陈悦 12311410. Mathematical Statistics. Assignment 1.

$\boxed{\text{Part 1. 6 questions from Q1.1} \sim \text{Q1.12.}}$

$\boxed{\text{Q1.1}}$

(a) $M_X(t) = E(e^{tX}) = \sum_{i=0}^{n} e^{ti} \binom{n}{i} p^i (1-p)^{n-i} = (p \cdot e^t + 1-p)^n.$

(b) $E(X) = \left.\frac{d M_X(t)}{dt}\right|_{t=0} = n(p \cdot e^t + 1-p)^{n-1} \cdot pe^t \Big|_{t=0} = np.$

$$E(X^2) = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = \left[n(n-1)p^2(p \cdot e^t + 1-p)^{n-2} e^{2t} + np(pe^t + 1-p)^{n-1} e^t\right]\Big|_{t=0}$$

$$= (n^2-n)p^2 + np.$$

$Var(X) = E(X^2) - (EX)^2 = np(1-p).$

(c) Since $X \perp\!\!\!\perp Y$, $F_{X,Y}(x,y) = F_X(x) \times F_Y(y).$

let $Z = X+Y$. $P_r(Z=z) = P_r(X+Y=z) = \sum_{k=0}^{n} P_r(X=k) P_r(Y=Z-X \mid X=k)$

$$= \sum_{k=0}^{n} P_r(X=k) P_r(Y=Z-X)$$

$$= \sum_{k=0}^{\min(n,z)} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^{z-k} e^{-\lambda}}{(z-k)!}$$

Thus, $P_r(Z=z) = \sum_{k=0}^{\min(n,z)} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^{z-k} e^{-\lambda}}{(z-k)!}$ (Support: $\mathbb{N}$).

$\boxed{\text{Q1.2}}$

(a) $P_r(X=1) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}.$ $\quad P_r(X=3) = \frac{3}{16} + \frac{1}{16} = \frac{1}{4}.$

$Pr(X=2) = \frac{2}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}$. $\quad Pr(X=4) = \frac{4}{16} = \frac{1}{4}$.

(b). $Z = X+Y$.

$Pr(Z=2) = \frac{1}{16}$. $\quad Pr(Z=3) = \frac{1}{16}$. $\quad Pr(Z=4) = \frac{1}{16} + \frac{2}{16} = \frac{3}{16}$.

$Pr(Z=5) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$. $\quad Pr(Z=6) = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$.

$Pr(Z=7) = \frac{1}{16}$. $\quad Pr(Z=8) = \frac{1}{4}$.

$\boxed{\text{Q 1.3}}$

(a) $f(x|y) = \frac{f(x,y)}{f_Y(y)}$. $\quad f(y|x) = \frac{f(x,y)}{f_X(x)}$. $\quad \frac{f_X(x)}{f_Y(y)} = \frac{y}{x} \frac{1-e^{-bx}}{1-e^{-by}}$.

$f_Y(y) = f_X(x) \cdot \frac{x}{y} \frac{1-e^{-by}}{1-e^{-bx}}$. $\quad \underbrace{\int_0^b f_Y(y)\,dy}_{=1} = f_X(x) \cdot \frac{x}{1-e^{-bx}} \cdot \int_0^b \frac{1-e^{-by}}{y}\,dy$.

$$f_X(x) = \frac{1-e^{-bx}}{x \cdot \int_0^b \frac{1-e^{-by}}{y}\,dy} \quad (0<x<b<+\infty)$$

(b). Suppose existence

Denote: $\lim_{b\to\infty} \int_0^b \frac{1-e^{-by}}{y}\,dy = \frac{1}{c}$.

$f_X(x) = c \cdot \lim_{b\to\infty} \frac{1-e^{-bx}}{x} = \frac{c}{x}$.

$\int_0^{+\infty} f_X(x)\,dx = c \cdot \ln|x| \Big|_{x=0}^{x=+\infty} = \infty \neq 1$. ↯.

$\Rightarrow$ Not exist.

$\boxed{Q1.4}$

(a). $Pr(X=x_1) \propto \dfrac{Pr(X=x_1 \mid Y=y_1)}{Pr(Y=y_1 \mid X=x_1)} = \dfrac{a_{11}}{b_{11}} = \dfrac{6}{7}.$

$$Pr(X=x_2) \propto \frac{Pr(X=x_2 \mid Y=y_1)}{Pr(Y=y_1 \mid X=x_2)} = \frac{a_{21}}{b_{21}} = 1.$$

$$Pr(X=x_3) \propto \frac{Pr(X=x_3 \mid Y=y_1)}{Pr(Y=y_1 \mid X=x_3)} = \frac{a_{31}}{b_{31}} = \frac{12}{7}.$$

$$\frac{6}{7} : 1 : \frac{12}{7} = 6 : 7 : 12.$$

$$\sum_{i=1}^{3} Pr(X=x_i) = 1 \Rightarrow Pr(X=x_1) = \frac{6}{6+7+12} = \frac{6}{25}$$

$$Pr(X=x_2) = \frac{7}{6+7+12} = \frac{7}{25}$$

$$Pr(X=x_3) = \frac{12}{6+7+12} = \frac{12}{25}$$

$$Pr(Y=y_1) \propto \frac{Pr(Y=y_1 \mid X=x_1)}{Pr(X=x_1 \mid Y=y_1)} = \frac{b_{11}}{a_{11}} = \frac{7}{6}.$$

$$Pr(Y=y_2) \propto \frac{Pr(Y=y_2 \mid X=x_1)}{Pr(X=x_1 \mid Y=y_2)} = \frac{b_{12}}{a_{12}} = \frac{2}{3}.$$

$$Pr(Y=y_3) \propto \frac{Pr(Y=y_3 \mid X=x_1)}{Pr(X=x_1 \mid Y=y_3)} = \frac{b_{13}}{a_{13}} = \frac{7}{6}.$$

$$Pr(Y=y_4) \propto \frac{Pr(Y=y_4 \mid X=x_1)}{Pr(X=x_1 \mid Y=y_4)} = \frac{b_{14}}{a_{14}} = \frac{7}{6}.$$

$$\frac{7}{6} : \frac{2}{3} : \frac{7}{6} : \frac{7}{6} = 7 : 4 : 7 : 7.$$

$$Pr(Y = y_1) = \frac{7}{7+4+7+7} = \frac{7}{25}.$$

$$Pr(Y = y_2) = \frac{4}{7+4+7+7} = \frac{4}{25}.$$

$$Pr(Y = y_3) = \frac{7}{7+4+7+7} = \frac{7}{25}.$$

$$Pr(Y = y_3) = \frac{7}{7+4+7+7} = \frac{7}{25}$$

In conclusion,

| $X$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $Pr(X = x_i)$ | $\frac{6}{25}$ | $\frac{7}{25}$ | $\frac{12}{25}$ |

| $Y$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $Pr(Y = y_i)$ | $\frac{7}{25}$ | $\frac{4}{25}$ | $\frac{7}{25}$ | $\frac{7}{25}$. |

(b) $Pr(X = x_i, Y = y_j) = Pr(X = x_i \mid Y = y_j) \cdot Pr(Y = y_j) = a_{ij} \cdot Pr(Y = y_j)$

The joint distribution of $(X, Y)$ is given by

$$P = \begin{pmatrix} \frac{1}{25} & \frac{1}{25} & \frac{3}{25} & \frac{1}{25} \\ \frac{2}{25} & \frac{2}{25} & \frac{1}{25} & \frac{2}{25} \\ \frac{4}{25} & \frac{1}{25} & \frac{3}{25} & \frac{4}{25} \end{pmatrix},$$

where the $(i,j)$-element of $P$ is $p_{ij} = Pr(X = x_i, Y = y_j)$

$\boxed{Q1.5}$

(a) $$E(|X-b|) = \int_{-\infty}^{\infty} |x-b| f(x)\,dx.$$

$$= \int_{-\infty}^{b} (b-x) f(x)\,dx + \int_{b}^{\infty} (x-b) f(x)\,dx.$$

$$= \int_{-\infty}^{m} (b-x) f(x)\,dx + \int_{m}^{b} (b-x) f(x)\,dx$$

$$+ \int_{b}^{m} (x-b) f(x)\,dx + \int_{m}^{\infty} (x-b) f(x)\,dx$$

$$= \int_{-\infty}^{m} (m-x) f(x)\,dx + \cancel{(b-m) \int_{-\infty}^{m} f(x)\,dx}$$

$$+ \int_{m}^{\infty} (x-m) f(x)\,dx + \cancel{(m-b) \int_{m}^{\infty} f(x)\,dx}$$

$$+ 2\int_{m}^{b} (b-x) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} |x-m| f(x)\,dx + 2\int_{m}^{b} (b-x) f(x)\,dx = E(|X-m|) + 2\int_{m}^{b} (b-x) f(x)\,dx.$$

(b). Since $m$ is the unique median (fixed), $E(|X-m|)$ is known.

$$E(|X-b|) = E(|X-m|) + 2\int_{m}^{b} (b-x) f(x)\,dx.$$

if $b > m$, $\int_{m}^{b} \underset{>0}{(b-x)} \underset{>0}{f(x)}\,dx > 0.$

else if $b < m$. $\int_{m}^{b} \underset{<0}{(b-x)} \underset{>0}{f(x)}\,dx > 0.$

else. $b = m$. $\int_{m}^{b} (b-x) f(x)\,dx = 0.$

Thus, $E(|X-b|) \geq E(|X-m|)$

where the equivalence holds when $b=m$.

$\Rightarrow b=m.$

## Q 1.6

(a) $\Pr\left(\frac{1}{4}<X<\frac{5}{8}\right) = \Pr\left(X<\frac{5}{8}\right) - \Pr\left(X\leq\frac{1}{4}\right)$

$= F\left(\frac{5}{8}\right) - F\left(\frac{1}{4}\right) = 1-2\left(1-\frac{5}{8}\right)^2 - 2\left(\frac{1}{4}\right)^2$

$= 1-\frac{9}{32}-\frac{1}{8} = \frac{19}{32}.$

(b) $f(x) = F'(x) = \begin{cases} 0, & x<0 \text{ or } x\geq 1 \\ 4x, & 0\leq x<\frac{1}{2} \\ -4x+4, & \frac{1}{2}\leq x<1. \end{cases}$

$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\frac{1}{2}} 4x^2\,dx + \int_{\frac{1}{2}}^1 (-4x^2+4x)\,dx$

$= 4\left(\frac{1}{3}x^3\Big|_{x=0}^{x=\frac{1}{2}} + \left(-\frac{1}{3}x^3+\frac{1}{2}x^2\right)\Big|_{x=\frac{1}{2}}^{x=1}\right)$

$= 4\cdot\left(\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{3}+\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{2}\cdot\frac{1}{4}\right) = \frac{1}{2}.$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\frac{1}{2}} 4x^3\,dx + \int_{\frac{1}{2}}^1 (-4x^3+4x^2)\,dx$

$= x^4\Big|_{x=0}^{x=\frac{1}{2}} + \left(-x^4+\frac{4}{3}x^3\right)\Big|_{x=\frac{1}{2}}^{x=1}$

$= \frac{1}{16} + \left(-1+\frac{4}{3}\right) - \left(-\frac{1}{16}+\frac{1}{6}\right) = \frac{1}{8}+\frac{1}{3}-\frac{1}{6} = \frac{7}{24}.$

$\mathrm{Var}(X) = E(X-EX)^2 = E(X^2)-(EX)^2 = \frac{7}{24}-\frac{1}{4} = \frac{1}{24}.$

Part 2. 3 questions from Q1.13 ~ Q1.17.

## Q1.13

**Proof:** Since $1_{X>\lambda\mu}$ is semi-positive, $X$ is positive

$$\Pr(X>\lambda\mu)\, E(X^2)$$

$$= E\left[1^2_{X>\lambda\mu}\right] E(X^2)$$

$$\geq \left\{ E\left[1_{X>\lambda\mu}\cdot X\right]\right\}^2.$$

(Cauchy-Schwarz)

$$= \left\{E\left[X-\lambda\mu+\lambda\mu+1_{X>\lambda\mu}\cdot X - X\right]\right\}^2.$$

$$= \left\{\underbrace{(1-\lambda)\mu+\lambda\mu}_{\geq 0} - E\left[1_{X\leq\lambda\mu}\cdot X\right]\right\}^2.$$

$$\geq \left\{(1-\lambda)\mu+\lambda\mu - E\left[1_{X\leq\lambda\mu}\cdot\lambda\mu\right]\right\}^2$$

$$\geq \left\{(1-\lambda)\mu+\lambda\mu-\lambda\mu\right\}^2 = (1-\lambda)^2\mu^2. \quad \square$$

## Q1.14.

(a). $$F(x) = \Pr(X\leq x) = \int_{-\infty}^{x} \frac{e^{-\frac{t-\mu}{\alpha}}}{\alpha\left(1+e^{-\frac{t-\mu}{\alpha}}\right)^2}dt. = \int_{-\infty}^{x} -\frac{d\left(1+e^{-\frac{t-\mu}{\alpha}}\right)}{\left(1+e^{-\frac{t-\mu}{\alpha}}\right)^2}$$

$$= \frac{1}{1+e^{-\frac{t-\mu}{\alpha}}}\Bigg|_{t=-\infty}^{t=x} = \frac{1}{1+e^{-\frac{x-\mu}{\alpha}}}.$$

$$\xi_q = F^{-1}(q) = -\alpha \ln\left(\frac{1}{q}-1\right)+\mu$$

$\xi_{0.5} = \mu.$

(b). $F(x) = \Pr(X \le x) = \int_0^x \frac{t}{a^2} e^{-\frac{t^2}{2a^2}} dt = \int_0^x -e^{-\frac{t^2}{2a^2}} d(-\frac{t^2}{2a^2})$

$= -e^{-\frac{t^2}{2a^2}} \Big|_{t=0}^{t=x} = -e^{-\frac{x^2}{2a^2}} + 1. \quad = q$

$\xi_q = F^{-1}(q) = \sqrt{2a^2 \ln(1-q)}$

$\xi_{0.5} = \sqrt{2a^2 \ln \frac{1}{2}}. = \sqrt{a^2 \ln 4}.$

$\boxed{\text{Q1.15.}}$

(a) $f(x) = \frac{x(2\alpha+x)}{\alpha(\alpha+x)^2} \mathbb{1}_{0<x<\alpha} + \frac{\alpha^2(\alpha+2x)}{x^2(\alpha+x)^2} \mathbb{1}_{x>\alpha}.$

$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\alpha} \frac{x(2\alpha+x)}{\alpha(\alpha+x)^2} dx + \int_{\alpha}^{\infty} \frac{\alpha^2(\alpha+2x)}{x^2(\alpha+x)^2} dx$

$= \int_0^{\alpha} \left[\frac{1}{\alpha} - \frac{\alpha}{(\alpha+x)^2}\right] dx + \int_{\alpha}^{\infty} \left[\frac{\alpha^2}{x^2(\alpha+x)} + \frac{\alpha^2}{x(\alpha+x)^2}\right] dx.$

$= 1 + \alpha \cdot \frac{1}{\alpha+x}\Big|_{x=0}^{x=\alpha} + (-\frac{\alpha}{x})\Big|_{x=\alpha}^{x=\infty} + (\frac{\alpha}{\alpha+x})\Big|_{x=\alpha}^{x=\infty}$

$= 1 + \frac{1}{2} - 1 + 1 - \frac{1}{2} = 1.$

(b). For $0 < x \le \alpha$

$F(x) = \int_0^x \frac{t(2\alpha+t)}{\alpha(\alpha+t)^2} dt = \frac{x}{\alpha} + \alpha \cdot \frac{1}{\alpha+t}\Big|_{t=0}^{t=x} = \frac{x}{\alpha} + \frac{\alpha}{\alpha+x} - 1.$

since $0<x\le\alpha$, $\propto F(x)\le\frac{1}{2}$.

$\Rightarrow \xi_{0.5}=F^{-1}(0.5)=\alpha$.

**Part 3. 4 questions ( Q 1.18 ~ Q 1.21 )**

$\boxed{Q\,1.18}$

(a) $f(x|y)=\dfrac{f(x,y)}{f_Y(y)}$, $f(y|x)=\dfrac{f(x,y)}{f_X(x)}$.

$$\Rightarrow \frac{f_X(x)}{f_Y(y)}=\frac{2(x+1)}{1+4y}. \qquad f_X(x)\int_0^1(1+4y)\,dy=2(x+1)\cdot\int_0^1 f_Y(y)\,dy.$$

$$f_X(x)=\frac{2(x+1)}{(y+2y^2)\Big|_{y=0}^{y=1}}=\frac{2}{3}(x+1). \qquad (0<x<1)$$

(b). $f(x,y)=f(y|x)\cdot f_X(x)=\frac{2}{3}(x+2y)$. $\quad(0<x<1,\ 0<y<1)$

$\boxed{Q\,1.19}$

$X\overset{d}{=}UY$.

$F(x)=P_r(X\le x)=P_r(UY\le x)$

① $x<0$, $UY\ge0$, $F(x)=0$. $\quad f_X(x)=\frac{d}{dx}F(x)=0$.

② $x\ge0$, $F(x)=P_r(UY\le x)=\int_0^1 P_r(UY\le x\,|\,U=u)\cdot 1\,du$.

$$\overset{U\perp\!\!\!\perp Y}{=}\int_0^1 P_r\left(Y\le\frac{x}{u}\right)du=\int_0^1\int_0^{\frac{x}{u}}f_Y(y)\,dy\,du$$

$$f_X(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \int_0^1 \int_0^{\frac{x}{u}} f_Y(y)\, dy\, du$$

$$= \int_0^1 \frac{d}{dx}\left( \int_0^{\frac{x}{u}} f_Y(y)\, dy \right) du = \int_0^1 \frac{1}{u} f_Y\left(\frac{x}{u}\right) du.$$

N-L

$\int_0^{\frac{x}{u}} f_Y(y)\, dy$ continuous on $(0,1) \times (0,+\infty)$

and $\int_0^{\frac{x}{u}} f_Y(y)\, dy \in C'(0,+\infty)$.

Thus $f_X(x) = \begin{cases} 0, & x < 0 \\ \int_0^1 \frac{1}{u} f_Y\left(\frac{x}{u}\right) du, & x \geq 0 \end{cases}$

$\boxed{\text{Q 1.20}}$

(a) $F_Y(y) = P_r(F_2(X_1) \leq y) = P_r(X_1 \leq F_2^{-1}(y))$

$$= \int_{-\infty}^{F_2^{-1}(y)} f_1(x)\, dx.$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\infty}^{F_2^{-1}(y)} f_1(x)\, dx = f_1(F_2^{-1}(y)) \cdot \frac{d}{dy} F_2^{-1}(y)$$

N-L.

$$= \frac{f_1(F_2^{-1}(y))}{f_2(F_2^{-1}(y))}. \qquad (0 < y < 1)$$

(b). $X_1 \sim IBeta(\alpha, \beta)$. $f_1(t) = \frac{1}{B(\alpha,\beta)} \cdot \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}$, $t > 0$.

$X_2 \sim IBeta(1,1)$. $f_2(t) = \frac{1}{(1+t)^2}$, $t > 0$.

$$F_2(x) = \int_0^x \frac{1}{(1+t)^2}\, dt = -\frac{1}{1+t}\Big|_{t=0}^{t=x} = 1 - \frac{1}{1+x}$$

$$F_2^{-1}(y) = \frac{1}{1-y} - 1 = \frac{y}{1-y}.$$

$$f_Y(y) = \frac{1}{B(\alpha,\beta)} \frac{y^{\alpha-1}}{(1-y)^{\alpha-1}} (1-y)^{\alpha+\beta} (1-y)^{-2}.$$

$$= \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}. \quad \Rightarrow Y \sim \text{Beta}(\alpha,\beta).$$

$\boxed{\text{Q 1.21}}$

Proof: $\forall x > 0, \quad x^{-1} = \int_0^\infty e^{-tx}\, dt.$

$$\Rightarrow x^{-1} f_X(x) = \int_0^\infty e^{-tx} f_X(x)\, dt$$

$$\Rightarrow \underbrace{\int_0^\infty x^{-1} f_X(x)\, dx}_{= E(X^{-1})} = \int_0^\infty \int_0^\infty e^{-tx} f_X(x)\, dt\, dx$$

$= \int_0^\infty \int_0^\infty e^{-tx} f_X(x)\, dx\, dt$ — Since $e^{-tx} f_X(x)$ continuous on $(0,+\infty) \times [0,+\infty)$. $\int_0^\infty e^{-tx} f_X(x)\, dt$ uniformly convergent.

$$= \int_0^\infty E(e^{-tX})\, dt$$

$$= \int_0^\infty M_X(-t)\, dt. \quad \square.$$