Problems 91-100 More Re-enforcements

91. Let X_1 and X_2 be independent standard normal random variables. Let $U\sim U(0,1)$ be independent of X_1 and X_2. Define $Z = UX_1 + (1-U)X_2$. (a) Find the conditional distribution of $Z|(U = u)$. (b) Find $\mathbb{E}(Z)$ and $\mathbb{V}(Z)$.

92. For a random variable $X\sim N(0,1)$, define $Y = X^2$. (a) Find the distribution of Y. (b) Find the correlation coefficient of X and Y. Are X and Y independent? (c) Find the conditional distribution of $Y|(X = x)$ and $X|(Y = y)$. (d) Find the joint cumulative distribution function of X and Y. (e) Can the identities "$f(x,y) = f(x)f(y|x) = f(y)f(x|y)$" be used to derive the joint density function of X and Y? Comment on the existence of the joint density function of X and Y in the two dimensional space.

93. Suppose that the conditional density of the random vector (X,Y) given the random variable Z is $f(x,y|z) = [z + (1-z)(x+y)]I_{(0,1)}(x)I_{(0,1)}(y)$ for $0 \le z \le 2$, and the density of the random variable Z is $f(z) = \frac{1}{2}I_{[0,2]}(z)$ where $I_A(x)$ denotes the indicator function, i.e., $I_A(x) = 1$ iff $x \in A$ and $I_A(x) = 0$ iff $x \notin A$. (a) Find the expectation $\mathbb{E}(X+Y)$. (b) Determine whether X and Y are independent or not. (c) Determine whether X and Z are independent or not. (d) Find the joint density of X and $X+Y$. (e) Find the distribution function of $\max(X,Y)\,|(Z = z)$.

94. Let $a_1, \dots, a_n$ be positive constants and $b_1, \dots, b_{n-1}$ be nonnegative constants. Suppose that random variables $Y_1, \dots, Y_{n-1}$ are mutually independent and $Y_j\sim\mathrm{Beta}\left(d_j, a_{j+1}\right)$, where $d_j = \sum_{k=1}^{j}(a_k + b_k)$, $j = 1, \dots, n-1$. Define

$$\begin{cases} X_i = (1 - Y_{i-1})\prod_{j=i}^{n-1} Y_j\,, & i = 1, \dots, n-1 \\ X_n = 1 - Y_{n-1} \end{cases}$$

where $Y_0 = 0$. Prove that the joint density of $X_1, \dots, X_{n-1}$ is given by

$$\frac{\prod_{i=1}^{n} x_i^{a_i - 1}\prod_{j=1}^{n-1}\left(\sum_{k=1}^{j} x_k\right)^{b_j}}{\prod_{j=1}^{n-1}\mathrm{B}\left(d_j, a_{j+1}\right)}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. [Hint: $X_1 + \cdots + X_i = Y_iY_{i+1}\cdots Y_{n-1}, i = 1, \dots, n-1$.]

95. Let X_1 and X_2 be two independent standard normal random variables. Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$. (a) Find the joint density of Y_1 and Y_2. (b) Find the marginal density of Y_2.

96. Let $X_1, \dots, X_n$ be a random sample from the uniform distribution on the unit interval $(0,1)$. The cumulative distribution function (cdf) of $X\sim U(0,1)$ is given by

$$\begin{cases} 0, & \text{iff } x \le 0 \\ x, & \text{iff } 0 < x < 1 \\ 1, & \text{iff } x \ge 1 \end{cases}$$

(a) Find the cdf and the density of $X_{(1)} = \min\{X_1, \dots, X_n\}$. (b) Find the cdf and the density of $X_{(n)} = \max\{X_1, \dots, X_n\}$. (c) Find the expression of $\mathbb{E}[X_{(n)} - X_{(1)}]$. (d) When $n = 2$, find the value of $\mathbb{V}[X_{(n)} - X_{(1)}]$. [Hint: If $Y \sim \text{Beta}(a, b)$ then $\mathbb{E}(Y) = \frac{a}{a+b}, \mathbb{E}(Y^2) = \frac{a(a+1)}{(a+b)(a+b+1)}, \mathbb{V}(Y) = \frac{ab}{(a+b)^2(a+b+1)}$.]

97. Let $X_1, \dots, X_n$ be a random sample from $N(\mu, \sigma^2)$. (a) Let $\sigma^2 = 1$. Find the most powerful test of size α for testing the null hypothesis $H_0: \mu = \mu_0$ against the alternative hypothesis $H_1: \mu = \mu_1 (< \mu_0)$. (b) Let μ be unknown. Use the likelihood ratio (LR) method to find the LR test of size α for testing $H_0: \sigma^2 = \sigma_0^2$ agianst the alternative $H_1: \sigma^2 \neq \sigma_0^2$.

98. Consider a random sample $X_1, \dots, X_n$ from the Poisson distribution $f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}, x = 0,1,2, \dots$, where $\theta > 0$. (a) Use the likelihood ratio test to find the general form of the critical region C for testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$, where $\theta_0 > 0$. (b) Given the critical region $C = \{(x_1, \dots, x_n): |\bar{x} - \theta_0| \geq k\}$, when $n \to \infty$, use the central limit theorem to find k, where the size is α.

99. A point is to be selected from the unit interval $(0,1)$ randomly. Let $A_1 = (0, 1/4]$, $A_2 = (1/4,\ 1/2]$, $A_3 = (1/2,\ 3/4]$ and $A_4 = (3/4,\ 1)$. Random experiments are repeated independently for 80 times under the same conditions. Observed frequencies that these points fall into A_1, A_2, A_3 and A_4 are 6, 18, 20, and 36, respectively. Test

$$H_0: \text{The cdf is Beta}(2,1)$$

against

$$H_1: \text{The cdf is not Beta}(2,1)$$

at the 0.025 significance level ($\chi^2(0.025,3) = 9.3484$).

100. The following are the numbers of passengers carried on flights 136 and 137 between Chicago and Phoenix on 12 days :

232 and 189,	265 and 230,	249 and 236,	250 and 261,
255 and 249,	236 and 218,	270 and 258,	247 and 253,
249 and 251,	240 and 233,	257 and 254,	239 and 249.

Use the paired-sample sign test at the 0.05 level of significance to test the null hypothesis $H_0: \mu_1 = \mu_2$ (that on the average the two flights carry equally many passengers) against the alternative hypothesis $H_1: \mu_1 > \mu_2$ by calculating the exact p-value.