1. Solution: (Red part means this part is not in $f(E)$).

(a). Consider $f(\partial E)$. Let $A=\{1+ki \mid 0\le k\le 1\}$, $B=\{t+i \mid 0\le t\le 1\}$.

$f(A)=\{1-k^2+2ki \mid 0\le k\le 1\}$. If $f(A)=\{x+yi\}$, then: $x=1-k^2$, $y=2k$.

$\Rightarrow x=1-\frac{y^2}{4}$, $y\in[0,2]$.

$f(B)=\{t^2+2ti-1 \mid 0\le t\le 1\}$. If $f(B)=\{a+bi\}$. then: $a=t^2-1$, $b=2t$.

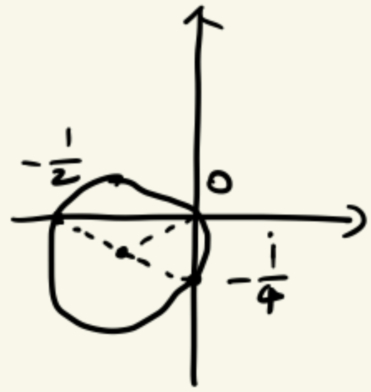
$\Rightarrow a=\frac{b^2}{4}-1$, $b\in[0,2]$.

$f(\{z \mid \operatorname{Re}z=0, 0\le \operatorname{Im}z\le 1\}) = -(\operatorname{Im}z)^2 \in [-1,0]$.

$f(\{z \mid \operatorname{Im}z=0, 0\le \operatorname{Re}z\le 1\}) = (\operatorname{Re}z)^2 = [0,1]$.

Hence: $f(E)=\{z \mid -1\le \operatorname{Re}z\le 1,\ \operatorname{Im}z\le 2\sqrt{1+\operatorname{Re}z}$ for $-1\le \operatorname{Re}z\le 0$ and $\operatorname{Im}z\le 2\sqrt{1-\operatorname{Re}z}$ for $0\le \operatorname{Re}z\le 1\}$. In graph:

(b). $f(z)=\frac{1}{z+i}$ is a linear-fraction function. It maps E as a line into a circle.



$f(0+3i)=\frac{1}{4i}=-\frac{i}{4}$. $f(-2-i)=-\frac{1}{2}$. $f(\infty)=0$.

Hence: $f(E)$ is a circle passing $-\frac{i}{4}, -\frac{1}{2}, 0$.

Hence: $f(E)=\{z \mid (\operatorname{Re}z+\frac{1}{4})^2+(\operatorname{Im}z+\frac{1}{8})^2=\frac{5}{64}\}$

(c). Since $f(z)$ is the branch of $\sqrt{z}$ determined by $f(i)=-\frac{1+i}{\sqrt{2}}$, then the $\arg z$ of the branch is $2\pi<\theta<4\pi$

Hence $f(E)=\pi^-\setminus\{z \mid \operatorname{Re}z=0, \operatorname{Im}z<0\}$. The image is:

(d). $\cot z = \frac{\cos z}{\sin z} = i\cdot\frac{e^{2iz}+1}{e^{2iz}-1}$. It is a linear fraction function of e^{2iz}.

for $E=\{0<\operatorname{Re}z<\frac{\pi}{4}\}$, $g(z)=e^{2iz}$, $h(z)=i\cdot\frac{z+1}{z-1}$. We have:

$g(E)=\{z\mid \operatorname{Re}z>0, \operatorname{Im}z>0\}$. $h(\{z\mid \operatorname{Re}z=0, \operatorname{Im}z>0\})=\partial B_1(0)\cap\{z\mid \operatorname{Re}z>0\}$

$h(\{z\mid \operatorname{Im}z=0, \operatorname{Re}z>0\})=(-\infty,-i)\cup(i,+\infty)$. And we also have:

$h(1+i)=i\cdot\frac{z+i}{1+i-1}=2+i$. Hence the image $f(E)$ is:

$f(E)=\{z\mid \operatorname{Re}z>0\}\setminus\overline{B_1(0)}$

(e). For $\operatorname{Im}z>(\operatorname{Re}z)^2+10$, if $\operatorname{Im}z=2k\pi$, then: $\operatorname{Re}z\in(-\sqrt{2k\pi-10},\sqrt{2k\pi-10})$

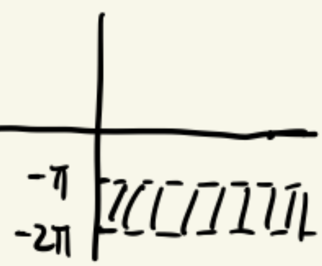
There is no restriction for k. Then: if $k\to+\infty$, we have that: $\operatorname{Re}z\in(-\infty,+\infty)$. Hence: $|f(z)|\in(0,+\infty)$. And for $2k\pi\le\operatorname{Im}z\le 2(k+1)\pi$, let $k\to+\infty$, we have that $\theta\in[0,2\pi)$, $r\in(0,+\infty)$.

Hence: $f(E)=\mathbb{C}\setminus\{0\}$.

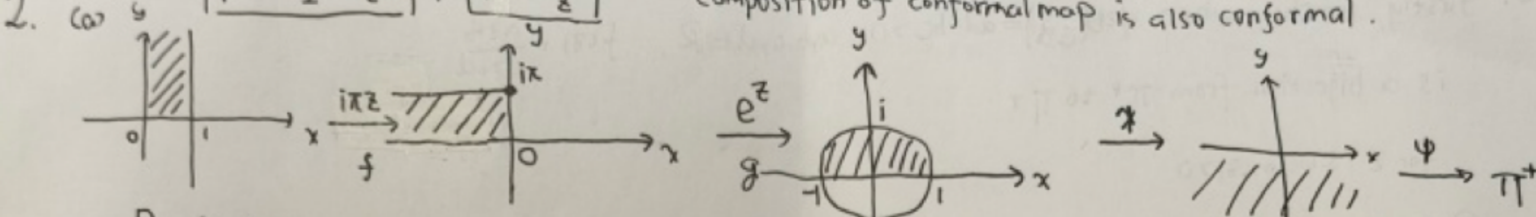
(f). From $\operatorname{Ln}z=\ln|z|+i\cdot\operatorname{Arg}z$, we have: $f(i)=i\cdot\operatorname{Arg}i=-\frac{3\pi}{2}i\Rightarrow\arg z=-\frac{3\pi}{2}$

Hence: $f(i)=\operatorname{Ln}z=\ln|z|+i(\operatorname{Arg}z-2\pi)$.

Hence: $f(E)=\{z\mid -2\pi<\operatorname{Im}z<\pi, \operatorname{Re}z\geq 0\}$.



2. (a) $\boxed{\ast(z)=\frac{1}{2}(z+\frac{1}{z})}$, $\boxed{\psi(z)=\frac{1}{z}}$ Composition of conformal map is also conformal.



$D=\{0<\mathrm{Re}z<1,\ \mathrm{Im}z>0\}$

$\{z \mid 0<|z|<1,\ \mathrm{Arg}z\in(0,\pi)\}$

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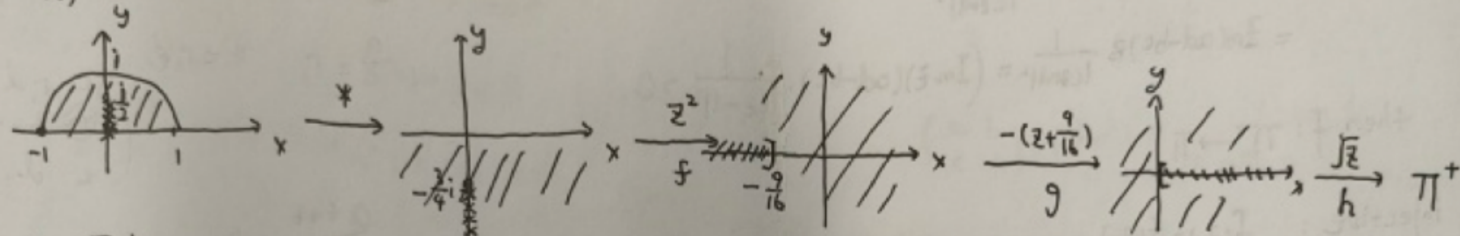
Hence, $\psi\circ\ast\circ g\circ f(z)=\frac{1}{2}(e^{i\pi z}+\frac{1}{e^{i\pi z}})$ is the conformal map we want.

(b)

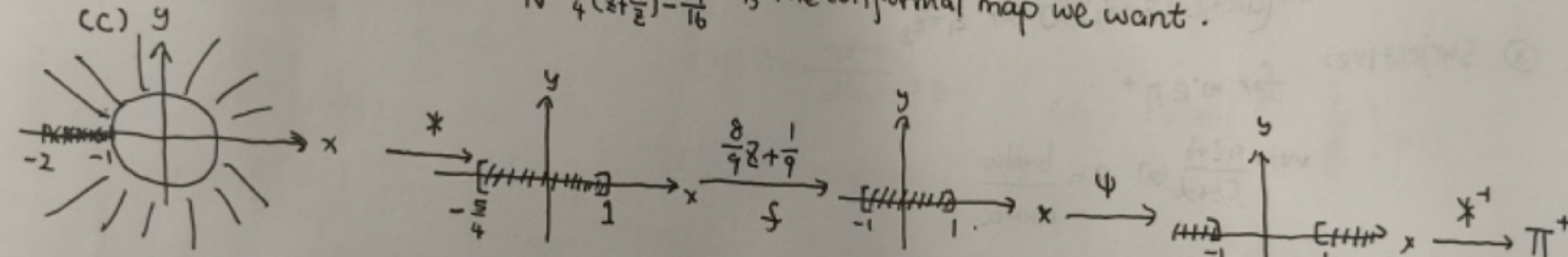


$D=G\setminus[0,\frac{i}{2}]$ $\quad \Pi^-\setminus(-\infty,-\frac{3}{4}i]$ $\quad \mathbb{C}\setminus(-\infty,-\frac{9}{16}]$ $\quad \mathbb{C}\setminus[0,+\infty)$

G:= the upper unit half disc.

Hence, $h\circ g\circ f\circ\ast(z)=\sqrt{-\frac{1}{4}(z+\frac{1}{z})^2-\frac{9}{16}}$ is the conformal map we want.

(c)



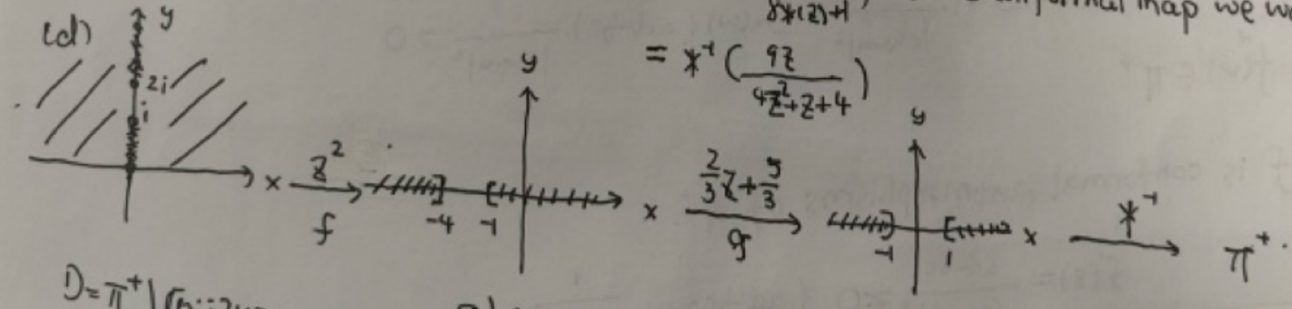
$D=G\setminus[-2,-1]$ $\quad \mathbb{C}\setminus[-\frac{5}{4},1]$ $\quad \mathbb{C}\setminus[-1,1]$ $\quad \mathbb{C}\setminus(-\infty,-1]\cup[1,+\infty)$

$G=\{|z|>1\}$

Hence $\ast^{-1}\circ\psi\circ f\circ\ast(z)=\ast^{-1}(\frac{9}{8\ast(z)+1})$ is the conformal map we want.

$=\ast^{-1}(\frac{9z}{4z^2+z+4})$

(d)



$D=\Pi^+\setminus([0,i]\cup[2i,+\infty))$ $\quad \mathbb{C}\setminus(-\infty,-4]\cup[-1,+\infty)$ $\quad \mathbb{C}\setminus(-\infty,-1]\cup[1,+\infty)$

Hence $\ast^{-1}\circ g\circ f(z)=\ast^{-1}(\frac{2}{3}z^2+\frac{5}{3})$ is the conformal map we want.

$\boxed{3.}$ <u>Proof:</u>

<u>Step 1:</u> If $M \in SL_2(\mathbb{R})$, then f_M maps $\mathbb{H}^+$ to itself.

Pf: Since $Im(f_M(z)) = \dfrac{(ad-bc)\, Im(z)}{|cz+d|^2} = \dfrac{Im(z)}{|cz+d|^2} > 0$, $\forall z \in \mathbb{H}^+$.
(*1)

<u>Step 2:</u> If $M, M' \in SL_2(\mathbb{R})$, then $f_M \circ f_{M'} = f_{MM'}$. (proved in class)

Thus, $\forall$ linear fractional map f_M, $f_M \in Aut(\mathbb{H}^+)$

Pf: $(f_M \circ f_{M^{-1}})(z) = f_{MM^{-1}}(z) = f_I(z) = z$

$\Rightarrow$ f_M has a simple inverse $f_{M^{-1}}$.

<u>Step 3:</u> $\forall z, w \in \mathbb{H}^+$, $\exists$ $M \in SL_2(\mathbb{R})$, s.t. $f_M(z) = w$.

(Therefore, $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}^+$).

Pf: It's sufficient to show: we can map any $z \in \mathbb{H}^+$ to i.

Set $d=0$ in (*1) $\Rightarrow$ $Im(f_M(z)) = \dfrac{Im(z)}{|cz|^2}$.

Choose $c \in \mathbb{R}$ s.t. $Im(f_M(z)) = 1$.

Define $M_1 = \begin{pmatrix} 0 & c^{-1} \\ c & 0 \end{pmatrix}$. $\Rightarrow$ $Im(f_{M_1}(z)) = 1$.

Define $M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbb{R}$.

$M = M_2 M_1. \Rightarrow f_M(z) = i.$

<u>Step 4</u>: For $\theta \in \mathbb{R}$, the mat. $M_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SL_2(\mathbb{R})$.

Let $F(z) = \dfrac{z-i}{z+i}$, $F : \Pi^+ \to B_1(0)$.

$\Rightarrow F \circ f_{M_\theta} \circ F^{-1}$ denotes the rotation of angle -2θ in the disc,

since $F \circ f_{M_\theta} = e^{-2i\theta} F(z)$.

<u>Step 5</u>: Suppose $f \in Aut(\Pi^+)$ with $f(\beta) = i$.

Consider $N \in SL_2(\mathbb{R})$ s.t. $f_N(i) = \beta$.

$\Rightarrow g := f \circ f_N$ satisfies $g(i) = i$.

$\Rightarrow F \circ g \circ F^{-1}$ is an automorphism of the disc that fixes the origin. $\Rightarrow F \circ g \circ F^{-1}$ is a rotation.

By step 4, there exists $\theta \in \mathbb{R}$ s.t.

$$F \circ g \circ F^{-1} = F \circ f_{M_\theta} \circ F^{-1}.$$

$\Rightarrow g = f_{M_\theta}. \quad \Rightarrow f = f_{M_\theta N^{-1}}.$

i.e. $\forall f \in Aut(\Pi^+)$, f is a linear fraction map. $\square$

4. Prove: Method 1: Consider computing with linear algebra:

$$f(z)=\frac{az+b}{cz+d}, \text{ Then } (z_1,z_2,z_3,z_4)=\frac{z_2-z_3}{z_2-z_4}:\frac{z_1-z_3}{z_1-z_4}$$

transforming into: $(f(z_1),f(z_2),f(z_3),f(z_4))=\frac{f(z_2)-f(z_3)}{f(z_2)-f(z_4)}:\frac{f(z_1)-f(z_3)}{f(z_1)-f(z_4)}$

Since $\frac{az_i+b}{cz_i+d}-\frac{az_j+b}{cz_j+d}=\frac{(az_i+b)(cz_j+d)-(az_j+b)(cz_i+d)}{(cz_i+d)(cz_j+d)}=\frac{(ad-bc)(z_i-z_j)}{(cz_i+d)(cz_j+d)}$

Hence: $(f(z_1),f(z_2),f(z_3),f(z_4))$

$$=\frac{\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}}{\frac{(ad-bc)(z_2-z_4)}{(cz_2+d)(cz_4+d)}}:\frac{\frac{(ad-bc)(z_1-z_3)}{(cz_1+d)(cz_3+d)}}{\frac{(ad-bc)(z_1-z_4)}{(cz_1+d)(cz_4+d)}}=\frac{z_2-z_3}{z_2-z_4}:\frac{z_1-z_3}{z_1-z_4}$$

$$=(z_1,z_2,z_3,z_4)$$

Q.E.D.

Method 2: Consider the property: There exists only a linear-frac mapping $T(z)$,

That matches $T(z_i)=w_i$ $(i=1,2,3)$

Consider $F(z)=(z,z_1,z_2,z_3)$, $G(w)=(w,w_1,w_2,w_3)$

Easily we find $F(z), G(w)$ are both linear-frac mapping,

and $(G^{-1}\circ F)(z_i)=w_i$ (Since F, G both matches: $z_1, w_1 \mapsto 1$, $z_2, w_2 \mapsto 0$, $z_3, w_3 \mapsto \infty$)

Thus $T(z)=(G^{-1}\circ F)(z)$ is a right linear-frac mapping

If there exists T_1, T_2, then $T_2^{-1}\circ T_1$ has three fixed points,

So $T_2^{-1}\circ T_1$ must be identity mapping, $T_1=T_2$

Thus we find $T(z)=(G^{-1}\circ F)(z)$ matches $(z,z_1,z_2,z_3)=(w,w_1,w_2,w_3)=(T(z),T(z_1),T(z_2),T(z_3))$

(with its uniqueness) Q.E.D.

$\boxed{5.}$ Proof:

1° Suppose $z_1 \sim z_4$ belong to the same gener-d cycle.

Let $L(z) = \frac{z_2 - z_3}{z_2 - z_4} : \frac{z - z_3}{z - z_4}$ be a linear fractional map.

$\Rightarrow L(z_2) = 1 \quad L(z_4) = \infty \quad L(z_3) = 0$.

$\Rightarrow L$ maps the gener-d cycle determined by z_2, z_3, z_4 to $\mathbb{R}$.

Since $z_1 \sim z_4$ belong to the same gener-d cycle,

$L(z_1) \in \mathbb{R}. \Rightarrow \frac{z_2 - z_3}{z_2 - z_4} : \frac{z_1 - z_3}{z_1 - z_4} \in \mathbb{R}$.

2° Suppose $\frac{z_2 - z_3}{z_2 - z_4} : \frac{z_1 - z_3}{z_1 - z_4} \in \mathbb{R}$. i.e $L(z_1) \in \mathbb{R}$.

Since $L(z_2), L(z_3), L(z_4) \in \mathbb{R}$.

$L(z_1) \sim L(z_4)$ belongs to the same gener-d cycle.

namely $\mathbb{R}$.

Since L^{-1} is also a linear fractional map.

by the circle prop. (linear frac. map maps

gener-d cycles to gener-d cycles).

$\Rightarrow z_1 \sim z_4$ belong to the same gener-d cycle. $\square$