Discrete Mathematics for Computer Science

Lecture 15-2: Relation

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This Lecture

Logic and Mathematical Proofs

Sets and Functions

Relations

Graphs I

Graphs II

Trees

Complexity of Algorithms

Number Theory — Cryptography

Mathematical Induction and Recursion

Counting

Discrete Probability*

Cartesian Product

Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$, the Cartesian product $A \times B$ is the set of pairs

$$\{(a_1, b_1), (a_2, b_2), ..., (a_1, b_n), ..., (a_m, b_n)\}.$$

Cartesian product defines a set of all ordered arrangements of elements in the two sets.

A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B.

Binary Relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product $A \times B$.

Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

We use the notation aRb to denote $(a, b) \in R$, and $a \not R b$ to denote $(a, b) \notin R$.

Example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$

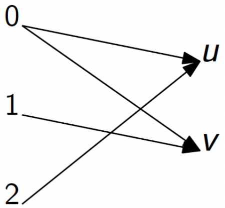
- Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B?
- Is $Q = \{(1, a), (2, b)\}$ a relation from A to B?
- Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A?

Representing Binary Relations

We can graphically represent a binary relation R as:

if aRb, then we draw an arrow from a to b: $a \rightarrow b$

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. $(R \subseteq A \times B)$



Representing Binary Relations

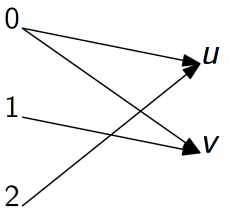
We can also represent a binary relation R by a table showing the ordered pairs of R.

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. $(R \subseteq A \times B)$

R	u	v
0	×	×
1	×	
2		×

Representing Binary Relations

Relations represent one to many relationships between elements in A and B.



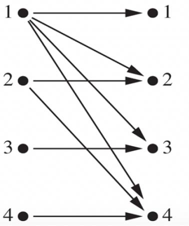
What is the difference between a relation and a function from A to B?

Relation on the Set

Definition: A relation on the set A is a relation from A to itself.

Example: Let $A = \{1, 2, 3, 4\}$ and $R_{div} = \{(a, b) : a \text{ divides } b\}$. What does R_{div} consist of?

$$R_{div} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Number of Binary Relations

Theorem: The number of binary relations on a set A, where $|A| = n$, is 2^{n^2}.

Proof: If $|A| = n$, then the cardinality of the Cartesian product $|A \times A| = n^2$.

R is a binary relation on A if $R \subseteq A \times A$ (R is subset).

The number of subsets of a set with k elements is 2^k.

Properties of Relations

- Reflexive Relation
- Irreflexive Relation
- Symmetric Relation
- Antisymmetric Relation
- Transitive Relation

Properties of Relations: Reflexive Relation

Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$:

$$R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Is R_{div} reflexive?

Yes. $(1, 1), (2, 2), (3, 3), (4, 4) \in R_{div}$.

Reflexive Relation

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$:

$$R_{div} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Is R_{div} reflexive?

Yes. $(1, 1), (2, 2), (3, 3), (4, 4) \in R_{div}$.

Relation Matrix (binary matrix):

$$\mathrm{MR}_{\mathrm{div}} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

A relation R is reflexive if and only if MR has 1 in every position on its main diagonal.

Examples

Consider the set of integers:

$$\begin{aligned} R_1 &= \{(a, b) \mid a \leq b\}, \\ R_2 &= \{(a, b) \mid a > b\}, \\ R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\ R_4 &= \{(a, b) \mid a = b\}, \\ R_5 &= \{(a, b) \mid a = b + 1\}, \\ R_6 &= \{(a, b) \mid a + b \leq 3\}. \end{aligned}$$

Which of these relations reflexive?

R_1, R_3, and R_4.

Number of Reflexive Relations

Theorem: The number of reflexive relations on a set A with $|A| = n$ is $2^{n(n-1)}$.

Proof: A reflexive relation R on A must contain all pairs (a, a) for every $a \in A$.

All other pairs in R are of the form (a, b) with $a \neq b$, s.t. $a, b \in A$.

How many of these pairs are there? $n(n-1)$

How many subsets on $n(n-1)$ elements are there? $2^{n(n-1)}$

Properties of Relations: Irreflexive Relation

Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

Is $R_{\neq}$ irreflexive?

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Yes. $(1,1),(2,2),(3,3),(4,4) \notin R_{\neq}$.

Irreflexive Relation

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

Is $R_{\neq}$ irreflexive?

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is irreflexive if and only if MR has 0 in every position on its main diagonal.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\}, \\
R_2 &= \{(a, b) \mid a > b\}, \\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\
R_4 &= \{(a, b) \mid a = b\}, \\
R_5 &= \{(a, b) \mid a = b + 1\}, \\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}
$$

Which of these relations irreflexive?

R_2 and R_5.

Properties of Relations: Symmetric Relation

Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

Is R_{div} symmetric?

No. $(1, 2) \in R_{div}$ but $(2, 1) \notin R$.

Symmetric Relation

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),\\ (3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Is $R_{\neq}$ symmetric?

Yes. If $(a, b) \in R_{\neq}$ then $(b, a) \in R_{\neq}$.

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

A relation R is symmetric if and only if MR is symmetric.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\}, \\
R_2 &= \{(a, b) \mid a > b\}, \\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\
R_4 &= \{(a, b) \mid a = b\}, \\
R_5 &= \{(a, b) \mid a = b + 1\}, \\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}
$$

Which of these relations symmetric?

R_3, R_4, and R_6.

Properties of Relations: Antisymmetric Relation

Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

Example: Assume that $R = \{(1, 2), (2, 2), (3, 3)\}$ on $A = \{1, 2, 3, 4\}$.

Is R antisymmetric? Yes.

$$\mathrm{MR} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

A relation R is antisymmetric if and only if $m_{ij} = 1$ implies $m_{ji} = 0$ for $i \neq j$.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a,b) \mid a \leq b\},\\
R_2 &= \{(a,b) \mid a > b\},\\
R_3 &= \{(a,b) \mid a = b \text{ or } a = -b\},\\
R_4 &= \{(a,b) \mid a = b\},\\
R_5 &= \{(a,b) \mid a = b + 1\},\\
R_6 &= \{(a,b) \mid a + b \leq 3\}.
\end{aligned}
$$

Which of these relations antisymmetric?

R_1, R_2, R_4 and R_5.

Symmetric Relation

The number of symmetric relations on set A, where A has n elements, is $2^{n(n+1)/2}$.

Proof: When relation R is symmetric, it contains two types of elements (or pair of elements) from $A \times A$:

- (a, a) with $a \in A$: n such tuples in $A \times A$
- both (a, b) and (b, a), with $a, b \in A$ and $a \neq b$: $C(n, 2)$ such tuples in $A \times A$

Each of these elements (or pair of elements) can be either in R or not. Thus, there are $2^{n(n-1)/2+n} = 2^{n(n+1)/2}$ symmetric relations.

Antisymmetric Relation

The number of antisymmetric relations on set A, where A has n elements, is $2^n 3^{n(n-1)/2}$.

Proof: Consider the following two types of elements in $A \times A$:

- (a, a) with $a \in A$: There are n such tuples in $A \times A$. Each tuple can be either in R or not in R. Thus, there are 2^n possibilities.
- (a, b) or (b, a), with $a, b \in A$ and $a \neq b$: There are $C(n, 2)$ pairs of a and b. For each of such pairs, there are three cases:
  - $(a, b) \in R$ and $(b, a) \notin R$;
  - $(a, b) \notin R$ and $(b, a) \in R$;
  - $(a, b) \notin R$ and $(b, a) \notin R$.

  Thus, there are $3^{n(n-1)/2}$ possibilities.

Using product rule, there are $2^n 3^{n(n-1)/2}$ such relations.

Properties of Relations: Transitive Relation

Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Example: Assume that $R_{div} = \{(a, b) : a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$:

$$R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

Is R_{div} transitive?

Yes. If $a|b$ and $b|c$, then $a|c$.

Transitive Relation

Example: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

$$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),$$
$$(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}.$$

Is $R_{\neq}$ transitive?

No. $(1,2),(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$.

Transitive Relation

Example: Assume that $R = \{(1,2),(2,2),(3,3)\}$ on $A = \{1,2,3,4\}$.

Is R transitive?

Yes.

Examples

Consider the set of integers:

$$
\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\}, \\
R_2 &= \{(a, b) \mid a > b\}, \\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\
R_4 &= \{(a, b) \mid a = b\}, \\
R_5 &= \{(a, b) \mid a = b + 1\}, \\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}
$$

Which of these relations transitive?

R_1, R_2, R_3 and R_4.

Summary on Properties of Relations

- Reflexive Relation: A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
- Irreflexive Relation: A relation R on a set A is called irreflexive if $(a, a) \notin R$ for every element $a \in A$.
- Symmetric Relation: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- Antisymmetric Relation: A relation R on a set A is called antisymmetric if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.
- Transitive Relation: A relation R on a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Combining Relations

Since relations are sets, we can combine relations via set operations.

Set operations: union, intersection, difference, etc.

Example: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and
$R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$,
$R_2 = \{(1, v), (3, u), (3, v)\}$

What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

Combining Relations

Example: $R_1 = \{(x, y) | x < y\}$ and $R_2 = \{(x, y) | x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R1 \oplus R_2$?

- $R_1 \cup R_2 = \{(x, y) | x \neq y\}$
- $R_1 \cap R_2 = \emptyset$
- $R_1 - R_2 = R_1$
- $R_2 - R_1 = R_2$
- $R_1 \oplus R_2 = \{(x, y) | x \neq y\}$

Composite of Relations

Definition: Let R be a relation from a set A to a set B and S be a relation from B to C. The composite of R and S is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

We denote the composite of R and S by $S \circ R$.

Example: Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$, and $C = \{a, b\}$:

- $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$
- $S = \{(0, b), (1, a), (2, b)\}$
- $S \circ R = \{(1, b), (3, a), (3, b)\}$

Power of a Relation

Definition: Let R be a relation on A. The powers R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Example: Let $A = \{1, 2, 3, 4\}$, and $R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$

- $R^1 = R$
- $R^2 = R \circ R = \{(1, 3), (1, 4), (2, 3), (3, 3)\}$
- $R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$
- $R^4 = R^3 \circ R = \{(1, 3), (2, 3), (3, 3)\}$
- $R^k =?$ for $k > 3$

Transitive Relation and R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Proof:

- "if" part: In particular, $R^2 \subseteq R$. If $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, we have $(a, c) \in R^2 \subseteq R$.
- "only if: part: by induction.
  - $n = 1$: $R^1 \subseteq R$
  - Suppose $R^n \subseteq R$:
    - Consider $(a, c) \in R^{n+1} \triangleq R^n \circ R$: there is a $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R^n \subseteq R$
    - Since R is transitive, $(a, b) \in R$ and $(b, c) \in R^n \subseteq R$ implies that $(a, c) \in R$.
    - Thus, $R^{n+1} \subseteq R$

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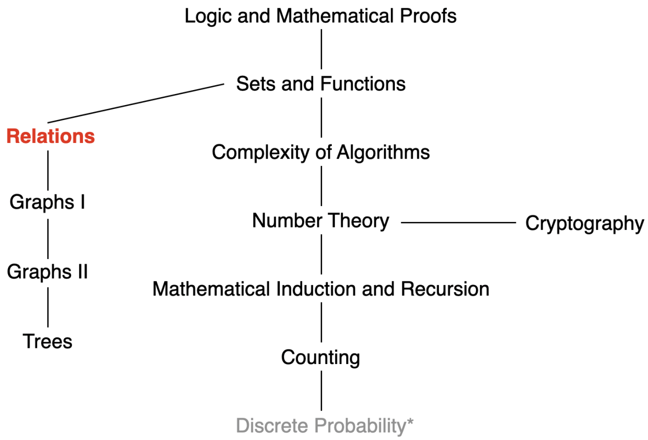
This Lecture



Relation, n-ary Relations, Representing Relations, Closures of Relations, ...

Representing Relations

Some ways to represent n-ary relations:

- with an explicit list or table of its tuples
- with a function from the domain to $\{T, F\}$

Some special ways to represent binary relations:

- with a zero-one matrix
- with a directed graph

Zero-One Matrix

$$m_{ij} = \begin{cases} 1, & (a_i, b_j) \in R \\ 0, & (a_i, b_j) \notin R \end{cases} \tag{1}$$

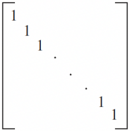
Example: Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$.

What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

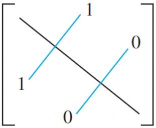
Solution: $R = \{(2, 1), (3, 1), (3, 2)\}$

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

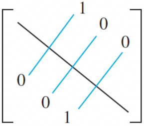
Zero-One Matrix



Reflexive



Symmetric



Antisymmetric

Example: Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Reflexive, symmetric. Not antisymmetric.

Zero-One Matrix: Join and Meet

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero–one matrices.

The join of A and B is the zero–one matrix with (i, j)-th entry $a_{ij} \vee b_{ij}$.
The join of A and B is denoted by $A \vee B$.

The meet of A and B is the zero–one matrix with (i, j)-th entry $a_{ij} \wedge b_{ij}$.
The meet of A and B is denoted by $A \wedge B$.

Zero-One Matrix: Join and Meet

Consider relations R_1 and R_2 on a set A:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Example: Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Zero-One Matrix: Composite of Relations

Let $A = [a_{ij}]$ be an $m \times k$ zero–one matrix and $B = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then, the Boolean product of A and B, denoted by $A \odot B$, is the $m \times n$ matrix with (i, j)-th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

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Zero-One Matrix: Composite of Relations

Suppose that R is a relation from A to B and S is a relation from B to C:

$$M_{S \circ R} = M_R \odot M_S.$$

The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S.

Example:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

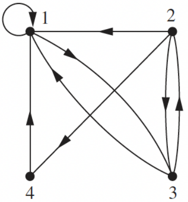
Directed Graph

A directed graph, or digraph, consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

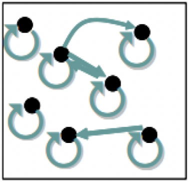
Example: Relation R is defined on $\{1, 2, 3, 4\}$:

$$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$$

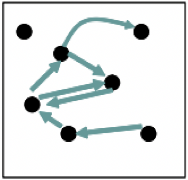


Directed Graph

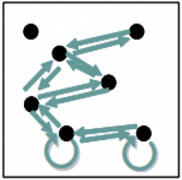
Reflexive, irreflexive, symmetric, antisymmetric?



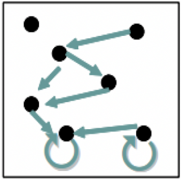
reflexive



irreflexive

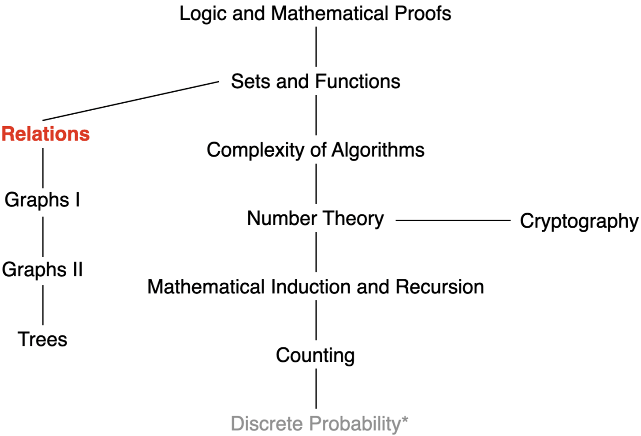


symmetric



antisymmetric

This Lecture



Relation, n-ary Relations, Representing Relations, Closures of Relations, ...