An Exposition of Euler's Proof of the Infinitude of Primes

Google Gemini 2.5 Pro (Preview)

June 11, 2025

Contents

1 Introduction: The Ancient Proposition of the Infinitude of Primes

1.1 Prime Numbers: An Unsolved Mystery Spanning Millennia

Prime numbers, being natural numbers greater than 1 that are divisible only by 1 and themselves, form the fundamental "atoms" of the integer world. According to the Fundamental Theorem of Arithmetic, any integer greater than 1 can be uniquely decomposed into a product of primes. This unique property places prime numbers at the core of number theory. However, the distribution of prime numbers appears mysterious and unpredictable; seemingly random, yet embodying profound mathematical structures that have captivated mathematicians for centuries, urging them to explore their secrets.

1.2 Euclid's Classic Proof: A Milestone in Mathematical History

Around 300 BC, the ancient Greek mathematician Euclid, in his monumental work *Elements*, Book IX, first provided a proof for the infinitude of primes.[1, 2] Euclid's proof is renowned for its simplicity and ingenuity, employing a method of **reductio ad absurdum** (proof by contradiction). Its core idea is: assume the number of primes is finite, say $p_1, p_2, \ldots, p_n$. Then construct a new number $N = (p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$. This number N is either prime itself or divisible by some prime. However, N divided by any prime p_i in the list leaves a remainder of 1, so N cannot be divisible by any of $p_1, p_2, \ldots, p_n$.[1, 3] This implies the existence of a prime not in this finite list (either N itself or a prime factor of N), which contradicts the initial assumption that primes are finite. Therefore, the number of primes must be infinite. Euclid's proof set a paradigm for later mathematics and laid the foundation for discussing the infinitude of primes.

1.3 Euler's Novel Analytic Approach

Over a thousand years later, in the 18th century, the Swiss mathematical giant Leonhard Euler adopted a method entirely different from Euclid's to once again prove the infinitude of primes. Euler's proof, published in 1737, ingeniously introduced tools from analysis—infinite series and infinite products—into number theory research.[2, 4] His central argument was to demonstrate that the sum of the reciprocals of all primes diverges, i.e., the series $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \ldots$ tends to infinity.[1] If the number of primes were finite, this sum would clearly be a finite value. Thus, the divergence of this sum directly leads to the conclusion that primes are infinite.

Euler's method not only provided a new proof for the infinitude of primes but, more importantly, marked a significant shift in mathematical thought. Euclid's proof relied on basic divisibility properties, a construction purely within the realm of algebra and number theory. Euler, however, introduced tools of continuous mathematics to study discrete number-theoretic objects, heralding the birth of a new mathematical branch—analytic number theory. This methodological shift was, in a sense, more profoundly impactful than the proof itself, opening up new perspectives and powerful tools for number theory research. Furthermore, Euler's result is actually stronger than Euclid's conclusion. Euclid merely proved that primes "do not stop appearing," whereas Euler's result (the divergence of the sum of prime reciprocals) implies that primes cannot be "too sparsely" distributed among natural numbers. This preliminary characterization of prime density was a level of insight untouched by Euclid's proof.

2 Preliminaries: Key Mathematical Concepts

Understanding Euler's proof requires grasping some fundamental mathematical concepts. These concepts are like pieces of a puzzle that, when skillfully combined by Euler, form the complete picture of the proof.

2.1 The Harmonic Series ($\sum \frac{1}{n}$) and its Divergence

The harmonic series is the sum of the reciprocals of all positive integers, i.e., $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, and its infinite form is $\sum_{n=1}^{\infty} \frac{1}{n}$. Although each term $\frac{1}{n}$ of the series approaches zero as n increases, surprisingly, the sum of this infinite series diverges, meaning it grows beyond any preset finite value.[5] Many beginners might mistakenly assume the harmonic series converges because its terms get progressively smaller.[6]

There are several methods to prove the divergence of the harmonic series, one of the most classic being the grouping comparison method proposed by the medieval mathematician Nicole Oresme around 1350.[5] The method is as follows:

$$\begin{aligned}
&1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots
\end{aligned}$$

Since the series on the right clearly diverges to infinity, and each term of the harmonic series is greater than or equal to the corresponding term (or sum of corresponding group) in the comparison series, the harmonic series must also diverge.

Another proof method is the integral test, which compares the sum of the series with the integral $\int_1^{\infty} \frac{1}{x} dx$. Since $\int_1^{\infty} \frac{1}{x} dx = [\ln x]_1^{\infty} = \infty$, the integral diverges, and therefore the harmonic series also diverges.[5]

It is noteworthy that the harmonic series diverges very slowly. For example, for its partial sum to exceed 100, approximately 2^{198} terms are needed![1] This slow divergence characteristic suggests that we should have a similar expectation for the divergence of the sum of prime reciprocals (if it holds): even if primes become increasingly sparse among natural numbers, the sum of their reciprocals could still accumulate to infinity.

2.2 The Concept of Geometric Series

A geometric series is a series of the form $a + ar + ar^2 + ar^3 + \dots$, where a is the first term and r is the common ratio. When the absolute value of the common ratio $|r| < 1$, the sum of an infinite geometric series converges to $S = \frac{a}{1-r}$.

In Euler's proof, a particularly important form of the geometric series is when $a = 1$ and the common ratio is x (where $|x| < 1$):

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

This formula is one of the key steps in deriving the Euler product formula, used to expand each factor involving the reciprocal of a prime.[2]

2.3 The Fundamental Theorem of Arithmetic (Unique Prime Factorization Theorem)

The Fundamental Theorem of Arithmetic is one of the cornerstones of number theory. It states that any integer greater than 1 can be uniquely expressed as a product of prime numbers (disregarding the order of factors). For example, $12 = 2^2 \cdot 3$, $30 = 2 \cdot 3 \cdot 5$.

This theorem plays a crucial role in Euler's proof. When Euler multiplied together the geometric series associated with each prime, the Fundamental Theorem of Arithmetic ensures that each term in the expansion (of the form $\frac{1}{n^s}$, where n is any positive integer) will appear exactly once, no more and no less.[2, 7] It is this uniqueness that allows Euler to establish an equality between the sum of the s-th powers of the reciprocals of all natural numbers (the Riemann ζ-function) and a certain product involving all primes.

Euler's genius lay in skillfully weaving together these three seemingly independent basic concepts—the divergence of the harmonic series, the summation formula for geometric series, and the Fundamental Theorem of Arithmetic. Geometric series helped him expand each prime factor in the Euler product into an infinite sum; the Fundamental Theorem of Arithmetic ensured that when these infinite sums were multiplied, they would cover the reciprocals (to the s-th power) of all natural numbers without repetition or omission; and the divergence of the harmonic series (or its behavior at $s = 1$) provided the "infinity" benchmark for the final contradiction argument or direct proof of the divergence of the sum of prime reciprocals. These tools, perhaps not complex individually, together revealed a profound conclusion about the infinitude of primes under Euler's integrated application.

3 Euler's Product Formula: Bridging Integers and Primes

The core tool Euler used to prove the infinitude of primes is a profound identity he discovered, known as the Euler product formula. This formula establishes a crucial bridge between the Riemann ζ-function and prime numbers.

3.1 Defining the Riemann ζ-function

The Riemann ζ-function (usually denoted by the Greek letter ζ) is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

In its initial discussion, the variable s can be considered a real number. This series converges when $s > 1$.[4, 8] For example, when $s = 2$, $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, a famous result (the Basel problem). When $s = 1$, $\zeta(1)$ is the divergent harmonic series.

3.2 Derivation of the Euler Product Formula: $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$

The Euler product formula asserts that for $s > 1$, the Riemann ζ-function can be expressed as an infinite product that ranges over all prime numbers p:

$$\zeta(s) = \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}$$

There are mainly two approaches to derive this formula:

Method 1: Euler's Sieve Method (Intuitive Approach) This method resembles the ancient Sieve of Eratosthenes but is applied to series operations.[4]

1. Start with the definition of $\zeta(s)$:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \dots$$

2. Multiply both sides of the above equation by $\frac{1}{2^s}$:

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \dots$$

(All terms whose denominators are multiples of 2)

3. Subtract the second equation from the first:
$$\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\dots$$
(All terms whose denominators contain the factor 2 have been "sieved out")
4. Next, for the remaining series, multiply both sides by $\frac{1}{3^s}$:
$$\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\dots$$
(All terms whose denominators are multiples of 3 (and not multiples of 2))
5. Subtract this from the result of $\left(1-\frac{1}{2^s}\right)\zeta(s)$:
$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\dots$$
(All terms whose denominators contain factors of 2 or 3 have been sieved out)
6. Repeat this process indefinitely for all prime numbers p. Each step multiplies the left side by a factor $\left(1-\frac{1}{p^s}\right)$ and sieves out terms on the right side whose smallest prime factor is p. After all primes have been considered, the series on the right side is left with only the first term, 1.[4] Thus, we get:
$$\left(\prod_{p\text{ is prime}}(1-p^{-s})\right)\zeta(s)=1$$
Rearranging gives the Euler product formula:
$$\zeta(s)=\prod_{p\text{ is prime}}\frac{1}{1-p^{-s}}$$

Method 2: Expansion using Geometric Series and the Fundamental Theorem of Arithmetic (Constructive Approach) Consider the right side of the infinite product: $\prod_p \frac{1}{1-p^{-s}}$. For each prime p, according to the geometric series summation formula (where $x=p^{-s}$, and since $s>1$ and $p\geq 2$, so $|p^{-s}|<1$):
$$\frac{1}{1-p^{-s}}=1+p^{-s}+p^{-2s}+p^{-3s}+\dots=1+\frac{1}{p^s}+\frac{1}{(p^2)^s}+\frac{1}{(p^3)^s}+\dots$$
Now, multiply these expansions for all primes:
$$\prod_p\left(1+\frac{1}{p^s}+\frac{1}{(p^2)^s}+\dots\right)=\left(1+\frac{1}{2^s}+\frac{1}{4^s}+\dots\right)\left(1+\frac{1}{3^s}+\frac{1}{9^s}+\dots\right)\left(1+\frac{1}{5^s}+\frac{1}{25^s}+\dots\right)\dots$$
When this infinite product is expanded, by the distributive law, each term is obtained by selecting one term from each parenthesis and multiplying them. For example, a typical term looks like:
$$\left(\frac{1}{p_1^{a_1 s}}\right)\cdot\left(\frac{1}{p_2^{a_2 s}}\right)\cdot\dots\cdot\left(\frac{1}{p_k^{a_k s}}\right)\cdot 1\cdot 1\cdot\ldots=\frac{1}{(p_1^{a_1}p_2^{a_2}\dots p_k^{a_k})^s}$$
where $p_1,p_2,\dots,p_k$ are distinct primes, and $a_1,a_2,\dots,a_k$ are non-negative integers. According to the Fundamental Theorem of Arithmetic, any positive integer n can be uniquely expressed as a product of prime powers $n=p_1^{a_1}p_2^{a_2}\dots p_k^{a_k}$. Therefore, each term $\frac{1}{n^s}$ in the above expansion appears exactly once.[2, 9] So, this infinite product equals the sum of all $\frac{1}{n^s}$, which is the Riemann ζ-function:
$$\prod_{p\text{ is prime}}\frac{1}{1-p^{-s}}=\sum_{n=1}^{\infty}\frac{1}{n^s}=\zeta(s)$$

3.3 Rigor and Convergence

The operations on series and infinite products in the above derivations are rigorously justified by the condition $s > 1$ (or for a complex variable s, its real part $\text{Re}(s) > 1$). Under this condition, all involved series and infinite products are absolutely convergent, which ensures the legality of steps like changing the order of summation and expanding products.[4, 8]

The Euler product formula itself is an extremely profound result. It is like the "Rosetta Stone" of number theory, connecting the additive structure of integers (manifested in the definition of the ζ-function—summing reciprocals of powers of all natural numbers n) with the multiplicative structure of integers (manifested in the Euler product—operating on all prime numbers p). This connection is the cornerstone of analytic number theory, enabling mathematicians to use tools of analysis (like calculus, complex analysis) to study the properties of prime numbers, and vice versa.

Euler's sieve method is not just a clever algebraic trick; it can be seen as a continuous analog of the Sieve of Eratosthenes in the realm of analysis. The Sieve of Eratosthenes finds primes by physically striking out composite numbers; Euler's method, on the level of series, "filters out" terms whose denominators contain specific prime factors through algebraic operations. This conceptual analogy makes the origin of the Euler product more intuitive.

Furthermore, the variable s in $\zeta(s)$ is not merely a placeholder. Its value determines the convergence and behavior of the series and product. The Euler product formula holds for a continuous range of s values ($s > 1$), which endows it with powerful analytical capabilities, allowing the use of analytical techniques such as taking limits (e.g., letting $s \to 1$), which is precisely the key step in Euler's proof of the infinitude of primes.

4 Euler's Proof: From Divergence to Infinitude

Based on the Euler product formula, Euler proved the infinitude of primes in two closely related ways.

4.1 Argument 1: Contradiction via the Harmonic Series (The case $s = 1$)

The Euler product formula is: $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$. Consider the case when s approaches 1 (strictly speaking, $s \to 1^+$, i.e., s approaches 1 from values greater than 1).

1. As $s \to 1^+$, the left side of the formula, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, approaches the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. We already know that the harmonic series diverges, and its sum is infinite.[1, 2, 5, 6] Therefore, $\lim_{s \to 1^+} \zeta(s) = \infty$.[8]

2. Now consider the right side of the formula, $\prod_p \frac{1}{1-p^{-s}}$. If the set of primes were finite, say there are only k primes $p_1, p_2, \ldots, p_k$. Then as $s \to 1^+$, this product would become:

$$\prod_{i=1}^{k} \frac{1}{1-p_i^{-1}} = \frac{1}{1-p_1^{-1}} \cdot \frac{1}{1-p_2^{-1}} \cdot \ldots \cdot \frac{1}{1-p_k^{-1}}$$

   This is a product of a finite number of finite positive numbers, and its result must be a finite positive constant.

3. This leads to a contradiction: the left side tends to infinity, while the right side (under the assumption of finitely many primes) is a finite value. That is, $\infty =$ finite value.

4. This contradiction implies that the initial assumption—that primes are finite—must be false. Therefore, there must be infinitely many primes.[2]

Rigorous treatment for $s = 1$: Since the defining series for $\zeta(s)$ converges only for $s > 1$, directly setting $s = 1$ encounters the divergence of the harmonic series. A more rigorous statement involves examining the limit as s approaches 1 from the right. As stated above, $\lim_{s\to 1^+} \zeta(s) = \infty$. If primes were finite, then $\lim_{s\to 1^+} \prod_p \frac{1}{1-p^{-s}}$ would be a finite value, and the contradiction still holds.[8]

4.2 Argument 2: Directly Proving the Divergence of the Sum of Reciprocals of Primes ($\sum \frac{1}{p}$)

This is often considered Euler's more direct and stronger result, as it not only proves the infinitude of primes but also reveals an important property of their distribution.

1. Take the natural logarithm of both sides of the Euler product formula $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$:

$$\ln \zeta(s) = \ln \left(\prod_p \frac{1}{1-p^{-s}} \right) = \sum_p \ln \left(\frac{1}{1-p^{-s}} \right) = -\sum_p \ln(1-p^{-s})$$

2. Use the Taylor series expansion $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$ (valid for $|x| < 1$). Let $x = p^{-s}$ (since $s > 1, p \geq 2$, thus $0 < p^{-s} < 1/2 \leq 1/2^s < 1/2$, satisfying the condition):

$$-\ln(1-p^{-s}) = p^{-s} + \frac{p^{-2s}}{2} + \frac{p^{-3s}}{3} + \dots = \frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots$$

3. Substitute back into the original equation:

$$\ln \zeta(s) = \sum_p \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots \right)$$

$$\ln \zeta(s) = \left(\sum_p \frac{1}{p^s} \right) + \left(\sum_p \frac{1}{2p^{2s}} \right) + \left(\sum_p \frac{1}{3p^{3s}} \right) + \dots$$

4. Now consider the behavior as $s \to 1^+$:

   - Left side: Since $\lim_{s\to 1^+} \zeta(s) = \infty$, it follows that $\lim_{s\to 1^+} \ln \zeta(s) = \infty$.
   - Right side: Apart from the first term $\sum_p \frac{1}{p^s}$, the sum of the remaining terms converges. For example,

$$\sum_p \left(\frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots \right) < \sum_p \left(\frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots \right)$$

$$= \sum_p \frac{p^{-2s}}{1-p^{-s}}$$

   When $s \geq 1$, $1 - p^{-s} \geq 1 - 1/2 = 1/2$ (since $p \geq 2$). So the above expression is less than $2\sum_p p^{-2s}$. And $\sum_p p^{-2s} < \sum_{n=1}^{\infty} n^{-2s} = \zeta(2s)$. When $s \geq 1$, $2s \geq 2 > 1$, so $\zeta(2s)$ is a finite value. Therefore, $\sum_p \left(\frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots \right)$ converges to a finite constant C as $s \to 1^+$. (More concisely, $\sum_p \sum_{k=2}^{\infty} \frac{1}{kp^{ks}} < \sum_p \sum_{k=2}^{\infty} \left(\frac{1}{p^s}\right)^k = \sum_p \frac{1/p^{2s}}{1-1/p^s} = \sum_p \frac{1}{p^s(p^s-1)}$. When $s = 1$, this is $\sum_p \frac{1}{p(p-1)}$, which is smaller than $\sum_n \frac{1}{n(n-1)}$ (which converges), and is therefore convergent.)

5. So, as $s \to 1^+$, we have:
$$\infty = \lim_{s\to 1^+} \left(\sum_p \frac{1}{p^s} \right) + C$$
This forces $\lim_{s\to 1^+} \left(\sum_p \frac{1}{p^s} \right)$ to tend to infinity. That is, the series $\sum_p \frac{1}{p}$ (the sum of the reciprocals of primes) diverges.[1, 2, 8]

6. If the number of primes were finite, then $\sum \frac{1}{p}$ would necessarily be a finite sum, which contradicts the conclusion that it diverges. Therefore, there must be infinitely many primes.

The limit process $s \to 1^+$ is key to analytical rigor. The Euler product formula and the ζ-function are well-behaved for $s > 1$. The point $s = 1$ is the critical point where the harmonic series (i.e., $\zeta(1)$) "explodes." By examining the behavior approaching this critical point, Euler was able to use the known divergence of the harmonic series to deduce important conclusions about primes. This use of limits cleverly connects the convergence domain of the Euler product formula with its divergent behavior at the boundary.

The divergence of $\sum \frac{1}{p}$ is a stronger statement than the infinitude of primes. It implies that primes are "relatively dense" among the natural numbers. In contrast, the sum of the reciprocals of square numbers $\sum \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$), even though there are infinitely many square numbers. This indicates that, from the perspective of reciprocal sums, primes are "more numerous" or "become sparse more slowly" than square numbers.[1] This quantitative perspective is a hallmark of analytic number theory, moving beyond simple "yes or no" questions to explore "how many" and "how they are distributed."

The two arguments presented by Euler (via the contradiction from $\zeta(1)$, and by directly proving the divergence of $\sum \frac{1}{p}$) essentially rely on the Euler product formula and properties of series. They demonstrate from different angles the depth of Euler's insight and the powerful utility of the Euler product as an analytical tool.

5 Comparison of the Two Proofs: Euler and Euclid

Euclid and Euler both proved the infinitude of primes, but their methods, the tools they relied on, and the deeper implications of their conclusions differ significantly.

5.1 Review of Euclid's Proof

Euclid's proof (circa 300 BC) is an elegant constructive proof by contradiction. It assumes primes are finite $(p_1, \ldots, p_k)$, then constructs the number $N = p_1 p_2 \ldots p_k + 1$. This N is either prime or has a prime factor p. Since N leaves a remainder of 1 when divided by any of $p_1, \ldots, p_k$, p cannot be any of them. Thus, p is a new prime, contradicting the assumption of finite primes.[1, 2, 3]

5.2 Review of Euler's Proof

Euler's proof (1737) employed tools from analysis. Its core is the Euler product formula $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$. By considering the case as $s \to 1^+$, $\zeta(s)$ tends towards the divergent harmonic series (infinity). If primes were finite, the product would be a finite value, leading to a contradiction. Alternatively, through logarithms and series expansions, he directly proved that the sum of the reciprocals of primes, $\sum \frac{1}{p}$, diverges, which also implies infinitely many primes.[1, 2]

5.3 Comparison of Euclid's and Euler's Proofs

To more clearly illustrate the differences between the two, the following table provides a comparison across multiple dimensions:

Feature	**Euclid's Proof**	**Euler's Proof**
Era	Circa 300 BC	1737 AD
Mathematical Field	Elementary Number Theory	Analytic Number Theory (invo ing calculus, infinite series)
Core Idea	Construct a new prime not in the finite list.	Prove that the sum of the rec rocals of primes diverges.
Key Tools	Divisibility, existence of prime factors.	Infinite series (geometric, h monic), infinite products, uni prime factorization, limits, lo rithms, Riemann ζ-function.
Nature of Argument	Proof by contradiction.	Proof by contradiction, or dir proof of divergence.
Complexity	Relatively simple concepts, fewer prerequisites.	More complex, requires und standing of calculus concepts.
Strength of Conclusion	There are infinitely many primes.	There are infinitely many prin + the sum of their reciproc ($\sum \frac{1}{p}$) diverges (a stronger c clusion).
Constructive?	Yes (shows how to find *a* new prime, though not necessarily $N+1$ itself).	No (does not directly constr new primes).
Dependent Fundamental Theorems	Every integer greater than 1 has at least one prime factor.[2, 7]	The unique prime factorizat theorem for integers (Fundam tal Theorem of Arithmetic).[2

Table 1: Comparison of Euclid's and Euler's Proofs

5.4 Highlighting Conceptual Differences and Strength of Conclusions

Euclid's proof is existential and, to some extent, constructive, indicating how to find a new prime. Euler's proof, on the other hand, is analytical and quantitative. It not only proves the infinitude of primes but its more profound conclusion—the divergence of the sum of prime reciprocals—implies that primes are "sufficiently dense" in the natural numbers for the sum of their reciprocals to reach infinity. This is much more profound than merely stating that primes do not cease to appear.[1] Euclid's proof relies on the basic fact that "every integer greater than 1 has a prime factor," while Euler's proof goes further, relying on "every integer greater than 1 can be uniquely decomposed into a product of primes"—the Fundamental Theorem of Arithmetic.[7]

This comparison reveals that the concept of "infinity" can have different "textures." Euclid tells us the sequence of primes never ends. Euler, through his analytical method, shows that primes do not become sparse "too quickly." This more nuanced characterization of the infinitude of primes foreshadowed later theories of prime distribution (such as the Prime Number Theorem, which quantifies the density of primes).

Simultaneously, these two proofs exemplify a common trade-off in mathematical proof techniques. Euclid's proof is elementary and easy to understand, requiring almost no advanced mathematical prerequisites. Euler's proof, however, requires more advanced analytical tools, but it thereby achieves a more profound result and opens up entirely new avenues of research.

This illustrates that in mathematics, more powerful tools often lead to deeper insights, though usually at the cost of some simplicity.

6 The Legacy of Euler's Proof: The Dawn of Analytic Number Theory

Euler's proof of the infinitude of primes signifies much more than just providing new evidence for this ancient proposition. More importantly, the methods and ideas he used opened up an entirely new field for number theory research.

6.1 The Birth of a New Field

Euler's paper on infinite series, published in 1737, is widely regarded as the seminal work of **analytic number theory**.[2] Analytic number theory is the branch of mathematics that applies methods of continuous analysis (such as calculus, complex analysis, theory of infinite series and products) to study discrete number-theoretic problems (like properties of integers, distribution of primes, etc.).[4] Euler's proof perfectly embodies this combination of ideas: using functions of a continuous variable (the Riemann ζ-function) and limit processes to explore properties of the discrete set of primes.

6.2 A New Way to "Measure" Primes

The divergence of the sum of prime reciprocals, $\sum \frac{1}{p}$, provided a new, albeit rough, way to "measure" the density of primes. It showed that primes are "much more numerous" than infinite sets whose reciprocal sums converge (e.g., square numbers, because $\sum \frac{1}{n^2}$ converges). This discovery spurred mathematicians' interest in more precise quantitative descriptions of prime distribution, eventually leading to the Prime Number Theorem, proved at the end of the 19th century. The Prime Number Theorem gives an asymptotic estimate for the prime-counting function $\pi(x)$ (the number of primes not exceeding x) as $\pi(x) \sim \frac{x}{\ln x}$, providing a solid mathematical basis for understanding the macroscopic distribution of primes.

6.3 Influence on Later Work

Euler's ideas and methods profoundly influenced later mathematicians.

- **Dirichlet's Theorem on Arithmetic Progressions**: The German mathematician Peter Gustav Lejeune Dirichlet was deeply inspired by Euler's work. He generalized Euler's methods, introduced Dirichlet L-functions (generalizations of the Riemann ζ-function), and used similar analytical techniques to prove the famous Dirichlet's Theorem in 1837: any arithmetic progression $a, a + d, a + 2d, \ldots$, where the first term a and common difference d are coprime, contains infinitely many primes.[2] This was a landmark achievement, solving a long-standing problem in number theory.

- **Riemann's Contributions**: Later, the German mathematician Bernhard Riemann extended Euler's study of the real-variable ζ-function to complex variables and conducted in-depth research into its properties. Riemann's paper "On the Number of Primes Less Than a Given Magnitude" laid the foundation for modern analytic number theory. The Riemann Hypothesis, proposed therein, remains one of the most important and difficult unsolved problems in mathematics, deeply connected to the fine distribution of primes.[4] The Euler product formula was the starting point for Riemann's work.

6.4 Comparison with Unsolved Problems

It is worth noting that while Euler's method of proving $\sum \frac{1}{p}$ diverges can be used to prove the infinitude of certain subsets of primes (e.g., primes in Dirichlet's theorem), it is not universally applicable. For example, the question of whether there are infinitely many twin primes (pairs of primes differing by 2, like 3 and 5, 11 and 13) remains unsolved to this day. The mathematician Viggo Brun proved that the sum of the reciprocals of all twin primes (known as Brun's constant) converges.[2] This means that even if there are infinitely many twin primes, Euler's method of inferring infinitude by proving the divergence of the reciprocal sum cannot be applied here. This demonstrates both the power of Euler's method and its limitations.

Euler's method provided mathematicians with a new "analytical lens" to observe and study the properties of integers. Analytical tools can reveal the asymptotic behavior and distribution laws of discrete sets on a macroscopic scale, which are difficult to achieve with purely algebraic methods. From this perspective, the *method* of Euler's proof of the infinitude of primes is perhaps even more important than re-proving this known result itself. The field it pioneered and the tools it provided enabled the resolution of many previously intractable problems (like Dirichlet's theorem). The charm of analytic number theory lies in its successful bridging of continuous mathematics (analysis) and discrete mathematics (number theory), with Euler being one of the earliest architects of this bridge.

7 Conclusion: The Elegance and Power of Euler's Insight

Euler's proof of the infinitude of primes is a brilliant gem in the history of mathematics. It not only confirmed an ancient truth in a completely new way but, more importantly, the ideas and methods it embodied profoundly changed the landscape of number theory research.

7.1 Summary of Euler's Achievement

Euler's core contribution was the ingenious use of the Euler product formula he discovered: $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$. This formula connects a sum over all natural numbers (manifested through the Riemann ζ-function) with an infinite product involving only prime numbers. By examining the behavior of this formula as $s \to 1^+$, and utilizing the divergence of the harmonic series (or the properties of $\zeta(s)$ near $s = 1$), Euler argued that a finite number of primes would lead to a contradiction, or more directly, proved that the sum of the reciprocals of primes, $\sum \frac{1}{p}$, itself diverges. Both paths unequivocally lead to the conclusion that primes are infinite.

7.2 Lasting Impact

This work by Euler marked the birth of analytic number theory, introducing powerful analytical tools to number theory research. It prompted mathematicians to move from qualitative questions like whether primes are infinite to more in-depth quantitative questions about how primes are distributed and what their density is. This shift in perspective has had a profound impact on the development of number theory and mathematics as a whole.

7.3 The Beauty of the Proof

Euler's proof showcases the inherent harmony and beauty of mathematics. He connected seemingly unrelated mathematical concepts—infinite series, infinite products, logarithms, and the unique factorization property of integers—in an unexpected way, not only solving an ancient problem but also imbuing the process itself with intellectual delight and logical power. Euler demonstrated extraordinary intuition and bold innovation in dealing with infinity.[9] Although the rigorous theoretical foundations for infinite series and products were still under construction in his time, he was often able to accurately grasp the essence of these infinite processes through his keen insight and arrive at correct conclusions.

Euler's work was not an endpoint but a new beginning. The path he opened led subsequent mathematicians into the vast realm of number theory research. Profound problems such as the fine regularities of prime distribution and the Riemann Hypothesis are all inextricably linked to Euler's foundational work. Therefore, understanding Euler's proof of the infinitude of primes is not just about learning a specific mathematical theorem; it is about appreciating a powerful mathematical idea, feeling the pulse of mathematical development, and drawing inspiration for exploring the unknown.