**CS201: Discrete Math for Computer Science**
**2025 Spring Semester   Written Assignment #2**

Please answer questions in English. Using any other language will lead to a zero point.

**Q. 1.** Suppose that $A$, $B$ and $C$ are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A - B = A) \to (B \subset A)$

(b) $(A \cap B \cap C) \subseteq (A \cup B)$

(c) $\overline{(A - B)} \cap (B - A) = B$

**Solution:**

(a) False. As an counterexample, let $A = \{1\}$, and $B = \{2\}$. Then $A - B = A$, but $B$ is not a subset of $A$.

(b) True. $A \cap B \cap C \subseteq A \cap B \subseteq A \cup B$.

(c) False. Let $A = B = \{1\}$. Then, $\overline{A - B} \cap (B - A) = U \cap \emptyset \neq B = \{1\}$.

$\square$

**Q. 2.** The symmetric difference of $A$ and $B$, denoted by $A \oplus B$, is the set containing those elements in either $A$ or $B$, but not in both $A$ and $B$.

(a) Determine whether the symmetric difference is associative; that is, if $A$, $B$ and $C$ are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that $A, B$ and $C$ are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

**Solution:**

(a) Using membership table, one can show that each side consists of the elements that are in an odd number of the sets $A, B$ and $C$. Thus, it follows.

(b) Yes. We prove that for every element $x \in A$, we have $x \in B$ and vice versa. We use proof by cases.

First, for elements $x \in A$ and $x \notin C$, since $A \oplus C = B \oplus C$, we know that $x \in A \oplus C$ and thus $x \in B \oplus C$. Since $x \notin C$, we must have $x \in B$. For elements $x \in A$ and $x \in C$, we have $x \notin A \oplus C$. Thus, $x \notin B \oplus C$. Since $x \in C$, we must have $x \in B$.

The proof of the other way around is similar.

$\square$

**Q. 3.** Prove or disprove that there exists an infinite set $A$ such that $|A| < |\mathbf{Z}^+|$.

**Solution:** This statement is false. Suppose there exists an infinite set $A$ such that $|A| < |\mathbf{Z}^+|$. This means that $|A| \le |\mathbf{Z}^+|$ and $|A| \ne |\mathbf{Z}^+|$.

- Since $|A| \ne |\mathbf{Z}^+|$, there does not exist any one-to-one correspondence that maps from $A$ to $\mathbf{Z}^+$. Thus, $A$ cannot be countable infinite.
- Since $|A| \le |\mathbf{Z}^+|$, there exists a one-to-one function maps from $A$ to $\mathbf{Z}^+$. There is a subset $S \subset \mathbf{Z}^+$ such that there exists a one-to-one correspondence that maps from $A$ to $S$. Since the subset of a countable set is also countable, $S$ is countable. Thus, $S$ is either finite or there exists a one-to-one correspondence from $S$ to $\mathbf{Z}^+$. This leads to the fact that $A$ is either finite or countable infinite.

Thus, contradiction occurs. This complete the disprove.

**Q. 4.** Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the composition function.

(a) If $f \circ g$ is one-to-one and $g$ is one-to-one, must $f$ be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and $f$ is one-to-one, must $g$ be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must $g$ be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must $f$ be onto? Explain your answer.

(e) If $f \circ g$ is onto, must $g$ be onto? Explain your answer.

(a) No. We prove this by giving a counterexample. Let $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = A$. Define the function $g$ by $g(1) = a$ and $g(2) = b$, and define the function $f$ by $f(a) = 1$, and $f(b) = f(c) = 2$. Then it is easily verified that $f \circ g$ is one-to-one and $g$ is one-to-one. But $f$ is not one-to-one.

(b) Yes. For any two elements $x, y \in A$ with $x \neq y$, assume to the contrary that $g(x) = g(y)$. On one hand, since $f \circ g$ is one-to-one, we have $f \circ g(x) \neq f \circ g(y)$. One the other hand, $f \circ g(x) = f(g(x)) = f(g(y)) = f \circ g(y)$. This leads to a contradiction. Thus, $g(x) \neq g(y)$, which means that $g$ must be one-to-one.

(c) Yes. Similar to (b), the condition that $f$ is one-to-one is in fact not used.

(d) Yes. Since $f \circ g$ is onto, we know that $f \circ g(A) = C$, which means that $f(g(A)) = C$. Note that $g(A)$ is a subset of $B$, thus, $f(B)$ must also be $C$. This means that $f$ is also onto.

(e) No. A counterexample is the same as that in (a).

□

**Q. 5.** Give an example of two uncountable sets $A$ and $B$ such that the difference $A - B$ is (a) finite, (b) countably infinite, (c) uncountable. Note: one example for each subquestion (a), (b), or (c).

**Solution:** In each case, let $A$ be the set of real numbers.

(a) Let $B$ be the set of real numbers as well, then $A - B = \emptyset$, which is finite.

(b) Let $B$ be the set of real numbers that are not positive integers, then $A - B = \mathbf{Z}^+$, which is countably infinite.

(c) Let $B$ be the set of positive real numbers. Then $A - B$ is the set of negative real numbers, which is uncountable.

□

**Q. 6.** If $A$ is an uncountable set and $B$ is a countable set, must $A - B$ be uncountable?

**Solution:** Since $A = (A-B) \cup (A \cap B)$, if $A - B$ is countable, the elements of $A$ can be listed in a sequence by alternating elements of $A - B$ and elements of $A \cap B$. This contradicts the uncountability of $A$.

$\square$

**Q. 7.** Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m + n - 2)(m + n - 1)/2 + m$ is one-to-one and onto.

**Solution:** It is clear from the formula that the range of values the function takes on for a fixed value of $m+n$, say $m+n = x$, is $(x-2)(x-1)/2+1$ through $(x-2)(x-1)/2+(x-1)$, because $m$ can assume the values $1, 2, 3, \ldots, (x-1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m+n$ is fixed. To show that this function is one-to-one and onto, we merely need to show that the range of values for $x + 1$ picks up precisely where the range of values for $x$ left off, i.e., that $f(x - 1, 1) + 1 = f(1, x)$. We have $f(x - 1, 1) + 1 = (x - 2)(x - 1)/2 + (x - 1) + 1 = (x^2 - x + 2)/2 = (x - 1)x/2 + 1 = f(1, x)$.

$\square$

**Q. 8.** Assume that $|S|$ denotes the cardinality of the set $S$. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

By definition, we have one-to-one and onto functions $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is a one-to-one and onto function from $A$ to $C$, so we have $|A| = |C|$.

$\square$

**Q. 9.** Suppose that $f(x), g(x)$ and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.

**Solution:** The definition of "$f(x)$ is $\Theta(g(x))$" is that $f(x)$ if both $O(g(x))$ and $\Omega(g(x))$. This means that there are positive constants $C_1, k_1, C_2$, and $k_2$ such that $|f(x)| \leq C_2|g(x)|$ for all $x > k_2$ and $|f(x) \geq C_1|g(x)|$ for all

$x > k_1$. Similarly, we have that there are positive constants $C_1', k_1', C_2'$, and $k_2'$ such that $|g(x)| \leq C_2'|h(x)|$ for all $x > k_2'$ and $|g(x)| \geq C_1'|h(x)|$ for all $x > k_1'$. We can combine these inequalities to obtain $|f(x)| \leq C_2 C_2'|h(x)|$ for all $x > \max(k_2, k_2')$ and $|f(x)| \geq C_1 C_1'|h(x)|$ for all $x > \max(k_1, k_1')$. This means that $f(x)$ is $\Theta(h(x))$.

$\square$

**Q. 10.** Suppose that $f_1(x)$ is $\Theta(g_1(x))$ and $f_2(x)$ is $\Theta(g_2(x))$. Prove or disprove that $f_1(x) - f_2(x)$ is $\Theta(g_1(x) - g_2(x))$.

**Solution:** Disprove. Let $f_1(x) = 2x$, $f_2(x) = g_1(x) = g_2(x) = x$. $f_1(x) - f_2(x) = x$ is not $\Theta(g_1(x) - g_2(x)) = 0$

$\square$