form with the upwind scheme, the definition of higher-order shape functions and the theoretical convergence rate for transport calculations. In Section C, we present the matrix-free transport sweep and the iterative solver based on it with both the Source Iteration (SI) and GMRes. Details on dealing with the sweeping cycles are explained with the introduction of the so-called Significant Angular Fluxes (SAF). In Section D, we present the results of higher-order transport calculations. Both the convergence rate with the uniform refinement and the grind time in the matrix-free transport sweep are presented. Finally, conclusions are provided in Section E.

B. DGFEM for the Multi-Group S_N Transport Equation

1. Multigroup S_N Transport Equations

The multigroup S_N transport equation is presented in detail in Appendix C. For the completeness, it is reproduced here. Given an angular quadrature set $\left\{\vec{\Omega}_m, w_m\right\}_{m=1,\ldots M}$ consisting of M directions $\vec{\Omega}_m$ and weights w_m, and a total number of G energy groups, the steady-state multigroup S_N transport equation in *one* direction indexed by m, for *one* group g, written for an open convex spatial domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$ is,

$$\begin{aligned}\left(\vec{\Omega}_m \cdot \vec{\nabla} + \sigma_{t,g}\right)\Psi_{m,g}(\vec{r}) &= \sum_{g'=1}^{G}\sum_{n=0}^{N_a}\frac{2n+1}{4\pi}\sigma_{s,n}^{g'\to g}(\vec{r})\sum_{k=-n}^{n}\Phi_{n,k}^{g'}(\vec{r})Y_{n,k}(\vec{\Omega}_m) \\ &\quad + \frac{\chi_g}{4\pi}\sum_{g'=1}^{G}\nu\sigma_{f,g'}(\vec{r})\Phi_{g'}(\vec{r}) + S_{m,g}^{ext}(\vec{r}) \qquad \text{for } \vec{r}\in\mathcal{D}\end{aligned} \tag{2.1}$$

with the general boundary condition

$$\begin{aligned}\Psi_{m,g}(\vec{r}_b) &= \Psi_{m,g}^{inc}(\vec{r}_b) + \sum_{g'=1}^{G}\sum_{\vec{\Omega}_{m'}\cdot\vec{n}_b>0}\beta_{m'\to m}^{g'\to g}(\vec{r}_b)\Psi_{m',g'}(\vec{r}_b) \\ &\qquad \text{for } \vec{r}_b\in\partial\mathcal{D}_m^- = \left\{\partial\mathcal{D},\ \vec{\Omega}_m\cdot\vec{n}_b<0\right\}\end{aligned} \tag{2.2}$$

Symbols used in the above equation are standard in textbooks and journals; their meanings are listed below for completeness:

$\vec{r}$	position variable [cm]
$\mathcal{D}$	$\in \mathbb{R}^d$ open convex spatial domain of dimension d
$\partial\mathcal{D}$	boundary of spatial domain $\mathcal{D}$
$\vec{n}_b$	$= \vec{n}(\vec{r}_b)$ outward unit normal vector on boundary $\partial\mathcal{D}$
$\vec{\Omega}_m$	unit vector for a steaming direction m in the angular quadrature set
m	index of streaming directions, from 1 to M
g	index of energy groups, from 1 to G

Angular fluxes and flux moments variables are given by:

$\Psi_{m,g}(\vec{r})$	$= \Psi_g(\vec{r}, \vec{\Omega}_m)$ neutron angular flux $[\frac{n}{cm^2 \cdot ster \cdot s}]$
$\Phi_g(\vec{r})$	$= \Phi^g_{0,0}(\vec{r}) = \sum_{m=1}^{M} w_m \Psi_{m,g}$ neutron scalar flux $[\frac{n}{cm^2 \cdot s}]$
$\Phi^g_{n,k}(\vec{r})$	$= \sum_{m=1}^{M} w_m Y_{n,k}(\vec{\Omega}_m) \Psi_{m,g}$ neutron flux moments $[\frac{n}{cm^2 \cdot s}]$
$Y_{n,k}(\vec{\Omega})$	spherical harmonics functions defined with Eq. (C.29) and Eq. (C.39)

The other terms are:

$S^{ext}_{m,g}(\vec{r})$ $= S^{ext}_{g}(\vec{r}, \vec{\Omega}_m)$ external source $[\frac{n}{cm^2 \cdot ster \cdot s}]$

$\sigma_{t,g}(\vec{r})$ macroscopic total cross section$[cm^{-1}]$

$\sigma^{g' \to g}_{s,n}(\vec{r})$ $= \int_{-1}^{1} d\mu \, \sigma^{g' \to g}_{s}(\vec{r}, \mu) P_n(\mu)$ macroscopic scattering cross section $[cm^{-1}]$

N_a truncation order of the P_N approximation, see Section A of Appendix C for more details

$\nu\sigma_{f,g}(\vec{r})$ fission cross section times the average number of neutrons emitted per fission $[cm^{-1}]$

$\chi_g(\vec{r})$ neutron fission spectrum

$\Psi^{inc}_{m,g}(\vec{r}_b)$ non-homogeneous incoming angular flux on boundary $[\frac{n}{cm^2 \cdot ster \cdot s}]$

$\beta^{g' \to g}_{m' \to m}(\vec{r}_b)$ boundary albedo, its definition depends on the quadrature set, also refer to the Eq. (C.28).

The multigroup equation is solved with the iterative solver presented in Section B of Appendix C, in which a sequence of one-group problems are solved. In addition, in the case of an eigenvalue problem, an additional (outer) iteration loop is added to update the eigenvalue estimate after each multigroup solve. For brevity's sake, it is suitable to solely present the spatial discretization and AMR technique method in the one-group case. The one-group S_N equation is,

$$\left(\vec{\Omega}_m \cdot \vec{\nabla} + \sigma_t(\vec{r})\right) \Psi_m(\vec{r}) = \sum_{n=0}^{N} \frac{2n+1}{4\pi} \sum_{k=-n}^{n} Y_{n,k}(\vec{\Omega}_m) \left[\sigma_{s,n}(\vec{r}) \Phi_{n,k}(\vec{r}) + Q_{n,k}(\vec{r})\right] \quad \text{for } \vec{r} \in \mathcal{D} \tag{2.3}$$

where the source terms (inscattering source, external source, fission source) have been lumped into a single source term $Q(\vec{r}, \vec{\Omega})$ which has already been expanded on a spherical harmonics basis to yields the source moments $Q_{n,k}(\vec{r})$ in Eq. (2.3).

The general boundary condition for the one-group case is:

$$\begin{aligned} \Psi_m(\vec{r}_b) &= \Psi_m^{inc}(\vec{r}_b) + \sum_{\vec{\Omega}_{m'}\cdot\vec{n}_b>0} \beta_{m'\to m}(\vec{r}_b)\Psi_{m'}(\vec{r}_b) \\ &\text{for } \vec{r}_b \in \partial\mathcal{D}_m^- = \left\{\partial\mathcal{D}, \vec{\Omega}_m \cdot \vec{n}_b < 0\right\} \end{aligned} \tag{2.4}$$

If there are only Dirichlet-type and reflecting boundaries, the boundary condition can be written as

$$\begin{aligned} \Psi_m(\vec{r}_b) &= \begin{cases} \Psi_m^{inc}(\vec{r}_b), & \vec{r}_b \in \partial\mathcal{D}^d \\ \Psi_{m'}(\vec{r}_b), & \vec{r}_b \in \partial\mathcal{D}^r \end{cases} \\ &\text{for } \vec{r}_b \in \partial\mathcal{D}_m^- \end{aligned} \tag{2.5}$$

where $\partial\mathcal{D} = \partial\mathcal{D}^d \cup \partial\mathcal{D}^r$, and the reflecting direction is given by

$$\vec{\Omega}_{m'} = \vec{\Omega}_m - 2(\vec{\Omega}_m \cdot \vec{n}_b)\vec{n}_b \tag{2.6}$$

We suppose the angular quadrature set is designed to satisfy the following two conditions regarding any outward unit normal vector on reflecting boundaries $\partial\mathcal{D}^r$:

1. $\forall m = 1, \cdots, M$, the reflected direction $\vec{\Omega}_{m'}$ is also in the quadrature set for any location on $\partial\mathcal{D}^r$.

2. the weights of the incident and reflected directions must be equal, i.e., $w_m = w_{m'},\ m = 1, \cdots, M$

This is usually not an issue when reflecting boundaries are on the x, y, z axes. Nevertheless, even the reflecting boundaries are not perpendicular or parallel to these axes, for example, in the case of a 2-D hexagon fuel assembly with reflecting boundaries on all six sides, a proper angular quadrature set can still be chosen to meet the above two conditions (e.g., product quadrature set for hexagonal fuel lattices).

2. Local Weighted-Residual Formula and DGFEM

We consider the DGFEM for one angular direction $\vec{\Omega}_m$ and one spatial element K of an unstructured mesh $\mathbb{T}_h$, such that the union of the all elements fully covers $\mathcal{D}$, i.e., $\bigcup_{K\in\mathbb{T}_h} K = \mathcal{D}$. We denote the local polynomial function space as $V(K) = \{\text{all polynomials on } K \text{ of degree equal to or lesser than } p_K\}$. The dimension of this local space is determined by the polynomial order p_K. Note that all the polynomial orders $\{p_K, K \in \mathbb{T}_h\}$ do not have to be the same for all elements. We then multiply the transport equation Eq. (2.3) with a test function $\Psi^*_m \in V(K)$ and integrate it over element K. After integrating by parts the streaming term $(\vec{\Omega}_m \cdot \vec{\nabla})$ and employing *the upwind scheme* for the flux values on the upwind boundary of K, we obtain the Galerkin weighted-residual formula for a given angular direction $\vec{\Omega}_m$:

$$\begin{aligned}
&\text{Find} \quad \Psi_m \in V(K), \quad \text{such that} \quad \forall \Psi^*_m \in V(K), \\
&\left(\Psi_m, (-\vec{\Omega}_m \cdot \vec{\nabla} + \sigma_t)\Psi^*_m\right)_K + \left\langle \Psi^-_m, \Psi^{*-}_m \right\rangle_{\partial K^+} - \left\langle \Psi^-_m, \Psi^{*+}_m \right\rangle_{\partial K^- \setminus \partial\mathcal{D}} \\
&\qquad - \left\langle \Psi^{inc}_m, \Psi^{*+}_m \right\rangle_{\partial K^- \cap \partial\mathcal{D}^d} - \left\langle \Psi^+_{m'}, \Psi^{*+}_m \right\rangle_{\partial K^- \cap \partial\mathcal{D}^r} \\
&\qquad = \sum_{n=0}^{N_a} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} Y_{n,k}(\vec{\Omega}_m)\left(\sigma_{s,n}\Phi_{n,k} + Q_{n,k}, \Psi^*_m\right)_K
\end{aligned} \tag{2.7}$$

where ∂K^- is the inflow boundary, ∂K^+ is the outflow boundary. The traces f^+ and f^- are defined with respect to the particle direction $\vec{\Omega}_m$, i.e., on an inflow boundary, f^+ (resp. f^-) is the value of f taken from within element K (resp. from the upwind neighbor) and on an outflow boundary, f^+ (resp. f^-) is the value of f taken from the downwind neighbor (resp. from within element K). These definitions are expressed

as follows:

$$\Psi_m^+ \equiv \lim_{s\to 0^+} \Psi_m(\vec{r}+s\vec{\Omega}_m) \tag{2.8}$$

$$\Psi_m^- \equiv \lim_{s\to 0^-} \Psi_m(\vec{r}+s\vec{\Omega}_m) \tag{2.9}$$

$$(f,g)_K \equiv \int_K f\, g\, \mathrm{d}\vec{r} \tag{2.10}$$

$$\langle f,g\rangle_e \equiv \int_e |\vec{\Omega}_m \cdot \vec{n}(\vec{r})| f\, g\, \mathrm{d}s \tag{2.11}$$

$$\partial K^+ = \left\{\vec{r}\in\partial K : \vec{n}(\vec{r})\cdot\vec{\Omega}_m \geq 0\right\} \tag{2.12}$$

$$\partial K^- = \left\{\vec{r}\in\partial K : \vec{n}(\vec{r})\cdot\vec{\Omega}_m < 0\right\} \tag{2.13}$$

$\vec{n}$ is the outward unit normal vector of element K, e represent any edge (or face in 3-D) of K. Note that (i) in the case of straight element boundaries, the quantity $\vec{\Omega}_m \cdot \vec{n}(\vec{r})$ can be factored out of the boundary integrals and that (ii) $\Phi_{n,k} \in V(K)$ because $\Psi_m \in V(K)$. This is a Galerkin scheme because the function space in which the solution is sought is also the function space of the test functions.

It needs to be pointed out that while the test functions on the edges are always taken from within the element, the primal functions on the edges are taken based on the upwind (upstream) with respect to the streaming direction; thus, for outgoing edges, the angular flux is taken from within the element K whereas for incoming edges, its value is taken from the upwind neighbor element, leading to the so-called upwind scheme.

This weighted-residual formula states that once we know the solution values on the upwind elements, we can solve the local system for the flux value within element K. The local balance is conserved, i.e., the total collision in the element is equal to the total source, including the scattering and external source, plus the net in-leakage through the upwind sides minus the net out-leakage through the downwind sides. This simple upwind scheme is the essence of DGFEM; for fluid conservation laws,

a more complicated treatment of the numerical fluxes on the inter-element edges is performed.

Finally, for completeness we also provide another (more common) variant for the weighted-residual formula. The formula below and its associated definitions are *only* given here because they are widely used in numerical fluid flows and would help a reader familiar with that notation understand the notation we have chosen.

$$\begin{aligned}
&\text{Find} \quad \Psi_m \in V(K), \quad \text{such that} \quad \forall \Psi_m^* \in V(K), \\
&\left(\Psi_m, (-\vec{\Omega}_m \cdot \vec{\nabla} + \sigma_t)\Psi_m^*\right)_K + \left\langle \Psi_m^+, \Psi_m^{*+} \right\rangle_{\partial K^+} - \left\langle \Psi_m^-, \Psi_m^{*+} \right\rangle_{\partial K^- \setminus \partial \mathcal{D}} \\
&\qquad - \left\langle \Psi_m^{inc}, \Psi_m^{*+} \right\rangle_{\partial K^- \cap \partial \mathcal{D}^d} - \left\langle \Psi_{m'}^+, \Psi_m^{*+} \right\rangle_{\partial K^- \cap \partial \mathcal{D}^r} \\
&\qquad\qquad = \sum_{n=0}^{N_a} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} Y_{n,k}(\vec{\Omega}_m) \left(\sigma_{s,n}\Phi_{n,k} + Q_{n,k}, \Psi_m^*\right)_K \qquad (2.14)
\end{aligned}$$

where ∂K^- is the inflow boundary, ∂K^+ is the outflow boundary, f^+ denotes the restriction (trace) of the function f taken from within the element K, and f^- represents the restriction of the function f taken from the neighboring element of K. These definitions are expressed as follows:

$$u = \text{sgn}\left(\vec{n}(\vec{r}) \cdot \vec{\Omega}_m\right) = +1 \text{ for outflow, or } -1 \text{ for inflow} \qquad (2.15)$$

$$\Psi_m^{\pm} \equiv \lim_{s \to 0^{\pm}} \Psi_m(\vec{r} - u\, s\, \vec{\Omega}_m) \qquad (2.16)$$

The differences between the two notations are related to the definitions of the numerical "traces". The advantage of Eq. (2.14) is that the + sign always denotes the value taken from within element K, be it for an inflow or outflow boundary, whereas this is not the case in Eq. (2.7). Again, Eqs. (2.14) through (2.16) will not be employed further here and have been given for completeness only.

3. Variational Form

Integrating by parts the streaming term of the local formula once again, multiplying the result with the angular weight w_m and summing over all elements and all directions, we obtain the variational form for the one-group S_N equation:

$$\begin{aligned}
&\sum_{m=1}^{M} w_m \left[\left((\vec{\Omega}_m \cdot \vec{\nabla} + \sigma_t)\Psi_m, \Psi_m^* \right)_{\mathcal{D}} + \left\langle [\![\Psi_m]\!], \Psi_m^{*+} \right\rangle_{E_h^i} \right] + \\
&\qquad \sum_{m=1}^{M} w_m \left\langle \Psi_m, \Psi_m^* \right\rangle_{\partial \mathcal{D}_m^-} - \sum_{m=1}^{M} w_m \left\langle \Psi_{m'}, \Psi_m^* \right\rangle_{\partial \mathcal{D}_m^{r-}} \\
&\quad = \sum_{m=1}^{M} w_m \sum_{n=0}^{N_a} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} Y_{n,k}(\vec{\Omega}_m) \left(\sigma_{s,n} \Phi_{n,k} + Q_{n,k}, \Psi_m^* \right)_K + \\
&\qquad \sum_{m=1}^{M} w_m \left\langle \Psi_m^{inc}, \Psi_m^* \right\rangle_{\partial \mathcal{D}_m^{d-}}
\end{aligned} \tag{2.17}$$

with the following definitions

$$[\![\Psi_m]\!] = \Psi_m^+ - \Psi_m^- \tag{2.18}$$

$$(f,g)_{\mathcal{D}} \equiv \sum_{K \in \mathbb{T}_h} (f,g)_K \tag{2.19}$$

$$\langle f,g \rangle_{E_h^i} \equiv \sum_{e \in E_h^i} \langle f,g \rangle_e , \tag{2.20}$$

where $E_h^i = \cup_{K \in \mathbb{T}_h} \partial K \backslash \partial \mathcal{D}$ is the set of all interior edges (more generally, the dimension of the interior edges is $d-1$ for a computational domain of dimension d). For simplicity, we have dropped the $\pm$ superscript for the angular fluxes appearing in the boundary terms later because the angular fluxes inside the domain are always used. More detailed derivations regarding DGFEM for the first-order equation can be found in Section B of Appendix B. If we change the sequence of summation over directions

and summation over elements for the boundary terms, we obtain, after some algebra:

Find $\Psi_m \in W_{\mathcal{D}}^h, \quad m = 1, \cdots, M$ such that:

$$b(\Psi, \Psi^*) - \sum_{e \in \partial \mathcal{D}^r} \sum_{\vec{\Omega}_m \cdot \vec{n}_b < 0} w_m \langle \Psi_{m'}, \Psi_m^* \rangle_e - \sum_{n=0}^{N} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} \left(\sigma_{s,n} \Phi_{n,k}, \Phi_{n,k}^*\right)_{\mathcal{D}} = l(\Psi^*) \tag{2.21}$$

$$\forall \Psi_m^* \in W_{\mathcal{D}}^h, \quad m = 1, \cdots, M$$

where the bilinear and linear forms are

$$\begin{aligned} b(\Psi, \Psi^*) = & \sum_{m=1}^{M} w_m \left((\vec{\Omega}_m \cdot \vec{\nabla} + \sigma_t)\Psi_m, \Psi_m^*\right)_{\mathcal{D}} + \sum_{m=1}^{M} w_m \left\langle [\![\Psi_m]\!], \Psi_m^{*+} \right\rangle_{E_h^i} \\ & + \sum_{e \in \partial \mathcal{D}} \sum_{\vec{\Omega}_m \cdot \vec{n}_b < 0} w_m \langle \Psi_m, \Psi_m^* \rangle_e \end{aligned} \tag{2.22}$$

$$l(\Psi^*) = \sum_{n=0}^{N} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} \left(Q_{n,k}, \Phi_{n,k}^*\right)_{\mathcal{D}} + \sum_{e \in \partial \mathcal{D}^d} \sum_{\vec{\Omega}_m \cdot \vec{n}_b < 0} w_m \left\langle \Psi_m^{inc}, \Psi_m^* \right\rangle_e \tag{2.23}$$

and $W_{\mathcal{D}}^h = \{\Psi \in L^2(\mathcal{D});\ \Psi|_K \in V(K), \forall K \in \mathbb{T}_h\}$ is the function space in which the solution is sought. Note that the functions in this space may not be continuous across the interior edges. There are three terms on the left-hand-side of Eq. (2.21): the streaming operators from M simple transport (e.g., one direction) equations $b(\Psi_m, \Psi_m^*)$, the reflecting-boundary term and the scattering term. For notational simplicity, we define the following bilinear form,

$$\begin{aligned} a(\Psi, \Psi^*) = b(\Psi, \Psi^*) - & \sum_{e \in \partial \mathcal{D}^r} \sum_{\vec{\Omega}_m \cdot \vec{n}_b < 0} w_m \langle \Psi_{m'}, \Psi_m^* \rangle_e - \\ & \sum_{n=0}^{N} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} \left(\sigma_{s,n} \Phi_{n,k}, \Phi_{n,k}^*\right)_{\mathcal{D}} \end{aligned} \tag{2.24}$$

The variational form for the multigroup S_N transport equation can be found in Section C of Appendix C.

With the same procedure, we obtain the variational form for the adjoint equation.

Find $\Psi_m^* \in W_{\mathcal{D}}^h, \quad m = 1, \cdots, M$ such that:

$$b^*(\Psi, \Psi^*) - \sum_{e \in \partial \mathcal{D}^r} \sum_{\vec{\Omega}_m \cdot \vec{n}_b > 0} w_m \left\langle \Psi_m, \Psi_{m'}^* \right\rangle_e - \sum_{n=0}^{N} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} \left(\Phi_{n,k}, \sigma_{s,n} \Phi_{n,k}^* \right)_{\mathcal{D}} = l^*(\Psi) \tag{2.25}$$

$$\forall \Psi_m \in W_{\mathcal{D}}^h, \quad m = 1, \cdots, M$$

where

$$\begin{aligned} b^*(\Psi, \Psi^*) = & \sum_{m=1}^{M} w_m \left(\Psi_m, (-\vec{\Omega}_m \cdot \vec{\nabla} + \sigma_t) \Psi_m^* \right)_{\mathcal{D}} - \sum_{m=1}^{M} w_m \left\langle \Psi_m^-, [\![\Psi_m^*]\!] \right\rangle_{E_h^i} \\ & + \sum_{e \in \partial \mathcal{D}} \sum_{\vec{\Omega}_m \cdot \vec{n}_b > 0} w_m \left\langle \Psi_m, \Psi_m^* \right\rangle_e \end{aligned} \tag{2.26}$$

$$l^*(\Psi) = \sum_{n=0}^{N} \sum_{k=-n}^{n} \frac{2n+1}{4\pi} \left(\Phi_{n,k}, Q_{n,k}^* \right)_{\mathcal{D}} + \sum_{e \in \partial \mathcal{D}^d} \sum_{\vec{\Omega}_m \cdot \vec{n}_b > 0} w_m \left\langle \Psi_m, \Psi_m^{*inc} \right\rangle_e \tag{2.27}$$

In Section B of Appendix B, we prove that $b(\Psi_m, \Psi_m^*) = b^*(\Psi_m, \Psi_m^*)$. Furthermore, it is easy to note that the scattering terms are self adjoint. Finally, if the quadrature set is properly designed, the reflecting terms are also self-adjoint. So the bilinear forms for the primal equation and the adjoint equation are the same, i.e., $a(\Psi_m, \Psi_m^*) = a^*(\Psi_m, \Psi_m^*)$.

The only difference between the spatially discretized bilinear form $a(\Psi_m, \Psi_m^*)$ and the form stemming from the continuous one-group S_N equation is the additional jump term on all interior edges. It is easy to see that this term vanishes in the continuous case, i.e., jumps along the streaming directions are zero. So we have the Galerkin orthogonality (the jumps in the exact solution are zero)

$$a(\Psi - \Psi_{exact}, \Psi^*) = 0, \quad \forall \Psi_m^* \in W_{\mathcal{D}}^h, \quad m = 1, \cdots, M \tag{2.28}$$

Because of the positive definiteness of the form a, i.e.,

$$a(\Psi, \Psi) > 0 \tag{2.29}$$

we can define a so-called DG-norm

$$\|\|\Psi\|\|_{DG} = a(\Psi, \Psi) \tag{2.30}$$

Although we do not have to write down the variational form to solve the S_N equation with DGFEM, it is very useful for the study of theoretical convergence. It has been proved that with the uniform polynomial order, i.e., $p_K = p, \forall K \in \mathbb{T}_h$ [88]

$$\|\|\Psi - \Psi_{exact}\|\|_{DG} \leq C \frac{h^q}{(p+1)^q} \tag{2.31}$$

where $q = \min(p + 1/2, 1)$ for continuous transport solution, or $q = \min(p + 1/2, 0)$ if discontinuities are present in the transport solution. h is the maximum diameter of all elements, which is fixed for a given mesh. C is a constant independent of the mesh. It can be proved the convergence holds also for the scalar flux measured in the L_2-norm,

$$\|\Phi - \Phi_{exact}\|_2 \leq C \frac{h^{q+\frac{1}{2}}}{(p+1)^q} \tag{2.32}$$

The variational form derived here will be employed to obtain conforming preconditioners for the DSA schemes, in Chapter III.

4. Higher-Order Shape Functions

A good choice of shape functions is important for an easy and efficient implementation of higher-order DGFEM. Over the course of the last decade, higher-order and spectral finite elements have enjoyed many theoretical and applied developments; for instance,