Toronto Lectures on Physics

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January 3, 2008

Contents

I have been invited by the mathematicians here at Toronto to give three talks describing my joint work with the late Prof. Yuval Ne'eman in elementary particle physics. This work uses some mathematical ideas such as super Lie algebras and their representations, and the Quillen theory of superconnections. But the work is in physics, and this puts me in a quandary as to the amount of physics background that I can assume.

In order not to chase away any physicists in the audience, I will start by listing the physics problems that our approach tries to partially address. This will involve some words that may not be familiar to mathematicians, to whom I apologize. I hope to elucidate the meaning of most of these words in the course of the lectures. All of the results in these lectures are contained in the review article I wrote with Prof. Ne'eman, and which appeared in Physics Reports **406** 2005, 303 -377.

I include here some expository material (mainly for mathematicians) which I did not have time to include in the lectures; for example a review of classical electromagnetism and material on the Dirac operator.

I thank Yael Karshon for helpful comments on this text and the associated slides.

1 The "Standard Model" and some of its ills.

The "Standard Model" of the physics of particles and fields (assumed to include all known fundamental interactions except for gravity) is enormously successful, with its predictions validated by all experimental tests. In particular, the electroweak interactions seem to be correctly described by the $su(2) \times u(1)$ spontaneously broken local gauge symmetry. Although the full implementation of this (Weinberg-Salam) theory requires quantum field theory, much of its basic structure can be phrased in terms of classical field theory, see for example, Kane *Modern Elementary Particle Physics* , or, for the more mathematically inclined reader, Derdzinski *Geometry of the Standard Model of Elementary Particles.* Note that a comprehensive review intended for particle (or high energy) physicists appeared in E.S. Abers and B.W. Lee, "Gauge Theories", *Physics Reports*, **9C** no. 1 1973. So this theory has been successfully around for a long time.

The very success of this theory prompted a number of questions relating to its structure, hypotheses and input. The unresolved issues include

- The large number of free parameters which must be experimentally determined to serve as input into the theory such as the various gauge coupling constants (including the Weinberg angle), the parameters of Higgs potential, the coupling constants of the matter fields, the eigenvalues of the weak isospin and weak hypercharge for the chiral leptons and fermions etc.
- As a result, the theory is unable to predict the value of the mass of the Higgs particle. This meson has therefore been searched for all over the accelerator-available spectrum, from a few GeV to the 115 GeV reached at Cern in October 2000, when 9 "events" were reported at the limit of the accelerator's energy range. (These "events" constituted 2.6 standard deviations above background level, whereas 5 standard deviations are considered necessary for an accepted result that could be interpreted as evidence for the Higgs particle.) All this was before the planned closure of the machine. However when the accelerator was granted another month of operation, no further evidence was found. Several machines are expected to renew the search in the next 1- 3 years, reaching into the 100-500 GeV range.
- The lack of correlation between the quantum numbers of left and right chiral leptons and fermions.

- The *ad hoc* introduction of Higgs fields to implement spontaneous symmetry breaking.
- The fact that these Higgs fields constitute a weak isospin doublet.
- No explanation of the origin of the Higgs potential needed to achieve Goldstone-Higgs spontaneous symmetry breaking.
- No explanation of the absence of right handed neutrinos. In fact, since we now know that the neutrino is massive, we know that right handed neutrinos do exist. So we can reformulate the question as follows: Why don't the right handed neutrinos participate in the Weinberg-Salam theory?

I wish to show in these lectures how using superconnections allows an answer to some of these difficulties.

2 Problems of translation between mathematics and physics.

There are several communications difficulties between mathematicians and physicists, some more serious than others. I want to get a few of these out into the open before one group or the other disappears:

2.1 Is there an i in the structure constants of a Lie algebra?

The first barrier between the mathematics literature and most of the physics literature is the ubiquitous factor of i: The mathematical definition of a Lie algebra is that it is a vector space $\mathfrak{k}$ with a bilinear map

$$\mathfrak{k} \times \mathfrak{k} \to \mathfrak{k}$$

which is anti-symmetric and satisfies Jacobi's identity.

So the set of self-adjoint matrices under commutator bracket is *not* a Lie algebra. Indeed the commutator of two self adjoint matrices is skew adjoint. So the Lie algebra of $u(n)$ is not the space of self-adjoint matrices but rather the space of skew adjoint matrices. Indeed, if A is a skew adjoint matrix then $\exp tA$ is a one parameter group of unitary matrices. The physicists prefer to write $\exp itH$ where H is self adjoint. This is of course due to the fact that self adjoint operators are the observables of quantum mechanics, and Noether's theorem suggests that elements of the Lie algebra should correspond to observables. But the price to pay for this is to put an i in front of all brackets.

For example, the three dimensional real vector space consisting of self adjoint two by two matrices of trace zero has, as a basis, $\tau_i,\ i = 1, 2, 3$ the "Pauli matrices", where, to be absolutely sure of the factors 1/2 etc.,

$$\tau_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \tau_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The physicists like to think of these as "generators" of $SU(2)$, i.e. as elements of the Lie algebra $su(2)$. Of course, we mathematicians would say that multiplying each of these three matrices by i gives a basis of $su(2)$. This distinction is relatively harmless, but is a nuisance for a mathematician reading a physics book or paper.

2.2 Ad invariant metrics on $u(2)$.

If we use the scalar product

$$(A, B) = 2 \operatorname{tr} AB$$

then the elements $\frac{1}{2}\tau_i$ form an orthonormal basis of our three dimensional space of self-adjoint matrices of trace zero. Since the algebra $su(2)$ is simple, the most general Ad invariant scalar product on our three dimensional space of self-adjoint matrices of trace zero must be a positive multiple of the above scalar product.

We will want to consider the four dimensional space of all two by two self adjoint matrices. (After multiplication by i this would yield the Lie algebra of $U(2)$.) So we must add the two by two identity matrix I to get a basis of this four dimensional real space. The algebra $u(2) \sim su(2) \oplus u(1)$ is not simple, but decomposes into the sum of two ideals consisting of $su(2)$ and all (real) multiples of iI. These ideals must be orthogonal under any Ad invariant metric. So there is a two parameter family of Ad invariant metrics on $u(2)$.

2.3 Ad invariant metrics, coupling constants, and the Weinberg angle.

Indeed, the most general Ad invariant metric on our four dimensional space of all self-adjoint two by two matrices can be written as

$$\frac{2}{g_2^2} \operatorname{tr}(A - \frac{1}{2}(\operatorname{tr} A)I)(B - \frac{1}{2}(\operatorname{tr} B)I) + \frac{1}{g_1^2} \operatorname{tr} A \operatorname{tr} B. \tag{1}$$

Relative to this scalar product the elements

$$\frac{g_2}{2}\tau_1,\ \frac{g_2}{2}\tau_2,\ \frac{g_2}{2}\tau_3, \frac{g_1}{2}I \tag{2}$$

form an orthonormal basis.

Notice that for traceless matrices the second term in (1) vanishes, and the first term reduces to a multiple of $2 \operatorname{tr} AB$; similarly, for multiples of I, the first term vanishes.

For mathematicians, the question is "why this strange notation, with the g_2 and g_1 occurring in the denominator?". The answer is that, in the physics literature, these constants are not regarded as parametrizing metrics on $u(2)$, but rather as "universal coupling constants". I will spend a chunk of today's lecture explaining why choosing a metric on a Lie algebra is important, and what is its physical significance.

In any event, however you want to interpret these parameters, the *Weinberg angle* θ_W is defined by

$$\frac{g_1^2}{g_2^2} = \tan^2 \theta_W.$$

It plays an important role in the theory.

2.4 What are classical fields?

A third difference between the mathematical literature and the physics literature is that in the physics literature all (classical) fields are regarded as scalar valued functions (or vector fields) or n-tuplets of scalar valued functions (or vector fields) . One must then discuss the "field transformations" under which, for example, the Lagrangian is invariant. The mathematical literature prefers a "basis free" formulation where many of the invariance properties of the Lagrangian are obvious - they are built into the formulation. The price to pay is that the fields are no longer scalar functions or n-tuplets of scalar functions but vector valued functions, or, more generally, sections of a vector bundle, or differential forms with values in a vector bundle.

This means that in the physics literature a basis of the vector space (or a basis of sections of the vector bundle) is chosen. Thus, for example, if we choose a basis $v_1, \ldots, v_n$ of a Lie algebra $\mathfrak{k}$ then the Lie bracket can be given in terms of the Cartan structure constants c_{jk}^{ℓ} where

$$[v_j, v_k] = \sum_{\ell} c_{jk}^{\ell} v_{\ell}.$$

As explained above, in the physics literature there will be an additional factor of i in front of the structure constants as understood by the mathematicians. For example, if we take the orthonormal basis of the space of traceless two by two self adjoint matrices consisting of the first three elements of (2), we find by direct computation that

$$\left[\frac{g}{2}\tau_1, \frac{g}{2}\tau_2\right] = i\frac{g^2}{2}\tau_3 = ig\frac{g}{2}\tau_3, \qquad g = g_2,$$

with a similar formula for the brackets of the remaining two elements. So relative to this basis, the structure constants are

$$C_{jk\ell} = ig\epsilon_{jk\ell}.$$

Up to an overall sign arising from slightly different conventions this is the statement about the structure constants of $SU(2)_L$ found in S. Weinberg, *The Quantum Theory of Fields*, Cambridge U. Press (1996), vol. 2. page 307 just after equation (21.3.11) giving the expression of the Lagrangian of the Yang-Mills field. So whereas for mathematicians the parameter g describes the scalar product on $su(2)$, for physicists, who write out the fields in terms of an orthonormal basis, the g appears in the structure constants and is interpreted as a "coupling constant", measuring the "strength of the interaction between the fields".

3 The permittivity of space-time is a metric on $u(1)$.

In order to bolster my contention that the metric on a Lie algebra has important physical significance, I want to review Maxwell's classical theory of electromagnetism, with special attention to units.

I will begin with two non-relativistic regimes:

3.1 Electrostatics.

The objects are:

3.1.1 The electric field.

This is a linear differential form, E, called the *electric field strength*. A point charge e experiences the force eE. The integral of E along any path gives the voltage drop along that path. The units of E are

$$\frac{\text{voltage}}{\text{length}} = \frac{\text{energy}}{\text{charge} \cdot \text{length}}.$$

Remember that force has the units energy/length and voltage has units energy/charge.

The fundamental law satisfied by E is

$$dE = 0.$$

In simply connected regions this implies the existence of a function u called the potential such that

$$E = -du.$$

3.1.2 The dielectric displacement.

This is a two form D on $\mathbb{R}^3$. Its physical significance is as follows. To determine the value of D on a (small) oriented plane element, insert two small metal plates of the shape of this plane element, touch them together and then separate them. Charges $\pm Q$ are acquired on the plates. The orientation of the plane together with the orientation of $\mathbb{R}^3$ determines which of these two separated plates is the "top" plate and the value of D is (the limit of)

$$4\pi \frac{\text{charge on the top plate}}{\text{area of the plates}}.$$

So the units of D are

$$\frac{\text{charge}}{\text{area}}.$$

Notice that this definition makes no mention of the electric field.

The fundamental law satisfied by D is **Gauss's law** which asserts that for any region U

$$\int_{\partial U} D = 4\pi \int_U \rho dx \wedge dy \wedge dz$$

where ρ is the electric charge density.

Stokes' theorem gives the infinitesimal version of Gauss's law as

$$dD = 4\pi \rho dx \wedge dy \wedge dz.$$

If E is an electric field strength and D is a dielectric displacement then $E \wedge D$ is a three form which we may integrate over $\mathbb{R}^3$ if it is of compact support or if it vanishes sufficiently rapidly at infinity. Then we can form

$$\langle D, E \rangle := \int_{\mathbf{R^3}} E \wedge D$$

which we can consider as a sort of pairing between the space of electric fields and the space of dielectric displacements. The value of this pairing has units

$$\text{volume} \cdot \frac{\text{force}}{\text{charge}} \cdot \frac{\text{charge}}{\text{area}} = \text{force} \cdot \text{length} = \text{energy}.$$

3.1.3 The dielectric operator and the dielectric coefficient.

This is a map C from the space of electric fields to the space of dielectric displacements. Later on we shall be more specific as to the form of C in terms of the three dimensional $\star$ operator. At the moment we can do with the following mild assumptions:

- C is linear.
- C is local in the sense that it E vanishes on an open set U so does $C(E)$.
- C is symmetric in the sense that

  $$\langle C(E), \hat{E} \rangle = \langle C(\hat{E}), E \rangle$$

  when both sides are defined. When both sides are defined, we set

  $$(E, \hat{E}) := \langle C(E), \hat{E} \rangle.$$

We can then define the energy of an electric field as

$$\frac{1}{2}(E, E).$$

3.1.4 The dielectric coefficient.

A more specific choice of the dielectric operator is to take

$$C(E) = \epsilon \star E$$

where $\star$ is the three dimensional star operator mapping one forms into two forms and ϵ is a function. Even more specifically, in many cases (such as for the vacuum) ϵ is a constant - called the **dielectric constant**. We will postpone the issue of units for the moment, assume that ϵ is indeed constant, and then choose our units of length so that it is absorbed into the star operator. Then the equations of electrostatics becomes

$$\begin{aligned} E &= -du \\ D &= \star E \\ dD &= 4\pi\rho dx \wedge dy \wedge dz \\ &\text{so} \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= -4\pi\rho. \end{aligned}$$

If $\rho = 0$ in some region, then in that region the last equation becomes Laplace's equation which we can write in coordinate free notation as

$$d \star du = 0.$$

3.1.5 Rotationally invariant solutions of Laplace's equation.

In polar coordinates we have

$$\star dr = r^2 \sin\theta d\theta \wedge d\phi.$$

So if $f = f(r)$ is defined for $r > 0$ we have

$$\begin{aligned} df &= f'(r)dr \\ \star df &= (r^2 f'(r)) \sin\theta d\theta \wedge d\phi \\ d \star df &= \left(r^2 f'(r)\right)' dr \wedge \sin\theta d\theta \wedge d\phi \\ &\text{so} \\ d \star df = 0 &\Rightarrow \left(r^2 f'(r)\right)' = 0 \\ &\Rightarrow r^2 f'(r) = -c \text{ (a constant)} \\ &\Rightarrow \\ f(r) &= \frac{c}{r} + A \end{aligned}$$

where c and A are constants.

The inverse square law of high-school physics drops out.

Exactly the same computation yields the Yukawa potential as the static fundamental solution of the Klein-Gordon equation: Indeed

$$d\left(\frac{e^{-mr}}{r}\right) = -e^{-mr}\frac{mr+1}{r^2}dr$$

so

$$\star d\left(\frac{e^{-mr}}{r}\right) = -e^{-mr}(mr+1)\sin\theta d\theta \wedge d\phi$$

and hence

$$\begin{aligned} d \star d\left(\frac{e^{-mr}}{r}\right) &= m^2 r e^{-mr}\sin\theta dr \wedge d\theta \wedge d\phi \\ &= m^2\left(\frac{e^{-mr}}{r}\right) r^2 \sin\theta dr \wedge d\theta \wedge d\phi. \end{aligned}$$

Thus if we think of a solution of the Klein-Gordon equation or of the Proca equation as a "transmitter" of a "force", then the value of m determines the range of this force.

3.2 Magnetoquasistatics.

3.2.1 Objects: The magnetic induction and the magnetic excitation.

The magnetic induction is a two form B on $\mathbb{R}^3$, which exerts a force on a current according to the following rule: if a charge e moves past the point P with velocity $\mathbf{v}$ then the force exerted on that charge is the covector

$$ei(\mathbf{v})B_P.$$

At each point of space the form B if $\neq 0$ will determine a direction in space: the line determined by the equation

$$i(\mathbf{w})B = 0.$$

Iron filings free to rotate but not to move will align themselves in these directions, producing the "magnetic lines of force" favored by Faraday. These are precisely the directions in which a current will feel no force.

The second item is a one form H known as the magnetic excitation. The second of Ampère's laws says that if S is any surface bounded by a curve γ, and if J is the two form representing the current flow, then

$$\int_\gamma H = 4\pi \int_S J.$$

3.2.2 The laws. 1: Faraday's law of induction.

This says that if S is a surface bounded by a curve γ then

$$-\frac{d}{dt}\int_S B = \int_\gamma E. \tag{3}$$

By Stokes the differential version of this law is

$$\frac{\partial B}{\partial t} = -dE. \tag{4}$$

If S is a closed surface bounding a region (so with no boundary curves) then Faraday's law implies that

$$\frac{d}{dt}\int_S B = 0.$$

In fact, a stronger law holds (Hertz), not only does the derivative of the integral of B over a closed surface vanish, but the integral itself does:

3.2.3 The laws. 2: There are no magnetic poles.

This says that

$$\int_S B = 0 \quad \text{for any closed surface} \quad S.$$

By Stokes, the differential version of this law is

$$dB = 0.$$

3.2.4 The laws. 3: Ampère's law.

Recall that this says that if S is any surface bounded by a curve γ, and if J is the two form representing the current flow, then

$$\int_\gamma H = 4\pi \int_S J.$$

By Stokes' theorem, the differential version of this law is

$$dH = 4\pi J.$$

3.2.5 The force on a moving charge.

The force exerted by a magnetic field B on a moving charge $\mathbf{I} = e\mathbf{v}$ is

$$i(\mathbf{I})B.$$

3.2.6 The permeability.

There is a relation between H and B given by

$$B = \mu \star H$$

where $\star$ is the three dimensional star operator and μ is known as the permeability.

3.2.7 The units of $\int_S B$.

By Faradays' law of induction, the time derivative of this integral over a surface bounded by a curve is equal to the negative of the integral of E around that curve which has units of voltage which is energy/charge. So

$$\frac{\text{units of } \int\int_S B}{\text{time}} = \frac{\text{energy}}{\text{charge}}.$$

In "natural units", where $\hbar = 1$, energy has units of inverse time. This implies that the integral of B over a surface has units of inverse charge.

3.3 The Maxwell equations.

The laws of quasi-magnetostatics take on a very suggestive form when written in four dimensions rather than three, and when an important modification to Ampère's law is made. This modification was introduced by Maxwell.

3.3.1 The equation $dF = 0$.

We can combine the laws

$$\begin{aligned} dB &= 0 \qquad \text{(Hertz)} \\ \frac{\partial B}{\partial t} &= -dE \qquad \text{(Faraday's law of induction)} \end{aligned}$$

into the single law

$$dF = 0 \tag{5}$$

if we set

$$F = B + E \wedge dt.$$

In (5) the operator d means the four dimensional (space-time) d. The coefficient of $dx \wedge dy \wedge dz$ in (5) says that

$$d_{space} B = 0 \qquad \text{(Hertz)}$$

while the coefficient of dt in (5) gives Faraday's law of induction.

The equation $dF = 0$ implies that locally we can find a one form A (called the four potential) such that

$$dA = F. \tag{6}$$

3.3.2 The equation $dG = 4\pi j$.

In electrostatics we assumed that $J = 0$ and that the charge density ρ did not depend on t. In quasi-magnetostatics we ignored ρ. For the full equations of electromagnetism one assumes that there is a charge density and a current, and so consider the three form

$$j := \rho dx \wedge dy \wedge dz - J \wedge dt$$

on space time. "Conservation of charge" then demands that

$$dj = 0.$$

Locally this says that there is two form G such that

$$dG = 4\pi j. \tag{7}$$

(The 4π is conventional.) If we write

$$G = D - H \wedge dt, \tag{8}$$

then the dt component of (7) is

$$d_{space} H = \frac{\partial D}{\partial t} + 4\pi J.$$

So we recover Ampère's law with the modification that the "displacement current"

$$\frac{\partial D}{\partial t}$$

is added to the right hand side of Ampère's original law. The "space component" of (7) is

$$d_{space} D = 4\pi \rho dx \wedge dy \wedge dz$$

as in electrostatics.

3.4 Units.

Let us work in natural units where $\hbar = 1$ so that energy has units of inverse time.

3.4.1 The units of the integral of F over a surface.

We have already observed that the integral of B over a surface has units of inverse charge. The integral of E over a curve has units of (energy)/(charge), so the integral of $E \wedge dt$ over a surface in space time has units of

$$\frac{(\text{energy}) \times (\text{time})}{(\text{charge})} = \frac{1}{(\text{charge})}.$$

In short, the integral of F over a surface in space time has units of inverse charge.

3.4.2 The units of the integral of G over a surface.

From its definition, or from Gauss's law $dD = 4\pi\rho dx \wedge dy \wedge dz$ we see that the units of the integral of D over a surface are charge. Ampère's law

$$d_{space}H = \frac{\partial D}{\partial t} + 4\pi J$$

together with Stokes' theorem says that the integral of H over a curve has the same units as the flux of current through a surface and this has units (charge)/(time). So the integral of $H \wedge dt$ over a surface in space time also has the units of (charge). In short, the integral of G over a surface in space time has units of charge.

3.4.3 The integral of $F \wedge G$ over a four dimensional region is a scalar.

This follows from the preceding two results. In particular, this means that G is in a sense "dual" to F, the duality being given by exterior multiplication followed by integration. Of course we can not expect that the integral over all of space time will converge. We will examine this"duality" in more detail further on.

Notice that until now we have not used the metric structure of space time.

3.4.4 The units of the permittivity.

The units D are (charge)/(area). The units of E are (energy)/(charge)×(length). If there is a point-wise matrix which expresses the coefficients of D in terms of those of E its entries will have units

$$\frac{\text{charge}}{\text{area}} \times \frac{(\text{charge}) \times (\text{length})}{\text{energy}} = \frac{(\text{charge})^2}{(\text{energy}) \times (\text{length})}.$$

Indeed, the permittivity of free space is a scalar ϵ_0 given by

$$\epsilon_0 = 8.854187... \times 10^{-12} \ \frac{\text{Farad}}{\text{meter}}$$

where the Farad is a unit of capacitance:

$$1 \text{ Farad} := 1\frac{\text{coulomb}}{\text{volt}}.$$

Since

$$1 \text{ volt} = 1 \ \frac{\text{joule}}{\text{coulomb}}$$

has units of (energy)/(charge) we see that ϵ_0 has units of

$$\frac{(\text{charge})^2}{(\text{energy}) \times (\text{length})}.$$

3.4.5 The units of the permeability.

According to Ampère's law the units of H are (charge)/(length)×(time).

According to Faraday's law the units of B are (energy)×(time) / (charge)×(length^2. If there is a point-wise matrix which expresses the coefficients of B in terms of those of H its entries will have units

$$\frac{\text{energy} \times (\text{time})^2}{(\text{charge})^2 \times (\text{length})}.$$

Indeed, the permeability of free space is a scalar μ_0 given by

$$\mu_0 = 12.566370 \ \times 10^{-7} \ \frac{\text{joule}}{(\text{amp})^2 \times (\text{meter})}.$$

Since one amp = one (coulomb)/(second) we see that μ_0 does have the above stated units.

3.4.6 $\epsilon_0 \times \mu_0 = 1/c^2$.

This was of course another of the great discoveries of Maxwell and verified by Hertz. We can see that the product of the units of the permittivity with those of the permeabilty yield units of $1/(\text{velocity})^2$, and doing the multiplication for the values of free space give the velocity of light, implying that light consists of electromagnetic propagation.

As a consequence, we can choose units in which $c = 1$ and lengths and times are measured in the same units. Special relativity with its Minkowski metric $dt^2 - dx^2 - dy^2 - dz^2$ is then an immediate consequence.

3.4.7 The permittivity and the permeability in natural units.

If we choose natural units so that $\hbar = 1$ and $c = 1$ then length has the same units as time and so energy has units of inverse length and the expression in the denominator for the units of the permeability is just a scalar. So the permittiviity has units of $(\text{charge})^2$.

Similarly, the units of the permeability become $(\text{charge})^{-2}$.

3.4.8 The fine structure constant.

The expression

$$\alpha := \frac{(\text{charge of the electron})^2}{4\pi\epsilon_0}$$

is a pure number in terms of our natural units where $\hbar = 1$ and $c = 1$ and is equal to

$$\frac{1}{137.0359...}.$$

In terms of conventional units we would write

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}.$$

4 Gauge theories.

Hermann Weyl had suggested that the true objects of general relativity should not be (semi-)Riemann metrics, but rather the associated Levi-Civita connection. And if we generalize this connection to be a conformal connection (i.e. if we enlarge the group from $O(1,3)$ to $\mathbb{R}^+ \times O(1,3)$) then we can incorporate electromagnetism.(See his classic *Raum Zeit Materie*, Springer, Berlin (1918)). The word "gauge" derives from Weyl's theory in which the length is changed by a conformal transformation.

Einstein rejected Weyl's proposal of considering a conformal connection as the underlying physical field, although Einstein himself considered the possibility that Riemannian geometry be replaced by conformal geometry as a basis for unified theories - see his article in *Preuss Akad.* 261 (1921) as well as the following notes on the "unified field theory": *loc. cit.* (1925) p. 414, (1928) p. 3, (1929) p. 3.

After the advent of quantum mechanics, Fritz London, in a short note in early 1927(F. London, "Die Theorie von Weyl und die Quantenmechanik", *Naturwiss.* **15** 187. and soon after in a longer paper, "Quantenmechanische Deutung der Theorie von Weyl," *Zeit. für Physik* **42**, 375-389 (1927), proposed a quantum mechanical interpretation of Weyl's attempt to unify electromagnetism and gravitation. The essential idea is to replace Weyl's $\mathbb{R}^+$ by $U(1)$ acting as phase transformations of the quantum mechanical state vector. The group $U(1)$ does not act on the tangent space of space time. It is "internal". The London theory for $U(1)$ was generalized to $SU(2)$ by Yang and Mills in 1954, C.N. Yang and R. Mills, "Conservation of isotopic spin and isospin gauge invariance," *Phys. Rev.* **96** 191-195 (1954).

The "field" in a Yang-Mills theory on space time is a connection on a principal bundle P.

Giving a connection on a principal bundle is the same as giving (consistently) the notion of covariant derivative on any associated bundle. The covariant derivative language is more popular in the standard physics texts. I will give a self contained review of the notions of connection and curvature in the more general setting of superconnections and supercurvature later on.

If $\mathcal{G}$ is the structure group of the bundle P and $\mathfrak{g}_0$ is the Lie algebra of $\mathcal{G}$, the curvature of such a connection is a 2-form on space-time with values in the vector bundle $\mathfrak{g}_0(P)$ associated to the adjoint representation of $\mathcal{G}$. If F is such a curvature form, and if $\star$ denotes the Hodge star operator of space time, then $\star F$ is another 2-form with values in $\mathfrak{g}_0(P)$, so

$$F \wedge \star F$$

is a 4-form with values in $\mathfrak{g}_0(P) \otimes \mathfrak{g}_0(P)$. In order to get a numerical valued 4-form which we can consider as a Lagrangian density, we need an Ad invariant scalar product on $\mathfrak{g}_0$.

For example, we have seen that the electromagnetic field F is a two form whose integral over any surface has units of inverse charge. So $F \wedge \star F$ is a 4-form with units of $1/(\text{charge})^2$. In order to get the correct Lagrangian density, we must multiply by ϵ_0, the permittivity of empty space which (in natural units) has units of $(\text{charge})^2$, so that

$$\frac{1}{2}\epsilon_0 F \wedge \star F$$

is the Lagrangian density for the electromagnetic field in empty space. If we want to consider F (strictly speaking iF) as the curvature of a connection on a $U(1)$ bundle, we see that we must consider ϵ_0 as determining a metric on $u(1)$ (different from the "natural" one regarding $u(1)$ as $i\mathbb{R}$), and this metric has deep physical significance.

In the Standard Model of the electroweak theory, the group under consideration is $U(2)$ or $SU(2) \times U(1)$ with Lie algebra $\mathfrak{g}_0 = u(2)$. As we have seen, there is a two parameter family of invariant metrics on $u(2)$ given by (1).

We repeat that we are regarding g_1 and g_2 as parameters describing possible Ad invariant scalar products on the Lie algebra $u(2)$. As such they have physical significance similar to that of the permittivity of free space in electromagnetic theory and are necessary to be able to formulate a Yang-Mills functional. In a general relativistic theory one would expect them to have a space time dependence just as the metric of space time does. The interpretation of g_1 and g_2 as "universal coupling constants" then derives from the interpretation as defining a metric.

5 The Higgs mechanism.

5.1 The Higgs mechanism in a nutshell.

The Higgs mechanism in the Standard Model of electroweak interactions is a device for breaking the $u(2) = su(2) \oplus u(1)$ symmetry of a $U(2)$ gauge theory in such a way that the three of the four components of a connection form (originally massless in a pure Yang-Mills theory) become differential forms with values in a vector bundle associated to $U(1)$ and which enter into a Lagrangian whose quadratic terms correspond to particles with positive mass. In mathematical terms this corresponds to a reduction of a principal $U(2)$ bundle to a $U(1)$ bundle.

The ingredients that go into this mechanism and into the computation of the acquired masses are the following:

- An Ad invariant positive definite metric on $u(2)$. This is needed for the original (unbroken) Yang-Mills theory. We have argued that the "universal coupling constants" that enter into the general formulation of this theory are in fact parameters which describe the possible Ad invariant metrics on $u(2)$. In general there is a two parameter family of such metrics. They are related by a certain angle θ_W known as the Weinberg angle. Our internal supersymmetry proposal will determine this angle as 30^0, or $\sin^2\theta_W = 0.25$, which is not too far from the measured value of 0.2312 ± 0.003.

- A two dimensional Hermitian vector bundle associated to the principal $U(2)$ bundle. In the general presentation of the Standard Model this vector bundle is an extraneous ingredient put in "by hand". In our theory this vector bundle is $\mathfrak{g}_1$, the odd component of a Lie super algebra bundle. The sections of this bundle are regarded as the exterior degree zero components of a superconnection. More details on this later.

- A degree-four polynomial on this vector bundle. In the general presentation this must also be provided by hand. In our theory, the quartic term of this polynomial is the super-Yang-Mills functional.

- The vector bundle $\mathfrak{g}_1$ is associated to the original $U(2)$ bundle, so $U(2)$ invariance determines the Hermitian metric up to a scalar factor. We proposed to fix this scalar by relating it to the choice of scale entering into the metric on $su(2)$. This is done by using the concept of a Hermitian Lie algebra, see S. Sternberg, J. Wolf, "Hermitian Lie algebras and metaplectic representations", *Trans. Amer. Math. Soc.* **231** 1 (1978) which relates certain Lie superalgebras to ordinary Lie algebras. Once the metric has been fixed, we can write the most general (invariant) degree four polynomial as
$$a\|\cdot\|^4 - b\|\cdot\|^2.$$

The next three steps are part of the standard Higgs mechanism, cf. for example A. Derdzinski, *Geometry of the standard model* Section 11. We summarize them here for the reader's convenience. Additional details will be given below.

- If a and b are both positive, then the quadratic polynomial
$$az^2 - bz$$
achieves its minimum at
$$z_0 = \frac{b}{2a}$$
and hence any section ψ of our vector bundle lying on the three-sphere bundle
$$\|\psi\|^2 = z_0$$
is a global minimum. Any such section is called a *vacuum state*. The reduction of the principal $U(2)$ bundle is achieved by fixing one such vacuum. For example, if the bundle is trivial and is given a trivialization which identifies it with the trivial $\mathbb{C}^2$ bundle then we may choose ψ of the form
$$\psi = \psi_0 := \begin{pmatrix} 0 \\ v \end{pmatrix}, v > 0$$
so
$$\|\psi_0\| = \sqrt{\frac{b}{2a}}.$$

- The mass of the W particle is then given as

$$m(W) = \frac{\|\psi_0\|}{\|i\tau_1\|_{u(2)}} \tag{9}$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

See the discussion in Section 5.2 below. In terms of the parameter g_2 entering into the definition of the metric on $su(2)$ (see (1)) this becomes

$$m(W) = \frac{1}{2} g_2 \|\psi_0\| = \frac{1}{2} g_2 \sqrt{\frac{b}{2a}}. \tag{10}$$

- The mass of the Higgs field (see Section 5.3 below) is given by

$$m(\text{Higgs}) = 2\sqrt{b}. \tag{11}$$

This gives the value of the Higgs mass in terms of parameters entering into the Higgs model. Notice that only the coefficient of the quadratic term (b) enters into this formula, but if we know the coefficient a of the quartic term, then we can get b from $\|\psi_0\| = \sqrt{b/2a}$.

As indicated above, we will derive the value of a from the supercurvature and the metric on the superalgebra coming from a corresponding Lie algebra, see equation (15) below. Thus we are able to predict the Higgs mass from the observed experimental value of the W mass using (10) and (11) and the value of a. We will find that m(Higgs)=2m(W).

To reiterate - we make no predictions about b. We do make a prediction of a coming from the interpretation of the quartic term in the Higgs field as arising from a super-Yang-Mills Lagrangian (to be explained below). No matter what b is, the knowledge of a determines the ratio of the mass of the Higgs to the mass of the W.

5.1.1 The Weinberg angle, again.

We return to equation (1) which gives the most general ad-invariant scalar product on $u(2)$. The Weinberg angle is then defined by

$$\frac{g_1^2}{g_2^2} = \tan^2 \theta_W.$$

Thus, for example, any choice of g_1 and g_2 which leads to a value of

$$\frac{g_1^2}{g_2^2} = 1/3$$

will yield a Weinberg angle of 30 degrees.

5.1.2 Scalar products from representations.

Any faithful unitary representation r of $u(2)$ will yield a positive definite scalar product by letting the scalar product of A and B be

$$-\operatorname{tr} r(A)r(B).$$

Under our identification of $u(2)$ with self adjoint rather than skew adjoint matrices, which involves multiplication by i, we can forget about the minus sign. But we do want to allow for an overall scale factor and so consider the metric

$$A \mapsto \frac{2}{g^2} \operatorname{tr}\left(r(A)^2\right) \tag{12}$$

as being associated to the representation r. Of course the Weinberg angle will be independent of the factor g.

So any theory which singles out a preferred faithful representation of $u(2)$ will give a prediction of the Weinberg angle. Our proposal is to regard $u(2)$ as the even part of the superalgebra $su(2/1) \subset sl(2/1)$. See Section 8.1 for the definition of the Lie superalgebras $sl(m/n)$. Each of these Lie superalgebras has a fundamental (defining) representation as described in Section 8.1. In particular, this picks out a preferred faithful representation of $u(2)$ and hence gives a prediction of the Weinberg angle. We do the computation in the next section.

5.1.3 The Weinberg angle of the fundamental representation of $sl(2/1)$.

In this representation the two by two matrix A is represented by the three by three matrix

$$r(A) = \begin{pmatrix} A & 0 \\ 0 & \operatorname{tr} A \end{pmatrix}.$$

If we take $A \in su(2)$ so $\operatorname{tr} A = 0$ in (12) we get $\operatorname{tr}(r(A)^2) = \operatorname{tr}(A^2)$ from which we see that the g_2 entering into formula (1) for the metric on $u(2)$ is given by $g_2^2 = g^2$. If we take $A = I$ in (12) we get

$$r(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

so $\operatorname{tr}(r(I)^2) = 6$. So

$$\frac{2}{g^2} \cdot 6 = \frac{4}{g_1^2} \quad \text{so} \quad \frac{g_1^2}{g_2^2} = \frac{1}{3}$$

yielding a Weinberg angle of 30 degrees.

5.2 Other quadratic forms.

Given a positive definite real scalar product $(\cdot,\cdot)$ on a real vector space, any other quadratic form is given by $x \mapsto (Sx, x)$ where S is a self-adjoint operator. We can then diagonalize S. If the second quadratic form is positive semi-definite, then these eigenvalues are non-negative, and S has a unique square root $S^{\frac{1}{2}}$ with non-negative eigenvalues. For reasons of differential geometry - essentially the reduction of a $U(2)$ bundle to a $U(1)$ bundle via the choice of section of an associated bundle - these eigenvalues are identified with the masses of certain spin 1 particles in the example I will now work out. I will do the elementary linear algebra now, so we can see what is needed for mass predictions, and discuss the geometry later.

For example, consider the standard action of $u(2)$ on $\mathbb{C}^2$ and define the "second" quadratic form on $u(2)$ to be

$$q(A) := \|A\psi_0\|^2_{\mathbb{C}^2} = (A\psi_0, A\psi_0)_{\mathbb{C}^2}$$

where ψ_0 is a fixed element of $\mathbb{C}^2$, and where $(\cdot,\cdot)_{\mathbb{C}^2}$ is some $U(2)$ invariant scalar product on $\mathbb{C}^2$ (and so is some positive multiple of the standard scalar product). The corresponding bilinear form on $u(2)$ is

$$\langle A, B\rangle = \mathrm{Re}(A\psi_0, B\psi_0)_{\mathbb{C}^2}.$$

In fact, let us take

$$\psi_0 := \begin{pmatrix} 0 \\ v \end{pmatrix}, v > 0$$

as above. Then

$$\tau_1\psi_0 = \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad \tau_2\psi_0 = \begin{pmatrix} -iv \\ 0 \end{pmatrix}, \quad \tau_3\psi_0 = \begin{pmatrix} 0 \\ -v \end{pmatrix}, \text{ and } \ I\psi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Then relative to any scalar product $(\cdot,\cdot)$ on $u(2)$ we have

$$(S\tau_1, X) = \langle \tau_1, X\rangle = 0 \quad \text{for} \quad X = \tau_2, \tau_3, I.$$

If $(\cdot,\cdot)$ is any of the invariant metrics (1), then $(\tau_1, X) = 0$ for $X = \tau_2, \tau_3, I$. This shows that τ_1 is an eigenvector of S with eigenvalue $\|\psi_0\|^2/\|\tau_1\|^2_{u(2)}$. Similarly for τ_2. Sections of the line bundles corresponding to these eigenvectors are identified with the W particles. This accounts for the mass of the W as given in equation (9) above.

We have $(\tau_3 + I)\psi_0 = 0$ so $\tau_3 + I$ is an eigenvector of S with eigenvalue 0. Expressed in terms of the orthonormal basis (2) and normalized so to have length one gives

$$\frac{1}{(g_1^2 + g_2^2)^{\frac{1}{2}}}\left(g_2\frac{g_1}{2}I + g_1\frac{g_2}{2}\tau_3\right).$$

The corresponding mass zero field is then identified with the electromagnetic field.

Taking the orthogonal complement of the three eigenvectors found so far (corresponding to the W's and the electromagnetic field) gives the field of the Z particle.

All of the material in this section is part of the standard repertoire of the Higgs mechanism and is not particular to the model we propose. For instance, equation (9) is the formula in equation (11.30) A. Derdzinski, *Geometry of the standard model* for the mass of the W up to differences in notation and the fact that we are computing in natural units.

But it might be instructive to see how all this is written out in the physics literature, where "fields" are always scalar valued. In terms of the basis $\tau_1, \tau_2, \tau_3, I$ we have verified that our quadratic form is given by

$$q(X_1\tau_1 + X_2\tau_2 + X_3\tau_3 + YI) = v^2(X_1^2 + X_2^2 + (Y - X_3)^2).$$

Let us express this in terms of the coordinates in the orthonormal basis written above (and taking the standard Hermitian form on $\mathbb{C}^2$). We have

$$X_i\tau_1 = \frac{2X_i}{g_2} \cdot \frac{g_2\tau_i}{2}$$

so the coefficient W_i of $X_i\tau_i$ in terms of the normalized basis element is

$$W_i = \frac{2X_i}{g_2}$$

and hence

$$X_i = \frac{g_2}{2}W_i, i = 1, 2, 3$$

and similarly $Y = \frac{g_1}{2}B$ where B is coefficient relative to the last normalized element. So

$$Q(W_1, W_2, W_3, B) = \frac{1}{4}v^2\left(g_2^2(W_1^2 + W_2^2) + (g_2W_3 - g_1B)^2\right).$$

The rotation

$$R_{\theta_W} = \frac{1}{\sqrt{g_1^2 + g_2^2}}\begin{pmatrix} g_2 & -g_1 \\ g_1 & g_2 \end{pmatrix}$$

in the W_3, B plane brings the quadratic form to diagonal form. This is the reason for angle terminology. The W_1, W_2 and Z are considered as transmitters of the weak interaction, while the massless field is identified with the photon.

5.2.1 Experimental determination of the coupling constant g_2.

The coupling constant g_2 enters into the definition of the metric on $u(2)$ as we have seen, and is observed via the "strength" of the electro-weak interaction. We have

$$g_2 = \frac{e}{\sin\theta_W}.$$

So if $\sin\theta_W = \frac{1}{2}$ we have $g_2 = 2e$. If

$$\frac{e^2}{4\pi} \doteq \frac{1}{137}$$

then $g_2 \doteq 0.6$.

5.3 The Higgs mass.

It is assumed that the Higgs field is a section of a Hermitian vector bundle with potential $\mathcal{V}$ which has the form

$$\mathcal{V}(\psi) = f(\langle\psi, \psi\rangle)$$

where

$$f : [0, \infty) \to \mathbb{R}$$

is a smooth function with a minimum at z_0. A particular section is ψ_0 chosen with $\langle\psi_0, \psi_0\rangle = z_0$. (If, as we shall assume, the Hermitian vector bundle is a two dimensional bundle associated to a principal $U(2)$ or $SU(2) \times U(1)$ bundle this has the effect of reducing the principal bundle to a $U(1)$ bundle.)

The most general section of our vector bundle is then written as $\psi_0 + \eta$ and we consider the quadratic term in the expansion of $f(\psi_0 + \eta)$ as a function of η. It will be given by

$$\frac{1}{2}\mathrm{Hess}(f)(\psi_0)(\eta) = 2f''(\langle\psi_0, \psi_0\rangle)(\mathrm{Re}\langle\psi, \eta\rangle)^2.$$

For η tangent to the orbit of the action of $U(2)$ this vanishes. But for $\eta \in \mathbb{R}\psi_0$ we have $\langle\psi_0, \eta\rangle = \pm\|\psi_0\|\|\eta\|$ so for such η (known as the Higgs field) the quadratic term is

$$2z_0 f''(z_0)\|\eta\|^2.$$

We want to consider this as a mass term, which means that we want to write this quadratic expression as $\frac{1}{2}m^2\|\eta\|^2$.

If

$$f(z) = az^2 - bz$$

with a and b positive constants, then the minimum of f is achieved at

$$z_0 = \frac{b}{2a}$$

and

$$f''(z_0) = 2a.$$

So

$$2z_0 f''(z_0) = 2b.$$

So we wish to write $2b\|\eta\|^2$ as $\frac{1}{2}m^2\|\eta\|^2$ where m is the mass of the Higgs. This gives

$$m(\mathrm{Higgs}) = 2\sqrt{b}$$

as in equation (11) above.

Once again, all of the material in this section is part of the standard repertoire of the Higgs mechanism and is not particular to the model we propose. Equation (11) is the formula in equation (11.30) of Derdzinski, *Geometry of the standard model* for the Higgs mass up to the fact that we are computing in natural units.

We will now revert to standard notation and write the Higgs field as ψ.

6 Using superconnections.

We assume that the Higgs field ψ is the degree zero piece of a superconnection for $su(2/1)$, and use this - together with an idea coming from the theory of Hermitian Lie algebras - to predict a value of a, namely

$$a = \frac{1}{8} g_2^2.$$

I will present a detailed exposition of the theory of superconnections later. But I want to get to the punch line in a hurry. So I will now show how the super-Yang-Mills Lagrangian for $su(2/1)$ makes a prediction of the factor a occurring in the f in the preceding section. In general, the Lagrangian of a super-Yang-Mills-Higgs theory will be of the form

$$(1/2)||F||^2 + ...$$

where F is the supercurvature and where .... involves the fermions, plus a quadratic term in the Higgs whose origin we leave open. The supercurvature is quadratic in the degree zero part of the superconnection, and hence the above Lagrangian, being quadratic in F, will be quartic in the degree zero part of the superconnection. So if we identify the Higgs field with this degree zero part, we get a quartic polynomial in the Higgs which derives from the underlying theory with no additional ad hoc assumptions. Here are the details of the computation:

If the Higgs field ψ is the degree zero piece of a superconnection for $su(2/1)$, then the supercurvature F will include a term $\frac{1}{2}[\psi, \psi]$ which is a section of $u(2)$ regarded as the even part of $su(2/1)$. If

$$\psi = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ \overline{x} & \overline{y} & 0 \end{pmatrix}$$

then

$$\frac{1}{2}[\psi, \psi] = \begin{pmatrix} |x|^2 & x\overline{y} & 0 \\ \overline{x}y & |y|^2 & 0 \\ 0 & 0 & |x|^2 + |y|^2 \end{pmatrix}.$$

To compute $\|F\|^2$, we need a metric on $u(2)$. In the computation of the Weinberg angle, we took the metric to be proportional to the metric induced by the fundamental representation of $sl(2/1)$. So we must use the metric

$$A \mapsto \frac{2}{g_2^2} \operatorname{tr} \left(\begin{pmatrix} A & 0 \\ 0 & \operatorname{tr} A \end{pmatrix}^2 \right)$$

so as to get the metric (1) on the $u(2)$ component. Applied to the $\frac{1}{2}[\psi, \psi]$ given above we get

$$\frac{4}{g_2^2}(|x|^2 + |y|^2)^2.$$

Taking $\frac{1}{2}$ of the above expression (as one half of the square length appears in the Lagrangian) gives the quartic term as

$$\frac{2}{g_2^2}(|x|^2+|y|^2)^2. \tag{13}$$

6.1 The metric on the Higgs.

We need to express (13) as $a\|\psi\|^4$. To do this we must say what $\|\psi\|^2$ is. We now use the paper S. Sternberg, J. Wolf, "Hermitian Lie algebras and metaplectic representations", *Trans. Amer. Math. Soc.* **231** 1 (1978) and propose that we think of $su(2/1)$ as the real part of the object whose imaginary part is $su(3)$.

If I have time, I will explain this later.

On $su(3)$ the only invariant metrics are scalar multiples of the Killing form, and since we want the metric to reduce to the above metric on $su(2)$ we must choose $\|\psi\|^2$ as

$$\psi \mapsto \frac{2}{g_2^2}\operatorname{tr}\psi^2 = \frac{4}{g_2^2}(|x|^2+|y|^2). \tag{14}$$

Comparing the two expressions (13) and (14) gives

$$a = \frac{1}{8}g_2^2. \tag{15}$$

Substituting this into (10) gives

$$m(W) = \sqrt{b}. \tag{16}$$

Comparing with (11) gives

$$\frac{m(\text{Higgs})}{m(W)} = 2. \tag{17}$$

This was the prediction in [N86]. For later versions of this prediction see [R98] and references cited there.

7 Superconnections.

In this section we give a self contained introduction to the theory of superconnections for the convenience of the reader. In the main, we follow the exposition given in [BGV91] with some changes in notation. For an alternative treatment see [MaSa2000].

7.1 Superspaces and superalgebras.

A **superspace** E is just a vector space with a $\mathbb{Z}_2$ grading:

$$E = E^+ \oplus E^-.$$

A **superalgebra** A is an algebra whose underlying vector space is a superspace and such that

$$A^+ \cdot A^+ \subset A^+, \quad A^- \cdot A^- \subset A^+, \quad A^+ \cdot A^- \subset A^-, \quad A^- \cdot A^+ \subset A^-.$$

The commutator of two homogeneous elements of A is defined as

$$[a, b] := ab - (-1)^{|a|\cdot|b|} ba.$$

We use the notation $|a| = 0$ if $a \in A^+$ and $|a| = 1$ if $a \in A^-$ and we do addition and multiplication mod 2.

A superalgebra is **commutative** if the commutator of any two elements vanishes. For example, the exterior algebra $\wedge(V)$ of a vector space is a commutative superalgebra where

$$\wedge(V)^+ := \wedge^0(V) \oplus \wedge^2(V) \oplus \wedge^4(V) \oplus \cdots,$$

and

$$\wedge(V)^- := \wedge^1(E) \oplus \wedge^3(V) \oplus \cdots.$$

7.2 The tensor product of two superalgebras.

If A and B are superspaces we make $A \otimes B$ into a superspace by

$$|a \otimes b| = |a| + |b|.$$

If A and B are superalgebras we make $A \otimes B$ into a superalgebra by

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b|\cdot|a'|} aa' \otimes bb'.$$

For example, the Clifford algebra of any vector space with a scalar product is a superalgebra, where $C(V)^+$ consists of those elements which can be written as a sum of products of an even number of elements of V and $C(V)^-$ consists of those elements which can be written as a sum of products of an odd number of elements of V. If V and W are two spaces with scalar products then the Clifford algebra of their orthogonal direct sum is the tensor product of their Clifford algebras:

$$C(V \oplus W) = C(V) \otimes C(W).$$

We will use the convention of the algebraists rather than that of the geometers in the definition of the Clifford algebra, W. Greub, *Multilinear algebra* Springer, Berlin (1978) . So if V is a vector space with a (not necessarily positive definite) scalar product then $C(V)$ is the universal algebra relative to the relations

$$uv + vu = 2(u, v)\mathbf{1}.$$

Chevalley, in is classic book does not have the factor 2 on the right hand side, because he considers fields of arbitrary characteristic, including characteristic rwo

(In N. Berline, E. Getzler, M. Vergne: *Heat Kernels and Dirac Operators*, Springer, Berlin 1991 the opposite convention (with a minus sign on the right hand side) is used.)

7.3 Lie superalgebras.

If A is an associative superalgebra the commutator of two homogeneous elements of A was defined as

$$[a,b] := ab - (-1)^{|a|\cdot|b|}ba.$$

This commutator satisfies the axioms for a **Lie superalgebra** which are

- $[a,b] + (-1)^{|a|\cdot|b|}[b,a] = 0$, and
- $[a,[b,c]] = [[a,b],c] + (-1)^{|a|\cdot|b|}[b,[a,c]]$.

It was proved in L. Corwin, Y. Ne'eman, S. Sternberg, "Graded Lie algebras in mathematics and physics", *Rev. Mod. Phy.* **47** 573 (1975) that every Lie superaglebra has a universal (associative) enveloping algebra and that the analogue of the Poincaré-Birkhoff-Witt theorem holds.

If A is a commutative superalgebra and L is a Lie superalgebra then $A \otimes L$ is again a Lie superalgebra under the usual definition:

$$[a \otimes X, b \otimes Y] := (-1)^{|X|\cdot|b|} ab \otimes [X,Y].$$

7.4 The endomorphism algebra of a superspace.

Let $E = E^+ \oplus E^-$ be a superspace. We make the algebra of all endomorphisms (= linear transformations) of E into a superalgebra by letting $\mathrm{End}(E)^+$ consist of those linear transformations which carry E^+ into E^+ and E^- into E^- while $\mathrm{End}(E)^-$ interchanges the two components. Thus a typical element of $\mathrm{End}(E)^+$ looks like

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad A \in \mathrm{End}(E^+), \quad D \in \mathrm{End}(E^-)$$

while a typical element of $\mathrm{End}(E)^-$ looks like

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad B : E^- \to E^+, \quad C : E^+ \to E^-.$$

An action (or a representation) of an associative algebra A on a superspace E is a (gradation preserving) homomorphism of A into $\mathrm{End}(E)$. We then also say that E is an A module.

Similarly, a representation of a Lie superalgebra L on a superspace E is a homomorphism of L into the commutator Lie superalgebra of $\mathrm{End}(E)$. This is the same as an action of the universal enveloping algebra $U(L)$ on E. We say that E is an L module.

7.5 Superbundles.

Let $\mathcal{E} \to M$ be a bundle of superspaces over a manifold M. We call such an object a **superbundle**. So $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ where $\mathcal{E}^+ \to M$ and $\mathcal{E}^- \to M$ are vector bundles over M. We will call a section of $\mathcal{E}^+$ an even section of $\mathcal{E}$ and a section of $\mathcal{E}^-$ an odd section of $\mathcal{E}$.

If $\mathcal{E}$ and $\mathcal{F}$ are superbundles, then $\mathcal{E} \otimes \mathcal{F}$ is a superbundle. In particular, $\wedge(T^*M)$ is a superbundle where

$$\wedge(T^*M)^+ := \wedge^0(T^*M) \oplus \wedge^2(T^*M) \oplus \wedge^4(T^*M) \oplus \cdots ,$$

$$\wedge(T^*M)^- := \wedge^1(T^*M) \oplus \wedge^3(T^*M) \oplus \wedge^5(T^*M) \oplus \cdots .$$

A section of $\wedge(T^*M) \otimes \mathcal{E}$ is called an **$\mathcal{E}$-valued differential form** and the space of all $\mathcal{E}$-valued differential forms will be denoted by $\mathcal{A}(M, \mathcal{E})$. Locally any element of $\mathcal{A}(M, \mathcal{E})$ is a sum of terms of the form $\alpha \otimes s$ where α is a differential form on M and s is a section $\mathcal{E}$.

7.6 The endomorphism bundle of a superbundle.

If $\mathcal{E} \to M$ is a superbundle, then we can consider the superbundle $\mathrm{End}(\mathcal{E})$ where, at each $m \in M$ we have $\mathrm{End}(\mathcal{E})_m := \mathrm{End}(\mathcal{E}_m)$. We have an action of any section of $\mathrm{End}(\mathcal{E})$ on any section of $\mathcal{E}$. By tensor product, any element of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$ acts on any element of $\mathcal{A}(M, \mathcal{E})$. In particular any element of $\mathcal{A}(M)$, i.e. any differential form acts on $\mathcal{A}(M, \mathcal{E})$ and (super)commutes with all elements of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$.

7.7 The centralizer of multiplication by differential forms.

Any element of $\mathcal{A}(M)$, i.e. any differential form, acts on $\mathcal{A}(M, \mathcal{E})$ and (super)commutes with all elements of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$.

There is an important converse to this last assertion. A differential operator on $\mathcal{A}(M, \mathcal{E})$ is by definition an operator which in local coordinates looks like

$$\sum_\gamma a_\gamma \partial^\gamma$$

where a_γ is a section of $\mathrm{End}\,\mathcal{A}(M, \mathcal{E})$ and $\partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$ is a partial differentiation operator in terms of the local coordinates. Leibnitz's rule implies that if such an operator commutes with all multiplications by functions then it can't really involve any differentiations. If furthermore it commutes with the action of all elements of $\mathcal{A}(M)$ it must be given by the action of some element of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$. In short: a differential operator on $\mathcal{A}(M, \mathcal{E})$ commutes with the action of $\mathcal{A}(M)$ if and only if it is given by an element of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$.

7.8 Bundles of Lie superalgebras.

If $\mathfrak{g}$ is a bundle of Lie superalgebras over M then $\mathcal{A}(M, \mathfrak{g})$ is a Lie superalgebra with bracket determined fiberwise (as we have seen) by

$$[\alpha \otimes X, \beta \otimes Y] = (-1)^{|X| \cdot |\beta|}(\alpha \wedge \beta) \otimes [X, Y].$$

If $\mathcal{E}$ is a superbundle on which $\mathfrak{g}$ acts, meaning that we have a fiberwise Lie superalgebra homomorphism ρ of $\mathfrak{g}$ into the Lie superalgebra bundle $\mathrm{End}(\mathcal{E})$

(under fiberwise bracket), then we have an action of $\mathcal{A}(M,\mathfrak{g})$ on $\mathcal{A}(M,\mathcal{E})$ determined by

$$\rho(\alpha\otimes X)(\beta\otimes v)=(-1)^{|X|\cdot|\beta|}(\alpha\wedge\beta)\otimes(\rho(X)v).$$

7.9 Superconnections.

A **superconnection** on a superbundle $\mathcal{E}$ is an odd first order differential operator

$$\mathbb{A}:\mathcal{A}^{\pm}(M,\mathcal{E})\to\mathcal{A}^{\mp}(M,\mathcal{E})$$

which satisfies

$$\mathbb{A}(\alpha\wedge\theta)=d\alpha\wedge\theta+(-1)^{|\alpha|}\alpha\wedge\mathbb{A}\theta,\qquad \forall\ \alpha\in\mathcal{A}(M),\ \theta\in\mathcal{A}(M,\mathcal{E}).$$

We can write this as

$$[\mathbb{A},e(\alpha)]=e(d\alpha) \tag{18}$$

where $e(\beta)$ denotes the operation of exterior multiplication by $\beta\in\mathcal{A}(M)$.

Let $\Gamma(\mathcal{E})$ denote the space of smooth sections of $\mathcal{E}$ which we can regard as a subspace of $\mathcal{A}(M,\mathcal{E})$. Then

$$\mathbb{A}:\Gamma(\mathcal{E}^{\pm})\to\mathcal{A}^{\mp}(M,\mathcal{E})$$

and $\mathbb{A}$ is completely determined by this map since

$$\mathbb{A}(\alpha\otimes s)=d\alpha\otimes s+(-1)^{|\alpha|}\alpha\otimes\mathbb{A}s$$

for all differential forms α and sections s of $\mathcal{E}$.

Conversely, suppose that $\mathbb{A}:\Gamma(\mathcal{E}^{\pm})\to\mathcal{A}^{\mp}(M,\mathcal{E})$ is a first order differential operator which satisfies

$$\mathbb{A}(fs)=df\otimes s+f\otimes\mathbb{A}s$$

for all functions f and sections s of $\mathcal{E}$. Then we can extend $\mathbb{A}$ to $\mathcal{A}(M,\mathcal{E})$ by setting

$$\mathbb{A}(\alpha\otimes s)=d\alpha\otimes s+(-1)^{|\alpha|}\otimes s$$

without fear of running into a contradiction.

7.10 Extending superconnections to the bundle of endomorphisms.

If $\gamma\in\mathcal{A}(M,\mathrm{End}(\mathcal{E}))$ define

$$\mathbb{A}\gamma:=[\mathbb{A},\gamma].$$

We claim that $[\mathbb{A},\gamma]$ belongs to $\mathcal{A}(M,\mathrm{End}(\mathcal{E}))$. To prove this, we must check that $[\mathbb{A},\gamma]$ commutes with all $e(\alpha),\ \alpha\in\mathcal{A}(M)$. For any $\alpha\in\mathcal{A}(M)$ we have

$$\mathbb{A}\circ\gamma\circ e(\alpha)=(-1)^{|\gamma|\cdot|\alpha|}\mathbb{A}\circ e(\alpha)\circ\gamma$$

$$= (-1)^{|\gamma|\cdot|\alpha|} e(d\alpha) \circ \gamma + (-1)^{|\alpha|+|\gamma|\cdot|\alpha|} e(\alpha) \circ \mathbb{A} \circ \gamma$$

while

$$\gamma \circ \mathbb{A} \circ e(\alpha) = \gamma \circ e(d\alpha) + (-1)^{|\alpha|} \gamma \circ e(\alpha) \circ \mathbb{A}$$
$$= (-1)^{|\gamma|+|\gamma|\cdot|\alpha|} e(d\alpha) \circ \gamma + (-1)^{|\alpha|+|\alpha|\cdot|\gamma|} e(\alpha) \circ \gamma \circ \mathbb{A}$$

so

$$[\mathbb{A}, \gamma] \circ e(\alpha) = A \circ \gamma \circ e(\alpha) - (-1)^{|\gamma|} \gamma \circ \mathbb{A} \circ e(\alpha)$$
$$= (-1)^{|\alpha|+|\alpha|\cdot|\gamma|} e(\alpha) \circ [\mathbb{A}, \gamma]$$

Since $|[\mathbb{A}, \gamma]| = |\gamma| + 1$ this shows that $[[\mathbb{A}, \gamma], e(\alpha)] = 0$ as desired.

7.11 Supercurvature.

Consider the even operator $\mathbb{A}^2$. We have, D. Quillen, "Superconnections and the Chern character", *Topology* **24** 89 (1985),

$$[\mathbb{A}^2, e(\alpha)] = \mathbb{A} \circ [\mathbb{A}, e(\alpha)] + (-1)^{|\alpha|} [\mathbb{A}, e(\alpha)] \circ \mathbb{A} =$$

$$\mathbb{A} \circ e(d\alpha) - (-1)^{|d\alpha|} e(d\alpha) \circ \mathbb{A} = [\mathbb{A}, e(d\alpha)] = e(dd(\alpha)) = 0.$$

So $\mathbb{A}^2 \in \mathcal{A}(M, \mathrm{End}(\mathcal{E}))$. We set

$$\mathbb{F} := \mathbb{A}^2$$

and call it the **curvature** of the superconnection $\mathbb{A}$.

The **Bianchi identity** says that

$$\mathbb{A}\mathbb{F} = 0.$$

Indeed $\mathbb{A}\mathbb{F}$ is defined as $[\mathbb{A}, \mathbb{F}]$ and since $\mathbb{F} := \mathbb{A}^2$ is even we have

$$[\mathbb{A}, \mathbb{A}^2] = \mathbb{A} \circ \mathbb{A}^2 - \mathbb{A}^2 \circ \mathbb{A} = 0$$

by the associative law.

7.12 The tensor product of two superconnections.

If $\mathcal{E}$ and $\mathcal{F}$ are superbundles recall that $\mathcal{E} \otimes \mathcal{F}$ is the superbundle with grading

$$\begin{aligned} (\mathcal{E} \otimes \mathcal{F})^+ &= \mathcal{E}^+ \otimes \mathcal{F}^+ \oplus \mathcal{E}^- \otimes \mathcal{F}^-, \\ (\mathcal{E} \otimes \mathcal{F})^- &= \mathcal{E}^+ \otimes \mathcal{F}^- \oplus \mathcal{E}^- \otimes \mathcal{F}^+. \end{aligned}$$

If $\mathbb{A}$ is a superconnection on $\mathcal{E}$ and $\mathbb{B}$ is a superconnection on $\mathcal{F}$ then $\mathbb{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbb{B}$ is a superconnection on $\mathcal{E} \otimes \mathcal{F}$. Thus

$$(\mathbb{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbb{B})(a \otimes b) := \mathbb{A}a \otimes b + (-1)^{|a|} a \otimes \mathbb{B}b.$$

A bit of computation shows that this definition is consistent and defines a superconnection on $\mathcal{E} \otimes \mathcal{F}$.

7.13 The exterior components of a superconnection.

If $\mathbb{A}$ is a superconnection on on a superbundle $\mathcal{E}$ we may break $\mathbb{A}$ into its homogeneous components $\mathbb{A}_{[i]}$ which map $\Gamma(M,\mathcal{E})$ into $\mathcal{A}^i(M,\mathcal{E})$, the space of i-forms with values in $\mathcal{E}$:

$$\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + A_{[2]} + \cdots .$$

Let s be a section of $\mathcal{E}$ and f a function. By the above decomposition and the defining property of a superconnection we have

$$\mathbb{A}(fs) = \sum_{i=0}^{n} \mathbb{A}_{[i]}(fs)$$

and

$$\mathbb{A}(fs) = df \otimes s + f \sum_{i=0}^{n} \mathbb{A}_{[i]} s$$

where n is the dimension of M. We see that

$$\mathbb{A}_1(fs) = df \otimes s + f\mathbb{A}_{[1]} s$$

which is the defining property of an ordinary connection. Furthermore, since $\mathbb{A}_{[1]}$ has total odd degree, we see that as an ordinary connection

$$\mathbb{A}_{[1]} : \Gamma(\mathcal{E}^+) \to \Gamma(T^*M \otimes \mathcal{E}^+) \quad \text{and} \quad \mathbb{A}_{[1]} : \Gamma(\mathcal{E}^-) \to \Gamma(T^*M \otimes \mathcal{E}^-).$$

It also follows from the above comparison of the two expressions for $\mathbb{A}(fs)$ that the remaining $\mathbb{A}_{[i]}$, $i \neq 1$ are given by the action of an element of $\mathcal{A}^i(M, \mathrm{End}(\mathcal{E}))$. For example $\mathbb{A}_{[0]}$ is given by an element of $\Gamma(M, \mathrm{End}^-(\mathcal{E}))$.

7.14 A local computation.

To see what the supercurvature computation looks like in terms of a local description, let us assume that our bundle $\mathcal{E}$ is trivial, i.e. $\mathcal{E} = M \times E$ where E is a superspace. Let us also assume that $\mathbb{A}$ has only components $\mathbb{A}_{[0]}$ and $\mathbb{A}_{[1]}$. This will be the case in the physical model that we will propose.

We may thus write $\mathbb{A}_{[0]} = L \in C^\infty(M, \mathrm{End}^-(E))$ so

$$L = \begin{pmatrix} 0 & L^- \\ L^+ & 0 \end{pmatrix}, \quad L^- \in C^\infty(M, \mathrm{Hom}(E^-, E^+)),$$

$$L^+ \in C^\infty(M, \mathrm{Hom}(E^+, E^-)).$$

We may also write

$$\mathbb{A}_{[1]} = d + A, \quad A \in \mathcal{A}^1(M, \mathrm{End}(E)^+).$$

Let ∇ denote the covariant differential corresponding to the ordinary connection $\mathbb{A}_{[1]}$ Then

$$\mathbb{F} := (\mathbb{A})^2 = \mathbb{A}_{[0]}^2 + [\mathbb{A}_{[1]}, \mathbb{A}_{[0]}] + \mathbb{A}_{[1]}^2 = \mathbb{A}_{[0]}^2 + \nabla \mathbb{A}_{[0]} + F$$

where F is the curvature of $\mathbb{A}_{[1]}$. In terms of the matrix decomposition above we have

$$\mathbb{F} = \begin{pmatrix} L^- L^+ + F^+ & \nabla L^- \\ \nabla L^+ & L^+ L^- + F^- \end{pmatrix}$$

where $F^\pm$ is the restriction of F to $E^\pm$. Notice that $\mathbb{F}$ is quadratic in L, and so any quadratic function of $\mathbb{F}$ will involve a quartic function of L. This will be our proposal for the quartic term entering into the Higgs mechanism.

7.15 Superconnections and principal bundles.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and G be a Lie group whose Lie algebra is $\mathfrak{g}_0$. Suppose that we have a representation of G as (even) automorphisms of $\mathfrak{g}$ whose restriction to $\mathfrak{g}_0$ is the adjoint representation of G on its Lie algebra.

We will denote the representation of G on all of $\mathfrak{g}$ by Ad.

Let $P = P_G \to M$ be a principal bundle with structure group G. Recall that this means the following:

- We are given an action of G on P. To tie in with standard notation we will denote this action by

  $$(p, a) \mapsto pa^{-1}, \quad p \in P, \quad a \in G$$

  so $a \in G$ acts on P by a diffeomorphism that we will denote by r_a:

  $$r_a : P \to P, \qquad r_a(p) = pa^{-1}.$$

  If $\xi \in \mathfrak{g}_0$, then $\exp(-t\xi)$ is a one parameter subgroup of G, and hence

  $$r_{\exp(-t\xi)}$$

  is a one parameter group of diffeomorphisms of P, and for each $p \in P$, the curve

  $$r_{\exp(-t\xi)} p = p(\exp t\xi)$$

  is a smooth curve starting at t at $t = 0$. The tangent vector to this curve at $t = 0$ is a tangent vector to P at p. In this way we get a linear map

  $$u_p : \mathfrak{g}_0 \to TP_p, \qquad u_p(\xi) = \frac{d}{dt} p(\exp t\xi)_{|t=0}. \tag{19}$$

- The action of G on P is free.

- The space P/G is a differentiable manifold M and the projection $\pi : P \to M$ is a smooth fibration.

- The fibration π is locally trivial consistent with the G action in the sense that every $m \in M$ has a neighborhood U such that there exists a diffeomorphism
$$\psi_U : \quad \pi^{-1}(U) \to U \times G$$
such that
$$\pi_1 \circ \psi = \pi$$
where
$$\pi_1 : \ U \times F \to U$$
is projection onto the first factor and if $\psi(p) = (m, b)$ then
$$\psi(r_a p) = (m, ba^{-1}).$$

Suppose that $\pi : P \to M$ is a principal fiber bundle with structure group G. Since π is a submersion, we have the sub-bundle Vert of the tangent bundle TP where $\mathrm{Vert}_p, p \in P$ consists of those tangent vectors which satisfy $d\pi_p v = 0$. From its construction, the subspace $\mathrm{Vert}_p \subset TP_p$ is spanned by the tangents to the curves $p(\exp t\xi)$, $\xi \in \mathfrak{g}_o$. In other words, u_p is a surjective map from $\mathfrak{g}_0$ to Vert_p. Since the action of G on P is free, we know that u_p is injective. Putting these two facts together we conclude that

If $\pi : P \to M$ is a principal fiber bundle with structure group G then u_p is an isomorphism of $\mathfrak{g}_0$ with Vert_p for every $p \in P$.

An (ordinary) connection on a principal bundle is a choice of a "horizontal" subbundle Hor complementary to the vertical bundle which is invariant under the action of G. At any p we can define the projection
$$\mathbf{V}_p : TP_p \to \mathrm{Vert}_p$$
along Hor_p, i.e. $\mathbf{V}_p$ is the identity on Vert_p and sends all elements of Hor_p to 0. Giving Hor_p is the same as giving $\mathbf{V}_p$ and condition of invariance under G translates into
$$d(r_b)_p \circ \mathbf{V}_p = \mathbf{V}_{r_b(p)} \circ d(r_b)_p \quad \forall\ b \in G, \ \ p \in P.$$
This then defines a one form ω on P with values in $\mathfrak{g}_0$:
$$\omega_p := u_p^{-1} \circ \mathbf{V}_p.$$
Invariance of the connection under G translates into
$$r_b^* \omega = \mathrm{Ad}_b\, \omega.$$
Let ξ_P be the vector field on P which is the infinitesimal generator of $r_{\exp t\xi}$. The the infinitesimal version of the preceding equation is
$$D_{\xi_P}\omega = [\xi, \omega].$$

In view of the definition of u_p as identifying ξ with the tangent vector to the curve $t \mapsto p(\exp t\xi) = r_{\exp -t\xi}p$ at $t = 0$, we see that

$$i(\xi_P)\omega = -\xi.$$

We now generalize this to superconnections: We define a **superconnection form** A to be an odd element of $\mathcal{A}(P, \mathfrak{g})$ which satisfies

$$\begin{aligned} r_b^* A &= \mathrm{Ad}_b\, A \quad \forall b \in G & (20)\\ i(\xi_P)A &= -\xi \quad\ \forall\, \xi \in \mathfrak{g}_0. & (21)\end{aligned}$$

The meaning of (21) is the following:

$$A = A_{[0]} + A_{[1]} + \cdots + A_{[n]}, \qquad n = \dim M$$

where $A_{[i]}$ is an i-form with values in $\mathfrak{g}_0$ if i is odd and with values in $\mathfrak{g}_1$ it i is even. Then $A_{[1]}$ is a connection form and all the other components satisfy

$$i(\xi_P)A_{[i]} = 0.$$

This condition together with (20) imply that these other components can be identified with odd i-forms on M with values in $\mathfrak{g}(P)$, the vector bundle over M associated to the representation Ad of G on $\mathfrak{g}$.

More generally, if the superspace E is a G module and also a $\mathfrak{g}$ module in a consistent way, then we can form the associated bundle

$$\mathcal{E}(M) = E(P)$$

which is a module for the associated bundle of superalgebras $\mathfrak{g}(P)$. A k-form on M with values in $\mathcal{E}$ is the same thing as a k-form σ on P with values in E which satisfies

1. $i(\xi_P)\sigma = 0 \quad \forall\, \xi \in \mathfrak{g}_0$ and
2. $r_a^*\sigma = \rho(a)\sigma$ where ρ denotes the action of G on E.

The bilinear map

$$\mathfrak{g} \times E \to E$$

given by the action of $\mathfrak{g}$ determines an exterior multiplication

$$\Omega(P, \mathfrak{g}) \times \Omega(P, E) \to \Omega(P, E)$$

which we will denote by $\diamond$. We then obtain a superconnection on $\mathcal{E}$ given by

$$\mathbb{A}\sigma = d\sigma + A \diamond \sigma. \tag{22}$$

7.16 The Higgs field and superconnections.

In the model that we proposed in [NS90] ,[NS91], we are given a bundle of Lie superalgebras $\mathfrak{g} = \mathfrak{g}(P) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as above. If we assume that the superconnection form A has only exterior terms of degree zero and one, then $\mathbb{A}_{[0]}$ is given by the action of a section of $\mathfrak{g}_1$. We take the sections of $\mathfrak{g}_1 = \mathfrak{g}_1(P)$ to be the Higgs fields. As described above, the supercurvature is then quadratic in the Higgs field, and hence a super-Yang-Mills functional which will be be quartic in the Higgs field.

7.17 Clifford Bundles and Clifford superconnections.

Suppose that M is a semi-Riemannian manifold so that we can form the bundle of Clifford algebras $C(TM)$. Suppose that $\mathcal{F}$ is a bundle of Clifford modules. We denote the action of a section a of $C(TM)$ on a section of $\mathcal{F}$ by $c(a)$. We extend this notation to denote the action of a Clifford bundle valued differential form, i.e. an element of $\mathcal{A}(M, C(TM))$ on $\mathcal{A}(M, \mathcal{F})$ by

$$c(\alpha \otimes a)(\beta \otimes s) = (-1)^{|a| \cdot |\beta|}(\alpha \wedge \beta) \otimes c(a)s$$

on homogeneous elements.

A superconnection $\mathbb{B}$ on $\mathcal{F}$ is called a **Clifford superconnection** [BGV91] if for all sections a of $C(T(M))$ we have

$$[\mathbb{B}, c(a)] = c(\nabla a)$$

where ∇ is the covariant differential on $C(T(M))$ coming from the Levi-Civita connection on M.

Suppose that $\mathbb{B}$ and $\mathbb{B}'$ are Clifford superconnections on $\mathcal{F}$. Then

$$[\mathbb{B} - \mathbb{B}', e(\alpha)] = 0 \quad \forall\ \alpha \in \mathcal{A}(M)$$

so $\mathbb{B} - \mathbb{B}' \in \mathcal{A}^-(M, \mathrm{End}(\mathcal{F})$. Also

$$[\mathbb{B} - \mathbb{B}', c(a)] = 0$$

implying that

$$\mathbb{B} - \mathbb{B}' \in \mathcal{A}^-(M, \mathrm{End}_{C(M)}(\mathcal{F})).$$

Conversely, if $\tau \in \mathcal{A}^-(M, \mathrm{End}_{C(M)}(\mathcal{F}))$ and $\mathbb{B}'$ is a Clifford superconnection then $\mathbb{B} = \mathbb{B}' + \tau$ is a Clifford superconnection. Thus the collection of all Clifford superconnections is an affine space modeled on the linear space $\mathcal{A}^-(M, \mathrm{End}_{C(M)}(\mathcal{F}))$.

If $\mathcal{E}$ is a superbundle and $\mathcal{F}$ is a bundle of Clifford modules then we can make $\mathcal{E} \otimes \mathcal{F}$ into a Clifford module by letting a section a of $C(TM)$ act as $1 \otimes c(a)$ where $c(a)$ denote the action of a on $\mathcal{F}$. If $\mathbb{A}$ is a superconnection on $\mathcal{E}$ then

$$[\mathbb{A} \otimes \mathbf{1}, \mathbf{1} \otimes c(a)] = 0$$

for all sections a of $C(TM)$ and so

$$[\mathbb{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbb{B}, \mathbf{1} \otimes c(a)] = \mathbf{1} \otimes c(\nabla a).$$

In other words, the tensor product of a superconnection with a Clifford superconnection is a Clifford superconnection.

7.18 The Dirac operator of a Clifford superconnection.

Let $\mathcal{E}$ be a Clifford module over the semi-Riemannian manifold M and let $\mathbb{A}$ be a Clifford superconnection on $\mathcal{E}$. We can associate to this data a certain first order differential operator on sections of M

$$\mathbb{D} = \mathbb{D}_{\mathbb{A}} : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E})$$

which generalizes the classical Dirac operator in the presence of an electromagnetic field. In order to define it we need to record a relation between the Clifford algebra and the exterior algebra.

7.18.1 The exterior algebra as a Clifford module.

Let V be a vector space with a non-degenerate scalar product $(\cdot, \cdot)$ which then defines an isomorphism of V with its dual space V^*: $v \mapsto (v, \cdot)$.

If $v \in V$ we will let $i(v) : \wedge(V) \to \wedge(V)$ denote interior product by the element $v^* \in V^*$ corresponding to V. Explicitly, $i(v)$ is the (odd) derivation on $\wedge(V)$ determined by

$$i(v)1 = 0, \qquad i(v)w = (v, w), \qquad w \in V.$$

We let $e(v) : \wedge(V) \to \wedge(V)$ denote exterior multiplication by v. If we put the standard scalar product on $\wedge(V)$ induced by the scalar product on V, it is easy to check that $i(v)$ is the transpose of $e(v)$. Since $e(v)^2 = 0$ it follows that $i(v)^2 = 0$ (as can also be checked directly from the definition) and that

$$(i(v) + e(v))^2 = i(v)e(v) + e(v)i(v) = (v, v)\mathrm{id}.$$

So $v \mapsto i(v) + e(v)$ is a Clifford map and so makes $\wedge(V)$ into a $C(V)$ module. Consider the linear map

$$\sigma : \; C(V) \to \wedge(V), \qquad x \mapsto x1$$

where $1 \in \wedge^0(V)$ under the identification of $\wedge^0(V)$ with the ground field. The element $x1$ on the extreme right means the image of 1 under the action of $x \in C(V)$. For elements $v_1, \ldots, v_k \in V$ this map sends

$$\begin{array}{rcl}
v_1 & \mapsto & v_1 \\
v_1 v_2 & \mapsto & v_1 \wedge v_2 + (v_1, v_2)1 \\
v_1 v_2 v_3 & \mapsto & v_1 \wedge v_2 \wedge v_3 + (v_1, v_2)v_3 - (v_1, v_3)v_2 + (v_2, v_3)v_1 \\
v_1 v_2 v_3 v_4 & \mapsto & v_1 \wedge v_2 \wedge v_3 \wedge v_4 + (v_2, v_3)v_1 \wedge v_4 - (v_2, v_4)v_1 \wedge v_3 \\
 & & +(v_3, v_4)v_1 \wedge v_2 + (v_1, v_2)v_3 \wedge v_4 - (v_1, v_3)v_1 \wedge v_4 \\
 & & +(v_1, v_4)v_2 \wedge v_3 + (v_1, v_4)(v_2, v_3) - (v_1, v_3)(v_2, v_4) \\
+(v_1, v_2)(v_3, v_4) & & \\
\vdots & & \vdots
\end{array}$$

If the v's form an "orthonormal" basis of V then the products

$$v_{i_1} \cdots v_{i_k}, \quad i_1 < i_2 \cdots < i_k, \; k = 0, 1, \ldots, n \tag{23}$$

form a basis of $C(V)$ while the

$$v_{i_1} \wedge \cdots \wedge v_{i_k}, \quad i_1 < i_2 \cdots < i_k, \; k = 0, 1, \ldots, n \tag{24}$$

form a basis of $\wedge(V)$, and in fact

$$v_1 \cdots v_k \mapsto v_1 \wedge \cdots \wedge v_k \quad \text{if } (v_i, v_j) = 0 \; \forall i \neq j. \tag{25}$$

In particular, the map σ given above is an isomorphism of vector spaces.

We will let

$$\mathbf{q} : \; \wedge(V) \to C(V) \tag{26}$$

denote the inverse of σ:

$$\mathbf{q} := \sigma^{-1}. \tag{27}$$

On a semi-Riemannian manifold we have an identification ℓ of $\Gamma(M, \wedge(T^*M))$ with $\Gamma(M, \wedge T(M))$ given by the metric. We can then apply the map $\mathbf{q}$ at each point so as to get a map (which we will also denote by $\mathbf{q}$):

$$\mathbf{q} : \Gamma(M, \wedge(TM)) \to \Gamma(M, C(M)).$$

7.18.2 The Dirac operator.

Let $\mathbb{A}$ be a Clifford superconnection on the Clifford module $\mathcal{E}$. We have the following sequence of maps:

$$\begin{array}{rcl} \mathbb{A} : \Gamma(M, \mathcal{E}) & \to & \mathcal{A}(M, \mathcal{E}) = \Gamma(M, \wedge(T^*M) \otimes \mathcal{E}) \\ \ell \otimes \mathrm{id} : \Gamma(M, \wedge(T^*M) \otimes \mathcal{E}) & \to & \Gamma(M, \wedge(TM) \otimes \mathcal{E}) \\ \mathbf{q} \otimes \mathrm{id} : \Gamma(M, \wedge(TM) \otimes \mathcal{E}) & \to & \Gamma(M, C(M) \otimes \mathcal{E}) \\ \mathbf{c} : \Gamma(M, C(M) \otimes \mathcal{E}) & \to & \Gamma(M, \mathcal{E}) \end{array}$$

where the last map $\mathbf{c}$ is given by the action of $C(M)$ on $\mathcal{E}$.

The composite of all these operators is the Dirac operator

$$\mathbb{D}_{\mathbb{A}} : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E}) \tag{28}$$

associated to the superconnection $\mathbb{A}$.

7.18.3 A local description of the Dirac operator.

Let $x^1, \ldots, x^n$ be a local coordinate system with $dx^1, \ldots, dx^n$ the corresponding differential forms and $\partial_1, \ldots, \partial_n$ the corresponding vector fields so that the exterior differential d is given by

$$d = \sum_{i=1}^{n} dx^i \otimes \partial_i.$$

Let $e_1, \dots, e_n$ be an "orthonormal" frame field over this coordinate neighborhood and $\theta^1, \dots, \theta^n$ the dual coframe field. The most general superconnection on $\mathcal{E}$ can then be written as

$$\mathbb{A} = \sum_{i=1}^{n} dx^i \otimes \partial_i + \sum_{I \subset \{1,\dots,n\}} \theta^I \otimes A_I$$

where

$$\theta^I := \theta_{i_1} \wedge \cdots \wedge \theta_{i_j} \quad \text{where } \; I = \{i_1, \dots, i_j\} \quad i_1 < i_2 < \cdots < i_j \tag{29}$$

and A_I is a section of $\mathrm{End}(\mathcal{E})$. Applying $\ell \otimes \mathrm{id}$ gives

$$\sum_{i=1}^{n} \ell(dx^i) \otimes \partial_i + \sum_{I \subset \{1,\dots,n\}} e^I \otimes A_I. \tag{30}$$

Applying $\mathbf{q} \otimes \mathrm{id}$ gives

$$\sum_{i=1}^{n} \mathbf{q}(\ell(dx^i) \otimes \partial_i + \sum_{I \subset \{1,\dots,n\}} \mathbf{q}(e^I) \otimes A_I$$

and the applying the Clifford action gives

$$\mathbb{D}_{\mathbb{A}} = \sum_{i=1}^{n} \mathbf{c}(\mathbf{q}(\ell(dx^i))\partial_i \; + \sum_{I \subset \{1\dots,n\}} \mathbf{c}(\mathbf{q}(e^I)) \circ A_I.$$

7.19 Clifford bundles and spinors.

So far, we have not made any assumptions about the dimension of M or about the signature of the semi-Riemann metric on M. On a complex vector space, all non-degenerate quadratic forms are equivalent. The Clifford algebra of an even dimensional complex vector space with non-degenerate quadratic form is isomorphic to $\mathrm{End}(S)$ where $S = S_+ \oplus S_-$ is known as the space of spinors. In the case of a real vector space with a negative definite scalar product, which we then complexify, there is a positive definite Hermitian form on S invariant under the group $Spin(V)$ which is the double cover in $C(V)$ of the group $SO(V)$. The spaces S_+ and S_- are orthogonal under the Hermitian form and give the (irreducible) half spin representations of $Spin(V)$. These are well known facts and can be found in standard texts such as [G78] or [BGV91].

The case of physical interest is where we are dealing with a four dimensional space with Lorentzian metric. The following is a summary of the well known facts. As it is hard to find a cogent presentation of these facts in the standard texts, we will give a more detailed presentation in the next section.

The (real) Clifford algebra $C(3,1)$ (spacelike positive, timelike negative) is isomorphic as an algebra to $\mathrm{End}(\mathbb{R}^4)$. Wedderburn's theorem then implies that this four dimensional real $C(3,1)$ module, known as the space of Majorana spinors, is unique up to canonical isomorphism, and that any $C(3,1)$ module is isomorphic to the tensor product of this module with a trivial module.

The element

$$\gamma = e_0 e_1 e_2 e_3$$

(where e_0, e_1, e_2, e_3 is an oriented orthonormal basis) satisfies

$$\gamma^2 = -1$$

and

$$\gamma a = a\gamma, \;\; a \in C_0(3,1), \qquad \gamma b = -b\gamma, \quad b \in C_1(3,1).$$

Thus γ defines a complex structure $\mathbf{J}$ on $\mathbb{R}^4$ and the even elements of $C(3,1)$ act as linear transformations (commute with $\mathbf{J}$) while the odd elements of $C(3,1)$ act as antilinear transformations (anti-commute with $\mathbf{J}$). This complex structure allows us identify the space $\mathbb{R}^4$ of Majorana spinors with $\mathbb{C}^2$.

The group $Sl(2,\mathbb{C})$ is simply connected and is the double cover of the connected component of the Lorentz group $O(3,1)$. It preserves a complex symplectic form (a non-degenerate anti-symmetric bilinear form) which is determined up to multiplication by a non-zero complex number. Let H be the two component group in $C(3,1)$ which (double) covers the two component subgroup of $O(3,1)$ consisting of those Lorentz transformations which preserve the forward light cone. (So H includes elements which project onto "parity transformations".) Then there is a real symplectic form s on $\mathbb{R}^4$ invariant under H which is determined up to a non-zero real scalar multiple and a bilinear map $\mathfrak{j}$ from $\mathbb{R}^4$ to Minkowski which is equivariant under the action of H.

The space of Dirac spinors is the complexification of the space of Majorana spinors. It decomposes into the direct sum of the $\pm i$ eigenvalues of $\mathbf{J}$ and these are the right and left handed spinors. This is the $\mathbb{Z}_2$ structure we will be using throughout this paper. If we extend s to be a sesquilinear form on the space of *Dirac* spinors, then is is a non-degenerate Hermitian form of signature $(2,2)$ and is uniquely determined up to real scalar multiple as being invariant under H. The space of right or left handed spinors is isotropic under this Hermitian form.

7.20 Facts about Dirac spinors.

The facts collected in this section are well known to physicists. For the convenience of the mathematical reader we collect them here.

7.20.1 The element γ in general.

Let V be a real vector space with a non-degenerate quadratic form of signature (p,q) and let C be the corresponding Clifford algebra. Let

$$v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}$$

be an "orthonormal" basis so that

$$(v_i, v_i) \quad = \quad \begin{array}{ll} 1 & 1 \le i \le p \\ & \\ -1 & p+1 \le i \le p+q. \end{array}$$

Let

$$\gamma := v_1 \cdot v_2 \cdot \dots \cdot v_{p+q}.$$

Notice that γ is determined up to sign (fixed by choosing an orientation of V) and satisfies

$$\begin{aligned} \gamma^2 &= (-1)^{\frac{1}{2}n(n-1)+q}\mathbf{1}_C \quad n = p+q = \dim V \\ \gamma v &= (-1)^{n-1} v\gamma, \quad v \in V. \end{aligned}$$

If $p = q+2$ then $n = 2(q+1)$ and

$$\frac{1}{2}n(n-1) + q = (q+1)(2q+1) + q = 2q^2 + 4q + 1$$

is odd hence

$$\gamma^2 = -\mathbf{1}_C \tag{31}$$

$$\gamma v = -v\gamma. \tag{32}$$

These equations will also hold if $p = q + r$ where $r \equiv 2 \bmod 4$.

7.20.2 Majorana spinors for $C(q+2,2)$.

By Bott periodicity (see for example [G78]) we have

$$\begin{aligned} C(p,q) \otimes C(2,0) &= C(q+2,p) \\ C(q,q) &= \mathrm{End}(\mathbb{R}^{2^q}) \\ C(2,0) &= \mathrm{End}(\mathbb{R}^2) \quad \text{hence} \\ C(q+2,q) &\cong \mathrm{End}(\mathbb{R}^{2^{q+1}}). \end{aligned}$$

Then (31) says that $\gamma \in \mathrm{End}(\mathbb{R}^{2^{q+1}})$ defines a complex structure on $\mathbb{R}^{2^{q+1}}$ and (32) implies that all the odd elements of $C = C(q+2,q)$ act as antilinear transformations and all the even elements act as linear elements on the space of **Majorana spinors**: $S = \mathbb{C}^{2^q} \sim \mathbb{R}^{2^{q+1}}$.

7.20.3 Majorana spinors in four dimensions.

We know that $Spin(3,1)$ is isomorphic to $sl(2,\mathbb{C})$. In fact, we will shortly give a an explicit realization of this fact. So there is an invariant anti-symmetric complex bilinear form on S which is invariant under $Spin(3,1)$. (Such an object is called a complex symplectic form.) In fact, there is a whole family of them determined up to multiplication by a complex number. If we enlarge the group $Spin(3,1)$ to include conjugation by time-like vectors we will find that we obtain a group G which double covers the subgroup of $O(3,1)$ which has two components consisting of the connected component $SO(3,1)$ and also the parity transformations. We will find that there is a real symplectic form s on S which is invariant under G. This will determine s up to multiplication by a non-zero real number. We will also find that s determines a quadratic map $\mathfrak{j}$ from S to vectors, and we will use this to associate a "current" to each pair of spinors.

Let e_0 be a "unit" time like vector so that $e_0^2 = -\mathbf{1}_C$. Hence e_0 is invertible in the Clifford algebra $C = C(3,1)$ and

$$e_0^{-1} = -e_0.$$

Consider the operation of conjugation by e_0 in the Clifford algebra:

$$a \mapsto e_0 a e_0^{-1} = -e_0 a e_0.$$

Acting on e_0 we get

$$e_0 \mapsto -e_0^3 = e_0.$$

Acting on a vector v perpendicular to e_0 we get

$$v \mapsto -e_0 v e_0 = +e_0^2 v = -v.$$

Thus conjugation by e_0 carries the subspace $\mathbb{R}^{3,1}$ into itself and acts there as the "parity transformation" $\mathbf{P}$:

$$\mathbf{P}e_0 = e_0, \qquad \mathbf{P}v = -v \;\text{ if } v \perp e_0.$$

For a general discussion of the "Pin group" using twisted conjugation rather than conjugation see [G78].

7.20.4 A model for the Majorana spinors.

We identify the space $V = \mathbb{R}^{1,3}$ with the space of two by two (complex) self adjoint matrices: if P and Q are self adjoint two by two matrices we define

$$||P||^2 = \det P, \quad (P,Q) = \frac{1}{2}\mathrm{tr}\ PQ^{\mathrm{a}} \tag{33}$$

where Q^{a} denotes the "adjoint" according to Cramer's rule

$$\mathrm{a}: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

so

$$QQ^{\mathrm{a}} = \det\ Q\ I.$$

We have

$$\det \begin{pmatrix} t-x & y+iz \\ y-iz & t+x \end{pmatrix} = t^2 - x^2 - y^2 - z^2 \tag{34}$$

so the space of self-adjoint two by two matrices is a model of $\mathbb{R}^{1,3}$.

Let A be a two by two complex matrix. If P is self-adjoint then so is $APA^\dagger$ and the map

$$P \mapsto APA^\dagger$$

is a real linear map of the space of two by two self adjoint matrices into itself. If $\det A = 1$ then

$$\det(APA^\dagger) = \det P.$$

This shows that we have a homomorphism from $Sl(2,\mathbb{C}) \to SO(1,3)$. It is not hard to show that this homomorphism is two to one and surjective and hence gives an identification of $Spin(1,3) = Spin(3,1)$ with $Sl(2,\mathbb{C})$. We will take the space of spinors to be $\mathbf{C}^2$ regarded as a real four dimensional space. Define the anti-linear operator

$$\star : \mathbf{C}^2 \to \mathbf{C}^2, \quad \star : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\overline{y} \\ \overline{x} \end{pmatrix}.$$

Then

$$\star^2 = -I$$

and

$$\langle \star u, u \rangle = 0, \quad \forall u \in \mathbf{C}^2$$

where $\langle\ ,\ \rangle$ denotes the standard Hermitian form on $\mathbf{C}^2$. A direct verification shows that

$$\star A = A^{\mathrm{a}\dagger} \star \tag{35}$$

for any two by two complex matrix, A.

Indeed, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\star A \begin{pmatrix} x \\ y \end{pmatrix} = \star \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} -\overline{cx} - \overline{d}\overline{y} \\ \overline{ax} + \overline{b}\overline{y} \end{pmatrix}, \quad A^{a\dagger} \star \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} -\overline{y} \\ \overline{x} \end{pmatrix}.$$

In particular, for self adjoint matrices, P, we have

$$\star P \star^{-1} = P^{\mathrm{a}}. \tag{36}$$

If we take

$$P = e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $P^{\mathrm{a}} = P$. On the other hand, if P is orthogonal to e_0, so that $\mathrm{tr}\ P = 0$, then $P = -P^{\mathrm{a}}$. Thus conjugation by $\star$ induces the "parity transformation" on Minkowski space.

Any $A \in Sl(2,\mathbb{C})$ satisfies

$$A^{\mathrm{a}} = A^{-1}$$

and therefore for $A \in Sl(2,\mathbb{C})$ we have

$$\begin{aligned} A\mu(P)A^{-1} &= AP \star A^{-1} \\ &= APA^{\dagger} \star \\ &= \mu(APA^{\dagger}). \end{aligned}$$

The transformation

$$P \mapsto APA^{\dagger}$$

gives the action of $A \in Sl(2,\mathbb{C})$ on $P \in \mathbb{R}^{1,3}$. Thus the equation

$$A\mu(P)A^{-1} = \mu(APA^{\dagger}) \tag{37}$$

asserts that the map $\mu : \mathbb{R}^{1,3} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^2)$ is an $Sl(2,\mathbb{C})$ morphism. Observe also that in this representation the element $\gamma \mapsto \pm i$, where i denotes the usual multiplication by the complex number i on $\mathbb{C}^2$, because γ commutes with with all even elements of $C(3,1)$ and its square is -1. The choice of sign reflects the indeterminacy in the choice of γ depending on the choice of orientation in Minkowski space. In order to avoid later confusion when we complexify the space $\mathbb{C}^2$ and hence have still another notion of multiplication by i, we shall denote the element γ in our case by the neutral symbol $\mathbf{J}$.

7.20.5 Bilinear covariants for Majorana spinors.

Define the real quadratic map

$$\mathsf{j} : S = \mathbb{C}^2 \mapsto \mathbb{R}^{1,3}, \qquad \mathsf{j}(u) := u \otimes u^{\dagger}. \tag{38}$$

We have

$$\mathsf{j}(Au) = A\mathsf{j}(u)A^{\dagger} \quad \forall A \in gl(2,\mathbb{C}), \tag{39}$$

implying the equivariance of the map j for the group $Sl(2,\mathbb{C})$. Also $(u,v) = (\star v, \star u)$ $\forall u,v \in \mathbb{C}^2$ hence

$$\begin{aligned} \mathsf{j}(\star u)v &= (v, \star u) \star u \\ &= \star\{(\star u, v)u\} \\ &= \star\{(\star v, \star\star u)u\} \\ &= \star\{(-\star v, u)u\} \\ &= \star\{(\star^{-1} v, u)u\} \quad \text{so} \\ \\ \mathsf{j}(\star u) &= \star \mathsf{j}(u) \star^{-1} . \end{aligned}$$

This equation, together with (39) has the following meaning: Let G denote the subgroup of the group of all invertible real linear transformations of $\mathbb{C}^2$ generated by $Sl(2,\mathbb{C})$ and $\star$. Since

$$\star A \star^{-1} = A^{\dagger -1} \quad \forall\ A \in Sl(2,\mathbb{C}), \tag{40}$$

we see that G consists of elements of the form B or $B\star$, $\quad B \in Sl(2,\mathbb{C})$. So the group G consists of two of the four components of the group $Pin(3,1)$, the double cover of $O(3,1)$ in the Clifford algebra. Indeed G consists of those elements of $Pin(3,1)$ which (in their action on $\mathbb{R}^{3,1}$) preserve the direction of time.

$$\mathfrak{j}(\star u) = \star\mathfrak{j}(u)\star^{-1} \tag{41}$$

thus asserts that $\mathfrak{j}$ is a morphism for the "parity" action of G on Minkowski space. (This is usually expressed by saying that $\mathfrak{j}$ defines a "vector current" as opposed to an "axial current".) Notice that the time component of $\mathfrak{j}(u)$ is always non-negative. Indeed

$$\mathrm{tr}\ \mathfrak{j}(u) = ||u||^2. \tag{42}$$

This result was important to Dirac in that it allowed the interpretation of the time component of $\mathfrak{j}(u)$ as a probability density, when $\mathfrak{j}(u)$ is interpreted as a current.

The map $\mathfrak{j}$, being quadratic, defines, by polarization, a real symmetric bilinear map from $\mathbb{C}^2$ to Minkowski space:

$$\mathfrak{j}(u,v) := \frac{1}{2}(u \otimes v^\dagger + v \otimes u^\dagger).$$

We can also consider the antisymmetric form

$$b : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}^{1,3} \qquad b(u,v) := \frac{1}{2}\mathbf{J}(u \otimes v^\dagger - v \otimes u^\dagger). \tag{43}$$

(Remember that the $\mathbf{J}$ in this equation is simply multiplication by i or by $-i$ depending on the orientation. So the matrix on the right is indeed self adjoint.) "Polarizing" the argument that we gave above shows that

$$(\star u) \otimes (\star v)^\dagger = \star[u \otimes v^\dagger]\,\star^{-1}.$$

But

$$\mathbf{J}\star = -\star\mathbf{J}$$

so

$$b(\star u, \star v) = -\star b(u,v)\star^{-1}. \tag{44}$$

One says that "$b(u,v)$ is an axial current". Now $\mathbb{C}^2$ carries a $\mathbb{C}$ valued symplectic form invariant under $Sl(2,\mathbb{C})$ (in fact a one complex dimensional space of them). We can use the symplectic form to identify $\mathbb{C}^2$ with its dual and so define a bilinear map

$$c : \mathbb{C}^2 \times \mathbb{C}^2 \to gl(2,\mathbb{C}), \quad c(u,v)w := \omega(v,w)u$$

where ω is (a choice of) symplectic form. One choice of the symplectic form is

$$\omega(v,w) := (w, \star v).$$

Explicitly

$$(w, \star v) = v_1 w_2 - v_2 w_1.$$

For this choice we have

$$c(u, v) = u \otimes (\star v)^{\dagger}. \tag{45}$$

So

$$c(u, v)w = (w, \star v)u.$$

Now $(w, \star v) = (v, \star^{-1} w) = \overline{(\star^{-1} w, v)}$ so we see that this choice of c satisfies

$$c(\star u, \star v) = \star c(u, v) \star^{-1}. \tag{46}$$

Under the conjugation action of $Sl(2, \mathbb{C})$ the space $gl(2, \mathbb{C})$ decomposes as

$$gl(2, \mathbb{C}) = sl(2, \mathbb{C}) \oplus \mathbb{C}.$$

Under the action of conjugation by $\star$ we have the further decomposition

$$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$$

which is the ± 1 eigenvector decomposition. We can thus write

$$c = a \oplus s \oplus iq$$

where a is the $sl(2, \mathbb{C})$ component, where s is a "scalar" (transforms according to the trivial representation of G) and where q is a "pseudoscalar" (transforms according the representation which assigns $+1$ to the identity component and -1 to the other component of G). Both s and q are real valued symplectic forms on $S = \mathbb{C}^2$.

Notice that for any $P \in \mathbb{R}^{1,3}$, $\mu(P)$ is in the symplectic algebra of the symplectic form s (as are the elements of $sl(2, \mathbb{C})$). Indeed,

$$\begin{aligned}
s(\mu(P)u, v) &= \frac{1}{2} \text{ Re tr } c(P \star u, v) \\
&= \frac{1}{2} \text{Re } (P \star u, \star v) \quad \text{while} \\
s(u, \mu(P)v) &= \text{Re } (u, \star P \star v) \\
&= \frac{1}{2} \text{Re } (\star u, \star \star P \star v) \\
&= -\frac{1}{2} \text{Re } (\star u, P \star v) \\
&= -\frac{1}{2} \text{Re } (P \star u, \star v)
\end{aligned}$$

since $\star\star = -1$ and P is self adjoint. Hence

$$s(\mu(P)u, v) + s(u, \mu(P)v) = 0.$$

Therefore $\mu(P)$ determines a quadratic form

$$u \mapsto s(\mu(P)u, u)$$

on $S = \mathbb{C}^2$ since

$$s(\mu(P)u, v) = -s(u, \mu(P)v) = s(\mu(P)v, u).$$

We claim that

$$s(\mu(P)u, u) = P \cdot \mathfrak{j}(u). \tag{47}$$

Indeed, by the definition of the scalar product, by (36), by (41), and by the definition (38) of $\mathfrak{j}$ we have,

$$\begin{aligned}
P \cdot \mathfrak{j}(u) &= \frac{1}{2}\mathrm{tr}\ P\mathfrak{j}(u)^a \\
&= \frac{1}{2}\mathrm{tr}\ P \star \mathfrak{j}(u) \star^{-1} \\
&= \frac{1}{2}\mathrm{tr}\ P\mathfrak{j}(\star u) \\
&= \frac{1}{2}(P \star u, \star u) \\
&= s(\mu(P)u, u)
\end{aligned}$$

since P is self adjoint implying that $(P\star u, \star u)$ is real and by definition, $s(\mu(P)u, u) = \frac{1}{2}\mathrm{Re}\ (P \star u, \star u)$.

We shall see later on that the representation of G on S is absolutely irreducible, that is, remains irreducible even after complexification. But this implies that (up to non-zero real scalars) there can exist at most one G invariant real symplectic form. Since we have expressed $\mathfrak{j}$ in terms of s, we see that s, and hence $\mathfrak{j}$ are determined (up to scalar factors) by the representation of G on S.

7.20.6 The Dirac equation for Majorana spinors.

We now explain how the general notion of the Dirac operator associated to a Clifford connection specializes to yield the Dirac operator on Majorana spinors when we take the trivial connection.

Let $\mathbf{S} \to M$ be the trivial vector bundle over Minkowski space, M whose fiber is S. Let ψ be a section of $\mathbf{S}$, so we can think of ψ as a function from $M \to S$. Then $d\psi$ is a section of $T^* \otimes S$ where T^* is the cotangent bundle of M. Using the Minkowski metric, we can identify T^* with $T \sim \mathbb{R}^{1,3}$ and then apply

$$\mu : T \otimes \mathbf{S} \to \mathbf{S}.$$

So

$$\mu(d\psi)$$

is a section of $\mathbf{S}$. The physicists write $\mu(\partial)\psi$ for $\mu(d\psi)$ since, if

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

is regarded as a $\mathbb{C}^2$ valued function, then

$$\mu(d\psi) = \begin{pmatrix} \partial_0 - \partial_3 & \partial_1 + i\partial_2 \\ \partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \star \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The (Majorana version of the) Dirac equation is

$$\mu(d\psi) = m\psi. \tag{48}$$

If ψ is a solution of this equation, the corresponding vector field, $j(\psi)$ is called the current associated to ψ. We claim that

$$\text{div } \mathfrak{j}(\psi) = 0. \tag{49}$$

Indeed

$$\begin{aligned} \text{div } \mathfrak{j}(\psi) &:= \partial \cdot \mathfrak{j}(\psi) \\ &= \frac{1}{2}\text{tr } (\partial)^a(\psi \otimes \psi^\dagger) \\ &= -\frac{1}{2}\text{tr } \star(\partial)\star(\psi \otimes \psi^\dagger) \\ &= -\frac{1}{2}\text{tr } \star\mu(\partial)\psi \otimes \psi^\dagger \\ &= -\frac{1}{2}m(\star\psi, \psi) \\ &= 0. \end{aligned}$$

Equation (49) expresses the "conservation of the current".

Notice that if we seek plane wave solutions to the Dirac equation

$$\psi(x) = \cos(P \cdot x + \alpha)u \quad u \in \mathbb{C}^2$$

then (48) implies that

$$||P||^2 = m^2$$

if $u \neq 0$.

We may think of d mapping sections of $\mathbf{S}$ to sections of $T^* \otimes \mathbf{S}$ as defining a flat connection on $\mathbf{S}$. We may modify this connection by considering $\mathbf{S}$ as a $U(1)$ bundle which has its own connection adding a one form and so consider the equation

$$\mu(d\psi + eA \otimes \psi) = m\psi.$$

This is the Dirac equation in the presence of an external electromagnetic field with four potential A.

7.20.7 Complexifying a vector space with a complex structure.

The space of Dirac spinors is the complexification of the space of Majorana spinors. This will involve us several times in the painful process of complexifying a real vector space with a complex structure, so we review the general construction.

Let V be a real vector space with a complex structure. That is, we are given an operator $\mathbf{J}$ on V such that $\mathbf{J}^2 = -I$. Any operator, A, on V extends as the operator $A \otimes \mathrm{id}$ on $V^{\mathbb{C}} = V \otimes \mathbb{C}$. When there is no danger of confusion we shall continue to denote this extended operator by A. Thus the (extended) operator $\mathbf{J}$ has eigenvalues $\pm i$ on $V^{\mathbb{C}}$. In other words $V^{\mathbb{C}}$ decomposes as

$$V^{\mathbb{C}} = V^{\mathbb{C}}_{+} \oplus V^{\mathbb{C}}_{-}$$

where

$$V^{\mathbb{C}}_{+} := \{u - i\mathbf{J}u, \quad u \in V\}$$

consists of all the $+i$ eigenvectors of $\mathbf{J}$ and

$$V^{\mathbb{C}}_{-} : \{u + i\mathbf{J}u, \quad u \in V\}$$

consists of all the $-i$ eigenvectors of $\mathbf{J}$.

Suppose that the operator $A$is $\mathbf{J}$ linear, meaning that $A\mathbf{J} = \mathbf{J}A$. Suppose that we choose a $\mathbf{J}$ basis of V. This means that we choose vectors $e_1, \ldots, e_n$ so that the vectors

$$e_1, \ldots, e_n, \mathbf{J}e_1, \ldots, \mathbf{J}e_n$$

form a basis of V . Relative to such a basis the assertion that A is $\mathbf{J}$ linear amounts to saying that A has the block matrix decomposition

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Now $e_1 - i\mathbf{J}e_1, \ldots, e_n - i\mathbf{J}e_n$ is a basis of $V^{\mathbb{C}}_{+}$ while $e_1 + i\mathbf{J}e_1, \ldots, e_n + i\mathbf{J}e_n$ is a basis of $V^{\mathbb{C}}_{-}$. It then follows immediately that in terms of the combined basis of $V^{\mathbb{C}}$ we have

$$A \otimes \mathrm{id} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix} \text{ if } A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ is } \mathbf{J} \text{ linear.}$$

Now suppose that A is anti-$\mathbf{J}$ linear, meaning that $A\mathbf{J} = -\mathbf{J}A$. This amounts to saying that A has the block decomposition

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

and it follows that

$$A = \begin{pmatrix} 0 & a + ib \\ a - ib & 0 \end{pmatrix} \text{ if } A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ is } \mathbf{J} \text{ anti-linear.}$$

For example, let us consider the case where $V = \mathfrak{g}$ is a Lie algebra in which the Lie bracket is $\mathbf{J}$ linear. This Lie bracket extends by complexification to $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{\mathbb{C}}$, and the two subspaces $\mathfrak{g}^{\mathbb{C}}_{+}$ and $\mathfrak{g}^{\mathbb{C}}_{-}$ are subalgebras each isomorphic to $\mathfrak{g}$ under the isomorphisms

$$\xi \mapsto \frac{1}{\sqrt{2}}(\xi - i\mathbf{J}\xi), \quad \xi \mapsto \frac{1}{\sqrt{2}}(\xi + i\mathbf{J}\xi).$$

Suppose that the Lie algebra $\mathfrak{g}$ has a representation on the vector space S which carries a complex structure, $\mathbf{J}_S$, and that the complex structure on $\mathfrak{g}$ is consistent with the complex structure on S in the sense that

$$\xi(\mathbf{J}_S u) = (\mathbf{J}_{\mathfrak{g}}\xi)u.$$

where $\mathbf{J}_{\mathfrak{g}}$ denotes the complex structure on $\mathfrak{g}$. We can drop the two subscripts and write this as

$$\xi\mathbf{J}u = \mathbf{J}\xi u.$$

Then

$$(\xi - i\mathbf{J}\xi)(u + i\mathbf{J}u) = \xi u + i\mathbf{J}\xi u - i\mathbf{J}\xi u - i^2\mathbf{J}^2\xi u = 0.$$

In other words $\mathfrak{g}_+^{\mathbb{C}}$ acts trivially on $S_-^{\mathbb{C}}$ and similarly $\mathfrak{g}_-^{\mathbb{C}}$ acts trivially on $S_+^{\mathbb{C}}$. Also the action of $\mathfrak{g}_+^{\mathbb{C}}$ on $S_+^{\mathbb{C}}$ is isomorphic to the action of $\mathfrak{g}$ on S and similarly for the other component.

In the case of interest to us we see that

$$sl(2,\mathbb{C}) \otimes \mathbb{C} = sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$$

and that the space of Dirac spinors, the complexification of the space of Majorana spinors, decomposes as

$$S \otimes \mathbb{C} = S_+^{\mathbb{C}} \oplus S_-^{\mathbb{C}} = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}),$$

where $\frac{1}{2}$ denotes the standard two dimensional representation of $sl(2,\mathbb{C})$ and 0 denotes the trivial representation.

Any $\mathbf{J}-$ antilinear map of S (where $\mathbf{J}$ is now γ) extends to a complex linear map of

$$\mathbf{D} := S \otimes \mathbb{C}$$

which switches the two components. In particular this applies to the operator $\star$. So we see that the group G acts irreducibly on $\mathbf{D}$ as claimed above.

Let us now consider the action of the real Lie algebra $sl(2,\mathbb{C})$ on Minkowski space, identified, as usual, with the space of self adjoint two by two matrices. The action is given by

$$P \mapsto \xi P + P\xi^{\dagger}.$$

Since every complex square matrix can be written as $P + iQ$ where P and Q are self adjoint, we see that the complexification of Minkowski space is just $gl(2,\mathbb{C})$, the space of all complex two by two matrices. Furthermore, recalling that the complex structure on $sl(2,\mathbb{C})$ is exactly multiplication by the scalar matrix, iI, we see that

$$(\mathbf{J}\xi)P = i\xi P = \xi(iP)$$

as two by two matrices and hence

$$(\xi + i\mathbf{J}\xi)P = 0.$$

Similarly
$$P(\xi^\dagger - i\mathbf{J}\xi^\dagger) = 0.$$
Thus $M^{\mathbb{C}}$ is irreducible under $sl(2,\mathbb{C})\otimes\mathbb{C}$ and is the representation $(\frac{1}{2},\frac{1}{2})$, the tensor product of the basic representation of each factor. Recall that $\mathbf{D} = S\otimes\mathbb{C}$ is the complexification of the space of Majorana spinors. We extend $\mu(P)$ by complex linearity to $\mathbf{D}$ and define
$$\gamma(P) = i\mu(P)$$
where i is now the good old fashioned complex number and so commutes with $\mu(P)$. Hence
$$\gamma(P)^2 = ||P||^2 I.$$
These are the defining relations for the Dirac "matrices". But notice that the Clifford algebra $C(1,3)$ is isomorphic to the algebra $H(2)$ of all two by two matrices over the quaternions. Hence its minimal module must have dimension eight over the real numbers. Thus the Dirac matrices have no realization as four by four real matrices. This is in contrast to the algebra $C(3,1)$ which we studied above in conjunction with the Majorana spinors. The Dirac equation is as before, namely
$$-i\gamma(\partial)\psi = \mu(\partial)\psi = m\psi.$$
But now ψ is a $\mathbf{D}$ valued function and $\mathbf{D}$ is a complex vector space so we can seek plane wave solutions of the form
$$\psi(x) = u(P)e^{iP\cdot x}.$$
Then we must have
$$\gamma(P)u(P) = mu(P)$$
which implies
$$||P||^2 = m^2$$
as before.

Thus if ψ is a general solution of the Dirac equation, its Fourier transform must be supported on the two sheeted hyperboloid $||P||^2 = m^2$. It is a fact that the space of ψ concentrated on the forward (or backward) sheet provides an irreducible unitary representation of the Poincaré group.

7.20.8 Sesquilinear covariants for Dirac spinors.

For each of the bilinear covariants defined on the space of Majorana spinors S we have a choice: we can extend it as a bilinear or as a sesquilinear form on $D\otimes D$. For example, let us extend $\mathfrak{j}$ so as to be sesquilinear. Then
$$\begin{aligned}\mathfrak{j}(u+iv) &= (u+iv)\otimes(u^\dagger - iv^\dagger)\\ &= u\otimes u^\dagger + v\otimes v^\dagger + i[v\otimes u^\dagger - u\otimes v^\dagger],\end{aligned}$$

where u and v are elements of S. The original group G acts as real linear transformations on $\mathbf{D} = S^{\mathbb{C}}$ and hence the relations

$$\mathsf{j}(Aw) = A\mathsf{j}(w)A^{\dagger}, \quad \mathsf{j}(\star w) = \mathsf{j}(w)^{\mathrm{a}}$$

continue to hold for $w \in \mathbf{D}$ and $A \in Sl(2, C)$. Also $\mu(\partial)$ is a real operator, so if ψ is a complex (i.e. $\mathbf{D}$ valued) solution of the Dirac equation we continue to have

$$\mathrm{div}\ \mathsf{j}(\psi) = 0.$$

Notice that

$$\begin{aligned} \mathrm{tr}\ \mathsf{j}(u+iv) &= ||u||^2 + ||v||^2 + 2i\mathrm{Im}\ (u,v) \\ &\geq ||u||^2 + ||v||^2 - 2||u||||v|| \\ &\geq 0. \end{aligned}$$

Similarly the real symplectic form s extends to $\mathbf{D}$ as a $\mathbf{C}$ valued anti Hermitian form:

$$s(v,u) = -\overline{s(u,v)}.$$

So we can define a G invariant Hermitian form by

$$\langle u, v\rangle := is(u,v). \tag{50}$$

Since the complexification of any (real two dimensional) Lagrangian subspace of S will be a null space for $\langle\ ,\ \rangle$ we see that $\langle\ ,\ \rangle$ has signature $(2,2)$. In fact we have the decomposition

$$\mathbf{D} = \mathbf{D}_+ \oplus \mathbf{D}_-$$

into two complex inequivalent irreducible representations of $sl(2,\mathbb{C})$ according to the $\pm i$ eigenvectors of $\mathbf{J}$. The restriction of $\langle\ ,\ \rangle$ to each component must be trivial since $\mathbb{C}^2$ admits no $sl(2,\mathbb{C})$ invariant Hermitian form. We can see this directly since

$$s(\mathbf{J}u, v) = s(u, \mathbf{J}v)$$

and

$$\mathbf{J}^2 = -I$$

imply that

$$\begin{aligned} s(u+i\mathbf{J}u, v+i\mathbf{J}v) &= s(u,v) + s(\mathbf{J}u, \mathbf{J}v) + i[s(\mathbf{J}u,v) - s(u,\mathbf{J}v)] \\ &= 0. \end{aligned}$$

Notice that

$$\begin{aligned} \langle\gamma(P)u, v\rangle &= -s(?(P)u, v) \\ &= s(u, ?(P)v) \\ &= \langle u, \gamma(P)v\rangle. \end{aligned}$$

In other words the operators $\gamma(P)$ are self adjointrelative to the Hermitian form $\langle\ ,\ \rangle$. It follows from equation (47) that

$$P \cdot \mathfrak{j}(w) = \langle \gamma(P)w, w\rangle. \tag{51}$$

The Hermitian form $\langle\ ,\rangle$ determines an antilinear map $\mathbf{D} \to \mathbf{D}^*$. The image of a spinor w is called the spinor adjoint to w and is denoted in the physics literature by putting a bar over w. Thus

$$\overline{w}(z) = \langle z, w\rangle.$$

8 Special representations of $sl(m/n)$.

8.1 The definition of the Lie superalgebras $sl(m/n)$.

We begin by recalling the definition of these superalgebras. For general facts about Lie superalgebras we refer to the book [Sch79] or the articles [CNS75] or [Kac77].

Let

$$V = V_0 \oplus V_1$$

be a supervector space with

$$\dim V_0 = m, \quad \text{and} \quad \dim V_1 = n.$$

The Lie superalgebra $sl(V_0/V_1)$ is the (commutator) Lie superalgebra of the superalgebra of all endomorphisms with supertrace zero. A typical such endomorphism has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \operatorname{tr} A = \operatorname{tr} D.$$

Here

$$A \in \operatorname{Hom}(V_0, V_0), \quad B \in \operatorname{Hom}(V_1, V_0), \quad C \in \operatorname{Hom}(V_0, V_1), \\ D \in \operatorname{Hom}(V_1, V_1).$$

Recall that those endomorphisms which preserve the grading (those with $B = C = 0$) are "even", i.e. belong to $sl(V_0/V_1)_0$ and those that reverse the grading (those with $A = D = 0$) are "odd", i.e. belong to $sl(V_0/V_1)_1$. We are assuming that the vector spaces V_0 and V_1 are finite dimensional. The structure of the Lie algebra clearly depends only on the dimensions of these spaces and hence the notation $sl(m/n)$.

Since our spaces are finite dimensional, we may identify $\operatorname{Hom}(V_1, V_0)$ with $V_0 \otimes V_1^*$. Under this identification, if $v \in V_0$ and $\xi \in V_1^*$ then $v \otimes \xi$ is identified with the rank one linear transformation given by

$$(v \otimes \xi)w = \langle \xi, w\rangle v$$

where $\langle \xi, w \rangle$ denotes the value of the linear function ξ on the vector w. These rank one linear transformations span $\mathrm{Hom}(V_1, V_0)$. Similar identifications will be made for each of the other three spaces corresponding to the entries of our block matrix. For example, we compute the (super)commutator

$$\left[\begin{pmatrix} 0 & v \otimes \xi \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ x \otimes \mu & 0 \end{pmatrix}\right] = \begin{pmatrix} \langle \xi, x \rangle v \otimes \mu & 0 \\ 0 & \langle \mu, v \rangle x \otimes \xi \end{pmatrix}.$$

Notice that the trace of the upper left block and the lower right block are both equal to $\langle \xi, x \rangle \cdot \langle \mu, v \rangle$. This proves that $sl(V_0, V_1)$ is indeed a Lie super subalgebra of the Lie superalgebra of $\mathrm{End}(V)$.

To save space we will write the above bracket relations (and similar ones) as follows: We write

$$sl(V_0/V_1)_0 = (V_0 \otimes V_0^*) \oplus (V_1 \otimes V_1^*)$$

and

$$sl(V_0/V_1)_1 = (V_0 \otimes V_1^*) \oplus (V_1 \otimes V_0^*).$$

Then we would write the preceding bracket relation as

$$[v \otimes \xi, x \otimes \mu] = \langle \xi, x \rangle v \otimes \mu \ \oplus \ \langle \mu, v \rangle x \otimes \xi.$$

8.2 The representation of $sl(V_0/V_1)$ on the super exterior algebra of V.

By definition, the super exterior algebra $\bigwedge(V)$ of a superspace V is

$$\bigwedge(V) := \wedge(V_0) \otimes S(V_1)$$

where $S(V_1)$ denotes the symmetric algebra of V_1 so

$$S(V_1) = \bigoplus_{k=0}^{\infty} S^k(V_1)$$

and $S^k(V_1)$ consists of homogeneous polynomials of degree k on V_1^*. The multiplication in $S(V_1)$ is the ordinary multiplication of polynomials so the elements of $S^k(V_1)$ are all declared to have even grading even if k is odd.

The Lie superalgebra $sl(V_0, V_1)$ has a natural representation on $\bigwedge(V)$. Perhaps the best way to realize this representation is by imbedding $sl(V_0, V_1)$ in the orthosymplectic algebra as the centralizer of a one dimensional subalgebra. This "Howe pair" point of view is explained by Howe in his original paper [H77]. In [NS82] we used this description in conjunction with the method of dimensional reduction. But here is a direct description:

Each $x \in V_1$ defines a multiplication operator on $S(V)$:

$$m_x : S^k(V_1) \to S^{k+1}(V_1)$$

given by

$$(m_x f)(\eta); = \langle \eta, x \rangle f(\eta), \quad \forall\, \eta \in V_1^*. \tag{52}$$

Each $\xi \in V_1^*$ defines a derivation D_ξ of $S(V_1)$ so

$$D_\xi(fg) = (D_\xi f)g + f D_\xi g$$

determined by

$$D_\xi 1 = 0 \quad \text{ and } \quad D_\xi x = \langle \xi, x \rangle \quad \forall\, x \in V_1 = S^1(V_1). \tag{53}$$

The standard Fock commutation relations hold, i.e.

$$D_\xi m_x - m_x D_\xi = \langle \xi, x \rangle \mathrm{id}. \tag{54}$$

Similarly, each $v \in V_0$ determines the operator of exterior multiplication by v which we currently denote by e_v and each $\mu \in V_0^*$ defines the operator on $\wedge(V_0)$ of interior multiplication by μ which we will denote by i_μ. So i_μ is the (odd) derivation of $\wedge(V_0)$:

$$i_\mu : \wedge^k(V_0) \to \wedge^{k-1}(V_0)$$

determined by

$$i_\mu(\omega_1 \wedge \omega_2) = i_\mu(\omega_1) \wedge \omega_2 + (-1)^{|\omega_1|} \omega_1 \wedge i_\mu(\omega_2)$$

on homogeneous elements,

$$i_\mu v = \langle \mu, v \rangle \quad \forall\, v \in V_0 = \wedge^1(V_0),$$

and

$$i_\mu 1 = 0.$$

We have the supercommutation relations

$$\begin{aligned} [e_{v_1}, e_{v_2}] &= 0, \\ [i_{\mu_1}, i_{\mu_2}] &= 0, \\ [e_v, i_\mu] &= \langle \mu, v \rangle \mathrm{id}. \end{aligned}$$

In short, m and D are Bose-Einstein creation and annihilation operators while e and i are Fermi-Dirac creation and annihilation operators.

If $x \in V_1$ and $\xi \in V_1^*$ then $m_x \circ D_\xi$ is again a derivation of $S(V_1)$ since a derivation followed by a multiplication is again a derivation. In fact, it is the derivation determined by the map

$$y \mapsto \langle \xi, y \rangle x$$

on V_1 and this is just the linear transformation $x \otimes \xi$. Similarly, $e_v \circ i_\mu$ is the derivation of $\wedge(V_0)$ determined by the linear transformation $v \otimes \mu$ on V_0.

If $v \in V_0$ and $\xi \in V_1^*$ then $e_v \circ D_\xi := (e_v \otimes 1) \circ (1 \otimes D_\xi)$ is an odd derivation of $\bigwedge(V)$:

$$(e_v \circ D_\xi)(\sigma \otimes f) = v \wedge \sigma \otimes D_\xi f$$

so that

$$\begin{aligned}
e_v \circ D_\xi \left((\sigma \otimes f)(\omega \otimes g)\right) &= e_v \circ D_\xi(\sigma \wedge \omega \otimes fg) \\
&= v \wedge \sigma \wedge \omega \otimes D_\xi(fg) \\
&= v \wedge \sigma \wedge \omega \otimes ((D_\xi f)g + f D_\xi g) \\
&= v \wedge \sigma \wedge \omega \otimes (D_\xi f)g + (-1)^{|\sigma|}\sigma \wedge v \wedge \omega \otimes f D_\xi g \\
&= (e_v \circ D_\xi(\sigma \otimes f))(\omega \otimes g) \\
&\qquad + (-1)^{|\sigma \otimes f|}(\sigma \otimes f) e_v \circ D_\xi(\omega \otimes g).
\end{aligned}$$

By definition

$$e_v \circ D_\xi : \wedge^p(V_0) \otimes S^k(V_1) \to \wedge^{p+1}(V_0) \otimes S^{k-1}(V_1). \tag{55}$$

Similarly we have the odd derivation $m_x \circ i_\mu$ on $\bigwedge(V)$ and

$$m_x \circ i_\mu : \ \wedge^p(V_0) \otimes S^k(V_1) \to \wedge^{p-1}(V_0) \otimes S^{k+1}(V_1). \tag{56}$$

Also we have the even derivations $m_x \circ D_\xi$ and $e_v \circ i_\mu$ which preserve all bidegrees. We have

$$\begin{aligned}
[e_{v_1} \circ D_{\xi_1}, e_{v_2} \circ D_{\xi_2}] &= e_{v_1} \circ D_{\xi_1} \circ e_{v_2} \circ D_{\xi_2} + e_{v_2} \circ D_{\xi_2} \circ e_{v_1} \circ D_{\xi_1} \\
&= (e_{v_1} e_{v_2} + e_{v_2} e_{v_1}) \otimes D_{\xi_1} D_{\xi_2} \quad \text{since} \quad D_{\xi_2} D_{\xi_1} = D_{\xi_1} D_{\xi_2} \\
&= 0
\end{aligned}$$

and similarly

$$[m_{x_1} \circ i_{\mu_1}, m_{x_2} \circ i_{\mu_2}] = 0$$

while

$$\begin{aligned}
[e_v \circ D_\xi, i_\mu \circ m_x] &= e_v \circ i_\mu \otimes D_\xi \circ m_x + i_\mu \circ e_v \otimes m_x \circ D_\xi \\
&= \langle \xi, x \rangle e_v \circ i_\mu \otimes 1 + e_v \circ i_\mu \otimes m_x D_\xi - e_v \circ i_\mu \otimes m_x \circ D_\xi + \langle \mu, v \rangle 1 \otimes m_x D_\xi \\
&= \langle \xi, x \rangle e_v \circ i_\mu \otimes 1 + \langle \mu, v \rangle 1 \otimes m_x D_\xi.
\end{aligned}$$

This shows that $sl(V_0/V_1)$ acts as derivations of $\bigwedge(V)$ where

$$\begin{aligned}
v \otimes \mu &\mapsto e_v \circ i_\mu && (57) \\
x \otimes \xi &\mapsto m_x \circ D_\xi && (58) \\
v \otimes \xi &\mapsto e_v \circ D_\xi && (59) \\
x \otimes \mu &\mapsto m_x \circ i_\mu. && (60)
\end{aligned}$$

Notice that for each integer k the finite dimensional subspace of $\bigwedge(V)$ given by

$$\wedge^0(V_0) \otimes S^k(V_1) \oplus \wedge^1(V_0) \otimes S^{k-1}(V_1) \oplus \cdots \oplus \wedge^n(V_0) \otimes S^{k-n}(V_1)$$

is invariant. In the above expression (and in contrast to our notation in the next section) the space $S^\ell(V_1)$ is taken to be 0 if $\ell < 0$. It is clear that each such subspace is irreducible under $sl(V_0, V_1)$. We have thus associated an irreducible representation of $sl(V_0, V_1)$ to each non-negative integer k.

If we replace the spaces of homogenous polynomials $S^k(V_1)$ by the spaces F^b of all smooth functions homogenous of degree b and defined on some fixed open cone in V_1^* with vertex at the origin (vertex not included), then we still have the multiplication operator $m_x : F^b \to F^{b+1}$ given by (52), the derivation operator $D_\xi : F^b \to F^{b-1}$ given by (53) and the commutation relations (54) continue to hold. If dim $V_1 > 1$ and the cone is non-empty these spaces are infinite dimensional. But if V_1 is one dimensional something special happens.

8.3 Special representations of $sl(m/1)$.

We suppose that $V_1 = \mathbb{C}$. We now let $S^b = S^b(V_1)$ denote the one dimensional space with basis element p_b. Now b can be any complex number. For $x \in V_1$ define

$$m_x : S^b \to S^{b+1}$$

by

$$m_x p_b = x p_{b+1}. \tag{61}$$

For $\xi \in V_1^*$ define

$$D_\xi : S^b \to S^{b-1}$$

by

$$D_\xi p_b = b\xi p_{b-1}. \tag{62}$$

The commutation relation (54) continues to hold (where $\langle \xi, x\rangle$ is simply the product ξx). So the ingredients that we needed to construct the representations of $sl(m/n)$ in the preceding section are all present. In this way, [NS80], we have associated a finite dimensional representation of $sl(m/1)$ on

$$\wedge^0(V_0) \otimes S^b \oplus \wedge^1(V_0) \otimes S^{b-1} \oplus \cdots \oplus \wedge^m(V_0)_\otimes S^{b-m} \tag{63}$$

for each complex number b and these representations are irreducible unless b is a non-negative integer with $0 < b < m$. Since all the spaces S^a are one dimensional, all of these representation are on a space of dimension 2^m, the same dimension as that of the exterior algebra.

Each of the summands in (63) is invariant and irreducible under $sl(m/1)_0$. It will be useful for future computations to record the action of a diagonal matrix on each of these components: The action of the diagonal matrix

$$\left(\begin{array}{cccc|c} u_1 & 0 & \cdots & 0 & \\ 0 & u_2 & \cdots & 0 & \\ \vdots & \vdots & \cdots & \vdots & 0 \\ 0 & 0 & \cdots & u_m & 0 \\ \hline 0 & 0 & \cdots & 0 & U \end{array}\right), \quad U = u_1 + u_2 + \cdots + u_m$$

is as follows:

On the one dimensional space $\wedge^0(V_0) \otimes S^b$ it is multiplication by

$$bU.$$

If $v_1, \ldots v_m$ is the basis in terms of which the above matrix is diagonal, the action on $\wedge^1(V_0) \otimes S^{b-1}$ is diagonal with basis $v_1 \otimes p_{b-1}, \ldots, v_m \otimes p_{b-1}$ with eigenvalues

$$u_1 + (b-1)U, \ldots, u_m + (b-1)U,$$

and in general, the action on $\wedge^q(V_0) \otimes S^{b-q}$ is diagonal with basis

$$(v_{i_1} \wedge \cdots \wedge v_{i_q}) \otimes p_{b-q}, \quad i_1 < \cdots < i_q \tag{64}$$

and corresponding eigenvalues

$$u_{i_1} + \cdots + u_{i_q} + (b-q)U. \tag{65}$$

In tabulating computations we will usually use some shorthand for the eigenvectors (64). For example we do not need to include the $\otimes p_{b-q}$ since this is determined by the representation. We will also shorten the notation for the wedge product and simply write

$$i_1 i_2 \ldots i_q$$

for the eigenvector (64).

9 $sl(2/1)$ and the electroweak isospins and hypercharges.

In [NS80] we showed how to derive the various values of the weak isospin and hypercharge by choosing the appropriate elements of $sl(2/1)$ and then choosing various parameters for b in (63). In particular, we predicted the existence of the right handed neutrino which occurs with weak isospin and hypercharge zero, and does not participate to first order in the weak interaction. With the recent discovery that the neutrino has positive mass [Fu98] this expectation has been justified.

The choice of the weak isospin and hyperchange elements of $sl(2/1)$ are (up to the pervasive factor of i):

$$I_3 = \left(\begin{array}{cc|c} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad Y = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right). \tag{66}$$

We will tabulate below the weak isospin and hypercharge values corresponding to the leptons ($b = 0$) and the quarks ($b = \frac{2}{3}$) and their anti-particles ($b = 1$ corresponding to the anti-leptons and $b = \frac{1}{3}$ corresponding to the anti-quarks).

In the full geometrical theory, we would take the tensor product of the superbundle associated to these representations $su(2/1)$ with the bundle of Dirac spinors which has the $\mathbb{Z}_2$ gradation according to chirality. From the tables below it will follow that all the particles have the same total degree (in the tensor product) which is opposite to the total degrees of the anti-particles.

9.1 $b = 0$ - the leptons.

We get the lepton assignments by choosing the parameter $b = 0$ in (63). For the reader's convenience we have also tabulated the electric charge

$$Q = I_3 + \frac{1}{2}Y.$$

leptons ($b = 0$)	$\wedge^0(V_0)$	$\wedge^1(V_0)$		$\wedge^2(V_0)$
basis elements	$\emptyset$	1	2	12
I_3	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
Y	0	-1	-1	-2
Q	0	0	-1	-1
particle	ν_R	ν_L	e_L	e_R

(67)

Notice that the gradation of the superspace on which the representation takes place corresponds to chirality - the first and third columns which correspond to $\wedge^+(V_0) = \wedge^0(V_0) \otimes S^0 \oplus \wedge^2(V_0) \otimes S^{-2}$ corresponds to right handed particles while $\wedge^-(V_0) = \wedge^1(V_0) \otimes S^{-1}$ corresponds to left handed particles. Notice also that the entire even subalgebra $sl(2/1)_0$ acts trivially on $\wedge^0(V_0) \otimes S^0$ corresponding to the right handed neutrino.

9.2 $b = \frac{2}{3}$ - the quarks.

The choice $b = \frac{2}{3}$ gives the electroweak isospin and hypercharge assignments for quarks:

quarks ($b = \frac{2}{3}$)	$\wedge^0(V_0)$	$\wedge^1(V_0)$		$\wedge^2(V_0)$
basis elements	$\emptyset$	1	2	12
I_3	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
Y	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$
Q	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
particle	u_R	u_L	d_L	d_R

(68)

Once again observe the relation between the gradation and chirality

9.3 $b = 1$ - the anti-leptons.

The choice $b = 1$ gives the anti-lepton assignment:

anti-leptons ($b = 0$)	$\wedge^0(V_{)})$	$\wedge^1(V_0)$		$\wedge^2(V_0)$
basis elements	$\emptyset$	1	2	12
I_3	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
Y	2	1	1	0
Q	1	1	0	0
particle	$(\overline{e_R})_L$	$(\overline{e_L})_R$	$(\overline{\nu}_L)_R$	$(\overline{\nu_R})_L$

(69)

Again there is a correspondence between gradation and chirality (the opposite from that of the leptons). Notice again that the entire even subalgebra acts trivially on $\wedge^2$.

9.4 $b = \frac{1}{3}$ - the anti-quarks.

Finally the choice $b = \frac{1}{3}$ gives the anti-quark assignment:

anti-quarks ($b = \frac{1}{3}$)	$\wedge^0(V_0)$	$\wedge^1(V_0)$		$\wedge^2(V_0)$
basis elements	$\emptyset$	1	2	12
I_3	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
Y	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$
Q	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
particle	$(\overline{d_R})_L$	$(\overline{d_L})_R$	$(\overline{u_L})_R$	$(\overline{u_R})_L$

(70)

10 Using $sl(m/1)$ for $m = 3, 5,$ and $5 + n$.

10.1 $m = 3$ - unifying quarks and leptons.

We showed in [NS80] that if we take

$$I_3 = \left(\begin{array}{ccc|c} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad Y = \left(\begin{array}{ccc|c} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} \end{array}\right) \tag{71}$$

then we get the correct isospins and hypercharges if we combine the anti-leptons and quarks into the single eight dimensional representation of $sl(3/1)$ with $b = \frac{2}{3}$ and if we combine the leptons and anti-quarks in the single eight dimensional representation with $b = \frac{1}{2}$. We refer to [NS80] for details.

10.2 $m = 5$ - including color.

We showed in [NS80] that if we choose

$$I_3 = \left(\begin{array}{ccccc|c} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad Y = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{72}$$

then the single 32 dimensional representation given by

$$b = 2$$

gives the correct isospin and hypercharge assignments to the right and left handed up and down quarks in three colors and the right and left handed electrons and neutrino (so 16 in all) and their antiparticles (yielding 32). Again the chirality and the gradation match up: All the elements of $\wedge^+$ have eigenvalues corresponding to left handed particles and all the elements of $\wedge^-$ have eigenvalues corresponding to right handed particles We refer to the Appendix in [NS80] for the list of all 32 eigenvalues.

There is something special about the value $b = \frac{m-1}{2}$ (for example the value $b = 2$ in our current case of $m = 5$. Indeed, as pointed out in the note added in proof in [NS80], the space $\wedge^m(V_0) \otimes S^{-1}$ is acted on trivially by the even part of $sl(m/1)$, i.e. has a canonical trivialization. This means that the natural multiplication

$$\left(\wedge^k \otimes S^{b-k}\right) \otimes \left(\wedge^{m-k} \otimes S^{b-m+k}\right) \to \wedge^m \otimes S^{2b-m}$$

can be thought of as invariant bilinear form on the space of the representation corresponding to $b = \frac{m-1}{2}$. Notice that the particles and the anti-particles of any given species occur in the components $\wedge^k$ and $\wedge^{5-k}$in the representation. If m is odd then either k or $m-k$ is even, so the above bilinear form is symmetric.

In this set up all the particles and anti-particles have the same total tensor degree. What the meaning of the opposite total degree is in this formulation (whether "ghosts" or some other meaning) was left open to speculation.

10.3 $m = 5 + n$ - accomodating 2^n generations.

It was shown in [NS80] that generational symmetry can be achieved if we enlarge the superalgebra $sl(5/1)$ to $sl(5 + n/1)$. This would be a theory with 2^n or 2^{n+1} generations . At the time, this seemed inappropriate since the number of generations was observed to be at least three, and was thought to be less than four based on arguments from the Z width. In [NS91] it was argued that if the neutrinos had positive mass, especially if the neutrinos in the higher generations were heavy, then a fourth generation is not excluded.

The idea is that the weak isospin $su(2)$ and the color $su(3)$ are regarded as commuting subalgebras of the even part of $sl(m/1)$ where $m = 5 + n$ while the generational behavior is produced by an $sl(n/1)$ sub Lie superalgebra.

The I_3 assignment for $sl(5 + n/1)$ is the diagonal matrix

$$\text{diag}(\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0|0) \qquad (n+4) \text{ zeros in all}$$

while the hypercharge assignment is

$$Y = \text{diag}\left(\frac{-n}{4+n}, \frac{-n}{4+n}, \left(\frac{4}{4+n}\right)_{n \text{ times}}, \left(\frac{4-2n}{3(4+n)}\right)_{3 \text{ times}} \middle| \frac{4}{4+n}\right). \quad (73)$$

and the preferred representation is given by $b = \frac{5+n-1}{2}$.

We will discuss the model with four generations in the next two sections.

11 $sl(7/1)$ - unifying color and four generations.

In this section we show how the value $b = 3$ can accommodate four generations of particles with the correct isospin and hypercharge values provided that we reverse the chirality assignments in two out of the four generations. Our fundamental superbundle will be the tensor product of the spin bundle with the bundle associated to this 128 dimensional representation. So this means that all particles will correspond to the same total degree as indicated above. The tables here follow the tables (42)-(45) in [NS91]. We need a name (or at least a letter) for the particles in the fourth generation, and we have tentatively chosen σ for the analogue of the electron and x and y for the analogue of the u and d quark. Also, we have made the choice that $\wedge^0 \otimes S^3$ has left handed chirality. This then determines that all the spaces with $\wedge^k \otimes S^{3-k}$ are

left handed when k is even and are right handed when k is odd. In [NS80] the choice of $m = 7$ was made in order to accommodate the possibility of ghost fields. An assignment of particles without ghosts and which fits better with the theory of Clifford superconnections will be presented in the next section.

As usual, the element I_3 is given by

$$\left(\begin{array}{ccccccc|c}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

In accordance with (73) the hypercharge is given by

$$\left(\begin{array}{ccccccc|c} -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{array}\right). \tag{74}$$

Then the eigenvalues on $\wedge^k(V_0) \otimes S^{b-k}$ (and particle assignments) are given as follows:

$\wedge^0 \otimes S^3$	$\emptyset$
Y	2
I_3	0
particle	$(\overline{e_R})_L$

$\wedge^1 \otimes S^2$	1	2	3	4	5	6	7
Y	1	1	2	2	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
I_3	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0
particle	$(\overline{e_L})_R$	$(\overline{\nu_{eL}})_R$	$(\overline{\mu_L})_R$	$(\overline{\tau_L})_R$	u_R	u_R	u_R

Notice the opposite chirality assignments (as compared to the electron) to the μ and τ. This is somewhat arbitrary at the moment. We could make this opposite assignment to the third and fourth generation as opposed to the second and third.

In the next tables we will conjoin the color entries, so write $_{2;5,6,7}$ instead of having three columns f_{25}, f_{26}, f_{27}.

$\wedge^2 \otimes S^1$	12	13	14	1:5,6,7	23	24	2;5,6,7	34	3;5,6,7	4;5,6,7	56,57,67
Y	0	1	1	$\frac{1}{3}$	1	1	$\frac{1}{3}$	2	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{2}{3}$
I_3	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0
particle	$(\overline{\nu_{eR}})_L$	$(\overline{\mu_R})_L$	$(\overline{\tau_R})_L$	u_L	$(\overline{\nu_{\mu R}})_L$	$(\overline{\nu_{\tau R}})_L$	d_L	$(\overline{\sigma_R})_L$	c_L	t_L	$(\overline{d_R})_L$

All 35 particle assignments in the next table of eigenvalues for $\wedge^3 \otimes S^0$ are right handed. To save space we no longer indicate this in the table.

$\wedge^3 \otimes S^0$	123	124	12;5,6,7	134	13;5,6,7	14;5,6,7	1;56,67,67	234	23:5,6,7	24;5,6,7	2;56,57,67
Y	0	0	$-\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
I_3	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
part.	$\overline{\nu_{\tau L}}$	$\nu_{\sigma R}$	y_R	$\overline{\sigma_L}$	c_R	t_R	$\overline{y_L}$	$\overline{\nu_{\sigma_L}}$	s_R	b_R	$\overline{x_L}$

$\wedge^3 \otimes S^0$	34:5,6,7	3:56,57,67	4,56,57,67	567
Y	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0
I_3	0	0	0	0
particle	x_R	$\overline{s_L}$	$\overline{b_L}$	$\overline{\nu_{\mu L}}$

The particles in the remaining four components of our 128 dimensional representation will be the anti-particles of the ones we have already seen, and paired with them under the bilinear form. So the 35 dimensional component $\wedge^4 \otimes S^{-1}$ gives following table of left handed particles:

$\wedge^4 \otimes S^{-1}$	1234	123;5,6,7	124:5,6,7	12;56,57,67	134;5,6,7	13:56,57,67	14;56,57,67	1567
Y	0	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1
I_3	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
particle	$\nu_{\mu L}$	b_L	s_L	$\overline{x_R}$	x_L	$\overline{b_R}$	$\overline{s_R}$	$\nu_{\sigma L}$

$\wedge^4 \otimes S^{-1}$	234;5,6,7	23;56.57,57	24;56,57,67	2567	34;56,57,67	3567	4567
Y	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	$\frac{2}{3}$	0	0
I_3	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
particle	y_L	$\overline{t_R}$	$\overline{c_R}$	σ_L	$\overline{y_R}$	$\overline{\nu_{\sigma R}}$	$\nu_{\tau L}$

The 21 dimensional component $\wedge^5 \otimes S^{-2}$ gives the following table of right handed particles:

$\wedge^5 \otimes S^{-2}$	1234;5,6,7	123:56,57,67	124;56,57,67	12567	134:56,57,67	13567	14567
Y	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	-2	$-\frac{1}{3}$	-1	-1
I_3	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
particle	d_R	$\overline{t_L}$	$\overline{c_L}$	σ_R	$\overline{d_L}$	$\nu_{\tau R}$	$\nu_{\mu R}$

$\wedge^5 \otimes S^{-2}$	234;56,57,67	23567	24567	34567
Y	$-\frac{1}{3}$	-1	-1	0
I_3	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
particle	$\overline{u_L}$	τ_R	μ_R	ν_{eR}

The 7 dimensional component $\wedge^6 \otimes S^{-3}$ gives the following table of left handed particles:

$\wedge^6 \otimes S^{-3}$	1234;56,57,67	123567	124567	134567	234567
Y	$-\frac{4}{3}$	-2	-2	-1	-1
I_3	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
particle	$\overline{u_R}$	τ_L	μ_L	ν_{eL}	e_L

Finally there is the one dimensional $\wedge^7 \otimes S^{-4}$ giving the right handed particle

$\wedge^7 \otimes S^{-4}$	1234567
Y	-2
I_3	0
particle	e_R

12 $sl(6/1)$.

If ghosts are not required, we use $sl(6/1)$ to accommodate four generations:

For $sl(6/1)$ we have $b = \frac{5}{2}$

$$I_3 = \operatorname{diag}(\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0 \,\big|\, 0)$$

and

$$Y = \operatorname{diag}(-\frac{1}{5}, -\frac{1}{5}, \frac{4}{5}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15} \,\Big|\, \frac{4}{5}).$$

We will assign both left and right handed spinors to each subrepresentation so that we get four families of particles with both even and odd total gradings:

$\wedge^0 \otimes S^{\frac{5}{2}}$	$\emptyset$
Y	2
I_3	0
particle (L)	$\overline{e_R}$
particle (R)	$\overline{\tau_L}$

$\wedge^1 \otimes S^{\frac{3}{2}}$	1	2	3	$4, 5, 6$
Y	1	1	2	$\frac{4}{3}$
I_3	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
particle (R)	$\overline{e_L}$	$\overline{\nu_{eL}}$	$\overline{\mu_L}$	u_R
particle (L)	$\overline{\tau_R}$	$\overline{\nu_{\tau R}}$	$\overline{\sigma_R}$	c_L

$\wedge^2 \otimes S^{\frac{1}{2}}$	12	13	$1; 4, 5, 6$	23	$2; 4, 5, 6$	$3; 4, 5, 6$	$45, 46, 56$
Y	0	1	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{2}{3}$
I_3	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
particle (L)	$\overline{\nu_{eR}}$	$\overline{\mu_R}$	u_L	$\overline{\nu_{\mu R}}$	d_L	t_L	$\overline{d_R}$
particle (R)	$\overline{\nu_{\tau L}}$	$\overline{\sigma_L}$	c_R	$\overline{\nu_{\sigma L}}$	s_R	x_R	$\overline{s_L}$

$\wedge^3 \otimes S^{-\frac{1}{2}}$	123	12;4,5,6	13;4,5,6	1;45,46,56	23:4,5,6	2;45,46,56	3,45,46,56	456
Y	0	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0
I_3	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
particle (R)	$\nu_{\sigma R}$	y_R	t_R	$\overline{y_L}$	b_R	$\overline{x_L}$	$\overline{b_L}$	$\overline{\nu_{\mu L}}$
particle (L)	$\nu_{\mu L}$	b_L	x_L	$\overline{b_R}$	y_L	$\overline{t_R}$	$\overline{y_R}$	$\overline{\nu_{\sigma R}}$

$\wedge^4 \otimes S^{-\frac{3}{2}}$	123;4,5,6	12;45,46,56	13;45,46,56	1456	23:45,46,56	2456	3456
Y	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	-1	$-\frac{1}{3}$	-1	0
I_3	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
particle (L)	s_L	$\overline{x_R}$	$\overline{s_R}$	$\nu_{\sigma L}$	$\overline{c_R}$	σ_L	$\nu_{\tau L}$
particle (R)	d_R	$\overline{t_L}$	$\overline{d_L}$	$\nu_{\mu R}$	$\overline{u_L}$	μ_R	ν_{eR}

$\wedge^5 \otimes S^{-\frac{5}{2}}$	123;45,46,56	12456	13456	23456
Y	$-\frac{4}{3}$	-2	-1	-1
I_3	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
particle (R)	$\overline{c_L}$	σ_R	$\nu_{\tau R}$	τ_R
particle (L)	$\overline{u_R}$	μ_L	ν_{eL}	e_L

$\wedge^6 \otimes S^{-\frac{7}{2}}$	123456
Y	-2
I_3	0
particle (L)	τ_L
particle (R)	e_R

Notice that the relation between these assignments and those of the preceding section are

$$\begin{aligned}
\wedge^0_7 &= {}_L\wedge^0_6, \\
\wedge^1_7 &= {}_R\wedge^0_6 \oplus {}_R\wedge^1_6, \\
\wedge^2_7 &= {}_L\wedge^1_6 \oplus {}_L\wedge^2_6,
\end{aligned}$$

etc.

13 Hermitian Lie algebras.

In this section we explain the notion of a Hermitian Lie algebra which was introduced in [SW78] and which we used above to determine the metric on the Higgs field.

13.1 The Lie superalgebra $su(2/1)$ and the Lie algebra $su(3)$.

We illustrate the notion by the relevant example. It is the special case of section 2A of [SW78] corresponding to the case $k = 0, \ell = 2, a = 0, b = 1$ of that section.

For

$$z = \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 & w_1 \\ 0 & 0 & w_2 \\ -\overline{w}_1 & -\overline{w}_2 & 0 \end{pmatrix}$$

we let

$$H(z, w) = izw,$$

and this equals

$$i \begin{pmatrix} -z_1\overline{w}_1 & -z_1\overline{w}_2 & 0 \\ -z_2\overline{w}_1 & -z_2\overline{w}_2 & 0 \\ 0 & 0 & -w_1\overline{z}_1 - w_2\overline{z}_2 \end{pmatrix} = i \begin{pmatrix} -z \otimes w^\dagger & 0 \\ 0 & -\overline{\langle z, w\rangle} \end{pmatrix}.$$

The right hand side is an element of $gl(2, \mathbb{C}) \oplus gl(1, \mathbb{C})$.

If we are given a hermitian form on $\mathbb{C}^n$ we define the complex conjugation on $gl(n, \mathbb{C})$ to be

$$\xi \mapsto \xi^* := -\xi^\dagger$$

where $\xi^\dagger$ denotes the adjoint of ξ relative to the hermitian form. Then the "real subspace", i.e. the set of matricies fixed by this complex conjugation is $u(n)$.

On $gl(2, \mathbb{C}) \oplus gl(1, \mathbb{C})$ we put the standard complex structure on $gl(2, \mathbb{C})$ but the conjugate complex structure on $gl(1, \mathbb{C})$. This means that we can write

$$H(z, w) = -iz \otimes w^\dagger \oplus i\langle z, w\rangle 1.$$

Then

$$H(z, w)^* = -iw \otimes z^\dagger \oplus i\langle w, z\rangle 1 = H(w, z).$$

So $H(z, w)$ is a hermitian form with values in the complexification of $u(2) \oplus u(1)$ and satisfies

$$H(w, z) = H(z, w)^*. \tag{75}$$

Since commutator is a derivation of multiplication (of matrices) we have $[M, zw] = [M, z]w + z[M, w]$ so if we define the action of $\xi \in u(2) \oplus u(1)$ on the space of z's to be commutator we have

$$[\xi, H(z, w)] = H(\xi z, w) + H(z.\xi w), \qquad \xi \in \mathfrak{g}_0, \quad z, w \in V \tag{76}$$

where

$$\mathfrak{g}_0 = u(2) \oplus u(1)$$

and where

$$V \sim \mathbb{C}^2$$

denotes the set of all matrices of the form

$$\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & 0 \end{pmatrix}.$$

Explicitly,

$$\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & z \\ -z^\dagger & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & Az - Bz \\ -(Az - Bz)^\dagger & 0 \end{pmatrix}.$$

We can write this more simply as an action on $\mathbb{C}^2$:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} z = Az - Bz, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

So

$$H(u, v)w = -i\langle w, v\rangle u + i\langle v, u\rangle w.$$

Therefore if we take the cyclic sum we get zero:

$$H(u, v)w + H(v, w)u + H(w, u)v = 0. \tag{77}$$

Now

$$2 \operatorname{Im} H(z, w) = \frac{1}{i}[H(z, w) - H(z, w)^*] = \frac{1}{i}[H(z, w) + H(w, z)^\dagger]$$

$$= \begin{pmatrix} -z \otimes w^\dagger + w \otimes z^\dagger & 0 \\ 0 & -\langle z, w\rangle + \langle w, z\rangle \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & w_1 \\ 0 & 0 & w_2 \\ -\overline{w}_1 & -\overline{w}_2 & 0 \end{pmatrix}\right].$$

Thus if we define $\mathfrak{g}_o := u(2) \oplus u(1)$ and $\mathfrak{g}_1 = V = \mathbb{C}^2$ then $2\,\mathrm{Im}\,H$ makes $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into the Lie algebra $u(3)$.

On the other hand,

$$2\mathrm{Re}H(z,w) = H(z,w) + H(w,z)^* = H(z,w) + H(w,z) = i(zw + wz)$$

is i times the anti-commutator of z and w. Since z and w are skew-adjoint their anti-commutator is self-adjoint, so multiplying by i gives a skew-adjoint matrix. So $\mathrm{Re}H$ makes $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into the super Lie algebra $u(2/1)$.

13.2 The general definition.

So the general definition of a **Hermitian Lie algebra** is as follows: We start with a real Lie algebra $\mathfrak{g}_0$ which is represented on a complex vector space $\mathfrak{g}_1$. We let $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \otimes \mathbb{C}$ which is a complex Lie algebra with a preferred complex conjugation $w \mapsto w^*$ so that $\mathfrak{g}_0$ consists of the real subspace, i.e. those w which are fixed under this complex conjugation. We assume that there is sesquilinear map

$$H : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0^{\mathbb{C}}$$

which satisfies (75), (76), and (77). For the convenience of the reader we collect these conditions here:

- (75): H is Hermitian - $H(w,z) = H(z,w)^*$.
- (76): H is equivariant - $[\xi, H(z,w)] = H(\xi z, w) + H(z.\xi w), \quad \xi \in \mathfrak{g}_0, \quad z, w \in \mathfrak{g}_1$, and
- (77): Complex Jacobi - $H(u,v)w + H(v,w)u + H(w,u)v = 0$.

When this happens we make $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into an ordinary Lie algebra using the imaginary part of H as the Lie bracket of two elements of $\mathfrak{g}_1$, and we make $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into a Lie superalgebra using the real part of H as the superbracket of two elements of $\mathfrak{g}_1$.

It is this relation between Lie algebras and Lie superalgebras that we use to fix the metric on the Higgs field regarded as sections of a bundle associated to $\mathfrak{g}_1$.

13.3 The unitary algebras.

Let $m = k + \ell$ be integers and let V_0 be an m-dimensional complex vector space endowed with a (pseudo) Hermitian form of signature (ℓ, k). For example we might take

$$V_0 = \mathbb{C}^{k,\ell}$$

be complex m space with the Hermitian form

$$\langle z, w\rangle = -\sum j = 1^k z_j \overline{w}_j + \sum_{j=k+1}^{m} z_j \overline{w}_j.$$

Let $c = a + b$ be integers and V_1 a c-dimensional vector space with a (pseudo) Hermitian form of signature (b, a). Put the direct sum Hermitian on $V = V_0 \oplus V_1$. Then

$$\mathfrak{g} = u(V),$$

the unitary algebra of V is an ordinary Lie algebra. Then we have the vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_0$ is the subalgebra

$$\mathfrak{g}_0 = u(V_0) \oplus u(V_1)$$

and $\mathfrak{g}_1$ can be identified with the complex vector space $\mathrm{Hom}_{\mathbb{C}}(V_1, V_0)$. (see [SW78] section 2). Then there is a structure of a Hermitian Lie algebra on $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ whose imaginary part gives $u(V_0 \oplus V_1)$.

The real part gives a class of Lie superalgebras which are called Hermitian superalgebras in [SS85]. They can be viewed as a real form of the complex Lie superalgebra $gl(V_0/V_1)$. If write the most general element of $gl(V_0/V_1) = \mathrm{End}(V)_0$ where $V = V_0 \oplus V_1$ in the block form as

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

then the condition to belong to our Hermitian superalgebra is that

$$A \in u(V_0) \quad \text{and} \qquad D \in u(V_1).$$

If we write the most general element of $\mathrm{End}(V)_1$ as

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

then the condition to belong to our superalgebra is

$$\langle Cv_0, v_1 \rangle_1 = i \langle v_0, Bv_1 \rangle_0 \qquad \forall\ v_0 \in V_0,\ v_1 \in V_1.$$

See [SS85] page 4.

13.4 $su(2, 2/1)$ and the superconformal superalgebra of Wess and Zumino.

The supersymmetry studied in this paper is purely internal and related to the chirality gradation as we have seen. So it is not of the “superspace” variety. Nevertheless we should point out that the superalgebra $su(2, 2/1)$ is nothing other than the superalgebra of Wess and Zumino [CNS75] and [GGRS83] where the odd part of the superalgebra is regarded as the “square root” of the conformal algebra of flat space time. We follow the presentation in [SW75] and [SS85].

Let $V_0 = \mathbb{C}^{2,2}$ be four dimenisonal complex space equipped with a Hermitian form of signature $(2,2)$. To fix the Ideas let us assume that the form is given by

$$\langle z, w \rangle = w^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} z = z_1\overline{w_3} + z_2\overline{w_4} + z_3\overline{w_1} + z_4\overline{w_2}$$

where I is the two by two identity matrix.

The condition that a four by four matrix A belongs to $u(V_0)$ is that

$$A \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A^\dagger.$$

If we break A up into blocks of two by two matrices we see that the condition is that A be of the form

$$\begin{pmatrix} X & P \\ Q & -X^\dagger \end{pmatrix}$$

where X is an arbitrary complex two by two matrix and where $P = -P^\dagger$ and $Q = -Q^\dagger$.

The fifteen dimensional algebra $su(2,2)$ is known to be isomorphic to the conformal algebra $o(2,4)$. Under the above description of the matrix A, the condition to belong to $su(2,2)$ is that $\mathrm{Im}\mathrm{tr}\, X = 0$. We can regard matrices of the form

$$\begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}, \quad P = -P^\dagger$$

as consisting of translations, and we may denote the set of all such matrices as $\mathfrak{g}^2$. We can regard the matrices of the form

$$\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}, \quad Q = -Q^\dagger$$

as consisting of those conformal vector fields whose expression is purely quadratic at a specified choice of origin and denote the set of such elements as $\mathfrak{g}^{-2}$. The set of elements of $su(2,2/1)$ of the form

$$\left(\begin{array}{cc|c} X & 0 & 0 \\ 0 & -X^\dagger & 0 \\ \hline 0 & 0 & 2i\mathrm{Im}\ \mathrm{tr}\ X \end{array}\right)$$

will be denoted by $\mathfrak{g}^0$. If we impose the additional condition that $\mathrm{tr}\, A = 0$ which is the same as $\mathrm{Im}\ \mathrm{tr}\, X = 0$ we get an element of $su(2,2)$ which acts as a linear conformal vector field on space time, i.e. as an infinitesimal Lorentz tranformation plus a scale transformation. The purely imaginary scalar matrices act trivially on space time but non-trivially on the odd part of the superalgebra which can be identified with the space of Dirac spinors.

The full algebra $su(2,2/1)$ consists of matrices of the form

$$\left(\begin{array}{cc|c} X & P & u \\ Q & -X^\dagger & v \\ \hline iv^\dagger & iu^\dagger & 2i\mathrm{Im}\,\mathrm{tr}\, J \end{array}\right), \quad P = -P^\dagger,\ Q = -Q^\dagger,\ \ u \in \mathbb{C}^2,\ \ v \in \mathbb{C}^2. \quad (78)$$

If $X \in sl(2,\mathbb{C})$ then

$$\left[\left(\begin{array}{cc|c} X & 0 & 0 \\ 0 & -X^\dagger & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & u \\ 0 & 0 & v \\ \hline iv^\dagger & iu^\dagger & 0 \end{array}\right)\right] = \left(\begin{array}{cc|c} 0 & 0 & Xu \\ 0 & 0 & -X^\dagger v \\ -iv^\dagger X & iu^\dagger X^\dagger & 0 \end{array}\right).$$

We see that u transforms as $u \mapsto Xu$ and v transforms as $v \mapsto -X^\dagger v$.

So we have a $\mathbb{Z}$ gradation more refined than the $\mathbb{Z}_2$ gradation:

$$\mathfrak{g}_0 = \mathfrak{g}^{-2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^2$$

and

$$\mathfrak{g}_1 = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$$

is identified with the right and left handed spinors. We refer to [SS85] for details.

14 Renormalization of the supergroup couplings and the Higgs mass.

For couplings given solely by the internal supergroup, i.e. by the quotient $su(2/1)/[su(2) \otimes u(1)]$, there is no known non-renormalization theorem. These couplings are θ_W and a, the coefficient of the quartic. In the sequel, we show that unitarity is preserved by appropriate BRST equations, so that we can apply the renormalization group (RG)equations to estimate the corrections. We follow a linearized treatment as an approximation [HLN96] .

In one case – the angle θ_W – we have the group value $(sin\theta_W)^2 = 0.25$ and may compare it to the experimentally observed value 0.229 ± 0.005. The supergroup prediction fits, but only very roughly. One therefore evaluates the energy level $q^2 = E_s$ at which the fit becomes precise, finding $E_s \sim 5$TeV. This may possibly be the level at which a larger symmetry structure breaks down, with SU(2/1) as the residual internal supersymmetry.

One can now invert the procedure, to estimate the renormalization effects for the Higgs potential quartic coefficient a. The supergroup value is assumed to hold at the energy $E_s = 5$TeV and one then evaluates the correction for a at $E \sim 100$GeV. This corrected value can then be used to reevaluate the predicted Higgs mass, i.e. obtain the value of that mass after the inclusion of renormalization effects.

The coefficients of the renormalization group equation depend only on the field contents of the theory, which is the same as in SU(2)×U(1). One can therefore apply the Standard Model calculation. For the gauge couplings, the renormalization group equations are given by [HLN96];

$$\frac{1}{[g_i(M)]^2} - \frac{1}{[g_i(M_0)]^2} + 2t_i \ln\frac{M}{M_0}\ , \qquad i = 1, 2, 3, \tag{79}$$

where

$$\begin{aligned}
t_1 &= \frac{1}{12\pi^2}(-\frac{5}{3}N_g - \frac{1}{8}), \\
t_2 &= \frac{1}{12\pi^2}(-N_g - \frac{1}{8} + \frac{11}{2}), \\
t_3 &= \frac{1}{16\pi^2}(-\frac{4}{3}N_g + 11),
\end{aligned}$$

N_g is the number of generations, and g_1, g_2, g_3 denote the gauge couplings of U(1), SU(2), SU(3), respectively.
For the (top-quark) Yukawa-Higgs coupling g_t and the quartic Higgs coupling a, RGE are given by [SZ86];

$$\frac{dg_t}{dM} = \frac{1}{16\pi^2 M}\left\{\frac{9}{2}g_t^3 - (\frac{17}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)g_t\right\}, \tag{80}$$

$$\begin{aligned}
\frac{da}{dM} &= \frac{1}{16\pi^2 M}\left\{24a^2 + 12ag_t^2 - 6g_t^4 - 3(g_1^2 + 3g_2^2)a\right. \\
&\qquad \left. + \frac{3}{8}\left[(g_1^2 + g_2^2)^2 + 2g_2^4\right]\right\},
\end{aligned} \tag{81}$$

These equations were solved numerically, setting the su(2/1) value of a as initial condition holding at $E_s = 5$TeV and taking $M_t = 174$GeV in The low energy range($E \sim 100$ to 200GeV). Assuming three generations ($N_g = 3$), with $\alpha_Q^{-1} = 128.80 \pm .05$, $\alpha_2^{-1} = 29.5 \pm .6$, $\alpha_3^{-1} = 8.332$, where $\alpha_i^{-1} = \frac{4\pi}{g_i^2}$ and $\frac{1}{g_Q^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2}$

In section 1.4 we discussed the mass of the Higgs field, as related to that of the W bosons gauging SU(2),

$$(M(\Phi))^2 = \frac{2a}{g^2}(M_W)^2 = 4(M_W)^2, \quad M(\Phi) = 2M_W \tag{82}$$

In solving the equations, the relation $g_t(M) = \frac{\sqrt{2}}{v}M_t = \frac{\sqrt{2}}{246}M_t$ was used, where $v =< 0|\Phi^0|0 >= 246GeV$. The outcome was a reduction of the predicted Higgs meson mass down to 130 ± 6GeV. Note that while there is at least one other theory predicting the Higgs mass - ordinary supersymmetry - su(2/1) is the only one that does not require the existence of a large number of new particles.

References

[AL73] E.S. Abers and B.W. Lee, “Gauge Theories”, *Physics Reports*, **9C** no. 1 1973

[B66] A. Berezin, *The Method of Second Quantization*, Academic Press, New York 1966

[Be75] Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, "Pseudo-particle solutions of Yang-Mills equations", *Phys. Lett.* **59B** 85 (1975).

[BGV91] N. Berline, E. Getzler, M. Vergne: *Heat Kernels and Dirac Operators*, Springer, Berlin 1991

[BRS75] C. Becchi, A. Rouet and R. Stora, "Renormalization of the Abelian Higgs-model", *Comm. Math. Phys.* **42** 127 (1975)

[BT82] L. Beaulieu and J. Thierry-Mieg, "The Principle of BRS Symmetry: An alternative approach to Yang-Mills theories", *Nucl. Phys.* **B197** 477 (1982)

[BT83] L. Beaulieu and J. Thierry-Mieg, "Covariant Quantization of Non-Abelian Antisymmetric Tensor Gauge Theories", *Nucl. Phys.* **B228** 259 (1983)

[C94] A. Connes, *Non-commutative geometry*, Academic Press, New York 1994

[Coq] R. Coquereaux, "Higgs fields and Superconnections", *Lecture Notes in Physics* **375**

[CL90] A. Connes and J. Lott, "Particle models and noncommutative geometry", *Nucl. Phy.* **18B** (Proc. Suppl.) 29 (1990)

[CNS75] L. Corwin, Y. Ne'eman, S. Sternberg, "Graded Lie algebras in mathematics and physics", *Rev. Mod. Phy.* **47** 573 (1975)

[CF75] G. Curci and R. Ferrari, "A Canonical and Lorentz-Covariant Quantization of Yang-Mills Theories" *Nuovo Cimento A* **30** 155-168 (1975)

[D92] A. Derdzinski, *Geometry of the standard model*, Springer, Berlin 1992

[DW67] B.S. DeWitt, "Quantum Theory of Gravity. II. The Manifestly Covariant Theory", *Phys. Rev.* **162** 1195 (1967)

[DiKi02] S. Dimopoulos and D.E. Kaplan, "The Weak Mixing angle from $SU(3)$ symmetry at a TEV", (2002) **hep-ph/0201148**
S. Dimopoulos, D.E. Kaplan and N. Weiner, "Electroweak Unification into a five dimensional $SU(3)$ at a TEV", (2002) **hep-ph/0202136**

[DJ79] P.H. Dondi and P.D. Jarvis, "A supersymmetric Weinberg-Salam model", *Physics Letters* **84B** Erratum **87B** 403 (1979)

[DJ80] P. Dondi and P.D. Jarvis, "Assignments in Strong, Electroweak unified models with an internal and space-time symmetry" (1980) *Zs.für Physik* **C4** 201.

[Ein21] A. Einstein, "On a reasonable extension of the basis of general relativity" *Preuss Akad.* 261 (1921)

[F78] D.B. Fairlie, "Two consistent calculations of the Weinberg angle", *J. Phys G.* **5** L55-L58 (1978)

[F79] D.B. Fairlie, "Higgs Fields and the Determination of the Weinberg Angle", *Phys. Lett.* **B82** 97 (1979)

[FM80] P. Forgacs and N. Manton, "Space-time symmetries in Gauge Theories", *Comm. Math. Phys.* **72** 15 (1980)

[FP67] L.D. Faddeev and V.N. Popov, "Feynman Diagrams for the Yang-Mills Field", *Phys. Lett.* **B25** 29 (1967)

[Fu98] Y. Fukuda et al., "Constraints on neutrino oscillation parameters from the measurement of day-night solar neutrino fluxes at Super-Kamionkande", *Phys. Rev. Lett.* **82** 1810 (1999)

[GGRS83] Gates, Grisaru, Rocek and Siegel, *Superspace; Frontiers in Physics* Lecture Notes Series **58** Benjamin Cummings 1983

[G78] W. Greub, *Multilinear algebra* Springer, Berlin (1978)

[GN63] H. Goldberg and Y. Ne'eman, "Baryon Charge and R-Inversion in the Octet Model", *Nuovo Cimento* **27** 1-5 (1963). Also pub. as Israel AEC Report 1A-725 (February 1962)

[GL71] T.A. Golfand and E.P. Likhtman, "Extension of the algebras of Poincaré group generators P-invariance", *J.E.T.P. Lett.* **13** 322-326 (1971)

[Hi64] P.W. Higgs, "Broken Symmetries, Massless Particles and Gauge Fields", *Phys. Lett.* **12** 132 (1964)

[HMMN95] F.W. Hehl, J. Dermott McCrea, E.W. Mielke and Y. Ne'eman, "Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance", in *Physics Reports (Elsevier)* **258** 1-171 (1995)

[H77] R. Howe, "Remarks on classical invariant theory", preprint 1977

[HLN96] D.S. Hwang, C.Y. Lee and Y. Ne'eman, "BRST Quantizatin of SU(2/1) Electroweak Theory in the Superconnection Approach and the Higgs Meson Mass" *Intern. Jour. Mod. Phys.* **A11** 3509-3522 (1996)

[IO96] B. Iochum and T. Schucker, "Yang-Mills-Higgs versus Connes Lott", *Comm. Math. Phys.* **178** 1-26 (1996)

[Kac77] V. Kac, "Lie superalgebras", *Advances in mathematics*, **26** 8 (1977)

[Kane93] G. Kane, *Modern elementary particle physics*, Perseus Publishing, Cambridge, Mass. 1993

[KS93] D. Kastler and T. Schucker, "A detailed account of Alain Connes' version of the Standard Model I and II", *Rev. Math. Phys.* **5** 477 (1993)

[KS96] D. Kastler and T. Schucker, "A detailed account of Alain Connes' version of the Standard Model III and IV", *Rev. Math. Phys.* **8** 103, 205-228 (1996)

[Li73] H.J. Lipkin, "Quarks for Pedestrians", *Phys. Reports* **8C** 175-263 (1973)

[LN90] C.-Y. Lee and Y. Ne'eman, "Renormalization of Gauge-Affine Gravity", *Physics Letters* **B242** 59-63 (1990)

[Lo27a] F. London, "Die Theorie von Weyl und die Quantenmechanik", *Naturwiss.* **15** 187.

[Lo27b] F. London, "Quantenmechanische Deutung der Theorie von Weyl," *Zeit. für Physik* **42**, 375-389 (1927)

[Loh78] M.A. Lohe, "Magnetic Monopoles and Bäcklund Transformations", *Nucl. Phys.* **B 142** 36 (1978).

[Mai91] L. Maiani, in *Proceedings of the NATO Advanced Study Institute on Z^0 Physics*, Cargese 1990, ed. M. Levy *et al.* (Plenum Press, New York, 1991).

[MaSa2000] L. Mangiarotti and G. Sardanashvily, *Connections in Classical and Quantum Filed Theory*, World Scientific, Singapore (2000)

[MQ86] V. Mathai, D. Quillen, "Superconnections, Thom Classes, and Equivariant Differential Forms", *Topology* **25**, 85 (1986)

[N61] Y. Ne'eman, "Derivation of Strong Interactions from a Gauge Invariance", *Nucl. Phys.* **26** (1961) p. 222-229. Reprinted in *The Eightfold Way*, p. 58 (in coll. with M. Gell-Mann) "Frontiers in Physics" series, W. A. Benjamin, New York, 1964

[N64] Y. Ne'eman, *"Algebraic Theory of Particle Physics"*, Frontiers in Physics Series, W. A. Benjamin, New York, 1964, 334 pp.

[N79a] Y. Ne'eman, "Ghost Fields, BRS and Extended Supergravity as Applications of Gauge Geometry", Proc. XIX Intern. Conf. on High Energy Physics (Tokyo 1978) Phys. Soc. Japan, 552 (1979)

[N79b] Y. Ne'eman, "Irreducible Gauge Theory of a Consolidated Salam-Weinberg Model", *Phys. Lett.* **B81**, 190 (1979)

[N80] Y. Ne'eman, "Gauge Theory Ghosts and Unitarity - Progress in the Geometrical Treatment", in *Group Theoretical Methods in Physics*, Proc. Cocoyoc (Mexico 1980), *Springer Verlag Lecture Notes in Physics* **15** 594 (1980)

[N86] Y. Ne'eman, "Internal supergroup prediction for the Goldstone-Higgs particle mass", *Physics Letters B* **181**, 308 (1986)

[N95] Y. Ne'eman, "Internal Supersymmetry, Superconnections and NCG", in *Group Theory and its Applications*, Proc. XXXth ELAF (Latin-American School of Physics, Mexico 1995) AIP Press no. **365** 311 (1996)

[N98] Y. Ne'eman, "A Superconnection for Riemannian Gravity as Spontaneously Broken SL(4,R) Gauge Theory", *Physics Letters* **B427** (1998) 19-25.

[N04] Y. Ne'eman, "Mathematics, Physics and Ping Pong", Award speech for the proceedings of the Marcel Grossman 10th, held in Rio de Janeiro, Brazil. 15 July 2003.

[NRT78] Y. Ne'eman, T. Regge and J. Thierry-Mieg, "Models of Extended Supergravity as Gauge Theories on Group Manifolds", Abstract for XIX Intern. Conf. on High Energy Physics, (Tokyo, 1978).

[NS80] Y. Ne'eman, S. Sternberg, "Internal supersymmetry and unification", *Proc. Nat. Acad. Sci. USA* **77**, 3127 (1980)

[NS82] Y. Ne'eman, S. Sternberg, "Internal supersymmetry and dimensional reduction", in *Gauge theories: fundamental ineractions and rigorous results*, eds. P. Dita, V. Georgescu, and R. Purice, 103 Birkhauser, Boston 1982

[NS88] Y. Ne'eman and Dj. Šijački, "Gravity from Symmetry Breakdown of a Gauge Affine Theory", *Phys. Lett.* **B200** 489-494 (1988)

[NS90] Y. Ne'eman, S. Sternberg, "Superconnections and internal supersymmetry dynamics", *Proc. Nat. Acad. Sci. USA* **87** 7875 (1990)

[NS91] Y. Ne'eman, S. Sternberg, "Internal Supersymmetry and Superconnections", in *Symplectic Geometry and Mathematical Physics*, (Proc. Intern. Conf. Diff. Geom. in Physics honoring J.M. Souriau, Marseilles, 1990) 326 Birrkhauser, Boston 1991

[NT80a] Y. Ne'eman and J. Thierry-Mieg, "Geometrical Gauge Theory of Ghost and Goldstone Fields and of Ghost Symmetries", *Proc. Nat. Acad. Sci., USA* **77** 720-723 (1980)

[NT80b] Y. Ne'eman and J. Thierry-Mieg, "Geometrical Theory of Ghost and Higgs Fields and $SU(2/1)$", *Proc. VIII Intern. Conf. on Group Theoretical Methods in Physics*, (Kiryat Anavim 1979) Annals of Israel Phys. Soc. **3** 100-116 (1980)

[Po76] A.M. Polyakov, "Particle spectrum in quantum field theory", *J.E.T.P. Lett.* **20** 194 (1974)

[Q85] D. Quillen, “Superconnections and the Chern character”, *Topology* **24** 89 (1985)

[R98] G. Roepstorff, “Superconnections and the Higgs Field”, arXiv:hep-th/9801040 v2 9 Apr 1998

[RS99a] L. Randall and R. Sundrum, “A Large Mass Hierarchy from a small extra Dimension”, *Phys. Rev. Letters* **83** 3370 (1999)

[RS99b] L. Randall and R. Sundrum, “An Alternative to Compactification”, *Phys. Rev. Letters* **83** 4690 (1999)

[Sa68] A. Salam, in *Elementary Particle Theory*, N. Svartholm, ed. Almquist Verlag A.B., Stockholm (1968)

[Scheck97] F. Scheck, The Standrd Model within Non-commutative Geometry: a comparison of models hep-th/9701073

[Sch79] M. Scheunert, *The theory of Lie superalgebras*, Lecture Notes in Mathematics **716** Springer, Berlin (1979)

[SN90] S. Sternberg and Y. Ne'eman, “Superconnections and Internal Supersymmetry Dynamics”, *Proc. Nat. Acad. Sci. USA* **87** 7875-7877 (1990)

[SS85] O. Sanchez-Valenzula, S. Sternberg, “The automorphism group of a Hermitian superalgebra”, *Lecture notes in Mathematics* **1251** 1-48 Springer, Berlin 1985

[SNR77] M. Scheunert, W. Nahm & V. Rittenberg, “Irreducible Representations of the osp(2/1) and sp1(2/1) graded Lie algebras”, *J. Math. Phys.* **18** 156-167 (1977)

[SW75] S. Sternberg and J. Wolf, “Charge conjugation and Segal's cosmology”, *Il Nuovo Cimento* **28** 253 (1975)

[SW78] S. Sternberg, J. Wolf, “Hermitian Lie algebras and metaplectic representations”, *Trans. Amer. Math. Soc.* **231** 1 (1978)

[SZ86] A. Sirlin and R. Zucchini, “Dependence of the Higgs Coupling $\overline{h}_{\overline{MS}}(M)$ on m_H and the Possible onset of New Physics”, *Nucl. Phys.* **B266** 389 (1986)

[T79] J.G. Taylor, “Gauging su(n/m)”, *Physics Letters* **84B** 79 (1979)

[TM78] J. Thierry-Mieg, These de Doctorat d'Etat (U. de Paris Sud, 1978)

[TM80] J. Thierry-Mieg, “Geometrical reinterpretation of Faddeev-Popov particles and BRS transformations”, *J. Math. Phys.* **21** 2834 (1980)

[TM85] J. Thierry-Mieg, "Classical Geometric Interpretation of Ghost Fields and Anomalies in Yang-Mills Theory and Quantum Gravity" in *Anomalies*, W.A. Bardeen & A.R. White, eds., 239-246, World Scientific, Singapore 1985

[TN80] J. Thierry-Mieg and Y. Ne'eman, "Gauge Asthenodynamics ($SU2/1$) (classical discussion)", *Methods in Mathematical Physics* (Proc. Aix en Provence and Salamanca 1979) *Lecture Notes in Mathematics* **836** 318-348 Springer, Berlin 1980

[TMN82] J. Thierry-Mieg and Y. Ne'eman, "Exterior Gauging of an Internal Supersymmetry and $SU(2/1)$ Quantum Asthenodynamics", in *Proc. of Nat. Acad. of Sciences, USA* **79** (1982) p. 7068-7072.

[TMN82a] J. Thierry-Mieg and Y. Ne'eman, "BRS Algebra of the SU(2/1) Electroweak Gauge Theory", *Il Nuovo Cimento* **71A** (1982) p. 104-118.

['tH79] G. 't Hooft, "Magnetic Monopoles in Unified Gauge Theories", *Nucl. Phys.* **B79** 276 (1974)

[Ty75] I.V. Tyutin, Lebedev Institute preprint (1975)

[Wei67] S. Weinberg, "A Model of Leptons", *Phys. Rev. Lett.* **19** (1967) 1264.

[Wei71] S. Weinberg, "Physical Processes in a Convergent Theory of the Weak and Electromagnetic Interactions", *Phys. Rev. Lett.* **27** 1688 (1971)

[Wei75] S. Weinberg, "The U(1) problem", *Phys. Rev.* **D11** (1975) 3583.

[Wei96] S. Weinberg, *The Quantum Theory of Fields*, Cambridge U. Press (1996), vol. 2.

[Wey18] H. Weyl, *Raum Zeit Materie*, Springer, Berlin (1918)

[YM54] C.N. Yang and R. Mills, "Conservation of isotopic spin and isospin gauge invariance," *Phys. Rev.* **96** 191-195 (1954)

[WY75] T.T. Wu and C.N. Yang, "Concept of nonintegrable phase factors and global formulation of gauge fields", *Phys. Rev.* **D12** 3845 (1975)