

S.-T. Yau College Student Mathematics Contests 2011

## Analysis and Differential Equations

Individual

2:30{5:00 pm, July 9, 2011  
(Please select 5 problems to solve)

- a) Compute the integral:  $\int_1^R \frac{x \cos x dx}{(x^2+1)(x^2+2)}$ ;

b) Show that there is a continuous function  $f: [0; +\infty) \rightarrow (-1; +1)$  such that  $f \not\equiv 0$  and  $f(4x) = f(2x) + f(x)$ .
- Solve the following problem:  

$$\begin{aligned} \frac{d^2 u}{dx^2} + u(x) &= 4e^{ix}; \quad x \in (0; 1); \\ u(0) &= 0; \quad \frac{du}{dx}(0) = 0; \end{aligned}$$
- Find an explicit conformal transformation of an open set  $U = \{z \in \mathbb{C} : |z| > 1, \operatorname{Im} z > 1\}$  to the unit disc.
- Assume  $f \in C^2[a; b]$  satisfying  $|f(x)| \leq A$ ;  $|f''(x)| \leq B$  for each  $x \in [a; b]$  and there exists  $x_0 \in [a; b]$  such that  $|f'(x_0)| \leq D$ , then  $|f'(x)| \leq \sqrt{AB} + D$ ;  $\forall x \in [a; b]$ .
- Let  $C([0; 1])$  denote the Banach space of real valued continuous functions on  $[0, 1]$  with the sup norm, and suppose that  $X \subset C([0; 1])$  is a dense linear subspace. Suppose  $I: X \rightarrow \mathbb{R}$  is a linear map (not assumed to be continuous in any sense) such that  $I(f) \geq 0$  if  $f \in X$  and  $f \geq 0$ . Show that there is a unique Borel measure  $\mu$  on  $[0, 1]$  such that  $I(f) = \int_0^1 f d\mu$  for all  $f \in X$ .
- For  $s \geq 0$ , let  $H^s(T)$  be the space of  $L^2$  functions  $f$  on the circle  $T = \mathbb{R}/(2\pi\mathbb{Z})$  whose Fourier coefficients  $\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} e^{inx} f(x) dx$  satisfy  $\sum (1+n^2)^s |\hat{f}_n|^2 < \infty$ ; with norm  $\|f\|_s^2 = \sum (1+n^2)^s |\hat{f}_n|^2$ .
  - Show that for  $r > s \geq 0$ , the inclusion map  $i: H^r(T) \hookrightarrow H^s(T)$  is compact.
  - Show that if  $s > 1/2$ , then  $H^s(T)$  includes continuously into  $C(T)$ , the space of continuous functions on  $T$ , and the inclusion map is compact.