formula:

$$(f,g)_{\mathcal{D}} = \sum_{i=1}^{N} (f,g)_i \tag{D.31}$$

$$(f,g)_i = \int_{x_{i-1/2}}^{x_{i+1/2}} f \cdot g \; dx \tag{D.32}$$

$$(f,g)_{E_h^i} = \sum_{i=1}^{N-1} (f,g)_{i+1/2} \tag{D.33}$$

$$(f,g)_{i+1/2} = f_{i+1/2} g_{i+1/2} \tag{D.34}$$

$$(f,g)_{i+1/2}^R = f_{i+1/2}^R g_{i+1/2}^R \tag{D.35}$$

$$(f,g)_{i+1/2}^L = f_{i+1/2}^L g_{i+1/2}^L \tag{D.36}$$

$$[\![\phi]\!]_{i+1/2} = \phi_{i+1/2}^R - \phi_{i+1/2}^L \tag{D.37}$$

$$[\![\phi^*]\!]_{i+1/2} = \phi_{i+1/2}^{*R} - \phi_{i+1/2}^{*L} \tag{D.38}$$

$$[\![D d_x \phi]\!]_{i+1/2} = \left[D \frac{d\phi}{dx}\right]_{i+1/2}^R - \left[D \frac{d\phi}{dx}\right]_{i+1/2}^L \tag{D.39}$$

$$[\![D d_x \phi^*]\!]_{i+1/2} = \left[D \frac{d\phi^*}{dx}\right]_{i+1/2}^R - \left[D \frac{d\phi^*}{dx}\right]_{i+1/2}^L \tag{D.40}$$

$$\{\!\{D d_x \phi\}\!\}_{i+1/2} = \frac{1}{2}(\left[D \frac{d\phi}{dx}\right]_{i+1/2}^R + \left[D \frac{d\phi}{dx}\right]_{i+1/2}^L) \tag{D.41}$$

$$\{\!\{D d_x \phi^*\}\!\}_{i+1/2} = \frac{1}{2}(\left[D \frac{d\phi^*}{dx}\right]_{i+1/2}^R + \left[D \frac{d\phi^*}{dx}\right]_{i+1/2}^L) \tag{D.42}$$

We copy the multi-dimensional diffusion conforming form to here:

$$
\begin{aligned}
b_{DCF}(\Phi,\Phi^*) =& (\sigma_a\Phi,\Phi^*)_{\mathcal{D}} + (D\vec{\nabla}\Phi,\vec{\nabla}\Phi^*)_{\mathcal{D}} \\
&+ \frac{1}{4}([\![\Phi]\!],[\![\Phi^*]\!])_{E_h^i} + ([\![\Phi]\!],\{\!\{D\partial_n\Phi^*\}\!\})_{E_h^i} + (\{\!\{D\partial_n\Phi\}\!\},[\![\Phi^*]\!])_{E_h^i} \\
&+ \frac{1}{4}(\Phi,\Phi^*)_{\partial\mathcal{D}^d} - \frac{1}{2}(\Phi,D\partial_n\Phi^*)_{\partial\mathcal{D}^d} - \frac{1}{2}(D\partial_n\Phi,\Phi)_{\partial\mathcal{D}^d} \\
&- \frac{9}{16}([\![D\vec{\nabla}\Phi]\!],[\![D\vec{\nabla}\Phi^*]\!])_{E_h^i} - \frac{9}{16}([\![D\partial_n\Phi]\!],[\![D\partial_n\Phi^*]\!])_{E_h^i} \\
&- \frac{9}{16}(D\vec{\nabla}\Phi,D\vec{\nabla}\Phi^*)_{\partial\mathcal{D}^d} - \frac{9}{16}(D\partial_n\Phi,D\partial_n\Phi^*)_{\partial\mathcal{D}^d} \\
l_{DCF}(\Phi^*) =& (Q_0,\Phi^*)_{\mathcal{D}} + (J^{inc},\Phi^*)_{\partial\mathcal{D}^d} - (\vec{\Upsilon}^{inc},D\vec{\nabla}\Phi^*)_{\partial\mathcal{D}^d}
\end{aligned}
\tag{D.43}
$$

where

$$
\begin{aligned}
J^{inc} &= \sum_{\vec{\Omega}_m\cdot\vec{n}(\vec{r}_b)<0} w_m \left|\vec{\Omega}_m\cdot\vec{n}(\vec{r}_b)\right| \Psi_m^{inc} \\
\vec{\Upsilon}^{inc} &= -\sum_{\vec{\Omega}_m\cdot\vec{n}(\vec{r}_b)<0} 3w_m\vec{\Omega}_m \left|\vec{\Omega}_m\cdot\vec{n}(\vec{r}_b)\right| \Psi_m^{inc}
\end{aligned}
\tag{D.44}
$$

We can see that, with following

$$D\vec{\nabla}\Phi = Dd_x\phi \tag{D.45}$$

$$D\partial_n\Phi = Dd_x\phi \qquad \text{on } E_h^i \tag{D.46}$$

$$D\partial_n\Phi = Dd_x\phi \qquad \text{on } x_{N+1/2} \tag{D.47}$$

$$D\partial_n\Phi = -Dd_x\phi \qquad \text{on } x_{1/2} \tag{D.48}$$

We can obtain the 1-D form from the multi-dimensional formula.

I will present a different way to assemble the system. Let us consider two element adjacent with each other noted with 1 for the left and 2 for the right.

$$
\begin{aligned}
(\sigma_a\phi,\phi^*)_D &= \phi_1^{*T}\sigma_{a,1}\Delta x_1\mathbf{M}\phi_1 + \phi_2^{*T}\sigma_{a,2}\Delta x_2\mathbf{M}\phi_2 \\
(Dd_x\phi, d_x\phi^*)_D &= \phi_1^{*T}\frac{D_1}{\Delta x_1}\mathbf{S}\phi_1 + \phi_2^{*T}\frac{D_2}{\Delta x_2}\mathbf{S}\phi_2 \\
\frac{1}{4}([\phi],[\phi^*])_{E_h^i} &= \frac{1}{4}(\phi_{3/2}^R - \phi_{3/2}^L)(\phi_{3/2}^{*R} - \phi_{3/2}^{*L}) \\
&= \frac{1}{4}\phi_{3/2}^L\phi_{3/2}^{*L} + \frac{1}{4}\phi_{3/2}^R\phi_{3/2}^{*R} - \frac{1}{4}\phi_{3/2}^R\phi_{3/2}^{*L} - \frac{1}{4}\phi_{3/2}^L\phi_{3/2}^{*R} \\
&= \frac{1}{4}\phi_1^{*T}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\phi_1 + \frac{1}{4}\phi_2^{*T}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\phi_2 \\
&\quad - \frac{1}{4}\phi_1^{*T}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\phi_2 - \frac{1}{4}\phi_2^{*T}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\phi_1 \\
&= \phi_1^{*T}\mathbf{E}_{11}^1\phi_1 + \phi_2^{*T}\mathbf{E}_{22}^1\phi_2 + \phi_1^{*T}\mathbf{E}_{12}^1\phi_2 + \phi_2^{*T}\mathbf{E}_{21}^1\phi_1
\end{aligned}
$$

$$
\begin{aligned}
(\{Dd_x\phi\},[\phi^*])_{E_h^i} &= \frac{1}{2}(\left[D\frac{d\phi}{dx}\right]_{3/2}^R + \left[D\frac{d\phi}{dx}\right]_{3/2}^L)(\phi_{3/2}^{*R} - \phi_{3/2}^{*L}) \\
&= -\frac{1}{2}\left[D\frac{d\phi}{dx}\right]_{3/2}^L\phi_{3/2}^{*L} + \frac{1}{2}\left[D\frac{d\phi}{dx}\right]_{3/2}^R\phi_{3/2}^{*R} \\
&\quad - \frac{1}{2}\left[D\frac{d\phi}{dx}\right]_{3/2}^R\phi_{3/2}^{*L} + \frac{1}{2}\left[D\frac{d\phi}{dx}\right]_{3/2}^L\phi_{3/2}^{*R} \\
&= -\frac{1}{2}\phi_1^{*T}\begin{bmatrix}0\\1\end{bmatrix}\frac{D_1}{\Delta x_1}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_1 + \frac{1}{2}\phi_2^{*T}\begin{bmatrix}1\\0\end{bmatrix}\frac{D_2}{\Delta x_2}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_2 \\
&\quad - \frac{1}{2}\phi_1^{*T}\begin{bmatrix}0\\1\end{bmatrix}\frac{D_2}{\Delta x_2}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_2 + \frac{1}{2}\phi_2^{*T}\begin{bmatrix}1\\0\end{bmatrix}\frac{D_1}{\Delta x_1}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_1 \\
&= \phi_1^{*T}\frac{D_1}{\Delta x_1}\mathbf{E}_{11}^2\phi_1 + \phi_2^{*T}\frac{D_2}{\Delta x_2}\mathbf{E}_{22}^2\phi_2 + \phi_1^{*T}\frac{D_2}{\Delta x_2}\mathbf{E}_{12}^2\phi_2 + \phi_2^{*T}\frac{D_1}{\Delta x_1}\mathbf{E}_{21}^2\phi_1
\end{aligned}
$$

$$
\begin{aligned}
([\phi]\,,\{Dd_x\phi^*\})_{E_h^i} &= \frac{1}{2}(\phi^R_{3/2}-\phi^L_{3/2})(\left[D\frac{d\phi^*}{dx}\right]^R_{3/2}+\left[D\frac{d\phi^*}{dx}\right]^L_{3/2}) \\
&= -\frac{1}{2}\left[D\frac{d\phi^*}{dx}\right]^L_{3/2}\phi^L_{3/2}+\frac{1}{2}\left[D\frac{d\phi^*}{dx}\right]^R_{3/2}\phi^R_{3/2} \\
&\quad +\frac{1}{2}\left[D\frac{d\phi^*}{dx}\right]^L_{3/2}\phi^R_{3/2}-\frac{1}{2}\left[D\frac{d\phi^*}{dx}\right]^R_{3/2}\phi^L_{3/2} \\
&= -\frac{1}{2}\frac{D_1}{\Delta x_1}\phi_1^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\phi_1+\frac{1}{2}\frac{D_2}{\Delta x_2}\phi_2^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\phi_2 \\
&\quad -\frac{1}{2}\frac{D_1}{\Delta x_1}\phi_1^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\phi_2+\frac{1}{2}\frac{D_2}{\Delta x_2}\phi_2^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\phi_1 \\
&= \phi_1^{*T}\frac{D_1}{\Delta x_1}\mathbf{E}^{2T}_{11}\phi_1+\phi_2^{*T}\frac{D_2}{\Delta x_2}\mathbf{E}^{2T}_{22}\phi_2+\phi_1^{*T}\frac{D_1}{\Delta x_1}\mathbf{E}^{2T}_{21}\phi_2+\phi_2^{*T}\frac{D_2}{\Delta x_2}\mathbf{E}^{2T}_{12}\phi_1
\end{aligned}
$$

$$
\begin{aligned}
-\frac{9}{8}([Dd_x\phi]\,,[Dd_x\phi^*])_{E_h^i} &= -\frac{9}{8}(\left[D\frac{d\phi}{dx}\right]^R_{3/2}-\left[D\frac{d\phi}{dx}\right]^L_{3/2})(\left[D\frac{d\phi^*}{dx}\right]^R_{3/2}-\left[D\frac{d\phi^*}{dx}\right]^L_{3/2}) \\
&= -\frac{9}{8}\left[D\frac{d\phi}{dx}\right]^L_{3/2}\left[D\frac{d\phi^*}{dx}\right]^L_{3/2}-\frac{9}{8}\left[D\frac{d\phi}{dx}\right]^R_{3/2}\left[D\frac{d\phi^*}{dx}\right]^R_{3/2} \\
&\quad +\frac{9}{8}\left[D\frac{d\phi}{dx}\right]^R_{3/2}\left[D\frac{d\phi^*}{dx}\right]^L_{3/2}+\frac{9}{8}\left[D\frac{d\phi}{dx}\right]^L_{3/2}\left[D\frac{d\phi^*}{dx}\right]^R_{3/2} \\
&= -\frac{9}{8}\frac{D_1}{\Delta x_1}\phi_1^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\frac{D_1}{\Delta x_1}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_1-\frac{9}{8}\frac{D_2}{\Delta x_2}\phi_2^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\frac{D_2}{\Delta x_2}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_2 \\
&\quad +\frac{9}{8}\frac{D_1}{\Delta x_1}\phi_1^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\frac{D_2}{\Delta x_2}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_2+\frac{9}{8}\frac{D_2}{\Delta x_2}\phi_2^{*T}\begin{bmatrix}-1\\1\end{bmatrix}\frac{D_2}{\Delta x_2}\begin{bmatrix}-1 & 1\end{bmatrix}\phi_1 \\
&= \phi_1^{*T}(\frac{D_1}{\Delta x_1})^2\mathbf{E}^3_{11}\phi_1+\phi_2^{*T}(\frac{D_2}{\Delta x_2})^2\mathbf{E}^3_{22}\phi_2 \\
&\quad +\phi_1^{*T}\frac{D_1}{\Delta x_1}\frac{D_2}{\Delta x_2}\mathbf{E}^3_{12}\phi_2+\phi_2^{*T}\frac{D_1}{\Delta x_1}\frac{D_2}{\Delta x_2}\mathbf{E}^3_{21}\phi_1
\end{aligned}
$$

where

$$\mathbf{M} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{E}^1_{11} = \frac{1}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{E}^1_{22} = \frac{1}{4}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{E}^1_{12} = -\frac{1}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\mathbf{E}^1_{21} = -\frac{1}{4}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{E}^2_{11} = \mathbf{E}^2_{12} = -\frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{E}^2_{22} = \mathbf{E}^2_{21} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{E}^3_{11} = \mathbf{E}^3_{22} = -\frac{9}{8}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{9}{8}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{E}^3_{12} = \mathbf{E}^3_{21} = \frac{9}{8}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{9}{8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Note that $\mathbf{E}_{12}^{1} = \mathbf{E}_{21}^{1T}$. So the final system without the boundary treatment is

$$\begin{bmatrix} \phi_1^* & \phi_2^* \end{bmatrix} \mathbf{A} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{D.49}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{D.50}$$

$$\mathbf{A}_{11} = \sigma_{a,1}\Delta x_1 \mathbf{M} + \frac{D_1}{\Delta x_1}\mathbf{S} + \mathbf{E}_{11}^{1} + \frac{D_1}{\Delta x_1}(\mathbf{E}_{11}^{2} + \mathbf{E}_{11}^{2T}) + (\frac{D_1}{\Delta x_1})^2 \mathbf{E}_{11}^{3} \tag{D.51}$$

$$\mathbf{A}_{12} = \mathbf{E}_{12}^{1} + \frac{D_2}{\Delta x_2}\mathbf{E}_{12}^{2} + \frac{D_1}{\Delta x_1}\mathbf{E}_{21}^{2T} + \frac{D_1}{\Delta x_1}\frac{D_2}{\Delta x_2}\mathbf{E}_{12}^{3} \tag{D.52}$$

$$\mathbf{A}_{21} = \mathbf{E}_{21}^{1} + \frac{D_1}{\Delta x_1}\mathbf{E}_{21}^{2} + \frac{D_2}{\Delta x_2}\mathbf{E}_{12}^{2T} + \frac{D_1}{\Delta x_1}\frac{D_2}{\Delta x_2}\mathbf{E}_{21}^{3} \tag{D.53}$$

$$\mathbf{A}_{22} = \sigma_{a,2}\Delta x_2 \mathbf{M} + \frac{D_2}{\Delta x_2}\mathbf{S} + \mathbf{E}_{22}^{1} + \frac{D_2}{\Delta x_2}(\mathbf{E}_{22}^{2} + \mathbf{E}_{22}^{2T}) + (\frac{D_2}{\Delta x_2})^2 \mathbf{E}_{22}^{3} \tag{D.54}$$

Matrix $\mathbf{A}$ is symmetric. If there is another cell on the right of cell 2 denoted with 3, the resulting matrix will be

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix} \tag{D.55}$$

where

$$\mathbf{A}_{11} = \sigma_{a,1}\Delta x_1\mathbf{M} + \frac{D_1}{\Delta x_1}\mathbf{S} + \mathbf{E}_{11}^1 + \frac{D_1}{\Delta x_1}(\mathbf{E}_{11}^2 + \mathbf{E}_{11}^{2T}) + (\frac{D_1}{\Delta x_1})^2\mathbf{E}_{11}^3 \tag{D.56}$$

$$\mathbf{A}_{12} = \mathbf{E}_{12}^1 + \frac{D_2}{\Delta x_2}\mathbf{E}_{12}^2 + \frac{D_1}{\Delta x_1}\mathbf{E}_{21}^{2T} + \frac{D_1}{\Delta x_1}\frac{D_2}{\Delta x_2}\mathbf{E}_{12}^3 \tag{D.57}$$

$$\mathbf{A}_{21} = \mathbf{E}_{21}^1 + \frac{D_1}{\Delta x_1}\mathbf{E}_{21}^2 + \frac{D_2}{\Delta x_2}\mathbf{E}_{12}^{2T} + \frac{D_1}{\Delta x_1}\frac{D_2}{\Delta x_2}\mathbf{E}_{21}^3 \tag{D.58}$$

$$\begin{aligned}\mathbf{A}_{22} = {} & \sigma_{a,2}\Delta x_2\mathbf{M} + \frac{D_2}{\Delta x_2}\mathbf{S} + \mathbf{E}_{22}^1 + \frac{D_2}{\Delta x_2}(\mathbf{E}_{22}^2 + \mathbf{E}_{22}^{2T}) + (\frac{D_2}{\Delta x_2})^2\mathbf{E}_{22}^3 + \mathbf{E}_{11}^1 + \\ & \frac{D_2}{\Delta x_2}(\mathbf{E}_{11}^2 + \mathbf{E}_{11}^{2T}) + (\frac{D_2}{\Delta x_2})^2\mathbf{E}_{11}^3\end{aligned} \tag{D.59}$$

$$\mathbf{A}_{32} = \mathbf{E}_{21}^1 + \frac{D_2}{\Delta x_2}\mathbf{E}_{21}^2 + \frac{D_3}{\Delta x_3}\mathbf{E}_{12}^{2T} + \frac{D_2}{\Delta x_2}\frac{D_3}{\Delta x_3}\mathbf{E}_{21}^3 \tag{D.60}$$

$$\mathbf{A}_{33} = \sigma_{a,3}\Delta x_3\mathbf{M} + \frac{D_3}{\Delta x_3}\mathbf{S} + \mathbf{E}_{22}^1 + \frac{D_3}{\Delta x_3}(\mathbf{E}_{22}^2 + \mathbf{E}_{22}^{2T}) + (\frac{D_3}{\Delta x_3})^2\mathbf{E}_{22}^3 \tag{D.61}$$

This is one way where we basically are considering a bunch of small 2-by-2 system and then summing them together. We can also consider each cell with two side vertices, i.e., each row of the global system. 11 is right self-coupling, 22 is left self-coupling, 12 is right vertex coupling, 21 is left vertex coupling. This way is better in multi-dimensional situation.

$$\mathbf{E}_{11} = \mathbf{E}_R \tag{D.62}$$

$$\mathbf{E}_{22} = \mathbf{E}_L \tag{D.63}$$

$$\mathbf{E}_{12} = \mathbf{E}_{RC} \tag{D.64}$$

$$\mathbf{E}_{21} = \mathbf{E}_{LC} \tag{D.65}$$

APPENDIX E

PRECONDITIONED CG METHOD WITH EISENSTAT TRICK