Abstract Algebra
: Lecture 8

Leo

2024.10.17

至[illegible]为 $y^p=1$.

Lemma 1. *Let G be a abelian group with $|G| = p^e$. Let $g \in G$ which has the largest order. Then $G =< g > \times H$ for some subgroup $H < G$.*

证明. Assume G is not cyclic. Do induction on $|G|$. $|G|=p$ ✓. Assume for $|G|=p^{e-1}$, $G=\langle g\rangle\times H$.

Claim: $\exists h \in G$ s.t. $< h > \cap < g >= 1$ and $o(h) = p$.

Since $G \neq< g >$, then $\exists y \in G- < g >$ s.t. $y^p \in< g >$. Then $\exists k \in \mathbb{Z}_{\geqslant 1}$ s.t. $y^p = g^{kp}$. Let $h = y^{-1}g^k$, then $h \neq 1$ and $h^p = 1$. Thus $o(h) = p$. And if $h \in< g >$ then $y^{-1}g^k \in< g >\Rightarrow y \in< g >$, contradiction.

$\langle g\rangle < G$

$\langle\bar g\rangle = \langle g\rangle\langle h\rangle/\langle h\rangle < G/\langle h\rangle \cong \langle g\rangle/\langle g\rangle\cap\langle h\rangle \cong \langle g\rangle.$

$|\bar G| = p^{e-1}$.

Let $\bar{G} = G/ < h >$. Then $|\bar{G}| < |G|$, and $o(g) = o(\bar{g})$ is the largest order of element in $\bar{G}$. By indction, Let $\bar{G} =< \bar{g} > \times \bar{H}$. Let H be the full preimage of $\bar{H}$ in G.

Claim: $< g > \cap H = 1$. $\bar{x}\in\langle\bar g\rangle\cap\bar H. \to x\in\langle g\rangle$, $x\in\langle h\rangle$.

Suppose there exists $x \in< g > \cap H$. $\neq 1$ Then $\bar{x} = \bar{1}$ as $< \bar{g} > \cap \bar{H} = \bar{1}$, so $x \in< h >$ and $\underset{\neq 1.}{x} \in< g >\cancel{\Rightarrow< h >\subseteq< g >}$, contradiction. $\langle h\rangle\cap\langle g\rangle=\{1\}$

Therefore $G =< g > \times H$.

$|G| = |g|\,|H| \leftarrow \frac{|G|}{|h|} = |g|\cdot\frac{|H|}{|h|} \leftarrow |\bar G| = |g|\,|\bar H|.$

□

Theorem 2. *(Fundemental Theorem of Finite Abelian Groups) Let G be a finite abelian group of order n. Let $n = p_1^{e_1} \dots p_r^{e_r}$. Then:*

(1). $G = G_1 \times \dots \times G_r$ where $|G_i| = p_i^{e_i}$.

(2). G is a direct product of cyclic groups.

证明. (1). Let $n = p^e m$ s.t. p is a prime and $(p, m) = 1$. Let $H = \{g^m | g \in G\}$. Then H is a subgroup and every element of H has order p-power. Moreover $|H| = p^e$, ✓ and $G = H \times K$ where K has order m. By induction on K we can prove (1). ✓.

(2). By lemma 1. $\to$ for every G_i. $G_i=\langle g\rangle\times H=\langle g\rangle\times\langle h\rangle\times H'=\cdots=\langle g\rangle\times\langle h\rangle\times\cdots\langle h^{(n)}\rangle$.

□

Solvable Groups:

Definition 3. *Let G be a finite group. For $x, y \in G$, let $[x, y] = x^{-1}y^{-1}xy$, called the commutator of x and y.*

$[x,y]=x^{-1}y^{-1}xy.$

$a^{-1}b^{-1}ab$ $\quad ba^2b \quad (a^{-1})^2$

$babbab$ $\quad a^{-2}$

Definition 4. *Let $H = \langle [x, y] | x, y \in G\rangle$. Then H is called the commutator subgroup of G, denoted by G'.*

G'.

Proposition 5. *$\cancel{H} \lhd G$.* $\quad g^{-1}(x^{-1}y^{-1}xy)g = \underline{g^{-1}x^{-1}g}\,\underline{g^{-1}y^{-1}g}\,\underline{g^{-1}xg}\,\underline{g^{-1}yg}$

证明. $\forall g \in G$, $[x, y]^g = g^{-1}[x, y]g = [x^g, y^g]$. $\quad = (x^g)^{-1}(y^g)^{-1}x^g y^g = [x^g, y^g] \in H.$ □

Example 6. *1. Let $G = \langle a, b\rangle = D_{2n}$, where $o(a) = n$, $o(b) = 2$ and $a^b = a^{-1}$. Then* $G' = \begin{cases} \langle a\rangle, if\ o(a)\ odd, \\ \langle a^2\rangle, if\ o(a)\ even. \end{cases}$ *and* $G/G' = \begin{cases} \langle \bar{b}\rangle = C_2, if\ o(a)\ odd, \\ \langle \bar{a}, \bar{b}\rangle = C_2 \times C_2, if\ o(a)\ even. \end{cases}$

$\langle b \rangle$

Example 7. *Let G be abelian. Then $G' = 1$.*

Lemma 8. *G/G' is abelian.*

证明. Let $\bar{x}, \bar{y} \in \bar{G}$, let x, y be preimages of $\bar{x}, \bar{y}$ under $\pi : G \to \bar{G}$ respectively. Then by definition $[x.y] = x^{-1}y^{-1}xy \in G'$, hence $[\bar{x}, \bar{y}] = \bar{x}^{-1}\bar{y}^{-1}\bar{x}\bar{y} = \overline{x^{-1}y^{-1}xy} = \bar{1} \Rightarrow \bar{G}$ is abelian. □

homo.

$a, a^2, \cdots, a^n$

$b, ab, \cdots, a^{n-1}b$.

Lemma 9. *For any $H \lhd G$, G/H is abelian $\Leftrightarrow G' \leqslant H$.*

证明. ($\Leftarrow$) If $G' \leqslant H$, then $G/H \simeq \frac{G}{G'}/\frac{H}{G'}$ is abelian.

($\Rightarrow$) Assume G/H is abelian, then for any $\bar{x}, \bar{y} \in \bar{G} = G/H$, $\bar{x}\bar{y} = \bar{y}\bar{x}$ i.e. $\underset{\overline{[x,y]}}{[\bar{x}, \bar{y}]} = \bar{1}$, it shows $\forall x, y \in G$, $[x, y] \in H$. Thus $G' \leqslant H$. □

We can come to the conclusion that G' is the smallest subgroup of G s.t. G/G' is abelian.

Definition 10. *$G \rhd G' \rhd G'' \rhd \cdots \rhd G^{(n)} \rhd \ldots$ where $G^{(n)} = (G^{(n-1)})'$. Since G is finite, there exists n s.t. $G^{(n)} = G^{(n+1)} = \ldots$.*

Example 11. *If G is nonabelian simple, then $G = G'$.*

Definition 12. *If $G = G'$, then G is called a perfect group.*

$G^{(n)} = 1.$

Definition 13. *A finite group G is called a solvable group if $\cancel{G = G^{(n)}}$ for some n. Otherwise, G is called a nonsolvable group.*

Proposition 14. *A group G is solvable iff there exists a subgroup chain:*

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_s = 1$$

s.t. G_{i+1} is normal in G_i and G_i/G_{i+1} is abelian for all i.

证明. ($\Rightarrow$) obviously.

($\Leftarrow$) Assume $G = G_0 \rhd G_1 \rhd \cdots \rhd G_s = 1$ s.t. G_i/G_{i+1} is abelian.

Claim: $G^{(i)} \leqslant G_i$ for all i.

First, $G' \leqslant G_1$ as G/G_1 is abelian. Suppose $G^{(i)} \leqslant G_i$ for some $i \geqslant 1$. Since G_i/G_{i+1} is abelian, $(G_i)' \leqslant G_{i+1}$. Thus $G^{(i+1)} = (G^{(i)})' \leqslant (G_i)' \leqslant G_{i+1}$. □

Definition 15. *Let $x, g \in G$, consider the conjugation action $x^g = g^{-1}xg$. It induces an automorphism of G s.t. $x \mapsto g^{-1}xg$, called the inner automorphism induced by g.*