MA204: Mathematical Statistics—Midterm Test

(7:50am–9:50am, 25 May 2024)

1. (**45 Marks**). Directly give your answers to the following questions:

1.1 Let X be an exponential *random variable* (r.v.) with *probability density function* (pdf) $f(x) = \beta\,\mathrm{e}^{-\beta x}$, $x \geqslant 0, \beta > 0$, denoted by $X \sim \text{Exponential}(\beta)$. Let the median of X be denoted by $\text{med}(X)$. Find the $\text{med}(X)$. [**2ms**]

1.2 Let the r.v. $X \sim \text{Gamma}(\alpha, \beta)$ have pdf $\beta^{\alpha} x^{\alpha-1}\,\mathrm{e}^{-\beta x}/\Gamma(\alpha)$, $x > 0, \alpha > 0, \beta > 0$, where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\,\mathrm{e}^{-x}\,\mathrm{d}x$ denotes the gamma function. Assume that $\alpha > 1$, find $E(X^{-1})$. [**2ms**]

1.3 Let the r.v. X have the following *stochastic representation* (SR):

$$X \stackrel{\mathrm{d}}{=} \mu + \frac{Z}{\sqrt{\xi/\nu}},$$

where $Z \sim N(0, \sigma^2)$, $\xi \sim \chi^2(\nu)$, $Z \perp\!\!\!\perp \xi$, $\mu \in \mathbb{R} \,\hat{=}\, (-\infty, \infty)$ and $\nu > 0$. Assume that $\nu > 2$, find $E(X)$ and $\text{Var}(X)$. [**2ms**]

[Hint: Using the result in Q1.2]

1.4 Let the r.v. $X \sim \text{Geometric}(p)$ have *probability mass function* (pmf) $p(1-p)^{x-1}$, where $0 < p < 1, x = 1, 2, \ldots, \infty$, compute its *moment generating function* (mgf). [**2ms**]

1.5 Let $Y \sim \text{Exponential}(\theta)$. Define $X = \lfloor Y \rfloor$, i.e., X is the largest integer less than or equal to Y. Find the distribution of X. [**2ms**]

1.6 Let $X_1, X_2 \stackrel{\text{iid}}{\sim} N(0, 1)$, $U \sim U(0, 1)$, and U be independent of $\{X_1, X_2\}$. Define $Z = UX_1 + (1 - U)X_2$. Find the conditional distribution of Z given $U = u$, compute $E(Z)$ and $\text{Var}(Z)$. [**6ms**]

1.7 Let $Y, Z \stackrel{\text{iid}}{\sim} N(0, 1)$. Find the distribution of $V = Y/|Z|$. [**2ms**]

1.8 Let $X_1, \ldots, X_n \stackrel{\text{ind}}{\sim} \text{Gamma}(\alpha_i, \beta)$, where $\alpha_i > 0, i = 1, \cdots, n$ and $\beta > 0$. Find the distribution of $Y = \sum_{i=1}^{n} X_i$. [**2ms**]

1.9 What distribution does the r.v. X follow such that $X \overset{\text{d}}{=} X^r$ for any positive integer r? Given an example such that $Y \overset{\text{d}}{=} -Y$. **[2ms]**

1.10 Let $X_1, X_2 \overset{\text{iid}}{\sim}$ Bernoulli(0.5), what is the distribution (or pmf) of $Y \hat{=} \min(X_1, X_2)$? **[2ms]**

1.11 Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$, what is the *maximum likelihood estimator* (MLE) of $\log(\sigma^2)$? **[2ms]**

1.12 Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} f(x; \theta) = (\theta + 1)x^\theta,\ 0 < x \leqslant 1,\ \theta > 0$, find the *moment estimator* of θ. **[2ms]**

1.13 In Bayesian statistics, let $X|\theta \sim \text{Binomial}(n, \theta)$ with known positive integer n and the prior distribution of θ be $\text{Beta}(a, b)$ with known $a, b\,(> 0)$. What is the posterior distribution $\theta|(X = x)$ and what is the marginal distribution of X? **[2ms]**

[Hint: The distribution of X is called beta-binomial distribution with parameters n, a, b, denoted by $X \sim \text{BBinomial}(n, a, b)$]

1.14 State three limiting properties of the MLE $\hat{\theta}_n$. **[2ms]**

1.15 Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} f(x; \theta) = \theta x^{\theta-1},\ 0 < x \leqslant 1, \theta > 0$. Find the *Fisher information* $I_n(\theta)$ of θ. **[2ms]**

1.16 Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} f(x; \theta) = \theta a^\theta x^{-(\theta+1)},\ x \geqslant a,\ a > 0\ ,\ \theta > 0$. Find a sufficient statistic of θ. **[2ms]**

1.17 Let $X_{11}, \dots, X_{1n_1} \overset{\text{iid}}{\sim} N(\mu_1, \sigma_1^2),\ X_{21}, \dots, X_{2n_2} \overset{\text{iid}}{\sim} N(\mu_2, \sigma_2^2)$, and the two samples be independent. What is the pivotal quantity for constructing the *confidence interval* (CI) of σ_1^2/σ_2^2? **[2ms]**

1.18 Let $X_1, \dots, X_n, X \overset{\text{iid}}{\sim} F(x; \theta)$, where $F(x; \theta)$ is the cdf of the continuous r.v. X. To construct a CI for the single parameter θ, what is the universal form of the pivot quantity? **[2ms]**

1.19 State the central limit theorem. **[2ms]**

1.20 Bernoulli distribution is one special case of four distributions. Please give at least three such distributions. **[3ms]**

2. (**10 Marks**). Let two conditional distributions be quadratic and linear restricted to the unit interval $(0, 1)$ and the interval $(0, 2)$, respectively;

that is,

$$f_{(X|Y)}(x|y) = \frac{6x(y+x)}{3y+2}, \quad 0 < x < 1,$$

$$f_{(Y|X)}(y|x) = \frac{y+x}{2(1+x)}, \quad 0 < y < 2.$$

Find the marginal distribution of X.

3. (**10 Marks**). Let $Y \sim \chi^2(\nu)$ with $\nu > 0$, $Z \sim N(0,1)$ and $Y \perp\!\!\!\perp Z$. Use the mixture technique to find the distribution of $T \hat{=} Z/\sqrt{Y/\nu}$.

4. (**20 Marks**). Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$, where $0 < \theta < 1$.

4.1 Show that $T = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ. [**10ms**]

4.2 Find the UMVUE of θ. [**5ms**]

4.3 Show that $\bar{X} = T/n$ is an efficient estimator of θ. [**5ms**]

5. (**15 Marks**). Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exponential}(\theta)$ and define $\mathbf{x} = (X_1, \ldots, X_n)^\top$.

5.1 Find the $100(1-\alpha)\%$ exact equal-tail CI of θ. [**5ms**]

5.2 Find the MLE $\hat{\theta}_n$ of θ and the $100(1-\alpha)\%$ approximate CI of θ based on the limiting property of the MLE:

$$\{nI(\theta)\}^{1/2}(\hat{\theta}_n - \theta) \dot{\sim} N(0,1),$$

where $nI(\theta)$ denotes the Fisher information. [**5ms**]

5.3 Find the $100(1-\alpha)\%$ approximate CI of θ based on the asymptotic normality of the score function: $S(\theta; \mathbf{x})/\sqrt{nI(\theta)} \dot{\sim} N(0,1)$, where $S(\theta; \mathbf{x})$ denotes the score function. [**5ms**]

=== **END OF THE PAPER** ===

1 Solution.

1.1 See **Example T1.6**. The cdf of exponential r.v. X is $F(x) = \int_0^x f(t)\,\mathrm{d}t = 1 - \mathrm{e}^{-\beta x}$ for $x \geqslant 0$. Let $F(\xi_q) = q$, then

$$\xi_q = F^{-1}(q) = -\frac{\log(1-q)}{\beta}, \quad 0 \le q < 1.$$

Thus, $\mathrm{med}(X) = \xi_{0.5} = \dfrac{\log(2)}{\beta}$.

1.2 Since $X \sim \mathrm{Gamma}(\alpha, \beta)$, we have

$$\begin{aligned} E(X^{-1}) &= \int_0^\infty x^{-1} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\,\mathrm{e}^{-\beta x}\,\mathrm{d}x \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(\alpha-1)-1}\,\mathrm{e}^{-\beta x}\,\mathrm{d}x \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}} = \frac{\beta}{\alpha-1}. \end{aligned} \tag{MT.1}$$

1.3 <u>Method I</u>: From $X \stackrel{\mathrm{d}}{=} \mu + \dfrac{Z}{\sqrt{\xi/\nu}}$ and $Z \sim N(0, \sigma^2)$, we obtain

$$\begin{aligned} E(X) &= \mu + E(Z) \times E\left(\frac{1}{\sqrt{\xi/\nu}}\right) = \mu, \\ E(X^2) &= \mu^2 + E(Z^2) \times E\left(\frac{1}{\xi/\nu}\right) + 2\mu E(Z) \times E\left(\frac{1}{\sqrt{\xi/\nu}}\right), \\ &= \mu^2 + \sigma^2 \nu E(\xi^{-1}). \end{aligned}$$

Since $\xi \sim \chi^2(\nu) \equiv \mathrm{Gamma}(\nu/2, 1/2)$, from (MT.1), we have

$$E(\xi^{-1}) = \frac{1}{\nu-2}, \tag{MT.2}$$

so that

$$E(X^2) = \mu^2 + \sigma^2 \frac{\nu}{\nu-2} \quad \text{and} \quad \mathrm{Var}(X) = \sigma^2 \frac{\nu}{\nu-2}.$$

<u>Method II</u>: From $X \stackrel{\mathrm{d}}{=} \mu + \dfrac{Z}{\sqrt{\xi/\nu}}$, we have

$$X|\xi \stackrel{\mathrm{d}}{=} \mu + \sqrt{\frac{\nu}{\xi}} Z \sim N(\mu, \nu\xi^{-1}\sigma^2),$$

so that

$$E(X|\xi) = \mu \quad \text{and} \quad \mathrm{Var}(X|\xi) = \nu\xi^{-1}\sigma^2. \tag{MT.3}$$

Therefore,

$$\begin{aligned} E(X) &= E[E(X|\xi)] \stackrel{\text{(MT.3)}}{=} \mu, \\ \mathrm{Var}(X) &= E[\mathrm{Var}(X|\xi)] + \mathrm{Var}[E(X|\xi)] \\ &\stackrel{\text{(MT.3)}}{=} \nu\sigma^2 E(\xi^{-1}) + 0 \\ &\stackrel{\text{(MT.2)}}{=} \sigma^2 \frac{\nu}{\nu - 2}. \end{aligned}$$

1.4 See **Table 1.2** on page 25 of the textbook "Mathematical Statistics". The mgf of the geometric distribution r.v. is

$$\frac{p\,\mathrm{e}^t}{(1 - q\,\mathrm{e}^t)}.$$

1.5 Let $F_Y(y) = 1 - \mathrm{e}^{-\theta y}$ denote the cdf of $Y \sim \text{Exponential}(\theta)$. From $X = \lfloor Y \rfloor$, we have the pmf of X as

$$\begin{aligned} \Pr(X = x) &= \Pr(x \leqslant Y < x + 1) = F_Y(x + 1) - F_Y(x) \\ &= \mathrm{e}^{-\theta x} - \mathrm{e}^{-\theta(x+1)} = \mathrm{e}^{-\theta x}(1 - \mathrm{e}^{-\theta}) \\ &= (1 - p)^x p, \quad x = 0, 1, \ldots, \infty, \end{aligned}$$

where $p \hat{=} 1 - \mathrm{e}^{-\theta}$. That is, X follows the geometric distribution.

1.6 See **Example T1.13**.
$Z|(U = u) = uX_1 + (1 - u)X_2 \sim N(0, u^2 + (1 - u)^2)$. Hence,

$$Z|U \sim N(0, U^2 + (1 - U)^2)$$

so that $E(Z|U) = 0$ and $\mathrm{Var}(Z|U) = U^2 + (1 - U)^2$.

Method I: We need to use the following conclusion that $X \perp\!\!\!\perp Y$ iff for any functions $f(\cdot)$ and $g(\cdot)$, $f(X) \perp\!\!\!\perp g(Y)$.

Since $Z = UX_1 + (1-U)X_2$, we have

$$\begin{aligned} E(Z) &= E[UX_1 + (1-U)X_2] = E(UX_1) + E[(1-U)X_2] \\ &= E(U)E(X_1) + E(1-U)E(X_2) = 0, \\ E(Z^2) &= E[U^2X_1^2 + (1-U)^2X_2^2 + 2U(1-U)X_1X_2] \\ &= E(U^2)E(X_1^2) + E[(1-U)^2]E(X_2^2) + 2E[U(1-U)]E(X_1)E(X_2) \\ &= E(U^2) + E[(1-U)^2] + 0 = \int_0^1 u^2 \ \mathrm{d}u + \int_0^1 (1-u)^2 \ \mathrm{d}u \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

Method II: $E(Z) = E[E(Z|U)] = 0$ and

$$\mathrm{Var}(Z) = E[\mathrm{Var}(Z|U)] + \mathrm{Var}[E(Z|U)] = E(U^2) + E[(1-U)^2] + 0 = \frac{2}{3}.$$

1.7 See **Example T3.5**.
Since $Z^2 \sim \chi^2(1)$, by, we have

$$V = \frac{Y}{|Z|} = \frac{Y}{\sqrt{Z^2}} = \frac{N(0,1)}{\sqrt{\chi^2(1)/1}} \sim t(1) = \mathrm{Cauchy}(0,1).$$

1.8 $Y = \sum_{i=1}^n X_i \sim \mathrm{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$.

1.9 $X \sim \mathrm{Bernoulli}(p)$. $Y \sim N(0,1)$.

1.10 $Y \sim$ **Bernoulli**(0.25).
The r.v. $Y = \min(X_1, X_2)$ only takes 0 or 1, so Y follows a Bernoulli distribution with probability of success:

$$\Pr(Y=1) = \Pr(X_1 = 1, X_2 = 1) = \Pr(X_1 = 1)\Pr(X_2 = 1) = 0.25.$$

1.11 The MLE of $\log(\sigma^2)$ is

$$\log\left\{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2\right\}.$$

1.12 The mean of the population is

$$E(X) = \int_0^1 x(\theta+1)x^\theta \, \mathrm{d}x = \frac{\theta+1}{\theta+2}.$$

Let the sample mean $\bar{X}$ equal $E(X)$, we obtain the moment estimator of θ as

$$\hat{\theta}^{\mathrm{M}} = \frac{1-2\bar{X}}{\bar{X}-1}.$$

1.13 (i) The joint density of (X, θ) is

$$f(x,\theta) = f(x|\theta)\cdot\pi(\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} \times \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)},$$

so that the posterior density is

$$p(\theta|x) = \frac{f(x,\theta)}{f_X(x)} \propto f(x,\theta) \propto \theta^{x+a-1}(1-\theta)^{n-x+b-1},$$

i.e., $\theta|(X=x) \sim \mathrm{Beta}(x+a, n-x+b)$.

(ii) The marginal density of X is

$$f_X(x) = \int_0^1 f(x,\theta)\,\mathrm{d}\theta = \binom{n}{x}\frac{B(x+a,n-x+b)}{B(a,b)}, \quad x = 0,1,\ldots,n,$$

i.e., $X \sim \mathrm{BBinomial}(n,a,b)$.

1.14 See Section 3.5.
(i) The MLE $\hat{\theta}_n$ weakly converges in probability to its true value.

(ii) The MLE $\hat{\theta}_n$ converges in distribution to its true value.

(iii) The MLE $\hat{\theta}_n$ is asymptotically normally distributed.

1.15 The Fisher information is $I_n(\theta) = \dfrac{n}{\theta^2}$.
From the pdf of the population, we have

$$\log f(x;\theta) = \log\theta + (\theta-1)\log x.$$

So

$$\frac{\mathrm{d}\log f(x;\theta)}{\mathrm{d}\theta} = \frac{1}{\theta} + \log x \quad \text{and} \quad \frac{\mathrm{d}^2\log f(x;\theta)}{\mathrm{d}\theta^2} = -\frac{1}{\theta^2}.$$

Hence, the Fisher information

$$I_n(\theta) = nI(\theta) = nE\left\{-\frac{\mathrm{d}^2 \log f(X;\theta)}{\mathrm{d}\theta^2}\right\} = \frac{n}{\theta^2}.$$

1.16 The joint density of $X_1, \ldots, X_n$ is

$$\begin{aligned} f(x_1, \ldots, x_n; \theta) &= \prod_{i=1}^{n} \left[\theta a^\theta x_i^{-(\theta+1)} \cdot I(x_i > a)\right] \\ &= \theta^n a^{n\theta} \left[\prod_{i=1}^{n} x_i I(x_i > a)\right]^{-\theta} \times \left[\prod_{i=1}^{n} x_i I(x_i > a)\right]^{-1}, \end{aligned}$$

so that $\prod_{i=1}^n [x_i I(x_i > a)]$ is a sufficient statistic of θ based on the factorization theorem.

1.17 See **Section 4.5**.

The pivot quantity is

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(n_1 - 1, n_2 - 1).$$

1.18 See **(4.3)** on page 165 of the textbook "Mathematical Statistics".

The pivot quantity is

$$-2\sum_{i=1}^{n} \log F(X_i;\theta) \stackrel{\mathrm{d}}{=} -2\sum_{i=1}^{n} \log[1 - F(X_i;\theta)] \sim \chi^2(2n).$$

1.19 See **Theorem 2.9** on page 94.

Let $\{X_n\}_{n=1}^\infty$ be a sequence of i.i.d. random variables with common mean μ and common variance $\sigma^2 \in (0, \infty)$. Let $\bar{X}_n = \sum_{i=1}^n X_i/n$ and $Y_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$, then $Y_n \xrightarrow{\mathrm{L}} N(0,1)$ as $n \to \infty$.

1.20 (i) Two-point distribution with $\Pr(X = x_1) = 1 - p$ and $\Pr(X = x_2) = p$. In particular, when $x_1 = 0$ and $x_2 = 1$, this two-point distribution reduces to Bernoulli(p).

[Two-point distribution is a special case of the finite discrete distribution $X \sim \text{FDiscrete}_n(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$, $\boldsymbol{p} = (p_1, \ldots, p_n)^\top \in \mathbb{T}_n$, see **Appendix A.1.1** on page 246]

(ii) Binomial(n, p) becomes Bernoulli(p) if $n = 1$.

(iii) The categorical distribution (a special multinomial distribution), $(Y_1, \ldots, Y_d)^\top \sim$ Multinomial$(1; p_1, \ldots, p_d)$, will become the two-category distribution, i.e., Bernoulli(p_1) if $d = 2$.

[see **Appendix A.1.5** on page 249]

(iv) If $n = 1$, then BBinomial(n, α, β) becomes Bernoulli(p) with $p = \alpha/(\alpha + \beta)$.

[see **Example T5.2** in Tutorial 5]

2. Solution. See **Example T2.3**.

Note that $\mathcal{S}_X = (0, 1)$ and $\mathcal{S}_Y = (0, 2)$. Let $y_0 = 1 \in \mathcal{S}_Y = (0, 2)$, we have

$$f_X(x) \propto \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)} = \frac{\dfrac{6x(y_0 + x)}{3y_0 + 2}}{\dfrac{y_0 + x}{2(1 + x)}} \propto x + x^2,$$

so that $f_X(x) = K^{-1} \cdot (x + x^2) \cdot I(0 < x < 1)$. From $1 = \int_0^1 f_X(x)\,\mathrm{d}x$, we obtain

$$K = \int_0^1 (x + x^2)\,\mathrm{d}x = \frac{x^2}{2}\bigg|_0^1 + \frac{x^3}{3}\bigg|_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Thus,

$$f_X(x) = \frac{6}{5} \cdot (x + x^2) \cdot I(0 < x < 1),$$

which is a quadratic pdf on the unit interval. ‖

3. Solution. See **Example T3.2**.

We first find the conditional distribution of $T|(Y = y)$. Note that

$$T|(Y = y) = \sqrt{\frac{\nu}{y}} \cdot Z \sim N\left(0, \frac{\nu}{y}\right),$$

i.e.,

$$f_{(T|Y)}(t|y) = \frac{1}{\sqrt{2\pi\nu/y}} \exp\left(-\frac{t^2}{2\nu/y}\right).$$

Hence, we have

$$\begin{aligned} f_T(t) &= \int_{\mathcal{S}_Y} f_{(T,Y)}(t,y)\,\mathrm{d}y = \int_{\mathcal{S}_Y} f_Y(y) f_{(T|Y)}(t|y)\,\mathrm{d}y \\ &= \int_0^\infty \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)} y^{\nu/2-1}\,\mathrm{e}^{-y/2} \cdot \frac{1}{\sqrt{2\pi\nu/y}} \exp\left(-\frac{yt^2}{2\nu}\right)\mathrm{d}y \\ &= \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)} \cdot \frac{1}{\sqrt{2\pi\nu}} \cdot \underbrace{\int_0^\infty y^{\frac{\nu+1}{2}-1} \exp\left[-y\left(\frac{1}{2}+\frac{t^2}{2\nu}\right)\right]\mathrm{d}y}_{\text{Using (1.41) in the textbook}} \\ &= \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)} \cdot \frac{1}{\sqrt{2\pi\nu}} \cdot \frac{\Gamma((\nu+1)/2)}{\left(\frac{1}{2}+\frac{t^2}{2\nu}\right)^{(\nu+1)/2}} \\ &= \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} \left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < t < \infty. \end{aligned}$$

indicating that T follows the t distribution with ν degrees of freedom, i.e., $T \sim t(\nu)$. ‖

4. Solution.

4.1 See **Example 3.28** on page 150 of the textbook "Math Statistics".

The joint pmf of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i} = \theta^t (1-\theta)^{n-t},$$

where $t = \sum_{i=1}^n x_i$. By using the factorization theorem, we know that $T = \sum_{i=1}^n X_i$ **is sufficient**, and $T \sim \text{Binomial}(n, \theta)$.

Now assume that a function $h(T)$ satisfies

$$E\{h(T)\} = \sum_{t=0}^n h(t)\,\Pr(T=t) = \sum_{t=0}^n h(t)\binom{n}{t}\theta^t(1-\theta)^{n-t} = 0, \quad \text{(MT.4)}$$

for $0 < \theta < 1$. Let $y = \theta/(1-\theta)$, then (MT.4) becomes

$$\sum_{t=0}^n h(t)\binom{n}{t} y^t = 0, \quad y > 0.$$

A polynomial is identical to zero, then all coefficients are zero. Thus

$$h(t)\binom{n}{t} = 0 \qquad \text{for } t = 0, 1, \ldots, n.$$

Hence $h(T) \equiv 0$. Then T **is also complete**.

4.2 See **Example 3.28** on page 150 of the textbook "Math Statistics".

Since $\bar{X} = T/n$ is unbiased for θ, it is the unique UMVUE for θ according to the Lehmann–Sheffé Theorem.

4.3 See **Example 3.17** on page 135 of the textbook "Math Statistics".

Let $X \sim \text{Bernoulli}(\theta)$, then the pmf of X is $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$. Then, the Fisher information is

$$\begin{aligned} I(\theta) &= E\left\{\frac{\mathrm{d}\log f(X;\theta)}{\mathrm{d}\theta}\right\}^2 = E\left(\frac{X}{\theta} - \frac{1-X}{1-\theta}\right)^2 \\ &= E\left\{\frac{X-\theta}{\theta(1-\theta)}\right\}^2 = \frac{\text{Var}(X)}{\theta^2(1-\theta)^2} = \frac{1}{\theta(1-\theta)} \end{aligned}$$

and

$$I_n(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}.$$

Now, $\bar{X}$ is unbiased and

$$\text{Var}(\bar{X}) = \frac{\text{Var}(X)}{n} = \frac{\theta(1-\theta)}{n} = \frac{1}{I_n(\theta)};$$

i.e., the variance attains the CR lower bound. Then $\bar{X}$ is an efficient estimator of θ. ‖

5. Solution.

5.1 See **Example 4.1** on page 164 of the textbook "Mathematical Statistics".

The pdf of $X \sim \text{Exponential}(\theta)$ is $\theta\,\mathrm{e}^{-\theta x}$, $x \geqslant 0$, $\theta > 0$. Using the property stated in Appendix A.2.3, we have $n\bar{X} = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. By Property 1 in Appendix A.2.4, we obtain

$$2\theta n\bar{X} \sim \text{Gamma}(2n/2, 1/2) = \chi^2(2n),$$

so that $2\theta n\bar{X}$ **is a pivotal quantity**.

Let $\chi^2(\alpha,\nu)$ denote the upper α-th quantile satisfying $\Pr\{\chi^2(\nu) > \chi^2(\alpha,\nu)\} = \alpha$. Thus, using the equal-probability (or equal-tail) method, we have

$$\begin{aligned} 1-\alpha &= \Pr\left\{\chi^2(1-\alpha/2,2n) \leqslant 2\theta n\bar{X} \leqslant \chi^2(\alpha/2,2n)\right\} \\ &= \Pr\left\{\frac{\chi^2(1-\alpha/2,2n)}{2n\bar{X}} \leqslant \theta \leqslant \frac{\chi^2(\alpha/2,2n)}{2n\bar{X}}\right\}; \end{aligned}$$

that is,

$$[L_p, U_p] = \left[\frac{\chi^2(1-\alpha/2,2n)}{2n\bar{X}}, \frac{\chi^2(\alpha/2,2n)}{2n\bar{X}}\right]$$

is a $100(1-\alpha)\%$ exact CI for θ. $\|$

5.2 See **Examples 4.5 & 4.6** on pages 176 & 178 of the textbook "Mathematical Statistics".

$\hat{\theta}_n = 1/\bar{X}_n$ and $I(\theta) = 1/\theta^2$. Hence,

$$[L_4, U_4] = \left[\frac{1/\bar{X}_n}{1+z_{\alpha/2}/\sqrt{n}}, \frac{1/\bar{X}_n}{1-z_{\alpha/2}/\sqrt{n}}\right]$$

is a large-sample $100(1-\alpha)\%$ CI for θ.

5.3 See **Example 4.5** on page 176 of the textbook "Mathematical Statistics".

The log-likelihood function is $\log L(\theta) = n\log\theta - \theta\sum_{i=1}^n x_i$. So the score function is give by

$$S(\theta;\mathbf{x}) = \frac{\mathrm{d}\log L(\theta)}{\mathrm{d}\theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i.$$

Since $I(\theta) = \theta^{-2}$, we have

$$1-\alpha = \Pr\left\{-z_{\alpha/2} \leqslant \frac{n/\theta - n\bar{X}_n}{\sqrt{n\theta^{-2}}} \leqslant z_{\alpha/2}\right\}.$$

Hence

$$[L_3, U_3] = \left[\frac{1-z_{\alpha/2}/\sqrt{n}}{\bar{X}_n}, \frac{1+z_{\alpha/2}/\sqrt{n}}{\bar{X}_n}\right]$$

is a large-sample $100(1-\alpha)\%$ CI for θ. $\|$