A central result in the study of unbiased estimates, the Cramér-Rao Inequality, shows the minimum variance achievable by an unbiased estimate. –D. R. Cox

Problems 41-50 on More Estimation Theory (STAT2802 Statistical Models Tutorial notes for the week of 5-NOV-2012)

(Assume all likelihoods subjected here are well behaved and are at least differentiable twice with respect to the parameter.)

41. If the likelihood is $L(\theta; x)$, what is its score function $S(\theta; x)$? Regarding S as a transformation of the random variable X (by capitalizing the data argument), and fix $\theta = \theta_0$, then $S(\theta_0; X)$ should have an expectation. Find this expectation.

42. If the likelihood is $L(\theta; x)$, what is its observed information $J(\theta; x)$? What is its Fisher information $I(\theta; x)$? Why do people say the Fisher information is the same as the variance of the score function?

43. If the likelihood is $L(\theta; x)$, let S and I denote the score function and the Fisher information and let $\hat{\theta}$ be an estimator for θ, show that $\mathbb{V}(\hat{\theta}) \geq \frac{\mathbb{E}(\hat{\theta}S)^2}{I}$. Then show that $\mathbb{E}(\hat{\theta}S) = 1 + \frac{\partial b(\hat{\theta})}{\partial \theta}$ where $b(\hat{\theta})$ is the bias of the estimator $\hat{\theta}$. Now if $\hat{\theta}$ is required to be unbiased for θ, then what is a lower bound for the variance of the unbiased $\hat{\theta}$?

44. What is a *uniformly minimum variance unbiased estimator (UMVUE)*? Why is it *unique*, if exists? What is a lower *bound* for that minimum variance? What is the role of the parameter's Ln-Likelihood to play with this UMVUE?

45. Describe the sufficiency of a statistic verbally without citing Neyman's factorization. Then, define the sufficiency of a statistic via Neyman's factorization.

46. Describe completeness of a statistic verbally based on the basic requirement of identifiability of any parameter's true value by the statistic. Then, define the completeness of a statistic, formally, using expectation.

47. Let X and Y be two random variables. Show that $\mathbb{E}(Y|X)$, as a random variable, reduces variance of Y while maintaining its mean, i.e., show that $\mathbb{V}\big(\mathbb{E}(Y|X)\big) \leq \mathbb{V}(Y)$ and that $\mathbb{E}\big(\mathbb{E}(Y|X)\big) = \mathbb{E}(Y)$. When is $\mathbb{V}\big(\mathbb{E}(Y|X)\big) = \mathbb{V}(Y)$?

~~48. Let T be a complete sufficient statistic on the data. Show that for any function h of T, its variance, $\mathbb{V}\big(h(T)\big)$, cannot be reduced by conditioning on T.~~

49. Establish a way for constructing the UMVUE by considering the connection between a complete sufficient statistic and UMVUE. That is, please prove the Lehmann-Scheffé Theorem.

50. Let $X \sim Poisson(\lambda)$, $\lambda > 0$. The statistic $T(X) := \mathbb{I}(X = 0)$ (i.e., T indicates whether $X = 0$) is used to estimate $q(\lambda) = e^{-\lambda}$. Show that T is the UMVUE (Hint: by Lehmann-Scheffé) and yet $\mathbb{V}(T)$ does not attain the Cramér-Rao Lower Bound. Note that the sample size here is only 1.