MA204: Mathematical Statistics

Assignment 4

You have a total of 12 questions in Assignment 4.

Submit your solutions to 9 questions (i.e., Q4.1–Q4.9) in Exercise 4 on pages 180–182 of the Textbook "Mathematical Statistics", plus 3 questions chosen from the following 4 questions.

4.10 Let $X_1, \ldots, X_n, X \stackrel{\text{iid}}{\sim} f(x; \mu)$, where X is called the population random variable,

$$f(x; \mu) = \frac{1}{\sigma_0} \mathrm{e}^{-\frac{x-\mu}{\sigma_0}} \exp(-\mathrm{e}^{-\frac{x-\mu}{\sigma_0}}), \quad x \in \mathbb{R} \triangleq (-\infty, \infty),$$

where $\mu \in \mathbb{R}$ is the location parameter and $\sigma_0 > 0$ is the known scale parameter.

(a) Find the cdf of X.

(b) Use the result (4.3) in page 165 of the textbook "Mathematical Statistics" to find the $100(1-\alpha)\%$ equal-tail CI of μ.

4.11 Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} f(x; \sigma^2)$, where

$$f(x; \sigma^2) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0, \ \sigma > 0.$$

Use the result in Q3.20(a) of Assignment 3 and the result in Example 4.1 of the textbook "Mathematical Statistics" to find the $100(1-\alpha)\%$ equal-tail CI of σ^2.

4.12 Let $X_1, \ldots, X_n, X \overset{\text{iid}}{\sim} U(0, \theta)$, where $\theta > 0$ is an unknown parameter.

(a) Show that $X_{(n)} = \max(X_1, \ldots, X_n)$ is a sufficient statistic of θ.

(b) Find the pdf of $U \triangleq X_{(n)}/\theta$. Is U a pivotal quantity?

(c) Given $\alpha \in (0, 1)$, find the upper $\alpha/2$-th quantile $h_{\alpha/2}$ and the upper $(1-\alpha/2)$-th quantile $h_{1-\alpha/2}$ of the distribution of U.

(d) Construct the $100(1-\alpha)\%$ equal-tail CI for θ based on the results of Q4.12(b)–(c).

(e) Let $x_1 = 4.2$, $x_2 = 3.5$, $x_3 = 1.7$, $x_4 = 1.2$, $x_5 = 2.4$. Calculate the 95% equal-tail CI for θ.

(f) Show that the interval $[X_{(n)}, X_{(n)}\alpha^{-1/n}]$ is the shortest $100(1-\alpha)\%$ CI of θ.

4.13 Let $X_1, \ldots, X_n, X \overset{\text{iid}}{\sim} f(x; \lambda)$, where

$$f(x; \lambda) = \frac{\lambda}{e^{\lambda} - 1} e^{\lambda x}, \quad 0 \leqslant x \leqslant 1,\ \lambda > 0.$$

(a) Use Newton's method to calculate the MLE of λ.

(b) Find the cdf of X.

(c) Use the result (4.3) in page 165 of the textbook "Mathematical Statistics" to find the $100(1-\alpha)\%$ equal-tail CI of λ.