**CS201: Discrete Math for Computer Science**
**2025 Spring Semester Written Assignment #4**

The assignment needs to be written in English. Assignments in any other language will get zero point. Any plagiarism behavior will lead to zero point.

**Q. 1.** Let $(x_i, y_i)$, $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the $xy$ plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates.

**Q. 2.** Use induction to prove that 3 divides $n^3 + 2n$ whenever $n$ is a positive integer.

**Q. 3.** Let $S$ be a set of $n$ distinct integers. Prove that there exists a non-empty subset $T \subseteq S$ such that the sum of the elements in $T$ is divisible by $n$.

**Q. 4.** The running time of an algorithm $A$ is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where $b$ is a positive constant and $n$ is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where $a$ and $c$ are positive constants and $n$ is a power of 4. For the rest of this problem, you may assume that $n$ is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in <u>closed form</u> (in terms of $b$ if necessary) and should <u>not</u> use summation.

(b) Find a solution for $T(n)$. Your solution should be in <u>closed form</u> (in terms of $a$ and $c$ if necessary) and should <u>not</u> use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

**Q. 5.** Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

**Q. 6.** Suppose that $p$ and $q$ are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding $n$ that are relatively prime to $n$, i.e., the Euler function $\phi(n)$.

**Q. 7.** How many ordered pairs of integers $(a, b)$ are needed to guarantee that there are two ordered pairs $(a_1, b_1)$ and $(a_2, b_2)$ such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

**Q. 8.** Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever $n$ and $r$ are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

**Q. 9.** Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when $n$ and $r$ are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

**Q. 10.** Solve the recurrence relation:

$$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$$

with initial conditions $a_0 = 2$, $a_1 = 3$, and $a_2 = 7$.