Appendix A:

Basic Statistical Distributions

A.1 Discrete Distributions

A.1.1 Finite discrete distribution

Notation: $X \sim \text{FDiscrete}_n(\boldsymbol{x}, \boldsymbol{p})$, $\boldsymbol{x} = (x_1, \dots, x_n)^\top$, $\boldsymbol{p} = (p_1, \dots, p_n)^\top \in \mathbb{T}_n \hat{=} \{(p_1, \dots, p_n)\colon\, p_i > 0, \sum_{i=1}^n p_i = 1\}$.

Density: $\Pr(X = x_i) = p_i, \quad i = 1, \dots, n.$

Moments: $E(X) = \sum_{i=1}^n x_i p_i$ and $\text{Var}(X) = \sum_{i=1}^n x_i^2 p_i - (\sum_{i=1}^n x_i p_i)^2$.

Note: The *uniform discrete* distribution is a special case of the finite discrete distribution with $p_i = 1/n$ for all i.

Sampling: `sample(x, size, replace = FALSE, prob = NULL)` takes a sample of the specified size from the elements of `x` using either with or without replacement.

Examples:

```
> sample(c(0,1), 100, replace= T, prob=c(0.8, 0.2))
> sample(1:20, 4)    # the default: replace= F
```

A.1.2 Hypergeometric distribution

Notation: $X \sim \text{Hgeometric}(m, n, k)$, m, n, k are positive integers.

Density: $\text{Hgeometric}(x|m, n, k) = \binom{m}{x}\binom{n}{k-x}/\binom{m+n}{k}$, where $x = \max(0, k-n), \dots, \min(m, k)$.

Moments: $E(X) = km/N'$ and $\text{Var}(X) = kmn(N'-k)/[N'^2(N'-1)]$, where $N' \hat{=} m + n$.

Computing: `> prod(5:1)` $= 5!$
`> prod(20:16)` $= 20 \times 19 \times 18 \times 17 \times 16$
`> choose(40,5)` $= \binom{40}{5}$

Functions: `dhyper(x, m, n, k)`
`phyper(q, m, n, k)`
`qhyper(p, m, n, k)`
`rhyper(nn, m, n, k)`

A.1.3 Poisson distribution

Notation: $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$

Density: $\text{Poisson}(x|\lambda) = \lambda^x \, \mathrm{e}^{-\lambda}/x!$, $x = 0, 1, \dots, \infty$.

Moments: $E(X) = \lambda$ and $\text{Var}(X) = \lambda$.

Properties: • If $\{X_i\}_{i=1}^n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda_i)$, then

$$\begin{aligned} \textstyle\sum_{i=1}^n X_i &\sim \text{Poisson}(\textstyle\sum_{i=1}^n \lambda_i), \quad \text{and} \\ (X_1, \dots, X_n)|(\textstyle\sum_{i=1}^n X_i = m) &\sim \text{Multinomial}_n(m, \boldsymbol{p}), \end{aligned}$$

where $\boldsymbol{p} = (\lambda_1, \dots, \lambda_n)^\top / \sum_{i=1}^n \lambda_i$;

• The Poisson and gamma distribution have relationship:

$$\textstyle\sum_{x=k}^{\infty} \text{Poisson}(x|\lambda) = \int_0^{\lambda} \text{Gamma}(y|k, 1) \, \mathrm{d}y.$$

Functions: `dpois(x, lambda)`
`ppois(q, lambda)`
`qpois(p, lambda)`
`rpois(n, lambda)`

```
===============================================================
> x <- 0:20
> plot(x, dpois(x, 4), type="h")              # histogram-like
                                                  # Figure A.1
***************************************************************
```

A.1.4 Binomial distribution

Notation: $X \sim \text{Binomial}(n, p)$, n is a positive integer, $p \in (0, 1)$.

Density: $\text{Binomial}(x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \dots, n$.

Moments: $E(X) = np$ and $\text{Var}(X) = np(1-p)$.

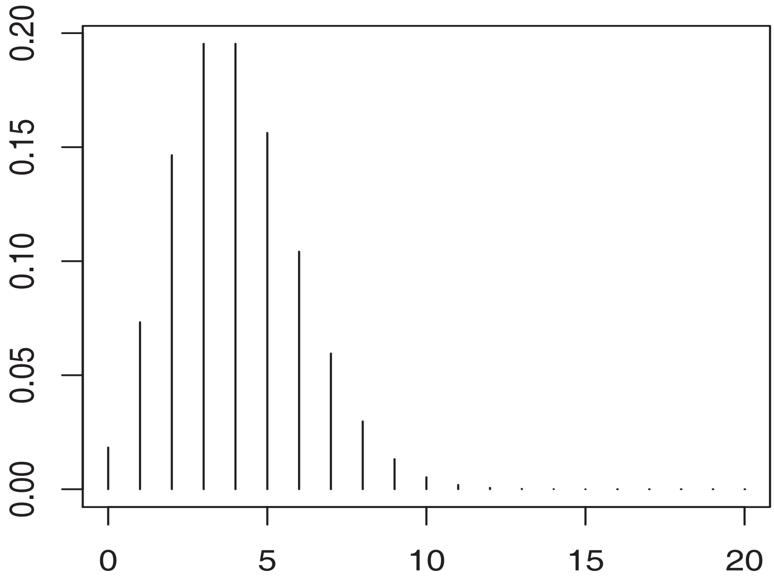

Figure A.1 Point probabilities of Poisson(4).

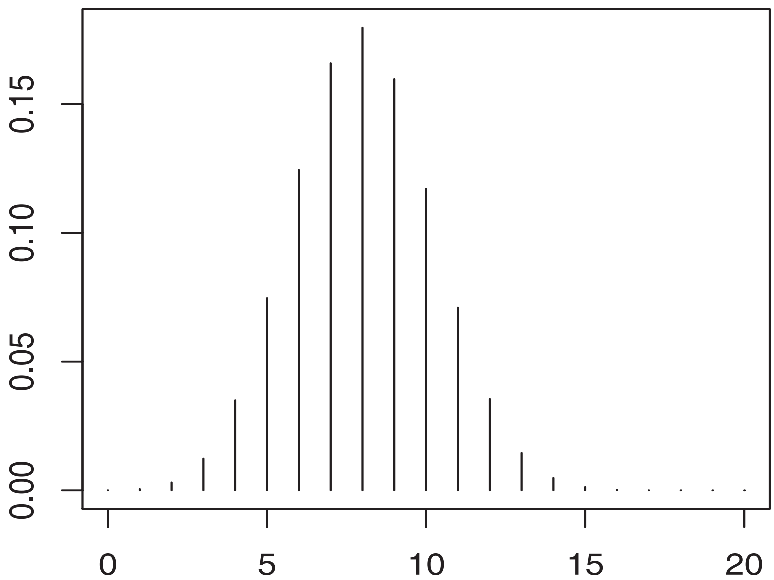

Figure A.2 Point probabilities of Binomial$(20, 0.4)$.

Properties: • If $\{X_i\}_{i=1}^d \overset{\text{ind}}{\sim} \text{Binomial}(n_i, p)$, then

$$\textstyle\sum_{i=1}^d X_i \sim \text{Binomial}(\sum_{i=1}^d n_i, p);$$

• The binomial and beta distribution have relationship:

$$\textstyle\sum_{x=0}^k \text{Binomial}(x|n,p) = \int_0^{1-p} \text{Beta}(x|n-k, k+1)\ \mathrm{d}x,$$

where $0 \leqslant k \leqslant n$.

Note: When $n = 1$, binomial distribution is called *Bernoulli* distribution.

Functions:
```
dbinom(x, size, prob)     # size= n,  prob= p
pbinom(q, size, prob)
qbinom(p, size, prob)
rbinom(nn, size, prob)
```

```
==============================================================
> x <- 0:20
> plot(x, dbinom(x, size=20, prob=0.4), type="h")
                                                 # Figure A.2
**************************************************************
```

A.1.5 Multinomial distribution

Notation: $\mathbf{x} = (X_1, \ldots, X_d)^\top \sim \text{Multinomial}(n; p_1, \ldots, p_d)$ or
$\mathbf{x} = (X_1, \ldots, X_d)^\top \sim \text{Multinomial}_d(n, \boldsymbol{p})$,
n is a positive integer, $\boldsymbol{p} = (p_1, \ldots, p_d)^\top \in \mathbb{T}_d$,

Density: $\text{Multinomial}_d(\boldsymbol{x}|n, \boldsymbol{p}) = \binom{n}{x_1, \ldots, x_d} \prod_{i=1}^d p_i^{x_i}$,
$\boldsymbol{x} = (x_1, \ldots, x_d)^\top$, $x_i \geqslant 0$, $\sum_{i=1}^d x_i = n$.

Moments: $E(X_i) = np_i$, $\text{Var}(X_i) = np_i(1-p_i)$ and $\text{Cov}(X_i, X_j) = -np_ip_j$.

Note: The binomial distribution is a special case of the multinomial with $d = 2$.

Functions:
```
dmultinom(x, size = NULL, prob) # size= n, prob= p
rmultinom(nn, size, prob)
```

A.2 Continuous Distributions

A.2.1 Uniform distribution

Notation: $X \sim U(a, b)$ or $X \sim U[a, b]$, $a < b$

Density: $U(x|a, b) = 1/(b - a)$, $x \in (a, b)$ or $x \in [a, b]$.

Moments: $E(X) = (a + b)/2$ and $\mathrm{Var}(X) = (b - a)^2/12$.

Properties: If $Y \sim U(0, 1)$, then $X = a + (b - a)Y \sim U(a, b)$.

Functions:
```
dunif(x, min= 0, max= 1)     # min= a,  max= b
punif(q, min= 0, max= 1)
qunif(p, min= 0, max= 1)
runif(n, min= 0, max= 1)
```

A.2.2 Beta distribution

Notation: $X \sim \mathrm{Beta}(a, b)$, $a > 0, b > 0$.

Density: $\mathrm{Beta}(x|a, b) = x^{a-1}(1 - x)^{b-1}/B(a, b)$, $0 < x < 1$.

Moments: $E(X) = a/(a + b)$, $E(X^2) = a(a + 1)/[(a + b)(a + b + 1)]$ and $\mathrm{Var}(X) = ab/[(a + b)^2(a + b + 1)]$.

Properties: If $Y_1 \sim \mathrm{Gamma}(a, 1)$, $Y_2 \sim \mathrm{Gamma}(b, 1)$, and $Y_1 \perp\!\!\!\perp Y_2$, then $Y_1/(Y_1 + Y_2) \sim \mathrm{Beta}(a, b)$.

Note: When $a = b = 1$, $\mathrm{Beta}(1, 1) = U(0, 1)$.

Functions:
```
dbeta(x, shape1, shape2)   # shape1= a,  shape2= b
pbeta(q, shape1, shape2)
qbeta(p, shape1, shape2)
rbeta(n, shape1, shape2)
```

A.2.3 Exponential distribution

Notation: $X \sim \mathrm{Exponential}(\beta)$, rate parameter $\beta > 0$.

Density: $\mathrm{Exponential}(x|\beta) = \beta\,\mathrm{e}^{-\beta x}$, $x \geqslant 0$.

Moments: $E(X) = 1/\beta$ and $\mathrm{Var}(X) = 1/\beta^2$.

Properties: • If $U \sim U(0,1)$, then $-\log(U)/\beta \sim \text{Exponential}(\beta)$;

• If $\{X_i\}_{i=1}^n \overset{\text{iid}}{\sim} \text{Exponential}(\beta)$, then $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \beta)$.

Functions:
```
dexp(x, rate= 1)          # rate= β
pexp(q, rate= 1)
qexp(p, rate= 1)
rexp(n, rate= 1)
```

A.2.4 Gamma distribution

Notation: $X \sim \text{Gamma}(\alpha, \beta)$, shape parameter $\alpha > 0$, rate parameter $\beta > 0$.

Density: $\text{Gamma}(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x}$, $x > 0$.

Moments: $E(X) = \alpha/\beta$ and $\text{Var}(X) = \alpha/\beta^2$.

Properties: • If $X \sim \text{Gamma}(\alpha, \beta)$ and $c > 0$, then $cX \sim \text{Gamma}(\alpha, \beta/c)$;

• If $\{X_i\}_{i=1}^n \overset{\text{ind}}{\sim} \text{Gamma}(\alpha_i, \beta)$, then $\sum X_i \sim \text{Gamma}(\sum \alpha_i, \beta)$;

• $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.

Note: $\text{Gamma}(1, \beta) = \text{Exponential}(\beta)$. $\text{Gamma}(\nu/2, 1/2) = \chi^2(\nu)$.

Functions:
```
dgamma(x, shape, rate= 1)     # shape= α,   rate= β
pgamma(q, shape, rate= 1)
qgamma(p, shape, rate= 1)
rgamma(n, shape, rate= 1)
```

A.2.5 Chi-square distribution

Notation: $X \sim \chi^2(\nu) \equiv \text{Gamma}(\frac{\nu}{2}, \frac{1}{2})$, degrees of freedom $\nu > 0$.

Density: $\chi^2(x|\nu) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)} x^{\nu/2-1} \mathrm{e}^{-x/2}$, $x > 0$.

Moments: $E(X) = \nu$ and $\text{Var}(X) = 2\nu$.

Properties: • If $Y \sim N(0,1)$, then $X = Y^2 \sim \chi^2(1)$;

• If $\{X_j\}_{j=1}^m \overset{\text{ind}}{\sim} \chi^2(\nu_j)$, then $\sum_{j=1}^m X_j \sim \chi^2(\sum_{j=1}^m \nu_j)$.

Functions:
```
dchisq(x, df)          # df = nu
pchisq(q, df)
qchisq(p, df)
rchisq(nn, df)
```

```
=================================================================
> x <- seq(0.01, 25, 0.1)
> par(mfrow=c(2, 2))                                  # Figure A.3
> curve(dchisq(x, df= 1), from=0.1, to = 25)
> curve(dchisq(x, df= 2), from=0.1, to = 25)
> curve(dchisq(x, df= 3), from=0.1, to = 25)
> curve(dchisq(x, df= 4), from=0.1, to = 25)
*****************************************************************
```

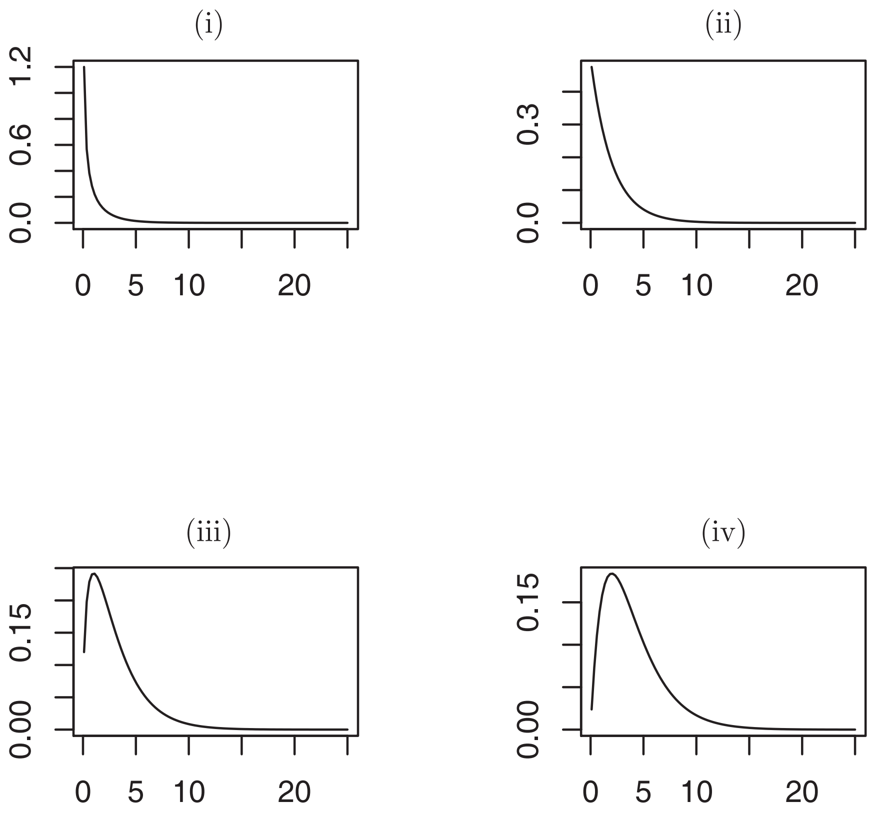

Figure A.3 Density functions of $\chi^2(\nu)$ for various ν. (i) $\nu = 1$; (ii) $\nu = 2$; (iii) $\nu = 3$; (iv) $\nu = 4$.

A.2.6 *t*- or Student's *t*-distribution

Notation: $X \sim t(\nu)$, $\nu > 0$ is a positive real number.

Density: $$t(x|\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < x < \infty.$$

Moments: $E(X) = 0$ (if $\nu > 1$) and $\mathrm{Var}(X) = \frac{\nu}{\nu-2}$ (if $\nu > 2$).

Properties: Let $Z \sim N(0,1)$, $Y \sim \chi^2(\nu)$, and $Z \perp\!\!\!\perp Y$, then

$$\frac{Z}{\sqrt{Y/\nu}} \sim t(\nu).$$

Note: When $\nu = 1$, $t(\nu) = t(1)$ is called *standard Cauchy distribution*, whose mean and variance are undefined.

Functions:
```
dt(x, df)            # df = nu
pt(q, df)
qt(p, df)
rt(nn, df)
```

A.2.7 *F* or Fisher's *F*-distribution

Notation: $X \sim F(n_1, n_2)$, n_1, n_2 are positive integers.

Density: $F(x|n_1,n_2) = \frac{(n_1/n_2)^{n_1/2}}{B(\frac{n_1}{2},\frac{n_2}{2})} x^{\frac{n_1}{2}-1}(1+\frac{n_1 x}{n_2})^{-\frac{n_1+n_2}{2}}$, $x > 0$.

Moments: $E(X) = \frac{n_2}{n_2-2}$ (if $n_2 > 2$), $\mathrm{Var}(X) = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-4)(n_2-2)^2}$ (if $n_2 > 4$).

Properties: Let $Y_i \sim \chi^2(n_i)$, $i = 1, 2$, and $Y_1 \perp\!\!\!\perp Y_2$, then

$$\frac{Y_1/n_1}{Y_2/n_2} \sim F(n_1, n_2).$$

Functions:
```
df(x, df1, df2)            # df1= n1,   df2= n2
pf(q, df1, df2)
qf(p, df1, df2)
rf(n, df1, df2)
```

A.2.8 Normal or Gaussian distribution

Notation: $X \sim N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$.

Density: $N(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-\frac{(x-\mu)^2}{2\sigma^2}]$, $-\infty < x < \infty$.

Moments: $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

Properties:
- If $\{X_i\} \overset{\text{ind}}{\sim} N(\mu_i, \sigma_i^2)$, then $\sum a_i X_i \sim N(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2)$;
- If $X_1|X_2 \sim N(X_2, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, then

$$X_1 \sim N(\mu_2, \sigma_1^2 + \sigma_2^2).$$

Functions:
```
dnorm(x, mean=0, sd= 1)   # mean= μ, sd= σ
pnorm(q, mean=0, sd= 1)
qnorm(p, mean=0, sd= 1)
rnorm(n, mean=0, sd= 1)
```

```
===============================================================
> x <- seq(-4, 4, 0.1)
> plot(x, dnorm(x), type="l",
                          ylab="Density function of N(0,1)")
# Note that this is the letter "l", not the digit "1"
                                                  # Figure A.4
---------------------------------------------------------------
# An alternative way of creating the plot is

> curve(dnorm(x), from=-4, to = 4,
                          ylab="Density function of N(0,1)")
***************************************************************
```

A.2.9 Multivariate normal or Gaussian distribution

Notation: $\mathbf{x} = (X_1, \ldots, X_d)^\top \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ or $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\mu} \in \mathbb{R}^d$, $\boldsymbol{\Sigma} > 0$.

Density: $N_d(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(\sqrt{2\pi})^d |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\}$, $\boldsymbol{x} \in \mathbb{R}^d$.

Moments: $E(\mathbf{x}) = \boldsymbol{\mu}$ and $\mathrm{Var}(\mathbf{x}) = \boldsymbol{\Sigma}$.

Functions: Producing one or more samples from the specified multivariate normal distribution

```
mvrnorm(n= 1, mu, Sigma, tol= 1e-6, empirical= F)
rmvn(n, mu, V)
```

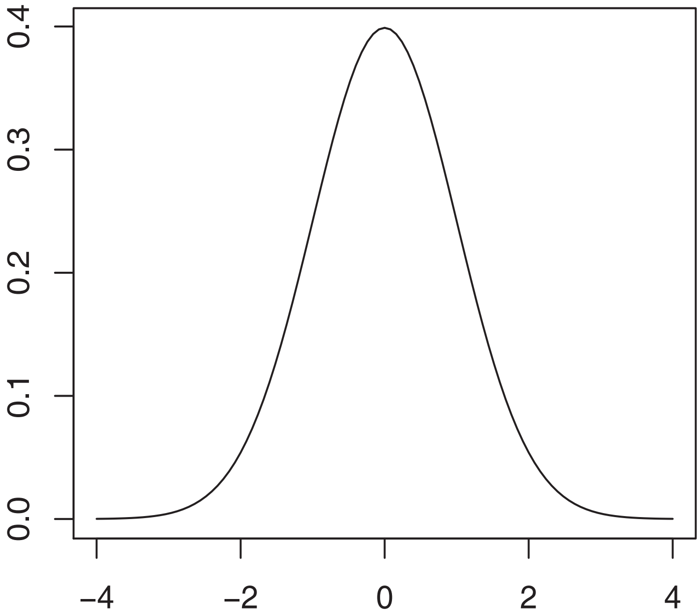

Figure A.4 Density functions of $N(0, 1)$.