Scheme of Riemann Surface of Arcsin z

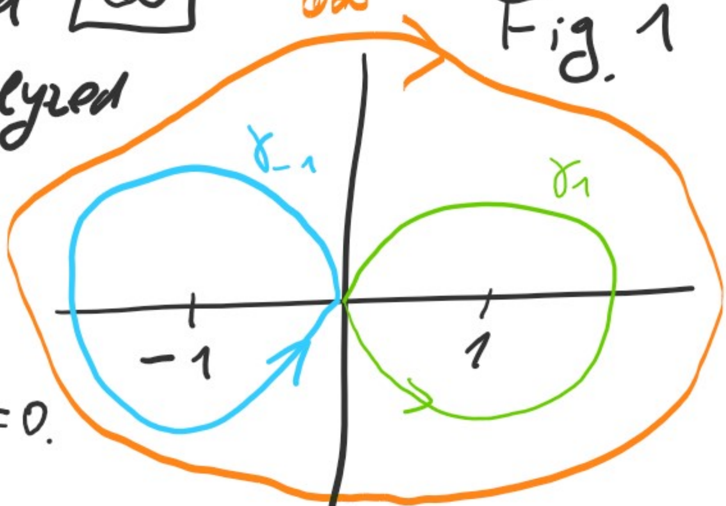
Wednesday, November 10, 2021 8:38 AM

$$W = F(z) = \operatorname{Arcsin} z = i\,\operatorname{Ln}\left(iz + \sqrt{1-z^2}\right)$$

Branch points: $iz + \sqrt{1-z^2} \neq 0$ in $\mathbb{C}$ ($\forall$ Branch of $\sqrt{\ }$),

so we get $\boxed{\pm 1}$ and $\boxed{\infty}$

Fig. 1



The point ∞ will be analyzed using ± 1, so we deal with these two.

We choose base point $z_0 = 0$.

All possible elements at 0 we get by fixing branch of $\sqrt{\ }$ and then branch of Ln.

For example, in the unit disc $B_1(0) = B$.

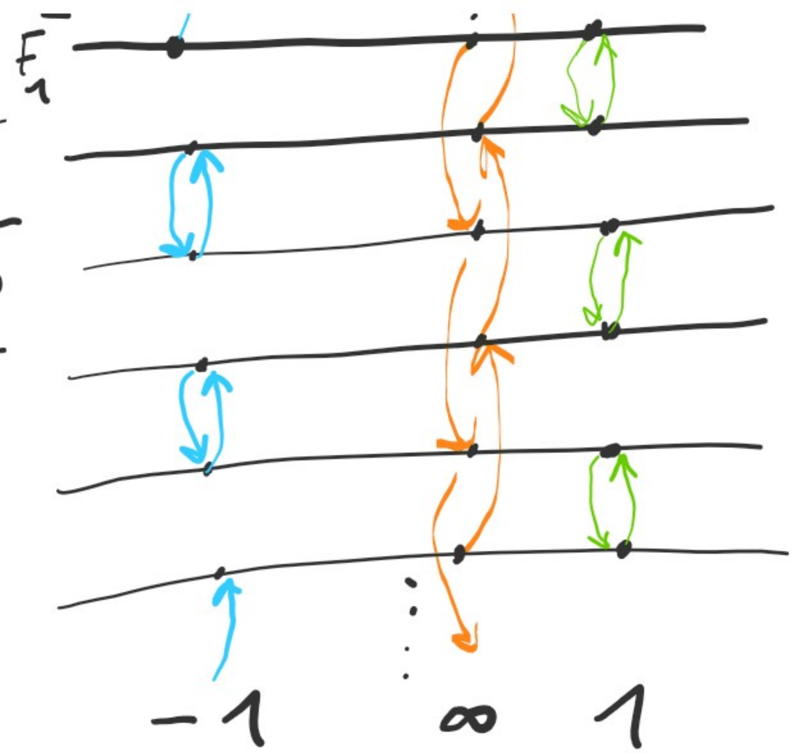
We will arrange the elements (they correspond) to sheets of the Riemann surface like that:

For the branch of $\sqrt{\ }$ with $\sqrt{1} = 1$, (so that $iz + \sqrt{1-z^2} = 1$ for $z=0$) we denote by F_k^+ the branch that corresponds to the choice $\operatorname{Ln} 1 = 2\pi k i$; for the choice $\sqrt{1} = -1$, we similarly define F_k^- corresponding to $\operatorname{Ln}(-1) = \pi(1+2k)i$.

In accordance with that, we display the sheets in the scheme:

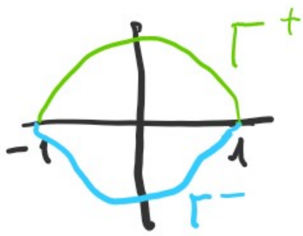
Fig. 2

the sheets in the scheme:
(They of course go infinitely up and down).
We fix simple loops γ_1, γ_{-1} as on fig 1.
To analyze the action of γ_1 on F_0^+, we



here we precisely mean the analytic exten of $\sqrt{1-z^2}$ along the parameterized curve $\gamma_1: t \to 1+e^{it}, 0 \le t \le 2\pi$

compute directly that the map $z \to iz + \sqrt{1-z^2}$ maps γ_1 onto a curve Γ^+ like that:



Performing the respective analytic extension of the Ln along Γ^+, we see that F_0^+ arrives finally to F_0^-; similarly, further extension along γ_1 (i.e. when considering $(\gamma_1)^2$), brings us back to the original element F_0^+ (we consider the curve Γ^- on the picture for this purpose).

Similarly, we get $F_k^+ \xrightarrow{\gamma_1} F_k^- \xrightarrow{\gamma_1} F_k^+$

According to this, we display on Fig. 2 the

According to this, we display on Fig. 2 the arrows above the point $z = 1$.
So, $z=1$ contains ∞ branch points of ord $= 2$.

We then argue completely analogously, and get the arrows at $z = -1$ (Fig. 2). This time, γ_{-1} is mapped on Γ^-, and analyzing the anal extens of Ln along Γ^-, we see that F_0^- arrives to F_1^+ (!)
This is because $Ln(-1) = \pi i$ "analytically extends" to $Ln\, 1 = 2\pi i$, when walking along Γ^-.
(So, $z=-1$ also contains ∞ branch points of ord$=2$).
Analyzing the ∞ is then easy: we note that the simple loop γ_∞ around ∞ (Fig. 1) is homotopic to the product $(\gamma_{-1})^{-1}(\gamma_1)^{-1}$

So, extension of an elem along γ_∞ amounts to two extensions (in the negative direction) along γ_1 and γ_{-1}, successively. This gives the arrows for $z = \infty$ on Fig. 2, and the final scheme. The point $z=\infty$ contains then two logarithmic branch points. The first corresponds to the starting element F_0^+, the second corresponds to the starting element F_0^+, the second to F_0^-.