SUPERSYMMETRY

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1. Introduction

The purpose of these notes is to give a short and (overly?)simple description of supersymmetry for Mathematicians. Our description is far from complete and should be thought of as a first pass at the ideas that arise from supersymmetry. Fundamental to supersymmetry is the mathematics of Clifford algebras and spin groups. We will describe the mathematical results we are using but we refer the reader to the references for proofs. In particular [4], [1], and [5] all cover spinors nicely.

2. Spin and Clifford Algebras

We will first review the definition of spin, spinors, and Clifford algebras. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with some nondegenerate quadratic form. The clifford algebra of V, $\mathcal{C}l(V)$, is the algebra generated by V and 1, subject to the relations $v \cdot v = \langle v, v\rangle \cdot 1$, or equivalently $v \cdot w + w \cdot v = 2\langle v, w\rangle$. Note that elements of $\mathcal{C}l(V)$ can be written as polynomials in V and this gives a splitting $\mathcal{C}l(V) = \mathcal{C}l(V)^0 \oplus \mathcal{C}l(V)^1$. Here $\mathcal{C}l(V)^0$ is the set of elements of $\mathcal{C}l(V)$ which can be written as a linear combination of products of even numbers of vectors from V, and $\mathcal{C}l(V)^1$ is the set of elements which can be written as a linear combination of products of odd numbers of vectors from V. Note that more succinctly $\mathcal{C}l(V)$ is just the quotient of the tensor algebra of V by the ideal generated by $v \otimes v - \langle v, v\rangle \cdot 1$.

If V is a $\mathbb{C}$ vector space then all nondegenerate quadratic forms are equivalent, so the clifford algebras are isomorphic, and in fact one finds that $\mathcal{C}l(\mathbb{C}^{2n}) = \mathbb{C}[2^n]$, (the algebra by 2^n by 2^n complex matrices), and $\mathcal{C}l(\mathbb{C}^{2n+1}) = \mathbb{C}[2^n] \oplus \mathbb{C}[2^n]$. Over $\mathbb{R}$ things are more complicated. One still gets matrix algebras or direct sums of matrix algebras, but depending on the signature these matrices will be over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

From our Clifford algebra we can construct a group. We define $\mathrm{Spin}(V)$ to be the group generated by $\{v_1 \dots v_{2l} | \langle v_i, v_i\rangle = \pm 1, l \in \mathbb{N}\}$. This is in fact a Lie group. Now let us denote by $SO(n)$ the special orthogonal group, and $SO(1, n-1)$ the special lorentz group, i.e. the group of determinant 1 linear transformations preserving the quadratic form $\langle x, x\rangle = x_1^2 - x_2^2 - \dots - x_{n-1}^2$. Also denote $\mathrm{Spin}(n)$ and $\mathrm{Spin}(1, n-1)$ the Spin groups associated to the clifford algebra on euclidean space and on a Minkowski space, i.e., on $\mathbb{R}^n$ with the form of signature $1, n-1$ described in the previous line. Then one finds there are maps $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ and $\pi : \mathrm{Spin}(1, n-1)^\circ \to \mathrm{SO}(1, n-1)^\circ$, where $\cdot^\circ$ denotes the connected component of the identity. These maps are both surjective and $2:1$. One can also show that Spin(n) and $\mathrm{Spin}(1, n-1)^\circ$ are simply connected. Thus $\mathrm{Spin}(n)$ and $\mathrm{Spin}(1, n-1)^\circ$ are the universal covering spaces of $\mathrm{SO}(n)$ and $\mathrm{SO}(1, n-1)^\circ$ which we see have fundamental group $\mathbb{Z}/2\mathbb{Z}$. The map π is basically the map that associates to $v_1 \dots v_{2l}$ the linear transformation that acts on a vector by succesively reflecting it over the hyperplane orthogonal to v_i.

Let us note that $\mathrm{Spin}(4) = \mathrm{SU}(2)\times\mathrm{SU}(2)$, $\mathrm{Spin}(1,3)^\circ = \mathrm{SL}(2,\mathbb{C})_\mathbb{R}$ and $\mathrm{Spin}(4,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$, where $\mathrm{Spin}(n,\mathbb{C})$ is the spin group of $\mathbb{C}^n$ with its standard quadratic form (or any quadratic form as they are all equivalent). We write $\mathrm{SL}(2,\mathbb{C})_\mathbb{R}$ to emphasize that we are considering $\mathrm{SL}(2,\mathbb{C})$ as a real Lie group.

3. Spin(1,3)

Let $g = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ be the lorentz inner product on $\mathbb{M}^4$, four dimensional Minkowski Space. The Lorentz group $SO(1,3)$ is the group of determinant 1 linear transformations preserving this inner product, i.e., $SO(1,3) = \{A : A \in GL(4,\mathbb{R}),\ |A| = 1,\ g(Av_1, Av_2) = g(v_1, v_2) \text{ for all v} \in V\}$. The lorentz group is not connected and the connected component containing the identity is called the proper Lorentz group. From now on whenever we say Lorentz group we mean the proper Lorentz group. We wish to gain a more concrete understanding the Lorentz group. First let us define the Pauli spin matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and the gamma matrices,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix}.$$

If also $g_{ij} = \mathrm{diag}(1,-1,-1,-1)$, the diagonal matrix with entries $1,-1,-,1-1$, so $g = g_{ij}dx^i dx^j$, then one finds $\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij}I$. Thus under the identification $e_i \mapsto \gamma^i$ the gamma matrices generate $\mathcal{C}l(\mathbb{M}^4)$ over $\mathbb{R}$ and $\mathcal{C}l(\mathbb{C}^4)$ over $\mathbb{C}$. Now if e_i is the standard basis of $\mathbb{M}^4$, then to any vector $x^i e_i$, associate the matrix

$$\sum_i x^i\sigma^i = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}.$$

Note that $\sum_i x^i\sigma^i$ is a Hermitian matrix, and $\det\left(\sum_i x^i\sigma^i\right) = x_0^2 - x_1^2 - x_2^2 - x_3^2$. Thus the norm of a vector is the determinant of its associated matrix. Now consider $A \in SL(2,\mathbb{C})$, where $SL(2,\mathbb{C})$ is complex 2 by 2 matrices of determinant 1. Note that if X is a hermitian matrix, then the transformation $X \mapsto AXA^\dagger$, where $\dagger$ denotes conjugate transpose, preserves the hermicity and determinant so is a lorentz transformation. This linear transformation lies in the proper lorentz group as $SL(2,\mathbb{C})$ is connected. Thus we have a map from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SO}(1,3)$. It is in fact surjective homomorphism with kernel ± 1 and since $\mathrm{SL}(2,\mathbb{C})$ is simply connected we see that $\mathrm{SL}(2,\mathbb{C})$ is $\mathrm{Spin}(1,3)^\circ$. In fact one finds using the gamma matrix description of the Clifford algebra that

$$\mathrm{Spin}(4,\mathbb{C}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Big| A, B \in \mathrm{SL}(2,\mathbb{C}) \right\} = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}),$$

and

$$\mathrm{Spin}(1,3) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^\dagger \end{pmatrix} \Big| A \in \mathrm{SL}(2,\mathbb{C}) \right\} = \mathrm{SL}(2,\mathbb{C})_\mathbb{R},$$

4. Spin Representations

We will be interested in representations of Spin(V) that in some sense factor through the clifford algebra. First note as stated above, over $\mathbb{C}$, $\mathcal{C}l(\mathbb{C}^{2n})$ is the algebra of 2^n by 2^n complex matrices $\mathbb{C}[2^n]$, so has a unique irreducible representation. This is just $\mathbb{C}^{2^n}$. Similarly $\mathcal{C}l(\mathbb{C}^{2n+1}) = \mathbb{C}[2^n] \oplus \mathbb{C}[2^n]$ so has two irreducible representations, the standard ones on each factor, whose underlying vector space is $\mathbb{C}^{2^n}$.

In the even dimensional case if we restrict the unique irreducible representation of $\mathcal{C}l(\mathbb{C}^{2n})$, (which is just $\mathbb{C}^{2^n}$) to $\mathrm{Spin}(n, \mathbb{C})$ or $\mathcal{C}l(\mathbb{C}^n)^0$ then $\mathbb{C}^{2^n}$ it splits into a direct sum of two irreducible representations. Let us denote $\mathbb{C}^{2^n}$ as a representation by $S_{\mathbb{C}}$, call these two irreducible pieces S^+ and S^-. The reason we have this splitting is that if $\omega_{\mathbb{C}} = Ae_1 \dots e_n$ where A is a constant depending on the dimension, and e_i an orthonormal basis, then $\omega_{\mathbb{C}}^2 = 1$ and S^+ is the plus eigenspace of $\omega_{\mathbb{C}}$, and S^- is the minus eigenspace. In the odd case we have the opposite situation, upon restricting the two representations of $\mathcal{C}l(\mathbb{C}^{2n+1})$ to $\mathcal{C}l(\mathbb{C}^{2n+1})^0$ or $\mathrm{Spin}(n, \mathbb{C})$ they become isomorphic, we also call this representations $S_{\mathbb{C}}$.

Thus $\mathrm{Spin}(2n, \mathbb{C})$ has two natural irreducible representations $S_{\mathbb{C}} = S^+ \oplus S^-$ that are the restriction of a representation of $\mathcal{C}l(2n)^0$ and $\mathrm{Spin}(2n+1)$ has one natural irreducible representation $S_{\mathbb{C}}$ that is the restriction of a representation of $\mathcal{C}l(2n+1)^0$. These are often called the spin representations. Note that elements of S^+ and S^- are often called half spin, right and left handed, right chiral and left chiral, or chiral and anti-chiral. More generally we call representations that are restrictions of representations of $\mathcal{C}l(m)^0$ spinorial.

Over $\mathbb{R}$ things are more complicated but a representation of a real spin group is spinorial if it essentially becomes spinorial after complexification (to make this precise we really need to work at the level of lie algebras and exponentiate). We won't be precise since we will only use one example. First note that for Spin(4,$\mathbb{C}$)=SL(2,$\mathbb{C}$)$\times$SL(2,$\mathbb{C}$), its spinor space is $S = \mathbb{C}^4 = S^+ \oplus S^-$ for $S^{\pm} = \mathbb{C}^2$ with the representation $(A, B) \cdot (v, w) = (Av, Bw)$. Consider the following complex representations of Spin(1,3)=SL$(2, \mathbb{C})$ viewed as a real lie group: $A \cdot v = Av$ and $A \cdot v = (A^{-1})^{\dagger} v$. Also denote these by S^+ and S^-. Note that viewed as real representations S^+ and S^- are isomorphic, while as complex representations they are complex conjugate. Thus there is a conjugate linear map of representations from S^+ to S^- and so we can think of them as conjugate vector spaces. We denote this map (which for us is not just complex conjugation) by $f \mapsto \bar{f}$ for $f \in S^+$. We denote the underlying real reprsentation of S^+ and S^- by S and then we have that the complexification of S, $S_{\mathbb{C}} = S^+ \oplus S^-$. The basic reason why these are spinorial is the fact that Spin(1,3) is embedded in Spin(4,$\mathbb{C}$) as $(A, (A^{-1})^{\dagger})$ and so these representations naturally come from $\mathrm{Spin}(4, \mathbb{C})$.

Physicists call elements of a full spin represntation $S_{\mathbb{C}}$, Dirac spinors. Elements of a half spin representations are called Weyl spinors. Elements of a real form of a complex spin representation are called Majorana spinors. For example for Spin(1,3), the Dirac spinors $S_{\mathbb{C}} = S^+ \oplus S^-$ are a spinorial representation, and the real subspace $S = (z, \bar{z})$ is the Majorana spinors.

5. Fermions

For all of last quarter we were only concerned with what are called bosonic fields. Essentially on a quantum level we describe bosonic fields using operators on a Hilbert space satisfying certain commutation relations. Fermions are particles described by fields that at the quantum level satisfy anti-commutation relations. Most particles in nature are fermionic, for examples electronic, protons, quarks, basically most things that you would think of as matter are fermions. In order to get to a quantum Theory we need something classical that naturally quantizes into anti-commuting operators. If we try and do this using standard "commutative" objects we are doomed to fail, instead we need geometric objects which anti-commute, i.e. we should work with supergeometry.

Recall that a (p, q) dimensional supermanifold M is a smooth manifold with a sheaf of rings $\mathcal{O}_M$ locally isomorphic to the sheaf on $\mathbb{R}^p$ which takes an open set U to $C^\infty(U) \otimes \bigwedge[\xi, \ldots, \xi_q]$ where $\bigwedge[\xi_1, \ldots, \xi_q]$ is the algebra generated by 1 and ξ_i subject to $\xi_i \xi_j = -\xi_j \xi_i$. Bundles, tensors, etc, on supermanifolds are described as locally free sheaves over $\mathcal{O}_M$.

6. Supersymmetry Abstractly

We would like to construct a theory that has a symmetry that interchanges bosonic and fermionic fields. Let V be a lorentzian space of some dimension, i.e. a vector space with a quadratic form of signature $(1, n-1)$. Let S be real spinorial representation for $\mathrm{Spin}(V)$. By general theory there will exist a nonzero equivariant symmetric map $\Gamma : S \otimes S \to V$ (we will describe it explicitly in the specific example we do). If Q_a is a basis for S, and P_μ a basis for V, then we can define Γ^μ_{ab} by $\Gamma(Q_a, Q_b) = \Gamma^\mu_{ab} P_\mu$. We define super Minkowski space to be the super vector space $\mathbb{M} = V \times \Pi S$, where Π denotes parity change. Thus the spinor directions become fermionic. We give it the structure of a super Lie algebra by defining brackets between two elements of V to be 0, brackets between elements of V and S to be 0, and defining $[s_1, s_2] = -2\Gamma(s_1, s_2)$. Note that viewing V as the lie algebra to the group of spatial translations, the fermionic directions bracket to translations. Let $\mathfrak{p} = V \times \Pi S \rtimes \mathfrak{spin}(V)$, this is the super poincare algebra. We define the brackets in $\mathfrak{p}$ using the action of $\mathfrak{spin}(V)$ and Γ as above. We wish exponentiate M into a Lie group. There is a abstract, general way to exponentiate a super Lie algebra into a super Lie group but let us just define it explicitly. $\mathbb{M}$ as a Lie group is the same underlying space but with the group operation

$$(x, \theta)(x', \theta') = (x^a + x'^a + \Gamma^\mu_{ab} \theta^a \theta'^b, \theta^a + \theta'^b).$$

It has right and left invariant vector fields

$$\tau_{Q_a} = \frac{\partial}{\partial \theta^a} + \theta^b \Gamma^\mu_{ab} \partial_\mu$$

$$D_a = \frac{\partial}{\partial \theta^a} - \theta^b \Gamma^\mu_{ab} \partial_\mu.$$

These satisfy $[D_a, D_b] = -2\Gamma^\mu_{ab}\partial_\mu$, $[\tau_{Q_a}, \tau_{Q_b}] = 2\Gamma^\mu_{ab}\partial_\mu$ and $[D_a, \tau_{Q_b}] = 0$.

Note that $\mathbb{M}$ as a Lie algebra is the Lie algebra of $\mathbb{M}$ as a lie group. We wish to construct a theory has $\mathbb{M}$ as its symmetry algebra. Let us be explicit about what we mean by this. Consider a super function $f(x, \theta) = \phi(x) + \psi_i(x)\theta^i + F_{ij}\theta^i\theta^j$. Elements of $\mathbb{M}$ as a Lie algebra should act on f by some infinitesimal transformation. Before we describe this we note that a useful trick in doing computations in superalgebra is to introduce formal odd parameters to make all objects even and then factor these out at the end. For example for η^a odd paramters, $\eta^a Q_a$ is an even element of the super Poincare algebra. We will instead identify $\eta^a Q_a$ with an infinitesimal symmetry. To see what we need first note that

$$[\eta^{1a}Q_a, \eta^{2b}Q_b] = -\eta^{1a}\eta^{2b}[Q_a, Q_b] = 2\eta^{1a}\eta^{2b}\Gamma^{\mu}_{ab}P_{\mu}.$$

Thus identifying P_μ with translations which should act by derivatives, we want to associate $\eta^a Q_a \mapsto \hat{\xi}$ so that

$$[\hat{\xi}, \hat{\xi}]f = 2\eta^{1a}\eta^{2b}\Gamma^{\mu}_{ab}\partial_\mu f.$$

The natural way to do this is to use the invariant vector fields on $\mathbb{M}$ viewed as a group, thus we want formally

$$\hat{\xi}f = \frac{d}{dt}\Big|_{t=0} \exp(-t\eta^a \tau_{Q_a})^* f = -\eta^a \tau_{Q_a} f.$$

We define the symmetry associated to $\eta^a Q_a$ by the right hand side and we see that it satisfy the correct bracket relations.

Suppose we wish to construct a supersymmetric theory on $\mathbb{M}^1$. The supersymmetry algebra here is just $\{Q, Q\} = P$. Consider $\mathbb{M}^{1|1}$ with coordinates (t, θ). Here $D = \partial_\theta - t\partial_t$, $\tau_Q = \partial_\theta + \theta\partial_t$. We will look at the sigma model of maps $\Phi : M^{1|1} \to \mathbb{R}$. We can describe such a map by the pull back of the coordinate function on $\mathbb{R}$, unfortunately $\Phi^*(x)$ must be even and so can't involve θ.

We immediately have a minor problem. In general we will want to look at maps from some superspace say $\mathbb{R}^{p|q}$ into a differentiable manifold M. Unfortunately, viewed as a supermanifold, a differentiable manifold M is purely even, so all functions on M pull back to even functions and we don't get any anticommutativity. What is the problem. Philosophically the resolution is the following, let $\mathcal{F}$ formally denote the space of all fields in our theory. For the moment lets think of it as some sort of infinite dimensional supermanifold. A purely even manifold is totally determined by its $\mathbb{R}^{0|0}$ points, i.e., maps from points into it which give just the underlying space. But for even a finite dimensional supermanifold to see the odd directions we have to probe it with objects having odd directions, i.e. maps from supermanifolds such as $R^{p|q}$ into it. Thus we should consider for all supermanifolds B, the B points of $\mathcal{F}$. Formally a B point of $\mathcal{F}$ is a map from B to $\mathcal{F}$, which should be viewed as a family of fields parameterized by B. Thus in the baby example above we need to consider for all B maps from $\Phi : B \times \mathbb{M}^{1|1} \to \mathbb{R}$. In this case $\Phi^* x = x(t) + \theta\psi(t)$ where $x(t)$ is some even function on $\mathbb{M} \times B$ and $\psi(t)$ is some odd function on $\mathbb{M} \times B$. Now we are able to have both even and odd components. Everything we do should be functorial in B and never depend explicitly on a specific B thus we will not mention it from now on, but it is implicitly there. Let us note that in general what we do is define $\mathcal{F}$, the space of maps from $\mathbb{R}^{p|q}$ to M, as the "geometric" object that represents the functor $B \to \mathrm{Hom}(\mathbb{R}^{p|q} \times B, M)$.

Continuing on with the example let η be an odd parameter, then for $\hat{\xi} = \eta\tau_Q$ we have

$$\hat{\xi}\Phi := \hat{\xi}\Phi^* x = -\eta(\partial_\theta + \theta\partial_t)(x(t) + \theta\psi(t)) = -\eta\theta\dot{x}(t) - \eta\psi(t).$$

Thus we find $\delta x = -\eta\psi(t), \delta\psi = \eta\dot{x}(t)$. Note that we have an additional minus sign as we view $\hat{\xi}$ as the infinitesmial transformation coming from $\exp -\eta\tau_Q$. It is easy to check that for $\hat{\xi}_i = \eta^i \tau_Q$ that $[\hat{\xi_1}, \hat{\xi_2}]x(t) = 2\eta^1\eta^2\dot{x}(t)$ and $[\hat{\xi_1}, \hat{\xi_2}] = 2\eta^1\eta^2\partial_t\psi(t)$ so this realizes the supersymmetry algebra. Consider the Lagrangian

$$L = (1/2)(\dot{x})^2 + (1/2)\psi\dot{\psi}.$$

Note that

$$\hat{\xi}L = -\eta\dot{\psi}\dot{x} + \frac{\eta}{2}\dot{x}\dot{\psi} - (1/2)\eta\psi\ddot{x}(t)$$
$$= -\eta\dot{\psi}\dot{x} + \eta\dot{x}\dot{\psi} + \text{exact} = \text{exact},$$

so this actually defines a symmetry.

7. A more complicated Example

Consider the sigma matrices defined earlier. Let us introduce some notation we shall need. We will be working with both S and $S_{\mathbb{C}} = S^+ \oplus S^-$. We wish to use indices without dots for S^+ and indices with dots for components of S^-. We also use the convention that if the same index occurs raised and lowered, once with and once without a dot, that we sum over that index. Recall the matrices σ^μ defined earlier, we write them in terms of indices as $\sigma^\mu_{a\dot{b}}$. Choose a basis Q_a for S^+ and the corresponding basis $\bar{Q}_{\dot{a}}$ for S^- (As explained earlier we are viewing S^+ and S^- as conjugate vector spaces). We can define the map $\Gamma : S \otimes S \to \mathbb{M}^4$ by $\Gamma(x^a Q_a, y^b Q_b) = \frac{1}{2}\mathrm{Re}(y^b \sigma^\mu_{b\dot{a}} \bar{x}^a) P_\mu$ where Re denotes the part invariant under conjugate transpose. Note in the above we identify S with the underlying real vector space of S^+. This map takes values in the set of hermitian matrices which we saw could be naturally identified with $\mathbb{M}^4$ earlier. It is a morphism of representations because as we saw the action of a matrix in $\mathrm{SL}(2,\mathbb{C})$ as a Lorentz transformation is $AMA^\dagger$. As we stated earlier we denote S the underlying real representation of $S^+ \cong S^-$. We also view S as the real subspace of $S_{\mathbb{C}}$ invariant under the map $(u,v) \mapsto (\bar{v},\bar{u})$, where again $\bar{\cdot}$ denotes the conjugate linear map identifing S^+ and S^- as conjugate representations. Define $\bar{\sigma}^\mu = (\sigma^0, -\sigma^{i\dot{a}b})$, where $\sigma^{\mu\dot{c}d} = \sigma^\mu_{a\dot{b}}\epsilon^{da}\epsilon^{\dot{c}\dot{b}}$ for ϵ antisymmetric. The important consequence of this definition is the fact that $\sigma^\mu_{a\dot{b}}\bar{\sigma}^{\nu\dot{b}c} = \delta_a{}^c g^{\mu\nu}$ as well as $(\sigma^{\mu\dot{c}d})^\dagger = \sigma^{\mu\dot{d}c}$. We will simply give a description of this theory in components although a supergeometry formulation exists. Our fields will be an even real valued function $\phi(x)$ and an odd spinor $\psi_a = (\psi_1,\psi_2)$ taking values in ΠS. The Lie algebra corresponding to super Minkowski space is $\mathbb{M}^4 \times \Pi S$. As we said we also view ΠS as a real subspace of $S_{\mathbb{C}}$. Super Minkowski space is $\mathbb{M}^4 \times \Pi S$. The Lie bracket is defined using Γ written above as $[\mathrm{Re}(\eta^{1a}Q_a), \mathrm{Re}(\eta^{2b}Q_b)] = \mathrm{Re}(\bar{\eta}_1^{\dot{b}}\sigma^\mu_{a\dot{b}}\eta_2^a)P_\mu$. So we need to associate an infinitesimal symmetry to $\hat{\xi} = (\eta^a Q_a - \bar{Q}_{\dot{a}}\bar{\eta}^{\dot{a}}) = 2\cdot\mathrm{Re}(\eta^a Q_a)$ satisfying

$$[\hat{\xi}_1, \hat{\xi}_2] = 4\cdot\mathrm{Re}(\bar{\eta}_1^{\dot{b}}\sigma^\mu_{a\dot{b}}\eta_2^a)\partial_\mu.$$

We use $\bar{\eta}^{\dot{a}} = \overline{\eta^a}$. First we need an action for our theory. Let us use

$$S(\phi,\psi_a) = \int g^{\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi^*\right) - \psi_b^*\sigma^{\mu b\dot{a}}\partial_\mu\psi_a \; d^4x,$$

where here $\cdot^*$ denotes regular complex conjugation of the components. We postulate the following action of $\hat{\xi}$, $\hat{\xi}\phi = -\sqrt{2}\eta^a\psi_a$, $\hat{\xi}\psi_a = \sqrt{2}\bar{\eta}^{\dot{b}}\partial_\mu\phi\sigma^\mu_{a\dot{b}}$, $\hat{\xi}\phi^* = -\sqrt{2}\eta^{\dot{a}}\psi^*_{\dot{a}}$, $\hat{\xi}\psi^*_{\dot{a}} = \sqrt{2}\sigma^\mu_{b\dot{a}}\eta^b\partial_\mu\phi^*$. Now if we compute the variation of the the Lagrangian we find

$$\hat{\xi}L = g^{\mu\nu}(-\sqrt{2}\eta^a\partial_\mu\psi_a\partial_\nu\phi^* - \sqrt{2}\bar{\eta}^{\dot{a}}\partial_\mu\phi\partial_\nu\bar{\psi}^*_{\dot{a}})$$
$$+\sqrt{2}\sigma^\nu_{b\dot{a}}\eta^b\partial_\nu\phi^*\bar{\sigma}^{\mu\dot{a}a}\partial_\mu\psi_a + \sqrt{2}\psi^*_{\dot{a}}\bar{\sigma}^{\mu a\dot{a}}\bar{\eta}^{\dot{b}}\partial_\mu\partial_\nu\phi\sigma^\nu_{a\dot{b}}.$$

Using $\sigma^{\mu}_{a\dot{b}}\bar{\sigma}^{\nu\dot{b}c} = \delta_a{}^c g^{\mu\nu}$ and integrating by parts we are left with an exact term, thus we have a symmetry. It remains to show that it realizes the supersymmetry algebra. Let us check this for ϕ. Note that

$$\hat{\xi}_1\hat{\xi}_2\phi = \hat{\xi}_1(\sqrt{2}\psi_a\eta_2^a) = 2\partial_\mu\phi\bar{\eta}_1^{\dot{b}}\sigma^{\mu}_{a\dot{b}}\eta_2^a.$$

Thus

$$[\hat{\xi}_1, \hat{\xi}_2] = 4\mathrm{Re}(\bar{\eta}_1^{\dot{b}}\sigma^{\mu}_{a\dot{b}}\eta_2^a)\partial_\mu\phi.$$

Things do not work so well for ψ. One finds that the supersymmetry algebra is realized but only on shell. What this means is that

$$[\hat{\xi}_1, \hat{\xi}_2] = \mathrm{Re}(\bar{\eta}_1^{\dot{b}}\sigma^{\mu}_{a\dot{b}}\eta_2^a)\partial_\mu,$$

holds only for ψ_a thats satisfy the equations of motion for this theory. It is possible to extend this to a theory that holds off shell, i.e. for all fields by adding an auxilary field which is a scalar superfunction whose equations of motion are $F = 0$. This extended theory will agree with our theory on shell and realizes the supersymmetry algebra off shell as well.

8. Mathematical Implications

The mathematics of spinors and Clifford algebras makes supersymmetry aesthetically pleasing but it has deeper mathematical content. Recall that the basic aspect of the supersymmetry algebra is that $[Q, Q] = P$ where Q is in ΠS and P in $\mathbb{M}$ is a translation. P will act by derivatives and thus Q is a sort of square root of a differential operator. Mathematically square roots of differential operators naturally occur as Dirac operators which are in some sense square roots of a laplacian. When a supersymmetric theory is quantized, the supercharges Q will generally quantize as Dirac operators acting on sections of some vector bundle. One can use the techniques of quantum field theory to study the geometry of Dirac operators. Witten uses this idea to give a proof of the Atiyah Singer index theorem.

References

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