MA204: Mathematical Statistics—Midterm Test

(7:50am–9:50am, 17 May 2025)

1. (**40 Marks**). Give your answers to the following questions:

1.1 Let $f(x, y)$ and $F(x, y)$ denote the joint *probability density function* (pdf) and *cumulative distribution function* (cdf) of the random vector (X, Y), $f(x)$ and $f(y)$ be their marginal pdfs, and $F(x)$ and $F(y)$ be their marginal cdfs. The *random variables* (r.v.'s) X and Y are said to be independent, denoted by $X \perp\!\!\!\perp Y$, if

$$f(x,y) \;=\; f(x) \times f(y), \quad \forall\,(x,y) \in \mathcal{S}_{(X,Y)}, \quad \text{or} \tag{1.1}$$

$$F(x,y) \;=\; F(x) \times F(y), \quad \forall\,(x,y) \in \mathcal{S}_{(X,Y)}, \tag{1.2}$$

where $\mathcal{S}_{(X,Y)} \triangleq \{(x,y)\colon f(x,y) > 0\}$ denotes the joint support of (X, Y). By comparing equations (1.1) and (1.2), which one is better to define the independency of X and Y? Why? [**2ms**]

1.2 Let the r.v. $X \sim \text{Gamma}(\alpha, \beta)$ with pdf $\beta^\alpha x^{\alpha-1}\,\mathrm{e}^{-\beta x}/\Gamma(\alpha)$, where $\alpha > 0$ and $\beta > 0$. The support of X is ______. [**2ms**]

1.3 Let $X_i \sim \text{Exponential}(\beta_i)$ for $i = 1, \dots, n$, and $\{X_i\}_{i=1}^n$ be independent. Then then cdf of $X_{(1)} = \min(X_1, \dots, X_n)$ is ______. [**2ms**]

1.4 To define a multivariate normal distribution, what is the advantage of using the *stochastic representation* (SR) rather than the joint pdf? [**2ms**]

1.5 Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$, $\bar{X} = (1/n)\sum_{i=1}^n X_i$ be the sample mean, then its *moment generating function* (mgf) $M_{\bar{X}}(t) =$______. [**2ms**]

1.6 Let the continuous r.v. X follow the logistic distribution with pdf

$$f_X(x) = \frac{\exp(-\frac{x-\mu}{\sigma})}{\sigma\{1 + \exp(-\frac{x-\mu}{\sigma})\}^2}, \quad x \in \mathbb{R} \hat{=} (-\infty, \infty),\ \mu \in \mathbb{R},\ \sigma > 0.$$

Given a $q \in (0, 1)$, the q-th quantile ξ_q of the X is ______. [**2ms**]

1.7 Let $X_1, X_2 \stackrel{\text{iid}}{\sim} U(0,1)$, the pdf of $Y = X_1X_2$ is ______. **[2ms]**

1.8 Let X follow the inverse gamma distribution, denoted by $X \sim \text{IGamma}(\alpha, \beta)$, if its pdf is $\beta^\alpha x^{-(\alpha+1)}\, \mathrm{e}^{-\beta/x}/\Gamma(\alpha)$, where $x > 0, \alpha > 0, \beta > 0$. Furthermore, let c be a positive constant, then the distribution of $Y = cX$ is ______. **[2ms]**

1.9 Let $X_i \sim \text{Poisson}(\lambda_i)$ for $i = 1, 2$ and $X_1 \perp\!\!\!\perp X_2$. What is the conditional distribution of $X_1|(X_1 + X_2 = n)$? **[2ms]**

1.10 Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} f(x;\theta)$ with $f(x;\theta) = \mathrm{e}^{-(x-\theta)}$ for $x \geqslant \theta$ and $\theta > 0$. Find the *maximum likelihood estimator* (MLE) and the moment estimator of θ? **[4ms]**

1.11 Let $X_1, \ldots, X_{18} \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$, what is the distribution of

$$Y = \frac{X_1^2 + X_2^2 + \cdots + X_{12}^2}{2(X_{13}^2 + X_{14}^2 + \cdots + X_{18}^2)}\ ?$$ **[2ms]**

1.12 Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$, where μ and σ^2 are unknown mean and variance parameters. Calculate the *mean square error* (MSE) of the MLE $\hat\sigma^2 = (1/n)\sum_{i=1}^n (X_i - \bar X)^2$ of σ^2. **[2ms]**

1.13 What is the relationship between an efficient estimator of θ and an *unified minimum variance unbiased estimator* (UMVUE) of θ? **[2ms]**

1.14 Let the pdf of the r.v. X be $f(x;\theta) = \theta a^\theta x^{-(\theta+1)}$ for $x > a$, $a > 0$ and $\theta > 0$, find the Fisher information $I(\theta)$? **[2ms]**

1.15 Let X be a discrete r.v. with *probability mass function* (pmf) $p_i = \Pr(X = x_i)$ for $i = 1, 2$ and Y be a discrete random variable with pmf $q_j = \Pr(Y = y_j)$ for $j = 1, 2$. Given two conditional distribution matrices

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 3/5 & 2/5 \end{pmatrix},$$

where the (i,j) element of $\boldsymbol{A}$ is $a_{ij} = \Pr(X = x_i|Y = y_j)$ and the (i,j) element of $\boldsymbol{B}$ is $b_{ij} = \Pr(Y = y_j|X = x_i)$. Find the marginal distribution of X. **[2ms]**

1.16 Let $X_1, \ldots, X_n \sim N(\mu, 3.3^2)$ with $n = 30$ and $\bar{x} = 27$.

(a) What is the sufficient statistic of μ? **[2ms]**

(b) What is the pivotal quantity? **[2ms]**

(c) Construct a 90% CI for μ, where $z_{0.05} = 1.645$. **[2ms]**

1.17 Assume we want to find the root x^* of the equation $0 = g(x)$ for $x \in \mathbb{X}$. What is Newton's method to iteratively calculate the root x^* ? **[2ms]**

2. (**20 Marks**).

(a) Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$, where $0 < \theta < 1$. Show that $T = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ. **[10ms]**

(b) Let $X_1, X_2, X_3 \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$, find the pmf of $T_{123} = X_1 X_2 + X_3$. **[5ms]**

(c) Show that T_{123} is not a sufficient statistic for θ. **[5ms]**

3. (**25 Marks**). Let $X_1, \ldots, X_n$ be a random sample from the population r.v. X with pdf $f(x; \theta) = \theta x^{\theta-1}$, where $\theta > 0$ and $0 < x < 1$.

(a) Let the prior distribution of θ be Gamma(a, b), where $a(> 0)$ and $b(> 0)$ are known constants. Find the posterior distribution of θ and the Bayesian estimator of θ. **[5ms]**

(b) Find a sufficient statistic of θ. **[5ms]**

(c) Find the MLE of $\tau(\theta) = 1/\theta$. **[5ms]**

(d) Let $Y = -\log(X)$, show that $Y \sim \text{Exponential}(\theta) = \text{Gamma}(1, \theta)$, and find $E(Y)$ and $\text{Var}(Y)$. **[5ms]**

(e) Find the efficient estimator of $\tau(\theta)$. **[5ms]**

4. (**15 Marks**). Let $X_1, \ldots, X_{n_1} \sim N(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2} \sim N(\mu_2, \sigma_2^2)$ with $n_1 = 18$, $\bar{x} = 13.5$, $s_1 = 5$ and $n_2 = 12$, $\bar{y} = 9.5$, $s_2 = 6$.

(a) Show that $f(1 - \alpha/2, v_1, v_2) = f^{-1}(\alpha/2, v_2, v_1)$, where $f(\alpha, v_2, v_1)$ denotes the upper α-quantile of the $F(v_2, v_1)$ distribution. **[5ms]**

(b) Construct a 95% CI for σ_1/σ_2. **[5ms]**

(c) Let $\sigma_1 = \sigma_2 = \sigma$ be unknown, construct a 95% CI for $\mu_1 - \mu_2$. [**5ms**]

[Hint: $f(0.025, 11, 17) = 2.8696$, $f(0.025, 17, 11) = 3.2816$ and $t(0.025, 28)$ $= 2.0484$, where $t(\alpha, n)$ denotes the upper α-quantile of the $t(n)$ distribution.]

=== **END OF THE PAPER** ===

1. Solution.

1.1 The equation (1.2) is better than (1.1) to define the independency of X and Y, because the joint cdf $F(x, y)$ always exists while the joint pdf $f(x, y)$ may not exist.

1.2 (i) When $\alpha = 1$, $\text{Gamma}(\alpha, \beta) = \text{Exponential}(\beta)$ with density $\beta\,\mathrm{e}^{-\beta x}$. Its support is $[0, \infty)$.

(ii) When $\alpha \neq 1$, the support of $X \sim \text{Gamma}(\alpha, \beta)$ is $(0, \infty)$.

1.3 $X_{(1)} \sim \text{Exponential}(\beta_+)$ **with** $\beta_+ = \sum_{i=1}^n \beta_i$.

<u>**Solution**</u>: The pdf of $X_i \sim \text{Exponential}(\beta_i)$ is $\beta_i\,\mathrm{e}^{-\beta_i x_i}$, so that its cdf is

$$F_i(x) = \Pr(X_i \leqslant x) = \int_0^x \beta_i\,\mathrm{e}^{-\beta_i t}\,\mathrm{d}t = 1 - \mathrm{e}^{-\beta_i x}.$$

Let $G_1(x)$ denote the cdf of the first order statistic $X_{(1)}$, then

$$\begin{aligned} G_1(x) &= \Pr(X_{(1)} \leqslant x) = 1 - \Pr\{\min(X_1, \dots, X_n) > x\} \\ &= 1 - \Pr(X_1 > x, \dots, X_n > x) \\ &= 1 - \prod_{i=1}^n \Pr(X_i > x) = 1 - \prod_{i=1}^n [1 - F_i(x)] \\ &= 1 - \prod_{i=1}^n \mathrm{e}^{-\beta_i x} = 1 - \mathrm{e}^{-\beta_+ x}, \end{aligned}$$

indicating that $X_{(1)} \sim \text{Exponential}(\beta_+)$ with $\beta_+ = \sum_{i=1}^n \beta_i$.

1.4 The variance-covariance matrix is not necessarily positive definite if we use the SR method to define a multivariate normal distribution. The variance-covariance matrix must be positive definite if we use the joint density method to define a multivariate normal distribution.

1.5 $M_{\bar{X}}(t) = \exp[\mu t + 0.5(\sigma^2/n)t^2]$.

Proof: The mgf of $\bar{X}$ is given by

$$\begin{aligned} M_{\bar{X}}(t) &= M_{\sum_{i=1}^{n} X_i/n}(t) = \prod_{i=1}^{n} M_{X_i/n}(t) = \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) \\ &= \left\{M_{X_1}\left(\frac{t}{n}\right)\right\}^n = \left\{\exp\left(\mu\frac{t}{n} + 0.5\sigma^2\frac{t^2}{n^2}\right)\right\}^n \\ &= \exp\left[\mu t + 0.5\left(\frac{\sigma^2}{n}\right)t^2\right]. \end{aligned}$$

1.6 The cdf of $X \sim \text{Logistic}(\mu, \sigma^2)$ with density

$$f_X(x) = \frac{\exp(-\frac{x-\mu}{\sigma})}{\sigma\{1+\exp(-\frac{x-\mu}{\sigma})\}^2}, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \sigma > 0,$$

is given by

$$F(x) = \left[1 + \exp\left(-\frac{x-\mu}{\sigma}\right)\right]^{-1}.$$

Based on the definition of the q-th quantile, we have $F(\xi_q) = q \in (0,1)$, so

$$\xi_q = F^{-1}(q) = \mu + \sigma \log\left(\frac{q}{1-q}\right).$$

1.7 $\mathcal{S}_{X_1} = \mathcal{S}_{X_2} = \mathcal{S}_Y = (0,1)$. The conditional distribution of $Y|(X_2 = x_2)$ is

$$Y|(X_2 = x_2) = x_2 \cdot X_1 \sim U(0, x_2), \quad 0 < x_2 < 1,$$

i.e.,

$$f_{(Y|X_2)}(y|x_2) = \frac{1}{x_2} \cdot I(0 < y < x_2), \quad 0 < x_2 < 1.$$

Hence, we have

$$\begin{aligned} f_Y(y) &= \int_{\mathcal{S}_{X_2}} f_{X_2}(x_2) \cdot f_{(Y|X_2)}(y|x_2)\,\mathrm{d}x_2 \\ &= \int_0^1 1 \cdot I(0 < x_2 < 1) \times \frac{1}{x_2} \cdot I(0 < y < x_2)\,\mathrm{d}x_2 \\ &= \int_y^1 \frac{1}{x_2}\,\mathrm{d}x_2 = \log(x_2)\Big|_y^1 \\ &= -\log(y), \quad 0 < y < 1. \end{aligned}$$

1.8 $Y = cX \sim \text{IGamma}(\alpha, c\beta)$.

Proof. Method I: Based on the result that $X \sim \text{IGamma}(\alpha, \beta)$ if and only if $X^{-1} \sim \text{Gamma}(\alpha, \beta)$. Thus, from $Y = cX$, we have

$$\begin{aligned} & Y^{-1} = c^{-1}X^{-1} \sim c^{-1}\text{Gamma}(\alpha, \beta) = \text{Gamma}(\alpha, c\beta) \\ \Leftrightarrow \quad & Y \sim \text{IGamma}(\alpha, c\beta). \end{aligned}$$

Method II: Using the transformation method, we obtain

$$\begin{aligned} f_Y(y) &= f_X(x) \cdot \left| \frac{\mathrm{d}x}{\mathrm{d}y} \right| = f_X(x) \cdot \frac{1}{c} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} \, \mathrm{e}^{-\beta/x} \cdot \frac{1}{c} = \frac{\beta^\alpha}{\Gamma(\alpha)} (y/c)^{-(\alpha+1)} \, \mathrm{e}^{-c\beta/y} \cdot \frac{1}{c} \\ &= \frac{(c\beta)^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} \, \mathrm{e}^{-(c\beta)/y}, \end{aligned}$$

implying $Y \sim \text{IGamma}(\alpha, c\beta)$.

1.9 $X_1|(X_1 + X_2 = n) \sim$ **Binomial**(n, p)**, where** $p = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$.

Proof: Since $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$, the conditional distribution of $X_1 = i|(X_1 + X_2 = n)$ is

$$\begin{aligned} & \Pr(X_1 = i | X_1 + X_2 = n) \\ = & \frac{\Pr(X_1 = i, X_1 + X_2 = n)}{\Pr(X_1 + X_2 = n)} \\ = & \frac{\Pr(X_1 = i) \Pr(X_2 = n - i)}{\Pr(X_1 + X_2 = n)} \\ = & \frac{\dfrac{\lambda_1^i}{i!} \mathrm{e}^{-\lambda_1} \dfrac{\lambda_2^{n-i}}{(n-i)!} \mathrm{e}^{-\lambda_2}}{\dfrac{(\lambda_1 + \lambda_2)^n}{n!} \mathrm{e}^{-(\lambda_1 + \lambda_2)}} \\ = & \binom{n}{i} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^i \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n-i}, \quad i = 0, 1, \ldots n. \end{aligned}$$

1.10 (i) The likelihood function

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \mathrm{e}^{-(x_i-\theta)} \cdot I_{[\theta,\infty)}(x_i) = \mathrm{e}^{-\sum_{i=1}^{n} x_i + n\theta} \prod_{i=1}^{n} I_{[\theta,\infty)}(x_i) \\ &= \mathrm{e}^{-n\bar{x}+n\theta} \cdot I_{[\theta,\infty)}(x_{(1)}) = \mathrm{e}^{-n\bar{x}+n\theta} \cdot I_{(0,x_{(1)}]}(\theta) \end{aligned}$$

Note that $L(\theta)$ is an increasing function of θ. When $\theta = x_{(1)}$, $L(\theta)$ reaches its maximum. Thus, the MLE of θ is $X_{(1)}$.

(ii) Let $y = x - \theta$, we obtain

$$E(X) = \int_{\theta}^{\infty} x\, \mathrm{e}^{-(x-\theta)}\, \mathrm{d}x = \int_{0}^{\infty} (y+\theta)\, \mathrm{e}^{-y}\, \mathrm{d}y = 1 + \theta.$$

The moment estimator of θ must satisfy

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = E(X) = 1 + \theta.$$

We have $\hat{\theta}^M = \bar{X} - 1$.

1.11 Since

$$Y_1 \hat{=} \frac{\sum_{i=1}^{12} X_i^2}{\sigma^2} \sim \chi^2(12), \quad Y_2 \hat{=} \frac{\sum_{i=13}^{18} X_i^2}{\sigma^2} \sim \chi^2(6),$$

and $Y_1 \perp\!\!\!\perp Y_2$, we have $Y = (Y_1/12)/(Y_2/6) \sim F(12, 6)$.

1.12 The MSE of $\hat{\sigma}^2$ is given by

$$\mathrm{MSE}(\hat{\sigma}^2) = \mathrm{Var}(\hat{\sigma}^2) + [E(\hat{\sigma}^2) - \sigma^2]^2.$$

Thus, we only need to find $E(\hat{\sigma}^2)$ and $\mathrm{Var}(\hat{\sigma}^2)$. Note that

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{n-1}{n} S^2,$$

where S^2 is the sample variance. Since

$$\frac{n\hat{\sigma}^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1),$$

we obtain

$$E\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = n - 1 \quad \text{and} \quad \mathrm{Var}\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = 2(n-1),$$

we have

$$E(\hat{\sigma}^2) = \frac{(n-1)\sigma^2}{n} \quad \text{and} \quad \mathrm{Var}(\hat{\sigma}^2) = \frac{2(n-1)\sigma^4}{n^2},$$

so that

$$\mathrm{MSE}(\hat{\sigma}^2) = \frac{2(n-1)\sigma^4}{n^2} + \left[\frac{(n-1)\sigma^2}{n} - \sigma^2\right]^2 = \frac{2n-1}{n^2}\sigma^4.$$

1.13 See **21.1• Efficient estimator versus UMVUE** on page 134 and **21.3• Is the UMVUE unique?** on page 138.

(i) An efficient estimator for θ is a UMVUE for θ; i.e.,

$$\text{efficient estimator} \implies \text{UMVUE}.$$

1.14 The Fisher information is $I(\theta) = \dfrac{1}{\theta^2}$.

Solution: From the pdf of X, we have

$$\log f(X;\theta) = \log\theta + \theta\log a - (\theta+1)\log X.$$

Then the first derivative is

$$\frac{\mathrm{d}\log f(X;\theta)}{\mathrm{d}\theta} = \frac{1}{\theta} + \log a - \log X.$$

The second derivative is

$$\frac{\mathrm{d}^2\log f(X;\theta)}{\mathrm{d}\theta^2} = -\frac{1}{\theta^2}.$$

So the Fisher information

$$I(\theta) = E\left\{-\frac{\mathrm{d}^2\log f(X;\theta)}{\mathrm{d}\theta^2}\right\} = \frac{1}{\theta^2}.$$

1.15 Note that $\mathcal{S}_X = \{x_1, x_2\}$ and $\mathcal{S}_Y = \{y_1, y_2\}$. By using point-wise IBF, the marginal distribution of X is given by

X	x_1	x_2
$p_i = \Pr(X = x_i)$	$3/8$	$5/8$

1.16 $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ is a sufficient statistic of μ.

The pivotal quantity is

$$P = \frac{\sqrt{n}(\bar{X} - \mu)}{3.3} \sim N(0, 1).$$

A $100(1-\alpha)\%$ equal-tail CI for μ can be constructed as

$$\begin{aligned}
&\Pr\left\{-z_{\alpha/2} \leqslant \frac{\sqrt{n}(\bar{X} - \mu)}{3.3} \leqslant z_{\alpha/2}\right\} = 1 - \alpha, \\
\Rightarrow\ &\Pr\left(\bar{X} - z_{\alpha/2}\frac{3.3}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} + z_{\alpha/2}\frac{3.3}{\sqrt{n}}\right) = 1 - \alpha, \\
\Rightarrow\ &\Pr\left(\bar{X} - z_{0.05}\frac{3.3}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} + z_{0.05}\frac{3.3}{\sqrt{n}}\right) = 0.9.
\end{aligned}$$

Therefore, a 90% CI for μ is given by

$$\left[27 - 1.645\frac{3.3}{\sqrt{30}},\ 27 + 1.645\frac{3.3}{\sqrt{30}}\right] = [26.0089,\ 27.9911]\,.$$

1.17 Newton's method to iteratively calculate the root x^* of the equation $g(x) = 0$ is

$$x^{(t+1)} = x^{(t)} - \frac{g(x^{(t)})}{g'(x^{(t)})}, \qquad t = 0, 1, 2, \ldots, \infty.$$

2. Solution.

(a) See **Example 3.28** on page 150 of the textbook "Math Statistics".

The joint pmf of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^t(1-\theta)^{n-t},$$

where $t = \sum_{i=1}^{n} x_i$. By using the factorization theorem, we know that $T = \sum_{i=1}^{n} X_i$ **is sufficient**, and $T \sim \text{Binomial}(n, \theta)$.

Now assume that a function $h(T)$ satisfies

$$E\{h(T)\} = \sum_{t=0}^{n} h(t)\Pr(T=t) = \sum_{t=0}^{n} h(t)\binom{n}{t}\theta^t(1-\theta)^{n-t} = 0, \quad \text{(MT.1)}$$

for $0 < \theta < 1$. Let $y = \theta/(1-\theta)$, then (MT.1) becomes

$$\sum_{t=0}^{n} h(t)\binom{n}{t} y^t = 0, \qquad y > 0.$$

A polynomial is identical to zero, then all coefficients are zero. Thus

$$h(t)\binom{n}{t} = 0 \qquad \text{for } t = 0, 1, \ldots, n.$$

Hence $h(T) \equiv 0$. Then T **is also complete**.

(b) The support of $T_{123} = X_1X_2 + X_3$ is $\{0, 1, 2\}$. We have

$$\begin{aligned}
& \Pr(T_{123} = 0) = \Pr(X_1X_2 + X_3 = 0) = \Pr(X_1X_2 = 0, X_3 = 0) \\
= \; & \Pr(X_1 = 0, X_2 = 0, X_3 = 0) + \Pr(X_1 = 0, X_2 = 1, X_3 = 0) \\
& + \Pr(X_1 = 1, X_2 = 0, X_3 = 0) \\
= \; & (1-\theta)^3 + \theta(1-\theta)^2 + \theta(1-\theta)^2 = (1-\theta)^2(1+\theta),
\end{aligned}$$

$$
\begin{aligned}
& \Pr(T_{123}=1) = \Pr(X_1X_2+X_3=1) \\
= \;& \Pr(X_1X_2=0, X_3=1) + \Pr(X_1X_2=1, X_3=0) \\
= \;& \Pr(X_1=0, X_2=0, X_3=1) + \Pr(X_1=0, X_2=1, X_3=1) \\
+ \;& \Pr(X_1=1, X_2=0, X_3=1) + \Pr(X_1=1, X_2=1, X_3=0) \\
= \;& (1-\theta)^2\theta + 3(1-\theta)\theta^2 = (1-\theta)\theta(1+2\theta),
\end{aligned}
$$

$$
\begin{aligned}
& \Pr(T_{123}=2) = \Pr(X_1X_2+X_3=2) = \Pr(X_1X_2=1, X_3=1) \\
= \;& \Pr(X_1=1, X_2=1, X_3=1) = \theta^3.
\end{aligned}
$$

(c) The conditional density

$$
\begin{aligned}
& \Pr(X_1=0, X_2=1, X_3=0 \mid T_{123}=0) \\
= \;& \frac{\Pr(X_1=0, X_2=1, X_3=0, X_1X_2+X_3=0)}{\Pr(T_{123}=0)} \\
= \;& \frac{\Pr(X_1=0, X_2=1, X_3=0)}{\Pr(T_{123}=0)} \\
= \;& \frac{(1-\theta)^2\theta}{(1-\theta)^2(1+\theta)} \\
= \;& \frac{\theta}{1+\theta}
\end{aligned}
$$

is a function of θ, indicating that T_{123} is not a sufficient statistic for θ.

3. Solution.

(a) The joint density of $X_1, \ldots, X_n$ and θ is

$$\begin{aligned} f(x_1, \ldots, x_n; \theta) &= \left\{ \theta^n \prod_{i=1}^n x_i^{\theta-1} \right\} \times \frac{b^a}{\Gamma(a)} \theta^{a-1} \,\mathrm{e}^{-b\theta} \\ &= \frac{b^a}{\Gamma(a)} \theta^{n+a-1} \exp\left\{ -\theta \left[b - \sum_{i=1}^n \log(x_i) \right] \right\} \times \prod_{i=1}^n x_i^{-1}. \end{aligned}$$

So the posterior density of θ is

$$p(\theta|\boldsymbol{x}) \propto \theta^{n+a-1} \exp\left\{ -\theta \left[b - \sum_{i=1}^n \log(x_i) \right] \right\},$$

so that $\theta|\boldsymbol{x} \sim \text{Gamma}(n+a, b - \sum_{i=1}^n \log(x_i))$, where $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$. Thus,

$$E(\theta|\boldsymbol{x}) = \frac{n+a}{b - \sum_{i=1}^n \log(x_i)}$$

is the Bayesian estimate of θ, and $(n+a)/[b - \sum_{i=1}^n \log(X_i)]$ is the Bayesian estimator of θ.

(b) See Example 3.25 on page 142 of Lecture Notes Chapter 3.

(c) See Example 3.29 on page 151 of Lecture Notes Chapter 3, that is

$$\hat{\tau}(\theta) = -\frac{1}{n} \sum_{i=1}^n \log(X_i).$$

(d) Let $Y = -\log(X)$, then

$$\begin{aligned} \Pr(Y \leqslant y) &= \Pr(-\log X \leqslant y) = \Pr(X \geqslant \mathrm{e}^{-y}) \\ &= \int_{\mathrm{e}^{-y}}^1 \theta x^{\theta-1} \,\mathrm{d}x = 1 - \mathrm{e}^{-\theta y}. \end{aligned}$$

Hence, $Y \sim \text{Exponential}(\theta) = \text{Gamma}(1, \theta)$, so that

$$E(Y) = \frac{1}{\theta} \quad \text{and} \quad \text{Var}(Y) = \frac{1}{\theta^2}.$$

(e) We can obtain $\hat{\tau}(\theta) \sim \text{Gamma}(n, n\theta)$, so that

$$E[\hat{\tau}(\theta)] = \frac{n}{n\theta} = \frac{1}{\theta} = \tau(\theta), \quad \text{and} \quad \text{Var}[\hat{\tau}(\theta)] = \frac{n}{(n\theta)^2} = \frac{1}{n\theta^2}.$$

To get the effective estimator of $\tau(\theta)$, we need to derive the Fisher Information. As $\log f(x;\theta) = \log\theta + (\theta-1)\log x$, it is easy to obtain

$$\frac{\mathrm{d}\log f(x;\theta)}{\mathrm{d}\theta} = \frac{1}{\theta} + \log x, \quad \text{and} \quad \frac{\mathrm{d}^2\log f(x;\theta)}{\mathrm{d}\theta^2} = -\frac{1}{\theta^2},$$

then we can obtain

$$I(\theta) = E\left[-\frac{d^2\log f(X;\theta)}{d\theta^2}\right] = \frac{1}{\theta^2}$$

and

$$I_n(\theta) = nI(\theta) = \frac{n}{\theta^2}.$$

The derivative of $\tau(\theta)$ is that $\tau'(\theta) = -1/\theta^2$, then the C-R lower bound is

$$\frac{[\tau'(\theta)]^2}{I_n(\theta)} = \frac{1}{n\theta^2}.$$

Therefore, $\hat{\tau}(\theta) = -\frac{1}{n}\sum_1^n \log(X_i)$ is unbiased and its variance reaches the C-R lower bound, it is an efficient estimator.

4. Solution.

(a) See **10.1•** on pages 173–174.

(b) See **Example T7.3**(a) in Tutorial 7.

(c) See **Example T7.3**(b) in Tutorial 7.