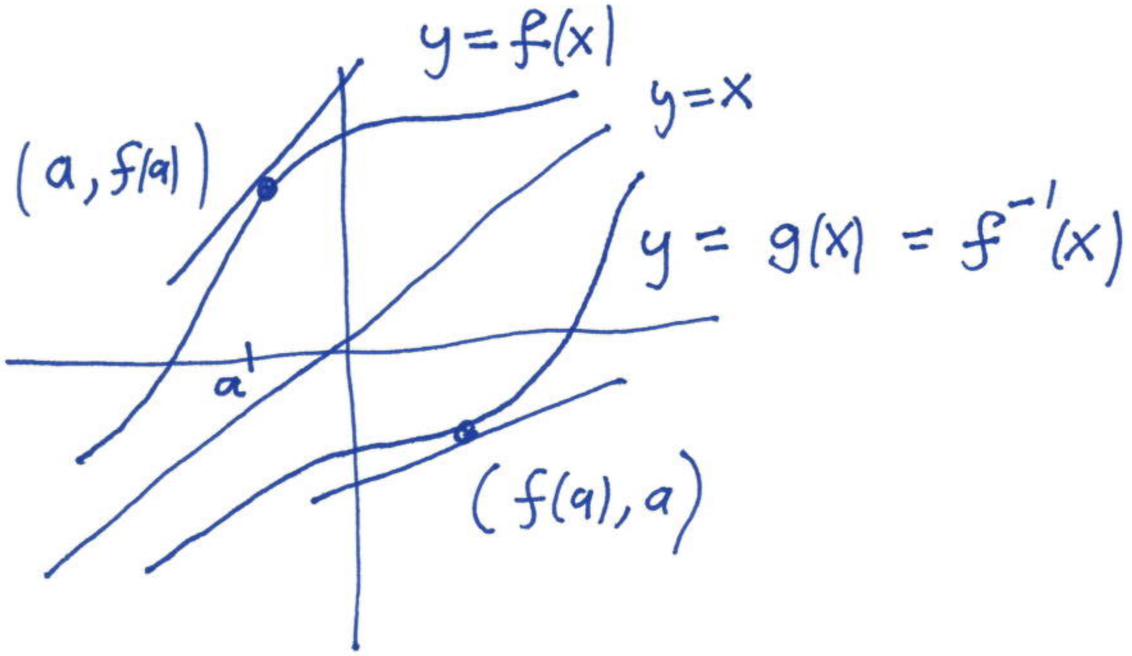
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$



$$g'(f(a)) = \frac{1}{f'(a)}$$

replace $a \to f^{-1}(x)$

Compute derivatives of inverse functions in 2 steps for example

$$f(x) = e^x \qquad f^{-1}(x) = \log x$$

write $y = f(x) = e^x \qquad x = f^{-1}(y)$

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{e^x} = \frac{1}{y}$$

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{y}$$

or $\boxed{\frac{d}{dx} \log x = \frac{1}{x}}$

Inverse trig Functions

$$f(x) = \sin x \qquad \cancel{f^{-1}(x) = \sin^{-1}}$$

$$y = \sin x \qquad x = \sin^{-1} y$$

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\boxed{\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}}$$

What is $\frac{d}{dx}\cos^{-1}x$?

$$f(x) = \tan x$$

$$y = \tan x \qquad x = \tan^{-1} y$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\sec^2 x}$$

$$= \frac{1}{1+\tan^2 x} \qquad \boxed{\sec^2 x = 1 + \tan^2 x}$$

$$= \frac{1}{1+y^2}$$

$$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}}$$

What is $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}x + c$

since $\dfrac{1}{1+x^2}$ is the derivative of $\tan^{-1}x$

Another example

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}x + c$$

2. Power Series and Limits

A power series is a function of form

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

$c_0, c_1, \dots$ real constants

"a polynomial of infinite degree"

Seen examples

$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Can add, multiply, compose, differentiate and integrate power series.

A simple power series is the infinite geometric series

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= \frac{1}{1-x} \quad \text{if } |x| < 1$$

(If $|x| \geq 1$ PS not convergent)

~~In~~ Integrate w.r.t x

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$= -\log(1-x) + C$$

$C = 0$ since setting $x = 0$
LHS $= 0$ RHS $= C$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

or (replace x with $-x$)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Another example

replace x with $-x^2$

$$1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$= \frac{1}{1+x^2} \qquad |x| < 1$$

Now integrate

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \tan^{-1} x + c$$

Setting $x=0$ gives $0 = c$

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \underset{?}{\uparrow}\, x^7$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{\cancel{\cos}\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

$$\frac{1}{\cos x} = \frac{1}{1 + X} = 1 - X + X^2 - \cdots$$

$$|X| < 1$$

$$X = -\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$$

$$\frac{1}{\cos x} = 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$

$$+ \left(-\frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)^2 + \cdots$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \cdots$$

$$+ \frac{x^4}{4} + \cdots$$

$$= 1 + \frac{x^2}{2} + \left(\frac{1}{4} - \frac{1}{24}\right)x^4 + \cdots$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots$$

to compute PS for $\tan x$ multiply PS for $\sin x$ and $\frac{1}{\cos x}$

Another example

$f(x) = \frac{1}{1+e^x}$ first few terms in PS expansion?

$$= \frac{1}{1+\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}$$

$$= \frac{1}{2\left(1+\underbrace{\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\cdots}_{X}\right)}$$

$$= \frac{1}{2(1+X)}$$

$$\left[\; X = \frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\cdots \;\right]$$

$$= \frac{1}{2}\left[1 - X + X^2 + \cdots\right.$$

$$= \frac{1}{2}\left[1 - \left(\frac{x}{2}+\frac{x^2}{4}+\cdots\right) + \left(\frac{x}{2}+\frac{x^2}{4}+\cdots\right)^2 - \left(\quad\right)^3\right.$$

$$= \frac{1}{2}\left[1 - \frac{x}{2} - \frac{x^2}{4} + \frac{x^2}{4}\right]$$

$$= \frac{1}{2} - \frac{x}{4} + \cancel{\left(-\frac{1}{8} + \frac{1}{8}\right)x^2} + x^3$$

Limits

The symbol

$$L = \lim_{x \to a} f(x)$$

indicates that $f(x)$ "approaches" the number L as x "tends to" a.

[Given $\epsilon > 0$ $\exists\, \delta > 0$ such that $|x-a| < \delta$ and $x \in \text{dom}(f)$ implies $|f(x) - L| < \epsilon$
see 2nd term analysis]

x can approach a from the left or right

One sided limits

$\lim_{x \to a^+} f(x)$

↑ x appoaches from right

$\lim_{x \to a^-} f(x)$

↑ x approaches from left

$$f(x) = \frac{x}{|x|}$$

$\lim_{x \to 0} f(x)$ does not exist

However, one-sided limits are well defined

$$\lim_{x \to 0^+} \frac{x}{|x|} = 1 \qquad \lim_{x \to 0^-} \frac{x}{|x|} = -1$$

Simple Examples

(i) $f(x) = x^2$

$\lim_{x \to 2} f(x) = 4$

here

$\lim_{x \to 2} f(x) = f(2)$

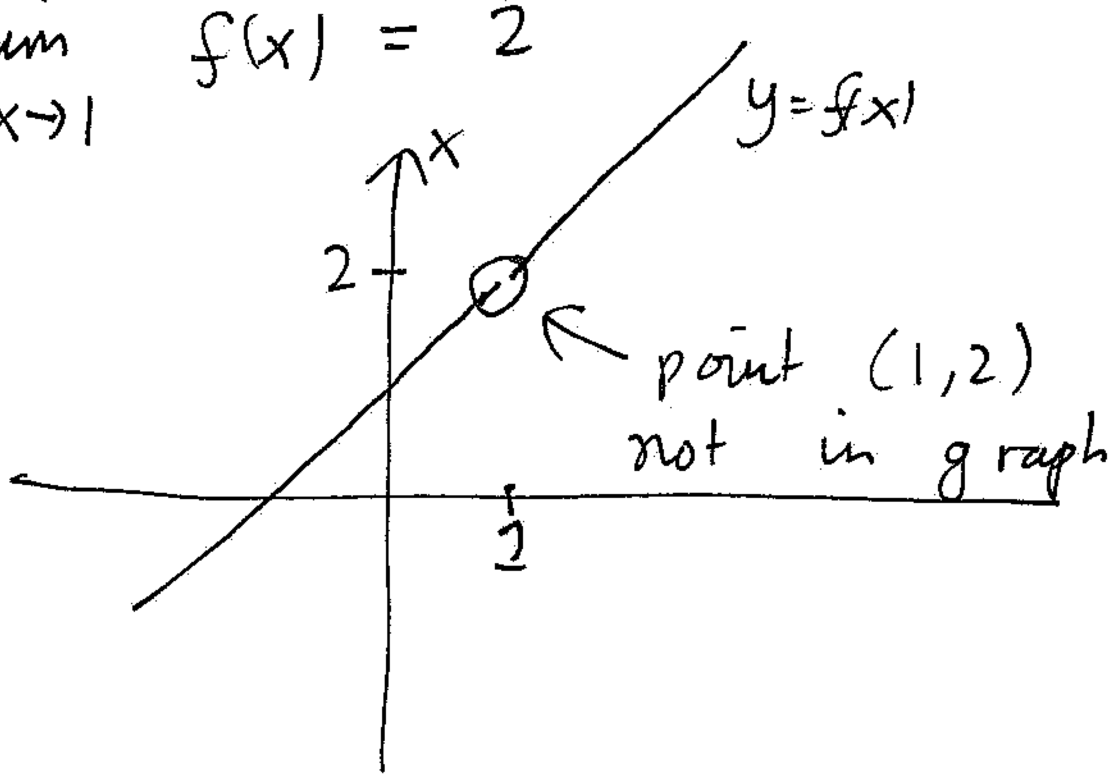
(ii) $f(x) = \dfrac{x^2-1}{x-1} \qquad x \neq 1$

$\lim\limits_{x \to 1} f(x)$ an indeterminate limit "of form $\frac{0}{0}$"

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1 \quad \text{if } x \neq 1$$

$\rightarrow 2$ as $x \to 1$

$\lim\limits_{x \to 1} f(x) = 2$



Not all indetermine limits well defined, e.g.

$$\lim_{x \to 1} \frac{x^2-1}{(x-1)^3}$$ does not exist

$$\frac{x^2-1}{(x-1)^3} = \frac{x+1}{(x-1)^2} \quad x \neq 1$$

$$\to \infty \quad \text{as} \quad x \to 1$$

Properties of Limits

(i) addition formula

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

provided $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist

(ii) product rule

$$\lim_{x\to a} f(x)\, g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

provided $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist

(iii) quotient rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

provided $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$

exist <u>and</u> $\lim\limits_{x\to a} g(x) \neq 0$

Infinite limits - consider behaviour of $f(x)$ as $x \to +\infty$

or as $x \to -\infty$

Examples

$$\lim_{x \to \infty} \frac{1}{1+x^2} = 0$$

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

The limit $x \to +\infty$ same as $\frac{1}{x} \to 0^+$

$x \to -\infty$ same as $\frac{1}{x} \to 0^-$

Computing Limits

Methods include

(i) Manipulate $f(x)$ so that limit is "obvious".

(ii) power series!

(iii) L Hôpital's rule

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$ of form $\frac{0}{0}$

Can manipulate function – multiply numerator and denominator by $\sqrt{1+x} + \sqrt{1-x}$

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)}$$

$$= \frac{2}{\sqrt{1+x} + \sqrt{1-x}} \qquad x \neq 0$$

$\to 1$ as $x \to 0$

A similar example

$$\lim_{x \to 0} \frac{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{3}}}{x} \quad \text{of form } \frac{0}{0}$$

Use power series

General Boinomial expansion

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \qquad |x| < 1$$

where p is a constant

If $p = -1$ recover geometric series

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

If p is a non-negative

integer expansion terminates

formula is the Binomial expansion of $(1+x)^p$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \cdots\cdots$$

$$= 1 + \frac{1}{3}x + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{-2}{3}x^2 + \cdots$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \cdots\cdots$$

$$(1-x)^{\frac{1}{3}} = 1 - \frac{1}{3}x - \frac{1}{9}x^2 + \cdots\cdots$$

$$\frac{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{3}}}{x} = \frac{(\cancel{1} + \frac{1}{3}x + \cdots) - (\cancel{1} - \frac{1}{3}x + \cdots)}{x}$$

$$= \frac{\frac{2}{3}x \cancel{+} - \cancel{\frac{2}{9}x^2} + \cdots}{x}$$

$$= \frac{2}{3} - \cancel{\frac{2}{9}x} + \cdots \rightarrow \frac{2}{3} \text{ as } x \to 0$$