NATURAL OPERATORS ON THE BUNDLE OF CARTAN CONNECTIONS

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Abstract. We show that Cartan connections of a given type can be viewed as sections of a gauge natural bundle. This fact enables us to study natural operators on Cartan connections as gauge natural invariants. We deduce a modification of the Utiyama theorem. The relation to the classical results on natural operators on affine connections is discussed in Section 7.

1. Cartan connections.

A Cartan connection on a principal B-bundle P over a m-dimensional manifold is a $\mathfrak{g}$-valued one-form ω ($\mathfrak{g}$ is the Lie algebra of G, B is a closed subgroup of G) with the following properties:

(1) $\omega(X^*) = X$, for $X \in \mathfrak{b}$, where X^* is the fundamental vector field corresponding to X
(2) $r_a^*\omega = \mathrm{Ad}(a^{-1})\omega$ for each $a \in B$, where r_a is the right principal action of an element a,
(3) $\omega : T_uP \to \mathfrak{g}$ is an isomorphism for each $u \in P$.

We point out, that $\dim G/B = m$.

Associating a group G via the left multiplication to a principal B-bundle we get a G-principal bundle $P' = P \times_B G$ with the right principal action $(u \times_B g) \cdot h = (u \times_B g \cdot h)$. Thus the category $\mathcal{PB}_m(B)$, the category of principal bundles with structure group B, can be seen as a "subcategory" of $\mathcal{PB}_m(G)$, the category of principal bundles with a structure group G. Namely each principal B-bundle P can be identified with a subbundle of $P \times_B G$ ($u \mapsto (u \times_B e)$), and a $\mathcal{PB}_m(B)$ morphism $\varphi : P \to R$ yields a morphism $\varphi' : P' \to R'$, $\varphi'(u \times_B g) = (\varphi(u) \times_B g)$. Further we can identify $(\mathbb{R}^m \times B) \times_B G$ with $\mathbb{R}^m \times G$ ($(x, b) \times_B g \mapsto (x, bg)$). Notice that the composition of the embeddings described above, $\mathbb{R}^m \times B \to (\mathbb{R}^m \times B) \times_B G \to \mathbb{R}^m \times G$, is just the canonical embedding $\mathbb{R}^m \times B \to \mathbb{R}^m \times G$.

2. Gauge natural bundle of Cartan connections.

Every principal connection can be interpreted as a section Γ of a fibred manifold $J^1(P') \to P'$ (with the projection on the target) invariant with respect to the right action of G on $J^1(P')$ (it is induced by the principal action of G on P'). Then the structure of fibred manifold can be introduced on the set $\mathcal{Q}P' = J^1(P')/G$ and this construction gives rise to a functor $\mathcal{Q} : \mathcal{PB}_m(G) \to \mathcal{FM}$, the bundle of principal connections (see [KMS, 17.4.]).

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Analogously we can study Cartan connections. There is a one-to-one correspondence between Cartan connections on P and principal connections on P' with certain properties. We claim that these connections are even all sections of a subbundle of $\mathcal{Q}P'$. Thus Cartan connections can be viewed as sections of a subbundle of the bundle functor $\mathcal{Q}(\_ \times_B G)$. We describe the correspondence in more detail.

Firstly we describe the correspondence between Cartan connections on P and principal connections on P'. A Cartan connection on P, that is a form $\omega \in \Omega(P; \mathfrak{g})$, can be spread as a pseudotensorial form of the type $(\mathrm{Ad}, \mathfrak{g})$ (KN, II.5) over the whole P' ($P \subset P'$, see above).

$$\omega_{(u,g)} = Ad(g^{-1})\pi_P^*\omega + \pi_G^*\omega_G,$$

where $\pi_p : P \times G \to P$ and $\pi_G : P \times G \to G$ are canonical projections, and ω_G is the left Maurer-Cartan form on G. Precisely speaking, we extended ω to a form on $P \times G$, but this extension is a pull-up of a connection form of a principal connection on $P' = P \times_B G$. This construction gives rise to principal connections on P', whose horizontal bundle doesn't meet the tangent bundle of P viewed as the subbundle of P' (see [S, Appendix A] for details). We will call this condition the condition †. Conversely the pull-back of a connection form of a principal connection which satisfies the condition †, under the canonical inclusion $i : P \to P \times_B G$ is a Cartan connection on P with values in $\mathfrak{g}$. These constructions are inverse to each other.

From now on we will use the term "Cartan connection" also in the sense of a principal connection on P' satisfying the condition †.

The condition † is obviously preserved by the action of $\mathcal{PB}_m(B)$-morphisms: there is the embedding of $\mathcal{PB}_m(B)$ into $\mathcal{PB}_m(G)$ (see above) and thus we can consider any $\mathcal{PB}_m(B)$-morphism as a $\mathcal{PB}_m(G)$-morphism. The image of the horizontal bundle of the given connection under the $\mathcal{PB}_m(G)$-morphism defines then the resulting connection. Since every $\mathcal{PB}_m(G)$-morphism is a local diffeomorphism, its tangent mapping is an isomorphism in every point of P', and consequently the new connection also satisfies the condition † (because of dimensions, the horizontal space of a Cartan connection in any point $u \in P$ and the tangent space of P at u generate the tangent space of P' at u, so must their images under any isomorphism and again because of dimensions their intersection has to be the zero vector only).

It makes sense to ask, whether a given principal connection satisfies the condition † in a given point of P', that is if an element of QP satisfies the condition †. If so, we will call this element an element of Cartan connection. The action of $\mathcal{PB}_m(B)$ morphisms on Cartan connections (i.e. principal connections with special properties) is given pointwise and we can as well consider the action just on the elements of Cartan connections. We have just proved, that these elements (i.e. values of Cartan connections in a given point of P') are closed under the action of $\mathcal{PB}_m(B)$ morphisms, moreover they form an open subset of $\mathcal{Q}P$ (because of the nature of the condition †). Thus these elements form a gauge natural subbundle of the gauge natural bundle $\mathcal{Q}(\_ \times_B G)$ over the category $\mathcal{PB}_m(B)$. We will call this subbundle the bundle of Cartan connections $\mathcal{C}$. That means the bundle of Cartan connections is a functor $\mathcal{C} : \mathcal{PB}_m(B) \to \mathcal{FM}$ on the category of principal B-bundles over m-dimensional manifolds to the category of fibred manifolds. Then Cartan connections are sections of the bundle of Cartan connections.

Now we can study natural operators on Cartan connections as gauge natural invariants.

3. Natural operators on gauge natural bundles.

In this paper we deal with the concept of a natural operator stated in [KMS].

Natural transformations (zero order natural operators) between two r-th order gauge natural bundles correspond to $W^r_m(B)$-equivariant maps between their standard fibres (where $W^r_m(B)$ is the group of r-jets of local isomorphisms of principal B-bundles). To get the higher order operators we have to work with the corresponding jet prolongation of the source bundle (see [KMS]). We are going to study first order operators on the first order gauge natural bundle, thus we will study the action of $W^2_m(B)$ on the standard fiber of the first jet prolongation of the bundle of Cartan connections. $W^2_m(B)$ is canonically embedded into $W^2_m(G)$ and the desired action is the restriction of the action of $W^2_m(G)$ on the standard fibre $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ of the first jet prolongation of the bundle of principal connections on $\mathbb{R}^m \times G$ (the standard fibre of $J^1(\mathcal{Q}(P'))$ is isomorphic to the standard fibre of $J^1(\mathcal{Q}(\mathbb{R}^m \times G))$ for any principal bundle P').

The following two paragraphs are quite technical ones.

4. The action of morphisms on principal connections. In this paragraph we will present the calculations made in [KMS, Section 52]: we will compute the action of $\mathcal{PB}_m(G)$-morphisms on the standard fibre of the bundle of principal connections. Knowing this action we will be able then to compute in the following paragraph the action of $\mathcal{PB}_m(G)$-morphisms on the standard fibre of Cartan connections quite easily.

For the standard fibre S of the bundle of principal connections we use identification $S = J^1_0(\mathbb{R}^m \times G)/G \simeq J^1_0(\mathbb{R}^m, G)_e \simeq \mathfrak{g} \otimes \mathbb{R}^{m*}$. In this identification we take a canonical representative $\Gamma(x, e) \in \mathbb{R}^{m*} \otimes \mathfrak{g}$ to represent a connection Γ in the fibre over x. In cordinates we get Γ^p_i, the Christoffel symbols (for a fixed base e_p in $\mathfrak{g}$); then the connection form ω of the connection Γ is related with the coordinates Γ^p_i in the standard fibre via the left Maurer-Cartan form ω_G of G by the formula $\Gamma^p_i dx^i = \omega_G(e) - \omega(x, e)$.

Now we describe coordinates on $W^2_m(G)$: for $\mathcal{PB}_m(G)$-isomorphism $\Phi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G$; $\Phi(x, g) = (f(x), \varphi(x) \cdot g)$, $f(0) = 0$, $\varphi : \mathbb{R}^m \to G$, $\varphi(x) = \Phi(x, e)$, the coresponding element in $W^2_m(G)$ has the coordinates $a = \varphi(0) \in G$, $(a^p_i, a^p_{ij}) \sim j^2_0(a^{-1} \cdot \varphi(x)) \in \mathfrak{g} \otimes \mathbb{R}^{m*} \oplus \mathfrak{g} \otimes S^2 R^{m*}$, $(a^i_j, a^i_{jk}) \sim j^2_0 f \in G^2_m$. The coordinates on $W^1_m(G)$ are then only the first order partial derivatives terms: (a, a^i_j, a^p_i).

The action of Φ on $\Gamma(x) \simeq j^1_x s \in \mathcal{Q}(\mathbb{R}^m \times G)$, is given by the formula $\mathcal{Q}\Phi(j^1_x s) = j^1_{f(x)}\big(y \mapsto (\rho^{(\varphi(x))^{-1}} \circ \Phi \circ s \circ f^{-1}(y))\big) = j^1_{f(x)}\big(y \mapsto (\mathrm{conj}(\varphi(x)) \circ \mu \circ (\lambda^{(\varphi(x))^{-1}} \circ \varphi, s) \circ f^{-1}(y))\big)$, where ρ^a is the right multiplication by an element a in G, λ is the left multiplication by a in G, and μ is the multiplication in G. This yields then in coordinates

$$\overline{\Gamma}^p_i(f(x)) = A^p_q(\varphi(x))\big(\Gamma^q_j(x) + a^q_j(x)\big)\widetilde{a}^j_i,$$

where $A^p_q(a)$ is a coordinate expression of the adjoint representation of G, $\widetilde{a}^j_i$ is the inverse matrix to a^j_i, that's $\widetilde{a}^i_j \sim j^1_0(f^{-1})$. Especially for $\Gamma(0) \sim j^1_0(s) \in S$ we get

$$\overline{\Gamma}^p_i = A^p_q(a)(\Gamma^q_j + a^q_j)\widetilde{a}^j_i.$$

On $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ we introduce coordinates Γ^p_j and $\Gamma^p_{jk} = \partial\Gamma^p_j/\partial x^k$.

The action of $W^2_m(G)$ on $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ is then given by the formula $(j^2_0\Phi)(\Gamma) = j^1_0(\mathcal{Q}\Phi \circ \Gamma \circ f^{-1}) \sim j^1_0(\mathcal{Q}\Phi \circ j^1 s \circ f^{-1})$.

In coordinates we get

$$\begin{aligned}\overline{\Gamma}^p_{ij} = \frac{\partial \overline{\Gamma}^p_i}{\partial x^j} &= \frac{\partial}{\partial x^j}(A^p_q(\varphi \circ f^{-1}(x))(\Gamma^q_k(f^{-1}(x)) + a^q_k(f^{-1}(x)))\widetilde{a}^k_i) \\ &= A^p_q(a)\Gamma^q_{kl}\widetilde{a}^k_i\widetilde{a}^l_j + A^p_q(a)a^q_{kl}\widetilde{a}^k_i\widetilde{a}^l_j + D^p_{qr}(a)\Gamma^q_k a^r_l \widetilde{a}^k_i\widetilde{a}^l_j \\ &\quad + E^p_{qr}(a)a^q_k a^r_l \widetilde{a}^k_i\widetilde{a}^l_j + A^p_q(a)(\Gamma^q_k + a^q_k)\widetilde{a}^k_{ij},\end{aligned}$$

where D^p_{qr} and E^p_{qr} are some function on G, which we won't need, cf. [KMS].

5. Action of morphisms on Cartan connections. As promissed we will use the preceeding computation to express the action of $\mathcal{PB}_m(G)$ morphisms on the standard fibre of Cartan connections.

The action of $W^2_m(B)$ on the $J^1_0(\mathcal{C}(\mathbb{R}^m \times B))$ is then given by the same formulae; what we are left with to do is to describe the restriction of $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ to $J^1_0(\mathcal{C}(\mathbb{R}^m \times B))$ and the embedding of $W^2_m(B)$ into $W^2_m(G)$ in coordinates.

5.1. Notation. We can choose a base e_p of $\mathfrak{g}$ in such a way, that e_p, $p \leq \dim\, \mathfrak{b}$ form a base of $\mathfrak{b}$, and we shall write $(e_p) = (e_s, e_t)$, $1 \leq s \leq \dim\, \mathfrak{b}$, $\dim\, \mathfrak{b} < t \leq \dim\, \mathfrak{g}$, where (e_s) is a base of $\mathfrak{b}$. Then (e_t) represents a base in $\mathfrak{g}/\mathfrak{b}$. Correspondingly for a vector $v \in \mathfrak{g}$ the ^{s} coordinates determine its $\mathfrak{b}$ part and ^{t} coordinates its $\mathfrak{g}/\mathfrak{b}$ part.

The horizontal bundle HP' of a principal connection on P' corresponding to a Cartan connection satisfies the condition †, that is $TP \cap HP' = P \times \{0\}$, this means that vectors Γ^t_i, $i = 1, \dots, m$ are linearly independent and consequently the matrix Γ^t_i is regular. The coordinates on $T := J^1_0(\mathcal{C}(\mathbb{R}^m \times B))$ are then Γ^p_i, Γ^p_{ij}, with Γ^t_i a regular matrix and Γ^p_{ij} arbitrary.

The coordinates corresponding to the embedding of $W^2_m(B)$ into $W^2_m(G)$ are (a, a^s_i, $a^t_i = 0$, a^s_{ij}, $a^t_{ij} = 0$, a^i_j, a^i_{jk}). Notice, that the action of $W^1_m(B)$ on the standard fibre of $\mathcal{Q}(P)$ really preserves the standard fibre of $\mathcal{C}(P)$ (it is exactly what it should do): it preserves the regularity of Γ^t_i.

6. The orbit reduction. Now we exploit the orbit reduction theorem (cf. [KMS, Section 28]) to simplify the action on the standard fibre. To be able to do this we introduce new coordinates on $J^1_0 C(\mathbb{R}^m \times B)$. These new coordinates are Γ^p_i, $\Gamma^p_{(ij)}$ (symetrizations of derivations of Christoffel symbols),

$$R^p_{ij} = \Gamma^p_{[ij]} + c^p_{qr}\Gamma^q_i\Gamma^r_j, \tag{2}$$

where c^p_{qr} are the structure constants of the group G. Recall that $1 \leq s \leq \dim\, \mathfrak{b}$, $\dim\, \mathfrak{b} < t, u \leq \dim\, \mathfrak{g}$. R^p_{ij} coordinates are just the coordinate expression of the curvature tensor.

Now the kernel of the projection $W^2_m B \to G^1_m \times B$, (with coordinates a^p_i, a^p_{ij} – symmetric, $a^i_j = \delta^i_j$ $a = e$, a^i_{jk}) acts on these in the following way:

$$\begin{aligned}\overline{\Gamma}^s_i &= \Gamma^s_i + a^s_i \\ \overline{\Gamma}^t_i &= \Gamma^t_i \\ \overline{\Gamma}^s_{(ij)} &= \Gamma^s_{(ij)} + a^s_{ij} + D^s_{qr}(e)\Gamma^q_k a^r_l \widetilde{a}^k_{(i}\widetilde{a}^l_{j)} + E^s_{qr}(e)a^q_k a^r_l \widetilde{a}^k_{(i}\widetilde{a}^l_{j)} + (\Gamma^s_k + a^s_k)\widetilde{a}^k_{ij} \\ \overline{\Gamma}^t_{(ij)} &= \Gamma^t_{(ij)} + D^t_{qr}(e)\Gamma^q_k a^r_l \widetilde{a}^k_{(i}\widetilde{a}^l_{j)} + E^t_{qr}(e)a^q_k a^r_l \widetilde{a}^k_{(i}\widetilde{a}^l_{j)} + \Gamma^t_k\widetilde{a}^k_{ij} \\ \overline{R}^p_{ij} &= R^p_{ij}\end{aligned}$$

The last equality results from the substance of the curvature tensor (the curvature is a natural operator). We can see that the kernel acts on Γ^s_i, Γ^p_{ij} transitively, and on Γ^t_i and R^p_{ij} trivially: it is obvious, that it acts transitively on Γ^s_i; as for $\Gamma^t_{(ij)}$ and $\Gamma^s_{(ij)}$ we can "adjust" Γ^t_{ij} by the term $\Gamma^t_k \widetilde{a}^k_{ij}$ (Γ^t_k is a regular matrix and a^k_{ij} arbitrary) and then we can choose freely a^s_{ij} and thus we get a transitive action on $\Gamma^s_{(ij)}$. That means the action of $W^2_m B$ on $T = J^1_0 \mathcal{C}(\mathbb{R}^m \times B)$ can be factorized through the action of $G^1_m \times B$ just on coordinates Γ^t_i, R^p_{ij}.

We conclude that every $W^2_m(B)$-map from T to $(G^1_m \times B)$-space Z (every $(G^1_m \times B)$-space can be viewed as a $W^2_m(B)$-space – we consider the action of the kernel of the projection $W^2_m B \to G^1_m \times B$ to be trivial) can be factorized through $G^1_m \times B$-map defined just on coordinates Γ^t_i and R^p_{ij} (with the above action) to Z.

If we denote by $\alpha : S \to \mathfrak{g}/\mathfrak{b} \otimes \mathbb{R}^{m*}$ the projection $(\Gamma^p_i, \Gamma^p_{ij}) \mapsto (\Gamma^t_i)$ and by $\beta : S \to \mathfrak{b} \otimes \Lambda^2 \mathbb{R}^{m*}$ the map (2), we can summarize:

6.1. Proposition. *Let Z be a $(G^1_m \times B)$-space. Then for every $W^2_m B$-map $f : S \to Z$ there exists a unique $(G^1_m \times B)$-map $g : \mathfrak{g}/\mathfrak{b} \otimes \mathbb{R}^{m*} \oplus \mathfrak{g} \otimes \Lambda^2 \mathbb{R}^{m*} \to Z$ satisfying $f = g \circ (\alpha, \beta)$.*

7. The main theorem and its application.

We can interpret Proposition 6.1. in terms of natural operators: The first order gauge natural operators on the bundle of Cartan connections with target at the gauge natural bundles, where just the subgroup $G^1_m \times B$ of $W^2_m B$ acts non-trivially (the value of the bundle (i.e. of the functor) on $\mathcal{PB}_m(B)$-morphism in a given point depends only on 1-jet of base mapping of the given morphism and only on the value of the given mapping in the fibre – the bundles of the order (1,0), see [KMS, Section 52]), depends only at the curvature form of the given connection and $\mathfrak{g}/\mathfrak{b}$-part of Christoffel symbols. The curvature form is so called tensorial 2-form of the type $(\mathfrak{g}, \mathrm{Ad})$, i.e. it is a horizontal 2-form and $r^{a*}\Omega = Ad^{-1}(a)\Omega$, thus it can be regarded as a section of the bundle $LP \otimes \Lambda^2 T(BP)^*$, where P is a principal G-bundle, B is the base functor, and LP is the asociated bundle $LP := P \times_{\mathrm{Ad}} \mathfrak{g}$.

We get the following result:

7.1. Theorem. *Let $\mathcal{K}$ be a gauge natural bundle of order* $(1, 0)$. *Then for every first order gauge natural operator $A : \mathcal{C} \to \mathcal{K}$ there exists a unique natural transformation $O : L \otimes \Lambda^2 T^* B \to \mathcal{K} \oplus \mathfrak{g}/\mathfrak{b} \otimes T^* B$ satisfying $A = O \circ (E, F)$, where $E : \mathcal{C} \to L \otimes \Lambda^2 T^* B$ is the curvature operator, and $F : \mathcal{C} \to \mathfrak{g}/\mathfrak{b} \otimes T^* B$ is the $\mathfrak{g}/\mathfrak{b}$-part of a Cartan connection (a $\mathfrak{g}$-valued form).*

7.2. Relations to natural operators on affine connections.

Now we are going to apply Theorem 7.1 in the situation, where $B \subset GL(n, \mathbb{R})$ and $G = B \rtimes \mathbb{R}^m$ to get the classical result on standard operators on affine connections. Let ρ be a Cartan connection with values in $\mathfrak{g}$ on the principal B-bundle P and $\pi : P \to BP$ the projection on a base manifold. Such a connection can be split to $\rho = \omega + \varphi$ according to the splitting $\mathfrak{g} = \mathfrak{b} + \mathbb{R}^m$. The $\mathfrak{b}$ part is just a principal connection on P, the $\mathbb{R}^m$ part represents a non-degenerated tensorial 1-form of the type $(GL(n, \mathbb{R}), \mathbb{R}^m)$ on P (see [KN]), that is it is a horizontal form and it is right equivariant with respect to the actions of B – see the second condition in the definition of the Cartan connection). We will write $\varphi_u : T^*_{\pi(u)} M \to \mathbb{R}^m$. This form identifies P with a subbundle of the bundle of linear frames $P^1 M$, namely we have an injection $i : P \to P^1M$, $i(u) = \varphi_u^{-1}$ (we consider a linear frame at $x \in M$ to be a linear mapping from $\mathbb{R}^m$ to T_xM). Then i is a principal bundle morphism:

$$i(u \cdot A) = \varphi_{u \cdot A}^{-1} = (A^{-1} \cdot \varphi_u)^{-1} = \varphi_u^{-1} \circ A = i(u) \cdot A,$$

for $A \in B$. Then the push out of the form φ under the morphism i is just the canonical soldering form θ on $i(P) \subset P^1M$:

$$i_*\varphi(X_E) = \varphi_{i^{-1}(E)}\pi(X_E) = E^{-1}(\pi(X_E)) = \theta(X_E),$$

for a linear frame $E \in i(P)$.

Let us explain, what we mean by an affine connection in this paper. An affine connection will be a "classical" affine connection a (a is 1-form on P^1M, $a = \omega + \theta$, where ω is a linear connection on P^1M and θ is the soldering form, [KN, Chapter VI]), whose restriction to $P \subset P^1M$ has values in $\mathfrak{g} = \mathfrak{b} + \mathbb{R}^m$. In other words it is a principal connection belonging to a B-structure – the realization of the B-structure as a subbundle of P^1M is enabled by the soldering form. Equivalently we can say, that an affine connection is a Cartan connection a on a B-subbundle P of the bundle of linear frames and $a = \omega + \theta$, where θ is the restriction of the soldering form θ on P^1M to the subbundle P.

We will show, that any first order gauge natural operator on Cartan connections on P of the discussed type (i.e. operator invariant under the action of $\mathcal{PB}_m(B)$ morphisms) can be restricted to a first order natural operator on affine connections (i.e. operator invariant under the action of morphisms of the form P^1f, $f : BP \to BP$) and conversely any first order natural operator on affine connections can be extended to a first order gauge natural operator on Cartan connections.

It is quite obvious, that each gauge natural operator on Cartan connections with values in $\mathfrak{g}$ defines a natural operator on affine connections: restricting the given operator just to affine connections (Cartan connections on a subbundle of P^1M with some extra properties) we obtain a natural operator on affine connections. Such an operator is then equivariant with respect to the action of $P^1(f)$ morphisms $f : \mathbb{R}^m \to \mathbb{R}^m$ (the action of morphism on sections of source or target bundles is given by pull-backs or equivalently by push-outs), because the original operator was equivariant to any principle bundle morphism.

Conversely, given a first order natural operator on affine connections, we can extend it to an operator over the whole $\mathcal{PB}_m(B)$. Since any first order gauge natural operator on Cartan connections is determined by its values on the standard fibre $J^1_0\mathcal{C}(\mathbb{R}^m \times B)$, we will carry on our considerations on the bundle $\mathbb{R}^m \times B$. Let $P = \mathbb{R}^m \times B$ be a principal bundle with structure group B and we have chosen a tensorial form θ_0 of the type $(GL(n, \mathbb{R}), \mathbb{R}^m)$ which identifies P with a subbundle of $P^1(BP)$. Further let $\mathcal{Z}$ be a gauge natural bundle of order (1,0) and $D : \mathcal{A} \to \mathcal{Z}$ be a natural operator on affine connections. Then we define a natural operator $\overline{D} : \mathcal{C} \to \mathcal{Z}$. Let $c = \omega + \theta_1$ be a splitting of a Cartan connection on P. Now the forms θ_0 and θ_1 are related in the following way: their values at $(0, b)$, $b \in B$ are given by values at $(0, e)$. If $E \in GL(n, \mathbb{R})$ is a matrix representing θ_0 at $(0, e)$ and $F \in GL(n, \mathbb{R})$ a matrix representing θ_1 in $(0, e)$ then $F = EE^{-1}F$. If we choose a $\mathcal{PB}_m(B)$ morphism $\varphi = (\varphi_0, \mathrm{const}_e)$ with $j^1_{(0,e)}\varphi_0 = E^{-1}F \in G^1_m$, Then (1) yields $\theta_1(0) = \varphi^*\theta_0(0)$ and we define (for a value of an operator $\overline{D}$ on a 1-jet of connection

c at $0 \in \mathbb{R}^m$; we denote the corresponding element in $J^1_0\mathcal{C}(\mathbb{R}^m \times B)$ by the same letter c)

$$\overline{D}(c) = \overline{D}(\omega+\theta_1) = \overline{D}((\varphi^*(\varphi_*\omega)+\varphi^*\theta_0) = \varphi^*\overline{D}(\varphi_*\omega+\theta_0) = \varphi^* i^* D(i_*(\varphi_*\omega+\theta_0)).$$

Since our choice of 1-jet at $(0, e)$ of the morphism φ is uniquely determined by θ_1 and θ_0, this definition is correct. Now we can decompose any morphism $\psi : \mathbb{R}^m \times B \to R^m \times B$ to $(\psi_0, \overline{\psi})$ (see paragraph 4) and we can write for a 1-jet of Cartan connection c in $0 \in \mathbb{R}^m$, $c = \omega + \theta_1$:

$$\begin{aligned}\overline{D}(\psi^* c) =& \overline{D}(\overline{\varphi} \circ \psi_0 \circ (\overline{\psi} \circ \overline{\varphi}^{-1}))^* c) \\ &= \overline{D}((\overline{\psi} \circ \overline{\varphi}^{-1})^* \circ \psi_0^* \circ \overline{\varphi}^*(c)) \\ &= (\overline{\psi} \circ \overline{\varphi}^{-1})^* D(\psi_0^* \circ \overline{\varphi}^*(c)) \\ &= (\overline{\psi} \circ \overline{\varphi}^{-1})^* \circ \psi_0^* D(\overline{\varphi}^*(c)) \\ &= (\overline{\psi} \circ \overline{\varphi}^{-1})^* \circ \psi_0^* \circ \overline{\varphi}^* \overline{D}(c)) \\ &= \psi^* \overline{D}(c),\end{aligned}$$

which proves $\overline{D}$ to be a gauge natural operator.

Taking a form θ_0' to identify $\mathbb{R}^m \times B$ with $P^1(\mathbb{R}^m)$ (we will write i' for the corresponding isomorphism) we would get the same operator: let $\theta_0' = \eta^*\theta_0$, $\theta_1 = \psi^*\theta_0'$, $\theta_1 = \varphi^*\theta_0$, that is $\eta \circ \psi = \varphi$. Then

$$\begin{aligned}\overline{D}(c) &= \overline{D}(\omega + \theta_1) = \overline{D}((\psi^*(\varphi_*\omega) + \psi^*\theta_0') \\ &= \psi^*\overline{D}(\psi_*\omega + \theta_0') = \psi^*(i')^* D(i'_*(\psi_*\omega + \theta_0')) \\ &= (\eta^{-1} \circ \varphi)^*(i \circ \eta)^* D((i \circ \eta)_*(\psi_*\omega + \theta_0')) \\ &= \varphi^* \circ (\eta^{-1})^* \eta^* i^* D(i_* \circ (\eta_*\psi_*\omega + \eta_*\theta_0')) \\ &= \varphi^* i^* D(i_*(\varphi_*\omega + \theta_0)).\end{aligned}$$

That means we have proved:

7.3. Theorem. *There is a one-to-one correspondence between natural operators on affine connections and gauge natural operators on Cartan connections.*

7.4. Example. We can apply the Theorems 7.1 and 7.3 to deduce, that natural operators on affine connections (that is Cartan connections on bundle of linear frames with values in $\mathfrak{af}(n, \mathbb{R})$) can be factorized through curvature form of an affine connection, that is of the curvature and torsion form of a corresponding linear connection. It is namely obvious, that natural operators on affine connections can't depend on the soldering form θ, since they should be invariant to the action of morphisms preserving θ, and θ plays a role of a "passive invariant": it is invariant by definition but it doesn't come up explicitly in any "real" invariant.

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