Discrete Mathematics for Computer Science

Lecture 14: Counting

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

Mathematical Induction

Use strong induction to prove that $\sqrt{2}$ is irrational.

- To prove the above, we need to first consider the statement $P(n)$ that we need to prove. Write this statement $P(n)$: ____________
- Prove the above statement using strong induction. Suppose we know $\sqrt{2} > 1$.

Solution:

- $P(n)$: there is no positive integer b such that $\sqrt{2} = n/b$.
- Basic step: $P(1)$ is true, because $\sqrt{2} > 1 > 1/b$.
- Inductive step: Suppose $P(j)$ is true for all $j \leq k$, where k is an arbitrary positive integer. We prove that $P(k+1)$ is true by contradiction.

Mathematical Induction

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- Prove the above statement using strong induction. Suppose we know $\sqrt{2} > 1$.

Solution:

- Inductive step: Suppose $P(j)$ is true for all $j \leq k$, where k is an arbitrary positive integer. We prove that $P(k+1)$ is true by contradiction.
  Suppose $P(k+1)$ is false. Then, there exists a b such that $\sqrt{2} = (k+1)/b$. Thus, $2b^2 = (k+1)^2$, so $(k+1)^2$ must be even, and hence $k+1$ must be even. This implies $k+1 = 2t$ for some positive integer t. Substituting $k+1$, we have $2b^2 = 4t^2$, so $b = 2s$ for some positive integer s. This implies that $\sqrt{2} = (k+1)/b = 2t/(2s) = t/s$. That is, when $n = t$, there exists an s such that $\sqrt{2} = t/s$. However, since $t < k+1$, this statement is contradict with $P(t)$ is true.

Mathematical Induction

Suppose there are n people in a group, each aware of a secrete no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all secrete each knows about. For example, consider three people A, B, C with secrete S_A, S_B, S_C, respectively:

- On the first call, A and B communicate. Then, both A and B know about secrets $\{S_A, S_B\}$.
- On the second call, A and C communicate. Then, both A and C knows about secrets $\{S_A, S_B, S_C\}$.

The gossip problem asks for $G(n)$, the minimum number of telephone calls that are needed for **all** n **people** to learn about **all the secrete**.

(a) Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

(b) Use mathematical induction to prove that $G(n) \leq 2n - 4$ for all $n \geq 4$.

Mathematical Induction

Solution:

(a) $G(1) = 0,\ G(2) = 1,\ G(3) = 3,\ G(4) = 4.$

(b) Let $P(n)$: $G(n) \leq 2n - 4$ for all $n \geq 4$.

- Basic Step: $G(4) = 4 \leq 2 \times 4 - 4 = 4$.
- Inductive Step: Suppose $G(n) \leq 2n - 4$ for all $n \geq 4$. Now consider $n + 1$ people. Let $\mathcal{N}$ denote a set of n people, and let r denote the remaining one person. Let k denote an arbitrary person in set $\mathcal{N}$.
  - r and k communicate by telephone: takes 1 calls; k knows the secret of S_r
  - the people in set $\mathcal{N}$ communicate: by inductive hypothesis, takes $G(n) \leq 2n - 4$ calls; everyone in set $\mathcal{N}$ knows every secrete of $\{S_i \mid i \in \mathcal{N} \cup \{r\}\}$
  - r and k communicate by telephone: takes 1 calls; r knows the secret of $\{S_i \mid i \in \mathcal{N} \cup \{r\}\}$

  In total, it takes $G(n+1) \leq 2n - 4 + 2 = 2(n+1) - 4$ calls. The proof completes.

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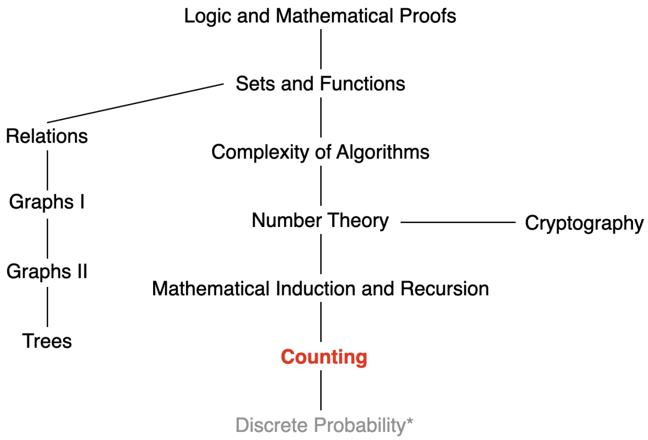
This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Generalized Permutations and Combinations, Generating Function, Solving Linear Recurrence Relations , ...

Generating Function

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k.$$

Example:

- The sequence $\{a_k\}$ with $a_k = 3$

$$\sum_{k=0}^{\infty} 3x^k$$

- The sequence $\{a_k\}$ with $a_k = 2^k$

$$\sum_{k=0}^{\infty} 2^k x^k$$

Generating Function

Generating function can be written in simpler forms:

- For $|x| < 1$, the function $G(x) = 1/(1-x)$ is the generating function of the sequence 1, 1, 1, 1, . . . ,

$$1/(1-x) = 1 + x + x^2 + ...$$

- For $|ax| < 1$, function $G(x) = 1/(1-ax)$ is the generating function of the sequence 1, a, a^2, a^3, . . . ,

$$1/(1-ax) = 1 + ax + a^2x^2 + ...$$

- For $|x| < 1$, $G(x) = 1/(1-x)^2$ is the generating function of the sequence 1, 2, 3, 4, 5, . . .

$$1/(1-x)^2 = 1 + 2x + 3x^2 + ...$$

Generating Function: Finite Series

A finite sequence $a_0, a_1, \ldots, a_n$ can be easily extended by setting $a_{n+1} = a_{n+2} = \ldots = 0$.

The generating function $G(x)$ of this sequence $\{a_n\}$ is a polynomial of degree n, i.e.,

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

Generating Function: Example

Example: What is the generating function for the sequence a_0, a_1, ... , a_m, with $a_k = C(m,k)$?

$$G(x) = C(m,0) + C(m,1)x + C(m,2)x^2 + ... + C(m,m)x^m.$$

Based on binomial theorem, this generating function has a simpler form: $G(x) = (1+x)^m = C(m,0) + C(m,1)x + C(m,2)x^2 + ... + C(m,m)x^m$.

Example: Generating function of 1,1,1,1,1,1?

$$1 + x^2 + x^3 + x^4 + x^5.$$

Based on the summation of geometric sequence,

$$1 + x^2 + x^3 + x^4 + x^5 = \frac{x^6 - 1}{x - 1}.$$

Operations of Generating Functions

Theorem: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$$

Example 1: To obtain the corresponding sequence of $G(x) = 1/(1-x)^2$: Consider $f(x) = 1/(1-x)$ and $g(x) = 1/(1-x)$. Since the sequence of $f(x)$ and $g(x)$ corresponds to 1, 1, 1, ..., we have

$$G(x) = f(x)g(x) = \sum_{k=0}^{\infty} (k+1) x^k.$$

Operations of Generating Functions

Theorem: Let $f(x) = \sum_{k=0}^{\infty} a_k x_k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x) g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Example 2: To obtain the corresponding sequence of $G(x) = 1/(1-ax)^2$ for $|ax| < 1$:

Consider $f(x) = 1/(1-ax)$ and $g(x) = 1/(1-ax)$. Since the sequence of $f(x)$ and $g(x)$ corresponds to 1, a, a^2, ..., we have

$$G(x) = f(x)g(x) = \sum_{k=0}^{\infty} (k+1) a^k x^k.$$

Useful Generating Functions

$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$

$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$

$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$

$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$

$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$

$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$

$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$

Useful Generating Functions

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$$
$$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$$
$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Extended Binomial Coefficient

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Here, u can be any real number, e.g., negative integers, non-integers, ...

Extended Binomial Coefficient

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Example: Find the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.

Taking $u = -2$ and $k = 3$

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

Taking $u = 1/2$ and $k = 3$

$$\begin{aligned}\binom{1/2}{3} &= \frac{(1/2)(1/2-1)(1/2-2)}{3!} \\ &= (1/2)(-1/2)(-3/2)/6 \\ &= 1/16.\end{aligned}$$

Extended Binomial Coefficient

When u is a negative integer:

$$\begin{aligned}
\binom{-n}{r} &= \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} \\
&= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!} \\
&= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!} \\
&= \frac{(-1)^r (n+r-1)!}{r!(n-1)!} \\
&= (-1)^r \binom{n+r-1}{r} \\
&= (-1)^r C(n+r-1, r).
\end{aligned}$$

Extended Binomial Theorem

Theorem: Let x be a real number with $|x| < 1$ and let u be a real number. Then,

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

Example:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

Generating Function

Generating function and counting ...

Generating Function and Combinations with Repetitions

Recall the following example:

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$ are nonnegative integers?

This type of counting problem can be solved with generating function.

Generating Function and Combinations with Repetitions

Formally, generating functions can also be used to solve counting problems of the following type:

$$e_1 + e_2 + \cdots + e_n = C,$$

where C is a constant and each e_i is a nonnegative integer that may be subject to a specified constraint.

Example 1

Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution: The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

By enumerating all possibilities, we have that the coefficient of x^{17} in this product is 3. (Note: In this example, you need to enumerate all possibilities; no better approach.)

Example 2

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Solution: This corresponds to the coefficient of x^8 of expansion

$$(x^2 + x^3 + x^4)^3$$

This coefficient equals 6. (Enumerate; we will see smarter ways in later examples.)

Example 3

Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in the cases

- Case 1: when the order does not matter

  E.g., three \$1 tokens; one \$1 token and a \$2 token

- Case 2: when the order does matter

  E.g., three \$1 tokens; a \$1 token and then a \$2 token; a \$2 token and then a \$1 token

Example 3

Case 1: when the order does not matter
The answer is the coefficient of x^r in the generating function

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots).$$

Case 2: when the order does matter
The number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of x^r in

$$(x + x^2 + x^5)^n$$

Because any number of tokens may be inserted,

$$1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \cdots$$

Generating Function and Counting

Use generating functions to find the number of *r*-combinations of a set with *n* elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1+x)^n$$

Why $(1+x)^n$?

r-combinations: select r balls from the following n balls

Not select or Select

$$(x^0 + x^1)\ (x^0 + x^1)\ (x^0 + x^1)\ \cdots\ (x^0 + x^1)$$

Product rule:
balls 1 and 2 are selected

$$x^1 x^1 x^0 \ldots x^0 = \boxed{x^2}$$ Two balls are selected

Generating Function and Counting

Use generating functions to find the number of r-combinations of a set with n elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1+x)^n$$

Why the coefficient of x^r?

r-combinations: select $r=2$ balls from the following $n=3$ balls

Sum rule:

$$\begin{aligned}(x^0+x^1)(x^0+x^1)(x^0+x^1) = {} & x^0x^0x^0 \\ & + x^1x^0x^0 + x^0x^1x^0 + x^0x^0x^1 \\ & \boxed{+ x^1x^1x^0 + x^0x^1x^1 + x^1x^0x^1} \\ & + x^1x^1x^1\end{aligned}$$

$3x^2$ Two balls are selected

Three possibilities

Example 4

Use generating functions to find the number of r-combinations of a set with n elements.

Solution: The answer is the coefficient of x^r in generating function

$$(1+x)^n$$

But by the binomial theorem, we have

$$f(x) = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Thus, $\binom{n}{r}$ is the answer.

Example 5

Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (1 + x + x^2 + \cdots)^n.$$

As long as $|x| < 1$, we have $1 + x + x^2 + \cdots = 1/(1 - x)$, so

$$G(x) = 1/(1 - x)^n = (1 - x)^{-n}.$$

Applying the extended binomial theorem

$$(1 - x)^{-n} = (1 + (-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

Hence, the coefficient of x^r equals $\binom{-n}{r}(-1)^r = C(n + r - 1, r)$.

Example 6

Use generating functions to find the number of ways to select r objects of n different kinds if we must select at least one object of each kind.

Solution: The answer is the coefficient of x^r in generating function

$$G(x) = (x + x^2 + x^3 + \cdots)^n = x^n(1 + x + x^2 + \cdots)^n = x^n/(1-x)^n.$$

$$\begin{aligned}
G(x) &= x^n/(1-x)^n \\
&= x^n \cdot (1-x)^{-n} \\
&= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r \\
&= x^n \sum_{r=0}^{\infty} (-1)^r C(n+r-1, r)(-1)^r x^r
\end{aligned}
\qquad
\begin{aligned}
&= \sum_{r=0}^{\infty} C(n+r-1, r) x^{n+r} \\
&= \sum_{t=n}^{\infty} C(t-1, t-n) x^t \\
&= \sum_{r=n}^{\infty} C(r-1, r-n) x^r.
\end{aligned}$$

Hence, there are $C(r-1, r-n)$ ways to select r objects of n different kinds if we must select at least one object of each kind.

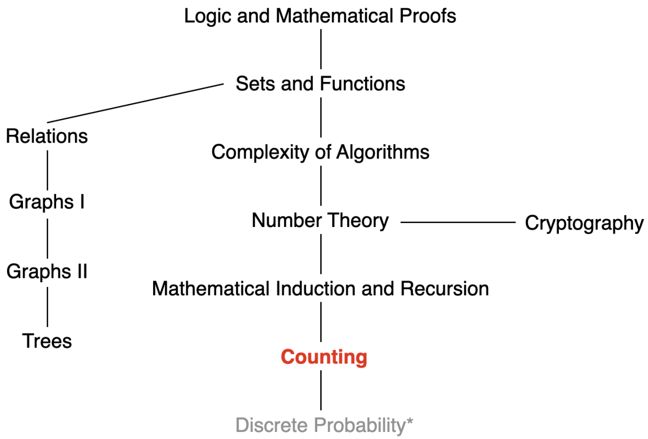
Generating Function and Combinations with Repetitions

- Based on the combination problem, transfer the problem as finding the coefficient of x^r of a generating function, e.g.,

$$G(x) = (1 + x + x^2 + x^3 + \cdots)^n$$

- Find the coefficient of x^r
  - Enumerate all possibilities or
  - Use useful generating functions (e.g., Binomial Theorem)

This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Generalized Permutations and Combinations, Generating Function, Solving Linear Recurrence Relations , ...

Solving Linear Recurrence Relations

Solve the closed-form:

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Solving Linear Recurrence Relations

One important class of recurrence relations can be explicitly solved in a systematic way.

Definition: A linear homogeneous relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- linear: it is a linear combination of previous terms
- homogeneous: all terms are multiples of a_j's
- degree k: a_n is expressed by the previous k terms
- constant coefficients: coefficients are constants

Example:

- $a_n = a_{n-1} + (a_{n-2})^2$: not linear.
- $H_n = 2H_{n-1} + 1$: not homogeneous.
- $B_n = nB_{n-1}$: not constant coefficients.
- $P_n = 1.11 \cdot P_{n-1}$: Yes; of degree 1
- $f_n = f_{n-1} + f_{n-2}$: Yes; of degree 2

Solving Linear Recurrence Relations

By induction, such a recurrence relation is uniquely determined by this recurrence relation and k initial conditions $a_0, a_1, \dots, a_{k-1}$.

Example: Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2},$$

Which of the following are solutions?

- $a_n = 3n$ Yes
- $a_n = 2^n$ No
- $a_n = 5$ Yes

Question: Why not unique?

Question: Any systematic way?

Solving Linear Recurrence Relations

Definition: A linear homogeneous relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where c_1, c_2, . . . , c_k are real numbers, and $c_k \neq 0$.

Basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant.

Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}.$$

Divide both sides by r^{n-k},

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0.$$

Characteristic equation of the recurrence relation.

Solving Linear Recurrence Relations

Why of the form of $a_n = r^n$?

https://math.stackexchange.com/questions/926112/
what-is-the-intuitive-idea-behind-looking-for-a-solution-of-the-form-an-rn-for

Solving Linear Recurrence Relations: Degree Two

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

The sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ with the initial condition $a_0 = C_0$ and $a_1 = C_1$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, ...,$$

where α_1 and α_2 are constants.

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is the solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and satisfy the initial conditions.

- $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation for any arbitrary constants α_1 and α_2.
- For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Since the solution is unique, the if and only if statement holds.

Solving Linear Recurrence Relations: Degree Two

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation.

Since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, it follows that $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$. Suppose $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$:

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n.
\end{aligned}$$

Solving Linear Recurrence Relations: Degree Two

For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Suppose that $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold.

$$a_0 = C_0 = \alpha_1 + \alpha_2, \qquad a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

Thus,

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \qquad \alpha_2 = C_0 - \alpha_1 = \frac{C_0 r_1 - C_1}{r_1 - r_2}.$$

α_1 and α_2 exist since $r_1 \neq r_2$.

Solving Linear Recurrence Relations: Degree Two

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and satisfy the initial conditions.

- $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation.
- For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Note that there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, so the if and only if statement holds.

Solving Linear Recurrence Relations: Degree Two

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, ...,$$

where α_1 and α_2 are constants.

Solve Linear Recurrence Relations:

- Solve r_1 and r_2 with $r^2 - c_1 r - c_2 = 0$.
- Solve α_1 and α_2 with the initial conditions.

Example 1: Fibonacci number

Fibonacci number: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

What is the closed-form expression of F_n?

To solve r_1 and r_2, consider $r^n = r^{n-1} + r^{n-2}$, i.e., $r^2 - r - 1 = 0$. There are two different roots:

$$r_1 = \frac{1+\sqrt{5}}{2}, \qquad r_2 = \frac{1+\sqrt{5}}{2}$$

Consider the form of $F_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. To solve α_1 and α_2, by $F_0 = 0$ and $F_1 = 1$, we have $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 r_1 + \alpha_1 r_2 = 1$.

Thus, $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -\alpha_1$. Hence,

$$F_n = \alpha_1 r_1^n + \alpha_2 r_2^n = \frac{r_1^n - r_2^n}{\sqrt{5}}.$$

Example 2

$a_n = a_{n-1} + 2a_{n-2}$, with $a_0 = 2$, $a_1 = 7$.

The characteristic equation is

$$r^2 - r - 2 = 0.$$

The two roots are 2 and -1. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n.$$

By the two initial conditions, we have

$$a_0 = \alpha_1 + \alpha_2 = 2, \qquad a_1 = 2\alpha_1 - \alpha_2 = 7.$$

We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Example 3

$a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 2$, $a_1 = 1$

The characteristic equation is

$$r^2 - 7r + 10 = 0.$$

Two roots are 2 and 5. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 5^n.$$

By the two initial conditions, we have

$$a_0 = \alpha_1 + \alpha_2 = 2, \qquad a_1 = 2\alpha_1 + 5\alpha_2 = 1.$$

We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

$$a_n = 3 \cdot 2^n - 5^n.$$

Solving Linear Recurrence Relations of Degree k

Consider an arbitrary linear homogeneous relation of degree k with constant coefficients:

$$a_n = \sum_{i-1}^{k} c_i a_{n-i}.$$

The characteristic equation (CE) is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0.$$

Theorem: If this CE has k distinct roots r_i, then the solutions to the recurrence are of the form

$$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for all $n \geq 0$, where the α_i's are constants.

Example

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

The characteristic equation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. So, assume that

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

By the three initial conditions, we have $a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$, $a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3$, $a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9$.

We get $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Thus,

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

The Case of Degenerate Roots: Degree Two

Theorem: If the $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0, then

$$a_n = (\alpha_1 + \alpha_2 n) r_0^n,$$

for all $n \geq 0$ and two constants α_1 and α_2.

Example

$a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$

The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

The only root is 2. So, assume that

$$a_n = (\alpha_1 + \alpha_2 n)2^n.$$

By the three initial conditions, we have

$$a_0 = \alpha_1 = 1, \qquad a_1 = 2 \cdot (\alpha_1 + \alpha_2) = 0.$$

We get $\alpha_1 = 1$, $\alpha_2 = -1$. Thus,

$$a_n = (1 - n)2^n.$$

The Case of Degenerate Roots: Degree k

Theorem: Suppose that there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then,

$$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n$$

for all $n \geq 0$ and constants $\alpha_{i,j}$.

$$\begin{aligned} a_n = \; & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

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Example

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, $a_2 = -1$.

The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

There is a single root $r = -1$ of multiplicity three of the characteristic equation. Thus, assume that

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n.$$

By the three initial conditions, we have ...

We get $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, $\alpha_{1,2} = -2$. Thus, $a_n = (1 + 3n - 2n^2)(-1)^n$.