Homework 11 (Due December 12)

Grade Distribution (Total=2+4+10+8=26).

1. If $Y = aX + b$, where a and b are constants, express the characteristic function of Y in terms of the characteristic function of X.

2. The positive random variable X is said to be a lognormal random variable with parameters μ and σ^2 if $\ln(X)$ is a normal random variable with mean μ and variance σ^2. Use the characteristic function to find the mean and variance of a lognormal random variable. [Hint: Use $X = e^{\ln X}$.]

3. Hint: Use

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_i,$$

In the above ξ_i is the number of offspring of individual i in generation $n-1$, and $P(\xi_i = j) = P_j$ for $j \geq 0$. Then condition on X_{n-1}.

7.44. Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that, by the end of its lifetime, each individual will have produced j new offspring with probability P_j, $j \geq 0$, independently of the number produced by any other individual. The number of individuals initially present, denoted by X_0, is called the size of the zeroth generation. All offspring of the zeroth generation constitute the first generation, and their number is denoted by X_1. In general, let X_n denote the size of the nth generation. Let $\mu = \sum_{j=0}^{\infty} jP_j$ and $\sigma^2 = \sum_{j=0}^{\infty}(j - \mu)^2 P_j$ denote, respectively, the mean and the variance of the number of offspring produced by a single individual. Suppose that $X_0 = 1$—that is, initially there is a single individual in the population.

(a) Show that

$$E[X_n] = \mu E[X_{n-1}]$$

$EX_n = E\left[\sum_{i=1}^{X_{n-1}} \xi_i\right].$

(b) Use part (a) to conclude that

$$E[X_n] = \mu^n$$

(c) Show that

$$\mathrm{Var}(X_n) = \sigma^2\mu^{n-1} + \mu^2\mathrm{Var}(X_{n-1})$$

(d) Use part (c) to conclude that

$$\mathrm{Var}(X_n) = \begin{cases} \sigma^2\mu^{n-1}\left(\dfrac{\mu^n - 1}{\mu - 1}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

The model just described is known as a *branching process*, and an important question for a population that evolves along such lines is the probability that the population will eventually die out. Let π denote this probability when the population starts with a single individual. That is,

$\pi = P\{\text{population eventually dies out}|X_0 = 1)$

(e) Argue that π satisfies

$$\pi = \sum_{j=0}^{\infty} P_j \pi^j$$

Hint: Condition on the number of offspring of the initial member of the population.

4. Prove that for a sequence of random variables X_n and X, the following are equivalent:

   (a) For any $\varepsilon > 0$,
   $$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

   (b) For any $\varepsilon > 0$,
   $$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0.$$