陈恺. 123140 Mathematical Statistics Assignment 4.

Part 1. Q.41 ~ Q.49

Q.41

Solution: $T \triangleq T_{welch} = \dfrac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$

$$\nu = \left(\frac{C^2}{n_1 - 1} + \frac{(1-C)^2}{n_2 - 1} \right)^{-1}, \quad C = \frac{S_1^2/n_1}{S_1^2/n_1 + S_2^2/n_2}. \quad 1 - C = \frac{S_2^2/n_2}{S_1^2/n_1 + S_2^2/n_2}.$$

$$T = \frac{(\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} = \frac{\dfrac{(\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}}{\dfrac{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}}$$

$$\frac{(\bar{X}_1 - \mu_1) - (\bar{X}_2 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} = \frac{\frac{1}{n_1}\sum_{i=1}^{n_1}(X_{1i} - \mu_1) - \frac{1}{n_2}\sum_{i=1}^{n_2}(X_{2i} - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \sim N(0,1)$$

$$\frac{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} = \frac{\sqrt{\dfrac{\sigma_1^2}{n_1(n_1-1)} \dfrac{(n_1-1)S_1^2}{\sigma_1^2} + \dfrac{\sigma_2^2}{n_2(n_2-1)} \dfrac{(n_2-1)S_2^2}{\sigma_2^2}}}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

$$\sim \frac{\sqrt{\dfrac{\sigma_1^2}{n_1(n_1-1)} \chi^2(n_1-1) + \dfrac{\sigma_2^2}{n_2(n_2-1)} \chi^2(n_2-1)}}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

Since $\dfrac{\sigma_1^2}{n_1(n_1-1)}$ and $\dfrac{\sigma_2^2}{n_2(n_2-1)}$ are generally unequal,

W is not exactly propotional to a chi-square r.v.

Approximete W by a scaled chi-squared r.v., say $c\chi^2(\nu_0)$

We identify c and ν_0 by matching the first two moments of W

and $c\chi^2(\nu_0)$.

$$E(W) = \frac{a_1^2}{n_1} + \frac{a_2^2}{n_2} = E(c\chi^2(\nu_0)) = c\nu_0$$

$$Var(W) = \frac{1}{n_1^2} Var(S_1^2) + \frac{1}{n_2^2} Var(S_2^2)$$

$$= 2\left(\frac{a_1^4}{n_1^2(n_1-1)} + \frac{a_2^4}{n_2^2(n_2-1)}\right) = Var(c\chi^2(\nu)) = c^2 \cdot 2\nu_0$$

$$\Rightarrow \quad c = \frac{\frac{a_1^4}{n_1^2(n_1-1)} + \frac{a_2^4}{n_2^2(n_2-1)}}{\frac{a_1^2}{n_1} + \frac{a_2^2}{n_2}}$$

$$\nu_0 = \frac{\left(\frac{a_1^2}{n_1} + \frac{a_2^2}{n_2}\right)^2}{\frac{a_1^4}{n_1^2(n_1-1)} + \frac{a_2^4}{n_2^2(n_2-1)}}$$

$$\Rightarrow \quad \frac{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}{\sqrt{a_1^2/n_1 + a_2^2/n_2}} = \sqrt{\frac{W}{\frac{a_1^2}{n_1} + \frac{a_2^2}{n_2}}} \sim \sqrt{\frac{\chi^2(\nu_0)}{\nu_0}}$$

$$\Rightarrow \quad \sqrt{\frac{W}{\frac{a_1^2}{n_1} + \frac{a_2^2}{n_2}}} \dot{\sim} \sqrt{\frac{\chi^2(\nu)}{\nu}}$$

with $\nu = \left(\frac{c^2}{n_1-1} + \frac{(1-c)^2}{n_2-1}\right)^{-1}$.

$$\Rightarrow \quad T_{welch} \dot{\sim} \frac{N(0,1)}{\sqrt{\frac{\chi^2(\nu)}{\nu}}} \sim t(\nu).$$

Q.4.2.

Solution:

(a) $1-\alpha \approx Pr\left\{-z_{\alpha/2} \le \frac{\sqrt{n}(\bar{X}_n - \lambda)}{a(\lambda)} \le z_{\alpha/2}\right\}$

$= Pr(L_1 \le \lambda \le U_1)$.

$$Pr\{-z_{\alpha/2} \leq \frac{\sqrt{n}(\bar{X}_n-\lambda)}{\sigma(\lambda)} \leq z_{\alpha/2}\}$$

$$= Pr\{-z_{0.025} \leq \frac{10\,(6.25-\lambda)}{\sqrt{\lambda}} \leq z_{0.025}\}$$

$$= Pr\{-1.96\sqrt{\lambda} \leq 62.5 - 10\lambda \leq 1.96\sqrt{\lambda}\}$$

$$= Pr\{5.78 \leq \lambda \leq 6.76\}.$$

$$\Rightarrow [L_1, U_1] = [5.78, 6.76].$$

(b)

$$1-\alpha \approx Pr\{-z_{1-\alpha+\alpha_2} \leq \frac{\sqrt{n}(\bar{X}_n-\lambda)}{\sigma(\lambda)} \leq z_{\alpha_2}\}.$$

$$= Pr\{-z_{1-\alpha+\alpha_2} \leq \frac{\sqrt{n}(\bar{X}_n-\lambda)}{\sqrt{\lambda}} \leq z_{\alpha_2}\} \xlongequal[\alpha_1\in[0,\alpha]]{\alpha_1+\alpha_2=\alpha} Pr(-z_{\alpha_1} \leq \frac{\sqrt{n}(\bar{X}_n-\lambda)}{\sqrt{\lambda}} \leq z_{\alpha-\alpha_1}).$$

To solve the inequality, we derive 2 cases:

① $-z_{\alpha_1} \leq \frac{\sqrt{n}(\bar{X}_n-\lambda)}{\sqrt{\lambda}} \leq 0.$

$$\Rightarrow \bar{X} \leq \lambda \leq \bar{X} + \frac{z_{\alpha_1}^2}{2n} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}.$$

② $0 \leq \frac{\sqrt{n}(\bar{X}_n-\lambda)}{\sqrt{\lambda}} \leq z_{\alpha-\alpha_1}$

$$\Rightarrow \bar{X} + \frac{z_{\alpha_1}^2}{2n} - z_{\alpha-\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha-\alpha_1}^2}{4n^2}} \leq \lambda \leq \bar{X}.$$

Take the union:

$$\bar{X} + \frac{z_{\alpha_1}^2}{2n} - z_{\alpha-\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha-\alpha_1}^2}{4n^2}} \leq \lambda \leq \bar{X} + \frac{z_{\alpha_1}^2}{2n} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}.$$

$\Rightarrow$ the length of $100(1-\alpha)\%$ CI for λ is

$$l(\alpha_1) = \frac{z_{\alpha_1}^2 - z_{\alpha-\alpha_1}^2}{2n} + z_{\alpha-\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha-\alpha_1}^2}{4n^2}} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}.$$

Thus, we could numerically find α_1^* (e.g. use grid-point method) s.t.

$$\alpha_1^* = \arg\min_{\alpha_1\in[0,\alpha]} l(\alpha_1)$$

$\Rightarrow$ the shortest $100(1-\alpha)\%$ CI for λ is

$$\left[\bar{X}+\frac{z_{\alpha_1^*}^2}{2n}-z_{\alpha-\alpha_1^*}\sqrt{\frac{\bar{X}}{n}+\frac{z_{\alpha-\alpha_1^*}^2}{4n^2}},\ \bar{X}+\frac{z_{\alpha_1^*}^2}{2n}+z_{\alpha_1^*}\sqrt{\frac{\bar{X}}{n}+\frac{z_{\alpha_1^*}^2}{4n^2}}\right].$$

Q.4.3

Solution:

(a) $1-\alpha=0.9 \quad \alpha=0.1 \quad \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma_0}\sim N(0,1).$

$$1-\alpha=Pr\left\{-z_{\alpha/2}\leq\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}\leq z_{\alpha/2}\right\}.$$

$$=Pr\left\{-z_{0.05}\leq\frac{2(0.375-\mu)}{3}\leq z_{0.05}\right\}$$

$$\approx Pr\left\{-1.65\leq\frac{0.75-2\mu}{3}\leq 1.65\right\}$$

$$=Pr\{2.1\leq\mu\leq 2.85\}$$

$\Rightarrow$ 90% CI is $[2.1, 2.85]$.

(b) $\frac{\sqrt{n}(\bar{X}-\mu)}{S}\sim t(n-1). \quad S_n=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2=3.863.$

$$1-\alpha=Pr\left(-t(\tfrac{\alpha}{2},n-1)\leq\frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n}\leq t(\tfrac{\alpha}{2},n-1)\right)$$

$$=Pr\left(-t(0.05,3)\leq\frac{2(0.375-\mu)}{3.863}\leq t(0.05,3)\right)$$

$$=Pr\left(\frac{0.75-3.863\,t(0.05,3)}{2}\leq\mu\leq\frac{0.75+3.863\,t(0.05,3)}{2}\right)$$

$\Rightarrow$ 90% CI is $\left[\frac{0.75-3.863\,t(0.05,3)}{2}, \frac{0.75+3.863\,t(0.05,3)}{2}\right]\approx[-1.94, 2.69]$.

Q.4.4.

Solution:

As argued in Q.4.3 or simply using (4.6).

the 90% CI of μ is :

$$\left[\bar{X}-t(0.05, n-1)\frac{S}{\sqrt{n}},\ \bar{X}+t(0.05, n-1)\frac{S}{\sqrt{n}}\right].$$

$$L = 2t(0.05, n-1)\frac{S}{\sqrt{n}}$$

$$\Rightarrow 0.95 = \Pr\left(L \le \frac{a}{5}\right)$$

$$= \Pr\left(2t(0.05, n-1)\frac{S}{\sqrt{n}} \le \frac{a}{5}\right)$$

$$= \Pr\left(4t^2(0.05, n-1)\frac{S^2}{n} \le \frac{a^2}{25}\right)$$

$$= \Pr\left(\frac{(n-1)S^2}{a^2} \le \frac{n(n-1)}{100\, t^2(0.05, n-1)}\right)$$

$$= \Pr\left(\chi^2(n-1) \le \frac{n(n-1)}{100\, t^2(0.05, n-1)}\right).$$

Meanwhile, $0.95 = 1 - 0.05 = \Pr\left(\chi^2(n-1) \ge \chi^2(0.05, n-1)\right)$

$$\Rightarrow \frac{n(n-1)}{100\, t^2(0.05, n-1)} = \chi^2(0.05, n-1)$$

The suitable n^* could be solved through numerical methods.

An intended solution is $n^* = 309$, while the error is minor.

Q.4.5

Solution: $A - B \sim N(\mu_A - \mu_B, \sigma_A^2 - \sigma_B^2)$.

Sample: 6, 8, -2, 2, 7, 11, 1, 13. $\bar{X} = 5.75$, $S = 5.12$.

Warning: A and B: 2 groups of samples are not indep.

We CANNOT use (4.8).

By (4.6), the $100(1-\alpha)\%$ CI for μ is

$$\left[\bar{X}-t(\tfrac{\alpha}{2}, n-1)\frac{S}{\sqrt{n}},\ \bar{X}+t(\tfrac{\alpha}{2}, n-1)\frac{S}{\sqrt{n}}\right] = [1.47, 10.03].$$

Q.4.6.

Solution:

(a) $f(x;\theta) = \theta x^{\theta-1}$. $\quad F(x;\theta) = \int_0^x \theta t^{\theta-1} dt = x^\theta$. $\quad x \in (0,1)$.

By (4.3),

$$-2\sum_{i=1}^n \log F(X_i;\theta) = -2\sum_{i=1}^n \theta \log X_i \sim \chi^2(2n)$$

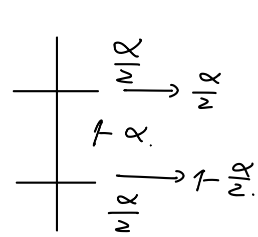
$\downarrow$

a pivot quantity.

$$1-\alpha = P_r\left(\chi^2(1-\tfrac{\alpha}{2}, 2n) \le -2\theta \sum_{i=1}^n \log X_i \le \chi^2(\tfrac{\alpha}{2}, 2n)\right)$$

$\log X_i < 0$, since $x \in (0,1)$.

$\chi^2(\alpha, \nu)$: $Pr\{\chi^2(\nu) > \chi^2(\alpha,\nu)\} = \alpha$.



$\Rightarrow$ $100(1-\alpha)\%$ CI is

$$\left[-\frac{\chi^2(1-\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^n \log X_i}, \; -\frac{\chi^2(\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^n \log X_i}\right].$$

(b)

$$l(\alpha) = \frac{\chi^2(1-\frac{\alpha}{2}, 2n) - \chi^2(\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^n \log X_i}.$$

α^* could be found numerically s.t

$$\alpha^* = \arg\min_\alpha \frac{\chi^2(1-\frac{\alpha}{2}, 2n) - \chi^2(\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^n \log X_i}.$$

$\Rightarrow$ the shortest $100(1-\alpha)\%$ CI for θ is

$$\left[-\frac{\chi^2(1-\frac{\alpha^*}{2}, 2n)}{2\sum_{i=1}^n \log X_i}, \; -\frac{\chi^2(\frac{\alpha^*}{2}, 2n)}{2\sum_{i=1}^n \log X_i}\right]$$

Q.4.7

Solution:

(a) By Example 4.1,

$$[L_p, U_p] = \left[\frac{\chi^2(1-\frac{\alpha}{2}, 2n)}{2n\bar{X}}, \frac{\chi^2(\frac{\alpha}{2}, 2n)}{2n\bar{X}} \right]$$

$$= \left[\frac{9.591}{22\times 55.087}, \frac{34.170}{22\times 55.087} \right] = [0.00871, 0.03101].$$

(b) Just take the bottom-top reverse

$$[L_p', U_p'] = \left[\frac{2n\bar{X}}{\chi^2(\frac{\alpha}{2}, 2n)}, \frac{2n\bar{X}}{\chi^2(1-\frac{\alpha}{2}, 2n)} \right] = [32.24766, 114.8106]$$

Q.4.8

Solution:

(a) Let $Y = \dfrac{\lambda(X-\mu)^2}{\mu^2 X}$.

$$E(g(Y)) = \int_0^{+\infty} g\left(\frac{\lambda(x-\mu)^2}{\mu^2 x}\right)\sqrt{\frac{\lambda}{2\pi}}\, x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2} dx$$

$$= \int_0^{\mu} + \int_{\mu}^{+\infty} g\left(\frac{\lambda(x-\mu)^2}{\mu^2 x}\right)\sqrt{\frac{\lambda}{2\pi}}\, x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2} dx$$

$$\overset{y=\frac{\lambda(x-\mu)^2}{\mu^2 x}}{=\!=\!=} \int_0^{+\infty} g(y)\frac{2^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \frac{\lambda(x_1-\mu)}{\mu x_1^2} \cdot \frac{\mu^2 x_1^2}{\lambda(x_1-\mu)(x_1+\mu)} dy$$

$$+ \int_0^{+\infty} g(y)\frac{2^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \frac{\lambda(x_2-\mu)}{\mu x_2^2} \cdot \frac{\mu^2 x_2^2}{\lambda(x_2-\mu)(x_2+\mu)} dy.$$

(x_1, x_2 are two roots of $y = \dfrac{\lambda(x-\mu)^2}{\mu^2 x} \Leftrightarrow x^2 - (2\mu + \dfrac{\mu^2 y}{\lambda})x + \mu^2 = 0$)

$$= \int_0^{+\infty} g(y)\frac{2^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \mu\left(\frac{1}{x_1+\mu} + \frac{1}{x_2+\mu}\right) dy.$$

$$\frac{1}{x_1+\mu} + \frac{1}{x_2+\mu} = \frac{x_1+x_2+2\mu}{x_1x_2+\mu(x_1+x_2)+\mu^2} = \frac{2\mu+\frac{\mu^2 y}{\lambda}+2\mu}{\mu^2+2\mu^2+\frac{\mu^3 y}{\lambda}+\mu^2} = \frac{1}{\mu}.$$

$$\Rightarrow E(g(Y)) = \int_0^{+\infty} g(y)\frac{2^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} dy.$$

By the expectation technique

$$f_Y(y) = \frac{2^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \quad \Rightarrow Y \sim \chi^2(1).$$

(b)

$$L(\mu,\lambda;\vec{x}) = \prod_{i=1}^{n} \sqrt{\frac{\lambda}{2\pi}} x_i^{-\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2}(x_i+\frac{\mu^2}{x_i}-2\mu)}$$

$$= \left(\sqrt{\frac{\lambda}{2\pi}}\right)^n e^{\left(-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}x_i - \frac{\lambda}{2}\sum_{i=1}^{n}x_i^{-1} + \frac{n\lambda}{\mu}\right)} \times \left(\prod_{i=1}^{n} x_i^{-\frac{3}{2}}\right).$$

$\Rightarrow \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^{-1}$ are jointly sufficient for (μ,λ).

(c).

$$L(\mu,\lambda_0;\vec{x}) = \left\{\left(\sqrt{\frac{\lambda_0}{2\pi}}\right)^n e^{\left(-\frac{\lambda_0}{2\mu^2}\sum_{i=1}^{n}x_i + \frac{n\lambda_0}{\mu}\right)}\right\} \times \left\{\left(\prod_{i=1}^{n} x_i^{-\frac{3}{2}}\right) e^{-\frac{\lambda_0}{2}\sum_{i=1}^{n}x_i^{-1}}\right\}.$$

$\Rightarrow \sum_{i=1}^{n} x_i$ is suifficient for μ.

(d). By Q.3.18(c),

$$\sum_{i=1}^{n} X_i \sim IG\left(n\mu, \frac{n\mu^3}{\lambda_0}\right). \qquad (\mu' = n\mu,\ \lambda' = n^2\lambda_0)$$

By (a),

$$\frac{n^2\lambda_0\left(\sum_{i=1}^{n}X_i - n\mu\right)^2}{n^2\mu^2\sum_{i=1}^{n}X_i} \sim \chi^2(1).$$

$1-\alpha = 0.95$, $\alpha = 0.05$.

$$1-\alpha = \Pr\left(\chi^2(1-\tfrac{\alpha}{2},1) \le \frac{n^2\lambda_0\left(\sum_{i=1}^{n}X_i - n\mu\right)^2}{n^2\mu^2\sum_{i=1}^{n}X_i} \le \chi^2(\tfrac{\alpha}{2},1)\right)$$

$$= \Pr\left(\frac{\mu^2}{\lambda_0}\chi^2(1-\tfrac{\alpha}{2},1)\sum_{i=1}^{n}X_i \le n^2\mu^2 - 2n\left(\sum_{i=1}^{n}X_i\right)\mu + \left(\sum_{i=1}^{n}X_i\right)^2 \le \frac{\mu^2}{\lambda_0}\chi^2(\tfrac{\alpha}{2},1)\sum_{i=1}^{n}X_i\right).$$

For $\frac{\mu^2}{\lambda_0}\chi^2(1-\frac{\alpha}{2},1)\sum_{i=1}^{n}X_i \le n^2\mu^2 - 2n\left(\sum_{i=1}^{n}X_i\right)\mu + \left(\sum_{i=1}^{n}X_i\right)^2$.

$$\Leftrightarrow \left(\frac{\chi^2(1-\frac{\alpha}{2},1)\sum_{i=1}^{n}X_i}{\lambda_0} - n^2\right)\mu^2 + 2n\left(\sum_{i=1}^{n}X_i\right)\mu - \left(\sum_{i=1}^{n}X_i\right)^2 \le 0.$$

$$\Leftrightarrow \left(\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}\right)\mu^2 + 2n\mu - n\bar{X} \le 0.$$

$$\Leftrightarrow \frac{-n-\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}} \le \mu \le \frac{-n+\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}}.$$

For $n^2\mu^2 - 2n(\sum_{i=1}^n X_i)\mu + (\sum_{i=1}^n X_i)^2 \le \frac{\mu^2}{\lambda_0}\chi^2(\frac{\alpha}{2},1)\sum_{i=1}^n X_i$

$$\Leftrightarrow \left(\frac{\chi^2(\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}\right)\mu^2 - 2n\mu - n\bar{X} \ge 0.$$

$$\Leftrightarrow \mu \le \frac{n-\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}} \text{ or } \mu \ge \frac{n+\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}}.$$

The equal-tail 95% CI of μ is

$$\left[\frac{-n-\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}}, \frac{-n+\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}}\right] \cap \left\{\left(-\infty, \frac{n-\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}}\right] \cup \left[\frac{n+\sqrt{n\bar{X}\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0}}}{\frac{\chi^2(1-\frac{\alpha}{2},1)}{\lambda_0} - \frac{n}{\bar{X}}}, +\infty\right)\right\} \cap (0,+\infty)$$

$\alpha = 0.05$.

Q.4.9.

Solution:

(a) $Pr(X=x) = Pr(X_1 + X_2 = x)$

$$= \sum_{k=0}^{\min(m,x)} Pr(X_1+X_2 = x \mid X_1 = k)\, Pr(X_1 = k)$$

$$\overset{X_1 \perp\!\!\!\perp X_2}{=} \sum_{k=0}^{\min(m,x)} Pr(X_2 = x-k)\, Pr(X_1 = k)$$

$$= \sum_{k=0}^{\min(m,x)} \binom{m}{k} p^k (1-p)^{m-k} \cdot \frac{\lambda^{x-k} e^{-\lambda}}{(x-k)!}. \qquad x = 0, 1, 2, \ldots\ldots$$

(b) $E(X) = E(X_1+X_2) = E(X_1) + E(X_2) = mp + \lambda.$

$$Var(X) = Var(X_1+X_2) \underset{X_1 \perp\!\!\!\perp X_2}{=} Var(X_1) + Var(X_2) = mp(1-p)+\lambda.$$

(c) $\mu \triangleq mp + \lambda = E(X)$, $Var(X) = \mu - mp^2 \triangleq \sigma^2(\mu)$.

By the central limit theorem,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma(\mu)} \;\dot{\sim}\; N(0,1).$$

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma(\mu)} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu - mp^2}}$$

$$\Rightarrow 1-\alpha \approx \Pr\left\{-z_{\alpha/2} \le \frac{\sqrt{n}(\bar{X}_n-\mu)}{\sqrt{\mu-mp^2}} \le z_{\alpha/2}\right\}.$$

$$= \Pr\left\{\frac{2n\bar{X}_n + z_{\alpha/2}^2 - \sqrt{(2n\bar{X}_n + z_{\alpha/2}^2) - 4n(n\bar{X}^2 + z_{\alpha/2}^2 mp^2)}}{2n} \le \mu \le \frac{2n\bar{X}_n + z_{\alpha/2}^2 + \sqrt{(2n\bar{X}_n + z_{\alpha/2}^2) - 4n(n\bar{X}^2 + z_{\alpha/2}^2 mp^2)}}{2n}\right\}$$

$\Rightarrow$ An approximate $100(1-\alpha)\%$ CI for the mean μ is

$$\left[\frac{2n\bar{X}_n + z_{\alpha/2}^2 - \sqrt{(2n\bar{X}_n + z_{\alpha/2}^2) - 4n(n\bar{X}^2 + z_{\alpha/2}^2 mp^2)}}{2n}, \frac{2n\bar{X}_n + z_{\alpha/2}^2 - \sqrt{(2n\bar{X}_n + z_{\alpha/2}^2) - 4n(n\bar{X}^2 + z_{\alpha/2}^2 mp^2)}}{2n}\right].$$

Q.4.10

Solution:

(a) $F(x) = \Pr(X \le x)$

$$= \int_{-\infty}^{x} \frac{1}{a_0} e^{-\frac{t-\mu}{a_0}} \exp\left(-e^{-\frac{t-\mu}{a_0}}\right) dt.$$

$$= \exp\left(-e^{-\frac{t-\mu}{a_0}}\right)\Bigg|_{t=-\infty}^{t=x} = \exp\left(-e^{-\frac{x-\mu}{a_0}}\right) \qquad (x \in \mathbb{R}).$$

(b) By (4.3),

$$-2\sum_{i=1}^{n} \log F(X_i; \mu) \sim \chi^2(2n)$$

$$= -2\sum_{i=1}^{n}\left(-e^{-\frac{X_i-\mu}{a_0}}\right) = 2\sum_{i=1}^{n} e^{-\frac{X_i-\mu}{a_0}} = 2e^{\frac{\mu}{a_0}}\sum_{i=1}^{n} e^{-\frac{X_i}{a_0}}$$

$$1-\alpha = \Pr\left(\chi^2(1-\tfrac{\alpha}{2}, 2n) \le 2e^{\frac{\mu}{a_0}}\sum_{i=1}^{n} e^{-\frac{X_i}{a_0}} \le \chi^2(\tfrac{\alpha}{2}, 2n)\right)$$

$$= \Pr\left(a_0 \log\frac{\chi^2(1-\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^{n} e^{-\frac{X_i}{a_0}}} \le \mu \le a_0 \log\frac{\chi^2(\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^{n} e^{-\frac{X_i}{a_0}}}\right)$$

$\Rightarrow$ The $100(1-\alpha)\%$ CI for μ is

$$\left[a_0 \log\frac{\chi^2(1-\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^{n} e^{-\frac{X_i}{a_0}}},\ a_0 \log\frac{\chi^2(\frac{\alpha}{2}, 2n)}{2\sum_{i=1}^{n} e^{-\frac{X_i}{a_0}}}\right].$$

Q.4.11

Solution:

By Q.3.20 (a), $X_i \sim f(x; a^2) = \frac{x}{a^2}\exp\left(-\frac{x^2}{2a^2}\right)$.

$$\Rightarrow X_i^2 \sim \text{Exponential}\left(\frac{1}{2a^2}\right).$$

By Example 4.1,

$$2\frac{1}{2a^2}\sum_{i=1}^{n} X_i^2 \sim \chi^2(2n).$$

$$\Rightarrow 1-\alpha = \Pr\left\{\chi^2(1-\tfrac{\alpha}{2}, 2n) \le \frac{1}{a^2}\sum_{i=1}^{n} X_i^2 \le \chi^2(\tfrac{\alpha}{2}, 2n)\right\}.$$

$$= \Pr\left\{\frac{\sum_{i=1}^{n} X_i^2}{\chi^2(\frac{\alpha}{2}, 2n)} \le a^2 \le \frac{\sum_{i=1}^{n} X_i^2}{\chi^2(1-\frac{\alpha}{2}, 2n)}\right\}.$$

$$\Rightarrow \left[\frac{\sum_{i=1}^{n} X_i^2}{\chi^2(\frac{\alpha}{2}, 2n)},\ \frac{\sum_{i=1}^{n} X_i^2}{\chi^2(1-\frac{\alpha}{2}, 2n)}\right] \text{ is a } 100(1-\alpha)\% \text{ CI for } a^2.$$

Q.4.12

Solution:

(a) $$L(\vec{X}; \theta) = \frac{1}{\theta^n}\cdot\prod_{i=1}^{n} 1(0 \le X_i \le \theta)$$

$$= \left\{\frac{1}{\theta^n}\cdot 1(X_{(n)} \le \theta)\right\} \times 1(X_{(1)} \ge 0). \Rightarrow X_{(n)} \text{ is sufficient.}$$

(b). $L(\vec{X};\theta) = \left\{ \frac{1}{\theta^n} \cdot 1(U \leq 1) \right\} \times 1(X_{(1)} \geq 0) \Rightarrow U$ is sufficient.

By (2.17), the p.d.f of $X_{(n)}$ is

$$n f(x;\theta) F^{n-1}(x) = n \frac{1}{\theta} 1(0 \leq x \leq \theta) \frac{x^{n-1}}{\theta^{n-1}}$$

$\Rightarrow$ the p.d.f of U is

$$n \frac{1}{\theta} 1(0 \leq U \leq 1) U^{n-1} \cdot \theta = n \cdot 1(0 \leq U \leq 1) \cdot U^{n-1}$$

does NOT depend on θ.

$\Rightarrow U$ is a pivotal quantity.

(c)

$$\frac{\alpha}{2} = Pr(U \geq h_{\alpha/2}) = \int_{h_{\frac{\alpha}{2}}}^{1} n u^{n-1} du = 1 - h_{\frac{\alpha}{2}}^{n} \Rightarrow h_{\frac{\alpha}{2}} = \sqrt[n]{1-\frac{\alpha}{2}}.$$

$$1-\frac{\alpha}{2} = Pr(U \geq h_{1-\frac{\alpha}{2}}) = \int_{h_{1-\frac{\alpha}{2}}}^{1} n u^{n-1} du = 1 - h_{1-\frac{\alpha}{2}}^{n} \Rightarrow h_{1-\frac{\alpha}{2}} = \sqrt[n]{\frac{\alpha}{2}}.$$

(d) $1-\alpha = Pr(h_{1-\frac{\alpha}{2}} \leq U \leq h_{\frac{\alpha}{2}})$

$$= Pr\left(\frac{X_{(n)}}{\sqrt[n]{1-\frac{\alpha}{2}}} \leq \theta \leq \frac{X_{(n)}}{\sqrt[n]{\frac{\alpha}{2}}}\right)$$

$\Rightarrow$ the $100(1-\alpha)\%$ equal-tail CI for θ is $\left[\frac{X_{(n)}}{\sqrt[n]{1-\frac{\alpha}{2}}}, \frac{X_{(n)}}{\sqrt[n]{\frac{\alpha}{2}}}\right]$.

(e).

the 95% equal-tail CI for θ is

$$\left[\frac{4.2}{\sqrt[5]{0.975}}, \frac{4.2}{\sqrt[5]{0.025}}\right] = [4.221, 8.783].$$

(f) Set $1-\alpha = Pr\left(\frac{X_{(n)}}{a} \leq \theta \leq \frac{X_{(n)}}{b}\right)$.

$$\alpha_1 = Pr\left(\frac{X_{(n)}}{\theta} \geq a\right) = \int_a^1 n u^{n-1} du = 1 - a^n \Rightarrow a = (1-\alpha_1)^{\frac{1}{n}}.$$

$$\alpha - \alpha_1 = Pr\left(\frac{X_{(n)}}{\theta} \leq b\right) = b^n \Rightarrow b = (\alpha-\alpha_1)^{\frac{1}{n}}.$$

$\alpha_1 \in [0, \alpha]$

$\Rightarrow$ the length of CI is

$$l(\alpha_1) = X_{(n)} \left(\frac{1}{(\alpha-\alpha_1)^{\frac{1}{n}}} - \frac{1}{(1-\alpha_1)^{\frac{1}{n}}} \right) > 0 \text{ , } \forall \alpha_1.$$

$$\Rightarrow \quad \alpha_1^* = \underset{\alpha_1}{\arg\min}\ l(\alpha_1) = 0.$$

$\Rightarrow$ the shortest $100(1-\alpha)\%$ CI is $\left[X_{(n)},\ X_{(n)}\,\alpha^{-\frac{1}{n}} \right]$.