HOMEWORK 2

1. For each of the following functions, determine all its zeroes and all its isolated singular points *in the extended complex plane* $\overline{\mathbb{C}}$ (that is, the point ∞ must be investigated as well!). As long as a point is a zero or a pole, determine the order. Explain your answer!

a) $f(z) = \frac{1-\cos(z^6)}{(\sin z)^5} \cdot e^{\frac{1}{\pi - z}}$

b) $f(z) = \sqrt{z}\sin\left(\frac{1}{\sqrt{z^7}}\right)$ (explain also why the function is holomorphic outside its isolated singular points; the choice of $\operatorname{Arg} z$ for the two roots is the same).

Remark. I remind that a removable singularity can be considered at the same time as an isolated zero! In this case, also determine the order.

2. Find Taylor/Laurent expensions of the function $f(z)$ in *all* (!) possible discs and annuli with the center $z = 0$:

$$f(z) = \frac{z^4}{z(z-1)(z-2)}$$

After doing so, determine the types of isolated singularity at the points $z = 0$ and $z = \infty$ (including the order, if applicable).

3. Calculate

$$\lim_{R\to\infty}\left(\int_{|z|=R} z^{100}e^{1/z}\right)$$

4. The *residue at infinity* for a function f holomorpic in $D_R = \{|z| > R\}$ is the number $\operatorname{Res}_\infty f := -c_1$, where $f = \sum_{n=-\infty}^{+\infty} c_n z^n$ is the Laurent expansion of f in D_R.

Prove that if $f \in \mathcal{O}(\overline{\mathbb{C}} \setminus \{a_1, ..., a_s\}$, then the following version of the Residue Theorem holds:

$$\sum_{j=1}^{s} \operatorname{Res}_{a_j} f = 0.$$

5. Using the residue calculus, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{(\cos(2x))^2}{x^2+2x+2}dx$$

6. Using the residue calculus, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin(3x)}{x(x^2+1)}dx.$$

7. Using the residue calculus, evaluate the integral

$$\int_{0}^{\infty} \frac{x+1}{\sqrt[3]{x}(x^2+1)}dx.$$

8. Using the residue calculus, evaluate the integral

$$\int_0^\infty \frac{\ln x}{\sqrt[4]{x}(x^2-1)}dx.$$

9. Using the residue calculus, evaluate the integral

$$\int_{|z|=101} \frac{1}{z+1} \cos\left(\frac{1}{(z-1)(z-2)\cdots(z-100)}\right) dz.$$

10. Evaluate the sum of the series $\sum_{n=1}^\infty \frac{1}{n^2+1}$.