

Abstract Algebra

: Lecture 12

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Definition 1. Let G be a group, we have following subnormal chain/ subnormal series:

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright \{1\}$$

Where $G_{i+1} \triangleleft G_i$ for all i . All G_i are subnormal subgroups of G , denoted by $G_i \triangleleft \triangleleft G$.

If G_i/G_{i+1} is simple for all i , then this chain is called a **composition series/chain** of G , and each factor group G_i/G_{i+1} is called a **composition factor** of G .

Example 2. Let $G = S_4$, $G = G_0 \triangleright A_4 \triangleright V_4 \triangleright Z_2 \triangleright 1$, we have $G_0/G_1 \simeq Z_2$, $G_1/G_2 \simeq Z_3$, $G_2/G_3 \simeq Z_2$, $G_3/G_4 \simeq Z_2$.

Example 3. Let $G = S_5$, $G = G_0 \triangleright A_5 \triangleright 1$. $G_0/G_1 \simeq Z_2$, $G_1/G_2 \simeq A_5$.

Theorem 4. The number and the set of composition factors of a finite group G is uniquely determined by G .

Example 5. Let $G = S_4 \times S_5$.

$$G = G_0 = S_4 \times S_5 \triangleright A_4 \times S_5 \triangleright A_4 \times A_5 \triangleright A_4 \triangleright Z_2 \times Z_2 \triangleright Z_2 \triangleright 1$$

$$G_0/G_1 \simeq Z_2, G_1/G_2 \simeq Z_2, G_2/G_3 \simeq A_5, G_3/G_4 \simeq Z_3, G_4/G_5 \simeq Z_2, G_5/G_6 \simeq Z_2$$

Composition factors set $\{Z_2, Z_2, Z_2, Z_2, Z_3, A_5\}$.

Choose another composition chain:

$$G = G_0 = S_4 \times S_5 \triangleright S_4 \times A_5 \triangleright S_4 \triangleright A_4 \triangleright Z_2 \times Z_2 \triangleright Z_2 \triangleright 1$$

$$G_0/G_1 \simeq Z_2, G_1/G_2 \simeq A_5, G_2/G_3 \simeq Z_2, G_3/G_4 \simeq Z_3, G_4/G_5 \simeq Z_2, G_5/G_6 \simeq Z_2$$

Composition factors set is the same.

Theorem 6. Any two subnormal series of a finite group G have subnormal refinements that are equivalent.

Example 7. Let $G = A_4 \rtimes Z_2$.

Exercise: Find (1234) by this semidirect product.

Definition 8. Let G be a group, and $A = \text{Aut}(G)$, Let $X = \{(g, \sigma) | g \in G, \sigma \in A\}$, $|X| = |G| |\text{Aut}(G)|$, $(g_1, \sigma_1)(g_2, \sigma_2) = (g_1 g_2^{\sigma_1^{-1}}, \sigma_1 \sigma_2)$, X is a group, called the holomorph of G , denoted by $\text{Hol}(G)$.

- $G \simeq G \times \{1\} = \{(g, 1) | g \in G\} \triangleleft \text{Hol}(G)$;
- $A \simeq \{1\} \times A = \{(1, \sigma) | \sigma \in A\} < \text{Hol}(G)$;
- $\text{Hol}(G) = G \rtimes A$.

Example 9. Let $G = Z_3 \times Z_5 = \langle a \rangle \times \langle b \rangle$. Let $\sigma \in \text{Aut}(G)$ s.t. $a^\sigma = a^{-1}, b^\sigma = b^{-1}$, $G \rtimes \langle \sigma \rangle = D_{30}$

Example 10. Let $G = Z_3 \times Z_5 = \langle a \rangle \times \langle b \rangle$, Let $\tau \in \text{Aut}(G)$ s.t. $a^\tau = a^{-1}, b^\tau = b$, then $G \rtimes \langle \tau \rangle = D_6 \times Z_5$.

Example 11. Let $\rho \in \text{Aut}(G)$ s.t. $a^\rho = a, b^\rho = b^{-1}$, $G \rtimes \langle \rho \rangle = Z_3 \times D_{10}$.

Example 12. Let $G = Z_5 = \langle a \rangle$, Let $\sigma \in \text{Aut}(G) = Z_4$, be such that $a^\sigma = a^{-1}$, then $G \rtimes \langle \sigma \rangle = D_{10}$.
Let $\tau \in \text{Aut}(G)$ s.t. $a^\tau = a^2$, then $G \rtimes \langle \tau \rangle = Z_5 \rtimes Z_4$.

How many different types of $Z_5 \rtimes_\gamma Z_4$?

- (1). $x^y = x$, Z_{20} ; *trivial - y is id.*
- (2). $x^y = x^{-1}$, $Z_2 \times D_{10}$;
- (3). $x^y = x^2$, $Z_5 \rtimes Z_4 = \langle x, y | x^5 = 1, y^4 = 1, y^{-1}xy = x^2 \rangle$.

Exercise 13. Identify $Z_{17} \rtimes Z_{16}$.

Exercise 14. Construct $Z^3 \rtimes Q_8$ s.t. $Z(G) = 1$.

Definition 15. Let $X = \{x_1, \dots, x_r\}$, $X' = \{x'_1, \dots, x'_r\}$, $Y = X \cup X'$. Let $\{\text{word on } Y\} = \text{Word}(Y)$.
Let $u, v \in \text{Word}(Y)$. If $u = vx_i x'_i$ then u, v are equivalent. Denoted by $u \sim v$.

If $u = v_1 x_i x'_i x_j x'_j v_2$ then u is equivalent to $v_1 v_2$. Denoted by $u \sim v_1 v_2$.

Thus we have an equivalent relation on $\text{Word}(Y)$.

Let \bar{W} be a equivalent class of $w \in \text{Word}(Y)$. Define $G = \{\bar{w} | w \in \text{Word}(Y)\}$. For $\bar{w}_1, \bar{w}_2 \in G$, let $\bar{w}_1 \bar{w}_2 = \overline{w_1 w_2}$. Then G is a group. Called a finite generated free group of rank r .

Example 16. Let $G = \langle a, \sigma \rangle = D_{20}$ There is a free group F of rank 2 s.t. G is a homomorphism image of F s.t. $\phi : x_1 \mapsto a, x_2 \mapsto \sigma, x'_1 \mapsto a^{-1}, x'_2 \mapsto \sigma^{-1}, \emptyset \mapsto 1$, $\ker(\phi) = \langle x_1^5, x_2^2, x_1 x_2 x_1 x_2 \rangle$. i.e. $F / \ker(\phi) \simeq G$.