

Theorem $H, K \trianglelefteq G$, consider $G/H, G/K$

If $H \leq K$, then G/H is homomorphic to G/K $G/H \rightarrow G/K$

with the kernel K/H , i.e. $G/H / K/H \cong G/K$ by the 1st isomorphism theorem.

And this is the 3rd isomorphism theorem.

Definition Let G, H be groups. Let $X = G \times H$

$$\text{i.e. } X = \{(g, h) \mid g \in G, h \in H\}$$

$$\text{S.e. } (g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

Then $G \times H$ is a grp. called a "direct product" of G and H , or "direct sum" (Just consider finitely many copies).

Definition Cyclic group. $G = \langle g \rangle = \{g^{\pm i} \mid i \in \mathbb{N}\}$

$$\text{E.g. } \textcircled{1} (\mathbb{Z}, +) = \{(\pm i) \cdot 1 \mid i \in \mathbb{N}\} = \{\pm i \mid i \in \mathbb{N}\}$$

$$\textcircled{2} (\mathbb{Z}_p, \oplus) = \{(\pm i) \cdot 1 \mid i \in \mathbb{N}\}$$

$$= \{\bar{0}, \bar{1}, \dots, \overline{p-1}\}$$

every element (except $\bar{0}$) is a generator of \curvearrowright

Definition Dihedral group

$$G = \{1, a, \dots, a^{n-1}, b, ab, \dots, a^{n-1}b \mid |a| = n, |b| = 2\}$$

where $bab = a^{-1}$

Claim. G is a group of order $2n$.

Denoted by D_{2n} , it has Geometric Background.

To prove it is a group you need check.

① $\forall x \in D_{2n}$. x can ^{be} written into the form

$$a^i b^j \text{ where } i=0, \dots, n-1, j=0,1$$

② $\forall i_1, i_2, j_1, j_2$

$$(a^{i_1} b^{j_1})^{-1} \in D_{2n} \text{ and } (a^{i_1} b^{j_1})(a^{i_2} b^{j_2}) \in D_{2n}.$$

Symmetric Group

Let $\Omega = \{1, \dots, n\}$. A 1-1 map from Ω to Ω is called a permutation on Ω .

Let $\text{Sym}(\Omega) = \{ \text{all permutations on } \Omega \}$ $|\text{Sym}(\Omega)| = n!$

Define multiplication "·" by composition. Then

$(\text{Sym}(\Omega), \cdot)$ is a group, called the symmetric group on Ω .

denoted by S_n

$$\text{E.g.: } \phi: \begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 4 \\ 3 \rightarrow 3 \\ 4 \rightarrow 1 \end{array} \quad \text{denote } (124)(3) \text{ or } (124)$$

HW. Write all elements of $S_4 = \text{Sym}(\Omega)$, $\Omega = \{1, 2, 3, 4\}$

Lemma Each permutation of $\Omega = \{1, 2, \dots, n\}$ can be written as a product of disjoint cycles.

Lemma 2 Each permutation can be written into a product of transpositions. (not uniquely)

The number of transpositions is unique (mod 2)

Def If the number of transpositions appear in a permutation is even, then the permutation is called even permutation.

Lemma All even permutations in $\text{Sym}(\Omega)$ form a subgroup of $\text{Sym}(\Omega)$ called alternating group. denoted by $A(\Omega)$ or A_n .

Claim $A_n \triangleleft S_n$ and $|S_n| = 2|A_n| \Rightarrow |A_n| = \frac{n!}{2}$