$\boxed{5.3}$

Solution:

(a) The n samples here should be independent.

$$Y = \sum_{i=1}^{n} X_i \sim Gamma(2n, \theta).$$

(b) $L(\theta) = \prod_{i=1}^{n}\left(\frac{\theta^2}{\Gamma(2)} x_i e^{-\theta x_i}\right) = \frac{\theta^{2n}}{(\Gamma(2))^n}\left(\prod_{i=1}^{n} x_i\right) e^{-\theta \sum_{i=1}^{n} x_i}.$

let $\frac{L(\theta_0)}{L(\theta_1)} = \frac{\theta_0^{2n}}{\theta_1^{2n}} e^{-\theta_0 \sum_{i=1}^{n} x_i + \theta_1 \sum_{i=1}^{n} x_i} \le k.$ $(*)$

Since $\theta_1 > 1 = \theta_0$. $\theta_1 - \theta_0 > 0$.

$(*) \Leftrightarrow \quad \sum_{i=1}^{n} x_i \le \frac{\log\left(\frac{\theta_1^{2n}}{\theta_0^{2n}} k\right)}{\theta_1 - \theta_0} \triangleq c.$

When H_0 is true we have $\sum_{i=1}^{n} X_i \sim Gamma(2n, \theta)$.

$$\alpha = Pr(\vec{x} \in \mathbb{C} \mid \theta = \theta_0 = 1) = Pr\left(\sum_{i=1}^{n} x_i \le c \mid \theta = \theta_0 = 1\right)$$

$$= Pr(Gamma(2n, 1) \le c) = Pr(\chi^2(4n) \le 2c)$$

$$= Pr(\chi^2(4n) \le \chi^2(1-\alpha, 4n))$$

$\Rightarrow c = \frac{1}{2}\chi^2(1-\alpha, 4n)$

$\Rightarrow$ The test with critical region $\mathbb{C} = \{\vec{x} : \sum_{i=1}^{n} x_i \le \frac{1}{2}\chi^2(1-\alpha, 4n)\}$.

is the most powerful test of size α.

(c)

$p(\theta) = Pr(\vec{x} \in \mathbb{C} \mid \theta)$

$$= Pr\left(\sum_{i=1}^{n} X_i \le Gamma(1-\alpha, 2n, 1) \mid \theta\right)$$

$$= \int_{-\infty}^{\chi^2(1-\alpha, 4n)} \frac{\theta^{2n}}{\Gamma(2n)} x^{2n-1} e^{-\theta x}\, dx.$$

$\boxed{5.4}$

Solution:

(a) $L(\theta) = \theta^n \prod_{i=1}^{n}\left[(1-x_i)^{\theta-1}\mathbb{1}(0<x_i<1)\right]$.

Let $\dfrac{L(\theta_0)}{L(\theta_1)} = \dfrac{\theta_0^n}{\theta_1^n}\prod_{i=1}^{n}(1-x_i)^{\theta_0-\theta_1} \leq k$

Since $\theta_1 > \theta_0 = 1$,

$$(\theta_0-\theta_1)\sum_{i=1}^{n}\log(1-x_i) \leq \log\left(\frac{k\,\theta_1^n}{\theta_0^n}\right).$$

$$\sum_{i=1}^{n}\log(1-x_i) \geq \frac{1}{\theta_0-\theta_1}\log\left(\frac{k\,\theta_1^n}{\theta_0^n}\right) \triangleq C.$$

Note that for $Y_i \triangleq -\log(1-x_i)$

$Y_i \sim \text{Exponential}(\theta)$.

$$\Rightarrow -\sum_{i=1}^{n}\log(1-x_i) = \sum_{i=1}^{n}Y_i \sim \text{Gamma}(n, \theta).$$

$$\alpha = \Pr(\vec{x}\in \mathbb{C} \mid \theta=\theta_0=1) = \Pr\left(\sum_{i=1}^{n}Y_i \leq -c \mid \theta=\theta_0=1\right)$$

$$= \Pr(\text{Gamma}(n,1) \leq -c) = \Pr(\chi^2(2n) \leq -2c) = \Pr(\chi^2(2n) \leq \chi^2(1-\alpha, 2n))$$

$$\Rightarrow -2c = \chi^2(1-\alpha, 2n)$$

$$c = -\frac{1}{2}\chi^2(1-\alpha, 2n)$$

$\Rightarrow$ The test with critical region $\left\{\vec{x}: \sum_{i=1}^{n}\log(1-X_i) \geq -\frac{1}{2}\chi^2(1-\alpha, 2n)\right\}$.

is the most powerful test of size α.

(b). $\Theta_0 = \{1\}$. $\Theta_1 = (0,1)\cup(1,+\infty)$ $\quad \Theta = (0,+\infty) \subseteq \mathbb{R} = \Theta^*$.

Step 1: Calculate $\lambda(\vec{x})$:

$$L(\theta) = \prod_{i=1}^{n}\left[\theta(1-x_i)^{\theta-1}\mathbb{1}(0<x_i<1)\right].$$

$$= \theta^n \exp\left\{(\theta-1)\sum_{i=1}^{n}\log(1-x_i)\right\}$$

$$\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log(1-x_i)$$

$$\frac{d\log L(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(1-x_i)$$

$$\Rightarrow \hat{\theta} = \frac{-n}{\sum_{i=1}^{n}\log(1-X_i)} \quad \left(\sum_{i=1}^{n}\log(1-X_i) < 0\right)$$

$$\frac{\sup\limits_{\theta\in\Theta_0} L(\theta)}{\sup\limits_{\theta\in\Theta} L(\theta)} = \frac{1}{\hat{\theta}^{n}\left(\prod\limits_{i=1}^{n}(1-x_i)\right)^{\hat{\theta}-1}}$$

Step 2: Find the critical region $\mathbb{C}$.

$$\mathbb{C} = \left\{ \frac{1}{\hat{\theta}^{n}\left(\prod\limits_{i=1}^{n}(1-x_i)\right)^{\hat{\theta}-1}} \le \lambda_\alpha \right\}. \qquad T(\vec{X}) = -\sum_{i=1}^{n} \log(1-X_i) \qquad \hat{\theta} = \frac{n}{t}.$$

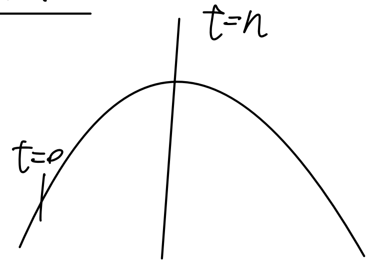
$$= \left\{ \frac{1}{\frac{n^n}{t^n} e^{-(\frac{n}{t}-1)t}} \le \lambda_\alpha \right\} = \left\{ \frac{1}{n^n} t^n e^{n-t} \le \lambda_\alpha \right\}.$$

$$= \left\{ \vec{x} : t>0,\ h(t) \le \lambda_\alpha \right\}.$$

Step 2(a): Check if or not $h(t)$ is log-concave.

$$H(t) \triangleq \log h(t) = n\log t + t - n - n\log n$$

$$H'(t) = \frac{n}{t} - 1 \qquad H''(t) = -\frac{n}{t^2} < 0$$



$\Rightarrow h(t)$ is strictly log-concave (attain its maximum at $t = n$).

Step 2(b): Find equivalent $\mathbb{C}$.

$$t>0,\ h(t) \le \lambda_\alpha \iff 0<t\le C_1 \ \text{ or } \ t \ge C_2.$$

$$\mathbb{C} = \left\{ \vec{x} : 0 < T \le C_1 \ \text{ or } \ T \ge C_2 \right\}.$$

Step 2(c): Determine C_1 & C_2

$$\alpha = \Pr\{T \le C_1 \mid \theta = \theta_0\} + \Pr\{T \ge C_2 \mid \theta = \theta_0\}$$

We use equal-tail approach.

Let $\frac{\alpha}{2} = \Pr\{T \le C_1 \mid \theta=\theta_0\} = \Pr\{T \ge C_2 \mid \theta = \theta_0\}$

Since $T \sim$ Gamma(n, θ) by (a)

Under H_0, $T \sim$ Gamma$(n, 1)$

$$2T \sim \text{Gamma}(n, \tfrac{1}{2}) = \chi^2(2n)$$

$$\Rightarrow \frac{\alpha}{2} = \Pr\{\chi^2(2n) \le 2C_1\} = \Pr\{\chi^2(2n) \le \chi^2(1-\tfrac{\alpha}{2}, 2n)\}.$$

$$= \Pr\{\chi^2(2n) \ge 2C_2\} = \Pr\{\chi^2(2n) \ge \chi^2(\tfrac{\alpha}{2}, 2n)\}.$$

$$\Rightarrow C_1 = \frac{1}{2}\chi^2(1-\tfrac{\alpha}{2}, 2n), \quad C_2 = \frac{1}{2}\chi^2(\tfrac{\alpha}{2}, 2n).$$

$\Rightarrow$ The LRT with size α is the test with critical region

$$C = \{\vec{x} : 0 < -\sum_{i=1}^{n} \log(1-X_i) \le \frac{1}{2}\chi^2(1-\tfrac{\alpha}{2}, 2n)$$

or

$$-\sum_{i=1}^{n} \log(1-X_i) \ge \frac{1}{2}\chi^2(\tfrac{\alpha}{2}, 2n)\}.$$

$\boxed{5.5}$

Solution: Step 1: Consider test

$$H_{0S}: \theta = \theta_0 \text{ against } H_{1S}: \theta = \theta_1 < \theta_0. \quad (*)$$

$$L(\theta) = (2\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}.$$

$$\frac{L(\theta_0)}{L(\theta_1)} = \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta_0)^2 + \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right\}.$$

$$= \exp\left\{\frac{1}{2}\sum_{i=1}^{n}(2x_i-\theta_0-\theta_1)(\theta_0-\theta_1)\right\}$$

$$= \exp\left\{(\theta_0-\theta_1)\sum_{i=1}^{n}x_i - \frac{n}{2}(\theta_0-\theta_1)^2\right\}. \le k.$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} x_i \le \frac{\log k + \frac{n}{2}(\theta_0 - \theta_1)^2}{(\theta_0 - \theta_1)} \triangleq C.$$

Since $\sum_{i=1}^{n} X_i \sim N(n\theta, n)$. $\frac{\sum_{i=1}^{n} X_i - n\theta}{\sqrt{n}} \sim N(0, 1)$

$$\alpha = P_r(\vec{x} \in \mathbb{C} \mid \theta = \theta_0) = P_r\left(\frac{\sum_{i=1}^{n} X_i - n\theta_0}{\sqrt{n}} \le \frac{c - n\theta_0}{\sqrt{n}} \right)$$

$$= P_r\left\{ \frac{\sum_{i=1}^{n} X_i - n\theta_0}{\sqrt{n}} \le Z_{1-\alpha} \right\}$$

$$\Rightarrow \quad \frac{c - n\theta_0}{\sqrt{n}} = Z_{1-\alpha} = -Z_\alpha. \quad \Rightarrow \quad c = n\theta_0 - \sqrt{n}\, Z_\alpha.$$

$\Rightarrow$ the test φ with critical region

$C = \left\{ \vec{x} : \sum_{i=1}^{n} x_i \le n\theta_0 - \sqrt{n} \cdot Z_\alpha \right\}$ is a MPT with size α.

<u>Step 2:</u> Consider the test

$H_{0s}: \theta = \theta_0$ against $H_1: \theta < \theta_0$.

Since the $\mathbb{C}$ above depends only on n, θ_0, α and the fact that $\theta_1 < \theta_0$, but not on the value of θ_1.

$\Rightarrow$ The test φ is also the UMPT of size α for testing

$H_{0s}: \theta = \theta_0$ against $H_1: \theta < \theta_0$

<u>Step 3:</u>

$$\sup_{\theta \ge \theta_0} P_\varphi(\theta) = \sup_{\theta \ge \theta_0} P_r\left\{ \sum_{i=1}^{n} X_i \ge c \mid \theta \right\}$$

$$= \sup_{\theta \ge \theta_0} P_r\left\{ \frac{\sum_{i=1}^{n} X_i - n\theta}{\sqrt{n}} \le \frac{c - n\theta}{\sqrt{n}} \right\}$$

$$= \sup_{\theta \ge \theta_0} P_r\left\{ N(0,1) \le \frac{c - n\theta}{\sqrt{n}} \right\} = P_r\left\{ N(0,1) \le \frac{c - n\theta_0}{\sqrt{n}} \right\}$$

$$= P_r\left\{ N(0,1) \le Z_{-\alpha} \right\} = \alpha$$

Then, the test φ is also the UMPT of size α for testing

$$H_0: \theta \geq \theta_0, \quad H_1: \theta < \theta_0.$$

$\boxed{5.6}$

Solution:

(a) We first consider $H_0: \sigma^2 = \sigma_0^2$ against $H_1: \sigma^2 \neq \sigma_0^2$.

$$L(\mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]}.$$

$$\Theta_0 = \{(\mu, \sigma^2): -\infty < \mu < +\infty, \ \sigma^2 = \sigma_0^2\}. \quad \Theta = \{(\mu, \sigma^2): -\infty < \mu < +\infty, \ \sigma^2 \geq 0\}.$$

$$l(\mu, \sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2.$$

let
$$\begin{cases} \dfrac{\partial l}{\partial \mu} = \dfrac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0 \\ \dfrac{\partial l}{\partial \sigma^2} = -\dfrac{n}{2}\dfrac{1}{\sigma^2} + \dfrac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 = 0 \end{cases} \Rightarrow \begin{cases} \mu = \dfrac{1}{n}\sum_{i=1}^{n}x_i = \bar{x}. \\ \sigma^2 = \dfrac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \end{cases}$$

$$\Rightarrow \lambda(\vec{x}) = \frac{(2\pi\sigma_0^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)}{\left[2\pi \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-\frac{n}{2}} \exp\left[-\frac{n}{2\cancel{\sum_{i=1}^{n}(x_i-\bar{x})^2}}\cancel{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]}$$

$$= \left[\frac{(n-1)S^2}{n\sigma_0^2}\right]^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_0^2}(n-1)S^2 + \frac{n}{2}\right) \triangleq f(S^2)$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2.$$

The critical region that H_0 is rejected is

$$C = \{\vec{x}: \lambda(\vec{x}) \leq \lambda_\alpha\}$$

$$= \left\{\vec{x}: \left[\frac{(n-1)S^2}{n\sigma_0^2}\right]^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_0^2}(n-1)S^2 + \frac{n}{2}\right) \leq \lambda_\alpha\right\}.$$

$$= \left\{\vec{x}: \frac{n}{2}\log\frac{n-1}{n\sigma_0^2} + \frac{n}{2}\log S^2 - \frac{1}{2\sigma_0^2}(n-1)S^2 + \frac{n}{2} \leq \log\lambda_\alpha\right\}$$

Let $g(s^2) = \log f(s^2)$. $\quad C = \{\vec{x} : g(s^2) \le \log \lambda_\alpha\}$.

$$\frac{d\,g(s^2)}{d\,s^2} = \frac{n}{2}\frac{1}{s^2} - \frac{1}{2\sigma_0^2}(n-1). \qquad \frac{d^2 g(s^2)}{d(s^2)^2} = -\frac{n}{2}\frac{1}{(s^2)^2} < 0.$$

$\Rightarrow f(s^2)$ is log-concave and attains its maximum at $\frac{n}{n-1}\sigma_0^2$.

$\Rightarrow \quad f(s^2) \le \lambda_\alpha \Leftrightarrow \quad 0 < s^2 \le C_1$ or $s^2 \ge C_2$.

We select equal-tail approach.

$$\frac{\alpha}{2} = \Pr(0 < s^2 \le C_1 \mid \sigma^2 = \sigma_0^2)$$

$$= \Pr\left(0 < \frac{(n-1)s^2}{\sigma_0^2} \le \frac{(n-1)C_1}{\sigma_0^2}\right)$$

$$= \Pr\left(0 < \chi^2(n-1) \le \frac{(n-1)C_1}{\sigma_0^2}\right).$$

$$= \Pr\left(0 < \chi^2(n-1) \le \chi^2(1-\tfrac{\alpha}{2}, n-1)\right).$$

$$\Rightarrow \frac{(n-1)C_1}{\sigma_0^2} = \chi^2(1-\tfrac{\alpha}{2}, n-1). \qquad C_1 = \frac{\sigma_0^2}{n-1}\chi^2(1-\tfrac{\alpha}{2}, n-1).$$

$$\frac{\alpha}{2} = \Pr(s^2 \ge C_2 \mid \sigma^2 = \sigma_0^2)$$

$$= \Pr\left(\chi^2(n-1) \ge \frac{(n-1)C_2}{\sigma_0^2}\right)$$

$$= \Pr\left(\chi^2(n-1) \ge \chi^2(\tfrac{\alpha}{2}, n-1)\right) \quad \Rightarrow C_2 = \frac{\sigma_0^2}{n-1}\chi^2(\tfrac{\alpha}{2}, n-1).$$

$\Rightarrow$ The critical region is

$$C = \left\{\vec{x} : 0 < s^2 \le \frac{\sigma_0^2}{n-1}\chi^2(1-\tfrac{\alpha}{2}, n-1) \text{ or } s^2 \ge \frac{\sigma_0^2}{n-1}\chi^2(\tfrac{\alpha}{2}, n-1)\right\}.$$

(b). Then we consider

$$H_0 : \sigma^2 = \sigma_0^2 \text{ against } H_1 : \sigma^2 > \sigma_0^2.$$

The critical region is $C = \left\{\vec{x} : s^2 \ge \frac{\sigma_0^2}{n-1}\chi^2(\alpha, n-1)\right\}$.

(c) Finally, consider

$$H_0 : \sigma^2 = \sigma_0^2 \text{ against } H_1 : \sigma^2 < \sigma_0^2.$$

The critical region is $\mathbb{C} = \{ \vec{x} : 0 < S^2 \le \frac{\sigma_0^2}{n-1} \chi^2(1-\alpha, n-1) \}$.

$\boxed{5.7}$

<u>Solution:</u>

$\vec{\theta} = (\mu_1, \mu_2, \sigma^2)^T$, $\Theta_0 = \{ (0, 0, \sigma^2)^T, \sigma^2 > 0 \}$, $\Theta = \{ (\mu_1, \mu_2, \sigma^2)^T, \mu_1, \mu_2 \in R, \sigma^2 > 0 \}$.

$H_0: \mu_1 = \mu_2 = 0$ against $H_1: \mu_1 \ne \mu_2$ or $\mu_1 = \mu_2 \ne 0$

$$L(\mu_1, \mu_2, \sigma^2) = (2\pi\sigma^2)^{-n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_1)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu_2)^2 \right].$$

$$\begin{aligned} l(\mu_1, \mu_2, \sigma^2) &= \log L(\mu_1, \mu_2, \sigma^2) \\ &= -n \log 2\pi - n \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} [(x_i - \mu_1)^2 + (y_i - \mu_2)^2]. \end{aligned}$$

$$\text{let} \begin{cases} \dfrac{\partial l}{\partial \mu_1} = \dfrac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu_1) = 0 \\ \dfrac{\partial l}{\partial \mu_2} = \dfrac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \mu_2) = 0 \\ \dfrac{\partial l}{\partial \sigma^2} = -\dfrac{n}{\sigma^2} + \dfrac{1}{2(\sigma^2)^2} \sum_{i=1}^{n} [(x_i - \mu_1)^2 + (y_i - \mu_2)^2] = 0. \end{cases}$$

$$\Rightarrow \hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}, \quad \hat{\mu}_2 = \bar{y}, \quad \hat{\sigma}^2 = \frac{1}{2} \frac{n-1}{n} (S_1^2 + S_2^2).$$

Under H_0:
$$\begin{aligned} \hat{\sigma}^{2R} &= \frac{1}{2n} \sum_{i=1}^{n} (x_i^2 + y_i^2) = \frac{1}{2n} \left[\sum_{i=1}^{n} (x_i - \bar{x})^2 + n\bar{x}^2 + \sum_{i=1}^{n} (y_i - \bar{y})^2 + n\bar{y}^2 \right]. \\ &= \frac{(n-1) S_1^2 + (n-1) S_2^2 + n(\bar{x}^2 + \bar{y}^2)}{2n}. \end{aligned}$$

$$\lambda(\vec{x}) = \frac{L(0, 0, \hat{\sigma}^{2R})}{L(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2)} = \frac{(2\pi\hat{\sigma}^{2R})^{-n} \cancel{\exp\left[-\frac{1}{2\hat{\sigma}^{2R}} \sum_{i=1}^{n} (x_i^2 + y_i^2) \right]}}{(2\pi\hat{\sigma}^2)^{-n} \cancel{\exp\left[-\frac{n}{(n-1)(S_1^2 + S_2^2)} (n-1)(S_1^2 + S_2^2) \right]}}$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\sigma}^{2R}} \right)^n = \left[\frac{(n-1)(S_1^2 + S_2^2)}{(n-1)(S_1^2 + S_2^2) + n(\bar{x}^2 + \bar{y}^2)} \right]^n$$

$$= \left(\frac{1}{1+F} \right)^n, \qquad F = \frac{n(\bar{x}^2+\bar{y}^2)}{(n-1)(S_1^2+S_2^2)}$$

$$= \frac{\frac{n}{\sigma^2}\bar{x}^2 + \frac{n}{\sigma^2}\bar{y}^2}{\frac{(n-1)S_1^2}{\sigma^2} + \frac{(n-1)S_2^2}{\sigma^2}} \underset{\text{Under } H_0}{\sim} \frac{\chi^2(2)}{\chi^2(2n-2)}$$

$$\Rightarrow (n-1)F \sim \frac{\chi^2(2)/2}{\chi^2(2n-2)/(2n-2)} = F(2, 2n-2)$$

under H_0.

$$\alpha = \Pr\{\lambda(\vec{x}) \le \lambda_\alpha \mid H_0\} = \Pr\{F \ge c \mid H_0\}$$

$$= \Pr\{(n-1)F \ge (n-1)c \mid H_0\} = \Pr\{F(2, 2n-2) \ge (n-1)c\}$$

$$= \Pr\{F(2, 2n-2) \ge f(\alpha, 2, 2n-2)\}.$$

$$\Rightarrow \quad c = \frac{1}{n-1} f(\alpha, 2, 2n-2).$$

$\Rightarrow$ The critical region is

$$\left\{ \vec{x} : \frac{n(\bar{x}^2+\bar{y}^2)}{(n-1)(S_1^2+S_2^2)} \ge \frac{1}{n-1} f(\alpha, 2, 2n-2) \right\}.$$

$\boxed{5.8}$

Solution:

(a) We first consider

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \text{against} \quad H_1 : \sigma_1^2 \ne \sigma_2^2.$$

$$\vec{\theta} = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \qquad \Theta_0 = \{\vec{\theta} : \sigma_1^2 = \sigma_2^2\}. \quad \Theta_1 = \Theta \setminus \Theta_0 = \Theta^* \setminus \Theta_0$$

$$L(\vec{\theta}) = (2\pi\sigma_1^2)^{-\frac{n_1}{2}} \exp\left(-\frac{1}{2\sigma_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2\right)$$
$$\times (2\pi\sigma_2^2)^{-\frac{n_2}{2}} \exp\left(-\frac{1}{2\sigma_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right).$$

$$l(\vec{\theta}) = \log L(\vec{\theta}) = -\frac{n_1}{2} \log 2\pi - \frac{n_1}{2} \log \sigma_1^2 - \frac{1}{2\sigma_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2$$

$$-\frac{n_2}{2}\log 2\pi - \frac{n_2}{2}\log \sigma_2^2 - \frac{1}{2\sigma_2^2}\sum_{i=2}^{n_2}(y_i-\mu_2)^2 .$$

$$\Rightarrow \hat{\mu}_1 = \bar{x},\ \hat{\mu}_2 = \bar{y},\ \hat{\sigma}_k^2 = \frac{(n_k-1)S_k^2}{n_k}. \quad k=1,2.$$

Under H_0 the restricted mles are

$$\hat{\mu}_1^R = \bar{x},\ \hat{\mu}_2^R = \bar{y},\ \hat{\sigma}^{2\,R} = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2}.$$

$$\lambda(\vec{x}) = \frac{(\hat{\sigma}_1^2)^{\frac{n_1}{2}} (\hat{\sigma}_2^2)^{\frac{n_2}{2}}}{(\hat{\sigma}^{2\,R})^{\frac{n_1+n_2}{2}}}$$

$$= \frac{[(n_1+n_2)(n_1-1)/n_1]^{\frac{n_1}{2}}\,[(n_1+n_2)(n_2-1)(S_2^2/S_1^2)/n_2]^{\frac{n_2}{2}}}{\left((n_1-1)+(n_2-1)\frac{S_2^2}{S_1^2}\right)^{\frac{n_1+n_2}{2}}}.$$

let $t = \frac{S_1^2}{S_2^2}.$

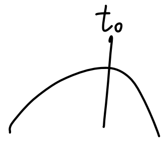
$$\Rightarrow \lambda(\vec{x}) \propto f(t) = \frac{t^{\frac{n_2}{2}}}{[n_1-1+(n_2-1)t]^{\frac{n_1+n_2}{2}}}.$$

$$\Rightarrow \log f(t) = \frac{n_2}{2}\log t - \frac{n_1+n_2}{2}\log[n_1-1+(n_2-1)t].$$

$$\frac{d\log f}{dt} = \frac{n_2}{2}\frac{1}{t} - \frac{(n_1+n_2)}{2}\frac{(n_2-1)}{[n_1-1+(n_2-1)t]}.$$

$$\frac{d^2\log f}{dt^2} < 0 \qquad f(t) \text{ is log-concave.}$$

$$t_0 = \frac{n_2(n_1-1)}{n_1(n_2-1)}.$$



$$\Rightarrow C = \{\vec{x} : t \le C_1 \text{ or } t \ge C_2\}.$$

We use equal-tail approach:

$$\frac{\alpha}{2} = \Pr\left\{\frac{S_2^2}{S_1^2} \le C_1 \mid H_0\right\}$$

Under H_0, $\sigma_1^2 = \sigma_2^2$, $\frac{S_2^2}{S_1^2} = \frac{\frac{(n_2-1)S_2^2}{\sigma_2^2}/(n_2-1)}{\frac{(n_1-1)S_1^2}{\sigma_1^2}/(n_1-1)} \sim F(n_2-1, n_1-1)$

$$\Rightarrow \frac{\alpha}{2} = Pr\{ F(n_2-1, n_1-1) \le C_1 \}$$

$$= Pr\{ F(n_2-1, n_1-1) \le f(1-\tfrac{\alpha}{2}, n_2-1, n_1-1) \}.$$

$$\Rightarrow C_1 = f(1-\tfrac{\alpha}{2}, n_2-1, n_1-1).$$

$\frac{\alpha}{2} = Pr\{ \frac{S_2^2}{S_1^2} \ge C_2 \mid H_0 \}$. Similarly, $C_2 = f(\frac{\alpha}{2}, n_2-1, n_1-1)$.

(b). Consider $H_0: \sigma_1^2 = \sigma_2^2$ against $H_1: \sigma_1^2 > \sigma_2^2$.

$$\mathbb{C} = \left\{ \vec{x} : \frac{S_2^2}{S_1^2} \le f(1-\alpha, n_2-1, n_1-1) \right\}.$$

(c). Consider $H_0: \sigma_1^2 = \sigma_2^2$ against $H_1: \sigma_1^2 < \sigma_2^2$.

$$\mathbb{C} = \left\{ \vec{x} : \frac{S_2^2}{S_1^2} \ge f(\alpha, n_2-1, n_1-1) \right\}.$$

$\boxed{5.9.}$

Solution:

(a) Consider $H_0: \theta = \frac{1}{2}$ against $H_1: \theta \ne \frac{1}{2}$.

$$\Theta_0 = \{\tfrac{1}{2}\}, \quad \Theta_1 = \Theta \setminus \Theta_0, \quad \Theta = \Theta^* = \mathbb{R}.$$

$$L(\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}.$$

$$l(\theta) = \log L(\theta) = \log\binom{n}{x} + x \log\theta + (n-x)\log(1-\theta).$$

$$\frac{dl}{d\theta} = \frac{x}{\theta} + \frac{x-n}{1-\theta} = 0 \Leftrightarrow x - x\theta = (-x+n)\theta \Leftrightarrow \theta = \frac{x}{n}.$$

$$\lambda(x) = \frac{\binom{n}{x} (\frac{1}{2})^x (\frac{1}{2})^{n-x}}{\binom{n}{x} (\frac{x}{n})^x (\frac{n-x}{n})^{n-x}}$$

$$= \frac{(\frac{1}{2}n)^n}{x^x (n-x)^{n-x}}.$$

$\Rightarrow$ The LR statistic is $\lambda(X) = \dfrac{(\frac{1}{2}n)^n}{X^X (n-X)^{n-X}}.$

(b) $\lambda(x) \leq \lambda_\alpha$

$$\Leftrightarrow \frac{(\frac{1}{2}n)^n}{x^x (n-x)^{n-x}} \leq \lambda_\alpha$$

$$\Leftrightarrow n\log(\tfrac{1}{2}n) - x\log x - (n-x)\log(n-x) \leq \log \lambda_\alpha.$$

$$\Leftrightarrow x\log x + (n-x)\log(n-x) \geq n\log\frac{n}{2} - \log\lambda_\alpha \triangleq c.$$

(c).

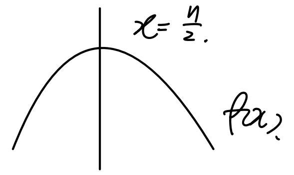
$$f'(x) = \log x + 1 - \log(n-x) - 1 = \log\frac{x}{n-x}.$$

$$f(x) = f(n-x) = x\log x + (n-x)\log(n-x)$$

$\Rightarrow$ $f(x)$ has symmetric axis $x = \frac{n}{2}$.

$f'(x) < 0$, $x < \frac{n}{2}$

$f'(x) > 0$, $x > \frac{n}{2}$. $\Rightarrow$



$\Rightarrow$ $f(x) \geq C_0 \Leftrightarrow |x - \frac{n}{2}| \geq c$

$\boxed{5.11}$

Solution: $H_0: p_1 = \cdots = p_6 = \frac{1}{6}$. against $H_1: \neg(p_1 = \cdots = p_6 = \frac{1}{6})$

$$Q_n = \sum_{i=1}^{6} \frac{(N_i - np_{i0})^2}{np_{i0}}$$

$$= \frac{(43-50)^2}{50} + \frac{(49-50)^2}{50} + \frac{(56-50)^2}{50}$$

$$+ \frac{(45-50)^2}{50} + \frac{(66-50)^2}{50} + \frac{(41-50)^2}{50}$$

$$= 8.96 < \chi^2(0.05, 5) = 11.07.$$

$\Rightarrow$ we CANNOT reject H_0.