Discrete Mathematics for Computer Science

Lecture 12: Recursion and Counting

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Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iterating recurrences: how to solve

$$T(n) = \begin{cases} \text{something given}, & \text{if } n \leq n_0 \\ rT(n/m) + a, & \text{if } n > n_0 \end{cases}$$

- Three different behaviors

Iterating Recurrences: Example 1

Consider

$$T(n) = \begin{cases} T(1), & \text{if } n = 1, \\ 2T(n/2) + n, & \text{if } n \geq 2. \end{cases}$$

This corresponds to solving a problem of size n, by

- solving 2 subproblems of size $n/2$ and
- doing n units of additional work

or using $T(1)$ work for "bottom" case of $n = 1$

Iterating Recurrences: Example 1

Algebraically iterating the recurrence
(assume that n is a power of 2):

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$= 4T\left(\frac{n}{4}\right) + 2n \quad = 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

Iterating Recurrences: Example 1

We just iterated the recurrence to derive that the solution to

$$T(n) = \begin{cases} T(1), & \text{if } n = 1, \\ 2T(n/2) + n, & \text{if } n \geq 2. \end{cases}$$

is $nT(1) + n\log_2 n$.

Note: Technically, we still need to use induction to prove that our solution is correct. Practically, we never explicitly perform this step, since it is obvious how the induction would work.

Iterating Recurrences: Example 2

Consider

$$T(n) = \begin{cases} 1, & \text{if } n = 1, \\ T(n/2) + 1, & \text{if } n \geq 2. \end{cases}$$

$T(n) = T\left(\frac{n}{2}\right) + 1$

$T(n) = T\left(\frac{n}{2}\right) + 1 \qquad = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$

Iterating Recurrences: Example 3

Consider

$$T(n) = \begin{cases} 1, & \text{if } n = 1, \\ T(n/2) + n, & \text{if } n \geq 2. \end{cases}$$

$$\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + n \\
&= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\
&= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\
&\quad \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&\quad \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 2n - 1 = \Theta(n)
\end{aligned}$$

Iterating Recurrences: Example 4

Consider

$$T(n) = \begin{cases} 1, & \text{if } n < 3, \\ 3T(n/3) + n, & \text{if } n \geq 3. \end{cases}$$

$$\begin{aligned} T(n) &= 3T\left(\tfrac{n}{3}\right) + n & &= 3\left(3T\left(\tfrac{n}{3^2}\right) + \tfrac{n}{3}\right) + n \\ &= 3^2 T\left(\tfrac{n}{3^2}\right) + 2n & &= 3^2\left(3T\left(\tfrac{n}{3^3}\right) + \tfrac{n}{3^2}\right) + 2n \\ &= 3^3 T\left(\tfrac{n}{3^3}\right) + 3n & & \\ &\;\vdots & &\;\vdots \\ &= 3^i T\left(\tfrac{n}{3^i}\right) + in & & \\ &\;\vdots & &\;\vdots \\ &= 3^{\log_3 n} T\left(\tfrac{n}{3^{\log_3 n}}\right) + n\log_3 n & &= n + n\log_3 n \end{aligned}$$

Iterating Recurrences: Example 5

Consider

$$T(n) = \begin{cases} 1, & \text{if } n = 1, \\ 4T(n/2) + n, & \text{if } n \geq 2. \end{cases}$$

$$\begin{aligned}
T(n) &= 4T\left(\tfrac{n}{2}\right) + n & &= 4\left(4T\left(\tfrac{n}{2^2}\right) + \tfrac{n}{2}\right) + n \\
&= 4^2 T\left(\tfrac{n}{2^2}\right) + \tfrac{4}{2}n + n & &= 4^2\left(4T\left(\tfrac{n}{2^3}\right) + \tfrac{n}{2^2}\right) + \tfrac{4}{2}n + n \\
&= 4^3 T\left(\tfrac{n}{2^3}\right) + \tfrac{4^2}{2^2}n + \tfrac{4}{2}n + n \\
&\quad\vdots \qquad \vdots \\
&= 4^i T\left(\tfrac{n}{2^i}\right) + \tfrac{4^{i-1}}{2^{i-1}}n + \qquad \tfrac{{}^{2}}{2^{2}}n + n \\
&\quad\vdots \qquad \vdots \\
&= 4^{\log_2 n} T\left(\tfrac{n}{2^{\log_2 n}}\right) + \tfrac{4^{\log_2 n - 1}}{2^{\log_2 n - 1}}n + \cdots + \tfrac{4}{2}n + n \\
&= 2n^2 - n
\end{aligned}$$

Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iterating recurrences
- Three different behaviors

Three Different Behaviors

Compare the iteration for the recurrences

- $T(n) = 2T(n/2) + n$ $\quad nT(1) + n\log_2 n$
- $T(n) = T(n/2) + n$ $\quad \Theta(n)$
- $T(n) = 4T(n/2) + n$ $\quad 2n^2 - n$

Anything in common?

In each case, size of subproblem in next iteration is half the size in the preceding iteration level.

All three recurrences iterate $\log_2 n$ times.

Three Different Behaviors

Theorem: Suppose that we have a recurrence of the form

$$T(n) = aT(n/2) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

- If $a < 2$, then $T(n) = \Theta(n)$.
- If $a = 2$, then $T(n) = \Theta(n \log_2 n)$.
- If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

We will now prove the case with $a > 2$.

Proof

$T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

$$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots \frac{a}{2} + 1\right) n$$

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Work at "bottom"

Iterated Work

Proof

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Since $a > 2$, the geometric series is Θ of the largest term.

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1 - (a/2)^{\log_2 n}}{1 - a/2} = n\Theta((a/2)^{\log_2 n - 1})$$

Proof

n times the largest term in the geometric series is

$$n\left(\frac{a}{2}\right)^{\log_2 n - 1} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{n} = \frac{2}{a}\cdot a^{\log_2 n}$$

Notice that

$$a^{\log_2 n} = \left(2^{\log_2 a}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 a} = n^{\log_2 a}$$

So the total work is

$$a^{\log_2 n} T(1) + n\sum_{i=0}^{\log_2 n - 1}\left(\frac{a}{2}\right)^i$$

$$a^{\log_2 n} T(1) + n\sum_{i=0}^{\log_2 n - 1}\left(\frac{a}{2}\right)^i \quad \Theta\left(n^{\log_2 a}\right) \quad \Theta\left(n^{\log_2 a}\right)$$

Example 5 Recap

$$T(n) = \begin{cases} 1, & \text{if } n = 1, \\ 4T(n/2) + n, & \text{if } n \geq 2. \end{cases}$$

$a = 4$, so the Theorem says that

$$T(n) = \Theta\left(n^{\log_2 a}\right) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

This matches with the exact answer of $2n^2 - n$.

Three Different Behaviors Recap

Theorem: Suppose that we have a recurrence of the form

$$T(n) = aT(n/2) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

- If $a < 2$, then $T(n) = \Theta(n)$.
- If $a = 2$, then $T(n) = \Theta(n \log_2 n)$.
- If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

The Master Theorem

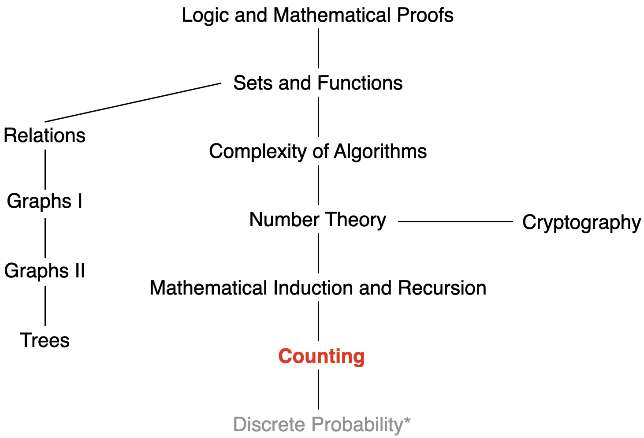
***Theorem:** Suppose that we have a recurrence of the form

$$T(n) = aT(n/b) + cn^d,$$

where a is a positive integer, $b \geq 1$, c, d are real numbers with c positive and d nonnegative, and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

- If $a < b^d$, then $T(n) = \Theta(n^d)$.
- If $a = b^d$, then $T(n) = \Theta(n^d \log_2 n)$.
- If $a > b^d$, then $T(n) = \Theta(n^{\log_b a})$.

This Lecture



Counting basis, Permutations, ...

Counting

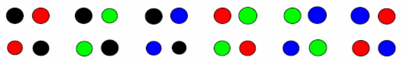
Assume we have a set of objects with certain properties

Counting is used to determine the number of these objects.

Example: How many different ways are there to choose 2 balls from

Unordered count?

Order counts?



Counting

Assume we have a set of objects with certain properties

Counting is used to determine the number of these objects.

Example:

- the number of passwords between 6 - 10 characters
- the number of telephone numbers with 8 digits

Counting may be very hard, not trivial.

Simplify the solution by decomposing the problem.

Basic Counting Rules

Product Rule:

- A count decomposes into a sequence of dependent counts.
- Each element in the first count is associated with all elements of the second count.

Sum Rule:

- A count decomposes into a set of independent counts.
- Elements of counts are alternatives.

The Product Rule

A count decomposes into a sequence of dependent counts.

The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks:

- There are n_1 ways to do the first task.
- For each of these ways of doing the first task, there are n_2 ways to do the second task.
- Then, there are $n_1 n_2$ ways to do the procedure.

Example: In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., A23). What is the total number of seats?
$26 \times 50 = 1300$

The Product Rule

The Product Rule: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and k-th count n_k elements, then the total number of elements is

$$n = n_1 \times n_2 \times ... \times n_k$$

Example:

- How many different bit strings of length seven are there? 2^7
- How many different functions are there from a set with m elements to a set with n elements? n^m
- How many one-to-one functions are there from a set with m elements to a set with n elements? $n(n-1)(n-2)...(n-m+1)$

The Product Rule: Example 1

What is the value of k after the following code, where n_1, n_2, . . ., n_m are positive integers, has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
  for $i_2 := 1$ **to** n_2
      .
      .
      .
    for $i_m := 1$ **to** n_m
      $k := k + 1$

$k = n_1 n_2 \ldots n_m$

The Product Rule: Example 2

If A_1, A_2, . . . , A_m are finite sets, then what is the number of elements in the Cartesian product of these sets?

$$|A_1 \times A_2 \times ... \times A_m| = |A_1||A_2|...|A_m|.$$

The Sum Rule

A count decomposes into a set of independent counts.

The Sum Rule:

- A task can be done either in one of n_1 ways or in one of n_2 ways
- None of the set of n_1 ways is the same as any of the set of n_2 ways

Then, there are $n_1 + n_2$ ways to do the task.

Example: You need to travel from city A to B. You may either fly, take a train, or a bus. There are 12 different flights, 5 different trains and 10 buses. How many options do you have to get from A to B? $12 + 5 + 10$

The Sum Rule

The Sum Rule: If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and k-th count n_k elements, then the total number of elements is

$$n = n_1 + n_2 + ... + n_k.$$

The Sum Rule: Example 1

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\quad k := k + 1 \\
\textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\quad k := k + 1 \\
\quad\quad \vdots \\
\textbf{for } i_m := 1 \textbf{ to } n_m \\
\quad k := k + 1
\end{array}
$$

$k = n_1 + n_2 + \ldots + n_m$.

The Sum Rule: Example 2

If A_1, A_2, . . . , A_m are pairwise disjoint finite sets, then what is the number of elements in the union of these sets?

$$|A_1 \cup A_2 \cup ... \cup A_m| = |A_1| + |A_2| + ... + |A_m|.$$

More Complex Counting

Typically requires a combination of the sum and product rules.

Example: Each password is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of possible passwords:

$$P = P_6 + P_7 + P_8.$$

Use P_6 as an example:

$$P_6 = (10 + 26)^6 - (26)^6 = 1,867,866,560.$$

The Subtraction Rule

The Subtraction Rule:

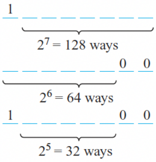
- A task can be done in either n_1 ways or n_2 ways

Then, the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common.

Principle of inclusion–exclusion:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

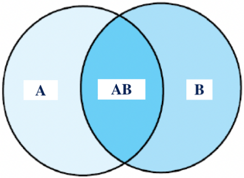
Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00 (inclusive or)?



$2^7 + 2^6 - 2^5$

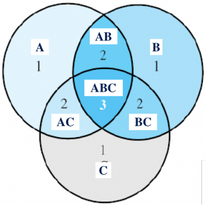
Inclusion-Exclusion Principle

Two sets A and B: $|A \cup B| = |A| + |B| - |A \cap B|$



Three sets A, B, and C:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$



Inclusion-Exclusion Principle

Let $E_1, E_2, \ldots, E_n$ be finite sets:

$$|E_1 \cup E_2 \cup \cdots \cup E_n| = \sum_{1\le i\le n} |E_i| - \sum_{1\le i<j\le n} |E_i \cap E_j| + \sum_{1\le i<j<k\le n} |E_i \cap E_j \cap E_k| - \ldots + (-1)^{n+1}|E_1 \cap E_2 \cap \ldots \cap E_n|.$$

Or equivalently,

$$|\cup_{i=1}^n E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

Proof by Induction.

An Alternative Form of Inclusion–Exclusion Principle

This form can be used to solve problems that ask for the number of elements in a set that have none of n properties P_1, P_2, . . . , P_n.

- A_i: the subset containing the elements that have property P_i.
- $N(P_{i_1}, P_{i_2}, ..., P_{i_k})$: The number of elements with all the properties.

$$|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = N(P_{i_1}, P_{i_2}, ..., P_{i_k}).$$

- $N(P'_{i_1}, P'_{i_2}, ..., P'_n)$: The number of elements with none of the properties P_1, P_2, ... , P_n.

$$N(P'_{i_1}, P'_{i_2}, ..., P'_n) = N - |A_1 \cup A_2 \cup ... \cup A_n|.$$

$$N(P'_1 P'_2 \ldots P'_n) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) - \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n).$$

Inclusion-Exclusion Principle: Example

How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3.

Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively.

Let A_1, A_2, A_3 be the corresponding subsets of functions.

$$N(P_1', P_2', ..., P_3') = N - |A_1 \cup A_2 \cup A_3|$$

$$N(P_1'P_2'P_3') = N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3),$$

Inclusion-Exclusion Principle: Example

How many onto functions are there from a set with six elements to a set with three elements?

$$N(P_1'P_2'P_3') = N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3),$$

- N: the total number of functions. $N = 3^6$
- $N(P_i)$: the number of functions that do not have b_i in their range. $N(P_i) = 2^6$
- $N(P_i, P_j)$: The number of functions that do not have b_i and b_j in their range. $N(P_i, P_j) = 1^6$
- $N(P_1, P_2, P_3) = 0$

The Division Rule

The Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Or equivalently, if f is a function from A to B, where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d-to-one), then $|B| = |A|/d$.

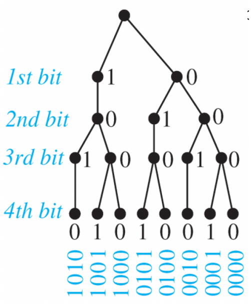
Example: How many different ways are there to seat four people around a circular table?
$4!/4 = 6$.

Tree Diagrams

A tree is a structure that consists of a root, branches and leaves.

Can be useful to represent a counting problem and record the choices we made for alternatives. The count appears on the leaves.

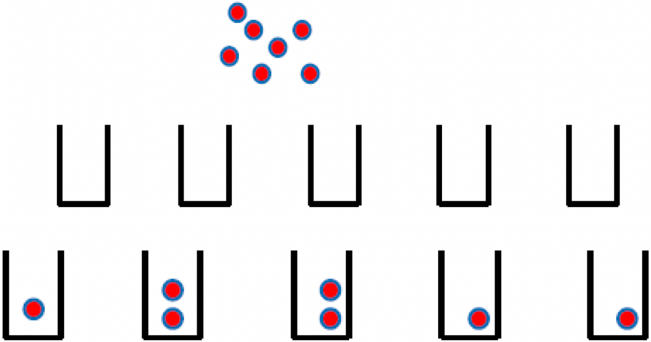
Example: What is the number of bit strings of length 4 that do not have two consecutive 1's?



Pigeonhole Principle

Assume that there are a set of objects and a set of bins to store them.

The Pigeonhole Principle: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.



Pigeonhole Principle

The Pigeonhole Principle: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof by Contradiction: Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least $k+1$ objects.

Example:

Assume that there are 367 students. Are there any two people who have the same birthday?

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Pigeonhole Principle: Example

Show that for every integer n, there is a multiple of n that has only 0s and 1s in its decimal expansion.

Proof: Let n be a positive integer. Consider the $n + 1$ integers

$$1, 11, 111, ..., 11...1,$$

where the last integer has $n + 1$ 1s in its decimal expansion.

Note that there are n possible remainders when an integer is divided by n.

Because there are $n + 1$ integers in this list, by the pigeonhole principle there must be two with the same remainder when divided by n.

The larger of these integers minus the smaller one is a multiple of n, which has a decimal expansion consisting entirely of 0s and 1s.

Generalized Pigeonhole Principle

There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?

If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example: Assume there are 100 students. How many of them were born in the same month?

Generalized Pigeonhole Principle

If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Proof: Suppose that none of the boxes contains more than $\lceil N/k \rceil$ objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

This is a contradiction because there are a total of N objects.

Pigeonhole Principle: Example 1

During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games.

Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let a_j be the number of games played on or before the j th day of the month. Then,

$$a_1, a_2, ..., a_{30},$$

which is an increasing sequence of distinct integers, with $1 \leq a_j \leq 45$.

Moreover, $a_1 + 14, a_2 + 14, ..., a_{30} + 14$ is also an increasing sequence of distinct integers, with $15 \leq a_j + 14 \leq 59$.

Pigeonhole Principle: Example 1

During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games.

Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: The 60 integers $a_1, a_2, ..., a_{30}, a_1 + 14, a_2 + 14, ..., a_{30} + 14$ are all less than or equal to 59. By the pigeonhole principle, two of these integers are equal.

Since the integers in each sequence are distinct, there must be indices i and j with $a_i = a_j + 14$.

Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of real numbers:

- A subsequence of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$, where $1 \leq i_1 < i_2 < \ldots < i_m \leq N$.
- A sequence is called strictly increasing if each term is larger than the one that precedes it.

Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Example: The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains 10 terms. Note that $10 = 3^2 + 1$.

There are four strictly increasing subsequences of length four:

$$1, 4, 6, 12 \qquad 1, 4, 6, 7$$

$$1, 4, 6, 10 \qquad 1, 4, 5, 7$$

There is also a strictly decreasing subsequence of length four:

$$11, 9, 6, 5$$

Pigeonhole Principle: Example 2

Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Proof: Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate (i_k, d_k) to the term a_k:

- i_k: the length of the longest increasing subsequence starting at a_k
- d_k: the length of the longest decreasing subsequence starting at a_k.

Suppose that there are no increasing or decreasing subsequences of length $n + 1$. I.e., $i_k \leq n$ and $d_k \leq n$ for $k = 1, 2, ..., n^2 + 1$.

By the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these $n^2 + 1$ ordered pairs are equal.

That is, there exist terms a_s and a_t with $s < t$ such that $i_s = i_t$ and $d_s = d_t$.

Pigeonhole Principle: Example 2

Theorem: Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Proof: There exist terms a_s and a_t with $s<t$ such that $i_s=i_t$ and $d_s=d_t$. We will show that this is impossible.

The terms of the sequence are distinct, either $a_s<a_t$ or $a_s>a_t$:

- $a_s<a_t$: an increasing subsequence of length i_t+1 can be built, i.e., a_s, a_t, ... (followed by an increasing subsequence of length i_t beginning at a_t); thus, $i_s>i_t$
- $a_s>a_t$: $d_s=d_t$; an decreasing sequence of length d_t+1 can be built, i.e., a_s, a_t, ...; thus, $d_s>d_t$;

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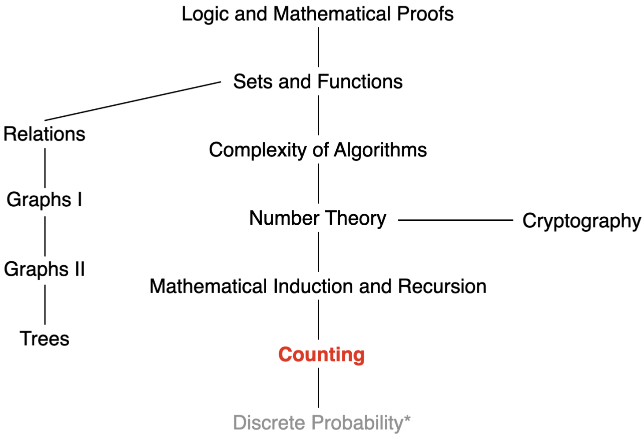

Counting basis, Permutations, ...

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an **r-permutation**.

Example: Let $S = \{a, b, c\}$. The 2-permutations of S are the ordered arrangements (a, b), (a, c), (b, a) (b, c), (c, a). (c, b).

Permutations

How many 3-permutations of $\{1, 2, ..., n\}$ are there?

Based on product rule:

- n choices for first number.
- For each way of choosing first number, there are $n - 1$ choices for the second.
- For each way of choosing first two numbers, there are $n - 2$ choices for the third number.

By product rule, there are $n(n-1)(n-2)$ ways to choose the permutation.

Permutations

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof by the Product Rule: The first element of the permutation can be chosen in n ways, because there are n elements in the set.

There are $n-1$ ways to choose the second element of the permutation.

...

Permutations

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Corollary: If n and r are integers with $0 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}.$$

Permutations: Example

Example 1: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$P(100, 3) = 100 \times 99 \times 98 = 970, 200.$

Example 2: How many permutations of the letters ABCDEFGH contain the string ABC?

The letters ABC must occur as a block. Thus, it is equivalent to finding the number of permutations of six objects:

ABC, D, E, F, G, H.

Thus, there are $P(6, 6) = 6! = 720$ permutations.

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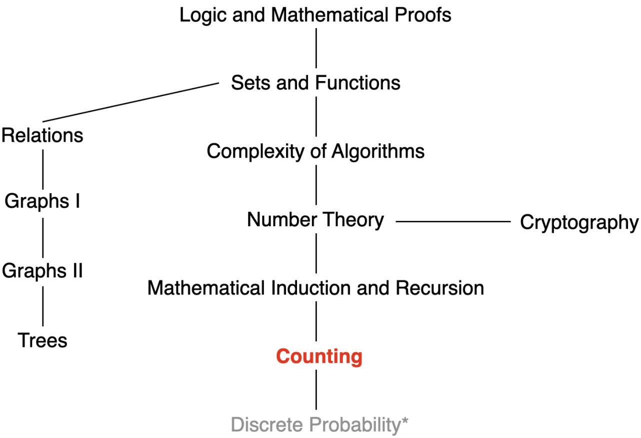

Counting basis, Permutations, Combinations, ...