1) NO. If such $f$ exists, then $f^{-1}$ is a hol map of $\mathbb{C}$ onto square $\Rightarrow f^{-1}$ is entire and bounded $\Rightarrow f^{-1} = \text{const}$ – contrad.

4) Consider $\forall$ disc $\overline{B_R(a)} \subset D$. Take $N$: $\forall j \geq N$, $f_j - f$ is nonsing on $K$ ($K = \overline{B_R(a)}$), and $f_j - f \overset{K}{\rightrightarrows} 0$. Since $f_k - f_\ell$ with $k, \ell \geq N$ are nonsing. on $K$, all $\{f_j\}_{j \geq N}$ have sing.-s on the same set $\Rightarrow$ have the same finite set $E$ of poles. So, in partic, $f_j \in \mathcal{O}(B_R(a) \setminus E) \Rightarrow$ since $f_j \overset{K}{\rightrightarrows} f$, we conclude that $f \in \mathcal{O}(B_R(a) \setminus E)$ by Weierstrass Thm. Since $f_N - f$ is nonsing on $K$, $f$ has poles (at most) on $E$. So, $f \in \mathrm{Mer}(D)$.

7) Take an integer $N > a$. Then $\forall$ disc $B_R(0)$ we apply the Cauchy inequalities and get:
$|c_n| \leq \frac{\max_{|z|=R} |f|}{R^n}$; using $\frac{|f(z)|}{|z|^a} \leq M$ for a constant $M > 0$, we have for $n \geq N$: $|c_n| \leq \frac{M \cdot R^a}{R^n} = M R^{a-n} \xrightarrow{R \to +\infty} 0$ $\Rightarrow c_n = 0$ for $n \geq N$. (Here $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in $\mathbb{C}$).
So, $f(z)$ is a polyn of deg $\leq N-1$

8) For $n = 2k$, $k \in \mathbb{N}$ we have: $f(\frac{1}{2k}) = \frac{1}{2k}$, so $f(z) = z|_{z \in \{\frac{1}{2k}\}_{k=1}^{\infty}}$. Hence by uniqueness Thm, $f(z) \equiv z$ (since $\frac{1}{2k} \xrightarrow{k \to \infty} 0 \in B_1(0)$). But

12) The choice $\operatorname{Arg} w \in (-\pi, \pi)$ in the domain $\mathbb{C} \setminus ((-\infty, 0])$ (where $w = 1+z$) allows for choosing a hol branch of $\operatorname{Ln} w$ in the same domain, which agrees with the real log-func. Then $f(x) = x \ln(1+x)$ extends using the chosen branch to $F(z) = z \ln(1+z)$, holomorphic in $\{z \notin (-\infty, -1]\}$.
By uniq. thm, the extension is unique.
However, $F(z)$ doesn't extend hol-ly to $\mathbb{C}$ (for example, since $F(z) \to \infty$ as $\underset{z \in \Pi^+}{z \to -1}$)

15) Note that if $|z| > 1$, then $1 + \frac{1}{z^2} \in \{w : \operatorname{Re} w > 0\} = \Omega$.
(since $\operatorname{Re}(1 + \frac{1}{z^2}) \geq 1 - \frac{1}{|z|^2} > 0$). Since $\Omega = S_{-\frac{\pi}{2}, \frac{\pi}{2}}$, it admits a single-valued branch $\varphi(w)$ of $\sqrt{w}$.
Set $f(z) := z\varphi(1 + \frac{1}{z^2}) \left(= z\sqrt{1 + \frac{1}{z^2}}\right)$.
Clearly, $f^2(z) = z^2(1 + \frac{1}{z^2}) = 1 + z^2$. So, $f$ is as desired. $\blacksquare$

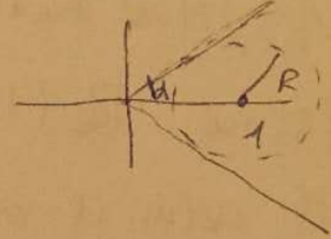
16) Consider $g(z) := \frac{1}{f(z)} \in \mathcal{O}(\{|z| > 1\})$.
We have $|g(z)| \leq \frac{1}{M}$, so that $z = \infty$ is an isolated singul. for $g(z)$. Hence
$\exists \lim_{z \to \infty} g(z) = \left[\begin{array}{l} 0 \\ A, |A| > 0 \end{array}\right. \Rightarrow \exists \lim_{z \to \infty} f(z) = \left[\begin{array}{l} \infty \\ \frac{1}{A}. \end{array}\right.$

this contradicts $f(\frac{1}{3}) = -\frac{1}{3}$. So, such $f$ doesn't exist.

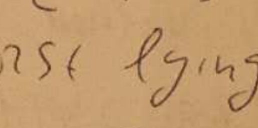
9) Consider $g(z) := f(z) - z\bar{z}$. Then $g \in C^1(\mathbb{C})$, and $\frac{\partial g}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}} - z = 0 \Rightarrow g \in \mathcal{O}(\mathbb{C}) \Rightarrow g \in C^\infty(\mathbb{C})$.

But $f = z\bar{z} + g \Rightarrow f \in C^\infty(\mathbb{C})$.

10) Let $w = z^2$. Then $E = \{|z^2 - 1| < R\}$ is the image of the disc $B_R(1)$ under the 2-valued map $z = \sqrt{w}$.




If $R \leq 1$, then $B_R(1) \subset S_{-\alpha,\alpha}$ for some $\alpha \in (0, \frac{\pi}{2}]$, hence $E$ has 2 connected comp-s, first lying in $S_{-\frac{\alpha}{2}, \frac{\alpha}{2}}$, second lying in $S_{\frac{\pi-\alpha}{2}, \frac{\pi+\alpha}{2}}$ (each is the conformal image under $\sqrt{w}$ of $B_R(1)$). So, $E$ is disconnected.

For $R > 1$, let's represent $B_R(1) = \Omega^+ \cup \Omega^- \cup I$, where $\Omega^\pm \subset \Pi^\pm$ are domains (in fact half-discs), and $I = (1-R, 1+R) \ni 0$. If $f(w) = \sqrt{w}$, then $f^{-1}(I)$ is a cross:  (with center $z = 0$), while $f^{-1}(\Omega^\pm)$ is the union of 4 domains having 0 on their boundaries. Hence $E = f^{-1}(B_R(1))$ is linearly conn-d $\Rightarrow$ conn-d.

17) Since $f \in C(\bar{D})$, we conclude that $z = 0$ is an isolated sing. for $f$. Put $g(z) := f(z) - 1$.
Then $g \in \mathcal{O}(B_1(0)) \cap C(\overline{B_1(0)})$, and $g|_{\partial B_1(0)} = 0$.
Hence, by the Max. Princ., $g \equiv 0$ in $B_1(0)$, so that $f \equiv 1$ in $D$.

2. Let $T$ be any closed Triangle ~~connected~~ contained in $D$.

10 Since $f$ is continuous in $D$, it is bounded on the compact set $T$

Case 1: $T$ doesn't intersect $\gamma$

since $T \subseteq D$, $T \cap \gamma = \phi$. $T$ is contained in the region $D \setminus \gamma$ where $f$ is holomorphic

by Cauchy Theorem, $\int_T f(z)\,dz = 0$

Case 2: $T$ intersects $\gamma$

since $\gamma$ is a finite union of line segements

we can subdivided $T$ into smaller trangles $\{T_i\}$ (finite)

such that each $T_i$ either avoid $\gamma$ or overlap with $\gamma$

at edges / vertices

by Cauchy Thm, $\int_{T_i} f(z)dz = 0$

for $T_i$ overlapping with $\gamma$.


in each condition, we can approximate $T_i$ by the perturbed trangle $T_i'$

by the continuity of $f$

the integral difference satifies:

$\left| \int_{T_i} f(z)dz - \int_{T_i'} f(z)dz \right| \to 0$ (as perturbations vanish)

since $\int_{T_i'} f(z)dz = 0$ (Cauchy Thm)

we have $\int_{T_i} f(z)dz = 0$ holds for $\forall T_j \in \{T_i\}$

summing over all sub-trangles (each with proper direction $(T_j^+ / T_j^-)$)

$\Rightarrow \int_T f(z)dz = \sum_{T_j \in \{T_i\}} \int_{T_j^\pm} f(z)dz = 0$

In conclude, $f \in C(D)$, satifying the Morera property

by a Theorem proved in class, $f \in \mathcal{O}(D)$

3. On the circle $\{|z|=1\}$ $\bar{z}=\frac{1}{z}$ $(z\cdot\bar{z}=|z|=1)$

$f(z)=\bar{z} \Leftrightarrow f(z)=\frac{1}{z}$

Suppose, there exists a sequence of polynomials $\underline{\{P_n(z)\}}$ that converges uniformly to $\frac{1}{z}$ on $\{|z|=1\}$ $\quad P_n(z) \overset{|z|=1}{\rightrightarrows} f(z)$

it's only related to $\varepsilon$.

Consider $\int_{|z|=1} P_n(z)dz$ and $\int_{|z|=1} f(z)dz$ $\quad (\Rightarrow)$ for $\forall \varepsilon>0$, $\exists\, \underline{N(\varepsilon)}$, when $n>N(\varepsilon)$,

for any polynomial $P_n(z)$, the integral $\quad$ we have $|f(z)-P_n(z)|<\varepsilon$

$\int_{|z|=1} P_n(z)dz=0$ by Cauchy's theorem. (since polynomials are entire functions)

$P_n(z)\in\mathcal{O}(\mathbb{C})$

however, $\int_{|z|=1} f(z)\,dz = \int_{|z|=1}\frac{1}{z}dz=\int_0^{2\pi}\frac{i\cdot e^{i\theta}}{e^{i\theta}}d\theta=2\pi i\neq 0$

for $\forall\varepsilon>0$, $\exists N(\varepsilon)$, when $n>N(\varepsilon)$, we have $|f(z)-P_n(z)|<\varepsilon$

but

$2\pi=|2\pi i-0| = \left|\int_{|z|=1}(f(z)-P_n(z))dz\right| \le \int_{|z|=1}|f(z)-P_n(z)|\,dz \le \int_{|z|=1}\varepsilon\, dz=0$ it's a contradiction.

10 pts

therefore, $f(z)=\bar{z}$ cannot be uniformly approximated by polynomials on $\{|z|=1\}$. $\square$

6. Let $g(z)=z\cdot f(z)$ since $f(z)$ is holomorphic in $B_1^*(0)$

$g(z)$ is also holomorphic there.

$|z|^{\frac{1}{2}}\cdot|f(z)|\le \underset{\text{constant}}{C}$ by the given condition

$|g(z)|=|f(z)\cdot z|=|f(z)|\cdot|z|\le C\cdot|z|^{\frac{1}{2}}$

let $z\to 0$, $|g(z)|\to 0$. So $g(z)$ is bounded and approaches 0 near $z=0$

$\Rightarrow \lim\limits_{z\to 0} g(z)=0 \Leftrightarrow z=0$ is removable for $g(z)$

$\Rightarrow g(z)$ extends $\underline{\text{holomorphically}}$ to $z=0$ with $g(0)=0$

$\Rightarrow g(z)$ has Taylor expansion at $z=0$:

constant term.

$g(z)=a_1z+a_2z^2+az^3+\cdots$ for $z\in B_1(0)$ $\Big($since $g(0)=0\Rightarrow a_0=0\Big)$

then $f(z)=\frac{g(z)}{z}=a_1+a_2z+a_3z^2+\cdots$ for $z\in(\cap(B_1^*(0)))$, $\underline{\underline{f(0)=a_1}}$

10 pts

This series converges $\underset{\substack{\Uparrow \\ \text{it's a classic power series.}}}{\underline{\text{in } B_1(0)}}$ (it's an equivalent definition of holomorphical)

So $f(z)$ is holomorphical in $B_1(0)$.

11. for any $z\in\overline{B_r(a)}$ (with $0<r<R$), let $\Gamma=\{|\xi-a|=\rho\}$ where $\rho=\frac{r+R}{2}$

$f_n'(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f_n(\xi)}{(\xi-z)^2}d\xi$ (CIF for derivatives)


for $\xi\in\Gamma$, $|\xi-z|\ge\rho-r=\frac{R-r}{2}$ since $|f_n(\xi)|\le \underset{\text{uniformly bounded.}}{C}$ $\Rightarrow \left|\frac{f_n(\xi)}{(\xi-z)^2}\right|\le\frac{C}{\left(\frac{R-r}{2}\right)^2}=\frac{4C}{(R-r)^2}$

$|f_n'(z)|=\left|\frac{1}{2\pi i}\int_\Gamma\frac{f_n(\xi)}{(\xi-z)^2}d\xi\right|\le\frac{1}{2\pi}\cdot\int_\Gamma\left|\frac{f_n(\xi)}{(\xi-z)^2}\right|d\xi\le\frac{1}{2\pi}\cdot 2\pi\cdot\rho\cdot\frac{4C}{(R-r)^2}=\frac{4C\rho}{(R-r)^2}$

for example, chosing $r=\frac{R}{2}$, $\rho=\frac{3}{4}R$

$|f_n'(z)|\le \underset{\text{uniformly bounded}}{\frac{12C}{R}}$ for all $z\in\overline{B_{\frac{R}{2}}(a)}$ $\Rightarrow$ $\{f_n'\}$ is also uniformly bounded in $\underline{\text{some}}$ $\overline{B_r(a)}$, $r\le R$. $\boxed{/\!/}$

10 pts

14. ① First prove holomorphic in $B_1(0)$

proof: $f$ is defined by the power series $f(z) = \sum_{k=1}^{\infty} a_k z^k$, where $a_k = \begin{cases} 1 & k = m!, m \geq 1 \\ 0 & \text{otherwise.} \end{cases}$

Consider its convergence radius $R$:

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{\frac{1}{k}} = \limsup_{m\to\infty} |a_{m!}|^{\frac{1}{m!}} = \limsup_{m\to\infty} 1^{\frac{1}{m!}} = 1$$

$\Rightarrow R = 1$

by Cauchy-Adamard Theorem and Weierstrass Theorem

$f(z) \in \mathcal{O}(B_1(0))$

② Then prove the second part

Consider points on $\partial B_1(0)$ of the form $a = e^{i2\pi \cdot \frac{p}{q}}$, where $p, q$ are integers and $q > 0$ ($\frac{p}{q}$ is a rational point)

These points are dense on $\partial B_1(0)$.

Then consider $z = ra = re^{i2\pi\frac{p}{q}}$, where $0 < r < 1$

$\Rightarrow f(z) = \sum_{n=1}^{\infty} r^{n!} (e^{i2\pi\frac{p}{q}})^{n!}$

for $n > q$, $q \mid n!$. let $n! = kq$, $k \in \mathbb{N}_+$.

$\Rightarrow (e^{i2\pi\frac{p}{q}})^{n!} = (e^{i2p\pi})^k = 1^k = 1$

$\Rightarrow f(z) = \sum_{n=1}^{q-1} r^{n!} (e^{i2\pi\frac{p}{q}})^{n!} + \sum_{n=q}^{\infty} r^{n!}$

As $r \to 1^-$, $\sum_{n=1}^{q-1} r^{n!} (e^{i2\pi\frac{p}{q}})^{n!}$ approaches a finite value $\sum_{n=1}^{q-1} (e^{i2\pi\frac{p}{q}})^{n!}$

real (since it's a finite sum of terms approaching finite values)

The second sum is $\sum_{n=q}^{\infty} r^{n!}$, assume it converges

$\Rightarrow \lim_{r\to 1^-} \sum_{n=q}^{\infty} r^{n!} = \sum_{n=q}^{\infty} (\lim_{r\to 1^-} r^{n!}) = \sum_{n=q}^{\infty} 1$, diverges to $+\infty$

$\Rightarrow \sum_{n=q}^{\infty} r^{n!}$ diverges

$\Rightarrow \sum_{n=q}^{\infty} r^{n!} = +\infty$ (since $r^{n!} > 0$, $\forall n \geq q$)

This implies $\lim_{r\to 1^-} |f(re^{i2\pi\frac{p}{q}})| = +\infty$

since the function becomes unbounded as $z$ approaches any $a = e^{i2\pi\frac{p}{q}}$ along the radius, these points must be singularities for $f(z)$

The set of points $\{e^{i2\pi p/q}\}$ is dense on the unit circle $\{|z| = 1\}$.

If a function could be holomorphically extends to a neighborhood of any boundary point $a_0$, it will contain points of the form $e^{i2\pi\frac{p}{q}}$ (singularity). It's a contradiction

QED.