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HIGHEST POSSIBLE ORDER OF ALGEBRAICALLY STABLE DIAGONALLY IMPLICIT RUNGE–KUTTA METHODS

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Abstract.

We prove that the highest possible order of an algebraically stable diagonally implicit RK-method is *four*; the highest possible order of a circle contractive singly diagonally implicit RK-method is *four*; the highest possible order of a circle contractive diagonally implicit RK-method is *six*.

A Runge–Kutta method

$$k_i = hf\left(x_0+c_ih,\ y_0+\sum_{j=1}^{s} a_{ij}k_j\right), \quad i=1,\ldots,s\,,$$

$$y_1 = y_0+\sum_{i=1}^{s} b_ik_i$$

is called *algebraically stable* [2] (or A-contractive [3]) if

i) $b_i \geqq 0$ for all i,
ii) $M \geqq 0$ (M is nonnegative),

where M is the matrix with elements

$$m_{ij} = b_ia_{ij}+b_ja_{ji}-b_ib_j\,.$$

If one of the b_i is zero, then $m_{ii}=0$ and algebraic stability implies that the corresponding a_{ji} are zero too. Thus the method is reducible. So for methods of practical interest condition i) can be replaced by the strict inequalities $b_i>0$ for all i.

LEMMA 1. *For a RK-method of order p ($p \geqq 3$) we assume that $b_i>0$ for $i=1,\ldots,s$. Then*

$$\sum_{j=1}^{s} a_{ij}c_j^{k-1} = c_i^k/k \quad \textit{for } k=1,\ldots,[(p-1)/2] \textit{ and all } i\,.$$

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PROOF. The proof is an extension of John Butcher's argument in [3]: The order conditions

$$\sum_{i,j,l} b_i a_{ij} c_j^{k-1} a_{il} c_l^{k-1} = 1/(2k+1)k^2$$

$$\sum_{i,j} b_i c_i^k a_{ij} c_j^{k-1} = 1/(2k+1)k$$

$$\sum_i b_i c_i^{2k} = 1/(2k+1)$$

for the trees of order $2k+1$ sketched in Fig. 1

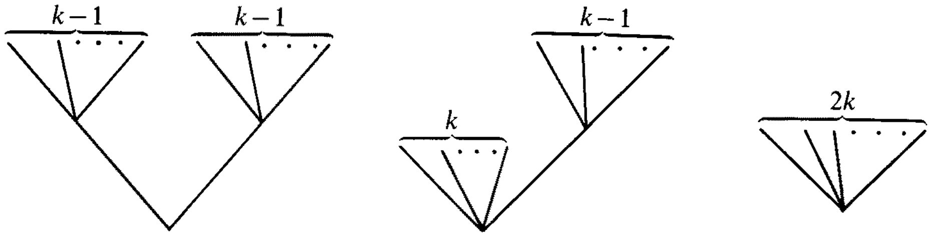

Fig. 1.

(see e.g. [4] for more explanations) imply that

$$\sum_{i=1}^{s} b_i \left(\sum_{j=1}^{s} a_{ij} c_j^{k-1} - c_i^k / k \right)^2 = 0$$

for $2k+1 \leqq p$. Since the b_i are positive, the individual terms must be zero. ∎

We recall that an RK-method is called *diagonally implicit* (DIRK) if $a_{ij}=0$ for $i<j$. These methods can be implemented much easier than fully implicit methods. Moreover, a diagonally implicit method is called *singly diagonally implicit* (SDIRK), if $a_{ii}=\gamma$ for all i.

Conjecture of Burrage [1]. There are no algebraically stable s-stage SDIRK's of order s for $s>7$.

Observe that the result of the following theorem, which justifies this conjecture, is independent of the number of stages s.

THEOREM 2. *A diagonally implicit algebraically stable RK-method has highest possible order four.*

PROOF. It is sufficient to consider irreducible methods, since the above mentioned reduction process leaves the class of diagonally implicit RK-methods invariant. Assume the method to be algebraically stable and of order greater than 4, then by Lemma 1

$$a_{11} = c_1, \; a_{11} c_1 = \tfrac{1}{2} c_1^2 .$$

This implies $a_{11}=0$. Inserting this into m_{11} yields $m_{11}=-b_1^2<0$, a contradiction to $M \geqq 0$. ∎

A weaker condition than algebraic stability is circle contractivity (see [3]). For irreducible RK-methods circle contractivity is equivalent to all b_i being positive.

THEOREM 3. *A circle contractive* SDIRK *has highest possible order four.*

PROOF. Assume an order greater than four. As in the proof of the last theorem we obtain $a_{11}=0$, so that the considered method must be explicit. This is a contradiction to a result in [3], where it is shown that the order of an explicit circle contractive method cannot exceed four. ∎

THEOREM 4. *A diagonally implicit circle contractive RK-method has highest possible order six.*

PROOF. Suppose an order greater than six and let i be the smallest number such that $c_i \neq 0$. Then by Lemma 1

$$a_{ii}c_i = \tfrac{1}{2}c_i^2, \quad a_{ii}c_i^2 = \tfrac{1}{3}c_i^3 .$$

This implies $c_i=0$ which is a contradiction. ∎

We remark that the stated order bounds are all optimal.

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