CS201: Discrete Math for Computer Science
2025 Spring Semester Written Assignment #1
Due: 23:55 on Mar. 17th, 2025, please submit through Blackboard

Please answer questions in English. Using any other language will lead to a zero point.

Q. 1. Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(a) You do not get 100 marks on the final.

(b) You get 100 marks on the final, but you do not get an A in this course.

(c) You will get an A in this course if you get 100 marks on the final.

(d) If you do not get 100 marks on the final, then you will not get an A in this course.

(e) Getting 100 marks on the final is sufficient for getting an A in this course.

(f) You get an A in this course, but you do not get 100 marks on the final.

(g) Whenever you get an A in this course, you got 100 marks on the final.

Solution:

(a) $\neg p$

(b) $p \land \neg q$

(c) $p \to q$

(d) $\neg p \to \neg q$

(e) $p \to q$

(f) $q \land \neg p$

(g) $q \to p$

$\square$

Q. 2. Construct a truth table for each of these compound propositions.

(a) $(p \oplus q) \to (p \wedge q)$

(b) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

Solution: The details are omitted. The final results are given as follows:

(a)

p	q	$(p \oplus q) \to (p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	T

(b)

p	q	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	T
T	F	T
F	T	T
F	F	T

$\square$

Q. 3. "Logic is difficult or not many students like logic."
"If mathematics is easy, then logic is not difficult."
Which of the following are valid conclusions?

(a) That mathematics is not easy, if many students like logic.

(b) That not many students like logic, if mathematics is not easy.

(c) That mathematics is not easy or logic is difficult.

(d) That logic is not difficult or mathematics is not easy.

(e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

Solution:

"Logic is difficult or not many students like logic." That is $p \vee \neg q$, where $p =$"Logic is difficult", $q =$"many students like logic"

"If mathematics is easy, then logic is not difficult." That is $r \to \neg p$, where $r =$"mathematics is easy"

Then express the conclusions:

(a) $q \to \neg r$. That is $\neg q \vee \neg r$. When p is true, r must be false according to $r \to \neg p$. When p is false, then q must be false according to $p \vee \neg q$. Thus $q \to \neg r$ is true.

(b) $\neg r \to \neg q$. When p, q are true and r is false, both prerequisites are satisfied, while $\neg r \to \neg q$ is false.

(c) $r \to \neg p$. When p is false, q is false and r is true, $p \vee \neg q$ and $r \to \neg p$, however $r \to \neg p$ is false.

(d) $\neg p \vee \neg r$. $\neg p \vee \neg r \leftrightarrow r \to \neg p$.

(e) $\neg q \to (\neg p \vee \neg r)$. Since $\neg p \vee \neg r$ is true, $\neg q \to (\neg p \vee \neg r) = q \vee \mathrm{T} = \mathrm{T}$.

Q. 4. Determine whether the following statements are correct or incorrect. Explain your answer. Assume that p, q and r are logical propositions, x and y are real numbers, and m and n are integers.

(1) $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology.

(2) $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$ are equivalent.

(3) Under the domain of all real numbers, the truth value of $\exists x \forall y (y \neq 0 \to xy = 1)$ is T.

(4) Under the domain of all integers, the truth value of $\exists n \exists m (n^2 + m^2 = 5)$ is T.

Solution:

(1) Incorrect. This can be proven using truth table.

Since the truth value of $(\neg p \wedge (p \to q)) \to \neg q$ is not always T, it is not a tautology. (Any proof is acceptable, as long as it explains that under some p and q, the truth value of $(\neg p \wedge (p \to q)) \to \neg q$ is false.)

p	q	$\neg p$	$p \to q$	$\neg p \wedge (p \to q)$	$\neg q$	$(\neg p \wedge (p \to q)) \to \neg q$
T	T	F	T	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

(2) Correct. This can be proven as follows:

$$\begin{aligned} (p \vee q) \to r &\equiv \neg(p \vee q) \vee r && \text{(Useful Law)} \\ &\equiv (\neg p \wedge \neg q) \vee r && \text{(De Moegan's Law)} \\ &\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{(Distributive Law)} \\ &\equiv (p \to r) \wedge (q \to r) && \text{(Useful Law)} \end{aligned}$$

Any proof that shows the equivalence is acceptable.

(3) Incorrect. This proposition means that there is a real number x for which $y \neq 0 \to xy = 1$ for every real number y. Consider an arbitrary x. Suppose $y_1 \neq 0$ and $xy_1 = 1$. Let $y_2 = 2y_1$. Then, $xy_2 = 2$, i.e., $y \neq 0 \to xy = 1$ does not hold for every y.

(4) Correct. When $n = 1$ and $m = 2$, $n^2 + m^2 = 5$.

Q. 5. For each of the following argument, determine whether it is valid or invalid. Explain using the validity of its argument form.

(1) Premise 1: If you did not finish your homework, then you cannot answer this question.

Premise 2: You finished your homework.

Conclusion: You can answer this question.

(2) Premise 1: If all students in this class has submitted their homework, then all students can get 100 in the final exam.

Premise 2: There is a student who did not submit his or her homework.

Conclusion: It is not the case that all student can get 100 in the final exam.

Solution:

(1) Invalid. Let p denote "you finished your homework". Let q denote "you can answer this question". Thus, premises 1 and 2 can be represented as $\neg p \to \neg q$ and p, respectively. Conclusion can be represented as q. This argument form is not valid, since $((\neg p \to \neg q) \wedge p) \to q$ is not a tautology. This is because when p is T and q is F, the truth value of $((\neg p \to \neg q) \wedge p) \to q$ is F.

(2) Invalid. Consider the domain of this class. Let $P(x)$ denote "student x has submitted his or her homework". Let $Q(x)$ denote "student x can get 100 in the final exam". Premises 1 and 2 can be represented as $\forall x P(x) \to \forall x Q(x)$ and $\exists x(\neg P(x))$, respectively. The conclusion can be represented as $\neg \forall x Q(x)$. This argument form is not valid, since $((\forall x P(x) \to \forall x Q(x)) \wedge \exists x(\neg P(x)) \to \neg \forall x Q(x)$ is not a tautology. Consider the case where both $\exists x(\neg P(x))$ and $\forall x Q(x)$ are T. Thus, $\forall x P(x)$ is F, since $\neg \forall x P(x) \equiv \exists x(\neg P(x))$ is T. Hence, $((\forall x P(x) \to \forall x Q(x)) \wedge \exists x(\neg P(x))$ is T. However, since $\neg \forall x Q(x)$ is F, the entire proposition is F.

Q. 6. Suppose that p, q, r, s are all logical propositions. You are given the following statement

$$(\neg r \vee (p \wedge \neg q)) \to (r \wedge p \wedge \neg q)$$

Prove that this implies $r \vee s$ using logical equivalences and rules of inference.

Solution:

$$\begin{array}{lll}
 & (\neg r \vee (p \wedge \neg q)) \to (r \wedge p \wedge \neg q) & \\
\equiv & \neg(\neg r \vee (p \wedge \neg q)) \vee (r \wedge p \wedge \neg q) & \text{Useful} \\
\equiv & (r \wedge \neg(p \wedge \neg q)) \vee (r \wedge p \wedge \neg q) & \text{De Morgan's} \\
\equiv & (r \wedge (\neg p \vee q)) \vee (r \wedge p \wedge \neg q) & \text{De Morgan's} \\
\equiv & (r \wedge (\neg p \vee q)) \vee (r \wedge (p \wedge \neg q)) & \text{Associative} \\
\equiv & r \wedge ((\neg p \vee q) \vee (p \wedge \neg q)) & \text{Distributive} \\
\equiv & r \wedge ((\neg p \vee q) \vee \neg(\neg(p \wedge \neg q))) & \text{Double negation} \\
\equiv & r \wedge ((\neg p \vee q) \vee \neg(\neg p \vee q))) & \text{De Morgan's} \\
\equiv & r \wedge T & \text{Negation} \\
\equiv & r & \text{Identity} \\
\to & r \vee s & \text{Addition}
\end{array}$$

Q. 7. Use logical equivalences to prove the following statements.

(a) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

(b) $\neg(p \rightarrow q) \rightarrow \neg q$ is a tautology.

(c) $(p \rightarrow q) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q))$ is a tautology.

Solution:

(a) We have

$$\begin{aligned}
&\neg(p \oplus q) \\
&\equiv \neg((p \wedge \neg q) \vee (\neg p \wedge q)) \quad \text{Definition} \\
&\equiv \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) \quad \text{De Morgan} \\
&\equiv (\neg p \vee q) \wedge (p \vee \neg q) \quad \text{De Morgan} \\
&\equiv (p \rightarrow q) \wedge (q \rightarrow p) \quad \text{Useful} \\
&\equiv p \leftrightarrow q \quad \text{Definition}
\end{aligned}$$

Thus, they are equivalent.

(b) We have

$$\begin{aligned}
&\neg(p \rightarrow q) \rightarrow \neg q \\
&\equiv \neg\neg(p \rightarrow q) \vee \neg q \quad \text{Useful} \\
&\equiv (p \rightarrow q) \vee \neg q \quad \text{Double negation} \\
&\equiv (\neg p \vee q) \vee \neg q \quad \text{Useful} \\
&\equiv \neg p \vee (q \vee \neg q) \quad \text{Associative} \\
&\equiv T \quad \text{Domination}
\end{aligned}$$

Therefore, it is a tautology.

(c) We have

$$
\begin{aligned}
&(p \to q) \to ((r \to p) \to (r \to q)) \\
&\equiv \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
&\equiv \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan} \\
&\equiv \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
&\equiv \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
&\equiv \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Complement} \\
&\equiv \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
&\equiv (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
&\equiv T \vee \neg r \quad \text{Complement} \\
&\equiv T \quad \text{Identity.}
\end{aligned}
$$

Thus, it is a tautology.

$\square$

Q. 8. Let $C(x)$ be the statement "x has a cat", let $D(x)$ be the statement "x has a dog" and let $F(x)$ be the statement "x has a ferret." Express each of these sentences in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

(a) A student in your class has a cat, a dog, and a ferret.

(b) All students in your class have a cat, a dog, or a ferret.

(c) Some student in your class has a cat and a ferret, but not a dog.

(d) No student in your class has a cat, a dog, and a ferret.

(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution:

(a) $\exists x(C(x) \wedge D(x) \wedge F(x))$

(b) $\forall x(C(x) \vee D(x) \vee F(x))$

(c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$

(d) $\neg \exists x(C(x) \wedge D(x) \wedge F(x))$

(e) $(\exists x C(x)) \wedge (\exists x D(x)) \wedge (\exists x F(x))$

$\square$

Q. 9. Prove that if $p \wedge q$, $p \to \neg(q \wedge r)$, $s \to r$, then $\neg s$.

Solution:

(1)	$p \wedge q$	Premise
(2)	p	Simplication of (1)
(3)	$p \to \neg(q \wedge r)$	Premise
(4)	$\neg(q \wedge r)$	Modens ponens (2) (3)
(5)	$\neg q \vee \neg r$	De Morgan
(6)	q	Simplication of (1)
(7)	$\neg r$	Disjunctive syllogism (6) (7)
(8)	$s \to r$	Premise
(9)	$\neg s$	Modus tollens (7) (8)

$\square$

Q. 10. (a) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \Rightarrow n \text{ is even}).$$

(b) Either the original statement in (a) or its negation is true. Which one is it and explain why?

Solution:

(a) The negation is

$$\exists x \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \wedge n \text{ is odd}).$$

(b) If n is odd then $n^3 + 6n + 5$ is even because n^3 is then odd and $6n$ is then even. Therefore, the original statement is true.

$\square$

Q. 11. Give a direct proof that: Let a and b be integers. If $a^2 + b^2$ is even, then $a + b$ is even.

Solution: Observe that $a^2+b^2 = (a+b)^2 - 2ab$. Thus, $(a+b)^2$ has the same parity as $a^2 + b^2$. So $(a + b)^2$ is even. Then $a + b$ is also even.

$\square$

Q. 12. Prove that $\sqrt[3]{2}$ is irrational.

Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

$\square$