[2.18].

1. Introduction. (ppt).

2. First Order DEs.

2.1. Linear eqs

method of integrating factors (积分因子法).

General 1st-order linear eqs: $y' + p(t)y = g(t)$. (2.1)

Method of integrating factors:

Multiply both sides of (2.1) by $\mu(t)$:

$$\mu y' + \mu p y = \mu g.$$

Choose $\mu(t) > 0$ s.t. $\mu p = \mu'$. $\Rightarrow \dfrac{\mu'}{\mu} = p$.

$(\ln \mu(t))' = p(t)$. let $\ln \mu(t) = \int p(t)\,dt$.

$\Rightarrow \mu(t) = e^{\int p(t)\,dt}$. (an anti-derivative of $p(t)$)

△ 取一个原函数即可.

Then $(\mu y)' = \mu' y + \mu y' = \mu g$

$\Rightarrow \mu y = \int \mu g\,dt + C$. (C: any const).

$$\Rightarrow y = \frac{1}{\mu}\int \mu g\,dt + \frac{C}{\mu} = e^{-\int p(t)\,dt}\int \mu g\,dt + e^{-\int p(t)\,dt}\cdot C. \quad (2.2)$$

(2.2) is the general solution of (2.1).

Now, consider

$$(\text{IVP})\ \begin{cases} y' + \overset{\text{coefficient}}{p(t)}\, y = g(t) & \leftarrow \text{driving term / input.} \\ y(t_0) = y_0 & \leftarrow \text{initial condition / initial datum.} \end{cases}$$

By (2.2): Let $P(t) = \int p(t)\,dt$ be any antiderivative of $p(t)$.

$$y(t) = e^{-P(t)}\left[\int_{t_0}^{t} g(s)\, e^{P(s)}\, ds + c\right]$$

$$y_0 = y(t_0) = c \cdot e^{-P(t_0)} \Rightarrow c = y_0 \cdot e^{P(t_0)}.$$

$$\Rightarrow \text{Sol of (IVP)}: \quad y(t) = \underbrace{e^{-P(t)} \int_{t_0}^{t} g(s)\, e^{P(s)}\, ds}_{\text{respond to the input}} + \underbrace{e^{P(t_0)}\, y_0 \cdot e^{-P(t)}}_{\text{respond to initial datum}}. \quad (2.3)$$

E.g. 1. Find the general sol. of

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{\frac{t}{3}} \qquad (2.4)$$

Sol: Multiply $\mu(t) > 0$:

$$\mu y' + \frac{1}{2}\mu y = \frac{1}{2}\mu e^{\frac{t}{3}}.$$

$$\overset{\parallel}{(\mu y)'}. \qquad \frac{1}{2}\mu = \mu' \Rightarrow \mu = e^{\frac{1}{2}t}.$$

$$(e^{\frac{1}{2}t} y)' = \frac{1}{2}e^{\frac{1}{2}t} \cdot e^{\frac{t}{3}} \Rightarrow y = \frac{3}{5}e^{\frac{t}{3}} + c e^{-\frac{t}{2}}. \quad c: \text{any const.}$$

[2.20].

Recall last time. $y' + p(t)y = g(t)$. (2.1).

If $g(t)=0$. i.e. $y' + p(t)y = 0$ (H) is called a homogeneous eq.
齐次的.

Otherwise, (2.1) is called an inhomogeneous eq.

The general sol. of (2.1) is

$$y(t) = \underbrace{e^{-\int p(t)dt} \cdot \int g(t) e^{\int p(t)dt} dt}_{\text{a particular sol. of (2.1)}} + \underbrace{c \cdot e^{-\int p(t)dt}}_{\text{a general sol. of (H)}}.$$

$y(t) = y_d(t) + y_h(t)$. ---------- common feature of linear eqs.

$$\left[\, y(t) = \boxed{e^{-\int p(t)dt} \cdot \left(\int g(t) e^{\int p(t)dt} dt + c \right)} \,\right].$$

Variation of Parameters (常数变易法). motivation: compare the sol. of homogeneous eqs and inhomogeneous ones.

First, solve the homogeneous eq. $y' + p(t)y = 0$.

If: $(\mu y)' = 0$. $\mu = e^{\int p(t)dt}$.

$\mu y = c$. c: any const.

$$\Rightarrow y = c \cdot \mu^{-1} = \boxed{c \cdot e^{-\int p(t)dt}.}$$

Second, we set the sol of (2.1) in the form

$$y(t) = a(t) \cdot e^{-\int p(t)dt}. \qquad (c \to a(t)).$$

Then, $y'(t) = a'(t) \cdot e^{-\int p(t)dt} + \underbrace{a(t) \cdot e^{-\int p(t)dt} \cdot [-p(t)]}$

$(2.1) \Rightarrow a'(t)\, e^{-\int p(t)dt} = g(t) \dashrightarrow -p(t)y(t),$

$\Rightarrow a'(t) = g(t) \cdot e^{\int p(t)dt}.$

$\Rightarrow a(t) = \int g(t) \cdot e^{\int p(t)dt}\, dt + C.$

$\Rightarrow$ general sol. of (2.1): $y(t) = e^{-\int p(t)dt} \cdot \left[\int g(t) \cdot e^{\int p(t)dt}\, dt + C \right].$

E.g. Use variation of parameters to solve $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{\frac{t}{3}}.$

Sol: First solve $y' = -\frac{1}{2}y.$ $y = c \cdot e^{-\frac{1}{2}t}.$ c: any const.

Second set $y = a(t)\, e^{-\frac{1}{2}t}.$

Then, $y' = a'(t)\, e^{-\frac{1}{2}t} + \underbrace{a(t)\, e^{-\frac{1}{2}t} \cdot (-\tfrac{1}{2})}_{-\frac{1}{2}y,}$

$\Rightarrow a'(t)\, e^{-\frac{1}{2}t} = \frac{1}{2} e^{\frac{t}{3}}$

$\Rightarrow a(t) = \frac{3}{5} e^{\frac{5}{6}t} + c.$

$\Rightarrow$ general sol: $y(t) = \left[\frac{3}{5} e^{\frac{5}{6}t} + c \right] \cdot e^{-\frac{1}{2}t}.$ $\square$.

Ex: Solve the IVP: $\begin{cases} y' = y + \cos t. \\ y(0) = -\frac{1}{2}. \end{cases}$

Sol.: First solve $y' = y.$ $y = c \cdot e^{t}.$

Second set $y = a(t)\,e^{t}$.

Then $y' = a'(t)e^{t} + a(t)e^{t}$.

Let $a'(t)\,e^{t} = \cos t \Rightarrow a(t) = \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t + C$

$y = \frac{1}{2}\sin t - \frac{1}{2}\cos t + c\cdot e^{t}$. $\quad y(0) = -\frac{1}{2} + C = -\frac{1}{2} \Rightarrow C = 0$.

$\rightarrow y(t) = \frac{1}{2}\sin t - \frac{1}{2}\cos t$. $\square$

2.2. Separable Differential Eqs. (变量分离的方程)

General separable DE.s: $\dfrac{dy}{dt} = f(t)\cdot g(y)$. $\quad f, g$ are continuous

Step 1: Solve $g(y) = 0$.

If $g(a) = 0$, then $y(t) \equiv a$ is a constant sol. (equilibrium sol.)

(uniqueness ?)

Step 2: Suppose $g(y) \neq 0$.

$$\frac{dy}{g(y)} = f(t)dt \Rightarrow \int \frac{dy}{g(y)} = \int f(t)dt + C.$$

Eg. Solve $\dfrac{dy}{dx} = \dfrac{x^3}{2+y^2} = f(x)\cdot g(y)$

Sol: 1. $\frac{1}{2+y^2} = 0$ has no sol.s.

2. $(2+y^2)\,dy = x^3 dx \Rightarrow 2y + \frac{1}{3}y^3 = \frac{1}{4}x^4 + C$.

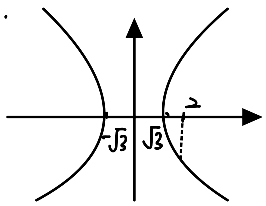
Rk. This defines a function $y=y(x)$ implicitly. We don't need to solve y in terms of x. which is not always possible anyway.

E.g. Solve $\begin{cases} y'y - x = 0 \\ y(2) = -1. \end{cases}$

Sol: $\frac{dy}{dx} y = x \Rightarrow y\,dy = x\,dx. \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C.$

$y(2) = -1: \frac{1}{2}\cdot 1 = \frac{1}{2}\cdot 4 + C \Rightarrow C = -\frac{3}{2}.$

$\Rightarrow$ Sol lies on $y^2 - x^2 = -3.$



$\Rightarrow y = -\sqrt{x^2 - 3}. \quad x \geq \sqrt{3}.$ (默认取初值点所在的最大区间).

E.g. Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right).$ ODEs with the form $\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$ is called homogeneous,

Sol: Let $u(x) = \frac{y(x)}{x}. \Rightarrow y = xu. \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}.$

$\Rightarrow u + x\frac{du}{dx} = u + \tan(u). \Rightarrow \frac{du}{dx} = \frac{\tan(u)}{x}.$

(Here this 'homogeneous' is different from that in P.3). Such ODEs can be solved by letting $u = \frac{y}{x}$.

1. Consider $\tan(u) = 0. \Rightarrow u = k\pi,\ k \in \mathbb{Z}. \Rightarrow y = k\pi x.\ k \in \mathbb{Z}.$

2. Consider $\tan(u) \neq 0. \Rightarrow u \neq k\pi,\ y \neq k\pi x.\ k \in \mathbb{Z}.$

$$\frac{du}{\tan(u)} = \frac{dx}{x} \Rightarrow \frac{\cos(u)\,du}{\sin(u)} = \frac{dx}{x} \Rightarrow \ln|\sin(u)| = \ln|x| + C.$$

$\Rightarrow |\sin(u)| = e^C|x|. \quad \sin(u) = c\cdot x. \quad c$: any constant. $c \neq 0.$

$\Rightarrow \sin\left(\frac{y}{x}\right) = c\cdot x. \quad c$: any constant.

(Note that the sol.s $y=kx$ corresponds to $c=0$).

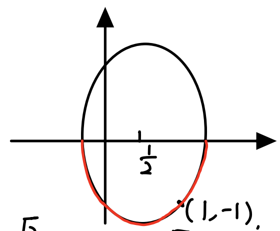
Separable eqs in differential form:

$$M(x)dx + N(y)dy = 0. \quad \left(\Leftrightarrow \frac{dy}{dx} = -\frac{M(x)}{N(y)}. \right)$$

E.g. Solve $\begin{cases} y\,dy = (1-2x)\,dx \\ y(1) = -1 \end{cases}$

Sol: $y\,dy = (1-2x)\,dx \Rightarrow \frac{1}{2}y^2 = x - x^2 + c.$

$y(1) = -1: \quad \frac{1}{2} = 1 - 1 + c \Rightarrow c = \frac{1}{2}. \Rightarrow y^2 = 2x - 2x^2 + 1.$



$\Rightarrow y = -\sqrt{2x - 2x^2 + 1}, \quad \frac{1-\sqrt{3}}{2} \le x \le \frac{1+\sqrt{3}}{2}.$

2.3. Modeling with First-Order ODEs

E.g. 1. Compound Interest (复利).

(a) Assume the initial deposit is S_0, the annual interest rate is r, and the interest is compounded continuously.

The balance at time t (year) is $S(t)$.

Then $\begin{cases} \frac{dS}{dt} = rS \\ S(0) = S_0. \end{cases}$ (model of continuous compound).

$\Rightarrow S(t) = S_0 \cdot e^{rt}.$

(b) Suppose, in addition, the yearly deposit is k.

Then $\begin{cases} \frac{dS}{dt} = rS + k. \\ S(0) = S_0. \end{cases}$

$\Rightarrow S(t) = S_0 \cdot e^{rt} + \frac{k}{r}(e^{rt} - 1).$

e.g. $r = 4\%$ (per year). $S_0 = 20000$, $k = 10000$/year.

Then $S(10) = 152300.$

$S_0 + 10k = 120000.$ gain: 32300.

(c) Mortgage (贷款).

Suppose the annual interest rate for a mortage is r.

If a person borrows S_0 and yearly payment is p, then

$$\frac{dS}{dt} = rS - p. \quad \Rightarrow \quad S(t) = S_0 \cdot e^{rt} - \frac{p}{r}(e^{rt} - 1).$$

Q: If the mortgage is for 20 years, $r = 6\%$, $S_0 = 2000\,k$.

Calculate the monthy payment.

Sol: $S(t) = S_0 \cdot e^{rt} - \frac{p}{r}(e^{rt} - 1).$

$S(20) = 0 \Rightarrow rt = 6\% \times 20 = 1.2. \Rightarrow p = \frac{S_0 \cdot e^{rt} \cdot r}{e^{rt} - 1} = \frac{120 \cdot e^{1.2}}{e^{1.2} - 1}$

$\Rightarrow \frac{10 \cdot e^{1.2}}{e^{1.2} - 1}$ per month.

$= 14.3\ k$/month.

E.g. 2. Predator - Prey Model

$x(t)$: population of prey at time t.

$y(t)$: population of predator at time t.

$\frac{x'(t)}{x} = r - by$. r: growth rate per capital of prey when $y=0$.

b: decrease rate per capital due to predator.

$\frac{y'(t)}{y} = -s + cx$ s: death rate per capital of predator when $x=0$.

c: increase rate due to prey.

$$\Rightarrow \begin{cases} x' = x(r - by) \\ y' = y(-s+cx) \end{cases} \cdots\cdots (LV).$$

(Lotka - Volterra equation).
(1925) (1926)

(LV) reduces to a single eq:

$$\frac{dy}{dx} = \frac{y(-s+cx)}{x(r-by)}.$$ (view y as a func. of x).
— separable.

$$\left(\frac{r}{y} - b\right) dy = \left(c - \frac{s}{x}\right) dx.$$

$$\Rightarrow r\ln y - by = cx - s\ln x + k.$$ k: any constant.

$$\underbrace{(s\ln x - cx) + (r\ln y - by)}_{V(x,y) = G(x) + H(y)} = \text{const}.$$

Such func. like $V(x,y)$ is called a constant of motion.

粒子运动

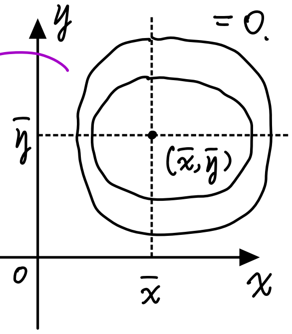
If $(x(t), y(t))$ is a sol of (LV), then

$$\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x} \cdot x'(t) + \frac{\partial V}{\partial y} \cdot y'(t).$$

$V(x(t), y(t))$
看作t时刻粒子的位置
$(x(t), y(t), V(x(t), y(t)))$

$$= \left(\frac{s}{x} - c\right) \cdot x(r - by) + \left(\frac{r}{y} - b\right) \cdot y(-s + cx)$$

$$= 0.$$

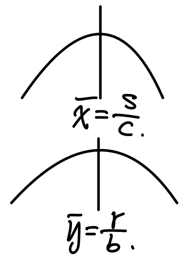


绕一圈
要符合(LV)
中x, y的
增减性.

$V(x,y) = G(x) + H(y)$

$G'(x) = \frac{s}{x} - c$, $G''(x) = -sx^{-2} < 0$.

$H'(y) = \frac{r}{y} - b$, $H''(x) = -ry^{-2} < 0$.



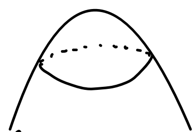
$(\bar{x}, \bar{y})$ is an equlibrium pt.

$x \equiv \bar{x}$, $y \equiv \bar{y}$ is a const. sol.

$$\text{Hessian}(V) = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{\substack{x_1 = x \\ x_2 = y}}$$

$$= \begin{pmatrix} G''(x) & 0 \\ 0 & H''(y) \end{pmatrix} < 0. \text{ (negative definite).}$$

$\Rightarrow V$ is strictly concave down.



$\Rightarrow$ all orbits $(x(t), y(t))$ are closed curves.

$\Rightarrow$ all sol.s $(x(t), y(t))$ are periodic in t.

2.4 Exact Differential Eqs. (Sect. 2.1) (恰当方程)

Consider 1st-order ODE in the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0. \quad \text{or equivalently, } M(x,y)dx + N(x,y)dy = 0 \quad (1).$$

<u>Def</u>: (1) is called an <u>exact diff. eq.</u> if $\exists\, \psi(x,y)$ s.t. $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$. (2).

If (1) is exact, let $y = y(x)$ be a sol of (1).

then $\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\cdot\frac{dy}{dx} = M + N\frac{dy}{dx} = 0.$

$\Rightarrow \psi(x, y(x)) \equiv$ const C, which defines y (Sol.) as a function of x implicitly for every C.

<u>Thm</u>: Let $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ be continuous in $D = \{(x,y)\in\mathbb{R}^2 : \alpha < x < \beta,$ $\gamma < y < \delta\}$.

Then, (1) is exact in D iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in D. (3)

the Thm works for any simply connected open region.

<u>Proof</u>: $(\Rightarrow)$: Suppose (1) is exact. i.e. $\exists\ \psi$, $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$.

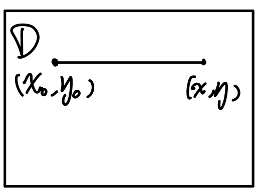
Since $\psi, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x}, \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial M}{\partial y}$ are all continuous in D.

By Clairau's Thm, $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$ in D. i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$(\Leftarrow)$: Suppose (3) holds. we want to find a $\psi(x,y)$

s.t. $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$.

Take a point $(x_0, y_0) \in D$.



Define $\psi(x,y) = \int_{x_0}^{x} M(s,y)\,ds + h(y)$,

$h(y)$ to be determined.

$$\frac{\partial \psi}{\partial y} = \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)\,ds + h'(y) \overset{(3)}{=\!=} \int_{x_0}^{x} \frac{\partial N}{\partial s}(s,y)\,ds + h'(y)$$

$$= N(x,y) - N(x_0, y) + h'(y) \overset{\text{hope}}{=\!=} N(x,y).$$

$\Rightarrow h'(y) = N(x_0, y)$. $\quad h(y) = \int_{y_0}^{y} N(x_0, t)\,dt$.

In this way, we construct $\psi(x,y) = \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt$. $\square$

E.g. 1. Show that $2tyy' + 2t + y^2 = 0$ is exact and find the sols.

Sol: It has the form $M(t,y) + N(t,y)\,y' = 0$.

with $M(t,y) = 2t + y^2$. $N(t,y) = 2ty$.

Since $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial t} = 2y$. we see it is exact by the Thm.

Next, we find $\psi(t,y)$ s.t $\frac{\partial \psi}{\partial t} = M = 2t + y^2$, $\frac{\partial \psi}{\partial y} = N = 2ty$. (4).

$(4)_1 \Rightarrow \psi(t,y) = t^2 + y^2 t + h(y)$. $h(y)$ to be determined.

$\Rightarrow \frac{\partial \psi}{\partial y} = 2ty + h'(y) \overset{(4)_2}{=} 2ty$. $\Rightarrow h'(y) = 0$. we may take $h(y) = 0$.

$\Rightarrow \psi(t,y) = t^2 + y^2 t$.

Thus the sol.s $y = y(t)$ are defined implicitly by

$\psi(t,y) = t^2 + y^2 t = C$. C is any constant.

Integrating factors:

Consider $M(x,y) + N(x,y)\,y' = 0$ with $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hope: $\mu(x,y) M(x,y) + \mu(x,y) N(x,y)\,y' = 0$ is exact.

i.e. $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$.

$$\Rightarrow \frac{\partial \mu}{\partial y}\cdot M + \mu \cdot \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}\cdot N + \mu \cdot \frac{\partial N}{\partial x}.$$

i.e. $M\dfrac{\partial \mu}{\partial y} - N\cdot\dfrac{\partial \mu}{\partial x} + \left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\mu = 0$

(we don't need to find all sol.s. one sol. is enough.)

Often, we may find μ as $\mu = \mu(x)$ or $\mu = \mu(y)$.

E.g. 2. Find an integrating factor and solve $\underset{M}{\underset{\parallel}{1}} + \underset{N}{\underset{\parallel}{\left(\frac{x}{y} - \sin y\right)}}y' = 0$. $(y \neq 0)$

Sol: $\mu + \mu\left(\frac{x}{y} - \sin y\right)y' = 0$.

Hope: $\dfrac{\partial \mu}{\partial y} = \dfrac{\partial\left(\mu\left(\frac{x}{y} - \sin y\right)\right)}{\partial x} = \dfrac{\partial \mu}{\partial x}\left(\frac{x}{y} - \sin y\right) + \mu\cdot\frac{1}{y}$.

Let $\mu = \mu(y)$. Then $\mu'(y) = \dfrac{\mu}{y} = \dfrac{d\mu}{dy}$. $\Rightarrow \dfrac{dy}{y} = \dfrac{d\mu}{\mu}$.

We may take $\mu = y$.

$\underbrace{y}_{M} + \underbrace{(x - y\sin y)}_{N}y' = 0$

$\dfrac{\partial M}{\partial y} = 1 = \dfrac{\partial N}{\partial x}$. it is exact.

Now, find $\psi(x,y)$ s.t

$\dfrac{\partial \psi}{\partial x} = y$, $\dfrac{\partial \psi}{\partial y} = x - y\sin y$.

$\Rightarrow \psi(x,y) = xy + y\cos y - \sin y$.

$\Rightarrow$ Sol.s are defined implicitly

by $xy + y\cos y - \sin y = \text{const}$.

$y \neq 0$.

2.5. Elementary Transformation Methods

E.g. $\dfrac{dy}{dx} = \dfrac{xy^2 + \sin x}{2y}$.

let $v = y^2$.

$$\frac{dv}{dx} = 2y\frac{dy}{dx} \Rightarrow \frac{dv}{dx} = xv + \sin x. \quad (\text{linear}).$$

E.g. Homogeneous Eq.

$\frac{dy}{dx} = f(x,y)$ is called homogeneous if $f(x,y) = g(\frac{y}{x})$.

different.

[linear: $\frac{dy}{dx} + p(x)y = q(x)$ is called homogeneous if $q(x) = 0$].

Let $u = \frac{y}{x}$. $\quad xu = y \Rightarrow u + x\frac{du}{dx} = \frac{dy}{dx} = g(u) \Rightarrow \frac{du}{dx} = \frac{g(u)-u}{x}$ separable.

(i) Solve $\frac{dy}{dx} = \frac{x+y}{x-y} \underset{x \neq 0.}{=} \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$. Let $u = \frac{y}{x}$. $x\frac{du}{dx} = \frac{1+u}{1-u} - u. = \frac{u^2+1}{1-u}$

$$\frac{1-u}{u^2+1}du = \frac{1}{x}dx \qquad\qquad = \frac{2}{1-u} + (1-u) - 2$$

$\arctan u - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$. $\quad C$: any const.

$$\Rightarrow \arctan\frac{y}{x} = \ln\sqrt{1+(\frac{y}{x})^2} + C.$$

(ii) Solve $\frac{dy}{dx} = \frac{x-y+1}{x+y-3} = \frac{X+Y}{X-Y}$. $\quad \begin{matrix} X = x-1 \\ Y = 2-y \end{matrix} \quad \frac{dY}{dX} = -\frac{dy}{dx}$.

(iii) Solve $\frac{dy}{dx} = f\left(\frac{ax+by+c}{mx+ny+l}\right)$, $a.b.c.m.n.l$ are constants.

Sol: If $c = l = 0$, let $\frac{y}{x} = u$.

Otherwise, we want to find ξ, η, s.t

$$\begin{cases} ax+by+c = a\xi + b\eta \\ mx+ny+l = m\xi + n\eta \end{cases} \quad (*1).$$

$$\Leftrightarrow \begin{cases} a(\xi - x) + b(\eta - y) = c \\ m(\xi - x) + n(\eta - y) = l \end{cases}$$

① $an - bm \neq 0$.

$$\begin{pmatrix} \xi - x \\ \eta - y \end{pmatrix} = A^{-1}\begin{pmatrix} c \\ l \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ m & n \end{pmatrix}.$$

$\Rightarrow \xi = x + \alpha.\quad \eta = y + \beta.$ Then $\dfrac{d\eta}{d\xi} = \dfrac{dy}{dx} = f\left(\dfrac{a\xi + b\eta}{m\xi + n\eta}\right)$

$$\overset{u = \frac{\eta}{\xi}}{=\!=} f\left(\frac{a+bu}{m+nu}\right).$$

② $an - bm = 0$. let $\dfrac{m}{a} = \dfrac{n}{b} = \lambda$.

Then $\dfrac{dy}{dx} = f\left(\dfrac{ax+by+c}{\lambda(ax+by)+l}\right)$. let $v = ax + by$ $\quad \dfrac{dv}{dx} = a + b\dfrac{dy}{dx}$

$$= a + b\, f\left(\frac{v+c}{\lambda v + l}\right).$$

Bernoulli Eq. 伯努利方程. (Jakob Bernoulli)

$$\frac{dy}{dx} + p(x)y = q(x)y^n. \quad n \in \mathbb{R}.\ n \neq 0,\ n \neq 1.$$

Multiply $(1-n)y^{-n}$: $(1-n)y^{-n}\cdot y' + p(x)\cdot(1-n)y^{1-n} = (1-n)q(x)$.

Let $z = y^{1-n}$: $z' + p(x)\cdot(1-n)\cdot z = (1-n)q(x)$. (1-order. linear)

Riccati Eq. 理卡蒂方程. (Jacopo Francesso Riccati).

$$\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x). \quad \text{(RE)}$$

- $r(x) \equiv 0$, Bernoulli Eq. with $n=2$.
- $p(x) \equiv 0$, 1st-order linear.
- $r(x) \not\equiv 0$, $p(x) \not\equiv 0$. Riccati.

Theorem: If (RE) has a particular sol. $y = \phi(x)$, then the general sol can be obtained by integral method.

Pf: $y = u + \phi(x)$: $\frac{dy}{dx} = \frac{du}{dx} + \frac{d\phi}{dx} \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - \frac{d\phi}{dx}$

$$= py^2 + qy - (p\phi^2 + q\phi)$$

$$\Rightarrow \frac{du}{dx} = [2p(x)\phi(x) + q(x)]u + p(x)u^2. \quad \text{(Bernoulli)}$$

let $z = u^{-1}$. $\Rightarrow \frac{dz}{dx} + [2p(x)\phi(x) + q(x)]z = -p(x)$. (1st-order linear)

Theorem: The Riccati Eq.:

$$\frac{dy}{dx} + ay^2 = bx^m, \quad a \neq 0,\ b \neq 0,\ m \text{ consts.} \quad (*2) \quad \left(\begin{array}{l} \text{RE:} \\ p(x) \equiv -a \\ q(x) \equiv 0 \\ r(x) = bx^m \end{array}\right).$$

can be reduced to a special eq. by some suitable transformation if

$$m = 0,\ -2,\ \frac{-4k}{2k-1},\quad k = 1, 2, \cdots. \qquad (\ast 3)$$

<u>Rmk:</u> • This thm was proved by Daniel Bernoulli in 1725.

• Liouville proved $(\ast 3)$ is neccessary for $(\ast 2)$ to be solved by an elementary integral method in 1841.

• Liouville told us it is very hard to solve the Riccati eq. of a very simple form: $y' = x^2 + y^2$.

$\boxed{\text{2.6. Qualitative Analysis for Autonomous DEs: } \frac{dy}{dx} = f(y).}$

<u>Malthusian model:</u> $\frac{dp}{dt} = rp.\quad r = b - d.$ (birth rate − death rate)
马尔萨斯.

$\Rightarrow p(t) = p(0)e^{rt}$: If $r>0$, then $p(t) \nearrow \infty$.

If $r<0$, then $p(t) \searrow 0$.

rate of change (per capital): $\dfrac{\frac{dp}{dt}}{p} = r.$

<u>Logistic model:</u> (Verhulst 1838).
逻辑斯谛

$$\frac{\frac{dp}{dt}}{p} = \begin{cases} +, & \text{if } p \text{ is not large } (p<k) \\ -, & \text{if } p \text{ is too large } (p>k). \end{cases}$$

<u>e.g.</u> $r\cdot(k-p)$.

$$\frac{dp}{dt} = r \cdot p \cdot (k-p) = \underbrace{rk}_{r.} p\left(1-\frac{p}{k}\right) = rp\left(1-\frac{p}{k}\right)$$

内禀的.本质的.

r: intrinsic growth rate.

k: saturation level / environmental carrying capacity.

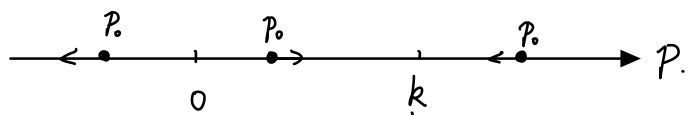
饱和的.

Phase line analysis:

Step 1: Find all equilibrium sol.s: solve $rp\left(1-\frac{p}{k}\right)=0$

$p=0$ or $p=k$.

Step 2: For any initial value $p(0)=p_0$, visualize the sol. $p(t)$ as a partical moving along the phace line p-axis.

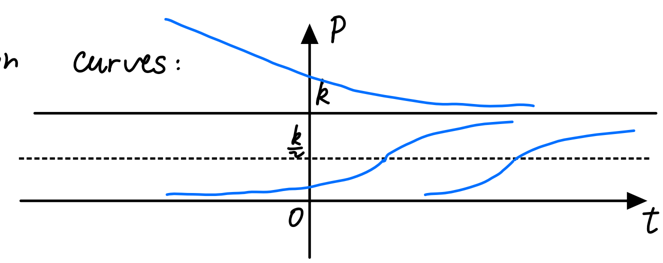


In k's nbd: slow down. approach k. but never reach k.

$0 < p(0) < k$: $p(t) \to k$ as $t \to \infty$.

$p(0) > k$: $p(t) \to k$ as $t \to \infty$.

Solution curves:

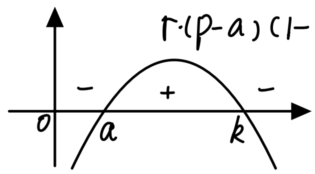


$0<p(0)<k$: $p'(t)>0$, $\forall t$.

$p' = rp(1-\frac{p}{k})$

$p'' = f'(p)\cdot p' = r^2 p(k-p)(k-2p)$.

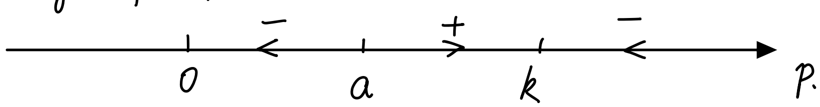
Allee effect: $\dfrac{\frac{dp}{dt}}{p}$:

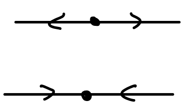


$$\frac{dp}{dt} = r\cdot p(p-a)(1-\frac{p}{k}).$$

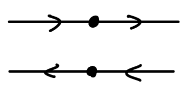
Phase line analysis:

sgn of $f(p)$:



$P(t)\equiv a$:  repellor (unstable)

$P(t)\equiv k$: attracor (stable). } node.

 } Saddle (unstable).

<u>Bifuration.</u>

$$\frac{dy}{dt} = f(y,c). \qquad c\text{: physical parameter.}$$

E.g. 1: Logistic eq. with harvesting

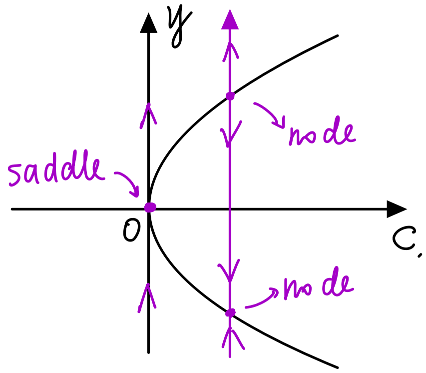
$$\frac{dp}{dt} = rp(1-\frac{p}{k}) - H = f(p,H). \qquad (1)$$

<u>Def</u>. As the parameter c varies, when equilibrium p.t.s

suddenly appear or disappear, we say <u>bifuration</u> occurs.

E.g. 2: $\frac{dy}{dt} = y^2 - c.$

Sol: Solve the equilibrium p.t.s: $y^2 - c = 0$. if $c<0$, no equil.



if $c=0$. 1 equil: $y=0$.

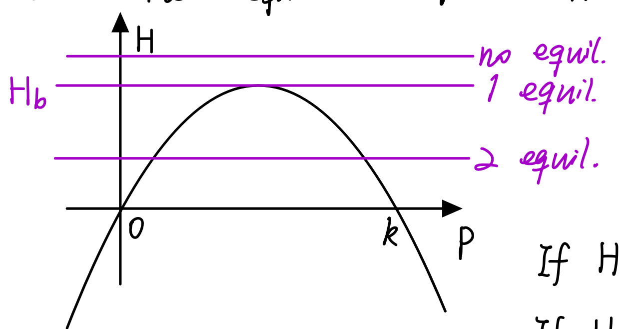
if $c>0$. 2 equil.s: $y = \pm\sqrt{c}$.

$c=0$: bifuration value.

$\rightarrow$ saddle - node bifuration.

<u>Bifuration analysis for E.g. 1</u>:

First, solve the equilibrium p.t.s: $H = rp(1-\frac{p}{k}) =: g(p)$.



$g'(p) = r(1-\frac{2p}{k}) = 0 \Rightarrow p = \frac{k}{2}$.

$H_b = g(\frac{k}{2}) = \frac{rk}{4}$.

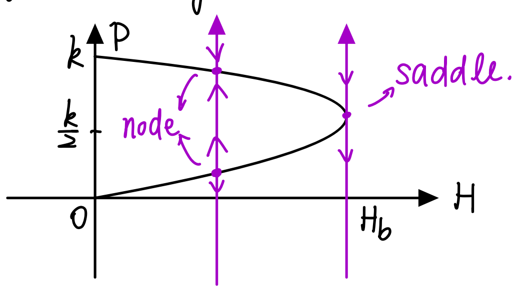
If $H > H_b$, no equil.

If $H = H_b$, 1 equil.

If $H < H_b$, 2 equil.s.

$\therefore H_b$ is a bifuration value.

bifuration diagram:



E.g. 3: $\frac{dy}{dt} = (c-y^2)y.$

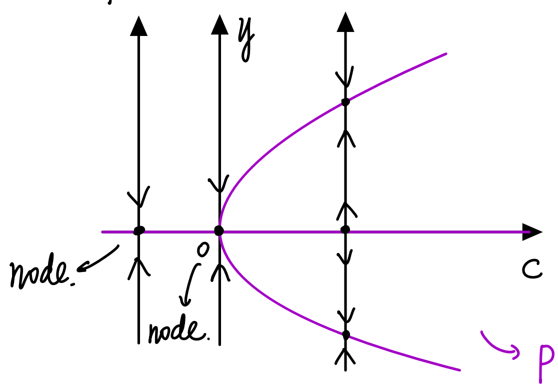
Sol: Step 1: $(c-y^2)\,y=0.$

$c<0$: 1 equil. $y=0.$

$c=0$: 1 equil. $y=0.$

$c>0$: 3 equil. $y=0,\ y=\pm\sqrt{c}.$

Step 2: bifuration diagram.



$c=0$ is a bifuration value.

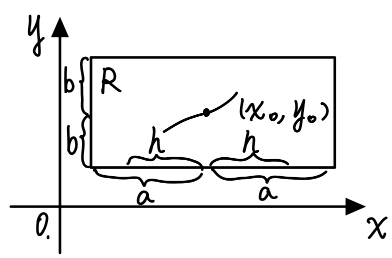
pitchfork bifuration.

2.7. Existence and Uniqueness Theorem (Picard's iteration method)

Theorem (E&U). Consider (IVP) $\begin{cases} \dfrac{dy}{dx}=f(x,y). \\ y(x_0)=y_0. \end{cases}$ (also called Cauchy prob.)

Let $R=\{(x,y)\in\mathbb{R}^2 \mid |x-x_0|\le a,\ |y-y_0|\le b\}$. Assume f is continuous on R and is Lipschitz continuous in y on R (that is $\exists\, L>0$ s.t. $\forall\,(x,y_1),(x,y_2)\in\mathbb{R}^2$, $|f(x,y_1)-f(x,y_2)|\le L|y_1-y_2|$).

Then, (IVP) $\exists!$ sol. on $I=[x_0-h,\ x_0+h]$, where $h=\min\{a,\frac{b}{M}\}$, $M=\max\limits_{R}|f(x,y)|$.



Pf: Step 1. Observation: The (IVP) on I is equivalent to the integral eq.:

$$\text{(IE)}: \quad y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\, dt, \quad x \in I, \; y(x) \in C(I).$$

Why? (IVP) $\Rightarrow$ (IE): Suppose $y(x)$, $x \in I$ is a sol of (IVP).

Then $y \in C(I)$, $\Rightarrow f(x, y(x)) \in C(I)$, $\frac{dy}{dx} \in C(I)$.

$\Rightarrow \forall x \in I$, $\int_{x_0}^{x} \frac{dy}{dx} \cdot dx = \int_{x_0}^{x} f(t, y(t))\, dt$.

$\|$ Fundemental Thm of Calculas.

$y(x) - y(x_0)$.

$\Rightarrow y(x)$ satisfies the (IE).

(IE) $\Rightarrow$ (IVP): Suppose $y = y(x)$, $x \in I$, $y \in C(I)$. s.t.

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\, dt, \quad \forall x \in I.$$

Then, $f(t, y(t)) \in C(I) \Rightarrow y \in C'(I)$

& $\frac{dy}{dx} = \frac{d}{dx} \int_{x_0}^{x} f(t, y(t))\, dt = f(x, y(x))$.

$y(x_0) = y_0$.

$\Rightarrow y(x)$ is a sol. of (IVP).

Step 2. Let $y_0(x) \equiv y_0$.

$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$. $y_1 \in C'(I)$, & $|y_1(x) - y_0| \le |x - x_0| \cdot M \le h \cdot M \le b$.

(*1).

$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt$. $y_2 \in C'(I)$, & $|y_2(x) - y_0| \le |x - x_0| \cdot M \le b$.

$\cdots\cdots$

$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$. $y_{n+1} \in C'(I)$, & $|y_{n+1} - y_0| \le b$.

$\{y_n(x)\}_{n=1}^{\infty}$ is the <u>Picard's sequence</u>: $\forall n,\ y_n \in C'(I)$ & $|y_n(x) - y_0| \leq b,\ \forall x \in I$.

<u>Step 3.</u> $y_n(x) \longrightarrow$ Some $y_\infty(x)$ uniformly on I, and $y_\infty(x)$ is a sol. of (IE).

(That is, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, s.t $|y_n(x) - y_\infty(x)| < \varepsilon,\ \forall x \in I,\ \forall n \geq N$).

<u>Claim</u>: $\left|y_{k+1}(x) - y_k(x)\right| \leq \frac{M}{L} \cdot \frac{(L|x-x_0|)^{k+1}}{(k+1)!}$, $\forall x \in I,\ \forall k \in \mathbb{N}$.

$\left(\leq \frac{M \cdot L^k \cdot h^{k+1}}{(k+1)!}\right)$.

For $k=0$. $|y_1(x) - y_0(x)| \leq \frac{M}{L} \cdot \frac{L|x-x_0|}{1}$. This is (✳1).

Suppose claim holds for $k=n$. W.T.S it holds for $k=n+1$.

$k=n+1$:

$$\left|y_{n+2}(x) - y_{n+1}(x)\right| = \left|\int_{x_0}^{x} \left[f(t, y_{n+1}(t)) - f(t, y_n(t))\right] dt\right|$$

$$\leq L \cdot \int_{x_0}^{x} \left|y_{n+1}(t) - y_n(t)\right| dt. \text{ (by the Lipchitz condition)}$$

WLOG, $x > x_0$.

$$\leq L \cdot \int_{x_0}^{x} \frac{M}{L} \cdot \frac{(L(t-x_0))^{n+1}}{(n+1)!} dt. \text{ (by assumption for } k=n\text{)}$$

$$= \frac{M \cdot L^{n+1}}{(n+1)!} \left.\frac{(t-x_0)^{n+2}}{n+2}\right|_{t=x_0}^{t=x} = \frac{M L^{n+1}}{(n+2)!}(x-x_0)^{n+2}.$$

If $x < x_0$, the proof is similar. Thus, the claim is verified. #.

The positive series $\sum_{k=1}^{\infty} \frac{M L^{k-1} h^k}{k!} =: a_k$.

Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{Lh}{k+1} \to 0$ as $k \to \infty$. $\Rightarrow$ The series is convergent.

$y_n(x) = y_0 + (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) = y_0 + \sum_{k=0}^{n-1}(y_{k+1} - y_k)$.

Now comparing $\sum_{k=1}^{\infty}\left(y_k(x)-y_{k-1}(x)\right)$ with $\sum_{k=1}^{\infty}\frac{ML^{k-1}h^k}{k!}$, we see

$y_n \to$ some y_∞ uniformly on I. (Weierstrass)

$\Rightarrow y_\infty \in C(I)$. (The convergence is uniform).

(IVP) $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$

E & U Thm: Picard's seq:

$$y_0(x) = y_0$$
$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt.$$
$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt.$$
$$\vdots$$
$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt.$$
$$\vdots$$

($y_n \in C^1(I)$, $I = [x_0-h, x_0+h]$.)

Last time: $y_n \longrightarrow y_\infty$ $(n\to\infty)$, uniformly on I.

$y_\infty \in C(I)$.

Lipschitz condition:

$$|f(x, y_n(x)) - f(x, y_\infty(x))| \le L\,|y_n(x) - y_\infty(x)|.$$

$\Rightarrow f(x, y_n(x)) \longrightarrow f(x, y_\infty(x))$ $(n\to\infty)$, uniformly on I. $\cdots (\ast 1)$

Thus, $\underbrace{\lim_{n\to\infty} y_{n+1}(x)}_{= y_\infty(x)} = y_0 + \lim_{n\to\infty}\int_{x_0}^{x} f(t, y_n(t))\, dt \overset{(\ast 1)}{=} y_0 + \int_{x_0}^{x} f(t, y_\infty(t))\, dt.$

$\Rightarrow y_\infty(x)$ satisfies the (IE), and hence also the (IVP) by step 1.

<u>Step 4:</u> Uniqueness: Let $w(x)$ be another sol of (IVP). (hence of (IE))

$$w(x) = y_0 + \int_{x_0}^{x} f(t, w(t))\, dt.$$

$$y_\infty(x) - w(x) = \int_{x_0}^{x} f(t, y_\infty(t)) - f(t, w(t))\, dt. \quad \forall x \in I.$$

$$\Rightarrow |y_\infty(x) - w(x)| \le \int_{x_0}^{x} |f(t, y_\infty(t)) - f(t, w(t))|\, dt. \quad \forall x \in [x_0, x_0+h].$$

$$\overset{\text{Lipschitz}}{\le} \int_{x_0}^{x} L\,|y_\infty(t) - w(t)|\,dt.$$

Let $z(x) = \int_{x_0}^{x} |y_\infty(t) - w(t)|\,dt$. $\quad x \in [x_0, x_0+h]$. $\qquad z(x) \ge 0, \ \forall x \in [x_0, x_0+h]$.

Special case of Gronwall's inequality. $\leftarrow \begin{cases} z'(x) \le L\,z(x) \\ z(x_0) = 0. \end{cases} \Rightarrow \left(e^{-Lx} z(x)\right)' = e^{-Lx}\left(z'(x) - L\,z(x)\right) \le 0$

$\Rightarrow e^{-Lx} z(x)$ is decreasing in x. $\Rightarrow \underbrace{e^{-Lx}}_{\ge 0}\underbrace{z(x)}_{\ge 0} \le e^{-Lx_0} z(x_0) = 0$. $\Rightarrow z(x) \equiv 0 \quad \forall x \in [x_0, x_0+h]$

$\Rightarrow y_\infty(t) \equiv w(t)$. $\Rightarrow$ Uniqueness.

Remark: ① If $f(x,y)$ satisfies $f_y(x,y)$ exists & is bounded in R. (in partic. f_y is continuous in R)

then, f is Lipschitz w.r.t. y in R.

Pf: $\forall (x, y_1), (x, y_2) \in R$,

$$|f(x,y_1) - f(x,y_2)| \xlongequal[\text{MVT.}]{\text{Lagrange}} \left|\frac{\partial f}{\partial y}(x, \bar{y})\right| |y_1 - y_2| \le L\,|y_2 - y_1|. \qquad L = \left\|\frac{\partial f}{\partial y}\right\|_{L^\infty(R)}$$

(L is the least upp bound of $\frac{\partial f}{\partial y}$ on R)

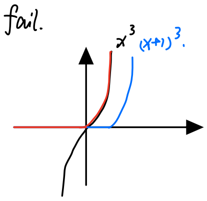
② If in the E&U Theorem, $f(x,y)$ is only continuous on $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ without the Lipschitz condition, then the existence part still holds, i.e. $\exists$ at least one sol.

$y = \phi(x)$, $x \in [x_0 - h, x_0 + h]$, $h = \min\{a, \frac{b}{M}\}$, $M = \max\limits_{(x,y)\in R} |f|$.

(See Peano's existence thm).

However, the uniqueness part may fail.

E.g.3 $\begin{cases} y' = 3y^{\frac{2}{3}} \\ y(0) = 0. \end{cases}$ ① $y \equiv 0$ ② $\frac{dy}{y^{\frac{2}{3}}} = 3\,dt$. $\quad y = t^3$.



③ For the linear case (IVP) $\begin{cases} y' + p(x) y = q(x). \\ y(x_0) = y_0. \end{cases}$ $\quad f(x,y) = q(x) - p(x) y$.

If $p(x)$ & $q(x)$ are continuous on some interval containing x_0, say $I = [\alpha, \beta]$

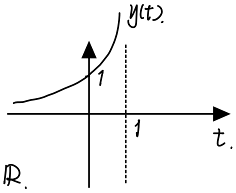
$R = [\alpha, \beta] \times \mathbb{R}$. Then the (IVP) $\exists!$ sol. $y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left[\int_{x_0}^{x} q(s)\, e^{\int_{x_0}^{s} p(t)\,dt}\,ds + y_0\right]$ on I.

However, in the non-linear case, the interval is usually smaller than the width of R, on which f is continuous & Lipschitz w.r.t. y.

e.g. $\begin{cases} \frac{dy}{dt} = y^2. \\ y(0) = 1. \end{cases}$ $\Rightarrow y = \frac{-1}{t-1}$

$\downarrow$

$R = \mathbb{R} \times \mathbb{R}$. $\qquad (-\infty, 1) \times \mathbb{R}$.



④ In the E&U Thm, the Lipschitz condition can be replaced by a more general Osgood condition:

$|f(x,y_1) - f(x,y_2)| \le F(|y_1 - y_2|)$. where $F(r) > 0$ & continuous for $r > 0$. & $\int_0^{r_1} \frac{dr}{F(r)} = \infty$ for a const $r_1 > 0$.

(Lipschitz: $F(r) = Lr$. $\int_0^{r_1} \frac{dr}{Lr} = \infty$).

Both the Lipschitz and the Osgood conditions are "sufficient" for "uniqueness" but not "necessary".

2.8. Extension of Sol. (解的延拓)

Eg. 1. Riccati eq. $\begin{cases} \frac{dy}{dx} = x^2 + y^2 \\ y(0) = 0 \end{cases}$. (✳1) $\quad$ · Liouville told us (✳1) cannot be solved by integral method.

· Picard's E&U Thm: $f(x,y) = x^2 + y^2$.

Take $R = [-1,1] \times [-1,1]$. $(a = b = 1)$. $M = \max_R |f(x,y)| = 2$. $h = \min\{a, \frac{b}{M}\} = \frac{1}{2}$.

$\Rightarrow \exists!$ sol. $y(x)$, $x \in [-\frac{1}{2}, \frac{1}{2}]$.

Take $R = [-2,2] \times [-2,2]$, $(a = b = 2)$. $M = \max_R |f(x,y)| = 8$. $h = \min\{a, \frac{b}{M}\} = \frac{1}{4}$.

Def: If $y = \phi_1(x)$, $x \in (\alpha_1, \beta_1)$ and $y = \phi_2(x)$, $x \in (\alpha_2, \beta_2)$ are both sol.s of $\frac{dy}{dx} = f(x,y)$, and

(1) $(\alpha_1, \beta_1) \subsetneqq (\alpha_2, \beta_2)$. $\qquad$ (2) $\phi_1(x) = \phi_2(x)$, $x \in (\alpha_1, \beta_1)$.

then, $\phi_2(x)$ is called an extension of $\phi_1(x)$.

If a sol. $y = \phi(x)$, $x \in (\alpha, \beta)$ has no extension, then it is called a <u>saturated sol.</u> (饱和解)

or non-extendible sol. The (α, β) is called the <u>saturated interval</u> (饱和区间) or the

<u>maximum existence interval.</u> (最大存在区间).

<u>Thm 1 (Extension theorem)</u>

Consider the (IVP) $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0. \end{cases}$ (✳2) where f is continuous in some domain $\Omega \subseteq \mathbb{R}^2$. (open, connected)

Then, every sol. curve Γ in Ω through $P_0(x_0, y_0)$ can be extended to the boundary of Ω ($\partial\Omega$)

☺ $\begin{cases} \text{That is, for every subdomain } \Omega_1 \subset\subset \Omega \ (\text{i.e. } \bar{\Omega}_1 \text{ is compact } \& \ \bar{\Omega}_1 \subset \Omega) \text{ that contains } P_0, \Gamma \text{ can be} \\ \text{prolonged through } \partial\Omega_1. \end{cases}$

<u>Remark</u>: If Ω is unbounded, the $\partial\Omega$ includes ∞.

<u>Pf</u>: By Peano, ($*2$) $\exists$ a sol. $y=\phi(x)$ on $I=[x_0-h, x_0+h]$.

Let I_{max} be the maximum existence interval of the sol.: $I^+_{max} = I_{max} \cap [x_0, +\infty)$.

$I^-_{max} = I_{max} \cap (-\infty, x_0]$.

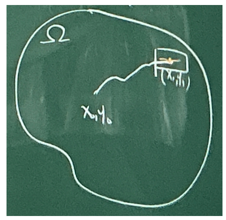
We discuss I^+_{max} for example.

Case 1. $I^+_{max} = [x_0, +\infty)$. done ($\infty \in \partial\Omega$).

Case 2. $I^+_{max} = [x_0, x_1]$. ($x_0 < x_1 < \infty$)

$(x, \phi(x)) \in \Omega \quad \forall x \in [x_0, x_1]$. Let $y_1 = \phi(x_1)$, then $(x_1, y_1) \in \Omega$.

Since Ω open, $\exists$ a rectangle $R: |x-x_1| \le a, \ |y-y_1| \le b$, st. $R \subset \Omega$.



the (IVP) $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_1) = y_1 \end{cases}$ has a sol. $y = \phi_1(x)$, $x \in [x_1 - h_1, x_1 + h_1]$.

Next let $y(x) = \begin{cases} \phi(x), & x \in [x_0, x_1] \\ \phi_1(x), & x \in [x_1, x_1+h_1] \end{cases}$

Then, $y(x) \in C^1([x_0, x_1+h_1])$ is an extension of ϕ $\times$ $I^+_{max} = [x_0, x_1]$

Case 3. $I^+_{max} = [x_0, x_1)$. $(x_0 < x_1 < \infty)$. $\qquad \Gamma = \{(x, \phi(x)) : x \in I\}$.

Argue by contradiction. Suppose ☺ is not true. Then $\exists$ bounded domain Ω_1,

$\overline{\Omega_1} \subset \Omega$, s.t. $(x, \phi(x)) \in \overline{\Omega_1}$, $\forall x \in [x_0, x_1)$

Since $\phi(x)$, $x \in [x_0, x_1)$, solves the I.V.P. we see

$$\phi(x) = y_0 + \int_{x_0}^{x} f(s, \phi(s))\, ds. \quad \forall x \in [x_0, x_1)$$

$f \in C(\Omega) \Rightarrow f \in C(\overline{\Omega_1}) \Rightarrow |f| \le k$ on $\overline{\Omega_1}$ for some $k > 0$.

Thus, $|\phi(x') - \phi(x'')| = \left| \int_{x''}^{x'} f(s, \phi(s))\, ds \right| \le k |x' - x''|$. $\forall x', x'' \in [x_0, x_1)$. $(*1)$

<u>Recall</u>: Cauchy criterion: The $\lim\limits_{x \to x^*} g(x)$ exists iff $\forall \varepsilon > 0 \ \exists \delta > 0$. st.

$x', x'' \in \hat{B}_\delta(x^*) \Rightarrow |g(x') - g(x'')| < \varepsilon$.

$(*1)$ + Cauchy criterion $\Rightarrow \lim\limits_{x \to x_1^-} \phi(x)$ exists $(:= y_1)$.

Now, define $\psi(x) = \begin{cases} \phi(x), & x \in [x_0, x_1) \\ y_1, & x = x_1. \end{cases}$

Then, $\psi(x)$ is continuous on $[x_0, x_1]$ and satisfies

$$\psi(x) = y_0 + \int_{x_0}^{x} f(s, \phi(s))\, ds. \quad x \in [x_0, x_1]$$

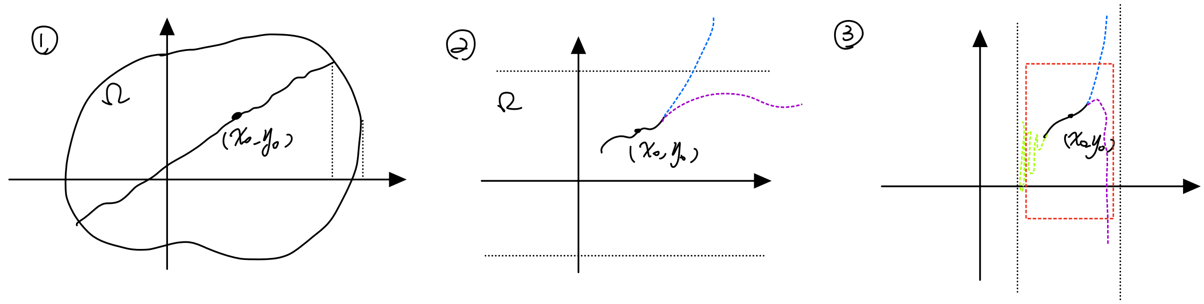
$\Rightarrow$ $\psi(x)$ solves the I.V.P. for every $x \in [x_0, x_1]$.

$\Rightarrow$ $\psi(x)$ is an extension of $\phi(x)$.

$\Rightarrow$ $I^+_{max} \supseteq [x_0, x_1]$ —X— $I^+_{max} = [x_0, x_1)$.

For I^-_{max}, the discussion is similar. $\square$.

E.g. ① ② ③



Def. (Local Lipschitz Condition)

We say $f(x,y)$ satisfies local Lipschitz condition w.r.t. variable y on a domain $\Omega \subseteq \mathbb{R}^2$.

if $\forall$ bounded rectangle R in Ω, $\exists$ $L = L(R) > 0$ s.t.

$$|f(x,y_1) - f(x,y_2)| \leq L\, |y_1 - y_2|, \quad \forall (x, y_1), (x, y_2) \in R.$$

Corollary 2: If in Thm 1 (Extension), we assume, in addition, that $f(x,y)$ satisfies local Lipschitz condition in Ω, then every sol. curve Γ through $P_0(x_0, y_0)$ can be uniquely extended to $\partial\Omega$.

E.g. Find the saturated interval.

(i) $\begin{cases} \dfrac{dy}{dx} = y \\ y(x_0) = y_0. \end{cases}$ $\quad \Omega = \mathbb{R}^2.$ $\quad y(x) = y_0 e^{(x-x_0)}.$ $\quad I_{max} = (-\infty, \infty).$

(ii) $\begin{cases} \dfrac{dy}{dx} = y^2. \\ y(0) = 1. \end{cases}$ $\quad \Omega = \mathbb{R}^2.$ $\quad y(x) = \dfrac{1}{1-x}.$ $\quad I_{max} = (-\infty, 1).$

(iii) $\begin{cases} \dfrac{dy}{dx} = 1+y^2 \\ y(0)=0 \end{cases}$ $\Omega = \mathbb{R}^2$. $y(x) = \tan x$. $I_{max} = (-\frac{\pi}{2}, \frac{\pi}{2})$.

<u>E.g.</u> Prove the I_{max} of every solution to the Riccati eq. $\begin{cases} \dfrac{dy}{dx} = x^2+y^2 \\ y(x_0)=y_0 \end{cases}$ is finite. $\Omega = \mathbb{R}^2$.

<u>P.f.</u> Suppose rather that $I^+_{max} = [x_0, +\infty)$.

Then take some $x_1 > \max\{x_0, 0\}$. $\dfrac{dy}{dx} = x^2+y^2 \geqslant x_1^2 + y^2$. $x \in [x_1, +\infty)$.

$$\Rightarrow \frac{1}{x_1}\left[\arctan \frac{y(x)}{x_1}\right]' = \frac{y'(x)}{x_1^2+y^2} \geqslant 1, \quad \forall x \in [x_1, +\infty).$$

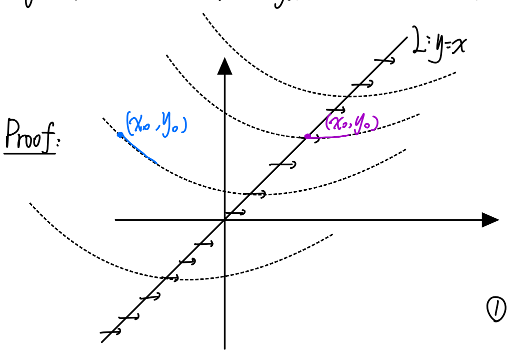
$$\Rightarrow \int_{x_1}^{x} \frac{1}{x_1}\left[\arctan \frac{y(t)}{x_1}\right]' dt = \int_{x_1}^{x} \frac{y'(t)}{x_1^2+y^2} dt \geqslant \int_{x_1}^{x} 1\, dt, \quad \forall x \in [x_1, +\infty).$$

$$\frac{1}{x_1}\left[\arctan \frac{y(x)}{x_1} - \arctan \frac{y(x_1)}{x_1}\right] \geqslant x - x_1, \quad \forall x \in [x_1, +\infty)$$

$$\Rightarrow \frac{\pi}{x_1} \geqslant x - x_1, \quad \forall x \in [x_1, +\infty). \quad \text{※}$$

<u>E.g.</u> Prove that $\forall (x_0, y_0) \in \mathbb{R}^2$, the sol. of $\begin{cases} \dfrac{dy}{dx} = (x-y)e^{xy^2} \\ y(x_0) = y_0 \end{cases}$ has $I^+_{max} = [x_0, +\infty)$. $\Omega = \mathbb{R}^2$.

<u>Proof:</u>



$$\frac{dy}{dx} = 0 \Leftrightarrow y = x.$$

observations:

① If (x_0, y_0) lies above $L: y=x$, then $y(x)$ is decreasing first, then it reaches the line L, and passes through it, and then becomes increasing, and remains below the line. Otherwise, if it returns to the line L at some pt. (x_1, y_1), then $\left.\dfrac{dy}{dx}\right|_{x=x_1} \geqslant 1$, but at the line L, we have $\dfrac{dy}{dx} \equiv 0$. ※

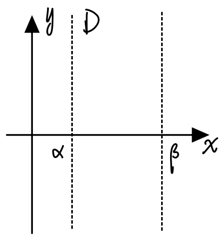
② If (x_0, y_0) lies on L, then it moves below L and remains there.

③ If (x_0, y_0) lies below L, then it remains below L.

By Extension thm, $x^2+y^2(x)\to\infty$. since $y(x)<x$ for x large, we see $x\to+\infty$.

Thus, $I^+_{max}=[0,+\infty)$.

<u>Thm 2 (Linear control.)</u> Consider $\begin{cases}\frac{dy}{dx}=f(x,y) \\ y(x_0)=y_0\end{cases}$ where $f\in C(D)$. $D=\left\{(x,y)\,\middle|\,\alpha<x<\beta,\ -\infty<y<\infty\right\}$.



If $\exists$ $A(x), B(x): (\alpha,\beta)\longrightarrow\mathbb{R}^+$ continuous. s.t.

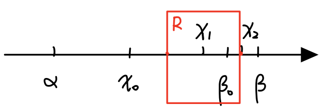
$$|f(x,y)|\le A(x)|y|+B(x),\quad \forall (x,y)\in D.$$

then $I_{max}=(\alpha,\beta)$.

<u>Rmk</u>: For $f(x,y)=p(x)y+q(x)$, $A(x)=|p(x)|$, $B(x)=|q(x)|$.

<u>E.g.</u> $\left|\frac{dy}{dx}\right|=\left|x^2e^{-y^2}\cdot y+\frac{1}{1+x^2}\right|\le \underbrace{x^2}_{A(x)}|y|+\underbrace{\frac{1}{1+x^2}}_{B(x)}$, $I_{max}=(-\infty,\infty)$.

<u>Pf</u>: Argue by contradiction. Assume $I^+_{max}=[x_0,\beta_0)$, $\alpha<x_0<\beta_0<\beta$.



For x_1, x_2. s.t $x_0<x_1<\beta_0<x_2<\beta$ & $x_2-x_1<x_1-x_0$.

Let $R=\left\{(x,y)\,\middle|\,|x-x_1|\le x_2-x_1,\ |y-y_1|\le b\right\}$. $b>0$, to be determined.

$R\subset D$. By the continuity of $A(x)$, $B(x)$ on (α,β) & of f on D,

we see $\exists$ $A_0, B_0>0$, s.t. $\max\limits_{|x-x_1|\le x_2-x_1}A(x)\le A_0$, $\max\limits_{|x-x_1|\le x_2-x_1}B(x)\le B_0$, $M:=\max\limits_{R}|f|\le A_0\max\limits_{R}|y|+B_0$

$\le A_0(|y_1|+b)+B_0$.

Since $\frac{b}{M}\ge\frac{b}{A_0(|y_1|+b)+B_0}\xrightarrow{b\to\infty}\frac{1}{A_0}$, we may take b sufficiently large s.t. $\frac{b}{M}>\frac{1}{2A_0}$.

By the flexibility of x_1 & x_2, we may assume $x_2-x_1<\frac{1}{2A_0}$.

$[x_1, x_1+h]$. $h=\min\left\{x_2-x_1, \frac{b}{M}\right\}=x_2-x_1$.

Peano $\Rightarrow$ the sol can be extended to $x=x_2$ $\longrightarrow\!\!\!\!\!\!\!\times$ $I^+_{max}=[x_0,\beta_0)$.

$\boxed{\text{2.9. Comparism Theorems.}}$

<u>Theorem 1.</u> Assume $f(x,y)$, $F(x,y)$: $\underset{\mathbb{R}^2}{\underset{\cap}{\overset{\text{open}}{G}}}\to\mathbb{R}$ are continuous & $f(x,y)<F(x,y)$ $\forall (x,y)\in G$. (* 1)

If $\{(x,\varphi(x)): x\in(a,b)\}\subset G$ and $\{(x,\Phi(x)): x\in(a,b)\}$ satisfy

$$\begin{cases} \dfrac{d\phi}{dx} = f(x, \phi(x)) \\ \phi(x_0) = y_0 \end{cases} \quad (*2) \quad \text{and} \quad \begin{cases} \dfrac{d\Phi}{dx} = F(x, \Phi(x)) \\ \Phi(x_0) = y_0 \end{cases} \quad (*3) \quad \text{with} \quad x_0 \in (a,b).$$

then $\begin{cases} \phi(x) < \Phi(x), \ x \in (x_0, b) \\ \phi(x) > \Phi(x), \ x \in (a, x_0) \end{cases}$ $(*4)$.

Pf: Let $w(x) = \Phi(x) - \phi(x)$, $x \in [x_0, b)$. Then $w(x_0) = 0$. $w'(x_0) = \Phi'(x_0) - \phi'(x_0) = F(x_0, \Phi(x_0)) - f(x_0, \phi(x_0)) > 0$.

$\Rightarrow w(x) > 0$, $x \in (x_0, x_0 + \delta)$ for some $\delta > 0$.

W.T.S. $w(x) > 0$, $\forall x \in (x_0, b)$. Suppose rather $\exists x_1 \in (x_0 + \delta, b)$. s.t. $w(x_1) = 0$.

$\beta = \inf\{x \mid w(x) = 0, \ x_0 \le x < b\}$. $\beta \in (x_0 + \delta, x_1]$.

$\Rightarrow w(\beta) = 0$, $w(x) > 0$, $\forall x \in (x_0, \beta)$. $\Rightarrow w'(\beta) \le 0$.

However, $w'(\beta) = F(\beta, \Phi(\beta)) - f(\beta, \phi(\beta)) > 0$ since $\Phi(\beta) = \phi(\beta)$. $\times$.

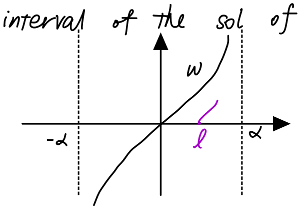
For $x \in (a, x_0]$, the proof is similar. $\square$

Rmk 1: If in Thm 1, we have $\phi(x_0) < \Phi(x_0)$ instead of $\phi(x_0) = \Phi(x_0)$ $\Rightarrow (*4)_1$ still holds.

$\phi(x_0) > \Phi(x_0)$. $\Rightarrow (*4)_2$ still holds.

E.g. 1: Discuss the staturated interval of the sol of

$$\begin{cases} \dfrac{dy}{dx} = x^2 + y^2. \\ y(0) = 0. \end{cases} \quad (*5).$$



Sol: $-y(-x) = y(x)$. The sol is an odd func. We set the $I_{max} = (-\alpha, \alpha)$, $\alpha > 0$. $y = w(x)$.

For every $l > 0$:

Consider (l) $\begin{cases} \dfrac{dy}{dx} = l^2 + y^2, \ x \ge l. \\ y(l) = 0. \end{cases}$ $\Rightarrow y = \phi(x) = l \tan[l(x - l)]$, $l \le x < l + \dfrac{\pi}{2l}$.

For every $L > 0$:

Consider (L) $\begin{cases} \dfrac{dy}{dx} = L^2 + y^2, \ 0 \le x < L. \\ y(0) = 0. \end{cases}$ $\Rightarrow y = \Phi(x) = L \tan(Lx)$, $0 \le x < \min\{L, \dfrac{\pi}{2L}\}$.

By Rmk1: $u(x) > \phi(x)$ as long as they both exist $\Rightarrow \alpha \le l + \frac{\pi}{2l}$.

By Thm1: $u(x) < \Phi(x)$ ——————||—————— $\Rightarrow \alpha \ge \min\{L, \frac{\pi}{2L}\}$.

$\Rightarrow \min\{L, \frac{\pi}{2L}\} \le \alpha \le l + \frac{\pi}{2l}, \quad \forall\, l, L > 0.$

$\Rightarrow \sqrt{\frac{\pi}{2}} \le \alpha \le 2\sqrt{\frac{\pi}{2}}$.

<u>Theorem 2</u>: Assume $f(x,y), F(x,y): \underset{\mathbb{R}^2}{\overset{\text{open}}{G}} \longrightarrow \mathbb{R}$ ($G \subset \mathbb{R}^2$) are continuous & locally Lipschitz conti. w.r.t. y, and

$$f(x,y) \le F(x,y), \quad \forall (x,y) \in G.$$

If $\{(x, \phi(x)) : a<x<b\} \subset G$, $\{(x, \Phi(x)) : a<x<b\} \subset G$,

$$\begin{cases} \phi'(x) = f(x, \phi(x)), & x \in (a,b) \\ \phi(x_0) = y_0 \end{cases} \qquad \begin{cases} \Phi'(x) = F(x, \Phi(x)), & x \in (a,b) \\ \Phi(x_0) = y_0 \end{cases}$$

then $\begin{cases} \phi(x) \le \Phi(x), & x_0 \le x < b \\ \phi(x) \ge \Phi(x), & a < x \le x_0 \end{cases}$

<u>Idea of pf</u>: Consider $\begin{cases} \phi_m'(x) = f(x, \phi_m(x)) - \varepsilon_m < F(x, \Phi_m(x)) \\ \phi_m(x_0) = y_0 \end{cases}$

$\varepsilon_m > 0$, $\varepsilon_m \to 0$ as $m \to \infty$.

By Thm 1. $\phi_m < \Phi$, $\forall x \in (x_0, b)$

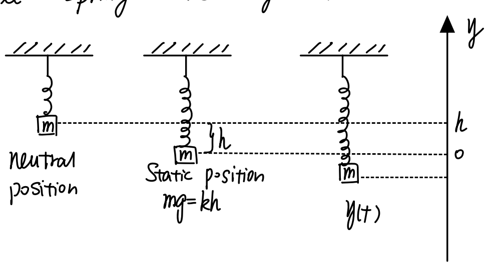
$\phi_m > \Phi$, $\forall x \in (a, x_0)$.

Let $m \to \infty$: $\phi_m \to \phi$. $\square$

$\boxed{3}$ Second-order Linear Eq.s

3.1 Homogeneous D.E.s with constant coefficients.

Example: Spring-mass System.



Newton's 2nd law of motion:

$$m\frac{d^2y}{dt^2} = -mg + F_{spring} + F_{friction} + F_{external}.$$

$F_{spring} = k(h-y(t))$, $k>0$.

$F_{friction} = -cv = -c\frac{dy}{dt}$, $c>0$

$F_{external} = F(t)$.

$\Rightarrow m\dfrac{d^2y}{dt^2} = -mg + k(h - y(t)) - c\dfrac{dy}{dt} + F(t)$. $\Rightarrow m\dfrac{d^2y}{dt^2} + c\dfrac{dy}{dt} + ky = F(t)$ 2nd-order, linear.

$m, c, k > 0$. #.

General 2nd-order DE: $y'' = f(t, y, y')$ (1) (Recall: 1st-order: $y' = f(t, y)$)

- (1) is <u>linear</u> if $y'' + p(t)y' + q(t)y = g(t)$ (2)
- (2) is <u>homogeneous</u> if $g(t) \equiv 0$. (otherwise: <u>nonhomogeneous</u>).

Second-order linear homogeneous D.E. with constant coefficients:

$ay'' + by' + cy = 0$, $a \neq 0$, b, c are real constants. (3)

<u>Trial sol:</u> $y(t) = e^{rt}$. $y' = re^{rt}$. $y'' = r^2e^{rt}$.

$\Rightarrow (ar^2 + br + c)e^{rt} = 0$. $\Rightarrow ar^2 + br + c = 0$. $\Rightarrow r$ must be a root of $ar^2 + br + c = 0$. (*)

(*) is called the <u>characteristic eq.</u>

<u>Case 1:</u> $\Delta = b^2 - 4ac > 0$. $r_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$. $r_1 \neq r_2$

Then the general sol. of (3) is <u>$y_g(t) = C_1e^{r_1t} + C_2e^{r_2t}$. C_1, C_2: consts</u> (4).

Easy to check every $y(t)$ in the form of (4) is a sol.

(Why a general sol? To be discussed in Sec 3.2.)

<u>E.g.</u> Solve $y'' + 2y' - 8y = 0$.

Sol: char. eq.: $r^2 + 2r - 8 = 0$. $\Delta = 36 > 0$. $r_{1,2} = \dfrac{-2 \pm 6}{2} = 2, -4$

$\Rightarrow$ the general sol. is $y_g(t) = C_1e^{2t} + C_2e^{-4t}$. C_1, C_2: consts.

<u>E.g.</u> Solve the (IVP) $\begin{cases} y'' + 2y' - 8y = 0. \\ y(0) = 3, \ y'(0) = -6. \end{cases}$

Sol: $y_g(t) = C_1e^{2t} + C_2e^{-4t}$. $\begin{cases} 3 = C_1 + C_2 \\ -6 = 2C_1 - 4C_2 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 2. \end{cases} \Rightarrow y(t) = e^{2t} + 2e^{-4t}$.

<u>Case 2:</u> $b^2-4ac=0$. $r_1=r_2$. (§3.4)

<u>Case 3:</u> $b^2-4ac>0$. $r_{1,2}=\lambda\pm i\mu$ (§3.5)

3.2. General Theory of Sol.s of 2nd-order Linear Homogeneous Eq.s (i.e. (2) with $g(t)\equiv 0$)

$I=\{t:\alpha<t<\beta\}$ $(-\infty\le\alpha<\beta\le\infty)$ open interval. $p(t)$ & $q(t)\in C(I)$.

Differential Operator: $L: C^2(I)\longrightarrow C(I)$, $L[\phi]=\phi''+p(t)\phi'+q(t)\phi$.

$L=D^2+p(t)D+q(t)I_d$. D – derivative operator. I_d – identity operator.

<u>Theorem 1 (E&U):</u> Consider

$$(IVP)\begin{cases} L[y](t)=y''+p(t)y'+q(t)y=g(t), \\ y(t_0)=y_0,\ y'(t_0)=y_0', \end{cases}$$

where $p(t)$, $q(t)$, $g(t)$ are continuous on I, $t_0\in I$. Then $\exists!$ sol. $y=\phi(t)$ existing on I.

<u>E.g.</u> Discuss the interval of the sol. of

$$\begin{cases} (t^2-3t)y''+ty'-(t+3)y=0. \\ y(1)=2,\ y'(1)=1. \end{cases}$$

Sol: Convert the eq. to the standard form $y''+\dfrac{t}{t(t-3)}y'-\dfrac{t+3}{t(t-3)}y=0$.

The maximum interval that contains $t=1$ on which both the coefficients are continuous is $(0,3)$. Thus, by Thm 1, the sol. exists on $(0,3)$.

<u>E.g.</u> $\begin{cases} y''+p(t)y'+q(t)y=0. \\ y(t_0)=0,\ y'(t_0)=0. \end{cases}$ $p(t)$, $q(t)\in C(I)$, $t_0\in I$.

Thm 1 $\Rightarrow$ $y(t)\equiv 0$ on I is the unique sol.

<u>Thm 2</u> (Principle of Superposition)

y_1, y_2 are two sol.s of $L[y]=0$ $\Rightarrow$ $c_1y_1+c_2y_2$, c_1 & c_2 coefficients, is also a sol.

<u>Pf:</u> $L[c_1y_1+c_2y_2]=c_1L[y_1]+c_2L[y_2]=0+0=0$. $\square$.

<u>Def:</u> The <u>Wronskian determinant</u>, or simply Wronskian of $y_1, y_2\in C'(I)$ is

$$W(t) \triangleq \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}, \quad \forall t \in I.$$

<u>Thm3</u>: Assume $p(t), q(t) \in C(I)$. If y_1 & y_2 are two sol.s of $L[y]=0$ on I, then either $W[y_1, y_2](t) \equiv 0$ on I or never $=0$ on I

<u>Pf</u>: $\begin{cases} y_1'' + p(t)y_1' + q(t)y_1 = 0 & (1) \\ y_2'' + p(t)y_2' + q(t)y_2 = 0 & (2) \end{cases}$

$(1) \times y_2 - (2) \times y_1$:

$$\underbrace{(y_1'' y_2 - y_2'' y_1)}_{= (y_1' y_2 - y_2 y_1')'} + p(t)(y_1' y_2 - y_2 y_1') = 0$$

$\Rightarrow w'(t) + p(t)\, w(t) = 0.$

$\Rightarrow \boxed{\begin{array}{l} W(t) = c \cdot e^{-\int p(t)\,dt}, \quad c: \text{any const.} \\ \text{either } c=0,\ W(t) \equiv 0. \\ \text{or } c \neq 0,\ W(t) \neq 0,\ \forall t \in I. \end{array}}$ Abel's Theorem.

<u>Def</u>: If $\exists\, C_1, C_2$, (not both $=0$) s.t. $C_1 y_1(t) + C_2 y_2(t) \equiv 0$ on I, then we say y_1 and y_2 are <u>linearly dependent</u> on I. Otherwise, we say they are <u>linearly independent</u>.

<u>Thm 4</u>. Suppose y_1 and y_2 are two sol.s of $L[y]=0$ on I, where $p(t)$ & $q(t) \in C(I)$.

Then $\exists\, t_0 \in I$ s.t. $W(t_0) = 0 \Leftrightarrow y_1$ and y_2 are linearly dependent.

<u>Pf</u>: $(\Leftarrow)$ $\exists\, C_1, C_2$ not both $=0$. s.t. $(C_1 y_1 + C_2 y_2)(t) \equiv 0$ $\forall t \in I$. WLOG, assume $C_1 \neq 0$.

$$\Rightarrow y_1 = -\frac{C_2}{C_1} y_2, \text{ and } W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}(t) = \begin{vmatrix} -\frac{C_2}{C_1} y_2 & y_2 \\ -\frac{C_2}{C_1} y_2' & y_2' \end{vmatrix}(t) \equiv 0.$$

$(\Rightarrow)$ $\exists\, t_0 \in I$, s.t. $W(t_0) = 0$.

$$\Rightarrow \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad \exists \text{ Sol. } \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

WTS: $C_1 y_1 + C_2 y_2 \equiv 0$. $L[C_1 y_1 + C_2 y_2] = 0$. $\begin{cases} (C_1 y_1 + C_2 y_2)(t_0) = 0. \\ (C_1 y_1' + C_2 y_2')(t_0) = 0. \end{cases} \xrightarrow{\text{Thm1}} C_1 y_1 + C_2 y_2 \equiv 0.$ $\square$

Thm 5: Suppose $p(t), q(t) \in C(I)$, and y_1, y_2 are two linearly independent sol.s.
Then the general sol. of $L[y]=0$ is given by

$$y_g = C_1 y_1 + C_2 y_2. \quad C_1, C_2 \text{ are constants.}$$

y_1 & y_2 are said to form a fundemental set of sol.s of $L[y]=0$.

Pf: Given any sol $y=\phi(t)$ of $L[y]=0$. WTS: $\exists\, C_1, C_2$, s.t $\phi(t) = C_1 y_1(t) + C_2 y_2(t)$.

Since y_1, y_2 are linearly independent, by Thm 4, $\exists\, t_0$ s.t. $W[y_1, y_2](t_0) \neq 0$.

Consider $\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \phi(t_0) \\ \phi'(t_0) \end{pmatrix}$. $\exists!$ sol $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, then $\phi(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0)$

$\Rightarrow \phi(t) = C_1 y_1(t) + C_2 y_2(t)$ by E&U thm.

E.g. In sect. 3.1, for $ay''+by'+cy=0$. $\Delta = b^2-4ac>0$. $r_1 \neq r_2$.

two sol.s $y_1(t) = e^{r_1 t}$ $y_2(t) = e^{r_2 t}$. $y_g(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

$$W[y_1, y_2](t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1+r_2)t} \neq 0, \ \forall t. \xRightarrow{\text{Th4. Th5}} y_g = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

E.g. Find the Wronskian of any pair of sols of

$$2t^2 y'' + 3ty' - y = 0. \quad t>0.$$

Sol: Convert the equation to the standard form

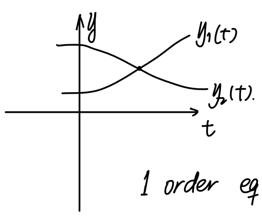
$$y'' + \frac{3}{2t} y' - \frac{1}{2t^2} y = 0, \quad t>0.$$

$p(t) = \frac{3}{2t}$. By Abel's Thm: Wronskian of any pair of sol's satisfies

$$W'(t) + \frac{3}{2t} W(t) = 0, \quad t>0.$$

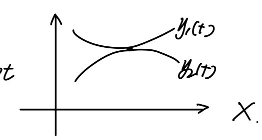
$$\Rightarrow W(t) = c \cdot \exp\left(-\int \frac{3}{2t}\, dt\right) = c\, t^{-\frac{3}{2}}, \quad t>0, \ c: \text{any const.}$$

E & U:



1 order eq: X

2 order eq: two sol.'s curves may intersect but cannot intersect tangently to each other.



$\boxed{3.3.\ \text{Complex Roots of the Characteristic Eq.s.}}$

$ay'' + by' + cy = 0.$ $a \neq 0,$ b, c: real numbers. (∗1)

Char. eq.: $ar^2 + br + c = 0.$ $\Delta = b^2 - 4ac < 0.$ $r_{1,2} = \lambda \pm i\mu.$ $\mu \neq 0.$

$$e^{(\lambda + i\mu)t} = e^{\lambda t}[\cos\mu t + i\sin\mu t]. \quad \left(\frac{d}{dz} e^{zt} = z \cdot e^{zt},\ z \in \mathbb{C}\right).$$
$$e^{(\lambda - i\mu)t} = e^{\lambda t}[\cos\mu t - i\sin\mu t].$$

They are two complex valued sol.s $\overset{(*)}{\Rightarrow}$ Their real part & image part give two real-valued sol.s.

Why (∗)?

Thm 1. Consider $L[y] = y'' + p(t)y' + q(t)y = 0.$ where p & q are conti. real-valued funcs.

If $y(t) = u(t) + iv(t)$ is a complex-valued sol. then $u(t), v(t)$ and $u - iv$ are also sol.s

Pf: $0 = L[u + iv] = L[u] + iL[v].$ $\Rightarrow L[u] = 0,$ $L[v] = 0.$

$$\Rightarrow L[u - iv] = L[u] - iL[v] = 0. \quad \square.$$

Next, consider $y_1(t) = e^{\lambda t} \cos(\mu t),$ $y_2(t) = e^{\lambda t} \cdot \sin(\mu t).$

They are linearly indep. $\Leftrightarrow$ $\cos(\mu t)$ and $\sin(\mu t)$ are linearly indep.

$$W[\cos(\mu t), \sin(\mu t)] = \begin{vmatrix} \cos(\mu t) & \sin(\mu t) \\ -\mu\sin(\mu t) & \mu\cos(\mu t) \end{vmatrix} = \mu \neq 0.$$

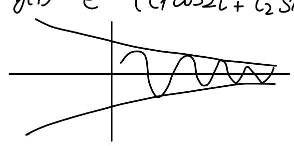
$\Rightarrow y_1(t)$ & $y_2(t)$ form a fundemental set of sol.s of (∗1)

Eg. 1: $y'' + 6y' + 13y = 0.$

Char eq: $r^2 + 6r + 13 = 0.$ $r_{1,2} = \dfrac{-6 \pm 4i}{2} = -3 \pm 2i.$

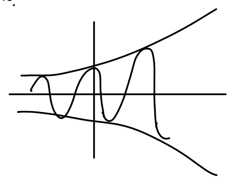
$e^{(-3+2i)t} = e^{-3t}(\cos 2t + i\sin 2t)$ $\quad u(t) = e^{-3t}\cos 2t$ $\quad y(t) = e^{-3t}(C_1\cos 2t + C_2 \sin 2t)$

$e^{(-3-2i)t} = e^{-3t}(\cos 2t - i\sin 2t).$ $\quad v(t) = e^{-3t}\sin 2t.$

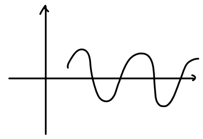


$y''+6y'+13y=0.$ $\quad my''+cy'+ky=F(t).$ $\quad m,c,k>0.$

$y''-6y'+13y=0$ $\quad y=e^{3t}(C_1\cos 2t+C_2\sin 2t).$



$y''+4y=0.$ $\quad y=C_1\cos(2t)+C_2\sin(2t).$



$ay''+by'+cy=0.$ $\quad a\neq 0,$ b,c: real numbers $\quad (*1)$

Char. eq: $ar^2+br+c=0.$ If $\Delta=b^2-4ac=0.$ $r_1=r_2=r=-\frac{b}{2a}.$

$y_1(t)=e^{rt}.$ $\quad y_2(t)=te^{rt}.$ $\quad ay_2''+by_2'+cy_2=e^{rt}[2ar+ar^2t+b+brt+ct]$

$$=e^{rt}(b+2ar)=0.$$

$$\begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}=1\neq 0. \qquad \Rightarrow y_g=C_1e^{rt}+C_2te^{rt}.$$

3.4 Repeated Roots; Order of Reduction. (Continued)

$ay''+by'+c=0.$ $\quad ar^2+br+c=0.$ $\quad \Delta=b^2-4ac=0,$ $r_1=r_2=r.$

$y_1(t)=e^{rt},$ $y_2(t)=te^{rt}.$

Rk: If the eq is non-homogeneous, then take y_1 as a sol. of the homogeneous eq.

Method of order of Reduction:

↑ Due to D'Alembert.

$y''+p(t)y'+q(t)y=0$

Assume $y_1(t)$ is a nontrivial sol. Look for a second sol. in the form

$y_2(t)=v(t)y_1(t).$

$y_2'=v'(t)y_1(t)+v(t)y_1'(t),$ $\quad y_2''=v''y_1+v'y_1'+v'y_1'+vy_1''.$

Plug y_2, y_2', y_2'' into the eq:

$$(v''y_1+2v'y_1'+vy_1'')+(v'y_1+vy_1')p(t)+vy_1q(t)=0.$$

$$v''y_1 + (2y_1' + y_1 p(t))v' + \underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{\overset{\|}{0} \text{ since } y_1 \text{ is a sol.}} v = 0.$$

Let $w(t) = v'(t) \Rightarrow w'(t)y_1 + (2y_1' + y_1 p(t))w(t) = 0.$ — a 1-order eq.

$\Rightarrow w \quad \Rightarrow v \Rightarrow y_2(t) = v(t)y_1(t).$

<u>E.g.</u> Given that $y_1(t) = t^{-1}$ is a sol. of

$$2t^2y'' + 3ty' - y = 0 \qquad t>0.$$

find a fundemental set of sol.s.

Sol: Look for $y_2(t) = v(t)\cdot y_1(t) = v(t)\cdot t^{-1}$.

$y_2'(t) = v'(t)t^{-1} - t^{-2}v. \qquad y_2''(t) = v''t^{-1} - t^{-2}v' - t^{-2}v' + 2t^{-3}v.$

$2t^2(v''t^{-1} - t^{-2}v' + 2t^{-3}v - t^{-2}v') + 3t(v't^{-1} - t^{-2}v) - vt^{-1} = 0.$

$\Rightarrow 2tv'' - v' = 0$

Let $w(t) = v'(t). \Rightarrow 2tw'(t) - w(t) = 0. \Rightarrow w = C_1\cdot t^{\frac{1}{2}}.$ C_1: any const.

$\Rightarrow v(t) = C_1 t^{\frac{3}{2}} + C_2.$ C_1, C_2: any const.

$\Rightarrow y_2(t) = v(t)\,y_1(t) = C_1 t^{\frac{1}{2}} + C_2 t^{-1}.$

we may take $y_2(t) = t^{\frac{1}{2}}$. $W[y_1, y_2](t) = \begin{vmatrix} t^{-1} & t^{\frac{1}{2}} \\ -t^{-2} & \frac{1}{2}t^{-\frac{1}{2}} \end{vmatrix} = \frac{3}{2}t^{-\frac{3}{2}} \neq 0.$ $(t>0)$

Thus, a f.s. of sol is $\{t^{\frac{1}{2}}, t^{-1}\}$.

3.5 Nonhomogeneous Eq.s: Method of undetermined coefficients.

Thm: Consider $L[y] = y'' + p(t)y' + q(t)y = g(t)$ (1)

& $L[y] = 0$ (2)

Then the general sol of (1) is $y_g = y_p + y_h$. (3)

where y_p is any particular sol. of (1) & y_h is the general sol. of (2).

Pf: Any $y(t) = y_p + y_h$, $L[y] = L[y_p] + L[y_h] = g(t) + 0 = g(t)$

Any sol. of (1), say $y = \phi(t)$, then

$L[\phi(t) - y_p] = L[\phi] - L[y_p] = g(t) - g(t) = 0$.

$\Rightarrow \phi(t) - y_p$ is a sol. of (2).

Eg. 1 $y'' - 3y' - 4y = 3e^{2t}$.

Sol: $y'' - 3y' - 4y = 0$. $r^2 - 3r - 4 = 0$. $(r-4)(r+1) = 0$. $r_1 = 4$, $r_2 = -1$.

$\Rightarrow y_h = C_1 e^{4t} + C_2 e^{-t}$.

Look for a $y_p = Ae^{2t}$.

$\Rightarrow (4A - 6A - 4A)e^{2t} = 3e^{2t} \Rightarrow A = -\frac{1}{2} \Rightarrow y_p = -\frac{1}{2}e^{2t}$.

$\Rightarrow$ the general sol. is $y_g = y_p + y_h = -\frac{1}{2}e^{2t} + C_1 e^{4t} + C_2 e^{-t}$.

Eg. 2 $y'' + 2y' - 8y = e^{2t}$.

Sol: $y_h = C_1 e^{2t} + C_2 e^{-t}$.

Trial Sol: $y_p = Ae^{2t} \cdot t$.

$y_p' = Ae^{2t} + 2Ae^{2t}\cdot t.$ $\quad y_p'' = 2Ae^{2t} + 2Ae^{2t} + 4Ae^{2t}\cdot t.$

$= 4Ae^{2t} + 4Ae^{2t}\cdot t.$

$Ae^{2t}(4+4t+2+4t-8t) = e^{2t}. \quad \Rightarrow A = \frac{1}{6}.$

$y_p = \frac{1}{6}te^{2t} \quad \Rightarrow y_g = y_p + y_h = \frac{1}{6}te^{2t} + C_1e^{2t} + C_2e^{-t}.$

E.g.3 $y'' + 2y' + y = e^{t}.$

Sol: $y_h = C_1e^{-t} + C_2te^{-t}.$

Trial Sol.: $y_p = At^2e^{-t} \Rightarrow A = \frac{1}{2}. \quad \Rightarrow y_g = y_p + y_h = \frac{1}{2}t^2e^{-t} + C_1e^{-t} + C_2te^{-t}.$

Alternative Sol: $(y'+y)' + y'+y = e^{t}$, let $u(t) = y'+y$.

Rk1: Moral of stories: if a trial sol. is covered by y_h, then multiply it by t.

E.g.4 $y'' + 2y' - 8y = t+1.$

Sol: $y_h = C_1e^{2t} + C_2e^{-4t}.$

Trial sol: $y_p = A_1t + A_0.$ $\quad 2A_1 - 8A_1t - 8A_0 = t+1 \Rightarrow A_1 = -\frac{1}{8}, A_0 = -\frac{5}{32}$

$\Rightarrow y_g = y_p + y_h = -\frac{1}{8}t - \frac{5}{32} + C_1e^{2t} + C_2e^{-4t}.$

E.g.5 $y'' + 2y' = t+1.$

Sol: $y_h = C_1 + C_2e^{-2t}.$

Trial sol: $y_p = A_1t + A_2t^2.$ $\quad 2A_2 + 2(2A_2t + A_1) = t+1 \Rightarrow A_1 = \frac{1}{4}, A_2 = \frac{1}{4}.$

$\Rightarrow y_g = y_p + y_h = \frac{1}{4}t + \frac{1}{4}t^2 + C_1 + C_2e^{-2t}.$

Eg. 6 $\quad y'' = t+1$

Sol: $y_h = c_1 + c_2 t$.

Trial sol: $y_p = A_1 t^3 + A_0 t^2 \Rightarrow A_1 = \frac{1}{6},\ A_0 = \frac{1}{2}$.

$\Rightarrow y_g = y_p + y_h = \frac{1}{6}t^3 + \frac{1}{2}t^2 + c_1 + c_2 t$.

Rk2: $y'' + ay' + by = h(t)$, $h(t)$ is a poly with $\deg h(t) = n$.

Case 1: $b \neq 0$, $\quad y_p = \sum_{k=0}^{n} A_k t^k$.

Case 2: $b = 0$, $a \neq 0$. $\quad y_p = \sum_{k=1}^{n+1} A_k t^k$.

Case 3: $b = a = 0$, $\quad y_p = \sum_{k=2}^{n+2} A_k t^k$.

Eg. 7 $\quad y'' - 3y' - 4y = 2\sin t \quad \mathrm{Im}(2e^{it})$.

Sol: $y_h = c_1 e^{-t} + c_2 e^{4t}$.

Trial sol: $y_p = A\sin t + B\cos t$

$$\Rightarrow \begin{cases} -5A + 3B = 2 \\ -5B - 3A = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{5}{17} \\ B = \frac{3}{17} \end{cases}$$

$\Rightarrow y_p = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$.

$y'' - 3y' - 4y = 2e^{it} = 2\cos t + i(2\sin t)$

Trial sol: $y_p(t) = Ae^{it}$.

$-Ae^{it} - 3Ae^{it} i - 4Ae^{it} = 2e^{it}$.

$Ae^{it}[-5 - 3i] = 2e^{it}$.

$A = \frac{2}{-5-3i} = \frac{2(-5+3i)}{34}$.

$\Rightarrow y_p = \frac{2(-5+3i)}{34}(\cos t + i\sin t)$.

$\mathrm{Im}\, y_p = \frac{3}{17}\cos t - \frac{5}{17}\sin t$.

Eg. 8. $\quad y'' - 3y' - 4y = -8e^t\cos(2t)$. $\quad \mathrm{Re}(-8e^{(1+2i)t})$.

Sol: $y_h = c_1 e^{-t} + c_2 e^{4t}$.

$y_p = Ae^t\cos(2t) + Be^t\sin(2t)$

$$\Rightarrow \begin{cases} A = \frac{10}{13} \\ B = \frac{2}{13}. \end{cases}$$

Trial: $y_p = Ae^{(1+2i)t}$. $\quad A \in \mathbb{C}$.

E.g. 9. Find a particular sol. of $y'' - 3y' - 4y = e^{2t} + \sin t - e^{t}\cos(2t)$.

If $L[y_i] = g_i$, $i = 1, \cdots, m$, then $L[\sum_{i=1}^{m} c_i y_i] = \sum_{i=1}^{m} c_i L[y_i] = \sum_{i=1}^{m} c_i g_i$.

In E.g 1, $L[y_p^{(1)}] = 3e^{2t}$, $y_p^{(1)} = -\frac{1}{2} e^{2t}$.

...

$$y_p = \frac{1}{3}\left(-\frac{1}{2}e^{2t}\right) + \frac{1}{2}\left(-\frac{5}{17}\sin t + \frac{3}{17}\cos t\right) + \frac{1}{8}\left(\frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t)\right).$$

3.6 Method of Variation of Parameters.

Consider $y'' + p(t)y' + q(t)y = g(t)$ (1)

$y_h(t) = C_1 y_1(t) + C_2 y_2(t)$. C_1, C_2 : any const.

Look for a particular sol. of (1) in the form $y_p(t) = C_1(t) y_1 + C_2(t) y_2$.

$$y_p'(t) = \underbrace{C_1'(t) y_1 + C_2'(t) y_2}_{=0} + C_1(t) y_1' + C_2(t) y_2'.$$

Choose $C_1(t)$, $C_2(t)$, s.t $C_1'(t) y_1 + C_2'(t) y_2 = 0$.

$$y_p''(t) = C_1'(t) y_1' + C_2'(t) y_2' + C_1(t) y_1'' + C_2(t) y_2''.$$

$$y_p''(t) + p(t) y_p'(t) + q(t) y_p(t)$$

$$= C_1'(t) y_1' + C_2'(t) y_2' + \cancel{C_1(t) y_1''} + \cancel{C_2(t) y_2''} + \cancel{C_1(t) p(t) y_1'} + \cancel{C_2(t) p(t) y_2'} + \cancel{C_1 q y_1} + \cancel{C_2 q y_2} = g(t)$$

Thus, $\begin{cases} C_1'(t) y_1 + C_2'(t) y_2 = 0 \\ C_1'(t) y_1' + C_2'(t) y_2' = g(t) \end{cases}$ $\underset{:= W(t)}{W[y_1, y_2](t) \neq 0}$ $\Rightarrow$ $\begin{cases} C_1'(t) = \dfrac{-g(t) y_2(t)}{W(t)} \\ C_2'(t) = \dfrac{g(t) y_1(t)}{W(t)}. \end{cases}$

Fix $t_0 \in I$, where $p(t)$, $q(t)$ and $g(t)$ are continuous.

Take $C_1(t) = \int_{t_0}^{t} \frac{-y_2(s) g(s)}{W(s)} ds$ $\quad C_2(t) = \int_{t_0}^{t} \frac{y_1(s) g(s)}{W(s)} ds.$

$\Rightarrow y_p(t) = C_1(t)\, y_1(t) + C_2(t)\, y_2(t).$

Rk. This method of finding y_p is called variation of param. / consts, which is due to Lagrange. It is a general-method and complements the method of undetermined coefficients very well.

E.g. 1: $y'' + y = \sec t = \dfrac{1}{\cos t}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$

Sol: $y_h = C_1 \underbrace{\cos(t)}_{y_1(t)} + C_2 \underbrace{\sin(t)}_{y_2(t)}, \quad W[y_1, y_2](t) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} \equiv 1.$

Trial sol: $y_p = C_1(t)\cos(t) + C_2(t)\sin(t).$

$$\begin{cases} C_1'(t)\cos(t) + C_2'(t)\sin(t) = 0 \\ -C_1'(t)\sin(t) + C_2'(t)\cos(t) = \dfrac{1}{\cos t} = g(t). \end{cases}$$

$\Rightarrow \quad C_1'(t) = -\dfrac{\sin(t)}{\cos(t)}, \quad C_2'(t) = 1.$

$$C_1(t) = \int_0^t \frac{-\sin(s)}{\cos(s)}\, ds \;\; \underset{du = -\sin(s)\, ds}{\overset{u = \cos(s)}{=\!=\!=}} \;\; \int_1^{\cos(t)} \frac{du}{u} = \ln|\cos(t)| = \ln \cos(t).$$

$$C_2(t) = \int_1^t 1\, ds = t.$$

$\Rightarrow y_p(t) = \cos(t) \ln\cos(t) + t\sin(t).$

E.g. 2: $y'' + 5y' + 6y = 4e^t.$

Sol: $y_h = C_1 e^{2t} + C_2 e^{3t}.$

$y_p = ce^t \Rightarrow \quad ce^t[1 - 5 + 6] = 4e^t \Rightarrow y_p = 2e^t.$

Alternative: trial sol: $y_p = C_1(t) e^{2t} + C_2(t) e^{3t}.$

$$\begin{cases} C_1'(t)\, e^{2t} + C_2'(t)\, e^{3t} = 0 \\ C_1'(t) \cdot 2e^{2t} + C_2'(t) \cdot 3e^{3t} = 4e^t \end{cases}$$

$\Rightarrow \quad C_1'(t) = -4e^{-t}, \qquad C_2'(t) = 4e^{-2t}$

$$C_1(t) = \int_0^t (-4e^{-s})\, ds = 4e^{-s}\Big|_{s=0}^{s=t} = 4e^{-t} - 4.$$

$$C_2(t) = \int_0^t 4e^{-2s}\, ds = -2e^{-2s}\Big|_{s=0}^{s=t} = -2e^{-2t} + 2.$$

$\Rightarrow y_p = (4e^{-t} - 4)\, e^{2t} + (-2e^{-2t} + 2)\, e^{3t}$

$= 4e^t - 4e^{2t} - 2e^t + 2e^{3t} = 2e^{3t} - 4e^{2t} + 2e^t.$

3.7 Mechanical Vibrations; Beats and Resonance.

Recall: Spring-mass system:

$my'' + cy' + ky = f(t)$, where m = mass, $0 \le c$ = damping coefficient.
k = spring const.

$\Rightarrow y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{1}{m} f(t).$

$\Rightarrow y'' + 2cy' + \omega_0^2 y = F(t), \quad \omega_0 = \sqrt{\frac{k}{m}} > 0, \quad c \ge 0.$

(i) Case of unforced system: $F(t) = 0.$

$y'' + 2cy' + \omega_0^2 y = 0. \qquad (*1)$

Char eq.: $r^2 + 2cr + \omega_0^2 = 0. \quad r = \frac{-2c \pm \sqrt{4c^2 - 4\omega_0^2}}{2} = -c \pm \sqrt{c^2 - \omega_0^2}.$

Case (a): $c > \omega_0$. $r_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2} < 0.$

$y_g(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \longrightarrow 0$ as $t \to \infty.$

Claim: $\nexists\, t_1 \ne t_2$, s.t. $y(t_1) = 0 = y(t_2)$ unless $y(t) \equiv 0.$

Suppose rather: $\exists\, t_1 \neq t_2$ s.t. $y(t_1) = 0 = y(t_2)$

$$\begin{cases} C_1 e^{r_1 t_1} + C_2 e^{r_2 t_1} = 0 \\ C_1 e^{r_1 t_2} + C_2 e^{r_2 t_2} = 0 \end{cases} \qquad A\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \det A = e^{r_1 t_1 + r_2 t_2} - e^{r_2 t_1 + r_1 t_2} = 0$$

$$\Updownarrow$$

$$r_1 t_1 + r_2 t_2 - r_2 t_1 - r_1 t_2 = 0$$

$$\Leftrightarrow \underbrace{(r_1 - r_2)}_{\neq 0}\underbrace{(t_1 - t_2)}_{\neq 0} = 0. \quad ↯$$

Case (b): $c = \omega_0$. $r_1 = r_2 = -c < 0$.

$$y_g(t) = C_1 e^{-ct} + C_2 t e^{-ct} \to 0 \quad \text{as } t \to \infty.$$

Case (c): $c < \omega_0$. $r_{1,2} = -c \pm \beta i$, $\beta = \sqrt{\omega_0^2 - c^2}$.

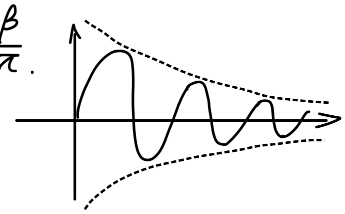
$$e^{(-c+\beta i)t} = e^{-ct}\left[\cos(\beta t) + i \sin(\beta t)\right].$$

$$\Rightarrow y_g(t) = C_1 e^{-ct} \cos(\beta t) + C_2 e^{-ct} \sin(\beta t) = e^{-ct} \cdot \sqrt{C_1^2 + C_2^2} \cos(\beta t - \varphi).$$

$$\tan \varphi = \frac{C_2}{C_1}.$$

amplitude: $\sqrt{C_1^2 + C_2^2}\, e^{-ct} \longrightarrow 0$ as $t \to \infty$.

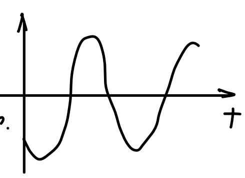
period of oscillation: $\frac{2\pi}{\beta}$. frequency: $\frac{\beta}{2\pi}$.



In particular, the undamped case: $c = 0$

$$\Rightarrow \beta = \omega_0. \quad y_g(t) = \sqrt{C_1^2 + C_2^2} \cos(\omega_0 t - \varphi).$$

$\to$ harmonic oscillation.



(ii) Case of undamped and forced system. ($c = 0$)

$$y'' + \omega_0^2 y = F(t) = \underline{A \cos(\omega t)}.$$

ω_0: natural freq.

ω: external freq.

$A \sin(\omega t)$ is similar.

Case (a) $w = w_0$. (phenomenon of resonance).

$$y_h = C_1\cos(w_0 t) + C_2\sin(w_0 t) = \sqrt{C_1^2+C_2^2}\cos(w_0 t - \varphi),\quad \tan\varphi = \frac{C_2}{C_1}.$$

Trial sol: $y_p = C_1 t\cos(w_0 t) + C_2 t\sin(w_0 t)$.

$$y_p'' + w_0^2 y_p = -2C_1 w_0\sin(w_0 t) + 2C_2 w_0\cos(w_0 t) = A\cos(w_0 t).$$

$$\Rightarrow C_1 = 0,\quad C_2 = \frac{A}{2w_0}.\quad \Rightarrow y_p = \frac{A}{2w_0} t\sin(w_0 t).$$

$$y_g = y_h + y_p = \underbrace{\boxed{\frac{A}{2w_0} t}}_{\to\infty \text{ as } t\to\infty}\sin(w_0 t) + \sqrt{C_1^2+C_2^2}\cos(w_0 t - \varphi).$$

Case (b) $w \neq w_0$.

$$y_h = \sqrt{C_1^2+C_2^2}\cos(w_0 t - \varphi)$$

trial sol: $y_p = B_1\cos(wt) + \underbrace{B_2\sin(wt)}$.

$-B_2 w^2 + B_2 w_0^2 = 0$. impossible

$$y_p = C\cdot\cos(wt).$$

$$y_p'' + w_0^2 y_p = -Cw^2\cos(wt) + w_0^2\cdot C\cdot\cos(wt) = A\cos(wt)$$

$$\Rightarrow C = \frac{A}{w_0^2 - w^2}.$$

$$y_p = \frac{A}{w_0^2-w^2}\cos(wt).\qquad y_g = y_p + y_h = \frac{A}{w_0^2-w^2}\cos(wt) + C_1\cos(w_0 t) + C_2\sin(w_0 t).$$

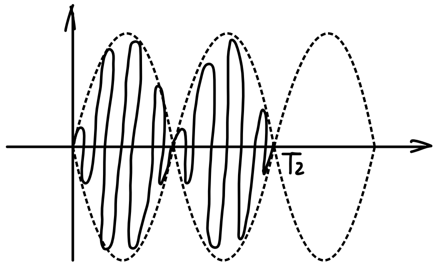
Initial condition: $y(0) = 0$, $y'(0) = 0$. (The mass is initially at rest.)

$$\Rightarrow \begin{cases} C_1 = -\dfrac{A}{w_0^2-w^2} \\ C_2 = 0 \end{cases} \Rightarrow y = \frac{A}{w_0^2-w^2}\left[\cos(wt) - \cos(w_0 t)\right].$$

$$= \frac{-2A}{w_0^2-w^2}\sin\left(\frac{w+w_0}{2}t\right)\sin\left(\frac{w-w_0}{2}t\right)$$

$$T_1 = \frac{4\pi}{w+w_0}.\qquad T_2 = \frac{4\pi}{w-w_0}.$$

- $w \neq w_0$, $w \approx w_0$: $T_1 \ll T_2$.



phenomenon of beats:
periodic variation of amplitude.

- $$\lim_{w\to w_0} y(t) = \lim_{w\to w_0} \frac{2A}{w_0+w} \sin\left(\frac{w+w_0}{2}t\right) \frac{\sin\left(\frac{w-w_0}{2}t\right)}{\frac{w-w_0}{2}t} \cdot \frac{t}{2}.$$

$$= \frac{A}{w_0} \sin(w_0 t) \frac{t}{2}.$$

- $$\lim_{w\to\infty} y_g(t) = \lim_{w\to\infty} \left[C_1 \cos(w_0 t) + C_2 \sin(w_0 t) + \frac{A}{w_0^2-w^2}\cos(wt) \right].$$

$$= C_1 \cos(w_0 t) + C_2 \sin(w_0 t).$$

(iii) $c>0$, $f(t) \neq 0$.

amplitude of input.

$$y'' + 2cy' + w_0^2 y = A \cos(wt).$$

$y_g = y_h + y_p \approx y_p$ for $t\to\infty$ (since $y_h \to 0$ as $t\to\infty$).

$\Rightarrow$ y_p : "steady state sol." y_h: transient sol

转瞬的.

Trial sol: $y_p = B_1 \cos(wt) + B_2 \sin(wt)$.

$$\Rightarrow y_p = \frac{A}{(w_0^2-w^2)^2 + (2cw)^2} \left[(w_0^2-w^2)\cos(wt) + 2cw \sin(wt) \right]$$

$$= \boxed{\frac{A}{\sqrt{(w_0^2-w^2)^2+4c^2w^2}}} \cos(wt-\gamma).$$

amplitude of output.

Q: How to choose w to maximize $M = \frac{1}{\sqrt{(w_0^2-w^2)^2+4c^2w^2}}$?

4. High Order Linear DEs

4.1 General Theory.

nth-order linear ODE has the form:

$$\underbrace{\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y}_{:= L(y).} = g(t). \quad (1)$$

$L: C^n(I) \to C^0(I)$. differential operator.

<u>Theorem 1</u>: (E & U)

The Cauchy problem

$$\begin{cases} y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_{n-1}(t)y' + P_n(t)y = g(t) \\ y(t_0) = y_0,\ y'(t_0) = y_0^{(1)},\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}. \end{cases} \quad (2)$$

with continuous $P_1, \cdots, P_n, g$ on I, $t_0 \in I$, $\exists!$ sol. $y = \phi(t)$, which exists throughout I.

<u>Pf</u>: Brief idea: $Y(t) \triangleq (y(t), y'(t), \cdots, y^{(n-1)}(t))$

$$\begin{matrix} y(t), & y'(t), & \cdots, & y^{(n-1)}(t) \\ \| & \| & & \| \\ y_1(t) & y_2(t), & \cdots, & y_n(t). \end{matrix}$$

$y_1'(t) = y_2$

$y_2'(t) = y_3$

$\vdots$

$y_{n-1}'(t) = y_n$

$y_n'(t) = y^{(n)}(t) = g(t) - P_1(t)y_n - \cdots - P_{n-1}(t)y_2 - P_n(t)y_1$

$\longrightarrow Y'(t) = F(t, Y)$, $F = (f_1(t, Y), \cdots, f_n(t, Y))$.

E&U: F is continuous in R: $|t-t_0|\le a$, $\|Y-Y_0\|\le b$.

and is Lipschitz continuous w.r.t. Y:

$\exists\ L>0$, s.t $\forall (t,Y_1), (t,Y_2)\in R$,

it holds $\|F(t,Y_1)-F(t,Y_2)\|\le L\|Y_1-Y_2\|$.

Then $\exists!$ sol. $Y(t)=\Phi(t)$ defined on $[t_0-h, t_0+h]$,

where h is given by $h=\min\{a, \frac{b}{M}\}$. $M=\max\limits_{(t,Y)\in R}\|F(t,Y)\|$.

Theorem 2:

Assume $y_1, y_2, \cdots, y_n$ are n sol.s of

$$L[y]=y^{(n)}(t)+p_1(t)\,y^{(n-1)}(t)+\cdots+p_{n-1}(t)\,y'(t)+p_n(t)\,y(t)=0.$$

where $p_1,\cdots,p_n\in C(I)$. Then TFAE:

(i) $y_g=C_1y_1(t)+C_2y_2(t)+\cdots+C_ny_n(t)$. $C_1,\cdots,C_n$: any consts.

(ii) $\exists\ t_0\in I$ s.t. the Wronskian:

$$W[y_1,y_2,\cdots,y_n](t_0)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}(t_0)\ne 0.$$

(iii) $W[y_1,y_2,\cdots,y_n](t)\ne 0\quad \forall\, t\in I$.

(iv) $y_1,\cdots,y_n$ are linearly independent on I:

i.e. $C_1y_1+\cdots+C_ny_n\equiv 0$ on $I\Rightarrow C_1=\cdots=C_n=0$.

Pf: (i) $\Rightarrow$ (iii) ($\Rightarrow$ (ii)):

Suppose rather: $\exists\ t_0\in I$ s.t $W[y_1,\cdots,y_n](t_0)=0$.

Then, one can select $a_1, \cdots, a_n$ s.t.

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = a_1 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) = a_2 \\ \cdots \cdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = a_n \end{cases}$$

has no sol. for $(c_1, \cdots, c_n)$.

$A_{n\times n}: \mathbb{R}^n \to \mathbb{R}^n$.

$\dim(\ker A) + \dim(\mathrm{Im} A) = n$.

For such a seq. of $a_1, \cdots, a_n$, the Cauchy problem $\begin{cases} L[y]=0 \\ y(t_0)=a_1, \cdots, y^{(n-1)}(t_0)=a_n \end{cases}$

$\exists!$ sol. $\phi(t)$ by Thm 1. However, $\phi(t)$ cannot be expressed as $c_1y_1 + \cdots + c_n y_n$ (Otherwise the above system has sol. for $(c_1, \cdots, c_n)$) which violets (i).

(ii) ⇒ (iv): Assume $c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0$. $\forall t \in I$.

Then $\sum_{i=1}^{n} c_i y_i^{(k)}(t) = 0 \quad \forall t \in I, \ \forall k = 1, \cdots, n-1.$

This shows $(c_1, \cdots, c_n)$ satisfies

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = 0. \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) = 0. \\ \cdots \cdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = 0. \end{cases} \quad (3)$$

t_0 is as in (ii).

$W[y_1, \cdots, y_n](t_0) \neq 0 \Rightarrow c_1 = \cdots = c_n = 0.$

(iv) ⇒ (iii): Suppose rather: $\exists\, t_0 \in I$ s.t $W[y_1, \cdots, y_n](t_0) = 0$

⇒ (3) has a nonzero sol $(c_1, c_2, \cdots, c_n)$.

Now consider $\phi(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$

satisfies $\begin{cases} L[y](t) = 0, & t \in I \\ y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0. \end{cases}$ $\xrightarrow{\text{Thm 1}}$ $\phi(t) \equiv 0$ on I.

(ii) $\Rightarrow$ (i): Let $y(t)$ be any sol. of $L[y](t) = 0$ on I.

WTS: $\exists\ c_1, \cdots, c_n$ s.t $y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$ on I.

By (ii), $W(t_0) \neq 0$. Then $\exists!$ sol. $(c_1, \cdots, c_n)$ s.t

$$B \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y(t_0) \\ y'(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{pmatrix} \qquad B\text{: the Wronskian mat. at } t_0.$$

Now consider $\phi(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$. $\begin{cases} L[\phi](t) = 0 \\ \phi(t_0) = y(t_0), \\ \phi'(t_0) = y'(t_0), \\ \vdots \\ \phi^{(n-1)}(t_0) = y^{(n-1)}(t_0) \end{cases}$

$\xrightarrow{\text{Thm 1}}$ $\phi(t) = y(t)$

$= c_1 y_1 + \cdots + c_n y_n$.

$\square$.

Remarks:

* (ii) $\Leftrightarrow$ (iii) Shows either $W[y_1, \cdots, y_n](t) = 0$ on I or $W(t) \neq 0\ \forall t \in I$.
  (Abel's Thm: see §4.1. #15).

* If $y_1, \cdots, y_n$ are not sol.s of $Lu = 0$, then (ii) $\Rightarrow$ (iii) may not be true.

* , then (iv) $\Rightarrow$ (iii), or even (iv) $\Rightarrow$ (ii) may not be true.
  (See §4.1. #18)

Theorem 3. The general sol. of $L[y] = g(t)$ can be written in the form $y = \phi(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t)$.

where $\{y_1, \cdots, y_n\}$ is a fundemental set of sol.s of $L[y] = 0$.

$Y(t)$ is any particular sol of $L[y] = g(t)$.

4.2 Homogeneous DEs with constant coefficients.

Consider $L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$ $\quad a_0, \cdots, a_n$ real consts. $a_0 \neq 0$.

Look for sol.s in the form: e^{rt}

$$L[e^{rt}] = e^{rt} \underbrace{[a_0 r^n + a_1 r^{n-1} + \cdots + a_n]}_{\text{Char. eq.}: \ \exists\, n \text{ roots } r_1, \cdots, r_n \in \mathbb{C}} = 0.$$

(1) Real and unequal roots:

$$y_g = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}.$$

$$W[e^{r_1 t}, \cdots, e^{r_n t}] = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} & \cdots & e^{r_n t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & \cdots & r_n e^{r_n t} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 t} & r_2^{n-1} e^{r_2 t} & \cdots & r_n^{n-1} e^{r_n t} \end{vmatrix}$$

$$= e^{(r_1 + r_2 + \cdots + r_n)t} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

$$= e^{(r_1 + \cdots + r_n)t} \prod_{1 \le i < j \le n} (r_i - r_j) \neq 0, \quad \forall t \in I.$$

$\Rightarrow e^{r_1 t}, \cdots, e^{r_n t}$ are linearly independent.

(2) Complex Roots.

If only r_{n-1} and r_n are complex conjugates $\lambda \pm i\mu$.

then $y_g = C_1 e^{r_1 t} + \cdots + C_{n-2} e^{r_{n-2} t} + C_{n-1} e^{\lambda t}\cos(\mu t) + C_n e^{\lambda t}\sin(\mu t)$.

. . . . . .

(3) Repeated Roots

If there is only one repeated r_1 of multiplicity s $(s \le n)$.

then $y_g = C_1 e^{r_1 t} + C_2 e^{r_1 t} \cdot \underset{\sim}{t} + \cdots + C_s e^{r_1 t} \cdot \underset{\sim}{t^{s-1}} + C_{s+1} e^{r_{s+1} t} + \cdots + C_n e^{r_n t}$.

4.3. The Method of undetermined coefficients

Eg. 1 Find the general sol. of

$$y''' - 3y'' + 3y' - y = 4e^t.$$

Sol: First solve $y''' - 3y'' + 3y' - y = 0$.

Char eq: $r^3 - 3r^2 + 3r - r = 0$. $(r-1)^3 = 0$.

$$r_1 = r_2 = r_3 = 1.$$

$$\Rightarrow y_h = C_1 e^t + C_2 e^t \cdot t + C_3 e^t \cdot t^2.$$

Trial sol: $y_p = A \cdot e^t \cdot t^3$. $\Rightarrow A = \frac{2}{3}$

$$\Rightarrow y_g = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

Eg. 2 Solve $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$.

Sol: $r^4 + 2y^2 + 1 = 0$. $(r^2 + 1)^2 = 0$. $r^2 = -1$ $r = \pm i$. $r_1 = r_3 = i$, $r_2 = r_4 = -i$.

$$\Rightarrow y_h = C_1 \cos t + C_2 \sin t + C_3 t\cos t + C_4 t \sin t.$$

$$y_p = At^2\cos t + Bt^2\sin t.$$

$$y_p' = A\,2t\cos t - At^2\sin t + B\,2t\sin t + Bt^2\cos t.$$

$$y_p'' = A\,2\cos t - A\,2t\sin t - A\,2t\sin t - At^2\cos t$$

$$+ B\,2\sin t + B\,2t\cos t + B\,2t\cos t - Bt^2\sin t$$

$$y_p''' = \cdots\cdots$$

$$y_p^{(4)} = \cdots\ \cdots \qquad \Rightarrow A = \frac{5}{8},\ B = -\frac{3}{8}.$$

$$\Rightarrow y_p = \frac{5}{8}t^2\cos t - \frac{3}{8}t^2\sin t.$$

4.4 The Method of Variation of Parameters.

Consider $L[y](t) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$.

Suppose $y_h = c_1y_1(t) + \cdots + c_ny_n(t)$. $c_1, \cdots, c_n$: any consts. general sol. of $L[y] \equiv 0$

Look for a $y_p(t)$ of $L[y] = g(t)$ in the form

$$y_p = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t). = \sum_{i=1}^{n} u_i(t)y_i(t).$$

Then
$$y_p'(t) = \underbrace{\sum_{i=1}^{n} u_i'(t)y_i(t)}_{=0\ (\text{Set})} + \sum_{i=1}^{n} u_i(t)y_i'(t)$$

$$y_p''(t) = \underbrace{\sum_{i=1}^{n} u_i'(t)\,y_i'(t)}_{=0\ (\text{Set})} + \sum_{i=1}^{n} u_i(t)y_i''(t)$$

$$\vdots$$

$$y_p^{(n-1)}(t) = \underbrace{\sum_{i=1}^{n} u_i'(t)\,y_i^{(n-2)}(t)}_{=0\ (\text{Set})} + \sum_{i=1}^{n} u_i(t)\,y_i^{(n-1)}(t).$$

$$y_p^{(n)}(t) = \sum_{i=1}^{n} u_i'(t) y^{(n-1)}(t) + \sum_{i=1}^{n} u_i(t) y_i^{(n)}(t).$$

Since $y_p^{(n)}(t) + \sum_{i=1}^{n} p_i y_p^{(n-i)} = g(t)$,

$$\sum_{i=1}^{n} u_i'(t) y^{(n-1)}(t) + \sum_{i=1}^{n} u_i(t) y_i^{(n)}(t) + \underline{\sum_{i=1}^{n} p_i y_p^{(n-i)}} = g(t)$$

$$= \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{n} u_j y_j^{(n-i)} \right)$$

$$= \sum_{j=1}^{n} u_j \sum_{i=1}^{n} p_i y_j^{(n-i)}.$$

$$\sum_{i=1}^{n} u_i' y_i^{(n-1)} + \sum_{j=1}^{n} u_j \underbrace{\left(y_j^{(n)} + \sum_{i=1}^{n} p_i y_j^{(n-i)} \right)}_{=0} = g(t)$$

Thus,
$$\begin{cases} \sum_{i=1}^{n} u_i'(t) y_i(t) = 0 \\ \sum_{i=1}^{n} u_i'(t) y_i'(t) = 0 \\ \vdots \\ \sum_{i=1}^{n} u_i'(t) y_i^{(n-2)}(t) = 0 \\ \sum_{i=1}^{n} u_i'(t) y_i^{(n-1)}(t) = g(t) \end{cases}$$

i.e. $B \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_n'(t) \end{pmatrix} = g(t) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

$|B| = W[y_1, \cdots, y_n](t) = W(t)$.

By Cramer's rule. we have

$$u_m'(t) = \frac{g(t) W_m(t)}{W(t)}, \quad m = 1, \cdots, n.$$

$W_m(t) = \det(B_m)$. B_m: replacing the m-th column by $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ in B.

with this notation: $y_p = \sum_{m=1}^{n} y_m(t) u_m(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} ds$.

$\boxed{5.}$ Systems of First-Order Linear Equations

$\boxed{5.1 \text{ Vector-valued and Matrix-valued Functions}}$

* Vector-valued func: $t \in (\alpha, \beta) \longmapsto \vec{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = (x_1(t), \cdots, x_n(t))^T.$

* Matrix-valued func: $t \in (\alpha, \beta) \longmapsto A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} = (a_{ij}(t))_{n \times n}$

$A(t)$ is continuous at $t_0 \in (\alpha, \beta)$ or on (α, β)

if every $a_{ij}(t)$ is continuous.

$A(t)$ is differentiable if every $a_{ij}(t)$ is differentiable and

$$\frac{dA(t)}{dt} = A'(t) \triangleq (a_{ij}'(t))_{n \times n} \quad \text{and} \quad \int_a^b A(t)\, dt = \left(\int_a^b a_{ij}(t)\, dt \right)_{n \times n}.$$

Suppose $A(t), B(t)$: $n \times n$ matrix-valued differentiable func.

$\vec{X}(t)$: n-dim. column vector-valued diff. func.

C : a const. mat. of size $n \times n$.

Then $[C A(t)]' = C \cdot A'(t),$

$[A(t) + B(t)]' = A'(t) + B'(t),$

$[A(t) B(t)]' = A'(t) B(t) + A(t) B'(t).$

$[A(t) \cdot \vec{X}(t)]' = A'(t) \vec{X}(t) + A(t) \cdot \vec{X}(t).$

5.2. The E&U Thm for a system of First-Order ODEs

Consider
$$\begin{cases} x_1'(t) = f_1(t, x_1, x_2, \cdots, x_n) \\ x_2'(t) = f_2(t, x_1, x_2, \cdots, x_n) \\ \quad \vdots \qquad\qquad \vdots \\ x_n'(t) = f_n(t, x_1, x_2, \cdots, x_n) \\ x_1(t) = x_1^0, \quad x_2(t) = x_2^0, \quad \cdots, \quad x_n(t_0) = x_n^0. \end{cases}$$

Let $\vec{X}(t) = (x_1(t), \cdots, x_n(t))^T$

$\vec{f}(t) = (f_1, \cdots, f_n)^T.$

$\vec{X}(t_0) = \vec{X}_0$

$$\Rightarrow \begin{cases} \vec{X}'(t) = \vec{f}(t, \vec{X}(t)) \\ \vec{X}(t_0) = \vec{X}_0 \end{cases} \quad (1)$$

Theorem 1

(i) E&U: For the Cauchy problem (1), if $\vec{f}(t,\vec{X})$ is continuous in $R: |t-t_0| \le a, \|\vec{X}-\vec{X_0}\| \le b$, and is Lipschitz continuous in $\vec{X}$: $\exists L>0$ s.t. $\forall (t,\vec{X_1}),(t,\vec{X_2}) \in R$, it holds:

$\|\vec{f}(t,\vec{X_1}) - \vec{f}(t,\vec{X_2})\| \le L \|\vec{X_1}-\vec{X_2}\|$.

then (1) $\exists!$ sol $\vec{X}=\vec{\phi}(t)$ defined on $[t_0-h, t_0+h]$.

where $h=\min\left\{a, \frac{b}{M}\right\}$, $M=\max\limits_{(t,\vec{X})\in R}\|\vec{f}(t,\vec{X})\|$.

(ii) Extension Thm: Let $\vec{f}(t,\vec{X})$ be continuous in an open subset $\Omega \subseteq \mathbb{R}^{n+1}$, $(t_0,\vec{X_0}) \in \Omega$, and $\vec{f}$ is Lipschitz continuous in $\vec{X}$ on many closed rectangle in Ω.

Then any sol. curve of (1) in Ω can be uniquely extended in both directions until it reaches $\partial\Omega$.

("∞" is included if Ω is unbouned).

(1) is called linear if each of $f_1, \cdots, f_n$ is a linear func. of $x_1, \cdots, x_n$.

namely, $\vec{X}'(t) = A(t)\vec{X}(t) + \vec{b}(t)$, $A(t) = (a_{ij}(t))_{n\times n}$.

If $\vec{b}(t) \equiv \vec{0}$, then (1) is called homogeneous.

Theorem 2 If $A(t)$ and $\vec{b}(t)$ are continuous on $I=(\alpha,\beta)$, then

$$\begin{cases} \vec{X}'(t) = A(t)\vec{X} + \vec{b}(t) \\ \vec{X}(t_0) = \vec{X_0}, \quad t_0 \in I \end{cases}$$

$\exists!$ sol. $\vec{X}=\vec{\phi}(t)$, which exists throughout I.

Pf: Observe that:

$*$ $\vec{f}(t,\vec{X}) = A(t)\vec{X} + \vec{b}(t)$ is continuous on $\Omega = I \times \mathbb{R}^n$.

$*$ $\vec{f}(t,\vec{X})$ is Lipschiz continuous in $\vec{X}$ on any closed rectangle

$R = [\alpha_1, \beta_1] \times [\vec{l_1}, \vec{l_2}]$, $[\alpha_1, \beta_1] \subset (\alpha, \beta)$.

$\forall (t, \vec{X_1}), (t, \vec{X_2}) \in R$: $\|\vec{f}(t,\vec{X_1}) - \vec{f}(t,\vec{X_2})\| = \|A(t)(\vec{X_1} - \vec{X_2})\|$

$$\le \|A(t)\| \cdot \|\vec{X_1} - \vec{X_2}\| \le L \cdot \|\vec{X_1} - \vec{X_2}\|. \quad L = \max_{t \in (\alpha,\beta)} \|A(t)\|.$$

Thm 1 $\Rightarrow \exists !$ sol. $\vec{X} = \vec{\phi}(t)$ of (IVP),

which can be extended in both directions until it reaches $\partial\Omega$.

Argue by contradiction & suppose $I^+_{max} = [t_0, \beta')$, $\beta' < \beta$.

$\Rightarrow \|\vec{\phi}(t)\| \to \infty$ as $t \to \beta'$. $(*1)$

We have $\forall t \in [t_0, \beta')$, $\vec{\phi}(t) = \vec{X_0} + \int_{t_0}^{t} \left[A(s)\vec{\phi}(s) + \vec{b}(s)\right] ds$.

$$\Rightarrow \|\vec{\phi}(t)\| \le \|\vec{X}_0\| + \int_{t_0}^{t} \|A(s)\vec{\phi}(s) + \vec{b}(s)\| \, ds.$$

$$\le \|\vec{X}_0\| + \int_{t_0}^{t} \left[\|A(s)\vec{\phi}(s)\| + \|\vec{b}(s)\|\right] ds. \qquad B = \max_{t \in [t_0, \beta']} \|\vec{b}(t)\|,$$

$$\le \|\vec{X}_0\| + L \int_{t_0}^{t} \|\vec{\phi}(s)\| \, ds + B(t - t_0) \qquad L = \max_{t \in [t_0, \beta']} \|A(t)\|.$$

$$\le \|\vec{X_0}\| + L Z(t) + B(\beta' - t_0). \ (*2) \qquad Z(t) = L \int_{t_0}^{t} \|\vec{\phi}(s)\| ds.$$

$\Rightarrow Z'(t) \le M + L Z(t)$. $\qquad M = \|\vec{X_0}\| + B(\beta' - t_0)$.

$$\Rightarrow (Ze^{-Lt})' \le M e^{-Lt} \xrightarrow{\int_{t_0}^{t}} Z(t) \le M e^{Lt} \int_{t_0}^{t} e^{-Ls} ds + \underset{0}{\underset{\parallel}{Z(t_0)}} e^{-Lt_0} e^{Lt}$$

$\Rightarrow Z(t)$ is uniformly bdd in $[t_0, \beta')$.

$\overset{(*2)}{\longrightarrow}$ $\|\vec{\phi}(t)\|$ is —||— $[t_0, \beta')$. —✗— $(*1)$. $\square$.

Rk. This pf. shows the linear control Thm works for 1st-order systems.

Assume $\vec{f}(t, \vec{X})$ is continuous on $\Omega = \{(t, \vec{X}) \mid \alpha < t < \beta,\ \vec{X} \in \mathbb{R}^n\}$ and

$\exists\ a(t), b(t) : (\alpha, \beta) \longrightarrow [0, \infty)$ continuous, s.t $\|\vec{f}(t, \vec{X})\| \le a(t)\|\vec{X}\| + \vec{b}(t)$ in Ω.

Then $\forall\ (t_0, \vec{X}_0) \in \Omega$, every sol. of (IVP) $\begin{cases} \vec{X}' = \vec{f}(t, \vec{X}) \\ \vec{X}(t_0) = \vec{X}_0 \end{cases}$ exists on (α, β).

5.3 Basic Theory of Systems of 1st-Order Linear Eq.s

Consider $\vec{X}'(t) = P(t)\vec{X}(t) + \vec{g}(t)$. $\quad P(t) = (P_{ij}(t))_{n\times n}$ $\quad$ (6)

homogeneous system: $\vec{X}'(t) = P(t)\vec{X}(t)$. $\quad$ (7)

Theorem 3. (Principle of Superposition).

$\vec{X}^{(1)}, \vec{X}^{(2)}$ are sol.s of (7) $\Rightarrow$ $c_1\vec{X}^{(1)} + c_2\vec{X}^{(2)}$ is also a sol.

Corollary 1. If $\vec{X}^{(1)}, \dots, \vec{X}^{(k)}$ are sol.s of (7)

$\Rightarrow$ So is $\vec{X} = c_1\vec{X}^{(1)} + \dots + c_k\vec{X}^{(k)}$.

Theorem 4. Let $\vec{X}^{(1)}, \dots, \vec{X}^{(n)}$ be n sol.s of $\underset{n\times 1}{\vec{X}'(t)} = \underset{n\times n}{P(t)}\underset{n\times 1}{\vec{X}(t)}$, where $P(t) \in C(\alpha, \beta)$.

TFAE:

(i) $\vec{X}^{(1)}, \dots, \vec{X}^{(n)}$ are linearly independent. $\vec{X}^{(i)} = (x_{1i}(t), x_{2i}(t), \dots, x_{ni}(t))^T$

(ii) $\forall t_0 \in (\alpha, \beta)$, the Wronskin $W[\vec{X}^{(1)}, \dots, \vec{X}^{(n)}](t_0) \neq 0$ $\quad \begin{vmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{vmatrix}$.

(iii) Each sol. $\vec{X} = \vec{\phi}(t)$ can be expressed as a linear combination of $\vec{X}^{(1)}, \dots, \vec{X}^{(n)}$.

that is: $\vec{\phi}(t) = c_1\vec{X}^{(1)}(t) + c_2\vec{X}^{(2)}(t) + \dots + c_n\vec{X}^{(n)}(t)$. $\forall\ t \in (\alpha, \beta)$ in exactly one way.

Pf: (i)$\Rightarrow$(ii): Suppose rather $\exists\, t_0 \in I = (\alpha,\beta)$, s.t. $W(t_0)=0$.

Then $\exists\, C_1,\cdots,C_n$: not all zero, s.t $C_1\vec{X}^{(1)}(t_0) + \cdots + C_n\vec{X}^{(n)}(t_0) = \vec{0}$.

$\Rightarrow \vec{\phi}(t) = \sum_{i=1}^{n} C_i\vec{X}^{(i)}(t)$ solves $\begin{cases}\vec{X}'(t) = P(t)\vec{X}(t) \\ \vec{X}(t_0) = \vec{0}\end{cases} \xRightarrow{E\&U} \vec{\phi}(t) = 0,\ \forall t\in I.$

$\Rightarrow \vec{X}^{(1)},\cdots,\vec{X}^{(n)}$ are linealy depend. $\times$

(ii)$\Rightarrow$(iii): Let $\vec{\phi}(t)$ be an arbitrary sol. of $\vec{X}' = P(t)\vec{X}$.

Since $W[\vec{X}^{(1)},\cdots,\vec{X}^{(n)}](t_0) \neq 0$. $\exists!\ C_1,\cdots,C_n$ s.t $\sum_{i=1}^{n} C_i\vec{X}^{(i)}(t_0) = \vec{\phi}(t_0)$.

Now, both $\vec{\phi}(t)$ and $\sum_{i=1}^{n} C_i\vec{X}^{(i)}(t)$ are sol.s of $\begin{cases}\vec{X}'(t) = P(t)\vec{X}(t) \\ \vec{X}(t_0) = \vec{\phi}(t_0)\end{cases}$

By E&U, $\vec{\phi}(t) = \sum_{i=1}^{n} C_i\vec{X}^{(i)}(t)$.

(iii)$\Rightarrow$(i): Suppose $\underbrace{C_1\vec{X}^{(1)}(t) + \cdots + C_n\vec{X}^{(n)}(t)}_{:=\vec{\phi}(t)} = 0,\ \forall t\in I.$

Then $\vec{\phi}(t)$ is a sol. of $\vec{X}' = P(t)\vec{X}$. Also, $\vec{\phi}(t) = 0\cdot\vec{X}^{(1)}(t) + \cdots + 0\cdot\vec{X}^{(n)}(t)$.

Since the expression of $\vec{\phi}(t)$ is unique, $C_1 = C_2 = \cdots = C_n = 0$.

$\Rightarrow \vec{X}^{(1)}(t),\cdots,\vec{X}^{(n)}(t)$ are linearly indep. $\square$.

Theorem 5. (Abel's Thm)

Assume the same conditions as in Thm 4. Then $W(t) = W[\vec{X}^{(1)},\cdots,\vec{X}^{(n)}](t)$ is either $\equiv 0$ or never $=0$ on I.

Pf: In fact, $\frac{dW}{dt} = [P_{11}(t) + P_{22}(t) + \cdots + P_{nn}(t)]\,W$. (§7.4 #8)

which implies the formula :

$$W[\vec{X}^{(1)},\cdots,\vec{X}^{(n)}](t) = C\cdot \exp\Big\{\int \overbrace{(P_{11}(t) + \cdots + P_{nn}(t))}^{tr[P(t)]}\, dt\Big\},\ C\ \text{any const.}\ \square.$$

Theorem 6. Let $\vec{e}^{(i)} = (0,\cdots,0,\underset{\substack{\uparrow \\ i\text{-th}}}{1},0,\cdots,0)^T$. Let $\vec{X}^{(i)}(t)$ be the sol. of

$\begin{cases}\vec{X}'(t) = P(t)\vec{X}(t) \\ \vec{X}^{(i)}(t_0) = \vec{e}^{(i)},\ t_0\in I.\end{cases}$ where $P(t)\in C(I)$.

Then $\{\vec{X}^{(i)}(t)\}$ form a fundemental set of sol.s in $\vec{X}' = P(t)\vec{X}$.

Pf: $W(t_0) = \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{vmatrix} = 1 \neq 0 \xrightarrow{\text{Thm 4, Thm 5}} \{\vec{X}^{(i)}(t)\}$ form a fundemental set of sol.s $\square$

Theorem 7. Consider the system $\vec{X}'(t) = P(t)\vec{X}(t)$. $P(t)$: real mat.-valued continu. func.

If $\vec{X} = \vec{U}(t) + i\vec{V}(t)$ is a complex-valued sol., then both $\vec{u}(t)$ and $\vec{v}(t)$ are sol.s of this system.

Pf: $[\vec{U}(t) + i\vec{V}(t)]' = P(t)[\vec{u}(t) + i\vec{v}(t)]$

$\Rightarrow \vec{U}'(t) + i\vec{V}'(t) = P(t)\vec{u}(t) + iP(t)\vec{v}(t)$

$\Rightarrow \vec{u}'(t) = P(t)\vec{u}(t)$, $\vec{v}'(t) = P(t)\vec{v}(t)$. $\square$.

5.4 Homogeneous Linear Systems with Constant coefficients.

Consider the system $\vec{X}'(t) = A\vec{X}(t)$. (* 1).

A is a constant $n \times n$ mat.

Eg. 1 $\vec{X}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \vec{X}$

Sol: $\vec{X}(t) = (X_1(t), X_2(t))^T$

$$\begin{cases} X_1' = 2X_1 \\ X_2' = -3X_2 \end{cases} \Rightarrow \begin{cases} X_1(t) = C_1 e^{2t} \\ X_2(t) = C_2 e^{-3t} \end{cases}$$

$$\Rightarrow \vec{X}(t) = \begin{pmatrix} C_1 e^{2t} \\ C_2 e^{-3t} \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In general, $\vec{X}'(t) = A\vec{X}(t)$, if $(r, \vec{v})$ is an eigen-pair of A. i.e. $A\vec{v} = r\vec{v}$, $\vec{v} \neq \vec{0}$.

$(A - rI)\vec{v} = \vec{0}$

$\det(A - rI) = 0$.

then $\vec{X}(t) = e^{rt}\vec{v}$ is a sol.: $\vec{X}'(t) = (e^{rt} \cdot \vec{v})' = re^{rt}\vec{v}$

$A\vec{X}(t) = A(e^{rt} \cdot \vec{v}) = e^{rt} A v = e^{rt} \cdot r\vec{v}$

Now, if all the eigenvalues of A are real, and it has n-linearly independent eigenvectors $\vec{v}^{(1)}, \dots, \vec{v}^{(n)}$, then $\{\vec{X}^{(i)}(t) = e^{r_i t} \vec{v}^{(i)}\}_{i=1}^{n}$ forms a fundemental set of sol.s.

$$W[\vec{X}^{(1)}(t), \cdots, \vec{X}^{(n)}(t)] = e^{(r_1+\cdots+r_n)t} \cdot \det\{\vec{v}^{(1)}, \cdots, \vec{v}^{(n)}\} \neq 0 \quad (\text{linearly indep.})$$

<u>E.g 2</u> $\vec{X}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{X}(t)$.

Sol: Solve the eigenvalue problem $(A - Ir)\vec{v} = \vec{0}$. i.e. $\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0 \Rightarrow r_1 = 3,\ r_2 = -1.$$

If $r_1 = 3$, $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = 2v_1 \Rightarrow \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

If $r_2 = -1$, $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = -2v_1 \Rightarrow \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

The corresponding sol.s are $\vec{X}^{(1)}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{X}^{(2)}(t) = C_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$W[\vec{X}^{(1)}(t), \vec{X}^{(2)}(t)] = e^{3t}e^{-t}\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -4e^{2t} \neq 0.$$

The general sol. is

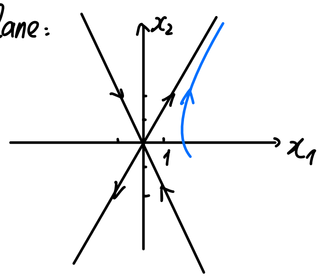
$$\vec{X}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}. \quad \vec{X}(0) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

<u>Rk:</u> An equilibrium sol. $\vec{X}$ satisfies $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{X} = \vec{0}$.

Since $\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -3$, $\vec{X} = \vec{0}$ is the only equilibrium sol.

$$\vec{X}^{(1)}(t) \xrightarrow{t\to\infty} \begin{pmatrix} \infty \\ \infty \end{pmatrix}, \quad \vec{X}^{(2)}(t) \xrightarrow{t\to\infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

phase plane:



5.5. Complex-valued Eigenvalues

Suppose $r_{1,2} = \lambda \pm i\mu$ is a pair of complex conjugate roots of $\det(A - rI) = 0$.

Let $\vec{v}=\vec{u}+i\vec{w}$ be an eigenvector of r_1. Then $\vec{X}(t)=e^{r_1 t}\vec{v}$ is a complex valued sol. of $\vec{X}'=A\vec{X}$.

$$\vec{X}(t)=e^{r_1 t}\vec{v}=e^{(\lambda+i\mu)t}(\vec{u}+i\vec{w})=e^{\lambda}\left[\cos(\mu t)+i\sin(\mu t)\right](\vec{u}+i\vec{w})$$

$$=\underbrace{e^{\lambda t}\left(\cos(\mu t)\vec{u}-\sin(\mu t)\vec{w}\right)}_{:=\vec{X}^{(1)}(t)}+\underbrace{e^{\lambda t}\left(\sin(\mu t)\vec{u}+\cos(\mu t)\vec{w}\right)}_{:=\vec{X}^{(2)}(t)}i$$

$$W[\vec{X}^{(1)},\vec{X}^{(2)}](t)=e^{2\lambda t}\left|\cos(\mu t)\vec{u}-\sin(\mu t)\vec{w},\ \sin(\mu t)\vec{u}+\cos(\mu t)\vec{w}\right|$$

$$=e^{2\lambda t}\underbrace{\left(\cos^2(\mu t)+\sin^2(\mu t)\right)}_{=1}|\vec{u},\vec{w}|\cdot\text{const}$$

Next, we W.T.S $\vec{u},\vec{w}$ are linearly indep. If rather $\vec{u}=c\vec{w}$, $\vec{v}=(1+ic)\vec{w}$.

$$A\vec{v}=(\lambda+i\mu)\vec{v}\Rightarrow A\vec{w}=(\lambda+i\mu)\vec{w}\Rightarrow \mu=0. \quad \times$$

<u>E.g. 1</u> $\vec{X}'=\begin{bmatrix}-\frac{1}{2} & 1\\ -1 & -\frac{1}{2}\end{bmatrix}\vec{X}=A\vec{X}$.

Sol: $|A-rI|=r^2+r+\frac{5}{4}=0\Rightarrow r_{1,2}=-\frac{1}{2}\pm i$.

solve $(A-r_1I)\vec{v}=\vec{0}$. i.e. $\begin{pmatrix}i & 1\\ -1 & i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\vec{v}=\begin{pmatrix}1\\ i\end{pmatrix}$

$$\vec{X}(t)=e^{r_1 t}\vec{v}=e^{(-\frac{1}{2}+i)t}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$=\underbrace{e^{-\frac{1}{2}t}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}}_{\vec{X}^{(1)}(t)}+\underbrace{e^{-\frac{1}{2}t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}}_{\vec{X}^{(2)}(t)}i.$$

Thus, $\vec{X}^{(1)}(t)$, $\vec{X}^{(2)}(t)$ form a fundemental set of sol.s.

§5.6 Fundamental Matrices

Consider $\vec{x}\,'(t) = P(t)\vec{x}$, $t\in(\alpha,\beta)$, $P(t)\in C(\alpha,\beta)$

If $\vec{x}^{(1)}(t),\dots,\vec{x}^{(n)}(t)$ form a fundamental set of sol's, then the matrix
$\boxed{\Psi(t) = \left(\vec{x}^{(1)}(t),\ \dots,\ \vec{x}^{(n)}(t)\right)}$ is called a fundament matrix.

· $\Psi(t)$ is nonsingular $\forall\, t\in(\alpha,\beta)$.

· The general sol is $\vec{x}_g(t) = c_1\vec{x}^{(1)}(t)+\dots+c_n\vec{x}^{(n)}(t) = \Psi(t)\vec{c}$, $\vec{c}=(c_1,\dots,c_n)^T$.

Then the particular sol with I.C.: $\vec{x}(t_0)=\vec{x}^0$, $\Psi(t_0)\vec{c}=\vec{x}^0 \Rightarrow \vec{c}=[\Psi(t_0)]^{-1}\vec{x}^0$

$\Rightarrow$ Sol to the (IVP) $\begin{cases}\vec{x}\,'=P(t)\vec{x}\\ \vec{x}(t_0)=\vec{x}^0\end{cases}$

$\vec{x}(t) = \Psi(t)\cdot\vec{c}$

$= \boxed{\Psi(t)[\Psi(t_0)]^{-1}}\vec{x}^0$, $\Psi(t)[\Psi(t_0)]^{-1} = \boxed{\Phi(t)}$

$\Phi(t)$: $\Phi(t_0)=I$, $\Phi(t)$ is called the standard fundamental matrix (S.F.M.)

E.g.1. Find the S.F.M. Φ s.t. $\Phi(0)=I$ ($t_0=0$) of $\vec{x}\,'=\begin{bmatrix}1&1\\4&1\end{bmatrix}\vec{x}$.

Sol: In E.g.2 of §5.4, we found $\vec{x}_g(t)=c_1\begin{bmatrix}e^{3t}\\2e^{3t}\end{bmatrix}+c_2\begin{bmatrix}e^{-t}\\-2e^{-t}\end{bmatrix}$

$\Rightarrow$ F.M. $\Psi(t)=\begin{bmatrix}e^{3t}&e^{-t}\\2e^{3t}&-2e^{-t}\end{bmatrix}$, $\Psi(0)=\begin{bmatrix}1&1\\2&-2\end{bmatrix}$, $[\Psi(0)]^{-1}=\frac{1}{-4}\begin{bmatrix}-2&-1\\-2&1\end{bmatrix}=\begin{bmatrix}\frac12&\frac14\\\frac12&-\frac14\end{bmatrix}$

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$\Rightarrow$ S.F.M. $\Psi(t)[\Psi(0)]^{-1}=\begin{bmatrix}\frac12e^{3t}+\frac12e^{-t}&\frac14e^{3t}-\frac14e^{-t}\\e^{3t}-e^{-t}&\frac12e^{3t}+\frac12e^{-t}\end{bmatrix}$

Recall: $\begin{cases}\frac{dx}{dt}=ax\\x(0)=x_0\end{cases}\Rightarrow x(t)=e^{at}x_0$

For $\begin{cases}\vec{x}\,'(t)=A_{n\times n}\vec{x}\\ \vec{x}(0)=\vec{x}^0\end{cases} \overset{?}{\Longrightarrow} \vec{x}(t)=e^{At}\vec{x}^0$ → S.F.M. s.t. $\Phi(0)=I$

If A has n linearly indep. e-vectors $\{\vec{v}^{(i)}\}_{i=1}^n$ $(\leftrightarrow\lambda_i)$, then $\vec{x}_g=c_1e^{\lambda_1 t}\vec{v}^{(1)}+\dots+c_ne^{\lambda_n t}\vec{v}^{(n)}$

If $\exists$ complex $\lambda=\alpha\pm i\beta$, take $\lambda=\alpha+i\beta \longleftrightarrow \vec{v}=\vec{u}+i\vec{w}$, then $Re(e^{\lambda t}\vec{v})$ & $Im(e^{\lambda t}\vec{v})$ are two linearly indep. realvalue sol's.

Def (Matrix exponential) $B_{n\times n}$ matrix

$$e^B := I + B + \frac{B^2}{2!}+\dots+\frac{B^n}{n!}+\dots \quad (*1)$$

norm: $B=(b_{ij})_{n\times n}$, $\|B\|_1=\sum_{i,j=1}^n|b_{ij}|$ or $\|B\|_2=\left(\sum_{i,j=1}^n b_{ij}^2\right)^{\frac12}$

Proposition: $\forall B$, the matrix series $(*1)$ converges under standard matrix norm.

Pf: $S_n=I+B+\frac{B^2}{2!}+\dots+\frac{B^n}{n!}$. We use Cauchy's criterion: $\forall \varepsilon>0$, $\exists N>0$ s.t. $\forall m,n$, $m>n>N$ $\Longrightarrow \|S_n-S_m\|<\varepsilon$

Observe: $\|S_m - S_n\| = \left\| \sum_{k=n+1}^{m} \frac{B^k}{k!} \right\| \le \sum_{k=n+1}^{\infty} \frac{\|B\|^k}{k!} \le \sum_{k=N+1}^{\infty} \frac{\|B\|^k}{k!} < \varepsilon$ if N is suff. large, because $\sum_{k=0}^{\infty} \frac{\|B\|^k}{k!}$ converges.

Then $e^{tA} = e^{At} \overset{(*1)}{=} I + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots$

$$e^{tA}\vec{c} = \vec{c} + tA\vec{c} + \frac{t^2A^2}{2!}\vec{c} + \cdots + \frac{t^nA^n}{n!}\vec{c} + \cdots$$

$$\frac{d}{dt}\left(e^{tA}\vec{c}\right) = A\vec{c} + tA^2\vec{c} + \cdots + \frac{t^{n-1}A^n\vec{c}}{(n-1)!} + \cdots = A\left(\vec{c} + tA\vec{c} + \cdots + \frac{t^{n-1}A^{n-1}\vec{c}}{(n-1)!} + \cdots\right)$$

$$= Ae^{tA}\vec{c}.$$

Pain: $e^{tA}\vec{c}$, in general, is an infinite sum, which is hard to compute.

Case 1. A is diagonal. $A = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $A^k = \mathrm{diag}(\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k)$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \\ & & & \lambda_n^k \end{bmatrix} = \begin{bmatrix} \sum \frac{t^k}{k!}\lambda_1^k & & \\ & \ddots & \\ & & \sum \frac{t^k}{k!}\lambda_n^k \end{bmatrix} = \boxed{\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}}$$

$$\vec{X}(t) = e^{At}\vec{x}_0 = \sum_{i=1}^{n} x_i^0 e^{\lambda_i t}\vec{e}_i$$

Case 2. A is diagonalizable

Thus, A has n linearly independent e-vectors: $\vec{v}^{(1)}, \vec{v}^{(2)}, \cdots, \vec{v}^{(n)}$, with e-values $\lambda_1, \cdots, \lambda_n$

Let $T = (\vec{v}^{(1)}, \cdots, \vec{v}^{(n)})$, T^{-1} exists.

Calculate: $AT = A(\vec{v}^{(1)}, \cdots, \vec{v}^{(n)}) = (A\vec{v}^{(1)}, \cdots, A\vec{v}^{(n)}) = (\lambda_1\vec{v}^{(1)}, \cdots, \lambda_n\vec{v}^{(n)}) = \underbrace{(\vec{v}^{(1)}, \cdots, \vec{v}^{(n)})}_{T} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$\Rightarrow T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad A = T\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}T^{-1}$$

$$A^k = T\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\underbrace{T^{-1}T}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}T^{-1} \cdots T\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}T^{-1}$$

$$= T\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}T^{-1}$$

$$\Rightarrow e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} T\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}T^{-1} = T\begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k t^k}{k!} & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty} \frac{\lambda_n^k t^k}{k!} \end{pmatrix}T^{-1} = \boxed{T\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}T^{-1}}$$

Case 3. A is not diagonalizable

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k} \qquad m_1 + \cdots + m_k = n$$

m_i: algebraic multiplicity of λ_i

k_i: number of linearly indep. e-vectors of λ_i, $k_i \le m_i$, called geometric multiplicity of λ_i

$V_{\lambda_i} = \mathrm{span}\{\vec{v} : (\lambda_i I - A)\vec{v} = \vec{0}\}$, eigenspace, $\dim V_{\lambda_i} = k_i$

$V_{\lambda_i}^{g} = \mathrm{span}\{\vec{v} : (\lambda_i I - A)^{m_i}\vec{v} = \vec{0}\}$, $\dim V_{\lambda_i}^{g} = m_i$

Suppose λ is an e-value of A with algebraic mult. m, let $\vec{v} \in V_\lambda^g$

Then $(\lambda I - A)^m \vec{v} = \vec{0}$.

Now compute $e^{tA}\vec{v} = e^{t\lambda I + t(A - \lambda I)}\vec{v} = e^{\lambda t} \cdot e^{t(A-\lambda I)}\vec{v}$

$$= e^{\lambda t}\left[\vec{v} + t(A - \lambda I)\vec{v} + \frac{t^2}{2!}(A - \lambda I)^2\vec{v} + \cdots + \frac{t^{m-1}}{(m-1)!}(A - \lambda I)^{m-1}\vec{v}\right]$$

To solve $\vec{x}' = A_{n\times n}\vec{x}$, we can find n linearly indep. generalized e-vectors $\vec{v}^{(i)}$, $i = 1, \cdots, n$

Then compute $e^{tA}\vec{v}^{(i)} \triangleq \vec{X}^{(i)}$. Then $\vec{X}_g(t) = C_1\vec{X}^{(1)} + \cdots + C_n\vec{X}^{(n)}$

Special case: $n=2$ $\quad A_{2\times 2}$

Suppose λ is a repeated e-value of A with $\dim V_\lambda = 1$

$\Rightarrow (A-\lambda I)^2 \vec{v} = \vec{0} \quad \forall \vec{v} \in \mathbb{R}^2$

If $(A-\lambda I)\vec{v} \neq \vec{0}$, then $\underbrace{(A-\lambda I)\vec{v}}_{:=\vec{v}^{(2)}} \in V_\lambda$

Let $\vec{v} := \vec{v}^{(1)}$. One sol of $\vec{x}' = A\vec{x}$ is $e^{tA}\vec{v}^{(1)} = e^{\lambda t} e^{t(A-\lambda I)}\vec{v}^{(1)} = e^{\lambda t}[\vec{v}^{(1)} + t(A-\lambda I)\vec{v}^{(1)}]$

$= e^{\lambda t}[\vec{v}^{(1)} + t\vec{v}^{(2)}] := \vec{x}^{(1)}(t)$

Another sol is $\vec{x}^{(2)}(t) = e^{\lambda t}\cdot \vec{v}^{(2)}$

$\Rightarrow \vec{x}_g(t) = c_1 e^{\lambda t}[\vec{v}^{(1)} + t\vec{v}^{(2)}] + c_2 e^{\lambda t}\vec{v}^{(2)}$

E.g. Solve $\vec{x}'(t) = A\vec{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\vec{x}$

Sol. $A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}$, $\det(A-\lambda I) = (\lambda-2)^2 \Rightarrow \lambda_1 = \lambda_2 = 2$

$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad m=3$, $k=1$

$(A-\lambda I)\vec{v} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 - v_2 \\ v_1 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow V_{\lambda=2} = \operatorname{Span}\{(1,-1)^T\}$ e-vec.

$\vec{v}^{(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Then $(A-2I)\vec{v}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \vec{x}^{(1)}(t) = e^{2t}[\vec{v}^{(1)} + t\vec{v}^{(2)}] = \begin{bmatrix} te^{2t} \\ -e^{2t} - te^{2t} \end{bmatrix}$

$\vec{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ generalized e-vec. $\quad \vec{x}^{(2)}(t) = e^{2t}\cdot\vec{v}^{(2)} = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$

$(A-\lambda I)\vec{v} \in V_{\lambda=2}$ $\quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad v_1 = 0,\ v_2 = -1 \Rightarrow \vec{v}_1^{(1)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\Rightarrow$ Two sols: e.v.

① $\vec{X}^{(2)}(t) = e^{2t}\cdot\vec{v}^{(1)} = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

② $\vec{X}^{(1)}(t) = e^{2t}\cdot(\vec{v}^{(1)} + t\vec{v}^{(2)})$

§5.8 Nonhomogeneous Linear Systems

Consider $(*)$ $\vec{x}'(t) = P(t)\vec{x}(t) + \vec{g}(t)$, $t \in I = (\alpha, \beta)$, $P_{n\times n}(t)$

$\vec{x}_g(t) = \vec{x}_h(t) + \vec{x}_p(t)$, $\quad \vec{x}_h(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) + \cdots + c_n\vec{x}^{(n)}(t)$ — general sol of $\vec{x}'(t) = P(t)\vec{x}(t)$

$\vec{x}_p(t)$ any particular sol of $(*)$

A few method for determining $\vec{x}_p(t)$

Method 1: Diagonalization.

E.g.1. $\vec{x}'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = A\vec{x} + \vec{g}(t) \quad (*2)$

Sol: First solve the homogeneous system: $\vec{x}'(t) = A\vec{x}$.

$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3$, $\lambda_1 = -3$, $\lambda_2 = -1$

$\lambda_1 = -3$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = -1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{X}^{(1)}(t) = e^{rt}\cdot\vec{v}^{(1)}$

$\vec{X}^{(2)}(t) = te^{rt}\cdot\vec{v}^{(1)} + e^{rt}\cdot\vec{v}^{(2)}$

$\vec{X}^{(3)}(t) = \frac{t^2}{2}e^{rt}\cdot\vec{v}^{(1)} + te^{rt}\vec{v}^{(2)} + e^{rt}\cdot\vec{v}^{(3)}$

$$\begin{cases} (A-rI)\vec{v}^{(1)} = 0 \\ (A-rI)\vec{v}^{(2)} = \vec{v}^{(1)} \\ (A-rI)\vec{v}^{(3)} = \vec{v}^{(2)} \end{cases}$$

Thus, $\vec{x}_h = c_1 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

Next, we determine an $\vec{x}_p(t)$.

Let $T = (\vec{v}^{(1)}, \vec{v}^{(2)}) = \begin{bmatrix}1&1\\-1&1\end{bmatrix}$. Then $T^{-1} = \begin{bmatrix}\frac12 & -\frac12\\ \frac12 & \frac12\end{bmatrix}$

& $T^{-1}AT = \begin{bmatrix}-3&0\\0&-1\end{bmatrix}$

Let $\vec{x} = T\vec{y}$, i.e. $\vec{y} = T^{-1}\vec{x}$.

Then $T\vec{y}'(t) = AT\vec{y} + \vec{g}(t)$

$$\Rightarrow \vec{y}'(t) = T^{-1}AT\vec{y} + T^{-1}\vec{g}(t) = \begin{bmatrix}-3&0\\0&-1\end{bmatrix}\vec{y} + \begin{bmatrix} e^{-t}-\frac32 t\\ e^{-t}+\frac32 t\end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1'(t) = -3y_1(t) + e^{-t} - \frac32 t \\ y_2'(t) = -y_2(t) + e^{-t} + \frac32 t\end{cases}$$

$$\Rightarrow \begin{cases} y_1 = \frac12 e^{-t} - \frac12\left(t-\frac13\right) + k_1 e^{-3t} \\ y_2 = t e^{-t} + \frac32(t-1) + k_2 e^{-t}\end{cases} \quad (\text{Take } k_1=k_2=0)$$

$$\Rightarrow \vec{x}_p(t) = T\vec{y} = \begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}\frac12 e^{-t}+te^{-t}+t-\frac43\\ -\frac12 e^{-t}+te^{-t}+2t-\frac53\end{bmatrix}$$

$$\Rightarrow \vec{x}_g(t) = \vec{x}_h + \vec{x}_p(t).$$

Rk: When A is not diagonalizable, one can obtain a Jordan form $J = T^{-1}AT$.

The eq. of $\vec{y} = T^{-1}\vec{x}$ is not decoupled. However, it can still be solved consecutively, starting with y_n. e.g. $\begin{bmatrix}2&1\\0&2\end{bmatrix}$, $\begin{bmatrix}2&1&&&\\0&2&&&\\&&3&1&0\\&&0&3&1\\&&0&0&3\end{bmatrix}$

Method 2 Undetermined coefficients.

For (*2): $\vec{g}(t) = e^{-t}\begin{bmatrix}2\\0\end{bmatrix} + t\begin{bmatrix}0\\3\end{bmatrix}$

Trial sol: $\vec{x}_p(t) = \vec{c}t + \vec{d} + \vec{b}e^{-t} + \vec{a}te^{-t}$

$\vec{x}_p'(t) = -\vec{a}te^{-t} + (\vec{a}-\vec{b})e^{-t} + \vec{c}$

$A\vec{x}_p + \vec{g} = (A\vec{a})te^{-t} + \left(A\vec{b} + \begin{bmatrix}2\\0\end{bmatrix}\right)e^{-t} + \left(A\vec{c} + \begin{bmatrix}0\\3\end{bmatrix}\right)t + A\vec{d}$

$$\Rightarrow \begin{cases} A\vec{a} = -\vec{a}\\ A\vec{b} = \vec{a}-\vec{b}-\begin{bmatrix}2\\0\end{bmatrix} \\ A\vec{c} = -\begin{bmatrix}0\\3\end{bmatrix} \\ A\vec{d} = \vec{c}\end{cases} \Rightarrow \vec{a} = \begin{bmatrix}1\\1\end{bmatrix},\ \vec{b} = k\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix},\ \vec{c} = \begin{bmatrix}1\\2\end{bmatrix},\ \vec{d} = \begin{bmatrix}-\frac43\\-\frac53\end{bmatrix}$$

(take $k=0$)

$$\Rightarrow \vec{x}_p(t) = \begin{bmatrix}1\\1\end{bmatrix}te^{-t} - \begin{bmatrix}0\\1\end{bmatrix}e^{-t} + \begin{bmatrix}1\\2\end{bmatrix}t - \frac13\begin{bmatrix}4\\5\end{bmatrix}$$

Method 3. Variation of Parameters

For (*2): $\vec{x}_h = c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-t} = \Psi(t)\cdot\vec{c}$, $\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{bmatrix}$ fundamental matrix

$\Rightarrow \vec{x}_h' = A\vec{x}_h$. i.e. $\Psi'(t)\cdot\vec{c} = A\Psi(t)\cdot\vec{c},\ \forall \vec{c}$.

Look for $\vec{x}_p$ of the form $\vec{x}_p = \Psi(t)\vec{u}(t)$, $\vec{u}(t) = \begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix}$

$$\vec{x}_p' = \underbrace{\Psi'(t)\cdot\vec{u}(t)}_{=A\Psi(t)\cdot\vec{u}(t)} + \Psi(t)\vec{u}'(t) = A\vec{x}_p + \vec{g}(t) \Rightarrow \Psi(t)\cdot\vec{u}'(t) = \vec{g}(t), \text{ i.e., } \begin{bmatrix}e^{-3t}&e^{-t}\\-e^{-3t}&e^{-t}\end{bmatrix}\begin{bmatrix}u_1'(t)\\u_2'(t)\end{bmatrix} = \begin{bmatrix}2e^{-t}\\3t\end{bmatrix}$$

$$\Rightarrow \begin{cases} u_1'(t) = e^{2t} - \frac32 te^{3t}\\ u_2'(t) = 1 + \frac32 te^{t}\end{cases} \Rightarrow \begin{cases} u_1(t) = \frac12 e^{2t} - \frac12 te^{3t} + \frac16 e^{3t} + k_1 \\ u_2(t) = t + \frac32 te^t - \frac32 e^t + k_2\end{cases}, \text{ take } k_1=k_2=0$$

$$\Rightarrow \vec{x}_p(t) = \Psi(t)\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix} = \begin{bmatrix}\frac12 e^{-t} + te^{-t} + t - \frac43\\ -\frac12 e^{-t} + te^{-t} + 2t - \frac53\end{bmatrix}. \ \square$$

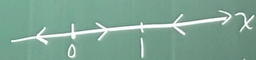
§6. Nonlinear 1st-order Systems and Stability

§6.1 Phase Portrait of <u>Linear</u> planar Systems.

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = A_{2\times 2}\begin{pmatrix}x_1\\x_2\end{pmatrix} \qquad (1)$$

Goal: geometric understanding of behavior of all Sol's as $t\to\infty$.

Recall: $\frac{dx}{dt} = x(1-x)$. phase-line analysis



Equilibrium pts of (1): Solve $A\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, if $|A|\neq 0$, then $\begin{pmatrix}0\\0\end{pmatrix}$ is the unique eqli

if $|A| = 0$, (Ex)

Let λ_1 & λ_2 be e-values.

Case 1 (a) $\lambda_1 < \lambda_2 < 0$

$$\vec{x}_g(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$$

Observe: · As $t\to\infty$: $\vec{x}_g(t) \to \vec{0}$

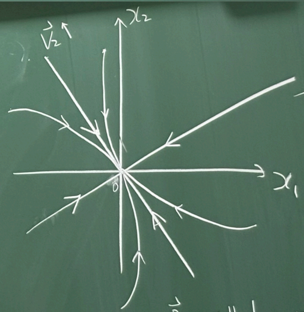
· Take $C_1 = 0$: $\vec{x}_2(t) = C_2 e^{\lambda_2 t}\cdot\vec{v}_2$, eigenline of λ_2

· Take $C_2 = 0$: $\vec{x}_1(t) = C_1 e^{\lambda_1 t}\vec{v}_1$

· If $C_2 \neq 0$: $\vec{x}_g(t) = e^{\lambda_2 t}\left(C_1\vec{v}_1 e^{\overbrace{(\lambda_1-\lambda_2)}^{<0}t} + C_2\vec{v}_2\right)$

$\sim \vec{x}_2(t)$ as $t\to\infty$

Thus, all orbits except $\vec{x}_1(t)$, are tangent to $\vec{v}_2$ at the origin $\vec{0}$.



$\boxed{\text{Asymptotically Stable}}$

Fig. 1

$\vec{0}$ is called a node or a nodal sink

结点(汇)

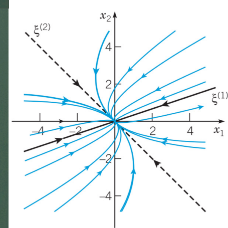


FIGURE 9.1.1 Trajectories in the phase plane when the origin is a node with $r_1 < r_2 < 0$. The solid black and dashed black curves show the fundamental solutions $\xi^{(1)}e^{r_1 t}$ and $\xi^{(2)}e^{r_2 t}$, respectively.

(b) $\lambda_1 > \lambda_2 > 0$

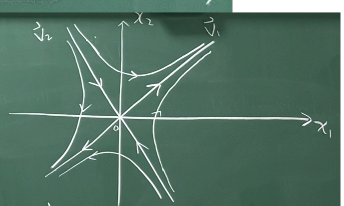
$$\vec{x}_g(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$$

- All orbits except $\vec{x}(t)\equiv\vec{0}$ diverge as $t\to\infty$.
- $\vec{x}_g(t) \to \vec{0}$ as $t\to-\infty$
- All orbits, except the ones on λ_1-eigen lines, are tangent to $\vec{v}_2$ at $\vec{0}$ (See Fig 1 with the arrows reversed) $\boxed{\text{unstable}}$
- $\vec{0}$ is called a node or a <u>nodal Source</u> 结点(源)

Case 2 $\lambda_2 < 0 < \lambda_1$

$$\vec{x}_g(t) = \underbrace{C_1 e^{\lambda_1 t}}_{\vec{x}_1(t)}\vec{v}_1 + \underbrace{C_2 e^{\lambda_2 t}}_{\vec{x}_2(t)}\vec{v}_2$$

· If $C_1\neq 0, C_2\neq 0$, $\vec{x}_g \approx \begin{cases}\vec{x}_1(t) & \text{if } t>0 \text{ large}\\ \vec{x}_2(t) & \text{if } t<0 \text{ large}\end{cases}$



$\vec{0}$ is called a <u>saddle point</u> (鞍点)

$\boxed{\text{Unstable}}$

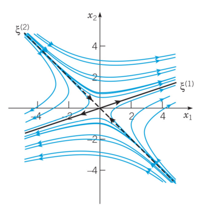
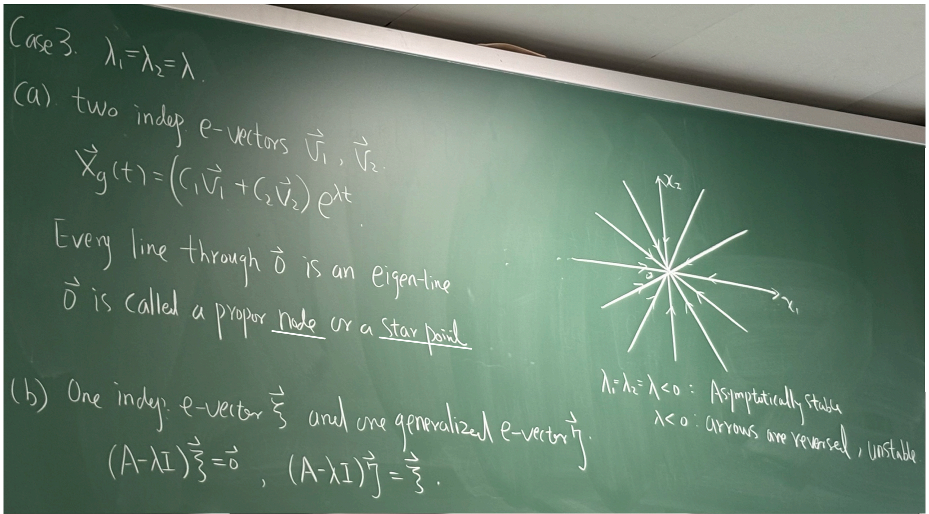
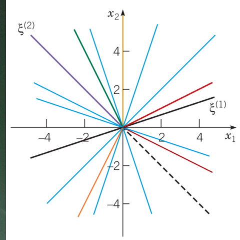
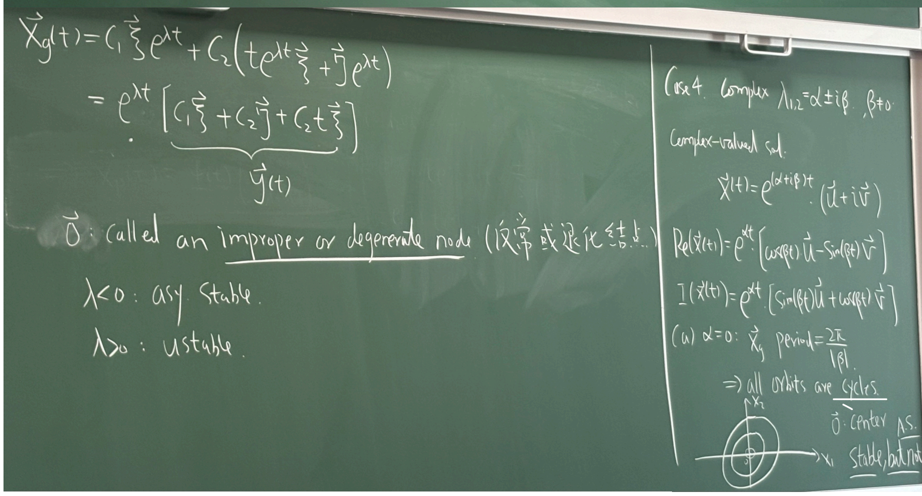
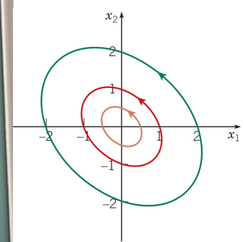


FIGURE 9.1.2 Trajectories in the phase plane when the origin is a saddle point with $r_1 > 0, r_2 < 0$. The solid black and dashed black curves show the fundamental solutions $\xi^{(1)}e^{r_1 t}$ and $\xi^{(2)}e^{r_2 t}$, respectively.









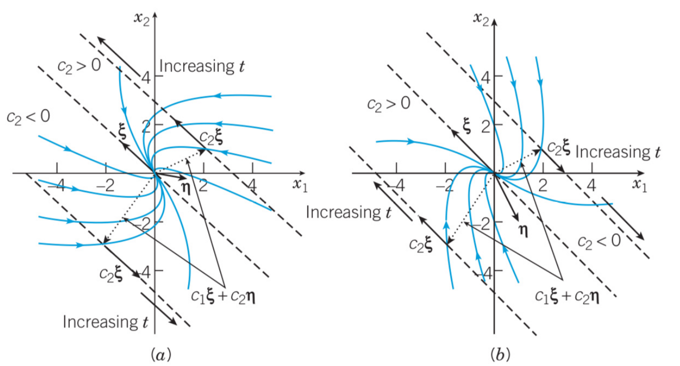


FIGURE 9.1.4 (*a*) The phase plane for an improper node with eigenvalues $r_1 = r_2 < 0$ and one independent eigenvector $\boldsymbol{\xi}$. (*b*) The phase plane for a system with the same eigenvalues $r_1 = r_2 < 0$ and eigenvector $\boldsymbol{\xi}$ but a different generalized eigenvector $\boldsymbol{\eta}$.

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_{2\times2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$\det(A-\lambda I)=0,\ \lambda_1,\ \lambda_2.$$

Case 4. Complex $\lambda_{1,2} = \alpha \pm i\beta$. $\beta \neq 0$.

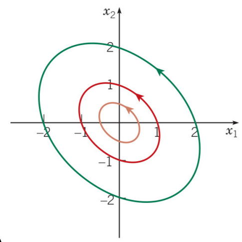
$\lambda_1 = \alpha + i\beta$. e-vector $\vec{v}_1 = \vec{u} + i\vec{v}$.

complex-value sol. $\vec{X}(t) = e^{(\alpha+i\beta)t}(\vec{u} + i\vec{v})$

$$= \underbrace{e^{\alpha t}\left[\cos(\beta t)\,\vec{u} - \sin(\beta t)\,\vec{v}\right]}_{\vec{X}^{(1)}(t)} + i\,\underbrace{e^{\alpha t}\left[\sin(\beta t)\,\vec{u} + \cos(\beta t)\,\vec{v}\right]}_{\vec{X}^{(2)}(t)}$$

(a) $\alpha = 0$. $\Rightarrow \vec{X}_g$ is periodic. $T = \dfrac{2\pi}{|\beta|}$.

$\Rightarrow$ All orbits are cycles.



$\vec{0}$: center (中心点).

stable (initial nearby sols remain close).

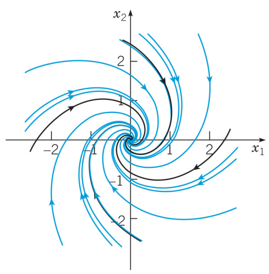
But NOT asymptotically stable ($\lim\limits_{t\to\infty} \vec{X}(t) \neq \vec{0}$)

E.g. $\begin{cases} X_1' = -X_2 \\ X_2' = X_1. \end{cases} \Rightarrow X_1'' = -X_1.$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\lambda = \pm i$.

Solutions: $X_1^2 + X_2^2 = C$. C: any const.

(b) $\alpha < 0$.

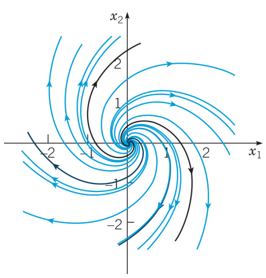


$\vec{X_g}(t) \to \vec{0}$ as $t \to \infty$ along a spiral

$\vec{0}$: a spiral sink. (稳定焦点)

asymptotically stable.

(c) $\alpha > 0$.



$\vec{X_g}(t)$ diverges as $t \to \infty$.

$\vec{0}$: a spiral source. (不稳定焦点)

unstable.

TABLE 9.1.1 Stability Properties of Linear Systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - r\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$		
$\lambda > 0$	Spiral point	Unstable
$\lambda < 0$	Spiral point	Asymptotically stable
$\lambda = 0$	Center	Stable

§6.2 Autonomous Systems and Stability

Autonomous 1st-order system:

$$\frac{d\vec{X}}{dt} = \vec{f}(\vec{X}) \qquad (*1).$$

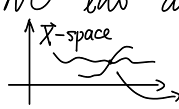
No t in $\vec{f}$. $\vec{f} = (f_1(\vec{X}), f_2(\vec{X}), \cdots, f_n(\vec{X}))^T$, $\vec{X} = (X_1(t), \cdots, X_n(t))^T$.

- $\vec{X}$ — space $(= \mathbb{R}^n)$ is called phase (state) space (相空间或状态空间).
  $n=1$: phase line; $n=2$: phase plane.
- visualized a sol. $\vec{X}(t)$ as position function of a particle moving in $\vec{X}$ — space
- If $\vec{X}(t)$ is a sol. of $(*1)$ on $I = (\alpha, \beta)$, so is $\vec{X}(t+a)$ on $(\alpha - a, \beta - a)$.
  (this prop. is the key of the concept of "Autonomous")

Suppose $\vec{f}(\vec{X})$ is nice: $\vec{f}$ is Lipschitz continuous in $\vec{X}$ on any rectangle in $\vec{X}$-space.

Then: 1. E&U Thm applies to the (IVP) $\begin{cases} \frac{d\vec{X}}{dt} = \vec{f}(\vec{X}) \\ \vec{X}(t_0) = \vec{X}^0 \end{cases}$ | $\frac{d\vec{X}}{dt} = \vec{f}(\cancel{t}, \vec{X})$

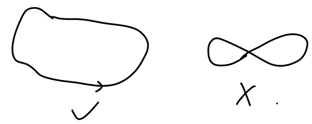
2. NO two distinct orbits intersect, even at different time.



use as initial condition. (Ex.22. §9.2)

3. $\vec{X}(t)$ is a periodic sol. $\Leftrightarrow$ Its orbit is a <u>simple closed curve.</u>

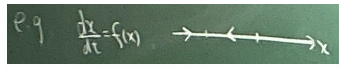
(Ex. 24 §9.2)



<u>Def.</u> equilibrium point = Const. sol. of $(*1)$ $\Leftrightarrow$ $\vec{f}$(equilibrium pt.) $= \vec{0}$.
(critical point of $(*1)$).

4. If $\vec{X}(t)$ is a sol. of $(*1)$ and $\vec{X}(0) \neq$ equilibrium pt. then it CANNOT reach a critical pt. in a finite time.

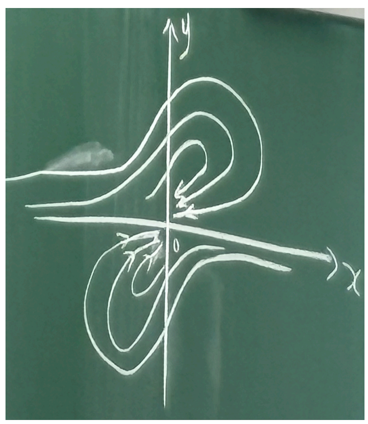
(Ex. 23 §9.2)



<u>Def.</u> An equilibrium pt. $\vec{X}^*$ (i.e. $\vec{f}(\vec{X}^*) = \vec{0}$) is said to be <u>stable</u> if $\forall\, \varepsilon>0$, $\exists\, \delta = \delta(\varepsilon) > 0$, s.t. every sol. $\vec{X}(t)$ of $(*1)$, with $\|\vec{X}(0) - \vec{X}^*\| < \delta$, exists for all $t>0$ and satisfies $\|\vec{X}(t) - \vec{X}^*\| < \varepsilon$ for all $t>0$.

- If $\vec{X}^*$ is not stable, then we say it is unstable
- If $\vec{X}^*$ is stable and in addition, $\exists\, \delta_0 > 0$, s.t. if a sol. $\vec{X}(t)$ with $\|\vec{X}(0) - \vec{X}^*\| < \delta_0$ $\Rightarrow \lim_{t\to\infty} \|\vec{X}(t) - \vec{X}^*\| = 0$. then we say it is <u>asymptotically stable.</u>

<u>Rk.</u> attractivity $\not\Rightarrow$ stability : W. Hahn, book, 1967



E.g. 1.
$$\begin{cases} \dfrac{dx}{dt} = -(x-y)(1-x-y) \\ \dfrac{dy}{dt} = x(2+y) \end{cases} \qquad \frac{dy}{dx} = \frac{-x(2+y)}{(x-y)(1-x-y)}.$$

Sol: Find equilibrium pt.s:

$$\begin{cases} (x-y)(1-x-y)=0 \\ x(2+y)=0 \end{cases}$$

$\Rightarrow$ $x=y$ or $x+y=1$

$\Rightarrow$ $x=0$ or $y=-2$.

$\Rightarrow (0,0),\ (0,1),\ (-2,-2),\ (3,-2).$

saddle unstable. spiral A.S. A.S. unstable.

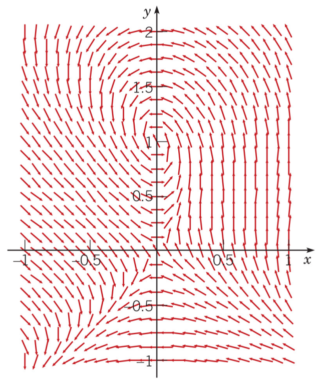


FIGURE 9.2.4 Direction field for the system (14) containing the critical points $(0,0)$ and $(0,1)$; the former is a saddle point and the latter is a spiral sink.

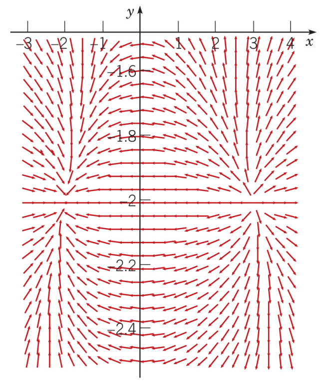


FIGURE 9.2.5 Direction field for the system (14) containing the critical points $(-2,-2)$ and $(3,-2)$; the former is a nodal sink and the latter is a nodal source.

Def. Let $\vec{X}^*$ be a critical pt. The basin of attraction or the region of asymptotic stability of $\vec{X}^*$ is the set of all points p that a trajectory 解曲线. passing through p ultimately approaches $\vec{X}^*$ as $t\to\infty$.

A trajectory that bounds a basin of attraction is called a separatrix.

More generally, a separatrix is a trajectory that separates trajectories of distinct topological structures, which usually contains a saddle point.

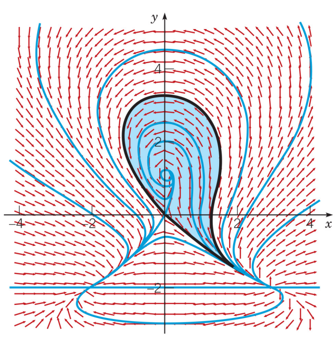


FIGURE 9.2.6 Direction field, trajectories, and critical points of the system (14). The separatrices are shown in black. The basis of attraction for the spiral point $(0,1)$ is shaded.

the trajectories of the system

$$\frac{dx}{dt} = 4-2y, \qquad \frac{dy}{dt} = 12-3x^2.$$

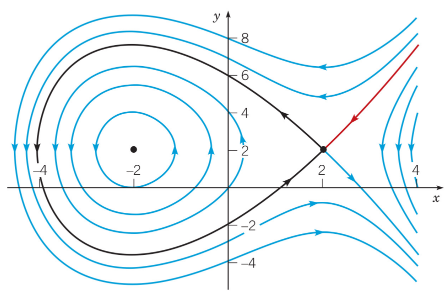


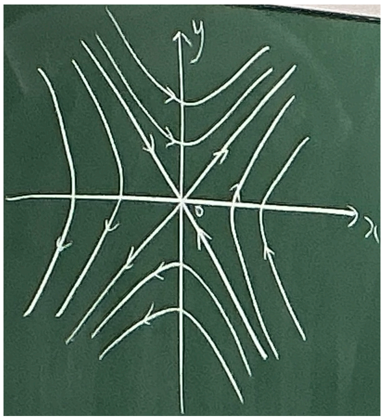
FIGURE 9.2.8 Trajectories of the system (22). The point $(-2,2)$ is a center, and the point $(2,2)$ is a saddle point. The black curve is a separatrix.

Determine trajectories of a two-dim autonomous system.

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \Rightarrow \frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$$

E.g. Find trajectories of $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x \end{cases}$

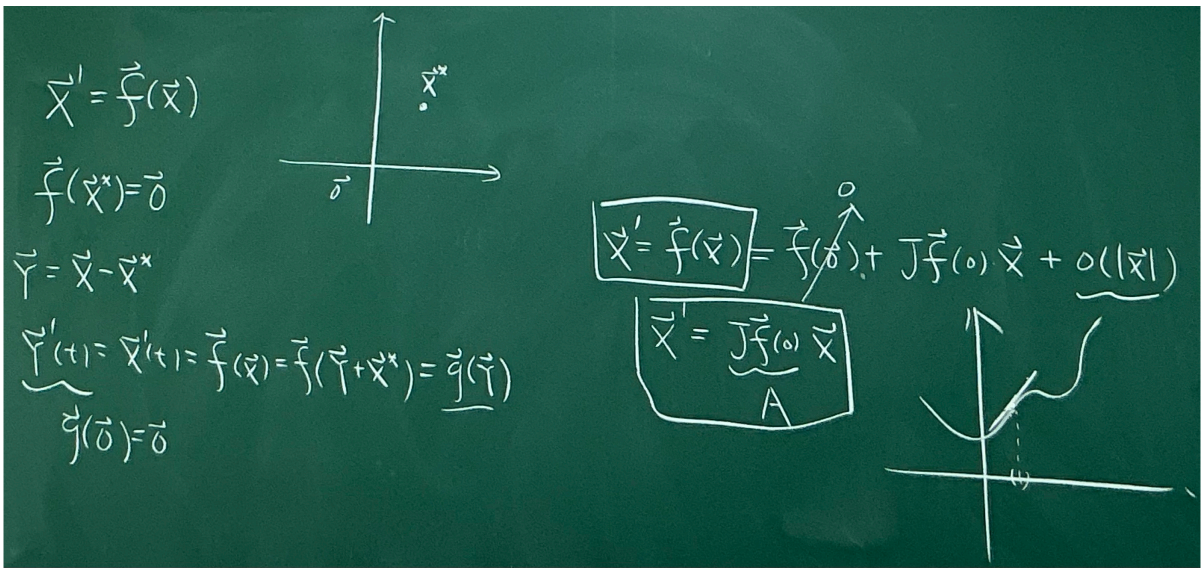
Sol. $\dfrac{dy}{dx} = \dfrac{x}{y} \Rightarrow y\,dy = x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C.$



$\Rightarrow H(x,y) = y^2 - x^2 = C.$ C: any const.

$(0,0)$ is the unique critical point.

saddle.



§6.3. Locally linear systems

Consider $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{f}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$, $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. non linear system. $(*1)$

Goal: to investigate the behaviour of trajectories near a critical pt $\vec{X}^*$

W.L.O.G: May assume $\vec{X}^* = \vec{0}$. otherwise let $\vec{Y} = \vec{X} - \vec{X}^*$ $\quad (\vec{f}(\vec{X}^*) = \vec{0})$

<u>Def</u>: For system $\vec{X}' = \overbrace{A\vec{X} + \vec{g}(\vec{X})}^{=\vec{f}(\vec{X})}$, $(*2)$

where $\vec{0}$ is an isolated critical pt. and $\det A \neq 0$.

If the components of $\vec{g}$ have continuous first partial derivatives in a neighbourhood, say N of $\vec{0}$, and $\lim_{\vec{X} \to \vec{0}} \frac{\|\vec{g}(\vec{X})\|}{\|\vec{X}\|} = 0$. $(*3)$

(i.e. $\vec{g}(\vec{X}) = o(\|\vec{X}\|)$)

Then, the system $(*2)$ is called a <u>locally linear system</u> in the neighbourhood of $\vec{X} = \vec{0}$.

<u>Rk</u>: If $\vec{f}(\vec{X})$ is C^1 in a neighbourhood of $\vec{X} = \vec{0}$, then

$$\vec{f}(\vec{X}) = \underbrace{\cancel{\vec{f}(\vec{0})}}_{\vec{0}} + \nabla \vec{f}(\vec{0})\vec{X} + o(\|\vec{X}\|), \quad \nabla \vec{f} = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_n \end{pmatrix} = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i, j \le n}$$

Jacobi mat.

Thus, locally linear $\Leftrightarrow C^1$.

Recall: Linear system $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A_{2\times 2}\begin{pmatrix} x \\ y \end{pmatrix}$, $\det A \neq 0$, $(*4)$

Let r_1, r_2 be eigenvalues of A.

<u>Thm1 (Table 9.1.1)</u> The only critical point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of $(*4)$ is

(i) asptotically stable if the real part of every e-value is negative

(ii) Stable but not A.S. if $r_{1,2} = \pm \beta i$, $\beta \neq 0$

(iii) unstable if at least one of the e-values has a positive real part.

$(\vec{x}(t) = e^{\lambda t} \cdot \vec{v})$

TABLE 9.1.1 Stability Properties of Linear Systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - r\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$		
$\lambda > 0$	Spiral point	Unstable
$\lambda < 0$	Spiral point	Asymptotically stable
$\lambda = 0$	Center	Stable

TABLE 9.3.1 Stability and Instability Properties of Linear and Locally Linear Systems

	Linear System		Locally Linear System	
Eigenvalues	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	*PN or IN*	Unstable	*N or SpP*	Unstable
$r_1 = r_2 < 0$	*PN or IN*	Asymptotically stable	*N or SpP*	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$\lambda = 0$	*C*	Stable	*C or SpP*	Indeterminate

Key: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

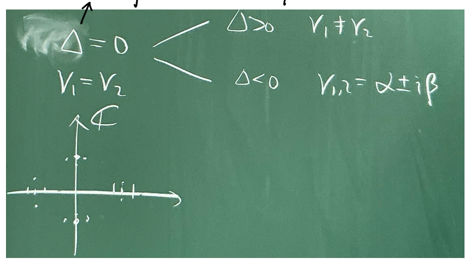
<u>Thm 2</u> (The principle of linearized stability)

For a locally linear system (*2), Thm 1 (i) & (iii) still hold.

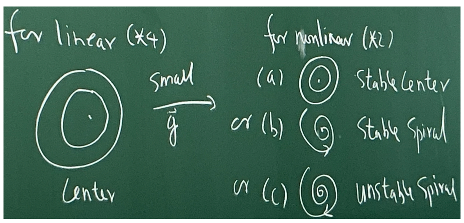
"Δ" of the char eq.

<u>Rk</u>: Table 9.3.1 contains more detailed information. Basically, it says that the small nonlinear term in (*2) does not alter the type of stability of the critical point, except two sensitive cases.

<u>E.g.</u>:



<u>Case 1</u>: $r_1, r_2 = \pm \mu i$.



<u>Case 2</u>: $r_1 = r_2$.

linear (*4) $\qquad$ nonlinear (*2)

$$\Delta = 0 \xrightarrow[\vec{g}]{\text{small}} \begin{cases} \Delta > 0 & \text{node} \\ \text{or } \Delta < 0 & \text{spiral pt.} \end{cases}$$

<u>Rk</u>: For $\vec{X}' = A\vec{X}$, if $\det A \neq 0$, then $\vec{X} = \vec{0}$ is the only critical pt.

If it's asymptotically stable, it is in fact globally asymptotically stable.

(i.e. the basin of attraction is the entire phase space)

This property of linear system is <u>not, in general, true</u> for nonlinear systems, even if it has only one asymp. stable critical pt.

<u>E.g.</u>

E.g: $\begin{cases} \dfrac{dx}{dt} = (2+y)(2y-x) \\[2ex] \dfrac{dy}{dt} = (2-x)(2y+x) \end{cases}$ (*5).

(a) Determine all critical pts.

(b) Find the corresponding linear system near each critical pt.

(c) Find the e-values of each linear system and draw conclusion about the non-linear system.

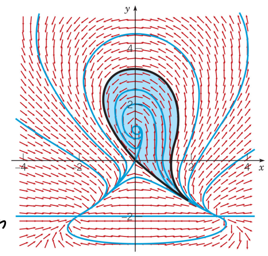


FIGURE 9.2.6 Direction field, trajectories, and critical points of the system (14). The separatrices are shown in black. The basis of attraction for the spiral point (0, 1) is shaded.

Sol: (a) $\begin{cases} (2+y)(2y-x)=0 \Rightarrow y=-2 \text{ or } x=2y \\ (2-x)(2y+x)=0 \Rightarrow x=2 \text{ or } x=-2y \end{cases}$ $\Rightarrow$ critical pts. $(2,-2)$, $(2,1)$, $(4,-2)$, $(0,0)$.

(b) & (c): $J(\vec{X}) = \begin{pmatrix} -(2+y) & 4y-x+4 \\ -2y+2-2x & 4-2x \end{pmatrix}$.

At $\vec{X}^* = (2,-2)$. $J(\vec{X}^*) = \begin{pmatrix} 0 & -6 \\ 2 & 0 \end{pmatrix}$. $\begin{cases} u = x - x^* \\ v = y - y^* \end{cases}$

$\Rightarrow$ linear system $\begin{cases} \dfrac{du}{dt} = -6v \\[2ex] \dfrac{dv}{dt} = 2u. \end{cases}$ (L_1)

$|\lambda I - J| = \begin{vmatrix} \lambda & 6 \\ -2 & \lambda \end{vmatrix} = \lambda^2 + 12 \Rightarrow \lambda = \pm 2\sqrt{3}\, i \Rightarrow (0,0)$ is a stable center for (L_1).

$\Rightarrow$ For (*5), $(2,-2)$ could be a center or a spiral point. interminate.

At $\vec{X}^* = (4,-2)$. $J(\vec{X}^*) = \begin{pmatrix} 0 & -8 \\ -2 & -4 \end{pmatrix} \Rightarrow \begin{cases} \dfrac{du}{dt} = -8v \\[2ex] \dfrac{dv}{dt} = -2u-4v \end{cases}$ (L_2)

$\Rightarrow \lambda = -2 \pm 2\sqrt{5}$

$\Rightarrow (0,0)$ is an unstable saddle pt. for (L_2).

$\Rightarrow$ For (*5), $(4,-2)$ is an unstable saddle pt.

At $\vec{X}^* = (2,1)$. $J(\vec{X}^*) = \begin{pmatrix} -3 & 6 \\ -4 & 0 \end{pmatrix} \Rightarrow \begin{cases} \dfrac{du}{dt} = -3u+6v \\[2ex] \dfrac{dv}{dt} = -4u \end{cases}$ (L_3).

$\Rightarrow \lambda = \dfrac{-3 \pm \sqrt{87}\, i}{2}$.

$\Rightarrow (0,0)$ is an A.S pt. for (L_3)

$\Rightarrow$ For (*5), $(2,1)$ is an A.S pt.

At $\vec{X}^* = (0,0)$, $J(\vec{X}^*) = \begin{pmatrix} -2 & 4 \\ 2 & 4 \end{pmatrix} \Rightarrow \begin{cases} \frac{du}{dt} = -2u + 4v \\ \frac{dv}{dt} = 2u + 4v \end{cases}$ (L4)

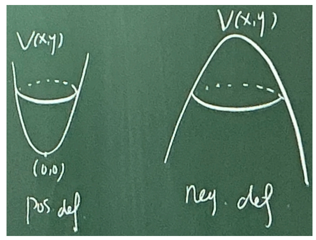
$\Rightarrow \lambda = 1 \pm \sqrt{17}$.

$\Rightarrow$ $(0,0)$ is an unstable saddle pt. for (L4)

$\Rightarrow$ For (*5), $(0,0)$ is an unstable saddle pt.

§6.4 Liapunov's Second Method

<u>Def.</u> $V: \underset{\mathbb{R}^2}{\overset{\cap}{D}}$ (domain, $\vec{0} \in D$) $\longrightarrow \mathbb{R}$ is said to be <u>positive (negative) definite</u> on D, ($\vec{0} = (0,0)$).



if (1) $V(\vec{0}) = \vec{0}$

(2) $V(x,y) > 0$ ($V(x,y) < 0$) on $D \setminus \{(0,0)\}$.

V is said to be <u>positive (negative) semi-definite</u> if „$>$" („$<$")

is replaced by „$\leq$" („$\geq$").

<u>E.g. 1:</u> $V(x,y) = \sin(x^2+y^2)$ is pos. definite on $D = \{(x,y) \mid x^2+y^2 < \frac{\pi}{2}\}$.

$V(x,y) = (x+y)^2$ is pos. semi-definite on $D = \mathbb{R}^2$.

Consider $\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$ (*1).

For any func. $V = V(x,y) \in C^1(D)$, we define

$$\dot{V}(x,y) := \frac{\partial V}{\partial x}(x,y)\, F(x,y) + \frac{\partial V}{\partial y}(x,y)\, G(x,y) : D \to \mathbb{R}.$$

Higher-dimention case:

$\dot{\vec{X}}(t) = \vec{f}(\vec{X}(t))$, $\vec{X} = (x_1, \cdots, x_n)^T$.

$\vec{f}(\vec{X}) = (f_1(\vec{X}), \cdots, f_n(\vec{X}))^T$.

$V: \mathbb{R}^n \to \mathbb{R}$.

$\dot{V}(\vec{X}) = \nabla V \cdot \vec{f}(\vec{X})$.

If $(x(t), y(t))$ is a sol of (*1), then

$$\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x} \cdot x'(t) + \frac{\partial V}{\partial y} \cdot y'(t) = \frac{\partial V}{\partial x} F(x,y) + \frac{\partial V}{\partial y} G(x,y).$$

$$= \dot{V}(x(t), y(t)).$$

Thus, $\dot{V}(x,y)$ can be identified as the rate of change of V along the trajectory of (*1) that passes through (x,y).

We call $\dot{V}$ the derivative of V w.r.t. system (*1).

<u>Theorem 1 (Liapunov's second method).</u> <u>A direct method.</u>

Suppose (*1) has an isolated critical pt. at the origin $(0,0)$.

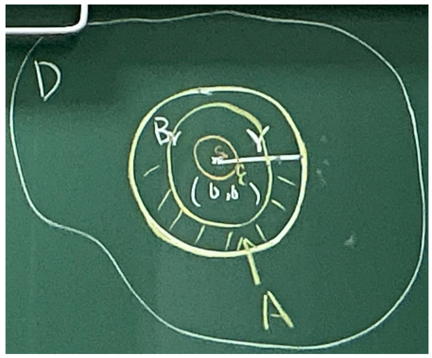
If $\exists\, V(x,y)$ that is continuous and has continuous 1st partial derivatives, and is pos. def. on some domain $D \ni (0,0)$, then the followings hold.

(a) If $\dot{V}$ is neg. semi-def. on D, then $(0,0)$ is <u>stable</u>.

(b) If $\dot{V}$ is neg. definite on D, then $(0,0)$ is <u>asymptotically stable</u>.

(c) If $\dot{V}$ is pos. definite on D, then $(0,0)$ is <u>unstable</u>.

<u>Pf of (a)</u>: Take a open ball $B_r(\vec{0})$ s.t. $\overline{B_r(\vec{0})} \subset D$.



$\forall\, \varepsilon \in (0,r)$, let $A = \{\vec{X} = (x,y) \mid \varepsilon \le |\vec{X}| \le r\}$.

Then $\min_A V \triangleq \mu > 0$.

Since $V(0,0) = 0$ and V is continuous at $(0,0)$,

we can find $\delta \in (0,\varepsilon)$ s.t. $V(\vec{X}) < \mu,\ \forall\, |\vec{X}| \le \delta$.

If $|\vec{X}_0| < \delta$, then we have $V(x(t),y(t)) \underset{(\because \dot{V} \le 0)}{\le} V(\vec{X}_0) < \mu$.

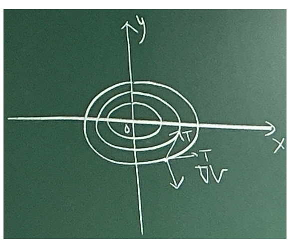
$\Rightarrow$ The sol. will never enter A.

$\Rightarrow$ It exists for $\forall\, t \in [0,\infty)$ and $|(x(t),y(t))| < \varepsilon \Rightarrow (0,0)$ is stable.

A geometric interpretation of Thm 1 (a)–(c):

V pos. def. $C^1 \Rightarrow$ For every $c > 0$ small, the level set

$L_c = \{(x,y) \in D \mid V(x,y) = c\}$ is a closed curve around $(0,0)$



$\dot{V}(x,y) = \nabla V(x,y) \cdot (x'(t), y'(t))$

$= \nabla V(x,y) \cdot T(x,y)$ (*2) direction of the orbit (x,y)

Note: At every pt. $(x,y) \in L_c$, $\nabla V(x,y)$ is outward normal to L_c.

(a) $\dot{V} \le 0 \Rightarrow T$ points inward or tangent to L_c.

$\Rightarrow$ If $\vec{X}_0$ is inside L_c, then sol. $(x(t), y(t))$ will never enter the region outside of L_c.

$\Rightarrow (0,0)$ is stable

(b) $\dot{V} < 0$ on $D \setminus \{\vec{0}\} \Rightarrow T$ points inward to L_c. $\Rightarrow (0,0)$ is asymp. stable.

(c) $\dot{V} > 0$ on $D \setminus \{\vec{0}\} \Rightarrow T$ points outward to L_c.

$\Rightarrow$ Sol. arrives L_c will always enter the outside of L_c.

$\Rightarrow (0,0)$ is unstable.

$*$ The function V in Thm 1 is called a <u>Liapunov func.</u>

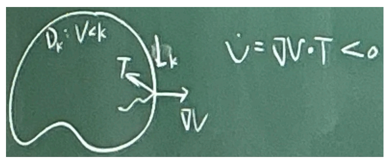
<u>Theorem 2. (estimate of the extent of the basin of attraction)</u>

Suppose $(0,0)$ is an isolated critical pt. of $(*1)$ $\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$ and $F, G \in C^1(\mathbb{R}^2)$.

Let $V \in C^1(\mathbb{R}^2)$.

If for some $k > 0$, the set $D_k = \{(x,y) \in \mathbb{R}^2 : V(x,y) < k\}$ is a bdd domain and V is pos. def. $\dot{V}$ is neg. def. on D_k, then $D_k \subseteq$ basin of attraction of $(0,0)$.

<u>Idea of Pf:</u> ① Trajectories starting in D_k cannot escape:



② No other critical pt. in D_k.

$(\bar{x}, \bar{y})$ is a critical pt. $\Rightarrow \dot{V}(\bar{x},\bar{y}) = \nabla V \cdot (\underset{0}{\underset{\parallel}{F(\bar{x},\bar{y})}}, \underset{0}{\underset{\parallel}{G(\bar{x},\bar{y})}}) = 0$

$\dot{V}$ is neg. def. $\dot{V}(\bar{x},\bar{y}) = 0$ iff $(\bar{x},\bar{y}) = (0,0)$.

③ No periodic sol. s of $(*1)$ in D_k.

Suppose $(x(t), y(t))$ is a periodic sol. with period $T > 0$:

$$\int_{t_0}^{t_0+T} \underbrace{\frac{d}{dt} V(x(t), y(t))}_{\dot{V}(x(t),\, y(t)) < 0.} \, dt = V(x(t_0+T), y(t_0+T)) - V(x(t_0), y(t_0)) = 0. \quad \times$$

Theorem 3. The func. $V(x,y)=ax^2+bxy+cy^2$ is $\begin{cases} \text{pos. def. iff } a>0 \ \& \ b^2-4ac<0. \\ \text{neg. def. iff } a<0 \ \& \ b^2-4ac<0. \end{cases}$

Pf: $V(x,y)=(x,y)\underbrace{\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix}$.

V pos. def $\Leftrightarrow A$ is pos. def.

neg. def $\Leftrightarrow A$ is neg. def.

($\mathrm{tr} = \sum$ e-values, $\det = \prod$ e-values)

E.g. 2 Determine the stability of (0,0) for system $\begin{cases} \frac{dx}{dt} = -y+x^3 \\ \frac{dy}{dt} = x+y^3 \end{cases}$

Sol: The corresponding linear system

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = x \end{cases} \quad A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad r_{1,2} = \pm i. \quad \text{indeterminate case.}$$

Construct: $V(x,y)=ax^2+bxy+cy^2$.

$$\dot{V}(x,y) = V_x \cdot x'(t) + V_y \cdot y'(t) = (2ax+by)(-y+x^3)+(bx+2cy)(x+y^3).$$

$$= 2(ax^4+cy^4) + 2(c-a)xy + b(x^2-y^2+x^3y+xy^3).$$

Let $\underset{\substack{\| \\ 1}}{a=c}$, $b=0$. $\underset{\text{pos. def}}{V(x,y)=x^2+y^2}$. $\underset{\text{pos. def.}}{\dot{V}(x,y)=2(x^4+y^4)}$. $\Rightarrow$ (0,0) is unstable.

Rks: 1. Thm 1 gives sufficient conditions for stability/instability of a critical pt but they are not necessary.

2. Thm 1 compliments the principle of linearized stability, especially when the linear system has pure imaginary e-values.

3. Our failure to find a suitable Liapunov func. does not mean that there is no such func.

4. No general methods for construction of Liapunov func.

Some related topics: $\frac{d\vec{X}}{dt} = \vec{f}(\vec{X})$. (✱3)

1. If $V \in C^1(\mathbb{R}^n)$, $V(\vec{X}(t))$ is strictly decreasing ↘ in t for any non-const sol. $\vec{X}(t)$ then (✱3) is called a dissipative system. 耗散系统.

Dissipative systems do NOT have cycles.

E.g. Gradient System.

$$\frac{d\vec{X}}{dt} = \nabla G(\vec{X}) \quad (*4).$$

$$\Rightarrow \frac{d}{dt}\left(-G(\vec{X}(t))\right) = -\frac{d}{dt} G(\vec{X}(t)) = -\nabla G(\vec{X}(t)) \cdot \vec{X}'(t) = -\left|\nabla G(\vec{X}(t))\right|^2 \le 0$$

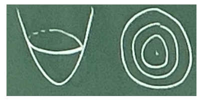
$\Rightarrow -G$ is a Liapunov func. & $(*3)$ is dissipative. & $=0$ only at critical pts

2. If $\exists V(\vec{x}) \in C^1(\mathbb{R}^n)$ s.t. (a) $V(\vec{X}(t)) = \text{const.}$ along every obit $\vec{X}(t)$.

(b) $V \neq$ const. in any ball in $\mathbb{R}^n$.

then $(*3)$ is called a <u>conservation system</u> and V is called "integral" of $(*3)$.

E.g. Loteka-Volterra. $\begin{cases} \dfrac{dx}{dt} = x(r-by), & r,b,s,c>0. \\ \dfrac{dy}{dt} = y(-s+cx). \end{cases}$



$V(x,y) = (s\ln x - cx) + (r\ln y - by).$

E.g. Hamiltonian system. $\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial y}(x,y) = F \\ \dfrac{dy}{dt} = -\dfrac{\partial H}{\partial x}(x,y) = G \end{cases}$ $\left(\dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} = 0\right)$

$$\Rightarrow \frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} + \frac{\partial H}{\partial y} \cdot -\left(\frac{\partial H}{\partial x}\right) = 0.$$

$\Rightarrow H$ is conservative along every orbit $\vec{X}(t)$.

Q: Conserative systems do not have attractors or repellers.

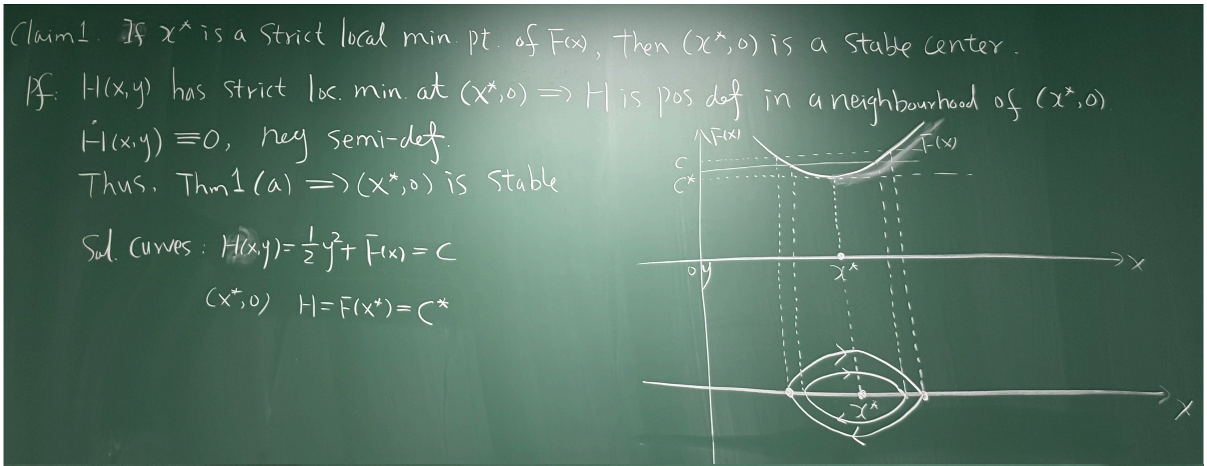
An application: $\dfrac{d^2x}{dt^2} + f(x) = 0. \quad f \in C^1(\mathbb{R})$

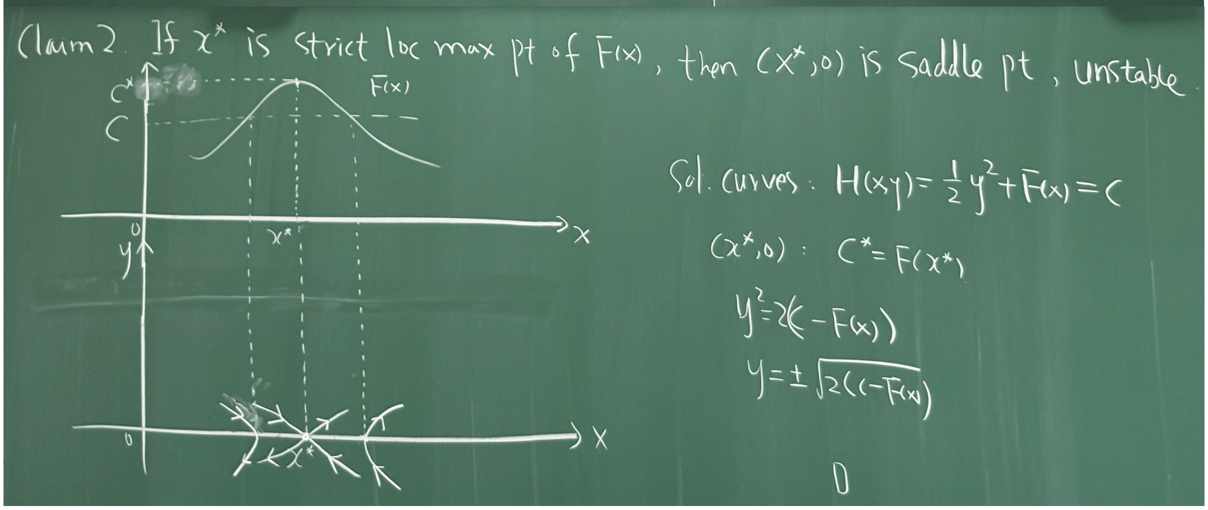
$$\Rightarrow \begin{cases} x' = y \\ y' = f(x) \end{cases} \quad (*5)$$

$\xRightarrow{\text{Hamiltonian.}} H(x,y) = \frac{1}{2}y^2 + F(x).$

$F(x) = \int_0^x f(s)\,ds, \quad F'(x) = f(x).$

Critical pt.s: $(x^*, 0)$ is an equilibrium pt. of $(*5)$ iff $f(x^*) = 0$, i.e. $F'(x^*) = 0$.





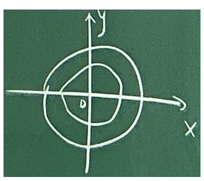
§6.5 Periodic Sol.s, limit cycles & Poincaré – Bendixson Theorem

Consider $\vec{X}'(t) = \vec{f}(\vec{X}(t))$

Periodic sol $\vec{X}(t)$:
- $\exists\, T > 0$ s.t. $\vec{X}(t+T) = \vec{X}(t)$, $\forall t$.
  smallest such T is called the period of $\vec{X}(t)$.
- Trajectory is a closed curve.
- A const sol. $\vec{X}(t) = \vec{X}_0$ is a critical pt., is a special periodic sol.

Example 1. $\vec{X}' = A\vec{X}$. $A_{2\times 2}$ mat. $\det A \neq 0$.

$\exists$ periodic sol.s $\Leftrightarrow$ $r_{1,2} = \pm \mu i$, $\mu > 0$.



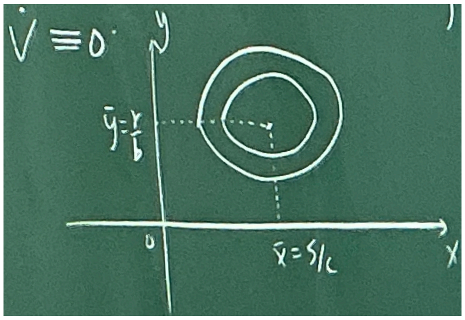
$\Leftrightarrow$ $\operatorname{tr} A = 0$, $\det A > 0$

$\Leftrightarrow$ $\vec{0}$ is a center.

$\Leftrightarrow$ Every orbit is a closed curve.

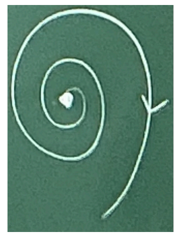
Example 2. $\begin{cases} \dot{x}(t) = x(r-by) & \text{(prey)} \\ \dot{y}(t) = y(-s+cx) & \text{(predator)} \end{cases}$ $\quad r, b, s, c > 0.$

$$V(x,y) = s\ln x - cx + r\ln y - by.$$



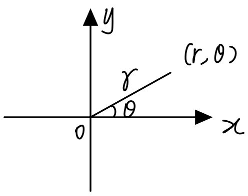
Example 3. $\begin{cases} \dot{x} = x+y - x(x^2+y^2) \\ \dot{y} = -x+y-y(x^2+y^2) \end{cases}$ $\quad (*1)$

Sol: critical pt.: $(x,y) = (0,0)$.



Linear system at $(0,0)$: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. $\quad r_{1,2} = 1 \pm i$

Polar coordinates (r, θ):



$\begin{cases} x = r\cos\theta, \quad y = r\sin\theta. \quad r \geq 0. \\ r^2 = x^2+y^2 \end{cases}$ $\quad (*2)$

$$(*2)_2 \Rightarrow 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}. \quad (*3)$$

$$\xRightarrow{(*1)} r\frac{dr}{dt} = x(x+y-x(x^2+y^2)) + y(-x+y-y(x^2+y^2))$$

$$= x^2+y^2-(x^2+y^2)^2 = r^2 - r^4.$$

$r = 0 \Leftrightarrow (x,y) = (0,0)$ critical pt.

$r \neq 0 \Leftrightarrow \dfrac{dr}{dt} = r(1-r^2)$ $\quad$ 0 $\rightarrow$ 1 $\leftarrow$

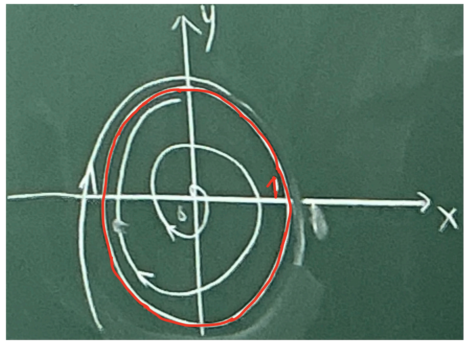
$$(*2)_1 \Rightarrow \frac{dx}{dt} = \frac{dr}{dt}\cos\theta + r(-\sin\theta)\frac{d\theta}{dt}, \quad \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}. \quad (*4)$$

$$\Rightarrow -y\frac{dx}{dt} + x\frac{dy}{dt} = r^2\frac{d\theta}{dt}. \quad (*5)$$

$$\overset{(*1)}{\Longrightarrow} r^2\frac{d\theta}{dt} = x\left(-x+y-y(x^2+y^2)\right) - y\left(x+y-x(x^2+y^2)\right)$$

$$= -x^2-y^2 = -r^2.$$

$$\Rightarrow \frac{d\theta}{dt} = -1.$$



Example 4. $\begin{cases} \dot{x} = -y + x r^2 \sin\left(\frac{\pi}{r}\right) \\ \dot{y} = x + y r^2 \sin\left(\frac{\pi}{r}\right) \end{cases}$, $r^2 = x^2+y^2$.

$$\text{RHS} = \begin{cases} (F, G), & \text{if } (x,y) \neq (0,0) \\ (0,0), & \text{if } (x,y) = (0,0). \end{cases}$$

Sol: critical pt. $(x,y) = (0,0)$.

$x = r\cos\theta$, $y = r\sin\theta$.

$r^2 = x^2+y^2$.

$$2r\frac{dr}{dt} = 2x\cdot\dot{x} + 2y\cdot\dot{y}$$

$$= 2x\left(-y + xr^2\sin\frac{\pi}{r}\right) + 2y\left(x + yr^2\sin\frac{\pi}{r}\right)$$

$$= 2r^4\sin\frac{\pi}{r}.$$

$r \neq 0$, $\frac{dr}{dt} = r^3\sin\frac{\pi}{r}$. $\frac{\pi}{r} = k\pi$, $k \in \mathbb{N}^*$. $r = \frac{1}{k}$.

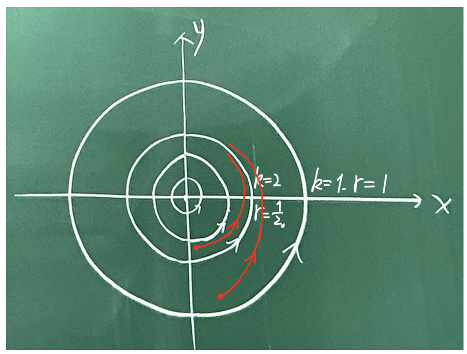
$$0 \cdots \tfrac{1}{4}\ \tfrac{1}{3}\ \tfrac{1}{2} \qquad 1$$

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}, \quad \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}.$$

$$y\frac{dx}{dt} - x\frac{dy}{dt} = -y^2 - x^2 = -r^2$$

$$= -r^2\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = 1. \text{ (counterclockwise.)}$$

$\begin{cases} \frac{dr}{dt} = r^3\sin\frac{\pi}{r} \\ \frac{d\theta}{dt} = 1. \end{cases}$ | k is odd. $\frac{1}{k+1} < r < \frac{1}{k}$, $r \downarrow \frac{1}{k+1}$.
k is even. $\frac{1}{k+1} < r < \frac{1}{k}$, $r \uparrow \frac{1}{k}$.



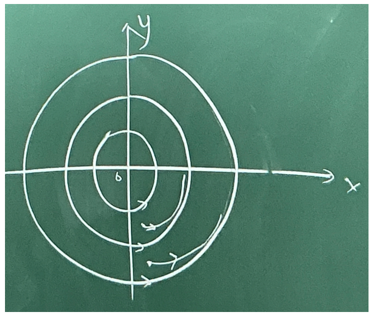
$\Rightarrow$ tend to even k. even: stable. odd: unstable.

Example 5. $\begin{cases} \dot{x} = -y + x\sin^2\left(\frac{\pi}{r}\right) \\ \dot{y} = x + y\sin^2\left(\frac{\pi}{r}\right) \end{cases}$

Sol: Critical pt. is $(0,0)$.

$$\begin{cases} \dfrac{dr}{dt} = r\sin^2\left(\dfrac{\pi}{r}\right) \\ \dfrac{d\theta}{dt} = 1. \end{cases}$$

periodic sols: $\frac{\pi}{r} = k\pi$, $r = \frac{1}{k}$, $k \in \mathbb{N}^*$.



Let $\vec{X}(t)$ be a periodic sol. we consider the following "orbit stability" of $\vec{X}(t)$:

① If all trajectories that start near $\vec{X}(t)$ (both inside and outside) spiral toward $\vec{X}(t)$ as $t \to \infty$, then $\vec{X}(t)$ is <u>asymptotically stable.</u>

(Ex. 3, Ex. 4 $(\frac{1}{k})$s, k even.)

② If only trajectories that start near $\vec{X}(t)$ on one side spiral toward $\vec{X}(t)$, while those on the outside spiral away from $\vec{X}(t)$ as $t \to \infty$, then $\vec{X}(t)$ is <u>semi-stable.</u>

(Ex. 5).

③ If all trajectories that start near $\vec{X}(t)$ (both inside and outside) spiral away from $\vec{X}(t)$ as $t \to \infty$, then $\vec{X}(t)$ is <u>unstable.</u>

(Ex. 4 $(\frac{1}{k})$s, k odd).

④ If trajectories nearby neither approach nor spiral away from $\vec{X}(t)$ as $t \to \infty$, then $\vec{X}(t)$ is <u>stable.</u>

(Ex. 1 & Ex. 2).

<u>Def</u> A periodic sol. $\vec{X}(t)$ is called a <u>limit cycle</u>, if $\exists$ an orbit spirals toward it from either the inside or outside, as $t \to \infty$.

(Ex. 3; Ex. 4 $(\frac{1}{k})$s, k even; Ex. 5).

Consider $\dfrac{d\vec{x}}{dt} = \vec{f}(\vec{x}),\ \vec{f} \in C^1$. (A).

Let $x(t; t_0, x_0)$ be the unique sol. of (A) with I.C. $x(t_0) = x_0$.

<u>Def</u> The (forward) <u>limit set</u> of a sol. $x(t; t_0, x_0)$ is the set of all limit pt.s
(backward)
of the sol. curve, denoted by $\omega(x_0)$
$(\alpha(x_0))$.

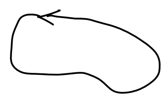
① $\bar{x} \in \omega(x_0)$ iff $\exists$ a seq. $t_0 < t_1 < t_2 < \cdots \longrightarrow \infty$ s.t. $x(t_k) \longrightarrow \bar{x}$ as $k \longrightarrow \infty$.

② If a sol. curve approaches a single pt., say, $\bar{x}$, then $\omega(x_0) = \{\bar{x}\}$.

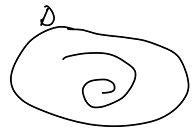
③ If a sol. curve is a closed orbit (i.e., the sol. is periodic)

then every pt. on the curve is in the limit set:

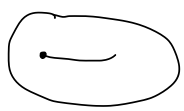
$x(t_0) = x(t_0+T) = x(t_0+2T) = \cdots,\quad t_0 + kT \longrightarrow \infty$.



④ If $x(t; t_0, x_0)$ stays in a bounded subset of D, then $\omega(x_0) \neq \emptyset$.



⑤ Any limit set is positively and negatively invariant.



Why? If $\bar{x} \in \omega(x_0)$, $\exists\ t_0 < t_1 < \cdots < t_k < \cdots \longrightarrow \infty$ s.t. $x(t_k) \longrightarrow \bar{x}$ as $k \longrightarrow \infty$.

WTS: $\forall\ \tilde{t}$, $x(\tilde{t}; 0, \bar{x}) \in \omega(x_0)$

$$x(t_k + \tilde{t}; t_0, x_0),\ k \in \mathbb{N}.$$

$$\| \qquad \searrow x(\tilde{t}, 0, \bar{x}) \text{ as } k \longrightarrow \infty\ ?$$

$$x(\tilde{t}; 0, x(t_k; t_0, x_0)) \nearrow_{k \to \infty}.$$

From now on, we consider the planer system.

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\[2ex] \dfrac{dy}{dt} = G(x,y) \end{cases} \quad (A_2)$$

where $F, G \in C^1(D)$, $D \subseteq \mathbb{R}^2$ – domain.

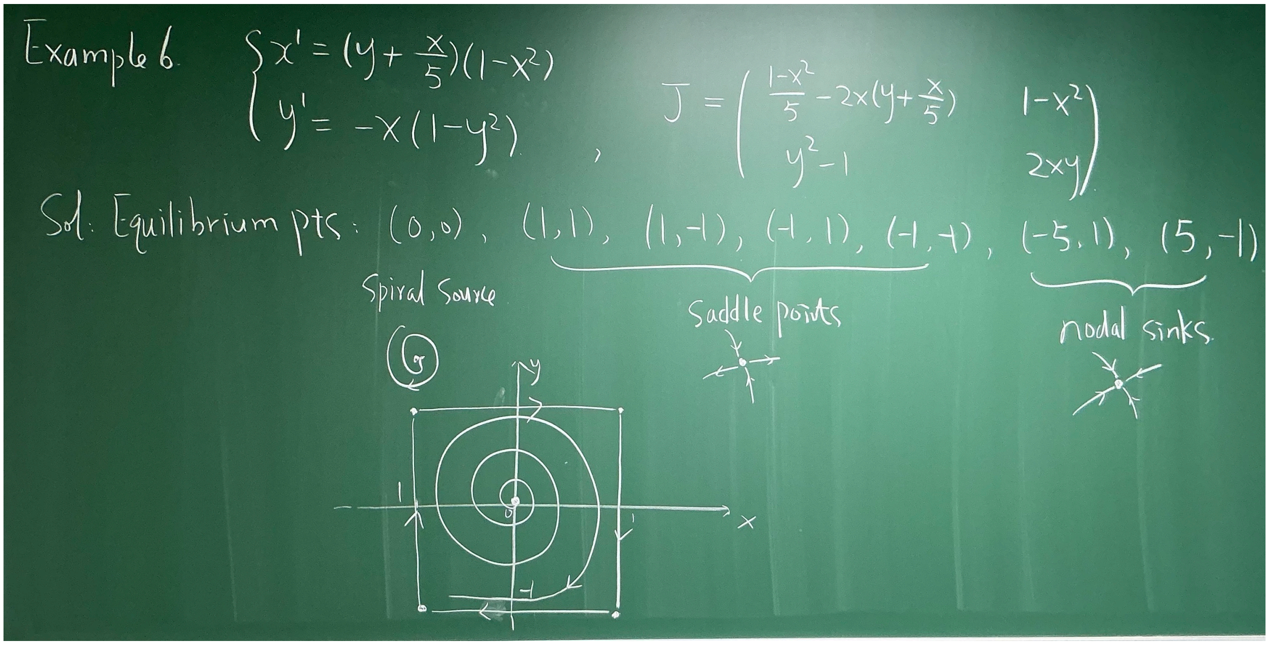
<u>Theorem 1. (Bendixon's Theorem)</u>

If ω is the limit set for a sol. to (A_2) in D.

then ω is one of the following:

- an equilibrium pt.
- a closed sol. curve (periodic sol.)
- a cyclic graph with vertices that are equilibrium pt.s and edges that are sol. curves, directed, forms a closed loop.

(Bendixon's Alternatives).

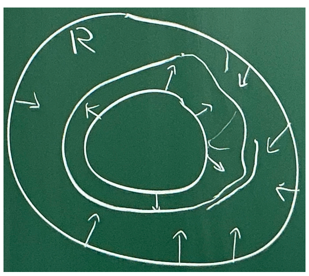


The boundary of the above square is a cyclic graph, which is the limit set of every sol. with $(x(0), y(0)) \neq (0,0)$ and $(x(0), y(0))$ inside the graph.

Theorem 2 (Poincaré–Bendixson Theorem).

Let $R \subseteq D$ be a closed and bounded region that is positively invariant for (A2).

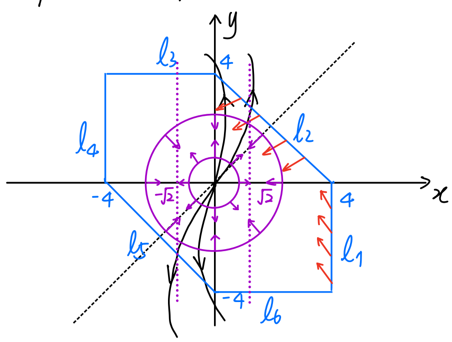
If R contains no equilibrium pt.s, then there is a closed sol. curve in R. (periodic sol.)



Recall Example 3: $\frac{dr}{dt} = r(1-r^2)$

<u>Example 7.</u> $\begin{cases} x' = 2x - y - x^3 \\ y' = x \end{cases}$ (the Vander Pol. system, to model the behaviour of an electric circuit).

Sol: Equilibrium pt.: $(0,0)$. unstable improper node. $\left(\lambda_{1,2} = 1,\ v = \begin{pmatrix}1\\1\end{pmatrix}\right)$



l_1: $x = 4,\ -4 \le y \le 0$

l_2: $x + y = 4,\ 0 \le x \le 4$

l_3: $y = 4,\ -4 \le x \le 0$

l_4: $x = -4,\ 0 \le y \le 4$

l_5: $x + y = -4,\ -4 \le x \le 0$

l_6: $y = -4,\ 0 \le x \le 4$

e.g. on l_1: $y' = 4 > 0$.

$x' = -56 - y < 0$.

on l_2: $y' = x \in [0, 4]$.

$x' = 2x - y - x^3$ $\quad (x+y)' = 3x - x^3 - y = -x^3 + 4x - 4 \le \frac{-8}{3\sqrt{3}} + \frac{8}{\sqrt{3}} - 4 < 0$

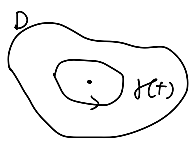
$$\frac{r\,dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = x^2(2 - x^2) > 0 \quad \text{if } x^2 < 2.$$

i.e. $\frac{dr}{dt} > 0$ if $x^2 < 2$.

Now, let R be the region enclosed by $l_1, \cdots, l_6$ with the unit ball centered at $(0,0)$ excluded.
Then R is pos. invariant & no equilibrium pts in R.

$\xRightarrow{\text{P-B}}$ $\exists$ a closed sol. curve in R.

<u>Theorem 3.</u> If $\gamma(t)$ is a closed orbit (sol. curve) in D, and the region enclosed by $\gamma(t)$ is also in D, then $\gamma(t)$ must enclose at least one critical pt. When it is unique, it can not be a saddle pt.



* This agrees with Example 1-5.

* No criticle pts in a simple connected $D \Rightarrow$ no periodic sol.s in D.

* The R in Thm 2 can not be simply connected.

<u>Theorem 4.</u> D– simply connected domain. $\vec{f}(\vec{X}) = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$.

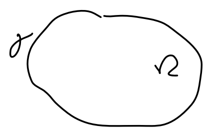
(Bendixson). If $\operatorname{div}\vec{f} = F_x + G_y < 0$ (or >0) in D, then $\nexists$ periodic sol.s in D.

(Dulac). If $\exists \mu(x,y): D \to \mathbb{R}$, C^1, s.t. $\operatorname{div}(\mu F, \mu G) < 0$ (or >0) in D then $\nexists$ periodic sol.s in D.

<u>Pf</u>: Argue by contradiction.

Suppose γ is a closed sol. curve in D. say $\gamma = \{(x(t), y(t)), t \in \mathbb{R}\}$.

Let Ω be the enclosed region by γ.



$$0 > \int_\Omega \operatorname{div}(\mu F, \mu G)\, dx dy \ \underline{\underline{\text{Green}}}\ \int_\gamma (-\mu G\, dx + \mu F\, dy)$$

(or $0 < \int_\Omega$)

$$= \int_0^T (-\mu G \underset{F}{\underset{\|}{\dot{x}}} dt + \mu F \underset{G}{\underset{\|}{\dot{y}}} dt) = 0. \quad \text{A contradiction.}$$

$\Rightarrow \nexists$ periodic sol. $\square$

<u>Example 8.</u> (Lotka–Volterra) $\begin{cases} \dot{x} = x(r - ax - by) \\ \dot{y} = y(-s + cx - dy) \end{cases}$ $\quad D = (0,\infty) \times (0,\infty)$, $r, a, b, s, c, d > 0$.

Sol: $\mu(x,y) = \frac{1}{xy}$. $\operatorname{div}(\mu\dot{x}, \mu\dot{y}) = \operatorname{div}\left(\frac{r - ax - by}{y}, \frac{-s + cx - dy}{x}\right)$

$$= -\frac{a}{y} - \frac{d}{x} < 0.$$

$\xRightarrow{\text{Dulac}}$ no periodic sol. in D.

§7 Sturm–Liouville <u>Boundary Value</u> Problems.

<u>E.g. 1</u> (∗1) $\begin{cases} y'' + \lambda y = 0 & x \in (0,1) \\ y(0) = y(1) = 0 \end{cases}$ Find λ & $y \not\equiv 0$ s.t. (∗1) holds.

IC: $y(0) = y_0$, $y'(0) = y_0'$

<u>Sol</u>: Claim: $\lambda > 0$

Why: $-y''y = \lambda y^2 \Rightarrow \int_0^1 [-(y'y)' + (y')^2]\, dx = \lambda \int_0^1 y^2 dx$.

$$-\underset{0}{y'y(1)} + \underset{0}{y'y(0)} + \underbrace{\int_0^1 (y')^2 dx}_{>0} = \lambda \underbrace{\int_0^1 y^2 dx}_{>0}. \Rightarrow \lambda > 0.$$

Now, char eq: $r^2 + \lambda = 0$, $r_{1,2} = \pm\sqrt{\lambda}\, i$.

$\Rightarrow y_g = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$.

B.C. $y(0) = C_1 = 0$, $y(1) = C_2 \sin\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = k\pi$, $k \in \mathbb{N}$, $k > 0$.

$\Rightarrow \lambda_k = k^2\pi^2$. $k = 1, 2, \dots$

e-value and $y_{\lambda_k} = \sin(k\pi x)$.

e-func

$A_{n\times n}\vec{y} = \lambda\vec{y}$, $\vec{y} \neq \vec{0}$.

$(\vec{y_1}, \vec{y_2}) = \vec{y_1} \cdot \overline{\vec{y_2}}$

$(A\vec{y_1}, \vec{y_2}) = (\vec{y_1}, A\vec{y_2})$ iff $A = A^\dagger$

Stum-Liouville boundary value problem

$$(SL)\begin{cases} (p(x)y')' - q(x)y + \lambda r(x) y = 0, & x \in (0,1). \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0, & \beta_1 y(1) + \beta_2 y'(1) = 0. \end{cases}$$

$p(x) \in C'([0,1])$, $q(x), r(x) \in C([0,1])$

$p(x) > 0$, $r(x) > 0$ in $[0,1]$.

$\alpha_2 = \beta_2 = 0$: $y(0) = y(1) = 0$. Dirichlet B.C.

$\alpha_1 = \beta_1 = 0$: $y'(0) = y'(1) = 0$. Neumann B.C.

$\alpha_i, \beta_i \neq 0$: Robin B.C.

(all requiring $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$)

In E.g. 1, $p(x) \equiv 1$, $q(x) \equiv 0$, $r(x) \equiv 1$.

$\alpha_2 = \beta_2 = 0$.

Differential operator:

$$L[y] := -(p(x)y')' + q(x)y.$$

Then $(SL)_1 \Leftrightarrow L[y] = \lambda r(x) y$. $r(x)$ – weight func.

Lagrange's Identity:

$\forall u, v \in C^1([0,1])$.

$$\boxed{\int_0^1 (L[u]v - uL[v])\, dx = -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_{x=0}^{x=1}} \quad (*2)$$

Pf: $L[u]v = -(p(x)u')' \cdot v + q(x)\cdot u\cdot v = -[p(x)u'\cdot v]' + p(x)u'v' + q(x)uv$

$uL[v] = -(p(x)v')'u + q(x)\cdot v\cdot u = -[p(x)v'u]' + p(x)v'u' + q(x)\cdot v\cdot u$.

$\int_0^1 (L[u]v - uL[v])\, dx = \int_0^1 -[p(x)u'v]' + [p(x)v'u]'\, dx$ = RHS of $(*2)$.

Now if u, v satisfy the B.C. $(SL)_2$: At $x = 1$: $-p(1)[u'(1)v(1) - u(1)v'(1)]$ 😊

$\beta_1^2 + \beta_2^2 \neq 0$: WLOG $\beta_1 \neq 0$, B.C. $\Rightarrow u(1) = -\frac{\beta_2}{\beta_1}u'(1)$. $v(1) = -\frac{\beta_2}{\beta_1}v'(1)$

$$\text{Then } ☺ = -p(1)\left[u'(1)\left(-\frac{\beta_2}{\beta_1}v'(1)\right) + \frac{\beta_2}{\beta_1}u'(1)\,v'(1)\right] = 0.$$

$\underline{(*2)} \Rightarrow$ $\underline{\int_0^1 L[u]\,v\,dx = \int_0^1 u\,L[v]\,dx, \ \forall u, v \in C^2([0,1]),\ u, v \text{ satisfy B.C. in (SL).}}$ $(*3)$

For $u, v \in C([0,1]; \mathbb{C})$, define inner product
$[0,1] \to \mathbb{C}$

$$(u, v) := \int_0^1 u(x)\cdot \bar{v}(x)\,dx, \quad \bar{v}(x): \text{conjugate of } v(x)$$

$(*3) \Rightarrow (L[u], v) = (u, L[v])$ still holds for complex-valued u, v, which satisfy the B.C. (SL).
$(*4)$

Such L is called self-adjoint.

<u>Theorem 1</u> All the eigenvalues of (SL) are real.

<u>Pf</u>: Suppose $L[\phi] = \lambda r(x)\phi$. λ, ϕ are possibly complex-valued & ϕ satisfies (SL).
(r is real-valued)

$(*4) \Rightarrow (L[\phi], \phi) = (\phi, L[\phi])$

$$\begin{array}{cc} \| & \| \\ (\lambda r(x)\phi, \phi) & (\phi, \lambda r(x)\phi) \\ \| & \| \\ \int_0^1 \lambda r(x)\phi\bar{\phi}\,dx & \int_0^1 \bar{\lambda} r(x)\phi\bar{\phi}\,dx. \end{array}$$

$$\Rightarrow (\lambda - \bar{\lambda})\underbrace{\int_0^1 \underset{>0}{r(x)}\, \underset{\geq 0,\ \not\equiv 0}{|\phi|^2}\,dx}_{>0} = 0. \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \text{ is real.}$$

<u>Theorem 2</u> The corresponding e-func. of (SL) are also real-valued and are of 1-dim.
(λ – simple).

<u>Pf</u>: Let (λ, ϕ) be an eigenpair and $\phi(x) = u(x) + i v(x)$.

Then $L[u + iv] = \lambda r(x)(u + iv)$. $\Rightarrow L[u] + iL[v] = \lambda r(x)u + i\lambda r(x)v$.

$\Rightarrow L[u] = \lambda r(x)u, \ L[v] = \lambda r(x)v.$

Both u & v are real-valued e-func of λ.

$$\text{Then} \begin{cases} p(x)u'' + p'(x)u' - q(x)u + \lambda r(x)u = 0. & ① \\ p(x)v'' + p'(x)v' - q(x)v + \lambda r(x)v = 0 & ② \end{cases}$$

② u − ① v

$p(x)(uv'' - u''v) + p'(x)(uv' - u'v) = 0$. $\quad W[u, v] := W(x)$
$= uv' - u'v$. $\quad W'(x) = uv'' - u''v$.

$\Rightarrow p(x)W'(x) + p'(x)W(x) = 0.$

$\Rightarrow [p(x)W(x)]' = 0 \Rightarrow W(x) = \dfrac{c}{p(x)}$
B.C. of $u, v \Rightarrow W(0) = W(1) = 0$

$\Rightarrow c = 0,\ W(x) \equiv 0$
$\Rightarrow (u, v)$'s wronskin is 0.
u, v are linearly dependent.

$\exists\, s \in \mathbb{R},\ u(x) = s\cdot v(x)$, $(s \neq 0)$ $\Rightarrow \phi(x) = s v(x) + i v(x) = (s + i)\,v(x).$

Theorem 3 (Othogonality).

Suppose (λ_m, ϕ_m) and (λ_n, ϕ_n) are eigen-pairs of (SL) with $\lambda_m \neq \lambda_n$.

Then $\int_0^1 \phi_m(x)\, \phi_n(x)\, r(x)\, dx = 0$.

Pf: $(*4) \Rightarrow \left(L[\phi_m], \phi_n\right) = (\phi_m, L[\phi_n])$

$$\left(L[\phi_m], \phi_n\right) = (\lambda_m r\phi_m, \phi_n) = \lambda_m \int_0^1 r(x)\phi_m\phi_n\, dx$$

$$(\phi_m, L[\phi_n]) = (\phi_m, \lambda_n r\phi_n) = \lambda_n \int_0^1 r(x)\phi_m\phi_n\, dx.$$

$$\Rightarrow \underbrace{(\lambda_m - \lambda_n)}_{\neq 0} \int_0^1 r(x)\phi_m\phi_n\, dx = 0. \quad \Rightarrow \int_0^1 r(x)\phi_m\phi_n\, dx = 0.$$

Theorem 4. All the eigen-values of (SL) can be listed as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lambda_n \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Theorem 5. ϕ_n changes sign exacted $(n-1)$ times in $(0,1)$

For each λ_n, may choose $\phi_n(x)$ s.t $\int_0^1 \phi_n^2(x)\, r(x)\, dx = 1$. normalization.

$\Rightarrow \{\phi_n\}_{n=1}^{\infty}$ othogonal set w.r.t the weight func. $r(x)$.

$$\int_0^1 \phi_m(x)\phi_n(x)r(x)\,dx = \delta_{mn} = \begin{cases} 1, & \text{if } m=n. \\ 0, & \text{if } m \neq n. \end{cases}$$

Theorem 6. Let $\{\phi_n\}_{n=1}^{\infty}$ be the normalized e-functions of (SL).

Let f, f' be piecewise continuous on $[0,1]$, then the series

$$\sum_{n=1}^{\infty} C_n \phi_n(x), \quad C_n = \int_0^1 f(x)\,\phi_n(x)\,r(x)\,dx$$

Converges to $\frac{1}{2}\left(f(x^+) + f(x^-)\right)$ at each $x \in (0,1)$.

Moreover, $\sum_{n=1}^{\infty} C_n \phi_n(x) \longrightarrow f(x)$ at each $x \in [0,1]$, provided f is continuous, f' is piecewise continuous on $[0,1]$, and f satisfies $f(0)=0$ when $\alpha_2 = 0$ and $f(1) = 0$ when $\beta_2 = 0$.