陈恺. 12311410 Complex Analysis (H) Homework 3.

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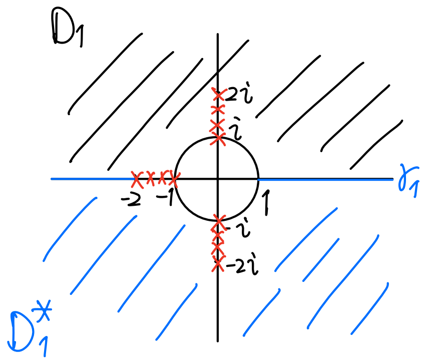
$\boxed{1.}$

Solution:

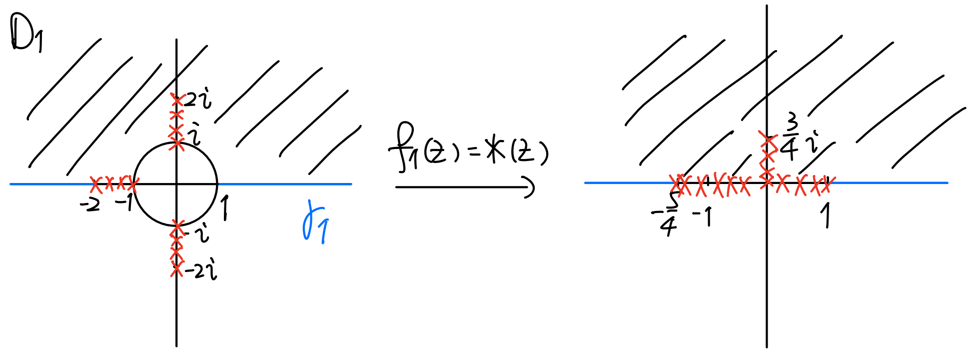



**1.** Find a conformal mapping, transforming the domain $D := \{|z| > 1\} \setminus ([-2,-1] \cup [i, 2i] \cup [-2i, -i])$ onto $\Pi^+$.

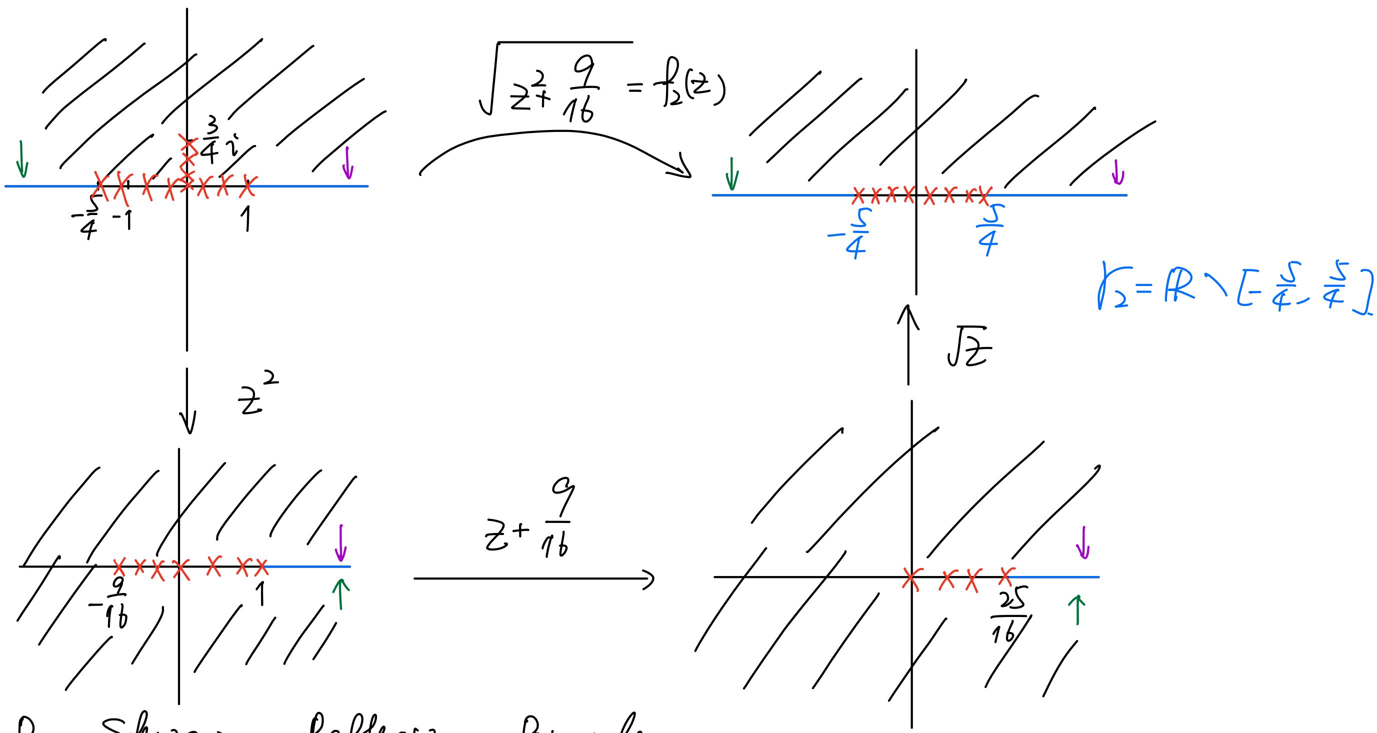
$\gamma_1 = \mathbb{R} \setminus [-2, 1]$.

Step 1: $*(\partial B_1(0)) = [-1, 1]$. $*([i, 2i]) = [0, \frac{3}{4}i]$

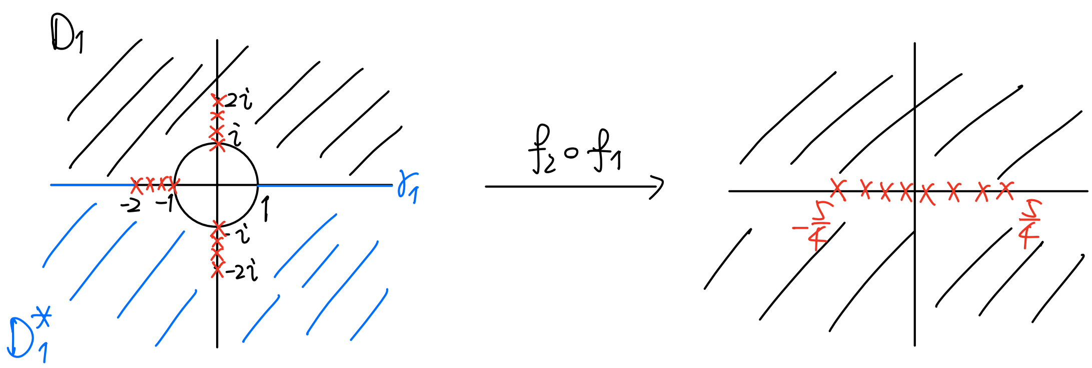
$*([-2,-1]) = [-\frac{5}{4}, -1]$. $*(\mathbb{R}) = \mathbb{R}$. $*(\gamma_1) = \mathbb{R} \setminus [-\frac{5}{4}, 1]$.



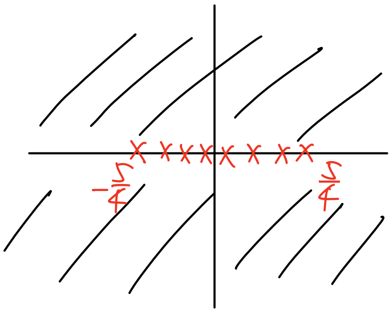

Step 2:



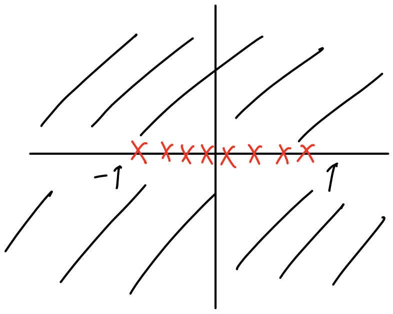

Step 3. By Schwarz Reflection Principle.



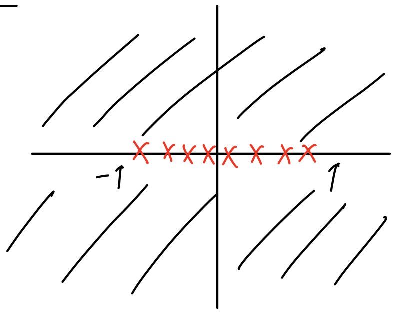
Step 4.



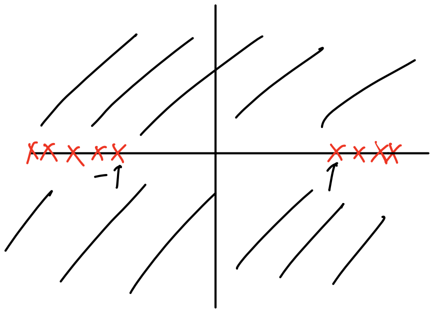

$\xrightarrow{f_3(z)=\frac{4}{5}z}$



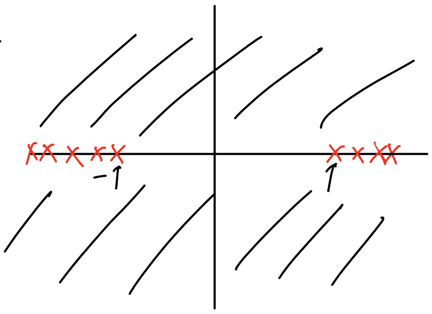

Step 5.



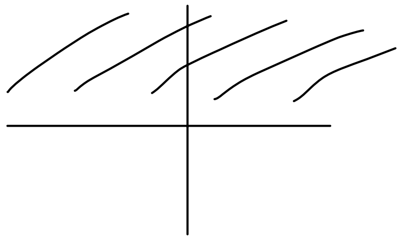

$\xrightarrow{f_4(z)=\frac{1}{z}.}$




Step 6.




$\xrightarrow{*^{-1}=f_5.}$



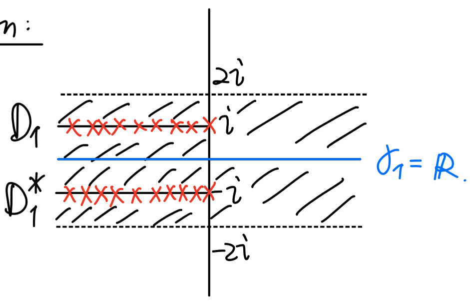
$\Pi^+$.

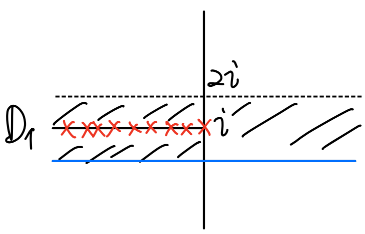
To sum up, $f=f_5\circ f_4\circ f_3\circ f_2\circ f_1$.

$\boxed{2.}$

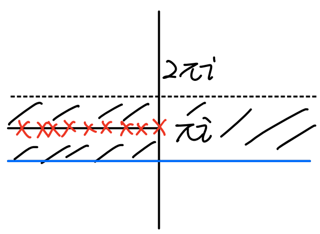
**2.** Find a conformal mapping, transforming the domain $D:=\{-2<\operatorname{Im} z<2\}\setminus\{\operatorname{Im} z=\pm1, \operatorname{Re} z\leq 0\}$ onto $\Pi^+$.

Solution:

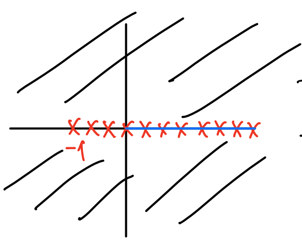


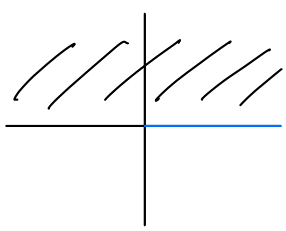

$\xrightarrow{f_1(z)=\pi z}$



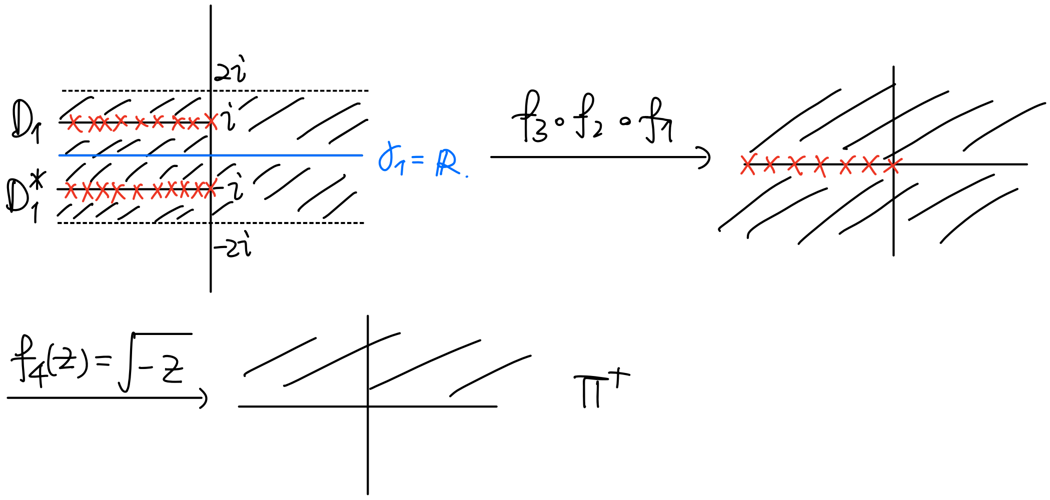

$\xrightarrow{f_2(z)=e^z}$




$\xrightarrow{f_3(z)=\sqrt{z+1}}$



By Schwarz Reflection Principle.




To sum up. $f = f_4 \circ f_3 \circ f_2 \circ f_1$.

$\boxed{4.}$ **4.** Detemine the group Aut $(\Omega)$, where $\Omega = \{1 < |z| < 2\}$ (you can use the Caratheodory Theorem for admissible domains).

<u>Solution:</u> $C_1 = \{|z| = 1\}$. $C_2 = \{|z| = 2\}$.

For $f \in Aut(\Omega)$, $f(\partial\Omega) = \partial\Omega$.

$\Rightarrow \begin{cases} f(C_1) = C_1 \\ f(C_2) = C_2 \end{cases}$ or $\begin{cases} f(C_1) = C_2 \\ f(C_2) = C_1. \end{cases}$

Since $f: \Omega \xrightarrow{conf} \Omega$, by Shwartz Reflection Principle,

we can reflect $\Omega$ across $C_1$ and $C_2$ repeatedly.

① Across $C_1$: $\Omega \to \Omega_{-1} = \{\frac{1}{2} < |z| < 1\}$

② Across $C_2$: $\Omega \to \Omega_2 = \{2 < |z| < 4\}$

By reflecting $\Omega$ repeatedly across $C_1$ and $C_2$ we finally get $\mathbb{C} \setminus \{0\}$.

i.e. $\forall f \in Aut(\Omega)$, it can be analytically continued to a func. that is hol. in $\mathbb{C} \setminus \{0\}$.

The biholomorphic automorphisms of $\mathbb{C} \setminus \{0\}$ are of two kinds:

① $g(z) = az$, $a \in \mathbb{C}$, $a \neq 0$. ② $g(z) = \frac{a}{z}$, $a \in \mathbb{C}$, $a \neq 0$.

Now, we restrict back to $\Omega$.

① $f(z) = az$.

1) $f(C_1)=C_1,\ f(C_2)=C_2.$

$\Rightarrow |a|=1 \Rightarrow a=e^{i\theta},\ \theta\in\mathbb{R}. \qquad f(z)=e^{i\theta}z.$

2) $f(C_1)=C_2,\ f(C_2)=C_1.$

$\Rightarrow |a|=2\ \&\ |a|=\frac{1}{2}.\quad ↯.$

② $f(z)=\frac{a}{z}.$

1) $f(C_1)=C_1,\ f(C_2)=C_2.$

$\Rightarrow |a|=1\ \&\ |a|=4.\quad ↯.$

2) $f(C_1)=C_2,\ f(C_2)=C_1.$

$\Rightarrow |a|=2. \quad \Rightarrow a=2e^{i\theta},\ \theta\in\mathbb{R}. \qquad f(z)=\frac{2e^{i\theta}}{z}.$

$\Rightarrow \mathrm{Aut}(\Omega)=\left\langle e^{i\theta_1}\cdot z,\ \frac{2e^{i\theta_2}}{z}\right\rangle,\quad \theta_1,\theta_2\in\mathbb{R}.$

$\boxed{5.}$ **5.** Find the index w.r.t. 0 of the parameterized curve $f(C)$, where $f(z)=z^3+2z$ and $C$ is the standardly parameterized circle $\{z=e^{it},\ 0\le t\le 2\pi\}$.

Solution: $z^3+2z=0 \iff z(z+i\sqrt{2})(z-i\sqrt{2})=0.$

3 distinct zeros: $z=0,\ z=i\sqrt{2},\ z=-i\sqrt{2}.$

In $B_1(0)$: only one zero: $z=0$. No poles.

$\mathrm{ind}_0 f(C)=\#\text{zeros}(f)-\#\text{poles}(f)=1-0=1.$

$\boxed{6.}$ **6.** How many roots (with multiplicities) does the equation $z^6-6z+10=0$ have in the domain $\{|z|>2\}$?

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Solution: $N := N_{|z|>2}+N_{|z|\le 2}=A+B.$

$A+B=6.$

Let $f(z)=z^6,\ g(z)=-6z+10.$

$\forall z\in\{|z|=2\}$

$|f(z)|=|z^6|=64$

$|g(z)|\le 12+10=22<64.$

By Rouche Thm, $\#\text{zeroes}(f+g)=\#\text{zeroes}(f)$ in $\{|z|\le 2\}.$

$\Rightarrow B=\#\text{zeros}(f)=6$ (with multiplicities)

$\Rightarrow A = 6 - 6 = 0.$

$\boxed{8.}$ **8.** Prove that the equation $\tan z = z$ has only real roots.

Proof: Assume $x+iy$ is a root. $\tan(x+iy) = x+iy$

WLOG, let $y>0$. $\Rightarrow \overline{\tan(x+iy)} = \overline{x+iy}$. $\Rightarrow \tan(x-iy) = x-iy$

$\Rightarrow x-iy$ is also a root.

$$\begin{cases} \tan(x+iy) = x+iy \\ \tan(x-iy) = x-iy \end{cases} \Rightarrow \begin{cases} \mathrm{Re}\tan(x+iy) = x \\ \mathrm{Im}\tan(x+iy) = y \end{cases} \Rightarrow \begin{cases} x = \dfrac{\sin(2x)}{\cos 2x + \cosh 2y} \\ y = \dfrac{\sinh 2y}{\cos 2x + \cosh 2y} \end{cases}$$

$$\Rightarrow x = \frac{\sin(2x)}{\cos 2x + \cosh 2y} = \frac{\sin 2x}{\frac{\sinh 2y}{y}} \Rightarrow \sin(2x) = x\frac{\sinh(2y)}{y}.$$

① $x=0$.

$\tan(iy) = iy \Rightarrow \tanh(y) = y. \quad y>0.$

$g(y) = \tanh(y) - y.$

$g'(y) = \tanh^2 y > 0 \quad g(0) = 0. \Rightarrow$ only $y=0$ satisfies $\times$

② $x \neq 0$.

$$\frac{\sin 2x}{2x} = \frac{\sinh 2y}{2y}$$

let $h(t) = \begin{cases} \frac{\sin(t)}{t}, & t\neq 0 \\ 1, & t=0. \end{cases}$ $k(t) = \begin{cases} \frac{\sinh t}{t}, & t\neq 0 \\ 1, & t=0. \end{cases}$

$\Rightarrow k(u) = \frac{1}{u}\left(u + \frac{u^3}{3!} + \cdots\right) = 1 + \frac{u^2}{3!} + \frac{u^4}{5!} + \cdots \quad , u\neq 0$

$h(u) = \frac{1}{u}\left(u - \frac{u^3}{3!} + \cdots\right) = 1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \cdots \quad , u\neq 0.$

since $\left|\frac{\sin x}{x}\right| < 1, x\neq 0, \quad h(2x) < 1. \Rightarrow k(2y) < 1.$

but $k(2y) = 1 + \frac{(2y)^2}{3!} + \cdots > 1. \ (y>0) \Rightarrow$ Contrad. $\times$

In conclusion, no solution for $x, y \in \mathbb{R}$. $\Rightarrow$ No complex roots. $\square$

$\boxed{7.}$ $z^4+z^3-4z+1=0.$

Pf: let $N_1 = \#$ roots in $\{|z|<1\}$.

$f(z)=-4z \qquad g(z)=z^4+z^3+1.$

On $|z|=1$:

$|f(z)|=4. \quad |g(z)| \leq |z^4|+|z^3|+1=3<4=|f(z)|.$

By Rouche Thm. $N_1 = \#$ zeros of $f$ in $\{|z|<1\}$

$= 1.$

let $N_2 = \#$ roots in $\{|z|<2\}$.

$f(z)=z^4+1. \qquad g(z)=z^3-4z.$

$|g(z)|^2 = |z|^2|z^2-4|^2 = 4|4e^{2i\theta}-4|^2$

$= 64|(\cos 2\theta-1)+i\sin 2\theta|^2 = 128(1-\cos 2\theta)$

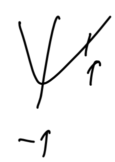
$|f(z)|^2 = |16e^{i4\theta}+1|^2 = |(16\cos 4\theta+1)+i16\sin 4\theta|^2$

$=257+32\cos 4\theta$

$|g(z)|^2<|f(z)|^2 \iff 128(1-u)<257+32(2u^2-1)$. $u=\cos 2\theta$. $u\in[-1,1]$.

$\iff 64u^2+128u+97>0.$ 对称轴: $-\frac{128}{128}=-1.$ LHS $\geq 64-128+97>0.$

the inequality holds.



By Roche Thm, $N_2 = \#$ zeros of $f(z)$ in $\{|z|<2\}$.

$=4.$

$\Rightarrow N_2-N_1=4-1=3.$

$\square.$