Discrete Mathematics for Computer Science

Lecture 18: Graph

Dr. Ming Tang

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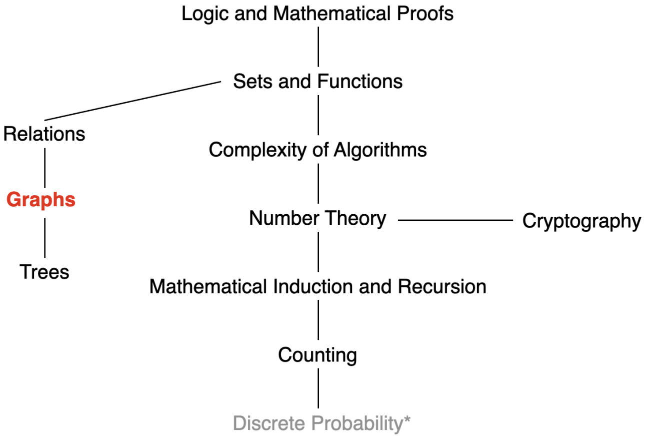
This Lecture



Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, ...

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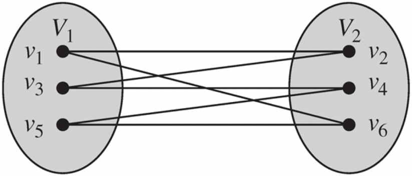

Bipartite Graphs

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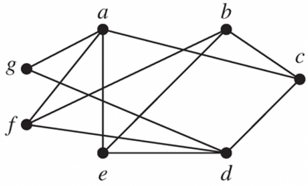

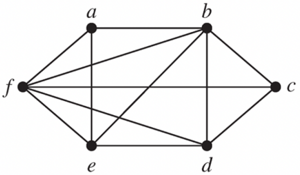

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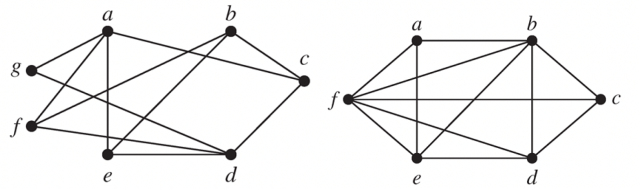

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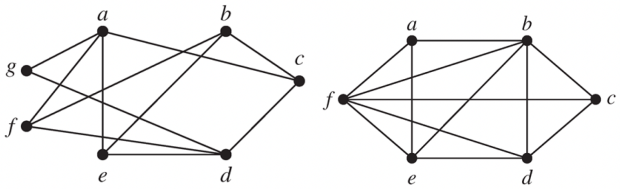
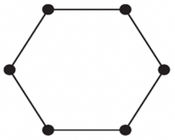


(a) Bipartite: Its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset.

(b) Not bipartite: Its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset.

Bipartite Graphs: Examples

Show that C_6 is bipartite.



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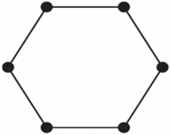

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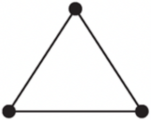
Show that C_3 is not bipartite.



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Complete Bipartite Graphs

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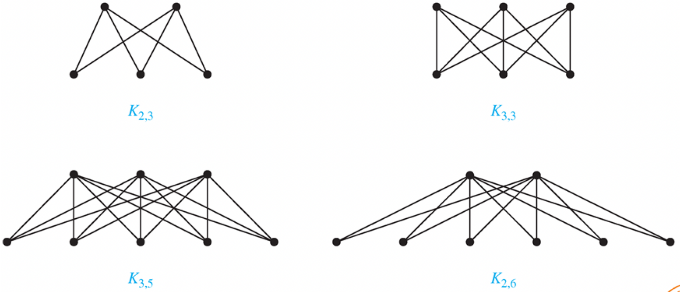

Bipartite Graphs and Matchings

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In other words, a matching is a subset of edges such that if $\{s, t\}$ and $\{u, v\}$ are distinct edges of the matching, then s, t, u, and v are distinct.

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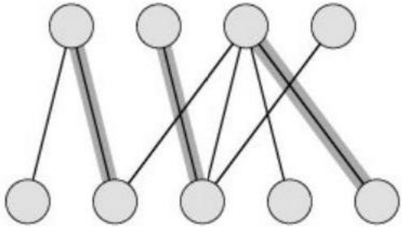

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Job assignments: vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A common goal is to match jobs to employees so that the most jobs are done.

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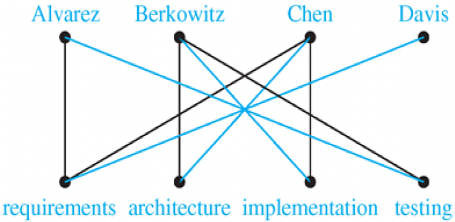

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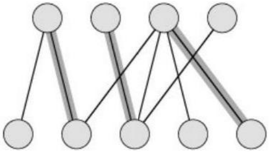

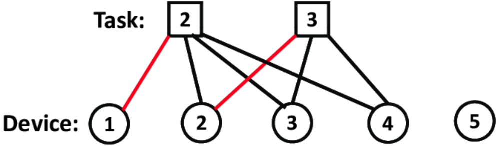

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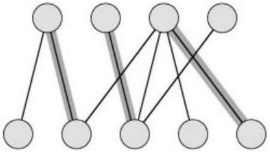
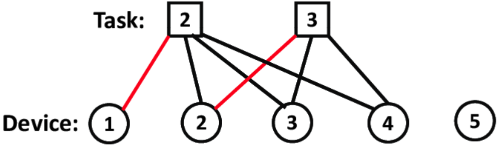
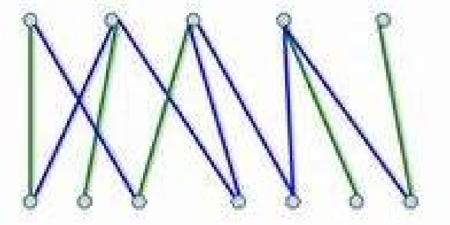
A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a complete matching from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

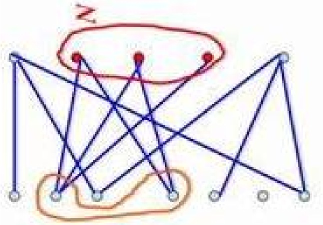




Hall's Theorem: Example

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.





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Hence, $|N(A)| \geq |A|$.

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Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2), and $|V_1| = j \leq k$, then there is a complete mathching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is met.

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(i) For all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2.

(ii) For some integer j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2.

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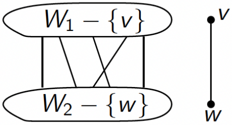

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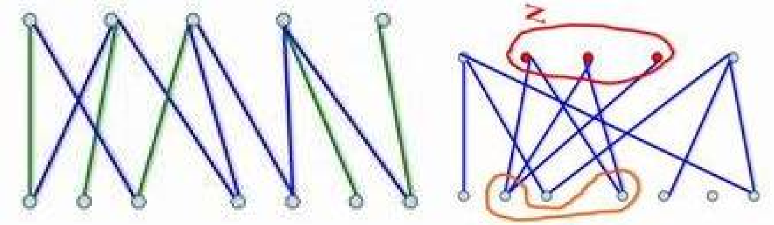

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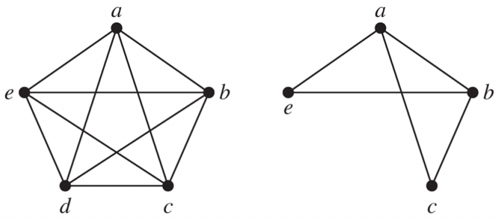

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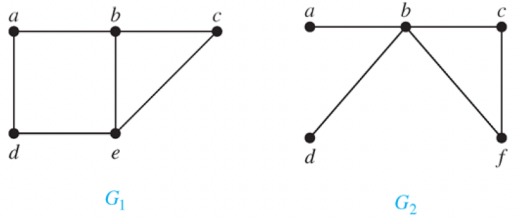

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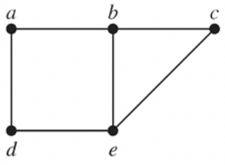
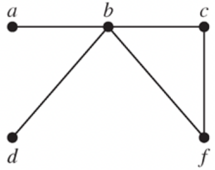
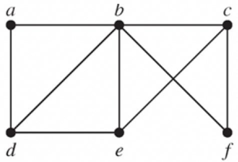


G_1

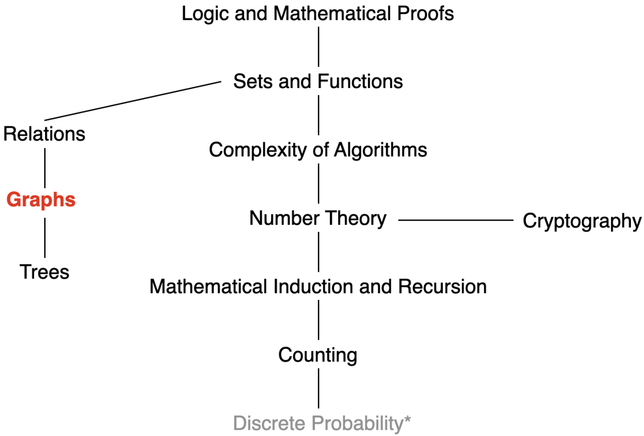


G_2



$G_1 \cup G_2$

This Lecture



Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, ...

Representation of Graphs

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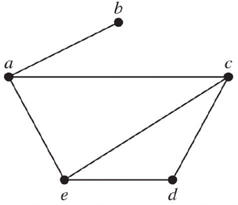

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

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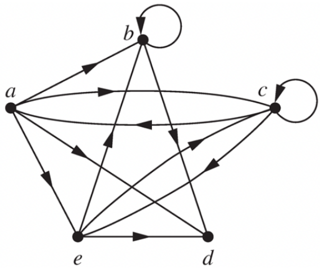

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrices

Definition: Suppose that $G = (V, E)$ is a simple graph with $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The adjacency matrix $\mathbf{A}_G$ of G, is the $n \times n$ zero-one matrix with 1 as its (i, j)-th entry when v_i and v_j are adjacent, and 0 as its (i, j)-th entry when they are not adjacent.

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$$\mathbf{A}_G = [a_{ij}]_{n\times n}, \text{ where}$$

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

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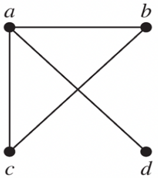

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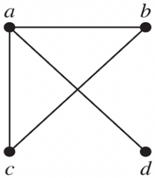

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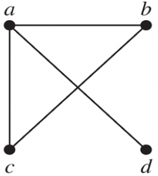

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Directed graph?

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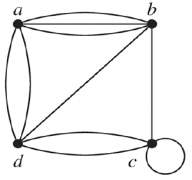

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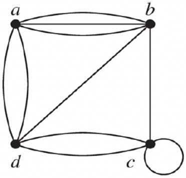

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

Definition: Let $G = (V, E)$ be an undirected graph with vertices v_1, v_2, . . . , v_n and edges e_1, e_2, . . . , e_m. The incidence matrix with respect to the ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]_{n \times m}$, where

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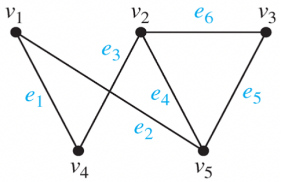

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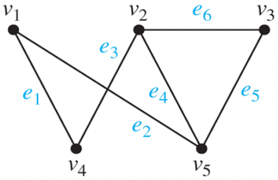

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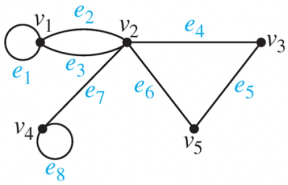

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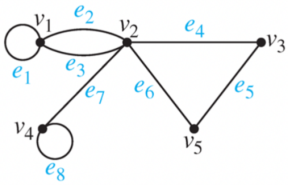

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Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an isomorphism.

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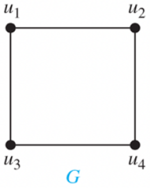

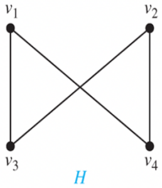

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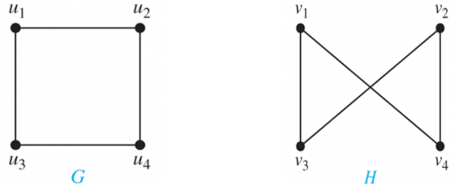

- Define a one-to-one correspondence: $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$.
- Check their adjacent matrics.

Isomorphism of Graphs

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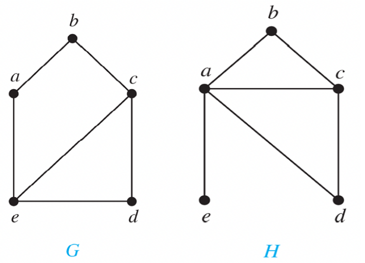
Sometimes it is not difficult to show that two graphs are not isomorphic. We can achieve this by checking some graph invariants.

Useful graph invariants include the number of vertices, number of edges, degree sequence, etc.

Isomorphism of Graphs: Example

Determine whether these two graphs are isomorphic.



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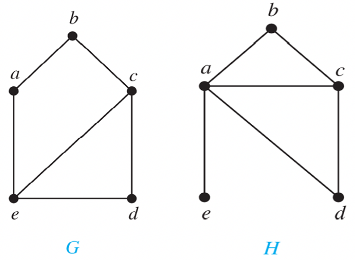

H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic.

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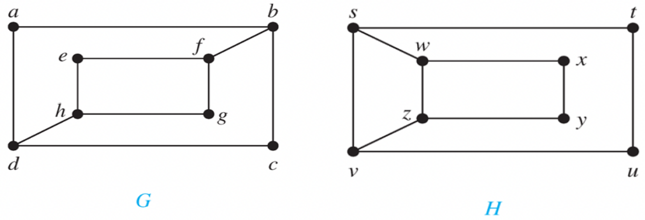

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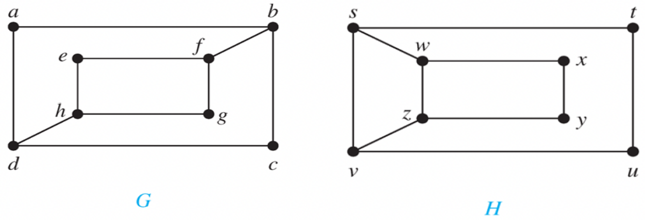

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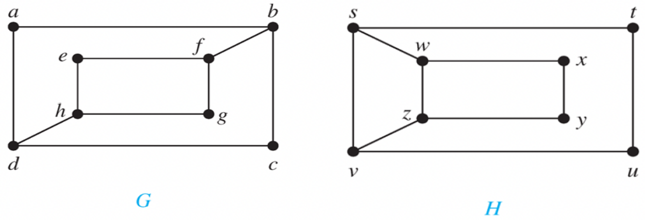

G and H are not isomorphic. This is because $deg(a) = 2$ in G, and a must correspond to either t, u, x, or y in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

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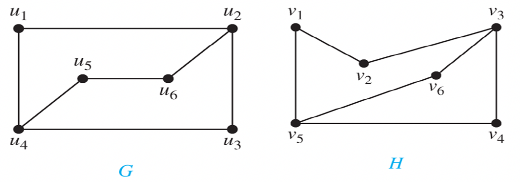

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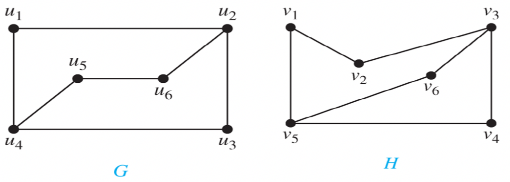

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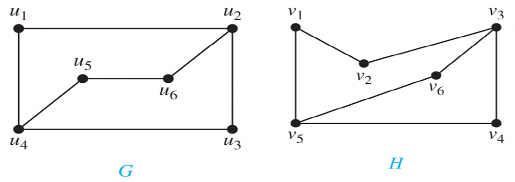

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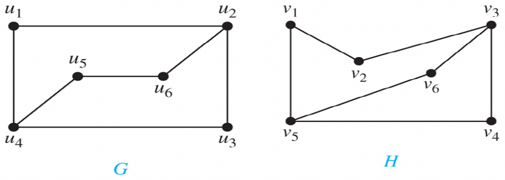

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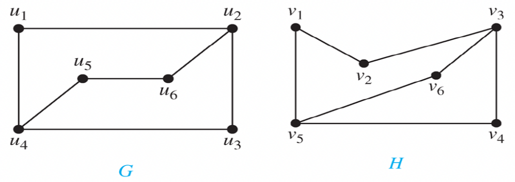

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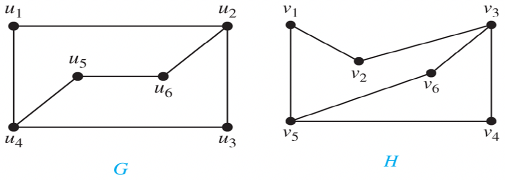

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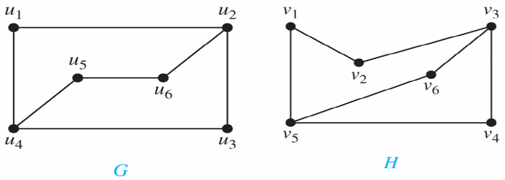
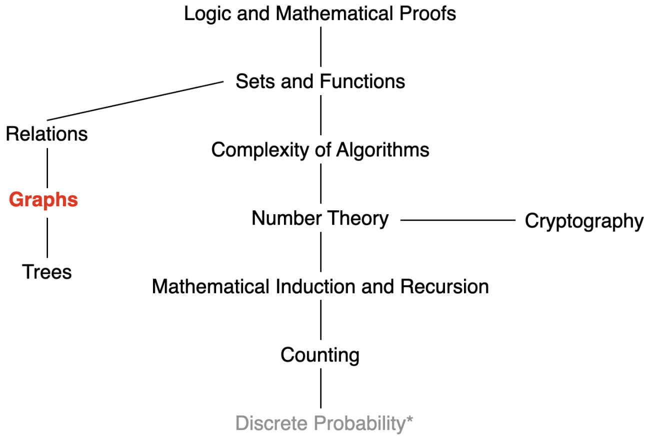


$f(u_1) = v_6,\ f(u_2) = v_3,\ f(u_3) = v_4,\ f(u_4) = v_5,\ f(u_5) = v_1,\ f(u_6) = v_2.$

$$\mathbf{A}_H = \begin{array}{cc} & \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{array}{c} v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}. \quad \mathbf{A}_G = \begin{array}{cc} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

We conclude that f is an isomorphism, so G and H are isomorphic.

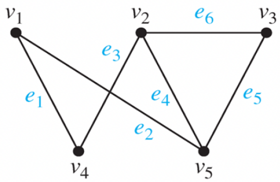
This Lecture



Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, ...

Path: Undirected Graph

Definition: Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i for $i = 1, \ldots, n$.



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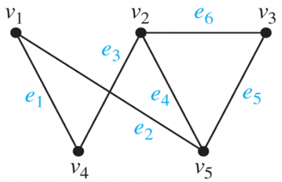

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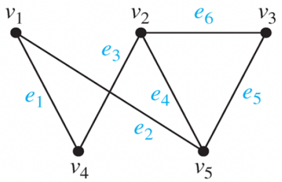

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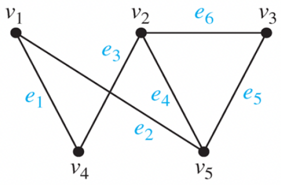

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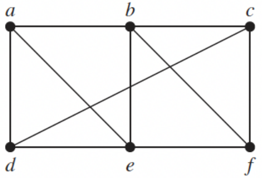

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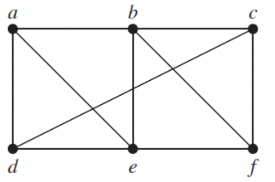

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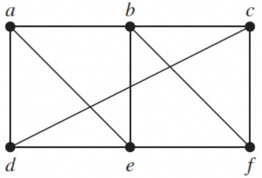

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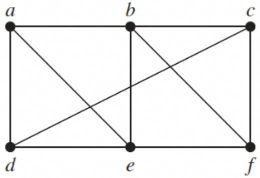

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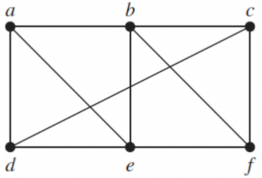
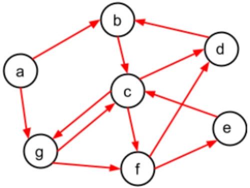


- a, d, c, f, e is a simple path of length 4.
- d, e, c, a is not a path, because $\{e, c\}$ is not an edge.
- b, c, f, e, b is a circuit of length 4.
- The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice.

Path: Directed Graph

Definition: Let n be a nonnegative integer and G an directed graph. A path of length n from u to v in G is a sequence of n edges $e_1, e_2, \ldots,$ e_n of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i is associated with initial vertex x_{i-1} and terminal vertex x_i for $i = 1, \ldots, n$.



A path of length greater than zero that begins and ends at the same vertex is called a circuit or cycle.

A path or circuit is called simple if it does not contain the same edge more than once.

Connectivity

An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

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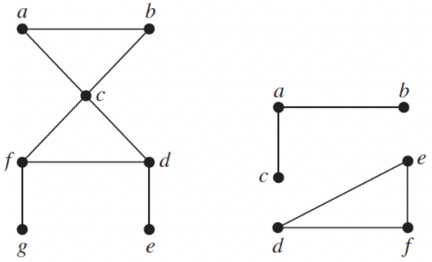

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Lemma: If there is a path between two distinct vertices x and y of a graph G, then there is a simple path between x and y in G.

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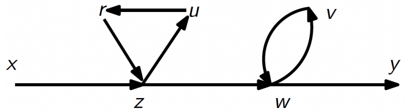

Path from x to y: $x, z, u, r, z, u, r, z, w, v, w, y$.
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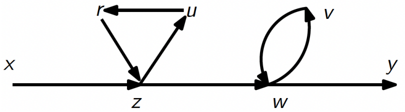

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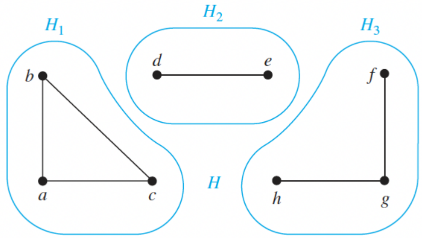
Path from x to y: x, z, w, y.

Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connectivity

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

A graph G that is not connected has two or more connected components that are disjoint and have G as their union.



Connectedness in Directed Graphs

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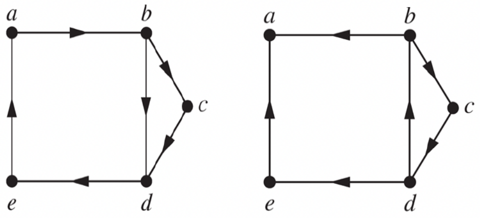

G H

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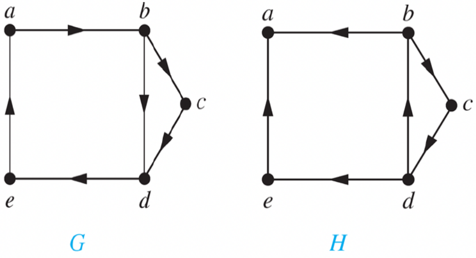

G is strongly connected; H is weakly connected.

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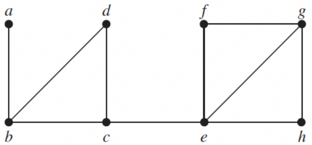

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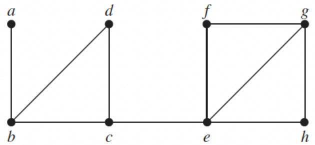

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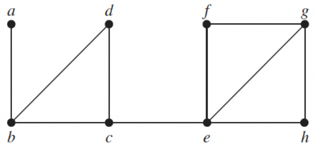

The cut vertices are b, c, and e.

The cut edges are $\{a, b\}$ and $\{c, e\}$.

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A set of edges E' is called an edge cut of G if the subgraph $G - E'$ is disconnected. The edge connectivity $\lambda(G)$ is the minimum number of edges in an edge cut of G.

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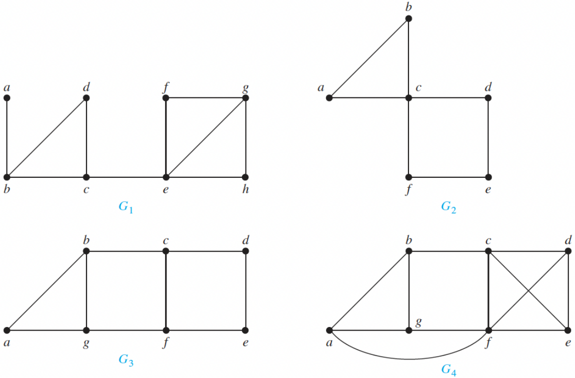

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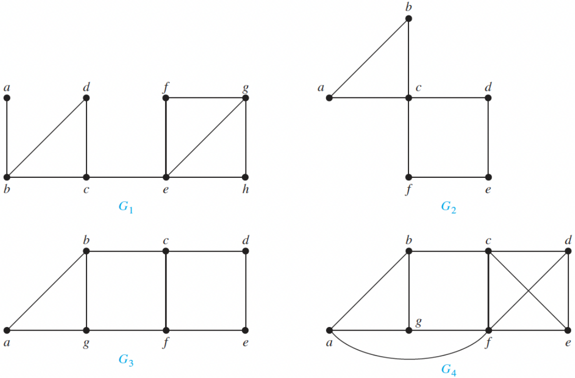

$\lambda(G_1) = 1$; $\lambda(G_2) = 2$; $\lambda(G_3) = 2$; $\lambda(G_4) = 3$

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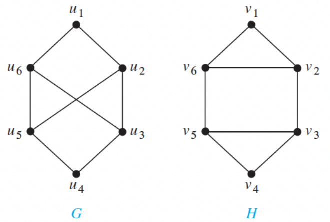

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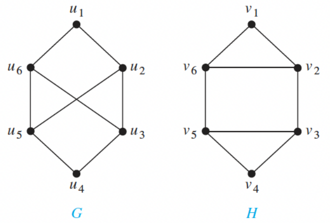

Not isomorphic. H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1, whereas G has no simple circuit of length three.

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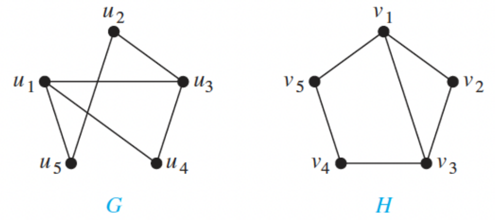

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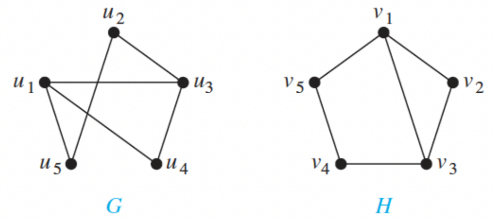

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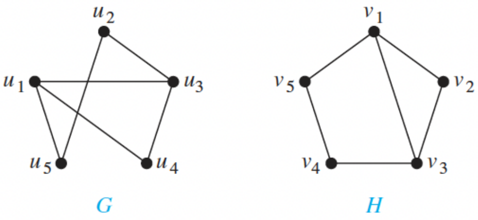


Because many isomorphic invariants (e.g., number of vertices/edges, degree, circuit) agree, G and H may be isomorphic. Let $f(u_1) = v_3$, $f(u_4) = v_2$, $f(u_3) = v_1$, $f(u_2) = v_5$, and $f(u_5) = v_4$. We can show that f is an isomorphism, so G and H are isomorphic.

Counting Paths between Vertices

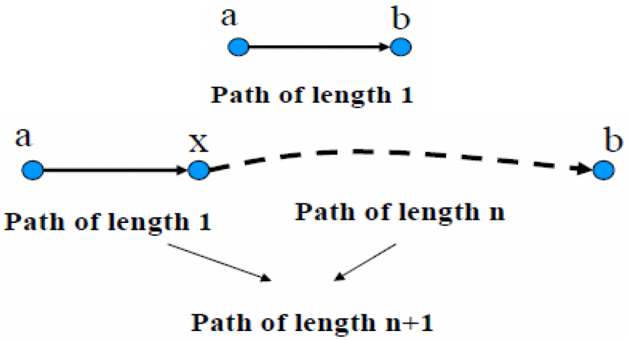
Theorem: Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)-th entry of $\mathbf{A}^r$.

Note: with directed or undirected edges, multiple edges and loops allowed

Recap: Path Length

Theorem: Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$. (Boolean product.)

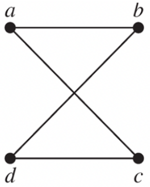
Proof (by induction):



Recall that $R^{n+1} = R^n \circ R$

Counting Paths between Vertices:

How many paths of length 4 are there from a to d in the graph G?



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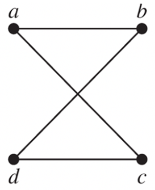

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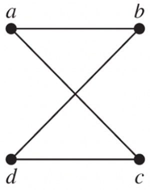

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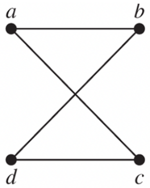

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a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d;
a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; a, c, d, c, d

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- Basic Step: The number of paths from v_i to v_j of length 1 is the (i, j)-th entry of $\mathbf{A}$.

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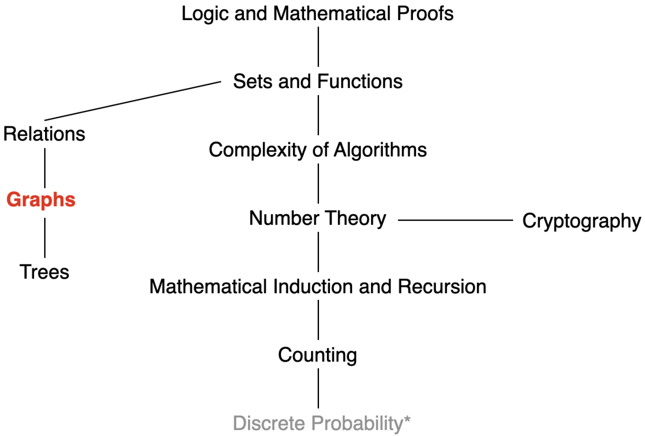
- Inductive Step: $\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$. The (i, j)-th entry of $\mathbf{A}^{r+1}$ equals

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj},$$

  where b_{ik} is the (i, k)-th entry of $\mathbf{A}^r$. By the inductive hypothesis, b_{ik} is the number of paths of length r from v_i to v_k.
- Inductive Conclusion: (i, j)-th entry of $\mathbf{A}^{r+1}$ counts all paths with length $r + 1$ for all possible intermediate vertices v_k.

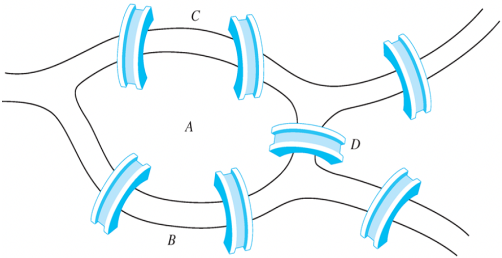
Next Lecture



Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, ...

Euler Paths

Königsberg seven-bridge problem: People wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



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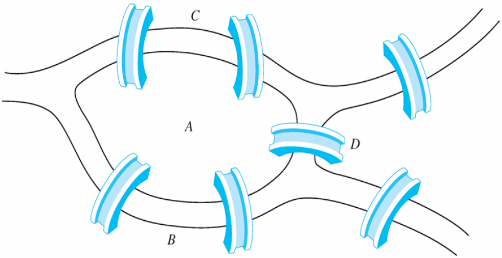

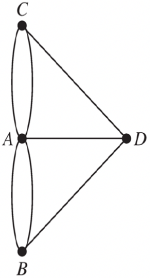

Euler Paths and Circuits

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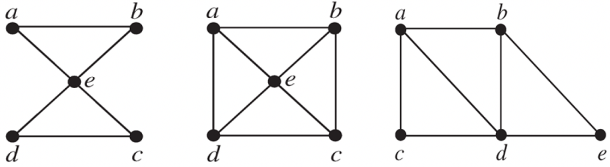

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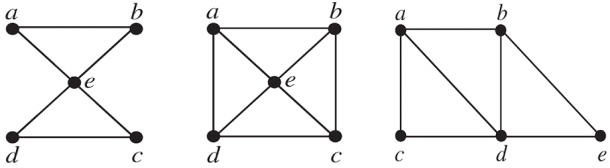

G_1 G_2 G_3

G_1: an Euler circuit, e.g., a, e, c, d, e, b, a;
G_2: neither; G_3: an Euler path, e.g., a, c, d, e, b, d, a, b

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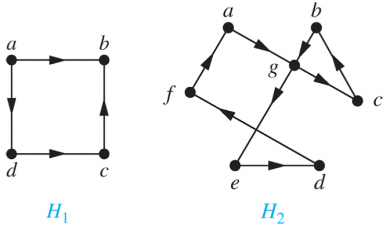
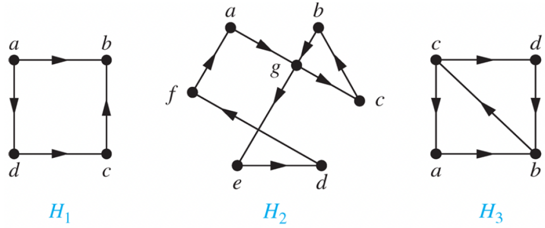


Euler Paths and Circuits

Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

Example: Which of the directed graphs have an Euler circuit? Of those that do not, which have an Euler path?



H_1: neither; H_2: an Euler circuit, e.g., a, g, c, b, g, e, d, f, a; H_3: an Euler path, e.g., c, a, b, c, d, b

Necessary Conditions for Euler Circuits and Paths

Euler Circuit $\Rightarrow$ The degree of every vertex must be even

- Each time the circuit passes through a vertex, it contributes two to the vertex's degree.
- The circuit starts with a vertex a and ends at a, then contributes two to $deg(a)$.

Euler Path $\Rightarrow$ The graph has exactly two vertices of odd degree

- The initial vertex and the final vertex of an Euler path have odd degree.