陈恺 12311410. Complex Analysis Midterm.

Problem 1

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NOT EXIST.

Proof: Suppose otherwise: $\exists$ invertible hol. $f: S \to \mathbb{C}$.

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$\Rightarrow \exists f^{-1}: \mathbb{C} \to S$.

Since f is bijective (invertible $\Rightarrow$ bij.), f is locally injective at a, $\forall a \in \mathbb{C}$.

Since f is also hol., $f'(a) \neq 0$ $\forall a \in \mathbb{C}$.

$\Rightarrow f^{-1}$ is also hol. (by our thm about the inverse of hol. func.)

(Note that f^{-1} is now entire!)

Since $f^{-1}(w) \in S$, $\forall w \in \mathbb{C}$, $|f^{-1}(w)| \leq \sqrt{1+1} = \sqrt{2}$, $\forall w \in \mathbb{C}$. $\Rightarrow f^{-1}$ is bounded.

Since f^{-1} is entire and bdd, by Liouville's Thm, $f^{-1} \equiv$ constant.

$\Rightarrow f^{-1}$ is not injective and surjective.

Since $f \circ f^{-1} = I_d$, f cannot be a map from S to $\mathbb{C}$! A contradiction. $\square$

Problem 2.

Proof: We want to use integral prop. implies differ. prop.

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We want to show: $\forall$ closed triangle $\bar{\Delta} \subset D$, $\int_{\partial\Delta} f(\zeta)d\zeta = 0$.

Case 1. The triangle Δ satisfies $\bar{\Delta} \cap \gamma = \emptyset$.

Then since $f \in \mathcal{O}(D \setminus \gamma)$, $\int_{\partial\Delta} f(\zeta)d\zeta = 0$.

Case 2. The triangle Δ satisfies $\bar{\Delta} \cap \gamma \neq \emptyset$.

By subdividing the triangle, we only need to prove: $\int_{\partial\Delta} f(\zeta)d\zeta = 0$ if the triangle has 2 sides in $D \setminus \gamma$ and 1 side in γ.

Since $\bar{\Delta}$ is bounded and f is continuous on $\bar{\Delta}$, f is uniformly continuous on $\bar{\Delta}$.

$\forall \varepsilon > 0, \exists \delta(\varepsilon)$: we could find a contour Δ' in $D \setminus \gamma$ arbitrarily close to Δ s.t.

$|f(z) - f(z')| < \varepsilon$, $\forall |z - z'| < \delta(\varepsilon)$, $z \in \Delta$, $z' \in \Delta'$. and the length difference between

Δ and Δ' is smaller than ε.

We could find a sequence of triangles Δ_n contained in $D\backslash\gamma$ that approximate Δ (e.g. using a grid refinement). Then by continuity of f, let $n\to\infty$, $(\varepsilon\to 0)$

$$0=\lim_{n\to\infty}\int_{\partial\Delta_n} f(\zeta)d\zeta = \int_{\partial\Delta} f(\zeta)d\zeta.$$

Thus, $\forall\, \bar{\Delta}\subset D$, $\int_{\partial\Delta} f(\zeta)d\zeta=0$. $\Rightarrow$ $f\in\mathcal{O}(D)$. $\square$.

Problem 3.

NO

Proof: Suppose rather. $\exists\,\{P_n(z)\}_{n=1}^{\infty}$ s.t. $P_n(z)$ is a poly and $P_n(z)\overset{n\to\infty}{\rightrightarrows} f(z)=\bar{z}$ on C.

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i.e. $\forall\,\varepsilon>0$, $\exists\, N=N(\varepsilon)\in\mathbb{N}^*$, s.t. $|P_n(z)-\bar{z}|<\varepsilon$, $\forall z\in C$.

Consider $\int_C \bar{z}dz$: $\int_C \bar{z}dz=\int_{|z|=1}\bar{z}\,dz=\int_{|z|=1}\frac{1}{z}dz=2\pi i$

$\forall\, P_n(z)$, suppose $P_n(z)=a_0+a_1z+\cdots+a_mz^m$.

$$\int_C P_n(z)\,dz=\sum_{i=0}^{m}\int_C a_iz^i dz=\sum_{i=0}^{m}0=0. \quad\Rightarrow\quad \forall\, P_n(z),\ \int_C P_n(z)=0.$$

Since $P_n(z)\overset{n\to\infty}{\rightrightarrows} f(z)=\bar{z}$ uniformly, $0=\lim_{n\to\infty}\int_C P_n(z)dz=\int_C\lim_{n\to\infty}P_n(z)dz=\int_C\bar{z}\,dz=2\pi i$.

$\Rightarrow 0=2\pi i$. a contradiction. $\square$.

Problem 4.

Proof: Observation: Since on any compact $K\subset D$, the sequence (f_j-f) doesn't have any singularities $\forall\, j\geq N=N(K)$, f_j and f must have the same singularities within K itself.

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$\forall\, z_0\in D$, define a compact $K=\overline{B_r(z_0)}$ s.t. $K\subset D$.

Since $\exists\, N=N(K)\in\mathbb{N}^*$, s.t. $\forall j\geq N$, $h_j(z)\triangleq f_j(z)-f(z)$ has no singularities.

$h_j(z)$ must be hol. on some open set containing K. why

In part., $h_j(z)$ is hol. in $B_r(z_0)$.

$\Rightarrow f(z) = f_j(z) - h_j(z)$. Since $f_j(z)$ is meromorphic in D, it's also meromorphic in $B_r(z_0)$

Since $h_j(z)$ is hol. in $B_r(z_0)$, $f(z)$ must be meromorphic in $B_r(z_0)$.

Therefore, $\forall z_0 \in D$, f is meromorphic in $B_r(z_0)$ for some $r>0$.

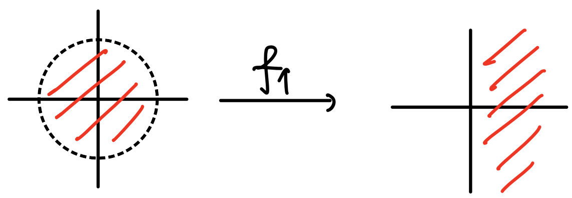
$\Rightarrow$ By def, f is meromorphic in D. $\square$

Problem 5.

(i) Solution: $f(B_1(0)) = \mathbb{C}$.

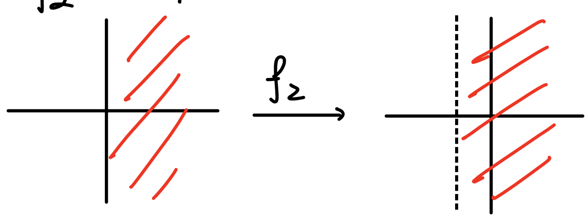
10 Step 1: $f_1 = \dfrac{1+z}{1-z}$

$1 \to \infty \quad i \to i \quad -i \to -i \quad 0 \to 1$

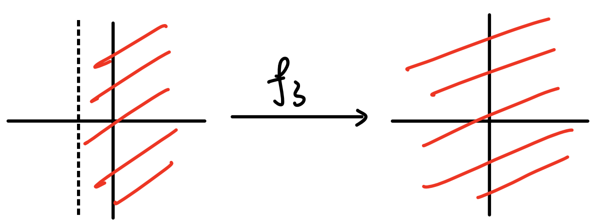


(right half plane)

Step 2: $f_2 = z - 1$



Step 3: $f_3 = z^2$.

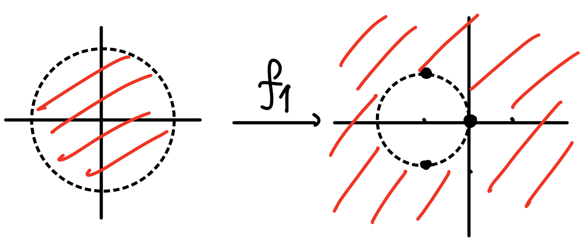


$f = f_3 \circ f_2 \circ f_1 : B_1(0) \longrightarrow \mathbb{C}$. ($\infty$ doesn't get a preimage)

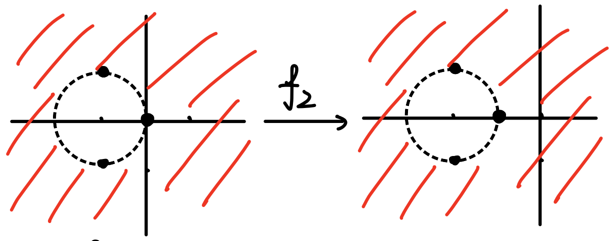
(ii) Solution: $f(B_1(0)) = \overline{\mathbb{C}}$.

Step 1: $f_1 = \dfrac{1-z}{z}$ (f_1 is $\mathbb{C}$-diff. at 0 since $\varphi(f_1(z))$ is $\mathbb{C}$-diff at 0, where $\varphi(z) = \frac{1}{z}$)

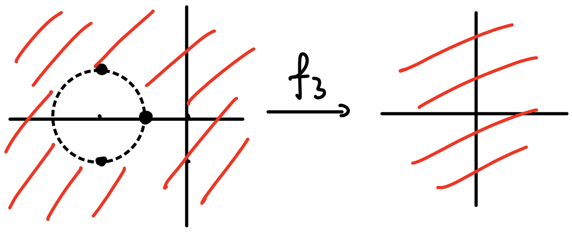
$1 \to 0 \quad i \to -i-1 \quad -i \to i-1 \quad 0 \to \infty$



Step 2: $f_2 = z-1$



Step 3: $f_3 = z^2$



$f = f_3 \circ f_2 \circ f_1 : B_1(0) \longrightarrow \overline{\mathbb{C}}$. ($\infty$'s preimage is 0).

Problem 6.

Proof: Since $|z|^{\frac{1}{2}} \cdot f(z)$ is bounded, $\exists M > 0$, s.t. $|f(z)| \le \frac{M}{|z|^{\frac{1}{2}}}$ $\forall z \in B_1^*(0)$

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$\Rightarrow \forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) = \varepsilon^2$, s.t. $\forall z \in B_\delta(0)$,

$$|z f(z) - 0| \le M |z|^{\frac{1}{2}} < M\varepsilon \Rightarrow \lim_{z \to 0} z f(z) = 0.$$

Define $g(z) = \begin{cases} z^2 f(z), & z \in B_1^*(0). \\ 0, & z = 0. \end{cases}$

For $z \in B_1^*(0)$, $g(z)$ is the product of z^2 and $f(z)$, which are both hol.

$\Rightarrow g(z)$ is hol. in $B_1^*(0)$.

Consider: $g'(0) = \lim_{z \to 0} \frac{g(z) - g(0)}{z - 0} = \lim_{z \to 0} z f(z) = 0$ exists.

$\Rightarrow g(z)$ is hol in $B_1^*(0)$ and $\mathbb{C}$-diff at $z=0$.

$\Rightarrow g(z)$ is hol. in $B_1^*(0)$.

$\Rightarrow g(z)$ has its Taylor expansion: $g(z) = C_0 + C_1 z + C_2 z^2 + \cdots\cdots$

where $C_0 = g(0) = 0$, $C_1 = g'(0) = 0$. $\Rightarrow g(z) = C_2 z^2 + C_3 z^3 + \cdots\cdots$

$\Rightarrow g(z) = z^2 (C_2 + C_3 z + C_4 z^2 + \cdots\cdots)$

For $z \in B_1^*(0)$, $g(z) = z^2 f(z)$ $\Rightarrow f(z) = C_2 + C_3 z + C_4 z^2 + \cdots\cdots$

Define $h(z) = C_2 + C_3 z + C_4 z^2 + \cdots\cdots$

$h(z)$ is hol. in $B_R(0)$, where R is the convergence diameter of $g(z)$'s Taylor series.

Since $h(z) = f(z)$ in $B_1^*(0)$ and $h(z)$ is hol. at $z=0$,

we have $h(z)$ is the holomorphic extension of $f(z)$ at $z=0$.

Thus, $f(z)$ extends holomorphically to $B_1(0)$. $\square$.

Problem 7.

Proof: Since f is entire it has a Taylor expansion around $z=0$ that converges $\forall z \in \mathbb{C}$:

7 $$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k.$$

Now, we WTS that $f^{(k)}(0)$ vanishes after some k.

Since $|f(z)| = O(|z|^a)$, there exists some $C>0$, $R_0>0$, s.t

$$|f(z)| < C \cdot |z|^a, \quad \forall |z| > R_0.$$

By the Cauchy Integral Formula for derivatives, $f^{(n)}(0) = \dfrac{n!}{2\pi i} \displaystyle\int_{|\xi| = R_0+1} \frac{f(\xi)}{\xi^{n+1}} d\xi$

$$|f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \int_{|\xi|=R} \frac{f(\xi)}{\xi^{n+1}} d\xi \right| \leq \frac{n!}{2\pi} \int_{|\xi|=R} \left| \frac{f(\xi)}{\xi^{n+1}} \right| d\xi$$

R here is a radius greater than R_0. (no other limits).

$$< \frac{n!}{2\pi} \cdot \int_{|\xi|=R} \frac{C \cdot R^a}{|\xi^{n+1}|} d\xi = \frac{n!}{2\pi} \cdot 2\pi R \cdot C \cdot R^{a-n-1} = n! \cdot C \cdot R^{a-n}.$$

For $n > a$, $|f^{(n)}(0)| < n! \cdot C \cdot R^{a-n}$, $\forall R > R_0$.

$$\Rightarrow |f^{(n)}(0)| \leq \lim_{R \to \infty} n! \cdot C \cdot R^{a-n} = 0 \quad \Rightarrow \quad f^{(n)}(0) = 0, \ \forall n > a.$$

Thus, the series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$ terminates and $f(z) = \displaystyle\sum_{k=0}^{\lfloor a \rfloor} \frac{f^{(k)}(0)}{k!} z^k$.

Thus, $f(z)$ is a polynomial. $\square$.

Problem 8

NO

Proof: Suppose rather. $f(\frac{1}{n}) = \dfrac{(-1)^n}{n}$

7 Consider: $n = 2k \Rightarrow f(\frac{1}{2k}) = \frac{1}{2k}$. Define $g(z) = z$.

Since $f, g \in hol(B_1(0))$, $E = \{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2k}, \cdots\}$ with accumulation point $z=0$.

$f \equiv g\big|_E$. By our uniqueness thm, $f \equiv g$ in $B_1(0)$.

Argueing similarly, we have $f \equiv h$ in $B_1(0)$, where $h(z) = -z$.

(Consider: $n = 2k-1 \Rightarrow f(\frac{1}{2k-1}) = \frac{-1}{2k-1}$.)

$\Rightarrow$ $z = f(z) = -z$, $\forall z \in B_1(0)$. a contradiction. $\square$.

Problem 9.

Proof: For the PDE: $\frac{\partial f}{\partial \bar{z}} = z$, the general sol. is composed of a particular

7 sol. and a homogeneous sol. (of $\frac{\partial f}{\partial \bar{z}} = 0$)

For the homogeneous sol. of $\frac{\partial f}{\partial \bar{z}} = 0$, we know: Cauchy-Riemann Eq.s hold.

Further, since $f \in C^1$, suppose $f = u + iv$, we know u, v are continuously differentiable.

Thus, the homogeneous sol. of $\frac{\partial f}{\partial \bar{z}} = 0$ here is all the holomorphic functions, denoted by $h(z)$.

Now, find a particular sol. of $\frac{\partial f}{\partial \bar{z}} = z$.

Note that for $f_0(z) = |z|^2 \overset{z = x+iy}{=} x^2 + y^2$, $f_0(z) \in C^1$.

$$\frac{\partial f_0}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f_0}{\partial x} - \frac{1}{i}\frac{\partial f_0}{\partial y}\right) = \frac{1}{2}\left(2x - \frac{1}{i}2y\right) = x + iy = z.$$ $f_0(z)$ is a particular sol.

Thus, all C^1 functions $f(z)$ in $\mathbb{C}$ such that $\frac{\partial f}{\partial \bar{z}} = z$ can be represented by $f(z) = f_0(z) + h(z)$, where $h(z)$ is hol. and $f_0(z)$ is a particular sol.

On one hand for $f_0(z) = |z|^2 = x^2 + y^2$, $u(x,y) = x^2 + y^2$, $v(x,y) = 0$.

$$\frac{\partial^k u}{\partial x^k} = \begin{cases} 2x, & k=1 \\ 2, & k=2 \\ 0, & k \geq 3 \end{cases}, \quad \frac{\partial^k u}{\partial y^k} = \begin{cases} 2y, & k=1 \\ 2, & k=2 \\ 0, & k \geq 3 \end{cases}, \quad \frac{\partial^k v}{\partial x^k} = \frac{\partial^k v}{\partial y^k} = 0 \Rightarrow f_0(z) \in C^\infty.$$

On the other hand, for holomorphic $h(z) \in \mathcal{O}(\overline{\mathbb{C}})$, by the Thm of Cauchy f-la for derivatives, we know $\exists\, h', h'', \dots, h^{(k)}, \dots$ all $\in \mathcal{O}(\overline{\mathbb{C}})$.

$\forall k \in \mathbb{N}^*$, $\exists\, h^{(k)}, h^{(k+1)}$. For $h^{(k)}$, $\exists$ complex derivative $h^{(k+1)}$.

$\Rightarrow h^{(k)} \in C^1$, $\forall k \in \mathbb{N}^* \Rightarrow h(z) \in C^\infty$.

Thus, we have $f(z) = f_0(z) + h(z)$ with $f_0(z), h(z) \in C^\infty \Rightarrow f(z) \in C^\infty$. $\square$.

Problem 10

Solution: Define $S = \{ z \in \mathbb{C} \mid |z^2 - 1| < R \}$, $f(z) = z^2 - 1$. $\Rightarrow S = f^{-1}(B_R(0))$.

10 Since $f(z) = z^2 - 1$ is a polynomial, $f(z) \in \mathcal{O}(\mathbb{C})$.

$f'(z) = 2z \Rightarrow$ critical point : $z = 0$. $f(0) = -1 \Rightarrow$ critical value: -1

Case 1: $R > 1$. the critical value -1 lies in the interior of the disc $B_R(0)$.

the critical point $z = 0 \in S$ since $|0-1| < R$.

Since f is continuous and $B_R(0)$ is a connected open set,

$S = f^{-1}(B_R(0))$ is also an open set.

To show S is connected, we only need to show that S is path-connected.

We prove this by proving: $\forall z \in S$, z can be connected to 0 by a path in S.

let $w = f(z) = z^2 - 1$. Since $z \in S$, $w \in B_R(0)$. Also, we know $-1 \in B_R(0)$

Since $B_R(0)$ is an open disc, it's convex and connected. $\Rightarrow$ it's path-connected.

In part., for w and -1, there exists $\gamma(t) = (1-t)w + t(-1)$, with $t \in [0,1]$.

Since $B_R(0)$ is convex, γ lies totally in $B_R(0)$'s interior.

Since f is continuous and is a covering map of $\mathbb{C} \setminus \{0\}$, $\gamma(t)$ could be lifted to another path $\tilde{\gamma}$, where $f(\tilde{\gamma}(t)) = \gamma(t)$ and $\tilde{\gamma}(0) = z$.

Since $\gamma(t) \to -1$ $(t \to 1)$, $f(\tilde{\gamma}(t)) = (\tilde{\gamma}(t))^2 - 1 \to -1$ $(t \to 1)$,

we have $(\tilde{\gamma}(t))^2 \to 0$ $(t \to -1) \Rightarrow \tilde{\gamma}(t) \to 0$ $(t \to -1)$

Since $z, 0 \in S$, the lifted path $\tilde{\gamma}(t)$ becomes a path in S connecting 0 and z.

Thus, any $z_1, z_2 \in S$ could be connected to 0 by paths in S.

Connecting this 2 paths, we get a path connecting z_1 and z_2.

$\Rightarrow$ S is path-connected $\Rightarrow$ S is connected.

<u>Case 2: $0 < R \leq 1$.</u> In this case, $-1 \notin B_R(0)$.

Since $|f(0)| = |-1| = 1$, $|f(0)| \geq R$. $\Rightarrow$ $0 \notin S$.

Consider $z^2 - 1 = w$. $w \in B_R(0)$. $\Leftarrow$ $z^2 = 1 + w$. let $u = 1 + w$, $u \in B_R(1)$.

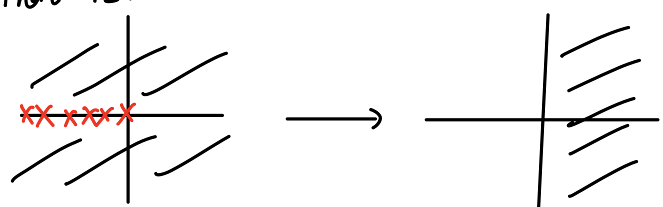
Since $R \leq 1$, $Re(u) > 1 - R \geq 0$. Thus, $B_R(1)$ totally lies in $\{z \mid Re(z) > 0\}$.

We now define two branches of $\sqrt{u}$:

In our case, we need to map $B_R(1)$ using $\sqrt{u}$. so we define $\sqrt{u}$ on $D = \mathbb{C} \setminus (-\infty, 0]$. That is:

$g_1(u) = \sqrt{u}^{(1)}$:

$= \sqrt{u}$

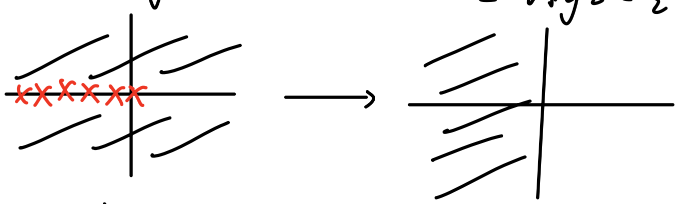


$-\pi < Arg\, z < \pi$ $\qquad$ $-\frac{\pi}{2} < Arg\, z < \frac{\pi}{2}$

$g_2(u) = \sqrt{u}^{(2)}$:

$= -\sqrt{u}^{(1)}$

$= -\sqrt{u}$



$\pi < Arg\, z < 3\pi$ $\qquad$ $\frac{\pi}{2} < Arg\, z < \frac{3}{2}\pi$. $\qquad$ where g_1, g_2 are both hol.

Define $S_1 = g_1(B_R(1))$, $S_2 = g_2(B_R(1))$

Since $\sqrt{u}$ is conformal, $B_R(1)$ is a domain, we have S_1, S_2 are also domains.

Also, since $S = \{z \mid |z^2 - 1| < R\} = \{z \mid z^2 \in B_R(1)\}$, we in fact have

$S = S_1 \cup S_2$. where S_1, S_2 are domains (open connected sets)

Now, we only need to show $S_1 \cap S_2 = \emptyset$ and $S_1 \neq \emptyset$, $S_2 \neq \emptyset$.

then S is not connected.

Suppose rather: $\exists z \in S_1 \cap S_2 \Rightarrow z = \sqrt{u}$ and $z = -\sqrt{u}$ $\Rightarrow$ $z = 0$. But $0 \notin S$ $\Rightarrow$ $S_1 \cap S_2 = \emptyset$.

Besides, since $1 \in B_R(1)\ \forall R>0$, $g_1(1)=\sqrt{1}=1 \in S_1$. $g_2(1) = -\sqrt{1} = -1 \in S_2$.

$\Rightarrow S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

Hence we have decomposite S as a disjoint union of 2 nonempty domains.

namely $S = S_1 \sqcup S_2$. $S_1 \neq \emptyset$, $S_2 \neq \emptyset$. $\Rightarrow$ S is not connected.

To sum up, the set $\{|z^2-1|<R\}$ is connected only if $R>1$.

Problem 11

Proof: $\forall r$, $0<r<R$. We can choose another radius ρ, s.t $r<\rho\le R$.

10 By the Cauchy Integral Formula for the derivative,

$$f_n'(z) = \frac{1}{2\pi i} \int_{|\xi-a|=\rho} \frac{f_n(\xi)}{(\xi-z)^2}\, d\xi, \quad \forall z \in \overline{B_r(a)}.$$

$$|f_n'(z)| = \left| \frac{1}{2\pi i} \int_{|\xi-a|=\rho} \frac{f_n(\xi)}{(\xi-z)^2}\, d\xi \right| \le \frac{1}{2\pi} \cdot 2\pi\rho \cdot \max_{|\xi-a|=\rho} \left| \frac{f_n(\xi)}{(\xi-z)^2} \right|.$$

$$|\xi-z| = |(\xi-a)-(z-a)| \ge \big||\xi-a|-|z-a|\big| = \big|\rho-|z-a|\big| = \rho-|z-a| > \rho-r>0.$$

$$|f_n'(z)| \le \frac{1}{2\pi}\cdot 2\pi\rho \cdot \frac{C}{(\rho-r)^2} = \frac{C\cdot\rho}{(\rho-r)^2}.$$

In partc., we can choose $\rho = \frac{R+r}{2}$, $|f_n'(z)| \le \frac{2\cdot C\cdot(R+r)}{(R-r)^2} \triangleq M$.

$\Rightarrow \forall r<R$, $\exists M = \frac{2C(R+r)}{(R-r)^2}$ s.t $|f_n'(z)| \le M$, $\forall n \in \mathbb{N}^*$, $\forall z \in \overline{B_r(a)}$.

Thus, the sequence f_n' is uniformly bdd in some $\overline{B_r(a)}$, $r<R$. $\square$.

Problem 12.

(i) It can be extended holomorphically to a domain in the complex plane.

Proof: Consider the complex function $F(z) = z\cdot \mathrm{Ln}(1+z)$, where $\mathrm{Ln}(w)$ is the principle

7 branch of the complex logrithm $(-\pi<\mathrm{Arg}\,w<\pi)$.

Here, $\mathrm{Ln}(w)$ is defined and holomorphic for $w \in \mathbb{C}\setminus(-\infty,0]$.

(To extend from $\mathbb{R}^+$, we choose $(-\infty,0]$ rather $[0,+\infty)$ for $x\ln(1+x)$ is defined on $\mathbb{R}^+$)

$\Rightarrow 1+z \in \mathbb{C} \setminus (-\infty, 0] \Rightarrow z \in \mathbb{C} \setminus (-\infty, -1]$

i.e. $\mathrm{Ln}(1+z)$ is hol. in $\mathbb{C} \setminus (-\infty, -1]$.

Since z is hol. in the whole complex plane, $z\,\mathrm{Ln}(1+z)$ is hol. in $\mathbb{C} \setminus (-\infty, -1]$.

Moreover, for $x \in \mathbb{R}^+$, $1+x \in (1, +\infty)$, $\mathrm{Ln}(1+x)$ is real and equals $\ln(1+x)$.

$\Rightarrow F(z) = z\,\mathrm{Ln}(1+z)$ doesn't coincide with $f(x) = x \ln(1+x)$.

$\Rightarrow f(x)$ can be extended holomorphically to $\mathbb{C} \setminus (-\infty, -1]$. $\square$.

(ii) It CANNOT be extended holomorphically to the entire complex plane.

<u>Proof</u>: Argue by contradiction.

A function need to be single-valued to be hol.

However, at the branch point $z=-1$, if we analytically continue $z\,\mathrm{Ln}(1+z)$ along a small circle centered at $z=-1$, namely $z(t) = -1 + \varepsilon \cdot e^{it}$ for $t \in [0, 2\pi]$, $\varepsilon \in (0, \frac{1}{2})$.

$\Rightarrow F(z(t)) = (-1 + \varepsilon e^{it})(\ln \varepsilon + it)$

$$F(z(0)) = (-1+\varepsilon) \ln \varepsilon \neq (-1+\varepsilon) \ln \varepsilon + 2\pi i(-1+\varepsilon) = F(z(2\pi)).$$

$\Rightarrow F(z)$ becomes multi-valued here. a contradiction. $\square$.

<u>Remark</u>: By uniqueness thm, if there $\exists$ another extension $G(z)$, $F(z) = G(z)$ for $z \in \mathbb{R}^+ \Rightarrow F(z) = G(z)\ \forall z$.

<u>Problem 13.</u>

<u>NO</u>

<u>Proof</u>: Suppose otherwise.

10 Since $f(z)$ is holomorphic and non-zero in $\mathbb{C}^* = \{|z| > 0\}$,

define $g(z) = \dfrac{1}{f(z)}$. $g(z)$ is hol. in $\mathbb{C}^*$.

Also, $|h(z)| = \dfrac{1}{|f(z)|} < e^{-\frac{1}{|z|}}\ \forall z \in \mathbb{C}^*$.

Since $e^{-\frac{1}{|z|}} \to 0$ $(z \to 0)$, we have $\lim\limits_{z\to 0} |h(z)| \leq \lim\limits_{z \to 0} e^{-\frac{1}{|z|}} = 0$. $\Rightarrow \lim\limits_{z\to 0} |h(z)| = 0$

$\Rightarrow z=0$ is a removable singularity of $h(z)$.

$\Rightarrow$ we can extend $h(z)$ to an entire function $\tilde{h}(z)$ by defining $\tilde{h}(0) = 0$.

Consider: $\lim\limits_{|z| \to \infty} |\tilde{h}(z)| = \lim\limits_{|z| \to \infty} |h(z)| \leq \lim\limits_{|z| \to \infty} e^{-\frac{1}{|z|}} = 1$.

$\Rightarrow \exists A > 0$, s.t. $|\tilde{h}(z)| < 1 + 1 = 2$, $\forall |z| > A$.

For $|z| \leq A$, $|\tilde{h}(z)| \leq \max_{|z|\leq A} e^{-\frac{1}{|z|}} \leq e^{-\frac{1}{|A|}}$.

$\Rightarrow |\tilde{h}(z)| \leq \max\{2, e^{-\frac{1}{|A|}}\}$, $\forall z \in \mathbb{C}$.

i.e $\tilde{h}(z)$ is an entire and bdd function on $\mathbb{C}$.

By Liouville's Thm, $\tilde{h}(z) \equiv$ const. Since $\tilde{h}(0) = 0$, $\tilde{h}(z) \equiv 0$ $\forall z \in \mathbb{C}$.

$\Rightarrow f(z)$ must be infinite. a contradiction. $\square$.

Problem 14.

Proof: Step 1: hol. in $B_1(0)$.

10 By Cauchy-Adamar Thm, $\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$.

For non-zero coefficients, $\lim_{n\to\infty} \sqrt[n!]{n!} = 1$. For zero coefficients, $\lim_{n\to\infty} \sqrt[n]{a_n} = 0$.

$\Rightarrow R = 1$.

$\Rightarrow f(z)$ is hol. in $B_1(0)$.

Step 2: CANNOT be extended holomorphically to a neighbourhood of any point a in the boundary of the disc.

We prove this by praving that every point on $\partial B_1(0)$ is a non-removable singularity point.

We only need to show it holds for all rational points on $\partial B_1(0)$ as these points are dense on $\partial B_1(0)$.

Denote the rational points on $\partial B_1(0)$ as $a = e^{2\pi i \cdot \frac{m}{n}}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}^*$.

Consider $z = ra = re^{2\pi i \cdot \frac{m}{n}}$, where $r \in (0,1)$.

$$f(z) = f(r \cdot e^{2\pi i \cdot \frac{m}{n}}) = \sum_{k=1}^{\infty} (r \cdot e^{2\pi i \frac{m}{n}})^{k!} = \sum_{k=1}^{\infty} r^{k!} \cdot e^{2\pi i \cdot \frac{m}{n} \cdot k!}$$

For $k \geq n$, $k! = n \cdot q$ for some $q \in \mathbb{N}^*$.

$$\Rightarrow e^{2\pi i \cdot \frac{m}{n} \cdot k!} = e^{2\pi i \cdot mq} = 1^{mq} = 1.$$

$$\Rightarrow f(z) = \sum_{k=1}^{n-1} r^{k!} e^{2\pi i \cdot \frac{m}{n} \cdot k!} + \sum_{k=n}^{\infty} r^{k!} \cdot 1 > \sum_{k=n}^{\infty} r^{k!} \to +\infty \quad (r \to 1^-)$$

$$\Rightarrow \lim_{r\to 1^-} f(r \cdot e^{2\pi i \cdot \frac{m}{n}}) = +\infty.$$

$\Rightarrow$ $f(z)$ becomes unbounded as z approaches any point $a = e^{2\pi i\cdot\frac{m}{n}}$.

Since these rational points are dense, if a function could be analytically to a neighbourhood of any points on $B_1(0)$, such region must contain rational points.

$\Rightarrow$ $f(z)$ CANNOT be analytically continued beyond $B_1(0)$. $\square$.

Problem 15.

Proof: First, check the branch point. Let $1+z^2=0$. $\Rightarrow$ $z=\pm i$.

7 Since $\pm i \notin \{|z|>1\} \triangleq D$, $g(z) \triangleq 1+z^2$ is never zero in D.

$g(z) = 1+z^2 = z^2(1+\frac{1}{z^2})$. Let $w = z^2$. For $z\in D$, $|w| = \frac{1}{|z^2|} < 1$.

We need to define $\sqrt{g(z)} = z\sqrt{1+w}$.

Let $h(w) = \sqrt{1+w}$ be the principle part of the square root for $1+w \in \mathbb{C}\setminus(-\infty, 0]$.

$\Rightarrow$ $\sqrt{1+w}$ is defined and holomorphic for $w \in \mathbb{C}\setminus(-\infty,-1]$.

Since $|w|<1$, $w\in\mathbb{C}\setminus(-\infty,-1]$ holds for $\forall z\in D$. $\Rightarrow$ $h(w)=\sqrt{1+w}$ choosing the principle branch is well-defined and hol for $|w|<1$.

Now define $f(z) = z\cdot h(\frac{1}{z^2}) = z\sqrt{1+\frac{1}{z^2}}$, where $\sqrt{\cdot}$ is the principle branch.

Since $w=\frac{1}{z^2}$ is hol. for $z\neq 0$, $h(w)$ is hol., z is hol.,

we have $f(z)$ is hol. in D.

$(f(z))^2 = z^2(1+\frac{1}{z^2}) = 1+z^2$ $\Rightarrow$ we have get the required function $f(z)$

Therefore, such function exists. $\square$.

Problem 16.

Proof: Let $w=\frac{1}{z}$. $g(w) = f(\frac{1}{w})$, $w\in B_1^*(0)$.

7 Since f and $\frac{1}{z}$ are hol. in $\{|z|>1\}$, $f(\frac{1}{z})$ is hol in $\{|z|>1\}$, i.e. $g(w)$ is hol. in $B_1^*(0)$

Since $|f(z)|\geq M$ $\forall |z|>1$, $|g(w)|\geq M$ $\forall w\in B_1^*(0)$.

Further, consider $h(w) = \frac{1}{g(w)}$. Since $g(w)$ is hol. and non-zero in $B_1^*(0)$,

$h(w)$ is hol in $B_1^*(0)$. Meanwhile, $|h(w)| = \frac{1}{|g(w)|} \leq \frac{1}{M}$, $\forall w\in B_1^*(0)$.

By Riemann's Thm, we have $w=0$ is a removable singularity of h.

$\Rightarrow \exists$ finite $\lim\limits_{w\to 0} h(w) \triangleq \lambda$.

$\Rightarrow$ We can extend $h(w)$ to a function $\tilde{h}(w)$ by defining $\tilde{h}(0)=\lambda$.

Then, $\tilde{h}(w)$ is hol. in $B_1^*(0)$.

Now, consider $g(w) = \dfrac{1}{\tilde{h}(w)}$.

<u>Case 1: $\lambda\neq 0$.</u> Then $\lim\limits_{w\to 0} g(w) = \dfrac{1}{\lim\limits_{w\to 0} \tilde{h}(w)} = \dfrac{1}{\lambda}$. The limit exists and is finite.

<u>Case 2: $\lambda=0$.</u> Since $\tilde{h}(w)$ is not identically zero, $\tilde{h}(w)$ has a zero of order $k\geq 1$ at $w=0$. $\Rightarrow \tilde{h}(w)=w^k\cdot\phi(w)$, where $\phi(w)$ is hol. and $\phi(0)\neq 0$.

$\Rightarrow \lim\limits_{w\to 0} g(w) = \lim\limits_{w\to 0}\dfrac{1}{w^k\phi(w)} = \infty$. $w=0$ is a pole of $g(w)$ and the limit is infinite.

To sum up. in both cases. the limit $\lim\limits_{w\to 0} g(w)$ exists, either finite or infinite. $\square$

<u>Problem 17.</u>

<u>Proof</u>: Since $f\in \mathcal{O}(D)\cap C(\overline{D})$, f is continuous near and at $z=0$.

7

$\Rightarrow f$ is bounded in some $B_\delta^*(0)$ for $\delta\in(0,1)$.

By Riemann's Thm, $z=0$ is a removable singularity of f.

Define $F(z)=\begin{cases} f(z), & 0<|z|<1 \\ 1, & z\in\partial D.\end{cases}$

Here $F(z)$ is the extension of $f(z)$ and $F(z)\in\mathcal{O}(B_1(0))$.

For $z\in B_1(0)$, $F(z)$ is hol., hence $F(z)$ is continuous.

For $z\in\partial B_1(0)$, $F(z)=f(z)=1$ is continuous.

Moreover, since $f\in C(\overline{D})$, when $z\to z_0$ ($|z_0|=1$) from the interior of $B_1(0)$, $f(z)\to f(z_0)=1$.

Thus, $F(z)\in C(\overline{B_1(0)})$.

Thus, $F(z)\in C(\overline{B_1(0)})\cap\mathcal{O}(B_1(0))$.

By the Maximum Modulus Principle, $|F(z)|\leq F(z)\big|_{\partial B_1(0)}=1$.

Since $F(0)=1=\max\limits_{z\in\overline{B_1(0)}} F(z)$, again by the Maximum Modulus Principle, $F(z)\equiv$ constant.

$\Rightarrow F(z)\equiv 1$ on $\overline{B_1(0)}$. $\Rightarrow f(z)=1$ $\forall z\in\overline{D}$.

Thus, f is constant. $\square$