Problems 81-90 on Pearson's Chisq Test for Goodness of Fit, Duality between Confidence Interval and Hypothesis Testing, and Re-enforcements

81. Four coins were tossed 160 times and 0,1,2,3, or 4 heads showed, respectively, 19, 54, 58, 23, and 6 times. Use the 0.05 level of significance to test whether it is reasonable to suppose that the coins are balanced and randomly tossed.

82. It is desired to test whether the number of gamma rays emitted per second by a certain radio-active substance is a random variable having the Poisson distribution with $\lambda = 2.4$. Use the following data obtained for 300 1-second intervals to test this null hypothesis at the 0.05 level of significance:

Number of gamma rays	Frequency	Expected Frequency
0	19	
1	48	
2	66	
3	74	
4	44	
5	35	
6	10	
7 or more	4	

83. The following is the distribution of the readings obtained with a Geiger counter of the numbers of particles emitted by a radioactive substance in 100 successive 40-second intervals:

Number of particles	Frequency	Expected Frequency
5-9	1	
10-14	10	
15-19	37	
20-24	36	
25-29	13	
30-34	2	
35-39	1	

Test at the 0.05 level of significance whether the data may be looked upon as a random sample from a normal population.

84. Suppose that for each value $\theta_0 \in \Theta$ there is a test at level α of the hypothesis $H_0: \theta = \theta_0$. Denote the acceptance region of the test by $A(\theta_0)$. Then the set $B(\mathbf{X}) = \{\theta: \mathbf{X} \in A(\theta)\}$ is a $100(1-\alpha)\%$ confidence region for θ.

85. Suppose that $B(\mathbf{X})$ is a $100(1-\alpha)\%$ confidence region for θ; that is, for each θ_0, $\mathbb{P}\{\theta_0 \in B(\mathbf{X})|\theta = \theta_0\} = 1-\alpha$. Then an acceptance region for a test at level α of the hypothesis $H: \theta = \theta_0$ is $A(\theta_0) = \{\mathbf{X}: \theta_0 \in B(\mathbf{X})\}$.

86. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu_0, \sigma^2)$, where μ_0 is known and σ^2 is unknown. (a) Using the Neyman-Pearson Lemma, find a most powerful test (MPT) of size α for testing the simple null hypothesis $H_0: \sigma^2 = \sigma_0^2$ against the simple alternative hypothesis $H_1: \sigma^2 = \sigma_1^2$, where $\sigma_1^2 > \sigma_0^2$. (b) Find a uniformly most powerful test (UMPT) of size α for testing the null hypothesis $H_1: \sigma^2 < \sigma_0^2$ against the alternative hypothesis $H_1: \sigma^2 > \sigma_0^2$.

87. Let $X_1, \ldots, X_n$ be a random sample from the following density $f(x;\lambda) = \lambda(1-x)^{\lambda-1}\,\mathbb{I}(0 < x < 1)$ where $\lambda > 0$. (a) Let $Y_i = -\ln(1-X_i)$ and $Q(X) = \prod_{i=1}^{n}(1-X_i)$. Prove that Y_i follows an exponential distribution with mean parameter $1/\lambda$ and $-2\lambda \ln Q(X) \sim \chi^2(2n)$. [Hint: $\chi^2(2n) = Gamma\left(n, \frac{1}{2}\right)$ and the density of $Gamma(w; m, \lambda) = \frac{\lambda^m}{\Gamma(m)} w^{m-1} e^{-\lambda w}$]. (b) Find the likelihood ratio test (LRT) for testing $H_0: \lambda = 1$ against $H_1: \lambda \neq 1$.

88. Let $X_1, \ldots, X_n$ be a random sample from the Bernoulli distribution with parameter $\theta = \mathbb{P}(X = 1)$. Define $\hat{\theta} = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Based on the convergence results $\frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\hat{\theta}(1-\hat{\theta})}} \xrightarrow{L} N(0,1)$ and $\frac{\sqrt{n}(\bar{X}_n - \theta)}{\sqrt{\theta(1-\theta)}} \xrightarrow{L} N(0,1)$, (a) Find two approximate $100(1-\alpha)\%$ confidence intervals for θ, denoted by CI_1 and CI_2. (b) Let $n = 10$, $(x_1, \ldots, x_{10}) = (1,1,1,1,0,1,1,0,0,1)$, $\alpha = 0.05$ and $z_{0.025} = 1.96$. Calculate CI_1 and CI_2 and compare their widths.

89. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with probability mass function $Poisson(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$ $(x = 0,1,2,\ldots)$. Prove that the sample mean $\bar{X}$ is the unique uniformly minimum variance unbiased estimator (UMVUE) of λ.

90. Let $\theta = (\theta_1, \theta_2, \theta_3)^{\top}$ be an unknown parameter vector, where $0 < \theta_i < 1$ for $i = 1, 2, 3$ and $\sum_{i=1}^{3}\theta_i = 1$. Let x_1, x_2, x_3, y_1 and y_2 be observed values of random variables X_1, X_2, X_3, Y_1 and Y_2, respectively. Assume that the likelihood function of θ is $L(\theta) = \left(\prod_{i=1}^{3}\theta_i^{x_i}\right)\theta_1^{y_1}(\theta_1+\theta_2)^{y_2}$. (a) Let $x_i > 0$ for $i = 1,2,3$ and $y_1 = y_2 = 0$. Find the maximum likelihood estimates (mles) of θ_i for $i = 1,2,3$ subject to the equality constraint $\sum_{i=1}^{3}\theta_i = 1$. (b) Let $x_i > 0$ for $i = 1,2,3$ and $y_j > 0$ for $j = 1,2$. Find the mles of θ_i with explicit expressions for $i = 1,2,3$ subject to the equality constraint $\sum_{i=1}^{3}\theta_i = 1$. (c) Let $x_1 = 100$, $x_2 = 50$, $x_3 = 20$, $y_1 = 10$ and $y_2 = 30$. Calculate the mles of θ_i for $i = 1,2,3$.