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Heat Kernel on Homogeneous Bundles over Symmetric Spaces

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We consider Laplacians acting on sections of homogeneous vector bundles over symmetric spaces. By using an integral representation of the heat semi-group we find a formal solution for the heat kernel diagonal that gives a generating function for the whole sequence of heat invariants. We show explicitly that the obtained result correctly reproduces the first non-trivial heat kernel coefficient as well as the exact heat kernel diagonals on two-dimensional sphere S^2 and the hyperbolic plane H^2. We argue that the obtained formal solution correctly reproduces the exact heat kernel diagonal after a suitable regularization and analytical continuation.

1 Introduction

The heat kernel is one of the most powerful tools in mathematical physics and geometric analysis (see, for example the books [24, 17, 26, 13, 27] and reviews [2, 18, 12, 14, 31]). The short-time asymptotic expansion of the trace of the heat kernel determines the spectral asymptotics of the differential operator. The coefficients of this asymptotic expansion, called the heat invariants, are extensively used in geometric analysis, in particular, in spectral geometry and index theorems proofs [24, 17].

There has been a tremendous progress in the explicit calculation of spectral asymptotics in the last thirty years [23, 2, 3, 4, 5, 30, 33]. It seems that further progress in the study of spectral asymptotics can be only achieved by restricting oneself to operators and manifolds with high level of symmetry, in particular, homogeneous spaces, which enables one to employ powerful algebraic methods. In some very special particular cases, such as group manifolds, spheres, rank-one symmetric spaces and split-rank symmetric spaces, it is possible to determine the spectrum of the Laplacian exactly and to obtain closed formulas for the heat kernel in terms of the root vectors and their multiplicities [1, 18, 19, 20, 26, 22]. The complexity of the method crucially depends on the global structure of the symmetric space, most importantly its rank. Most of the results for symmetric spaces are obtained for rank-one symmetric spaces only [18].

It is well known that heat invariants are determined essentially by local geometry. They are polynomial invariants in the curvature with universal constants that do not depend on the global properties of the manifold [24]. It is this universal structure that we are interested in this paper. Our goal is to compute the heat kernel asymptotics of the Laplacian acting on homogeneous vector bundles over symmetric spaces. Related problems in a more general context are discussed in [7, 9, 11].

2 Geometry of Symmetric Spaces

2.1 Twisted Spin-Tensor Bundles

In this section we introduce the basic concepts and fix notation. Let (M, g) be an n-dimensional Riemannian manifold without boundary. We assume that it is complete simply connected orientable and spin. We denote the local coordinates on M by x^μ, with Greek indices running over $1, \dots, n$. Let $e_a{}^\mu$ be a local orthonormal frame defining a basis for the tangent space T_xM so that

$$g^{\mu\nu} = \delta^{ab} e_a{}^\mu e_b{}^\nu\,, \tag{2.1}$$

We denote the frame indices by low case Latin indices from the beginning of the alphabet, which also run over $1,\dots,n$. The frame indices are raised and lowered by the metric δ_{ab}. Let $e^a{}_\mu$ be the matrix inverse to $e_a{}^\mu$, defining the dual basis in the cotangent space T^*_xM, so that,

$$g_{\mu\nu} = \delta_{ab} e^a{}_\mu e^b{}_\nu\,. \tag{2.2}$$

The Riemannian volume element is defined as usual by

$$d\mathrm{vol}\; = dx\,|g|^{1/2}\,, \tag{2.3}$$

where

$$|g| = \det g_{\mu\nu} = (\det e_a{}^\mu)^2\,. \tag{2.4}$$

The spin connection $\omega^{ab}{}_\mu$ is defined in terms of the orthonormal frame by

$$\begin{aligned}\omega^{ab}{}_\mu &= e^{a\mu} e^b{}_{\mu;\nu} = -e^a{}_{\mu;\nu} e^{b\mu}\\ &= e^{a\nu}\partial_{[\mu} e^b{}_{\nu]} - e^{b\nu}\partial_{[\mu} e^a{}_{\nu]} + e_{c\mu} e^{a\nu} e^{b\lambda}\partial_{[\lambda} e^c{}_{\nu]}\,,\end{aligned} \tag{2.5}$$

where the semicolon denotes the usual Riemannian covariant derivative with the Levi-Civita connection. The curvature of the spin connection is

$$R^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}\,. \tag{2.6}$$

The Ricci tensor and the scalar curvature are defined by

$$R_{\alpha\nu} = e_a{}^\mu e^b{}_\alpha R^a{}_{b\mu\nu}\,, \qquad R = g^{\mu\nu} R_{\mu\nu} = e_a{}^\mu e_b{}^\nu R^{ab}{}_{\mu\nu}\,. \tag{2.7}$$

Let $\mathcal{T}$ be a spin-tensor bundle realizing a representation Σ of the spin group $\mathrm{Spin}(n)$, the double covering of the group $SO(n)$, with the fiber Λ. Let Σ_{ab} be the generators of the orthogonal algebra $\mathcal{SO}(n)$, the Lie algebra of the orthogonal group $SO(n)$, satisfying the following commutation relations

$$[\Sigma_{ab}, \Sigma_{cd}] = -\delta_{ac}\Sigma_{bd} + \delta_{bc}\Sigma_{ad} + \delta_{ad}\Sigma_{bc} - \delta_{bd}\Sigma_{ac}\,. \tag{2.8}$$

The spin connection induces a connection on the bundle $\mathcal{T}$ defining the covariant derivative of smooth sections φ of the bundle $\mathcal{T}$ by

$$\nabla_\mu\varphi = \left(\partial_\mu + \frac{1}{2}\omega^{ab}{}_\mu \Sigma_{ab}\right)\varphi\,. \tag{2.9}$$

The commutator of covariant derivatives defines the curvature of this connection via

$$[\nabla_\mu, \nabla_\nu]\varphi = \frac{1}{2}R^{ab}{}_{\mu\nu}\Sigma_{ab}\varphi\,. \tag{2.10}$$

As usual, the orthonormal frame, $e^a{}_\mu$ and $e_a{}^\mu$, will be used to transform the coordinate (Greek) indices to the orthonormal (Latin) indices. The covariant derivative along the frame vectors is defined by $\nabla_a = e_a{}^\mu\nabla_\mu$. For example, with our notation, $\nabla_a\nabla_b T_{cd} = e_a{}^\mu e_b{}^\nu e_c{}^\alpha e_d{}^\beta \nabla_\mu\nabla_\nu T_{\alpha\beta}$.

The metric δ_{ab} induces a positive definite fiber metric on tensor bundles. For Dirac spinors, the fiber metric is defined as follows. First, one defines the Dirac matrices, γ_a, as generators of the Clifford algebra, (represented by $2^{[n/2]}\times 2^{[n/2]}$ complex matrices),

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}\mathbb{I}_S\,, \tag{2.11}$$

where $\mathbb{I}_S$ is the identity matrix in the spinor representation. Then one defines the anti-symmetrized products of Dirac matrices

$$\gamma_{a_1\dots a_k} = \gamma_{[a_1}\cdots\gamma_{a_k]}\,. \tag{2.12}$$

Then the matrices

$$\Sigma_{ab} = \frac{1}{2}\gamma_{ab} \tag{2.13}$$

are the generators of the orthogonal algebra $\mathcal{SO}(n)$ in the spinor representation. The Hermitian conjugation of Dirac matrices defines a Hermitian matrix β [1] by

$$\gamma_a^\dagger = \beta\gamma_a\beta^{-1}\,, \tag{2.14}$$

which defines a Hermitian inner product $\overline{\psi}\varphi = \psi^\dagger\beta\varphi$ in the vector space of spinors. We also find the following important relation

$$R^{ab}{}_{cd}\gamma_{ab}\gamma^{cd} = -2R^{ab}{}_{ab}\mathbb{I}_S = -2R\,\mathbb{I}_S\,, \tag{2.15}$$

where R is the scalar curvature.

In the present paper we will further assume that M is a *locally symmetric space* with a Riemannian metric with the parallel curvature

$$\nabla_\mu R_{\alpha\beta\gamma\delta} = 0\,, \tag{2.16}$$

[1] The Dirac matrices γ_{ab} and the spinor metric β should not be confused with the matrices γ_{AB} and β_{ij} defined below.

which means, in particular, that the curvature satisfies the integrability constraints

$$R^{fg}{}_{ea}R^{e}{}_{bcd} - R^{fg}{}_{eb}R^{e}{}_{acd} + R^{fg}{}_{ec}R^{e}{}_{dab} - R^{fg}{}_{ed}R^{e}{}_{cab} = 0\,. \tag{2.17}$$

Let G_{YM} be a compact Lie group (called a gauge group). It naturally defines the principal fiber bundle over the manifold M with the structure group G_{YM}. We consider a representation of the structure group G_{YM} and the associated vector bundle through this representation with the same structure group G_{YM} whose typical fiber is a k-dimensional vector space W. Then for any spin-tensor bundle $\mathcal{T}$ we define the twisted spin-tensor bundle $\mathcal{V}$ via the twisted product of the bundles $\mathcal{W}$ and $\mathcal{T}$. The fiber of the bundle $\mathcal{V}$ is $V = \Lambda \otimes W$ so that the sections of the bundle $\mathcal{V}$ are represented locally by k-tuples of spin-tensors.

Let $\mathcal{A}$ be a connection one form on the bundle $\mathcal{W}$ (called Yang-Mills or gauge connection) taking values in the Lie algebra $\mathcal{G}_{YM}$ of the gauge group G_{YM}. Then the total connection on the bundle $\mathcal{V}$ is defined by

$$\nabla_\mu\varphi = \left(\partial_\mu + \frac{1}{2}\omega^{ab}{}_\mu\Sigma_{ab}\otimes\mathbb{I}_W + \mathbb{I}_\Lambda\otimes\mathcal{A}_\mu\right)\varphi\,, \tag{2.18}$$

and the total curvature Ω of the bundle $\mathcal{V}$ is defined by

$$[\nabla_\mu, \nabla_\nu]\varphi = \Omega_{\mu\nu}\varphi\,, \tag{2.19}$$

where

$$\Omega_{\mu\nu} = \frac{1}{2}R^{ab}{}_{\mu\nu}\Sigma_{ab} + \mathcal{F}_{\mu\nu}\,, \tag{2.20}$$

and

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\mu] \tag{2.21}$$

is the curvature of the Yang-Mills connection.

We also consider the bundle of endomorphisms of the bundle $\mathcal{V}$. The covariant derivative of sections of this bundle is defined by

$$\nabla_\mu X = \left(\partial_\mu + \frac{1}{2}\omega^{ab}{}_\mu\Sigma_{ab}\right)X + [\mathcal{A}_\mu, X]\,, \tag{2.22}$$

and the commutator of covariant derivatives is equal to

$$[\nabla_\mu, \nabla_\nu]X = \frac{1}{2}R^{ab}{}_{\mu\nu}\Sigma_{ab}X + [\mathcal{F}_{\mu\nu}, X]\,. \tag{2.23}$$

In the following we will consider *homogeneous vector bundles* with parallel bundle curvature

$$\nabla_\mu \mathcal{F}_{\alpha\beta} = 0\,, \tag{2.24}$$

which means that the curvature satisfies the integrability constraints

$$[\mathcal{F}_{cd}, \mathcal{F}_{ab}] - R^f{}_{acd}\mathcal{F}_{fb} - R^f{}_{bcd}\mathcal{F}_{af} = 0\,. \tag{2.25}$$

2.2 Normal Coordinates

Let x' be a fixed point in M and $\mathcal{U}$ be a sufficiently small coordinate patch containing the point x'. Then every point x in $\mathcal{U}$ can be connected with the point x' by a unique geodesic. We extend the local orthonormal frame $e_a{}^\mu(x')$ at the point x' to a local orthonormal frame $e_a{}^\mu(x)$ at the point x by parallel transport

$$e_a{}^\mu(x) = g^\mu{}_{\nu'}(x, x')e_a{}^{\nu'}(x')\,, \tag{2.26}$$

$$e^a{}_\mu(x) = g_\mu{}^{\nu'}(x, x')e^a{}_{\nu'}(x')\,, \tag{2.27}$$

where $g^\mu{}_{\nu'}(x, x')$ is the operator of parallel transport of vectors along the geodesic from the point x' to the point x. Of course, the frame $e_a{}^\mu$ depends on the fixed point x' as a parameter. Here and everywhere below the coordinate indices of the tangent space at the point x' are denoted by primed Greek letters. They are raised and lowered by the metric tensor $g_{\mu'\nu'}(x')$ at the point x'. The derivatives with respect to x' will be denoted by primed Greek indices as well.

The parameters of the geodesic connecting the points x and x', namely the unit tangent vector at the point x' and the length of the geodesic, (or, equivalently, the tangent vector at the point x' with the norm equal to the length of the geodesic), provide normal coordinate system for $\mathcal{U}$. Let $d(x, x')$ be the geodesic distance between the points x and x' and $\sigma(x, x')$ be a two-point function defined by

$$\sigma(x, x') = \frac{1}{2}[d(x, x')]^2\,. \tag{2.28}$$

Then the derivatives $\sigma_{;\mu}(x, x')$ and $\sigma_{;\nu'}(x, x')$ are the tangent vectors to the geodesic connecting the points x and x' at the points x and x' respectively pointing in opposite directions; one is obtained from another by parallel transport

$$\sigma_{;\mu} = -g_\mu{}^{\nu'}\sigma_{;\nu'}\,. \tag{2.29}$$

Here and everywhere below the semicolon denotes the covariant derivative.

The operator of parallel transport satisfies the equation

$$\sigma^{;\mu}\nabla_\mu g^\alpha{}_{\beta'} = 0\,, \tag{2.30}$$

with the initial conditions

$$g^\alpha{}_{\beta'}\Big|_{x=x'} = \delta^\alpha_\beta\,. \tag{2.31}$$

It can be expressed in terms of the local parallel frame

$$g^\mu{}_{\nu'}(x,x') = e_a{}^\mu(x)e^a{}_{\nu'}(x')\,, \tag{2.32}$$

$$g_\mu{}^{\nu'}(x,x') = e^a{}_\mu(x)e_a{}^{\nu'}(x')\,. \tag{2.33}$$

Now, let us define the quantities

$$y^a = e^a{}_\mu\sigma^{;\mu} = -e^a{}_{\mu'}\sigma^{;\mu'}\,, \tag{2.34}$$

so that

$$\sigma^{;\mu} = e_a{}^\mu y^a \qquad \text{and} \qquad \sigma^{;\mu'} = -e_a{}^{\mu'}y^a\,. \tag{2.35}$$

Notice that $y^a = 0$ at $x = x'$. Further, we have

$$\frac{\partial y^a}{\partial x^\nu} = -e^{a\mu'}\sigma_{;\nu\mu'}\,, \tag{2.36}$$

so that the Jacobian of the change of variables is

$$\det\left(\frac{\partial y^a}{\partial x^\nu}\right) = |g|^{-1/2}(x')\det[-\sigma_{;\nu\mu'}(x,x')]\,. \tag{2.37}$$

The geometric parameters y^a are nothing but the normal coordinates. By using the Van Vleck-Morette determinant defined by [2]

$$\Delta(x,x') = |g|^{-1/2}(x')|g|^{-1/2}(x)\det[-\sigma_{;\nu\mu'}(x,x')]\,, \tag{2.38}$$

we can write the Riemannian volume element in the form

$$d\text{vol} = dy\,\Delta^{-1}(x,x')\,. \tag{2.39}$$

Let $\mathcal{P}(x,x')$ be the operator of parallel transport of sections of the bundle $\mathcal{V}$ from the point x' to the point x. It satisfies the equation

$$\sigma^{;\mu}\nabla_\mu\mathcal{P} = 0\,, \tag{2.40}$$

[2] Do not confuse it with the Laplacian Δ defined below.

with the initial condition

$$\mathcal{P}\Big|_{x=x'} = \mathbb{I}_V\,. \qquad (2.41)$$

Any spin-tensor φ can be now expanded in the covariant Taylor series

$$\varphi(x) = \mathcal{P}(x,x')\sum_{k=0}^{\infty}\frac{1}{k!}\left[\nabla_{(c_1}\cdots\nabla_{c_k)}\varphi\right](x')y^{c_1}\cdots y^{c_k}\,. \qquad (2.42)$$

Therefrom it is clear, in particular, that the frame components of a parallel spin-tensor are simply constant.

In symmetric spaces one can compute the Van Vleck-Morette determinant explicitly in terms of the curvature. Let K be a $n\times n$ matrix with the entries

$$K^a{}_b = R^a{}_{cbd}y^c y^d\,. \qquad (2.43)$$

Then [5, 2, 13]

$$\frac{\partial y^a}{\partial x^\nu} = \left(\frac{\sqrt{K}}{\sin\sqrt{K}}\right)^a{}_b\, e^b{}_\nu\,, \qquad (2.44)$$

and, therefore,

$$\Delta(x,x') = \det{}_{TM}\left(\frac{\sqrt{K}}{\sin\sqrt{K}}\right). \qquad (2.45)$$

Thus, the Riemannian volume element in symmetric spaces takes the following form

$$d\text{vol} = dy\;\det{}_{TM}\left(\frac{\sin\sqrt{K}}{\sqrt{K}}\right). \qquad (2.46)$$

The matrix $(\sin\sqrt{K})/\sqrt{K}$ determines the orthonormal frame in normal coordinates, and the square of this matrix determines the metric tensor in normal coordinates,

$$ds^2 = \left(\frac{\sin^2\sqrt{K}}{K}\right)_{ab} dy^a\,dy^b\,. \qquad (2.47)$$

Let us define an endo-morphism valued 1-form $\tilde{\mathcal{A}}_a$ by the equation

$$\nabla_\nu\mathcal{P} = \mathcal{P}\tilde{\mathcal{A}}_a e^a{}_{\mu'}\sigma^{;\mu'}{}_\nu\,. \qquad (2.48)$$

Then for bundles with parallel curvature over symmetric spaces one can find it explicitly [5, 2, 13]

$$\tilde{\mathcal{A}}_a = -\mathcal{F}_{bc}y^c\left(\frac{\mathbb{I}-\cos\sqrt{K}}{K}\right)^b{}_a\,. \qquad (2.49)$$

This object determines the gauge connection in normal coordinates,

$$\mathcal{A} = -\mathcal{F}_{bc}y^c \left(\frac{\mathbb{I} - \cos\sqrt{K}}{K}\right)^b{}_a \, dy^a \,. \tag{2.50}$$

This means that all connections on a homogeneous bundle are essentially the same. In particular, the spin connection one-form in normal coordinates has the form

$$\omega^a{}_b = -R^a{}_{bcd}y^d \left(\frac{\mathbb{I} - \cos\sqrt{K}}{K}\right)^c{}_e dy^e \,. \tag{2.51}$$

Remarks. Two remarks are in order here. First, strictly speaking, normal coordinates can be only defined locally, in geodesic balls of radius less than the injectivity radius of the manifold. However, for symmetric spaces normal coordinates cover the whole manifold except for a set of measure zero where they become singular [18]. This set is precisely the set of points conjugate to the fixed point x' (where $\Delta^{-1}(x, x') = 0$) and of points that can be connected to the point x' by multiple geodesics. In any case, this set is a set of measure zero and, as we will show below, it can be dealt with by some regularization technique. Thus, we will use the normal coordinates defined above for the whole manifold. Second, for compact manifolds (or for manifolds with compact submanifolds) the range of some normal coordinates is also compact, so that if one allows them to range over the whole real line $\mathbb{R}$, then the corresponding compact submanifolds will be covered infinitely many times.

2.3 Curvature Group of a Symmetric Space

We assumed that the manifold M is locally symmetric. Since we also assume that it is simply connected and complete, it is a globally symmetric space (or simply symmetric space) [32]. A symmetric space is said to be compact, non-compact or Euclidean if all sectional curvatures are positive, negative or zero. A generic symmetric space has the structure

$$M = M_0 \times M_s \,, \tag{2.52}$$

where $M_0 = \mathbb{R}^{n_0}$ and M_s is a semi-simple symmetric space; it is a product of a compact symmetric space M_+ and a non-sompact symmetric space M_-,

$$M_s = M_+ \times M_- \,. \tag{2.53}$$

Of course, the dimensions must satisfy the relation $n_0+n_s = n$, where $n_s = \dim M_s$.

Let Λ_2 be the vector space of 2-forms on M at a fixed point x'. It has the dimension $\dim\Lambda_2 = n(n-1)/2$, and the inner product in Λ_2 is defined by

$$\langle X, Y\rangle = \frac{1}{2}X_{ab}Y^{ab}\,. \tag{2.54}$$

The Riemann curvature tensor naturally defines the curvature operator

$$\text{Riem} : \Lambda_2 \to \Lambda_2 \tag{2.55}$$

by

$$(\text{Riem}\,X)_{ab} = \frac{1}{2}R_{ab}{}^{cd}X_{cd}\,. \tag{2.56}$$

This operator is symmetric and has real eigenvalues which determine the principal sectional curvatures. Now, let Ker (Riem) and Im (Riem) be the kernel and the range of this operator and

$$p = \dim \text{Im}(\text{Riem}) = \frac{n(n-1)}{2} - \dim \text{Ker}\,(\text{Riem})\,. \tag{2.57}$$

Further, let λ_i, $(i = 1,\dots,p)$, be the non-zero eigenvalues, and $E^i{}_{ab}$ be the corresponding orthonormal eigen-two-forms. Then the components of the curvature tensor can be presented in the form [10]

$$R_{abcd} = \beta_{ik}E^i{}_{ab}E^k{}_{cd}\,, \tag{2.58}$$

where β_{ik} is a symmetric, in fact, diagonal, nondegenerate $p\times p$ matrix

$$(\beta_{ik}) = \text{diag}\,(\lambda_1,\dots,\lambda_p)\,. \tag{2.59}$$

Of course, the zero eigenvalues of the curvature operator correspond to the flat subspace M_0, the positive ones correspond to the compact submanifold M_+ and the negative ones to the non-compact submanifold M_-. Therefore, Im (Riem) = T_xM_s.

In the following the Latin indices from the middle of the alphabet will be used to denote tensors in Im(Riem); they should not be confused with the Latin indices from the beginning of the alphabet which denote tensors in M. They will be raised and lowered with the matrix β_{ik} and its inverse

$$(\beta^{ik}) = \text{diag}\,(\lambda_1^{-1},\dots,\lambda_p^{-1})\,. \tag{2.60}$$

Next, we define the traceless $n \times n$ matrices $D_i = (D^a{}_{ib})$, where

$$D^a{}_{ib} = -\beta_{ik} E^k{}_{cb} \delta^{ca} \,. \tag{2.61}$$

Then

$$R^a{}_{bcd} = -D^a{}_{ib} E^i{}_{cd} \,, \tag{2.62}$$

$$R^a{}_b{}^c{}_d = \beta^{ik} D^a{}_{ib} D^c{}_{kd} \,, \tag{2.63}$$

$$R^a{}_b = -\beta^{ik} D^a{}_{ic} D^c{}_{kb}, \tag{2.64}$$

$$R = -\beta^{ik} D^a{}_{ic} D^c{}_{ka} \,. \tag{2.65}$$

Also, we have identically,

$$D^a{}_{j[b} E^j{}_{cd]} = 0 \,. \tag{2.66}$$

The matrices D_i are known to be the generators of the holonomy algebra, $\mathcal{H}$, i.e. the Lie algebra of the restricted holonomy group, H,

$$[D_i, D_k] = F^j{}_{ik} D_j \,, \tag{2.67}$$

where $F^j{}_{ik}$ are the structure constants of the holonomy group. The structure constants of the holonomy group define the $p \times p$ matrices F_i, by $(F_i)^j{}_k = F^j{}_{ik}$, which generate the adjoint representation of the holonomy algebra,

$$[F_i, F_k] = F^j{}_{ik} F_j \,. \tag{2.68}$$

These commutation relations follow directly from the Jacobi identities

$$F^I{}_{j[k} F^j{}_{ml]} = 0 \,. \tag{2.69}$$

For symmetric spaces the introduced quantities satisfy additional algebraic constraints. The most important consequence of the eq. (2.17) is the equation [10]

$$E^i{}_{ac} D^c{}_{kb} - E^i{}_{bc} D^c{}_{ka} = F^i{}_{kj} E^j{}_{ab} \,. \tag{2.70}$$

It is this equation that makes a generic Riemannian manifold a symmetric space.

Now, by using the eqs. (2.67) and (2.70) one can prove the following:

Proposition 1 *The matrix β_{ik} is H-invariant and satisfies the equation*

$$\beta_{ik} F^k{}_{jl} + \beta_{lk} F^k{}_{ji} = 0 \,. \tag{2.71}$$

This means that the matrices F_i satisfy the transposition rule

$$(F_i)^T = -\beta F_i \beta^{-1}\,, \tag{2.72}$$

which simply means that the adjoint and the coadjoint representations of the holonomy algebra $\mathcal{H}$ are equivalent. In particular, this means that the matrices F_i are traceless. Such an algebra is called *compact* [16].

Another consequence of the eq. (2.70) are the identities

$$D^a{}_{i[b}R_{c]ade} + D^a{}_{i[d}R_{e]abc} = 0, \tag{2.73}$$

$$R^a{}_c D^c{}_{ib} = D^a{}_{ic} R^c{}_b\,. \tag{2.74}$$

This means, in particular, that the Ricci tensor matrix commutes with all matrices D_i and is, therefore, an invariant matrix of the holonomy algebra. Thus,

$$R^a{}_b = \frac{1}{n_s} h^a{}_b R\,, \tag{2.75}$$

where $h^a{}_b$ is a projection (a symmetric idempotent parallel tensor) to the subspace $T_x M_s$ of the tangent space of dimension n_s, that is,

$$h_{ab} = h_{ba}\,, \qquad h^a{}_b h^b{}_c = h^a{}_c\,, \qquad h^a{}_a = n_s\,. \tag{2.76}$$

It is easy to see that the tensor h_{ab} is nothing but the metric tensor on the semi-simple subspace $T_x M_s$.

Since the curvature exists only in the semi-simple submanifold M_s, the components of the curvature tensor R_{abcd}, as well as the tensors $E^i{}_{ab}$, are non-zero only in the semi-simple subspace $T_x M_s$. Let

$$q^a{}_b = \delta^a{}_b - h^a{}_b \tag{2.77}$$

be the projection tensor to the flat subspace $\mathbb{R}^{n_0}$ such that

$$q_{ab} = q_{ba}\,, \qquad q^a{}_b q^b{}_c = q^a{}_c\,, \qquad q^a{}_a = n_0\,, \qquad q^a{}_b h^b{}_c = 0\,. \tag{2.78}$$

Then

$$R_{abcd} q^a{}_e = R_{ab} q^a{}_e = E^i{}_{ab} q^a{}_e = D^a{}_{ib} q^b{}_e = D^a{}_{ib} q_a{}^e = 0\,. \tag{2.79}$$

Now, we introduce a new type of indices, the capital Latin indices, $A, B, C, \ldots,$ which split according to $A = (a, i)$ and run from 1 to $N = p + n$. We define new quantities $C^A{}_{BC}$ by

$$C^i{}_{ab} = E^i{}_{ab}, \qquad C^a{}_{ib} = -C^a{}_{bi} = D^a{}_{ib}, \qquad C^i{}_{kl} = F^i{}_{kl}\,, \tag{2.80}$$

all other components being zero. Let us also introduce rectangular $p \times n$ matrices T_a by $(T_a)^j{}_c = E^j{}_{ac}$ and the $n \times p$ matrices $\bar{T}_a$ by $(\bar{T}_a)^b{}_i = -D^b{}_{ia}$. Then we can define $N \times N$ matrices $C_A = (C_a, C_i)$

$$C_a = \begin{pmatrix} 0 & \bar{T}_a \\ T_a & 0 \end{pmatrix}, \qquad C_i = \begin{pmatrix} D_i & 0 \\ 0 & F_i \end{pmatrix}, \tag{2.81}$$

so that $(C_A)^B{}_C = C^B{}_{AC}$.

Theorem 1 *The quantities $C^A{}_{BC}$ satisfy the Jacobi identities*

$$C^A{}_{B[C}C^C{}_{DE]} = 0 \,. \tag{2.82}$$

This means that the matrices C_A satisfy the commutation relations

$$[C_A, C_B] = C^C{}_{AB}C_C \,, \tag{2.83}$$

or, in more details,

$$\begin{aligned} [C_a, C_b] &= E^i{}_{ab}C_i, & (2.84)\\ [C_i, C_a] &= D^b{}_{ia}C_b, & (2.85)\\ [C_i, C_k] &= F^j{}_{ik}C_j \,, & (2.86) \end{aligned}$$

and generate the adjoint representation of a Lie algebra $\mathcal{G}$ with the structure constants $C^A{}_{BC}$.

Proof. This can be proved by using the eqs. (2.66), (2.67), (2.69) and (2.70) [10].

For the lack of a better name we call the algebra $\mathcal{G}$ the *curvature algebra*. As it will be clear from the next section it is a subalgebra of the total isometry algebra of the symmetric space. It should be clear that the holonomy algebra $\mathcal{H}$ is the subalgebra of the curvature algebra $\mathcal{G}$. The curvature algebra exists only in symmetric spaces; it is the eq. (2.70) that closes this algebra.

Next, we define a symmetric nondegenerate $N \times N$ matrix

$$(\gamma_{AB}) = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix} = \mathrm{diag}\left(\underbrace{1, \dots, 1}_{n}, \lambda_1, \dots, \lambda_p\right). \tag{2.87}$$

This matrix and its inverse $(\gamma^{AB}) = \begin{pmatrix} \delta^{ab} & 0 \\ 0 & \beta^{ik} \end{pmatrix} = \mathrm{diag}\,(\underbrace{1, \dots, 1}_{n}, \lambda_1^{-1}, \dots, \lambda_p^{-1})$ will be used to lower and to raise the capital Latin indices.

Finally, by using the eqs. (2.70) and (2.71) one can show the following:

Proposition 2 *The matrix γ_{AB} is G-invariant and satisfies the equation*

$$\gamma_{AB}C^B{}_{CD} + \gamma_{DB}C^B{}_{CA} = 0\,. \tag{2.88}$$

In matrix notation this equation takes the form

$$(C_A)^T = -\gamma C_A \gamma^{-1}\,, \tag{2.89}$$

which means that the adjoint and the coadjoint representations of the curvature group are equivalent. In particular, the matrices C_A are traceless.

Thus the curvature algebra $\mathcal{G}$ is compact; it is a direct sum of two ideals,

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_s, \tag{2.90}$$

an Abelian center $\mathcal{G}_0$ of dimension n_0 and a semi-simple algebra $\mathcal{G}_s$ of dimension $p + n_s$.

It is worth mentioning that although the holonomy algebra $\mathcal{H}$ is compact the (indefinite, in general) metric, β_{ij}, introduced above is *not* equal to the (positive definite) Cartan-Killing form, ρ_{ij}, defined by

$$\mathrm{tr}\,_{TM} D_i D_k = D^a{}_{ib} D^b{}_{ka} = -\rho_{ik}\,, \tag{2.91}$$

so that

$$\rho_{ik} = \mathrm{diag}\,(\lambda_1^2, \dots, \lambda_p^2)\,, \tag{2.92}$$

and

$$\beta^{ik}\rho_{ik} = R\,. \tag{2.93}$$

Similarly, the generators F_i satisfy

$$\mathrm{tr}\,_H F_i F_k = F^j{}_{im} F^m{}_{kj} = -4\frac{R_H}{R}\rho_{ik}\,, \tag{2.94}$$

where

$$R_H = -\frac{1}{4}\beta^{ik} F^j{}_{im} F^m{}_{kj}\,. \tag{2.95}$$

The Killing-Cartan form $\mathrm{tr}\,_G C_A C_B$ for the curvature algebra $\mathcal{G}$ is defined by

$$\mathrm{tr}\,_G C_a C_b = -\frac{2}{n_s} h_{ab} R\,, \tag{2.96}$$

$$\mathrm{tr}\,_G C_i C_j = -\left(1 + 4\frac{R_H}{R}\right)\rho_{ij}\,, \tag{2.97}$$

$$\mathrm{tr}\,_G C_a C_i = 0\,. \tag{2.98}$$

Notice that it is degenerate and is not equal to the metric γ_{AB}.

2.4 Killing Vectors Fields

We will use extensively the isometries of the symmetric space M. We follow the approach developed in [10, 2, 5, 13]. The generators of isometries are the Killing vector fields ξ defined by the equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \,. \tag{2.99}$$

The integrability conditions for this equation are

$$R_{\alpha\beta\mu[\lambda}\nabla_{\nu]}\xi^\mu + R_{\lambda\nu\mu[\beta}\nabla_{\alpha]}\xi^\mu = 0 \,. \tag{2.100}$$

By differentiating this equation, commuting derivatives and using curvature identities we obtain

$$\nabla_\mu \nabla_\nu \xi^\lambda = -R^\lambda{}_{\nu\alpha\mu}\xi^\alpha \,, \tag{2.101}$$

which means, in particular,

$$\Delta \xi^\lambda = -R^\lambda{}_\alpha \xi^\alpha \,. \tag{2.102}$$

By induction we obtain

$$\begin{aligned}
\nabla_{\mu_{2k}} \cdots \nabla_{\mu_1}\xi^\lambda &= (-1)^k R^\lambda{}_{\mu_1\alpha_1\mu_2} R^{\alpha_1}{}_{\mu_3\alpha_2\mu_4} \cdots R^{\alpha_{k-1}}{}_{\mu_{2k-1}\alpha_k\mu_{2k}}\xi^{\alpha_k} \,, && (2.103)\\
\nabla_{\mu_{2k+1}} \cdots \nabla_{\mu_1}\xi^\lambda &= (-1)^k R^\lambda{}_{\mu_1\alpha_1\mu_2} R^{\alpha_1}{}_{\mu_3\alpha_2\mu_4} \cdots R^{\alpha_{k-1}}{}_{\mu_{2k-1}\alpha_k\mu_{2k}}\nabla_{\mu_{2k+1}}\xi^{\alpha_k} \,. && (2.104)
\end{aligned}$$

These derivatives determine all coefficients of the covariant Taylor series (2.42) for the Killing vectors, and therefore, every Killing vector in a symmetric space has the form

$$\xi^a(x) = \left(\cos\sqrt{K}\right)^a{}_b \xi^b(x') + \left(\frac{\sin\sqrt{K}}{\sqrt{K}}\right)^a{}_b y^c \xi^b{}_{;c}(x') \,, \tag{2.105}$$

or

$$\xi(x) = \left\{\left(\sqrt{K}\cot\sqrt{K}\right)^a{}_b \xi^b(x') + \xi^a{}_{;c}(x')y^c\right\}\frac{\partial}{\partial y^a} \,. \tag{2.106}$$

Thus, Killing vector fields at any point x are determined by their values $\xi^a(x')$ and the values of their derivatives $\xi^a{}_{;c}(x')$ at the fixed point x'.

Similarly we can obtain the derivatives of the Killing vectors

$$\begin{aligned}
\xi^a{}_{;b}(x) &= \xi^a{}_{;b}(x') - R^a{}_{bcd}y^d\left(\frac{1-\cos\sqrt{K}}{K}\right)^c{}_e y^f \xi^e{}_{;f}(x') \\
&\quad - R^a{}_{bcd}y^d\left(\frac{\sin\sqrt{K}}{\sqrt{K}}\right)^c{}_e \xi^e(x') \,. && (2.107)
\end{aligned}$$

The set of all Killing vector fields forms a representation of the isometry algebra, the Lie algebra of the isometry group of the manifold M. We define two subspaces of the isometry algebra. One subspace is formed by Killing vectors satisfying the initial conditions

$$\nabla_\mu \xi^\nu \Big|_{x=x'} = 0\,, \tag{2.108}$$

and another subspace is formed by the Killing vectors satisfying the initial conditions

$$\xi^\nu \Big|_{x=x'} = 0\,. \tag{2.109}$$

We will call the Killing vectors from the first subspace *translations* and the Killing vectors from the second group *rotations*. However, this should not be understood literally.

One can easily show that the initial values $\xi^a(x')$ are independent and, therefore, there are n such parameters. Thus, there are n linearly independent translations, which can be chosen in the form

$$P_a = \left(\sqrt{K}\cot\sqrt{K}\right)^b{}_a \frac{\partial}{\partial y^b}\,, \tag{2.110}$$

so that

$$e^b{}_\mu P_a{}^\mu \Big|_{x=x'} = \delta^b{}_a\,, \qquad P_a{}^\mu{}_{;\nu}\Big|_{x=x'} = 0, \tag{2.111}$$

It is worth poiting out that the nature of the lower index of the Killing vectors $P_a{}^\mu$ is different from the frame indices. This means, in particular, that the covariant derivative of $P_a{}^\mu$ does not include the spin connection associated with the lower index. In other words, $P_a{}^\mu$ are just n vectors and not the components of a $(1,1)$ tensor.

On the other hand, the initial values of the derivatives $\xi^a{}_{;c}(x')$ are not independent because of the constraints (2.100). These constraints are valid only in the semi-simple subspace T_xM_s. However, in this subspace, due to the identity (2.73), it should be clear that there are p linearly independent rotations

$$L_i = -D^b{}_{ia} y^a \frac{\partial}{\partial y^b}\,, \tag{2.112}$$

satisfying the initial conditions

$$L_i{}^\mu \Big|_{x=x'} = 0\,, \qquad e^a{}_\mu e_b{}^\nu L_i{}^\mu{}_{;\nu}\Big|_{x=x'} = -D^a{}_{ib}\,. \tag{2.113}$$

More generally, by using (2.107) we also obtain

$$e^a{}_\mu e_b{}^\nu P_e{}^\mu{}_{;\nu} = -R^a{}_{bcd} y^d \left(\frac{\sin\sqrt{K}}{\sqrt{K}}\right)^c{}_e\,, \tag{2.114}$$

$$e^a{}_\mu e_b{}^\nu L_i{}^\mu{}_{;\nu} = -D^a{}_{ib} + R^a{}_{bcd} y^d \left(\frac{1-\cos\sqrt{K}}{K}\right)^c{}_e y^f D^e{}_{if}\,. \tag{2.115}$$

This means, in particular, that the derivatives of all Killing vectors have the form

$$\xi_A{}^a{}_{;b} = -D^a{}_{ib}\eta_A{}^i\,, \tag{2.116}$$

where $\eta_A{}^i$ are defined by

$$\eta_A{}^i = \alpha^{ij}\xi_A{}^a{}_{;b} D^b{}_{ja}\,, \tag{2.117}$$

and the matrix $\alpha^{ij} = (\rho_{ij})^{-1}$ is the inverse matrix of the Cartan-Killing form ρ defined by (2.91). Notice that

$$\eta_a{}^i\Big|_{x=x'} = 0\,, \qquad \eta_j{}^i\Big|_{x=x'} = \delta^i_j\,. \tag{2.118}$$

Then, from the eq. (2.101) we also immediately obtain

$$\eta_A{}^i{}_{;b} = -E^i{}_{ab}\xi_A{}^a\,. \tag{2.119}$$

By adding the trivial Killing vectors for flat subspaces we find that the dimension of the rotation subspace is equal to

$$p + n_0 n_s + \frac{n_0(n_0-1)}{2}\,. \tag{2.120}$$

Here $n_0 n_s$ is the number of mixed rotations between M_0 and M_s and $n_0(n_0-1)/2$ is the number of rotations of M_0. Since $p \le n_s(n_s-1)/2$, then the above number of rotations is less or equal to $n(n-1)/2$ as it should be (recall that $n = n_0 + n_s$).

In the following we will need only the Killing vectors P_a and L_i defined above. We introduce the following notation $(\xi_A) = (P_a, L_i)$.

Theorem 2 *The Killing vector fields ξ_A satisfy the commutation relations*

$$[\xi_A, \xi_B] = C^C{}_{AB}\xi_C\,, \tag{2.121}$$

or, in more detail,

$$[P_a, P_b] = E^i{}_{ab} L_i, \tag{2.122}$$

$$[L_i, P_a] = D^b{}_{ia} P_b, \tag{2.123}$$

$$[L_i, L_k] = F^j{}_{ik} L_j\,. \tag{2.124}$$

Proof. This can be proved by using the explicit form of the Killing vector fields obtained above [10].

Notice that they *do not* generate the complete isometry algebra of the symmetric space M but rather they form a representation of the curvature algebra $\mathcal{G}$ introduced in the previous section, which is a subalgebra of the total isometry algebra.

It is clear that the Killing vector fields L_i form a representation of the holonomy algebra $\mathcal{H}$, which is the isotropy algebra of the semi-simple submanifold M_s, and a subalgebra of the total isotropy algebra of the symmetric space M.

Proposition 3 *There holds*

$$\xi^c_{A;a}\xi_B{}^b{}_{;c} - \xi^c_{B;a}\xi_A{}^b{}_{;c} = C^C{}_{AB}\xi_C{}^b{}_{;a} - R^b{}_{acd}\xi_A{}^c\xi_B{}^d\,, \tag{2.125}$$

and

$$F^j{}_{ik}\eta_A{}^i\eta_B{}^k = C^C{}_{AB}\eta_C{}^j - E^j{}_{cd}\xi^c_A\xi_B{}^d\,. \tag{2.126}$$

Proof. By differentiating the eq. (2.121) and using (2.101) we obtain (2.125). Finally, by using (2.116) and the holonomy algebra (2.67) we obtain (2.126).

Now, we derive some bilinear identities that we will need in the present paper.

Proposition 4 *The Killing vector fields satisfy the equation*

$$\gamma^{AB}\xi_A{}^\mu\xi_B{}^\nu = \delta^{ab}P_a{}^\mu P_b{}^\nu + \beta^{ik}L_i{}^\mu L_k{}^\nu = g^{\mu\nu}\,, \tag{2.127}$$

Proof. This can be proved by using the explicit form of the Killing vectors.

Proposition 5 *There holds*

$$\gamma^{AB}\xi_A{}^\alpha\xi_B{}^\mu{}_{;\nu\lambda} = R^\alpha{}_{\lambda\nu\mu}\,. \tag{2.128}$$

Proof. This follows from eqs. (2.101) and (2.127).

Proposition 6 *There holds*

$$\gamma^{AB}\xi_A{}^\mu\xi_B{}^\nu{}_{;\beta} = 0\,, \tag{2.129}$$

$$\gamma^{AB}\xi_A{}^\mu{}_{;\alpha}\xi_B{}^\nu{}_{;\beta} = R^\mu{}_\alpha{}^\nu{}_\beta\,. \tag{2.130}$$

Proof. Let

$$\tau^{\mu\alpha}{}_{\nu} = \gamma^{AB}\xi_A{}^{\mu}\xi_B{}^{\alpha}{}_{;\nu} \tag{2.131}$$

and

$$\theta^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} = \gamma^{AB}\xi_A{}^{\mu}{}_{;\alpha}\xi_B{}^{\nu}{}_{;\beta}\,. \tag{2.132}$$

We compute

$$\nabla_\beta\tau_{\mu\alpha\nu} = \theta_{\mu\beta\nu\alpha} - R_{\mu\beta\nu\alpha}\,, \tag{2.133}$$

and

$$\nabla_\gamma\nabla_\beta\tau_{\mu\alpha\nu} = R_{\mu\beta\gamma\rho}\tau^{\rho}{}_{\nu\alpha} + R_{\nu\alpha\gamma\rho}\tau^{\rho}{}_{\mu\beta}\,. \tag{2.134}$$

All higher derivatives of $\tau_{\mu\nu\alpha}$ are expressed linearly in terms of $\tau_{\mu\nu\alpha}$ and its first derivative $\nabla_\beta\tau_{\mu\alpha\nu}$ with coefficients polynomial in curvature.

Let x' be a fixed point. We will show that the tensor $\tau_{\mu\nu\alpha}$ together with all its covariant derivatives is equal to zero at $x = x'$. This will then mean that $\tau_{\mu\nu\alpha} = 0$ identically and, therefore, from eq. (2.133) that $\theta^{\mu}{}_{\alpha\nu\beta} = R^{\mu}{}_{\alpha\nu\beta}$.

We have

$$\tau^{\mu\alpha}{}_{\nu} = \delta^{ab}P_a{}^{\mu}P_b{}^{\alpha}{}_{;\nu} + \beta^{ij}L_i{}^{\mu}L_j{}^{\alpha}{}_{;\nu}\,. \tag{2.135}$$

and

$$\theta^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} = \delta^{ab}P_a{}^{\mu}{}_{;\alpha}P_b{}^{\nu}{}_{;\beta} + \beta^{ij}L_i{}^{\mu}{}_{;\alpha}L_j{}^{\nu}{}_{;\beta}\,. \tag{2.136}$$

Therefore,

$$\tau^{\mu}{}_{\alpha\nu}\Big|_{x=x'} = 0 \tag{2.137}$$

and

$$\theta^{\mu}{}_{\alpha\nu\beta}\Big|_{x=x'} = R^{\mu}{}_{\alpha\nu\beta}\,. \tag{2.138}$$

Therefore,

$$\nabla_\beta\tau^{\mu}{}_{\alpha\nu}\Big|_{x=x'} = 0\,. \tag{2.139}$$

Thus, by induction, all derivatives of $\tau_{\mu\nu\alpha}$ vanish, and, therefore, $\tau_{\mu\nu\alpha} = 0$ identically. This also proves (2.130) by making use of (2.133).

Let i, j be non-negative integers. We define the tensors $X_{(i,j)}$ which are bilinear in Killing vectors by

$$X_{(i,j)}{}^{\mu\nu}{}_{\alpha_1\ldots\alpha_i\beta_1\ldots\beta_j} = \gamma^{AB}\nabla_{\alpha_1}\cdots\nabla_{\alpha_i}\xi_A{}^{\mu}\nabla_{\beta_1}\cdots\nabla_{\beta_j}\xi_B{}^{\nu}\,. \tag{2.140}$$

Theorem 3 *1. The tensors $X_{(i,j)}$ are G-invariant and parallel, that is,*

$$\nabla_\lambda X_{(i,j)} = 0\,. \tag{2.141}$$

2. For even $(i + j)$ the tensors $X_{(i,j)}$ are polynomial in the curvature tensor.

3. *For odd* $(i + j)$ *the tensors* $X_{(i,j)}$ *are identically equal to zero.*

Proof. First of all, we notice that (1) follows from (2) and (3).

There are three cases: a) both $i = 2k$ and $j = 2m$ are even, b) both $i = 2k + 1$ and $j = 2m + 1$ are odd, and c) $i = 2k$ is even and $j = 2m + 1$ is odd.

In the case (a), when both i and j are even, by using the eqs. (2.103) and (2.127) we immediately obtain a polynomial in the curvature.

In the cases (b) and (c) by using the eqs. (2.103) and (2.104) we reduce it to the tensors $\gamma^{AB}\xi_A{}^\mu{}_{;\alpha}\xi_B{}^\nu{}_{;\beta}$ and $\gamma^{AB}\xi_A{}^\mu\xi_B{}^\nu{}_{;\beta}$. Now, by using the lemma we prove the theorem.

Proposition 7 *There holds*

$$\gamma^{AB}\xi_A{}^\mu\eta_B{}^i = 0\,, \tag{2.142}$$

$$\gamma^{AB}\eta_A{}^i\eta_B{}^j = \beta^{ij}\,. \tag{2.143}$$

Proof. This follows from the definition of $\eta_A{}^i$ (2.117) and eqs. (2.129) and (2.130).

2.5 Homogeneous Vector Bundles

Equation (2.25) imposes strong constraints on the curvature of the homogeneous bundle $\mathcal{W}$. We define

$$\mathcal{B}_{ab} = \mathcal{F}_{cd}q^c{}_a q^d{}_b\,, \tag{2.144}$$

$$\mathcal{E}_{ab} = \mathcal{F}_{cd}h^c{}_b h^d{}_b\,, \tag{2.145}$$

so that

$$\mathcal{B}_{ab}h^a{}_c = 0\,, \qquad \mathcal{E}_{ab}q^a{}_c = 0\,. \tag{2.146}$$

Then, from eq. (2.25) we obtain

$$[\mathcal{B}_{ab}, \mathcal{B}_{cd}] = [\mathcal{B}_{ab}, \mathcal{E}_{cd}] = 0\,, \tag{2.147}$$

and

$$[\mathcal{E}_{cd}, \mathcal{E}_{ab}] - R^f{}_{acd}\mathcal{E}_{fb} - R^f{}_{bcd}\mathcal{E}_{af} = 0\,. \tag{2.148}$$

This means that $\mathcal{B}_{ab}$ takes values in an Abelian ideal of the gauge algebra $\mathcal{G}_{YM}$ and $\mathcal{E}_{ab}$ takes values in the holonomy algebra. More precisely, eq. (2.148) is only possible if the holonomy algebra $\mathcal{H}$ is an ideal of the gauge algebra $\mathcal{G}_{YM}$. Thus, the gauge group G_{YM} must have a subgroup $Z \times H$, where Z is an Abelian group and H is the holonomy group.

We proceed in the following way. The matrices $D^a{}_{ib}$ provide a natural embedding of the holonomy algebra $\mathcal{H}$ in the orthogonal algebra $\mathcal{SO}(n)$ in the following sense. Let X_{ab} be the generators of the orthogonal algebra $\mathcal{SO}(n)$ is some representation satisfying the commutation relations (2.8). Let T_i be the matrices defined by

$$T_i = -\frac{1}{2}D^a{}_{ib}X^b{}_a\,. \tag{2.149}$$

Proposition 8 *The matrices T_i satisfy the commutation relations*

$$[T_i, T_k] = F^j{}_{ik}T_j \tag{2.150}$$

and form a representation T of the holonomy algebra $\mathcal{H}$.

This can be proved by taking into account the orthogonal algebra (2.8).

Thus T_i are the generators of the gauge algebra $\mathcal{G}_{YM}$ realizing a representation T of the holonomy algebra $\mathcal{H}$. Since $\mathcal{B}_{ab}$ takes values in the Abelian ideal of the algebra of the gauge group we also have

$$[\mathcal{B}_{ab}, T_j] = 0\,. \tag{2.151}$$

Then by using eq. (2.70) one can show that [3]

$$\mathcal{E}_{ab} = \frac{1}{2}R^{cd}{}_{ab}X_{cd} = -E^i{}_{ab}T_i\,. \tag{2.152}$$

Proposition 9 *The two form*

$$\begin{aligned}\mathcal{F}_{ab} &= -E^i{}_{ab}T_i + \mathcal{B}_{ab}\\ &= \frac{1}{2}R^{cd}{}_{ab}X_{cd} + \mathcal{B}_{ab}\end{aligned} \tag{2.153}$$

satisfies the constrains (2.25), and, therefore, gives the curvature of the homogeneous bundle $\mathcal{W}$.

Now, we consider the representation Σ of the orthogonal algebra defining the spin-tensor bundle $\mathcal{T}$ and define the matrices

$$G_{ab} = \Sigma_{ab} \otimes \mathbb{I}_X + \mathbb{I}_\Sigma \otimes X_{ab}\,. \tag{2.154}$$

[3] We correct here a sign misprint in eq. (3.24) in [10].

Obviously, these matrices are the generators of the orthogonal algebra in the product representation $\Sigma \otimes X$.

Next, the matrices

$$Q_i = -\frac{1}{2} D^a{}_{ib} \Sigma^b{}_a \tag{2.155}$$

form a representation Q of the holonomy algebra $\mathcal{H}$. and the matrices

$$\begin{aligned} \mathcal{R}_i &= Q_i \otimes \mathbb{I}_T + \mathbb{I}_\Sigma \otimes T_i \\ &= -\frac{1}{2} D^a{}_{ib} G^b{}_a \end{aligned} \tag{2.156}$$

are the generators of the holonomy algebra in the product representation $\mathcal{R} = Q \otimes T$.

Then the total curvature, that is, the commutator of covariant derivatives, (2.20) of a twisted spin-tensor bundle $\mathcal{V}$ is

$$\begin{aligned} \Omega_{ab} &= -E^i{}_{ab} \mathcal{R}_i + \mathcal{B}_{ab} \\ &= \frac{1}{2} R^{cd}{}_{ab} G_{cd} + \mathcal{B}_{ab} \,. \end{aligned} \tag{2.157}$$

Finally, we define the Casimir operators of the holonomy algebra in the representations Q, T and $\mathcal{R}$

$$T^2 = C_2(H, T) = \beta^{ij} T_i T_j = \frac{1}{4} R^{abcd} X_{ab} X_{cd} \,, \tag{2.158}$$

$$Q^2 = C_2(H, Q) = \beta^{ij} Q_i Q_j = \frac{1}{4} R^{abcd} \Sigma_{ab} \Sigma_{cd} \,, \tag{2.159}$$

$$\mathcal{R}^2 = C_2(H, \mathcal{R}) = \beta^{ij} \mathcal{R}_i \mathcal{R}_j = \frac{1}{4} R^{abcd} G_{ab} G_{cd} \,. \tag{2.160}$$

They commute with all matrices T_i, Q_i and $\mathcal{R}_i$ respectively.

2.6 Twisted Lie Derivatives

Let φ be a section of a twisted homogeneous spin-tensor bundle $\mathcal{T}$. Let ξ_A be the basis of Killing vector fields. Then the covariant (or generalized, or twisted) Lie derivative of φ along ξ_A is defined by

$$\mathcal{L}_A \varphi = \mathcal{L}_{\xi_A} \varphi = \left(\nabla_{\xi_A} + S_A \right) \varphi \,, \tag{2.161}$$

where $\nabla_{\xi_A} = \xi_A{}^\mu \nabla_\mu$,

$$S_A = \eta_A{}^i \mathcal{R}_i = \frac{1}{2}\xi_A{}^a{}_{;b} G^b{}_a\,, \tag{2.162}$$

and $\eta_A{}^i$ are defined by (2.117). Note that

$$S_a q^a{}_b = 0\,. \tag{2.163}$$

Proposition 10 *There hold*

$$[\nabla_{\xi_A}, \nabla_{\xi_B}]\varphi = \left(C^C{}_{AB}\nabla_{\xi_C} - \mathcal{R}_{AB} + \mathcal{B}_{AB}\right)\varphi\,, \tag{2.164}$$

$$\nabla_{\xi_A} S_B = \mathcal{R}_{AB}\,, \tag{2.165}$$

$$[S_A, S_B] = C^C{}_{AB} S_C - \mathcal{R}_{AB}\,, \tag{2.166}$$

where

$$\begin{aligned} \mathcal{R}_{AB} &= \xi_A{}^a \xi_B{}^b E^i{}_{ab} \mathcal{R}_i \\ &= -\frac{1}{2} R^{cd}{}_{ab} \xi_A{}^a \xi_B{}^b G_{cd}\,, \end{aligned} \tag{2.167}$$

$$\mathcal{B}_{AB} = \xi_A{}^a \xi_B{}^b \mathcal{B}_{ab}\,. \tag{2.168}$$

Proof. By using the properties of the Killing vectors described in the previous section and the eq. (2.157) we obtain first (2.164). Next, by using the eqs. (2.119) we obtain (2.165), and, further, by using the eq. (2.126) we get (2.166).

Notice that from the definition (2.144) we have

$$P_c{}^a L_i{}^b \mathcal{B}_{ab} = L_i{}^a L_j{}^b \mathcal{B}_{ab} = 0\,, \tag{2.169}$$

and

$$P_c{}^a P_d{}^b \mathcal{B}_{ab} = \mathcal{B}_{cd}\,. \tag{2.170}$$

This means that the matrix $\mathcal{B}_{AB}$ has the form

$$\mathcal{B}_{AB} = \begin{pmatrix} \mathcal{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.171}$$

and, therefore,

$$C^A{}_{BC}\mathcal{B}_{AD} = \gamma^{BD} C^A{}_{BC} \mathcal{B}_{DE} = 0\,. \tag{2.172}$$

We define the operator

$$\mathcal{L}^2 = \gamma^{AB}\mathcal{L}_A\mathcal{L}_B\,. \tag{2.173}$$

Theorem 4 *The operators $\mathcal{L}_A$ and $\mathcal{L}^2$ satisfy the commutation relations*

$$[\mathcal{L}_A, \mathcal{L}_B] = C^C{}_{AB}\mathcal{L}_C + \mathcal{B}_{AB}, \qquad (2.174)$$

or, in more details,

$$\begin{aligned} [\mathcal{L}_a, \mathcal{L}_b] &= E^i{}_{ab}\mathcal{L}_i + \mathcal{B}_{ab}, & (2.175)\\ [\mathcal{L}_i, \mathcal{L}_a] &= D^b{}_{ia}\mathcal{L}_b, & (2.176)\\ [\mathcal{L}_i, \mathcal{L}_j] &= F^k{}_{ij}\mathcal{L}_k\,, & (2.177) \end{aligned}$$

and

$$[\mathcal{L}_A, \mathcal{L}^2] = 2\gamma^{BC}\mathcal{B}_{AB}\mathcal{L}_C\,. \qquad (2.178)$$

Proof. This follows from

$$[\mathcal{L}_A, \mathcal{L}_B] = [\nabla_{\xi_A}, \nabla_{\xi_B}] + [\nabla_{\xi_A}, S_B] - [\nabla_{\xi_B}, S_A] + [S_A, S_B] \qquad (2.179)$$

and eqs. (2.164), (2.165), and (2.166). The eq. (2.178) follows directly from (2.174).

The operators $\mathcal{L}_A$ form an algebra that is a direct sum of a nilpotent ideal and a semisimple algebra. For the lack of a better name we call this algebra *gauged curvature algebra* and denote it by $\mathcal{G}_{\rm gauge}$.

Proposition 11 *There hold*

$$\begin{aligned} \gamma^{AB}\xi_A{}^\mu S_B &= 0\,, & (2.180)\\ \gamma^{AB}\nabla_{\xi_A} S_B &= 0\,, & (2.181)\\ \gamma^{AB}S_A S_B &= \mathcal{R}^2\,. & (2.182) \end{aligned}$$

Proof. This can be proved by using the eqs. (2.142), (2.165) and (2.143).

Theorem 5 *The Laplacian Δ acting on sections of a twisted spin-tensor bundle $\mathcal{V}$ over a symmetric space has the form*

$$\Delta = \mathcal{L}^2 - \mathcal{R}^2\,. \qquad (2.183)$$

$$[\mathcal{L}_A, \Delta] = 2\gamma^{BC}\mathcal{B}_{AB}\mathcal{L}_C\,. \qquad (2.184)$$

Proof. We have

$$\gamma^{AB}\mathcal{L}_A\mathcal{L}_B = \gamma^{AB}\nabla_{\xi_A}\nabla_{\xi_B} + \gamma^{AB}S_A\nabla_{\xi_B} + \gamma^{AB}\nabla_{\xi_A}S_B + \gamma^{AB}S_A S_B\,. \qquad (2.185)$$

Now, by using eqs. (2.127) and (2.129) we get

$$\gamma^{AB}\nabla_{\xi_A}\nabla_{\xi_B} = \Delta\,. \qquad (2.186)$$

Next, by using the eqs. (2.165), (2.181) and (2.182), we obtain (2.183). The eq. (2.184) follows from the commutation relations (2.174).

2.7 Isometries and Pullbacks

Let ω^i be the canonical coordinates on the holonomy group and $(k^A) = (p^a, \omega^i)$ be the canonical coordinates on the gauged curvature group. We fix a point x' so that the basis Killing vectors fields ξ_A satisfy the initial conditions (2.111)-(2.113) and are given by (2.110)-(2.112). Let $\xi = \langle k, \xi\rangle = k^A\xi_A = p^a P_a + \omega^i L_i$ be a Killing vector field and let $\psi_t : M \to M$ be the one-parameter diffeomorphism (the isometry) generated by the vector field ξ. Let $\hat{x} = \psi_t(x)$, so that

$$\frac{d\hat{x}}{dt} = \xi(\hat{x}) \tag{2.187}$$

and

$$\hat{x}\Big|_{t=0} = x\,. \tag{2.188}$$

The solution of this equation depends on the parameters t, p, ω, x and x', that is,

$$\hat{x} = \hat{x}(t, p, \omega, x, x')\,. \tag{2.189}$$

We will be interested mainly in the case when the points x and x' are close to each other. In fact, at the end of our calculations we will take the limit $x = x'$. In this case, as we will show below, the Jacobian

$$\det\left(\frac{\partial \hat{x}^\mu}{\partial p^a}\right) \neq 0 \tag{2.190}$$

is not equal to zero, and, therefore, coordinates p can be used to parametrize the point $\hat{x}$, that is, the eq. (2.189) defines the function

$$p = p(t, \omega, \hat{x}, x, x')\,. \tag{2.191}$$

We will be interested in those trajectories that reach the point x' at the time $t = 1$. So, we look at the values $\hat{x}(1, p, \omega, x, x')$ when the parameters p are varied. Then, as we will show below, there is always a value of the parameters p that we call $\bar{p}$ such that

$$\hat{x}(1, \bar{p}, \omega, x, x') = x'\,. \tag{2.192}$$

Thus, eq. (2.192) defines a function $\bar{p} = \bar{p}(\omega, x, x')$. Therefore, the parameters $\bar{p}$ can be used to parameterize the point x. Of course,

$$\bar{p}(\omega, x, x') = p(1, \omega, x', x, x')\,. \tag{2.193}$$

Now, we choose the normal coordinates y^a of the point defined above and the normal coordinates $\hat{y}^a$ of the point $\hat{x}$ with the origin at x', so that the normal coordinates y' of the point x' are equal to zero, $y'^a = 0$. Recall that the normal coordinates are equal to the components of the tangent vector at the point x' to the geodesic connecting the points x' and the current point, that is, $y^a = -e^a{}_{\mu'}(x')\sigma^{;\mu'}(x, x')$ and $\hat{y}^a = -e^a{}_{\mu'}(x')\sigma^{;\mu'}(\hat{x}, x')$. Then by taking into account eqs. (2.110) and (2.112) the equation (2.187) becomes

$$\frac{d\hat{y}^a}{dt} = \left(\sqrt{K(\hat{y})}\cot\sqrt{K(\hat{y})}\right)^a{}_b p^b - \omega^i D^a{}_{ib}\hat{y}^b\,, \tag{2.194}$$

with the initial condition

$$\hat{y}^a\Big|_{t=0} = y^a\,. \tag{2.195}$$

The solution of this equation defines a function $\hat{y} = \hat{y}(t, p, \omega, y)$.

Proposition 12 *The Taylor expansion of the solution $\hat{y} = \hat{y}(t, p, \omega, y)$ of the eq. (2.194) in t reads*

$$\hat{y}^a = y^a + \left[\left(\sqrt{K(y)}\cot\sqrt{K(y)}\right)^a{}_b p^b - \omega^i D^a{}_{ib} y^b\right] t + O(t^2)\,. \tag{2.196}$$

The Taylor expansion of the function $\hat{y} = \hat{y}(t, p, \omega, y)$ in p and y reads

$$\hat{y}^a = (\exp[-tD(\omega)])^a{}_b y^b + \left(\frac{1 - \exp[-tD(\omega)]}{D(\omega)}\right)^a{}_b p^b + O(y^2, p^2, py)\,. \tag{2.197}$$

There holds

$$\det\left(\frac{\partial\hat{y}^a}{\partial p^b}\right)\Bigg|_{p=y=0,t=1} = \det{}_{TM}\left(\frac{\sinh[\,D(\omega)/2]}{D(\omega)/2}\right)\,. \tag{2.198}$$

Proof. The eq. (2.196) trivially follows from the eq. (2.194).

Let us expand the function $\hat{y}(t, p, \omega, y)$ in Taylor series in p and y restricting ourselves to linear terms, that is,

$$\hat{y}^a = \hat{y}^a\Big|_{p=y=0} + \frac{\partial\hat{y}^a}{\partial p^b}\Big|_{p=y=0} p^b + \frac{\partial\hat{y}^a}{\partial y^b}\Big|_{p=y=0} y^b + O(p^2, y^2, py)\,. \tag{2.199}$$

First of all, for $p = 0$ the eq. (2.194) becomes

$$\frac{d\hat{y}^a}{dt} = -\omega^i D^a{}_{ib}\hat{y}^b\,. \tag{2.200}$$

The solution of this equation with the initial condition $\hat{y} = 0$ is trivial, therefore,

$$\hat{y}\Big|_{p=y=0} = \hat{y}(t, 0, \omega, 0) = 0\,. \tag{2.201}$$

Next, by differentiating the eq. (2.200) with respect to y^b and setting $y = 0$ we obtain the equation

$$\frac{d}{dt}\frac{\partial\hat{y}^a}{\partial y^b}\Big|_{p=y=0} = -\omega^i D^a{}_{ic}\frac{\partial\hat{y}^c}{\partial y^b}\Big|_{p=y=0}\,. \tag{2.202}$$

with the initial condition

$$\frac{\partial\hat{y}^a}{\partial y^b}\Big|_{p=y=t=0} = \delta^a{}_b\,. \tag{2.203}$$

The solution of this equation is

$$\frac{\partial\hat{y}^a}{\partial y^b}\Bigg|_{p=y=0} = \left(\exp[-tD(\omega)]\right)^a{}_b\,, \tag{2.204}$$

where

$$D(\omega) = \omega^i D_i\,. \tag{2.205}$$

Let

$$Z^a{}_b = \frac{\partial\hat{y}^a}{\partial p^b}\Big|_{p=y=0}\,. \tag{2.206}$$

Then by differentiating the eq. (2.194) with respect to p^b and setting $p = 0$, we obtain

$$\frac{d}{dt}Z^a{}_b = \delta^a{}_b - \omega^i D^a{}_{ic}Z^c{}_b\,, \tag{2.207}$$

with the initial condition

$$Z^a{}_b\Big|_{t=0} = 0\,. \tag{2.208}$$

The solution of this equation is

$$Z = \frac{1 - \exp[-tD(\omega)]}{D(\omega)}\,. \tag{2.209}$$

By substituting the eqs. (2.201), (2.204) and (2.209) in (2.199) we get the desired result (2.197).

Finally, by taking into account that the matrix $D(\omega)$ is traceless, we find first $\det\exp[tD(\omega)] = 1$, and, then by using eq. (2.209) we obtain (2.198).

The function $\hat{y} = \hat{y}(t, p, \omega, y)$ implicitly defines the function

$$p = p(t, \omega, \hat{y}, y)\,. \tag{2.210}$$

The function $\bar{p} = \bar{p}(\omega, y)$ is now defined by the equation

$$\hat{y}(1, \bar{p}, \omega, y) = 0\,, \tag{2.211}$$

or

$$\bar{p}(\omega, y) = p(1, \omega, 0, y)\,. \tag{2.212}$$

Proposition 13 *The Taylor expansion of the function $\bar{p}(\omega, y)$ in y has the form*

$$\bar{p}^a = -\left(D(\omega)\frac{\exp[-D(\omega)]}{1-\exp[-D(\omega)]}\right)^a{}_b y^b + O(y^2)\,. \tag{2.213}$$

Therefore,

$$\det\left(-\frac{\partial \bar{p}^a}{\partial y^b}\right)\Bigg|_{y=0} = \det{}_{TM}\left(\frac{\sinh[\,D(\omega)/2]}{D(\omega)/2}\right)^{-1}\,. \tag{2.214}$$

Proof. We expand $\bar{p}$ in Taylor series in y

$$\bar{p}^a = \bar{p}^a\Big|_{y=0} + \frac{\partial \bar{p}^a}{\partial y^b}\Big|_{y=0} y^b + O(y^2)\,. \tag{2.215}$$

Next, by taking into account (2.201) we have

$$\bar{p}\Big|_{y=0} = 0\,. \tag{2.216}$$

Further, by differentiating (2.211) with respect to y^c and setting $y = 0$ we get

$$\frac{\partial \hat{y}^a}{\partial y^b}\Big|_{p=y=0,t=1} + \frac{\partial \hat{y}^a}{\partial p^c}\Big|_{p=y=0,t=1}\frac{\partial \bar{p}^c}{\partial y^b}\Big|_{y=0} = 0\,, \tag{2.217}$$

and, therefore,

$$\frac{\partial \bar{p}^a}{\partial y^b}\Big|_{y=0} = -\left(D(\omega)\frac{\exp[-D(\omega)]}{1-\exp[-D(\omega)]}\right)^a{}_b\,. \tag{2.218}$$

This leads to both (2.213) and (2.214).

Now, we define

$$\Lambda^{\hat{\mu}}{}_\nu = \frac{\partial \hat{x}^\mu}{\partial x^\nu}\,. \tag{2.219}$$

The pullback of the metric by the diffeomorphism ψ_t is defined by

$$(\psi_t^* g)_{\mu\nu}(x) = \Lambda^{\hat{\alpha}}{}_\mu \Lambda^{\hat{\beta}}{}_\nu g_{\hat{\alpha}\hat{\beta}}(\hat{x})\,. \tag{2.220}$$

Since ψ_t is an isometry, we have

$$(\psi_t^* g)_{\mu\nu}(x) = g_{\mu\nu}(x)\,. \tag{2.221}$$

Therefore, the inverse matrix Λ^{-1} is equal to

$$(\Lambda^{-1})^{\mu}{}_{\hat{\alpha}} = g^{\mu\nu}(x)\Lambda^{\hat{\beta}}{}_{\nu} g_{\hat{\beta}\hat{\alpha}}(\hat{x})\,. \tag{2.222}$$

Let $e^a{}_\mu$ and $e_a{}^\mu$ be a local orthonormal frame that is obtained by parallel transport along geodesics from a point x'. Then the action of the pullback ψ_t^* on the orthonormal frame is

$$(\psi_t^* e^a)_\mu(x) = \Lambda^{\hat{\alpha}}{}_\mu e^a{}_{\hat{\alpha}}(\hat{x})\,. \tag{2.223}$$

Since ψ_t is an isometry, we have

$$\delta_{ab}(\psi_t^* e^a)_\alpha(x)(\psi_t^* e^b)_\beta(x) = \delta_{ab} e^a{}_\alpha(x) e^b{}_\beta(x)\,. \tag{2.224}$$

Therefore, the frames of 1-forms e^a and $\psi_t^* e^a$ are related by an orthogonal transformation

$$(\psi_t^* e^a)(x) = O^a{}_b e^b(x)\,, \tag{2.225}$$

where the matrix $O^a{}_b$ is defined by

$$O^a{}_b = e^a{}_{\hat{\alpha}}(\hat{x})\Lambda^{\hat{\alpha}}{}_\mu e_b{}^\mu(x)\,. \tag{2.226}$$

Proposition 14 *For $p = y = 0$ the matrix O has the form*

$$O\Big|_{p=y=0} = \exp\left[-tD(\omega)\right]\,. \tag{2.227}$$

Proof. We use normal coordinates $\hat{y}^a$ and y^a. Then the matrix O takes the form

$$O^a{}_b = e^a{}_{\hat{\alpha}} \frac{\partial \hat{x}^\alpha}{\partial \hat{y}^c}\frac{\partial \hat{y}^c}{\partial y^d}\frac{\partial y^d}{\partial x^\mu} e_b{}^\mu\,. \tag{2.228}$$

Now, by using the Jacobian matrix (2.44) and recalling that $\hat{y} = 0$ for $p = y = 0$ we obtain

$$\frac{\partial y^a}{\partial x^\mu} e_b{}^\mu\Big|_{p=y=0} = e^a{}_{\hat{\alpha}}\frac{\partial \hat{x}^\alpha}{\partial \hat{y}^b}\Big|_{p=y=0} = \delta^a{}_b\,. \tag{2.229}$$

Therefore,

$$O^a{}_b\Big|_{p=y=0} = \frac{\partial \hat{y}^a}{\partial y^b}\Big|_{p=y=0}\,, \tag{2.230}$$

and, finally (2.204) gives the desired result (2.227).

Let φ be a section of the twisted spin-tensor bundle $\mathcal{V}$. Let V_x be the fiber at the point x and $V_{\hat{x}}$ be the fiber at the point $\hat{x} = \psi_t(x)$. The pullback of the diffeomorphism ψ_t defines the map, that we call just the pullback,

$$\psi_t^* : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}) \tag{2.231}$$

on smooth sections of the twisted spin-tensor bundle $\mathcal{V}$. The pullback of tensor fields of type (p, q) is defined by

$$(\psi_t^*\varphi)^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q}(x) = \Lambda^{\hat{\beta}_1}{}_{\nu_1}\cdots\Lambda^{\hat{\beta}_q}{}_{\nu_q}(\Lambda^{-1})^{\mu_1}{}_{\hat{\alpha}_1}\cdots(\Lambda^{-1})^{\mu_p}{}_{\hat{\alpha}_p}\varphi^{\hat{\alpha}_1\dots\hat{\alpha}_p}_{\hat{\beta}_1\dots\hat{\beta}_q}(\hat{x})\,. \tag{2.232}$$

We define the twisted pullback (a combination of a proper pullback and a gauge transformation) of a tensor of type (p, q) by

$$(\psi_t^*\varphi)^{a_1\dots a_p}_{b_1\dots b_q}(x) = O^{c_1}{}_{b_1}\cdots O^{c_q}{}_{b_q}O_{d_1}{}^{a_1}\cdots O_{d_p}{}^{a_p}\varphi^{d_1\dots d_p}_{c_1\dots c_q}(\hat{x})\,. \tag{2.233}$$

Since the matrix O is orthogonal, it can be parametrized by

$$O = \exp\theta\,, \tag{2.234}$$

where θ_{ab} is an antisymmetric matrix. The orthogonal transformation of the frame pulled back causes the transformation of spinors

$$(\psi_t^*\varphi)(x) = \exp\left(-\frac{1}{4}\theta_{ab}\gamma^{ab}\right)\varphi(\hat{x})\,. \tag{2.235}$$

More generally, we have

Proposition 15 *Let φ be a section of a twisted spin-tensor bundle V. Then*

$$(\psi_t^*\varphi)(x) = \exp\left(-\frac{1}{2}\theta_{ab}G^{ab}\right)\varphi(\hat{x})\,. \tag{2.236}$$

In particular, for $p = y = 0$ (or $x = x'$)

$$(\psi_t^*\varphi)(x)\Big|_{p=y=0} = \exp\left[t\mathcal{R}(\omega)\right]\varphi(x')\,, \tag{2.237}$$

where

$$\mathcal{R}(\omega) = \omega^i\mathcal{R}_i\,. \tag{2.238}$$

Proof. First, from the eq. (2.227) we see that

$$\theta^a{}_b\Big|_{p=y=0} = -t\omega^i D^a{}_{ib}\,. \tag{2.239}$$

Then, from the definition (2.156) of the matrices $\mathcal{R}_i$ we get (2.237).

It is not very difficult to check that the Lie derivatives are nothing but the generators of the pullback, that is,

$$\mathcal{L}_\xi\varphi = k^A\mathcal{L}_A\varphi = \frac{d}{dt}(\psi_t^*\varphi)\Big|_{t=0}\,. \tag{2.240}$$

We will use this fundamental fact to compute the heat kernel diagonal below.

3 Heat Semigroup

3.1 Geometry of the Curvature Group

Let $G_{\rm gauge}$ be the gauged curvature group and H be its holonomy subgroup. Both these groups have compact algebras. However, while the holonomy group is always compact, the curvature group is, in general, a product of a nilpotent group, G_0, and a semi-simple group, G_s,

$$G_{\rm gauge} = G_0\times G_s\,. \tag{3.1}$$

The semi-simple group G_s is a product $G_s = G_+\times G_-$ of a compact G_+ and a non-compact G_- subgroups.

Let ξ_A be the basis Killing vectors, k^A be the canonical coordinates on the curvature group G and $\xi(k) = k^A\xi_A$. The canonical coordinates are exactly the normal coordinates on the group defined above. Let C_A be the generators of the curvature group in adjoint representation and $C(k) = k^AC_A$. In the following ∂_M means the partial derivative $\partial/\partial k^M$ with respect to the canonical coordinates. We define the matrix $Y^A{}_M$ by the equation

$$\exp[-\xi(k)]\partial_M\exp[\xi(k)] = Y^A{}_M\xi_A\,, \tag{3.2}$$

which is well defined since the right hand side lies in the Lie algebra of the curvature group. This can be written in the form

$$\exp[-\xi(k)]\partial_M\exp[\xi(k)] = \exp[-Ad_{\xi(k)}]\partial_M \tag{3.3}$$

where the operator Ad_X is defined by $Ad_X Z = [X, Z]$. This enables us to compute the matrix $Y = (Y^A{}_M)$ explicitly, namely,

$$Y = \frac{1 - \exp[-C(k)]}{C(k)} . \tag{3.4}$$

Let $X = (X_A{}^M) = Y^{-1}$ be the inverse matrix of Y. Then we define the 1-forms Y^A and the vector fields X_A on the group G by

$$Y^A = Y^A{}_M dk^M , \qquad X_A = X_A{}^M \partial_M . \tag{3.5}$$

Proposition 16 *There holds*

$$X_A \exp[\xi(k)] = \exp[\xi(k)]\xi_A . \tag{3.6}$$

Proof. This follows immediately from the eq. (3.2).

Next, by differentiating the eq. (3.2) with respect to k^L and alternating the indices L and M we obtain

$$\partial_L Y^A{}_M - \partial_M Y^A{}_L = -C^A{}_{BC} Y^B{}_L Y^C{}_M , \tag{3.7}$$

which, of course, can also be written as

$$dY^A = -\frac{1}{2} C^A{}_{BC} Y^B \wedge Y^C . \tag{3.8}$$

Proposition 17 *The vector fields X_A satisfy the commutation relations*

$$[X_A, X_B] = C^C{}_{AB} X_C . \tag{3.9}$$

Proof. This follows from the eq. (3.7).

The vector fields X_A are nothing but the right-invariant vector fields. They form a representation of the curvature algebra.

We will also need the following fundamental property of Lie groups.

Proposition 18 *Let G be a Lie group with the structure constants $C^A{}_{BC}$, $C_A = (C^B{}_{AC})$ and $C(k) = C_A k^A$. Let $\gamma = (\gamma_{AB})$ be a symmetric non-degenerate matrix satisfying the equation*

$$(C_A)^T = -\gamma C_A \gamma^{-1} . \tag{3.10}$$

Let $X = (X_A{}^M)$ be a matrix defined by

$$X = \frac{C(k)}{1 - \exp[-C(k)]} . \tag{3.11}$$

Then

$$(\det X)^{-1/2}\gamma^{AB}X_A{}^M\partial_M X_B{}^N\partial_N(\det X)^{1/2} = -\frac{1}{24}\gamma^{AB}C^C{}_{AD}C^D{}_{BC}\,. \tag{3.12}$$

Proof. It is easy to check that this equation holds at $k = 0$. Now, it can be proved by showing that it is a group invariant. For a detailed proof for semisimple groups see [25, 18, 20].

It is worth stressing that this equation holds not only on semisimple Lie groups but on any group with a compact Lie algebra, that is, when the structure constants $C^A{}_{BC}$ and the matrix γ_{AB}, used to define the metric G_{MN} and the operator X^2, satisfy the eq. (2.88). Such algebras can have an Abelian center as in eq. (2.90).

Now, by using the right-invariant vector fields we define a metric on the curvature group G

$$G_{MN} = \gamma_{AB}Y^A{}_M Y^B{}_N\,, \qquad G^{MN} = \gamma^{AB}X_A{}^M X_B{}^N\,. \tag{3.13}$$

This metric is bi-invariant and satisfies, in particular, the equation

$$\mathcal{L}_{X_A}G_{BC} = X_A{}^M\partial_M G_{BC} + G_{BM}\partial_C X_A{}^M + G_{MC}\partial_B X_A{}^M = 0\,. \tag{3.14}$$

This equation is proved by using eqs. (2.88) and (3.9). This means that the vector fields X_A are the Killing vector fields of the metric G_{MN}. One can easily show that this metric defines the following natural affine connection ∇^G on the group

$$\nabla^G_{X_C}X_A = -\frac{1}{2}C^A{}_{BC}X_B\,, \qquad \nabla^G_{X_C}Y^A = \frac{1}{2}C^B{}_{AC}Y^B\,, \tag{3.15}$$

with the scalar curvature

$$R_G = -\frac{1}{4}\gamma^{AB}C^C{}_{AD}C^D{}_{BC}\,. \tag{3.16}$$

Since the matrix $C(k)$ is traceless we have $\det\exp[C(k)/2] = 1$, and, therefore, the volume element on the group is

$$|G|^{1/2} = (\det G_{MN})^{1/2} = |\gamma|^{1/2}\det{}_{\mathcal{G}}\left(\frac{\sinh[C(k)/2]}{C(k)/2}\right)\,, \tag{3.17}$$

where $|\gamma| = \det\gamma_{AB}$. Notice that this function is precisely the inverse Van Vleck-Morette determinant (2.45) on the group in normal coordinates.

It is not difficult to see that

$$k^M Y^A{}_M = k^M X_M{}^A = k^A\,. \tag{3.18}$$

By differentiating this equation with respect to k^B and contracting the indices A and B we obtain

$$k^M \partial_A X_M{}^A = N - X_A{}^A\,. \tag{3.19}$$

Now, by contracting the eq. (3.15) with G^{BC} we obtain the zero-divergence condition for the right-invariant vector fields

$$|G|^{-1/2}\partial_M\left(|G|^{1/2} X_A{}^M\right) = 0\,. \tag{3.20}$$

Next, we define the Casimir operator

$$X^2 = C_2(G, X) = \gamma^{AB} X_A X_B\,. \tag{3.21}$$

By using the eq. (3.20) one can easily show that X^2 is an invariant differential operator that is nothing but the scalar Laplacian on the group

$$X^2 = |G|^{-1/2}\partial_M |G|^{1/2} G^{MN}\partial_N = G^{MN}\nabla^G_M\nabla^G_N\,. \tag{3.22}$$

Then, by using the eqs. (2.88) and (2.83) one can show that the operator X^2 commutes with the operators X_A,

$$[X_A, X^2] = 0\,. \tag{3.23}$$

Since we will actually be working with the gauged curvature group, we introduce now the operators (covariant right-invariant vector fields) J_A by

$$J_A = X_A - \frac{1}{2}\mathcal{B}_{AB} k^B\,, \tag{3.24}$$

and the operator

$$J^2 = \gamma^{AB} J_A J_B\,. \tag{3.25}$$

Proposition 19 *The operators J_A and J^2 satisfy the commutation relations*

$$[J_A, J_B] = C^C{}_{AB} J_C + \mathcal{B}_{AB}\,, \tag{3.26}$$

and

$$[J_A, J^2] = 2\mathcal{B}_{AB} J^B\,. \tag{3.27}$$

Proof. By using the eqs. (2.169)-(2.172) we obtain

$$X_B{}^A \mathcal{B}_{AM} = \gamma_{BN}\gamma^{AC} X_C{}^N \mathcal{B}_{AM} = \mathcal{B}_{BM}\,, \tag{3.28}$$

and, hence,

$$\gamma^{AB} X_B{}^M \mathcal{B}_{AM} = 0\,, \tag{3.29}$$

and, further, by using (3.9) we obtain (3.26). By using the eqs. (3.28) we get (3.27).

Thus, the operators J_A form a representation of the gauged curvature algebra. Now, let $\mathcal{L}_A$ be the operators of Lie derivatives satisfying the commutation relations (2.174) and $\mathcal{L}(k) = k^A \mathcal{L}_A$.

Proposition 20 *There holds*

$$J_A \exp[\mathcal{L}(k)] = \exp[\mathcal{L}(k)]\mathcal{L}_A\,. \tag{3.30}$$

and, therefore,

$$J^2 \exp[\mathcal{L}(k)] = \exp[\mathcal{L}(k)]\mathcal{L}^2\,. \tag{3.31}$$

Proof. Similarly to (3.3) we have

$$\exp[-\mathcal{L}(k)]\partial_M \exp[\mathcal{L}(k)] = \exp[-Ad_{\mathcal{L}(k)}]\partial_M\,. \tag{3.32}$$

By using the commutation relations (2.174) and eq. (2.172) we obtain

$$\exp[-\mathcal{L}(k)]\partial_M \exp[\mathcal{L}(k)] = Y^A{}_M \mathcal{L}_A + \frac{1}{2}\mathcal{B}_{MN}k^N\,. \tag{3.33}$$

The statement of the proposition follows from the definition of the operators J_A, J^2 and $\mathcal{L}^2$.

3.2 Heat Kernel on the Curvature Group

Let $\mathcal{B}$ be the matrix with the components $\mathcal{B} = (\gamma^{AB}\mathcal{B}_{BC})$. Let k^A be the canonical coordinates on the curvature group G and $A(t;k)$ be a function defined by

$$A(t;k) = \det{}_{\mathcal{G}}\left(\frac{\sinh\,[C(k)/2 + t\mathcal{B}]}{C(k)/2 + t\mathcal{B}}\right)^{-1/2}\,. \tag{3.34}$$

By using the eqs. (3.28) one can rewrite this in the form

$$A(t;k) = \det{}_{\mathcal{G}}\left(\frac{\sinh[C(k)/2]}{C(k)/2}\right)^{-1/2} \det{}_{\mathcal{G}}\left(\frac{\sinh[t\mathcal{B}]}{t\mathcal{B}}\right)^{-1/2} . \tag{3.35}$$

Notice also that due to (2.171)

$$\det{}_{\mathcal{G}}\left(\frac{\sinh[t\mathcal{B}]}{t\mathcal{B}}\right)^{-1/2} = \det{}_{TM}\left(\frac{\sinh[t\mathcal{B}]}{t\mathcal{B}}\right)^{-1/2} , \tag{3.36}$$

where $\mathcal{B}$ is now regarded as just the matrix $\mathcal{B} = (\mathcal{B}^a{}_b)$.

Let $\Theta(t;k)$ be another function on the group G defined by

$$\Theta(t;k) = \frac{1}{2}\left\langle k, \gamma\hat{\Theta}k\right\rangle , \tag{3.37}$$

where $\hat{\Theta}$ is the matrix

$$\hat{\Theta} = t\mathcal{B}\coth(t\mathcal{B}) \tag{3.38}$$

and $\langle u, \gamma v\rangle = \gamma_{AB}u^A v^B$ is the inner product on the algebra $\mathcal{G}$.

Theorem 6 *Let $\Phi(t;k)$ be a function on the group G defined by*

$$\Phi(t;k) = (4\pi t)^{-N/2} A(t;k) \exp\left(-\frac{\Theta(t;k)}{2t} + \frac{1}{6}R_G t\right) , \tag{3.39}$$

Then $\Phi(t;k)$ satisfies the equation

$$\partial_t \Phi = J^2 \Phi , \tag{3.40}$$

and the initial condition

$$\Phi(0;k) = |\gamma|^{-1/2}\delta(k) . \tag{3.41}$$

Proof. We compute first

$$\partial_t \Theta = \frac{1}{t}\Theta - \frac{1}{2t}\left\langle k, \gamma\hat{\Theta}^2 k\right\rangle + \frac{t}{2}\left\langle k, \gamma\mathcal{B}^2 k\right\rangle \tag{3.42}$$

and

$$\partial_t A = \frac{1}{2t}\left(N - \operatorname{tr}_{\mathcal{G}}\hat{\Theta}\right) A . \tag{3.43}$$

Therefore,

$$\partial_t\Phi = \left[\frac{1}{6}R_G - \frac{1}{2t}\mathrm{tr}_{\mathcal{G}}\hat{\Theta} + \frac{1}{4t^2}\left\langle k, \gamma\hat{\Theta}^2 k\right\rangle - \frac{1}{4}\left\langle k, \gamma\mathcal{B}^2 k\right\rangle\right]\Phi\,. \tag{3.44}$$

Next, we have

$$J^2 = X^2 - \gamma^{AB}\mathcal{B}_{AC}k^C X_B + \frac{1}{4}\gamma^{AB}\mathcal{B}_{AC}\mathcal{B}_{BD}k^C k^D\,. \tag{3.45}$$

By using the eqs. (3.28) and (2.172) and the anti-symmetry of the matrix $\mathcal{B}_{AB}$ we show that

$$\gamma^{AB}\mathcal{B}_{AC}k^C X_B\Theta = 0\,, \tag{3.46}$$

and

$$\gamma^{AB}\mathcal{B}_{AC}k^C X_B A = 0\,, \tag{3.47}$$

and, therefore,

$$\mathcal{B}_{AC}k^C X_B\Phi = 0\,. \tag{3.48}$$

Thus,

$$\begin{aligned} J^2\Phi \quad = \quad & \Bigg[A^{-1}(X^2 A) - \frac{1}{2t}(X^2\Theta) + \frac{1}{4t^2}\gamma^{AB}(X_A\Theta)(X_B\Theta) \\ & -\frac{1}{t}A^{-1}\gamma^{AB}(X_B A)(X_A\Theta) - \frac{1}{4}\left\langle k, \gamma\mathcal{B}^2 k\right\rangle\Bigg]\Phi\,. \end{aligned} \tag{3.49}$$

Further, by using 3.28) we get

$$\gamma^{AB}(X_A\Theta)(X_B\Theta) = \left\langle k, \gamma\hat{\Theta}^2 k\right\rangle \tag{3.50}$$

$$X^2\Theta = \mathrm{tr}_{\mathcal{G}}X + \mathrm{tr}_{\mathcal{G}}\hat{\Theta} - N\,. \tag{3.51}$$

Now, by using the eq. (3.20) in the form

$$A^2\partial_M(A^{-2}X_B{}^M) = 0 \tag{3.52}$$

and eqs. (2.172) and (3.19) we show that

$$A^{-1}\gamma^{AB}(X_A\Theta)X_B A = \frac{1}{2}\left(N - \mathrm{tr}_{\mathcal{G}}X\right)\,. \tag{3.53}$$

Finally, by using eq. (3.12) we obtain

$$A^{-1}X^2 A = \frac{1}{6}R_G\,. \tag{3.54}$$

Finally, substituting the eqs. (3.50)-(3.54) into eq. (3.49) and comparing it with eq. (3.44) we prove the eq. (3.40). The initial condition (3.41) follows easily from the well known property of the Gaussian. This completes the proof of the theorem.

3.3 Regularization and Analytical Continuation

In the following we will complexify the gauged curvature group in the following sense. We extend the canonical coordinates $(k^A) = (p^a, \omega^i)$ to the whole complex Euclidean space $\mathbb{C}^N$. Then all group-theoretic functions introduced above become analytic functions of k^A possibly with some poles on the real section $\mathbb{R}^N$ for compact groups. In fact, we replace the actual real slice $\mathbb{R}^N$ of $\mathbb{C}^N$ with an N-dimensional subspace $\mathbb{R}^N_{\rm reg}$ in $\mathbb{C}^N$ obtained by rotating the real section $\mathbb{R}^N$ counterclockwise in $\mathbb{C}^N$ by $\pi/4$. That is, we replace each coordinate k^A by $e^{i\pi/4}k^A$. In the complex domain the group becomes non-compact. We call this procedure the *decompactification*. If the group is compact, or has a compact subgroup, then this plane will cover the original group infinitely many times.

Since the metric $(\gamma_{AB}) = \operatorname{diag}(\delta_{ab}, \beta_{ij})$ is not necessarily positive definite, (actually, only the metric of the holonomy group β_{ij} is non-definite) we analytically continue the function $\Phi(t; k)$ in the complex plane of t with a cut along the negative imaginary axis so that $-\pi/2 < \arg\, t < 3\pi/2$. Thus, the function $\Phi(t; k)$ defines an analytic function of t and k^A. For the purpose of the following exposition we shall consider t to be *real negative*, $t < 0$. This is needed in order to make all integrals convergent and well defined and to be able to do the analytical continuation.

As we will show below, the singularities occur only in the holonomy group. This means that there is no need to complexify the coordinates p^a. Thus, in the following we assume the coordinates p^a to be real and the coordinates ω^i to be complex, more precisely, to take values in the p-dimensional subspace $\mathbb{R}^p_{\rm reg}$ of $\mathbb{C}^p$ obtained by rotating $\mathbb{R}^p$ counterclockwise by $\pi/4$ in $\mathbb{C}^p$ That is, we have $\mathbb{R}^N_{\rm reg} = \mathbb{R}^n \times \mathbb{R}^p_{\rm reg}$.

This procedure (that we call a regularization) with the nonstandard contour of integration is necessary for the convergence of the integrals below since we are treating both the compact and the non-compact symmetric spaces simultaneously. Remember, that, in general, the nondegenerate diagonal matrix β_{ij} is not positive definite. The space $\mathbb{R}^p_{\rm reg}$ is chosen in such a way to make the Gaussian exponent purely imaginary. Then the indefiniteness of the matrix β does not cause any problems. Moreover, the integrand does not have any singularities on these contours. The convergence of the integral is guaranteed by the exponential growth of the sine for imaginary argument. These integrals can be computed then in the following way. The coordinates ω^j corresponding to the compact directions are rotated further by another $\pi/4$ to imaginary axis and the coordinates ω^j corresponding to the non-compact directions are rotated back to the real axis. Then, for $t < 0$ all the integrals below are well defined and convergent and define an analytic function of

t in a complex plane with a cut along the negative imaginary axis.

3.4 Heat Semigroup

Theorem 7 *The heat semigroup* $\exp(t\mathcal{L}^2)$ *can be represented in form of the integral*

$$\exp(t\mathcal{L}^2) = \int\limits_{\mathbb{R}^N_{\rm reg}} dk\, |G|^{1/2}(k)\Phi(t;k)\exp[\mathcal{L}(k)]\,. \qquad (3.55)$$

Proof. Let

$$\Psi(t) = \int\limits_{\mathbb{R}^N_{\rm reg}} dk\, |G|^{1/2}\Phi(t;k)\exp[\mathcal{L}(k)]\,. \qquad (3.56)$$

By using the previous theorem we obtain

$$\partial_t\Psi(t) = \int\limits_{\mathbb{R}^N_{\rm reg}} dk\, |G|^{1/2}\exp[\mathcal{L}(k)]J^2\Phi(t;k)\,. \qquad (3.57)$$

Now, by integrating by parts we get

$$\partial_t\Psi(t) = \int\limits_{\mathbb{R}^N_{\rm reg}} dk\, |G|^{1/2}\Phi(t;k)J^2\exp[\mathcal{L}(k)]\,, \qquad (3.58)$$

and, by using eq. (3.31) we obtain

$$\partial_t\Psi(t) = \Psi(t)\mathcal{L}^2\,. \qquad (3.59)$$

Finally from the initial condition (3.41) for the function $\Phi(t;k)$ we get

$$\Psi(0) = 1\,, \qquad (3.60)$$

and, therefore, $\Psi(t) = \exp(t\mathcal{L}^2)$.

Theorem 8 *Let* Δ *be the Laplacian acting on sections of a homogeneous twisted spin-tensor vector bundle over a symmetric space. Then the heat semigroup*

$\exp(t\Delta)$ *can be represented in form of an integral*

$$\begin{aligned}\exp(t\Delta) &= (4\pi t)^{-N/2} \det{}_{TM}\left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}}\right)^{-1/2} \exp\left(-t\mathcal{R}^2 + \frac{1}{6}R_G t\right) \\ &\quad \int\limits_{\mathbb{R}^N_{\rm reg}} dk\, |\gamma|^{1/2} \det{}_{\mathcal{G}}\left(\frac{\sinh[C(k)/2]}{C(k)/2}\right)^{1/2} \\ &\quad \times \exp\left\{-\frac{1}{4t}\langle k, \gamma t\mathcal{B}\coth(t\mathcal{B})k\rangle\right\} \exp[\mathcal{L}(k)]\,. \end{aligned} \tag{3.61}$$

Proof. By using the eq. (2.183) we obtain

$$\exp(t\Delta) = \exp\left(-t\mathcal{R}^2\right)\exp\left(t\mathcal{L}^2\right)\,. \tag{3.62}$$

The statement of the theorem follows now from the eqs. (3.55), (3.39), (3.35)-(3.38) and (3.17).

4 Heat Kernel

4.1 Heat Kernel Diagonal and Heat Trace

The heat kernel diagonal on a homogeneous bundle over a symmetric space is parallel. In a local parallel local frame it is just a constant matrix. The fiber trace of the heat kernel diagonal is just a constant. That is why, it can be computed at any point in M. We fix a point x' in M such that the Killing vectors satisfy the initial conditions (2.111)-(2.113) and are given by the explicit formulas above (2.110)-(2.112). We compute the heat kernel diagonal at the point x'.

The heat kernel diagonal can be obtained by acting by the heat semigroup $\exp(t\Delta)$ on the delta-function, [8, 10]

$$\begin{aligned} U^{\rm diag}(t) &= \exp(t\Delta)\delta(x,x')\Big|_{x=x'} \\ &= \exp\left(-t\mathcal{R}^2\right)\int\limits_{\mathbb{R}^N_{\rm reg}} dk\, |G|^{1/2}\Phi(t;k)\exp[\mathcal{L}(k)]\delta(x,x')\Big|_{x=x'}\,. \end{aligned} \tag{4.1}$$

To be able to use this integral representation we need to compute the action of the isometries $\exp[\mathcal{L}(k)]$ on the delta-function.

Proposition 21 *Let φ be a section of the twisted spin-tensor bundle $\mathcal{V}$, $\mathcal{L}_A$ be the twisted Lie derivatives, $k^A = (p^a, \omega^i)$ be the canonical coordinates on the group and $\mathcal{L}(k) = k^A\mathcal{L}_A$. Let $\xi = k^A\xi_A$ be the Killing vector and ψ_t be the corresponding one-parameter diffeomorphism. Then*

$$\exp\left[\mathcal{L}(k)\right]\varphi(x) \quad = \quad \exp\left(-\frac{1}{2}\theta_{ab}G^{ab}\right)\varphi(\hat{x})\Big|_{t=1}\,, \tag{4.2}$$

where $\hat{x} = \psi_t(x)$ and the matrix θ is defined by (2.234). In particular, for $p = 0$ and $x = x'$

$$\exp[\mathcal{L}(k)]\varphi(x)\Big|_{p=0,x=x'} \quad = \quad \exp\left[\mathcal{R}(\omega)\right]\varphi(x)\,. \tag{4.3}$$

Proof. This statement follows from eqs. (2.236) and (2.237) and the fact that the Lie derivative is nothing but the generator of the pullback.

Proposition 22 *Let ω^i be the canonical coordinates on the holonomy group H and $(k^A) = (p^a, \omega^i)$ be the natural splitting of the canonical coordinates on the curvature group G. Then*

$$\exp[\mathcal{L}(k)]\delta(x,x')\Big|_{x=x'} = \det{}_{TM}\left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2}\right)^{-1}\exp[\mathcal{R}(\omega)]\delta(p)\,. \tag{4.4}$$

Proof. Let $\hat{x}(t,p,\omega,x,x') = \psi_t(x)$. By making use of the eq. (4.2) we obtain

$$\exp[\mathcal{L}(k)]\delta(x,x')\Big|_{x=x'} \quad = \quad \exp\left(-\frac{1}{2}\theta_{ab}G^{ab}\right)\delta(\hat{x}(1,p,\omega,x,x'),x')\Big|_{x=x',t=1}\,. \tag{4.5}$$

Now we change the variables from x^μ to the normal coordinates y^a to get

$$\delta(\hat{x}(1,p,\omega,x,x'),x')\Big|_{x=x'} = |g|^{-1/2}\det\left(\frac{\partial y^a}{\partial x^\mu}\right)\delta(\hat{y}(1,p,\omega,y))\Big|_{y=0}\,. \tag{4.6}$$

This delta-function picks the values of p that make $\hat{y} = 0$, which is exactly the functions $\bar{p} = \bar{p}(\omega, y)$ defined by the eq. (2.211). By switching further to the variables p we obtain

$$\delta(\hat{x}(1,p,\omega,x,x'),x')\Big|_{x=x'} = |g|^{-1/2}\det\left(\frac{\partial y^a}{\partial x^\mu}\right)\det\left(\frac{\partial\hat{y}^b}{\partial p^c}\right)^{-1}\delta(p-\bar{p}(\omega,y))\Big|_{y=0,t=1}\,. \tag{4.7}$$

Now, by recalling from (2.216) that $\bar{p}|_{y=0} = 0$ and by using (2.44) and (2.198) we evaluate the Jacobians for $p = y = 0$ and $t = 1$ to get the eq. (4.4).

Remarks. Some remarks are in order here. We implicitly assumed that there are no closed geodesics and that the equation of closed orbits of isometries

$$\hat{y}^a(1, \bar{p}, \omega, 0) = 0 \tag{4.8}$$

has a unique solution $\bar{p} = \bar{p}(\omega, 0) = 0$. On compact symmetric spaces this is not true: there are infinitely many closed geodesics and infinitely many closed orbits of isometries. However, these global solutions, which reflect the global topological structure of the manifold, will not affect our local analysis. In particular, they do not affect the asymptotics of the heat kernel. That is why, we have neglected them here. This is reflected in the fact that the Jacobian in (4.4) can become singular when the coordinates of the holonomy group ω^i vary from $-\infty$ to ∞. Note that the exact results for compact symmetric spaces can be obtained by an analytic continuation from the dual noncompact case when such closed geodesics are absent [18]. That is why we proposed above to complexify our holonomy group. If the coordinates ω^i are complex taking values in the subspace $\mathbb{R}^p_{\rm reg}$ defined above, then the equation (4.8) should have a unique solution and the Jacobian is an analytic function. It is worth stressing once again that the canonical coordinates cover the whole group except for a set of measure zero. Also a compact subgroup is covered infinitely many times. We will show below how this works in the case of the two-sphere, S^2.

Now by using the above lemmas and the theorem we can compute the heat kernel diagonal. We define the matrix $F(\omega)$ by

$$F(\omega) = \omega^i F_i\,. \tag{4.9}$$

Theorem 9 *The heat kernel diagonal of the Laplacian on twisted spin-vector bundles over a symmetric space has the form*

$$\begin{aligned} U^{\rm diag}(t) &= (4\pi t)^{-n/2} \det{}_{TM}\left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}}\right)^{-1/2} \exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H - \mathcal{R}^2\right)t\right\} \\ &\quad\times \int\limits_{\mathbb{R}^n_{\rm reg}} \frac{d\omega}{(4\pi t)^{p/2}}\, |\beta|^{1/2} \exp\left\{-\frac{1}{4t}\langle\omega,\beta\omega\rangle\right\} \cosh\left[\mathcal{R}(\omega)\right] \\ &\quad\times \det{}_{\mathcal{H}}\left(\frac{\sinh\left[F(\omega)/2\right]}{F(\omega)/2}\right)^{1/2} \det{}_{TM}\left(\frac{\sinh\left[D(\omega)/2\right]}{D(\omega)/2}\right)^{-1/2}\,, \end{aligned} \tag{4.10}$$

where $|\beta| = \det\beta_{ij}$ *and* $\langle\omega,\beta\omega\rangle = \beta_{ij}\omega^i\omega^j$.

Proof. First, we have $dk = dp\ d\omega$ and $|\gamma| = |\beta|$. By using the equations (4.1) and (4.4) and integrating over p we obtain the heat kernel diagonal

$$U^{\rm diag}(t) = \int\limits_{\mathbb{R}^p_{\rm reg}} d\omega\ |G|^{1/2}(0,\omega)\Phi(t;0,\omega)\det{}_{TM}\left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2}\right)^{-1}\exp[\mathcal{R}(\omega) - t\mathcal{R}^2]\,. \tag{4.11}$$

Further, by using the eq. (2.81) we compute the determinants

$$\det{}_{\mathcal{G}}\left(\frac{\sinh[C(\omega)/2]}{C(\omega)/2}\right) = \det{}_{TM}\left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2}\right)\det{}_{\mathcal{H}}\left(\frac{\sinh[F(\omega)/2]}{F(\omega)/2}\right). \tag{4.12}$$

Now, we by using (2.171) we compute (3.37)

$$\Theta(t;0,\omega) = \frac{1}{2}\langle\omega,\beta\omega\rangle\ , \tag{4.13}$$

and, finally, by using eq. (3.39), (3.35), (3.16) and (2.95) we get the result (4.10).

By using this theorem we can also compute the heat trace for compact manifolds

$$\begin{aligned}
\mathrm{Tr}\,{}_{L^2}\exp(t\Delta) \quad = \quad & \int\limits_M d\mathrm{vol}\ (4\pi t)^{-n/2}\mathrm{tr}\,{}_V\det{}_{TM}\left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}}\right)^{-1/2} \\
& \times\exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H - \mathcal{R}^2\right)t\right\} \\
& \times\int\limits_{\mathbb{R}^p_{\rm reg}}\frac{d\omega}{(4\pi t)^{p/2}}\ |\beta|^{1/2}\exp\left\{-\frac{1}{4t}\langle\omega,\beta\omega\rangle\right\}\cosh\left[\mathcal{R}(\omega)\right] \\
& \times\det{}_{\mathcal{H}}\left(\frac{\sinh\left[F(\omega)/2\right]}{F(\omega)/2}\right)^{1/2}\det{}_{TM}\left(\frac{\sinh\left[D(\omega)/2\right]}{D(\omega)/2}\right)^{-1/2}\ ,
\end{aligned} \tag{4.14}$$

where $\mathrm{tr}\,{}_V$ is the fiber trace.

4.2 Heat Kernel Asymptotics

It is well known that there is the following asymptotic expansion as $t \to 0$ of the heat kernel diagonal [24]

$$U^{\rm diag}(t) \sim (4\pi t)^{-n/2}\sum_{k=0}^{\infty} t^k a_k\,. \tag{4.15}$$

The coefficients a_k are called the local heat kernel coefficients. On compact manifolds, there is a similar asymptotic expansion of the heat trace with the global heat invariants A_k defined by

$$A_k = \int_M d\text{vol}\ \text{tr}_V a_k\,. \tag{4.16}$$

In symmetric spaces the heat invariants do not contain any additional information since the local heat kernel coefficients define the heat invariants A_k up to a constant equal to the volume of the manifold,

$$A_k = \text{vol}\,(M)\text{tr}_V a_k\,. \tag{4.17}$$

We introduce a Gaussian average over the holonomy algebra by

$$\langle f(\omega)\rangle = \int\limits_{\mathbb{R}^p_{\text{reg}}} \frac{d\omega}{(4\pi)^{p/2}}\ |\beta|^{1/2} \exp\left(-\frac{1}{4}\langle\omega,\beta\omega\rangle\right) f(\omega) \tag{4.18}$$

Then we can write

$$U^{\text{diag}}(t) = (4\pi t)^{-n/2} \det{}_{TM}\left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}}\right)^{-1/2} \exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H - \mathcal{R}^2\right)t\right\} \tag{4.19}$$

$$\times\left\langle \cosh\left[\sqrt{t}\,\mathcal{R}(\omega)\right] \det{}_{\mathcal{H}}\left(\frac{\sinh\left[\sqrt{t}\,F(\omega)/2\right]}{\sqrt{t}\,F(\omega)/2}\right)^{1/2} \det{}_{TM}\left(\frac{\sinh\left[\sqrt{t}\,D(\omega)/2\right]}{\sqrt{t}\,D(\omega)/2}\right)^{-1/2}\right\rangle$$

This equation can be used now to generate all heat kernel coefficients a_k for any locally symmetric space simply by expanding it in a power series in t. By using the standard Gaussian averages [4]

$$\left\langle \omega_1^{i} \cdots \omega^{i_{2k+1}} \right\rangle = 0\,, \tag{4.20}$$

$$\left\langle \omega^{i_1} \cdots \omega^{i_{2k}} \right\rangle = \frac{(2k)!}{k!}\beta^{(i_1 i_2} \cdots \beta^{i_{2k-1} i_{2k})}\,, \tag{4.21}$$

one can obtain now all heat kernel coefficients in terms of traces of various contractions of the matrices $D^a{}_{ib}$ and $F^j{}_{ik}$ with the matrix β^{ik}. All these quantities are curvature invariants and can be expressed directly in terms of the Riemann tensor.

[4] We have corrected here a misprint in the eq. (4.68) of [10].

There is an alternative representation of the Gaussian average in purely algebraic terms. Let b^j and b^*_k be operators, called creation and annihilation operators, acting on a Hilbert space, that satisfy the following commutation relations

$$[b^j, b^*_k] = \delta^j_k\,, \tag{4.22}$$

$$[b^j, b^k] = [b^*_j, b^*_k] = 0\,. \tag{4.23}$$

Let $|0\rangle$ be a unit vector in the Hilbert space, called the vacuum vector, that satisfies the equations

$$\langle 0|0\rangle = 1\,, \tag{4.24}$$

$$b^j|0\rangle = \langle 0|b^*_k = 0\,. \tag{4.25}$$

Then the Gaussian average is nothing but the vacuum expectation value

$$\langle f(\omega)\rangle = \langle 0|f(b)\exp\langle b^*, \beta b^*\rangle|0\rangle\,, \tag{4.26}$$

where $\langle b^*, \beta b^*\rangle = \beta^{jk} b^*_j b^*_k$. This should be computed by the so-called normal ordering, that is, by simply commuting the operators b_j through the operators b^*_k until they hit the vacuum vector giving zero. The remaining non-zero commutation terms precisely reproduce the eqs. (4.20), (4.21).

4.2.1 Calculation of the Coefficient a_1

As an example let us calculate the lowest heat kernel coefficients: a_0 and a_1. Let X be a matrix. Then by using

$$\det\left(\frac{\sinh\left(\sqrt{t}\,X\right)}{\sqrt{t}\,X}\right)^m = \exp\left(m\ \mathrm{tr}\ \log\frac{\sinh\left(\sqrt{t}\,X\right)}{\sqrt{t}\,X}\right) \tag{4.27}$$

and [21]

$$\log\frac{\sinh\left(\sqrt{t}\,X\right)}{\sqrt{t}\,X} = \sum_{k=1}^{\infty}\frac{2^{2k-1}B_{2k}}{k(2k)!}t^k X^{2k}\,, \tag{4.28}$$

where B_k are Bernoulli numbers, in particular,

$$B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \tag{4.29}$$

we obtain

$$\det\left(\frac{\sinh\left(\sqrt{t}\,X\right)}{\sqrt{t}\,X}\right)^{\pm 1/2} = 1 \pm \frac{1}{12}t\,\mathrm{tr}\,X^2 + O(t^2)\,. \tag{4.30}$$

Now, by using eq. (4.19) we obtain

$$U^{\rm diag}(t) = (4\pi t)^{-n/2}\left[a_0 + ta_1 + O(t^2)\right], \tag{4.31}$$

where

$$a_0 = \mathbb{I}\,, \qquad a_1 = \langle b_1\rangle\ , \tag{4.32}$$

and

$$b_1 = \left[\frac{1}{8}R + \frac{1}{6}R_H + \frac{1}{48}\mathrm{tr}\,F(\omega)^2 - \frac{1}{48}\mathrm{tr}\,D(\omega)^2\right]\mathbb{I} - \mathcal{R}^2 + \frac{1}{2}\,\mathcal{R}(\omega)^2\,. \tag{4.33}$$

Next, bu using (4.21), in particular,

$$\left\langle \omega^i\omega^j\right\rangle = 2\beta^{ij}, \tag{4.34}$$

we obtain

$$\left\langle \mathcal{R}(\omega)^2\right\rangle = 2\mathcal{R}^2\,, \tag{4.35}$$

$$\left\langle \mathrm{tr}\,F(\omega)^2\right\rangle = 2\mathrm{tr}\,F_iF^i = 2F^j{}_{il}F^{li}{}_j = -8R_H\,, \tag{4.36}$$

$$\left\langle \mathrm{tr}\,D(\omega)^2\right\rangle = 2\mathrm{tr}\,D_iD^i = 2D^a{}_{ib}D^{bi}{}_a = -2R\,, \tag{4.37}$$

and, therefore,

$$a_1 = \left[\frac{1}{8}R + \frac{1}{6}R_H - \frac{1}{6}R_H + \frac{1}{24}R\right]\mathbb{I} - \mathcal{R}^2 + \mathcal{R}^2 = \frac{1}{6}R\mathbb{I}\,. \tag{4.38}$$

This confirms the well know result for the coefficient a_1 [24, 5].

4.3 Heat Kernel on S^2 and H^2

Let us apply our result to a special case of a two-sphere S^2 of radius r, which is a compact symmetric space equal to the quotient of the isometry group, $SO(3)$, by the isotropy group, $SO(2)$,

$$S^2 = SO(3)/SO(2). \tag{4.39}$$

The two-sphere is too small to incorporate an additional Abelian field $\mathcal{B}$; therefore, we set $\mathcal{B} = 0$.

Let y^a be the normal coordinates defined above. On the 2-sphere of radius r they range over $-r\pi \le y^a \le r\pi$. We define the polar coordinates ρ and φ by

$$y^1 = \rho\cos\varphi, \qquad y^2 = \rho\sin\varphi\,, \tag{4.40}$$

so that $0 \le \rho \le r\pi$ and $0 \le \varphi \le 2\pi$.

The orthonormal frame of 1-forms is

$$e^1 = d\rho\,, \qquad e^2 = r\sin\left(\frac{\rho}{r}\right)d\varphi\,, \tag{4.41}$$

which gives the spin connection 1-form

$$\omega_{ab} = -\varepsilon_{ab}\cos\left(\frac{\rho}{r}\right)d\varphi \tag{4.42}$$

with ε_{ab} being the antisymmetric Levi-Civita tensor, and the curvature

$$R_{abcd} = \frac{1}{r^2}\varepsilon_{ab}\varepsilon_{cd} = \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\,, \tag{4.43}$$

$$R_{ab} = \frac{1}{r^2}\delta_{ab}\,, \qquad R = \frac{2}{r^2}\,. \tag{4.44}$$

Since the holonomy group $SO(2)$ is one-dimensional, it is obviously Abelian, so all structure constants $F^i{}_{jk}$ are equal to zero, and therefore, the curvature of the holonomy group vanishes, $R_H = 0$. The metric of the holonomy group β_{ij} is now just a constant, $\beta = 1/r^2$. The only generator of the holonomy group in the vector representation is

$$D_{ab} = -\frac{1}{r^2}E_{ab} = -\frac{1}{r^2}\varepsilon_{ab}\,. \tag{4.45}$$

The irreducible representations of $SO(2)$ are parametrized by α, which is either an integer, $\alpha = m$, or a half-integer, $\alpha = m + \frac{1}{2}$. Therefore, the generator $\mathcal{R}$ of the holonomy group and the Casimir operator $\mathcal{R}^2$ are

$$\mathcal{R} = i\frac{\alpha}{r^2}\,, \tag{4.46}$$

$$\mathcal{R}^2 = \beta^{ij}\mathcal{R}_i\mathcal{R}_j = -\frac{\alpha^2}{r^2}\,. \tag{4.47}$$

The extra factor r^2 here is due to the inverse metric $\beta^{-1} = r^2$ of the holonomy group.

The Lie derivatives $\mathcal{L}_A$ are given by

$$\mathcal{L}_1 = \cos\varphi\partial_\rho - \frac{\sin\varphi}{r}\cot\left(\frac{\rho}{r}\right)\partial_\varphi + i\frac{\sin\varphi}{r\sin(\rho/r)}\alpha\,, \tag{4.48}$$

$$\mathcal{L}_2 = \sin\varphi\partial_\rho + \frac{\cos\varphi}{r}\cot\left(\frac{\rho}{r}\right)\partial_\varphi - i\frac{\cos\varphi}{r\sin(\rho/r)}\alpha\,, \tag{4.49}$$

$$\mathcal{L}_3 = \frac{1}{r^2}\partial_\varphi\,, \tag{4.50}$$

and form a representation of the $SO(3)$ algebra

$$[\mathcal{L}_1, \mathcal{L}_2] = -\mathcal{L}_3\,, \qquad [\mathcal{L}_3, \mathcal{L}_1] = -\frac{1}{r^2}\mathcal{L}_2 \qquad [\mathcal{L}_3, \mathcal{L}_2] = \frac{1}{r^2}\mathcal{L}_1\,. \tag{4.51}$$

The Laplacian is given by

$$\Delta = \partial_\rho^2 + \frac{1}{r}\cot\left(\frac{\rho}{r}\right)\partial_\rho + \frac{1}{r^2\sin^2(\rho/r)}\left[\partial_\varphi - i\alpha\cos\left(\frac{\rho}{r}\right)\right]^2 \tag{4.52}$$

Now, we need to compute the determinant

$$\det{}_{TM}\left(\frac{\sinh[\omega D]}{\omega D}\right)^{-1/2} = \frac{\omega/(2r^2)]}{\sin[\omega/(2r^2)]}\,. \tag{4.53}$$

The contour of integration over ω in (4.10) should be the real axis rotated counterclockwise by $\pi/4$. Since S^2 is compact, we rotate it further to the imaginary axis and rescale ω for $t<0$ by $\omega \to r\sqrt{-t}\,\omega$ to obtain an analytic function of t

$$\begin{aligned} U^{\rm diag}(t) &= \frac{1}{4\pi t}\exp\left[\left(\frac{1}{4}+\alpha^2\right)\frac{t}{r^2}\right] \\ &\times\int\limits_{-\infty}^{\infty}\frac{d\omega}{\sqrt{4\pi}}\exp\left(-\frac{\omega^2}{4}\right)\frac{\omega\sqrt{-t}/(2r)}{\sinh\left[\omega\sqrt{-t}/(2r)\right]}\cosh\left(\alpha\omega\sqrt{-t}/r\right)\,. \end{aligned} \tag{4.54}$$

If we would have rotated the contour to the real axis instead then we would have obtained after rescaling $\omega \to r\sqrt{t}\,\omega$ for $t>0$,

$$\begin{aligned} U^{\rm diag}(t) &= \frac{1}{4\pi t}\exp\left[\left(\frac{1}{4}+\alpha^2\right)\frac{t}{r^2}\right] \\ &\times\fint\limits_{-\infty}^{\infty}\frac{d\omega}{\sqrt{4\pi}}\exp\left(-\frac{\omega^2}{4}\right)\frac{\omega\sqrt{t}/(2r)}{\sin\left[\omega\sqrt{t}/(2r)\right]}\cos\left(\alpha\omega\sqrt{t}/r\right)\,, \end{aligned} \tag{4.55}$$

where $\fint$ denotes the Cauchy principal value of the integral. This can also be written as

$$\begin{aligned} &U^{\rm diag}(t) = \frac{1}{4\pi t}\exp\left[\left(\frac{1}{4}+\alpha^2\right)\frac{t}{r^2}\right] \\ &\times\sum_{k=-\infty}^{\infty}(-1)^k\int\limits_0^{2\pi r/\sqrt{t}}\frac{d\omega}{\sqrt{4\pi}}\exp\left[-\frac{1}{4}\left(\omega+\frac{2\pi r}{\sqrt{t}}k\right)^2\right]\frac{\sqrt{t}}{2r}\frac{\left(\omega+\frac{2\pi r}{\sqrt{t}}k\right)}{\sin\left[\omega\sqrt{t}/(2r)\right]} \\ &\times\cos\left(\alpha\omega\sqrt{t}/r\right)\,. \end{aligned} \tag{4.56}$$

This is nothing but the sum over the closed geodesics of S^2. Note that the factor $\cos\left(\alpha\omega\sqrt{t}/r\right)$ is either periodic (for integer α) or anti-periodic (for half-integer α).

The non-compact symmetric space dual to the 2-sphere is the hyperbolic plane H^2 of pseudo-radius a. It is equal to the quotient of the isometry group, $SO(1,2)$, by the isotropy group, $SO(2)$,

$$H^2 = SO(1,2)/SO(2). \tag{4.57}$$

Let y^a be the normal coordinates defined above. On H^2 they range over $-\infty \le y^a \le \infty$. We define the polar coordinates u and φ by

$$y^1 = u\cos\varphi, \qquad y^2 = u\sin\varphi\,, \tag{4.58}$$

so that $0 \le u \le \infty$ and $0 \le \varphi \le 2\pi$.

The orthonormal frame of 1-forms is

$$e^1 = du\,, \qquad e^2 = a\sinh\left(\frac{u}{a}\right)d\varphi\,, \tag{4.59}$$

which gives the spin connection 1-form

$$\omega_{ab} = -\varepsilon_{ab}\cosh\left(\frac{u}{a}\right)d\varphi\ , \tag{4.60}$$

and the curvature

$$R_{abcd} = -\frac{1}{a^2}\varepsilon_{ab}\varepsilon_{cd} = -\frac{1}{a^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\,, \tag{4.61}$$

$$R_{ab} = -\frac{1}{a^2}\delta_{ab}\,, \qquad R = -\frac{2}{a^2}\,. \tag{4.62}$$

The metric of the isotropy group β_{ij} is just a constant, $\beta = -1/a^2$, and the only generator of the isotropy group in the vector representation is given by

$$D_{ab} = \frac{1}{a^2}E_{ab} = \frac{1}{a^2}\varepsilon_{ab}\,. \tag{4.63}$$

The Lie derivatives $\mathcal{L}_A$ are now

$$\mathcal{L}_1 = \cos\varphi\partial_u - \frac{\sin\varphi}{a}\coth\left(\frac{u}{a}\right)\partial_\varphi + i\frac{\sin\varphi}{a\sinh(u/a)}\alpha\,, \tag{4.64}$$

$$\mathcal{L}_2 = \sin\varphi\partial_u + \frac{\cos\varphi}{a}\coth\left(\frac{u}{a}\right)\partial_\varphi - i\frac{\cos\varphi}{a\sinh(u/a)}\alpha\,, \tag{4.65}$$

$$\mathcal{L}_3 = -\frac{1}{a^2}\partial_\varphi\,, \tag{4.66}$$

and form a representation of the $SO(1,2)$ algebra

$$[\mathcal{L}_1,\mathcal{L}_2] = -\mathcal{L}_3\,, \qquad [\mathcal{L}_3,\mathcal{L}_1] = \frac{1}{a^2}\mathcal{L}_2 \qquad [\mathcal{L}_3,\mathcal{L}_2] = -\frac{1}{a^2}\mathcal{L}_1\,. \tag{4.67}$$

The Laplacian is given by

$$\Delta = \partial_u^2 + \frac{1}{a}\coth\left(\frac{u}{a}\right)\partial_u + \frac{1}{a^2\sinh^2(u/a)}\left[\partial_\varphi - i\alpha\cosh\left(\frac{u}{a}\right)\right]^2 \tag{4.68}$$

The contour of integration over ω in (4.10) for the heat kernel should be the real axis rotated counterclockwise by $\pi/4$. Since H^2 is non-compact, we rotate it back to the real axis and rescale ω for $t > 0$ by $\omega \to a\sqrt{t}\,\omega$ to obtain the heat kernel diagonal for the Laplacian on H^2

$$\begin{aligned} U^{\rm diag}(t) &= \frac{1}{4\pi t}\exp\left[-\left(\frac{1}{4}+\alpha^2\right)\frac{t}{a^2}\right] \\ &\quad\times\int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{4\pi}}\exp\left(-\frac{\omega^2}{4}\right)\frac{\omega\sqrt{t}/(2a)}{\sinh\left[\omega\sqrt{t}/(2a)\right]}\cosh\left(\alpha\omega\sqrt{t}/a\right)\,. \end{aligned} \tag{4.69}$$

We see that the heat kernel in the compact case of the two-sphere, S^2, is related with the heat kernel in the non-compact case of the hyperboloid, H^2, by the analytical continuation, $a^2 \to -r^2$, or $a \to ir$, or, alternatively, by replacing $t \to -t$ (and $a = r$). One can go even further and compute the Plancherel (or Harish-Chandra) measure in the case of H^2 and the spectrum in the case of S^2.

For H^2 we rescale the integration variable in (4.69) by $\omega \to \omega a/\sqrt{t}$, substitute

$$\frac{a}{\sqrt{4\pi t}}\exp\left(-\frac{a^2}{4t}\omega^2\right) = \int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\exp\left(-\frac{t}{a^2}\nu^2 + i\omega\nu\right)\,, \tag{4.70}$$

integrate by parts over ν, and use

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\,e^{i\omega\nu}\frac{\cosh(\alpha\omega)}{\sinh(\omega/2)} = \frac{1}{2}\{\tanh[\pi(\nu+i\alpha)] + \tanh[\pi(\nu-i\alpha)]\} \tag{4.71}$$

(and the fact that α is a half-integer) to represent the heat kernel for H^2 in the form

$$U^{\rm diag}(t) = \frac{1}{4\pi a^2}\int\limits_{-\infty}^{\infty} d\nu\, \mu(\nu)\exp\left\{-\left(\frac{1}{4}+\alpha^2+\nu^2\right)\frac{t}{a^2}\right\}, \qquad (4.72)$$

where

$$\mu(\nu) = \nu\tanh\nu \qquad (4.73)$$

for integer $\alpha = m$, and

$$\mu(\nu) = \nu\coth\nu \qquad (4.74)$$

for half-integer $\alpha = m + \frac{1}{2}$.

For S^2 we proceed as follows. We cannot just substitute $a^2 \to -r^2$ in (4.72). Instead, first, we deform the contour of integration in (4.72) to the V-shaped contour that consists of two segments of straight lines, one going from $e^{i3\pi/4}\infty$ to 0, and another going from 0 to $e^{i\pi/4}\infty$. Then, after we replace $a^2 \to -r^2$, we can deform the contour further to go counterclockwise around the positive imaginary axis. Then we notice that the function $\mu(\nu)$ is a meromorphic function with simple poles on the imaginary axis at $\nu_k = id_k$, where

$$d_k = \left(k+\frac{1}{2}\right), \qquad k = 0, \pm 1 \pm 2, \dots, \qquad \text{for integer } \alpha = m\,, \qquad (4.75)$$

and at

$$d_k = k, \qquad k = \pm 1, \pm 2, \dots, \qquad \text{for half-integer } \alpha = m + \frac{1}{2}\,. \qquad (4.76)$$

Therefore, we can compute the integral by residue theory to get

$$U^{\rm diag}(t) = \frac{1}{4\pi r^2}\sum_{k=0}^{\infty} d_k \exp\left(-\lambda_k t\right), \qquad (4.77)$$

where

$$\lambda_k = \frac{1}{r^2}\left[\left(k+\frac{1}{2}\right)^2 - \frac{1}{4} - m^2\right] \qquad \text{for integer } \alpha = m, \qquad (4.78)$$

and

$$\lambda_k = \frac{1}{r^2}\left[k^2 - \frac{1}{4} - \left(m+\frac{1}{2}\right)^2\right] \qquad \text{for half-integer } \alpha = m + \frac{1}{2}\,. \qquad (4.79)$$

Our results for the heat kernel on the 2-sphere S^2 and the hyperbolic plane H^2 coincide with the exact heat kernel of scalar Laplacian (when $\mathcal{R} = \alpha = 0$) reported in [18] and obtained by completely different methods.

4.4 Index Theorem

We can now apply this result for the calculation of the index of the Dirac operator on spinors on compact manifolds

$$D = \gamma^\mu \nabla_\mu \,. \tag{4.80}$$

Let the dimension n of the manifold be even and

$$\Gamma = \frac{1}{n!} i^{n(n-1)/2} \varepsilon^{a_1 \dots a_n} \gamma_{[a_1} \cdots \gamma_{a_n]} \tag{4.81}$$

be the chirality operator of the spinor representation so that

$$\Gamma^2 = \mathbb{I}_S \tag{4.82}$$

and

$$\Gamma \gamma_a = -\gamma_a \Gamma \,. \tag{4.83}$$

Then the index of the Dirac operator is equal to

$$\mathrm{Ind}\,(D) = \mathrm{Tr}_{\,L^2} \Gamma \exp(tD^2) \,. \tag{4.84}$$

We compute the square of the Dirac operator by using the eqs. (2.13), (2.19), (2.15) and (2.153)

$$\begin{aligned} D^2 &= \Delta - \frac{1}{4} R\, \mathbb{I}_S + \frac{1}{2} \mathcal{F}_{ab} \gamma^{ab} \\ &= \Delta - \frac{1}{4} R\, \mathbb{I}_S - \frac{1}{2} E^i{}_{ab} T_i \gamma^{ab} + \frac{1}{2} \gamma^{ab} \mathcal{B}_{ab} \,. \end{aligned} \tag{4.85}$$

In this case the generators $\mathcal{R}_i$ have the form

$$\mathcal{R}_i = -\frac{1}{4} D^a{}_{ib} \gamma^b{}_a \otimes \mathbb{I}_T + \mathbb{I}_S \otimes T_i \tag{4.86}$$

and the Casimir operator of the holonomy group in the spinor representation is obtained by using (2.15)

$$\mathcal{R}^2 = \frac{1}{8} R\, \mathbb{I}_S + \mathbb{I}_S \otimes T^2 - \frac{1}{2} E^j{}_{ab} \gamma^{ab} \otimes T_j \,. \tag{4.87}$$

Thus, we obtain the index

$$\begin{aligned}
\mathrm{Ind}\,(D) \quad = \quad & \int\limits_M d\mathrm{vol}\ \ (4\pi t)^{-n/2} \mathrm{tr}\,_V \Gamma \det{}_{TM} \left(\frac{\sinh(t\mathcal{B})}{t\mathcal{B}}\right)^{-1/2} \\
& \times \exp\left\{\left(-\frac{1}{4}R + \frac{1}{6}R_H - T^2 + \frac{1}{2}\mathcal{B}_{ab}\gamma^{ab}\right) t\right\} \\
& \times \int\limits_{\mathbb{R}^p_{\mathrm{reg}}} \frac{d\omega}{(4\pi t)^{p/2}}\ |\beta|^{1/2} \exp\left\{-\frac{1}{4t}\left\langle\omega, \beta\omega\right\rangle\right\} \\
& \times \cosh\left(-\frac{1}{4}\omega^i D^a{}_{ib}\gamma^b{}_a + \omega^i T_i\right) \\
& \times \det{}_{\mathcal{H}} \left(\frac{\sinh\left[F(\omega)/2\right]}{F(\omega)/2}\right)^{1/2} \det{}_{TM}\left(\frac{\sinh\left[D(\omega)/2\right]}{D(\omega)/2}\right)^{-1/2}\ . \quad (4.88)
\end{aligned}$$

Since the index does not depend on t, the right-hand side of this equation does not depend on t. By expanding it in an asymptrotic power series in t, we see that the index is equal to

$$\mathrm{Ind}\,(D) = (4\pi)^{-n/2} \int_M d\mathrm{vol}\ \mathrm{tr}\,_V\, \Gamma a_{n/2}\,. \qquad (4.89)$$

5 Conclusion

We have continued the study of the heat kernel on homogeneous spaces initiated in [6, 7, 8, 9, 10]. In those papers we have developed a systematic technique for calculation of the heat kernel in two cases: a) a Laplacian on a vector bundle with a parallel curvature over a flat space [6, 9], and b) a scalar Laplacian on manifolds with parallel curvature [8, 10]. What was missing in that study was the case of a non-scalar Laplacian on vector bundles with parallel curvature over curved manifolds with parallel curvature.

In the present paper we considered the Laplacian on a homogeneous bundle and generalized the technique developed in [10] to compute the corresponding heat semigroup and the heat kernel. It is worth pointing out that our formal result applies to general symmetric spaces by making use of the regularization and the analytical continuation procedure described above. Of course, the heat kernel coefficients are just polynomials in the curvature and do not depend on this kind of analytical continuation (for more detail, see [10]).

As we mentioned above, due to existence of multiple closed geodesics the obtained form of the heat kernel for compact symmetric spaces requires an additional regularization, which consists simply in an analytical continuation of the result from the complexified noncompact case. In any case, it gives a generating function for all heat invariants and reproduces correctly the whole asymptotic expansion of the heat kernel diagonal. However, since there are no closed geodesics on non-compact symmetric spaces, it seems that the analytical continuation of the obtained result for the heat kernel diagonal should give the *exact result* for the non-compact case, and, even more generally, for the general case too. We have seen on the example of the two-sphere that our method gives not just the asymptotic expansion of the heat kernel diagonal but, after an appropriate regularization, in fact, an exact result for the heat kernel diagonal.

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