MA204: Mathematical Statistics

Assignment 1

You have a total of **13 questions** in Assignment 1.

Submit your solutions for **6 questions** randomly selected from Q1.1–Q1.12 in Exercise 1 (pages 53–56) of the Textbook "Mathematical Statistics", plus **3 questions** chosen from the following Q1.13–Q1.17, plus **4 questions** Q1.18–Q1.21 below.

1.13 Let X be a positive random variable with expectation $E(X) = \mu$ and $E(X^2) < \infty$. Assume that $\lambda \in (0, 1)$ is a real number, show that

$$\Pr(X > \lambda\mu)E(X^2) \geqslant (1 - \lambda)^2\mu^2.$$

[Hint: Utilize Cauchy–Schwarz inequality in Theorem 1.5]

1.14 Given a $q \in (0, 1)$, find the q-th quantile ξ_q of the continuous r.v. X with the following pdfs, and calculate the median $\xi_{0.5}$:

(a) Logistic density

$$f(x) = \frac{\exp(-\frac{x-\mu}{\sigma})}{\sigma\{1 + \exp(-\frac{x-\mu}{\sigma})\}^2}, \quad x \in \mathbb{R} \mathrel{\hat{=}} (-\infty, \infty),$$

where $\mu \in (-\infty, \infty)$ is the location parameter and $\sigma > 0$ is the scale parameter.

(b) Rayleigh density $f(x) = \sigma^{-2}x\exp(-\frac{x^2}{2\sigma^2})$, $x > 0$, $\sigma > 0$.

1.15 For $\alpha > 0$, define

$$f(x) = \frac{x(2\alpha + x)}{\alpha(\alpha + x)^2} I_{(0,\alpha)}(x) + \frac{\alpha^2(\alpha + 2x)}{x^2(\alpha + x)^2} I_{(\alpha,\infty)}(x),$$

where $I_{\mathbb{A}}(x)$ is the indicator function, i.e.,

$$I_{\mathbb{A}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{A}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $f(x)$ is a density function.

(b) Let a continuous r.v. $X \sim f(x)$, find the median of X.

1.16 Find the median $\xi_{0.5}$ of the discrete r.v. X with pmf $p_i = \Pr(X = i)$ for $i = 1, 2, 3, 4$, where $p_1 = 0.20$, $p_2 = 0.15$, $p_3 = 0.25$ and $p_4 = 0.40$.

1.17 In Theorem 1.14 on page 20 of the Textbook, let $g(x) = -\log(x)$ for $x > 0$, we have

$$E\{-\log(X)\} \geqslant -\log\{E(X)\}, \tag{A1.1}$$

for any positive r.v. X taking values in $\mathbb{R}_+ \hat{=} (0, \infty)$. Define a discrete r.v. X as follows:

X	$x_1, \ldots, x_i, \ldots, x_n$
$\Pr(X = x_i)$	$p_1, \ldots, p_i, \ldots, p_n$

where the probabilities $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. From (A1.1), we obtain

$$\log\left(\sum_{i=1}^n p_i x_i\right) \geqslant \sum_{i=1}^n p_i \log(x_i), \tag{A1.2}$$

where $x_i > 0$ for all $i = 1, \ldots, n$. Use (A1.2), prove the following harmonic, geometric and arithmetic mean inequalities:

$$\underbrace{\left(\frac{1}{n}\sum_{i=1}^n x_i^{-1}\right)^{-1}}_{H(n)} \leqslant \underbrace{\left(\prod_{i=1}^n x_i\right)^{1/n}}_{G(n)} \leqslant \underbrace{\frac{1}{n}\sum_{i=1}^n x_i}_{A(n)}. \tag{A1.3}$$

where $x_i > 0$ for all $i = 1, \ldots, n$, $H(n), G(n)$ and $A(n)$ are called harmonic, geometric and arithmetic means, respectively.

1.18 Let two conditional distributions be given by

$$\begin{aligned} f_{(X|Y)}(x|y) &= \frac{2(x+2y)}{1+4y}, \quad 0 < x < 1, \\ f_{(Y|X)}(y|x) &= \frac{x+2y}{x+1}, \qquad 0 < y < 1. \end{aligned}$$

(a) Find the marginal distribution of X.

(b) Find the joint distribution of $(X, Y)^\top$.

1.19 Let a positive random variable $X \stackrel{\mathrm{d}}{=} UY$, where $U \sim U(0,1)$, $Y \sim f_Y(y) \cdot I(y > 0)$ and $U \perp\!\!\!\perp Y$. Find the pdf of X.

1.20 Let $f_i(x)$ and $F_i(x)$ denote the pdf and cdf of the continuous r.v. X_i for $i = 1, 2$. Define $Y \stackrel{\mathrm{d}}{=} F_2(X_1)$.

(a) Show that the pdf of Y is given by

$$f_Y(y) = \frac{f_1(F_2^{-1}(y))}{f_2(F_2^{-1}(y))}, \quad y \in (0,1).$$

(b) Let X_1 follow an inverted beta distribution (see **Example T2.6**) with parameters α and β, denoted by $X_1 \sim \mathrm{IBeta}(\alpha, \beta)$. Its pdf is

$$f_1(x_1) = \frac{1}{B(\alpha,\beta)} \cdot \frac{x_1^{\alpha-1}}{(1+x_1)^{\alpha+\beta}}, \quad x_1 > 0. \tag{A1.4}$$

Furthermore, let $X_2 \sim \mathrm{IBeta}(1,1)$. Show that $Y \sim \mathrm{Beta}(\alpha, \beta)$.

1.21 Let $M_X(t)$ denote the *moment generating function* (mgf) of the positive r.v. X with pdf $f_X(x)$ for $x > 0$. Prove that

$$E(X^{-1}) = \int_0^\infty M_X(-t)\,\mathrm{d}t. \tag{A1.5}$$

[Hint: Utilize the identity $\beta^{-1} = \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t$ for $\beta > 0$]