Discrete Mathematics for Computer Science

Lecture 15-1: Counting

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

Solving Linear Recurrence Relations

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Linear Nonhomogeneous Recurrence Relations

Definition: A linear nonhomogeneous relation with constant coefficients may contain some terms $F(n)$ that depend only on n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n).$$

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

Example:

- $a_n = a_{n-1} + 2^n$ $\quad a_n = a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ $\quad a_n = a_{n-1} + a_{n-2}$
- $a_n = 3a_{n-1} + n3^n$ $\quad a_n = 3a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ $\quad a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Linear Nonhomogeneous Recurrence Relations

Every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation.

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$.

Note: $a_n^{(p)}$ does not need to satisfy the initial conditions.

Linear Nonhomogeneous Recurrence Relations

Proof: Suppose $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation,

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \dots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation,

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + \dots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$.
Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n.

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Linear Nonhomogeneous Recurrence Relations

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$.

The key is to find the particular solution to the linear nonhomogeneous relation. However, there is no general method for finding such a solution.

Example 1

There are techniques that work for certain types of functions $F(n)$, such as polynomials and powers of constants.

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

- Compute $a_n^{(h)}$
- Compute $a_n^{(p)}$
- Initial condition

Example 1

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

To compute $a_n^{(h)}$:

The characteristic equation is

$$r^2 - 3r = 0.$$

The roots are $r_1 = 3$ and $r_2 = 0$. By So, assume that

$$a_n^{(h)} = \alpha 3^n.$$

To compute $a_n^{(p)}$: Try $a_n^{(p)} = cn + d$. Thus,

$$cn + d = 3(c(n-1) + d) + 2n.$$

We get $c = -1$ and $d = -3/2$. Thus, $a_n^{(p)} = -n - 3/2$.

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Example 1

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

To compute $a_n^{(h)}$: $a_n^{(h)} = \alpha 3^n$.

To compute $a_n^{(p)}$: $a_n^{(p)} = -n - 3/2$.

Initial condition:

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha 3^n - n - 3/2.$$

Base on the initial condition $a_1 = 3$. We have $3 = -1 - 3/2 + 3\alpha$, which implies $\alpha = 11/6$. Thus, $a_n = -n - 3/2 + (11/6)3^n$.

Example 2

Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$. (Since we do not provide the initial conditions, obtain the general form would be sufficient.)

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- Try $a_n^{(p)} = C \cdot 7^n$:

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n.$$

Thus, $C = 49/20$, and $a_n^{(p)} = (49/20)7^n$.

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

Linear Nonhomogeneous Recurrence Relations

For the previous two examples, we made a guess that there are solutions of a particular form. This was not an accident.

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Taken care when $s = 1$!

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2 + 1)3^n$.

To compute $a_n^{(h)}$:
The characteristic equation is

$$r^2 - 6r + 9 = 0.$$

This characteristic equation has a single root $r = 3$ of multiplicity $m = 2$.

$$a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n.$$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2+1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = n^2 2^n$:

Since $s = 2$ is not a root of the characteristic equation, we have

$$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0)2^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2, p_1, and p_0:

$$\begin{aligned}(p_2 n^2 + p_1 n + p_0)2^n = 6(p_2(n-1)^2 + p_1(n-1) + p_0)2^{n-1} \\ - 9(p_2(n-2)^2 + p_1(n-2) + p_0)2^{n-2} + n^2 2^n.\end{aligned}$$

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + (p_2 n^2 + p_1 n + p_0)2^n.$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2+1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = (n^2+1)3^n$:

Since $s = 3$ is a root of the characteristic equation
with multiplicity $m = 2$, we have

$$a_n^{(p)} = n^2(p_2 n^2 + p_1 n + p_0)3^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2, p_1, and p_0:

...

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + n^2(p_2 n^2 + p_1 n + p_0)3^n.$

Example 2: The Term n^m

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n$$

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- $a_n^{(p)}$ should be in the form of $np_0 2^n$.
- Try $a_n^{(p)} = p_0 \cdot 2^n$ instead:

$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2} + 2^n.$$

Since $s = 2$ is a root of the characteristic equation,

$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2}$$

always holds. Thus, we obtain $0 = 4$. Contradiction.

Generating Function

Generating function and recurrent relation ...

Useful Generating Functions

$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$
$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$
$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$$
$$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, ...$ and initial condition $a_0 = 2$.

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. We aim to first derive the formulation of $G(x)$.

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1})x^k \\ &= 2, \end{aligned}$$

Thus, $G(x) - 3xG(x) = (1-3x)G(x) = 2$:

$$G(x) = \frac{2}{(1-3x)}.$$

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Solution: We aim to first derive the formulation of $G(x)$.

$$G(x) = \frac{2}{(1-3x)}.$$

Then, derive a_k using the identity $1/(1-ax) = \sum_{k=0}^{\infty} a^k x^k$. That is,

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

Consequently, $a_k = 2 \cdot 3^k$.

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: We extend this sequence by setting $a_0 = 1$. We have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\begin{aligned}
G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8xG(x) + x/(1 - 10x),
\end{aligned}$$

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Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$.

Solution: Thus,

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)} = G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

Thus, $a_n = \frac{1}{2}(8^n + 10^n)$.

Generating function to solve recurrence relations

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Based on the recurrence relations, derive the formulation of $G(x)$.
- Using identities (or the useful facts of generating functions), derive sequence $\{a_k\}$.

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Next Lecture