CS201: Discrete Math for Computer Science
2025 Spring Semester   Written Assignment #4

The assignment needs to be written in English. Assignments in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1. Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates.

Solution: The midpoint of the segment whose endpoints are (a, b) and (c, d) is $((a + c)/2, (b + d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (both odd or both even) and b and d have the same parity. There are four possible pairs of parities: (odd, odd), $(odd, even)$, $(even, odd)$, $(even, even)$. Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

$\square$

Q. 2. Use induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Solution: **Base case**: $n = 1$, $n^3 + 2n = 3$, which is divisible by 3.

Inductive hypothesis: Suppose that 3 divides $n^3 + 2n$.

Inductive step: We now prove that 3 divides $(n + 1)^3 + 2(n + 1)$. We have

$$\begin{aligned}(n+1)^3 + 2(n+1) &= (n^3 + 2n) + (3n^2 + 3n + 3) \\ &= (n^3 + 2n) + 3(n^2 + n + 1).\end{aligned}$$

Since $n^3 + 2n$ is divisible by 3 by i.h., and also $3(n^2 + n + 1)$ is divisible by 3. it follows that $(n + 1)^3 + 2(n + 1)$ is divisible by 3.

Conclusion: By mathematical induction, we prove the result.

$\square$

Q. 3. Let S be a set of n distinct integers. Prove that there exists a non-empty subset $T \subseteq S$ such that the sum of the elements in T is divisible by n.

Solution We use the **Pigeonhole Principle** on the partial sums of S. Let $S = \{a_1, a_2, \ldots, a_n\}$ be the given set of integers. Define the *partial sums*:

$$s_k = \sum_{i=1}^{k} a_i \quad \text{for } k = 1, 2, \ldots, n.$$

Consider these sums modulo n:

$$r_k = s_k \mod n \quad \text{where } 0 \le r_k < n.$$

There are two cases:

1. **Case 1:** If $r_k = 0$ for some k, then s_k is divisible by n. Thus, the subset $T = \{a_1, a_2, \ldots, a_k\}$ satisfies the condition.

2. **Case 2:** If no $r_k = 0$, then the n remainders $r_1, r_2, \ldots, r_n$ must lie in the $n-1$ possible nonzero residues $\{1, 2, \ldots, n-1\}$. By the **Pigeonhole Principle**, at least two remainders must be equal, say $r_i = r_j$ where $i < j$. Then:

$$s_j - s_i = \sum_{m=i+1}^{j} a_m \equiv 0 \pmod{n}.$$

   Hence, the subset $T = \{a_{i+1}, a_{i+2}, \ldots, a_j\}$ has a sum divisible by n.

In both cases, such a subset T exists. $\square$

Q. 4. The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in closed form (in terms of b if necessary) and should not use summation.

(b) Find a solution for $T(n)$. Your solution should be in closed form (in terms of a and c if necessary) and should not use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Solution:

(a) By repeated substitution, we get

$$\begin{aligned}
S(n) &= 9S(n/2) + n^2 \\
&= 9\left[9S\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 \\
&= 9^2 S\left(\frac{n}{2^2}\right) + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^2\left[9S\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^3 S\left(\frac{n}{2^3}\right) + \left(\frac{9}{4}\right)^2 n^2 + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= \cdots \\
&= 9^{\log_2 n} S(1) + n^2 \sum_{i=0}^{\log_2 n - 1} \left(\frac{9}{4}\right)^i \\
&= bn^{\log_2 9} + \frac{4}{5} n^{\log_2 9} - \frac{4}{5} n^2 \\
&= \left(b + \frac{4}{5}\right) n^{\log_2 9} - \frac{4}{5} n^2,
\end{aligned}$$

where we are using the fact that

$$\left(\frac{9}{4}\right)^{\log_2 n} = \frac{9^{\log_2 n}}{n^2} = \frac{n^{\log_2 9}}{n^2}.$$

(b) Similar to (a), we get

$$\begin{aligned}
T(n) &= aT\left(\frac{n}{4}\right) + n^2 \\
&= a\left[aT\left(\frac{n}{4^2}\right) + \left(\frac{n}{4}\right)^2\right] + n^2 \\
&= a^2T\left(\frac{n}{4^2}\right) + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^2\left[aT\left(\frac{n}{4^3}\right) + \left(\frac{n}{4^2}\right)^2\right] + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^3T\left(\frac{n}{4^3}\right) + \left(\frac{a}{16}\right)^2 n^2 + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= \cdots \\
&= a^{\log_4 n}T(1) + n^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{a}{16}\right)^i \\
&= cn^{\log_4 a} + \frac{16}{a-16}n^{\log_4 a} - \frac{16}{a-16}n^2 \\
&= \left(c + \frac{16}{a-16}\right)n^{\log_4 a} - \frac{16}{a-16}n^2,
\end{aligned}$$

where we are using the fact that

$$\left(\frac{a}{16}\right)^{\log_4 n} = \frac{a^{\log_4 n}}{n^2} = \frac{n^{\log_4 a}}{n^2}.$$

(c) For $T(n) = O(S(n))$, we should have

$$\begin{aligned}
n^{\log_4 a} &\leq n^{\log_2 9} \\
\log_4 a &\leq \log_2 9 \\
a &\leq 9^2 = 81.
\end{aligned}$$

So the range of values is $16 < a \leq 81$.

$\square$

Q. 5. Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Solution:

(a) There are 3^n functions.

(b) If $n \leq 3$, there are $P(3, n)$ one-to-one functions. Hence, there are 3 when $n = 1$, 6 when $n = 2$, and 6 when $n = 3$. If $n > 3$, then there are 0 injective functions; there cannot be a one-to-one function from A to B if $|A| > |B|$.

(c) By the Inclusion-Exclusion Principle, we have

$$\begin{aligned}
\# &= \#\{f : f(A) \subseteq \{1,2,3\}\} - \#\{f : f(A) \subseteq \{1,2\}\} - \#\{f : f(A) \subseteq \{1,3\}\} \\
&\quad -\#\{f : f(A) \subseteq \{2,3\}\} + \{f : f(A) \subseteq \{1\}\} + \{f : f(A) \subseteq \{2\}\} \\
&\quad \#\{f : f(A) \subseteq \{3\}\} \\
&= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 \\
&= 3^n - 3 \cdot 2^n + 3.
\end{aligned}$$

□

Q. 6. Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding n that are relatively prime to n, i.e., the Euler function $\phi(n)$.

Solution: Let P be the set of numbers in $\{1, 2, 3, \ldots, n\}$ that are divisible by p, and similarly define the set Q. We want to count the numbers not divisible by either p or q, so we want $n - |P \cup Q|$. By the principle of inclusion-exclusion, $|P \cup Q| = |P| + |Q| - |P \cap Q|$. Every pth number is divisible by p, so $|P| = \lfloor n/p \rfloor = q$. Similarly $|Q| = \lfloor n/q \rfloor = q$. Clearly, n is the only positive integer not exceeding n that is divisible by both p and q, so $|P \cap Q| = 1$. Therefore, the number of positive integers not exceeding n that are relatively prime to n is $n - p - q + 1$.

□

Q. 7. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Solution:

Working modulo 5 there are 25 pairs: $(0,0),(0,1),\ldots,(4,4)$. Thus, we could have 25 ordered pairs of integers (a,b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

$\square$

Q. 8. Prove the hockeystick identity

$$\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Solution:

(a) $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternatively, suppose that the $(j+1)$st term is the last term equal to 1, so that $n \le j \le n+r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j-n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r}\binom{j}{j-n}=\sum_{k=0}^{r}\binom{n+k}{k}$ ways to this.

(b) Let $P(r)$ be the statement to be proved. The basis step is the equation $\binom{n}{0}=\binom{n+1}{0}$, which is just $1=1$. Assume that $P(r)$ is true. Then

$$\begin{aligned}
&\sum_{k=0}^{r+1}\binom{n+k}{k}\\
&\quad=\sum_{k=0}^{r}\binom{n+k}{k}+\binom{n+r+1}{r+1}\\
&\quad=\binom{n+r+1}{r}+\binom{n+r+1}{r+1}\\
&\quad=\binom{n+r+2}{r+1},
\end{aligned}$$

using the inductive hypothesis and Pascal's identity.

$\square$

Q. 9. Use generating functions to prove Pascal's identity: $C(n,r) = C(n-1,r) + C(n-1,r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

Solution:

First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$. Expanding both sides of this equality using the binomial theorem, we have

$$\begin{aligned}\sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r.\end{aligned}$$

Thus,

$$1+\left(\sum_{r=1}^{n-1} C(n,r)x^r\right)+x^n = 1+\left(\sum_{r=1}^{n-1}(C(n-1,r)+C(n-1,r-1))x^r\right)+x^n.$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n,r)$ must equal $C(n-1,r) + C(n-1,r-1)$ for $1 \leq r \leq n-1$, as desired.

$\square$

Q. 10. Solve the recurrence relation:

$$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$$

with initial conditions $a_0 = 2$, $a_1 = 3$, and $a_2 = 7$.

Solution:

The characteristic equation (CE) is:

$$r^3 - 4r^2 + 5r - 2 = 0.$$

Factoring the CE:

$$(r-1)^2(r-2) = 0.$$

The roots are $r = 1$ (double root) and $r = 2$. Hence, the general solution is:

$$a_n = (\alpha_1 + \alpha_2 n) \cdot 1^n + \alpha_3 \cdot 2^n.$$

Simplify:

$$a_n = \alpha_1 + \alpha_2 n + \alpha_3 2^n.$$

To find $\alpha_1, \alpha_2, \alpha_3$, use the initial conditions:

$$\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_3, \\ a_1 &= 3 = \alpha_1 + \alpha_2 + 2\alpha_3, \\ a_2 &= 7 = \alpha_1 + 2\alpha_2 + 4\alpha_3. \end{aligned}$$

Solving the system: 1. From a_0: $\alpha_1 = 2 - \alpha_3$. 2. Substitute into a_1:

$$3 = (2 - \alpha_3) + \alpha_2 + 2\alpha_3 \implies \alpha_2 + \alpha_3 = 1.$$

3. Substitute into a_2:

$$7 = (2 - \alpha_3) + 2\alpha_2 + 4\alpha_3 \implies 2\alpha_2 + 3\alpha_3 = 5.$$

4. Solve the system $\alpha_2 + \alpha_3 = 1$ and $2\alpha_2 + 3\alpha_3 = 5$: - From the first equation: $\alpha_2 = 1 - \alpha_3$. - Substitute into the second:

$$2(1 - \alpha_3) + 3\alpha_3 = 5 \implies 2 - 2\alpha_3 + 3\alpha_3 = 5 \implies \alpha_3 = 3.$$

- Then $\alpha_2 = -2$ and $\alpha_1 = -1$.
The final solution is:

$$a_n = -1 - 2n + 3 \cdot 2^n.$$

□