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TEST FOR INTERSECTION BETWEEN CIRCLE AND RECTANGLE

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Abstract—A test for whether a circle intersects a rectangle is developed. The test applies interval arithmetic tools and an observation regarding the relationship between a point and a circle. The test is robust and easy to implement and it gives a guaranteed answer (modulo rounding errors). Some numerical examples are given.

A geometric primitive for the construction of a Voronoi diagram in the plane in computational geometry (see [1]) is the test for whether a given point is inside a circle defined by three points or not (see [2]). In this note, we apply interval analysis to the test and we show that the interval analysis version of the test can be applied to decide whether a circle and an axis parallel rectangle intersect or not. If machine interval arithmetic is used and if the numerical result shows that the circle and the rectangle does not intersect, then the non-intersection is guaranteed also for the exact (non-rounded) result. If the numerical result shows intersection, then the exact result will, in general, be the same, however, there is a small probability in certain bordering cases that the result has been falsified by rounding errors.

We refer to [3] for a short overview of the field of interval arithmetic. A short list of references for further applications and uses of interval arithmetic is also given there. Applications of interval arithmetic to the computations of the range of functions are discussed in [4,5]. This note is a simple application of these techniques. We denote the set of compact intervals $X = [x_L, x_R]$ over R by $I(R)$. The arithmetic for these intervals is, for example, defined in [3]. We also require the notation $\overline{f}(X) = \{f(x) \mid x \in X\}$, i.e., the range of $f(x)$ over an interval X. In contrast, the natural interval extension of f to X is denoted by $f(X)$, that is the interval expression that arises from $f(x)$ by replacing x by X. This notation is also used if x is a multivariate variable and X a box of corresponding dimension.

Let $P_i = (x_i, y_i)$, $i = 1, \dots, 3$ be points in the plane defining a circle (if co-linear then a line) and let $Q = (x_Q, y_Q)$ be a fourth point. It was shown in [2] that testing for Q inside the circle was equivalent to the test

$$\mathcal{D}(P_1, P_2, P_3, Q) = \begin{vmatrix} x_1 & y_1 & x_1^2 + y_1^2 & 1 \\ x_2 & y_2 & x_2^2 + y_2^2 & 1 \\ x_3 & y_3 & x_3^2 + y_3^2 & 1 \\ x_Q & y_Q & x_Q^2 + y_Q^2 & 1 \end{vmatrix} > 0. \tag{1}$$

Q outside the circle was equivalent to $\mathcal{D}(P_1, P_2, P_3, Q) < 0$, on the circle to $\mathcal{D}(P_1, P_2, P_3, Q) = 0$, where it has been assumed that the points P_1, P_2, P_3 lie counterclockwise on the circle. (If P_1,

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P_2, P_3 lie clockwise on the circle then the signs of the test have to be reversed). In the collinear case, the test degenerated to $\mathcal{D}(P_1, P_2, P_3, Q) \neq 0$ iff Q is not on the line.

Here we apply the computation of this determinant to testing whether a circle defined by three points intersects an axis-parallel rectangle or not.

We note that expressions that are algebraically identical might result in different interval evaluations due to subdistributivity and dependency widths [3]. However, the interval evaluation of any expression for a function over an interval will always include the range of the function.

If we now let $D = X \times Y$ with $X, Y \in I(R)$, a rectangle, then the range of $\mathcal{D}(P_1, P_2, P_3, Q)$ with respect to $Q \in D$ is $\overline{\mathcal{D}}(P_1, P_2, P_3, D)$. Computing any one of the algebraically identical forms of (1) results in an interval $\mathcal{D}(P_1, P_2, P_3, D)$ that includes the range because of inclusion isotony. In some cases, the interval arithmetic computation is equal to the range (except for rounding errors). We will show that this is the case for (1) for at least one of the algebraically identical forms provided the computations are arranged appropriately.

We first expand the determinant in (1) as follows

$$\mathcal{D}(P_1, P_2, P_3, Q) = -x_Q \begin{vmatrix} y_1 & x_1^2 + y_1^2 & 1 \\ y_2 & x_2^2 + y_2^2 & 1 \\ y_3 & x_3^2 + y_3^2 & 1 \end{vmatrix} + y_Q \begin{vmatrix} x_1 & x_1^2 + y_1^2 & 1 \\ x_2 & x_2^2 + y_2^2 & 1 \\ x_3 & x_3^2 + y_3^2 & 1 \end{vmatrix} \tag{2}$$

$$-(x_Q^2 + y_Q^2) \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & x_1^2 + y_1^2 \\ x_2 & y_2 & x_2^2 + y_2^2 \\ x_3 & y_3 & x_3^2 + y_3^2 \end{vmatrix} \tag{3}$$

$$= -x_Q w_1 + y_Q w_2 - (x_Q^2 + y_Q^2) w_3 + w_4$$

$$= -x_Q w_1 - x_Q^2 w_3 + y_Q w_2 - y_Q^2 w_3 + w_4 \tag{4}$$

where the coefficients w_1, w_2, w_3 and w_4 are the values of the minors appearing in (2) and (3).

We also need a small observation. Let

$$f(x) = ax^2 + bx. \tag{5}$$

Then the range of $f(x)$ over $X = [x_L, x_R]$, denoted by $\overline{f}(X)$ can be found by first computing $a_1 = f(x_L)$, $a_2 = f(x_R)$ and if $a \neq 0$ $a_3 = f(-b/(2a))$ then obtaining

$$f(X) = \begin{cases} [\min(a_1, a_2, a_3), \max(a_1, a_2, a_3)], & \text{if } -b/(2a) \in X \text{ and } a \neq 0, \\ [\min(a_1, a_2), \max(a_1, a_2)], & \text{otherwise.} \end{cases}$$

We also note that (4) is of the form

$$h(x, y) = \psi(x) + \psi(y) + q, \tag{6}$$

where $\psi(x)$ does not depend on y, where $\psi(y)$ does not depend on x and q not on x and y. Hence, if $\overline{\psi}(X)$ and $\overline{\psi}(Y)$ are the ranges of $\psi(x)$ and $\psi(y)$ over X and Y respectively, then

$$\overline{h}(X, Y) = \overline{\psi}(X) + \overline{\psi}(Y) + q.$$

We now note that (4) has the form (6) where each occurring function has the form (5). This means that if

$$\mathcal{D}^*(P_1, P_2, P_3, D) = -X w_1 - X^2 w_3 + Y w_2 - Y^2 w_3 + w_4 \tag{7}$$

is evaluated using the result from (5) and interval arithmetic, then

$$\overline{\mathcal{D}}(P_1, P_2, P_3, D) = \mathcal{D}^*(P_1, P_2, P_3, D),$$

that is, the exact range is computed! Hence, we get in the counterclockwise case one of three results:

1. $\mathcal{D}^*(P_1, P_2, P_3, D) > 0$. The circle defined by P_1 ,P_2, P_3 does not intersect the rectangle D. The interior of the circle contains the rectangle in the non-collinear case.

2. $\mathcal{D}^*(P_1, P_2, P_3, D) \ni 0$. The circle defined by P_1, P_2, P_3 intersects the rectangle D.

3. $\mathcal{D}^*(P_1, P_2, P_3, D) < 0$. The circle defined by P_1, P_2, P_3 does not intersect the rectangle D. The exterior of the circle contains the rectangle in the non-collinear case.

If machine interval arithmetic is used and the numerical evaluation of $\mathcal{D}(P_1, P_2, P_3, D)$ is positive or negative then Cases 1 and 3 give guaranteed results due to outward rounding. In Case 2, the result is uncertain, i.e., the rectangle might be included, intersect or be outside the circle.

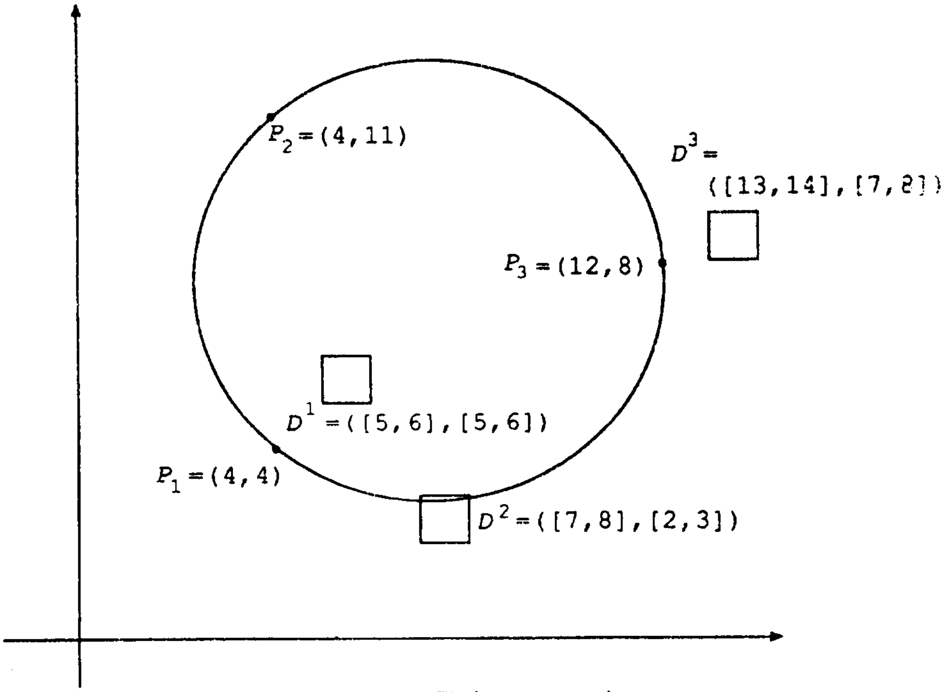

Figure 1. Circle test examples.

As an example, consider the circle defined by $P_1 = (4,4)$, $P_2 = (4,11)$ and $P_3 = (12,8)$. Note that the points P_1 ,P_2, P_3 lie clockwise on the circle. We display three cases in Figure 1. In Case 1, $D^1 = ([5,6],[5,6])$ and $\mathcal{D}^*(P_1, P_2, P_3, D^1) = [-1064, -644] < 0$, i.e, the rectangle is contained in the circle. In Case 2, $D^2 = ([7,8],[2,3])$ and $\mathcal{D}^*(P_1, P_2, P_3, D^2) = [-140, 448] \ni 0$, i.e, the rectangle intersects the circle. In Case 3, $D^3 = ([13,14],[7,8])$ and $\mathcal{D}^*(P_1, P_2, P_3, D^3) = [588, 1288] > 0$, i.e, the rectangle is outside the circle.

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