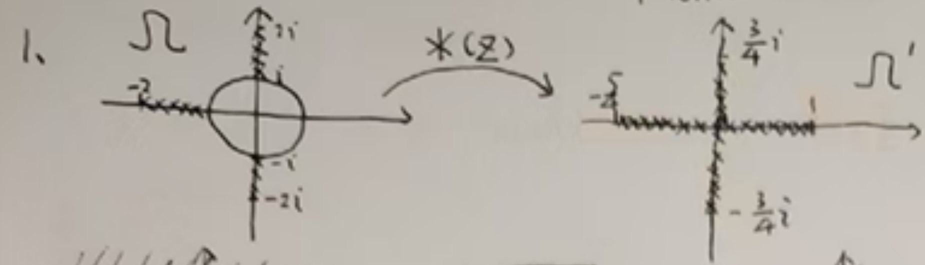
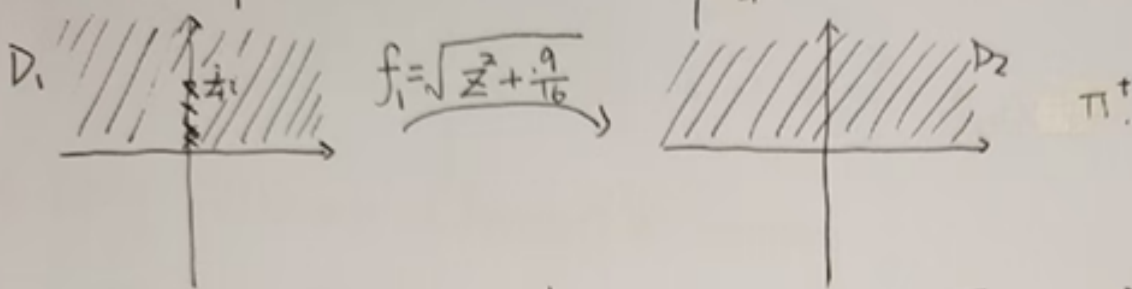
Homework 3

1.



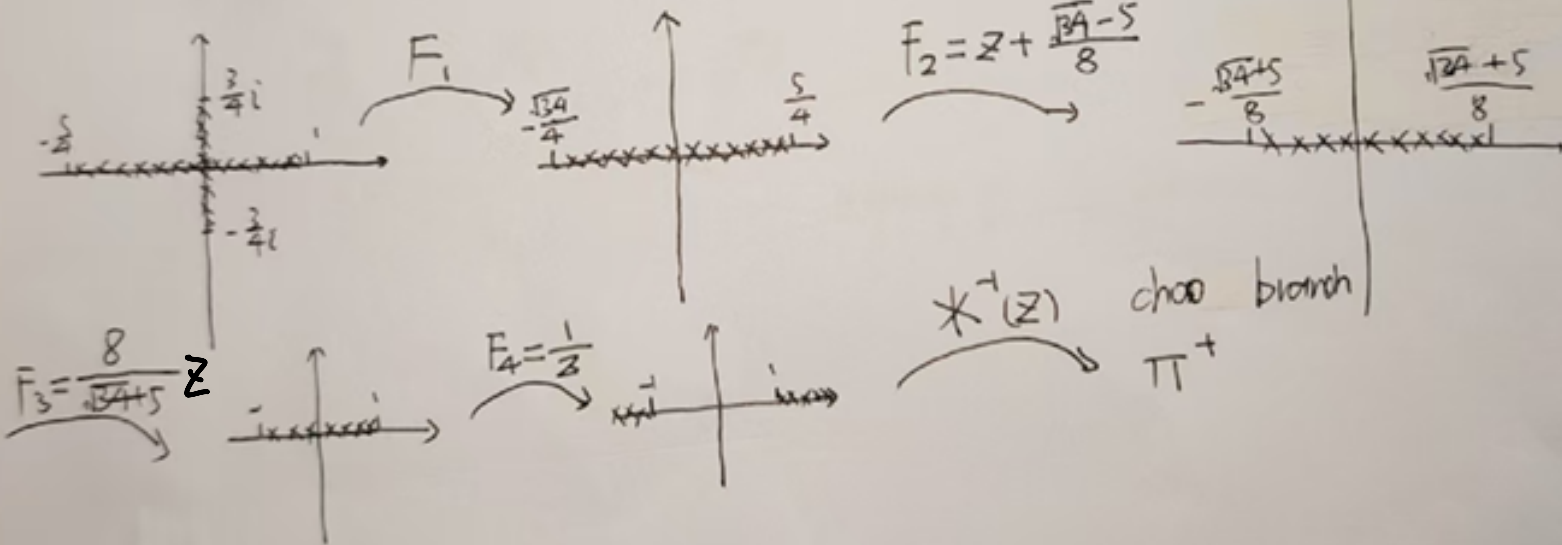


$\gamma_1 = \{(-\infty, -\frac{5}{4}) \cup (1, +\infty)\}$. $\gamma_2 = \{(-\infty, -\frac{\sqrt{34}}{4}) \cup (\frac{5}{4}, +\infty)\}$

$\therefore$ f extends to a homeom. of $D_1 \cup \gamma_1$ onto $D_2 \cup \gamma_2$.

By Refl. Princ.

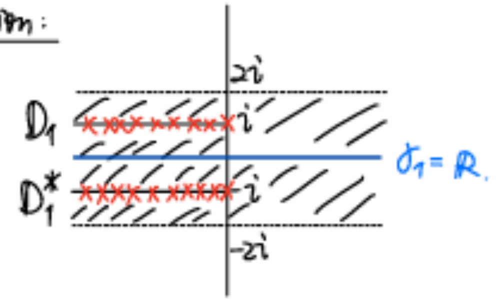
$\therefore$ f_1 extents to a conformal map $F_1 : \Omega \to \mathbb{C} \setminus [-\frac{\sqrt{34}}{4}, \frac{5}{4}]$

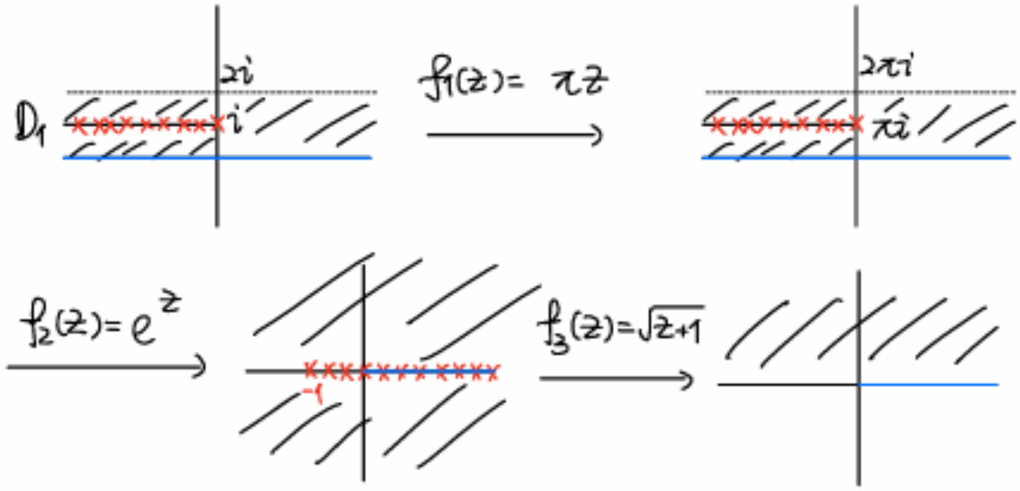


$\therefore F = *^{-1} \circ F_4 \circ F_3 \circ F_2 \circ F_1 \circ *(z)$

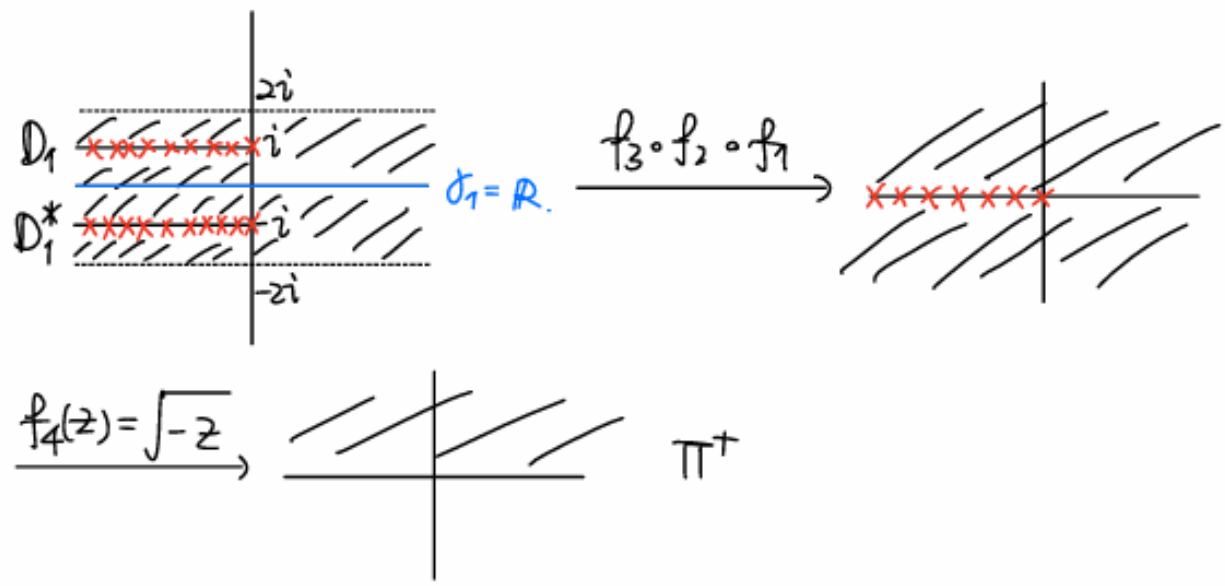
$\boxed{2.}$ **2.** Find a conformal mapping, transforming the domain $D := \{-2 < \operatorname{Im} z < 2\} \setminus \{\operatorname{Im} z = \pm 1, \operatorname{Re} z \le 0\}$ onto Π^+.

<u>Solution:</u>





By Schwarz Reflection Principle.



To sum up. $f = f_4 \circ f_3 \circ f_2 \circ f_1$.

4. Method 1

4. Let $f \in Aut(\Omega)$

By Carathéodory Thm for admissible domain,

f extends to a homeomorphism from the closure $\bar{\Omega} = \{1 \le |z| \le 2\}$ onto itself.

$\Rightarrow f$ maps $\partial\Omega = \{|z|=1\} \cup \{|z|=2\}$ to itself.

Since f is a homeomorphism, it must map connected components of $\partial\Omega$ to connected components of $\partial\Omega$

$\Rightarrow$ ① $C_1 = \{|z|=1\} \xrightarrow{f} C_1$

$C_2 = \{|z|=2\} \xrightarrow{f} C_2$

$\Rightarrow$ if $|z|=1$, $|f(z)|=1$.

if $|z|=2$, $|f(z)|=2$

Consider $h(z) = f(z)/z$. (h is analytic and $\neq 0$ in Ω)

$\Rightarrow$ on C_1, C_2, $|h(z)| = |f(z)|/|z| = 1$

By the Maximum Modulus Principle for an annulus, $|h(z)| = 1$ throughout Ω

$\Rightarrow f(z) = e^{i\theta} z$ (for some constant $\theta \in \mathbb{R}$)

② $f(C_1) = C_2$, $f(C_2) = C_1$

Consider $g(z) = f(z) \cdot z$, It is analytic in Ω

$\Rightarrow$ on C_1, C_2, $|g(z)| = |f(z)| \cdot |z| = 2$

By the Maximum Modulus Principle for an annulus, $|g(z)| = 2$ throughout Ω

$\Rightarrow f(z) = \frac{2e^{i\varphi}}{z}$, (for some constant $\varphi \in \mathbb{R}$)

$\Rightarrow Aut(\Omega)$ consists of all maps of the form $\boxed{f(z) = e^{i\theta} z \text{ and } f(z) = \frac{2e^{i\varphi}}{z}}$ ($\theta, \varphi \in \mathbb{R}$, constants)

Method 2

4. since $\partial\Omega$ is Jordan curve, by Caratheodory theorem. any $f\in Aut(\Omega)$ can be extended continuously to $Aut(\overline{\Omega})$

denote $C_r=\{|z|=r\}$. $\Omega_n=\{2^{-n}<|z|<2^{-n+1}\}$

then $f(C_1)=C_1$ or $f(C_1)=C_2$.

suppose $f(C_1)=C_1$, otherwise replace f by $\frac{2}{f}$

denote $g_0=f$.

construct $g_n(z)=\frac{1}{2^n}f(2^n z)$ on $\overline{\Omega_n}$

then we can easily check $g_n(C_{2^{-n}})=C_{2^{-n}}$, $g_n(C_{2^{-n+1}})=C_{2^{-n+1}}$

and g_n is holomorphic

consider $g(z)=g_n(z)$ for $z\in\overline{\Omega_n}$, g is well-defined since g_n and g_{n+1} are equal on $C_{2^{-n}}$, apply reflection principle.

$g\in \mathcal{O}(\{0<|z|<2\})$

since $|g(z)|<2^{-n}$ if $|z|<2^{-n}$, $\lim_{z\to 0} g(z)=0$

thus we can extend g to $z=0$ by $g(0)=0$

so $g\in Aut(\{|z|<2\})$ and $g(0)=0$, by Schwarz lemma. $g(z)=cz$ for some $c\neq 0$, since g is bijective, $|c|=1$

hence $Aut(\Omega)=\{e^{i\theta}z\}\cup\{e^{i\theta}\cdot\frac{2}{z}\}$

8. Method 1

$\tan z = z \iff \sin z - z\cos z = 0$

On $\partial B_{n\pi}(R)$, $\sin z = 0$

$|z\cos z| > |\sin z|$

So # zeros $(\sin z - z\cos z)$ = # zeros $(z\cos z)$

In $B_{n\pi}(R)$, # zeros $(z\cos z) = 2n+1$

So in $B_{n\pi}(R)$, # zeros $(\sin z - z\cos z) = 2n+1$

But $\sin z - z\cos z = 0$ has $2n+1$ real roots. (See below)

thus $\sin z - z\cos z = 0$ only has real roots. Let $n \to \infty$,

$\tan z = z$ only has real roots.

Method 2

8. Let $f = z$, $g = -\tan z$ on the circle $|z| = \pi n$.

$\tan z$ is bounded outside the ε-neighbourhood of it's polar $\frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

∴ Choose M s.t. $M \geq |\tan z|$ holds outside the ε-neighbourhood of $\frac{\pi}{2} + n\pi$ $n \in \mathbb{Z}$

∴ on the circle $|z| = \pi n$ for large enough $n = [\frac{M}{\pi}] + 1$

s.t. $|z| = n\pi > M \geq |\tan z|$ on $B_{n\pi}(0)$

∴ By Roche Thm for meromorphic function:

#Zeros $(z - \tan z)$ − #poles $(z - \tan z)$ = #zero (z) − #poles (z) in $B_{n\pi}(0)$

∴ #zeros $(z - \tan z) = 2n + 1 - 0 = 2n+1$ i.e. $z = \tan z$ has $2n+1$ roots in $B_{n\pi}(0)$ for n large enough

and we can find $2n+1$ real roots.

3 roots on $(-\frac{\pi}{2}, \frac{\pi}{2})$, one roots in $(\frac{\pi}{2} + k\pi, \frac{3\pi}{2} + k\pi)$ for each $k \in \mathbb{N}$

one roots in $(-\frac{3\pi}{2} - k\pi, -\frac{\pi}{2} - k\pi)$ for each $k \in \mathbb{N}$

totally $2n+1$ real roots in $B_{n\pi}(0)$ ∴ all roots in $B_{n\pi}(0)$ are real roots.

Let $n \to \infty$ ∴ $z = \tan z$ only has real roots.