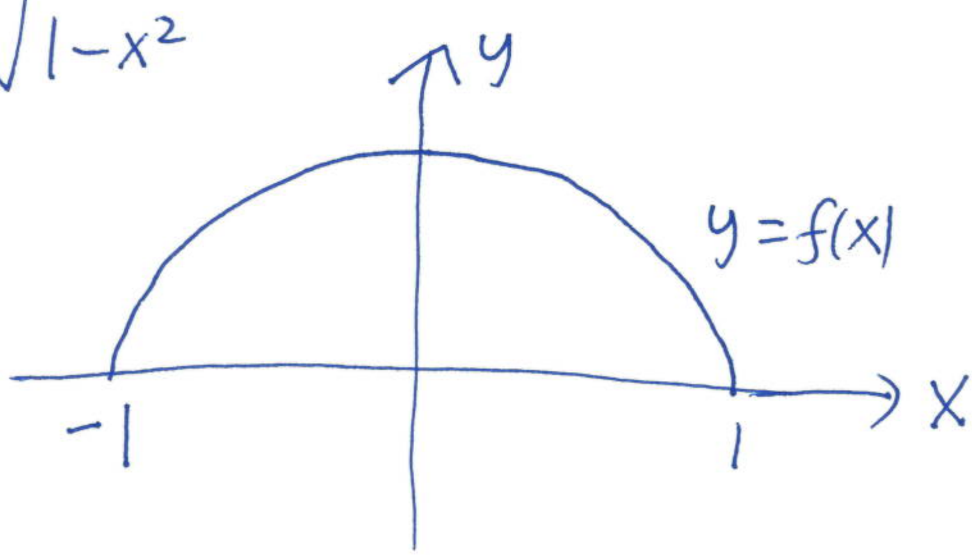
$$\mathrm{dom}(P) = \mathbb{R}$$

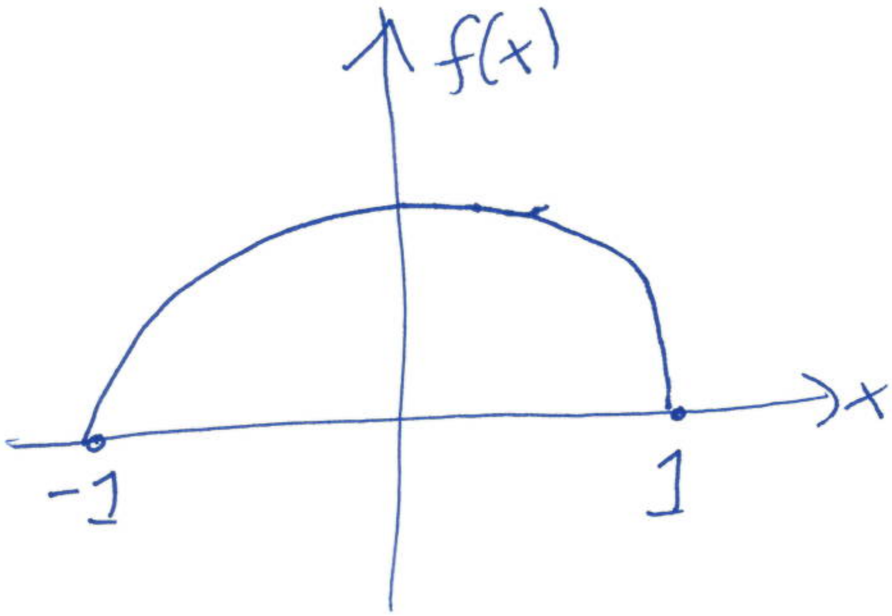
$$f(x) = \sqrt{1-x^2}$$

$$\mathrm{dom}(f) = [-1, 1]$$

The graph of a function f is the set of points (x,y) in the xy plane defined by $y = f(x)$, $x \in \mathrm{dom}(f)$

$$f(x) = \sqrt{1-x^2}$$





Rational Functions

are ratios of polynomials

$$f(x) = \frac{P(x)}{Q(x)}$$ where P, Q polynomial

eg. $$f(x) = \frac{x^5}{1-x^2}$$ rational

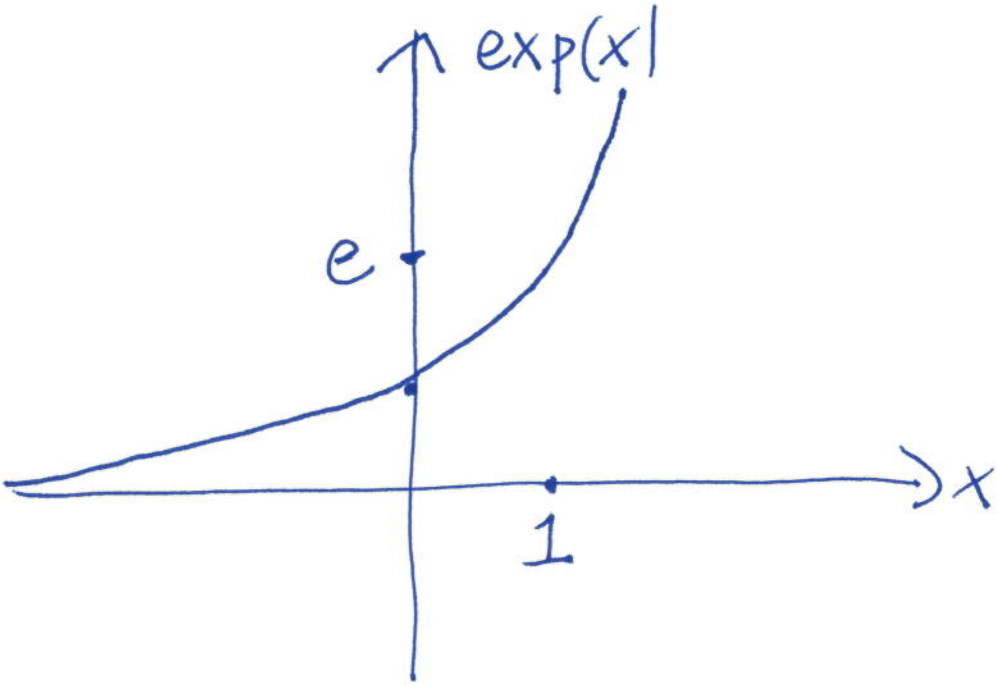
$$\mathrm{dom}(f) = \mathbb{R} \setminus \{-1, 1\}$$

$$f(x) = \frac{1}{1-x^2}$$

rational

here P has degree zero

Exponential Function



Definitions?

(i) $\exp(x) = e^x$

must define constant e

(ii) power series definition

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad 0! = 1$$

(iii) ODE definition

$$\frac{d}{dx} \exp(x) = \exp(x)$$

$$\exp(0) = 1$$

(iv) exp inverse of logarithm. Can define

$$\log x = \int_1^x \frac{du}{u}$$

(v) as a limit

$$\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Addition formula

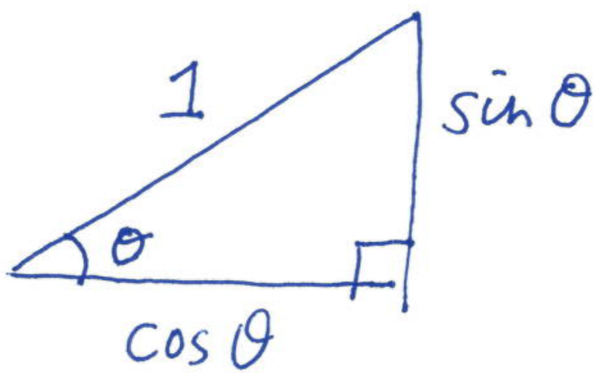
$$\exp(x+y) = \exp(x)\exp(y)$$

Trig Functions

sine and cosine

Definition?

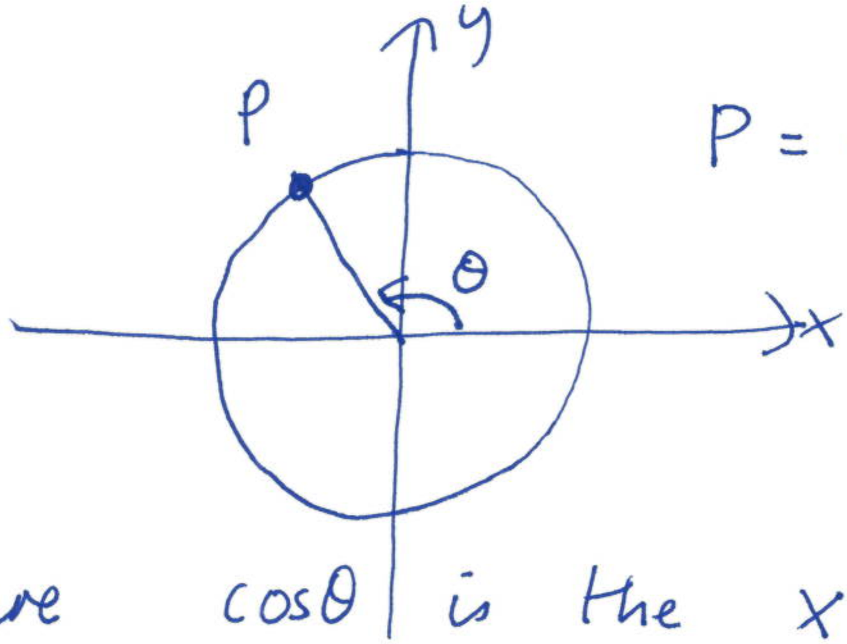
(i) elementary geometrical approach



defines $\cos\theta$ and $\sin\theta$ for $0 < \theta < \frac{\pi}{2}$

(ii) Polar construction

Let P be a point on the unit circle centred at origin



$P = (\cos\theta, \sin\theta)$

Here $\cos\theta$ is the x coordinate of P, $\sin\theta$ is the y coordinate of P. θ measured anti-clockwise from positive x-axis defines $\cos\theta$ and $\sin\theta$ for any real θ.

(iii) As power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

(iv) as a system of ODEs

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\sin(0) = 0, \quad \cos(0) = 1$$

(v) complex definition

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

more later

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

(vi) as infinite products

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)$$

$$\times\left(1 - \frac{x^2}{9\pi^2}\right) \ldots$$

or

$$\sin x = x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

There is a similar formula for $\cos x$

$$\cos x = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right)$$

Addition formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Shift properties

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

$$\cos\left(x + \frac{\pi}{2}\right) = -\sin x$$

$$\sin(x+\pi) = -\sin x$$

$$\cos(x+\pi) = -\cos x$$

and

$$\sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x$$

sin and cos are <u>periodic</u> functions with period 2π

If $f(x+a) = f(x)$ for all x where $a \neq 0$ is constant then f is called <u>periodic</u>.
Period of function is smallest positive a for which this holds

Special values

$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$

e.g. $\sin \frac{\pi}{6} = \frac{1}{2}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Other trig functions

$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$

$\sec x = \frac{1}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x}$

Trig identities

$\cancel{1 = \cos(1-1)}$

$$
\begin{aligned}
1 &= \cos(x-x) \\
&= \cos(x)\cos(-x) - \sin(x)\sin(-x) \\
&= \cos^2 x + \sin^2 x
\end{aligned}
$$

See problems for more trig identities

Even and Odd Functions

A function f is <u>even</u> if $f(-x) = f(x)$ for all $x \in \mathrm{dom}(f)$

A function f is <u>odd</u> if $f(-x) = -f(x)$

(definition assumes $\mathrm{dom}(f)$ is symmetric: $-x \in \mathrm{dom}(f)$ if $x \in \mathrm{dom}(f)$)

Cosine is even
sine is odd

A function can be neither odd nor even

eg. exp(x) is not odd and not even. Can decompose any function into a sum of an even function and an odd function. Proof

$$f(x) = \underbrace{\frac{1}{2}\left(f(x) + f(-x)\right)}_{f_e(x) \text{ even}} + \underbrace{\frac{1}{2}\left(f(x) - f(-x)\right)}_{f_o(x) \text{ odd}}$$

Examples

(i) $f(x) = \exp(x) = e^x$

$$e^x = \underbrace{\tfrac{1}{2}\left(e^x + e^{-x}\right)}_{\text{even}} + \underbrace{\tfrac{1}{2}\left(e^x - e^{-x}\right)}_{\text{odd}}$$

hyperbolic functions!

$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right)$$

$$\sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right)$$

(ii) decompose

$$f(x) = \log\left(x + \sqrt{1+x^2}\right)$$

into even and odd parts

Hyperbolic Functions

As power series

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\frac{d}{dx} \cosh x = \sinh x$$

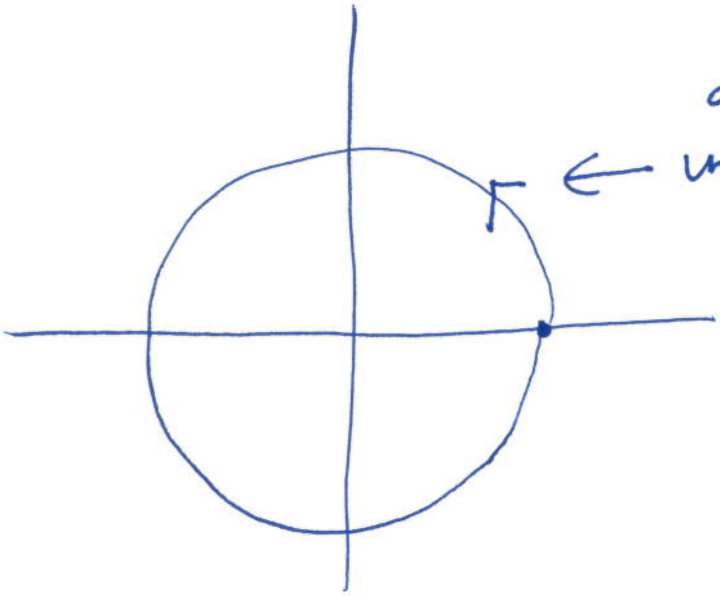
$$\frac{d}{dx} \sinh x = \cosh x$$

sin and cos sometimes called circular functions

$x = \cos\theta$

$y = \sin\theta$

$0 \le \theta \le 2\pi$

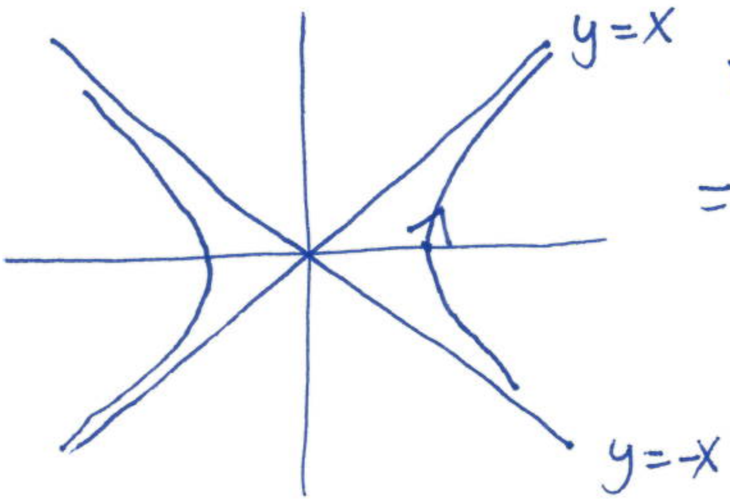


arrow indicates
increasing θ

$x = \cosh t$

$y = \sinh t$

where t
is real



$x^2 - y^2$

$= \cosh^2 t - \sinh^2 t = 1$

equation of
a hyperbola

$x = \cosh t$, $y = \sinh t$

parametrizes (right) branch

of hyperbola $x^2 - y^2 = 1$

Addition formulas

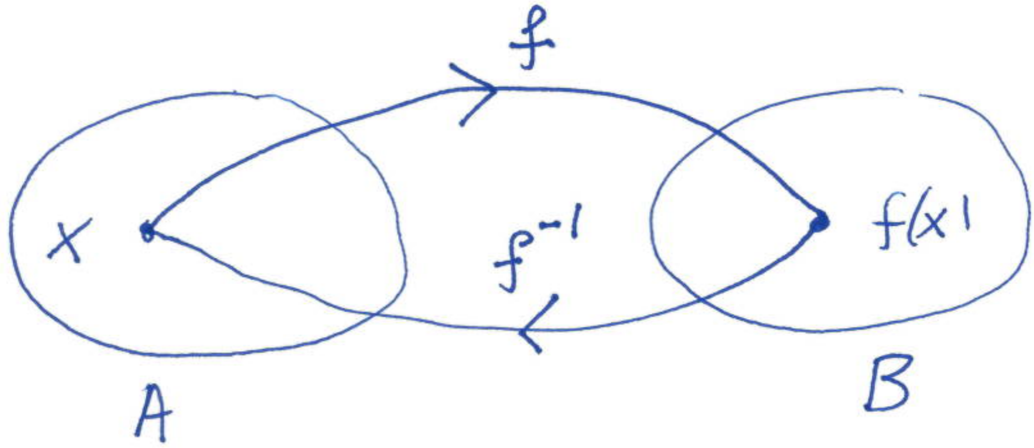
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

note

Inverse Functions

A function is a rule assigning a number $f(x)$ to every $x \in \mathrm{dom}(f)$



The inverse function denoted f^{-1} reverses action of f

$$f^{-1}(f(x)) = x \quad \text{for any } x \in \operatorname{dom} f$$

$$f(f^{-1}(y)) = y \quad \text{for any } y \in \operatorname{codom}(f)$$

Not all functions are invertible!

$f(x) = x^2$ $\quad$ $\operatorname{dom}(f) = [0, \infty)$

$\operatorname{codom}(f) = [0, \infty)$

is invertible

$f^{-1}(x) = \sqrt{x}$.

$f(x) = x^2$ $\quad$ $f : [0, \infty) \to \mathbb{R}$

A necessary condition for f to be invertible is that f <u>injective</u>

~~If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f$~~

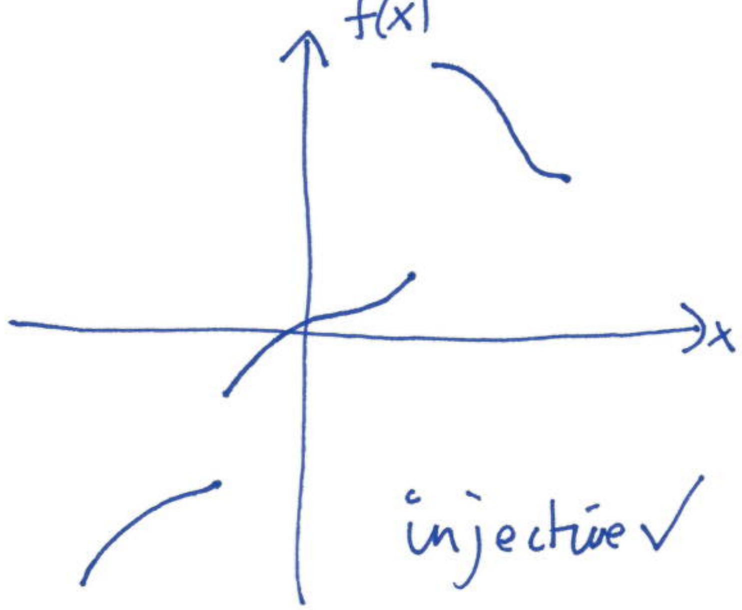
If ~~$x_1 \neq x_2 \Rightarrow f($~~

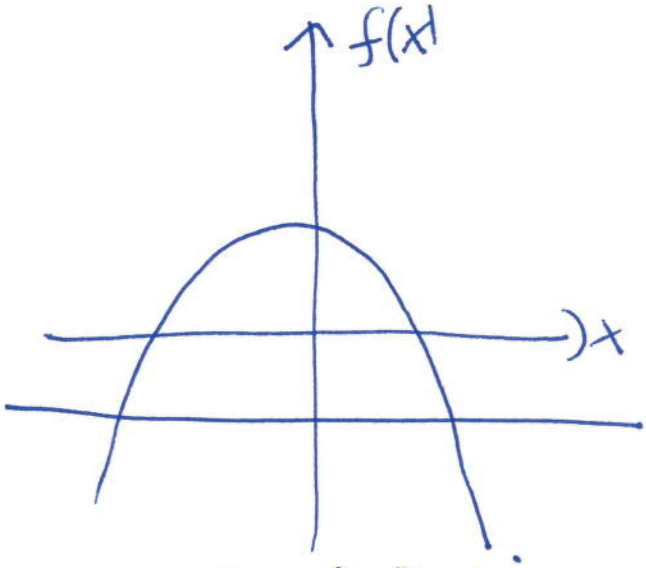
$$f(x_1) = f(x_2) \overset{\text{only if}}{\Longrightarrow} x_1 = x_2$$

$$f(x_1) \neq f(x_2) \underset{\text{if}}{\Longleftarrow} x_1 \neq x_2$$

This means that the outputs of the function ~~in~~ are distinct.

Graphically – the graph of an injective function satisfies the horizontal line test – the graph intersects any horizontal line at most once





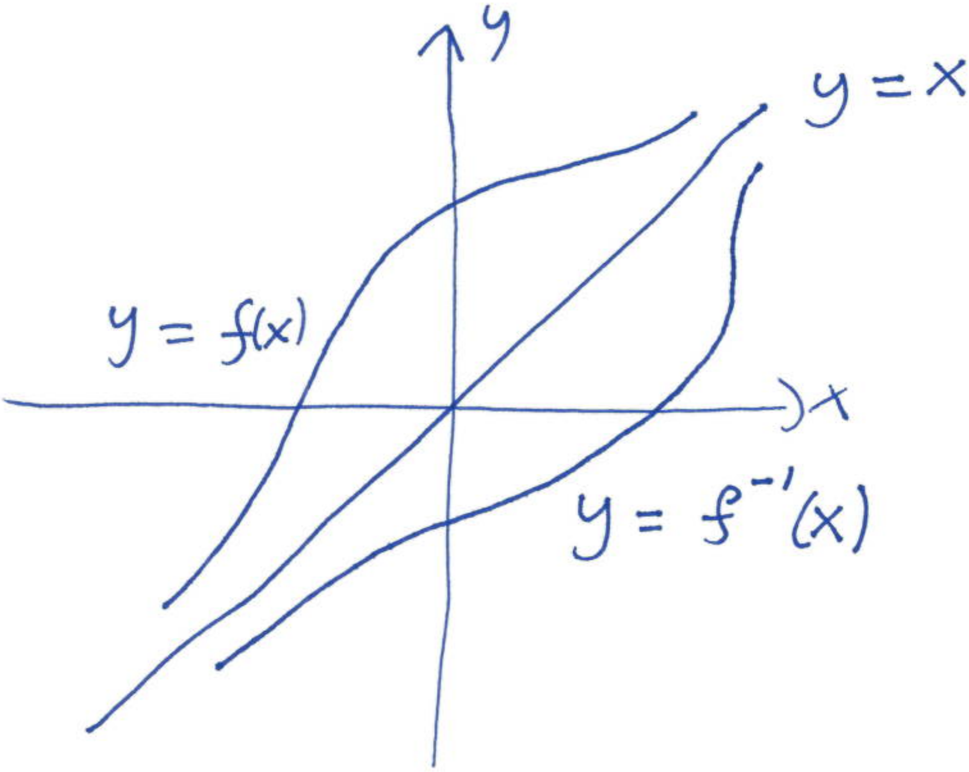
not injective
fails horizontal line test

injective aka one-to-one

MIF : f invertible iff
it injective and surjective
(or onto)
For more information
ask Professor Buzzard, in MIMI
assume functions are surjective

The graph of the
inverse function f^{-1} obtained
by swapping x and
y coordinates in graph
of f

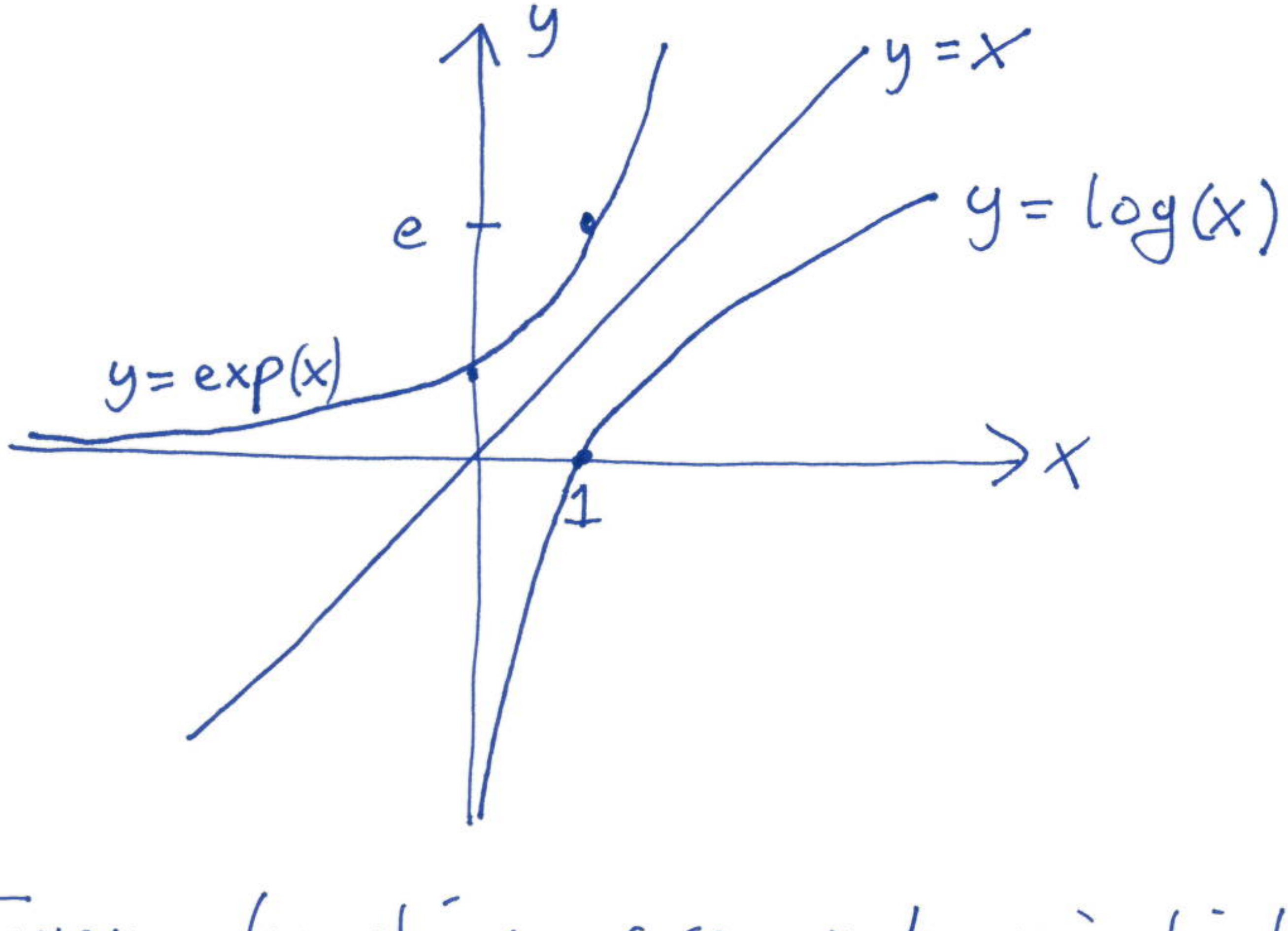
Graph of f^{-1} obtained by reflecting graph of f about line $y = x$



<u>Example</u> Can define the natural logarithm $\log x$ (or $\ln x$)

as the function of the exponential function

(injective)



Even functions are not injective!

Periodic functions are not injective!