

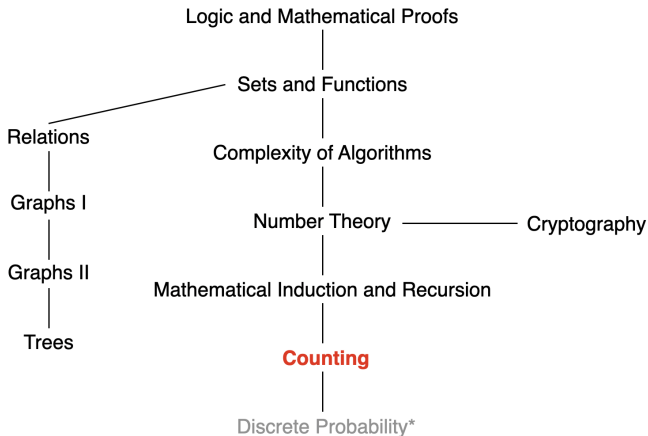
Discrete Mathematics for Computer Science

Lecture 13: Counting

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This Lecture



Counting basis, Permutations, **Combinations**, ...

Combinations

An r -combination of elements of a set is an unordered selection of r elements from the set.

The number of r -combinations of a set with n distinct elements is denoted by $C(n, r)$.

Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Example: The 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$. Thus, $C(4, 2) = 6$.

Binomial Coefficient

Theorem: For integers n and r with $0 \leq r \leq n$, the number of r -element subsets of an n -element set is

$$\binom{n}{r} = C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Proof: The $P(n, r)$ r -permutations of the set can be obtained by

- forming the $C(n, r)$ r -combinations of the set.
- ordering the elements in each r -combination, which can be done in $P(r, r)$ ways.

By the product rule,

$$P(n, r) = C(n, r)P(r, r).$$

Some Properties of Binomial Coefficients

- $C(n, 0) = 1$: one set of size 0.
- $C(n, n) = 1$: one set of size n .
- $C(n, r) = C(n, n - r)$

$$C(n, r) = \frac{n!}{r!(n - r)!}$$

$$C(n, n - r) = \frac{n!}{(n - r)!(n - (n - r))!} = \frac{n!}{r!(n - r)!}$$

Any other ideas to prove?

We will address this later.

Combinations: Example

Example 1: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

Example 2: There are 9 faculty members in the mathematics department and 11 in the computer science department.

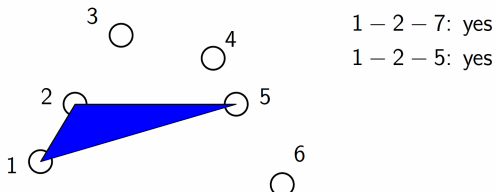
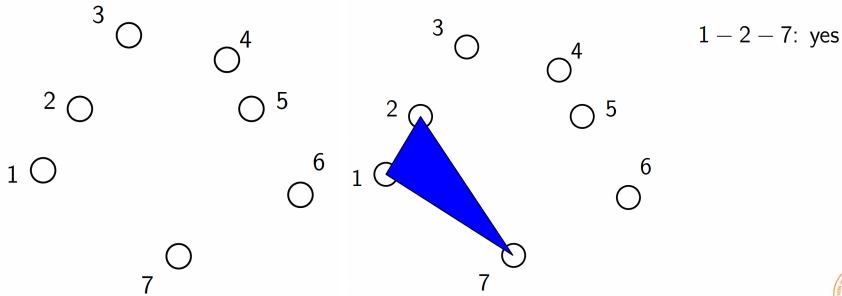
How many ways are there to form a committee with 3 faculty members from the mathematics department and 4 from the computer science department?

$$C(9, 3) \cdot C(11, 4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 27,720.$$

Example: Counting Triangle

Design an algorithm to count the number of triangles formed by n points in the plane:

- 3 points form a triangle if and only if they are non-collinear



Example: Counting Triangle

The following loop is a part of the program to determine the number of triangles formed by n points in the plane:

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

Question: Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

This corresponds to the total number of combinations. Why?

Example: Counting Triangle

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

- First loop begins with $i = 1$ and i increases up to n .
- Second loop begins with $j = i + 1$ and j increases up to n .
- Third loop begins with $k = j + 1$ and k increases up to n .

Thus each triple i, j, k with $i < j < k$ is examined **exactly once**.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, and $(2, 3, 4)$.

Example: Counting Triangle

Want to compute the number of **increasing triples** (i, j, k) with $1 \leq i < j < k \leq n$.

Claim: The number of **increasing triples** is exactly the **same** as the number of **3-combinations** from $\{1, 2, \dots, n\}$. **Why?**

- X : set of increasing triples
- Y : set of 3-combinations from $\{1, 2, \dots, n\}$

Define: $f : X \rightarrow Y$ by $f((i, j, k)) = \{i, j, k\}$

Claim: f is a **bijection**, so $|X| = |Y|$.

Example: Counting Triangle

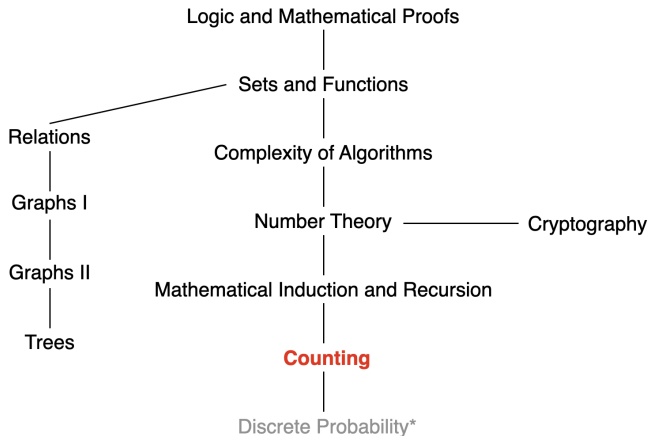
- X : set of increasing triples
- Y : set of 3-combinations from $\{1, 2, \dots, n\}$

Define: $f : X \rightarrow Y$ by $f((i, j, k)) = \{i, j, k\}$

Claim: f is a **bijection**, so $|X| = |Y|$.

- **One-to-one:** if $f((i, j, k)) = f((i', j', k'))$, then $(i, j, k) = (i', j', k')$.
- **Onto:** for any $\{i, j, k\}$, there exists a (i, j, k) such that $f((i, j, k)) = \{i, j, k\}$.

This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients,



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Combinatorial Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Definition: A **combinatorial proof** of an identity is

- a proof that uses counting arguments to prove that **both sides** of the identity **count the same objects** but in different ways
- **or** a proof that is based on showing that there is a **bijection between the sets of objects** counted by the two sides of the identity.

These two types of proofs are called **double counting proofs** and **bijective proofs**, respectively.

Combinatorial Proof: Bijective Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Bijective Proof: Suppose that S is a set with n elements.

The function that maps a subset A of S to \bar{A} is a bijection between subsets of S with r elements and subsets with $n - r$ elements.

- X : the set of all possible A , where $|X| = C(n, r)$
- Y : the set of all possible \bar{A} , where $|Y| = C(n, n - r)$
- $f : X \rightarrow Y$ is defined as $f(A) = \bar{A}$
 - ▶ **One-to-one:** if $f(A_1) = f(A_2)$, then $A_1 = A_2$.
 - ▶ **Onto:** for any \bar{A} , there exists an A such that $f(A) = \bar{A}$.

Since there is a bijection between two finite sets X and Y , they must have the same number of elements. Thus, $C(n, r) = C(n, n - r)$.



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Combinatorial Proof: Double Counting Proof

Theorem: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Double Counting Proof:

- **Left-hand side $C(n, r)$:** The number of subsets A of S with r elements.
- **Right-hand side $C(n, n - r)$:** The number of subsets \bar{A} (i.e., the complement of A) of S with $n - r$ elements.

Each subset A of S is also determined by specifying which elements are **not** in A , so are in \bar{A} . Thus, both sides count the same thing

It follows that $C(n, r) = C(n, n - r)$.

The Binomial Theorem

Let x and y be variables, and let n be a nonnegative integer:

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: The terms in the product when it is expanded are of the form $x^{n-j} y^j$ for $j = 0, 1, 2, \dots, n$.

To count the number of terms of the form $x^{n-j} y^j$, it is necessary to choose $n - j$ x s from the n sums (so that the other j terms in the product are y s).

Therefore, the coefficient of $x^{n-j} y^j$ is $\binom{n}{n-j}$, which is $\binom{n}{j}$.

The Binomial Theorem: Example

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Example 1: What is the expansion of $(x + y)^4$?

$$\begin{aligned}(x + y)^4 &= \sum_{j=0}^4 \binom{4}{j} x^{4-j} y^j \\&= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\&= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4.\end{aligned}$$

The Binomial Theorem: Example

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Example 2: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

First, note that this expression equals $(2x + (-3y))^{25}$

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

The coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$:

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13! 12!} 2^{12} 3^{13}.$$

The Binomial Theorem

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

This is proven by substituting $x = 1$ and $y = 1$.
Any other ideas to prove?

The Binomial Theorem

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Double Counting Proof: Let P denote the set of all subsets of $\{1, 2, \dots, n\}$.

- **Left-hand side:** Let S_k be the set of all subsets of $\{1, 2, \dots, n\}$ with k elements.

$$|P| = \sum_{k=0}^n |S_k| = \sum_{k=0}^n \binom{n}{k}$$

- **Right-hand side:** A set with n elements has a total of 2^n different subsets, i.e., $|P| = 2^n$.

The Binomial Theorem

Corollary: Let n be a nonnegative integer,

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Bijective Proof: Let P denote the set of all subsets of $\{1, 2, \dots, n\}$. Let S_k be the set of all subsets of $\{1, 2, \dots, n\}$ with k elements.

$$|P| = \sum_{i=0}^n |S_i| = \sum_{i=0}^n \binom{n}{i}$$

Consider $L = L_1 L_2 \dots L_n$ be a list of size n from $\{0, 1\}$. Let \mathcal{L} be the set of all such lists, we have $|\mathcal{L}| = 2^n$.

Objective: there is a **bijection** between \mathcal{L} and P ,
so $|P| = |\mathcal{L}| = 2^n$.

The Binomial Theorem

Define the following function $f : \mathcal{L} \rightarrow P$

- If $L \in \mathcal{L}$, then $f(L)$ is the set $S \subset \{1, 2, \dots, n\}$ defined by

$$i \in S, \text{ for } L_i = 1$$

f is a **bijection** between \mathcal{L} and P .

- **one-to-one**: If $f(L_1) = f(L_2)$, then $L_1 = L_2$
- **onto**: for any S , there exists an L such that $f(L) = S$

The Binomial Theorem

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary: Let n be a positive integer.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$$

This is proven by substituting $x = -1$ and $y = 1$.

Corollary: Let n be a nonnegative integer.

$$\sum_{k=0}^n 2^k \binom{n}{k} = 3^n$$

This is proven by substituting $x = 1$ and $y = 2$.

Pascal's Identity

Theorem: Let n and k be positive integers with $n \geq k$. Then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: Suppose that T is a set containing $n+1$ elements.

- let a be an element in T
- let $S = T - a$.

Left-hand side counts the number of subsets of T containing k elements, i.e., $\binom{n+1}{k}$.

Note that a subset of T with k elements either **contains a** together with $k-1$ elements of S , or contains k elements of S and **does not contain a** .

Right-hand side counts

- the subsets of $k-1$ elements of S , i.e., $\binom{n}{k-1}$
- the subsets of k elements of T , i.e., $\binom{n}{k}$.

Pascal's Identity

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$$

Consider $S = \{A, B, C, D, E\}$.

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts:

- S_2 : the 2-subsets that contain E
- S_3 : the set of 2-subsets that **do not** contain E

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

Pascal's Triangle

$\binom{0}{0}$		1
$\binom{1}{0} \binom{1}{1}$		1 1
$\binom{2}{0} \binom{2}{1} \binom{2}{2}$	By Pascal's identity:	1 2 1
$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$	$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$	1 3 3 1
$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$		1 4 6 4 1
$\binom{5}{0} \binom{5}{1} \binom{5}{2} \binom{5}{3} \binom{5}{4} \binom{5}{5}$		1 5 10 10 5 1
$\binom{6}{0} \binom{6}{1} \binom{6}{2} \binom{6}{3} \binom{6}{4} \binom{6}{5} \binom{6}{6}$		1 6 15 20 15 6 1
$\binom{7}{0} \binom{7}{1} \binom{7}{2} \binom{7}{3} \binom{7}{4} \binom{7}{5} \binom{7}{6} \binom{7}{7}$		1 7 21 35 35 21 7 1
$\binom{8}{0} \binom{8}{1} \binom{8}{2} \binom{8}{3} \binom{8}{4} \binom{8}{5} \binom{8}{6} \binom{8}{7} \binom{8}{8}$		1 8 28 56 70 56 28 8 1
...		...

Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n , can be used to **recursively** define binomial coefficients.

Other Identities Involving Binomial Coefficients

Let n and r be nonnegative integers with $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}.$$

Proof: Consider bit strings of length $n+1$.

The **left-hand side**, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.

We show that the **right-hand side** counts the same objects by considering the cases corresponding to **the possible locations of the final 1** in a string with $r+1$ ones.

Other Identities Involving Binomial Coefficients

Let n and r be nonnegative integers with $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}.$$

Proof: We show that the **right-hand side** counts the same objects by considering the cases corresponding to **the possible locations of the final 1** in a string with $r+1$ ones.

- This **final one** must occur at position $r+1, r+2, \dots$, or $n+1$.
- If the last one is the k -th bit there must be r ones **among the first $k-1$ positions**. There are $\binom{k-1}{r}$ such bit strings.

Summing over k with $r+1 \leq k \leq n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^n \binom{j}{r}.$$



Combinatorial Proof: Example

$$\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}.$$

Hint: Consider a set of people, and count the number of ways to select a committee and select one leader within the committee.

- **Left-hand side:** Suppose there are k people in a committee.
 - ▶ First select k people from the n people to form a committee.
 - ▶ Then, given the people in the committee, select a leader from the committee, i.e., k ways.

Thus, there are a total of $\sum_{k=1}^n k \binom{n}{k}$ ways.

- **Right-hand side:**
 - ▶ There are n ways to choose a leader.
 - ▶ Then, each person other than the leader can be either in the committee or not, i.e., 2^{n-1} ways.

Hence, using product rule, there are $n2^{n-1}$ ways.



Combinatorial Proof: Example

Use combinatorial proof to show that when integer $n \geq 2$, then

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^n 1 = \binom{n}{2}.$$

Labelling and Trinomial Coefficients

Suppose we have k_1 labels of one kind (e.g., red) and $k_2 = n - k_1$ labels of another (e.g., blue). How many different ways to label n distinct objects?

$$C(n, k_1) = \frac{n!}{k_1!k_2!}$$

If we have k_1 labels of one kind (e.g., red), k_2 labels of a second kind (e.g., blue), and $k_3 = n - k_1 - k_2$ labels of a third kind (e.g., green). How many different ways to label n distinct objects?

- There are $\binom{n}{k_1}$ ways to choose the red items
- There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$.

Labelling and Trinomial Coefficients

How many different ways to label n distinct objects?

- There are $\binom{n}{k_1}$ ways to choose the red items
- There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$.

$$\begin{aligned}\binom{n}{k_1} \binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}\end{aligned}$$

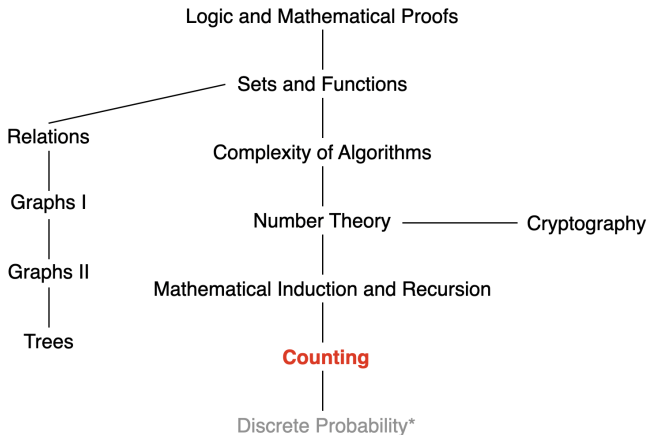
This is called a **trinomial coefficient** and denote it as

$$\binom{n}{k_1 \quad k_2 \quad k_3} = \frac{n!}{k_1!k_2!k_3!},$$

where $k_1 + k_2 + k_3 = n$.

What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x + y + z)^n$?

This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients,
The Birthday Paradox, Solving Linear Recurrence Relations, ...

The Birthday Paradox

Suppose that 25 students are in a room. What is the probability that **at least two of them share a birthday**?

It's greater than $1/2$! (only need 23).

A_n - “there are n students in a room and at least two of them share a birthday.”

We may assume that a year has **365 days** and there are **no twins** in the room.

This will be very similar to the analysis of **hashing n keys** into a table of size 365. ‘

The Birthday Paradox

A_n - “there are n students in a room and at least two of them share a birthday.”

Sample space: $|S| = 365^n$

B_n - “there are n students in a room and **none** of them share a birthday.”

$$\#B_n = 365 \times 364 \times \dots \times (365 - (n - 1))$$

$$\#A_n + \#B_n = 365^n$$

The Birthday Paradox

The Birthday Paradox

Event A: at least two people in the room have the same birthday

Event B: no two people in the room have the same birthday

$$\Pr[A] = 1 - \Pr[B]$$

$$\begin{aligned}\Pr[B] &= \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \dots \cdot \left(1 - \frac{n-1}{365}\right) \\ &= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right).\end{aligned}$$

$$\Pr[A] = 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

“Birthday” Attacks

Given a function f , the goal of the attack is to find **two different inputs** x_1 and x_2 such that $f(x_1) = f(x_2)$. Such a pair x_1 and x_2 is called a **collision**.

Collision in Hashing Functions: A good hashing function yields few collisions (i.e., which are mappings of two different keys to the same memory location).

$$p(n; H) := 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{H}\right)$$

“Birthday” Attacks

$$p(n; H) := 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{H}\right)$$

Note that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, for $|x| \ll 1$, $e^x \approx 1 + x$.

Thus, we have $e^{-i/H} \approx 1 - \frac{i}{H}$.

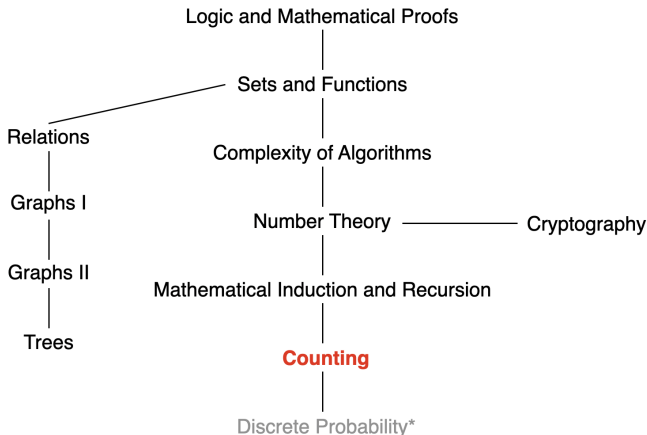
This probability can be approximated as

$$p(n; H) \approx 1 - e^{-n(n-1)/(2H)} \approx 1 - e^{-n^2/(2H)}.$$

Let $n(p; H)$ be the **smallest number** of values we have to choose, such that the probability for finding a collision is **at least p** . By inverting the expression above, we have

$$n(p; H) \approx \sqrt{2H \ln \frac{1}{1-p}}$$

This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients,
The Birthday Paradox, **Generalized Permutations and Combinations**,
Generating Function, Solving Linear Recurrence Relations , ...



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Generalized Permutations and Combinations

- Permutations with repetition
- Permutations with indistinguishable objects
- Combinations with repetition

Repetition: Distinct objects; each object can be selected multiple times

Indistinguishable objects: E.g., “SUCCESS”

Permutations with Repetition

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet? 26^r

Theorem: The number of r -permutations of a set of n objects **with repetition** allowed is n^r .

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word SUCCESS?

Solution:



- The three S's can be placed among the seven positions in $C(7, 3)$ different ways.
- The two C's can be placed in $C(4, 2)$ ways.
- The U can be placed in $C(2, 1)$ ways.
- The E can be placed in $C(1, 1)$ way.

From the product rule,

$$C(7, 3)C(4, 2)C(2, 1)C(1, 1) = \frac{7!}{3!2!1!1!} = 420.$$



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Permutations with Indistinguishable Objects



Theorem: The number of different permutations of n objects, where there are n_1 **indistinguishable objects** of type 1, n_2 indistinguishable objects of type 2, . . . , and n_k indistinguishable objects of type k , is

$$C(n, n_1)(n - n_1, n_2) \cdots C(n - n_1 - \cdots n_{k-1}, n_k) = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

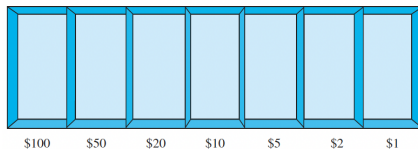
Combinations with Repetition

Example: How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills?

Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

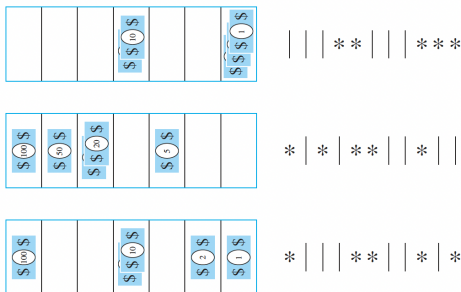
Combinations with Repetition

Solution: Suppose that a cash box has seven compartments, one to hold each type of bill.



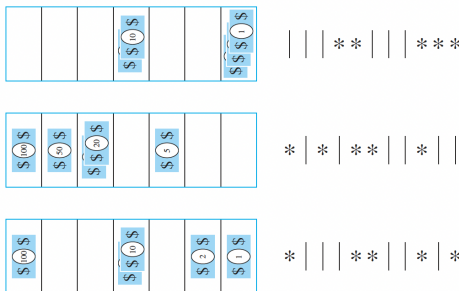
- These compartments are separated by **six dividers**
- The choice of five bills corresponds to placing **five markers** in the compartments holding different types of bills.

Combinations with Repetition



The number of ways to select five bills corresponds to the number of ways to **arrange six bars and five stars** in a row with a total of **11 positions**.

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Consequently, the number of ways to select the five bills is the number of ways to select **the positions of the five stars** from the 11 positions.

$$C(11, 5) = \frac{11!}{5!6!} = 462$$

Combinations with Repetition

Theorem: There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r -combinations from a set with n elements when repetition of elements is allowed.

In the previous example:

- Selecting five bills: $r = 5$
- Seven types of bills: $n = 7$

Combinations with Repetition: Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 , and x_3 are nonnegative integers?

Solution: This is equivalent to finding the number of ways of selecting 11 items from three types of items, so that x_1 items of type one, x_2 items of type two, and x_3 items of type three:

$$C(3 + 11 - 1, 11) = 78$$

Combinations with Repetition: Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$ are nonnegative integers?

Solution: A solution corresponds to a selection of 11 items with x_1 items of type one, x_2 items of type two, and x_3 items of type three:

- **At least** one item of type one, two items of type two, and three items of type three.

A solution corresponds to a choice of **one** item of type one, **two** of type two, and **three** of type three, together with a choice of **five additional items** of any type.

$$C(3 + 5 - 1, 5) = 21$$

Combinations with Repetition: Example

How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 15,$$

where $x_1 \geq 2$, $x_2 \geq 2$, and $x_3 \geq 3$ are nonnegative integers? Please provide the details for deriving the results.

Solution: Let $y_1 = x_1 - 2$, $y_2 = x_2 - 2$, and $y_3 = x_3 - 3$. Introducing a new variable y_4 , the original question is equivalent to finding the number of solutions to the equality

$$y_1 + y_2 + y_3 + y_4 = 8,$$

where $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, and $y_4 \geq 0$.

Thus, the number of solutions is equal to $C(8 + 4 - 1, 8) = 165$.



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