Homework-11

November 28, 2024

Remember, the most important thing is to understand how to solve these problems, not just to know their answers. When assigning homework, I won't change these questions. However, while you're working on them, always think about how you would solve the problem if I were to change some numbers or conditions?

1. Find $\mathrm{Irr}(a, F)$:
(1). $a = \sqrt{2} + \sqrt{3}$, $F = \mathbb{Q}(\sqrt{6})$;
(2). $a = \sqrt{2} + \sqrt{3}$, $F = \mathbb{Q}(\sqrt{2})$;
(3). $a = \sqrt{2} + \sqrt{3}$, $F = \mathbb{Q}$.

2. Let K/F be an extension of feilds.
(1). Let $a \in K$, if $a \in F(a^m)$ where $m > 1$. Prove a is algebraic over F;
(2). If $a \in K$ is a algebraic element over F of odd degree. Prove $F(a) = F(a^2)$.

3. Let u be a real root of $x^3 - 6x^2 + 9x + 3$.
(1). Prove $[\mathbb{Q}(u) : \mathbb{Q}] = 3$;
(2). Represent u^4, $(u+1)^{-1}$, $(u^2 - 6u + 8)^{-1}$ as $\mathbb{Q}-$linear combination of $\{1, u, u^2\}$.

4. Let K be a field. If $x^n - a \in K[x]$ is irreducible, prove for any positive factor m of n, $x^m - a$ is also irreducible in $K[x]$.

5. Let K be a field, x is transcendental over K, $u \in K(x)$, $u \notin K$. Prove x is algebraic over $K(u)$.

6. Prove $\mathrm{Aut}(\mathbb{R}) = \{id\}$. i.e. If σ is a field automorphism of $\mathbb{R}$ then $\sigma = id$.

7. Let L/F be a field extension. E, K be two intermediate fields of this extension, prove:
(1). $[EK : F]$ is finite iff $[E : F]$ and $[K : F]$ are all finite;
(2). $[EK : F] \leq [E : F][K : F]$;
(3). If $[E : F]$ and $[K : F]$ are coprime, then $[EK : F] = [E : F][K : F]$.

8. Construct a finite field with 8 elements and write out its addition table and multiplication table.

9. Let $f(x) = x^2 + 1$ and $g(x) = x^2 - x - 1$.
(1). Prove f, g are all irreducible in $\mathrm{GF}(3)[x]$;
(2). Let α, β denote a root of $f(x)$ and $g(x)$ in $\mathrm{GF}(9)$ respectively. Provide an isomorphism from $\mathrm{GF}(3)(\alpha)$ to $\mathrm{GF}(3)(\beta)$.

10. (1). Prove $\mathrm{GF}(p^m) \subseteq \mathrm{GF}(p^n)$ iff $m \mid n$;
(2). In $\mathrm{GF}(p)[x]$, prove $x^{p^m} - x \mid x^{p^n} - x$ iff $m \mid n$.