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FINITE-DIFFERENCE APPROXIMATIONS AND SUPERCONVERGENCE FOR THE DISCRETE-ORDINATE EQUATIONS IN SLAB GEOMETRY*

EDWARD W. LARSEN† AND PAUL NELSON‡

Abstract. A unified framework is developed for calculating the order of the error for a class of finite-difference approximations to the monoenergetic linear transport equation in slab geometry. In particular, the global discretization errors for the step characteristic, diamond, and linear discontinuous methods are shown to be of order two, while those for the linear moments and linear characteristic methods are of order three, and that for the quadratic method is of order four. A superconvergence result is obtained for the three linear methods, in the sense that the cell-averaged flux approximations are shown to converge at one order higher than the global errors.

1. Introduction. Several recent articles ([1], [3], [4]–[8], [10], [12]–[14]) have considered various finite-difference approximations to the discrete-ordinate equations for linear particle transport in slab geometry. In particular, Larsen and Miller [6] and Lee and Vaidyanathan [7] have obtained estimates of the order of the discretization errors for several of these methods as applied to transport along a fixed direction without scattering. These authors raised the question of whether their error estimates remain valid for the monoenergetic discrete-ordinates approximation with scattering. The purpose of this paper is to answer this question affirmatively, at least under appropriate restrictions and for several difference approximations which are linear in nature. The specific difference methods we consider are the step characteristic, diamond, linear characteristic, linear discontinuous and quadratic methods, as defined in [6], and the linear moments method introduced by Vaidyanathan [12]–[14].

In § 2 we outline the problem and introduce procedures which we subsequently employ in § 3 to obtain global error estimates for the various methods. For most methods these estimates are one order higher than the error corresponding to the associated piecewise polynomial approximation to the source function; in § 4 we show, in agreement with the results in [6], that some of the methods gain yet an additional order of accuracy for the cell-average and some of the cell-edge fluxes. In combination these superconvergence results are strongly reminiscent of those encountered in the study of H^{-1} Galerkin methods [4], [11], even in terms of the actual orders involved. However, the results of these studies are not directly applicable to the present problem, and our error estimates are obtained in a substantially different manner.

2. Preliminaries. We write the transport equation for slab geometry in the form

$$\mu\frac{\partial\psi}{\partial x}+\sigma(x)\psi(x,\mu)=c(x)\sigma(x)\int_{-1}^{1}k(x,\mu',\mu)\psi(x,\mu')\,d\mu'+q(x,\mu). \tag{1}$$

Here $\psi=\psi(x,\mu)$ is the angular flux, and the given functions σ, c and q are nonnegative and piecewise *constant* in $x\in[0,a]$ for some $a>0$. The scattering kernel (k) is

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† Theoretical Division, University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544.

‡ Consultant from Department of Mathematics, Texas Tech University, Lubbock, Texas 79423. The work of this author was partially supported by the National Science Foundation under grant ENG77-06766.

nonnegative, piecewise constant in x, symmetric in μ and μ', and satisfies

$$\int_{-1}^{1} k(x, \mu', \mu)\, d\mu \equiv 1. \tag{2}$$

For convenience in the following we define

$$c_{\max} = \sup \{c(x): 0 \leqq x \leqq a\}.$$

We write the discrete-ordinates approximation to (1) as

$$\mu_i \frac{d\psi_i^e}{dx} + \sigma(x)\psi_i^e(x) = c(x)\sigma(x) \sum_{j=1}^{N} \omega_j k(x, \mu_j, \mu_i)\psi_j^e(x) + q_i(x). \tag{3}$$

Here $\{\mu_i: 1 \leqq i \leqq N\}$ are the quadrature points (we assume $\mu_i \neq 0$), with associated weights $\{\omega_i: 1 \leqq i \leqq N\}$; $\psi_i^e(x)$ is the discrete-ordinates approximation to $\psi(x, \mu_i)$, and $q_i(x) = q(x, \mu_i)$. For simplicity we shall consider vacuum boundary conditions,

$$\psi_i(0) = 0, \quad \mu_i > 0, \qquad \psi_i(a) = 0, \quad \mu_i < 0. \tag{4}$$

Our analysis extends readily to problems involving a specified angular flux incident upon the slab faces, and it probably can be extended to even more general boundary conditions with a bit of effort.

Let $\mathscr{C}^N([0, a])(\mathscr{C}_p^N([0, a]))$ denote the completion under the max-sup norm

$$\|(f_1, f_2, \cdots, f_N)\| = \max_{1 \leqq i \leqq N} \sup_{0 \leqq x \leqq a} |f_i(x)| \tag{5}$$

of the class of N-tuples of real-valued functions which are continuous (respectively, piecewise continuous) on $[0, a]$. We define operators S and L, respectively mapping $\mathscr{C}^N([0, a])$ into $\mathscr{C}_p^N([0, a])$ and vice versa, by

$$(Sf)_i(x) = c(x) \sum_{j=1}^{N} \omega_j k(x, \mu_j, \mu_i) f_j(x), \tag{6}$$

and

$$(Lf)_i(x) = \frac{1}{\mu_i} \int_{a_i}^{x} E_i(x, x')\sigma(x') f_i(x')\, dx', \tag{7}$$

where

$$E_i(x, x') \equiv \exp\left[-\frac{1}{\mu_i} \int_{x'}^{x} \sigma(s)\, ds\right] \tag{8}$$

and $a_i = 0$ or a, according respectively as μ_i is positive or negative.

The two-point boundary-value problem defined by (3) and (4) is equivalent to

$$\psi^e = L(S\psi^e + \bar{q}), \tag{9}$$

where $\bar{q}_i = q_i/\sigma$. We shall always suppose that the spectral radius of LS is less than unity, so that (9) is a well-posed problem in $\mathscr{C}^N([0, a])$. The following result leads to a sufficient condition for this "subcriticality" condition to hold.

THEOREM 1. *If the quadrature weights are nonnegative, and the quadrature rules are in a family such that an arbitrary continuous function on* $[-1, 1]$ *is integrated exactly in the limit of infinitely many quadrature points, then*

$$\|S\| \leqq c_{\max}(1 + \varepsilon), \tag{10}$$

where ε can be made arbitrarily small by taking the quadrature rule sufficiently fine. If the scattering kernel is anisotropic of order n (i.e., has a representation of the form

$$k(x, \mu, \mu') = \sum_{l=0}^{n} b_l(x) P_l(\mu') P_l(\mu), \tag{11}$$

where the P_l are Legendre polynomials), and the quadrature rule integrates exactly polynomials of degree n, then (10) *holds with $\varepsilon = 0$.*

Proof. Positivity of the scattering kernel and quadrature weights implies the estimate

$$\|S\| \leqq c_{\max} \cdot \max \left\{ \sum_{j=1}^{N} \omega_j k(x, \mu_j, \mu_i) \colon 0 \leqq x \leqq a, i = 1, \cdots, N \right\}. \tag{12}$$

Under the assumptions of the first statement of the theorem it follows from (2), the symmetry of k, and [9, Lemma 1], that the quantity in braces in (12) is not greater than $1+\varepsilon$, where ε can be made arbitrarily small by taking the quadrature rule sufficiently fine. The second assertion follows immediately from (2), which implies $b_0(x) = \frac{1}{2}$. □

COROLLARY. *There exists $\delta > 0$ such that if $c_{\max} < (1+\delta)(1+\varepsilon)^{-1}$, where ε is as in Theorem* 1, *then $\|LS\| < 1$ (and therefore the spectral radius of LS is less than unity).*

Proof. One easily shows $\|L\| < 1$, and then it suffices to take $\delta = \|L\|^{-1} - 1$. □

Remarks. Most commonly used quadrature rules satisfy the assumptions of the first part of Theorem 1. The second assertion applies to isotropic scattering (i.e., $n = 0$ in (11)) provided only that the quadrature rule integrates constants exactly. The second assertion holds only if the scattering kernel as given by (11) is nonnegative; the physical kernel itself must be nonnegative, but this nonnegativity need not be shared by the associated truncated expansion of the form (11), as often used in calculations.

The class of finite-difference approximations we wish to consider are equivalent to equations of the form

$$\psi = L_a M (S\psi + \bar{q}). \tag{13}$$

Here ψ is the finite-difference approximation to ψ^e, and L_a and M are termed respectively the *inverse of the finite-difference streaming operator* and the *source-distribution operator* for the particular method at hand. The latter are intended respectively as approximations to L and to the identity injection on $\mathscr{C}_p^N([0, a])$. For the specific methods considered here the operators M map $\mathscr{C}_p^N([0, a])$ into itself, and M always commutes with S.

If $(I - L_a MS)^{-1}$ exists and is bounded, then standard techniques lead from (9) and (13) to the discretization-error estimate

$$\|\psi - \psi^e\| \leqq \|(I - L_a MS)^{-1}\| \cdot \|(L_a M - L)(S\psi^e + \bar{q})\|. \tag{14}$$

Let h be the maximum cell width, and suppose one has established a stability inequality

$$\|(I - L_a MS)^{-1}\| \leqq A_s \quad \text{for } h \leqq h_0 \tag{15}$$

and a consistency estimate

$$\|(L_a M - L)(S\psi^e + \bar{q})\| \leqq A_t h^n \quad \text{for } h \leqq h_0, \tag{16}$$

where A_s and A_t are constants, with A_t possibly depending on ψ^e and q but not on the particular net (within some class of nets). We then obtain the desired error estimate

$$\|\psi - \psi^e\| \leqq A_s A_t h^n \quad \text{for } h \leqq h_0. \tag{17}$$

We now briefly discuss general considerations relative to stability and to estimation of truncation errors.

a. Stability. We shall usually establish stability by showing that over some suitable class of nets L_aM converges pointwise to L in the limit as $h \to 0$, and that $\{L_aM\}$ is collectively compact. It then follows from the general theory of collectively compact operators (e.g., Anselone [2]) that for all sufficiently small h the operators $\{(I - L_aMS)^{-1}$ exist and are uniformly bounded on $\mathscr{C}_p^N([0, a])$.

Henceforth we consider a spatial net with mesh-cell edges at the points $0 = x_{1/2} < x_{3/2} < \cdots < x_{J+1/2} = a$. The center of the cell $C_j = [x_{j-1/2}, x_{j+1/2}]$ is x_j, the width of C_j is h_j. and $h \equiv \max\{h_j: 1 \leqq j \leqq J\}$. By a *material interface* we mean a point $x \in [0, a]$ at which at least one of σ, k or q is discontinuous. We shall consider only nets such that all material interfaces are also cell edges.

The M-operators for the various methods we consider are defined by

$$(M_0 f)_i(x) = \frac{1}{h_j} \int_{C_j} f_i(s)\, ds, \qquad x \in C_j^0, \tag{18}$$

$$(M_{\mathrm{D}} f)_i(x) = \frac{f(x_{j+1/2^-}) + f(x_{j-1/2^+})}{2}, \qquad x \in C_j^0, \tag{19}$$

$$(M_{\mathrm{LM}} f)_i(x) = (M_0 f)_i(x) + \frac{12(x - x_j)}{h_j^3} \int_{C_j} f_i(x')(x' - x_j)\, dx', \qquad x \in C_j^0, \tag{20}$$

$$(M_{\mathrm{LC}} f)_i(x) = (M_0 f)_i(x) + \frac{f_i(x_{j+1/2^-}) - f_i(x_{j-1/2^+})}{h_j}(x - x_j), \qquad x \in C_j^0, \tag{21}$$

and

$$\begin{aligned}(M_{\mathrm{Q}} f)_i(x) &= (M_{\mathrm{LC}} f)_i(x) \\ &\quad + [f_i(x_{j+1/2^-}) + f_i(x_{j-1/2^+}) - 2(M_0 f)_i(x_j)]\left[\frac{3}{h_j^2}(x - x_j)^2 - \frac{1}{4}\right], \qquad x \in C_j^0,\end{aligned} \tag{22}$$

where $C_j^0 = (x_{j-1/2}, x_{j+1/2})$ denotes the interior of C_j. The above subscripts 0, D, LM, LC and Q refer respectively to the step characteristic, diamond difference, linear moments, linear characteristic, and quadratic methods, which are discussed in specific detail in § 3. In all cases the above operators $\{M\}$ constitute a uniformly bounded family on $\mathscr{C}_p^N([0, a])$ into itself. (The assumption that all material interfaces are also cell edges is crucial to this assertion.)

THEOREM 2. *If* $\{M\}$ *is any one of the families of operators defined by* (18)–(22), *then the family* $\{LM\}$ *is collectively compact and converges pointwise to* L *on* $\mathscr{C}_p^N([0, a])$ *as* $h \to 0$.

Proof. Let $\varepsilon > 0$ be given, and suppose $f \in \mathscr{C}_p^N([0, a])$ and $x \in [0, a]$ are both arbitrary but given. One can clearly find δ, independent of x, such that for any partition of $[0, a]$ having norm less than δ the contributions to each $|(Lf)_i(x)|$ and $|(LMf)_i(x)|$ from integrating over subintervals having discontinuities of f within their interior is less than $\varepsilon/3$. But by choosing δ smaller, if necessary, one has $|f_i(y) - (Mf)_i(y)| \leqq \varepsilon/3a$ for all i and any y in the interior of a subinterval on which f_i is continuous; therefore the contribution to $|(Lf)_i(x) - (LMf)_i(x)|$ from integrating over such subintervals is less than $\varepsilon/3$. This shows that $\{LM\}$ converges pointwise to L as $h \to 0$. The collective compactness follows from the readily proved fact that L maps any bounded set in

$\mathscr{C}_p^N[(0, a])$ into a bounded set having equicontinuous components. This completes the proof. □

Theorem 2 already gives the desired stability result for those methods in which $L_a = L$ (i.e., the step characteristic, linear moments and linear characteristic methods). In the cases that L_a is an approximation to L our approach will be to establish the corresponding analog of Theorem 2.

b. Truncation errors. By $\mathscr{P}_n\mathscr{C}_p^N([0, a])$ we denote the subset of $\mathscr{C}_p^N([0, a])$ whose components have derivatives of order n which are piecewise continuous with (simple) discontinuities only at material interfaces. Recall that a family of spatial nets is *quasi-uniform* if there exists a constant C such that $h \leqq C \cdot \min\{h_j: 1 \leqq j \leqq J\}$ for all nets in the family. In the remainder of this paper all families of nets considered are always assumed to be quasi-uniform.

THEOREM 3. *The following estimates hold as $h \to 0$ through any family of nets having the property that each net has all material interfaces as cell edges*:

(i) $\|L(I - M_0)f\| = O(h^2)$ *for* $f \in \mathscr{P}_1\mathscr{C}_p^N([0, a])$;

(ii) $\|L(I - M_{\rm D})f\| = O(h^2)$ *for* $f \in \mathscr{P}_2\mathscr{C}_p^N([0, a])$;

(iii) $\|L(I - M_{\rm LM})f\| = O(h^3)$ *for* $f \in \mathscr{P}_2\mathscr{C}_p^N([0, a])$;

(iv) $\|L(I - M_{\rm LC})f\| = O(h^3)$ *for* $f \in \mathscr{P}_3\mathscr{C}_p^N([0, a])$;

(v) $\|{\rm L}({\rm I} - {\rm M}_{\rm Q})f\| = O(h^4)$ *for* $f \in \mathscr{P}_3\mathscr{C}_p^N([0, a])$.

Proof. We give a detailed proof only for $L(I - M_{\rm LC})$. The arguments in the other cases are similar, and all are based on the calculations of Larsen and Miller [6]. In particular we find it convenient to use the first few (unnormalized) cell-based Legendre polynomials p_{jl} introduced by these investigators, namely,

$$p_{j0}(x) = 1, \tag{23a}$$

$$p_{j1}(x) = \frac{2}{h_j}(x - x_j), \tag{23b}$$

and

$$p_{j2}(x) = \frac{6}{h_j^2}(x - x_j)^2 - \frac{1}{2}. \tag{23c}$$

If $f \in \mathscr{P}_3\mathscr{C}_p^N([0, a])$, then we can write

$$f_i(x) = \sum_{n=0}^{2} f_i^{(n)}(x_j)\frac{(x - x_j)^n}{n!} + f_i^{(3)}(\beta_j(x))\frac{(x - x_j)^3}{6}, \qquad x \in C_j^0, \tag{24}$$

where $\beta_j(x) \in C_j^0$. But we can also write

$$f_i(x) = \sum_{n=0}^{2} S_{jn}p_{jn}(x) + R_{j3}(x), \tag{25}$$

where

$$S_{jl} = \frac{2l+1}{h_j}\int_{C_j} f_i(x)p_{jl}(x)\,dx = \frac{2l+1}{h_j}\int_{C_j}\left[\sum_{n=0}^{2} f_i^{(n)}(x_j)\frac{(x - x_j)^n}{n!}\right]p_{jl}(x)\,dx + O(h_j^3), \tag{26}$$

and (24) is used in obtaining the last equality in (26). But it follows from (24)–(26) along with $p_{jl}(x) = O(1)$ that

$$R_{j3}(x) = O(h_j^3). \tag{27}$$

Note that $S_{j0}p_{j0}(x) = (M_0 f)_i(x)$. From (24) and (26) we obtain

$$S_{j1} = \frac{f_i'(x_j)}{2} h_j + O(h_j^3) = \frac{1}{2}[f_i(x_{j+1/2^-}) - f_i(x_{j-1/2^+})] + O(h_j^3), \tag{28}$$

$$S_{j2} = \frac{f_i^{(2)}(x_j)}{12} h_j^2 + O(h_j^3) = O(h_j^2). \tag{29}$$

From (25) and (27)–(29) we compute

$$\begin{aligned}
&\int_{C_j} E_i(x, x')[f_i(x') - (M_{\mathrm{LC}} f)_i(x')]dx' \\
&\qquad = \int_{C_j} [E_i(x, x') - 1] \cdot [f_i(x') - (M_{\mathrm{LC}} f)_i(x')]dx' \\
&\qquad = \int_{C_j} [E_i(x, x') - 1][S_{j2}p_{j2}(x') + R_{j3}(x')]\, dx' + O(h_j^5) \\
&\qquad = \int_{C_j} \frac{\partial^2 E_i}{\partial (x')^2}(x, \alpha(x)) \cdot \frac{(x' - x_j)^2}{2} S_{j2}p_{j2}(x')\, dx' + O(h_j^5) \\
&\qquad = O(h_j^5).
\end{aligned}$$

Therefore we have, for $\mu_i > 0$ and $x \in C_j^0$,

$$\begin{aligned}
[L(I - M_{\mathrm{LC}})f](x) &= \int_{x_{j-1/2}}^{x} E_i(x, x')[f_i(x') - (M_{\mathrm{LC}} f)_i(x')]\, dx' \\
&\quad + \sum_{j'=1}^{j-1} \int_{C_{j'}} E_i(x, x')[f_i(x') - (M_{\mathrm{LC}} f)_i(x')]\, dx' \\
&= O(h_j^3) + \sum_{j'=1}^{j-1} O(h_{j'}^5) = O(h^3),
\end{aligned}$$

where the quasi-uniformity has been used in obtaining the last equality. A similar estimate for $\mu_i < 0$ completes the proof of Theorem 3. □

Remarks. Because the data in (3) are piecewise constant, it follows that $S\psi^e + \bar{q}$ is in $\mathcal{P}_n\mathcal{C}_p^N([0, a])$ for arbitrary $n \geqq 0$. Therefore, Theorem 3 gives immediately the desired truncation error for those methods such that $L_a = L$; the truncation errors for the other methods will be estimated using these results along with appropriate estimates for $\|(L_a - L)M(S\psi^e + \bar{q})\|$.

3. Global error estimates. In this section we define the various methods considered, and obtain the appropriate estimates of the global discretization errors. The results are summarized in a concluding Theorem 4 (see § 3g).

a. Step characteristic. This method is defined by the integrodifferential system

$$\mu_i \frac{d\psi_i}{dx} + \sigma\psi_i = M_0(\sigma S\psi + q)_i, \qquad i = 1, \cdots, N. \tag{30}$$

Under the vacuum boundary conditions (4), this is equivalent to (13) (i.e., $\psi = L_a M(S\psi + \bar{q})$) with $L_a = L$ and $M = M_0$. It then follows from the results of the preceding section that the corresponding global discretization errors satisfy $\|\psi - \psi^e\| = O(h^2)$ as $h \to 0$.

b. Diamond difference. This finite-difference method is defined by

$$(31)\quad \mu_i \frac{\psi_i(x) - \psi_i(x_{j-1/2})}{x - x_{j-1/2}} + \sigma(x)(M_D\psi)_i(x) = \sigma(x)[M_D(S\psi + q)]_i(x), \qquad x \in C_j, \quad \mu_i > 0,$$

along with a corresponding relation (with $x_{j+1/2}$ replacing $x_{j-1/2}$) for $\mu_i < 0$. Under the vacuum boundary conditions (4), these approximations are equivalent to a system of the form $\psi = L_D M_D(s\psi + \bar{q})$, with L_D defined by

$$(32)\quad \begin{aligned} (L_D f)_1(x) = {} & \left[1 - \frac{2\sigma(x_j)(x - x_{j-1/2})}{(2+\varepsilon_{ji})\mu_i}\right] \sum_{j'=1}^{j-1} \left[\prod_{j=j'+1}^{j-1} \frac{2 - \varepsilon_{mi}}{2 + \varepsilon_{mi}}\right] \frac{2\varepsilon_{j'i}}{(2+\varepsilon_{j'i})h_{j'}} \int_{C_{j'}} f_i(s)\,ds \\ & + \frac{2\sigma(x_j)(x - x_{j-1/2})}{(2+\varepsilon_{ji})\mu_i h_j} \int_{C_j} f_i(s)\,ds, \qquad x \in C_j, \quad \mu_i > 0. \end{aligned}$$

along with an analogous defining relation for $\mu_i < 0$. Here we have introduced the notation $\varepsilon_{mi} = \sigma(x_m)h_m/\mu_i$. Upon using $e^{-y} = (2-y)/(2+y) + O(y^3)$ we find, using (7), that for arbitrary $f \in \mathscr{C}_p^N([0, a])$,

$$\|(L - L_D)M_D f\| = O(h^2)$$

as $h \to 0$. Upon using part (ii) of Theorem 3, we then conclude that the truncation error for the diamond difference method is $O(h^2)$ as $h \to 0$ through any quasi-uniform family of nets. Furthermore, it is readily seen that the components of $L_D f$ have derivatives which are bounded uniformly over any family of nets and any uniformly bounded family of elements $f \in \mathscr{C}_p^N([0, a])$. It follows that the family $\{L_D M_D\}$ corresponding to an arbitrary family of nets is collectively compact. The results outlined in § 2 then imply that the global discretization error for the diamond difference method satisfies $\|\psi - \psi^e\| = O(h^2)$ as $h \to 0$.

Remark. In the diamond difference method, the angular fluxes are defined in cell interiors by linear interpolation between the cell-edge values, as shown by (31). However, only the cell-edge and cell-average approximations usually are computed, and there are other interpolations to the cell interiors which lead to the same values of these quantities. An example of such an alternative interior interpolation would be obtained by replacing the second term on the left-hand side of (31) by $\sigma(x)[\psi_i(x) + \psi_i(x_{j-1/2})]/2$; this method also has $O(h^2)$ global discretization error.

c. Linear moments. This method was proposed by Vaidyanathan [12]–[14]. It is defined by the integrodifferential system

$$(33)\qquad \mu_i \frac{d\psi_i}{dx} + \sigma\psi_i = M_{LM}(\sigma S\psi + q)_i, \qquad i = 1, \cdots, N,$$

where M_{LM} is defined by (20). This method falls into our standard form (13), with $M = M_{LM}$ and $L_a = L$. It follows from the results of the preceding section, especially Theorem 2 and part iii of Theorem 3, that the linear moments method has a global discretization error which is $O(h^3)$ as $h \to 0$.

d. Linear characteristic. This method is defined by the integrodifferential system

$$\mu_i \frac{d\psi_i}{dx} + \sigma\psi_i = M_{\mathrm{LC}}(\sigma S\psi + q)_i, \qquad i = 1, \cdots, N, \tag{34}$$

where M_{LM} is defined by (21). Precisely as in the preceding linear moments method (except that now we use part (iv) of Theorem 3, rather than part (iii)), we conclude that this method has $O(h^3)$ global discretization error as $h \to 0$.

e. Linear discontinuous. The analysis of this method requires a bit of effort, primarily because the approximations ψ are allowed to be discontinuous at the cell edges. The equations defining this approximation are

$$\psi_i(x) = \frac{1}{h_j}[(x_{j+1/2} - x)\psi_i(x_{j-1/2^+}) + (x - x_{j-1/2})\psi_i(x_{j+1/2^-})], \qquad x \in C_j^0, \tag{35}$$

$$\begin{aligned} &\frac{\mu_i}{h_j}[\psi_i(x_{j-1/2^+}) - \psi_i(x_{j-1/2^+})] + \sigma(x_j)\frac{\psi_i(x_{j+1/2^-}) + \psi_i(x_{j-1/2^+})}{2} \\ &= \sigma(x_j)\left[\frac{(S\psi)_i(x_{j+1/2^-}) + (S\psi_i)(x_{j-1/2^+})}{2}\right] + \frac{q_i(x_{j+1/2^-}) + q_i(x_{j-1/2^+})}{2}, \qquad \mu_i > 0 \end{aligned} \tag{36}$$

and

$$\begin{aligned} &\frac{\mu_i}{h_j}[\psi_i(x_{j+1/2^-}) - \psi_i(x_{j-1/2^+})] + \sigma(x_j)\frac{2\psi_i(x_{j+1/2^-}) + \psi_i(x_{j-1/2^+})}{3} \\ &= \sigma(x_j)\left[\frac{2(S\psi)_i(x_{j+1/2^-}) + (S\psi)_i(x_{j-1/2^+})}{3}\right] + \frac{2q_i(x_{j+1/2^-}) + q_i(x_{j-1/2^+})}{3}, \\ &\qquad \mu_i > 0. \end{aligned} \tag{37}$$

The corresponding equations for negative μ_i are obtained by changing (36) and (37) so as to interchange $x_{j+1/2^-}$ with $x_{j-1/2^+}$ and to replace $x_{j-1/2^-}$ by $x_{j+1/2^+}$. See [1] for details of the derivation of this method.

The linear discontinuous method can be written in the form $\psi = L_a M(S\mu + \bar{q})$, with $M = M_{\mathrm{LM}}$ as defined by (20) and $L_a = L_{\mathrm{LD}}$ defined by

$$\begin{aligned} (L_{\mathrm{LD}}f)_i(x) = &\left[\frac{6 + \varepsilon_{ji} - 6\sigma(x_j)(x - x_j)/\mu_i}{6 + 4\varepsilon_{ji} + \varepsilon_{ji}^2}\right] \cdot \sum_{j'=1}^{j-1} \left[\prod_{m=j'+1}^{j-1} \frac{6 - 2\varepsilon_{mi}}{6 + 4\varepsilon_{mi} + \varepsilon_{mi}^2}\right] \\ &\cdot \frac{\varepsilon_{j'i}}{6 + 4\varepsilon_{j'i} + \varepsilon_{j'i}^2}[(6 + \varepsilon_{j'i})(M_0 f)_i(x_{j'}) + \varepsilon_{j'i}(R_1 f)_i(x_j)] \\ &+ \frac{\varepsilon_{ji}}{6 + 4\varepsilon_{ji} + \varepsilon_{ji}^2}\left\{\left[3 + \varepsilon_{ji} + 6\frac{(x - x_j)}{h_j}\right](M_0 f)_i(x_j)\right. \\ &\left. + \left[-1 + 2(1 + \varepsilon_{ji})\frac{(x - x_j)}{h_j}\right](R_1 f)_i(x_j)\right\}, \qquad x \in C_j^0, \quad \mu_i > 0, \end{aligned} \tag{38}$$

along with a similar expression for $\mu_i < 0$. Here R_1 is defined by

$$(R_1 f)_i(x_j) = \frac{6}{h_j^2}\int_{C_j}(x - x_j)f_i(x)\,dx. \tag{39}$$

In order both to establish stability and to estimate the truncation error for this method, we need to estimate $\|(L_{\mathrm{LD}} - L)M_{\mathrm{LM}}\|$. If one notes that $M_{\mathrm{LM}}f(x) =$

$M_0f(x_j)+[2(x-x_j)/h_j]R_1f(x_j)$ for $x\in C_j^0$, then it is fairly easy to show that

$$(LM_{\mathrm{LM}}f)_i(x)=\exp\left[-\sigma(x_j)\frac{(x-x_{j-1/2})}{\mu_i}\right]\cdot\sum_{j'=1}^{j-1}\left[\prod_{m=j'+1}^{j-1}e^{-\varepsilon_{mi}}\right]$$
$$\cdot\left\{(1-e^{-\varepsilon_{j'i}})(M_0f)_i(x_{j'})+\left[1+e^{-\varepsilon_{j'i}}-\frac{2}{\varepsilon_{j'i}}(1-e^{-\varepsilon_{j'i}}\right](R_1f)_i(x_{j'})\right\}$$
$$+\left\{1-\exp\left[-\sigma(x_j)\frac{(x-x_{j-1/2})}{\mu_i}\right]\right\}(M_0f)_i(x_j) \tag{40}$$
$$+\left\{2\left(\frac{x-x_j}{h_j}\right)-\frac{2}{\varepsilon_{ji}}+\left(1+\frac{2}{\varepsilon_{ji}}\right)\exp\left[-\sigma(x_j)\frac{(x-x_{j-1/2})}{\mu_i}\right]\right\}(R_1f)_i(x_j)$$
$$\text{for } x\in C_j^0,\quad \mu_i>0.$$

Of course, a similar expression holds for $\mu_i<0$. Now note that

$$\frac{6+\varepsilon_{ji}-6\sigma(x_j)(x-x_j)/\mu_i}{6+4\varepsilon_{ji}+\varepsilon_{ji}^2}=\frac{6+4\varepsilon_{ji}-6\sigma(x_j)(x-x_{j-1/2})/\mu_i}{6+4\varepsilon_{ji}+\varepsilon_{ji}^2}$$
$$=\exp\left[-\sigma(x_j)\frac{(x-x_{j-1/2})}{\mu_i}\right]+O(h^2),\qquad x\in C_j. \tag{41}$$

Similarly, one establishes

$$\frac{6-2\varepsilon_{mi}}{6+4\varepsilon_{mi}+\varepsilon_{mi}^2}=e^{-\varepsilon_{mi}}+O(h^4), \tag{42}$$

$$\frac{\varepsilon_{j'i}(6+\varepsilon_{j'i})}{6+4\varepsilon_{j'i}+\varepsilon_{j'i}^2}=1-e^{-\varepsilon_{j'i}}+O(h^4), \tag{43}$$

$$\frac{\varepsilon_{j'i}^2}{6+4\varepsilon_{j'i}+\varepsilon_{j'i}^2}=1+e^{-\varepsilon_{j'i}}-\frac{2}{\varepsilon_{j'i}}(1-e^{-\varepsilon_{j'i}})+O(h^3), \tag{44}$$

$$\frac{\varepsilon_{ji}[3+\varepsilon_{ji}+6(x-x_j)/h_j]}{6+4\varepsilon_{ji}+\varepsilon_{ji}^2}=1-\exp\left[-\frac{\sigma(x_j)(x-x_{j-1/2})}{\mu_i}\right]+O(h^2),\qquad x\in C_j, \tag{45}$$

and

$$\frac{\varepsilon_{ji}[-1+2(1+\varepsilon_{ji})(x-x_j)/h_j]}{6+4\varepsilon_{ji}+\varepsilon_{ji}^2}$$
$$=\frac{2}{h_j}(x-x_j)-\frac{2}{\varepsilon_{ji}}+\left(1+\frac{2}{\varepsilon_{ji}}\right)\cdot\exp\left[-\sigma(x_j)\frac{(x-x_{j-1/2})}{\mu_i}\right]+O(h),\qquad x\in C_j. \tag{46}$$

The last equality of (41), and (42), (45), and (46) can be established by comparing the first few terms of the respective Taylor expansions about $\varepsilon_{ji}=0$. The estimates (43) and (44) are most easily established by using (42).

Now if we use (41)–(46) to compare (38) and (40) termwise, and note that $|(M_0f)_i(x_j)|\leqq\|f\|$, $|(R_1f)_i(x_j)|\leqq 3\|f\|$, we conclude that $\|(L_{\mathrm{LD}}-L)M_{\mathrm{LM}}\|=O(h)$ as $h\to 0$ over any quasi-uniform family of nets. As we have already shown that $(I-LM_{\mathrm{LM}})^{-1}$ is uniformly bounded as $h\to 0$ over any quasi-uniform family of nets (Theorem 2), relatively straightforward estimates imply that the same conclusion holds for $(I-L_{\mathrm{LD}}M_{\mathrm{LM}})^{-1}$. Now if $f\in\mathcal{P}_1\mathcal{C}_p^N([0,a])$, then $R_1f(x_j)=O(h)$ as $h\to 0$ over any quasi-uniform family of nets. If this is used in conjunction with (46), then we have $\|(L_{\mathrm{LD}}-L)M_{\mathrm{LM}}f\|=O(h^2)$ for $f\in\mathcal{P}_1\mathcal{C}_p^N([0,a])$. If this estimate is used in conjunction with part (iii) of Theorem 3, we conclude that the linear discontinuous method has $O(h^2)$ truncation error. From the previously established stability we finally reach the conclusion that the linear discontinuous method has global discretization error which is $O(h^2)$ as $h \to 0$ through any quasi-uniform family of nets.

f. Quadratic. This method is due to Gopinath et al. [5]. It is defined by the integro-differential system

$$\mu_i \frac{d\mu_i}{dx} + \sigma\psi_i = M_Q(\sigma S\psi + q)_i, \qquad i = 1, \cdots, N, \tag{47}$$

where M_Q is defined by (22). This method can be put in the form (13), with $M = M_Q$ and $L_a = L$. It follows from the results of § 2, particularly Theorem 2 and part (v) of Theorem 3, that the quadratic method has a global discretization error which is $O(h^4)$ as $h \to 0$.

g. Summary. We summarize the results obtained in this section in the following theorem:

THEOREM 4. *The finite-difference approximations to the discrete-ordinates equations which were defined above have the following orders of global discretization error as $h \to 0$ through any quasi-uniform family of nets*:

(i) $O(h^2)$ *for the step characteristic, diamond and linear discontinuous methods*;
(ii) $O(h^3)$ *for the linear moments and linear characteristic methods*;
(iii) $O(h^4)$ *for the quadratic method.*

As noted in the introduction, these orders of accuracy, save for the linear discontinuous method, are one order higher than the best possible accuracy for the associated source distribution. For example, the source distribution $M_{LC}(S\psi + \bar{q})$ for the linear characteristic method is piecewise linear, and therefore can be expected to approximate the true source only to within an error of $O(h^2)$, but the approximation to the angular flux produced by this method is globally accurate to within $O(h^3)$. Even so, the accuracies described in Theorem 4 for the "linear" methods are one order *lower* then those found by Larsen and Miller [6] in the nonscattering case for the cell-average and cell-edge fluxes. The basis of this apparent discrepancy lies in the fact that these methods actually do produce an additional order of accuracy in some quantities, particularly including cell-average fluxes and some cell-edge fluxes. We devote the next section to the study of this "superconvergence" phenomenon.

4. Some higher-order convergence results. Our objective here is to show that, for the "linear" methods defined in the preceding section, the cell-averaged fluxes $M_0\psi$ converge to $M_0\psi^e$ one order of h faster than the corresponding global rate of convergence of ψ to ψ^e as described in Theorem 4. As a corollary of this result, we also have the same degree of faster convergence for those cell-edge fluxes which are determined by the conservation (or "balance") equation and the cell-averaged fluxes $M_0\psi$.

As a starting point we write the source-distribution operator for each of the linear methods in the form

$$M = M_0 + M_1, \tag{48}$$

where M_0 is given by (18) and M_1 is peculiar to the particular method under consideration. From (9) and (13) it follows that the cell-averaged discretization error $M_0(\psi - \psi^e)$ satisfies

$$M_0(\psi - \psi^e) = M_0L_aSM_0(\psi - \psi^e) + M_0L_aM_1S(\psi - \psi^e) + M_0(L_aM - L)(S\psi^e + \bar{q}). \tag{49}$$

If we can establish the *cell-averaged stability estimates*

$$\|(I-M_0L_aS)^{-1}\|\leqq A_s \quad \text{for } h\leqq h_0 \tag{50}$$

for some suitable constant A_s, and the consistency estimates

$$\|M_0L_aM_1S(\psi-\psi^e)\|=O(h^n) \quad \text{as } h\to 0, \tag{51a}$$

$$\|M_0(L_aM-L)(S\psi^e+q)\|=O(h^n) \quad \text{as } h\to 0, \tag{51b}$$

it will then follow from (49) that the cell-average discretization error $\|M_0(\psi-\psi^e)\|$ is $O(h^n)$ as $h\to 0$.

THEOREM 5. *If the exact discrete-ordinates problem* (9) *is subcritical, then the operators* $(I-M_0LS)^{-1}$ *exist for all sufficiently small h and are uniformly bounded as* $h\to 0$.

Proof. From the subcriticality and Theorem 2 (with $M=M_0$) we conclude that the Neumann series $\sum_n(LSM_0)^n$ converges and is uniformly bounded for sufficiently small h. Therefore $M_0\sum_n(LSM_0)^nLS=\sum_n(M_0LS)^{n+1}$ likewise converges and is uniformly bounded for sufficiently small h. But it is readily shown that

$$(I-M_0LS)^{-1}=I+\sum_{n=0}^{\infty}(M_0LS)^{n+1}$$

whenever the series on the right converges. This completes the proof. □

Theorem 5 already established the desired cell-averaged stability for those methods for which $L_a=L$ (i.e., the linear moments and linear characteristic methods). We now consider the remaining aspects of proving (50) and (51), along with the consequences of these results, in separate subsections for each of the three linear methods. The ultimate results are summarized in Theorem 6 (§ 4d).

a. Linear moments. For this method we have

$$(M_1f)_i(x)=\frac{12(x-x_j)}{h_j^3}\int_{C_j} f_i(x')(x'-x_j)\,dx', \qquad x\in C_j^0. \tag{52}$$

We wish to establish (51a) and (51b) with $n=4$, for $L_a=L$ and M_1 given by (52), where $h\to 0$ through any quasi-uniform family of nets.

In order to prove (51a), first note that our previous results as summarized in Theorem 4 give $\|\psi-\psi^e\|=O(h^3)$. Therefore it suffices to prove $\|M_0LM_1\|=O(h)$. But for $x\in C_j$ and $\mu_i>0$ (similar results hold for $\mu_i<0$) we have

$$\begin{aligned}(LM_1)_if(x)&=\frac{1}{\mu_i}\int_{x_{j-1/2}}^{x}E_i(x,x')(M_1f)_i(x')\,dx'\\&\quad+\frac{1}{\mu_i}\sum_{j'=1}^{j-1}E_i(x,x_{j'+1/2})\int_{C_{j'}}E_i(x_{j'+1/2},x')(M_1f)_i(x')\,dx'.\end{aligned} \tag{53}$$

The first term on the right here clearly is $O(h)\cdot\|f\|$. If we expand $E_i(x_{j'+1/2},x')$ in a first-order Taylor polynomial about $x'=x_{j'}$, with second-order differential remainder, the second term can be written as

$$\sum_{j'=1}^{j-1}\left\{\left[\frac{E_i(x,x_{j'+1/2})}{\mu_i}\right]\cdot\left[\frac{E_i(x_{j'+1/2},x_{j'})}{\mu_i}\right]\sigma(x_{j'})\int_{C_{j'}}f_i(x')(x'-x_{j'})\,dx'+O(h_{j'}^3)\cdot\|f\|\right\}$$
$$=O(h)\cdot\|f\|,$$

where we have used quasi-uniformity in obtaining the last equality. Therefore we have $\|LM_1\|=O(h)$, and as $\|M_0\|=1$ we have the desired estimate $\|M_0LM_1\|=O(h)$.

For convenience in proving (51b) we denote $S\psi^e+\bar{q}$ by g. Then for $x\in C_j$ we compute

$$(M-I)g(x)=\frac{1}{h_j}\int_{C_j}g(x')\,dx+\frac{12}{h_j^3}(x-x_j)\int_{C_j}(x'-x_j)g(x')\,dx'-g(x)$$
$$=g''(x_j)\left[\frac{h_j^2}{24}-\frac{(x-x_j)^2}{2}\right]+g'''(x_j)\left[\frac{(x-x_j)}{40}h_j^2-\frac{(x-x_j)^3}{6}\right]+O(h_j^4),$$

where we obtain the last equality from a Taylor expansion of g about $x=x_j$. Now if we substitute this into

$$[L(M-I)g]_i(x)=\frac{1}{\mu_i}\left\{\int_{x_{j-1/2}}^{x}E_i(x,x')[(M-I)g]_i(x')\,dx'\right.$$
$$\left.+\sum_{j'=1}^{j-1}E_i(x,x_{j'+1/2})\int_{C_{j'}}E_i(x_{j'+1/2},x')[(M-I)g]_i(x')\,dx'\right\},\qquad x\in C_j,$$

which is valid for $\mu_i>0$, and use

$$E_i(x,x')=E_i(x,x_j)\left[1+\frac{\sigma(x_j)}{\mu_i}(x-x_j)\right]+O(h_j^2),\qquad x'\in C_j,$$

along with quasi-uniformity, there results

$$[L(M-I)g]_i(x)=\frac{1}{\mu_i}\left[\frac{h_j^2}{24}(x-x_j)-\frac{(x-x_j)^3}{6}\right]+O(h^4).$$

But M_0 acting on the first term on the right is identically zero. Similar manipulations for $\mu_i<0$ yield the desired estimate

$$\|M_0L(M-I)g\|=O(h^4),\qquad h\to 0.$$

It follows from the above that the cell-averaged discretization error $\|M_0(\psi-\psi^e)\|$ associated with the linear moments method is $O(h^4)$ as $h\to 0$ through any quasi-uniform family of nets. If we now integrate each of (3) and (33) over the cell C_j and take the difference of the resulting equations, we obtain

$$\psi_i(x_{j+1/2})-\psi_i^e(x_{j+1/2})-[\psi_i(x_{j-1/2})-\psi_i^e(x_{j-1/2})]+\frac{\sigma(x_j)h_j}{\mu_i}[M_0(\psi-\psi^e)]_i(x_j)$$
$$=\frac{\sigma(x_j)h_j}{\mu_i}[SM_0(\psi-\psi^e)_i(x_j)],\qquad i=1,\cdots,N,\quad j=1,\cdots,J. \tag{54}$$

As either $\psi_i(x_{1/2})$ or $\psi_i(x_{J+1/2})$ is known to be exactly zero, according respectively as $\mu_i>0$ or $\mu_i<0$, it follows that maximum error in the cell-edge angular fluxes

$$\max\{|\psi_i(x_{j+1/2})-\psi_i^e(x_{j+1/2})|: i=1,\cdots,N, j=1,\cdots,J\}$$

is also $O(h^4)$ as $h\to 0$ through a quasi-uniform family of nets. Thus the cell-edge approximations inherit the fourth-order accuracy of the cell-average approximations as a consequence of the fact that the finite-difference approximation satisfies the same law of conservation of particles as does the exact discrete-ordinates solution.

b. Linear characteristic. For the linear characteristic method we have

$$(55)\qquad (M_1f)_i(x)=\frac{f_i(x_{j+1/2^-})-f_i(x_{j-1/2^+})}{h_j}(x-x_j),\qquad x\in C_j^0.$$

The representation (53) is again valid with M_1 as given by (55), and again the first term on the right is readily seen to be $O(h)\|f\|$. If we proceed as in the linear moments case, the corresponding second term on the right-hand side of (53) is

$$\sum_{j'-1}^{j-1}\left\{\left[\frac{E_i(x,x_{j'+1/2})}{\mu_i}\right]\cdot\left[\frac{E_i(x_{j'+1/2},x_{j'})}{\mu_i}\right]\right.$$
$$\left.\cdot\frac{h_{j'}^2}{12}\left[f_i(x_{j+1/2^-})-f_i(x_{j'-1/2})\right]+O(h_{j'}^3)\cdot\|f\|\right\}=O(h)\cdot\|f\|.$$

Thus we again have $\|M_0LM_1\|=O(h)$ which gives (51a) with $L_a=L$, M_1 given by (55) and $n=4$.

If we again let $g=S\psi^e+\bar{q}$, and proceed essentially as in the linear moments case, we find

$$(M-I)g(x)=g''(x_j)\left[\frac{h_j^2}{24}-\frac{(x-x_j)^2}{2}\right]+g'''(x_j)\left[\frac{(x-x_j)}{24}h_j^2-\frac{(x-x_j)^3}{6}\right]+O(h_j^4).$$

If we now compute $L(M-I)g$, as in the linear moments case, we ultimately conclude that $\|M_0L(M-I)g\|=O(h^4)$. We omit the details. □

It follows from the preceding that the cell-averaged discretization error associated with the linear characteristic method is $O(h^4)$ as $h\to 0$ through any quasi-uniform family of nets. Again equation (54) is valid for the linear characteristic method, and therefore the approximations to the cell-edge fluxes produced by this method also have an error which is $O(h^4)$.

c. Linear discontinuous. For this method M_1 is given by (52) and $L_a=L_{\mathrm{LD}}$ is defined by (38). Here $L_a\neq L$; therefore, Theorem 5 is inapplicable and our first task is to establish the cell-averaged stability inequality (50). Now $(L_{\mathrm{LD}}M_0f)_i(x)$ and $(LM_0f)_i(x)$ are given respectively by (38) and (40), except with R_1f set identically equal to zero. If we use (41)–(46) to compare these expressions termwise, as in § 3d, we conclude that $\|(L_{\mathrm{LD}}-L)M_0\|=O(h)$ as $h\to 0$ over a quasi-uniform family of nets. As we have already shown that $(I-LM_0S)^{-1}$ is uniformly bounded as $h\to 0$ over such a family of nets (Theorem 2), it follows that the same conclusion holds for $(I-L_{\mathrm{LD}}M_0S)^{-1}$. By applying the type of argument used in the proof of Theorem 5, along with commutivity of M_0 and S, we conclude that $(I-M_0L_{\mathrm{LD}}S)^{-1}$ is uniformly bounded as $h\to 0$ over any quasi-uniform family of nets. This is precisely the desired stability inequality (50) for the method at hand.

We now wish to show that the truncation error estimates (51) hold with $n=3$. For (51a) we know from Theorem 3 that $\|\psi-\psi^e\|=O(h^2)$. As $\|M_0\|=1$ and $\|S\|=c_{\max}$, it suffices to show that $\|L_{\mathrm{LD}}M_1\|=O(h)$. Now $(L_{\mathrm{LD}}M_1f)_i(x)$ is given by (38), except with M_0f set to zero. The desired estimate for $L_{\mathrm{LD}}M_1$ follows immediately from this representation.

In order to establish (51b), let us again write $g=S\psi^e+\bar{q}$. Note that the estimate $\|M_0L(M_{\mathrm{LM}}-I)g\|=O(h^4)$ was obtained in § 4a. Therefore we need only show that $\|M_0(L_{\mathrm{LD}}-L)M_{\mathrm{LD}}g\|=O(h^3)$. Now for $\mu_i>0$, $L_{\mathrm{LD}}M_{\mathrm{LD}}g$ and $LM_{\mathrm{LD}}g$ are given respectively by (38) and (40), with f replaced by g in each case. If we operate on these by

M_0, the results are

$$(M_0L_{\mathrm{LD}}M_{\mathrm{LM}}g)_i(x)=\left[\frac{6+\varepsilon_{ji}}{6+4\varepsilon_{ji}+\varepsilon_{ji}^2}\right]\sum_{j'=1}^{j-1}\left[\prod_{m=j'+1}^{j-1}\frac{6-2\varepsilon_{mi}}{6+4\varepsilon_{mi}+\varepsilon_{mi}^2}\right]$$
$$\cdot\frac{\varepsilon_{j'i}}{6+4\varepsilon_{j'i}+\varepsilon_{j'i}^2}[(6+\varepsilon_{j'i})(M_0g)_i(x_{j'})+\varepsilon_{j'i}(R_1g)_i(x_j)]$$
$$+\frac{\varepsilon_{ji}}{6+4\varepsilon_{ji}+\varepsilon_{ji}^2}[(3+\varepsilon_{ji})(M_0g)_i(x_j)-(1+\varepsilon_{ji})(R_1f)_i(x_j)],\qquad x\in C_j^0,$$

and

$$(M_0LM_{\mathrm{LM}}g)_i(x)=\frac{1}{\varepsilon_{ji}}(1-e^{-\varepsilon_{ji}})\cdot\sum_{j'=1}^{j-1}\left[\prod_{m=j'-1}^{j-1}e^{-\varepsilon_{mi}}\right]$$
$$\cdot\left\{(1-e^{-\varepsilon_{j'i}})(M_0g)_i(x_j)+\left[\frac{2}{\varepsilon_{j'i}}-1-\left(\frac{2}{\varepsilon_{j'i}}+1\right)e^{-\varepsilon_{j'i}}\right](R_1g)_i(x_j)\right\}$$
$$+\left\{1-\frac{1}{\varepsilon_{ji}}(1-e^{-\varepsilon_{ji}})\right\}(M_0g)_i(x_j)$$
$$+\left\{\left(-1+\frac{3}{\varepsilon_{ji}}-\frac{4}{\varepsilon_{ji}^2}\right)+\left(\frac{1}{\varepsilon_{ji}}+\frac{4}{\varepsilon_{ji}^2}\right)e^{-\varepsilon_{ji}}\right\}(R_1g)_i(x_j).$$

Now note that (42) gives the estimate

$$-1+\frac{3}{\varepsilon_{ji}}-\frac{4}{\varepsilon_{ji}^2}+\left(\frac{1}{\varepsilon_{ji}}+\frac{4}{\varepsilon_{ji}^2}\right)e^{-\varepsilon_{ji}}=-\frac{\varepsilon_{ji}(1+\varepsilon_{ji})}{6+4\varepsilon_{ji}+\varepsilon_{ji}^2}+O(h^2).\tag{56}$$

If we compare the preceding expressions for $M_0L_{\mathrm{LD}}M_{\mathrm{LM}}g$ and $M_0LM_{\mathrm{LM}}g$ termwise, and use (42)–(44) along with (56) and the fact that $|(R_1g)_i(x_j)|=O(h)\cdot\|g\|$, we conclude that $\|M_0(L_{\mathrm{LD}}-L)M_{\mathrm{LD}}g\|=O(h^3)$, as was desired.

From the preceding we conclude that the linear discontinuous method produces cell-averaged angular fluxes which are accurate to within $O(h^3)$. Now for the linear discontinuous method the conservation equation holds only in the modified form (36) for ordinates with $\mu_i>0$, and in the corresponding form with left and right limits interchanged for $\mu_i<0$. Thus we conclude only that the left-hand approximations to the cell-edge angular fluxes are accurate to within $O(h^3)$ for $\mu_i>0$, and similarly the right-hand approximations for $\mu_i<0$.

d. Summary. We summarize the major results obtained in this section in the following theorem.

THEOREM 6. *The linear moments and linear characteristic methods produce approximations to the cell-average and cell-edge angular fluxes which are accurate to within $O(h^4)$ as $h\to 0$ through any quasi-uniform family of nets. Under similar circumstances the linear discontinuous method produces approximations to the cell-averaged angular fluxes and to the cell-edge angular fluxes taken in the limit corresponding to the direction of particle flow (i.e., left-hand limit for $\mu_i>0$ and right-hand limit for $\mu_i<0$) which are accurate to within $O(h^3)$.*

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REFERENCES

[1] R. E. Alcouffe, E. W. Larsen, W. F. Miller, Jr. and B. R. Wienke, *Computational efficiency of numerical methods for the multigroup discrete-ordinates neutron transport equation: The slab geometry case*, Nuclear Sci. Engrg., 71 (1979), pp. 111–127.

[2] P. M. Anselone, *Collectively Compact Operator Approximation Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[3] P. Barbucci and F. D. Pasquantionio, *Exponential supplementary equations for S_N methods: The one dimensional case*, Nuclear Sci. Engrg., 63 (1977), pp. 179–187.

[4] T. Du Pont, *A unifiea theory of superconvergence for Galerkin methods for two-point boundary problems*, this Journal, 13 (1976), pp. 362–368.

[5] D. V. Gopinath, A. Natarajan and V. Sundararaman, *Improved interpolation schemes in anisotropic source-flux iteration techniques*, Nuclear Sci. Engrg., 75 (1980), pp. 181–184.

[6] E. W. Larsen and W. F. Miller, Jr., *Convergence rates of spatial difference equations for the discrete-ordinates neutron transport equations in slab geometry*, Nuclear Sci. Engrg, 73 (1980), pp. 76–81.

[7] S. M. Lee and R. Vaidyanathan, *Comparison of the order of approximation in several spatial difference schemes for the discrete ordinates transport equation in one-dimensional plane geometry*, Nuclear Sci. Engrg., 76 (1980), pp. 1–9.

[8] W. F. Miller, Jr. and E. W. Larsen, *Comparison of two spatial differencing schemes for the slab transport equation*, Rep. LA-UR-79-1986, Los Alamos Scientific Laboratory, 1979; abstract in Trans. Amer. Nuclear Soc., 33 (1979). pp. 316–317.

[9] P. Nelson, *Convergence of the discrete-ordinates method for anisotropically scattering multiplying particles in a subcritical slab*, this Journal, 10 (1973), pp. 175–181.

[10] J. Pitkaranta, *On the spatial differencing of the discrete ordinate neutron transport equation*, this Journal, 15 (1978), pp. 859–869.

[11] H. H. Rachford, Jr. and M. F. Wheeler, *An H^{-1}-Galerkin procedure for the two-point boundary value problem*, in *Mathematical Aspects of Finite Elements in Partial Differential Equations*, C. de Boor, ed., Academic Press, New York, 1974, pp. 353–382.

[12] R. Vaidyanathan, *Effectiveness of a modified source interpolation scheme in discrete integral transport methods*, Atomkernenergie, 29 (1977), pp. 301–302.

[13] ———, *A coarse mesh algorithm to solve neutron transport equation in thick shields*, in Proc. Fifth Int. Conf. on Reactor Shielding, Knoxville, Tenn., April 17–23, 1977, Science Press, pp. 754–761.

[14] ———, *A finite moments algorithm for particle transport problems*, Nuclear Sci. Engrg., 71 (1979), pp. 46–54.