

S.-T. Yau College Student Mathematics Contests 2011

Applied Math., Computational Math., Probability and Statistics

Individual

6:30–9:00 pm, July 9, 2011

(Please select 5 problems to solve)

1. Given a weight function $w(x) > 0$, let the inner-product corresponding to $w(x)$ be defined as follows:

$$(f, g) := \int_a^b w(x) f(x) g(x) dx;$$

and let $\|f\| := \sqrt{(f, f)}$.

- (1) Define a sequence of polynomials as follows:

$$\begin{aligned} p_0(x) &= 1; \quad p_1(x) = x - a_1; \\ p_n(x) &= (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x); \quad n = 2, 3, \dots \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{(x p_{n-1}, p_{n-1})}{(p_{n-1}, p_{n-1})}; \quad n = 1, 2, \dots \\ b_n &= \frac{(x p_{n-1}, p_{n-2})}{(p_{n-2}, p_{n-2})}; \quad n = 2, 3, \dots \end{aligned}$$

Show that $\{p_n(x)\}$ is an orthogonal sequence of monic polynomials.

- (2) Let $\{q_n(x)\}$ be an orthogonal sequence of monic polynomials corresponding to the w inner product. (A polynomial is called *monic* if its leading coefficient is 1.) Show that $\{q_n(x)\}$ is unique and it minimizes $\|q_n\|$ amongst all monic polynomials of degree n .
- (3) Hence or otherwise, show that if $w(x) = \frac{1}{\sqrt{1-x^2}}$ and $[a, b] = [-1, 1]$, then the corresponding orthogonal sequence is the Chebyshev polynomials:

$$T_n(x) = \cos(n \arccos x); \quad n = 0, 1, 2, \dots$$

and the following recurrent formula holds:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x); \quad n = 1, 2, \dots$$

- (4) Find the best quadratic approximation to $f(x) = x^3$ on $[-1, 1]$ using $w(x) = \frac{1}{\sqrt{1-x^2}}$.

2. If two polynomials $p(x)$ and $q(x)$, both of fifth degree, satisfy

$$p(i) = q(i) = \frac{1}{i}; \quad i = 2; 3; 4; 5; 6;$$

and

$$p(1) = 1; \quad q(1) = 2;$$

find $p(0)$ and $q(0)$.

3. Lay aside m black balls and n red balls in a jug. Suppose $1 \leq r \leq k \leq n$. Each time one draws a ball from the jug at random.

- 1) If each time one draws a ball without return, what is the probability that in the k -th time of drawing one obtains exactly the r -th red ball?
- 2) If each time one draws a ball with return, what is the probability that in the first k times of drawings one obtained totally an odd number of red balls?

4. Let X and Y be independent and identically distributed random variables. Show that

$$E[jX + Y] \leq E[jX]:$$

Hint: Consider separately two cases: $E[X^+] \leq E[X^-]$ and $E[X^+] > E[X^-]$.

5. Suppose that X_1, \dots, X_n are a random sample from the Bernoulli distribution with probability of success p_1 and Y_1, \dots, Y_n be an independent random sample from the Bernoulli distribution with probability of success p_2 .

- (a) Give a minimum sufficient statistic and the UMVU (uniformly minimum variance unbiased) estimator for $\mu = p_1$ and p_2 .
- (b) Give the Cramer-Rao bound for the variance of the unbiased estimators for $v(p_1) = p_1(1 - p_1)$ or the UMVU estimator for $v(p_1)$.
- (c) Compute the asymptotic power of the test with critical region

$$\sqrt{n}(\hat{p}_1 - \hat{p}_2) = \frac{\sqrt{p_1(1-p_1) + p_2(1-p_2)}}{2} Z_{1-\alpha/2}$$

when $p_1 = p$ and $p_2 = p + n^{-1/2} \delta$, where $\delta = 0.5p_1 + 0.5p_2$.

6. Suppose that an experiment is conducted to measure a constant μ . Independent unbiased measurements y of μ can be made with either of two instruments, both of which measure with normal errors: for $i = 1, 2$, instrument i produces independent errors with a $N(0, \sigma_i^2)$ distribution. The two error variances σ_1^2 and σ_2^2 are known. When a measurement y is made, a record is kept of the instrument used so that after n measurements the data is $(a_1; y_1), \dots, (a_n; y_n)$, where $a_m = i$ if y_m is obtained using instrument i . The choice between instruments is made independently for each observation in such a way that

$$P(a_m = 1) = P(a_m = 2) = 0.5; \quad 1 \leq m \leq n:$$

Let x denote the entire set of data available to the statistician, in this case $(a_1, y_1), \dots, (a_n, y_n)$, and let $l_\mu(x)$ denote the corresponding log likelihood function for μ . Let $a = \sum_{m=1}^n (2 - y_m a_m)$.

- (a) Show that the maximum likelihood estimate of μ is given by

$$\hat{\mu} = \frac{\sum_{m=1}^n y_m a_m}{\sum_{m=1}^n a_m}.$$

- (b) Express the expected Fisher information I_μ and the observed Fisher information I_x in terms of n , $\sum_{m=1}^n a_m^2$, and a . What happens to the quantity $I_\mu = I_x$ as $n \rightarrow \infty$?
- (c) Show that a is an ancillary statistic, and that the conditional variance of $\hat{\mu}$ given a equals $1/I_x$. Of the two approximations

$$\hat{\mu} \stackrel{\ell}{\approx} N(\mu; 1/I_\mu)$$

and

$$\hat{\mu} \stackrel{\ell}{\approx} N(\mu; 1/I_x);$$

which (if either) would you use for the purposes of inference, and why?