Proof of Hall's Marriage Theorem By Mathematical Induction

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Theorem 1 (Hall's Marriage Condition)**.** *Given a family of finite sets $S_1, S_2, \ldots, S_n$. If for every subset of indices $I \subseteq \{1, 2, \ldots, n\}$, the following condition holds:*

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|$$

Then the family has a ***System of Distinct Representatives (SDR)****, i.e., there exist n distinct elements $a_1, a_2, \ldots, a_n$ such that $a_i \in S_i$ for all $i \in \{1, \ldots, n\}$.*

Proof. *We proceed by mathematical induction on n, the number of sets.*

Base Case: $n = 1$ *When $n = 1$, we have a single set S_1. The Hall condition for $I = \{1\}$ implies $|S_1| \geq 1$. Thus, S_1 is non-empty, and we can choose an element $a_1 \in S_1$ as its representative. The base case holds.*

Inductive Hypothesis *Assume that the theorem holds for any collection of k sets, where $k < n$.*

Inductive Step *We now prove the theorem for a collection of n sets, $S_1, \ldots, S_n$, that satisfies the Hall condition. We consider two cases.*

Case 1: The Strong Hall Condition *Assume that for every proper, non-empty subset $I \subset \{1, \ldots, n\}$ (where $|I| < n$), the condition holds strictly: $|\bigcup_{i \in I} S_i| > |I|$.*

1. *Choose an arbitrary element $a_n \in S_n$ (this is possible since $|S_n| \geq 1$).*
2. *For the remaining $n - 1$ sets, define new sets $S'_i = S_i \setminus \{a_n\}$ for $i \in \{1, \ldots, n-1\}$.*
3. *We show that this new collection $\{S'_i\}$ satisfies the Hall condition. For any non-empty subset $I \subseteq \{1, \ldots, n-1\}$:*

$$\left| \bigcup_{i \in I} S'_i \right| = \left| \left(\bigcup_{i \in I} S_i \right) \setminus \{a_n\} \right| \geq \left| \bigcup_{i \in I} S_i \right| - 1$$

   By our assumption for Case 1, $|\bigcup_{i \in I} S_i| \geq |I| + 1$. Therefore:

$$\left| \bigcup_{i \in I} S'_i \right| \geq (|I| + 1) - 1 = |I|$$

4. *By the inductive hypothesis, the collection $\{S'_i\}$ has an SDR, say $\{a_1, \ldots, a_{n-1}\}$. By construction, none of these representatives is a_n. Thus, $\{a_1, \ldots, a_{n-1}, a_n\}$ is an SDR for the original family.*

Case 2: The Critical Case *Assume there exists a proper, non-empty subset $J \subset \{1, \ldots, n\}$ (let $|J| = k$, where $1 \le k < n$) for which the condition holds exactly:*

$$\left| \bigcup_{j \in J} S_j \right| = |J| = k$$

1. *Consider the family of k sets $\{S_j\}_{j \in J}$. This family satisfies the Hall condition. Since $k < n$, by the inductive hypothesis, there exists an SDR for this family, say $\{a_j\}_{j \in J}$. Let $U_J = \bigcup_{j \in J} S_j$. Note that all representatives $\{a_j\}$ are in U_J and $|U_J| = |J| = k$.*

2. *Now consider the remaining $n - k$ sets. Let J' be the complement of J in $\{1, \ldots, n\}$. For each $i \in J'$, define a new set $S_i'' = S_i \setminus U_J$.*

3. *We show that this new family $\{S_i''\}_{i \in J'}$ satisfies the Hall condition. For any non-empty subset $K \subseteq J'$:*

$$\begin{aligned} \left| \bigcup_{i \in K} S_i'' \right| &= \left| \left(\bigcup_{i \in K} S_i \right) \setminus U_J \right| \\ &= \left| \left(\bigcup_{i \in K} S_i \right) \cup U_J \right| - |U_J| \\ &= \left| \bigcup_{i \in K \cup J} S_i \right| - |J| \end{aligned}$$

   By the original Hall condition on the family $\{S_i\}$, the last expression is $\ge (|K| + |J|) - |J| = |K|$.

4. *Since $n - k < n$, by the inductive hypothesis, the family $\{S_i''\}$ has an SDR, say $\{a_i\}_{i \in J'}$.*

5. *By construction, the representatives $\{a_i\}_{i \in J'}$ are disjoint from U_J, and thus disjoint from the representatives $\{a_j\}_{j \in J}$ found in the first step. Combining these two sets of representatives gives a complete SDR for the original family.*

Having proven both cases, the inductive step is complete. By the principle of mathematical induction, the theorem holds.