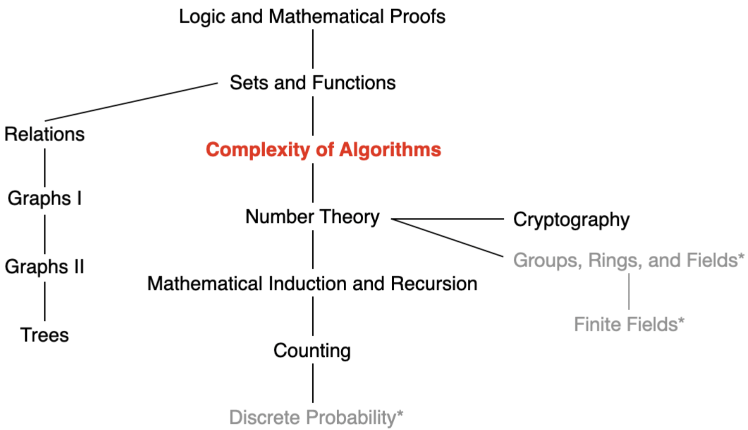
Discrete Mathematics for Computer Science

Lecture 6: Complexity of Algorithms

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

This Lecture



The growth of functions, complexity of algorithm, P and NP problem, ...

Algorithm

An algorithm is a finite sequence of precise instructions for performing a computation or for solving a problem.

ALGORITHM 4 The Bubble Sort.

procedure *bubblesort*$(a_1, \ldots, a_n :$ real numbers with $n \geq 2)$
for $i := 1$ **to** $n - 1$
  for $j := 1$ **to** $n - i$
    if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}
$\{a_1, \ldots, a_n$ is in increasing order$\}$

ALGORITHM 5 The Insertion Sort.

procedure *insertion sort*$(a_1, a_2, \ldots, a_n:$ real numbers with $n \geq 2)$
for $j := 2$ **to** n
  $i := 1$
  while $a_j > a_i$
    $i := i + 1$
  $m := a_j$
  for $k := 0$ **to** $j - i - 1$
    $a_{j-k} := a_{j-k-1}$
  $a_i := m$
$\{a_1, \ldots, a_n$ is in increasing order$\}$

How to compare algorithms with the same functionality?

- Time complexity: number of machine operations (e.g., additions)
- Space complexity: amount of memory needed

Before we get into details, the growth of functions ...

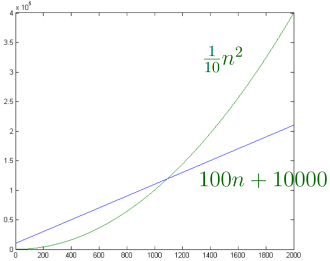
The Growth of Functions

Which function is "bigger", $\frac{1}{10}n^2$ or $100n + 10000$?

It depends on the value of n.

In Computer Science, we are usually interested in what happens when our problem input size gets large.

Notice that when n is "large enough", $\frac{1}{10}n^2$ gets much bigger than $100n + 10000$ and stays larger.



The Growth of Functions

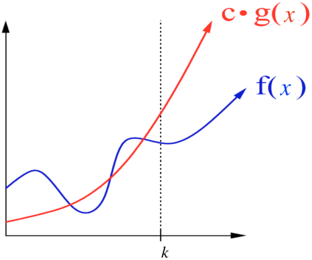
- Big-O notation, e.g., $O(n^2)$
- Big-Omega notation, e.g., $\Omega(n^2)$
- Big-Theta notation, e.g., $\Theta(n^2)$

Big-O Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|,$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]



Big-O Notation: Example

Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$.

Proof: We can readily estimate the size of $f(x)$ when $x > 1$:

$$0 \leq x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2.$$

This is because when $x > 1$, $x < x^2$ and $1 < x^2$. Thus, let $C = 4$, $k = 1$:

$$|f(x)| \leq C|x^2|, \text{ whenever } x > k.$$

Hence, $f(x) = O(x^2)$.

Note that there are multiple ways for proving this. Alternatively, we can estimate the size of $f(x)$ when $x > 2$:

$$0 \leq x^2 + 2x + 1 \leq x^2 + x^2 + x^2 = 3x^2.$$

It follows that $C = 3$, $k = 2$. ...

Big-O Notation: Example

Examples: The following formulas are all $O(x^2)$:

- $4x^2$
- $8x^2 + 2x - 3$
- $x^2/5 + \sqrt{x} - \log(x)$

Observe that in the relationship "$f(x)$ is $O(x^2)$," x^2 can be replaced by any function with "larger values" than x^2. For example,

- $f(x)$ is $O(x^3)$
- $f(x)$ is $O(x^2 + x + 7)$, ...

When $f(x)$ is $O(g(x))$, and $h(x)$ is a function that has larger absolute values than $g(x)$ does for sufficiently large values of x, it follows that

$$f(x) \text{ is } O(h(x)).$$

Big-O Estimates for Polynomials

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where $a_0, a_1, ..., a_n$ are real numbers. Then, $f(x) = O(x^n)$.

Proof:

Assuming $x > 1$, we have

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0| \\
&\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\
&= x^n (|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_0|/x^n) \\
&\leq x^n (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|).
\end{aligned}$$

The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-O Estimates for Some Functions

$$1 + 2 + \cdots + n = O(n^2)$$

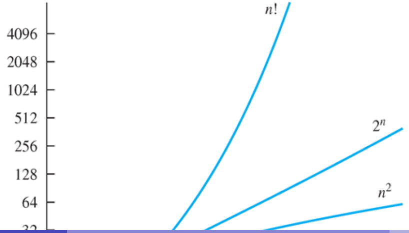
$$n! = O(n^n)$$

$$\log n! = O(n \log n)$$

$\log_a n = O(n)$ for an integer $a \geq 2$

$n^a = O(n^b)$ for integers $a \leq b$

$n^a = O(2^n)$ for an integer a



Big-O Estimates for Some Functions

Prove $\log_a n = O(n)$ for an integer $a \geq 2$.

Proof: We always have $\log_a n \leq n$ for $n \geq 1$. This can be proven using mathematical induction. ...

- $n = 1$: $log_a 1 = 0 < 1$
- Suppose $\log_a n \leq n$ for $n > 1$:

$$\log_a(n+1) \leq \log_a(an) = \log_a n + 1 \leq n + 1$$

Big-O Estimates for Some Functions

Prove $n^a = O(2^n)$ for an integer a.

Proof: According to L'Hopital's rule,

$$\lim_{n\to\infty} \frac{n^a}{2^n} = 0$$

Thus, $n^a \le 2^n$ for large enough n.

Note: If f and g are functions such that

$$\lim_{n\to\infty} \frac{|f(x)|}{|g(x)|} = C < \infty,$$

then $|f(x)| \le (C+1)|g(x)|$ for large enough x. So $f(n) = O(g(n))$. If that limit is ∞, then $f(n)$ is not $O(g(n))$.

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Combinations of Functions

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$.

Proof:

By definition, there exist constants C_1, C_2, k_1, k_2 such that $|f_1(x)| \leq C_1|g_1(x)|$ when $x > k_1$ and $|f_2(x)| \leq C_2|g_2(x)|$ when $x > k_2$. Then

$$
\begin{aligned}
|(f_1 + f_2)(x)| &= |f_1(x) + f_2(x)| \\
&\leq |f_1(x)| + |f_2(x)| \\
&\leq C_1|g_1(x)| + C_2|g_2(x)| \\
&\leq C_1|g(x)| + C_2|g(x)| \\
&= (C_1 + C_2)|g(x)| \\
&= C|g(x)|,
\end{aligned}
$$

where $g(x) = \max(|g_1(x)|, |g_2(x)|)$ and $C = C_1 + C_2$.

$k = \max\{k_1, k_2\}$.

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Combinations of Functions

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then $(f_1 f_2)(x) = O(g_1(x)g_2(x))$.

Proof:

When k $> \max(k_1, k_2)$,

$$\begin{aligned} |(f_1 f_2)(x)| &= |f_1(x)||f_2(x)| \\ &\leq C_1|g_1(x)|C_2|g_2(x)| \\ &\leq C_1 C_2|(g_1 g_2)(x)| \\ &\leq C|(g_1 g_2)(x)|, \end{aligned}$$

where $C = C_1 C_2$.

Big-Omega Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-Omega of $g(x)$."]

Big-O gives an upper bound on the growth of a function, while Big-Ω gives a lower bound.

Big-Ω tells us that a function grows at least as fast as another.

Note: $f(x)$ is $\Omega(g(x))$ if and only if $g(x)$ is $O(f(x))$.

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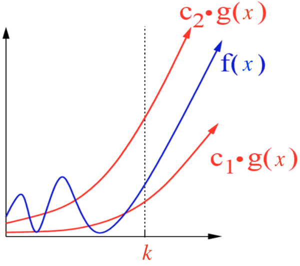
Big-Theta Notation (Big-O & Big-Omega)

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Theta(g(x))$ if

- $f(x)$ is $O(g(x))$ and
- $f(x)$ is $\Omega(g(x))$.

When $f(x)$ is $\Theta(g(x))$, we say that $f(x)$ is big-Theta of $g(x)$, that $f(x)$ is of order $g(x)$, and that $f(x)$ and $g(x)$ are of the same order.



Big-Theta Notation

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. Then $f(x)$ is of order x^n.

- $f(x) = O(x^n)$
- $f(x) = \Omega(x^n)$

Note: If f and g are functions such that

$$\lim_{n \to \infty} \frac{|f(x)|}{|g(x)|} = C < \infty,$$

and

$$C \neq 0,$$

then $f(n) = \Theta(g(n))$.

Big-Theta Notation: Examples

$3n^2 + 4n = \Theta(n)$?

$3n^2 + 4n = \Theta(n^2)$?

$3n^2 + 4n = \Theta(n^3)$?

$n/5 + 10n \log n = \Theta(n^2)$?

$n^2/5 + 10n \log n = \Theta(n \log n)$?

$n^2/5 + 10n \log n = \Theta(n^2)$?

$3n^2 + 4n = \Theta(n)$?	No
$3n^2 + 4n = \Theta(n^2)$?	Yes
$3n^2 + 4n = \Theta(n^3)$?	No, but $O(n^3)$
$n/5 + 10n \log n = \Theta(n^2)$?	No, but $O(n^2)$
$n^2/5 + 10n \log n = \Theta(n \log n)$?	No
$n^2/5 + 10n \log n = \Theta(n^2)$?	Yes

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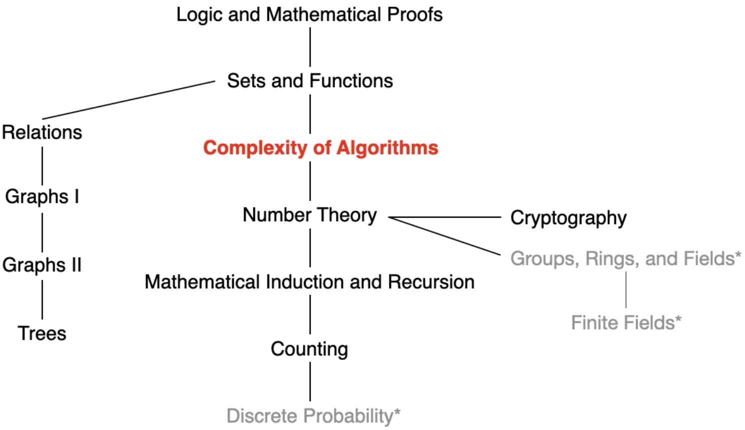
The Growth of Functions

- Big-O notation, e.g., $O(n^2)$
  - Upper bound
- Big-Omega notation, e.g., $\Omega(n^2)$
  - Lower bound
- Big-Theta notation, e.g., $\Theta(n^2)$
  - Of the same order

This Lecture



The growth of functions, complexity of algorithm, P and NP, ...

Algorithms

An algorithm is a finite sequence of precise instructions for performing a computation or for solving a problem.

A computational problem is a specification of the desired input-output relationship.

Example (Computational Problem and Algorithm):

- Computational Problem: Input n numbers $a_1, a_2, ..., a_n$; Output the sum of the n numbers.
- Algorithm: the following procedures

  Step 1: set $S = 0$

  Step 2: for $i = 1$ to n, replace S by $S + a_i$

  Step 3: output S

Instance

An instance of a problem is a realization of all the inputs needed to compute a solution to the problem.

Example: $8, 3, 6, 7, 1, 2, 9$

A correct algorithm halts with the correct output for every input instance. We can then say that the algorithm solves the problem.

Time and Space Complexity

- Time complexity: The number of machine operations (addition, multiplication, comparison, replacement, etc) needed in an algorithm.
- Space complexity: the amount of memory needed.

Example (Algorithm)

Step 1: set $S = 0$

Step 2: for $i = 1$ to n, replace S by $S + a_i$

Step 3: output S

Time Complexity:

- Steps 1 and 3 take one operation.
- Step 2 takes $2n$ operations.

Therefore, altogether this algorithm takes $1 + 2n + 1$ operations. The time complexity is $O(n)$.

Horner's Algorithm and Its Complexity

Example: Consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$. Direct computation takes 3 additions and 6 multiplications.

Can we do better?

Another way is $f(x) = 1 + x(2 + x(3 + 4x))$, which takes 3 additions and 3 multiplications.

Horner's algorithm for computing $f(x) = a_0 + a_1x + ... + a_{n-1}x^{n-1} + a_nx^n = a_0 + x(a_1 + ... + x(a_{n-1} + a_nx))$ at a particular x:

Step 1: set $S = a_n$

Step 2: for $i = 1$ to n, replace S by $a_{n-i} + Sx$

Step 3: output S

The number of operations needed in this algorithm is $1 + 3n + 1 = 3n + 2$. So the time complexity of this algorithm is $O(n)$.

Note: Operations: addition, multiplication, comparison, replacement, etc.

Time Complexity: Example

Determine the time complexity of the following algorithm:

for $i := 1$ to n

for $j := 1$ to n

$a := 2 * n + i * j;$

end for

end for

- Computing $a := 2 \times n + i \times j$ takes 4 operations (two multiplications, one addition, and one replacement).
- For each i, it takes $4n$ operations to complete the second loop.
- Thus, this algorithm takes $n \times 4n = 4n^2$ operations to complete the two loops. The time complexity of this algorithm is $O(n^2)$.

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Time Complexity: Example

Determine the time complexity of the following algorithm:

$S := 0$

for $i := 1$ to n

  for $j := 1$ to i

    $S := S + i * j;$

  end for

end for

- Computing $S := S + i \times j$ takes 3 operations.
- For each i , completing the second loop takes $3i$ operations.
- Thus, this algorithm takes

$$1 + \sum_{i=1}^{n} 3i = 1 + 3\frac{n(n+1)}{2}$$

So the complexity of this algorithm is $O(n^2)$.

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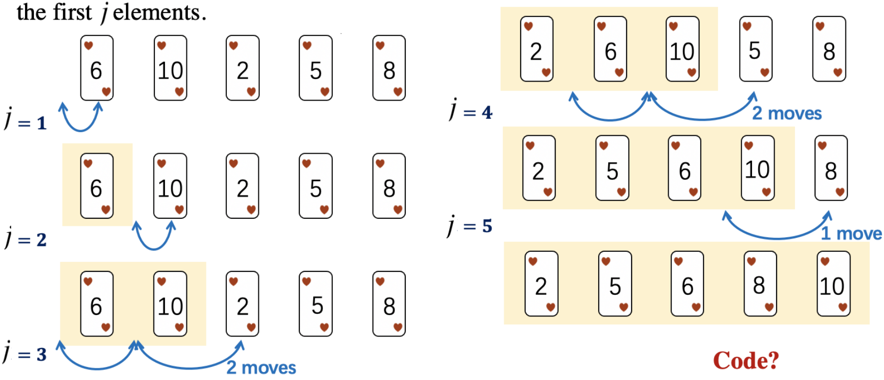
Time Complexity: Example - Insertion Sort

In iteration j, we move the j-th element left until its correct place is found among the first j elements.



Code?

Time Complexity: Example - Insertion Sort

Input: $A[1 \dots n]$ is an array of numbers

for $j := 2$ to n

  $key = A[j]$;

  $i = j - 1$;

  while $i \geq 1$ and $A[i] > key$ do

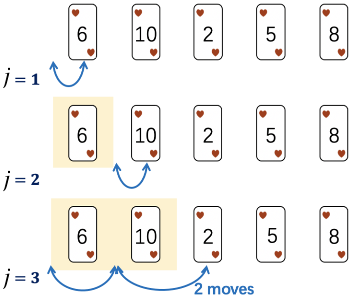
    $A[i+1] = A[i]$;

    $i - -$;

  end while

  $A[i+1] = key$;

end for



The time complexity depends on the input array $A[1, ..., n]$.

Consider only the number of comparisons

Three Cases of Analysis: Best-Case

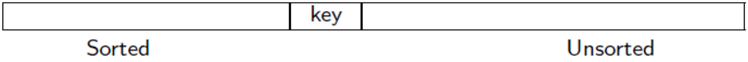
Best-Case Complexity: The smallest number of operations needed to solve the given problem using this algorithm on input of specified size.

Example: (Insertion Sort)

$$A[1] \le A[2] \le A[3] \le \cdots \le A[n]$$

The number of comparisons needed is

$$\underbrace{1 + 1 + 1 + \cdots + 1}_{n-1} = n - 1 = \Theta(n)$$



"key" is compared to only the element right before it.

Three Cases of Analysis: Worst-Case

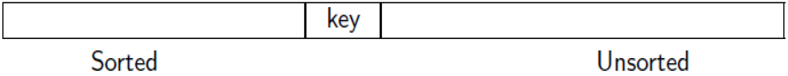
Worst-Case Complexity: The largest number of operations needed to solve the given problem using this algorithm on input of specified size.

Example: (Insertion Sort)

$$A[1] \geq A[2] \geq A[3] \geq \cdots \geq A[n]$$

The number of comparisons needed is

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} = \Theta(n^2)$$



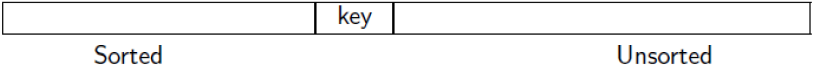
"key" is compared to everything element before it.

Three Cases of Analysis: Average-Case

Average-Case Complexity: The average number of operations used to solve the problem over all possible inputs of a given size is found in this type of analysis.

Example: (Insertion Sort)

$\Theta(n^2)$ assuming that each of the $n!$ instances are equally likely



On average, "key" is compared to half of the elements before it.

- For a particular instance, compute the number of comparisons
- Since we assume equal probability, take the average

Average-case complexity is usually difficult to compute.

Inversion

Definition

An **inversion** in an array of numbers is any ordered pair $(a[i], a[j])$ having the property that

$$i < j \text{ but } a[i] > a[j]$$

Example: $A = (34, 8, 64, 51, 32, 21)$ has nine inversions:
(34,8), (34,32), (34,21), (64,51), (64,32), (64,21), (51,32), (51,21), (32,21).

The number of inversions is exactly the number of swaps in insertion sort.

- swapping two adjacent elements that are out of place removes exactly one inversion
- a sorted array has no inversion

The running time of insertion sort is $O(I + N)$

- There is $O(N)$ other work involved in the algorithm.
- I is the number of inversions in the original array.
- Best case: the array in ascending order -> no inversion -> $O(N)$.

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Average Running Time

Average number of inversions -> Average running time of insertion sort

Assumption
- There is no duplicate elements.
- At any certain position, each integer appears with equal probability.

Theorem
The average number of inversions in an array of N distinct integers is $N(N-1)/4$.

Average Running Time

Average number of inversions -> Average running time of insertion sort

Theorem
The average number of inversions in an array of N distinct integers is $N(N-1)/4$.

Proof:

- Consider an array A and the array in reverse order, denoted by A_r. For example, if $A = (34, 8, 64, 51, 32, 21)$, then $A_r = (21, 32, 51, 64, 8, 34)$.
- Consider any ordered pair $(a[i], a[j])$ with $i < j$. This pair is an inversion in either A and A_r with equal probability.
- There are a total of $C(N, 2) = N(N-1)/2$ such pairs.
- Thus, on average, array A has $C(N, 2)/2 = N(N-1)/4$ inversions.

Average running time: $O(I + N) = O(N^2)$.

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Some Thoughts on Algorithm Design

Algorithm Design is mainly about designing algorithms that have small Big-O running time.

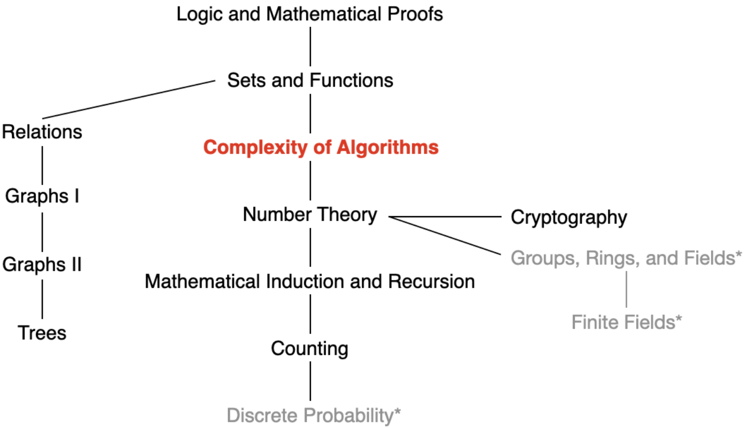
Being able to do good algorithm design lets you identify the hard parts of your problem and deal with them effectively.

Too often, programmers try to solve problems using brute force techniques and end up with slow complicated code!

- The most straightforward manner based on the statement of the problem and the definitions of terms

A few hours of abstract thought devoted to algorithm design could speed up the solution substantially and simplified it!

This Lecture



The growth of functions, complexity of algorithm, P and NP problem, ...

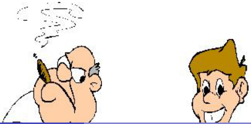
Dealing with Hard Problems

What happens if you cannot find an efficient algorithm for a given problem?

Blame yourself.

I couldn't find a polynomial-time algorithm.
I guess I am too dumb.

Show that no-efficient algorithm exists.



Dealing with Hard Problems

Showing that a problem has an efficient algorithm is, relatively easy:

- Design such an algorithm.

Proving that no efficient algorithm exists for a particular problem is difficult:

How can we prove the non-existence of something?

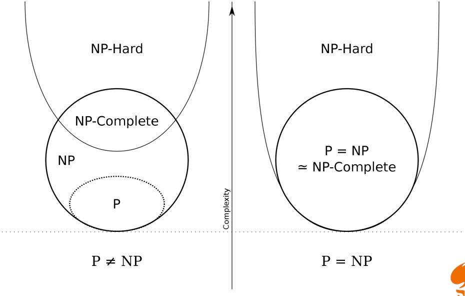
We will now learn about NP-Complete problems, which provides us with a way to approach this question.

NP-Complete

P: Problems that are solvable using an algorithm with polynomial worst-case complexity

NP: Problems for which a solution can be checked in polynomial time.

NP-Complete: If any of these problems can be solved by a polynomial worst-case time algorithm, then all problems in the class NP can be solved by polynomial worst-case time algorithms.



NP-Complete

Researchers have spent many years trying to find efficient solutions to these problems but failed.

NP-Complete and NP-Hard problems are very likely to be hard.

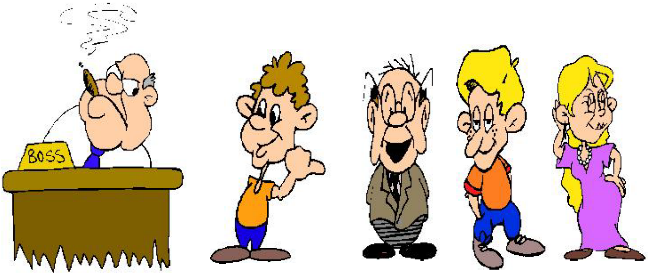
Thus, to proving that no efficient algorithm exists for a particular problem?

Prove that your problem is NP-Complete or even NP-Hard:

- Show that your problem can be reduced to a typical (well-known) NP-Complete or NP-Hard problem.

NP-Complete

What do you actually do:



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people!