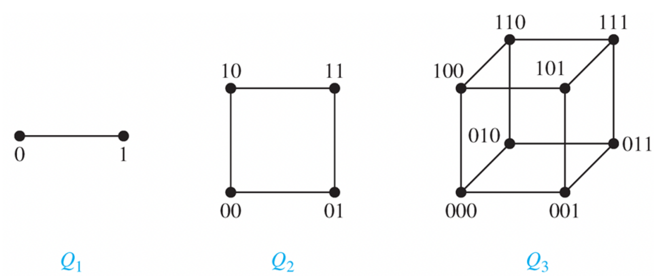
CS201: Discrete Math for Computer Science
2025 Spring Semester Written Assignment #5
Due: May 26th, 2025

The assignment needs to be written in English. Assignments in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1. An n-dimensional hypercube, or n-cube, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position. Let $l(n)$ denote the number of edges of Q_n.



(a) What is the initial condition of $l(n)$?

(b) What is the recursive function of $l(n)$?

(c) Derive the closed-form of $l(n)$ using the general approach we have learned for solving linear recurrence relation. Please provide the derivation details. Please do NOT use mathematical induction.

Solution:

(a) $l(1) = 1$;

(b) $l(n) = 2l(n-1) + 2^{n-1}$;

(c) First, compute the solution to the associated homogeneous recurrence relation $l^{(h)}(n)$. Since the characteristic equation is $r - 2 = 0$. Thus, $l^{(h)}(n) = \alpha 2^n$. Second, compute the particular solution $l^{(p)}(n)$. According to the formulation of $l(n)$, the particular solution is in the following form: $l^{(p)}(n) = pn2^n$. Substituting $l^{(p)}(n)$ into the recurrence relation of $l(n)$, we have

$$pn2^n = 2p(n-1)2^{n-1} + 2^{n-1}.$$

Thus, $p = 1/2$. Thus, the closed-form solution

$$l(n) = l^{(h)}(n) + l^{(p)}(n) = \alpha 2^n + \frac{1}{2} n 2^n.$$

By substituting the initial condition,

$$l(1) = 2\alpha + 1 = 1.$$

Thus, $\alpha = 0$. As a result $l(n) = n2^{n-1}$.

Q. 2. Consider 10 identical balloons (i.e., non-distinguishable balloons). We aim to give these balloons to four children, and each child should receive at least one balloons.

(a) How many ways to give these balloons to the children? Explain the reason.

(b) The answer to the above question is the coefficient of term ____________ of generating function ________________.

Solution:

(a) $C(9, 6)$

(b) x^{10}; $(x^1 + x^2 + x^3 + \cdots)^4$ or $(x^1 + x^2 + x^3 + \cdots + x^{10})^4$ or $(x^1 + x^2 + x^3 + \cdots + x^7)^4$ (any of these functions is ok)

Or alternatively, x^6; $(1 + x^1 + \cdots)^4$ or $(1 + x^1 + \cdots + x^6)^4$ (any of these functions is ok)

Q. 3. How many relations are there on a set with n elements that are

a) antisymmetric?

b) irreflexive?

c) neither reflexive nor irreflexive?

d) symmetric, antisymmetric and transitive?

Please explain your answer.

Solution:

a) $2^n 3^{n(n-1)/2}$

b) $2^{n(n-1)}$

c) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

d) 2^n

$\square$

Q. 4. Prove or disprove the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.

Solution: Counterexample: Consider $A = \{1, 2, 3\}$ and

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}.$$

Then R is symmetric and reflexive, but not transitive.

$\square$

Q. 5. Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

Solution: Suppose that $(a, b) \in R$. Because $(b, b) \in R$, it then follows that $(a, b) \in R^2$. Thus, R is a subset of R^2.

$\square$

Q. 6. Let R_1 and R_2 be symmetric relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also be symmetric? Explain your answer.

Solution: Yes. Yes. For both R_1 and R_2, the corresponding 0-1 matrices are both symmetric. Thus. the two matrices representing $R_1 \cap R_2$ and $R_1 \cup R_2$ are also symmetric.

$\square$

Q. 7. Show that $\{(x, y)|x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Solution: This relation is reflexive, since $x - x = 0 \in \mathbb{Q}$. To see that it is symmetric, suppose that $x-y \in \mathbb{Q}$. Then $y-x = -(x-y)$ is again a rational number. For transitivity, if $x-y \in \mathbb{Q}$ and $y-z \in \mathbb{Q}$, then their sum, namely $x-z$, is also rational (the rational numbers are closed under addition). The equivalence class of 1 and of 1/2 are both just the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number, in other words, $\{\pi + r | r \in \mathbb{Q}\}$.

$\square$

Q. 8. Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preceq$ on $\mathbf{R}(S)$ by $R_1 \preceq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $\mathbf{R}(s), \preceq)$ is a poset.

Solution: The subset relation is a partial ordering on any collection of sets, because it is reflexive, antisymmetric, and transitive. Here the collection of sets is $\mathbf{R}(S)$.

$\square$

Q. 9. Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both equivalence relations, then T is also an equivalence relation.

Solution: We need to show that T is reflexive, symmetric, and transitive.

Reflexive: For any x, we have $(x, x) \in R$ and $(x, x) \in S$, then $(x, x) \in T$.

Symmetric: Suppose that $(x, y) \in T$. This means $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both symmetric, we have $(y, x) \in R$ and $(y, x) \in S$. Then $(y, x) \in T$.

Transitive: Suppose that $(x, y) \in T$ and $(y, z) \in T$. Then $(x, y) \in R$ and $(y, x) \in R$ imply that $(x, z) \in R$. Similarly, we have $(x, z) \in S$. This will imply that $(x, z) \in T$.

$\square$

Q. 10. Suppose that the relation R is symmetric. Show that R^* is symmetric.

Solution: The result follows from

$$(R^*)^{-1} = (\cup_{n=1}^{\infty} R^n)^{-1} = \cup_{n=1}^{\infty} (R^n)^{-1} = \cup_{n=1}^{\infty} R^n = R^*.$$

$\square$