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The Use of Numerical Integration in Finite Element Methods for Solving Parabolic Equations

P. A. Raviart

I. Introduction and Preliminaries

Let Ω be a bounded *polyhedral* open subset of $\mathbb{R}^n$ with boundary Γ. We consider the parabolic model problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) = f \text{ in } & Q = \Omega \times \,]0,T[, \\ u = 0 \qquad\qquad\qquad\qquad \text{on} & \Sigma = \Gamma \times \,]0,T[, \\ u(x,0) = u_0(x) \qquad\qquad \text{in} & \Omega \end{cases} \tag{1.1}$$

where the functions a_{ij} and f are continuous over $\bar{Q}$. We assume that the 2nd order differential operator

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}\right) \tag{1.2}$$

satisfies the usual ellipticity property: for some constant $\alpha > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{n} \xi_i^2 \quad \text{for all} \quad (x,t) \in \bar{Q} \quad \text{and} \quad \xi \in \mathbb{R}^n. \tag{1.3}$$

We shall study in this paper a continuous time approximation of (1.1) by finite element methods using simplicial or quadrilateral elements. A numerical quadrature scheme is used to compute the coefficients of the resulting linear system of differential equations. The primary object of this paper is to investigate the effect of numerical integration on the error estimates.

We shall denote by (,) the scalar product in $L^2(\Omega)$. Given any integer $m \geqslant 0$, let

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega);\, \partial^\alpha v \in L^p(\Omega) \quad \text{for all} \quad |\alpha| \leqslant m\},\dagger$$

$$H^m(\Omega) = W^{m,2}(\Omega)$$

denote, for $1 \leqslant p \leqslant +\infty$, the usual Sobolev spaces provided with the norms

$$\| v \|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leqslant m} \| \partial^\alpha v \|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{for} \quad 1 \leqslant p \leqslant +\infty$$

and the usual modification for $p = +\infty$. Let

$$W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega);\, v|_\Gamma = 0\},$$

$$H_0^1(\Omega) = W_0^{1,2}(\Omega),$$

$$H^{-1}(\Omega) = \text{dual space of } H_0^1(\Omega) \text{ provided with the dual norm.}$$

We extend the scalar product (,) in $L^2(\Omega)$ to represent the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

If X is a Banach space with norm $\| \|_X$, let $L^p(0, T; X) = L^p(X)$, $1 \leqslant p \leqslant +\infty$, denote the space of functions $t \to v(t)$ which are L^p on $(0, T)$ with values in X provided with the norm

$$\| v \|_{L^p(X)} = \left(\int_0^T \| v(t) \|_X^p \, dt \right)^{1/p} \quad \text{for} \quad 1 \leqslant p < +\infty$$

and the usual modification for $p = +\infty$.

Let

$$a(t; u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x, t) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, dx, \quad u, v \in H^1(\Omega) \tag{1.4}$$

be the bilinear form associated with the operator $A(t)$. A weak form of (1.1) is as follows: Find a function u such that

$$\begin{cases} u \in L^2(H_0^1(\Omega)), \quad \dfrac{\partial u}{\partial t}\, L^2(H^{-1}(\Omega)), \\ \left(\dfrac{\partial u}{\partial t}(t), v \right) + a(t; u(t), v) = (f(t), v) \quad \text{for all} \quad v \in H_0^1(\Omega) \\ \qquad \text{and} \quad t \in \,]0, T[, \\ u(0) = u_0 \in L^2(\Omega)\ddagger \end{cases} \tag{1.5}$$

† If $\alpha = (\alpha_1, \ldots, \alpha_n) \in {}^n$, we let $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$ $\mathcal{N}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

‡ $u(t)$ denotes the function $x \to u(x, t)$.

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$$\begin{cases} \tilde{u}_h, \dfrac{\partial \tilde{u}_h}{\partial t} \in L^2(V_h), \\ \left(\dfrac{\partial \tilde{u}_h}{\partial t}(t), v_h \right) + a(t; \tilde{u}_h(t) \\ \qquad \text{and} \quad t \in \,]0, T[, \\ \tilde{u}_h(0) = u_{h,0} \in V_h. \end{cases}$$

Problem (1.7) has been exte
[5], Wheeler [16], Dupont [7]
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$$\begin{cases} u_h, \dfrac{\partial u_h}{\partial t} \in L^2(V_h), \\ \left(\dfrac{\partial u_h}{\partial t}(t), v_h \right)_h + a_h(t; u \\ \qquad \text{and} \quad t \in \,]0, T[, \\ u_h(0) = u_{h,0} \in V_h \end{cases}$$

(see for instance Lions and Magenes [9]). To approximate problem (1.5), we first construct a triangulation $\mathscr{T}_h$ of the set $\bar{\Omega}$ with finite elements K of simplicial type or of quadrilateral type (if $n = 2$) with diameters $\leqslant h$. With this triangulation, we associate a space V_h of trial functions such that

$$V_h \text{ is a finite dimensional subspace of } H_0^1(\Omega) \cap C^\circ(\bar{\Omega}). \tag{1.6}$$

The precise way in which V_h is defined will be explained in § 3.

Next, the usual manner of defining the semi-discrete problem associated with the space V_h consists in finding a function $\tilde{u}_h$ such that

$$\begin{cases} \tilde{u}_h, \dfrac{\partial \tilde{u}_h}{\partial t} \in L^2(V_h), \\ \left(\dfrac{\partial \tilde{u}_h}{\partial t}(t), v_h\right) + a(t; \tilde{u}_h(t), v_h) = (f(t), v_h) \quad \text{for all} \quad v_h \in V_h \\ \qquad \text{and} \quad t \in]0, T[, \\ \tilde{u}_h(0) = u_{h,0} \in V_h. \end{cases} \tag{1.7}$$

Problem (1.7) has been extensively studied: see Douglas and Dupont [5], Wheeler [16], Dupont [7] where optimal estimates of the error $\tilde{u}_h - u$ in the spaces $L^2(H^1(\Omega))$ and $L^\infty(L^2(\Omega))$ are derived.

Since it is either too costly or simply impossible to calculate exactly the integrals over Ω which appear in (1.7), we must take into account the fact that numerical integration is used for evaluating these integrals. Thus, for any finite element K belonging to the triangulation $\mathscr{T}_h$, we introduce a quadrature formula over K:

$$\int_K \varphi(x)\, dx \text{ is approximated by } \sum_{l=1}^{L} \omega_{l,K}\, \varphi(b_{l,K}) \tag{1.8}$$

for some specified points $b_{l,K} \in K$ and weights $\omega_{l,K} \in \mathbb{R}$, $1 \leqslant l \leqslant L$. By using these quadrature formulas (1.8), we replace the semi-discrete problem (1.7) by the following one: Find a function u_h such that

$$\begin{cases} u_h, \dfrac{\partial u_h}{\partial t} \in L^2(V_h), \\ \left(\dfrac{\partial u_h}{\partial t}(t), v_h\right)_h + a_h(t; u_h(t), v_h) = (f_h(t), v_h)_h \quad \text{for all} \quad v_h \in V_h \\ \qquad \text{and} \quad t \in]0, T[, \\ u_h(0) = u_{h,0} \in V_h \end{cases} \tag{1.9}$$

where, for each $u_h, v_h \in V_h$, we have

$$(u_h, v_h)_h = \sum_{K \in \mathscr{T}_h} \sum_{l=1}^{L} \omega_{l,K}(u_h, v_h)(b_{l,K}), \tag{1.10}$$

$$a_h(t; u_h, v_h) = \sum_{K \in \mathscr{T}_h} \sum_{l=1}^{L} \omega_{l,K} \left(\sum_{i,j=1} a_{ij}(., t) \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} \right)(b_{l,K}), \tag{1.11}$$

$$(f_h(t), v_h)_h = \sum_{K \in \mathscr{T}_h} \sum_{l=1}^{L} \omega_{l,K}(f(., t)v_h)(b_{l,K}). \tag{1.12}$$

The effect of numerical integration in finite element methods for solving elliptic equations has been recently analyzed: see Strang [14], Strang and Fix [15], Ciarlet and Raviart [4]. In the following, for studying problem (1.9), we shall make frequent use of results and techniques given in [4].

An outline of the paper is as follows: In § 2 we shall derive both $L^2(H^1(\Omega))$ and $L^\infty(L^2(\Omega))$ estimates for the error $u_h - u$. Under general hypotheses on the space V_h, the integration schemes (1.8), and on the smoothness of the exact solution u, we shall obtain

$$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{r+1}) \tag{1.13}$$

and

$$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{r+2}) \tag{1.14}$$

for some integer $r \geqslant 0$ (For another method for deriving the estimate (1.13) when the coefficients a_{ij} do not depend on t, we refer to Fix [8]). The beginning of § 3 will be devoted to a general description of finite element methods using simplicial or quadrilateral elements. Then, by applying the general results of § 2, we shall show how to choose the integration schemes in order to obtain error estimates which are optimal in the exponent of the parameter h. Finally, we shall study in § 4 some special cases of practical interest.

Several remarks are now in order. Just for the sake of simplicity, we have confined ourselves to polyhedral domains Ω, but all our results can be extended to the case of general curved domains by using curved isoparametric finite elements and the analysis given in Ciarlet and Raviart [3], [4].

Before practical calculations can be made, it is necessary to discretize problem (1.9) in time. This can be done in various ways and we only want to mention here that the proofs of convergence of this paper can be easily extended to a number of fully discretized schemes by using the author's stability results [10], [11], [12] and the techniques given in Douglas and Dupont [5] and Wheeler [16].

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$a_{ij}(., t) \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j}\Big)(b_{l,K})$, (1.11)

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2. General Error Estimates

In order to derive $L^2(H^1(\Omega))$ and $L^\infty(L^2(\Omega))$ estimates for the error, we use and generalize a now classical method of proof which has been given by Wheeler [16]. The first step consists in comparing the exact solution $u(t)$ of (1.5) with some appropriate "$H^1(\Omega)$-projection" $z_h(t)$ of $u(t)$ onto V_h. To do this, let us introduce some general hypotheses on the finite dimensional space V_h and on the integration schemes (1.8). Assume that:

$$\begin{cases} \text{for any function } v_h \in V_h \text{ and any (closed) finite element} \\ K \in \mathscr{T}_h,\, v_h \mid K \in C^{k+1}(K) \text{ for some integer } k \geqslant 1, \end{cases} \tag{2.1}$$

and that the space V_h satisfies the following *approximation property*:

$$\begin{cases} \text{for any integer } s \text{ with } 2 \leqslant s \leqslant k+1 \text{ and any real number } q \text{ with} \\ 2 \leqslant q \leqslant +\infty, \text{ there exists a linear operator} \\ r_h \in \mathscr{L}(W^{s,q}(\Omega) \cap W_0^{1,q}(\Omega);\, V_h) \text{ such that} \\ \Big(\sum\limits_{K \in \mathscr{T}_h} \| r_h v - v \|^q_{W^{m,q}(K)}\Big)^{1/q} \leqslant C h^{s-m} \| v \|_{W^{s,q}(\Omega)}, \quad 0 \leqslant m \leqslant s \\ \text{for all } v \in W^{s,q}(\Omega) \cap W_0^{1,q}(\Omega), \text{ where the constant } C \text{ is} \\ \text{independent of } h. \end{cases} \tag{2.2}$$

Next, we assume that the quadrature formulas (1.8) satisfy, the following *r-accuracy conditions* for some integer r with $0 \leqslant r \leqslant k-1$: for each u_h, $v_h \in V_h$ and any real number q with $2 \leqslant q \leqslant +\infty$ and $r + 1 - n/q > 0$ (so that $W^{r+1,q}(\Omega) \subset C^0(\Omega)$), we have

$$\begin{aligned} |(u_h, v_h) - (u_h, v_h)_h| \leqslant C h^{r+\mu} &\Big(\sum_{K \in \mathscr{T}_h} \| u_h \|^q_{W^{r+\mu,q}(K)}\Big)^{1/q} \times \\ &\times \Big(\sum_{K \in \mathscr{T}_h} \| v_h \|^2_{H^\mu(K)}\Big)^{1/2}, \quad \mu = 1, 2; \end{aligned} \tag{2.3}$$

$$\begin{aligned} |a(t; u_h, v_h) - a_h(t; u_h, v_h)| &\leqslant C h \max_{1 \leqslant i,j \leqslant n} \| a_{ij}(t) \|_{W^{1,\infty}(\Omega)} \times \\ &\times \| u_h \|_{H^1(\Omega)} \| v_h \|_{H^1(\Omega)} \quad \text{if } a_{ij}(t) \in W^{1,\infty}(\Omega), \quad 1 \leqslant i, j \leqslant n; \end{aligned} \tag{2.4}$$

$$|a(t; u_h, v_h) - a_h(t; u_h, v_h)| \leqslant$$

$$\begin{cases} \leqslant C h^{r+\mu} \max\limits_{1 \leqslant i,j \leqslant n} \| a_{ij}(t) \|_{W^{r+\mu,\infty}(\Omega)} \Big(\sum\limits_{K \in \mathscr{T}_h} \| u_h \|^q_{W^{r+2,q}(\Omega)}\Big)^{1/q} \times \\ \qquad \times \Big(\sum\limits_{K \in \mathscr{T}_h} \| v_h \|^2_{H^\mu(K)}\Big)^{1/2} \quad \text{if } a_{ij}(t) \in W^{r+\mu,\infty}(\Omega), \\ 1 \leqslant i, j \leqslant n, \quad \mu = 1, 2; \end{cases} \tag{2.5}$$

$$|(f(t), v_h) - (f_h(t), v_h)_h| \leqslant Ch^{r+\mu} \| f(t) \|_{W^{r+\mu,q}(\Omega)} \times$$

$$\times \left(\sum_{K \in \mathscr{T}_h} \| v_h \|^2_{H^\mu(K)} \right)^{2} \quad \text{if } f(t) \in W^{r+\mu,q}(\Omega), \quad \mu = 1, 2, \tag{2.6}$$

where the various constants C which appear in (2.3), . . ., (2.6) are independent of h. We begin with

Lemma 1

We assume that hypotheses (2.1), (2.2), (2.3), (2.5), (2.6) hold for some integers k, r, with $0 \leqslant r \leqslant k-1$ and $\mu = 1$ and that there exists a constant $\beta > 0$ independent of h such that

$$a_h(t; v_h, v_h) \geqslant \beta \| v_h \|^2_{H^1(\Omega)} \text{ for all } v_h \in V_h \text{ and all } t \in [0, T]. \tag{2.7}$$

Moreover, assume that, for some real number q with $2 \leqslant q \leqslant \infty$ and $r + 1 - n/q > 0$, we have

$$\begin{cases} u \in L^2(W^{r+2,q}(\Omega)), \quad \dfrac{\partial u}{\partial t} \in L^2(W^{r+1,q}(\Omega)), \\ f \in L^2(W^{r+1,q}(\Omega)), \\ a_{ij} \in L^\infty(W^{r+1,\infty}(\Omega)), \quad 1 \leqslant i, j \leqslant n. \end{cases} \tag{2.8}$$

Then, the unique solution $z_h \in L^2(V_h)$ of

$$a_h(t; z_h(t), v_h) = \left(f_h(t) - r_h \frac{\partial u}{\partial t}(t), v_h \right)_h, \quad t \in (0, T) \tag{2.9}$$

satisfies

$$\| z_h - u \|_{L^2(H^1(\Omega))} \leqslant Ch^{r+1} \left(\| u \|_{L^2(W^{r+2,q}(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(W^{r+1,q}(\Omega))} + \right.$$

$$\left. + \| f \|_{L^2(W^{r+1,q}(\Omega))} \right) \tag{2.10}$$

where the constant C is independent of h, u and f.

Proof

First, the assumption (2.7) ensures that the discrete problem (2.9) has a unique solution $z_h(t) \in V_h$ for almost every t. Moreover, it is clear that $z_h : t \to z_h(t)$ belongs to $L^2(V_h)$. Next, since u is the solution of equation (1.5), we may write for all $v_h \in V_h$

$$a_h(t; z_h(t), v_h) = a(t; u(t), v_h) + \left(\frac{\partial u}{\partial t}(t), v_h \right) - \left(r_h \frac{\partial u}{\partial t}(t), v_h \right)_h -$$

$$- (f(t), v_h) + (f_h(t), v_h)_h \quad \text{a.e.}$$

and

$$a_h(t; z_h(t) - r_h u(t), v_h) = a($$

$$+ \left(\frac{\partial u}{\partial t}(t) - r_h \frac{\partial u}{\partial t}(t), \right.$$

$$+ \left(r_h \frac{\partial u}{\partial t}(t), v_h \right) - \Big(r$$

We let $v_h = z_h(t) - r_h u(t)$. Then, hypothesis (2.7), we obtain (C d h)

$$\beta \| z_h - r_h u \|_{L^2(H^1(\Omega))} \leqslant C \|$$

$$+ \Big(\sup_{w_h \in L^2(V_h)} \| w_h \|^{-1}_{L^2(H^1}$$

$$- a_h(t; r_h u(t), w_h(t))]$$

$$- \left(r_h \frac{\partial u}{\partial t}(t), w_h(t) \right)_h \Big]$$

Thus, applying hypotheses (2.2) (2.6) with $\mu = 1$ and (2.8), we ob

$$\| z_h - u \|_{L^2(H^1(\Omega))} \leqslant \| z_h - r$$

$$\leqslant Ch^{r+1} \Big\{ \| u \|_{L^2(H^{r+2}(\Omega)}$$

$$+ \max_{1 \leqslant i, j \leqslant n} \| a_{ij} \|_{L^\infty(W^{r+1,}}$$

$$+ \left\| \frac{\partial u}{\partial t} \right\|_{L^2(W^{r+1,q}(\Omega))} + \|$$

since

$$\Big(\int_0^T \Big(\sum_{K \in \mathscr{C}_h} \| r_h u(t) \|^q_{W^{r+2,q}(\Omega)}$$

$f(t)\|_{W^{r+\mu,q}(\Omega)}\times$

$\in W^{r+\mu,q}(\Omega), \quad \mu = 1, 2,$ (2.6)

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$(t), v_h\Big) - \Big(r_h \frac{\partial u}{\partial t}(t), v_h\Big)_h -$

and

$$a_h(t; z_h(t) - r_h u(t), v_h) = a(t; u(t) - r_h u(t), v_h) + \left(\frac{\partial u}{\partial t}(t) - r_h \frac{\partial u}{\partial t}(t), v_h\right) + a(t; r_h u(t), v_h) - a_h(t; r_h u(t), v_h) + \left(r_h \frac{\partial u}{\partial t}(t), v_h\right) - \left(r_h \frac{\partial u}{\partial t}(t), v_h\right)_h - (f(t), v_h) + (f_h(t), v_h)_h.$$

We let $v_h = z_h(t) - r_h u(t)$. Then, integrating from $t = 0$ to $t = T$ and using hypothesis (2.7), we obtain (C denoting various constants independent of h)

$$\beta \| z_h - r_h u \|_{L^2(H^1(\Omega))} \leqslant C \| u - r_h u \|_{L^2(H^1(\Omega))} + \left\| \frac{\partial u}{\partial t} - r_h \frac{\partial u}{\partial t} \right\|_{L^2(L^2(\Omega))} + \left(\sup_{w_h \in L^2(V_h)} \| w_h \|^{-1}_{L^2(H^1(\Omega))} \right) \left\{ \left| \int_0^T [a(t; r_h u(t), w_h(t)) - a_h(t; r_h u(t), w_h(t))] \, dt \right| + \left| \int_0^T \left[\left(r_h \frac{\partial u}{\partial t}(t), w_h(t) \right) - \left(r_h \frac{\partial u}{\partial t}(t), w_h(t) \right)_h \right] dt \right| + \left| \int_0^T [(f(t), w_h(t)) - (f_h(t), w_h(t))_h] \, dt \right| \right\}.$$

Thus, applying hypotheses (2.2) with $s = r + 1, r + 2$ and (2.3), (2.5), (2.6) with $\mu = 1$ and (2.8), we obtain

$$\| z_h - u \|_{L^2(H^1(\Omega))} \leqslant \| z_h - r_h u \|_{L^2(H^1(\Omega))} + \| r_h u - u \|_{L^2(H^1(\Omega))} \leqslant$$

$$\leqslant C h^{r+1} \left\{ \| u \|_{L^2(H^{r+2}(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(H^{r+1}(\Omega))} + \max_{1 \leqslant i,j \leqslant n} \| a_{ij} \|_{L^\infty(W^{r+1,\infty}(\Omega))} \| u \|_{L^2(W^{r+2,q}(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(W^{r+1,q}(\Omega))} + \| f \|_{L^2(W^{r+1,q}(\Omega))} \right\}$$

since

$$\left(\int_0^T \left(\sum_{K \in \mathscr{C}_h} \| r_h u(t) \|^q_{W^{r+2,q}(\Omega)} \right)^{2/q} dt \right)^{1/2} \leqslant C \| u \|_{L^2(W^{r+2,q}(\Omega))}$$

and

$$\left(\int_0^T \left(\sum_{K\in\mathscr{C}_h} \left\| r_h \frac{\partial u}{\partial t}(t)\right\|^q_{W^{r+1,q}(\Omega)}\right)^{2/q} dt\right)^{1/2} \leqslant C \left\|\frac{\partial u}{\partial t}\right\|_{L^2(W^{r+1,q}(\Omega))}$$

The inequality (2.10) is then proved.

We want now to estimate $\| z_h - u \|_{L^2(L^2(\Omega))}$. To do this, we need the following regularity property for the operator

$$A^*(t) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ji}(x,t)\frac{\partial}{\partial x_i}\right):$$

$$\| v \|_{H^2(\Omega)} \leqslant C \| A^*(t)v \|_{L^2(\Omega)} \quad \text{for all } v \in H^2(\Omega) \cap H^1_0(\Omega)$$

$$\text{and all } t \in [0, T]. \tag{2.11}$$

Property (2.11) is satisfied if $a_{ij} \in L^\infty(W^{1,\infty}(\Omega))$, $1 \leqslant i, j \leqslant n$, and if the polyhedral domain Ω is convex.

Lemma 2

We assume that hypotheses (2.1) . . . (2.7), (2.11) hold for some integers k, r with $0 \leqslant r \leqslant k-1$ and $\mu = 1, 2$. Moreover, assume that, for some real number q with $2 \leqslant q \leqslant +\infty$ and $r + 1 - n/q > 0$, we have

$$\begin{cases} u, \dfrac{\partial u}{\partial t} \in L^2(W^{r+2,q}(\Omega)), \\ f \in L^2(W^{r+2,q}(\Omega)), \\ a_{ij} \in L^\infty(W^{r+2,\infty}(\Omega)), \quad 1 \leqslant i, j \leqslant n. \end{cases} \tag{2.12}$$

Then, the solution $z_h \in L^2(V_h)$ of equation (2.9) satisfies

$$\| z_h - u \|_{L^2(L^2(\Omega))} \leqslant Ch^{r+2}\left(\| u \|_{L^2(W^{r+2,q}(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))} + \right.$$
$$\left. + \| f \|_{L^2(W^{r+2,q}(\Omega))}\right) \tag{2.13}$$

where the constant C is independent of h, u and f.

Proof

We follow the method of proof given in [4, §5] which is a generalization of the classical Aubin-Nitsche's duality argument. We have

$$\| z_h - u \|_{L^2(L^2(\Omega))} = \sup_{g\in L^2(L^2(\Omega))} \frac{\left|\int_0^T (z_h(t) - u(t), g(t))\, dt\right|}{\| g \|_{L^2(L^2(\Omega))}}. \tag{2.14}$$

Given $g \in L^2(L^2(\Omega))$, for almc

of the Dirichlet problem

$$\begin{cases} A^*(t)\varphi(t) = g(t) \text{ in } \Omega, \\ \varphi(t) = 0 \text{ on } \Gamma. \end{cases}$$

Using hypothesis (2.11), we k

and

$$\| \varphi \|_{L^2(H^2(\Omega))} \leqslant C \| g \|_{L^2(}$$

Then, we may write

$$(z_h(t) - u(t), g(t)) = a(t;$$

On the other hand, by using e

$\varphi_h \in L^2(V_h)$.

$$a(t; z_h(t) - u(t), \varphi_h(t)) =$$
$$-\left(\frac{\partial u}{\partial t}(t), \varphi_h(t)\right) +$$
$$-(f_h(t), \varphi_h(t))_h$$

Thus, we have

$$\begin{cases} (z_h(t) - u(t), g(t)) = a(\\ -r_h \dfrac{\partial u}{\partial t}(t), \varphi_h(t)\Big) + \\ -\left(r_h \dfrac{\partial u}{\partial t}(t), \varphi_h(t)\right) \\ -(f_h(t), \varphi_h(t))_h \end{cases}$$

We let $\varphi_h(t) = r_h\varphi(t)$. Then, us

$$\left|\int_0^T a(t; z_h(t) - u(t), \varphi(t) -\right.$$
$$\leqslant Ch \| z_h - u \|_{L^2(H^1(}$$
$$\left|\int_0^T \left(\frac{\partial u}{\partial t}(t) - r_h \frac{\partial u}{\partial t}(t), \varphi_h\right.\right.$$
$$\leqslant Ch^{r+2}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^{r+}}$$

Given $g \in L^2(L^2(\Omega))$, for almost every $t \in (0, T)$ we let $\varphi(t)$ be the solution of the Dirichlet problem

$$\begin{cases} A^*(t)\varphi(t) = g(t) \text{ in } \Omega, \\ \varphi(t) = 0 \text{ on } \Gamma. \end{cases}$$

Using hypothesis (2.11), we know that $\varphi: t \to \varphi(t) \in L^2(H^2(\Omega) \cap H_0^1(\Omega))$ and

$$\| \varphi \|_{L^2(H^2(\Omega))} \leqslant C \| g \|_{L^2(L^2(\Omega))}. \tag{2.15}$$

Then, we may write

$$(z_h(t) - u(t), g(t)) = a(t; z_h(t) - u(t), \varphi(t)) \quad \text{a.e.}$$

On the other hand, by using equation (2.9), we get for any function $\varphi_h \in L^2(V_h)$.

$$\begin{aligned} a(t; z_h(t) - u(t), \varphi_h(t)) &= a(t; z_h(t), \varphi_h(t)) - a_h(t; z_h(t), \varphi_h(t)) - \\ &- \left(\frac{\partial u}{\partial t}(t), \varphi_h(t)\right) + \left(r_h \frac{\partial u}{\partial t}(t), \varphi_h(t)\right)_h + (f(t), \varphi_h(t)) - \\ &- (f_h(t), \varphi_h(t))_h \quad \text{a.e.} \end{aligned}$$

Thus, we have

$$\left\{ \begin{aligned} &(z_h(t) - u(t), g(t)) = a(t; z_h(t), \varphi(t) - \varphi_h(t)) + \left(\frac{\partial u}{\partial t}(t) - \right. \\ &\quad \left. - r_h \frac{\partial u}{\partial t}(t), \varphi_h(t)\right) + a(t; z_h(t), \varphi_h(t)) - a_h(t; z_h(t), \varphi_h(t)) - \\ &\quad - \left(r_h \frac{\partial u}{\partial t}(t), \varphi_h(t)\right) + \left(r_h \frac{\partial u}{\partial t}(t), \varphi_h(t)\right)_h + (f(t), \varphi_h(t)) - \\ &\quad - (f_h(t), \varphi_h(t))_h \quad \text{a.e.} \end{aligned} \right. \tag{2.16}$$

We let $\varphi_h(t) = r_h \varphi(t)$. Then, using hypotheses (2.2) and (2.12), we obtain

$$\begin{aligned} &\left| \int_0^T a(t; z_h(t) - u(t), \varphi(t) - \varphi_h(t)) \, dt \right| \leqslant \\ &\quad \leqslant Ch \| z_h - u \|_{L^2(H^1(\Omega))} \| \varphi \|_{L^2(H^2(\Omega))} \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\left| \int_0^T \left(\frac{\partial u}{\partial t}(t) - r_h \frac{\partial u}{\partial t}(t), \varphi_h(t)\right) dt \right| \leqslant \\ &\quad \leqslant Ch^{r+2} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(H^{r+2}(\Omega))} \| \varphi \|_{L^2(H^2(\Omega))}. \end{aligned} \tag{2.18}$$

From hypotheses (2.2), (2.4) and (2.12), we get

$$\left|\int_0^T [a(t; z_h(t) - r_h u(t), \varphi_h(t)) - a_h(t; z_h(t) - r_h u(t), \varphi_h(t))]\,dt\right| \leqslant Ch \max_{1\leqslant i,j\leqslant n} \|a_{ij}\|_{L^\infty(W^{1,\infty}(\Omega))} \times \|z_h - r_h u\|_{L^2(H^1(\Omega))} \|\varphi\|_{L^2(H^2(\Omega))}.$$

Likewise, by hypotheses (2.2), (2.5) with $\mu = 2$ and (2.12)

$$\left|\int_0^T [a(t; r_h u(t), \varphi_h(t)) - a_h(t; r_h u(t), \varphi_h(t))]\,dt\right| \leqslant Ch^{r+2} \max_{1\leqslant i,j\leqslant n} \|a_{ij}\|_{L^\infty(W^{r+2,\infty}(\Omega))} \|u\|_{L^2(W^{r+2,q}(\Omega))} \times \|\varphi\|_{L^2(H^2(\Omega))}.$$

Thus, we obtain

$$\begin{cases} \left|\int_0^T [a(t; z_h(t), \varphi_h(t)) - a_h(t; z_h(t), \varphi_h(t))]\,dt\right| \leqslant \\ \leqslant Ch^{r+2} \max_{1<i,j<n} \|a_{ij}\|_{L^\infty(W^{r+2,\infty}(\Omega))} (\|u\|_{L^2(W^{r+2,q}(\Omega))} + \\ + h^{-(r+1)} \|z_h - u\|_{L^2(H^1(\Omega))}) \|\varphi\|_{L^2(H^2(\Omega))}. \end{cases} \tag{2.19}$$

Finally, from hypotheses (2.3), (2.6) with $\mu = 2$ and (2.12), we get

$$\begin{cases} \left|\int_0^T \left[\left(r_h \frac{\partial u}{\partial t}(t), \varphi_h(t)\right) - \left(r_h \frac{\partial u}{\partial t}(t), \varphi_h(t)\right)_h\right] dt\right| \leqslant \\ \leqslant Ch^{r+2} \left\|\frac{\partial u}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))} \|\varphi\|_{L^2(H^2(\Omega))}, \end{cases} \tag{2.20}$$

$$\begin{cases} \left|\int_0^T [(f(t), \varphi_h(t)) - (f_h(t), \varphi_h(t))_h]\,dt\right| \leqslant \\ \leqslant Ch^{r+2} \|f\|_{L^2(W^{r+2,q}(\Omega))} \|\varphi\|_{L^2(H^2(\Omega))}. \end{cases} \tag{2.21}$$

By combining (2.15), . . ., (2.2

$$\left|\int_0^T (z_h(t) - u(t), g(t))\,dt\right| \ldots + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))} \ldots + h^{-(r+1)} \|z_h - u\|_{L^2} \ldots$$

where the constant C is indepe
(2.13) follows from (2.14), (2.
We want now to derive $L^2(H$

$$\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}.$$

Lemma 3
We assume that hypotheses (2.
with $0 \leqslant r \leqslant k - 1$ and $\mu = 1$.
q with $2 \leqslant q \leqslant +\infty$ and $r + 1 -$

$$\begin{cases} u, \frac{\partial u}{\partial t} \in L^2(W^{r+2,q}(\Omega)), \\ f, \frac{\partial f}{\partial t} \in L^2(W^{r+1,q}(\Omega)), \\ a_{ij}, \frac{\partial a_{ij}}{\partial t} \in L^2(W^{r+1,\infty}(\Omega \ldots \end{cases}$$

Then the solution $z_h \in L^2(V_h$

$$\frac{\partial z_h}{\partial t} \in L^2(V_h)$$

and

$$\begin{cases} \left\|\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}\right\|_{L^2(H^1(\Omega))} \leqslant \ldots \\ + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))} + \ldots \\ + \|f\|_{L^2(W^{r+1,q}(\Omega))} + \ldots \end{cases}$$

where the constant C is indep

, we get

$; z_h(t) -$

$\underset{i,j \leqslant n}{\text{ıax}} \; \| a_{ij} \|_{L^\infty(W^{1,\infty}(\Omega))} \times$

$^2(\Omega))$.

h $\mu = 2$ and (2.12)

$, \varphi_h(t))] \, dt \Big| \leqslant$

$(\Omega)) \; \| u \|_{L^2(W^{r+2,q}(\Omega))} \times$

$, \varphi_h(t))] \, dt \Big| \leqslant$

$(\Omega)) \; (\| u \|_{L^2(W^{r+2,q}(\Omega))} +$

$\|_{L^2(H^2(\Omega))}$. (2.19)

:h $\mu = 2$ and (2.12), we get

$), \varphi_h(t) \Big)_h \Big] \, dt \Big| \leqslant$

$\|_{L^2(H^2(\Omega))}$, (2.20)

$] \, dt \Big| \leqslant$

$L^2(H^2(\Omega))$. (2.21)

By combining (2.15), . . ., (2.21), we obtain

$$\left| \int_0^T (z_h(t) - u(t), g(t)) \, dt \right| \leqslant C h^{r+2} \{ \| u \|_{L^2(W^{r+2,q}(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(W^{r+2,q}(\Omega))} + \| f \|_{L^2(W^{r+2,q}(\Omega))} + h^{-(r+1)} \| z_h - u \|_{L^2(H^1(\Omega))} \} \| g \|_{L^2(L^2(\Omega))}, \tag{2.22}$$

where the constant C is independent of h, u and f. The desired inequality (2.13) follows from (2.14), (2.22) and Lemma 1.

We want now to derive $L^2(H^1(\Omega))$ and $L^2(L^2(\Omega))$ estimates for

$$\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}.$$

Lemma 3

We assume that hypotheses (2.1), . . ., (2.7) hold for some integers k, r with $0 \leqslant r \leqslant k - 1$ and $\mu = 1$. Moreover, assume that, for some real number q with $2 \leqslant q \leqslant +\infty$ and $r + 1 - n/q > 0$, we have

$$\begin{cases} u, \dfrac{\partial u}{\partial t} \in L^2(W^{r+2,q}(\Omega)), \quad \dfrac{\partial^2 u}{\partial t^2} \in L^2(W^{r+1,q}(\Omega)), \\ f, \dfrac{\partial f}{\partial t} \in L^2(W^{r+1,q}(\Omega)), \\ a_{ij}, \dfrac{\partial a_{ij}}{\partial t} \in L^2(W^{r+1,\infty}(\Omega)), \quad 1 \leqslant i, j \leqslant n. \end{cases} \tag{2.23}$$

Then the solution $z_h \in L^2(V_h)$ of equation (2.9) satisfies

$$\frac{\partial z_h}{\partial t} \in L^2(V_h) \tag{2.24}$$

and

$$\begin{cases} \left\| \dfrac{\partial z_h}{\partial t} - \dfrac{\partial u}{\partial t} \right\|_{L^2(H^1(\Omega))} \leqslant C h^{r+1} \Big(\| u \|_{L^2(W^{r+2,q}(\Omega))} + \\ \quad + \left\| \dfrac{\partial u}{\partial t} \right\|_{L^2(W^{r+2,q}(\Omega))} + \left\| \dfrac{\partial^2 u}{\partial t^2} \right\|_{L^2(W^{r+1,q}(\Omega))} + \\ \quad + \| f \|_{L^2(W^{r+1,q}(\Omega))} + \left\| \dfrac{\partial f}{\partial t} \right\|_{L^2(W^{r+1,q}(\Omega))} \Big) \end{cases} \tag{2.25}$$

where the constant C is independent of h, u and f.

Proof

The proof is similar to that of Lemma 1 and, for this reason, it will be only sketched. We let:

$$a'(t;u,v) = \sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial a_{ij}}{\partial t}(x,t)\,\frac{\partial u}{\partial x_i}(x)\,\frac{\partial v}{\partial x_j}(x)\,dx, \quad u, v \in H^1(\Omega), \tag{2.26}$$

$$a'_h(t;u_h,v_h) = \sum_{K\in\mathcal{T}_h} \sum_{l=1}^{L} \omega_{l,K}\left(\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial t}(.,t)\,\frac{\partial u_h}{\partial x_i}\,\frac{\partial v_h}{\partial x_j}\right)(b_{l,K}),$$

$$u_h, v_h \in V_h, \tag{2.27}$$

$$\left(\frac{\partial f_h}{\partial t}(t), v_h\right)_h = \sum_{K\in\mathcal{T}_h} \sum_{l=1}^{L} \omega_{l,K}\left(\frac{\partial f}{\partial t}(.,t)v_h\right)(b_{l,K}), v_h \in V_h. \tag{2.28}$$

Clearly $(\partial z_h/\partial t) \in L^2(V_h)$. Differentiating equation (2.9) with respect to t, we obtain for all $v_h \in V_h$

$$a_h\left(t;\frac{\partial z_h}{\partial t}(t), v_h\right) = \left(\frac{\partial f_h}{\partial t}(t) - r_h\frac{\partial^2 u}{\partial t^2}(t), v_h\right)_h -$$

$$- a'_h(t;z_h(t),v_h) \quad \text{a.e.} \tag{2.29}$$

Thus we may write

$$a_h\left(t;\frac{\partial z_h}{\partial t}(t) - r_h\frac{\partial u}{\partial t}(t), v_h\right) = a\left(t;\frac{\partial u}{\partial t}(t) - r_h\frac{\partial u}{\partial t}(t), v_h\right) +$$

$$+ a'(t;u(t) - z_h(t), v_h) + \left(\frac{\partial^2 u}{\partial t^2}(t) - r_h\frac{\partial^2 u}{\partial t^2}(t), v_h\right) +$$

$$+ a\left(t;r_h\frac{\partial u}{\partial t}(t), v_h\right) - a_h\left(t;r_h\frac{\partial u}{\partial t}(t), v_h\right) + a'(t;z_h(t) - r_h u(t), v_h) -$$

$$- a'_h(t;z_h(t) - r_h u(t), v_h) + a'(t;r_h u(t), v_h) -$$

$$- a'_h(t;r_h u(t), v_h) + \left(r_h\frac{\partial^2 u}{\partial t^2}(t), v_h\right) - \left(r_h\frac{\partial^2 u}{\partial t^2}(t), v_h\right)_h -$$

$$- \left(\frac{\partial f}{\partial t}(t), v_h\right) + \left(\frac{\partial f_h}{\partial t}, v_h\right)_h \quad \text{a.e.}$$

By applying hypotheses (2.1),

$$\left\|\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}\right\|_{L^2(H^1(\Omega))} \leqslant Ch$$

$$+ \left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^{r+2}(\Omega))} +$$

$$+ h^{-r}\|z_h - u\|_{L^2(H^1($$

$$+ \left\|\frac{\partial u}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))}$$

$$+ \left\|\frac{\partial f}{\partial t}\right\|_{L^2(W^{r+1,q}(\Omega))}$$

where the constant C is indep

is an easy consequence of (2.

Lemma 4

We assume that hypotheses (2

k, r with $0 \leqslant r \leqslant k-1$ and μ

number q with $2 \leqslant q \leqslant +\infty$ a

$$u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2} \in L^2(W^{r+2,q}$$

$$f, \frac{\partial f}{\partial t} \in L^2(W^{r+2,q}(\Omega)),$$

$$a_{ij}, \frac{\partial a_{ij}}{\partial t} \in L^\infty(W^{r+2,\infty}(\Omega$$

Then, the solution z_h of equa

$$\left\|\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}\right\|_{L^2(L^2(\Omega))} \leqslant Ch$$

$$+ \sum_{l=0}^{1}\left\|\frac{\partial^l f}{\partial t^l}\right\|_{L^2(W^{r+2,q}$$

where the constant C is indep

By applying hypotheses (2.1), . . ., (2.7) with $\mu = 1$ and (2.23), we get

$$
\left\{
\begin{aligned}
&\left\| \frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(H^1(\Omega))} \leqslant C h^{r+1} \Big\{ h^{-(r+1)} \| z_h - u \|_{L^2(H^1(\Omega))} + \\
&\quad + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(H^{r+2}(\Omega))} + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(H^{r+1}(\Omega))} + \\
&\quad + h^{-r} \| z_h - u \|_{L^2(H^1(\Omega))} + \| u \|_{L^2(W^{r+2,q}(\Omega))} + \\
&\quad + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(W^{r+2,q}(\Omega))} + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(W^{r+1,q}(\Omega))} + \\
&\quad + \left\| \frac{\partial f}{\partial t} \right\|_{L^2(W^{r+1,q}(\Omega))} \Big\},
\end{aligned}
\right. \tag{2.30}
$$

where the constant C is independent of h, u and f. Then, the inequality (2.25 is an easy consequence of (2.30) and Lemma 1.

Lemma 4

We assume that hypotheses (2.1), . . ., (2.7), (2.11) hold for some integers k, r with $0 \leqslant r \leqslant k - 1$ and $\mu = 1, 2$. Moreover, assume that, for some real number q with $2 \leqslant q \leqslant +\infty$ and $r + 1 - n/q > 0$, we have

$$
\left\{
\begin{aligned}
&u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2} \in L^2(W^{r+2,q}(\Omega)), \\
&f, \frac{\partial f}{\partial t} \in L^2(W^{r+2,q}(\Omega)), \\
&a_{ij}, \frac{\partial a_{ij}}{\partial t} \in L^\infty(W^{r+2,\infty}(\Omega)), \quad 1 \leqslant i, j \leqslant n.
\end{aligned}
\right. \tag{2.31}
$$

Then, the solution z_h of equation (2.9) satisfies

$$
\begin{aligned}
\left\| \frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(L^2(\Omega))} &\leqslant C h^{r+2} \left(\sum_{l=0}^{2} \left\| \frac{\partial^l u}{\partial t^l} \right\|_{L^2(W^{r+2,q}(\Omega))} + \right. \\
&\quad \left. + \sum_{l=0}^{1} \left\| \frac{\partial^l f}{\partial t^l} \right\|_{L^2(W^{r+2,q}(\Omega))} \right)
\end{aligned} \tag{2.32}
$$

where the constant C is independent of h, u and f.

Proof

Here again the proof is similar to that of Lemma 2 and will be only sketched. Using equation (2.29), we may write for any function $\varphi \in L^2(H^2(\Omega) \cap H_0^1(\Omega))$ and any function $\varphi_h \in L^2(V_h)$

$$
\left\{
\begin{aligned}
& a\left(t; \frac{\partial z_h}{\partial t}(t) - \frac{\partial u}{\partial t}(t), \varphi(t)\right) = a\left(t; \frac{\partial z_h}{\partial t}(t) - \frac{\partial u}{\partial t}(t), \varphi(t) - \varphi_h(t)\right) - \\
& \quad - a'(t; z_h(t) - u(t), \varphi_h(t)) + \left(\frac{\partial^2 u}{\partial t^2}(t) - r_h \frac{\partial^2 u}{\partial t^2}(t), \varphi_h(t)\right) + \\
& \quad + a\left(t; \frac{\partial z_h}{\partial t}(t), \varphi_h(t)\right) - a_h\left(t; \frac{\partial z_h}{\partial t}(t), \varphi_h(t)\right) + \\
& \quad + a'(t; z_h(t), \varphi_h(t)) - a_h'(t; z_h(t), \varphi_h(t)) + \\
& \quad + \left(r_h \frac{\partial^2 u}{\partial t^2}(t), \varphi_h(t)\right) - \left(r_h \frac{\partial^2 u}{\partial t^2}(t), \varphi_h(t)\right)_h - \\
& \quad - \left(\frac{\partial f}{\partial t}(t), \varphi_h(t)\right) + \left(\frac{\partial f_h}{\partial t}(t), \varphi_h(t)\right)_h \quad \text{a.e.}
\end{aligned}
\right.
\tag{2.33}
$$

We let $\varphi_h(t) = r_h \varphi(t)$. Then, using the approximation property (2.2), we obtain:

$$
\begin{aligned}
& \left| \int_0^T a\left(t; \frac{\partial z_h}{\partial t}(t) - \frac{\partial u}{\partial t}(t), \varphi(t) - \varphi_h(t)\right) dt \right| \leqslant \\
& \quad \leqslant Ch \left\| \frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(H^1(\Omega))} \|\varphi\|_{L^2(H^2(\Omega))};
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
& \left| \int_0^T \left(\frac{\partial^2 u}{\partial t^2}(t) - r_h \frac{\partial^2 u}{\partial t^2}(t), \varphi_h(t)\right) dt \right| \leqslant \\
& \quad \leqslant Ch^{r+2} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(H^{r+2}(\Omega))} \|\varphi\|_{L^2(H^2(\Omega))};
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
& \left| \int_0^T a'(t; z_h(t) - u(t), \varphi_h(t))\, dt \right| \leqslant \\
& \quad \leqslant \left| \int_0^T (z_h(t) - u(t), B(t)\varphi(t))\, dt \right| + \\
& \quad + \left| \int_0^T a'(t; z_h(t) - u(t), \varphi_h(t) - \varphi(t))\, dt \right|
\end{aligned}
$$

emma 2 and will be only
vrite for any function
$\varphi_h \in L^2(V_h)$

$(t) - \frac{\partial u}{\partial t}(t), \varphi(t) - \varphi_h(t)\Big) -$

$\frac{\partial^2 u}{\partial t^2}(t) - r_h \frac{\partial^2 u}{\partial t^2}(t), \varphi_h(t)\Big) +$

$\frac{\partial z_h}{\partial t}(t), \varphi_h(t)\Big) +$

$), \varphi_h(t)) +$

$(t), \varphi_h(t)\Big)_h -$

$(t)\Big)_h$ a.e. (2.33)

oximation property (2.2), we

$dt\Big| \leqslant$

$(H^2(\Omega))$; (2.34)

$^2(H^2(\Omega))$; (2.35)

$t\Big| +$

$\varphi(t))\, dt\Big|$

where

$$B(t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial a_{ji}}{\partial t}(x, t) \frac{\partial}{\partial x_i}\right)$$

and therefore

$$\left|\int_0^T a'(t; z_h(t) - u(t), \varphi_h(t))\, dt\right| \leqslant C(\| z_h - u \|_{L^2(L^2(\Omega))} +$$

$$+ h \| z_h - u \|_{L^2(H^1(\Omega))}) \| \varphi \|_{L^2(H^2(\Omega))}. \tag{2.36}$$

Now, by using the properties of the quadrature formulas, we get:

$$\begin{cases} \left|\int_0^T \left[\left(r_h \frac{\partial^2 u}{\partial t^2}(t), \varphi_h(t)\right) - \left(r_h \frac{\partial^2 u}{\partial t^2}(t), \varphi_h(t)\right)_h\right] dt\right| \leqslant \\ \leqslant C h^{r+2} \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(W^{r+2,q}(\Omega))} \| \varphi \|_{L^2(H^2(\Omega))}; \end{cases} \tag{2.37}$$

$$\begin{cases} \left|\int_0^T \left[\left(\frac{\partial f}{\partial t}(t), \varphi_h(t)\right) - \left(\frac{\partial f_h}{\partial t}(t), \varphi_h(t)\right)_h\right] dt\right| \leqslant \\ \leqslant C h^{r+2} \left\|\frac{\partial f}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))} \| \varphi \|_{L^2(H^2(\Omega))}; \end{cases} \tag{2.38}$$

$$\left|\int_0^T \left[a\left(t; \frac{\partial z_h}{\partial t}(t), \varphi_h(t)\right) - a_h\left(t; \frac{\partial z_h}{\partial t}(t), \varphi_h(t)\right)\right] dt\right| \leqslant$$

$$\leqslant \left|\int_0^T \left[a\left(t; \frac{\partial r_h u}{\partial t}(t), \varphi_h(t)\right) - a_h\left(t; \frac{\partial r_h u}{\partial t}(t), \varphi_h(t)\right)\right] dt\right| +$$

$$+ \left|\int_0^T \left[a\left(t; \frac{\partial z_h}{\partial t}(t) - \frac{\partial r_h u}{\partial t}(t), \varphi_h(t)\right) - \right.\right.$$

$$\left.\left. - a_h\left(t; \frac{\partial r_h u}{\partial t}(t) - \frac{\partial r_h u}{\partial t}(t), \varphi_h(t)\right)\right] dt\right|$$

and therefore

$$\left\{\begin{aligned} &\left|\int_0^T \left[a\left(t;\frac{\partial z_h}{\partial t}(t),\varphi_h(t)\right) - a_h\left(t;\frac{\partial z_h}{\partial t}(t),\varphi_h(t)\right)\right]dt\right| \leqslant \\ &\leqslant Ch^{r+2}\left(\left\|\frac{\partial u}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))} + \right.\\ &\left. + h^{-(r+1)}\left\|\frac{\partial z_h}{\partial t}-\frac{\partial u}{\partial t}\right\|_{L^2(H^1(\Omega))}\right)\|\varphi\|_{L^2(H^2(\Omega))}.\end{aligned}\right. \tag{2.39}$$

Similarly

$$\left\{\begin{aligned} &\left|\int_0^T [a'(t;z_h(t),\varphi_h(t)) - a_h'(t;z_h(t),\varphi_h(t))]\,dt\right| \leqslant \\ &\leqslant Ch^{r+2}(\|u\|_{L^2(W^{r+2,q}(\Omega))} + h^{-(r+1)}\|z_h-u\|_{L^2(H^1(\Omega))}) \times \\ &\times \|\varphi\|_{L^2(H^2(\Omega))}.\end{aligned}\right. \tag{2.40}$$

Combining (2.33), . . ., (2.40), we obtain

$$\left\{\begin{aligned} &\left|\int_0^T a\left(t;\frac{\partial z_h}{\partial t}(t)-\frac{\partial u}{\partial t}(t),\varphi(t)\right)dt\right| \leqslant \\ &\leqslant Ch^{r+2}\left\{\sum_{l=0}^{2}\left\|\frac{\partial^l u}{\partial t^l}\right\|_{L^2(W^{r+2,q}(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2(W^{r+2,q}(\Omega))} + \right.\\ &+ h^{-(r+1)}\left(\sum_{l=0}^{1}\left\|\frac{\partial^l z_h}{\partial t^l}-\frac{\partial^l u}{\partial t^l}\right\|_{L^2(H^1(\Omega))}\right) + \\ &\left. + h^{-(r+2)}\|z_h-u\|_{L^2(L^2(\Omega))}\right\}.\end{aligned}\right. \tag{2.41}$$

The conclusion follows from (2.41) and Lemmas 1, 2, 3.

We now come to the desired $L^2(H^1(\Omega))$ and $L^\infty(L^2(\Omega))$ estimates for the error $u_h - u$.

Theorem 1

We assume that $v_h \to |v_h|_h = (v_h, v_h)_h^{1/2}$ is a norm over V_h and that there exists a constant $D > 0$ independent of h such that

$$|v_h|_h \leqslant D\|v_h\|_{L^2(\Omega)} \quad \text{for all} \quad v_h \in V_h. \tag{2.42}$$

If, in addition, we assume th

u_h of Problem (1.9) satisfies.

$$\|u_h - u\|_{L^2(H^1(\Omega))} \leqslant C \quad + h^{r+1}\left(\sum_{l=0}^{1}\left\|\frac{\partial^l u}{\partial t^l}\right\| \quad + \left(\sum_{l=0}^{1}\left\|\frac{\partial^l f}{\partial t^l}\right\|_{L^2(W^r}\right.\right.$$

where the constant C is indep

Proof

First, since $v_h \to |v_h|_h$ is a no

the semi-discrete problem (1.

$w_h = u_h - z_h$ where z_h is defi

we may write

$$\left(\frac{\partial w_h}{\partial t}(t), w_h(t)\right)_h + a_h(t \quad = \left(r_h\frac{\partial u}{\partial t}(t) - \frac{\partial z_h}{\partial t}\right.$$

and for $0 \leqslant s \leqslant T$

$$|w_h(s)|_h^2 + 2\int_0^s a_h(t; w_h \quad + 2\int_0^s\left(r_h\frac{\partial u}{\partial t}(t) - \right.$$

By using hypotheses (2.7) an

$$|w_h(s)|_h^2 + 2\beta\int_0^s \|w_h(t) \quad + 2D^2\int_0^s\left\|r_h\frac{\partial u}{\partial t}(t\right.$$

$(t), \varphi_h(t)\Big)\Big]\, dt \Big| \leqslant$

$\varphi \|_{L^2(H^2(\Omega))}.$ (2.39)

$_h(t))]\, dt \Big| \leqslant$

$^{r+1)} \| z_h - u \|_{L^2(H^1(\Omega))}) \times$

(2.40)

$+ \left\| \frac{\partial f}{\partial t} \right\|_{L^2(W^{r+2,q}(\Omega))} +$

$(\Omega))\Big) +$

(2.41)

mmas 1, 2, 3.

and $L^\infty(L^2(\Omega))$ estimates for

norm over V_h and that there

uch that

$_h.$ (2.42)

If, in addition, we assume the hypotheses of Lemma 3, the unique solution u_h of Problem (1.9) satisfies.

$$\left\{\begin{aligned} & \| u_h - u \|_{L^2(H^1(\Omega))} \leqslant C \Bigg\{ \| u_{h,0} - u_0 \|_{L^2(\Omega)} + \\ & \quad + h^{r+1} \left(\sum_{l=0}^{1} \left\| \frac{\partial^l u}{\partial t^l} \right\|_{L^2(W^{r+2,q}(\Omega))} + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(W^{r+1,q}(\Omega))} \right) + \\ & \quad + \left(\sum_{l=0}^{1} \left\| \frac{\partial^l f}{\partial t^l} \right\|_{L^2(W^{r+1,q}(\Omega))} \right) \Bigg\} \end{aligned}\right. \tag{2.43}$$

where the constant C is independent of h, u, f and $u_{h,0}$.

Proof

First, since $v_h \to |v_h|_h$ is a norm over V_h, the assumption (2.7) ensures that the semi-discrete problem (1.9) has a unique solution u_h. We let $w_h = u_h - z_h$ where z_h is defined by (2.9). Using equations (1.9) and (2.9), we may write

$$\left(\frac{\partial w_h}{\partial t}(t), w_h(t) \right)_h + a_h(t; w_h(t), w_h(t)) =$$

$$= \left(r_h \frac{\partial u}{\partial t}(t) - \frac{\partial z_h}{\partial t}(t), w_h(t) \right)_h \quad \text{a.e.}$$

and for $0 \leqslant s \leqslant T$

$$| w_h(s) |_h^2 + 2 \int_0^s a_h(t; w_h(t), w_h(t))\, dt = | w_h(0) |_h^2 +$$

$$+ 2 \int_0^s \left(r_h \frac{\partial u}{\partial t}(t) - \frac{\partial z_h}{\partial t}(t), w_h(t) \right)_h dt.$$

By using hypotheses (2.7) and (2.42), we get

$$| w_h(s) |_h^2 + 2\beta \int_0^s \| w_h(t) \|_{H^1(\Omega)}^2\, dt \leqslant D^2 \| w_h(0) \|_{L^2(\Omega)} +$$

$$+ 2D^2 \int_0^s \left\| r_h \frac{\partial u}{\partial t}(t) - \frac{\partial z_h}{\partial t}(t) \right\|_{L^2(\Omega)} \| w_h(t) \|_{L^2(\Omega)}\, dt$$

and

$$\left\{\begin{aligned} & |w_h(s)|_h^2 + \beta \int_0^s \|w_h(t)\|_{H^1(\Omega)}^2\,dt \leqslant D^2 \|w_h(0)\|_{L^2(\Omega)}^2 + \\ & + \frac{D^4}{\beta}\int_0^s \left\| r_h \frac{\partial u}{\partial t}(t) - \frac{\partial z_h}{\partial t}(t)\right\|_{L^2(\Omega)}^2 dt. \end{aligned}\right. \tag{2.44}$$

Then, we obtain

$$\|w_h\|_{L^2(H^1(\Omega))} \leqslant C\left(\|w_h(0)\|_{L^2(\Omega)} + \left\| r_h \frac{\partial u}{\partial t} - \frac{\partial z_h}{\partial t}\right\|_{L^2(L^2(\Omega))}\right) \tag{2.45}$$

for some constant $C = C(\beta, D)$. Since

$$\|z_h - u\|_{L^\infty(L^2(\Omega))} \leqslant C\left(\|z_h - u\|_{L^2(L^2(\Omega))} + \left\|\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}\right\|_{L^2(L^2(\Omega))}\right), \tag{2.46}$$

we get from (2.45)

$$\left\{\begin{aligned} & \|w_h\|_{L^2(H^1(\Omega))} \leqslant C\Bigg\{\|u_{h,0} - u_0\|_{L^2(\Omega)} + \|z_h - u\|_{L^2(L^2(\Omega))} + \\ & + \left\|\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}\right\|_{L^2(L^2(\Omega))} + \left\| r_h \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t}\right\|_{L^2(L^2(\Omega))}\Bigg\}. \end{aligned}\right. \tag{2.47}$$

Assuming the hypotheses of Lemma 3, writing

$$\|u_h - u\|_{L^2(H^1(\Omega))} \leqslant \|w_h\|_{L^2(H^1(\Omega))} + \|z_h - u\|_{L^2(H^1(\Omega))}$$

and using (2.2), (2.10), (2.25) and (2.47), we obtain the inequality (2.43).

Theorem 2

We assume that there exist two constants $\delta, D > 0$ independent of h such that

$$\delta \|v_h\|_{L^2(\Omega)} \leqslant |v_h|_h \leqslant D\|v_h\|_{L^2(\Omega)} \quad \text{for all} \quad v_h \in V_h. \tag{2.48}$$

If, in addition, we assume the hypotheses of Lemma 4, then u_h satisfies

$$\left\{\begin{aligned} & \|u_h - u\|_{L^\infty(L^2(\Omega))} \leqslant C\Bigg\{\|u_{h,0} - u_0\|_{L^2(\Omega)} + \\ & + h^{r+2}\left(\sum_{l=0}^{2}\left\|\frac{\partial^l u}{\partial t^l}\right\|_{L^2(W^{r+2,q}(\Omega))} + \sum_{l=0}\left\|\frac{\partial^l f}{\partial t^l}\right\|_{L^2(W^{r+2,q}(\Omega))}\right)\Bigg\} \end{aligned}\right. \tag{2.49}$$

where the constant C is independent of h, u, f and $u_{h,0}$.

Proof

Using (2.44) and (2.48), we o

$$\|w_h\|_{L^\infty(L^2(\Omega))} \leqslant C\Big(\|w_h$$

for some constant $C = (\beta, \delta, L$

$$\|u_h - u\|_{L^\infty(L^2(\Omega))} \leqslant C$$

$$+ \left\|\frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t}\right\|_{L^2(L}$$

Assume the hypotheses of Le
consequence of (2.2), (2.13),

Remark 1

To obtain the estimates

$$\|u_h - u\|_{L^2(H^1(\Omega))} = 0(h^r$$

it is sufficient to choose $u_{h,0}$

$$\|r_h u_0 - u_0\|_{L^2(\Omega)} \leqslant Ch^r$$

$$\leqslant Ch^{r+2}\Big(\sum_{l=0}^{1}\Big\|\frac{\partial^l u}{\partial t^l}$$

3. Application to Finite Ele
Quadrilateral Elements

For the sake of brevity, we s
corresponding to Lagrange in
equally well to finite elemen
Moreover, finite elements of
by the Engineers for solving

First, we are given:

(i) a set $\hat{\Sigma} = \{\hat{a}_i\}_{i=1}^N$ of N d
is denoted by $\hat{K}$;

(ii) a finite dimensional sp
dim $\hat{P} = N$ and such that the
polation problem: "Find $\hat{p} \in$
unique solution for any give

Proof
Using (2.44) and (2.48), we obtain

$$\| w_h \|_{L^\infty(L^2(\Omega))} \leqslant C \left(\| w_h(0) \|_{L^2(\Omega)} + \left\| r_h \frac{\partial u}{\partial t} - \frac{\partial z_h}{\partial t} \right\|_{L^2(L^2(\Omega))} \right) \tag{2.50}$$

for some constant $C = (\beta, \delta, D)$. We get from (2.46) and (2.50)

$$\begin{cases} \| u_h - u \|_{L^\infty(L^2(\Omega))} \leqslant C \Big\{ \| u_{h,0} - u_0 \|_{L^2(\Omega)} + \| z_h - u \|_{L^2(L^2(\Omega))} + \\ \qquad + \left\| \frac{\partial z_h}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(L^2(\Omega))} + \left\| r_h \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(L^2(\Omega))} \Big\}. \end{cases} \tag{2.51}$$

Assume the hypotheses of Lemma 4. Then, inequality (2.49) is an easy consequence of (2.2), (2.13), (2.32) and (2.51).

Remark 1
To obtain the estimates

$$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(h^{r+1}), \quad \| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(h^{r+2}),$$

it is sufficient to choose $u_{h,0} = r_h u_0$. In fact, we get from (2.2)

$$\| r_h u_0 - u_0 \|_{L^2(\Omega)} \leqslant C h^{r+2} \| u_0 \|_{H^{r+2}(\Omega)} \leqslant$$
$$\leqslant C h^{r+2} \left(\sum_{l=0}^{1} \left\| \frac{\partial^l u}{\partial t^l} \right\|_{L^2(W^{r+2,q}(\Omega))} \right).$$

3. Application to Finite Element Methods Using Simplicial or Quadrilateral Elements

For the sake of brevity, we shall confine ourselves to finite elements corresponding to Lagrange interpolation but the present analysis applies equally well to finite elements corresponding to Hermite interpolation. Moreover, finite elements of Lagrange type are the most commonly used *by the Engineers* for solving Problem (1.1).

First, we are given:

(i) a set $\hat{\Sigma} = \{\hat{a}_i\}_{i=1}^N$ of N distinct points of $\mathbb{R}^n$ whose closed convex hull is denoted by $\hat{K}$;

(ii) a finite dimensional space $\hat{P}$ of C^1-functions defined over $\hat{K}$ with $\dim \hat{P} = N$ and such that the set $\hat{\Sigma}$ is $\hat{P}$-unisolvent, i.e. the Lagrange interpolation problem: "Find $\hat{p} \in \hat{P}$ such that $\hat{p}(\hat{a}_i) = \alpha_i$, $1 \leqslant i \leqslant N$," has a unique solution for any given set $\{\alpha_i\}_{i=1}^n$ of real numbers.

In practice, the space $\hat{P}$ will contain $\hat{P}(k)$ or $\hat{Q}(k)$ for some integer $k \geqslant 1$ where $\hat{P}(k)$ is the space of the restrictions over the set K of all polynomials of degree $\leqslant k$ in the n variables $x_1, \ldots, x_n$ and $\hat{Q}(k)$ is the space of the restriction over $\hat{K}$ of all polynomials of the form

$$\hat{q}(\hat{x}) = \sum_{i_1,\ldots,i_n \leqslant k} c_{i_1,\ldots,i_n} \hat{x}_1^{i_1} \ldots \hat{x}_n^{i_n}.$$

In the following, by *the reference finite element* $\hat{K}$, we shall mean the set $\hat{K}$ provided with $(\hat{\Sigma}, \hat{P})$. We shall assume that $\hat{K}$ is a C^0-*finite element*, i.e. given any $(n-1)$-dimensional closed face $\hat{K}'$ of $\hat{K}$, the set $\hat{\Sigma}' = \hat{\Sigma} \cap \hat{K}'$ contains a $\hat{P}'$-unisolvent set where $\hat{P}'$ denotes the space of the restrictions over $\hat{K}'$ of the functions of $\hat{P}$.

Next, we are given a quadrature formula over the set $\hat{K}$

$$\int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x} \simeq \sum_{l=1}^{L} \hat{\omega}_l \hat{\varphi}(\hat{b}_l) \tag{3.1}$$

for some specified points $\hat{b}_l \in \hat{K}$ and weights $\hat{\omega}_l$ which will be assumed *once and for all to satisfy*

$$\hat{\omega}_l > 0, \quad 1 \leqslant l \leqslant L. \tag{3.2}$$

Now, let K be a closed set of $\mathbb{R}^n$ such that there exists a C^1-diffeomorphism F_K from $\hat{K}$ onto K. We let:

$$\Sigma_K = \{a_{i,K}\}_{i=1}^{N}, \quad a_{i,K} = F_K(\hat{a}_i), \tag{3.3}$$

$$P_K = \{p; p = \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{P}\}. \tag{3.4}$$

Then, clearly, the set Σ_K is P_K-unisolvent. In the following, *by the finite element* K, we shall mean the set K provided with (Σ_K, P_K).

The quadrature formula (3.1) over $\hat{K}$ induces the quadrature formula over K

$$\int_K \varphi(x)\, dx \simeq \sum_{l=1}^{L} \omega_{l,K} \varphi(b_{l,K}) \tag{3.5}$$

with

$$b_{l,K} = F_K(\hat{b}_l), \quad \omega_{l,K} = \hat{\omega}_l |J_K(\hat{b}_l)|, \quad 1 \leqslant l \leqslant L, \tag{3.6}$$

where $J_K(\hat{x})$ is the Jacobian determinant of the mapping F_K at the point $\hat{x} \in \hat{K}$.

We now come to simplicial and quadrilateral finite elements.

Example 1: Simplicial finite elements
Here $\hat{K}$ is a nondegenerate n-simplex of $\mathbb{R}^n$. Let $\mathcal{T}_h$ be a triangulation of the set $\bar{\Omega}$ with nondegenerate n-simplices K of $\mathbb{R}^N$ with diameters $\leqslant h$. For any

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$1 \leqslant l \leqslant L,$ (3.6)

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teral finite elements.

. Let $\mathscr{T}_h$ be a triangulation of the
$\mathbb{R}^N$ with diameters $\leqslant h$. For any

$K \in \mathscr{T}_h$, we take for F_K an affine (i.e. $F_K \in (\hat{P}(1))^n$) invertible mapping such that $K = F_K(\hat{K})$. Given a family $(\mathscr{T}_h)$ of such triangulations of $\bar{\Omega}$, we shall say that $(\mathscr{T}_h)$ is a *regular family* if there exists a constant $\sigma > 0$ independent of h such that

$$h(K) < \sigma\rho(K) \quad \text{for all} \quad K \in \mathscr{T}_h \tag{3.7}$$

where

$h(K)$ = diameter of K,
$\rho(K)$ = diameter of the inscribed sphere of K.

Example 2: Quadrilateral finite elements
For simplicity, we consider only the case $n = 2$. Here $\hat{K}$ is the unit square $[0, 1]^2$ of $\mathbb{R}^2$. Let $\mathscr{T}_h$ be a "triangulation" of the set $\bar{\Omega}$ with nondegenerate *convex* quadrilaterals K with diameters $\leqslant h$. For any quadrilateral $K \in \mathscr{T}_h$, we take for F_K a bilinear (i.e. $F_K \in (\hat{Q}(1))^2$) invertible mapping such that $K = F_K(\hat{K})$ (Cf. [3, §6]). Notice that F_K degenerates to an affine invertible mapping if and only if K is a parallelogram. Given a family $(\mathscr{T}_h)$ of such "triangulations" of $\bar{\Omega}$ with quadrilateral elements, we shall say that $(\mathscr{T}_h)$ is a *regular family* if there exist two constants $\sigma > 0$ and γ with $0 < \gamma < 1$ both independent of h such that

$$h(K) \leqslant \sigma\rho(K) \quad \text{for all} \quad K \in \mathscr{T}_h, \tag{3.8}$$

$$\max_{1 \leqslant i \leqslant 4} |\cos\theta_i(K)| \leqslant \gamma \quad \text{for all} \quad K \in \mathscr{T}_h \tag{3.9}$$

where

$h(K)$ = diameter of K,
$\rho(K)$ = sup{diameter of the spheres contained in K}.
$\theta_i(K)$, $1 \leqslant i \leqslant 4$ = angles of the quadrilateral K.

Notice that (3.8) implies (3.9) when all the quadrilaterals K of the triangulation $\mathscr{T}_h$ are parallelograms.

For more explicit examples of simplicial and quadrilateral finite elements, we refer for instance of Zienkiewicz's book [17] and to [1], [2], [3] (See also the examples of §4).

Now, for any triangulation $\mathscr{T}_h$ of the set $\bar{\Omega}$ with finite elements K (of simplicial type or quadrilateral type) with diameters $\leqslant h$, we define V_h to be the space of functions v_h which satisfy:

(i) $v_h \in C^0(\bar{\Omega})$;
(ii) $v_h|_K \in P_K$ for all $K \in \mathscr{T}_h$;
(iii) $v_h = 0$ on the boundary Γ.

This definition makes sense since the reference finite element $\hat{K}$ is a C^0-element. A function $v_h \in V_h$ is then entirely determined by its values at the points $a_{i,K}$, $1 \leqslant i \leqslant N$, $K \in \mathscr{T}_h$. Thus, for each example considered previously, problem (1.9) is completely determined by the data of the triangulation $\mathscr{T}_h$, the reference finite element $\hat{K}$ and the quadrature formula (3.1).

Before applying Theorems 1 and 2, we need two preliminary results which will give us practical sufficient conditions for the hypotheses (2.7) and (2.48) to hold.

Lemma 5

Assume that $(\mathscr{T}_h)$ is a regular family of triangulations of $\bar{\Omega}$. Then, there exists a constant D independent of h such that (2.42) holds. Assume that, in addition, the quadrature formula (3.1) satisfies

$$\sum_{l=1}^{L} \hat{\omega}_l |\hat{p}(\hat{b}_l)|^2 \geqslant \hat{\delta}^2 \|\hat{p}\|^2_{L^2(\hat{K})} \quad \text{for all} \quad \hat{p} \in \hat{P} \tag{3.10}$$

for some constant $\hat{\delta} > 0$. Then, there exists a constant $\delta > 0$ independent of h such that the first inequality (2.48) holds.

Proof

First, there exists a constant $\hat{D}$ such that

$$\sum_{l=1}^{L} \hat{\omega}_l |\hat{p}(\hat{b}_l)|^2 \leqslant \hat{D}^2 \|\hat{p}\|^2_{L^2(\hat{K})} \quad \text{for all} \quad \hat{p} \in \hat{P}$$

If $p = \hat{p}_0 F_K^{-1} \in P_K$, we get from (3.6)

$$\sum_{l=1}^{L} \omega_{l,K} |p(b_{l,K})|^2 = \sum_{l=1}^{L} \hat{\omega}_l |J_K(\hat{b}_l)| \, |\hat{p}(\hat{b}_l)|^2 \leqslant$$

$$\leqslant \hat{D}^2 (\sup_{\hat{x} \in K} |J_K(\hat{x})|) \|\hat{p}\|^2_{L^2(\hat{K})}$$

and therefore

$$\sum_{l=1}^{L} \omega_{l,K} |p(b_{l,K})|^2 \leqslant \hat{D}^2 (\sup_{\hat{x} \in \hat{K}} |J_K(\hat{x})| / \inf_{\hat{x} \in \hat{K}} |J_K(\hat{x})|) \|p\|^2_{L^2(K)}$$

for all $p \in P_K$. Notice now that, for a regular family $(\mathscr{T}_h)$ of triangulations of $\bar{\Omega}$, there exist two constants $\gamma_0, \gamma_1 > 0$ independent of h such that

$$\gamma_0 \leqslant \frac{J_K(\hat{x})}{J_K(\hat{y})} \leqslant \gamma_1 \quad \text{for all} \quad \hat{x}, \hat{y} \in \hat{K}, \quad K \in \mathscr{T}_h.$$

This is obvious for simplicial elements and is a consequence of [3, Lemma 8] for quadrilateral elements. Thus, we obtain for a regular family $(\mathscr{T}_h)$ of triangulations of $\bar{\Omega}$

$$|v_h|_h^2 = \sum_{K \in \mathscr{T}_h} \sum_{l=1} \omega_{l,K} |v_h(b_{l,K})|^2 \leqslant \gamma_1 \hat{D}^2 \|v_h\|_{L^2(\Omega)}^2 \quad \text{for all} \quad v_h \in V_h$$

so that (2.42) holds with $D = \gamma_1^{1/2}\hat{D}$.

Next, using assumption (3.10), we similarly get

$$\sum_{l=1}^{L} \omega_{l,K} |p(b_{l,K})|^2 \geqslant \delta^2 \big(\inf_{\hat{x} \in \hat{K}} |J_K(\hat{x})| / \sup_{\hat{x} \in \hat{K}} |J_K(\hat{x})|\big) \|p\|_{L^2(K)}^2$$

for all $p \in P_K$ and therefore

$$|v_h|_h^2 \geqslant \hat{\gamma}_0 \delta^2 \|v_h\|_{L^2(\Omega)}^2 \quad \text{for all} \quad v_h \in V_h$$

so that the first inequality (2.48) holds with $\delta = \gamma_0^{1/2}\hat{\delta}$.

Remark 2

First, the inequality (3.10) is trivially satisfied if the quadrature formula (3.1) is exact for functions of the form $\hat{\varphi} = \hat{p}^2, \hat{p} \in \hat{P}$ since (3.10) becomes an equality with $\hat{\delta} = 1$. Next, since the weights $\hat{\omega}_l$ are > 0, $1 \leqslant l \leqslant L$, the assumption (3.10) can be equivalently stated as follows: the set $\{\hat{b}_l\}_{l=1}^{L}$ contains a $\hat{P}$-unisolvent subset.

Lemma 6

Assume that $(\mathscr{T}_h)$ is a regular family of triangulation of $\bar{\Omega}$ and that the quadrature formula (3.1) satisfies

$$\sum_{l=1}^{L} \hat{\omega}_l \sum_{i=1}^{n} \left| \frac{\partial \hat{p}}{\partial \hat{x}_i}(\hat{b}_l) \right|^2 \geqslant \hat{\gamma} \sum_{i=1}^{n} \left\| \frac{\partial \hat{p}}{\partial \hat{x}_i} \right\|_{L^2(\hat{K})}^2 \quad \text{for all} \quad \hat{p} \in \hat{P} \tag{3.11}$$

for some constant $\hat{\gamma} > 0$. Then, there exists a constant $\beta > 0$ independent of h such that (2.7) holds.

Proof

We get from [4, Theorem 3]

$$\sum_{K \in \mathscr{T}_h} \sum_{l=1}^{L} \omega_{l,K} \sum_{i=1}^{n} \left| \frac{\partial v_h}{\partial x_i}(b_{l,K}) \right|^2 \geqslant \gamma \sum_{i=1} \left\| \frac{\partial v_h}{\partial x_i} \right\|_{L^2(\Omega)}^2 \quad \text{for all}$$

$$v_h \in V_h, \tag{3.12}$$

for some constant $\gamma > 0$ independent of h. Thus, since $\omega_{l,K}$ is > 0, the desired inequality (2.7) follows from (3.12), the ellipticity condition and from Friedrichs' inequality.

Remark 3

First, if the quadrature formula (3.1) is exact for functions of the form $\hat{\varphi} = (\partial\hat{p}/\partial\hat{x}_i)^2$, $\hat{p} \in \hat{P}$, $i = 1, \ldots, n$, the inequality (3.11) becomes an equality with $\hat{\gamma} = 1$. Next, we let

$$\hat{P}_i = \left\{ \frac{\partial \hat{p}}{\partial \hat{x}_i} ; \hat{p} \in \hat{P} \right\}, \quad i = 1, \ldots, n. \tag{3.13}$$

Then, the assumption (3.11) is satisfied if $\{\hat{b}\}_{l=1}^{L}$ contains a $\hat{P}_i$-unisolvent subset for all $i = 1, \ldots, n$. Finally, notice that condition (3.10) is strictly more restrictive than condition (3.11) if $\hat{P} = \hat{P}(k)$ for instance.

We now want to apply the general results of § 2 to finite element methods using simplicial or quadrilateral elements. For the sake of simplicity, we shall describe our results only when $\hat{P}$ is a space of polynomials. We begin with *simplicial elements.*

Theorem 3

Let $(\mathscr{T}_h)$ be a regular family of triangulations of $\bar{\Omega}$ with simplicial elements We assume that the reference finite element $\hat{K}$ and the quadrature formula (3.1) satisfy the following conditions:

(i) $v_h \to |v_h|_h$ is a norm over V_h;
(ii) inequality (3.11) holds;
(iii) (3.13) $\hat{P}(k) \subset \hat{P} \subset \hat{P}(l)$ for some integers k, l with $1 \leqslant k \leqslant l$;
(iv) there exists an integer r with $0 \leqslant r \leqslant k - 1$ such that

$$\hat{E}(\hat{p}) = \int_{\hat{K}} \hat{p}\, d\hat{x} - \sum_{l=1}^{L} \hat{\omega}_l \hat{p}(\hat{b}_l) = 0 \quad \text{for all} \quad \hat{p} \in \hat{P}(r + l - 1). \tag{3.14}$$

Then, under the regularity assumptions (2.23) (with $2 \leqslant q \leqslant +\infty$, $r + 1 - n/q > 0$), the conclusion of Theorem 1 holds, i.e.

$$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{r+1}) \tag{3.15}$$

where u_h (resp. u) is the unique solution of Problem 1.9 (resp. Problem (1.5)).

Proof

We have to check the hypotheses of Theorem 1. As inequality (2.42) is a consequence of Lemma 5, it remains to show that (2.1), . . ., (2.7) hold with $\mu = 1$. First, since $\hat{P} \subset P(l)$, $v_h|_K$ is a polynomial of degree $\leqslant l$ for all $v_h \in V_h$ and all $K \in \mathscr{T}_h$ so that assumption (2.1) is trivially satisfied.

Consider next assumption (
vanishes over Γ, we may assoc

$r_h v \in V_h$ and $r_h v(a_{i,K}$

Then, for $n = 3$, $r_h \in \mathscr{L}(W^{s,q}($
any number $q \geqslant 2$ and proper
6]. For $n > 3$, a suitable opera
procedure (Cf. Strang [13]).

Now we remark that prope
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6. Thus, we may apply Theor

Theorem 4

Let $(\mathscr{T}_h)$ be a regular family c
We assume that

(i) inequality (3.10) hold
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$\hat{E}(\hat{P}) = 0$ for all $\hat{p} \in \hat{P}$

Then, under the regularity ass
of Theorem 2 holds, i.e.

$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\|$

Proof

In order to apply Theorem 2,
(2.3), (2.5), (2.6) with $\mu = 2$ a
follows from inequality (3.10
$\mu = 2$ are direct consequences
property (2.5) with $\mu = 2$ foll
we may apply Theorem 2 and

We can now solve the follo
the quadrature formula (3.1)
obtain error estimates which a
parameter h:

$\hat{P}(k) \subset \hat{P} \Rightarrow \{ \| u_h - u \|_{L^2} \ldots$; $\| u_h - u \|_{L^\infty} \ldots$

Consider next assumption (2.2). With any function $v \in C^0(\bar{\Omega})$ which vanishes over Γ, we may associate its V_h-interpolate which satisfies

$$r_h v \in V_h \quad \text{and} \quad r_h v(a_{i,K}) = v(a_{i,K}), \quad 1 \leqslant i \leqslant N, \quad K \in \mathscr{T}_h.$$

Then, for $n = 3$, $r_h \in \mathscr{L}(W^{s,q}(\Omega) \cap W_0^{1,q}(\Omega); V_h)$ for any integer $s \geqslant 2$ and any number $q \geqslant 2$ and property (2.2) follows from (3.13) and [2, Theorem 6]. For $n > 3$, a suitable operator r_h can be constructed by using a smoothing procedure (Cf. Strang [13]).

Now we remark that properties (2.3) and (2.6) are obvious consequences of (3.14) and [4, Theorem 4] when $\mu = 1$. Likewise, property (2.4) follows from (3.14) and [4, Theorem 7], while property (2.5) with $\mu = 1$ follows from (3.14) and [4, Theorem 6]. Finally, (2.7) is a consequence of Lemma 6. Thus, we may apply Theorem 1 and the desired conclusion follows.

Theorem 4

Let $(\mathscr{T}_h)$ be a regular family of triangulations of $\bar{\Omega}$ with simplicial elements. We assume that

(i) inequality (3.10) holds;
(ii) inequality (3.11) holds;
(iii) (3.13) $\hat{P}(k) \subset \hat{P} \subset \hat{P}(l)$ for some integers k, l with $1 \leqslant k \leqslant l$;
(iv) there exists an integer r with $0 \leqslant r \leqslant k - 1$ such that

$$\hat{E}(\hat{P}) = 0 \quad \text{for all} \quad \hat{p} \in \hat{P}(\max(r + l - 1, r + 1)). \tag{3.16}$$

Then, under the regularity assumptions (2.11) and (2.31), the conclusion of Theorem 2 holds, i.e.

$$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{r+2}). \tag{3.17}$$

Proof

In order to apply Theorem 2, it is necessary to check only assumptions (2.3), (2.5), (2.6) with $\mu = 2$ and (2.48). First, by Lemma 5, (2.48) follows from inequality (3.10). Next, properties (2.3) and (2.6) with $\mu = 2$ are direct consequences of (3.16) and [4, Theorem 5], while property (2.5) with $\mu = 2$ follows from (3.16) and [4, Theorem 8]. Thus we may apply Theorem 2 and we obtain the estimate (3.17).

We can now solve the following problem: How to choose *practically* the quadrature formula (3.1) over the reference finite element $\hat{K}$ to obtain error estimates which are optimal in the exponent of the parameter h:

$$\hat{P}(k) \subset \hat{P} \Rightarrow \begin{cases} \| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^k), \\ \| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{k+1}). \end{cases} \tag{3.18}$$

Using Theorems 3 and 4 and Remarks 2 and 3, *we obtain* (3.18) *if the quadrature formula* (3.1)

(i) *is such that* $\{\hat{b}_l\}_{l=1}^{L}$ *contains a* $\hat{P}$*-unisolvent subset and a* $\hat{P}_i$*-unisolvent subset for all* $i = 1, \ldots, n$;
(ii) *is exact for polynomials of degree* $\leqslant \max(k + l - 2, k)$ *if* $\hat{P}(k) \subset \hat{P} \subset \hat{P}(l)$.

Note that $\hat{P}_i \subset \hat{P}$, $i = 1, \ldots, n$, if $\hat{P}(k) \subset \hat{P} \subset \hat{P}(k + 1)$. Then, condition (i) is satisfied if the set $\{b_l\}_{l=1}^{L}$ contains a $\hat{P}$-unisolvent subset.

Remark 4
In Theorem 4, the condition (3.10) has to be considered only as a *practical* sufficient condition for obtaining inequality (2.48). Moreover this condition does not appear in the study of the corresponding elliptic problem (see [4], [15]). However, it is worthwhile to notice that we need some kind of assumption (3.10) to guarantee an optimal $L^\infty(L^2(\Omega))$ estimate as the following simple example shows.

Let Ω be the open interval $]0, 1[$ of $\mathbb{R}$ and let

$$x_0 = 0 < x_1 < \cdots < x_I < x_{I+1} = 1, \quad x_{i+1} - x_i = h, \quad i = 0, \ldots, I$$

be a subdivision of $[0, 1]$ with meshwidth h. We let:

$$\mathcal{T}_h = \{[x_i, x_{i+1}]\}_{i=0}^{I},$$
$$\hat{K} = [0, 1], \quad \hat{\Sigma} = \{0, 1\}, \quad \hat{P} = \hat{P}(1).$$

We choose for the quadrature formula over $\hat{K}$

$$\int_0^1 \hat{\varphi}(\hat{x})\, d\hat{x} \simeq \hat{\varphi}(\tfrac{1}{2})$$

so that condition (3.10) is not satisfied. Then

$$v_h \to |v_h|_h = \left(h \sum_{i=0}^{I} |v_h(x_{i+1/2})|^2\right)^{1/2}, \quad x_{i+1/2} = (i + \tfrac{1}{2})h,$$

is a norm over V_h, as is easily seen, but the first inequality (2.48) does not hold (take $v_h(x_i) = (-1)^i$, $1 \leqslant i \leqslant I$, for instance). Moreover, by an elementary calculation, one can show that

$$|v_h|_h \geqslant 2 \sin\left(\frac{\pi h}{2}\right)\left(h \sum_{i=1}^{I} |v_h(x_i)|^2\right)^{1/2}$$

so that there exists a constant $C > 0$ independent of h such that

$$|v_h|_h \geqslant Ch \|v_h\|_{L^2(\Omega)} \quad \text{for all} \quad v_h \in V_h.$$

Thus, by Theorem 3, we get

$\|u_h - u\|_{L^2(H^1(\Omega))} = 0(\|$

and by using the proof of Th

$\|u_h - u\|_{L^\infty(L^2(\Omega))} = 0(\|$

as a *direct* verification shows.

This suggests that it might
schemes for evaluating the L
$a_h(t; u_h, v_h)$.

We now come to *quadrilat*

Theorem 5
Let $(\mathcal{T}_h)$ be a regular family
elements. We assume that:

(i) $v_h \to |v_h|_h$ is a norm
(ii) inequality (3.11) hol
(iii) (3.19) $\hat{Q}(k) \subset P \subset \hat{Q}$
(iv) there exists an intege

$\hat{E}(\hat{p}) = 0$ for all $\hat{p} \in$

Then the same conclusion as

Proof
The proof is very similar to t
property (2.2) is now a cons
§ 6].

Similarly, we can prove th

Theorem 6
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(iv) there exists an intege

$\hat{E}(\hat{p}) = 0$ for all $\hat{p} \in$

Then the same conclusion as
By using Theorems 5 and

$\hat{Q}(k) \subset \hat{P} \Rightarrow \begin{cases} \|u_h - u\|_L \\ \|u_h - u\|_L \end{cases}$

if the quadrature formula (3.

Thus, by Theorem 3, we get for sufficiently smooth data and solution u

$$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h)$$

and by using the proof of Theorem 2, we obtain only

$$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h)$$

as a *direct* verification shows.

This suggests that it might be of interest to *use different integration schemes* for evaluating the L^2-scalar product (u_h, v_h) and the bilinear form $a_h(t; u_h, v_h)$.

We now come to *quadrilateral elements* (with $n = 2$).

Theorem 5

Let $(\mathscr{T}_h)$ be a regular family of "triangulations" of $\bar{\Omega}$ with quadrilateral elements. We assume that:

(i) $v_h \to | v_h |_h$ is a norm over v_h;
(ii) inequality (3.11) holds;
(iii) (3.19) $\hat{Q}(k) \subset P \subset \hat{Q}(l)$ for some integers k, l with $1 \leqslant k \leqslant l$;
(iv) there exists an integer r with $0 \leqslant r \leqslant k - 1$ such that

$$\hat{E}(\hat{p}) = 0 \quad \text{for all} \quad \hat{p} \in \hat{Q}(r + l). \tag{3.20}$$

Then the same conclusion as that of Theorem 3 holds.

Proof

The proof is very similar to that of Theorem 3. Notice however that property (2.2) is now a consequence of (3.19) and [3, Theorem 6 and § 6].

Similarly, we can prove the following result.

Theorem 6

Let $(\mathscr{T}_h)$ be a regular family of "triangulations" of $\bar{\Omega}$ with quadrilateral elements. We assume that

(i) inequality (3.10) holds;
(ii) inequality (3.11) holds;
(iii) (3.19) $\hat{Q}(k) \subset \hat{P} \subset \hat{Q}(l)$ for some integers k, l with $1 \leqslant k \leqslant l$;
(iv) there exists an integer r with $0 \leqslant r \leqslant k - 1$ such that

$$\hat{E}(\hat{p}) = 0 \quad \text{for all} \quad \hat{p} \in \hat{Q}\,(\max(r + l, r + 2)). \tag{3.21}$$

Then the same conclusion as that of Theorem 4 holds.

By using Theorems 5 and 6, we get the optimal estimates

$$\hat{Q}(k) \subset \hat{P} \Rightarrow \begin{cases} \| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^k) \\ \| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{k+1}) \end{cases} \tag{3.22}$$

if the quadrature formula (3.1)

(i) is such that $\{\hat{b}_l\}_{l=1}^{L}$ contains a $\hat{P}$-unisolvent subset and a $\hat{P}_i$-unisolvent subset for all $i = 1, \ldots, n$;
(ii) is exact for polynomials of $\hat{Q}(\max(k + l - 1, k + 1))$ if $\hat{Q}(k) \subset \hat{P} \subset \hat{Q}(l)$.

4. Miscellaneous Remarks

4.1. A special case of practical interest
Let us consider a general finite element method using finite elements K corresponding to Lagrange interpolation as has been described in § 3. Denote by $U_h(t)$ (resp. $F_h(t)$) the vector whose components are the values of $u_h(t)$ (resp. $f(t)$) at the points $a_{i,K}$, $1 \leqslant i \leqslant N$, $K \in \mathcal{T}_h$. Then, problem (1.9) is equivalent to the linear differential system

$$\begin{cases} B_h \cdot \dfrac{dU_h}{dt}(t) + A_h(t) \cdot U_h(t) = B_h \cdot F_h(t), \\ U_h(0) \text{ given.} \end{cases} \tag{4.1}$$

Consider the following problem: How to choose the quadrature formula (3.1) over the reference finite element $\hat{K}$ *so that the matrix B_h is diagonal.* In such a case, by using a suitable discrete analogue of the time derivative, we can generate *purely explicit schemes* for solving numerically the parabolic problem (1.1). For instance, by using forward differencing in time we can easily obtain the usual explicit difference scheme for the heat equation.

Clearly, the matrix B_h is diagonal if and only if $L = N$ and the nodes $\hat{b}_l$, $1 \leqslant l \leqslant L$, of the quadrature formula (3.1) coincide with the points $\hat{a}_i$, $1 \leqslant i \leqslant N$, of the interpolation set $\hat{\Sigma}$. Denote by $\hat{p}_i$, $1 \leqslant i \leqslant N$ the basis functions over the reference finite element $\hat{K}$, i.e. $\hat{p}_i \in \hat{P}$, $\hat{p}_i(\hat{a}_j) = \delta_{ij}$, $1 \leqslant j \leqslant N$. Then, *the Newton-Cotes type quadrature formula*

$$\int \hat{\varphi}(\hat{x})\, d\hat{x} \simeq \sum_{i=1}^{N} \left(\int_{\hat{K}} \hat{p}_i(\hat{x})\, d\hat{x} \right) \hat{\varphi}(\hat{a}_i) \tag{4.2}$$

satisfies the desired property if $\int_{\hat{K}} \hat{p}_i(\hat{x})\, d\hat{x} \neq 0$, $1 \leqslant i \leqslant N$. Let us now apply the results of § 3 when we use the quadrature formula (4.2). We shall always assume that

$$\hat{\omega}_i = \int_{\hat{K}} \hat{p}_i(\hat{x})\, d\hat{x} > 0, \quad 1 \leqslant i \leqslant N. \tag{4.3}$$

Begin with simplicial elements. For ease of exposition, we shall only consider the case $\hat{P} = \hat{P}(k)$ for some integer $k \geqslant 1$. Condition (4.3) implies that hypotheses (3.10) and (3.11) hold. Moreover, we have

$$\hat{E}(\hat{p}) = 0 \quad \text{for all} \quad \hat{p} \in \hat{P}(k). \tag{4.4}$$

Then, by applying Theorems
conditions

$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(\|$
$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\|$

Notice that the estimates are

Example 3
Let $\hat{a}_i$, $1 \leqslant i \leqslant n + 1$, be the ve

$\hat{\Sigma}(k) = \{ \hat{x} \in \hat{K}; \hat{x} = \sum_{i=1}^{n+1} \lambda$
$1 \leqslant i \leqslant n + 1 \}$

denote the principal lattice of
$\hat{\Sigma}(k)$ is $\hat{P}(k)$-unisolvent for an
element $\hat{K}$ associated with $(\hat{\Sigma}$
that condition (4.3) holds for
to the vertices $\hat{a}_i$ of $\hat{K}$ are zer
$k = 1, 3$. We conclude that the
only for $k = 1$.

Consider now quadrilateral
integer $k \geqslant 1$. Hypotheses (3.

$\hat{E}(\hat{p}) = 0$ for all $\hat{p} \in \hat{Q}$

Then, by applying Theorems
hypotheses

$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(\|$
$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\|$

The first estimate is optimal
optimal.

Example 4
Let $\hat{a}_i$, $1 \leqslant i \leqslant 4$, be the vertic

$\hat{\Xi}(k) = \{ \hat{x} \in \hat{K}; \hat{x} = (\hat{x}_1, \hat{x}$
$i = 1, 2 \}.$

Then, by applying Theorems 3 and 4, we get under suitable regularity conditions

$$\begin{cases} \| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{\min(2,k)}), \\ \| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{\min(3,k+1)}). \end{cases} \tag{4.5}$$

Notice that the estimates are optimal only for $k = 1, 2$.

Example 3

Let $\hat{a}_i$, $1 \leqslant i \leqslant n+1$, be the vertices of the reference n-simplex K. We let

$$\hat{\Sigma}(k) = \left\{ \hat{x} \in \hat{K}; \hat{x} = \sum_{i=1}^{n+1} \lambda_i \hat{a}_i, \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \in \left\{ 0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1 \right\}, \right.$$
$$\left. 1 \leqslant i \leqslant n+1 \right\}$$

denote the principal lattice of order k of the n-simplex $\hat{K}$. Then the set $\hat{\Sigma}(k)$ is $\hat{P}(k)$-unisolvent for any integer $k \geqslant 1$ and the reference finite element $\hat{K}$ associated with $(\hat{\Sigma}(k), \hat{P}(k))$ is a C^0-element. It is readily seen that condition (4.3) holds for $k = 1, 3$ while the weights $\hat{\omega}_i$ corresponding to the vertices $\hat{a}_i$ of $\hat{K}$ are zero for $k = 2$. Thus, we may apply (4.5) for $k = 1, 3$. We conclude that the quadrature formula (4.2) is fully satisfactory only for $k = 1$.

Consider now quadrilateral elements and assume that $\hat{P} = \hat{Q}(k)$ for some integer $k \geqslant 1$. Hypotheses (3.10), (3.11) hold again and we have

$$\hat{E}(\hat{p}) = 0 \quad \text{for all} \quad \hat{p} \in \hat{Q}(k). \tag{4.6}$$

Then, by applying Theorems 5 and 6, we get under suitable regularity hypotheses

$$\begin{cases} \| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h), \\ \| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^{\min(2,k)}). \end{cases} \tag{4.7}$$

The first estimate is optimal only for $k = 1$ and the second estimate is never optimal.

Example 4

Let $\hat{a}_i$, $1 \leqslant i \leqslant 4$, be the vertices of the unit square $\hat{K}$ of $\mathbb{R}^2$. We let

$$\hat{\Xi}(k) = \left\{ \hat{x} \in \hat{K}; \hat{x} = (\hat{x}_1, \hat{x}_2) \quad \text{with} \quad \hat{x}_i \in \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, k\}, \right.$$
$$\left. i = 1, 2 \right\}.$$

Then $\hat{\Xi}(k)$ is $\hat{Q}(k)$-unisolvent and the reference finite element K associated with $(\Xi(k), \hat{Q}(k))$ is a C^0-element. Obviously, condition (4.3) holds for $k = 1, 2$ for instance and we may apply (4.7).

4.2. A remark on finite element collocation methods

Now we assume that the space V_h associated with the triangulation $\mathscr{T}_h$ of $\bar{\Omega}$ with finite elements K satisfies the following properties:

(i) V_h is a finite dimensional subspace of $H^2(\Omega) \cap H_0^1(\Omega)$;
(ii) $v_h|_K \in C^2(K)$ for all $K \in \mathscr{T}_h$ *and all* $v_h \in V_h$.

If the coefficients $a_{ij}(t) \in C^1(\bar{\Omega})$, $1 \leqslant i, j \leqslant n$, we may choose for each u_h, $v_h \in V_h$

$$a_h(t; u_h, v_h) = \sum_{K \in \mathscr{T}_h} \sum_{l=1}^{L} \omega_{l,K}(A(t)u_h v_h)(b_{l,K}). \tag{4.8}$$

Again, we consider the semi-discrete problem (1.9) but with (1.11) replaced by (4.8).

Choose the quadrature nodes $b_{l,K}$ so that

(iii) $b_{l,K} \in \mathring{K}$, $\quad 1 \leqslant l \leqslant L$, $\quad$ for any $\quad K \in \mathscr{T}_h$,
(iv) a function $v_h \in V_h$ is uniquely determined by its values at the points $b_{l,K}$, $1 \leqslant l \leqslant L$, $K \in \mathscr{T}_h$.

Then problem (1.9) can be equivalently stated as follows: Find a function $u_h : [0, T] \to V_h$ such that

$$\begin{cases} \left(\dfrac{\partial u_h}{dt}(t) + A(t)u_h(t)\right)(b_{l,k}) = f(b_{l,K}, t), \quad 1 \leqslant l \leqslant L, \quad K \in \mathscr{T}_h, \\ u_h(0) = u_{h,0}. \end{cases} \tag{4.9}$$

Thus, we obtain a *finite element collocation method* where the collocation points are the points $b_{l,K}$, $1 \leqslant l \leqslant L$, $K \in \mathscr{T}_h$. So, when conditions (i), . . ., (iv) are satisfied, *the collocation method* (4.9) *appears to be a special case of the finite element method using numerical integration.*

Clearly, the results of § § 2 and 3 can be easily extended when the bilinear form $a_h(t; u_h, v_h)$ is given by (4.8). But, instead of stating the corresponding results, we shall consider a simple but significant example which has been analyzed by Douglas and Dupont [6].

Let Ω be the open interval $]0, 1[$ of $\mathbb{R}$ and let

$$0 = x_0 < x_1 < \cdots < x_I < x_{I+1} = 1$$

be a subdivision of $[0, 1]$. Let

$$V_h = \{v_h \in C^1(0, 1); \quad v_h|_{[x_i, x_{i+1}]} \text{ is a cubic polynomial}, i = 0, \ldots, I,$$
$$v_h(0) = v_h(1) = 0\}$$

and choose the collocation po

$$b_{i,l} = \tfrac{1}{2}(x_i + x_{i+1}) + (-1)$$

Note that the point $b_{i,l}$, $l =$
by using [6, Lemma 2.2] and

$$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(\|$$

which is not optimal. Let us
Theorem 4 since $v_h \to |v_h|_h =$
hypothesis (2.48).

However, it is shown in [6

$$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\|$$

holds. This is not surprising s
properties of the approximat
has to be refined in some cas

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erence finite element K
nt. Obviously, condition (4.3)
y apply (4.7).

ion methods
ited with the triangulation $\mathscr{T}_h$ of
owing properties:
of $H^2(\Omega) \cap H_0^1(\Omega)$;
$l\ v_h \in V_h$.
$\leqslant n$, we may choose for each u_h,

$v_h)(b_{l,K})$. (4.8)

lem (1.9) but with (1.11) replaced

at

$K \in \mathscr{T}_h$,
ermined by its values at the

ated as follows: Find a function

$, t), \quad 1 \leqslant l \leqslant L, \quad K \in \mathscr{T}_h,$
(4.9)

on method where the collocation
$\mathscr{T}_h$. So, when conditions (i), . . .,
4.9) *appears to be a special case*
ical integration.
easily extended when the
But, instead of stating the
simple but significant example
Dupont [6].
nd let

cubic polynomial, $i = 0, \ldots, I$,

and choose the collocation points

$$b_{i,l} = \tfrac{1}{2}(x_i + x_{i+1}) + (-1)^l \frac{x_{i+1} - x_i}{2\sqrt{3}}, \quad l = 1, 2, \quad 0 \leqslant i \leqslant I.$$

Note that the point $b_{i,l}$, $l = 1, 2$, are Gaussian quadrature points. Then, by using [6, Lemma 2.2] and an analogue of Theorem 3, we get the estimate

$$\| u_h - u \|_{L^2(H^1(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^2)$$

which is not optimal. Let us remark that we cannot use an analogue of Theorem 4 since $v_h \to | v_h |_h = (h \sum_{i=0}^{I} \sum_{l=1}^{2} | v_h(b_{i,l}) |^2)^{1/2}$ does not satisfy hypothesis (2.48).

However, it is shown in [6] that the optimal estimate

$$\| u_h - u \|_{L^\infty(L^2(\Omega))} = 0(\| u_{h,0} - u_0 \|_{L^2(\Omega)} + h^4)$$

holds. This is not surprising since the proof given in [6] uses refined properties of the approximation and this indicates that our general analysis has to be refined in some cases in order to obtain the best possible results.

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Error Estima
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Introduction

If we want to compare the dis
the exact solution of the diffe
either to discretize the exact s
from the mesh to the domain
we are led to discrete converg
This paper deals with the latte
for non-linear parabolic equat

The basic tool in our proof
Properties of this operator suc
and behaviour near the bound
make it possible to apply an e
approximate solutions of non-
estimate one can prove order
and also convergence without
One advantage of this approac
equations too. This fact is use
in order not to encumber the
regarding the difference probl

Finally let us remark that t
makes use of extensions of th
functional spaces without ind
case of non-rectangular domai

1. An Estimate for Approxim

We are concerned with the nu
problem:

$$L^i[z] \equiv z_t - f^i(t, x, z, z_x$$