MA204: Mathematical Statistics

Suggested Solutions to Assignment 2

2.1 Solution. (a) We first calculate $E(T)$. From the SR

$$T \stackrel{\mathrm{d}}{=} \frac{Z}{\sqrt{Y/n}},$$

we have

$$E(T) = E(Z) \times \sqrt{n} E(Y^{-1/2}) = 0$$

since $Z \sim N(0,1)$ and $Z \perp\!\!\!\perp Y$.

(b) We next calculate $\mathrm{Var}(T) = E(T^2) - [E(T)]^2 = E(T^2)$. The density of $Y \sim \chi^2(n)$ is

$$g(y) = \frac{2^{-n/2}}{\Gamma(n/2)} y^{n/2-1} \mathrm{e}^{-y/2}, \quad y > 0.$$

Hence, we have

$$\begin{aligned}
E(T^2) &= E(Z^2) \times nE(Y^{-1}) = 1 \times n \int_0^\infty y^{-1} g(y)\, \mathrm{d}y \\
&= n \frac{2^{-n/2}}{\Gamma(n/2)} \int_0^\infty y^{(n-2)/2-1} \mathrm{e}^{-y/2}\, \mathrm{d}y \\
&= n \frac{2^{-n/2}}{\Gamma(n/2)} \cdot \frac{\Gamma(\frac{n-2}{2})}{2^{-(n-2)/2}} \\
&= \frac{n}{n-2}, \qquad n > 2,
\end{aligned}$$

where we used the formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

2.2 Solution. Let $X \sim \text{Beta}(a, b)$, where $a = 3$ and $b = 2$. Then, the pdf and cdf of X are given by

$$\begin{aligned} f(x) &= \frac{1}{B(a,b)}\, x^{a-1}(1-x)^{b-1} \cdot I_{(0,1)}(x) \\ &= \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)}\, x^2(1-x) \cdot I_{(0,1)}(x) \\ &= 12(x^2 - x^3) \cdot I_{(0,1)}(x), \quad \text{and} \\ F(x) &= \begin{cases} 0, & \text{if } x \leqslant 0, \\ \displaystyle\int_0^x f(t)\ \mathrm{d}t, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1, \end{cases} \\ &= \begin{cases} 0, & \text{if } x \leqslant 0, \\ 4x^3 - 3x^4, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1. \end{cases} \end{aligned}$$

Thus, the cdf and pdf of $X_{(1)} = \min(X_1, \ldots, X_n)$ are given by

$$\begin{aligned} G_1(x) &= 1 - [1 - F(x)]^n \\ &= \begin{cases} 0, & \text{if } x \leqslant 0, \\ 1 - (1 - 4x^3 + 3x^4)^n, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1, \qquad \text{and} \end{cases} \\ g_1(x) &= nf(x)[1 - F(x)]^{n-1} \\ &= 12nx^2(1-x)[1 - 4x^3 + 3x^4]^{n-1} \cdot I_{(0,1)}(x). \end{aligned}$$

Similarly, the cdf and pdf of $X_{(n)} = \max(X_1, \ldots, X_n)$ are given by

$$\begin{aligned} G_n(x) &= [F(x)]^n \\ &= \begin{cases} 0, & \text{if } x \leqslant 0, \\ (4x^3 - 3x^4)^n, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1, \qquad \text{and} \end{cases} \end{aligned}$$

$$\begin{aligned} g_n(x) &= nf(x)[F(x)]^{n-1} \\ &= 12nx^2(1-x)[4x^3-3x^4]^{n-1} \cdot I_{(0,1)}(x). \end{aligned}$$

2.3 Solution. Define $Y_i = X_{(i)}$ for $i = 1, \ldots, n$. The joint density of $Y_1, \ldots, Y_n$ is given by

$$\begin{aligned} f(y_1, \ldots, y_n) &= n! f(y_1) \cdots f(y_n) \\ &= n! \mathrm{e}^{-\sum_{i=1}^n y_i}, \quad 0 < y_1 < \cdots < y_n. \end{aligned}$$

(a) Taking transformation

$$\begin{cases} z_1 &= ny_1 \\ z_2 &= (n-1)(y_2 - y_1) \\ &\vdots \\ z_n &= y_n - y_{n-1}, \end{cases}$$

we have $z_i > 0$ for $i = 1, \ldots, n$, and the inverse transformation is given by

$$\begin{cases} y_1 &= \dfrac{z_1}{n} \\ y_2 &= \dfrac{z_1}{n} + \dfrac{z_2}{n-1} \\ \vdots & \\ y_n &= \dfrac{z_1}{n} + \dfrac{z_2}{n-1} + \cdots + z_n. \end{cases}$$

Since the Jacobian is

$$J = \frac{\partial(y_1, \ldots, y_n)}{\partial(z_1, \ldots, z_n)} = \det \begin{pmatrix} \frac{1}{n} & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 \end{pmatrix} = \frac{1}{n!},$$

the joint density of $Z_1, \dots, Z_n$ is

$$\begin{aligned} g(z_1, \dots, z_n) &= f(y_1, \dots, y_n)|J| \\ &= \mathrm{e}^{-\sum_{i=1}^n z_i}, \quad z_i > 0, \quad i = 1, \dots, n. \end{aligned}$$

Therefore, the marginal density of Z_i is Exponential(1). Furthermore, note that

$$g(z_1, \dots, z_n) = g(z_1) \cdots g(z_n),$$

then $Z_1, \dots, Z_n$ are mutually independent.

(b) We can write

$$\begin{aligned} \sum_{i=1}^n a_i Y_i &= \sum_{i=1}^n a_i \left(\sum_{k=0}^{i-1} \frac{Z_{k+1}}{n-k} \right) \\ &= \sum_{k=0}^{n-1} \left(\sum_{i=k+1}^n a_i \right) \frac{Z_{k+1}}{n-k} \\ &= \sum_{j=1}^n \left(\sum_{i=j}^n a_i \right) \frac{Z_j}{n-j+1}, \end{aligned}$$

which is a linear function of independent random variables $Z_1, \dots, Z_n$.

2.4 Solution. Let $Y_n = X_1 + \cdots + X_n$. Making transformation

$$\begin{cases} y_1 &= x_1/y_n, \\ & \vdots \\ y_{n-1} &= x_{n-1}/y_n, \\ y_n &= x_1 + \cdots + x_n, \end{cases}$$

we have $y_i \geqslant 0$ for $i = 1, \dots, n-1$, $y_1 + \cdots + y_{n-1} \leqslant 1$, $y_n \geqslant 0$, and the inverse transformation is given by

$$\begin{cases} x_1 &= y_1 y_n \\ & \vdots \\ x_{n-1} &= y_{n-1} y_n \\ x_n &= (1 - y_1 - \cdots - y_{n-1}) y_n. \end{cases}$$

Since the Jacobian is

$$\begin{aligned}
J &= \frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)} \\
&= \det\begin{pmatrix} y_n & 0 & \cdots & 0 & y_1 \\ 0 & y_n & \cdots & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & y_n & 0 \\ -y_n & -y_n & \cdots & -y_n & 1-\sum_{i=1}^{n-1} y_i \end{pmatrix} \\
&= y_n^{n-1},
\end{aligned}$$

the joint density of $Y_1,\ldots,Y_{n-1},Y_n$ is

$$\begin{aligned}
& g(y_1,\ldots,y_{n-1},y_n) \\
= \; & f(x_1,\ldots,x_n)|J| \\
= \; & \left[\prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} x_i^{a_i-1} \mathrm{e}^{-x_i}\right] \cdot y_n^{n-1} \\
= \; & \left[\frac{\Gamma(a_+)}{\Gamma(a_1)\cdots\Gamma(a_n)} y_1^{a_1-1}\cdots y_{n-1}^{a_{n-1}-1}\left(1-\sum_{j=1}^{n-1} y_j\right)^{a_n-1}\right] \\
& \times \frac{1}{\Gamma(a_+)} y_n^{a_+-1}\mathrm{e}^{-y_n},
\end{aligned}$$

where $a_+ = \sum_{i=1}^{n} a_i$. Therefore,

$$(Y_1,\ldots,Y_{n-1})^\top \sim \text{Dirichlet}(a_1,\ldots,a_{n-1};\; a_n),$$

$Y_n \sim \text{Gamma}(a_+,1)$, and $(Y_1,\ldots,Y_{n-1})^\top \perp\!\!\!\perp Y_n$.

2.5 Solution. Let $U = \log(X)$ and $V = \log(Y)$. The mgf of U is

$$\begin{aligned} M_U(t) &= E(\mathrm{e}^{tU}) \\ &= \frac{1}{\Gamma(p)} \int_0^\infty \mathrm{e}^{t\log(x)} \cdot x^{p-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \frac{1}{\Gamma(p)} \int_0^\infty x^{p+t-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \frac{\Gamma(p+t)}{\Gamma(p)} \end{aligned}$$

and the mgf of V is

$$\begin{aligned} M_V(t) &= E(\mathrm{e}^{tV}) \\ &= \frac{1}{B(q,p-q)} \int_0^\infty \mathrm{e}^{t\log(y)} \cdot y^{q-1}(1-y)^{p-q-1}\,\mathrm{d}y \\ &= \frac{1}{B(q,p-q)} \int_0^\infty y^{q+t-1}(1-y)^{p-q-1}\,\mathrm{d}y \\ &= \frac{B(q+t,p-q)}{B(q,p-q)} = \frac{\Gamma(q+t)\Gamma(p)}{\Gamma(q)\Gamma(p+t)}. \end{aligned}$$

So the mgf of $\log(XY) = U + V$ is

$$M_{U+V}(t) = M_U(t) \cdot M_V(t) = \frac{\Gamma(q+t)}{\Gamma(q)},$$

which implies that $XY \sim \text{Gamma}(q, 1)$.

2.6 Solution. The joint pmf of $\mathbf{y} = Z\mathbf{x}$ is denoted by

$$f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) = \Pr(\mathbf{y} = \boldsymbol{y}) = \Pr(ZX_1 = y_1, \ldots, ZX_m = y_m).$$

If $\boldsymbol{y} = \mathbf{0}_m$, we have

$$\begin{aligned} f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) &= \Pr(ZX_1 = 0, \ldots, ZX_m = 0) \\ &= \Pr(Z = 0) + \Pr(Z = 1,\, X_1 = 0, \ldots, X_m = 0) \\ &= \phi + (1-\phi)\mathrm{e}^{-\lambda_+}, \end{aligned}$$

where $\lambda_+ = \sum_{i=1}^m \lambda_i$. If $\boldsymbol{y} \neq \mathbf{0}_m$, we have

$$
\begin{aligned}
f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) &= \Pr(ZX_1 = y_1, \ldots, ZX_m = y_m) \\
&= \Pr(Z = 1,\ X_1 = y_1, \ldots, X_m = y_m) \\
&= (1-\phi)\mathrm{e}^{-\lambda_+} \prod_{i=1}^m \frac{\lambda_i^{y_i}}{y_i!}.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
& f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) = \Pr(\mathbf{y} = \boldsymbol{y}) \\
= \ & [\phi + (1-\phi)\mathrm{e}^{-\lambda_+}] I(\boldsymbol{y} = \mathbf{0}) + \left[(1-\phi)\mathrm{e}^{-\lambda_+} \prod_{i=1}^m \frac{\lambda_i^{y_i}}{y_i!}\right] I(\boldsymbol{y} \neq \mathbf{0}) \\
= \ & \phi \Pr(\boldsymbol{\xi} = \boldsymbol{y}) + (1-\phi) \Pr(\mathbf{x} = \boldsymbol{y}),
\end{aligned}
$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)^\top$ and $\{\xi_i\}_{i=1}^m \stackrel{\text{iid}}{\sim} \text{Degenerate}(0)$.

2.7 Solution. (a) It is easy to know that

$$X_1 + X_2 \sim N(0, 2\sigma^2) \quad \text{and} \quad X_1 - X_2 \sim N(0, 2\sigma^2).$$

Since

$$
\begin{aligned}
\text{Cov}(X_1 + X_2, X_1 - X_2) &= E[(X_1 + X_2)(X_1 - X_2)] \\
&= E(X_1^2) - E(X_2^2) \\
&= 2\sigma^2 - 2\sigma^2 = 0,
\end{aligned}
$$

from the result 3) of Theorem 2.1, we have $(X_1 + X_2) \perp\!\!\!\perp (X_1 - X_2)$. Let

$$Z_1 \ \hat{=} \ \frac{X_1 + X_2}{\sqrt{2}\sigma} \quad \text{and} \quad Z_2 \ \hat{=} \ \frac{X_1 - X_2}{\sqrt{2}\sigma},$$

then $Z_1 \sim N(0,1)$, $Z_2 \sim N(0,1)$ and $Z_1 \perp\!\!\!\perp Z_2$. Therefore,

$$\frac{(X_1 - X_2)^2}{(X_1 + X_2)^2} = \frac{Z_2^2}{Z_1^2} \sim \frac{\chi^2(1)/1}{\chi^2(1)/1} = F(1,1).$$

(b) Since

$$
\begin{aligned}
& \Pr\left\{\frac{(X_1+X_2)^2}{(X_1+X_2)^2+(X_1-X_2)^2} > k\right\} \\
= & \Pr\left(\frac{Z_1^2}{Z_1^2+Z_2^2} > k\right) \\
= & \Pr\left(\frac{Z_2^2}{Z_1^2} < \frac{1-k}{k}\right) = 0.1,
\end{aligned}
$$

we obtain $(1-k)/k = 0.02508563$ so that $k = 0.9755283$.

2.8 Solution. Note that

$$\text{Exponential}(1) = \text{Gamma}(1,1) = \frac{1}{2}\text{Gamma}\left(\frac{2}{2}, \frac{1}{2}\right) = \frac{1}{2}\chi^2(2),$$

then, we obtain

$$\frac{X}{Y} \sim \frac{\chi^2(2)/2}{\chi^2(2)/2} = F(2,2).$$

2.9 Solution. (a) The cdf of X is

$$
\begin{aligned}
F(x) &= \Pr(X \leqslant x) = \Pr\{\max(aW, -bW) \leqslant x\} \\
&= \Pr(aW \leqslant x, -bW \leqslant x) \\
&= \Pr\left(-\frac{x}{b} \leqslant W \leqslant \frac{x}{a}\right) \\
&= \Pr\left(\frac{-x/b-\mu}{\sigma} \leqslant \frac{W-\mu}{\sigma} \leqslant \frac{x/a-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{x}{a\sigma} - \lambda\right) - \Phi\left(\frac{-x}{b\sigma} - \lambda\right), \quad x \geqslant 0,
\end{aligned}
$$

and then the pdf of X is

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}\sigma}\left\{\frac{1}{a}\exp\left[-\frac{1}{2}\left(\frac{x}{a\sigma} - \lambda\right)^2\right] + \frac{1}{b}\exp\left[-\frac{1}{2}\left(\frac{x}{b\sigma} + \lambda\right)^2\right]\right\} \\
&= \frac{1}{a}N\left(\frac{x}{a}\middle|\mu, \sigma^2\right) + \frac{1}{b}N\left(-\frac{x}{b}\middle|\mu, \sigma^2\right), \quad x \geqslant 0,
\end{aligned}
$$

where $N(x|\mu,\sigma^2)$ denotes the pdf of the $N(\mu,\sigma^2)$ distribution.

(b) From

$$X = \max(aW, -bW) = \begin{cases} aW, & \text{if } W > 0, \\ -bW, & \text{if } W \leqslant 0, \end{cases}$$

we know that W given $X = x$ has the following two-point distribution:

$$W|(X = x) = \begin{cases} x/a, & \text{with probability } p, \\ -x/b, & \text{with probability } 1-p, \end{cases}$$

where

$$p = \frac{(1/a)N(x/a|\mu,\sigma^2)}{f(x)} = \frac{1}{1 + \exp\left[c_1(c_2x^2 - 2x\mu)/(2\sigma^2)\right]}$$

and

$$c_1 \hat{=} \frac{1}{a} + \frac{1}{b} \quad \text{and} \quad c_2 \hat{=} \frac{1}{a} - \frac{1}{b}.$$

2.10 Solution. (a) From the solution to Q1.8, we known that the pmf of $X \sim \text{ZTP}(\lambda)$ is

$$\Pr(X = x) = c \cdot \frac{\lambda^x \mathrm{e}^{-\lambda}}{x!}, \quad x = 1, 2, \ldots,$$

where

$$c = \frac{1}{1 - \mathrm{e}^{-\lambda}} = \frac{\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda} - 1} \quad \text{so that} \quad c(1-c) = -\frac{\mathrm{e}^{\lambda}}{(\mathrm{e}^{\lambda} - 1)^2}.$$

In addition, $E(X) = c\lambda$ and $\text{Var}(X) = c\lambda[1 + (1-c)\lambda]$. From $Y \stackrel{\text{d}}{=} X + Z$ and $X \perp\!\!\!\perp Z$, we have

$$\begin{aligned} E(Y) &= E(X) + E(Z) = c\lambda + \rho\lambda = \frac{\lambda}{1 - \mathrm{e}^{-\lambda}} + \rho\lambda, \quad \text{and} \\ \text{Var}(Y) &= \text{Var}(X) + \text{Var}(Z) = c\lambda[1 + (1-c)\lambda] + \rho\lambda \\ &= E(Y) + c(1-c)\lambda^2 \\ &= E(Y) - \mathrm{e}^{\lambda}\left(\frac{\lambda}{\mathrm{e}^{\lambda} - 1}\right)^2. \end{aligned}$$

(b) Note that the support of Y is $\{1, 2, \ldots, \infty\}$. First, we consider the case of $y = 1$, the pmf of Y is

$$\begin{aligned}\Pr(Y=1) &= \Pr(X+Z=1) = \Pr(X=1)\Pr(Z=0)\\ &= c \cdot \lambda \mathrm{e}^{-\lambda} \cdot \mathrm{e}^{-\rho\lambda}\\ &= \frac{\lambda}{\exp(\rho\lambda)(\mathrm{e}^{\lambda}-1)}.\end{aligned}$$

Next, for $y \geqslant 2$, we have

$$\begin{aligned}\Pr(Y=y) &= \Pr(X+Z=y)\\ &= \sum_{z=0}^{\infty}\Pr(X+Z=y|Z=z)\Pr(Z=z)\\ &= \sum_{z=0}^{\infty}\Pr(X=y-z|Z=z)\Pr(Z=z)\\ &= \sum_{z=0}^{y-1}\Pr(X=y-z)\Pr(Z=z) \qquad [\because y-z \geqslant 1]\\ &= \sum_{z=0}^{y-1}\frac{c\lambda^{y-z}\mathrm{e}^{-\lambda}}{(y-z)!}\cdot\frac{(\rho\lambda)^z\mathrm{e}^{-\rho\lambda}}{z!}\\ &= \frac{c\mathrm{e}^{-\lambda}\mathrm{e}^{-\rho\lambda}\lambda^y}{y!}\sum_{z=0}^{y-1}\binom{y}{z}\rho^z\\ &= \frac{\lambda^y}{\exp(\rho\lambda)(\mathrm{e}^{\lambda}-1)y!}\left[\sum_{z=0}^{y}\binom{y}{z}\rho^z 1^{y-z}-\rho^y\right]\\ &= \frac{\lambda^y}{\exp(\rho\lambda)(\mathrm{e}^{\lambda}-1)y!}\left[(\rho+1)^y-\rho^y\right],\end{aligned}$$

which completes the proof.

2.11 Solution. (a) Since $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$ and

$X_1 \perp\!\!\!\perp X_2$, we have

$$\begin{aligned}
\Pr(Y=y) &= \Pr(X_2 - X_1 = y) \\
&= \sum_{x_1=0}^{\infty} \Pr(X_2 - X_1 = y | X_1 = x_1) \cdot \Pr(X_1 = x_1) \\
&= \sum_{x_1=0}^{\infty} \Pr(X_2 = x_1 + y | X_1 = x_1) \cdot \Pr(X_1 = x_1) \\
&= \sum_{x_1=\max(0,-y)}^{\infty} \Pr(X_2 = x_1 + y) \cdot \Pr(X_1 = x_1) \\
&= \sum_{x_1=\max(0,-y)}^{\infty} \frac{\lambda_2^{x_1+y} e^{-\lambda_2}}{(x_1+y)!} \cdot \frac{\lambda_1^{x_1} e^{-\lambda_1}}{x_1!} \\
&= \lambda_2^y e^{-(\lambda_1+\lambda_2)} \sum_{x_1=\max(0,-y)}^{\infty} \frac{(\lambda_1\lambda_2)^{x_1}}{(x_1+y)! x_1!},
\end{aligned}$$

for $y = -\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \infty$.

(b) $E(Y) = E(X_2) - E(X_1) = \lambda_2 - \lambda_1$ and $\mathrm{Var}(Y) = \mathrm{Var}(X_2) + \mathrm{Var}(X_1) = \lambda_2 + \lambda_1$.

2.12 Solution. (a) Let $y_1 = x_1 + x_2$ and $y_2 = x_1/x_2$. From Example 2.8 on page 65–66, we know that the Jacobian determinant is

$$J(x_1, x_2 \to y_1, y_2) = -\frac{y_1}{(1+y_2)^2}$$

so that the joint density of Y_1 and Y_2 is

$$\begin{aligned}
g(y_1, y_2) &= f(x_1, x_2) \times |J(x_1, x_2 \to y_1, y_2)| \\
&= \lambda^2 e^{-\lambda(x_1+x_2)} \times \frac{y_1}{(1+y_2)^2} \\
&= \lambda^2 e^{-\lambda y_1} \times \frac{y_1}{(1+y_2)^2} \\
&= \frac{\lambda^2}{\Gamma(2)} y_1^{2-1} e^{-\lambda y_1} I(y_1 \geqslant 0) \times \frac{1}{(1+y_2)^2} I(y_2 \geqslant 0).
\end{aligned}$$

(b) From the formula of $g(y_1, y_2)$, we obtain $Y_1 \sim \text{Gamma}(2, \lambda)$. Alternatively, since $X_1, X_2 \overset{\text{iid}}{\sim} \text{Exponential}(\lambda) = \text{Gamma}(1, \lambda)$, we have $Y_1 = X_1 + X_2 \sim \text{Gamma}(2, \lambda)$.

(c) From the formula of $g(y_1, y_2)$, the density of Y_2 is

$$g(y_2) = \frac{1}{(1+y_2)^2} I(y_2 \geqslant 0).$$

Alternatively, from the formula in the line -6 of page 31 of the textbook "Mathematical Statistics", we have

$$2\lambda X_i \sim \text{Gamma}(2/2, 1/2) = \chi^2(2),$$

so

$$Y_2 = \frac{X_1}{X_2} = \frac{2\lambda X_1}{2\lambda X_2} \overset{\text{d}}{=} \frac{\chi^2(2)/2}{\chi^2(2)/2} \sim F(2, 2).$$

2.13 Solution. (a) From the formula (2.25) on page 85 of the Textbook, we know that the joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$g_{1n}(x_{(1)}, x_{(n)}) = n(n-1) f(x_{(1)}) f(x_{(n)}) [F(x_{(n)}) - F(x_{(1)})]^{n-2},$$

for $x_{(1)} < x_{(n)}$. Making the transformation $r = x_{(n)} - x_{(1)}$ and $t = (x_{(1)} + x_{(n)})/2$, we obtain

$$x_{(1)} = t - \frac{r}{2} \quad \text{and} \quad x_{(n)} = t + \frac{r}{2}.$$

The Jacobian determinant is

$$\begin{aligned} J(x_{(1)}, x_{(n)} \to r, t) &= \frac{\partial(x_{(1)}, x_{(n)})}{\partial(r, t)} \\ &= \det \begin{pmatrix} -0.5 & 1 \\ 0.5 & 1 \end{pmatrix} = -1 \end{aligned}$$

so that the joint pdf of R and T is

$$\begin{aligned} f_{R,T}(r, t) &= g_{1n}(x_{(1)}, x_{(n)}) \times |J(x_{(1)}, x_{(n)} \to r, t)| \\ &= n(n-1) f(t - 0.5r) f(t + 0.5r) \\ &\quad \times [F(t + 0.5r) - F(t - 0.5r)]^{n-2}, \quad r > 0. \end{aligned}$$

(b) The marginal pdf of R is

$$f_R(r) = \int_{-\infty}^{\infty} f_{R,T}(r,t)\ \mathrm{d}t, \quad r > 0.$$

(c) The marginal pdf of T is

$$f_T(t) = \int_0^{\infty} f_{R,T}(r,t)\ \mathrm{d}r, \quad t \in \mathbb{R}.$$

2.14 Solution. (a) The cdf of the Bernoulli(p) distribution with $p \in (0,1)$ is

$$\begin{aligned} F(x) &= \Pr(X \leqslant x) \\ &= 0 \times I(x < 0) + (1-p) \times I(0 \leqslant x < 1) + 1 \times I(x \geqslant 1). \end{aligned}$$

(b) Let the mgf of the random variable X is $M_X(t) = \exp(c\,t)$, where c is a constant, then $X \sim \text{Degenerate}(c)$.

2.15 Solution. The pmf of $X = \lceil Y \rceil$ is (for $x = 0, 1, \ldots, \infty$)

$$\Pr(X = x) = \Pr(x \leqslant Y < x+1) = F_Y(x+1) - F_Y(x).$$

(a) The pmf of the discrete half logistic distribution is

$$\begin{aligned} p(x) &= F_Y(x+1) - F_Y(x) \\ &= \frac{2[\exp(1/\sigma) - 1]\exp(x/\sigma)}{[1 + \exp(x/\sigma)]\{1 + \exp[(x+1)/\sigma)]\}}, \end{aligned}$$

for $x = 0, 1, \ldots, \infty$.

(b) The pmf of the discrete Gamma distribution is

$$\begin{aligned} p(x) &= F_{Y}(x+1) - F_{Y}(x) \\ &= \int_0^{x+1} \frac{\beta\alpha}{\Gamma(\alpha)} z^{\alpha-1} \mathrm{e}^{-\beta z}\,\mathrm{d}z - \int_0^{x} \frac{\beta\alpha}{\Gamma(\alpha)} z^{\alpha-1} \mathrm{e}^{-\beta z}\,\mathrm{d}z \\ &= \int_x^{x+1} \frac{\beta\alpha}{\Gamma(\alpha)} z^{\alpha-1} \mathrm{e}^{-\beta z}\,\mathrm{d}z \\ &= \int_{\beta x}^{\beta(x+1)} \frac{w^{\alpha-1}\mathrm{e}^{-w}}{\Gamma(\alpha)}\,\mathrm{d}w, \end{aligned}$$

for $x = 0, 1, \ldots, \infty$.

(c) The pmf of the discrete Lindley distribution is

$$\begin{aligned} & p(x) = F_{Y}(x+1) - F_{Y}(x) \\ = & \frac{(1+\theta+\theta x)\exp(-\theta x)}{1+\theta} - \frac{[1+\theta+\theta(x+1)]\exp[-\theta(x+1)]}{1+\theta} \\ = & \frac{\mathrm{e}^{-\theta x}}{1+\theta}\Big[(1+\theta+\theta x)(1-\mathrm{e}^{-\theta}) - \theta\mathrm{e}^{-\theta}\Big] \\ = & \frac{\lambda^x}{1-\log\lambda}\Big\{\lambda\log\lambda + (1-\lambda)\big[1-\log(\lambda^{1+x})\big]\Big\}, \end{aligned}$$

for $x = 0, 1, \ldots, \infty$, where $\lambda \mathrel{\hat{=}} \mathrm{e}^{-\theta} > 0$.

2.16 Solution. The mgf of $Y = X^2$ is

$$\begin{aligned} M_Y(t) &= E(\mathrm{e}^{tY}) = E(\mathrm{e}^{tX^2}) \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-0.5x^2(1-2t)}\,\mathrm{d}x \qquad [\text{let } y = x\sqrt{1-2t}\,] \\ &= \frac{1}{\sqrt{1-2t}} = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{1/2}, \qquad t < 0.5. \end{aligned}$$

From Table 1.3 on page 26 of the Textbook, we know that the mgf of Gamma(α, β) is $[\beta/(\beta - t)]^{\alpha}$, $t < \beta$. Thus,

$$Y \sim \text{Gamma}(1/2, 1/2) = \chi^2(1).$$

2.17 Solution. (a) $\mathcal{S}_{X_1} = \mathcal{S}_{X_2} = \mathcal{S}_Y = (0, 1)$. The conditional distribution of $Y|(X_2 = x_2)$ is

$$Y|(X_2 = x_2) = x_2 \cdot X_1 \sim U(0, x_2), \quad 0 < x_2 < 1,$$

i.e.,

$$f_{(Y|X_2)}(y|x_2) = \frac{1}{x_2} \cdot I(0 < y < x_2), \quad 0 < x_2 < 1.$$

Hence, we have

$$\begin{aligned} f_Y(y) &= \int_{\mathcal{S}_{X_2}} f_{X_2}(x_2) \cdot f_{(Y|X_2)}(y|x_2)\, \mathrm{d}x_2 \\ &= \int_0^1 1 \cdot I(0 < x_2 < 1) \times \frac{1}{x_2} \cdot I(0 < y < x_2)\, \mathrm{d}x_2 \\ &= \int_y^1 \frac{1}{x_2}\, \mathrm{d}x_2 = \log(x_2)\Big|_y^1 \\ &= -\log(y), \quad 0 < y < 1. \end{aligned}$$

(b) $\mathcal{S}_Z = (0, \infty)$. The conditional distribution of $Z|(X_2 = x_2)$ is

$$Z|(X_2 = x_2) = x_2^{-1} \cdot X_1 \sim U(0, x_2^{-1}), \quad 0 < x_2 < 1.$$

i.e.,

$$f_{(Z|X_2)}(z|x_2) = x_2 \cdot I(0 < z < x_2^{-1}), \quad 0 < x_2 < 1.$$

Hence, we have

$$\begin{aligned} f_Z(z) &= \int_{\mathcal{S}_{X_2}} f_{X_2}(x_2) \cdot f_{(Z|X_2)}(z|x_2)\, \mathrm{d}x_2 \\ &= \int_0^1 1 \cdot I(0 < x_2 < 1) \times x_2 \cdot I(0 < z < x_2^{-1})\, \mathrm{d}x_2 \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\min(1,1/z)} x_2 \,\mathrm{d}x_2 \\
&= \left(\int_0^1 x_2 \,\mathrm{d}x_2\right) \cdot I(0 < z < 1) + \left(\int_0^{1/z} x_2 \,\mathrm{d}x_2\right) \cdot I(z \geqslant 1) \\
&= \frac{x_2^2}{2}\bigg|_0^1 \cdot I(0 < z < 1) + \frac{x_2^2}{2}\bigg|_0^{1/z} \cdot I(z \geqslant 1) \\
&= \frac{1}{2} \cdot I(0 < z < 1) + \frac{1}{2z^2} \cdot I(z \geqslant 1).
\end{aligned}
$$

2.18 Solution. We first find the conditional distribution of $\mathbf{x}|(Z = z)$,

$$
\mathbf{x}|(Z = z) = \boldsymbol{\mu} + \sqrt{\nu z^{-1}} \cdot \mathbf{y} \sim N_d(\boldsymbol{\mu}, \nu z^{-1}\boldsymbol{\Sigma}), \quad z > 0,
$$

i.e.,

$$
\begin{aligned}
f_{(\mathbf{x}|Z)}(\boldsymbol{x}|z) &= \frac{1}{(\sqrt{2\pi})^d |\nu z^{-1}\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{z}{2\nu}(\boldsymbol{x}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\} \\
&= \frac{z^{d/2}}{(\sqrt{2\pi\nu})^d |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-z\frac{\delta(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right\} \\
&= c_1^{-1} \cdot z^{d/2} \exp\left\{-z\frac{\delta(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right\},
\end{aligned}
$$

where $c_1 \,\hat{=}\, (\sqrt{2\pi\nu})^d |\boldsymbol{\Sigma}|^{1/2}$, $\delta(\boldsymbol{x}-\boldsymbol{\mu}) \,\hat{=}\, (\boldsymbol{x}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})/\nu$. Hence, we have

$$
\begin{aligned}
& f_{\mathbf{x}}(\boldsymbol{x}) \\
&= \int_{\mathcal{S}_Z} f_Z(z) \cdot f_{(\mathbf{x}|Z)}(\boldsymbol{x}|z) \,\mathrm{d}z \\
&= \int_0^{\infty} \frac{2^{-\nu/2}}{\Gamma(\nu/2)} z^{\nu/2-1} \mathrm{e}^{-z/2} \times c_1^{-1} z^{d/2} \exp\left\{-z\frac{\delta(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right\} \mathrm{d}z \\
&= c_2 \cdot \int_0^{\infty} z^{\frac{\nu+d}{2}-1} \exp\left\{-z\frac{1+\delta(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right\} \mathrm{d}z \\
&\overset{(1.41)}{=} c_2 \cdot \Gamma\left(\frac{\nu+d}{2}\right) \left\{\frac{1+\delta(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right\}^{-\frac{\nu+d}{2}}
\end{aligned}
$$

$$\overset{\text{(SA2.1)}}{=} \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})(\sqrt{\pi\nu}\,)^d|\mathbf{\Sigma}|^{\frac{1}{2}}} \left\{1 + \frac{(\boldsymbol{x}-\boldsymbol{\mu})^\top\mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{\nu}\right\}^{-\frac{\nu+d}{2}}$$

for $\boldsymbol{x} \in \mathbb{R}^d$, which is the density of d-dimensional t-distribution, where

$$c_2 \hat{=} \frac{2^{-\frac{\nu}{2}}}{(2\pi\nu)^{\frac{d}{2}}|\mathbf{\Sigma}|^{\frac{1}{2}}\Gamma(\frac{\nu}{2})} = \frac{1}{2^{\frac{\nu+d}{2}}\Gamma(\frac{\nu}{2})(\sqrt{\pi\nu}\,)^d|\mathbf{\Sigma}|^{\frac{1}{2}}}. \tag{SA2.1}$$

2.19 Proof. (a) Define $g(x) = \alpha f(x)[F(x)]^{\alpha-1}$, we only need to show that

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1.$$

In fact,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \alpha f(x)[F(x)]^{\alpha-1}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \alpha[F(x)]^{\alpha-1}\,\mathrm{d}F(x) \qquad [\text{let } u = F(x)] \\
&= \int_0^1 \alpha u^{\alpha-1}\,\mathrm{d}u = u^\alpha|_0^1 = 1.
\end{aligned}$$

(b) The cdf of $X \sim g(x)$ is $G(x) = [F(x)]^\alpha$. Let $U \sim U(0,1)$, we have

$$U \overset{\mathrm{d}}{=} G(X) = [F(X)]^\alpha \quad \Rightarrow \quad X \overset{\mathrm{d}}{=} F^{-1}(U^{1/\alpha}).$$

2.20 Proof. The pdf of $W \sim \text{Gamma}(\alpha, \beta)$ is

$$\frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha-1}\mathrm{e}^{-\beta w}, \quad w > 0,\ \alpha > 0,\ \beta > 0.$$

We know that for any $c > 0$,

$$c \cdot \text{Gamma}(\alpha, \beta) \overset{\mathrm{d}}{=} \text{Gamma}(\alpha, \beta/c), \tag{SA2.2}$$

and

$$\text{Gamma}(\nu/2, 1/2) = \chi^2(\nu), \tag{SA2.3}$$

We have the following SR:

$$\begin{aligned}
Y &\overset{\text{d}}{=} \mu + \frac{Z}{\sqrt{\tau}} = \mu + \frac{N(0,\sigma^2)}{\sqrt{\text{Gamma}(\alpha,\beta)}} \\
&\overset{\text{(SA2.2)}}{=} \mu + \frac{N(0,\sigma^2)}{\sqrt{(2\beta)^{-1}\cdot \text{Gamma}(\alpha,1/2)}} \\
&\overset{\text{(SA2.3)}}{=} \mu + \frac{N(0,\sigma^2)}{\sqrt{2\alpha(2\beta)^{-1}\cdot \chi^2(2\alpha)/(2\alpha)}} \\
&= \mu + \frac{N(0,\beta\boldsymbol{\sigma^2}/\alpha)}{\sqrt{\chi^2(2\alpha)/(2\alpha)}} \sim t(\mu,\sigma_*^2,\nu_*),
\end{aligned}$$

where $\sigma_*^2 = \beta\sigma^2/\alpha$ and $\nu_* = 2\alpha$.