

# TRIALITIES AND EXCEPTIONAL LIE ALGEBRAS: DECONSTRUCTING THE MAGIC SQUARE

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A construction of the magic square, and hence of exceptional Lie algebras, is carried out using trialities rather than division algebras. By way of preparation, a comprehensive discussion of trialities is given, incorporating a number of novel results and proofs. Many of the techniques are closely related to, or derived from, ideas which are commonplace in theoretical physics. The importance of symmetric spaces in the magic square construction is clarified, allowing the Jacobi property to be verified for each algebra in a uniform and transparent way, with no detailed calculations required. A variation on the construction, corresponding to other symmetric spaces, is also given.

## 1. Introduction

An understanding of the construction and classification of Lie groups and Lie algebras is essential in many branches of theoretical or mathematical physics, and certainly in gauge theory, string theory, supersymmetry and integrable systems. Valuable insights into the Cartan-Killing classification [1,2] can be gained by approaching it via the normed division algebras of real numbers  $\mathbb{R}$ , complex numbers  $\mathbb{C}$ , quaternions  $\mathbb{H}$  and octonions  $\mathbb{O}$  (and their classification in turn by Hurwitz's Theorem)—see e.g. [3] and references therein. The occurrence of the  $so$ ,  $su$  and  $sp$  families of classical Lie algebras can then be attributed directly to the existence of  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$ , respectively, and it is natural to suppose that the five simple exceptional algebras should arise from  $\mathbb{O}$  in some fashion. But while  $G_2$  duly emerges as the automorphism group of the octonions, the remaining exceptional algebras pose more of a problem: how, exactly, should these things be understood in terms of octonions, or in terms of some related ‘exceptional’ structure?

The magic square provides an answer to this question. It has appeared in various incarnations over the years, and these have been reconciled only comparatively recently, with the work of Barton and Sudbery [4]. The common starting point in any version of the magic square is a pair of division algebras (which are used to label the rows and columns of the square) from which a Lie algebra can be constructed, with the following results:

$$\begin{array}{cccc}
 & \mathbb{R} & \mathbb{C} & \mathbb{H} \\
 \mathbb{R} & so(3) & su(3) & sp(3) & F_4 \\
 \mathbb{C} & su(3) & su(3) \oplus su(3) & su(6) & E_6 \\
 \mathbb{H} & sp(3) & su(6) & so(12) & E_7 \\
 \mathbb{O} & F_4 & E_6 & E_7 & E_8
 \end{array} \tag{1.1}$$

Some versions of the construction are not obviously symmetrical in the two division algebras and then part of the magic of the square is its symmetry. For references to the earlier literature, see [4,3].

The principal aim of this paper is to give a description of the magic square, and hence of exceptional Lie algebras, in terms of *trialities* rather than division algebras. A triality is an algebraic structure—actually a multilinear map on three vector spaces to the reals—with an associated notion of symmetry [1]. There is a one-to-one correspondence between (normed) trialities and (normed) division algebras and, in particular, the octonions are associated with the celebrated symmetry properties of  $so(8)$  [1,3]. One can therefore regard trialities and division algebras as being manifestations of a common, exceptional kind of mathematical structure.

The symmetry of a triality—the triality group or algebra—is central to the magic square, as is emphasised in the Barton-Sudbery approach [4]. This group is present whether one works with the triality as an algebraic concept in its own right, or whether one chooses to express things using division algebras. But the passage from a triality to a division algebra involves relinquishing some manifest symmetry, at least to the extent that a particular element of each vector space must be selected to play the role of the identity (the first step in defining a multiplication on the vector space and turning it into a copy of the division algebra) [1]. For this reason alone it is natural to ask whether there is something to be gained from carrying out a construction of the magic square using just trialities and avoiding division algebras explicitly. There is no doubt this should be possible, but the question is whether it is illuminating. We shall endeavour to show that it is.

Using trialities instead of division algebras also seems particularly natural from the standpoint of mathematical physics, since trialities can be thought of as rather special examples of Dirac gamma matrices or invariant tensors [1,3,5]. Their distinguishing feature is an enhanced symmetry between vectors and spinors in the Euclidean dimensions concerned, namely  $n = 1, 2, 4$ , or  $8$ . This is interesting in its own right, but it also provides a link between the classification of trialities, or division algebras, and super-Poincare symmetry in dimensions  $3, 4, 6$  and  $10$  [5,6,7] (see e.g. [8,9,10] for background on supersymmetric systems). The early papers in this area considered both super Yang-Mills [5,6,7] and classical superstrings or extended objects [5,11,12,13] but interesting links between supersymmetry and exceptional structures of various kinds continue to be explored (see [10] and also [14,15,16] for a few more recent examples).

We shall give a thorough account of trialities and their symmetries in sections 3, 4 and 5 below. Many of these results are well-known in some form, but we will provide a number of new derivations, including a simple proof of what is sometimes called the *Principle of Triality* [1,3,4,17]. Our intention will be to make the connections with spinors as clear as possible, and although our treatment will be mathematically self-contained, many of the key ideas should be familiar in the context of theoretical physics.

Once we have laid the necessary foundations by investigating trialities, the construction of the magic square can be carried out remarkably simply, as we will see in section 6. Our work will also fully reconcile the magic square with other, superficially different ways of building Lie algebras which can be found in the literature. The techniques in question have been developed and applied in typically powerful fashion in the notes of Adams [1], and similar ideas have been popularised for physicists in e.g. Green, Schwarz and Witten [8]. These ideas really amount to a general method of approach rather than a uniform construction, however. We will review this in some generality in section 2, but we will also outline some aspects of it now, so that we can better explain the role it will play later in the paper.

Consider a Lie algebra  $\mathbf{h}$  and a representation of it on a vector space  $\mathbf{p}$ . We can attempt to define a new Lie algebra with vector space  $\mathbf{g} = \mathbf{h} \oplus \mathbf{p}$  by first extending the Lie bracket, using the representation, and then checking the Jacobi identity, with all definitions covariant under  $\mathbf{h}$ . Success is not automatic, and depends on certain properties of the representation chosen, but there are plenty of examples where it works. We can always extend  $so(n)$  by its vector representation to obtain  $so(n+1)$ , for instance. This is an example of a symmetric pair,  $\mathbf{g} \supset \mathbf{h}$ , meaning that the quotient of the corresponding groups  $G/H$  is a symmetric space [2]. Any symmetric pair can be constructed in this way, provided we know the right representation to choose. The same general approach can be pursued even if  $\mathbf{g} \supset \mathbf{h}$  is not a symmetric pair, but this leaves

much greater freedom in the definition of the bracket, and verification of Jacobi is usually significantly more involved.

The symmetric space approach works nicely for certain exceptional algebras: we can build  $\mathbf{g} = F_4$  or  $\mathbf{g} = E_8$  by starting from  $\mathbf{h} = so(9)$  or  $\mathbf{h} = so(16)$  and adding  $\mathbf{p}$  in a suitable spinor representation, for example. In either case, the Jacobi identity can be checked by using a Fierz rearrangement involving the  $so(9)$  or  $so(16)$  gamma matrices [8]—a standard technique in QFT—or by various other means [1]. The calculations are not too difficult, but they do not provide much insight into why things work for these particular choices of  $\mathbf{h}$  and  $\mathbf{p}$  (as remarked upon in e.g. [3]). In contrast to the ‘generic’ behaviour of our previous example, in which  $\mathbf{g} = so(n+1)$  is constructed from  $\mathbf{h} = so(n)$  for any  $n$ , there is something ‘exceptional’ in these new examples, as befits the exceptional groups.

A natural suspicion is that the triality corresponding to  $so(8)$  somehow lies behind the success of the  $F_4$  and  $E_8$  symmetric space constructions, and we can investigate this by looking at the simpler of these examples in more detail. If we try to build  $F_4$  directly from  $\mathbf{h} = so(8)$  we need to add  $\mathbf{p} = V \oplus S_+ \oplus S_-$ , where  $V$ ,  $S_+$ ,  $S_-$  denote vector and spinor representations, which are related by triality symmetries. The apparent disadvantage of this approach is that  $\mathbf{g} \supset \mathbf{h}$  is not a symmetric pair, the representation  $\mathbf{p}$  is reducible, and so completing the definition of the bracket and then checking Jacobi would seem to involve considerable further effort. By contrast, the symmetric pair approach amounts to re-grouping summands by taking  $\mathbf{h} = so(8) \oplus V = so(9)$  and  $\mathbf{p} = S_+ \oplus S_-$ , which becomes irreducible as a representation of this larger algebra. The definition of the bracket is then fixed, but in re-grouping we have relinquished some symmetry by treating  $V$  differently from  $S_\pm$ , and so the significance of the triality in verifying the Jacobi property is obscured.

The key to a more satisfactory understanding is the realisation that we can combine the best of both points of view, as follows. We know that certain simplifications occur in building  $F_4$  from  $so(8)$  if we first extend the subalgebra to  $so(9)$ ; we should also note that this is an example of the ‘generic’ sort of extension discussed above for  $so(n)$  algebras—there is nothing very special about this in itself. But the triality symmetry of  $so(8)$  implies that we can carry out such an extension in three inequivalent ways, by adding in  $V$ ,  $S_+$  or  $S_-$ , and this is certainly special. Despite the fact that for  $\mathbf{g} = F_4$  and  $\mathbf{h} = so(8)$  we are not dealing with a symmetric pair, the existence of these three intermediate  $so(9)$  extensions fixes much of the Lie bracket on  $\mathbf{p} = V \oplus S_+ \oplus S_-$ , as well as ensuring that large parts of the Jacobi identity are automatic. It turns out that the triality structure also provides an entirely natural way of completing the definitions of the brackets, and we need do little more than write these down to check the remaining parts of the Jacobi identity.

To re-iterate: we can build

$$F_4 \supset so(9) \supset so(8) \tag{1.2}$$

by considering *three* intermediate algebras, related by triality symmetry, with each inclusion corresponding to a symmetric pair. Since there are essentially no calculations required in checking the Jacobi property (no Fierz-style computations, for instance) the success of the construction is as transparent as possible. We can also see clearly how its ‘generic’ aspects, involving orthogonal algebras, have been combined with the more ‘exceptional’ properties of  $so(8)$ . We will describe in section 6 how a similar approach works for each triality and produces the first row or column of the magic square. We will then explain how to generalise it to a *pair* of trialities, leading to the complete magic square. The full construction will be entirely uniform, treating all entries of the square on the same footing, with no special case-by-case analysis required. It will also share the desirable feature of the example above by allowing a clear separation of ‘generic’ and ‘exceptional’ features, so that we can see at which points the trialities are really put to use.

The paper concludes, in section 7, with a short summary followed by a few additional developments. The first of these offers a slightly different perspective on the magic square, simplifying some aspects of it yet further. Next we give a sketch of the relationship between trialities and division algebras. This enables

us to discuss, in passing, how  $G_2$  arises in the context of trialities—thus completing the set of exceptional algebras—but our main concern is to link the construction in this paper to the work in [4]. The last topic is a variation on the magic square: from one point of view, it corresponds to applying similar ideas to different symmetric pairs, but it also has an attractive interpretation as a ‘folding’ of the entries on the diagonal of the square. This leads to an alternative, triality-based description of  $E_7$ .

## 2. General and Generic Constructions

Throughout this paper we will confine our attention to real Lie algebras equipped with an invariant, positive-definite inner-product. Every such algebra is a direct sum of simple algebras and a centre, with the inner-product restricting to some multiple of the Killing form on each simple factor [2,18]. Lie algebra elements in the abstract will be denoted by  $T$  or  $X$ , whereas  $M$  will be reserved for concrete representations in terms of matrices. We shall frequently work with component or index notation corresponding to orthonormal bases, and the summation convention will apply to any repeated index. In keeping with our use of terminology from theoretical physics, we will also frequently refer to basis elements of a Lie algebra as *generators*. Other standard notation includes e.g.  $\oplus$  to denote the orthogonal direct sum of vector spaces (but not necessarily a direct sum of algebras) and  $\wedge^p$  to denote the antisymmetrised tensor product of rank  $p$ .

### 2.1 General constructions: cosets and symmetric pairs

Let  $\mathbf{h}$  be a real Lie algebra with orthonormal basis  $\{T_A\}$  and Lie bracket

$$\wedge^2 \mathbf{h} \rightarrow \mathbf{h} , \quad [T_A, T_B] = f_{ABC} T_C \quad (2.1)$$

The structure constants  $f_{ABC}$  are real numbers, antisymmetric in the indices  $ABC$ , which is to say that the bracket and inner-product (denoted by round brackets) combine to give a map

$$f : \wedge^3 \mathbf{h} \rightarrow \mathbb{R} , \quad f_{ABC} = (T_A, [T_B, T_C]) \quad (2.2)$$

The Jacobi identity for  $\mathbf{h}$  is a condition on double brackets of elements of the form  $TTT$ , namely

$$[T_A, [T_B, T_C]] + [T_B, [T_C, T_A]] + [T_C, [T_A, T_B]] = 0 \quad (2.3a)$$

which is equivalent to

$$f_{BCD} f_{ADE} + f_{CAD} f_{BDE} + f_{ABD} f_{CDE} = 0 \quad (2.3b)$$

This is the adjoint representation of the algebra, expressed in components; it is also the statement that the map in (2.2) is invariant under the adjoint action.

Let  $\mathbf{p}$  be a real vector space with a positive-definite inner-product and  $\{X_i\}$  an orthonormal basis. Suppose that there is a representation of  $\mathbf{h}$  on this space by real matrices  $(M_A)_{ij}$  antisymmetric in  $ij$ , implying that the inner-product is invariant. This provides us with an extension of the Lie bracket because we can define

$$\mathbf{h} \otimes \mathbf{p} \rightarrow \mathbf{p} , \quad [T_A, X_i] = -(M_A)_{ij} X_j \quad (2.4)$$

The fact that we have a representation

$$(M_A)_{ij} (M_B)_{jk} - (M_B)_{ij} (M_A)_{jk} = f_{ABC} (M_C)_{ik} \quad (2.5a)$$

is then equivalent to the Jacobi property for elements of the form  $TTX$ , namely

$$[T_A, [T_B, X_i]] - [T_B, [T_A, X_i]] = [[T_A, T_B], X_i] \quad (2.5b)$$

Our aim is to understand the conditions under which  $\mathbf{g} = \mathbf{h} \oplus \mathbf{p}$  can be made into a Lie algebra. To do this we must complete the definition of the Lie bracket between elements in  $\mathbf{p}$  and then check the Jacobi property for the remaining combinations of generators  $TXX$  and  $XXX$ . The representation of  $\mathbf{h}$  on  $\mathbf{p}$  may be reducible, but it must decompose as a sum of real irreducible pieces. Using the freedom to scale the basis elements and inner-products on each irreducible representation, we can ensure that  $\{T_A, X_i\}$  is orthonormal with respect to an invariant inner-product on  $\mathbf{g}$ . Thus, without loss of generality,

$$(T_A, T_B) = \delta_{AB}, \quad (T_A, X_i) = 0, \quad (X_i, X_j) = \delta_{ij} \quad (2.6)$$

It then follows that the structure constants of the algebra we are seeking to construct are totally antisymmetric, and (2.4) implies

$$(T_A, [X_i, X_j]) = (X_j, [T_A, X_i]) = -(M_A)_{ij} \quad (2.7)$$

The most general form for the remaining part of the Lie bracket is therefore

$$\wedge^2 \mathbf{p} \rightarrow \mathbf{g}, \quad [X_i, X_j] = -(M_A)_{ij} T_A + c_{ijk} X_k \quad (2.8)$$

Here, the representation matrices  $(M_A)_{ij}$  are being re-interpreted as a map  $\wedge^2 \mathbf{p} \rightarrow \mathbf{h}$  (by dualising spaces using the inner-products) and this is covariant under  $\mathbf{h}$  by virtue of (2.5). The second term in the bracket sends  $\wedge^2 \mathbf{p} \rightarrow \mathbf{p}$ , but since  $c_{ijk}$  must be antisymmetric in  $ijk$  it actually defines a map

$$c : \wedge^3 \mathbf{p} \rightarrow \mathbb{R}, \quad c_{ijk} = (X_i, [X_j, X_k]) \quad (2.9)$$

It is easy to check that the condition for this map to be  $\mathbf{h}$ -invariant

$$(M_A)_{i\ell} c_{\ell jk} + (M_A)_{j\ell} c_{i\ell k} + (M_A)_{k\ell} c_{ij\ell} = 0 \quad (2.10a)$$

is necessary and sufficient for the Jacobi property to hold for combinations of elements  $TXX$ , or

$$[T_A, [X_i, X_j]] = [[T_A, X_i], X_j] + [X_i, [T_A, X_j]] \quad (2.10b)$$

The final step is to check the Jacobi property for elements of type  $XXX$ , and the definition of  $\mathbf{g}$  will be complete iff

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0 \quad (2.11)$$

This amounts to a non-trivial quadratic relationship between  $(M_A)_{ij}$  and  $c_{ijk}$ , in general. The situation becomes significantly simpler in the special case  $c_{ijk} = 0$ , however, with (2.8) then reducing to

$$\wedge^2 \mathbf{p} \rightarrow \mathbf{h}, \quad [X_i, X_j] = -(M_A)_{ij} T_A \quad (2.12)$$

In these circumstances the entire construction is determined by the choice of representation of  $\mathbf{h}$  on  $\mathbf{p}$ , and the final part of the Jacobi condition (2.11) holds iff

$$(M_A)_{ij}(M_A)_{k\ell} + (M_A)_{jk}(M_A)_{i\ell} + (M_A)_{ki}(M_A)_{j\ell} = 0 \quad (2.13)$$

Note the  $\mathbb{Z}_2$  grading corresponding to  $\mathbf{g} = \mathbf{h} \oplus \mathbf{p}$  which is manifest in the form of the brackets (2.1), (2.4) and (2.12); this is what is meant by  $\mathbf{g} \supset \mathbf{h}$  being a symmetric pair [2].

Whether we are attempting to build a new algebra as part of a symmetric pair or by using the more elaborate possibility involving a non-zero map  $c$ , the identification of  $\mathbf{g}$  is usually fairly straightforward. In particular, if  $\mathbf{h}$  is a subalgebra of maximal rank, then the roots of  $\mathbf{g}$  are the roots of  $\mathbf{h}$  together with the weights of  $\mathbf{p}$ . This approach can be used to build up each of the classical series of Lie algebras by adding in the defining or fundamental representations of smaller algebras to make larger ones. There are some well-known families of symmetric pairs of orthogonal algebras which we will need to describe in detail, in preparation for the work to follow.

## 2.2 The cases $so(n+1) \supset so(n)$ and $so(n+n') \supset so(n) \oplus so(n')$

Let  $V$  be a real vector space of dimension  $n$  with a positive-definite inner-product (denoted by a dot). There is a natural action

$$\wedge^2 V : V \rightarrow V, \quad (u \wedge w)(v) = u(w \cdot v) - w(u \cdot v) \quad (2.14)$$

which enables us to identify

$$so(n) = \wedge^2 V \quad (2.15)$$

with  $V$  the defining, or fundamental, or vector representation of  $so(n)$ . This is consistent with regarding the right-hand side as the adjoint representation, which can always be identified with the algebra itself. Now by taking  $V \oplus \mathbb{R}$  as the fundamental representation of  $so(n+1)$  we have, similarly,

$$so(n+1) = \wedge^2(V \oplus \mathbb{R}) = \wedge^2 V \oplus V = so(n) \oplus V \quad (2.16)$$

The direct sum corresponds to a  $\mathbb{Z}_2$  grading of the resulting Lie brackets and so  $so(n+1) \supset so(n)$  is a symmetric pair. It will also be very useful for us to understand this in a slightly different way, however.

Following the procedure of the subsection above, consider  $\mathbf{h} = so(n)$  and  $\mathbf{p} = V$  with representation matrices  $(M_A)_{ij}$ . The normalisation of the generators of  $so(n)$  can be fixed by demanding

$$\frac{1}{2}(M_A)_{ij}(M_B)_{ij} = \delta_{AB} \quad (2.17)$$

and this in turn fixes the multiple of the Killing form we are using, since the inner-product of basis elements is  $(T_A, T_B) = \delta_{AB}$ , by assumption. Now any antisymmetric  $n \times n$  matrix can be written  $K_{ij} = K_A(M_A)_{ij}$  and, because (2.17) then determines the coefficients to be  $K_A = \frac{1}{2}(M_A)_{kl}K_{kl}$ , we must have

$$(M_A)_{ij}(M_A)_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \quad (2.18)$$

This is enough to ensure that the symmetric pair construction works, since the key condition (2.13) follows immediately. Thus,  $\mathbf{g} = \mathbf{h} \oplus \mathbf{p}$  is a Lie algebra with brackets defined by (2.4) and (2.12) and on closer inspection we find  $\mathbf{g} = so(n+1)$  once again (see also below).

To understand how these two lines of argument are related, recall that the representation matrices  $(M_A)_{ij}$  define invariant maps:  $\wedge^2 V \rightarrow so(n)$  and its transpose  $so(n) \rightarrow \wedge^2 V$  (as pointed out after (2.8)). The conditions (2.17) and (2.18) say precisely that these maps are mutually inverse (and hence orthogonal) and so together they implement the identification (2.15). In scrutinising the details, we note that when an antisymmetric pair of indices  $ij$  is used as a basis, a sum over all values of  $i$  and  $j$  amounts to summing over the basis twice. The coefficient of  $\frac{1}{2}$  included in (2.17) is therefore just what we need to compensate for the double counting implicit in such a sum. Furthermore, the expression on the right of (2.18) is exactly the identity map on  $\wedge^2 V$  when a similar convention is adopted in writing down its action on this space.

The identification (2.15) also lies behind another standard description of  $so(n)$  in which basis elements are labelled by antisymmetric pairs of vector indices:

$$T_{ij} = (M_A)_{ij}T_A \quad (2.19)$$

With this definition, a short calculation (using (2.1), (2.5) and (2.18)) yields the algebra in familiar form

$$[T_{ij}, T_{k\ell}] = -\delta_{ik}T_{j\ell} + \delta_{jk}T_{i\ell} - \delta_{j\ell}T_{ik} + \delta_{i\ell}T_{jk} \quad (2.20)$$

and on taking matrix representations of both generators  $T$  in (2.19) we deduce

$$\begin{aligned} (M_{ij})_{k\ell} &= (M_A)_{ij}(M_A)_{k\ell} \\ &= \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} \end{aligned} \quad (2.21)$$

The new basis in (2.19) must be orthonormal, given our interpretation of (2.17) and (2.18) as orthogonality conditions, but this can be checked:

$$(T_A, T_B) = \delta_{AB} \quad \Rightarrow \quad (T_{ij}, T_{k\ell}) = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} \quad (2.22)$$

It is also instructive to re-write the brackets (2.4) and (2.12) as

$$[T_{ij}, X_k] = -\delta_{ik}X_j + \delta_{kj}X_i, \quad [X_i, X_j] = -T_{ij} \quad (2.23)$$

We need only take the additional generators to be  $T_{n+1,i} = X_i$  in order to cast  $\mathbf{g} = so(n+1)$  in the same standard form (2.20) as the original algebra  $\mathbf{h} = so(n)$ .

It is straightforward to extend the discussions above to symmetric pairs  $so(n+n') \supset so(n) \oplus so(n')$ . If  $V$  and  $V'$  are the fundamental representations of  $so(n)$  and  $so(n')$  then

$$so(n) = \wedge^2 V, \quad so(n') = \wedge^2 V' \quad (2.24)$$

according to (2.15). But we can also regard  $V \oplus V'$  as the fundamental representation of  $so(n+n')$  so that, similarly,

$$so(n+n') = \wedge^2(V \oplus V') \quad (2.25)$$

Then since

$$\wedge^2(V \oplus V') = \wedge^2V \oplus \wedge^2V' \oplus (V \otimes V') \quad (2.26)$$

we deduce that

$$so(n+n') = so(n) \oplus so(n') \oplus (V \otimes V') \quad (2.27)$$

and the Lie brackets are readily seen to be consistent with a  $\mathbb{Z}_2$  grading.

Once again, it will be very helpful for what follows later in the paper to see how the same conclusion is reached by applying the general approach of the last subsection. We will modify our notation slightly, to reflect the fact that we are now starting with a direct sum, writing

$$\mathbf{h} = so(n) \oplus so(n') \quad \text{with basis } \{T_A, T_{A'}\} \quad (2.28)$$

If  $V$  and  $V'$  have bases labelled by  $i$  and  $i'$  then we take

$$\mathbf{p} = V \otimes V' \quad \text{with basis } \{X_{ii'}\} \quad (2.29)$$

All bases are orthonormal and the representation matrices  $(M_A)_{ij}$  and  $(M_{A'})_{i'j'}$  for  $so(n)$  and  $so(n')$  will be chosen to satisfy

$$\frac{1}{2}(M_A)_{ij}(M_B)_{ij} = \delta_{AB}, \quad \frac{1}{2}(M_{A'})_{i'j'}(M_{B'})_{i'j'} = \delta_{A'B'} \quad (2.30)$$

It follows, as before, that they also obey

$$(M_A)_{ij}(M_A)_{k\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}, \quad (M_{A'})_{i'j'}(M_{A'})_{k'\ell'} = \delta_{i'k'}\delta_{j'\ell'} - \delta_{i'\ell'}\delta_{j'k'} \quad (2.31)$$

Given our slight change in notation, the definition (2.4) now takes the form

$$[T_A, X_{ii'}] = -(M_A)_{ij} X_{ji'}, \quad [T_{A'}, X_{ii'}] = -(M_{A'})_{i'j'} X_{i'j'} \quad (2.32a)$$

If we convert to bases of type (2.19) for both  $so(n)$  and  $so(n')$ , then Lie algebra indices  $A$  and  $A'$  are replaced by antisymmetric pairs  $k\ell$  and  $k'\ell'$  and we get

$$\begin{aligned} [T_{k\ell}, X_{ii'}] &= -(M_{k\ell})_{ij} X_{ji'} = -\delta_{ki} X_{\ell i'} + \delta_{\ell i} X_{ki'} \\ [T_{k'\ell'}, X_{ii'}] &= -(M_{k'\ell'})_{i'j'} X_{ij'} = -\delta_{k'i'} X_{i\ell'} + \delta_{\ell'i'} X_{ik'} \end{aligned} \quad (2.32b)$$

The definition of the bracket is completed as in (2.12),

$$\begin{aligned} [X_{ii'}, X_{jj'}] &= -(M_A)_{ij} \delta_{i'j'} T_A - \delta_{ij} (M_{A'})_{i'j'} T_{A'} \\ &= -\frac{1}{2}(M_{k\ell})_{ij} \delta_{i'j'} T_{k\ell} - \frac{1}{2}\delta_{ij} (M_{k'\ell'})_{i'j'} T_{k'\ell'} \\ &= -T_{ij} \delta_{i'j'} - \delta_{ij} T_{i'j'} \end{aligned} \quad (2.33)$$

To verify the Jacobi property, it is sufficient to check generators of the form  $XXX$  and for the case at hand a typical term is

$$\begin{aligned} [[X_{ii'}, X_{jj'}], X_{kk'}] &= (M_A)_{ij} (M_A)_{k\ell} \delta_{i'j'} X_{\ell k'} + (M_{A'})_{i'j'} (M_{A'})_{k'\ell'} \delta_{ij} X_{k\ell'} \\ &= \delta_{ik} \delta_{i'j'} X_{jk'} - \delta_{jk} \delta_{i'j'} X_{ik'} + \delta_{i'k'} \delta_{ij} X_{kj'} - \delta_{j'k'} \delta_{ij} X_{ki'} \end{aligned} \quad (2.34)$$

It is easy to see that cycling in the pairs of indices  $ii'$ ,  $jj'$  and  $kk'$  and adding gives zero, as desired.

We have shown, therefore, that there exists a Lie algebra  $\mathbf{g} = \mathbf{h} \oplus \mathbf{p}$  with brackets defined by (2.32) and (2.33); the result is precisely (2.27) once again, so  $\mathbf{g} = so(n+n')$ . Because this works for all positive integers  $n$  and  $n'$ , we will refer to it as a ‘generic’ type of construction. We will see in section 6 how it can be combined with certain ‘exceptional’ features which arise only when  $n, n' = 1, 2, 4$  or  $8$  and how this leads to the magic square. But first we must discuss trialities.

### 3. Trialities and Spinors

The term ‘triality’ is commonly used to refer to the exceptional outer automorphism symmetries of  $so(8)$  (exceptional by comparison with all other simple Lie algebras) and the ensuing relationships amongst its representations [17]. An alternative way of capturing this idea was advocated by Adams [1], who defined trialities as algebraic structures involving three  $n$ -dimensional vector spaces and then went on to establish a one-to-one correspondence with  $n$ -dimensional division algebras. Any triality, in this sense, has a symmetry algebra  $tri(n)$  with just the same kinds of outer automorphisms as  $so(8)$ , thereby setting this most famous example in context, as the head of a family. Its special features then arise because  $tri(8) = so(8)$ , which also allows them to be related in a natural fashion to the corresponding division algebra of octonions  $\mathbb{O}$ . For lower values of  $n$  one finds  $tri(4) = so(4) \oplus su(2)$ , corresponding to  $\mathbb{H}$ , and  $tri(2) = so(2) \oplus so(2)$ , corresponding to  $\mathbb{C}$ , and in these cases the symmetries of  $tri(n)$  depend on the fact that the algebra is no longer simply  $so(n)$ . The only other case allowed is rather degenerate, with  $tri(1)$  trivial, corresponding to  $\mathbb{R}$ . (We will see how to derive the properties of each triality in section 5.)

The approach to trialities in this paper rests firmly on the ideas and results of [1], but there will be a few significant departures and developments in both content and presentation. We start by giving a new

definition of a triality, which is easily shown to be equivalent to the notion of *normed triality* introduced in [1] but which seems to offer some advantages (for our purposes, at least). To avoid confusion, we should also point out that Adams reserves the unqualified term *triality* for a distinct, weaker concept in which the vector spaces are not equipped with inner-products—we shall not consider such things here. Our treatment of trialities and their symmetries will be self-contained and will occupy us for the next three sections of the paper.

Let  $V, S_+, S_-$  be real vector spaces, each with a positive-definite inner-product (the choice of names will be explained by what follows) and consider a trilinear map

$$\gamma : V \times S_+ \times S_- \rightarrow \mathbb{R} \quad (3.1a)$$

If we fix a vector in any one of these spaces, then  $\gamma$  will provide us with linear maps between the remaining two, using the non-degeneracy of their inner-products. Thus, for any elements  $u_V, u_+, u_-$  in  $V, S_+, S_-$ , we have maps

$$\sigma(u_V) : S_- \rightarrow S_+, \quad \sigma(u_+) : V \rightarrow S_-, \quad \sigma(u_-) : S_+ \rightarrow V \quad (3.1b)$$

which are defined by demanding that the inner-products (denoted by dots) satisfy

$$u_+ \cdot (\sigma(u_V)u_-) = u_- \cdot (\sigma(u_+)u_V) = u_V \cdot (\sigma(u_-)u_+) = \gamma(u_V, u_+, u_-) \quad (3.1c)$$

for all  $u_V, u_+, u_-$ . But it is also natural to require that these maps respect the inner-products in some appropriate sense, which prompts the following definition.

A **triality** is a map  $\gamma$  as in (3.1) such that for any  $u_V, u_+, u_-$  of unit length, the corresponding maps  $\sigma(u_V), \sigma(u_+), \sigma(u_-)$  are orthogonal, or length preserving.

An immediate consequence is that  $V$  and  $S_{\pm}$  must all have the same dimension  $n$ , but the concept of a triality is much more restrictive than this. Suppose

$$\gamma' : V' \times S'_+ \times S'_- \rightarrow \mathbb{R} \quad (3.2)$$

is also a triality, with each space of dimension  $n'$ . We say that  $\gamma$  and  $\gamma'$  are isomorphic if there exist invertible linear maps

$$R_V : V \rightarrow V', \quad R_+ : S_+ \rightarrow S'_+, \quad R_- : S_- \rightarrow S'_- \quad (3.3a)$$

(implying  $n = n'$ ) under which the inner-products are invariant, and for which

$$\gamma'(R_V u_V, R_+ u_+, R_- u_-) = \gamma(u_V, u_+, u_-) \quad (3.3b)$$

(Note that the composite map on the left hand side is indeed a triality, for any set of orthogonal maps  $R$ .) Armed with this notion of equivalence, we have the following fundamental result.

**Classification Theorem:** Trialities exist iff  $n = 1, 2, 4$  or  $8$  and they are unique up to isomorphism.

Proof: see the proposition, lemma and corollary below, and the appendix.

The correspondence with division algebras means that the existence of just four trialities matches exactly with the existence and uniqueness of  $\mathbb{R}, \mathbb{C}, \mathbb{H}$  and  $\mathbb{O}$ . In [1], the link with division algebras is also utilised by writing down triality maps explicitly for  $n = 1, 2, 4$  in terms of  $\mathbb{R}, \mathbb{C}, \mathbb{H}$ . But the  $n = 8$  case is dealt with differently (at least to begin with) by applying general results on spin representations. One motivation for treating the  $n = 8$  triality in this fashion is that it allows one to circumvent some of the complications inherent in the non-associative multiplication of  $\mathbb{O}$ .

In this paper we will work in terms of trialities throughout, returning to division algebras only briefly, in section 7, to outline connections with previous results. From this point of view, the classification of division algebras can be regarded as a corollary of the Classification Theorem for trialities, rather than the other way around. We shall give a uniform treatment of all four cases, with spinors playing a central role. The manner in which trialities can be realised in terms of spin representations is explained in [1,3,5], but the relationship is even more intimate than these accounts might suggest. We will show that spinors emerge *inevitably* from the concept of a triality, with  $\gamma$  becoming an  $so(n)$ -invariant map.

Let  $u$  be a vector belonging to any one of the spaces  $V, S_+, S_-$ . The map  $\sigma(u)$  between the other two spaces is orthogonal whenever  $u$  is of unit length, by definition. But since  $\gamma$  is multilinear this implies

$$\sigma(u)\sigma(u)^T = |u|^2 1, \quad \sigma(u)^T\sigma(u) = |u|^2 1 \quad (3.4a)$$

for  $u$  with *any* squared length  $|u|^2 = u \cdot u$  (the identity operator on each space being denoted by 1). Equivalent forms are obtained by polarising: combining the relations above as applied to  $u, w$  (any other vector in the same space) and  $u + w$  gives

$$\begin{aligned} \sigma(u)\sigma(w)^T + \sigma(w)\sigma(u)^T &= 2u \cdot w 1 \\ \sigma(u)^T\sigma(w) + \sigma(w)^T\sigma(u) &= 2u \cdot w 1 \end{aligned} \quad (3.4b)$$

Thus, within our definition of a triality lies a structure which closely resembles a Clifford algebra. Rather than building spin representations from this as it stands, however, it will be convenient for us to re-express things using index or component language.

Consider orthonormal bases for  $V, S_+, S_-$  with labels of type  $a, \alpha, \dot{\alpha}$  (each taking  $n$  values) reserved for each of these spaces. Since the nature of the index then also identifies the space to which a vector belongs, we will write the components of  $u_V, u_+, u_-$  simply as  $u_a, u_\alpha, u_{\dot{\alpha}}$ , omitting the additional subscripts to avoid cluttering our expressions. With respect to these bases,  $\gamma$  has a set of real components  $\gamma_{a\alpha\dot{\alpha}}$  and the matrices for the maps in (3.1b) are

$$\sigma(u_V)_{a\dot{\alpha}} = u_a \gamma_{a\alpha\dot{\alpha}}, \quad \sigma(u_+)_{\dot{\alpha}a} = u_\alpha \gamma_{a\alpha\dot{\alpha}}, \quad \sigma(u_-)_{a\alpha} = u_{\dot{\alpha}} \gamma_{a\alpha\dot{\alpha}} \quad (3.5)$$

Clearly, (3.4) holds for any  $u$  and  $w$ , in any one of the spaces, iff

$$\gamma_{a\alpha\dot{\alpha}} \gamma_{b\beta\dot{\alpha}} + \gamma_{b\alpha\dot{\alpha}} \gamma_{a\beta\dot{\alpha}} = 2\delta_{ab} \delta_{\alpha\beta} \quad (3.6a)$$

$$\gamma_{a\alpha\dot{\alpha}} \gamma_{b\alpha\dot{\beta}} + \gamma_{b\alpha\dot{\alpha}} \gamma_{a\alpha\dot{\beta}} = 2\delta_{ab} \delta_{\dot{\alpha}\dot{\beta}} \quad (3.6b)$$

$$\gamma_{a\alpha\dot{\alpha}} \gamma_{a\beta\dot{\beta}} + \gamma_{a\beta\dot{\alpha}} \gamma_{a\alpha\dot{\beta}} = 2\delta_{\alpha\beta} \delta_{\dot{\alpha}\dot{\beta}} \quad (3.6c)$$

Together, these conditions constitute the definition of a triality in component form.

There is also something very important to be learnt about the logical connections between the individual equations in (3.6), which can be uncovered as follows. For any  $u_V$  in  $V$  with  $u_V \cdot u_V = u_a u_a = 1$  we have

$$\sigma(u_V)\sigma(u_V)^T = 1 \quad \text{or} \quad \sigma(u_V)_{a\dot{\alpha}} \sigma(u_V)_{\beta\dot{\alpha}} = u_a \gamma_{a\alpha\dot{\alpha}} u_b \gamma_{b\beta\dot{\alpha}} = \delta_{\alpha\beta} \quad (3.7a)$$

which is true iff (3.6a) holds; but also

$$\sigma(u_V)^T \sigma(u_V) = 1 \quad \text{or} \quad \sigma(u_V)_{a\dot{\alpha}} \sigma(u_V)_{\alpha\dot{\beta}} = u_a \gamma_{a\alpha\dot{\alpha}} u_b \gamma_{b\alpha\dot{\beta}} = \delta_{\dot{\alpha}\dot{\beta}} \quad (3.7b)$$

which is true iff (3.6b) holds. Now (3.7a) is equivalent to (3.7b) simply because the matrix for  $\sigma(u_V)$  is square. This means that (3.6a) is equivalent to (3.6b) just by virtue of  $S_+$  and  $S_-$  having the same dimension. We can obviously apply identical reasoning with the spaces permuted, so  $\sigma(u_+)$  is orthogonal for any unit vector  $u_+$  iff (3.6a) and (3.6c) hold, and  $\sigma(u_-)$  is orthogonal for any unit vector  $u_-$  iff (3.6b) and (3.6c) hold. The full significance of these remarks will become clear shortly, in justifying the Classification Theorem.

Now to discuss spinors, using the component definition (3.6). The group  $SO(n)$  which preserves the inner-product on  $V$  has a Lie algebra which we will denote by  $so(n)_V = \wedge^2 V$ . Its generators can be written  $T_{cd}$ , labelled by antisymmetric pairs of vector indices, with the fundamental representation

$$(M_{cd})_{ab} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \quad (3.8)$$

all as in section 2. But once  $a$  and  $b$  are interpreted as vector indices, the conditions (3.6a) and (3.6b) are recognisable as Dirac/Clifford-like relations which allow the construction of spin representations of  $so(n)_V$  on  $S_\pm$  along familiar lines. To be specific, the real antisymmetric matrices

$$\begin{aligned} (M_{cd})_{\alpha\beta} &\equiv \frac{1}{2}(\gamma_{cd})_{\alpha\beta} \equiv \frac{1}{4}(\gamma_{c\alpha\dot{\alpha}}\gamma_{d\beta\dot{\alpha}} - \gamma_{d\alpha\dot{\alpha}}\gamma_{c\beta\dot{\alpha}}) \\ (M_{cd})_{\dot{\alpha}\dot{\beta}} &\equiv \frac{1}{2}(\gamma_{cd})_{\dot{\alpha}\dot{\beta}} \equiv \frac{1}{4}(\gamma_{c\alpha\dot{\alpha}}\gamma_{d\alpha\dot{\beta}} - \gamma_{d\alpha\dot{\alpha}}\gamma_{c\alpha\dot{\beta}}) \end{aligned} \quad (3.9)$$

each obey the same algebra as  $(M_{cd})_{ab}$  by virtue of (3.6a) and (3.6b). It is also simple to check that

$$(M_{cd})_{ab}\gamma_{b\alpha\dot{\alpha}} + (M_{cd})_{\alpha\beta}\gamma_{a\beta\dot{\alpha}} + (M_{cd})_{\dot{\alpha}\dot{\beta}}\gamma_{a\alpha\dot{\beta}} = 0 \quad (3.10)$$

so that the triality is invariant under  $so(n)_V$  provided that each space  $V$ ,  $S_\pm$  transforms appropriately.

Let us compare this with the construction of spin representations in general Euclidean dimension  $n$ . Spinors are commonly described using complex vector spaces as opposed to the real spaces  $S_\pm$  we have been using, but this is just a matter of notation: we can choose to write everything in real language if we wish, by regarding any complex vector space as a real space of twice the dimension. Once this has been done, the Dirac/Clifford construction for general  $so(n)$  can always be expressed in terms of a map  $\gamma : V \times S_+ \times S_- \rightarrow \mathbb{R}$  with  $V$  of dimension  $n$ , but with  $S_\pm$  now each having real dimension  $N$ , say. The conditions (3.6a) and (3.6b) still apply and they ensure that the spin representations can be defined by (3.9), as before.

To translate this into more conventional language, consider the maps

$$\sigma_a : S_- \rightarrow S_+, \quad \sigma_a^T : S_+ \rightarrow S_- \quad (3.11)$$

defined by (3.1b) for each basis vector in  $V$ , as specified by the label  $a$ . In matrix notation,

$$(\sigma_a)_{\alpha\dot{\alpha}} = \gamma_{a\alpha\dot{\alpha}} \quad (3.12)$$

and the conditions (3.6a) and (3.6b) are equivalent to

$$\sigma_a \sigma_b^T + \sigma_b \sigma_a^T = 2\delta_{ab} \quad (3.13a)$$

$$\sigma_a^T \sigma_b + \sigma_b^T \sigma_a = 2\delta_{ab} \quad (3.13b)$$

(this also follows directly from (3.4)). Standard Dirac matrices are then given by

$$\Gamma_a : S \rightarrow S \quad \text{with} \quad \Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a^T & 0 \end{pmatrix} \quad \text{on} \quad S = S_+ \oplus S_- \quad (3.14)$$

since (3.13) is clearly equivalent to

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} \quad (3.15)$$

The representation of  $so(n)_V$  on  $S$  is reducible, with generators  $M_{cd} = \frac{1}{2}\Gamma_{cd}$  having a block structure

$$\Gamma_{cd} \equiv \frac{1}{2}(\Gamma_c\Gamma_d - \Gamma_d\Gamma_c) = \frac{1}{2} \begin{pmatrix} \sigma_c\sigma_d^T - \sigma_d\sigma_c^T & 0 \\ 0 & \sigma_c^T\sigma_d - \sigma_d^T\sigma_c \end{pmatrix} \quad (3.16)$$

From this more general perspective, we can derive a simple criterion for the existence of a triality. In discussing (3.7) above we noted that (3.6a) and (3.6b) are equivalent whenever  $S_+$  and  $S_-$  have the same dimension, because  $\sigma(u_V)$  can then be expressed as a square matrix, with a left inverse equal to its right inverse. The same argument can be applied to  $\sigma(u_\pm)$ : these maps will correspond to square matrices, and hence (3.6a) or (3.6b) will be equivalent to (3.6c), whenever  $S_\mp$  have the same dimension as  $V$ . Such a matching of vector and spinor dimensions,  $N = n$ , is therefore both necessary *and* sufficient for the existence of a triality since all three conditions in (3.6) will be satisfied.<sup>(1)</sup> We record this important conclusion in the following form.

**Proposition:** If  $\sigma_a$  with  $a = 1, \dots, n$  are real  $N \times N$  matrices obeying (3.13a) or (3.13b) (one implies the other) then (3.12) defines a triality  $\gamma$  iff  $N = n$ .

The possible values of  $N$  which arise for each  $n$  can be found by consulting one of the extensive treatments of Clifford algebras and spin representations in the literature (e.g. [6,9,10,17]) and these sources duly confirm that  $N = n$  for precisely the four cases given in the Classification Theorem. But it is also worth pointing out that only a rather small part of the whole edifice of Clifford technology is really required to reach this conclusion, and so it is just as well to state and prove what we need directly.

**Lemma:** If  $\Gamma_a$  with  $a = 1, \dots, 2\ell$  are  $2N \times 2N$  real symmetric matrices obeying (3.15) then  $2^{\ell-1} \mid N$ .

Proof: (3.15) implies that each matrix  $\Gamma_{cd} = \frac{1}{2}(\Gamma_c\Gamma_d - \Gamma_d\Gamma_c)$  is real and antisymmetric with  $(\Gamma_{cd})^2 = -1$ . Its eigenvalues are therefore  $\pm i$ , and a mutually commuting set such as  $\{\Gamma_{12}, \Gamma_{34}, \dots, \Gamma_{2\ell-1, 2\ell}\}$  can be simultaneously diagonalised to produce a set of such eigenvalues. But given any joint eigenvector, we can apply matrices  $\Gamma_a$  to it and obtain new eigenvectors with any desired set of eigenvalues—applying  $\Gamma_1$  or  $\Gamma_2$ , for instance, gives a new eigenvector with the eigenvalue of  $\Gamma_{12}$  reversed but every other eigenvalue unchanged (using (3.15) again). All  $2^\ell$  possible signs are therefore allowed for the eigenvalues, and since the eigenspaces have a common dimension (there are invertible maps between them)  $2N$  is a multiple of  $2^\ell$ .

**Corollary:** If  $\gamma$  is a triality with  $n > 1$ , then  $n$  is even and  $2^{n/2} \mid 2n$ , implying  $n = 2, 4$  or  $8$ .

Proof: Given a triality  $\gamma$ , there exist matrices  $\sigma_a$  and  $\Gamma_a$  with  $N = n$ , by the proposition. If  $n > 1$  there is at least one matrix  $\Gamma_{ab}$  with a block form (3.16), and since  $(\Gamma_{ab})^2 = -1$  the size  $n$  of its blocks must be even. The rest follows immediately from the lemma.

Having shown that there are only four cases to consider, existence can be established in a similarly elementary fashion. By appealing to the proposition once again, it is sufficient to construct sets of matrices  $\sigma_a$  obeying (3.13) with  $N = n$ , which can easily be done. We will relegate further discussion to the appendix, however, which contains a fuller account of the relationship with the usual approach to Clifford algebra representations.

Treating  $V$  differently from  $S_\pm$  is helpful when comparing trialities with spinors in general dimensions, and we chose our notation with this in mind. But there is nothing in the definition of a triality which distinguishes any one of the spaces  $V, S_+, S_-$  from the remaining two. Moreover, equations such as (3.6) are related to one another under permutations of these spaces, and the arguments above involving  $\sigma_a$ , for instance, could equally well be given in terms of maps  $\sigma_\alpha$  or  $\sigma_{\dot{\alpha}}$  corresponding to unit vectors in  $S_\pm$  instead of  $V$ . The idea that all three vector spaces appear in the triality on exactly the same footing (our choice of notation notwithstanding) is the key to everything which follows. We will investigate this more thoroughly in the next section by studying symmetries of trialities.

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<sup>(1)</sup> A similar argument was used in [5] in establishing a correspondence with simple supersymmetric Yang-Mills.

#### 4. Triality Symmetries

Given any triality  $\gamma$ , there is a group  $O(n) \times O(n) \times O(n)$  which acts on  $V \times S_+ \times S_-$  and preserves the inner-product on each space. The **triality group** is the subgroup of elements  $(R_V, R_+, R_-)$  under which  $\gamma$  is invariant<sup>(2)</sup> so that

$$\gamma(R_V u_V, R_+ u_+, R_- u_-) = \gamma(u_V, u_+, u_-) \quad (4.1)$$

for all  $u_V, u_+, u_-$ . We will concentrate on the corresponding **triality algebra**, denoted by

$$tri(n) \subset so(n) \oplus so(n) \oplus so(n) \quad (4.2)$$

and defined as the subalgebra of elements

$$T \leftrightarrow (M_V, M_+, M_-) \text{ or } (M_{ab}, M_{\alpha\beta}, M_{\dot{\alpha}\dot{\beta}}) \quad (4.3a)$$

which obey

$$\gamma(M_V u_V, u_+, u_-) + \gamma(u_V, M_+ u_+, u_-) + \gamma(u_V, u_+, M_- u_-) = 0 \quad (4.3b)$$

or, in components,

$$M_{ab} \gamma_{b\alpha\dot{\alpha}} + M_{\alpha\beta} \gamma_{a\beta\dot{\alpha}} + M_{\dot{\alpha}\dot{\beta}} \gamma_{a\alpha\dot{\beta}} = 0 \quad (4.3c)$$

Note that when indices are written explicitly on any matrix  $M$  they also serve to identify the space on which it acts, allowing us to omit the subscripts  $V, \pm$ , just as we have chosen to do when writing vectors in components.

The definition of  $tri(n)$  as a subalgebra of  $so(n) \oplus so(n) \oplus so(n)$  means that it inherits a positive-definite, invariant inner-product. It also means that the algebra comes equipped with representations on each of the spaces  $V, S_+, S_-$  given by the matrices  $M$ . Let  $\{T_A\}$  be any orthonormal basis for  $tri(n)$ , with corresponding representation matrices

$$T_A \leftrightarrow ((M_A)_{ab}, (M_A)_{\alpha\beta}, (M_A)_{\dot{\alpha}\dot{\beta}}) \quad (4.4)$$

The overall scale of the inner-product can be fixed in terms of these representations by imposing

$$(T_A, T_B) = k \left\{ (M_A)_{ab} (M_B)_{ab} + (M_A)_{\alpha\beta} (M_B)_{\alpha\beta} + (M_A)_{\dot{\alpha}\dot{\beta}} (M_B)_{\dot{\alpha}\dot{\beta}} \right\} = \delta_{AB} \quad (4.5)$$

The value of the coefficient  $k$  is a detail to which we will return below.

Our earlier result (3.10) can now be expressed by saying that

$$so(n)_V = \wedge^2 V \text{ with basis } \{T_{cd}\} \quad (4.6a)$$

and representations (3.8) and (3.9) is a subalgebra of  $tri(n)$ . But from the permutation symmetry of (3.6) it is clear that we can equally well construct other subalgebras

$$so(n)_\pm = \wedge^2 S_\pm \text{ with bases } \{T_{\gamma\delta}\} \text{ or } \{T_{\dot{\gamma}\dot{\delta}}\} \quad (4.6b)$$

simply by interchanging the roles of the three spaces. To carry this out explicitly we introduce a little more notation, defining

$$(\gamma_{ab})_{\alpha\beta} \equiv (\gamma_{\alpha\beta})_{ab} \equiv \frac{1}{2}(\gamma_{a\alpha\dot{\alpha}}\gamma_{b\beta\dot{\alpha}} - \gamma_{b\alpha\dot{\alpha}}\gamma_{a\beta\dot{\alpha}}) \quad (4.7a)$$

$$(\gamma_{\dot{\alpha}\dot{\beta}})_{ab} \equiv (\gamma_{ab})_{\dot{\alpha}\dot{\beta}} \equiv \frac{1}{2}(\gamma_{a\alpha\dot{\alpha}}\gamma_{b\beta\dot{\alpha}} - \gamma_{b\alpha\dot{\alpha}}\gamma_{a\beta\dot{\alpha}}) \quad (4.7b)$$

$$(\gamma_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} \equiv (\gamma_{\dot{\alpha}\dot{\beta}})_{\alpha\beta} \equiv \frac{1}{2}(\gamma_{a\alpha\dot{\alpha}}\gamma_{b\beta\dot{\alpha}} - \gamma_{b\alpha\dot{\alpha}}\gamma_{a\beta\dot{\alpha}}) \quad (4.7c)$$

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<sup>(2)</sup> This is the automorphism group, with isomorphism of trialities defined in (3.3); automorphism groups of isomorphic trialities are conjugate in  $O(n) \times O(n) \times O(n)$ .

These antisymmetrised combinations of  $\gamma$  components generalise those in (3.9) which were used to introduce spin representations of (4.6a); the results can be summarised

$$so(n)_V : \quad (M_{cd})_{ab} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}, \quad (M_{cd})_{\alpha\beta} = \frac{1}{2}(\gamma_{cd})_{\alpha\beta}, \quad (M_{cd})_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\gamma_{cd})_{\dot{\alpha}\dot{\beta}} \quad (4.8a)$$

acting on  $V, S_+, S_-$ . Now we define, similarly, the algebras (4.6b) and their representations

$$so(n)_+ : \quad (M_{\gamma\delta})_{\alpha\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}, \quad (M_{\gamma\delta})_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\gamma_{\gamma\delta})_{\dot{\alpha}\dot{\beta}}, \quad (M_{\gamma\delta})_{ab} = \frac{1}{2}(\gamma_{\gamma\delta})_{ab} \quad (4.8b)$$

acting on  $S_+, S_-, V$  respectively; and

$$so(n)_- : \quad (M_{\dot{\gamma}\dot{\delta}})_{\dot{\alpha}\dot{\beta}} = \delta_{\dot{\alpha}\dot{\gamma}}\delta_{\dot{\beta}\dot{\delta}} - \delta_{\dot{\alpha}\dot{\delta}}\delta_{\dot{\beta}\dot{\gamma}}, \quad (M_{\dot{\gamma}\dot{\delta}})_{ab} = \frac{1}{2}(\gamma_{\dot{\gamma}\dot{\delta}})_{ab}, \quad (M_{\dot{\gamma}\dot{\delta}})_{\alpha\beta} = \frac{1}{2}(\gamma_{\dot{\gamma}\dot{\delta}})_{\alpha\beta} \quad (4.8c)$$

acting on  $S_-, V, S_+$  respectively. The invariance of the triality under any one of these  $so(n)$  algebras is easy to verify; on the other hand, the permutation symmetry of (3.6) implies that  $\gamma$  must be invariant under  $so(n)_\pm$  once we know it is invariant under  $so(n)_V$ .

The algebras and representations in (4.8) can also be described in a basis-independent way, of course. Given  $u$  and  $w$  belonging to one of the spaces  $V, S_+, S_-$ , we can regard  $u \wedge w$  as acting in the fundamental representation of the appropriate  $so(n)$  algebra, as discussed in section 2; the relations (3.4) then ensure that we also have corresponding actions on the other two spaces given by maps

$$\begin{aligned} &\frac{1}{4}(\sigma(u)\sigma(w)^T - \sigma(w)\sigma(u)^T) \\ &\frac{1}{4}(\sigma(u)^T\sigma(w) - \sigma(w)^T\sigma(u)) \end{aligned} \quad (4.9)$$

The component versions (4.8), with the accompanying notation introduced in (4.7), prove to be extremely useful and versatile, however, so we shall continue to deal mainly with these forms. To help keep the meanings clear despite the proliferation of indices attached to each  $M$  or  $\gamma$ , observe that in (4.8) pairs of indices inside brackets label Lie algebra generators, while those outside refer to the vector spaces on which the generators act. We also note for future reference that the defining relations for a triality in (3.6) can now be re-expressed

$$\gamma_{a\alpha\dot{\alpha}}\gamma_{b\beta\dot{\alpha}} = \delta_{ab}\delta_{\alpha\beta} + (\gamma_{ab})_{\alpha\beta} \quad (4.10a)$$

$$\gamma_{a\alpha\dot{\alpha}}\gamma_{b\alpha\dot{\beta}} = \delta_{ab}\delta_{\dot{\alpha}\dot{\beta}} + (\gamma_{\dot{\alpha}\dot{\beta}})_{ab} \quad (4.10b)$$

$$\gamma_{a\alpha\dot{\alpha}}\gamma_{a\beta\dot{\beta}} = \delta_{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}} + (\gamma_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} \quad (4.10c)$$

There is still a technical detail to clear up before we move on. In section 2 we fixed normalisation conventions for  $so(n)$  and its inner-product, expressing these in terms of the fundamental representation by equations (2.17) through to (2.21). We will insist on exactly the same conventions for  $so(n)_V$  and  $so(n)_\pm$ , and by comparing their fundamental representations on  $V$  and  $S_\pm$  in (4.8) with (2.21), we see that this corresponds to the bases in (4.6) being orthonormal. But we also want these inner-products to arise as the restrictions of the inner-product on  $tri(n)$  given in (4.5). The normalisations work out correctly provided we choose

$$k = (\frac{1}{2}n + 2)^{-1} \quad (4.11)$$

as can be checked from (4.8).

It remains to actually determine  $tri(n)$  and to understand how its three  $so(n)$  subalgebras interlock. Since there are only four cases to consider, it is tempting just to analyse each of them in turn, but there are benefits to delaying such a case-by-case approach for as long as possible. We will now derive some simple results on the structure of  $tri(n)$ , for any  $n$ , which will eventually be crucial for our uniform construction of the algebras in the magic square. We will also see, in the next section, how these results can be applied to reveal the specific properties of each triality rather easily.

A natural way to investigate  $tri(n)$  is to take each  $so(n)$  subalgebra and ask what is left over. Considering  $so(n)_V$ , for instance, we can plainly express *any* antisymmetric matrix acting on  $V$  as a linear combination of the generators in (4.8a). So, by linearity, it suffices to consider the subalgebra of  $tri(n)$  which fixes all elements in  $V$ , consisting of what might reasonably be called *internal* transformations (as far as  $V$  is concerned). We will therefore define  $int(n)_V$  to be the subalgebra of  $tri(n)$  with  $M_V = 0$ ; or

$$T_A \text{ is a generator of } int(n)_V \iff (M_A)_{ab} = 0 \quad (4.12a)$$

Similarly, we define  $int(n)_\pm$  as the subalgebras of  $tri(n)$  which fix all elements in  $S_\pm$ , so that  $M_\pm = 0$ ; or

$$T_A \text{ is a generator of } int(n)_\pm \iff (M_A)_{\alpha\beta} = 0 \text{ or } (M_A)_{\dot{\alpha}\dot{\beta}} = 0 \quad (4.12b)$$

From these definitions we can deduce the following.

**Proposition:** As orthogonal direct sums of Lie algebras

$$\begin{aligned} tri(n) &= so(n)_V \oplus int(n)_V \\ &= so(n)_+ \oplus int(n)_+ \\ &= so(n)_- \oplus int(n)_- \end{aligned} \quad (4.13)$$

In addition  $int(n)_V$ ,  $int(n)_+$ ,  $int(n)_-$  are commuting subalgebras with trivial pairwise intersections

$$int(n)_V \cap int(n)_\pm = int(n)_+ \cap int(n)_- = \{0\} \quad (4.14)$$

**Proof:** We can fix attention on  $V$ , with the understanding that similar arguments will also apply to  $S_\pm$ . We have already noted that any element of  $tri(n)$  is a sum of elements in  $so(n)_V$  and  $int(n)_V$  and their intersection is clearly trivial. Now if  $(0, \lambda_{\alpha\beta}, \lambda_{\dot{\alpha}\dot{\beta}})$  belongs to  $int(n)_V$  then (4.3) states that

$$\lambda_{\alpha\beta} \gamma_{a\beta\dot{\alpha}} + \lambda_{\dot{\alpha}\dot{\beta}} \gamma_{a\alpha\dot{\beta}} = 0 \quad (4.15)$$

and it is easy to deduce (by contracting with  $\gamma_{b\gamma\dot{\alpha}}$  or  $\gamma_{b\alpha\dot{\gamma}}$ ) that  $\lambda_{\alpha\beta}$  and  $(\gamma_{ab})_{\alpha\beta}$  commute, as do  $\lambda_{\dot{\alpha}\dot{\beta}}$  and  $(\gamma_{ab})_{\dot{\alpha}\dot{\beta}}$ . Referring to (4.8a), this means that  $so(n)_V$  and  $int(n)_V$  are commuting subalgebras, and so  $tri(n)$  is indeed their direct sum. A further consequence of (4.15), which can be obtained in a similar fashion, is

$$\lambda_{\alpha\beta} (\gamma_{ab})_{\alpha\beta} + \lambda_{\dot{\alpha}\dot{\beta}} (\gamma_{ab})_{\dot{\alpha}\dot{\beta}} = 0 \quad (4.16)$$

and so the subalgebras are orthogonal with respect to the inner-product in (4.5). The remaining assertions in the proposition are also simple consequences of the definitions; for example, the intersection of any pair of subalgebras is trivial since if two of the matrices  $M$  vanish in (4.3) then so must the third.

**Corollary:** The subalgebras  $so(n)_V$ ,  $so(n)_+$ ,  $so(n)_-$  arise as projections of  $tri(n)$ , with their generators (4.6) given in terms of the corresponding representation matrices by

$$T_{ab} = (M_A)_{ab} T_A, \quad T_{\alpha\beta} = (M_A)_{\alpha\beta} T_A, \quad T_{\dot{\alpha}\dot{\beta}} = (M_A)_{\dot{\alpha}\dot{\beta}} T_A \quad (4.17)$$

**Proof:** Compare with the treatment of  $so(n)$  in section 2, leading to (2.19). Since we have been careful to adopt the same normalisation conventions for all copies of  $so(n)$ , the only difference lies in the nature of the sum over  $A$ . In (4.17), as throughout this section,  $A$  labels a basis for  $tri(n)$ , and this is strictly larger than each  $so(n)$  subalgebra if  $int(n)$  is non-trivial. But the orthogonal decompositions in the proposition above imply that each sum in (4.17) can be split into a contribution from the appropriate  $so(n)$  subalgebra

and another from the companion  $\text{int}(n)$  factor, and the definitions (4.12) imply that the latter contributions vanish. The results therefore follow, just as in section 2.

It is instructive to check this by showing it is consistent with the results in section 2 in a slightly different way. Taking the appropriate representations of each generator  $T$  in (4.17) and then using (4.8) gives

$$\begin{aligned} (M_A)_{ab}(M_A)_{cd} &= (M_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \\ (M_A)_{\alpha\beta}(M_A)_{\gamma\delta} &= (M_{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} \\ (M_A)_{\dot{\alpha}\dot{\beta}}(M_A)_{\dot{\gamma}\dot{\delta}} &= (M_{\dot{\alpha}\dot{\beta}})_{\dot{\gamma}\dot{\delta}} = \delta_{\dot{\alpha}\dot{\gamma}}\delta_{\dot{\beta}\dot{\delta}} - \delta_{\dot{\alpha}\dot{\delta}}\delta_{\dot{\beta}\dot{\gamma}} \end{aligned} \quad (4.18)$$

But we can also arrive at the end results directly, by comparison with (2.21). The sums here have a larger range, in general, since  $A$  runs over a basis for  $\text{tri}(n)$ , but once again the sum can be split into a contribution from an  $so(n)$  subalgebra and another from its orthogonal complement  $\text{int}(n)$ , with the latter making no contribution. Other, similar relations can also be deduced from (4.17) by taking representations of  $T$  other than the fundamental, and we will find this idea useful later.

Returning to the proposition itself, there is also another way to understand the direct sums in (4.13), based on the following general observation. If a Lie algebra with an invariant positive-definite inner-product has a unitary representation  $\rho$ , then the algebra is isomorphic to  $\text{Ker}\rho \oplus \text{Imp}\rho$ . We can apply this to  $\text{tri}(n)$  and each of its representations on  $V, S_{\pm}$ : it is easy to see that the image of each representation is isomorphic to the corresponding copy of  $so(n)$ , while the kernel is the relevant  $\text{int}(n)$  algebra. Hence, we obtain the three orthogonal decompositions in (4.13).

The dominant theme in our discussion of triality symmetries has been the permutation symmetry which is manifest in all key relations, from the definitions (3.1) or (3.6), through the construction of the algebras (4.6) and (4.8), to the proposition above and the decompositions in (4.13). This symmetry entails nothing more than the ability to interchange, wholesale, the roles of the underlying spaces  $V, S_+, S_-$  in everything we do, whether in constructions or proofs. There is a much more sophisticated and elegant version of this idea, which the concept of a triality was specifically designed to capture, and which we will present in the form of a theorem. This will be the logical culmination of our discussion of symmetries, and it provides a new perspective on the results we have obtained so far. Nevertheless, these earlier conclusions are quite sufficient, in themselves, to allow us to construct the magic square, without appealing directly to the more sophisticated result which now follows.

To begin with, we must prepare the ground.

**Lemma:** Given a triality  $\gamma$ , let  $e_V$  be any unit vector in  $V$  and consider the associated orthogonal maps

$$\tau(e_V) : V \rightarrow V, \quad \sigma(e_V) : S_- \rightarrow S_+, \quad \sigma(e_V)^T : S_+ \rightarrow S_- \quad (4.19)$$

with  $\sigma(e_V)$  defined in (3.1) and  $\tau(e_V)$  a standard reflection

$$\tau(e_V)u_V = u_V - 2(u_V \cdot e_V)e_V \quad \text{or} \quad \tau(e_V)_{ab} = \delta_{ab} - 2e_ae_b \quad (4.20)$$

the latter equation giving its matrix form. Then

$$\gamma(\tau(e_V)u_V, \sigma(e_V)u_-, \sigma(e_V)^Tu_+) = -\gamma(u_V, u_+, u_-) \quad (4.21)$$

for all  $u_V, u_+, u_-$ .

**Proof:** This is easily confirmed by a short calculation to unwind the definitions, either in components or using a basis-independent approach.

Since composing reflections gives rotations, the lemma provides an alternative way to investigate the invariance of  $\gamma$ , and one which allows the study the triality *group* rather than just its algebra. From the result as stated, we can deduce invariance under a subgroup corresponding to  $so(n)_V$ , but by considering reflections on  $S_{\pm}$  and associated maps between the remaining pairs of spaces we can equally well construct the groups corresponding to  $so(n)_{\pm}$ . Our concern here, however, is to put the lemma to a different use.

**Permutation Theorem:** For  $n > 1$ ,  $tri(n)$  has a permutation group  $S_3$  of outer automorphisms which acts on its representations  $V$ ,  $S_+$ ,  $S_-$ , or equivalently permutes its subalgebras  $so(n)_V$ ,  $so(n)_+$ ,  $so(n)_-$  and also  $int(n)_V$ ,  $int(n)_+$ ,  $int(n)_-$ .

Proof: Outer automorphisms are equivalence classes of automorphisms (see e.g. [1,2]) and we can define representatives in each class. It suffices to find an automorphism which exchanges any pair of spaces, and so we consider  $S_+$  and  $S_-$ . Let  $\tau$  and  $\sigma$  be the maps defined by some unit vector  $e_V$  (the dependence on which we now choose to suppress, to simplify notation) as in the lemma above. It is straightforward to check that (4.21) implies that

$$(M_V, M_+, M_-) \mapsto (\tau M_V \tau, \sigma M_- \sigma^T, \sigma^T M_+ \sigma) \quad (4.22)$$

is an automorphism of  $tri(n)$ . It certainly exchanges the representations on  $S_{\pm}$  up to similarity, and different choices of the unit vector  $e_V$  are related by rotations in  $V$  and correspond to the same equivalence class of automorphisms.

In looking back over our general treatment of trialities, we can underscore the close links with ideas in theoretical physics. Although we are dealing here with Euclidean signature, the construction of spin representations of  $so(n)$  follows the same route as any elementary introduction to the Dirac equation. The invariance of the triality (3.10) is also obtained in just the same way that one checks Lorentz-invariance of the Dirac lagrangian, and the lemma leading to the Permutation Theorem can be viewed as a discussion of parity for spinors. At a slightly less elementary level, the  $int(n)$  algebras arise in the same fashion as  $R$ -symmetries in supersymmetric theories, and our proof of the Classification Theorem has strong links with the treatment of supersymmetric Yang-Mills (or classical superstrings) given in [5].

## 5. Trialities Case by Case

We will now use the general results of sections 3 and 4 to derive the distinctive properties of each of the four trialities. Our strategy will be to consider

$$tri(n) \subset so(n) \oplus so(n) \oplus so(n) \quad (5.1)$$

in conjunction with the results (4.13) and (4.14) of the proposition in the last section. Together these impose strong constraints, but with different consequences for each value of  $n$ . As far as our principal aim of constructing the magic square is concerned, none of the results in this section are needed to understand how or why this works; they are relevant to identifying the algebras which emerge from the construction, however.

We begin with the case  $n = 8$ . Since  $so(8)$  is a simple Lie algebra, we can deduce immediately from (4.13) and (4.14) that

$$tri(8) = so(8)_V = so(8)_+ = so(8)_- \quad \text{with} \quad int(8) \text{ trivial} \quad (5.2)$$

This is a striking result, sometimes referred to as *Principle of Triality* (see [17] for references to the earlier literature). It states that if the action of  $tri(8)$  is given on any one of  $V$ ,  $S_+$  or  $S_-$ , then its action on the other spaces is determined by (4.3). (We are always working locally, at the level of algebras; the statement for the corresponding groups must take account of finite, discrete ambiguities associated with their global structure.)

Since *any* two copies of  $so(8)$  are isomorphic, it is important to realise that the equalities above tell us far more: each of these  $so(8)$  algebras consists of precisely the same set of transformations, despite their differing actions on  $V$ ,  $S_+$ ,  $S_-$ . It must be possible, therefore, to relate the orthonormal bases defined in (4.6) and (4.8), and this has effectively been achieved already by the corollary of the last section. To be more specific, if we replace the index  $A$  in (4.17) by various kinds of antisymmetric pairs, we can obtain, for instance,

$$T_{\alpha\beta} = \frac{1}{2}(M_{ab})_{\alpha\beta} T_{ab}, \quad T_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(M_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} T_{\alpha\beta}, \quad T_{ab} = \frac{1}{2}(M_{\dot{\alpha}\dot{\beta}})_{ab} T_{\dot{\alpha}\dot{\beta}} \quad (5.3)$$

(recall the convention explained in section 2, according to which a factor  $\frac{1}{2}$  naturally accompanies such sums over antisymmetric pairs). The representation matrices  $M$  have now taken on a new interpretation (via implicit dualisations, yet again)—they provide the changes of base which allow us to identify

$$\wedge^2 V = \wedge^2 S_+ = \wedge^2 S_- \quad \text{for } n = 8 \quad (5.4)$$

The Permutation Theorem is manifest in the symmetry of the Dynkin diagram of  $so(8)$ , with  $S_3$  acting on the outer nodes corresponding to the representations  $V$ ,  $S_+$ ,  $S_-$ .

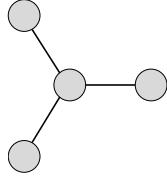


Figure 1: The Dynkin diagram of  $tri(8) = so(8)$

The central node corresponds to the adjoint representation, which is exactly what appears in (5.4). As the vector and spinor representation are permuted, the adjoint representation is inert, but only up to the changes of base, as in (5.3).

Of the remaining trialities, the case with  $n = 1$  is rather degenerate. With  $V$  and  $S_\pm$  one-dimensional, we can identify each of them with  $\mathbb{R}$  and take the triality map to be

$$\gamma(x, y, z) = xyz \quad (5.5)$$

without loss of generality. The Lie algebras  $tri(n)$ ,  $so(n)$  and  $int(n)$  are all trivial when  $n = 1$ , of course, and so this case does not even qualify for inclusion in the Permutation Theorem (although this could be adapted to apply to the discrete triality *group* instead [1]). The cases  $n = 2$  and  $n = 4$  are more interesting. They each have  $tri(n)$  strictly larger than  $so(n)$ , and the presence of non-trivial  $int(n)$  factors is essential in understanding both the existence of the triality and its  $S_3$  permutation symmetry. We will consider each case in turn.

The major difference in moving from  $n = 8$  to  $n = 4$  is that we now have a semi-simple algebra

$$so(4) = su(2) \oplus su(2) \quad \text{with} \quad int(4) = su(2) \quad (5.6)$$

This is the only possibility compatible with (5.1), (4.13) and (4.14); more precisely, we can write

$$tri(4) = su(2)_+ \oplus su(2)_- \oplus su(2)_v \quad (5.7)$$

where the factors on the right are defined so that

$$\begin{aligned} so(4)_V &= su(2)_+ \oplus su(2)_-, & int(4)_V &= su(2)_V \\ so(4)_\pm &= su(2)_\mp \oplus su(2)_V, & int(4)_\pm &= su(2)_\pm \end{aligned} \quad (5.8)$$

As representations of  $tri(4)$  we have

$$\begin{aligned} V &= (\mathbf{2}_+, \mathbf{2}_-, \mathbf{1}_V) \\ S_+ &= (\mathbf{1}_+, \mathbf{2}_-, \mathbf{2}_V) \\ S_- &= (\mathbf{2}_+, \mathbf{1}_-, \mathbf{2}_V) \end{aligned} \quad (5.9)$$

where **2** and **1** denote the fundamental and trivial representations of each  $su(2)$ , as usual. The Dynkin diagram of  $tri(4)$  is disconnected, but still exhibits the  $S_3$  symmetry required by the Permutation Theorem.

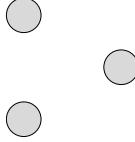


Figure 2: The Dynkin diagram of  $tri(4) = su(2) \oplus su(2) \oplus su(2)$

Its nodes correspond to the representations  $\mathbf{2}_V$  and  $\mathbf{2}_\pm$ , so pairs of nodes correspond to  $V$  and  $S_\pm$ .

To elaborate on the various statements above, let us first recall that  $so(4) = su(2) \oplus su(2)$  can be understood as a decomposition of the space of antisymmetric  $4 \times 4$  matrices into self-dual and anti-self-dual (or  $\pm$ -self-dual) subspaces. Consider, for definiteness, a matrix  $M_{ab}$  acting on  $V$ , and define this to be  $\pm$ -self-dual iff

$$M_{ab} = \pm \frac{1}{2} \varepsilon_{abcd} M_{cd} \quad (5.10)$$

where  $\varepsilon_{abcd}$  is the usual alternating symbol in four dimensions. Any antisymmetric matrix can be written as a sum of its self-dual and anti-self-dual parts, and it is easy to check that the  $\pm$ -self-dual subspaces are each closed under commutation and mutually commuting. This can also be expressed

$$so(4)_V = \wedge^2 V = \wedge_+^2 V \oplus \wedge_-^2 V \quad (5.11)$$

corresponding to the generators in the vector representation being decomposed

$$(M_{cd})_{ab} = \frac{1}{2} \{ \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} + \frac{1}{2} \varepsilon_{abcd} \} + \frac{1}{2} \{ \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} - \frac{1}{2} \varepsilon_{abcd} \} \quad (5.12)$$

Each of the quantities in braces above are  $\pm$ -self-dual in *both* sets of indices  $ab$  and  $cd$ . The other subalgebras can obviously be decomposed in a similar fashion

$$so(4)_\pm = \wedge^2 S_\pm = \wedge_+^2 S_\pm \oplus \wedge_-^2 S_\pm \quad (5.13)$$

Now consider the implications for the representations of  $so(4)_V$  on  $S_\pm$  (and, similarly, with spaces permuted). A useful elementary fact is that any antisymmetric matrix whose square is a non-zero multiple of the identity must be either self-dual or anti-self-dual. The symbols such as

$$(\gamma_{ab})_{\alpha\beta} \quad (\gamma_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} \quad (\gamma_{\dot{\alpha}\dot{\beta}})_{ab} \quad (5.14)$$

can be regarded as matrices in *either* pair of indices, and (3.6) implies that, as such, they always square to  $-1$  (with the other pair of indices fixed throughout). They must, therefore, be either self-dual or anti-self-dual in *every* pair of indices. Which possibility occurs in to some extent a matter of convention, because the conditions for  $\pm$ -self-duality are reversed under a change in orientation (so under any change of base with negative determinant). But we do not have complete freedom of choice, since the algebra (3.6) also implies that if  $(\gamma_{ab})_{\alpha\beta}$  is self-dual in  $\alpha\beta$ , then  $(\gamma_{ab})_{\dot{\alpha}\dot{\beta}}$  is anti-self-dual in  $\dot{\alpha}\dot{\beta}$ , and vice versa. This means that in the decomposition (5.11) the subalgebras  $\wedge_{\pm}^2 V$  each act non-trivially on just one of the spaces  $S_{\pm}$ , which is of course a familiar feature of the spin representations for  $n = 4$ .

To fix the freedom consistently while preserving as much symmetry as possible, we will adopt the convention that the  $\gamma$  symbols as written in (5.14) are anti-self-dual on each pair of indices inside the brackets and self-dual on each pair of indices outside. This can always be achieved by re-ordering the bases in  $V$  and  $S_{\pm}$ , if necessary, and once done it allows us to identify the  $\pm$ -self-dual parts of different copies of  $so(4)$  within  $tri(4)$ , according to

$$T_{\alpha\beta} = \frac{1}{4}(\gamma_{ab})_{\alpha\beta} T_{ab}, \quad T_{\dot{\alpha}\dot{\beta}} = \frac{1}{4}(\gamma_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} T_{\alpha\beta}, \quad T_{ab} = \frac{1}{4}(\gamma_{\dot{\alpha}\dot{\beta}})_{ab} T_{\dot{\alpha}\dot{\beta}} \quad (5.15)$$

These relations follow from (4.17), just like (5.3) for  $so(8)$ , but we have written things in terms of  $\gamma$  symbols to emphasise the consequences of our choice of convention; each relation is invertible, given the  $\pm$ -duality properties of the index pairs. Taking these identifications into account, we can now define  $su(2)$  algebras:

$$\begin{aligned} su(2)_V &\equiv \wedge_{-}^2 S_{+} = \wedge_{+}^2 S_{-} \\ su(2)_{+} &\equiv \wedge_{+}^2 V = \wedge_{-}^2 S_{-} \\ su(2)_{-} &\equiv \wedge_{-}^2 V = \wedge_{+}^2 S_{+} \end{aligned} \quad (5.16)$$

and then (5.11) and (5.13) produce (5.7) and (5.8).

The last triality to consider has  $n = 2$ . This is rather elementary since all the algebras concerned are abelian and we can identify

$$so(2) = int(2) = u(1) \quad \text{and} \quad tri(2) = u(1) \oplus u(1) \quad (5.17)$$

It is easiest to describe everything embedded in  $so(2) \oplus so(2) \oplus so(2) = u(1) \oplus u(1) \oplus u(1)$  acting on  $V \times S_{+} \times S_{-}$ , so each generator can be specified by a triple of real numbers or weights. The subalgebra whose weights add to zero is exactly  $tri(2)$ , and the various subalgebras it contains correspond to weights as follows

$$\begin{aligned} so(2)_V &\sim (2, -1, -1), & int(2)_V &\sim (0, 1, -1) \\ so(2)_{+} &\sim (-1, 2, -1), & int(2)_{+} &\sim (-1, 0, 1) \\ so(2)_{-} &\sim (-1, -1, 2), & int(2)_{-} &\sim (1, -1, 0) \end{aligned} \quad (5.18)$$

Finally, note that the appearance of non-trivial algebras  $int(2) = u(1)$  and  $int(4) = sp(1)$  reflects the fact that the spin representations of  $so(2)$  and  $so(4)$  are complex and quaternionic (or pseudo-real) respectively. In our approach, using real vector spaces, the generators of  $int(n)$  are real, antisymmetric matrices which square to  $-1$  when suitably normalised, allowing them to be interpreted as complex structures.

## 6. The Magic Square

We are now ready to return to the generic constructions of section 2 and use our understanding of triality to convert them into something a little more magical. The additional ingredients we need can actually be listed very concisely and so we will repeat the key equations here, for convenience. Starting with the definitions of the  $so(n)$  and  $int(n)$  algebras, the proposition in section 4 states

$$\begin{aligned} tri(n) &= so(n)_v \oplus int(n)_v \\ &= so(n)_+ \oplus int(n)_+ \\ &= so(n)_- \oplus int(n)_- \end{aligned} \tag{6.1}$$

The corollary which follows tells us how to project onto each  $so(n)$  algebra using the representation matrices

$$T_{ab} = (M_A)_{ab} T_A, \quad T_{\alpha\beta} = (M_A)_{\alpha\beta} T_A, \quad T_{\dot{\alpha}\dot{\beta}} = (M_A)_{\dot{\alpha}\dot{\beta}} T_A \tag{6.2}$$

We will also continue to refer to the triality permutation symmetry which has pervaded our work so far and which is evident in the results above, but *without* needing to appeal directly to the Permutation Theorem of section 4.

### 6.1 The first row or column

To begin, we will fill in the details for the example discussed in the introduction, adopting this as a prototype. The aim is to show that a Lie algebra

$$F_4 = so(8) \oplus V \oplus S_+ \oplus S_- \tag{6.3}$$

can be obtained by following the approach of subsection 2.1 with

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{p}, \quad \mathbf{h} = so(8)_v = so(8)_+ = so(8)_-, \quad \mathbf{p} = V \oplus S_+ \oplus S_- \tag{6.4}$$

Such a reducible representation of  $\mathbf{h}$  does not define a symmetric pair, so we need, firstly, to decide how to complete the definition of the bracket on  $\mathbf{p}$  and, secondly, to prove that it satisfies the Jacobi identity. But the choice of representation does suggest that we should consider the following as subalgebras of  $\mathbf{g}$ ,

$$so(9)_v = so(8)_v \oplus V, \quad so(9)_+ = so(8)_+ \oplus S_+, \quad so(9)_- = so(8)_- \oplus S_- \tag{6.5}$$

by comparison with (2.16).

Producing  $so(n+1) \supset so(n)$  by adding in the fundamental representation is a generic result in the sense that such a symmetric pair exists for all  $n$ , as discussed in detail in section 2. Here we are taking advantage of the fact that  $V$  and  $S_\pm$  can each play the role of the fundamental representation, depending on how we choose to think of  $tri(8)$ . So, while there is nothing remarkable about the existence of any one of these three  $so(9)$  algebras in isolation, collectively they have a very special character, being related by triality permutations. This symmetry ensures the existence of all three  $so(9)$  extensions given any one of them, e.g. the one corresponding to  $V$ .

To specify Lie brackets, let  $\{T_A\}$  be an orthonormal basis for  $tri(8)$  and  $\{X_a\}$ ,  $\{X_\alpha\}$ ,  $\{X_{\dot{\alpha}}\}$  be orthonormal bases for  $V$ ,  $S_+$ ,  $S_-$  respectively (in keeping with the notation of sections 3 to 5). The  $so(9)$  extensions above are given by

$$\begin{aligned} [T_A, X_a] &= -(M_A)_{ab} X_b, & [X_a, X_b] &= -(M_A)_{ab} T_A \\ [T_A, X_\alpha] &= -(M_A)_{\alpha\beta} X_\beta, & [X_\alpha, X_\beta] &= -(M_A)_{\alpha\beta} T_A \\ [T_A, X_{\dot{\alpha}}] &= -(M_A)_{\dot{\alpha}\dot{\beta}} X_{\dot{\beta}}, & [X_{\dot{\alpha}}, X_{\dot{\beta}}] &= -(M_A)_{\dot{\alpha}\dot{\beta}} T_A \end{aligned} \tag{6.6}$$

as in (2.4) and (2.12), but with the matrices  $M$  now the familiar representations of  $tri(8)$  on  $V$  and  $S_{\pm}$ . With  $so(9)_V$  and  $so(9)_{\pm}$  as subalgebras of  $\mathbf{g}$ , it is sufficient to consider elements drawn from different subspaces  $V, S_+, S_-$  in order to complete the definition of the Lie bracket. But since we also require invariance under  $tri(8)$ , the triality map itself allows one to write down the following natural expressions for the brackets

$$\begin{aligned} [X_a, X_\alpha] &= \kappa \gamma_{a\alpha\dot{\alpha}} X_{\dot{\alpha}} \\ [X_\alpha, X_{\dot{\alpha}}] &= \kappa \gamma_{a\alpha\dot{\alpha}} X_a \\ [X_{\dot{\alpha}}, X_a] &= \kappa \gamma_{a\alpha\dot{\alpha}} X_\alpha \end{aligned} \quad (6.7)$$

with an overall constant,  $\kappa$ , yet to be determined.

In fact, we can arrive at these expressions far more systematically, by referring back to the approach of section 2. We saw there that the general way to complete the definition of  $\mathbf{g}$  as a Lie algebra is to choose an antisymmetric, trilinear,  $\mathbf{h}$ -invariant map  $c$  on  $\mathbf{p}$ , as in (2.9). If we require that the brackets of the  $so(9)$  algebras above are unmodified, then  $c$  must give zero when acting on elements belonging to any two of the subspaces  $V, S_+, S_-$ . But this means that the antisymmetric map on  $\mathbf{p} \times \mathbf{p} \times \mathbf{p}$  is determined by a multilinear map on  $V \times S_+ \times S_-$ , and this is exactly the triality, up to the overall constant. By comparison with (2.9) we have

$$(X_a, [X_\alpha, X_{\dot{\alpha}}]) = (X_\alpha, [X_{\dot{\alpha}}, X_a]) = (X_{\dot{\alpha}}, [X_a, X_\alpha]) = \kappa \gamma_{a\alpha\dot{\alpha}} \quad (6.8)$$

and this reproduces (6.7).

Do the brackets (6.6) and (6.7) satisfy the Jacobi identity? The remarks in section 2 ensure that this works automatically except for generators of type  $XXX$ , all belonging to  $\mathbf{p}$ . But the existence of the  $so(9)$  subalgebras also means that Jacobi is guaranteed whenever the generators are taken from the same subspace of  $\mathbf{p}$ . It is therefore sufficient to carry out checks for the combinations  $X_a X_\alpha X_{\dot{\alpha}}$ , with each generator drawn from a different subspace, and for  $X_a X_b X_\alpha$  or similar combinations involving two of the three subspaces  $V, S_+, S_-$ .

For the first combination, we need only write down the definitions to see that the Jacobi condition reduces to invariance of  $\gamma$ :

$$\begin{aligned} &[X_a, [X_\alpha, X_{\dot{\alpha}}]] + [X_\alpha, [X_{\dot{\alpha}}, X_a]] + [X_a, [X_\alpha, X_{\dot{\alpha}}]] \\ &= -\kappa \{ (M_A)_{ab} \gamma_{b\alpha\dot{\alpha}} + (M_A)_{\alpha\beta} \gamma_{a\beta\dot{\alpha}} + (M_A)_{\dot{\alpha}\dot{\beta}} \gamma_{a\alpha\dot{\beta}} \} T_A = 0 \end{aligned} \quad (6.9)$$

For the second combination to work, we need the expressions

$$\begin{aligned} [[X_a, X_b], X_\alpha] &= (M_A)_{ab} (M_A)_{\alpha\beta} X_\beta \\ \text{and } [X_a, [X_b, X_\alpha]] - [X_b, [X_a, X_\alpha]] &= 2\kappa^2 (\gamma_{ab})_{\alpha\beta} X_\beta \end{aligned} \quad (6.10)$$

to match, which is equivalent to

$$(M_A)_{ab} (M_A)_{\alpha\beta} = 2\kappa^2 (\gamma_{ab})_{\alpha\beta} \quad (6.11)$$

One way to confirm this is to use the identification  $tri(8) = so(8)_V = \wedge^2 V$  to re-label the sum on the left, replacing  $A$  with an antisymmetric pair of vector indices  $cd$  (along with the standard factor of  $\frac{1}{2}$ ); the definitions in (4.8a) then reveal that the expressions match for  $\kappa = \pm 1/2$ . More satisfactorily, we can reach the same conclusion using (6.2): simply by taking matrix representations of the generators  $T$  on each side of these relations we obtain<sup>(3)</sup>

$$\begin{aligned} (M_A)_{ab} (M_A)_{\alpha\beta} &= \frac{1}{2} (\gamma_{ab})_{\alpha\beta} \\ (M_A)_{\alpha\beta} (M_A)_{\dot{\alpha}\dot{\beta}} &= \frac{1}{2} (\gamma_{\alpha\beta})_{\dot{\alpha}\dot{\beta}} \\ (M_A)_{\dot{\alpha}\dot{\beta}} (M_A)_{ab} &= \frac{1}{2} (\gamma_{\dot{\alpha}\dot{\beta}})_{ab} \end{aligned} \quad (6.12)$$

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<sup>(3)</sup> The same idea was used to arrive at (4.18) but using the fundamental representation in each case.

These identities imply (6.11) and all other, similar conditions obtained by permuting the spaces  $V$ ,  $S_\pm$ , provided  $\kappa = \pm 1/2$ , and then the proof of the Jacobi identity is complete.

Having understood our prototype, we can proceed to give an analogous construction based on any one of the four trialities by defining Lie algebras

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{p}, \quad \mathbf{h} = tri(n), \quad \mathbf{p} = V \oplus S_+ \oplus S_- \quad (6.13)$$

with  $n = 1, 2, 4$  or  $8$ . From the general structure (6.1) we now have the novel possibility of non-trivial  $int(n)$  factors. However, we can still define

$$so(n+1)_V = so(n)_V \oplus V, \quad so(n+1)_+ = so(n)_+ \oplus S_+, \quad so(n+1)_- = so(n)_- \oplus S_- \quad (6.14)$$

related by triality permutations, and whose Lie brackets are again given by (6.6) with  $\{T_A\}$  a basis for  $tri(n)$ . Because of the direct sums in (6.1) we obtain three inequivalent extensions of  $tri(n)$  within  $\mathbf{g}$ , which we can summarise by the following pattern of inclusions

$$\begin{array}{ccccc} & & so(n+1)_+ \oplus int(n)_+ & & \\ & \swarrow & & \nearrow & \\ \mathbf{g} & \leftarrow & so(n+1)_V \oplus int(n)_V & \leftarrow & tri(n) \\ & \nwarrow & & \searrow & \\ & & so(n+1)_- \oplus int(n)_- & & \end{array} \quad (6.15)$$

The three intermediate subalgebras specify much of the bracket on  $\mathbf{g}$ , as well as ensuring that large parts of the Jacobi identity hold. The triality map can be used to define the remaining brackets by (6.7), as before, and the proof of the Jacobi identity goes through unaltered for any  $n$  provided we use (6.12) and set  $\kappa = \pm 1/2$ .

## 6.2 The complete magic square

Drawing together the various lines of development will now bring us to the magic square. The main idea is to generalise the discussion of the last subsection to two trialities, in much the same way that  $so(n+1) \supset so(n)$  can be generalised to  $so(n+n') \supset so(n) \oplus so(n')$ . Given a pair of trialities

$$\gamma : V \times S_+ \times S_- \rightarrow \mathbb{R}, \quad \gamma' : V' \times S'_+ \times S'_- \rightarrow \mathbb{R} \quad (6.16)$$

with  $n, n' = 1, 2, 4$  or  $8$  we will define a Lie algebra using the method of section 2, with

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{p}, \quad \mathbf{h} = tri(n) \oplus tri(n'), \quad \mathbf{p} = (V \otimes V') \oplus (S_+ \otimes S'_+) \oplus (S_- \otimes S'_-) \quad (6.17)$$

Notice that we are matching up the underlying vector spaces in a certain way, consistent with the notation. This means that we should henceforth restrict attention to triality permutations applied to  $tri(n)$  and  $tri(n')$  simultaneously—i.e. to the diagonal subgroup of the product of the permutation groups for  $\gamma$  and  $\gamma'$ .

Let  $\{T_A, T_{A'}\}$  be an orthonormal basis for  $tri(n) \oplus tri(n')$  and  $\{X_{aa'}\}, \{X_{\alpha\alpha'}\}, \{X_{\dot{\alpha}\dot{\alpha}'}\}$  be orthonormal bases for  $V \otimes V'$ ,  $S_+ \otimes S'_+$ ,  $S_- \otimes S'_-$  respectively. From (6.1) we have

$$\begin{aligned} tri(n) \oplus tri(n') &= so(n)_V \oplus so(n')_V \oplus int(n)_V \oplus int(n')_V \\ &= so(n)_+ \oplus so(n')_+ \oplus int(n)_+ \oplus int(n')_+ \\ &= so(n)_- \oplus so(n')_- \oplus int(n)_- \oplus int(n')_- \end{aligned} \quad (6.18)$$

which leads us to consider the algebras

$$\begin{aligned} so(n+n')_v &= so(n)_v \oplus so(n')_v \oplus (V \otimes V') \\ so(n+n')_+ &= so(n)_+ \oplus so(n')_+ \oplus (S_+ \otimes S'_+) \\ so(n+n')_- &= so(n)_- \oplus so(n')_- \oplus (S_- \otimes S'_-) \end{aligned} \quad (6.19)$$

Each is constructed as in section 2, with brackets (2.32) and (2.33) but with  $\{X_{ii'}\}$  replaced by the appropriate set of basis elements above. Because of the direct sums in (6.18), these algebras lead to three different ways to extend  $tri(n) \oplus tri(n')$  within  $\mathbf{g}$ , and hence to embeddings

$$\begin{array}{ccccc} & & so(n+n')_+ \oplus int(n)_+ \oplus int(n')_+ & & \\ & \swarrow & & \nearrow & \\ \mathbf{g} & \leftarrow & so(n+n')_v \oplus int(n)_v \oplus int(n')_v & \leftarrow & tri(n) \oplus tri(n') \\ & \nwarrow & & \searrow & \\ & & so(n+n')_- \oplus int(n)_- \oplus int(n')_- & & \end{array} \quad (6.20)$$

The three intermediate algebras are related by triality permutations for  $\gamma$  and  $\gamma'$  simultaneously, so we could also infer the existence of all three from the existence of any one of them.

The definition of the Lie bracket on  $\mathbf{g}$  must now be completed by using an antisymmetric trilinear map on  $\mathbf{p}$  which is invariant under  $tri(n) \oplus tri(n')$ . But given the three  $so(n+n')$  subalgebras in (6.20), the map must vanish when restricted to any two of the three subspaces of  $\mathbf{p}$ , since brackets within each subspace are already determined. As with our prototype, this implies that an antisymmetric map on  $\mathbf{p} \times \mathbf{p} \times \mathbf{p}$  reduces to a multilinear map on its distinct subspaces, i.e. on  $(V \otimes V') \times (S_+ \otimes S'_+) \times (S_- \otimes S'_-)$ . When the trialities are combined as  $\gamma \otimes \gamma'$  they have exactly the properties we need; they produce a map (compare with (2.9))

$$(X_{aa'}, [X_{\alpha\alpha'}, X_{\dot{\alpha}\dot{\alpha}'}]) = (X_{\alpha\alpha'}, [X_{\dot{\alpha}\dot{\alpha}'}, X_{aa'}]) = (X_{\dot{\alpha}\dot{\alpha}'}, [X_{aa'}, X_{\alpha\alpha'}]) = \frac{1}{2}\gamma_{a\alpha\dot{\alpha}}\gamma'_{a'\alpha'\dot{\alpha}'} \quad (6.21)$$

and the resulting brackets are

$$\begin{aligned} [X_{aa'}, X_{\alpha\alpha'}] &= \frac{1}{2}\gamma_{a\alpha\dot{\alpha}}\gamma'_{a'\alpha'\dot{\alpha}'} X_{\dot{\alpha}\dot{\alpha}'} \\ [X_{\alpha\alpha'}, X_{\dot{\alpha}\dot{\alpha}'}] &= \frac{1}{2}\gamma_{a\alpha\dot{\alpha}}\gamma'_{a'\alpha'\dot{\alpha}'} X_{aa'} \\ [X_{\dot{\alpha}\dot{\alpha}'}, X_{aa'}] &= \frac{1}{2}\gamma_{a\alpha\dot{\alpha}}\gamma'_{a'\alpha'\dot{\alpha}'} X_{\alpha\alpha'} \end{aligned} \quad (6.22)$$

The only freedom is the choice of an overall constant, which we have fixed in advance, for simplicity. By comparison with our earlier discussion (recovered by taking  $n' = 1$ ) the normalisation in (6.21) and (6.22) corresponds to fixing the sign and setting  $\kappa = 1/2$  in (6.7) and (6.8).

It remains to show that all brackets on  $\mathbf{g}$  obey the Jacobi condition, and the arguments are very similar to those of the last subsection. Within each intermediate subalgebra in (6.20), Jacobi is automatic, and so it is sufficient to check it for elements belonging to two or more different subspaces of  $\mathbf{p}$ . Up to permutations, therefore, the relevant combinations are  $X_{aa'}X_{\alpha\alpha'}X_{\dot{\alpha}\dot{\alpha}'}$ , and  $X_{aa'}X_{bb'}X_{\alpha\alpha'}$ . The first possibility reduces immediately, on applying the definitions, to invariance of  $\gamma$  and  $\gamma'$ , generalising (6.9). For the second combination we need to compare

$$\begin{aligned} [[X_{aa'}, X_{bb'}], X_{\alpha\alpha'}] &= (M_A)_{ab}(M_A)_{\alpha\beta}\delta_{a'b'}X_{\beta\alpha'} + (M_{A'})_{a'b'}(M_{A'})_{\alpha'\beta'}\delta_{ab}X_{\alpha\beta'} \\ \text{with } [X_{aa'}, [X_{bb'}, X_{\alpha\alpha'}]] - [X_{bb'}, [X_{aa'}, X_{\alpha\alpha'}]] &= \frac{1}{2}(\gamma_{ab})_{\alpha\beta}\delta_{a'b'}X_{\beta\alpha'} + \frac{1}{2}(\gamma'_{a'b'})_{\alpha'\beta'}\delta_{ab}X_{\alpha\beta'} \end{aligned} \quad (6.23)$$

where the last expression requires just a minor rearrangement after applying the definitions:

$$\begin{aligned} [X_{bb'}, [X_{aa'}, X_{\alpha\alpha'}]] &= -\frac{1}{4}\gamma_{a\alpha\dot{\alpha}}\gamma'_{a'\alpha'\dot{\alpha}'}\gamma_{b\beta\dot{\alpha}}\gamma'_{b'\beta'\dot{\alpha}'}X_{\beta\beta'} \\ &= -\frac{1}{4}(\delta_{ab}\delta_{\alpha\beta} + (\gamma_{ab})_{\alpha\beta})(\delta_{a'b'}\delta_{\alpha'\beta'} + (\gamma'_{a'b'})_{\alpha'\beta'})X_{\beta\beta'} \end{aligned} \quad (6.24)$$

using (4.10), but then we antisymmetrise in  $aa' \leftrightarrow bb'$  to get the expression given. The quantities in (6.23) now indeed coincide, using

$$(M_A)_{ab}(M_A)_{\alpha\beta} = \frac{1}{2}(\gamma_{ab})_{\alpha\beta} \quad \text{and} \quad (M_{A'})_{a'b'}(M_{A'})_{\alpha'\beta'} = \frac{1}{2}(\gamma'_{a'b'})_{\alpha'\beta'} \quad (6.25)$$

Thus, the Jacobi identity holds by virtue of the relations (6.12), which follow from (6.2), for each triality.

By taking all four possibilities for  $\gamma$  and  $\gamma'$  in turn, we obtain a  $4 \times 4$  square of Lie algebras. The identification of each entry can be carried out using the results in section 5 and considering roots and weights, for instance. So with  $n = n' = 8$ , for example, we have

$$E_8 = so(8) \oplus so(8) \oplus (V \otimes V') \oplus (S_+ \otimes S'_+) \oplus (S_- \otimes S'_-) \quad (6.26)$$

We will not discuss such details any further, except to note that the result of the construction is, in general, obviously symmetrical in  $\gamma$  and  $\gamma'$ . Finally, we can record our conclusions as follows.

**Magic Square Theorem:** For any trialities  $\gamma$  and  $\gamma'$  with  $n, n' = 1, 2, 4$  or  $8$ , there exist Lie algebras

$$\mathbf{g}(n, n') = tri(n) \oplus tri(n') \oplus (V \otimes V') \oplus (S_+ \otimes S'_+) \oplus (S_- \otimes S'_-) \quad (6.27)$$

The results are given by the magic square (1.1), with  $n$  and  $n'$  the dimensions of the division algebras labelling rows and columns.

The nature of the relationship with division algebras will be described in a little more detail below.

## 7. Summary and Further Developments

### 7.1 Summary

The construction of the magic square given above has the desirable feature of being entirely uniform, producing each entry from a single, general set of definitions. The Lie brackets also arise naturally from the work on trialities in sections 3 and 4, and there is very little calculation required to verify the Jacobi identity once the definitions have been applied. In all these respects the construction seems about as straightforward as one could hope for. The end result is also very much in keeping with the intentions expressed in the introduction. With  $G_2$  aside, the exceptional algebras have been understood as emerging within some general scheme, yet at the same time there is something special in their nature in that they all owe their existence to the  $n = 8$  triality.

We also claimed in the introduction that our approach would give a better picture of how the ‘exceptional’ properties of trialities were to be combined with more ‘generic’ aspects of the construction. This distinction can be seen very clearly for the three embeddings of  $tri(n) \oplus tri(n')$  in (6.20), which each correspond to a symmetric pair

$$so(n+n') \supset so(n) \oplus so(n') \quad (7.1)$$

There is nothing remarkable about any one of these extensions individually, since such symmetric pairs occur for all values of  $n$  and  $n'$ . But the existence of three, related symmetric pairs is certainly something special, and depends on the underlying triality permutations for  $n, n' = 1, 2, 4$  or  $8$ .

The trialities are then used once more to complete the definition of the Lie bracket. The maps  $\gamma$  and  $\gamma'$  provide a natural way to do this, up to an overall constant which is determined via the Jacobi condition. This amounts to specifying the second set of embeddings in (6.20), which also correspond to symmetric pairs

$$\mathbf{g}(n, n') \supset so(n+n') \oplus int(n) \oplus int(n') \quad (7.2)$$

The algebra  $\mathbf{g}(n, n')$  therefore possesses three  $\mathbb{Z}_2$  gradings which are compatible, in fact, so that together they constitute a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading. It is easy to see this explicitly from the construction: if  $tri(n) \oplus tri(n')$  and exactly one of the subspaces  $V \otimes V'$  or  $S_{\pm} \otimes S'_{\pm}$  are taken to have grade zero and the two remaining subspaces are taken to have grade one, then this assignment is consistent with all Lie brackets, and in particular with (6.22).

The final step—checking the Jacobi identity—is simple and transparent (compare with the calculations required for isolated examples in [1,8] or for the entire square in [4]). The reason is that we can exploit the existence of the three copies of  $so(n+n')$  in (6.20) to limit drastically the combinations of brackets which need to be considered. It turns out that even this simple final step can be dispensed with, in a certain sense, if we are willing to appeal to some general properties of spinors of  $so(n+n')$ . We will explain this briefly, to give another perspective on how the magic square works.

## 7.2 An additional perspective

The spaces  $V \otimes V'$  and  $S_{\pm} \otimes S'_{\pm}$  begin life as representations of the algebras  $so(n) \oplus so(n')$  in the magic square construction, so it is natural to ask how they behave with respect to the larger algebras  $so(n+n')$ . Considering  $so(n)_V \oplus so(n')_V$ , for instance, we can combine representations and arrange that

$$so(n+n')_V \text{ acts on } (S_+ \otimes S'_+) \oplus (S_- \otimes S'_-) \quad (7.3)$$

but only if the action of  $V \otimes V'$  is defined appropriately. This is exactly what the brackets in (6.22) achieve, and it fixes their normalisation up to a sign. The Jacobi condition as discussed in (6.23) is then an expression of the fact that the representation has been extended nicely from the smaller to the larger algebra.

Once again, this behaviour is not in the least special for any single symmetric pair, since (7.3) amounts to a well-known decomposition property of spin representations which holds in arbitrary dimensions. We can verify this without even changing our notation, just by relaxing some assumptions to allow general values of  $n$  and  $n'$ . The spin representations  $S_{\pm}$  and  $S'_{\pm}$  will then have real dimensions  $N$  and  $N'$ , say, but they can still be described in terms of maps  $\gamma$  and  $\gamma'$  as discussed at the end of section 3. For any  $n$  and  $n'$ , the action of  $so(n)_V \oplus so(n')_V$  can always be extended as in (7.3) by using brackets

$$[X_{aa'}, X_{\alpha\alpha'}] = \pm \frac{1}{2} \gamma_{a\alpha\dot{\alpha}} \gamma'_{a'\alpha'\dot{\alpha}'} X_{\dot{\alpha}\dot{\alpha}'} , \quad [X_{aa'}, X_{\dot{\alpha}\dot{\alpha}'}] = \mp \frac{1}{2} \gamma_{a\alpha\dot{\alpha}} \gamma'_{a'\alpha'\dot{\alpha}'} X_{\alpha\alpha'} \quad (7.4)$$

which are thereby determined, up to an overall sign, just as in (6.22).

The really special feature of the brackets (6.22) is that they also ensure

$$so(n+n')_{\pm} \text{ acts on } (V \otimes V') \oplus (S_{\mp} \otimes S'_{\mp}) \quad (7.5)$$

in addition to (7.3), which requires  $n, n' = 1, 2, 4$  or  $8$ . The existence of all three representations is guaranteed by the existence of any one of them using the fact that the map (6.21) is totally antisymmetric under triality permutations (in the sense of merely interchanging the roles of  $V, S_+, S_-$ ). But if we follow the arguments of section 2, and keep in mind the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading mentioned above, we find that the relations amongst the brackets required for all three representations exhaust the Jacobi condition. Previously, we were able to dispense with much of the Jacobi property simply by noting the existence of three  $so(n+n')$  algebras. Now

we have learnt that what remains is nothing more nor less than the realisation of the representations (7.3) and (7.5) via Lie brackets within  $\mathbf{g}(n, n')$ .

This brings us back, full circle, to the discussion in the introduction about alternative approaches to our prototype,  $F_4$ . There we noted that  $so(9)_V = so(8) \oplus V$  had a representation on  $S = S_+ \oplus S_-$  but a further (un-illuminating) calculation was required to show directly that we ended up with a Lie algebra. Now that we have understood that these Lie brackets originate from (6.8), triality permutations imply that they must also give rise to representations of  $so(9)_\pm$  on  $V \oplus S_\mp$ . The Jacobi identity then follows in its entirety.

With hindsight, we can also see how the following specialisation of the approach in section 2 provides a template for the magic square. Consider building a Lie algebra

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{p} \quad \text{with} \quad \mathbf{p} = \mathbf{p}_1 \oplus \mathbf{p}_2 \oplus \mathbf{p}_3 \quad (7.6)$$

and a representation of  $\mathbf{h}$  given on each subspace  $\mathbf{p}_k$ . Suppose these representations define subalgebras

$$\mathbf{g}_1 = \mathbf{h} \oplus \mathbf{p}_1 , \quad \mathbf{g}_2 = \mathbf{h} \oplus \mathbf{p}_2 , \quad \mathbf{g}_3 = \mathbf{h} \oplus \mathbf{p}_3 \quad (7.7)$$

such that  $\mathbf{g}_k \supset \mathbf{h}$  is a symmetric pair for each  $k$ . This restricts the  $\mathbf{h}$ -invariant map  $\wedge^3 \mathbf{p} \rightarrow \mathbb{R}$  used to complete the definition of the Lie bracket: it must give zero when applied to two or more elements drawn from the same subspace  $\mathbf{p}_k$  and it is therefore obtained by antisymmetrising some  $\mathbf{h}$ -invariant multilinear map  $\mathbf{p}_1 \times \mathbf{p}_2 \times \mathbf{p}_3 \rightarrow \mathbb{R}$ . But this in turn implies that the resulting bracket gives  $\mathbf{g}$  a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading, with each  $\mathbf{g} \supset \mathbf{g}_k$  also being a symmetric pair. Moreover, the arguments of section 2 show that once the representations and multilinear map have been specified, the Jacobi property holds iff (i) the symmetric pairs  $\mathbf{g}_k \supset \mathbf{h}$  exist; and (ii) the representations of  $\mathbf{h}$  extend to representations of  $\mathbf{g}_k$  via the bracket, so that

$$\mathbf{g}_1 \text{ acts on } \mathbf{p}_2 \oplus \mathbf{p}_3 , \quad \mathbf{g}_2 \text{ acts on } \mathbf{p}_3 \oplus \mathbf{p}_1 , \quad \mathbf{g}_3 \text{ acts on } \mathbf{p}_1 \oplus \mathbf{p}_2 \quad (7.8)$$

From this standpoint we see, once again, how the triality permutation symmetry of the multilinear map  $\gamma \otimes \gamma'$  underlies the construction of  $\mathbf{g}(n, n')$ . It is this symmetry which ensures that (6.21) (with the normalisation chosen correctly) leads to three interlocking representations (7.3) and (7.5) and the Jacobi identity then holds automatically, from the arguments above. Furthermore, the existence of all three representations follows from the existence of any one of them. This is another instance of how a ‘generic’ property, in this case the decomposition or extension of spin representations, can be combined with triality permutations when  $n, n' = 1, 2, 4$  or  $8$  to provide a simple understanding of a rather non-trivial result.

### 7.3 Division algebras and $G_2$

Although  $G_2$  is the only simple exceptional algebra which does not appear in the magic square, it too can be understood in terms of trialities, and in terms of the  $n = 8$  triality in particular. This turns out to be closely related to how we make the transition from a triality to a division algebra—something we also need to discuss to make contact with the body of work in [4] and to justify labelling the entries of the square by division algebras in the first place. These matters are dealt with in [1] and [3], so our treatment will be brief. In consulting these sources, however, it may help to compare the new definition adopted in this paper with the original version, which we have not needed up till now. A *normed triality* is defined in [1] to be a map  $\gamma$  as in (3.1a) which obeys  $|\gamma(u_V, u_+, u_-)| \leq |u_V| |u_+| |u_-|$  and such that, if any two vectors are given, there is a non-zero choice of the third vector for which the bound is attained. It is a simple matter to show that this is equivalent to our definition of a triality, and with this understood there should be no difficulty in filling in the details in the following summary.

Given a triality  $\gamma$  and any element  $e_V$  of unit norm in  $V$ , the subgroup of  $tri(n)$  which fixes this element is  $so(n-1)_V \oplus int(n)_V \subset so(n)_V \oplus int(n)_V = tri(n)$ . Since the map  $\sigma(e_V)$  is invertible, the choice of  $e_V$

also gives us a way of identifying  $S_+$  and  $S_-$ . Now pick an additional element of unit norm  $e_-$  in  $S_-$ . The subalgebra of  $tri(n)$  which fixes both  $e_V$  and  $e_-$  also fixes a unit vector  $e_+$  in  $S_+$ , where

$$\begin{aligned} e_+ = \sigma(e_V)e_- &\iff e_V = \sigma(e_-)e_+ \iff e_- = \sigma(e_+)e_V \\ &\iff \gamma(e_V, e_+, e_-) = 1 \end{aligned} \tag{7.9}$$

Any two of these elements determines the third, and the subalgebra which fixes two of them therefore fixes all three. For  $n = 1$  or  $2$ , the corresponding Lie algebra is trivial, but in the remaining cases we obtain

$$su(2) \subset tri(4) \quad \text{and} \quad G_2 \subset tri(8) \tag{7.10}$$

The choice of corresponding elements  $e_V, e_\pm$  allows all three vector spaces to be identified, by making use of the maps  $\sigma(e_V), \sigma(e_\pm)$ . It also gives a way of turning the result into a division algebra

$$V \leftrightarrow S_+ \leftrightarrow S_- \leftrightarrow \mathbb{K} \tag{7.11}$$

with the distinguished, identified elements, now denoted simply by  $e$ , playing the role of the identity. Using the same symbol for identified elements in any space, the multiplication on  $\mathbb{K}$  can be chosen to obey

$$\gamma(xy, e, z) = \gamma(x, y, z) \tag{7.12}$$

so that it is defined in terms of  $\gamma$ . There is some freedom here, depending on the details of how spaces are identified (up to conjugation in each copy of the division algebra, for instance) but the multiplication can be shown to obey the normed division algebra axioms [1]. With appropriate choices, the original triality map then takes the form

$$\gamma(x, y, z) = \text{Re}((xy)z) \tag{7.13}$$

(where the real part of an element in  $\mathbb{K}$  is given by its inner-product with  $e$ ). Conversely, if we are given a division algebra then the map above can be shown to define a triality.

In this manner, we obtain an equivalence between trialities with  $n = 1, 2, 4, 8$  and division algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ . The Lie algebras in (7.9) then correspond to the automorphism groups of the division algebras  $\mathbb{H}$  and  $\mathbb{O}$  (the automorphism group for  $\mathbb{C}$  is  $\mathbb{Z}_2$  and there are no automorphisms of  $\mathbb{R}$ ; in either case the Lie algebra is trivial). Furthermore, the magic square construction (6.27) can now be re-expressed

$$\mathbf{g}(\mathbb{K}, \mathbb{K}') = tri(\mathbb{K}) \oplus tri(\mathbb{K}') \oplus 3(\mathbb{K} \otimes \mathbb{K}') \tag{7.14}$$

and it was studied in detail in this form by Barton and Sudbery. For the smaller square in which  $\mathbb{K}$  and  $\mathbb{K}'$  are both associative, the entire algebra  $\mathbf{g}(\mathbb{K}, \mathbb{K}')$  can be identified with the anti-hermitian  $3 \times 3$  matrices over  $\mathbb{K} \otimes \mathbb{K}'$  but for the octonionic entries in the square, the interpretation using division algebras requires much more care and ingenuity. We refer to [4] for full details.

This paper has been devoted to trialities rather than division algebras, but it is not our intention to advocate the use of one, rather than the other, irrespective of the circumstances. It can certainly be very attractive to see results expressed in a compact way using division algebra multiplication. In some respects, however, this structure can be almost too special: it is versatile enough to play many different roles, and so translating back into more conventional language is not always easy. Dealing with division algebras may also involve relinquishing some manifest symmetry (through the choice of the identity elements) as we have seen. Ultimately the two approaches are equivalent so one can choose whichever offers greater insights, or use them both to get complementary ways of understanding the same things.

## 7.4 Folding on the diagonal

The magic square arises by combining the special properties of triality algebras  $tri(n) = so(n) \oplus int(n)$  for  $n = 1, 2, 4, 8$  with symmetric pairs  $so(n+1) \supset so(n)$  or  $so(n+n') \supset so(n) \oplus so(n')$ . It is natural to ask whether similar ideas can be applied to other symmetric pairs, but if we want orthogonal subalgebras, to allow a link to trialities, then the only other possibilities are [2]

$$su(n) \supset so(n) \quad (7.15)$$

The choice of representation required here is the traceless symmetric tensor, so we could hope to give a direct construction

$$\tilde{\mathbf{g}}(n) = tri(n) \oplus \tilde{V} \oplus \tilde{S}_+ \oplus \tilde{S}_- \quad (7.16)$$

for  $n = 1, 2, 4$  or  $8$ , where  $\tilde{V}$  and  $\tilde{S}_{\pm}$  denote the traceless, symmetric tensor squares of  $V$  and  $S_{\pm}$ . Following this route, we could use (6.1) and appeal to the existence of three intermediate algebras

$$su(n)_v = so(n)_v \oplus \tilde{V}, \quad su(n)_+ = so(n)_+ \oplus \tilde{S}_+, \quad su(n)_- = so(n)_- \oplus \tilde{S}_- \quad (7.17)$$

in discussing the bracket and its Jacobi property. By comparison with (6.15) or (6.20), such an approach could be summarised

$$\begin{array}{ccccc} & & su(n)_+ \oplus int(n)_+ & & \\ & \swarrow & & \nearrow & \\ \tilde{\mathbf{g}}(n) & \leftarrow & su(n)_v \oplus int(n)_v & \leftarrow & tri(n) \\ & \nwarrow & & \searrow & \\ & & su(n)_- \oplus int(n)_- & & \end{array} \quad (7.18)$$

The algebras  $\tilde{\mathbf{g}}(n)$  can indeed be constructed in this way, but they can also be obtained with minimal extra work from the results we have found already. Consider any  $\mathbf{g}(n, n)$  on the diagonal of the magic square, and the map  $\tau$  which *folds* it by exchanging the  $tri(n)$  factors and the representations  $V \leftrightarrow V'$  and  $S_{\pm} \leftrightarrow S'_{\pm}$ . It is not difficult to see that  $\tau$  is an automorphism of  $\mathbf{g}(n, n)$ , of order 2, and so its fixed-point set is a subalgebra. It consists of the diagonal  $tri(n) \subset tri(n) \oplus tri(n)$  in addition to the symmetrised tensor squares of  $V$  and  $S_{\pm}$ , since the primed and un-primed spaces are identified as representations of the diagonal  $tri(n)$ . We have almost reached (7.16) by folding  $\mathbf{g}(n, n)$  in this way, except that the tensor representations are not yet traceless. To understand this last step we will make use of the explicit forms of the brackets for  $\mathbf{g}(n, n')$ .

On identifying  $V \leftrightarrow V'$  and  $S_{\pm} \leftrightarrow S'_{\pm}$  there are bases for the tensor product spaces  $\{X_{ab}\}$ ,  $\{X_{\alpha\beta}\}$ ,  $\{X_{\dot{\alpha}\dot{\beta}}\}$  and we have standard decompositions into trace-free and pure-trace parts:

$$\begin{aligned} \frac{1}{2}(X_{ab} + X_{ba}) &= \tilde{X}_{ab} + \frac{1}{n}\delta_{ab}X_{cc} & \text{where} & \tilde{X}_{ab} = \tilde{X}_{ba} & \text{and} & \tilde{X}_{cc} = 0 \\ \frac{1}{2}(X_{\alpha\beta} + X_{\beta\alpha}) &= \tilde{X}_{\alpha\beta} + \frac{1}{n}\delta_{\alpha\beta}X_{\gamma\gamma} & \text{where} & \tilde{X}_{\alpha\beta} = \tilde{X}_{\beta\alpha} & \text{and} & \tilde{X}_{\gamma\gamma} = 0 \\ \frac{1}{2}(X_{\dot{\alpha}\dot{\beta}} + X_{\dot{\beta}\dot{\alpha}}) &= \tilde{X}_{\dot{\alpha}\dot{\beta}} + \frac{1}{n}\delta_{\dot{\alpha}\dot{\beta}}X_{\dot{\gamma}\dot{\gamma}} & \text{where} & \tilde{X}_{\dot{\alpha}\dot{\beta}} = \tilde{X}_{\dot{\beta}\dot{\alpha}} & \text{and} & \tilde{X}_{\dot{\gamma}\dot{\gamma}} = 0 \end{aligned} \quad (7.19)$$

The sets of traceless combinations  $\{\tilde{X}_{ab}\}$ ,  $\{\tilde{X}_{\alpha\beta}\}$ ,  $\{\tilde{X}_{\dot{\alpha}\dot{\beta}}\}$  provide bases for  $\tilde{V}$ ,  $\tilde{S}_+$ ,  $\tilde{S}_-$  respectively. The pure trace combinations  $\{X_{aa}, X_{\alpha\alpha}, X_{\dot{\alpha}\dot{\alpha}}\}$  constitute an  $so(3)$  subalgebra, since the brackets (6.22) imply

$$\begin{aligned} [X_{aa}, X_{\alpha\alpha}] &= \frac{1}{2}nX_{\dot{\alpha}\dot{\alpha}} \\ [X_{\alpha\alpha}, X_{\dot{\alpha}\dot{\alpha}}] &= \frac{1}{2}nX_{aa} \\ [X_{\dot{\alpha}\dot{\alpha}}, X_{aa}] &= \frac{1}{2}nX_{\alpha\alpha} \end{aligned} \quad (7.20)$$

But, in addition, this subalgebra actually commutes with everything in  $tri(n)$  or in  $\tilde{V}, \tilde{S}_\pm$ ; for example

$$\begin{aligned} [X_{cc}, \tilde{X}_{\alpha\beta}] &= \tfrac{1}{2}\gamma_{c\alpha\dot{\alpha}}\gamma_{c\beta\dot{\beta}}\tilde{X}_{\dot{\alpha}\dot{\beta}} = \tfrac{1}{2}\delta_{\alpha\beta}\tilde{X}_{\dot{\alpha}\dot{\alpha}} = 0 \\ [X_{cc}, \tilde{X}_{ab}] &= -(M_A)_{ca}\delta_{cb}T_A - (M_A)_{cb}\delta_{ca}T_A = 0 \end{aligned} \quad (7.21)$$

We therefore reach the following conclusion.

**Proposition:** The subalgebra of  $\mathbf{g}(n, n)$  which is fixed by the folding automorphism is a direct sum:

$$\mathbf{g}(n, n) \supset \tilde{\mathbf{g}}(n) \oplus so(3) \quad (7.22)$$

with  $\tilde{\mathbf{g}}(n)$  given by (7.16) and (7.18).

For  $n = 1$  this is all rather trivial, but for the other three trialities we obtain interesting results. Labelling the cases by the corresponding division algebra, in the traditional way, we have

$$\begin{array}{cccc} \mathbb{K} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\ \tilde{\mathbf{g}}(\mathbb{K}) & su(3) & so(9) & E_7 \end{array} \quad (7.23)$$

In particular, this gives a nice triality description of

$$E_7 = so(8) \oplus \tilde{V} \oplus \tilde{S}_+ \oplus \tilde{S}_- \quad (7.24)$$

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## Appendix

We have already shown in section 3 that trialities exist only if  $n = 1, 2, 4$  or  $8$ . Here we elaborate on how existence and uniqueness in these cases is related to standard results in the theory of spin and Clifford algebra representations, deriving what we need from first principles.

Any triality  $\gamma$  can be related to real matrices  $\sigma_a$  by

$$(\sigma_a)_{\alpha\dot{\alpha}} = \gamma_{a\alpha\dot{\alpha}} \quad (\text{A.1})$$

with the conditions (3.6a) and (3.6b) equivalent to

$$\sigma_a \sigma_b^T + \sigma_b \sigma_a^T = 2\delta_{ab} \quad a, b = 1, 2, \dots, n \quad (\text{A.2})$$

Conversely, any set of  $n \times n$  real matrices which obey this define a triality via (A.1). It is convenient to consider the real symmetric matrices

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \sigma_a^T & 0 \end{pmatrix} \quad \text{on} \quad S = S_+ \oplus S_- \quad (\text{A.3})$$

so that (A.2) is holds iff

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} \quad a, b = 1, 2, \dots, n \quad (\text{A.4})$$

We will also consider more general, hermitian matrices

$$\Gamma_a = \begin{pmatrix} 0 & \Sigma_a \\ \Sigma_a^\dagger & 0 \end{pmatrix} \quad \text{on} \quad \Omega = \Omega_+ \oplus \Omega_- \quad (\text{A.5})$$

where  $\Omega_\pm$  are complex vector spaces, and then (A.4) is equivalent to

$$\Sigma_a \Sigma_b^\dagger + \Sigma_b \Sigma_a^\dagger = 2\delta_{ab} \quad a, b = 1, 2, \dots, n \quad (\text{A.6})$$

There are two distinct ways to relate (A.6) to (A.2). It may be simply that the matrices  $\Sigma_a$  can be chosen to be real, enabling them to be identified with  $\sigma_a$  (in some basis) with  $\Omega_\pm$  the complexifications of  $S_\pm$ . If this is not possible, then we can still resort to re-writing each complex  $\Sigma_a$  as a larger real matrix  $\sigma_a$  by choosing to regard  $\Omega_\pm$  as real vector spaces  $S_\pm$  of twice the dimension. We can also carry this out in reverse provided there exist suitable complex structures on  $S_\pm$  which are respected by the maps  $\sigma_a$ , so ensuring that they can be re-expressed as complex linear maps on  $\Omega_\pm$ . From our discussion of  $\text{int}(n)_V$  in section 4, we know that its generators can be regarded as pairs of antisymmetric matrices  $\lambda_\pm$  acting on  $S_\pm$ , and obeying

$$\lambda_+ \sigma_a = \sigma_a \lambda_- \quad (\text{A.7})$$

(in (4.15)  $\lambda_{\alpha\beta}$  and  $\lambda_{\dot{\alpha}\dot{\beta}}$  are the components of  $\lambda_\pm$ ). From section 5, we also know that these matrices can be normalised so that  $\lambda_\pm^2 = -1$ , and we therefore have exactly the kinds of complex structures just discussed whenever  $\text{int}(n)$  is non-trivial, i.e. for  $n = 2$  or  $4$ .

**Proposition:** (a) There exist  $2N \times 2N$  complex hermitian matrices  $\Gamma_a$  which obey (A.4) with  $N = 2^{\ell-1}$  for any even integer  $n = 2\ell$ . Any two sets of such matrices are related by a transformation

$$\Gamma_a \mapsto U \Gamma_a U^{-1} \quad \text{where} \quad U^\dagger = U^{-1} \quad (\text{A.8})$$

(b) The matrices in part (a) have a block form (A.5) with  $\Omega_{\pm}$  eigenspaces of

$$\hat{\Gamma} \equiv \Gamma_1 \Gamma_2 \dots \Gamma_{2\ell} = \xi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{where} \quad \xi = \pm i^{\ell} \quad (\text{A.9})$$

Any two sets of such matrices with the same sign for  $\xi$  are related by (A.8) with  $U$  block diagonal, or

$$\Sigma_a \mapsto U_+ \Sigma_a U_-^{-1} \quad \text{where} \quad U_{\pm}^{-1} = U_{\pm} \quad (\text{A.10})$$

(c) If, in addition,  $n = 2\ell$  is a multiple of 8, then we can choose the matrices  $\Gamma_a$  and  $\Sigma_a = \sigma_a$ , real. Any two sets of such matrices with the same sign for  $\xi$  are related by (A.10) with  $U_{\pm} = R_{\pm}$  real, or

$$\sigma_a \mapsto R_+ \sigma_a R_-^{-T} \quad \text{where} \quad R_{\pm}^{-T} = R_{\pm}^{-1} \quad (\text{A.11})$$

Proof: (a) Existence and uniqueness up to transformations (A.8) can be established by constructing joint eigenvectors for the mutually commuting anti-hermitian matrices

$$\Gamma_{12}, \Gamma_{34}, \dots, \Gamma_{2\ell-1, 2\ell} \quad (\text{A.12})$$

which each have eigenvalues  $\pm i$ . This basis will define a standard complex form for matrices obeying (A.4) and thereby also show that such matrices exist. First note that the combinations

$$A_r^{\pm} = \frac{1}{2}(\Gamma_{2r-1} \pm i\Gamma_{2r}) \quad \text{obey} \quad \{A_r^{\pm}, A_s^{\pm}\} = 0 \quad \text{and} \quad \{A_r^+, A_s^-\} = \delta_{rs} \quad r, s = 1, 2, \dots, \ell \quad (\text{A.13})$$

(and so behave like fermion creation and annihilation operators). A unit vector  $\Psi$  in  $\Omega$  is a joint eigenvector of the matrices (A.13) with all eigenvalues  $-i$  iff  $A_r^- \Psi = 0$  (it is the analogue of the ground state for the multi-fermion system) and this vector can be extended to a basis for  $\Omega$  consisting of

$$(A_1^+)^{m_1} \dots (A_{\ell}^+)^{m_{\ell}} \Psi \quad m_r = 0 \text{ or } 1 \quad (\text{A.14})$$

(each  $m_r$  is like a fermion occupation number). The vectors (A.14) are orthonormal by virtue of (A.13) and the fact that every  $\Gamma_a$  is hermitian. The action of each  $A_r^{\pm}$  on the expressions in (A.14) determines their matrices with respect to this basis—we note that they are all real—and the standard form for  $\Gamma_a$  is then obtained from (A.13). Any two sets of matrices  $\Gamma_a$  are unitarily equivalent to this standard form and hence to one another.

(b) Consider the subspaces spanned by vectors in (A.14) for which  $\sum_r m_r$  is even or odd; define one to be  $\Omega_+$  and the other to be  $\Omega_-$ , in some order. These subspaces are mapped to one another by every  $A_r^{\pm}$  and hence by each  $\Gamma_a$ ; they are also eigenspaces of  $\hat{\Gamma}$ , with the value of  $\xi$  consistent with  $\hat{\Gamma}^2 = (-1)^{\ell}$ . Which of the subspaces  $\Omega_{\pm}$  corresponds to which eigenvalue of  $\hat{\Gamma}$  cannot be determined from the block form for  $\Gamma_a$  alone, leaving the sign ambiguity in  $\xi$ . For any two sets of matrices related by (A.8) we have  $U\hat{\Gamma}U^{-1} = \hat{\Gamma}$ , but then if each set has the same value of  $\xi$ , this implies that  $U$  must be block diagonal.

(c) If matrices  $\Gamma_a$  meet the conditions in part (a) then so do their complex conjugates. The uniqueness result therefore implies that there exists a unitary matrix  $B$  with

$$B\Gamma_a^* B^{-1} = \Gamma_a \quad \text{and} \quad B \mapsto UBU^T \quad (\text{A.15})$$

with the latter transformation accompanying (A.8), for consistency. With respect to the basis (A.14),  $\Gamma_a$  is real and symmetric if  $a$  is odd, and imaginary and antisymmetric if  $a$  is even (since each  $A_r^{\pm}$  is real) and so we may choose  $B = \Gamma_2 \Gamma_4 \dots \Gamma_{2\ell}$ , because  $\ell$  is even. This commutes with  $\hat{\Gamma}$  and both matrices are

real, symmetric and square to 1 since  $\ell$  is in fact a multiple of 4. They can therefore be simultaneously diagonalised by (A.8) and (A.15) with  $U$  real and orthogonal in these circumstances. But having made  $B$  diagonal with entries  $\pm 1$ , a further transformation (A.8) and (A.15) with  $U$  complex and diagonal can be chosen to produce  $B = 1$ , which gives the result.

**Corollary:** There exist trialities with  $n = 2, 4$  or  $8$ , each unique up to isomorphism, as in the Classification Theorem.

Proof: When  $n = 2\ell = 8$ , part (c) above implies there exist real  $N \times N$  matrices  $\sigma_a$  obeying (A.2) with  $N = 2^{\ell-1} = 8$  and so (A.1) defines a triality. Conversely, from any two trialities with  $n = 8$  we can construct sets of matrices  $\sigma_a$  as in part (c). An orthogonal transformation on  $V$  which exchanges two basis elements will change the sign of  $\xi$ , so by using this if necessary we can assume that the signs match for each set. But then part (c) implies that the trialities are related by

$$\gamma_{a\alpha\dot{\alpha}} \mapsto R_{\alpha\beta} R_{\dot{\alpha}\dot{\beta}} \gamma_{a\beta\dot{\beta}} \quad (\text{A.16})$$

with  $R_{\alpha\beta}$  and  $R_{\dot{\alpha}\dot{\beta}}$  the entries of the matrices  $R_{\pm}$ .

When  $n = 2\ell = 2$  or  $4$ , part (b) implies that there exist complex  $N \times N$  matrices  $\Sigma_a$  obeying (A.6) which in turn define real  $2N \times 2N$  matrices  $\sigma_a$  obeying (A.2), with  $2N = 2^\ell = n$  for these two values. Hence (A.1) again defines a triality in these cases. When the dimensions of the spaces are doubled to pass from complex to real notation in this fashion, any unitary maps  $U_{\pm}$  on  $\Omega_{\pm}$  clearly become orthogonal maps  $R_{\pm}$  on  $S_{\pm}$ , and so we can use (A.10) to establish uniqueness by following the same argument as in the  $n = 8$  case. The only subtlety is that if we are given two trialities expressed as sets of real matrices  $\sigma_a$ , then we must be able to re-write these as complex matrices  $\Sigma_a$  in order to apply the uniqueness result in part (b). But this is ensured by our remarks on the existence of complex structures which the maps  $\sigma_a$  respect, as in (A.7), when  $n = 2$  and  $n = 4$ .

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