Homework 8 (Due November 21)

Grade Distribution (Total=12+8+12+10=42).

1. If X and Y are independently and identically distributed uniform random variables on $(0, 1)$, compute the joint density of

   (a) $U = X + Y, V = X/Y$;

   (b) $U = X, V = X/Y$;

   (c) $U = X + Y, V = X/(X + Y)$.

2. If X, Y, and Z are independent random variables having identical density functions $f(x) = e^{-x}, 0 < x < \infty$, derive the joint distribution of $U = X + Y, V = X + Z, W = Y + Z$.

3. Let $X_1, \cdots, X_n$ be a set of independent and identically distributed continuous random variables having cumulative distribution function $F(x)$, and let $X_{(i)}$, $i = 1, \cdots, n$ denote their ordered values. If X, independent of the X_i , $i = 1, \cdots, n$, also has cumulative distribution function F, determine

   (a) $P(X > X_{(n)})$;

   (b) $P(X > X_{(1)})$;

   (c) $P(X_{(i)} < X < X_{(j)}), 1 \leq i < j \leq n$.

   [Hint: For any $1 \leq i \leq n$, $P(X_i = X_{(n)}) = P(X_i \text{ is the max}) = 1/n$ by symmetry.]

4. Let $X_{(1)}, X_{(2)}, X_{(3)}$ be the ordered values of 3 independent uniform $(0, 1)$ random variables. Prove that for $1 \leq k \leq 4$,

$$P(X_{(k)} - X_{(k-1)} > t) = (1 - t)^3, \quad \forall t \in (0, 1),$$

   where $X_{(0)} = 0$ and $X_{(4)} = 1$.