Ordinary Differential Equations A-H (MA230)

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Q: Which of the following equations are DEs?

(a) $y'(t) + t - y = 0$ (b) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin t$ (c) $\sin y(t) = t$.

Motivations and Goals

Motivations: Many natural phenomena and processes in physics, chemistry, biology, and other disciplines could be modeled and simulated by differential equations.

Our Goals: We mainly study ODEs in this course.

- To learn the methods that have proved effective in finding solutions.
- To learn the tools that help us to know the qualitative properties of solutions without solving them explicitly. (Direction fields, phase-line analysis, phase-plane analysis, etc.)
- Existence, uniqueness, comparison, and stability of solutions.
- To solve application problems, i.e., modeling.

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• General form of an ODE of order n with independent variable t and unknown function y:

$$F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0,$$

often simplified as

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- General form of a first-order ODE:

$$F(t, y, y') = 0 \ \text{ or } \ y' = F(t, y).$$

• General form of a first-order ODE system with independent variable t and unknown functions $\vec{y}(t) = (y_1(t), \ldots, y_m(t))^T$:

$$\begin{cases} y_1'(t) = F_1(t, y_1, \ldots, y_m) \\ y_2'(t) = F_2(t, y_1, \ldots, y_m) \\ \cdots \\ y_m'(t) = F_m(t, y_1, \ldots, y_m) \end{cases}$$

or, in short,

$$\vec{y}\,'(t) = \vec{F}(t, \vec{y}), \quad \text{where} \quad \vec{F} = (F_1, \ldots, F_m)^T.$$

Def 1.5. The ODE $F(t, y, y', \ldots, y^{(n)}) = 0$ is said to be linear 线性 if F is a linear function of $y, y', \ldots, y^{(n)}$. Otherwise, the ODE is nonlinear 非线性.

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In vector form: $\vec{a}(t) := (a_0(t), \ldots, a_n(t))^T$, $\vec{y}(t) := (y^{(n)}, y^{(n-1)}, \ldots, y)^T$,

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$$\begin{aligned}
&3t^2y'' + 2[\sin(t)]y' + e^t y = \ln t \\
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L, L, NL, NL.

Direction Fields. I

Def 1.6. Consider $\frac{dy}{dt} = f(t, y)$. The direction field 方向场/线素场 or slope field 斜率场 of this eq. is the picture in $ty-$plane that assigns each point (t, y) a short arrow or line segment with slope $f(t, y)$. Nullclines 零斜率线/零增长等值线 are the curves of zero inclination, i.e., $f(t, y) = 0$.

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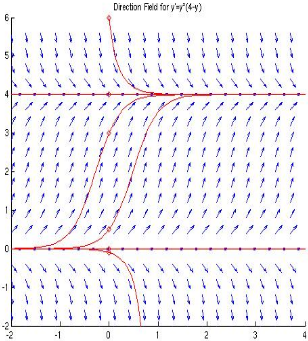
Example 1. Direction field of $y' = y(4 - y)$.



Direction Fields. II

Example 2.

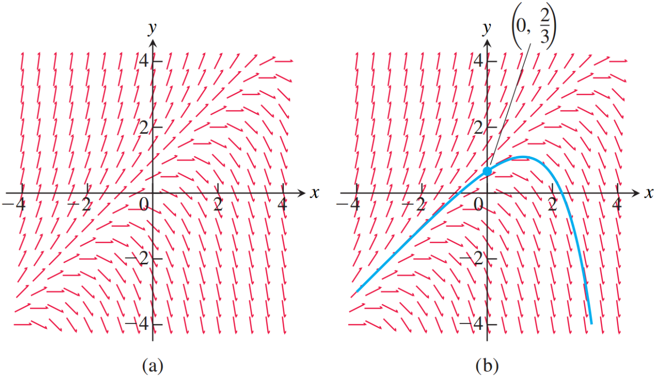


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

For $\frac{dy}{dt} = f(t, y)$:

The set of all the solutions is called the general solution.

The solution curves in $ty-$plane are called integral curves.

Modeling with ODEs

Strategy:

1. Identify the independent and dependent variables. Choose the units of measurement for each variable.

Dimensionally consistent: all the terms in the equation should have the same units.

2. Articulate the basic principle that underlies or governs the problem, and express the principle into an equation.

3. Solve the equation.

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M: mass of the object in unit kg

$v(t)$: velocity in unit m/s

$a(t) = \frac{dv}{dt}$: acceleration in unit m/s^2

$G = Mg$: gravity, $g = 9.8\, m/s^2$

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Step 2. Set up the ODE according to Newton's second law
$F = Ma$,
we come to

$$M \frac{dv}{dt} = Mg - \gamma v.$$

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Dimensionally consistent:

$$kg \cdot \frac{m}{s^2} = (\text{unit of } \gamma) \cdot \frac{m}{s} \Rightarrow (\text{unit of } \gamma) = kg/s$$

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$p(t)$: mouse population at t month

$r = 0.5$ per month: growth rate of mouse without owls, i.e., $\frac{dp}{dt} = rp$.

Assume that the owls kill 15 mice per day, and thus kill $15 \times 30 = 450$ mice per month.

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(rate of change of population = birth rate − death rate) □

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Assume the volume of the pond is $V = 10^6\,L$. Assume initially the pond is full of solution which contains this chemical y_0 (g).

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Step 2. Set up the ODE:

$$\frac{dy}{dt} = 0.01 \times 300 - \frac{y}{10^6} \times 300, \quad y(0) = y_0.$$

(rate of change = rate in − rate out) $\square$

Solving initial value problem of the type:

$$\begin{cases} \frac{dy}{dt} = ay - b, \\ y(0) = y_0, \end{cases} \tag{1}$$

where a, b are constants and y_0 is a given initial value.

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Sol: $1.\ a = 0.$

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Let $x(t) = y(t) - \frac{b}{a}$. Then $\frac{dx}{dt} = ax$. Thus, $x(t) = Ce^{at}$, where C is any constant.
The general solution is $y(t) = Ce^{at} + \frac{b}{a}$, where C is any constant.
The special solution satisfying the given initial value is the one with $C = y_0 - \frac{b}{a}$. $\square$