Discrete Mathematics for Computer Science

Lecture 21: Review

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Lecture Schedule

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Propositional Logic

Proposition: a declarative sentence that is either true or false (not both).

- Conventional letters used for propositional variables are p, q, r, s, ...
- **Truth value** of a proposition: true, denoted by T; false, denoted by F.

Compound propositions are build using logical connectives:

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Exclusive or $\oplus$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Tautology and Logical Equivalences

- Tautology: A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.
  - E.g., $p \vee \neg p$
- Contradiction: A compound proposition that is always false.

The compound propositions p and q are called logically equivalent, denoted by $p \equiv q$, if $p \leftrightarrow q$ is a tautology.

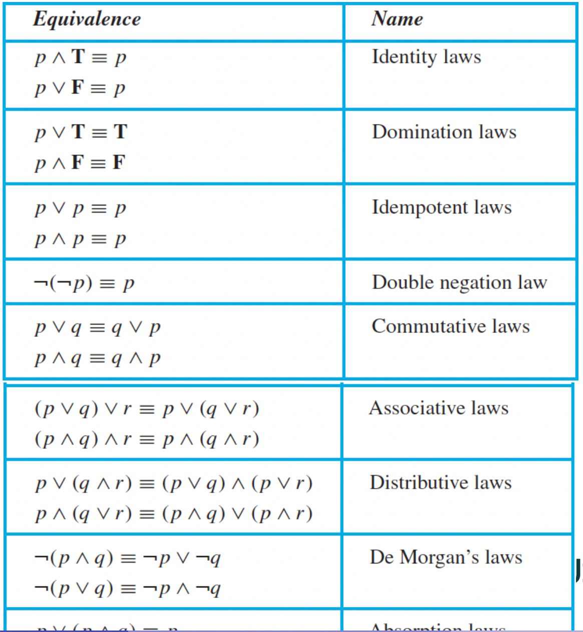
- E.g., $\neg(p \vee q)$ and $\neg p \wedge \neg q$

That is, two compound propositions are equivalent if they always have the same truth value.

Determine logically equivalent propositions using:

- Truth table
- Logical Equivalences

Important Logical Equivalences



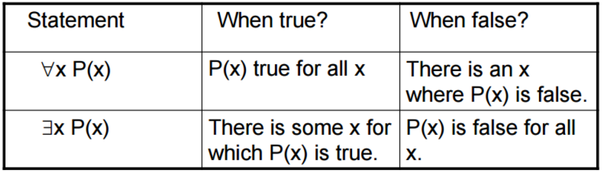
Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$	Absorption laws

Predicate Logic and Quantified Statements

Predicate Logic: make statements with variables: $P(x)$.

Propositional function $P(x) \overset{\text{specify } x}{\Longrightarrow}$ Proposition

Quantified Statements: Universal quantifier $\forall x P(x)$; Existential quantifier $\exists x P(x)$



Statement	When true?	When false?
∀x P(x)	P(x) true for all x	There is an x where P(x) is false.
∃x P(x)	There is some x for which P(x) is true.	P(x) is false for all x.

Propositional function $P(x) \overset{\text{for all/some } x \text{ in domain}}{\Longrightarrow}$ Proposition

Negation and Nest Quantifier

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x\ P(x)$	$\forall x\ \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x\ P(x)$	$\exists x\ \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

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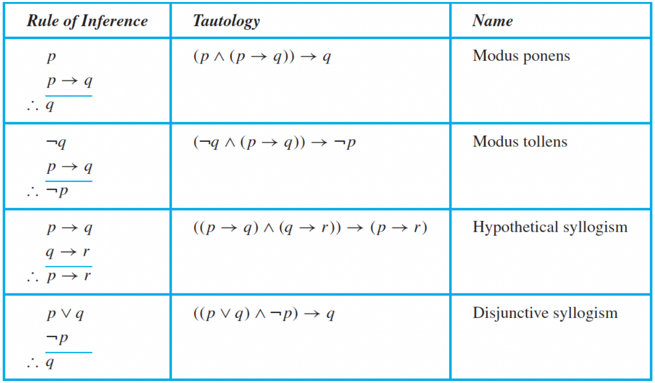
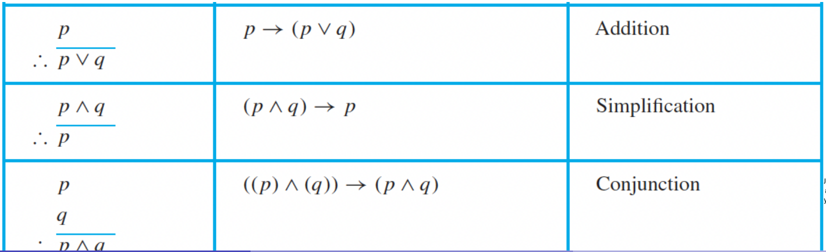
Validity of Argument Form:

The argument form with premises $p_1, p_2, ..., p_n$ and conclusion q is valid, if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q \text{ is a tautology.}$$

Note: According to the definition of $p \rightarrow q$, we do not worry about the case where $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ is false.

Rules of Inference for Propositional Logic

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction

Methods of Proving Theorems

A proof is a valid argument that establishes the truth of a mathematical statement.

- **Direct proof**

  $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Proof by contrapositive**

  show the contrapositive $\neg q \rightarrow \neg p$
- **Proof by contradiction**

  show that $(p \wedge \neg q)$ contradicts the assumptions
- **Proof by cases**

  give proofs for all possible cases
- **Proof of equivalence**

  $p \leftrightarrow q$ is replaced with $(p \rightarrow q) \wedge (q \leftarrow p)$

Proof Exercises

Prove that $\sqrt{2}$ is irrational. (Rational numbers are those of the form $\frac{m}{n}$, where m and n are integers.)

Prove that there are infinitely many prime numbers.

Show that there exist irrational numbers x and y such that x^y is rational.

Lecture Schedule

Sets

A set is an unordered collection of objects.

- listing (enumerating) the elements
- if enumeration is hard, use ellipses (...)
- definition by property, using the set builder

$$\{x \mid x \text{ has property } P \text{ or property } P(x))\}$$

Proof of Subset:

- Showing $A \subseteq B$: if x belongs to A, then x also belongs to B.
- Showing $A \nsubseteq B$: find a single $x \in A$ such that $x \notin B$.

Prove $A = B$?

Cardinality, Power Set, Tuples, and Cartesian Product

Cardinality: If there are exactly n distinct elements in S, where n is a nonnegative integer, we say that S is a finite set and n is the cardinality of S, denoted by $|S|$.

Power Set: Given a set S, the power set of S is the set of all subsets of the set S, denoted by $\mathcal{P}(S)$.

Tuples: The ordered n-tuple $(a_1, a_2, ..., a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on.

Cartesian Product: Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Set Operations

Union: Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set $\{x \mid x \in A \vee x \in B\}$.

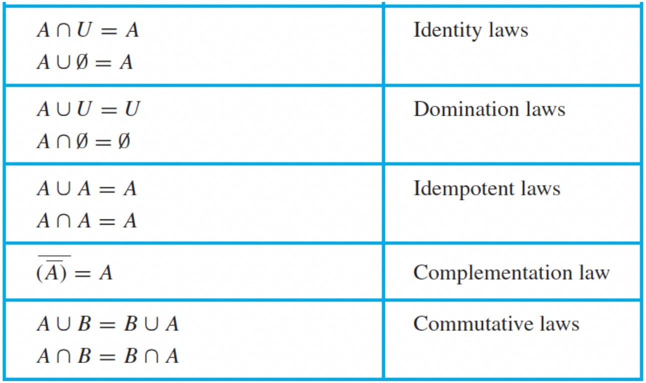
Intersection: The intersection of the sets A and B, denoted by $A \cap B$, is the set $\{x \mid x \in A \wedge x \in B\}$.

Complement: If A is a set, then the complement of the set A (with respect to U), denoted by $\bar{A}$ is the set $U - A$, $\bar{A} = \{x \in U \mid x \notin A\}$

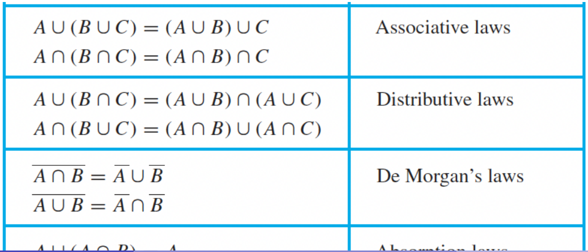
Difference: Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.
$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$.

Principle of inclusion–exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$

Set Identities



$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws



$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$	Absorption laws

Proof of Set Identities

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof 1: using membership tables.
Proof 2: by showing that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$
Proof 3: Using set builder and logical equivalences

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}$$

Function

Let A and B be two sets. A function from A to B, denoted by $f : A \to B$, is an assignment of exactly one element of B to each element of A.

- **One-to-one (injective) function**:
  - A function f is called one-to-one or injective if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f.
- **Onto (surjective) function**:
  - A function f is called onto or surjective if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.
- **One-to-one (bijective) correspondence**
  - One-to-one and onto

Proof for One-to-One and Onto

Suppose that $f : A \to B$.

To show that f is *injective*	Show that if $f(x) = f(y)$ for all $x, y \in A$, then $x = y$
To show that f is not *injective*	Find specific elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$
To show that f is *surjective*	Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$
To show that f is not *surjective*	Find a specific element $y \in B$ such that $f(x) \neq y$ for all $x \in A$

Inverse Function and Composition of Functions

Inverse function: Let f be a one-to-one correspondence (bijection) from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.

Let f be a function from B to C and let g be a function from A to B. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.

The floor function assigns a real number x the largest integer that is $\leq x$, denoted by $\lfloor x \rfloor$. E.g., $\lfloor 3.5 \rfloor = 3$.

The ceiling function assigns a real number x the smallest integer that is $\geq x$, denoted by $\lceil x \rceil$. E.g., $\lceil 3.5 \rceil = 4$.

Sequences

A sequence is a function from a subset of the set of integers (typically the set $\{0, 1, 2, ...\}$ or $\{1, 2, 3, ...\}$) to a set S.

We use the notation a_n to denote the image of the integer n. $\{a_n\}$ represents the ordered list $\{a_1, a_2, a_3, ...\}$

Recursively Defined Sequences: provide

- One or more initial terms
- A rule for determining subsequent terms from those that precede them.

Cardinality of Sets

A set that is either finite or has the same cardinality as the set of positive integers $\mathbf{Z}^+$ is called countable.

If there is a one-to-one function from A to B, the cardinality of A is less than or equal to the cardinality of B, denoted by $|A| \leq |B|$.

Theorem: If there is a one-to-one correspondence between elements in A and B, then the sets A and B have the same cardinality.

Theorem: If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

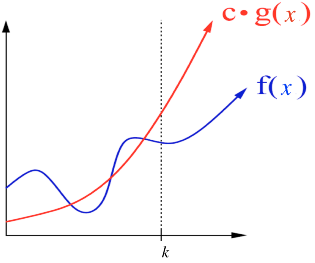
Lecture Schedule

Big-O Notation

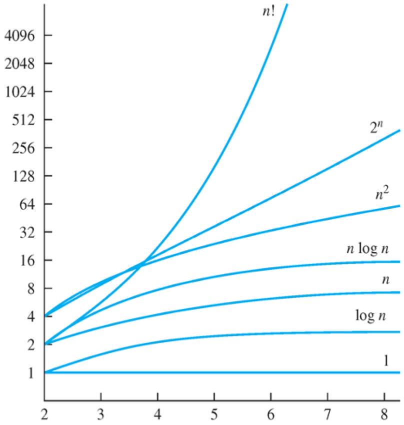
Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|,$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]



Big-O Estimates for Some Functions



Big-Omega Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-Omega of $g(x)$."]

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Theta(g(x))$ if

- $f(x)$ is $O(g(x))$ and
- $f(x)$ is $\Omega(g(x))$.

When $f(x)$ is $\Theta(g(x))$, we say that $f(x)$ is big-Theta of $g(x)$, that $f(x)$ is of order $g(x)$, and that $f(x)$ and $g(x)$ are of the same order.

We will NOT let you compute the complexity of an algorithm.

Lecture Schedule

Division

Divisibility: We say that a divides b if there is an integer c such that $b = ac$, or equivalently b/a is an integer.

- If a, b, c are integers, where $a \neq 0$, such that $a|b$ and $a|c$, then $a|(mb + nc)$ whenever m and n are integers.

Congruence Relation: If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$, denoted by $a \equiv b \pmod{m}$. Equivalently,

$$a \mod m = b \mod m$$

The integers a and b are congruent modulo m if and only if there is an integer k such that

$$a = b + km$$

.

Congruence: Properties

Theorem: Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m}$$

$$ac \equiv bd \pmod{m}$$

Corollary: Let m be a positive integer and let a and b be integers. Then,

$$(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$

$$ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$$

How to compute $b^n \mod m$?
If $ca \equiv cb \pmod{m}$, then $a \equiv b \pmod{m}$? Any condition?

Primes

An integer p that is greater than 1 is called a prime if the only positive factors of p are 1 and p.

- If n is composite, then n has a prime divisor less than or equal to $\sqrt{n}$.

Let a and b be integers, not both 0. The largest integer d such that $d|a$ and $d|b$ is called the greatest common divisor of a and b, denoted by $\gcd(a, b)$. Let $a = p_1^{a_1} p_2^{a_2} ... p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} ... p_n^{b_n}$. Then,

$$\gcd(a, b) = p^{\min(a_1, b_1)} p^{\min(a_2, b_2)} ... p^{\min(a_n, b_n)}$$

The least common multiple of a and b is the smallest positive integer that is divisible by both a and b, denoted by $\mathrm{lcm}(a, b)$.Let $a = p_1^{a_1} p_2^{a_2} ... p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} ... p_n^{b_n}$. Then,

$$\mathrm{lcm}(a, b) = p^{\max(a_1, b_1)} p^{\max(a_2, b_2)} ... p^{\max(a_n, b_n)}.$$

Euclidean Algorithm

Computing the greatest common divisor of two integers directly from the prime factorizations can be time consuming since we need to find all factors of the two integers.

For two integers 287 and 91, we want to find $\gcd(287, 91)$.

Step 1: $287 = 91 \cdot 3 + 14$

Step 2: $91 = 14 \cdot 6 + 7$

Step 3: $14 = 7 \cdot 2 + 0$

$\gcd(287, 91) = \gcd(91, 14) = \gcd(14, 7) = 7$

How to prove?

GCD as Linear Combinations

Bezout'S Theorem: If a and b are positive integers, then there exist integers s and t such that

$$\gcd(a, b) = sa + tb.$$

This equation is called Bezout's identity.

We can use extended Euclidean algorithm to find Bezout's identity.

Lemma: If a, b, c are positive integers such that $\gcd(a, b) = 1$ and $a|bc$, then $a|c$.

Lemma: If p is prime and $p|a_1a_2...a_n$, then $p|a_i$ for some i.

Linear Congruences

A congruence of the form $ax \equiv b \pmod{m}$, where m is a positive integer, a and b are integers, and x is a variable, is called a linear congruence.

The solutions to a linear congruence $ax \equiv b \pmod{m}$ are all integers x that satisfy the congruence.

Modular Inverse: An integer $\bar{a}$ such that $\bar{a}a \equiv 1 \pmod{m}$ is said to be an inverse of a modulo m.

Solve the congruence $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.

$$x \equiv \bar{a}b \pmod{m}.$$

Modular Inverse

Modular Inverse: An integer $\bar{a}$ such that $\bar{a}a \equiv 1 \pmod{m}$ is said to be an inverse of a modulo m.

When does inverse exist?

Theorem: If a and m are relatively prime integers and $m > 1$, then an inverse of a modulo m exists. The inverse is unique modulo m. That is,

- there is a unique positive integer $\bar{a}$ less than m that is an inverse of a modulo m and
- every other inverse of a modulo m is congruent to $\bar{a}$ modulo m.

Modular Inverse

How to find inverses?

Using extended Euclidean algorithm:

Example: Find an inverse of 101 modulo 4620. That is, find $\bar{a}$ such that $\bar{a} \cdot 101 \equiv 1 \pmod{4620}$.

With extended Euclidean algorithm, we obtain $\gcd(a, b) = sa + tb$, i.e., $1 = -35 \cdot 4620 + 1601 \cdot 101$. It tells us that -35 and 1601 are Bezout coefficients of 4620 and 101. We have

$$1 \ \mathbf{mod}\ 4620 = 1601 \cdot 101 \ \mathbf{mod}\ 4620.$$

Thus, 1601 is an inverse of 101 modulo 4620.

The Chinese Remainder Theorem

Theorem (The Chinese Remainder Theorem): Let $m_1, m_2, \ldots, m_n$ be pairwise relative prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then, the system

$$x \equiv a_1 \pmod{m_1}$$
$$x \equiv a_2 \pmod{m_2}$$
$$\ldots$$
$$x \equiv a_n \pmod{m_n}$$

has a unique solution modulo $m = m_1 m_2 \ldots m_n$.
(That is, there is a solution x with $0 \leq x < m$, and all other solutions are congruent modulo m to this solution.)

The Chinese Remainder Theorem: Example

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

1. Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, and $M_3 = m/7 = 15$.
2. Compute y_k, i.e., the inverse of M_k modulo m_k:
    - $35 \cdot 2 \equiv 1 \pmod{3}$ $y_1 = 2$
    - $21 \equiv 1 \pmod{5}$ $y_2 = 1$
    - $15 \equiv 1 \pmod{7}$ $y_3 = 1$
3. Compute a solution $x = a_1 M_1 y_1 + \ldots + a_n M_n y_n$:
   $x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \pmod{105}$
4. The solutions are all integers x that satisfy $x \equiv 23 \pmod{105}$.

How to prove $x \equiv 23 \pmod{105}$ is the solution?

Back Substitution

We may also solve systems of linear congruences with pairwise relatively prime moduli $m_1, m_2, \ldots m_n$ by back substitution.

Example:

(1) $x \equiv 1 \pmod{5}$
(2) $x \equiv 2 \pmod{6}$
(3) $x \equiv 3 \pmod{7}$

According to (1), $x = 5t + 1$, where t is an integer.

Substituting this expression into (2), we have $5t + 1 \equiv 2 \pmod{6}$, which means that $t \equiv 5 \pmod{6}$. Thus, $t = 6u + 5$, where u is an integer.

Substituting $x = 5t + 1$ and $t = 6u + 5$ into (3), we have $30u + 26 \equiv 3 \pmod{7}$, which implies that $u \equiv 6 \pmod{7}$. Thus, $u = 7v + 6$, where v is an integer.

Thus, we must have $x = 210v + 206$. Translating this back into a congruence,

$$x \equiv 206 \pmod{210}.$$

The Chinese Remainder Theorem

What if $m_1, m_2, \dots, m_n$ are positive integers greater than 1, but they are not pairwise relatively prime?

$x \equiv a_1 \pmod{m_1}$

$x \equiv a_2 \pmod{m_2}$

...

$x \equiv a_n \pmod{m_n}$

Translate these congruence into a set of congruence that together are equivalent to the given congruence:

For $x \equiv a_k \pmod{m_k}$, suppose m_k can be written as $m_k = b_k^1 b_k^2 \cdot b_k^r$, where b_k^1, b_k^2, ..., b_k^r are all primes. Then, $x \equiv a_k \pmod{m_k}$ is equivalent to the following set of congruence:

$x \equiv a_k \pmod{b_k^1}$

$x \equiv a_k \pmod{b_k^2}$

...

$x \equiv a_k \pmod{b_k^r}$

RSA Cryptosystem

Pick two large primes p and q. Let $n = pq$. **Encryption key** (n, e) and **decryption key** (n, d) are selected such that

(1) $\gcd(e, (p-1)(q-1)) = 1$

(2) $ed \equiv 1 \pmod{(p-1)(q-1)}$

RSA encryption: $C = M^e \textbf{ mod } n$;
RSA decryption: $M = C^d \textbf{ mod } n$.

How to prove RSA decryption can recover the message?

Digital Signature?

Lecture Schedule

The Principle of Mathematical Induction

Well-Ordering Property: Every nonempty set of nonnegative integers has a least element.

Principle. (Weak Principle of Mathematical Induction)

(a) Basic Step: the statement $P(b)$ is true

(b) Inductive Step: the statement $P(n-1) \rightarrow P(n)$ is true for all $n > b$

Thus, $P(n)$ is true for all integers $n \geq b$.

Principle (Strong Principle of Mathematical Induction):

(a) Basic Step: the statement $P(b)$ is true

(b) Inductive Step: for all $n > b$, the statement

$$P(b) \wedge P(b+1) \wedge \ldots \wedge P(n-1) \rightarrow P(n) \text{ is true.}$$

Then, $P(n)$ is true for all integers $n \geq b$.

Lecture Schedule

Recurrence

To specify a function on the basis of a recurrence:

- Basis step (initial condition): Specify the value of the function at zero.
- Recursive step: Give a rule for finding its value at an integer from its values at smaller integers.

How to formulate a recursive function?

Lecture Schedule

Counting

Product Rule: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and k-th count n_k elements, then the total number of elements is

$$n = n_1 \times n_2 \times ... \times n_k$$

Sum Rule:

- A task can be done either in one of n_1 ways or in one of n_2 ways
- None of the set of n_1 ways is the same as any of the set of n_2 ways

The Subtraction Rule:

- A task can be done in either n_1 ways or n_2 ways
- **Principle of inclusion–exclusion**:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Pigeonhole Principle

Assume that there are a set of objects and a set of bins to store them.

The Pigeonhole Principle: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Use pigenhole principle in proofs.

Permutations and Combinations

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Theorem: For integers n and r with $0 \leq r \leq n$, the number of r-element subsets of an n-element set is

$$\binom{n}{r} = C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Combinatorial Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n,r) = C(n,n-r)$.

Definition: A combinatorial proof of an identity is

- a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways
- or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity.

These two types of proofs are called double counting proofs and bijective proofs, respectively.

Use combinatorial proof to prove.

The Binomial Theorem

Let x and y be variables, and let n be a nonnegative integer:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Theorem: Let n and k be positive integers with $n \geq k$. Then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Generalized Permutations and Combinations

- Permutations with repetition

  Repetition: Distinct objects; each can be selected multiple times

  Theorem: The number of r-permutations of a set of n objects with repetition allowed is n^r.

- Permutations with indistinguishable objects

  Indistinguishable objects: E.g., "SUCCESS"

  Theorem: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, . . . , and n_k indistinguishable objects of type k, is

$$C(n, n_1)(n - n_1, n_2) \cdots C(n - n_1 - \cdots n_{k-1}, n_k) = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

- Combinations with repetition

Combinations with Repetition

Example: How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills?

Theorem: There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r-combinations from a set with n elements when repetition of elements is allowed.

Solving Linear Homogeneous Recurrence Relations

Definition: A linear homogeneous relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k},$$

where c_1, c_2, . . . , c_k are real numbers, and $c_k \neq 0$.

By induction, such a recurrence relation is uniquely determined by this recurrence relation and k initial conditions a_0, a_1, ... , a_{k-1}.

Solving Linear Homogeneous Recurrence Relations

The characteristic equation (CE) is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0.$$

Theorem: Suppose that there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then,

$$\begin{aligned} a_n = \ & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

- Solving the roots with CE
- Solving the α_i for all i using initial conditions

Linear Nonhomogeneous Recurrence Relations

Definition: A linear nonhomogeneous relation with constant coefficients may contain some terms $F(n)$ that depend only on n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n).$$

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$.

Linear Nonhomogeneous Recurrence Relations

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Pay attention to the case when $s = 1$.

Generating function

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k.$$

How to use generating function to solve counting problems?

- Based on the combination problem, transfer the problem as finding the coefficient of x^r of a generating function, e.g.,

$$G(x) = (1 + x + x^2 + x^3 + \cdots)^n$$

- Find the coefficient of x^r
  - Enumerate all possibilities or
  - Sum of geometric progression, binomial theorem

How to use generating function to solve linear recurrence relation?

Lecture Schedule

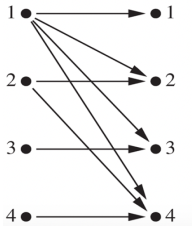
Cartesian Product

Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$, the Cartesian product $A \times B$ is the set of pairs $\{(a_1, b_1), (a_2, b_2), ..., (a_1, b_n), ..., (a_m, b_n)\}$.

Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product $A \times B$.

A relation on the set A is a relation from A to itself.

We use the notation aRb to denote $(a, b) \in R$, and $a \not R b$ to denote $(a, b) \notin R$.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Summary on Properties of Relations

- Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
- Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.
- Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.
- Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

The number of relations?

Composite of Relations

Definition: Let R be a relation from a set A to a set B and S be a relation from B to C. The composite of R and S is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$:

- $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$
- $S = \{(0, b), (1, a), (2, b)\}$
- $S \circ R = \{(1, b), (3, a), (3, b)\}$

Power of a Relation

Definition: Let R be a relation on A. The powers R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

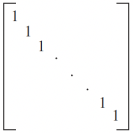
How to prove?

Physical meaning of R^n in the associated directed graph?

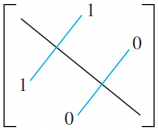
Representing Relations

Some special ways to represent binary relations:

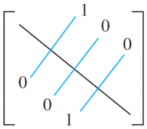
- with a zero-one matrix
- with a directed graph



Reflexive



Symmetric



Antisymmetric

Zero-One Matrix

Consider relations R_1 and R_2 on a set A:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Suppose that R is a relation from A to B and S is a relation from B to C:

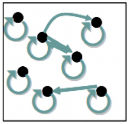
$$M_{S \circ R} = M_R \odot M_S.$$

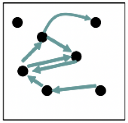
The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S.

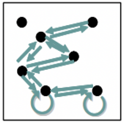
Directed Graph

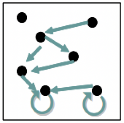
A directed graph, or digraph, consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.


reflexive


irreflexive


symmetric


antisymmetric

Closures of Relations

The set S is called the reflexive closure of R if it:

- contains R
- is reflexive
- is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Proof?

Relations can have different properties:

- Reflexive closures
- Symmetric closures
- Transitive closures: Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path.

Paths in Directed Graphs

Definition: A path from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), . . . , (x_{n-1}, x_n) in G, where n is nonnegative and $x_0 = a$ and $x_n = b$.

A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.

Theorem: Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Proof (by induction)

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R^* consists of all pairs (a, b) such that there is a path (of any length) between a and b in R:

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Prove R^* is a transitive closure of R?

Find Transitive Closure

Lemma: If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.

Thus,

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n.$$

Theorem: Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]},$$

where $M_R^{[n]} = \underbrace{M_R \odot M_R \odot \cdots \odot M_R}_{n\ M_R's}$

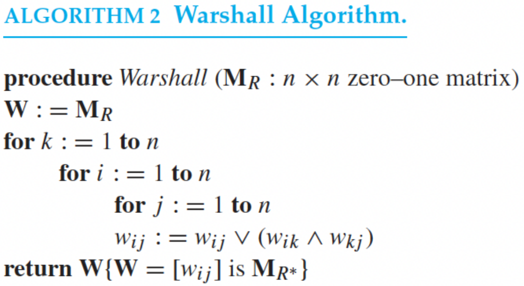
Roy-Warshall Algorithm

Consider a list of vertices $v_1, v_2, ..., v_k, ..., v_n$. Define a zero-one matrix

$$\mathbf{W}_k = [w_{ij}^{(k)}],$$

where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, ..., v_k\}$ and is 0 otherwise.

Warshall's algorithm computes M_{R^*} by efficiently computing $\mathbf{W}_0 = M_R, W_1, W_2, ..., \mathbf{W}_n = M_{R^*}$.



ALGORITHM 2 Warshall Algorithm.

```
procedure Warshall ($\mathbf{M}_R$ : $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ to $n$
    for $i := 1$ to $n$
        for $j := 1$ to $n$
        $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$
```

Equivalence Relation

Definition: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i)aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Partition of a Set S

Definition: Let S be a set. A collection of nonempty subsets of S, i.e A_1, A_2, . . . , A_k, is called a partition of S if:

$$A_i \cap A_j = \emptyset, i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$

Theorem: The equivalence classes form a partition of A.

Theorem: Let $\{A_1, A_2, ..., A_i, ...\}$ be a partition of S. Then, there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Partial Ordering

Definition: A relation R on a set S is called a partial ordering, or partial order, if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a partially ordered set, or poset, denoted by (S, R).

The notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R).

The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$.

Comparability

Definition: The elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, a and b are called incomparable.

Definition: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order.

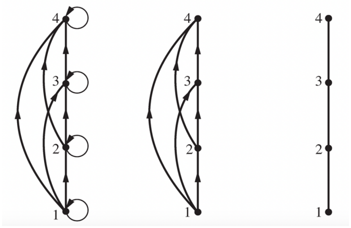
$(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

The Principle of Well-Ordered Induction

Hasse Diagram

Start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property.
- Remove all edges (x, y) for which there is an element $z \in S$ s.t. $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
- Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.



Maximal and Minimal Elements

Definition: a is a maximal (resp. minimal) element in poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ (resp. $b \prec a$).

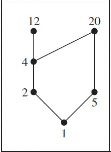
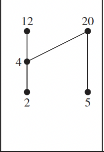
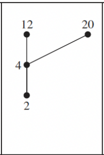
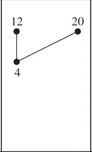
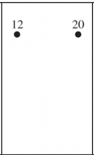
Definition: a is the greatest (resp. least) element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ (resp. $a \preccurlyeq b$) for all $b \in S$.

Definition: Let A be a subset of a poset $(S, \preccurlyeq)$.

- $u \in S$ is called an upper bound (resp. lower bound) of A if $a \preccurlyeq u$ (resp. $u \preccurlyeq a$) for all $a \in A$.
- $x \in S$ is called the least upper bound (resp. greatest lower bound) of A if x is an upper bound (resp. lower bound) that is less than any other upper bounds (resp. lower bounds) of A.

Topological Sorting for Finite Posets

Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever aRb. $\preccurlyeq$ is said compatible with R.
Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

					
Minimal element chosen 1	5	2	4	20	12

This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$$

Lecture Schedule

Definition of a Graph

A graph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to be incident to (or connect) its endpoints.

Simple graph: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.

Two vertices u, v in an undirected graph G are called adjacent (or neighbors) in G if there is an edge e between u and v. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

Undirected Graphs

Definition: The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v.

If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

Definition: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $deg(v)$.

Theorem (Handshaking Theorem): If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} deg(v)$$

Directed Graphs

Definition: Let (u, v) be an edge in G. Then u is the initial vertex of the edge and is adjacent to v and v is the terminal vertex of this edge and is adjacent from u. The initial and terminal vertices of a loop are the same.

Definition: The in-degree of a vertex v, denoted by $deg^-(v)$, is the number of edges which terminate at v. The out-degree of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex.

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

Theorem: Let $G = (V, E)$ be a graph with directed edges. Then,

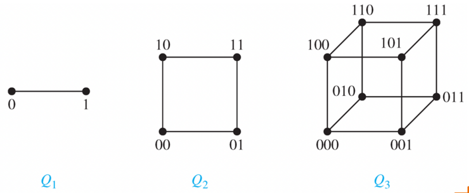
$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v)$$

Special Graphs

A complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

An *n*-dimensional hypercube, or *n*-cube, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.



Bipartite Graphs

Definition: A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.

Definition: A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

Bipartite Graphs and Matchings

Matching the elements of one set to elements in another. A matching is a subset of edges such that no two edges are incident with the same vertex.

A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a complete matching from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

Theorem (Hall's Marriage Theorem): The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.
Proof?

Subgraphs

Definition: A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subseteq V$ and $F \subseteq E$.

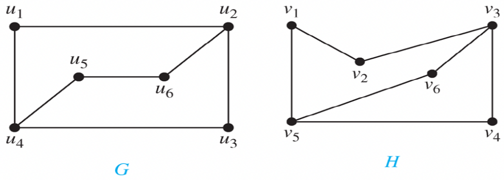
Definition: The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, denoted by $G_1 \cup G_2$.

To represent a graph, we may use adjacency lists, adjacency matrices, and incidence matrices.

Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an isomorphism.

Useful graph invariants include the number of vertices, number of edges, degree sequence, existence circuit with certain length, etc.



Path: Undirected Graph

Definition: Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges e_1, e_2, . . . , e_n of G for which there exists a sequence $x_0 = u$, x_1, . . . , x_{n-1}, $x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i for $i = 1, ..., n$.

The path is a circuit if it begins and ends at the same vertex, i.e., if $u = v$, and has length greater than zero.

A path or circuit is simple if it does not contain the same edge more than once.

Length of a path = the number of edges on path

Connectivity

An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

Lemma: If there is a path between two distinct vertices x and y of a graph G, then there is a simple path between x and y in G.

Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connectivity

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

Definition: A directed graph is strongly connected if there is a path from a to b and a path from b to a whenever a and b are vertices in the graph.

Definition: A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Cut Vertices and Cut Edges

Sometimes the removal from a graph of a vertex and all incident edges disconnect the graph. Such vertices are called cut vertices.

Similarly, we define cut edges.

A set of edges E' is called an edge cut of G if the subgraph $G - E'$ is disconnected. The edge connectivity $\lambda(G)$ is the minimum number of edges in an edge cut of G.

Counting Paths between Vertices

Theorem: Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering v_1, v_2, . . . , v_n of vertices. The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)-th entry of $\mathbf{A}^r$.

Proof (by induction)

Note: with directed or undirected edges, multiple edges and loops allowed

Euler Paths

Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

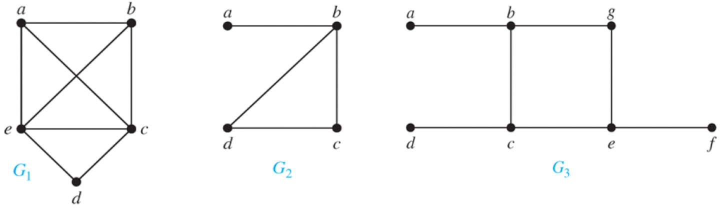
Theorem: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Hamilton Paths and Circuits

Definition: A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

Example: Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?



- G_1 has a Hamilton circuit: a, b, c, d, e, a;
- G_2 has no Hamilton circuit (because containing every vertex must contain the edge a, b twice), but it has a Hamilton path;
- G_3 has neither, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

Planar Graphs

Definition: A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

A planar representation of a graph splits the plane into regions, including an unbounded region.

Theorem (Euler's Formula): Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then, $r = e - v + 2$.

Proof?

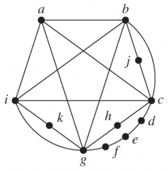
Kuratowski's Theorem

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.

The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.

Theorem: A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Example:



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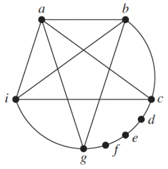

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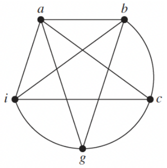


K_5

Lecture Schedule

Trees

Definition: A tree is a connected undirected graph with no simple circuits.

Theorem: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Definition: A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

m-Ary Trees

Definition: A rooted tree is called an m-ary tree if every internal vertex has no more than m children.

The tree is called a full m-ary tree if every internal vertex has exactly m children.

In particular, an m-ary tree with $m = 2$ is called a binary tree.

Definition: An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered. Ordered rooted trees are drawn so that the children of each internal vertex are shown in order from left to right.

Counting Vertices in a Full m-Ary Trees

Theorem: A tree with n vertices has $n-1$ edges.

Theorem: A full m-ary tree with i internal vertices has $n = mi + 1$ vertices.

Level and Height

The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.

The height of a rooted tree is the maximum of the levels of the vertices.

A rooted m-ary tree of height h is balanced if all leaves are at levels h or $h-1$.

Tree Traversal

The procedures for systematically visiting every vertex of an ordered tree are called traversals.

Definition Let T be an ordered rooted tree with root r. If T consists only of r, then r is the preorder traversal of T.

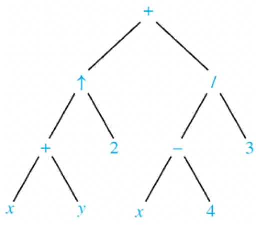
Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the inorder traversal of T.

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the postorder traversal of T.

Expression Trees

Complex expressions can be represented using ordered rooted trees.

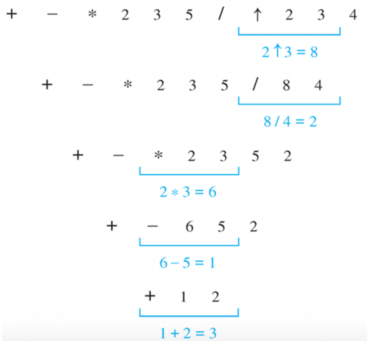
- the internal vertices represent operations
- the leaves represent the variables or numbers



Prefix Notation

Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

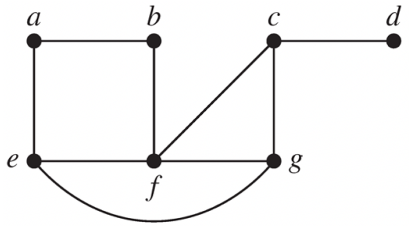
Example: $+ \; - \; * \; 2 \; 3 \; 5 \; / \uparrow 2 \; 3 \; 4$



The postorder traversal of expression trees leads to the postfix form of the expression (reverse Polish notation).

Spanning Trees

Definition: Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

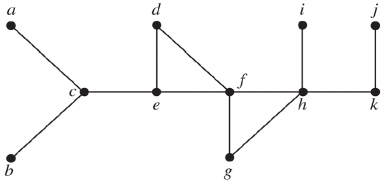


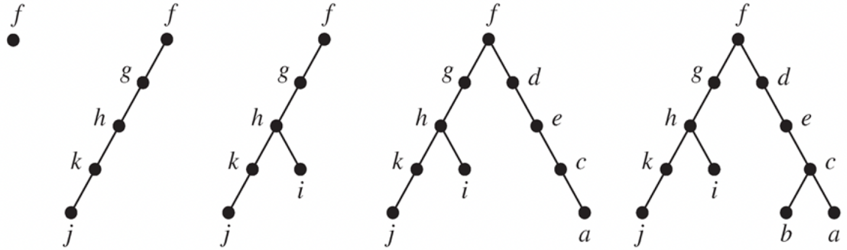
Remove edges to avoid circuits.

Depth-First Search

- First, arbitrarily choose a vertex of the graph as the root.
- Form a path by successively adding vertices and edges. Continue adding to this path as long as possible.
- If the path goes through all vertices of the graph, the tree is a spanning tree.
- Otherwise, move back to some vertex to repeat this procedure (backtracking).

Depth-First Search: Example



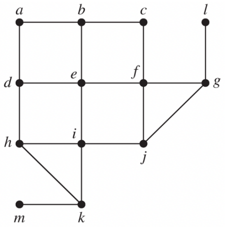


Breadth-First Search

This is the second algorithm that we build up spanning trees by successively adding edges.

- First arbitrarily choose a vertex of the graph as the root.
- Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges
- For each vertex added at the previous level, add edge incident to this vertex, as long as it does not produce a simple circuit.
- Continue in this manner until all vertices have been added.

Breadth-First Search: Example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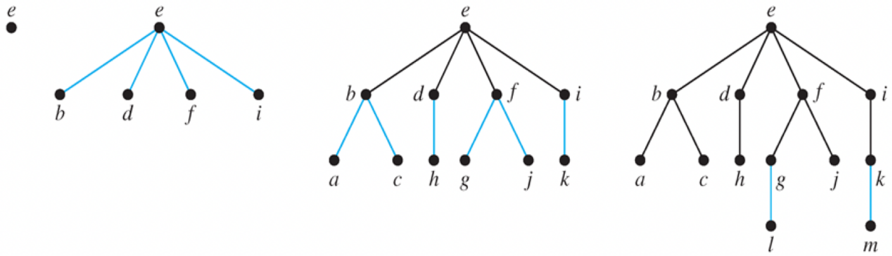
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