Real Polynomials

A real polynomial

$P(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$

$c_0, c_1, \dots, c_n$ real

if $c_n \neq 0$ degree n

can also be written

in form

$P(x) = c_n (x - a_1)(x - a_2) \dots (x - a_n)$

where $a_1, \dots, a_n$ are

complex roots of

$P(z) = 0$

Here any complex roots appear in complex conjugate pairs. In factorisation have products of form $(x-a)(x-\bar{a})$ which is a real quadratic. Any real polynomial can be factored into a product of real linear and quadratic factors

<u>Example</u> $P(x) = x^6 - 7x^3 - 8$

Can write

$$P(x) = (x+1)(x-2)(x-e^{i\pi/3})(x-e^{-i\pi/3})$$

$$\times \; (x-2e^{2i\pi/3})(x-2e^{-2i\pi/3})$$

—

$$(x-e^{+i\pi/3})(x-e^{-i\pi/3})$$

$$= x^2 - (e^{i\pi/3}+e^{-i\pi/3})x + 1$$

$$= x^2 - 2\cos\frac{\pi}{3}\, x + 1 = x^2 - x + 1$$

—

$$(x-2e^{2i\pi/3})(x-2e^{-2\pi i/3})$$

$$= x^2 - 2(e^{2\pi i/3}+e^{-2\pi i/3})x + 4$$

$$= x^2 - 4\underbrace{\cos(2\pi/3)}_{-\frac{1}{2}}\, x + 4$$

$$= x^2 + 2x + 4$$

Complex Functions

A complex function f is rule assigning a complex number $f(z)$ to every z in the domain of f.

Now $\mathrm{dom}(f)$ is $\mathbb{C}$ or a subset of $\mathbb{C}$.

Examples

(i) Complex polynomials

(ii) Complex Power series

$$f(z) = \sum_{m=0}^{\infty} c_m z^m \qquad |z| < R$$

$R =$ radius of convergence of power series

(iii) Complex Exponential

$$f(z) = \exp(z) = e^z$$

(iv) Complex trigonometric Functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

as power series

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$R = \infty$ both cases

Can define related functions in the usual way

$$\tan z = \frac{\sin z}{\cos z} \quad \text{etc.}$$

(v) Hyperbolic Functions

$$\cosh z = \cancel{e^{iz}} \; \frac{e^{z} + e^{-z}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

As power series

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Note

$$\cosh(iz) = \cos z$$

$$\sinh(iz) = i \sin z$$

$\cos z$ and $\sin z$ have zeros on real line

$\cosh z$ and $\sinh z$ have zeros on imaginary axis

(vi) Complex Logarithm

For reals logarithm defined through

$$\exp(\log x) = x \quad \text{for } x > 0$$

Define complex logarithm through

$$\exp(\log z) = z \qquad z \neq 0$$

Can 'solve' this using polar form $z = re^{i\theta}$

$$z = re^{i\theta} = e^{\log r + i\theta}$$

Therefore $\log z = \log r + i\theta$

However $\operatorname{Im}(\log z)$ ambiguous as replacing θ with $\theta + 2\pi$ has no effect

Recall that $e^{2\pi i} = 1$

Normally have $\log 1 = 0$ but it is also $2\pi i, 4\pi i, \ldots$

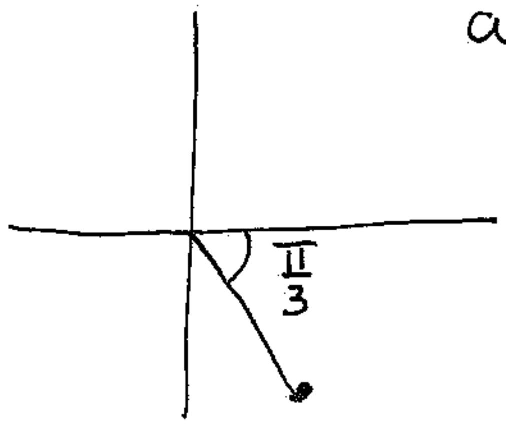
Can write

$$\boxed{\log z = \log |z| + i \arg(z)}$$

Example

$$\log(1 - i\sqrt{3}) = \log(2) - i\frac{\pi}{3}$$

$$|1 - i\sqrt{3}| = 2$$



$$\arg(1 - i\sqrt{3}) = -\frac{\pi}{3}$$

or $\frac{5\pi}{3}$, $\frac{11\pi}{3}$

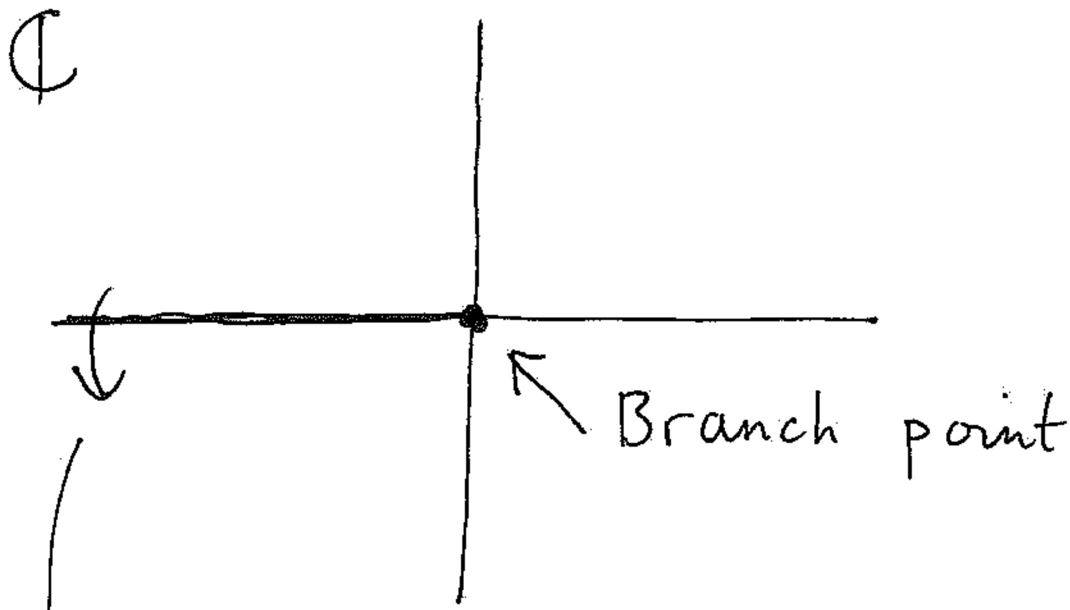
Can fix ambiguity by restricting values of θ.

The principal value of logarithm defined through

$$\operatorname{Log}(z) = \log|z| + i\operatorname{Arg}(z)$$

Here $-\pi < \mathrm{Arg}(z) \leq \pi$

$\mathrm{Log}(z)$ not ~~d~~ defined at $z=0$ and is discontinuous on negative real axis

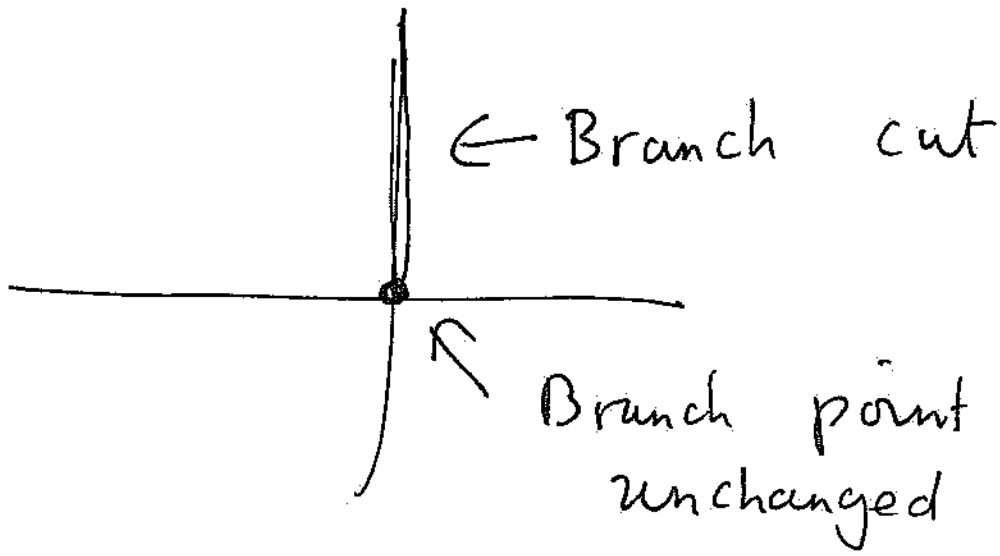


crossing negative real axis $\mathrm{Log}(z)$ jumps by $-2\pi i$ (Branch cut)

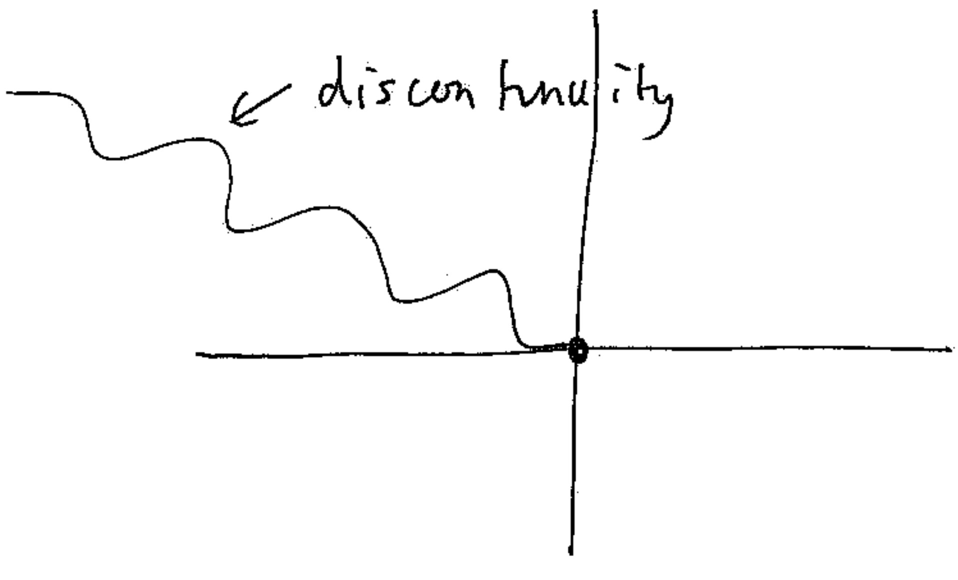
Discontinuity is moveable

eg instead restrict $\arg(z)$

to $+\frac{\pi}{2} < \arg(z) \leq \frac{5\pi}{2}$



Other prescriptions possible!



(vii) Powers

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= e^{n(\log r + i\theta)}$$

$$= e^{n \log z}$$

But $\log z = \log |z| + i \arg(z)$

ambiguous. Ambiguity drops out if $n \in \mathbb{Z}$

$$e^{2\pi i n} = 1 \qquad (n \in \mathbb{Z})$$

However $z^p = e^{p \log z}$

is ambiguous if p not integer

Taking

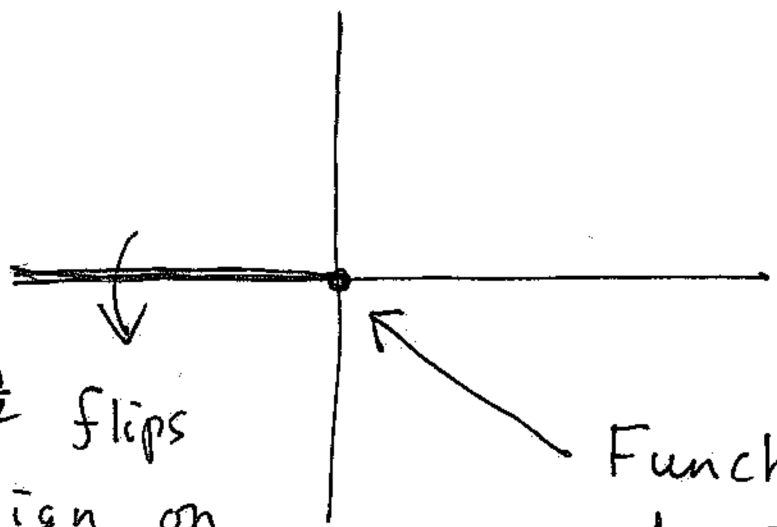
$$z^p = e^{p \operatorname{Log} z}$$

has a branch cut on negative real axis

Example $p = \frac{1}{2}$

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\theta/2}$$

now restrict $-\pi < \theta \leq \pi$



$z^{\frac{1}{2}}$ flips sign on crossing branch cut

Function defined at $z=0$ (Branch point)

7 Integration

3 approaches to integration

(i) geometrical approach

(ii) Analytical approach

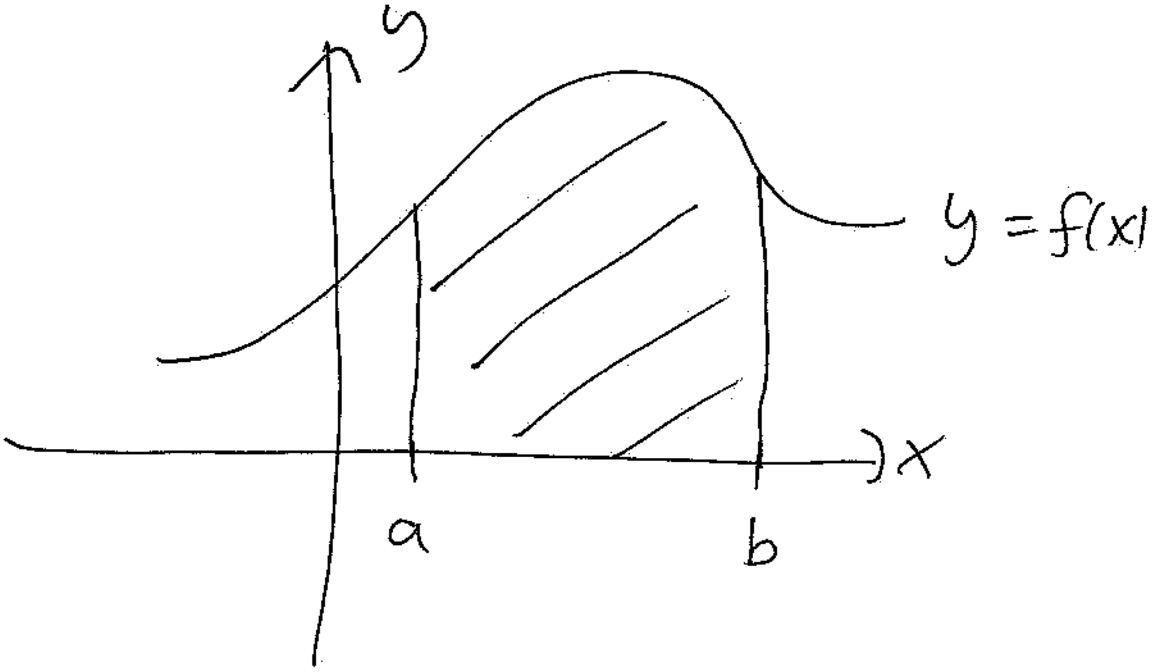
(iii) Fundamental Theorem of Calculus

(i) Geometrical Approach

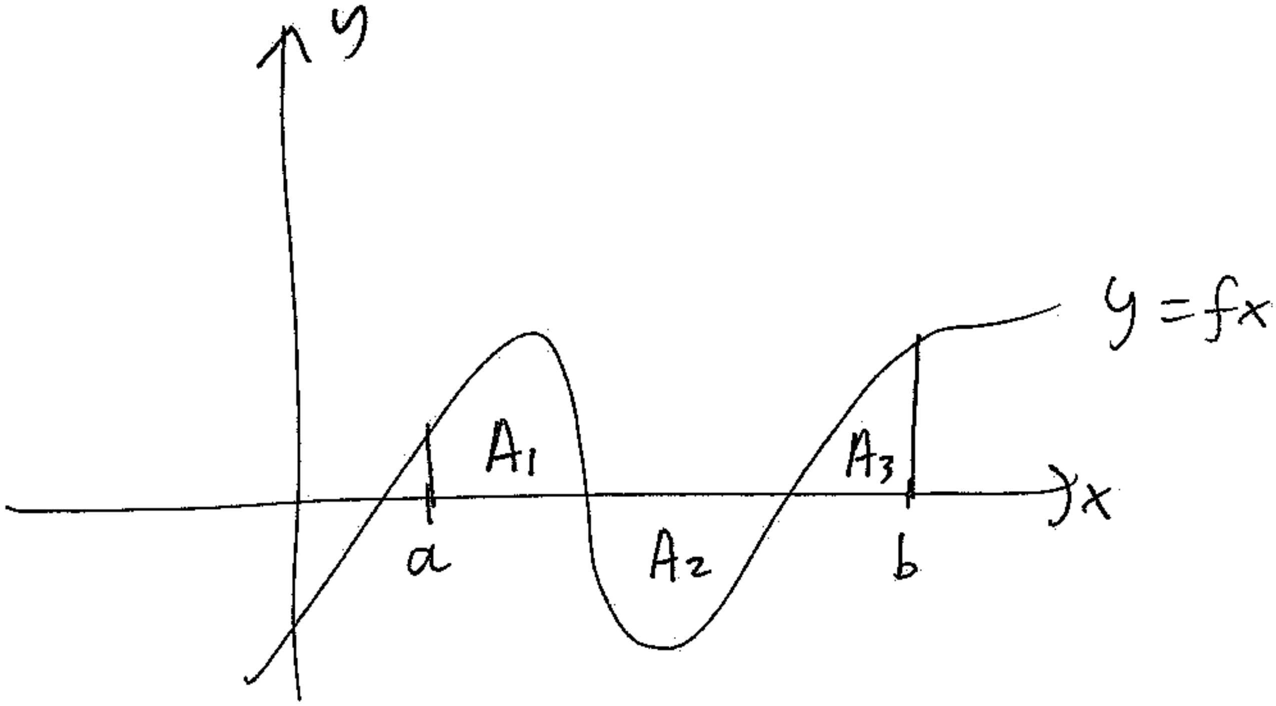
The symbol $\int_a^b f(x)\,dx$

(for now assume $b > a$)

denotes the area under graph $y = f(x)$ and above x-axis between $x = a$ and $x = b$



If graph dips below x-axis area counts negatively



Here $\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

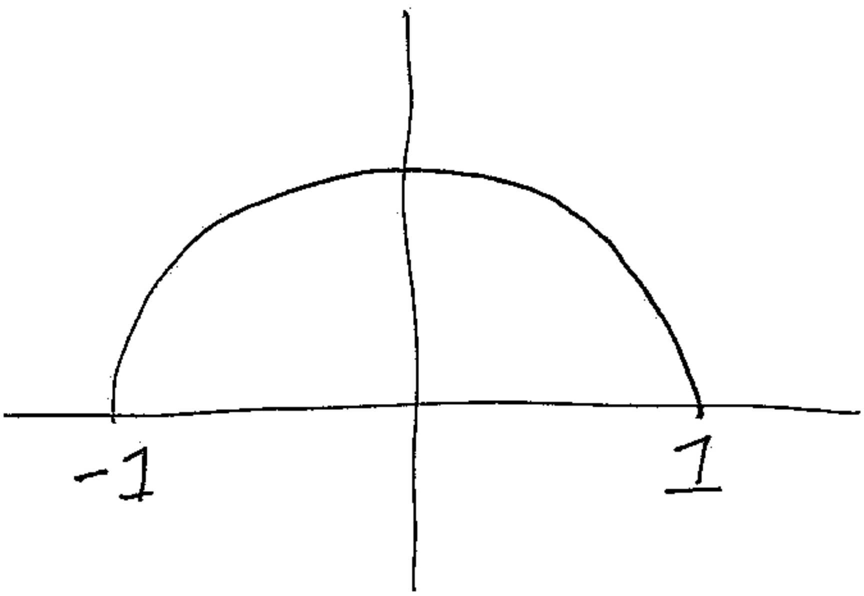
(A_1, A_2, A_3 all positive)

Sometimes can use simple geometrical reasoning to compute integrals

Examples

(a) $$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$$

$y = \sqrt{1-x^2}$ gives a semi-circle of unit radius



(b) $$\int_{-\pi}^{\pi} \sin x\, dx = 0$$

by symmetry

area above x-axis cancels area below

(ii) Analytical Approach

(a) Riemann Integral

(b) Lebesgue Integral

Riemann Integral

Aim is to give an analytical definition of

$$\int_a^b f(x)\,dx$$

Let P be a partition of $[a, b]$

That is consider numbers $x_1, x_2, \ldots, x_{N-1}$

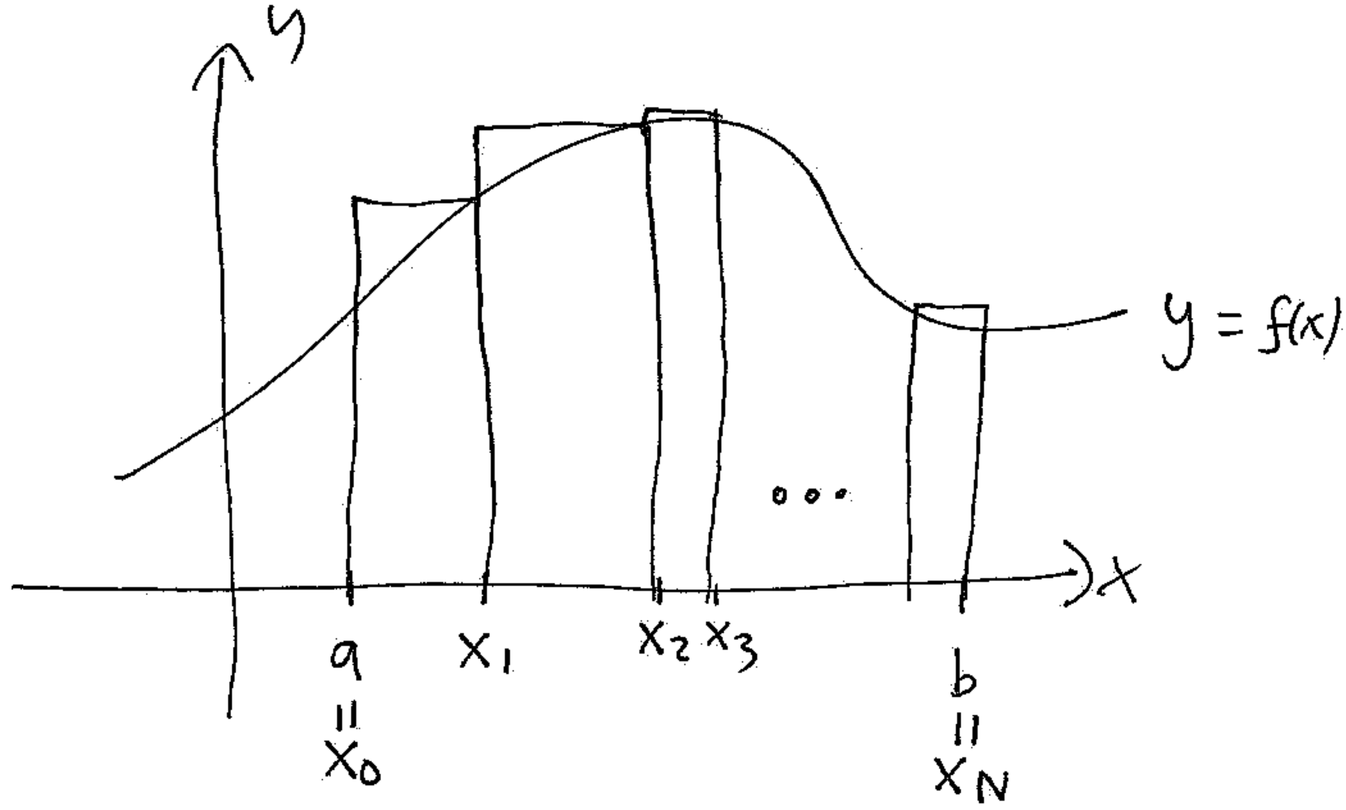
such that

$$a < x_1 < x_2 < \ldots < x_{N-1} < b$$

Now define the upper Riemann sum

$$\mathcal{U}(f, P) = \sum_{i=1}^{N} \underbrace{p_i}_{\text{height}} \underbrace{(x_i - x_{i-1})}_{\text{width}}$$

$$p_i = \underbrace{\sup}_{\text{maximum if } f \text{ continuous}} \{ f(x) : x_i \geq x \geq x_{i-1} \}$$



Lower Riemann sum

$$L(f, P) = \sum_{i=1}^{N} q_i \, (x_i - x_{i-1})$$

$$q_i = \inf \{ f(x) : x_i \geq x \geq x_{i-1} \}$$

(here rectangles are below curve)

Clearly $\quad U(f,P) \geq L(f,P)$

Define upper Riemann integral

$$\overline{\int_a^b} f(x)\,dx = \inf_P U(f,P)$$

infimum taken over all possible partitions

Define lower Riemann integral

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(f,P)$$

If both upper and lower Riemann integral exist and are equal f is said to be Riemann-integrable over $[a,b]$ and write the integral as

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

'Most' bounded functions

(meaning $|f(x)| <$ constant for $x \in [a, b]$) are

R- integrable

An exception is

$$f(x) = \begin{cases} 1, & x \notin \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$$

$\int_0^1 f(x)\,dx$ not defined as a Riemann integral

Easy to see that

$$\overline{\int_0^1} f(x)\,dx = 1 \qquad \underline{\int_0^1} f(x)\,dx = 0$$

Riemann integral provides a definition of symbol $\int_a^b f(x)\,dx$. However, it is not a practical method for computing integrals

Can be done in simple cases. For example

$$\int_0^1 x^2\,dx = \frac{1}{3} \quad \text{use FToC}$$

Divide $[0,1]$ into N subintervals of equal length

$$x_i = \frac{i}{N} \qquad i = 0, \ldots, N$$

$$U(f, P) = \sum_{i=1}^{N} \underbrace{\frac{1}{N}}_{\text{width}} \underbrace{\left(\frac{i}{N}\right)^2}_{p_i}$$

$$= \frac{1}{N^3} \sum_{i=1}^{N} i^2$$

$$= \frac{1}{N^3} \left(1^2 + 2^2 + \cdots + N^2\right)$$

$$= \frac{1}{N^3} \frac{1}{6} N(2N+1)(N+1)$$

$$\to \frac{1}{3} \quad \text{as} \quad N \to \infty$$