MA204: Mathematical Statistics

Suggested Solutions to Assignment 1

1.1 Solution. (a) The mgf of X is given by

$$\begin{aligned} M_X(t) &= E(\mathrm{e}^{tX}) = \sum_{x=0}^{n} \mathrm{e}^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (p\mathrm{e}^t)^x (1-p)^{n-x} = (p\mathrm{e}^t + 1 - p)^n. \end{aligned}$$

(b) We have

$$\begin{aligned} M'_X(t) &= \frac{\mathrm{d}M_X(t)}{\mathrm{d}t} = np\mathrm{e}^t(p\mathrm{e}^t + 1 - p)^{n-1} \quad \text{and} \\ M''_X(t) &= n(n-1)(p\mathrm{e}^t)^2(p\mathrm{e}^t + 1 - p)^{n-2} \\ &\quad + np\mathrm{e}^t(p\mathrm{e}^t + 1 - p)^{n-1}. \end{aligned}$$

Hence

$$\begin{aligned} E(X) &= M'_X(0) = np \quad \text{and} \\ \mathrm{Var}(X) &= E(X^2) - [E(X)]^2 = M''_X(0) - (np)^2 \\ &= n(n-1)p^2 + np - (np)^2 = np(1-p). \end{aligned}$$

(c) Let $Z = X + Y$. For any $z = 0, 1, 2, \ldots, \infty$, we define $m = \min(n, z)$. Then, the pmf of Z is

$$\begin{aligned} \Pr(Z = z) &= \Pr(X + Y = z) \\ &= \sum_{x=0}^{m} \Pr(X = x,\ Y = z - x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x=0}^{m} \Pr(X = x) \cdot \Pr(Y = z - x) \\
&= \sum_{x=0}^{m} \binom{n}{x} p^x (1-p)^{n-x} \cdot \frac{\lambda^{z-x}}{(z-x)!} \mathrm{e}^{-\lambda} \\
&= (1-p)^n \lambda^z \mathrm{e}^{-\lambda} \sum_{x=0}^{m} \binom{n}{x} \left[\frac{p}{\lambda(1-p)}\right]^x \frac{1}{(z-x)!}.
\end{aligned}
$$

1.2 Solution. (a) The marginal distribution of X is

$$\Pr(X = 1) = \sum_{y=1}^{4} \Pr(X = 1, Y = y) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4};$$

similarly, we have

$$\Pr(X = i) = \frac{1}{4}, \quad i = 2, 3, 4.$$

(b) The pmf of $Z = X + Y$ is

$$
\begin{aligned}
\Pr(Z = 2) &= \Pr(X = 1, Y = 1) = \frac{1}{16}, \\
\Pr(Z = 3) &= \Pr(X = 1, Y = 2) = \frac{1}{16}, \\
\Pr(Z = 4) &= \Pr(X = 1, Y = 3) + \Pr(X = 2, Y = 2) \\
&= \frac{1}{16} + \frac{2}{16} = \frac{3}{16}, \\
\Pr(Z = 5) &= \Pr(X = 1, Y = 4) + \Pr(X = 2, Y = 3) \\
&= \frac{1}{16} + \frac{1}{16} = \frac{2}{16}, \\
\Pr(Z = 6) &= \Pr(X = 2, Y = 4) + \Pr(X = 3, Y = 3) \\
&= \frac{1}{16} + \frac{3}{16} = \frac{4}{16}, \\
\Pr(Z = 7) &= \Pr(X = 3, Y = 4) = \frac{1}{16},
\end{aligned}
$$

$$\Pr(Z=8) \;=\; \Pr(X=4, Y=4) = \frac{4}{16}.$$

1.3 Solution. (a) Note that

$$f_{(Y|X)}(y|x) = \frac{x\mathrm{e}^{-xy}}{1-\mathrm{e}^{-bx}}, \quad 0 \leqslant y < b$$

by applying the sampling-wise formula

$$f_X(x) \propto \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)}, \tag{SA1.1}$$

and setting $y_0 = b/2$, the marginal distribution of X is given by

$$f_X(x) \propto \frac{1-\exp(-bx)}{x} \mathrel{\hat{=}} h(x), \quad 0 \leqslant x < b < +\infty. \tag{SA1.2}$$

We first prove

$$h(x) \leqslant b \quad \text{for any } x \in [0, b). \tag{SA1.3}$$

For any continuous and twice differentiable function $g(x)$ with $g''(x) > 0$, the second Taylor expansion of $g(x)$ around x_0 is

$$\begin{aligned} g(x) &= g(x_0) + (x-x_0)g'(x_0) + \frac{(x-x_0)^2}{2}g''(\xi) \\ &\geqslant g(x_0) + (x-x_0)g'(x_0), \end{aligned}$$

where ξ is a point between x and x_0. Now let $g(x) = \mathrm{e}^{-bx}$ and $x_0 = 0$. Since $g'(x) = -b\mathrm{e}^{-bx}$ and $g''(x) = b^2\mathrm{e}^{-bx} > 0$ for any $x \in [0, b)$, we have

$$\mathrm{e}^{-bx} \geqslant 1 - bx \quad \text{or} \quad b \geqslant \frac{1-\mathrm{e}^{-bx}}{x} = h(x),$$

implying (SA1.3). From (SA1.3), we obtain

$$\int_0^b h(x)\,\mathrm{d}x \leqslant \int_0^b b\,\mathrm{d}x = b^2 < +\infty,$$

which implies $f_X(x)$ exists.

(b) If let $b = +\infty$, then from (SA1.2),

$$f_X(x) \propto 1/x, \quad 0 \leqslant x < +\infty.$$

Obviously, $f_X(x)$ is not a density.

1.4 Solution. Note that $\mathcal{S}_X = \{x_1, x_2, x_3\}$ and $\mathcal{S}_Y = \{y_1, \ldots, y_4\}$. By using point-wise IBF, the marginal distribution of X is given by

X	x_1	x_2	x_3
$p_i = \Pr(X = x_i)$	0.24	0.28	0.48

Similarly, the marginal distribution of Y is given by

Y	y_1	y_2	y_3	y_4
$q_j = \Pr(Y = y_j)$	0.28	0.16	0.28	0.28

The joint distribution of $(X, Y)^\top$ is given by

$$\boldsymbol{P} = \begin{pmatrix} 0.04 & 0.04 & 0.12 & 0.04 \\ 0.08 & 0.08 & 0.04 & 0.08 \\ 0.16 & 0.04 & 0.12 & 0.16 \end{pmatrix}.$$

1.5 Proof. (a)

$$\begin{aligned} E(|X-b|) &= \int_{-\infty}^{\infty} |x-b| f(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{b} (b-x) f(x)\,\mathrm{d}x + \int_{b}^{\infty} (x-b) f(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{m} (b-m+m-x) f(x)\,\mathrm{d}x + \int_{m}^{b} (b-x) f(x)\,\mathrm{d}x \\ &\quad + \int_{m}^{\infty} (x-m+m-b) f(x)\,\mathrm{d}x + \int_{b}^{m} (x-b) f(x)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{m} (m-x) f(x)\ \mathrm{d}x + (b-m) \int_{-\infty}^{m} f(x)\ \mathrm{d}x \\
&\quad + \int_{m}^{\infty} (x-m) f(x)\ \mathrm{d}x + (m-b) \int_{m}^{\infty} f(x)\ \mathrm{d}x \\
&\quad + 2 \int_{m}^{b} (b-x) f(x)\ \mathrm{d}x \\
&= E(|X-m|) + 2 \int_{m}^{b} (b-x) f(x)\ \mathrm{d}x \\
&\quad + (b-m)\Big[\Pr(X \leqslant m) - \Pr(X \geqslant m) \Big] \\
&= E(|X-m|) + 2 \int_{m}^{b} (b-x) f(x)\ \mathrm{d}x.
\end{aligned}
$$

(b) Since $\int_m^b (b-x) f(x)\ \mathrm{d}x \geqslant 0$ for all b, $E(|X-b|)$ is minimised if and only if $b = m$.

1.6 Solution. (a) It is easy to obtain

$$
\begin{aligned}
\Pr(1/4 < X < 5/8) &= \int_{1/4}^{5/8} \mathrm{d}F(x) = F(5/8) - F(1/4) \\
&= 1 - 2(1-5/8)^2 - 2(1/4)^2 = \frac{19}{32}.
\end{aligned}
$$

(b) The pdf of X is

$$
f(x) = \begin{cases} 4x, & \text{if } 0 \leqslant x < 1/2, \\ 4(1-x), & \text{if } 1/2 \leqslant x < 1, \\ 0, & \text{otherwise.} \end{cases}
$$

then

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\ \mathrm{d}x \\
&= \int_0^{1/2} 4x^2\, \mathrm{d}x + \int_{1/2}^{1} 4x(1-x)\, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4}{3}x^3\Big|_0^{1/2} + \left(2x^2 - \frac{4}{3}x^3\right)\Big|_{1/2}^{1} \\
&= \frac{1}{6} - 0 + \left(2 - \frac{4}{3}\right) - \left(\frac{2}{4} - \frac{4}{3\cdot 8}\right) \\
&= \frac{1}{6} + \frac{2}{3} - \frac{1}{3} = \frac{1}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x \\
&= \int_0^{1/2} 4x^3\,\mathrm{d}x + \int_{1/2}^{1} 4x^2(1-x)\,\mathrm{d}x \\
&= x^4\Big|_0^{1/2} + \left(\frac{4}{3}x^3 - x^4\right)\Big|_{1/2}^{1} \\
&= \frac{1}{16} - 0 + \left(\frac{4}{3} - 1\right) - \left(\frac{1}{6} - \frac{1}{16}\right) \\
&= \frac{7}{24},
\end{aligned}
$$

Therefore,

$$
\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{24}.
$$

1.7 Solution. (a) Let $\mathbf{x} = (X_1, \ldots, X_d)^\top$ and $\boldsymbol{t} = (t_1, \ldots, t_d)^\top$. Note that $M_{\mathbf{x}}(\boldsymbol{t}) = E[\exp(t_1X_1 + \cdots + t_dX_d)]$, then

$$
\frac{\partial M_{\mathbf{x}}(\boldsymbol{t})}{\partial t_i} = E[X_i \exp(t_1X_1 + \cdots + t_dX_d)]
$$

and

$$
\frac{\partial M_{\mathbf{x}}(\boldsymbol{t})}{\partial t_i}\Big|_{t_1=\cdots=t_d=0} = E(X_i), \quad i = 1, \ldots, d.
$$

(b) Note that

$$
\frac{\partial^2 M_{\mathbf{x}}(\boldsymbol{t})}{\partial t_i \partial t_j} = E[X_iX_j \exp(t_1X_1 + \cdots + t_dX_d)],
$$

we obtain

$$\left.\frac{\partial^2 M_{\mathbf{x}}(\boldsymbol{t})}{\partial t_i \partial t_j}\right|_{t_1=\cdots=t_d=0} = E(X_i X_j), \quad i, j = 1, \ldots, d.$$

(c) If the joint density of X and Y is

$$f(x, y) = \begin{cases} \mathrm{e}^{-x-y}, & \text{for } x > 0,\ y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

then, the joint mgf is

$$\begin{aligned} M_{(X,Y)}(t_1, t_2) &= E[\exp(t_1 X + t_2 Y)] \\ &= \int_0^\infty \int_0^\infty \mathrm{e}^{t_1 X + t_2 Y} \mathrm{e}^{-x} \mathrm{e}^{-y} \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{(1-t_1)(1-t_2)}, \qquad t_1 < 1,\ t_2 < 1. \end{aligned}$$

Now

$$\frac{\partial M_{(X,Y)}(t_1, t_2)}{\partial t_1} = \frac{1}{(1-t_1)^2(1-t_2)},$$

then

$$E(X) = \left.\frac{\partial M_{(X,Y)}(t_1, t_2)}{\partial t_1}\right|_{t_1=t_2=0} = 1.$$

Similarly, we have $E(Y) = 1$. Furthermore, since

$$\frac{\partial^2 M_{(X,Y)}(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{1}{(1-t_1)^2(1-t_2)^2},$$

we have

$$E(XY) = \left.\frac{\partial^2 M_{(X,Y)}(t_1, t_2)}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0} = 1.$$

Note that X and Y are independent, we obtain

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.$$

1.8 Solution. A(a) Since

$$c^{-1} = \sum_{x=1}^{\infty} \frac{\lambda^x \mathrm{e}^{-\lambda}}{x!} = 1 - \mathrm{e}^{-\lambda},$$

we have

$$c = \frac{1}{1 - \mathrm{e}^{-\lambda}}.$$

A(b)

$$\begin{cases} E(X) & = & c\lambda, \\ E(X^2) & = & c(\lambda^2 + \lambda), \\ \mathrm{Var}(X) & = & c\lambda[1 + (1-c)\lambda]. \end{cases}$$

A(c) The mgf of X is

$$M_X(t) = E(\mathrm{e}^{tX}) = c\mathrm{e}^{-\lambda}[\exp(\lambda \mathrm{e}^t) - 1].$$

B(d) Let $c_i = 1/(1 - \mathrm{e}^{-\lambda_i})$ for $i = 1, 2$. The pmf of $X_1 + X_2$ is

$$\begin{aligned}
& \Pr(X_1 + X_2 = x) \\
= \; & \sum_{i=1}^{x-1} \Pr(X_1 = i, X_2 = x - i) \\
= \; & \sum_{i=1}^{x-1} \Pr(X_1 = i) \Pr(X_2 = x - i) \\
= \; & c_1 c_2 \sum_{i=1}^{x-1} \frac{\lambda_1^i \mathrm{e}^{-\lambda_1}}{i!} \cdot \frac{\lambda_2^{x-i} \mathrm{e}^{-\lambda_2}}{(x-i)!} \\
= \; & c_1 c_2 \cdot \frac{\mathrm{e}^{-(\lambda_1 + \lambda_2)}}{x!} \sum_{i=1}^{x-1} \binom{x}{i} \lambda_1^i \lambda_2^{x-i} \\
= \; & c_1 c_2 \cdot \frac{\mathrm{e}^{-(\lambda_1 + \lambda_2)}}{x!} [(\lambda_1 + \lambda_2)^x - \lambda_2^x - \lambda_1^x], \quad x = 2, 3, \ldots, \infty.
\end{aligned}$$

B(e) The conditional distribution of $X_1|(X_1+X_2=x)$ is

$$
\begin{aligned}
& \Pr(X_1=x_1|X_1+X_2=x) \\
= \; & \frac{\Pr(X_1=x_1, X_2=x-x_1)}{\Pr(X_1+X_2=x)} \\
= \; & \frac{\dfrac{c_1\lambda_1^{x_1}\mathrm{e}^{-\lambda_1}}{x_1!}\cdot\dfrac{c_2\lambda_2^{x-x_1}\mathrm{e}^{-\lambda_2}}{(x-x_1)!}}{c_1c_2\cdot\dfrac{\mathrm{e}^{-(\lambda_1+\lambda_2)}}{x!}[(\lambda_1+\lambda_2)^x-\lambda_2^x-\lambda_1^x]} \\
= \; & \frac{\binom{x}{x_1}\lambda_1^{x_1}\lambda_2^{x-x_1}}{(\lambda_1+\lambda_2)^x-\lambda_2^x-\lambda_1^x}, \quad x_1=1,2,\ldots,x-1.
\end{aligned}
$$

1.9 Solution. (a)

$$
\begin{aligned}
\mathrm{Var}(X) &= \lambda_0+\lambda, \\
E(Y) &= E(Z)\cdot[E(U)+E(W)] = (1-\phi)(\lambda_0+\beta\lambda), \\
\mathrm{Var}(Y) &= E[Z^2(U^2+W^2+2UW)]-[E(Y)]^2 \\
&= (1-\phi)[E(U^2)+E(W^2)+2E(U)E(W)]-[E(Y)]^2 \\
&= (1-\phi)[\lambda_0+\lambda_0^2+\beta\lambda+\beta^2\lambda^2+2\lambda_0\beta\lambda] \\
&\quad -(1-\phi)^2(\lambda_0+\beta\lambda)^2 \\
&= (1-\phi)(\lambda_0+\beta\lambda)[1+\phi(\lambda_0+\beta\lambda)], \\
\mathrm{Cov}(X,Y) &= E(XY)-E(X)E(Y) \\
&= E(Z)\cdot E[U^2+U(W+V)+VW]-E(X)E(Y) \\
&= (1-\phi)[\lambda_0+\lambda_0^2+\lambda_0(\beta\lambda+\lambda)+\beta\lambda^2] \\
&\quad -(\lambda_0+\lambda)(1-\phi)(\lambda_0+\beta\lambda) \\
&= (1-\phi)\lambda_0.
\end{aligned}
$$

Alternatively

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathrm{Cov}(U+V, ZU+ZW) = \mathrm{Cov}(U, ZU)\\ &= E(ZU^2) - E(U)E(ZU)\\ &= (1-\phi)\lambda_0.\end{aligned}$$

(b) When $y = 0$, the joint distribution of X and Y is

$$\begin{aligned}&\Pr(X = x, Y = y = 0)\\ =\ & \Pr\{U+V = x, Z(U+W) = 0\}\\ =\ & \Pr(U+V = x, Z = 0) + \Pr(U+V = x, Z = 1, U+W = 0)\\ =\ & \Pr(Z = 0)\Pr(U+V = x)\\ &+\ \Pr(Z = 1)\Pr(U+V = x, U+W = 0)\\ =\ & \phi\Pr(U+V = x) + (1-\phi)\Pr(U = 0, V = x, W = 0)\\ =\ & \phi\frac{(\lambda_0+\lambda)^x \mathrm{e}^{-\lambda_0-\lambda}}{x!} + (1-\phi)\frac{\lambda^x \mathrm{e}^{-\lambda_0-\lambda-\beta\lambda}}{x!}.\end{aligned}$$

When $y > 0$, the joint distribution of X and Y is

$$\begin{aligned}&\Pr(X = x, Y = y)\\ =\ & \Pr\{U+V = x, Z(U+W) = y\}\\ =\ & \Pr(U+V = x, Z = 1, U+W = y)\\ =\ & \Pr(Z = 1)\cdot\Pr(U+V = x, U+W = y)\\ =\ & (1-\phi)\sum_{k=0}^{\min(x,y)} \Pr(U = k, V = x-k, W = y-k)\\ =\ & (1-\phi)\sum_{k=0}^{\min(x,y)} \frac{\lambda_0^k \mathrm{e}^{-\lambda_0}}{k!}\cdot\frac{\lambda^{x-k}\mathrm{e}^{-\lambda}}{(x-k)!}\cdot\frac{(\beta\lambda)^k \mathrm{e}^{-\beta\lambda}}{(y-k)!}\end{aligned}$$

$$= \quad (1-\phi)\mathrm{e}^{-\lambda_0-\lambda-\beta\lambda}\frac{\lambda^x(\beta\lambda)^y}{x!y!}\sum_{k=0}^{\min(x,y)}\binom{x}{k}\binom{y}{k}k!\left(\frac{\lambda_0}{\beta\lambda^2}\right)^k.$$

1.10 Solution. Note that the mgf of $V \sim N(\mu, \sigma^2)$ is

$$M_V(t) = \exp(\mu t + 0.5\sigma^2 t^2),$$

we have $X \sim N(0,1)$ and $Y \sim N(-1,4)$. Hence,

$$W = 3X + 2Y \sim N(-2, 25)$$

since $X \perp\!\!\!\perp Y$.

(a) Let $Z = [W-(-2)]/5$, then $Z \sim N(0,1)$. Thus,

$$\Pr(-12 < W < 3) = \Pr(-2 < Z < 1) = \Phi(1) - \Phi(-2) = 0.8185.$$

(b) $E(W^2) = \mathrm{Var}(W) + [E(W)]^2 = 25 + 4 = 29.$

1.11 Solution. (a) The expectation of $Y = |X|$ is given by

$$\begin{aligned}
E(Y) &= E|X| = \int_{-\infty}^{+\infty} |x| \cdot \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\ \mathrm{d}x \\
&= 2\int_0^{+\infty} x \cdot \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\ \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \exp(-x^2/2)\ \mathrm{d}x^2 \qquad [\text{Let } t = x^2] \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \exp(-t/2)\ \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}(-2\mathrm{e}^{-t/2})\Big|_0^{+\infty} = \sqrt{\frac{2}{\pi}}.
\end{aligned}$$

Since $E(Y^2) = E(X^2) = \mathrm{Var}(X) + [E(X)]^2 = 1$, we have

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1 - \frac{2}{\pi}.$$

(b) Let $\Phi(\cdot)$ denote the cdf of the standard normal distribution. For any $y \geqslant 0$, the cdf of Y is

$$\begin{aligned} F(y) &= \Pr(Y \leqslant y) = \Pr(|X| \leqslant y) \\ &= \Pr(-y \leqslant X \leqslant y) = \Phi(y) - \Phi(-y) \\ &= \Phi(y) - [1 - \Phi(y)] = 2\Phi(y) - 1. \end{aligned}$$

Let $\phi(x) = (1/\sqrt{2\pi}\,)\mathrm{e}^{-x^2/2}$ denote the pdf of the standard normal distribution. The pdf of Y is given by

$$f(y) = F'(y) = \phi(y) + \phi(-y) = 2\phi(y), \quad y \geqslant 0.$$

1.12 Proof. Since the cdf of $U \sim U(0,1)$ is given by

$$\begin{aligned} \Pr(U \leqslant u) &= \begin{cases} 0, & \text{if } u \leqslant 0, \\ u, & \text{if } 0 < u < 1, \\ 1, & \text{if } u \geqslant 1 \end{cases} \\ &= 0 \cdot I(u \leqslant 0) + u \cdot I(0 < u < 1) + 1 \cdot I(u \geqslant 1), \end{aligned}$$

where $I(u \in \mathbb{A})$ is the indicator function. Define $Y = F^{-1}(U)$, then, the cdf of Y is

$$\Pr(Y \leqslant y) = \Pr\{F^{-1}(U) \leqslant y\} = \Pr\{U \leqslant F(y)\} = F(y),$$

indicating that $Y \sim F(\cdot)$. Thus, $Y \stackrel{\mathrm{d}}{=} X$.

1.13 Proof. Define a random variable Y as

$$Y = \begin{cases} 1, & \text{if } X > \lambda\mu, \\ 0, & \text{if } X \leqslant \lambda\mu, \end{cases}$$

then $Y \sim \text{Bernoulli}(p)$, where $p = \Pr(X > \lambda\mu)$. Of course, we have $Y^2 \sim \text{Bernoulli}(p)$, so that

$$E(Y^2) = p = \Pr(X > \lambda\mu). \tag{SA1.4}$$

From the definitions of X and Y, we obtain

$$XY = \begin{cases} X, & \text{if } X > \lambda\mu, \text{ its probability is } p, \\ 0, & \text{if } X \leqslant \lambda\mu, \text{ its probability is } 1-p. \end{cases}$$

Subtracting $X - \lambda\mu$ from both sides, we have

$$XY-(X-\lambda\mu) = \begin{cases} \lambda\mu > 0, & \text{if } X > \lambda\mu, \text{ with prob. } p, \\ \lambda\mu - X \geqslant 0, & \text{if } X \leqslant \lambda\mu, \text{ with prob. } 1-p. \end{cases}$$

In other words, $XY - (X - \lambda\mu) \geqslant 0$ so that

$$E(XY) - E(X - \lambda\mu) = E[XY - (X - \lambda\mu)] \geqslant 0.$$

Hence,

$$E(XY) \geqslant E(X - \lambda\mu) = \mu - \lambda\mu = (1 - \lambda)\mu. \qquad \text{(SA1.5)}$$

By Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} & \Pr(X > \lambda\mu)E(X^2) \overset{\text{(SA1.4)}}{=} E(Y^2)E(X^2) \\ \geqslant \quad & \{E(XY)\}^2 \qquad \text{[Cauchy-Schwarz inequality]} \\ \overset{\text{(SA1.5)}}{\geqslant} \quad & (1-\lambda)^2\mu^2. \end{aligned}$$

1.14 Solution. (a) The cdf of $X \sim \text{Logistic}(\mu, \sigma^2)$ with density

$$f(x) = \frac{\exp(-\frac{x-\mu}{\sigma})}{\sigma\{1 + \exp(-\frac{x-\mu}{\sigma})\}^2}, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \sigma > 0,$$

is given by

$$F(x) \quad = \quad \left[1 + \exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{-1}.$$

Based on the definition of the q-th quantile, we have $F(\xi_q) = q \in (0, 1)$, so

$$\xi_q = F^{-1}(q) = \mu + \sigma \log\left(\frac{q}{1-q}\right).$$

Especially, the median of X is $\xi_{0.5} = \mu$.

(b) The cdf of the Rayleigh distribution with pdf

$$f(x) = \sigma^{-2} x \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0, \quad \sigma > 0,$$

is given by

$$\begin{aligned} F(x) &= \int_0^x \sigma^{-2} y \exp\left(-\frac{y^2}{2\sigma^2}\right) \mathrm{d}y \\ &= -\exp\left(-\frac{y^2}{2\sigma^2}\right)\Bigg|_0^x = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right). \end{aligned}$$

Thus, $F(\xi_q) = q \in (0,1)$ implies

$$\xi_q = F^{-1}(q) = \sigma\sqrt{-2\log(1-q)}\,.$$

Especially, the median of X is $\xi_{0.5} = \sigma\sqrt{2\log 2}$.

1.15 Proof. (a) Since $\alpha > 0$ and $x > 0$, we have $f(x) > 0$. Now

$$\begin{aligned} &\int_0^\infty f(x)\,\mathrm{d}x \\ = &\int_0^\alpha \frac{x(2\alpha+x)}{\alpha(\alpha+x)^2}\,\mathrm{d}x + \int_\alpha^\infty \frac{\alpha^2(\alpha+2x)}{x^2(\alpha+x)^2}\,\mathrm{d}x \\ = &\int_0^\alpha \left[\frac{1}{\alpha} - \frac{\alpha}{(\alpha+x)^2}\right] \mathrm{d}x + \alpha\int_\alpha^\infty \left[\frac{1}{x^2} - \frac{1}{(\alpha+x)^2}\right] \mathrm{d}x \\ = &\left.\frac{x}{\alpha}\right|_0^\alpha + \left.\frac{\alpha}{\alpha+x}\right|_0^\alpha - \left.\frac{\alpha}{x}\right|_\alpha^\infty + \left.\frac{\alpha}{\alpha+x}\right|_\alpha^\infty = 1, \end{aligned}$$

indicating that $f(x)$ is a pdf.

(b) Let $X \sim f(x)$. When $0 < x \leqslant \alpha$, the cdf of X is

$$F(x) = \int_0^x \frac{t(2\alpha+t)}{\alpha(\alpha+t)^2}\,\mathrm{d}t = \frac{x}{\alpha} + \frac{\alpha}{\alpha+x} - 1. \tag{SA1.6}$$

When $x > \alpha$, we have

$$\begin{aligned} F(x) &= \int_0^x f(t)\,\mathrm{d}t = \int_0^\alpha \frac{t(2\alpha+t)}{\alpha(\alpha+t)^2}\,\mathrm{d}t + \int_\alpha^x \frac{\alpha^2(\alpha+2t)}{t^2(\alpha+t)^2}\,\mathrm{d}t \\ &= \frac{1}{2} - \frac{\alpha}{t}\Big|_\alpha^x + \frac{\alpha}{\alpha+t}\Big|_\alpha^x \\ &= \frac{\alpha}{\alpha+x} - \frac{\alpha}{x} + 1. \end{aligned} \tag{SA1.7}$$

The median of X, denoted by med(X), satisfies $F(\text{med}(X)) = 1/2$. From (SA1.7),

$$\frac{\alpha}{\alpha+x} - \frac{\alpha}{x} + 1 = \frac{1}{2} \Rightarrow 2\alpha^2 - \alpha x - x^2 = (\alpha - x)(2\alpha + x) = 0,$$

we have $x = \alpha \notin (\alpha, \infty)$ or $x = -2\alpha \notin (\alpha, \infty)$. From (SA1.6),

$$\frac{x}{\alpha} + \frac{\alpha}{\alpha+x} - 1 = \frac{1}{2} \Rightarrow 2x^2 - x\alpha - \alpha^2 = (x-\alpha)(2x+\alpha) = 0,$$

we have $x = \alpha \in (0, \alpha]$ or $x = -\alpha/2 \notin (0, \alpha]$. Thus, med$(X) = \alpha$.

1.16 Solution. The median of X satisfies

$$\Pr\{X \leqslant \text{med}(X)\} \geqslant 0.5 \quad \text{and} \quad \Pr\{X \geqslant \text{med}(X)\} \geqslant 0.5.$$

We have, med$(X) = 3$ because

$$\begin{aligned} \Pr(X \leqslant 3) &= 0.20 + 0.15 + 0.25 = 0.60 \geqslant 0.5 \quad \text{and} \\ \Pr(X \geqslant 3) &= 0.25 + 0.4 = 0.65 \geqslant 0.5. \end{aligned}$$

1.17 Proof. (a) First we prove the second inequality in (A1.3), which is equivalent to

$$\frac{1}{n}\log(x_1 \cdots x_n) \leqslant \log\left(\frac{x_1 + \cdots + x_n}{n}\right),$$

i.e.,

$$\log\left(\frac{x_1 + \cdots + x_n}{n}\right) \geqslant \frac{\log(x_1) + \cdots + \log(x_n)}{n}. \tag{SA1.8}$$

This is a special case of (A1.2) if we set all $p_i = 1/n$.

(b) Now, It can be seen that the first inequality in (A1.3) is just a transformation of the second one in (A1.3). The first inequality $H(n) \leqslant G(n)$ is equivalent to

$$
\begin{aligned}
& -\log\left(\frac{x_1^{-1}+\cdots+x_n^{-1}}{n}\right) \leqslant \frac{1}{n}\Big[\log(x_1)+\cdots+\log(x_n)\Big] \\
\Leftrightarrow\quad & \log\left(\frac{x_1^{-1}+\cdots+x_n^{-1}}{n}\right) \geqslant \frac{1}{n}\Big[\log(x_1^{-1})+\cdots+\log(x_n^{-1})\Big].
\end{aligned}
$$

Finally, replacing x_i^{-1} with x_i to yield

$$
\log\left(\frac{x_1+\cdots+x_n}{n}\right) \geqslant \frac{1}{n}\Big[\log(x_1)+\cdots+\log(x_n)\Big],
$$

which is just (SA1.8).

1.18 Solution. (a) Note that $\mathcal{S}_X = \mathcal{S}_Y = (0,1)$. Let $y_0 = 0.5 \in \mathcal{S}_Y = (0,1)$, we have

$$
f_X(x) \propto \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)} = \frac{\dfrac{2(x+2y_0)}{1+4y_0}}{\dfrac{x+2y_0}{x+1}} \propto x+1,
$$

so that $f_X(x) = K^{-1}\cdot(x+1)\cdot I(0<x<1)$. From $1 = \int_0^1 f_X(x)\,\mathrm{d}x$, we obtain

$$
K = \int_0^1 (x+1)\,\mathrm{d}x = \frac{x^2}{2}\bigg|_0^1 + x\bigg|_0^1 = \frac{1}{2}+1 = \frac{3}{2}.
$$

Thus,

$$
f_X(x) = \frac{2}{3}\cdot(x+1)\cdot I(0<x<1),
$$

which is a linear pdf on the unit interval.

(b) The joint distribution of $(X, Y)^\top$ is

$$\begin{aligned} f_{(X,Y)}(x, y) &= f_X(x) \cdot f_{(Y|X)}(y|x) \\ &= \frac{2}{3} \cdot (x + 2y) \cdot I(0 < x, y < 1). \end{aligned}$$

1.19 Solution. Since $U \sim U(0, 1)$, we have

$$X|(Y = y) = Uy \sim U(0, y);$$

i.e., $f_{(X|Y)}(x|y) = y^{-1} \cdot I(0 < x < y)$. Thus,

$$\begin{aligned} f_X(x) &= \int f_{(X,Y)}(x, y)\,\mathrm{d}y = \int f_Y(y) \cdot f_{(X|Y)}(x|y)\,\mathrm{d}y \\ &= \int_x^\infty f_Y(y) y^{-1}\,\mathrm{d}y, \quad x > 0. \end{aligned}$$

1.20 Proof. (a) The cdf of Y is

$$\Pr(Y \leqslant y) = \Pr[F_2(X_1) \leqslant y] = \Pr[X_1 \leqslant F_2^{-1}(y)] = F_1(F_2^{-1}(y)),$$

so that the pdf of Y is

$$f_Y(y) = f_1(F_2^{-1}(y)) \cdot \frac{\mathrm{d}F_2^{-1}(y)}{\mathrm{d}y} \mathrel{\hat{=}} f_1(F_2^{-1}(y)) \cdot \frac{\mathrm{d}z}{\mathrm{d}y},$$

where $F_2^{-1}(y) \mathrel{\hat{=}} z$. Thus, we have $y = F_2(z)$, so

$$\frac{\mathrm{d}y}{\mathrm{d}z} = f_2(z) = f_2(F_2^{-1}(y)) \quad \text{or} \quad \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{1}{f_2(F_2^{-1}(y))}.$$

Hence,

$$f_Y(y) = \frac{f_1(F_2^{-1}(y))}{f_2(F_2^{-1}(y))}. \tag{SA1.9}$$

(b) From (A1.4), we obtain that the pdf and cdf of $X_2 \sim \mathrm{IBeta}(1, 1)$ are given by

$$\begin{aligned} f_2(x_2) &= \frac{1}{B(1,1)} \cdot \frac{1}{(1 + x_2)^2} = \frac{1}{(1 + x_2)^2} \quad \text{and} \\ F_2(x_2) &= \int_0^{x_2} f_2(t)\,\mathrm{d}t = -\frac{1}{1+t}\bigg|_0^{x_2} = 1 - \frac{1}{1 + x_2}, \quad x_2 > 0, \end{aligned}$$

respectively. Set $F_2(x_2) = y$, we have

$$x_2 = F_2^{-1}(y) = \frac{y}{1-y}. \tag{SA1.10}$$

From (SA1.9), we obtain

$$\begin{aligned} f_Y(y) &= \frac{f_1(F_2^{-1}(y))}{f_2(F_2^{-1}(y))} \overset{\text{(SA1.10)}}{=} \frac{f_1(y/(1-y))}{f_2(y/(1-y))} \\ &\overset{\text{(A1.4)}}{=} \frac{\dfrac{1}{B(\alpha,\beta)} \cdot \dfrac{[y/(1-y)]^{\alpha-1}}{[1+y/(1-y)]^{\alpha+\beta}}}{[1+y/(1-y)]^{-2}} \\ &= \frac{1}{B(\alpha,\beta)} \cdot y^{\alpha-1}(1-y)^{\beta-1}, \quad y \in (0,1), \end{aligned}$$

that is $Y \sim \text{Beta}(\alpha, \beta)$.

1.21 Proof. By using the identity

$$\beta^{-1} = \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t$$

or

$$x^{-1} = \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{(-t)x}\,\mathrm{d}t, \tag{SA1.11}$$

we have

$$\begin{aligned} E(X^{-1}) &= \int_0^\infty x^{-1} \cdot f_X(x)\,\mathrm{d}x \\ &\overset{\text{(SA1.11)}}{=} \int_0^\infty \left[\int_0^\infty \mathrm{e}^{(-t)x}\,\mathrm{d}t\right] f_X(x)\,\mathrm{d}x \\ &= \int_0^\infty \left[\int_0^\infty \mathrm{e}^{(-t)x} f_X(x)\,\mathrm{d}x\right]\mathrm{d}t \\ &= \int_0^\infty M_X(-t)\,\mathrm{d}t, \end{aligned}$$

which completes the proof.