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Assignment 2.

Part 1: 7 questions chosen from Q2.1 ~ Q2.10

Q2.1 Sol: Since $T \overset{d}{=} \frac{Z}{\sqrt{Y/n}}$, $E(T) = E(Z)E\left(\frac{1}{\sqrt{Y/n}}\right) = 0$.

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{2^{-\frac{n}{2}}}{\Gamma(\frac{n}{2})} y^{\frac{n}{2}-2} e^{-\frac{y}{2}} dy = \frac{1}{\Gamma(\frac{n}{2})} \cdot \frac{1}{2} \int_0^\infty \left(\frac{y}{2}\right)^{(\frac{n}{2}-1)-1} e^{-\frac{y}{2}} d\left(\frac{y}{2}\right)$$

$$= \frac{1}{2} \times [\Gamma(\tfrac{n}{2})]^{-1} \Gamma(\tfrac{n}{2}-1) = \frac{1}{2} \cdot \frac{1}{\frac{n}{2}-1} = \frac{1}{n-2}$$

$$E(T^2) \underset{Z \perp\!\!\!\perp Y}{=} nE(Z^2)E\left(\frac{1}{Y}\right) = \frac{n}{n-2}.$$

Thus, $Var(T) = E(T^2) - (E(T))^2 = \frac{n}{n-2}$.

Q2.2 Sol: $X_1, \dots, X_n \overset{iid}{\sim} Beta(3,2)$.

$$\Rightarrow f(x) = \frac{x^2(1-x)}{B(3,2)} = \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} x^2(1-x) = \frac{4!}{2!\,1!} x^2(1-x) = 12x^2(1-x)$$

Support $S_X = (0,1)$.

$$F(x) = \int_0^x 12t^2(1-t)\,dt = \int_0^x (12t^2 - 12t^3)\,dt = (4t^3 - 3t^4)\Big|_{t=0}^{t=x}$$
$$= 4x^3 - 3x^4.$$

By (2.17), for $X_{(n)}$,

$$g_n(x) = nf(x) \cdot F^{n-1}(x)$$
$$= n \cdot 12x^2(1-x) \cdot (4x^3 - 3x^4)^{n-1}$$
$$= 12n x^2(1-x)(4x^3 - 3x^4)^{n-1} \quad (\text{Support: } 0<x<1).$$

By (2.19), for $X_{(1)}$,

$$g_1(x) = nf(x)\{1 - F(x)\}^{n-1}$$
$$= n\,12x^2(1-x)\{1 - 4x^3 + 3x^4\}^{n-1}$$
$$= 12nx^2(1-x)(3x^4 - 4x^3 + 1)^{n-1} \quad (\text{Support: } 0<x<1).$$

Q2.3. (a) $f(x) = e^{-x}$ $(x \geq 0)$. $F(x) = \int_0^x e^{-t}dt = (-e^{-t})\Big|_{t=0}^{t=x} = 1 - e^{-x}$.

$$Z_i = \{X_{(i)} - X_{(i-1)}\} \times (n-i+1)$$
$$= (n-i+1)X_{(i)} - (n-i+1)X_{(i-1)} \quad i = 1, 2, \dots, n. \quad (X_{(0)} := 0)$$

By (2.23), for $X_{(i)}$

$$g_i(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x) F^{i-1}(x) \times \{1 - F(x)\}^{n-i}$$
$$= \frac{n!}{(i-1)!\,(n-i)!} e^{-x}(1-e^{-x})^{i-1}(e^{-x})^{n-i}$$
$$= \frac{n!}{(i-1)!\,(n-i)!} (e^{-x})^{n-i+1}(1-e^{-x})^{i-1}.$$

Define $Y_i = (n-i+1)^{-1} Z_i = X_{(i)} - X_{(i-1)}$.

By (2.25), the joint density of $X_{(i)}, X_{(i-1)}$ is

$$g_{i-1,i}(x_{i-1}, x_i) = \frac{n!}{(i-2)!\,0!\,(n-i)!} f(x_i) f(x_{i-1}) \times F^{i-2}(x_{i-1}) \{F(x_i) - F(x_{i-1})\}^0 \{1 - F(x_i)\}^{n-i}$$

$$= \frac{n!}{(i-2)!\,(n-i)!} e^{-x_i} e^{-x_{i-1}} \times (1 - e^{-x_{i-1}})^{i-2} (e^{-x_i})^{n-i}$$

$$= \frac{n!}{(i-2)!\,(n-i)!} (e^{-x_i})^{n-i+1} e^{-x_{i-1}} (1 - e^{-x_{i-1}})^{i-2}$$

where $0 \le x_{i-1} \le x_i$.

Let $c := \frac{n!}{(i-2)!\,(n-i)!}$, $g_{i-1,i}(x_{i-1}, x_i) = c (e^{-x_i})^{n-i+1} e^{-x_{i-1}} (1 - e^{-x_{i-1}})^{i-2}$.

Making the transformation: $z = x_i - x_{i-1}$, $x = x_{i-1}$ we have

$$J(z, x \to x_{i-1}, x_i) = \frac{\partial(z,x)}{\partial(x_{i-1}, x_i)} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Hence the joint density of $Y_i = X_{(i)} - X_{(i-1)}$ and $X = X_{(i-1)}$ is

$$h(z,x) = g_{i-1,i}(x_{i-1}, x_i) \,/\, |J(z, x \to x_{i-1}, x_i)|$$

$$= c \cdot (e^{-x-z})^{n-i+1} e^{-x} (1 - e^{-x})^{i-2}. \qquad (0 \le x \le x+z)$$

The marginal density of $Y_i = X_{(i)} - X_{(i-1)}$ is given by

$$h(z) = \int_0^\infty h(z,x)\, dx$$

$$= c \int_0^\infty e^{-(n-i+1)z} (e^{-x})^{n-i+2} (1 - e^{-x})^{i-2} dx.$$

$$= c e^{-(n-i+1)z} \int_0^\infty (e^{-x})^{n-i+2} (1 - e^{-x})^{i-2} dx$$

$1 \to 1$

let $t = e^{-x}$, $dx = -\frac{1}{t} dt$

$$= c \cdot e^{-(n-i+1)z} \int_1^0 t^{n-i+2} (1-t)^{i-2} \left(-\frac{1}{t}\right) dt$$

$$= c \cdot e^{-(n-i+1)z} \int_0^1 t^{n-i+1} (1-t)^{i-2} dt.$$

$$= c \cdot e^{-(n-i+1)z} \times B(n-i+2, i-1)$$

$$= c \cdot e^{-(n-i+1)z} \times \frac{(n-i+1)!\,(i-2)!}{n!}$$

$$= \frac{n!}{(i-2)!\,(n-i)!} \times e^{-(n-i+1)z} \times \frac{(n-i+1)!\,(i-2)!}{n!}$$

$= (n-i+1)\, e^{-(n-i+1)z}$

Thus, the pdf of $Z_i = (n-i+1) Y_i$ is

$$f_i(z) = (n-i+1)\, e^{-(n-i+1)y} \times \left|\frac{dy}{dz}\right| = e^{-z} \Rightarrow Z_i \text{ is of expotantial distribution. } \forall i.$$

with $f_i(z_i) = e^{-z_i}$, $\forall i$.

By (2.24), the joint density of $X_{(1)}, \cdots, X_{(n)}$ is

$$g_{1\cdots n}(x_1, \cdots, x_n) = n! \left\{\prod_{i=1}^{n} f(x_i)\right\} \prod_{i=1}^{n+1} \frac{\{F(x_i) - F(x_{i-1})\}^0}{0!}$$

$$= n! \prod_{i=1}^{n} e^{-x_i}.$$

Make the following transformation: $z_1 = n(x_1 - x_0) = n x_1$

$z_2 = (n-1)(x_2 - x_1)$

$\cdots$

$z_n = x_n - x_{n-1}$

$$J(z_1, \cdots, z_n \longrightarrow x_1, \cdots, x_n) = \begin{vmatrix} n & & & \\ n-1 & n-1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{vmatrix} = n!$$

upper-triangular

$\Rightarrow$ the joint density of $Z_1, \cdots, Z_n$ is

$$f_{1\cdots n}(z_1, \cdots, z_n) = \frac{n!\; e^{-\frac{1}{n}z_1} \times e^{-(\frac{1}{n}z_1 + \frac{1}{n-1}z_2)} \times \cdots \times e^{-(\frac{1}{n}z_1 + \cdots + \frac{1}{1}z_n)}}{|J|}$$

$$= \frac{n! \prod_{i=1}^{n} e^{-z_i}}{n!} = \prod_{i=1}^{n} e^{-z_i} = \prod_{i=1}^{n} f_i(z_i)$$

Thus, $Z_1, \cdots, Z_n$ are independent.

(b). In fact, $\sum_{i=1}^{n} a_i X_{(i)} = \sum_{i=1}^{n} \left[a_i \frac{1}{n-i+1} \left(\sum_{j=1}^{i} Z_j\right)\right]$.

i.e. $\sum_{i=1}^{n} a_i X_{(i)}$ can be written as a linear combination of $Z_1, \cdots, Z_n$. By (a), $Z_1 \sim Z_n$ are independent.

$\Rightarrow$ The statement holds.

Q.2.4 (a) Since $\{X_i\}_{i=1}^{n} \overset{ind.}{\sim}$ Gamma $(a_i, 1)$,

the joint density of $X_1 \sim X_n$ is

$$f_{1,\cdots,n}(x_1,\cdots,x_n) = \prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} x_i^{a_i-1} e^{-x_i} = \prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} \times \prod_{i=1}^{n} x_i^{a_i-1} \times e^{-\sum_{i=1}^{n} x_i}$$

Make the following transformation: $\begin{cases} y_i = \dfrac{x_i}{x_1+\cdots+x_n}, & i=1,\cdots,n-1 \\ S = x_1+\cdots+x_n, & S>0 \end{cases}$

$$\Rightarrow J(y_1,\cdots,y_{n-1}, S \to x_1,\cdots,x_n) = \begin{vmatrix} \frac{1}{S} & & & \\ & \frac{1}{S} & & \\ & & \ddots \frac{1}{S} & \\ & & & 1 \end{vmatrix} = \frac{1}{S^{n-1}}$$

$\Rightarrow$ the joint density of $Y_1,\cdots,Y_{n-1}, S$ is

$$g_{1,\cdots,n-1,S}(y_1,\cdots,y_{n-1},S) = \prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} \times \prod_{i=1}^{n-1} (y_i S)^{a_i-1} \times \left[S\left(1-\sum_{i=1}^{n-1} y_i\right)\right]^{a_n-1} \times e^{-S} \times S^{n-1}$$

$$= \prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} \times \prod_{i=1}^{n-1} y_i^{a_i-1} \times S^{\sum_{i=1}^{n}(a_i-1)} \times \left(1-\sum_{i=1}^{n-1} y_i\right)^{a_n-1} \times e^{-S} \times S^{n-1}$$

$$= \prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} \times \prod_{i=1}^{n-1} y_i^{a_i-1} \times \left(1-\sum_{i=1}^{n-1} y_i\right)^{a_n-1} \times S^{\sum_{i=1}^{n} a_i - 1} \times e^{-S}.$$

$\Rightarrow$ the joint density of $Y_1,\cdots,Y_{n-1}$ is

$$g_{1,\cdots,n-1}(y_1,\cdots,y_{n-1}) = \left(\int_0^{\infty} S^{\sum_{i=1}^{n} a_i - 1} \times e^{-S}\, dS\right) \times \prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} \times \prod_{i=1}^{n-1} y_i^{a_i-1} \times \left(1-\sum_{i=1}^{n-1} y_i\right)^{a_n-1}$$

$$= \frac{\Gamma\left(\sum_{i=1}^{n} a_i\right)}{\prod_{i=1}^{n} \Gamma(a_i)} \times \prod_{i=1}^{n-1} y_i^{a_i-1} \times \left(1-\sum_{i=1}^{n-1} y_i\right)^{a_n-1}.$$

(b) Since $\{X_i\}_{i=1}^{n} \overset{ind}{\sim}$ Gamma $(\alpha_i, 1)$

$$\Rightarrow \sum_{i=1}^{n} X_i \sim \text{Gamma}\left(\sum_{i=1}^{n} a_i, 1\right).$$

Thus, the density of $X_1+\cdots+X_n$ is

$$f(x) = \frac{1}{\Gamma\left(\sum_{i=1}^{n} a_i\right)} x^{\sum_{i=1}^{n} a_i - 1} e^{-x}, \quad (x>0).$$

Q2.5 Sol: Set $Z = h(X,Y) = XY$ with $\overset{x>0}{X \sim Gamma(p,1)}$, $\overset{0<y<1}{Y \sim Beta(q,p-q)}$

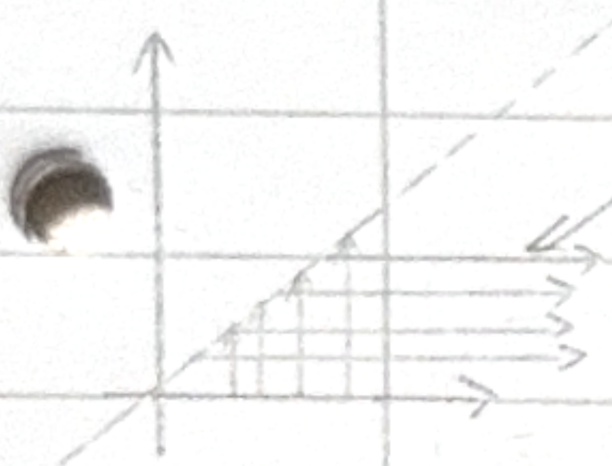
For any nonnegative and measurable $g(\cdot)$, we have

$$E\{g(Y)\} = \int_0^\infty\int_0^1 g(xy)\cdot f_{(X,Y)}(x,y)\,dx\,dy$$

$$= \int_0^\infty\int_0^1 g(xy)\cdot\frac{1}{\cancel{\Gamma(p)}}x^{p-1}e^{-x}\cdot y^{q-1}(1-y)^{p-q-1}\frac{\cancel{\Gamma(p)}}{\Gamma(q)\Gamma(p-q)}\,dy\,dx$$

$$= \int_0^\infty dx\int_0^1 g(xy)\,x^{p-1}e^{-x}y^{q-1}(1-y)^{p-q-1}\frac{1}{\Gamma(q)\Gamma(p-q)}\,dy$$

$y=\frac{z}{x}$, $dy=\frac{1}{x}dz$
$$= \int_0^\infty dx\int_0^x g(z)\,\cancel{x^{p-1}}e^{-x}\frac{z^{q-1}}{\cancel{x^{q-1}}}\frac{(x-z)^{p-q-1}}{\cancel{x^{p-q-1}}}\frac{1}{\Gamma(q)\Gamma(p-q)}\cancel{x^{-1}}\,dz$$



$$= \frac{1}{\Gamma(q)\Gamma(p-q)}\int_0^\infty g(z)\,z^{q-1}dz\int_z^\infty e^{-x}(x-z)^{p-q-1}dx$$

$$= \frac{1}{\Gamma(q)\Gamma(p-q)}\int_0^\infty g(z)\,z^{q-1}e^{-z}dz\int_z^\infty e^{-(x-z)}(x-z)^{p-q-1}d(x-z)$$

$t=x-z$, $0<t<\infty$
$$= \frac{1}{\Gamma(q)\Gamma(p-q)}\int_0^\infty g(z)\,z^{q-1}e^{-z}dz\int_0^\infty e^{-t}t^{p-q-1}dt$$

$$= \frac{1}{\Gamma(q)\cancel{\Gamma(p-q)}}\int_0^\infty g(z)\,z^{q-1}e^{-z}dz\,\cancel{\Gamma(p-q)}$$

$$= \int_0^\infty g(z)\frac{1}{\Gamma(q)}z^{q-1}e^{-z}dz.$$

Thus, the p.d.f of Z is $f_Z(z) = \frac{1}{\Gamma(q)}z^{q-1}e^{-z}$ with support: $\{z|z>0\}$.
($Z \sim Gamma(q)$)

Q2.6. Sol:

z	0	1
$Pr(Z=z)$	ϕ	$1-\phi$

Poisson$(x_i|\lambda_i) = \lambda_i^{x_i}\frac{e^{-\lambda_i}}{x_i!}$, $x_i = 0,1,\cdots,\infty$.

the joint p.m.f of $\vec{y}$ is:

$$P_{\vec{y}}(y_1,y_2,\cdots,y_m) = Pr(ZX_1=y_1, ZX_2=y_2,\cdots, ZX_m=y_m)$$

$$= Pr(ZX_1=y_1,\cdots,ZX_m=y_m \mid Z=0)\,Pr(Z=0)$$

$$+ Pr(ZX_1=y_1,\cdots,ZX_m=y_m \mid Z=1)\,Pr(Z=1)$$

$$= \phi\times 1(y_1=\cdots=y_m=0) + (1-\phi)\times Pr(X_1=y_1,\cdots,X_m=y_m)$$

(mutually independence implies pairwise independence.)

$$= \phi\times 1(y_1=\cdots=y_m=0) + (1-\phi)\times Pr(X_1=y_1)\times\cdots\times Pr(X_m=y_m)$$

$$= \phi\times 1(y_1=\cdots=y_m=0) + (1-\phi)\times\prod_{i=1}^m\lambda_i^{y_i}\frac{e^{-\lambda_i}}{y_i!}$$

i.e. the joint p.m.f of $\vec{y}$ is:

$$P_{\vec{y}}(y_1,y_2,\cdots,y_m) = \phi\times 1(y_1=\cdots=y_m=0) + (1-\phi)\times\prod_{i=1}^m\lambda_i^{y_i}\frac{e^{-\lambda_i}}{y_i!}$$

Q2.7. (a) Sol: $X_1, X_2 \overset{iid}{\sim} N(0, a^2)$.

$$\Rightarrow \begin{cases} (X_1-X_2) \sim N(0, 2a^2) \\ (X_1+X_2) \sim N(0, 2a^2) \end{cases} \Rightarrow \begin{cases} \frac{X_1-X_2}{\sqrt{2}a} \sim N(0,1) \\ \frac{X_1+X_2}{\sqrt{2}a} \sim N(0,1) \end{cases}$$

$$\Rightarrow \begin{cases} \left(\frac{X_1-X_2}{\sqrt{2}a}\right)^2 = \frac{(X_1-X_2)^2}{2a^2} \sim \chi^2(1) \\ \left(\frac{X_1+X_2}{\sqrt{2}a}\right)^2 = \frac{(X_1+X_2)^2}{2a^2} \sim \chi^2(1) \end{cases}$$

the joint density of X_1, X_2 is

$$f_{1,2}(x_1, x_2) \underset{X_1 \perp X_2}{\overset{\uparrow}{=}} f_1(x_1) \times f_2(x_2) = \frac{1}{2\pi a^2} e^{-\frac{x_1^2}{2a^2} - \frac{x_2^2}{2a^2}}.$$

Make the transformation $Z_1 = X_1 - X_2$, $Z_2 = X_1 + X_2$.

$$\Rightarrow h_{1,2}(z_1, z_2) = \frac{1}{2\pi a^2} e^{-\frac{1}{2a^2}\left[\left(\frac{z_1+z_2}{2}\right)^2 + \left(\frac{z_2-z_1}{2}\right)^2\right]} \times \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{\sqrt{2\pi \times 2a^2}} e^{-\frac{z_1^2}{2(\sqrt{2}a)^2}} \times \frac{1}{\sqrt{2\pi \times 2a^2}} e^{-\frac{z_2^2}{2(\sqrt{2}a)^2}}$$

$$= h_1(z_1) \times h_2(z_2)$$

$\Rightarrow$ $X_1 - X_2$ and $X_1 + X_2$ are independent.

$\Rightarrow$ $(X_1-X_2)^2$ and $(X_1+X_2)^2$ are independent.

$\Rightarrow$ $\frac{(X_1-X_2)^2}{2a^2}$ and $\frac{(X_1+X_2)^2}{2a^2}$ are independent.

$$\Rightarrow \frac{(X_1-X_2)^2}{(X_1+X_2)^2} = \frac{\frac{(X_1-X_2)^2}{2a^2}}{\frac{(X_1+X_2)^2}{2a^2}} \sim F(1,1).$$

Thus, $\frac{(X_1-X_2)^2}{(X_1+X_2)^2} \sim F(1,1)$

(b) Sol: $Pr\left\{\frac{(X_1+X_2)^2}{(X_1+X_2)^2 + (X_1-X_2)^2} > k\right\} = 0.1$

$$\Leftrightarrow Pr\left\{\frac{(X_1-X_2)^2}{(X_1+X_2)^2} < \frac{1}{k} - 1\right\} = 0.1$$

Since $\frac{(X_1-X_2)^2}{(X_1+X_2)^2} \sim F(1,1)$ $\Leftrightarrow Pr\{F(1,1) < \frac{1}{k} - 1\} = 0.1$

Since $F(1,1)$ is distributed continuously, strictly increasingly, and $Pr(F(1,1) < 0.0251) = 0.1$,

$$\frac{1}{k} - 1 = 0.0251 \Rightarrow k = \frac{1}{1.0251} \approx 0.9755$$

$\boxed{\text{Part 2. 3 questions chosen from Q2.11} \sim \text{Q2.15.}}$

Q2.11. (a) $X_i \sim Poisson(\lambda_i)$ $\qquad Poisson(x_i|\lambda_i) = \lambda_i^{x_i} e^{-\lambda_i}/x_i!$, $x_i = 0, 1, \cdots, \infty$

$Y = X_2 - X_1$, $\quad y = 0, \pm 1, \pm 2, \cdots\cdots$

the pmf of $Y = X_2 - X_1$ is

$$Pr(Y=m) = Pr(X_2 = X_1+m) = \sum_{i=\max\{0,-m\}}^{\infty} Pr(X_2 = X_1+m \mid X_1 = i)\, Pr(X_1 = i)$$

$X_1 \perp\!\!\!\perp X_2$

$$= \sum_{i=\max\{0,-m\}}^{\infty} Pr(X_2 = i+m)\, Pr(X_1 = i)$$

$$= \sum_{i=\max\{0,-m\}}^{\infty} \lambda_2^{i+m} e^{-\lambda_2} \frac{1}{(i+m)!} \lambda_1^{i} e^{-\lambda_1} \frac{1}{i!}$$

$$= \lambda_2^{m} e^{-\lambda_1-\lambda_2} \sum_{i=\max\{0,-m\}}^{\infty} \frac{\lambda_1^i \lambda_2^i}{(i+m)!\, i!}. \qquad m = 0, \pm 1, \pm 2, \cdots\cdots$$

(b) $E(Y) = E(X_2 - X_1) = E(X_2) - E(X_1) = \lambda_2 - \lambda_1$.

$$E(Y^2) = E(X_2^2 - 2X_1X_2 + X_1^2)$$

$$= E(X_2^2) + E(X_1^2) - 2E(X_1)E(X_2)$$

$$= Var(X_2) + (EX_2)^2 + Var(X_1) + (EX_1)^2 - 2E(X_1)E(X_2)$$

$$= \lambda_2 + \lambda_2^2 + \lambda_1 + \lambda_1^2 - 2\lambda_1\lambda_2$$

$$Var(Y) = E(Y^2) - (EY)^2$$

$$= \lambda_2 + \cancel{\lambda_2^2} + \lambda_1 + \cancel{\lambda_1^2} - \cancel{2\lambda_1\lambda_2} - \cancel{\lambda_2^2} - \cancel{\lambda_1^2} + \cancel{2\lambda_1\lambda_2}$$

$$= \lambda_1 + \lambda_2.$$

Q2.22 (a) $X_1, X_2 \overset{i.i.d}{\sim}$ Exponential (λ).

the joint p.d.f of X_1, X_2 is

$$f_{1,2}(x_1, x_2) = f_1(x_1) \cdot f_2(x_2) = \lambda^2 e^{-\lambda(x_1+x_2)} \qquad (x_1 \geq 0,\ x_2 \geq 0)$$

Make the following transformation: $Y_1 = X_1 + X_2$, $Y_2 = X_1/X_2$.

the joint p.d.f. of Y_1, Y_2 is

$$g_{1,2}(y_1, y_2) = f_{1,2}(x_1, x_2) \times |J(x_1, x_2 \longrightarrow y_1, y_2)|.$$

$$= \lambda^2 \cdot e^{-\lambda y_1} \times \begin{vmatrix} \dfrac{y_2}{y_2+1} & y_1 \dfrac{1}{(y_2+1)^2} \\ \dfrac{1}{y_2+1} & -\dfrac{y_1}{(y_2+1)^2} \end{vmatrix}$$

$$= \lambda^2 \cdot e^{-\lambda y_1} \frac{y_1}{(1+y_2)^2} \qquad (\text{Support: } y_1 > 0,\ y_2 > 0)$$

(b) the marginal p.d.f of Y_1 is

$$g_1(y_1) = \int_0^{\infty} \lambda^2 e^{-\lambda y_1} \frac{y_1}{(1+y_2)^2}\, dy_2$$

$$= \lambda^2 e^{-\lambda y_1} y_1 \left(-\frac{1}{1+y_2}\right)\Big|_{y_2=0}^{y_2=\infty}$$

$$= \lambda^2 y_1 \cdot e^{-\lambda y_1} \quad (\text{Support}: y_1 \geq 0)$$

(c) the marginal p.d.f of Y_2 is

$$g_2(y_2) = \int_0^{\infty} \lambda^2 e^{-\lambda y_1} \frac{y_1}{(1+y_2)^2}\, dy_1 = \frac{1}{(1+y_2)^2} \quad (\text{Support}: y_2 \geq 0)$$

Q 2.13 (a) By (2.25), the joint pdf of $X_{(1)}, X_{(n)}$ is

$$f_{1,n}(x_1, x_n) = \frac{n!}{0!(n-2)!0!} f(x_1) f(x_n) \{F(x_n) - F(x_1)\}^{n-2}$$

$$= n(n-1) f(x_1) f(x_n) \{F(x_n) - F(x_1)\}^{n-2}$$

Make the following transformation: $R = X_{(n)} - X_{(1)}$, $T = \frac{X_{(1)} + X_{(n)}}{2}$

$$g(r,t) = f_{1,n}(x_1, x_n) \times |J(x_1, x_n \longrightarrow r, t)|$$

$$= n(n-1) f\left(\frac{2t-r}{2}\right) f\left(\frac{2t+r}{2}\right) \times \text{abs}\left(\begin{vmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{vmatrix}\right) \times \left\{F\left(\frac{2t+r}{2}\right) - F\left(\frac{2t-r}{2}\right)\right\}^{n-2}$$

$$= n(n-1) f\left(\frac{2t-r}{2}\right) f\left(\frac{2t+r}{2}\right) \left\{F\left(t+\frac{r}{2}\right) - F\left(t-\frac{r}{2}\right)\right\}^{n-2}$$

Denote the support of $f(x)$ as S, then $g(r,t)$'s support is $[0, \max\{S\} - \min\{S\}] \times S$.

(b) the marginal p.d.f of R is

$$g_r(r) = \int_S n(n-1) f\left(\frac{2t-r}{2}\right) f\left(\frac{2t+r}{2}\right) \left\{F\left(t+\frac{1}{2}r\right) - F\left(t-\frac{1}{2}r\right)\right\}^{n-2} dt$$

$$= \int_S n(n-1) f\left(t-\frac{1}{2}r\right) f\left(t+\frac{1}{2}r\right) \left\{F\left(t+\frac{1}{2}r\right) - F\left(t-\frac{1}{2}r\right)\right\}^{n-2} dt$$

(c) the marginal p.d.f of T is

$$g_t(t) = \int_{[0, \max\{S\} - \min\{S\}]} n(n-1) f\left(t-\frac{1}{2}r\right) f\left(t+\frac{1}{2}r\right) \left\{F\left(t+\frac{1}{2}r\right) - F\left(t-\frac{1}{2}r\right)\right\}^{n-2} dt$$

Part 3. the last 5 questions

Q2.16. Sol: $X \sim N(0,1)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} d(x-t)$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

$$= e^{\frac{1}{2}t^2}$$

$$M_Y(t) = M_{X^2}(t) = \underbrace{\int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx}$$

$t < \frac{1}{2}$, otherwise the integral doesn't converge.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(t-\frac{1}{2})x^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\sqrt{\frac{\frac{1}{2}-t}{\frac{1}{2}}}\,x\right)^2} d\left(\sqrt{\frac{\frac{1}{2}-t}{\frac{1}{2}}}\,x\right) \sqrt{\frac{\frac{1}{2}}{\frac{1}{2}-t}}$$

$$= \left[\frac{1}{2} \Big/ \left(\frac{1}{2}-t\right)\right]^{\frac{1}{2}} \qquad (t<\tfrac{1}{2})$$

by Table 1.3, $Y \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

i.e. $Y \sim \chi^2(1)$

Q2.17. (a) Sol: $Y \mid (X_2 = x_2) = x_2 X_1 \sim U(0, x_2)$

i.e. $f_{(Y|X_2)}(y|x_2) = \frac{1}{x_2}$.

$$f_Y(y) = \int_y^1 f_{X_2}(x_2) \times f_{(Y|X_2)}(y|x_2)\, dx_2$$

for any y, $1 \geq x_2 = \frac{y}{x_1} \geq y$

$$= \int_y^1 1 \cdot \frac{1}{x_2} dx_2 = \ln|x_2| \Big|_{x_2=y}^{x_2=1} = -\ln y.$$

(b) Sol: $Z \mid (X_2 = x_2) = x_2^{-1} X_1 \sim U(0, x_2^{-1})$

i.e. $f_{(Z|X_2)}(z|x_2) = x_2$.

① $Z > 1$, $f_Z(z) = \int_0^{z^{-1}} f_{X_2}(x_2) \times f_{(Z|X_2)}(z|x_2)\, dx_2$

for any z, $0 < x_2 = \frac{x_1}{z} \leq z^{-1}$

$$= \int_0^{z^{-1}} 1 \cdot x_2\, dx_2 = \frac{1}{2} z^{-2}. \qquad (Z>1)$$

② $0<z\le 1$, $f_Z(z)=\int_0^1 x_2\,dx_2=\frac{1}{2}$.

Thus, $f_Z(z)=\begin{cases}\frac{1}{2}, & 0<z\le 1.\\ \frac{1}{2}z^{-2}, & z>1.\end{cases}$

Q2.18. Sol: $\vec{X}\mid(Z=z)=\vec{\mu}+\frac{\vec{Y}}{\sqrt{z/\nu}}\sim N_d(\vec{\mu},\frac{\nu}{z}\Sigma)$

i.e. $f_{(\vec{X}\mid Z)}(\vec{x}\mid z)=\frac{1}{(\sqrt{2\pi})^d\left|\frac{\nu}{z}\Sigma\right|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\frac{z}{\nu}\Sigma^{-1}(\vec{x}-\vec{\mu})\right\}$

Thus, $f_{\vec{X}}(\vec{x})=\int_0^\infty\frac{1}{(\sqrt{2\pi})^d\nu^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}}z^{\frac{d}{2}}\exp\left\{-z\frac{(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}{2\nu}\right\}\frac{2^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}z^{\frac{\nu}{2}-1}e^{-\frac{z}{2}}dz$

$=\int_0^\infty\frac{(\frac{1}{2})^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}z^{\frac{\nu}{2}+\frac{d}{2}-1}e^{-(\frac{1}{2}+\frac{1}{2\nu}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu}))z}\frac{1}{(\sqrt{2\pi})^d\nu^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}}dz$

$=\frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})(\sqrt{\pi\nu})^d|\Sigma|^{\frac{1}{2}}}\left\{1+\frac{(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}{\nu}\right\}^{-\frac{\nu+d}{2}}\quad(\vec{x}\in\mathbb{R}^d.)$

Q2.19. (a) Proof: $\alpha>0$, $\alpha f(x)[F(x)]^{\alpha-1}\ge 0$.

$\int_{-\infty}^{\infty}\alpha f(x)[F(x)]^{\alpha-1}dx$

$=\int_{-\infty}^{\infty}\alpha[F(x)]^{\alpha-1}d(F(x))$

$=[F(x)]^\alpha\Big|_{x=-\infty}^{x=\infty}$

$=1-0=1.$

Thus, $\alpha f(x)[F(x)]^{\alpha-1}$ is a pdf of some r.v. □

(b) Sol: let $F_X(x)$ be X's c.d.f. $F_X(x)=\int_{-\infty}^{x}\alpha f(t)[F(t)]^{\alpha-1}dt=[F(x)]^\alpha$

then, define $F^{-1}(u)$ as the quantile function defined as:

$F_X^{-1}(u)=\inf\{x\in\mathbb{R}\mid F_X(x)\ge u\}$, $u\in(0,1)$.

$\Rightarrow X\overset{d}{=}F_X^{-1}(U).=F^{-1}(u^{\frac{1}{\alpha}})$.

Q2.20. Sol: Firstly, we prove that $Y\sim t(\mu,\alpha_*^2,\nu_*)$.

Since $T\sim Gamma(\alpha,\beta)$,

$2\beta T\sim Gamma(\alpha,\frac{1}{2})$

i.e. $2\beta T\sim\chi^2(2\alpha)$.

$Y\overset{d}{=}\mu+\frac{Z}{\sqrt{T}}=\mu+\frac{1}{\sqrt{\frac{2\beta T}{2\alpha}}}\times\sqrt{\frac{\beta}{\alpha}}Z\sim t(\mu,\frac{\beta}{\alpha}\alpha^2,2\alpha)$. where $\alpha_*^2=\frac{\beta}{\alpha}\alpha^2$. $\nu_*=2\alpha$.

$\sqrt{\frac{\beta}{\alpha}}Z\sim N(0,\frac{\beta}{\alpha}\alpha^2)$