$\boxed{\text{Intros and Hooks.}}$

$w = f(z)$. $z, w \in \mathbb{C}$.

surprising results:

eg. ① if $\exists f' \Rightarrow \exists f^{(n)}, \forall n$!

② $\exists f'(z)$ $\forall z \in \mathbb{C}$, and $\exists C$, $|f| \le C$. $\Rightarrow$ $f =$ const !

Dirichlet Thm : $\{a+bn\}_{n\in\mathbb{N}}$. $a, b \in \mathbb{Z}$.

if $(a,b)=1 \Rightarrow$ the sequence contains ∞ primes.

the proof of the thm is by complex analysis.

$ax^3 + bx^2 + cx + d = 0$.

1° $a \Rightarrow a = 1$ $\quad x^3 + bx^2 + cx + d = 0$.

2° $x + \frac{b}{3} = y$. $\Rightarrow b = 0$.

3° reduced equation : $x^3 + px + q = 0$.

4° $x = \alpha + \beta$ $\quad \alpha^3 + \beta^3 + \underbrace{3\alpha\beta(\alpha+\beta) + p(\alpha+\beta)}_{(3\alpha\beta + p)(\alpha+\beta)} + q = 0$.

5° force $3\alpha\beta + p = 0$.

$$\Rightarrow \begin{cases} \alpha^3 + \beta^3 = -q \\ \alpha^3\beta^3 = -\frac{p^3}{27} \end{cases} \Rightarrow \alpha^3, \beta^3 \text{ roots of the equation:}$$

$$t^2 + qt - \frac{p^3}{27} = 0. \Rightarrow \alpha, \beta.$$

$\boxed{\text{Syllabus}}$

HW + Mid + Final.

HW ↓ about 5 pieces

Mid ↓ take home

Final ↓ in class.

textbooks:

① Ahlfors.

② Ellias ?

③ 沙巴特.

$\boxed{\text{Complex plane}}$

$\mathbb{C}_z \simeq \mathbb{R}^2_{(x,y)}$ $\boxed{z = x+iy}$

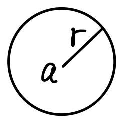
a field.

$\|(x,y)\| = \sqrt{x^2+y^2}$

$= |z|$.

1) Domains

$B_r(a)$: open disc



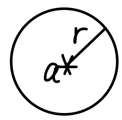
B_r : means $a=0$.

$\overline{B_r(a)}$: closed disc.

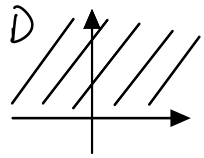
$B_r(a) = \{\, \underbrace{|z-a|}_{\text{dist}(z,a)} < r \,\}$

$\overline{B_r(a)} = \{\, |z-a| \leq r \,\}$.

$B^*_r(a) = B_r(a) \setminus \{a\}$.

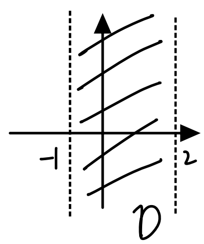


$= \{\, 0 < |z-a| < r \,\}$.

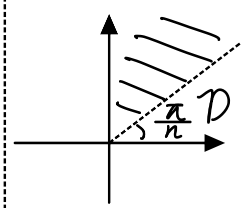


$D = \{\, \operatorname{Im} z > 0 \,\}$.

(denoted as Π^+).

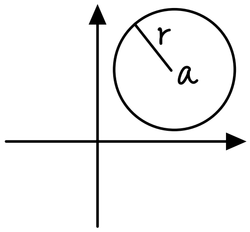


$D = \{\, -1 < \operatorname{Re} z < 2 \,\}$.

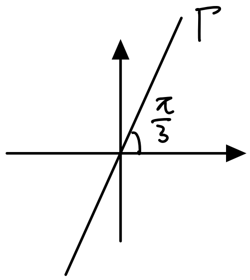


$D = \{\, \frac{\pi}{n} < \operatorname{Arg} z < \frac{\pi}{2} \,\}$.

2) Curves.



$C_r(a) = \{ |z-a| = r \}.$



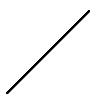
$\Gamma = \{\operatorname{Arg} z = \frac{\pi}{3}\} \cup \{\operatorname{Arg} z = \frac{4}{3}\pi\} \cup \{0\}$

$\quad \parallel$

$\{\operatorname{Im} z = \sqrt{3}\, \operatorname{Re} z\}.$

$\boxed{\text{Concepts arriving automatically from the analysis in } \mathbb{R}^2.}$

* domain = connected, open sets.
连通 开集.



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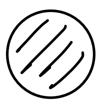

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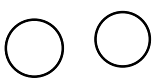

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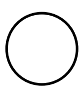
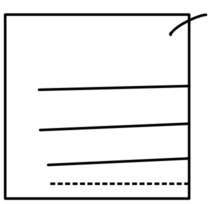


✓.



domain

∞ cuts.

✓.

* boundary: D: domain.

then $\partial D = \overline{D} \setminus D$

e.g. $C_r(a) = \partial B_r(a)$.

$D = \{ r < |z-a| < R \}$, $\Rightarrow$ $\partial D = C_r(a) \cup C_R(R)$.

annulus (圆环).

* convergence: $z_n \to z$ live in $\mathbb{R}^2$.

* complex-valued functions: $z = x + iy$

$w = u + iv$.

$$W = f(z) \longleftrightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

$\mathbb{C}$ $\qquad\qquad\qquad\qquad$ $\mathbb{R}^2$.

* Hence, continuity, differientiativity, smoothness (C^∞), etc arrive automatically from $\mathbb{R}^2$.

Linear functions

Two-fields: $\mathbb{R}$, $\mathbb{C}$.

* $\mathbb{R}$-linear function (map): $\mathbb{R}^2 \to \mathbb{R}^2$ ($\mathbb{C} \to \mathbb{C}$).
* $\mathbb{C}$-linear function: $\mathbb{C} \to \mathbb{C}$. $\quad W = az$. (to get a, e.g. let $z = 1$).

$\mathbb{R}$-linear: $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. $\mathbb{R}^2 \to \mathbb{R}^2$.

$z = x+iy$ $\quad x = \dfrac{z+\bar{z}}{2}$ $\qquad u = \frac{1}{2}(w+\bar{w})$

$\bar{z} = x-iy$. $\quad y = \dfrac{z-\bar{z}}{2i}$. $\qquad v = \frac{1}{2i}(w-\bar{w})$.

$w = u+iv = (\alpha x+\beta y) + i(\gamma x+\delta y)$.

$= z\cdot\left(\frac{1}{2}\alpha + \frac{1}{2i}\beta + \frac{i}{2}\gamma + \frac{1}{2}\delta\right) + \bar{z}\cdot\left(\frac{1}{2}\alpha - \frac{1}{2i}\beta + \frac{i}{2}\gamma - \frac{1}{2}\delta\right)$.

$= az+b\bar{z}$. $\neq$ az !

in general

So, an $\mathbb{R}$-linear func. is rarely $\mathbb{C}$-linear.

$\Updownarrow$

$b=0 \iff \frac{1}{2}\alpha - \frac{1}{2i}\beta + \frac{i}{2}\gamma - \frac{1}{2}\delta = 0$

$\iff \begin{cases} \alpha = \delta \\ \beta = -\gamma. \end{cases} \iff \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & -\gamma \\ \gamma & \alpha \end{pmatrix}}_{=: A}$.

$a = \alpha + i\gamma$.

In particular. $\det A = \alpha^2+\gamma^2 = \|a\|^2 \geq 0$.

and if $a \neq 0$, $\det A > 0$.

$w = az = \underbrace{r}_{|a|} \cdot e^{i\underbrace{\varphi}_{\operatorname{Arg} a}} \cdot |z|\, e^{i \operatorname{Arg} z} = r|z|\, e^{i(\varphi + \operatorname{Arg} z)}$.

$|z| \to r|z|$. $\quad$ homotopy $(z \neq 0)$

$\operatorname{Arg} z \to \operatorname{Arg} z + \varphi$. $\quad$ rotation.

So, $\mathbb{C}$-linear transformations are compositions of a homotopy and a rotation.

$$\begin{pmatrix} \alpha & -\gamma \\ \gamma & \alpha \end{pmatrix} = \sqrt{\alpha^2+\gamma^2} \begin{pmatrix} \frac{\alpha}{\sqrt{\ }} & \frac{-\gamma}{\sqrt{\ }} \\ \frac{\gamma}{\sqrt{\ }} & \frac{\alpha}{\sqrt{\ }} \end{pmatrix} = r \cdot \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

rotation operator

Def: the above linear maps

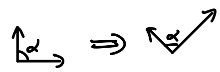
($= \mathbb{C}$-linear maps when $a \neq 0$)

are called conformal linear maps.

保角.

Fact: a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ is conformal

$\Leftrightarrow$ it preserves angles and orientation.

 α不变.

Def: Let $f(z)$ be a complex func. in a domain $D \subset \mathbb{C}$.

($f : D \to \mathbb{C}$).

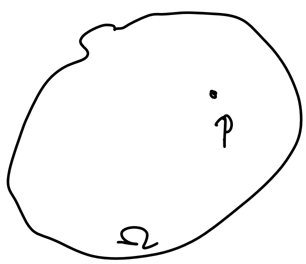
Assume that f is $\mathbb{R}$-differentiable at a point $p \in D$.

Then we say f is $\mathbb{C}$-diff at p, if $df|_p$ (= the differential of f at p), which is a map $\mathbb{R}^2 \to \mathbb{R}^2$, is actually a $\mathbb{C}$-linear map. $W = U + iV$. $f : \begin{cases} U = U(x,y) \\ V = V(x,y) \end{cases}$ $df|_p = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{pmatrix}$.

Furthermore, we say that f is holomorphic in D
全纯.
if f is $\mathbb{C}$-diff $\forall$ point $p \in D$.

(Notation: $f \in \mathcal{O}(D)$)

hol = diff. in the real sense + $df|_p$ is $\mathbb{C}$-linear.



$f: \Omega \to \mathbb{C}$.

Ω: domain in $\mathbb{C}$.

f is $\mathbb{C}$-diff at p, if

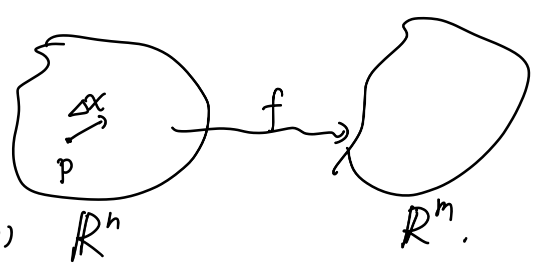
(i) f is $\mathbb{R}$-diff. (diff in real sense)

(ii) $df|_p$ is $\mathbb{C}$-linear.

f is holomorphic in Ω, if f is $\mathbb{C}$-diff $\forall p \in \Omega$.

Notation: $\boxed{f \in \mathcal{O}(\Omega)}$ (f is hol. in Ω)

Reminder: $f: \mathbb{R}^n \to \mathbb{R}^m$



$\Delta x \in \mathbb{R}^n$, $\Delta f \in \mathbb{R}^m$.

$\mathbb{R}$-diff at p: (just diff) $\Delta f = \underbrace{\overset{\in \mathrm{Mat}(m,n)}{A}}_{\text{linear map}} \Delta x + \underbrace{\bar{o}(\|\Delta x\|)}_{= \alpha(\Delta x) \cdot \|\Delta x\|}$

$\alpha(\Delta x) \to 0 \quad (\Delta x \to 0)$.

If holds, replace Δx by dx.

$$\boxed{df|_p := A\,dx}$$ df is a linear map in dx.

$$A = \left[\left(\frac{\partial f_i}{\partial x_j} \right) \Big|_p \right]_{i=1,\cdots,m}^{j=1,\cdots,n}$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n .$$

Go back to the complex content.

$$f : \Omega \longrightarrow \mathbb{C} \simeq \mathbb{R}^2 . \qquad \Omega \subset \mathbb{C} \cong \mathbb{R}^2$$

$f = u + iv$. $\mathbb{R}^2 \to \mathbb{R}^2$.

$z = x + iy$. $x, y \in \mathbb{R}$.

$w = f(z) = u(x,y) + i\,v(x,y)$.

<u>Goal</u>: understand $\mathbb{C}$-diff and hol.

$$df = du + i\,dv = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy .$$

$$\left[\begin{array}{ll} z = x+iy . & dz = dx + i\,dy \\ \bar{z} = x - iy . & d\bar{z} = dx - i\,dy . \end{array} \Rightarrow dx = \frac{1}{2}(dz + d\bar{z}),\ dy = \frac{1}{2i}(dz - d\bar{z}) \right].$$

$$\star \quad \boxed{\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right). \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)}$$

$$\Rightarrow \quad df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \checkmark$$

df is $\mathbb{C}$-diff func. of dz $\Leftrightarrow$ $\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$ $\rightarrow$ $\bar{\partial}$ - equation

holomorphic condition.

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i \cdot \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

$$\Rightarrow \quad \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \boxed{\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}}$$

$\rightarrow$ Cauchy - Riemann Equations.

df is $\mathbb{C}$-linear $\Leftrightarrow$ $df = \frac{\partial f}{\partial z} dz.$ $\Leftrightarrow$

$$\Delta f = \left.\frac{\partial f}{\partial z}\right|_p \cdot \Delta z + \bar{\bar{o}}(|\Delta z|). \Leftrightarrow \underset{\substack{\uparrow \\ \text{division}}}{(:\Delta z)} \quad \frac{\Delta f}{\Delta z} = \left.\frac{\partial f}{\partial z}\right|_p + \bar{\bar{o}}\Big(\underbrace{\frac{|\Delta z|}{\Delta z}}_{\text{has module } 1.}\Big).$$

$$\Leftrightarrow \exists \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \left.\frac{\partial f}{\partial z}\right|_p.$$

(inverse direc: if $\exists \lim = A \Rightarrow \frac{\Delta f}{\Delta z} = A + \bar{\bar{o}}(1) \Leftrightarrow \Delta f = A\Delta z + \bar{\bar{o}}(|\Delta z|)$)

Hence,

<u>Theorem:</u> TFAE:

(1) f is $\mathbb{R}$-diff at p and $df|_p$ is $\mathbb{C}$-linear. ($= f$ is $\mathbb{C}$-diff)

(2) f is $\mathbb{R}$-diff at p and $\left.\frac{\partial f}{\partial \bar{z}}\right|_p = 0$.

(3) —— $\|$ ———— the Cauchy-Riemann eqs hold at p.

(4) $\exists$ a complex derivative $f'(p) := \lim_{\Delta z \to 0} \left.\frac{\Delta f}{\Delta z}\right|_p = \left.\frac{\partial f}{\partial z}\right|_p$.

<u>Rmk:</u> An analog thm holds if replacing "$\mathbb{C}$-diff at p" by "hol in D" and the other and-s to hold $\forall p \in D$.

<u>Ex.</u> $f(z) = |z|^2 = x^2 + y^2$.

$u = x^2 + y^2$. $v = 0$.

$u_x = 2x$, $u_y = 2y$, $v_x = 0$, $v_y = 0$

So, f is $\mathbb{C}$-diff at $p \in \mathbb{C} \Leftrightarrow p = (x,y) = (0,0)$. $(p = 0)$.

No domains D where $f \in \mathcal{O}(D)$.

<u>Ex.</u> $f = z^2 = x^2 - y^2 + 2ixy$.

$u = x^2 - y^2$. $v = 2xy$.

$u_x = 2x$, $u_y = -2y$. $v_x = 2y$, $v_y = 2x$.

Cauchy–Riemann Eqs. hold $\forall\, p \in \mathbb{C}$! $\Rightarrow f \in \mathcal{O}(\mathbb{C})$.

Jacobian Mat. of a $\mathbb{C}$-diff map.

$$\begin{array}{ccccc} & \mathbb{C} & & \\ & \cup & & \\ f: & \Omega & \longrightarrow & \mathbb{C} \\ & \wr\| & & \wr\| \\ & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \end{array}$$

The real Jacobian mat of f is: $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

By applying Cauchy–Riemann eqs.

$$\Rightarrow \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} \triangleq \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

$$f'(p) = \left.\frac{\partial f}{\partial z}\right|_p = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

$$= \frac{1}{2}\left(u_x + i v_x - i u_y + v_y\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. = \alpha + i\beta.$$

$$J_f = \det\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha^2 + \beta^2 = |f'(p)|^2.$$

det. is a real number.

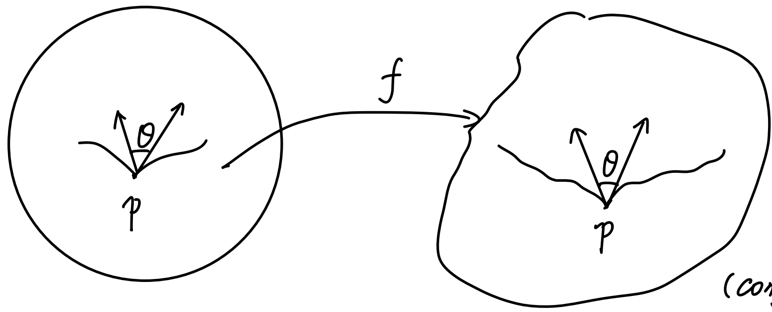
$\boxed{J_f = |f'(p)|^2}$ In partic., $J_f^{\mathbb{R}} \geq 0$ and $\boxed{J_f^{\mathbb{R}} = 0 \Leftrightarrow f'(p) = 0}$

det. of Jacobian mat of f.

Continuing to the discussion: if $\exists f'(p) \neq 0$, then df is a conformal map.
(in the last lecture)

Informally, this means that a $\mathbb{C}$-diff map with $f' \neq 0$ is "conformal in small".

Precisely,



(conform the angle but not conform the curves).

Theorem: Let $f, g : \Omega \to \mathbb{C}$.

Assume f, g are $\mathbb{C}$-diff at $p \in \Omega$.

(or $f, g \in \mathcal{O}(\Omega)$)

Then, (i) $\alpha f, \alpha g$ are $\mathbb{C}$-diff at p, and $(\alpha f)' = \alpha f'$.

(usp $\forall p \in \Omega$)

(ii) $f \pm g$ ——— || ———, and $(f \pm g)' = f' \pm g'$.

(iii) $f \cdot g$ ——— || ———, and $(fg)' = f'g + fg'$.

(iv) $\frac{f}{g}$ ——— || ———, and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$,

provided $g(p) \neq 0$.

(usp. $g \neq 0$ in Ω).

Proof: Since f is $\mathbb{C}$-diff $\Leftrightarrow \exists f'(p)$.

the proof is word-by-word like in *Mathematical Analysis*.

Cor 1: $\mathcal{O}(\Omega)$ form an algebra. (abelian).

Cor 2: Complex polys are all hol in $\mathbb{C}$

($p(z) = a_0 + a_1 z + \cdots + a_n z^n$), while complex rational func.s are hol.

in any Ω not containing their singularities, $f = \frac{P(z)}{Q(z)}$. $\Omega \cap \{Q(z)=0\} = \emptyset$.

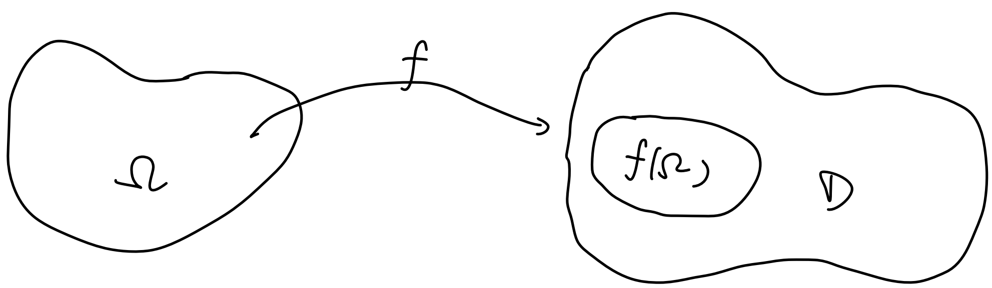
Proof: follows from the thm and the fact that

$$\begin{array}{cc} f(z) = \text{const.} & f(z) = z \in \mathcal{O}(\mathbb{C}) \\ | & | \\ df = 0. & df = dz. \end{array}$$

In both cases, f' is computed in the usual way.

Thm. (Composition)

Let $f \in \mathcal{O}(\Omega)$. $f(\Omega) \subset D$. $g \in \mathcal{O}(D)$.



Then the composition $h := g \circ f \in \mathcal{O}(\Omega)$

and $\boxed{h'(z) = g'(f(z))\, f'(z)}$.

Rmk: holds also pointwise.

Proof: $w = f(z)$. $p = f(q)$,

$$\Delta g = g'(p)\Delta w + \bar{\bar{o}}(\Delta w) = g'(p)\left(f'(q)\Delta z + \bar{\bar{o}}(\Delta z)\right) + \bar{\bar{o}}\left(f'(q)\Delta z + \bar{\bar{o}}(\Delta z)\right)$$

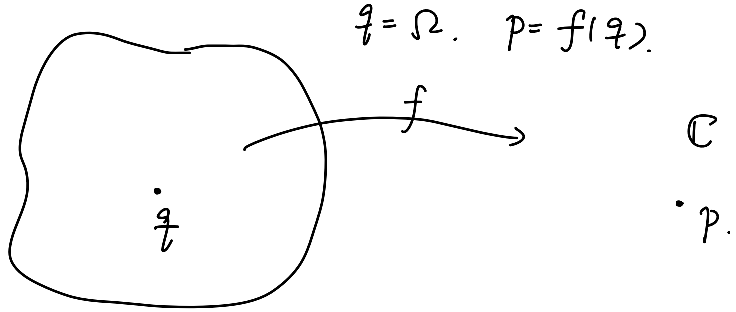
$$= g'(p) f'(q)\Delta z + \bar{\bar{o}}(\Delta z)$$

$\Rightarrow$ by def $\boxed{\exists\ h'(q) = g'(p) f'(q)}$ $\Rightarrow h \in \mathcal{O}(\Omega)$

and the desired chain rule holds. $\boxed{\Sigma}$.

<u>Theorem</u> (inverse func). Let $f: \Omega \to \mathbb{C}$, $f \in \mathcal{O}(\Omega)$.

$q = \Omega$, $p = f(q)$.



Assume that <u>$f'(q) \neq 0$</u>, then $\exists\ B_\varepsilon(p)^{C \mathbb{C}}$ and a func. $g \in \mathcal{O}(B_\varepsilon(p))$

with $g \circ f(z) = z$. $(g = f^{-1})$ and $g' = \frac{1}{f'}$.

<u>Proof</u>: By the above, $J_f^{R} = |f'(q)|^2 \neq 0$.

$\Rightarrow$ the inverse function g is $B_\varepsilon(p)$ exists.

(this is the real version of the thm).

let $w = f(z)$, $\Delta w = f'(q)\Delta z + \bar{\bar{o}}(\Delta z)$

$$\Rightarrow \quad \Delta z = \frac{\Delta w}{f'(q) + \bar{\bar{o}}(1)} = \frac{1}{f'(q)} \Delta w + \bar{\bar{o}}(\Delta w)$$

$$\Rightarrow \exists \lim_{\Delta w \to 0} \frac{\Delta z}{\Delta w} = \boxed{g'(p) = \frac{1}{f'(q)}}. \qquad \square.$$

<u>Def</u>. A map $f: \Omega \to \mathbb{C}$ is called conformal, if

(i) $f \in \mathcal{O}(\Omega)$

(ii) $f' \neq 0$ in Ω (i.e. $df|_p$ is a $\mathbb{C}$-linear conformal map).

(iii) f is injective ($\Leftrightarrow$ bijective onto image).

From the above Thm (inverse func).

we conclude that:

* a conf. map is a homeomorphism. (同胚映射)

(in partic. the image D is a domain too.

so, essentially, $f: \Omega \xleftrightarrow{\text{biject}} D$.)

conformal map between domain

Actually, f is a diffeomorphism. (微分同胚映射)

* Always $\exists f^{-1} = g: D \to \Omega$ which is also conf.

* By the composition thm, composition of conf. maps is conf. too.

* $\{f: \Omega \xrightarrow{\text{conf}} \Omega\} = \mathrm{Aut}(\Omega)$ — always a group, called the automorphism group of a domain.

Elementary functions and transformations of domains.

1) $f(z) = az + b$. $a \neq 0$. ($\mathbb{C}$- affine func.)

$f'(z) = a \neq 0$. f is conf. in $\mathbb{C}$.

$f(z) = g \circ h$

h: $z \mapsto az$. — linear conf. map.

g: $w \mapsto w + b$ — shift in $\mathbb{R}^2$ by the vec. b.

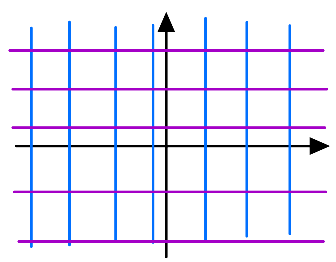
同伦.

composition of homotopy $k = |a|$ and rotation with $\theta = \operatorname{Arg} a$.

2) $f(z) = z^n$, $n \geq 2$, $n \in \mathbb{N}$.

$|z^n| = |z|^n$. $\operatorname{Arg}(z^n) = n \cdot \operatorname{Arg}(z)$.

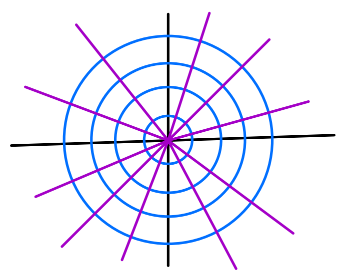
Cartesian net:



$\operatorname{Re} z = \text{const}$.

$\operatorname{Im} z = \text{const}$.

Polar net:



$|z| = \text{Const}$

$\operatorname{Arg} z = \text{const}$.

Under z^n: polar net $\longrightarrow$ polar net.

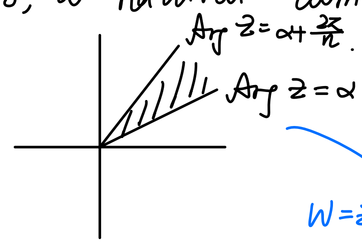
$z^n \in \mathcal{O}(\mathbb{C})$ as a poly.

$w' = (z^n)' = n z^{n-1}$, $w' = 0 \Leftrightarrow z = 0$.

<u>Goal:</u> determine "maximal" domains of conformality.

When $\underset{z_1^n}{\overset{\|}{w_1}} = \underset{z_2^n}{\overset{\|}{w_2}}$? Ans: $\begin{cases} |z_1| = |z_2| \\ \operatorname{Arg} z_1 = \operatorname{Arg} z_2 + \frac{2\pi}{n} \cdot k, \ k \in \mathbb{Z}. \end{cases}$

So, a natural domain: $\Omega_{\alpha,\beta} = \{ z \in \mathbb{C} : \alpha < \operatorname{Arg} z < \beta,\ \beta = \alpha + \frac{2\pi}{n} \}$.



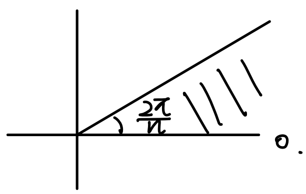
$\| \triangle$

$S_{\alpha,\beta}$. a max. dom. of conf.

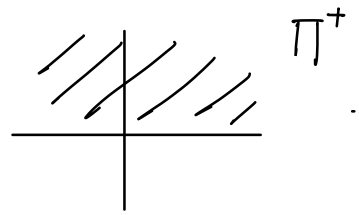
$w = z^n$

$w = \sqrt[n]{z}$.

For ex:

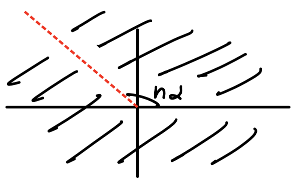


$n=2$:



$f(S_{\alpha,\beta}) = ?$ Ans: $S_{n\alpha, n\alpha} = \mathbb{C} \setminus \{ \operatorname{Arg} z = n\alpha \}$.

Image of $S_{\alpha,\beta}$:



<u>z^n maps conformally $S_{\alpha,\beta}$ to $S_{n\alpha,n\alpha}$</u>

In particular: $S_{0,\frac{2\pi}{n}} \longrightarrow \mathbb{C} \setminus [0, +\infty)$.

3). $w = \frac{1}{z^n}$. $n \geq 2$. somewhat similar.

$$\frac{1}{z^n} \in \mathcal{O}(\mathbb{C}\setminus\{0\})$$

$w' = -\frac{n}{z^{n+1}}$. Same max conf domains!

polar net $\longrightarrow$ polar net. $r \to \frac{1}{r^n}$. $\theta \to -n\theta$

$$S_{\alpha,\alpha+\frac{2\pi}{n}} \xrightarrow{\text{conf}} S_{-n\alpha,-n\alpha} = \mathbb{C}\setminus\{\operatorname{Arg} z = -n\alpha\}.$$

4) $w = \sqrt[n]{z}$, $n \geq 2$.

multiple value functions.

$z_1 = z_2 \iff w_1^n = w_2^n$. doesn't force $w_1 = w_2$.

How we deal with this func.?

Ans: this func is single-valued and hol (actually conformal) in any domain $D = \varphi(\Omega)$, where $\varphi(w) = w^n$, Ω is a domain of conformality of φ.

Then $f(z) := \varphi^{-1}(z)$

$\quad = \sqrt[n]{z}$ a single-valued branch of the func. $\sqrt[n]{z}$.

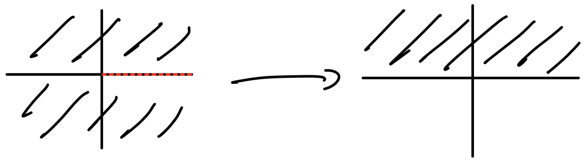
In partic, a branch of $\sqrt[n]{z}$ can be chosen in

$$S_{n\alpha,n\alpha} = \mathbb{C}\setminus\{\operatorname{Arg} z = n\alpha\}.$$

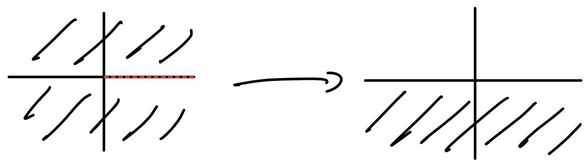
different branches of $\sqrt[n]{z}$ differ by multiplying by units of 1. ($\sqrt[n]{1}$)

For ex: $D = \mathbb{C} \setminus [0, +\infty)$.

$n=2$, $\alpha = 0$ or π.

$\sqrt{z}^{(1)}$:  Π^+

$0 < \operatorname{Arg} z < 2\pi$. $\quad\quad 0 < \operatorname{Arg} z < \pi$.

$\sqrt{z}^{(2)}$:  Π^-.

$2\pi < \operatorname{Arg} z < 4\pi \quad\quad \pi < \operatorname{Arg} z < 2\pi$.

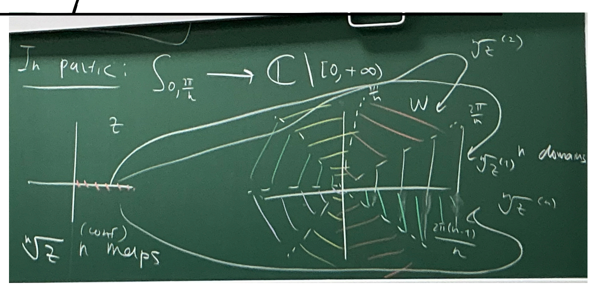
<u>Computationality</u>: $|\sqrt[n]{z}| = \sqrt[n]{|z|}$.

$$\operatorname{Arg} \sqrt[n]{z} = \frac{1}{n} \operatorname{Arg} z. = \begin{cases} \theta_1 + 2\pi k \\ \theta_2 + 2\pi k \\ \vdots \\ \theta_n + 2\pi k. \end{cases}$$

$$\theta_{j+1} - \theta_j = \frac{2\pi}{n}.$$

<u>So, branches of $\sqrt[n]{z}$ are 1-to-1 correspondence with branches of $\operatorname{Arg} z$.</u>

$|w| = \sqrt[n]{|z|}$ — real. , $\operatorname{Arg} w = \frac{1}{n} \operatorname{Arg} z$.



(partial) polar net $\to$ (partial) polar net.

5). $w = e^z$ — complex exponential.

Euler formula: $\boxed{\begin{array}{l} z = x + iy \Rightarrow \\ e^z = e^x(\cos y + i\sin y) \end{array}}$

Simple observations:

- $e^z \in \mathbb{R}$ if $z \in \mathbb{R}$. (and concide with the real exponent: $e^z = e^x$).
- $|e^z| = e^x = e^{Re\, z}$.

  $\operatorname{Arg} e^z = \operatorname{Im} z = y$.

In particular, $e^z \neq 0$. (complex expotential never vanishes).

$Re(e^z) = e^{Re\, z} \cos(\operatorname{Im} z)$

$\operatorname{Im}(e^z) = e^{Re\, z} \sin(\operatorname{Im} z)$.

* $e^z \in O(\mathbb{C})$

$u = e^x \cos y$. $v = e^x \sin y$. $u_x = e^x \cos y$. $u_y = -e^x \sin y$

$v_x = e^x \sin y$. $v_y = e^x \cos y$. $\Rightarrow$ C-R eqs. hold.

* Using $f'(z) = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$, we deduce $\boxed{(e^z)' = e^z}$.

* Direct calculation gives: $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$.

In partic. $(e^{-z})' = \frac{1}{e^z}$.

* Since $e^{2\pi i k} = 1$. $k \in \mathbb{Z}$.

$\Rightarrow \boxed{e^{z+2\pi ik} = e^{z}.}$ $\Rightarrow$ e^{z} is periodic in $\mathbb{C}$!

(with min $T = 2\pi i$).

* Variant small facts:

$e^{\pi i} = -1.$ $\boxed{z = |z| \cdot e^{i \operatorname{Arg} z}}$

* Values of $w = e^{z}$.

<u>Claim</u>: every $w \neq 0$ is a value.

<u>Proof</u>: $|w| = e^{\operatorname{Re} z}$. so choose $\operatorname{Re} z := \ln |w|$.

$\operatorname{Arg} w = \operatorname{Im} z$. so choose $z = \ln|w| + i \operatorname{Arg} w$.

By def. $e^{z} = w$. any value.

(we see that $f^{-1}(w) = \{z_0 + 2\pi k \cdot i\}_{k \in \mathbb{Z}}$).

What are the maximal domains of conformality of e^{z}?

$e^{z} \in \mathcal{O}(\mathbb{C})$, $(e^{z})' = e^{z} \neq 0$. $\forall z \in \mathbb{C}$.

So, only need to check injectivity.

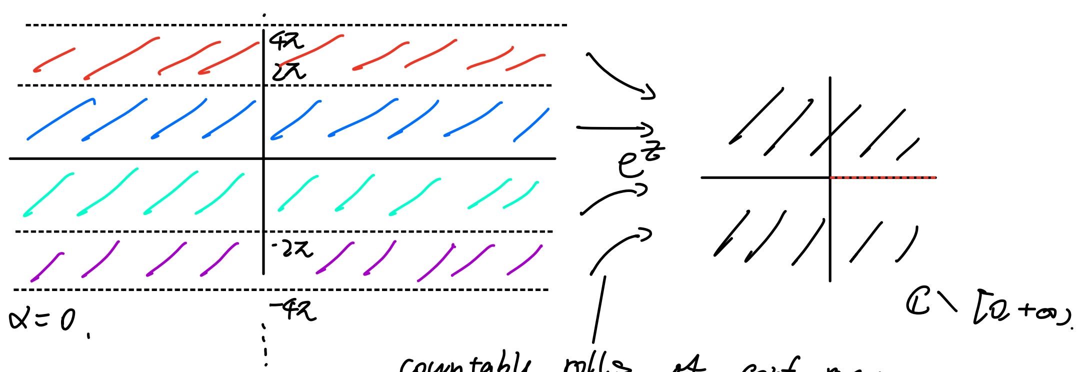
Study the equation: $e^{z_1} = e^{z_2} = w$. $\Leftrightarrow$ $\boxed{z_2 = z_1 + 2\pi k i.}$

we need domains where such a relation is forbiddened.

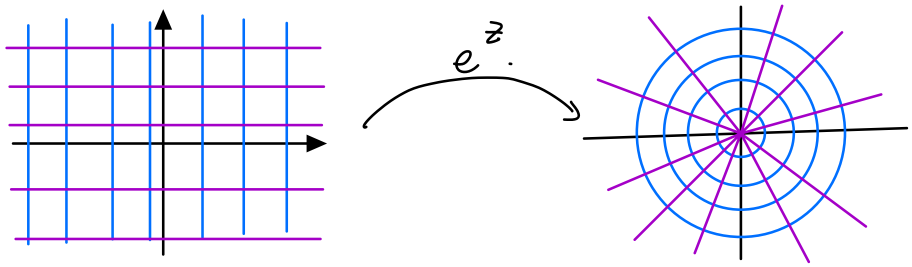
条带.

max domains: strips $\{\alpha < \operatorname{Im} z < \alpha + 2\pi i\}$.

For ex:



*. e^z maps the Cartesian net onto the polar net!



6). Complex Ln.

$$\boxed{e^{Ln z} = z}$$

$\Uparrow$

Def: $\boxed{Ln\, z := \ln|z| + i\,Arg(z).}$ $z \neq 0$.

formal inverse of e^z.

multiple (∞)-valued func. defined up to $2\pi k i$. $k \in \mathbb{Z}$.

$Re\, Ln\, z$ is single-valued though.

We are interested in single-valued branches (continuous / hol).

Recall: $\Omega_\alpha = \{\alpha < Im\, w < \alpha + 2\pi\}$. $\xrightarrow[\text{conf.!}]{e^z}$ $\mathbb{C} \setminus \{Arg\, z = \alpha\}$.

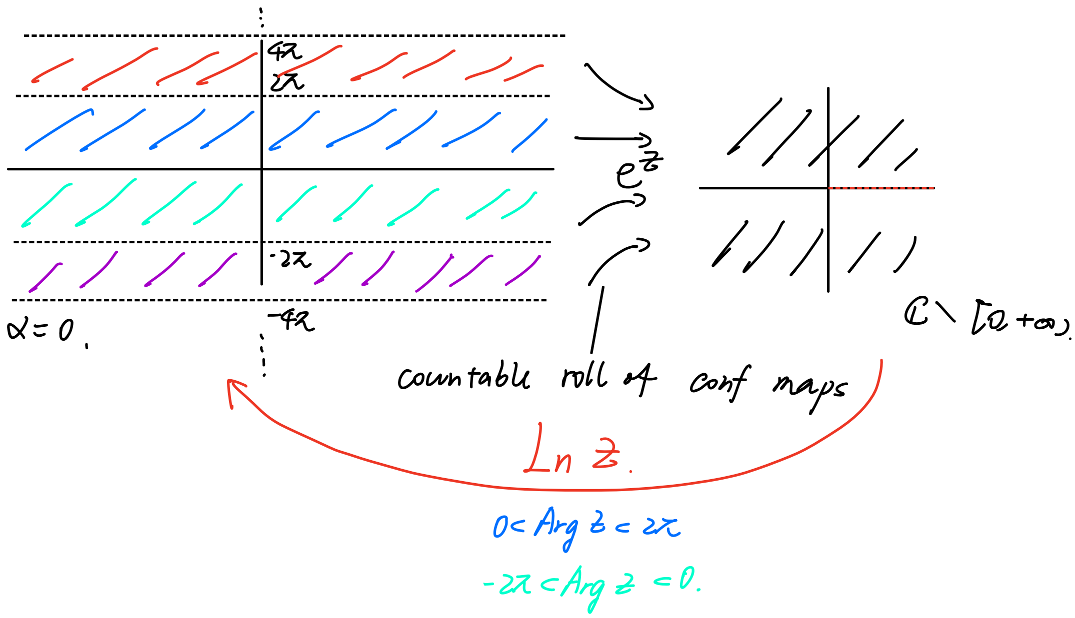
$\Rightarrow$ in $\mathbb{C} \setminus \{Arg\, z = \alpha\}$ we find an inverse map.

it corresponds to a hol. branch of Ln.

But actually all $\{\Omega_\alpha + 2\pi k i\}_{k\in\mathbb{Z}}$ get mapped by e^w to the same domain.

$\Rightarrow$ in this way we get countably many branches of Ln.

For ex:



Same as with a power function, choosing a (continuous / hol) branch of $Ln\, z$ corresponds to choosing a continuous branch of $Arg\, z$.

$\forall$ branch, $\boxed{(Ln\, z)' = \frac{1}{z} = \frac{1}{e^{Ln\, z}}}$

7) Complex power func: $w = z^a$. $a \in \mathbb{C}$

$\boxed{z^a := e^{a\, Ln\, z}}$ — multiple-valued unless $a \in \mathbb{Z}$.

∞ — valued unless $a \in \mathbb{Q}$.

$$i^i = e^{i \operatorname{Ln} i} = e^{i(0+i(\frac{\pi}{2}+2\pi k))} = \{e^{-(\frac{\pi}{2}+2\pi k)}\}_{k\in\mathbb{Z}} \in \mathbb{R}.$$

z^a has single-valued branches in same domains as $\operatorname{Ln} z$.

Extended complex plane.

Def: $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ — extended complex plane.

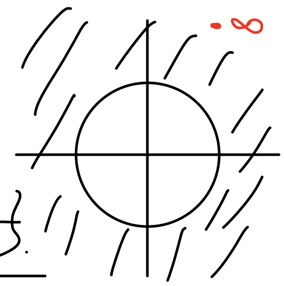
Introduced in order to obtain a compactification of $\mathbb{C}$

(1-point compactification).

Topology in $\overline{\mathbb{C}}$: Base of topology is

it means the building blocks of the whole $\overline{\mathbb{C}}$.

$$\{\text{discs in } \mathbb{C}\} \cup \underbrace{\{\{|z|>R\} \cup \{\infty\}\}}_{\text{basic neighbourhood of } \infty}.$$



Easy to check then: open set in $\overline{\mathbb{C}}$ is always of the kind (if contains ∞)

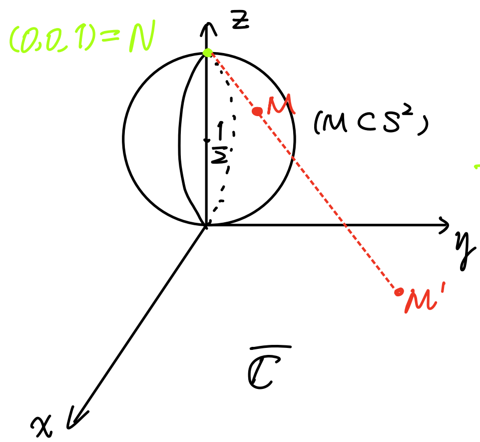
$$\underset{\substack{\text{open set}\\ \text{in } \mathbb{C}}}{U} \cup \underset{\text{nbd. of } \infty}{V}$$

Main fact: the new topology space $\overline{\mathbb{C}}$ is compact.

every infinite sequence in $\overline{\mathbb{C}}$ has a subsequence with a limit.

Stereographic projection

— A special 1-to-1 map from $S^2 \subset \mathbb{R}^3$ onto $\overline{\mathbb{C}}$.



$S^2 = \partial B_{(0,0,\frac{1}{2})}(\frac{1}{2})$

This S^2 is often called the Riemann Sphere.

$M' = \varphi(M)$

stereographic proj. map

pnt $\varphi(N) := \infty$.

* φ is a bijection $S^2 \to \overline{\mathbb{C}}$.
* φ is a homeomorphism from S^2 onto $\overline{\mathbb{C}}$.

同胚映射. (with the above topology)

$M = (x, y, z)$. $M' = \left(\frac{x}{1-z}, \frac{y}{1-y}\right)$

Continuity at $M = \infty$: φ maps "spherical hat" onto nbd. of ∞

φ^{-1}: if $M' = (u, v) \in \mathbb{R}^2$.

then $\varphi^{-1}(M') = \left(\frac{u}{1+u^2+v^2}, \frac{v}{1+u^2+v^2}, \frac{u^2+v^2}{1+u^2+v^2}\right)$

Amusing fact: "circle property".

φ maps circles in S^2 onto

generalized circles = $\begin{cases} \text{circle in } \mathbb{C}. \\ \text{line in } \mathbb{C} \cup \{\infty\} \end{cases}$

(closed line)

Fact: $\overline{\mathbb{C}}$ is a complex manifold.

Holomorphicity involving ∞.

always formally assume $\frac{1}{\infty}=0$ and $\frac{1}{0}=\infty$.

employee the map $\varphi(z)=\frac{1}{z}$. $0 \overset{\varphi}{\longleftrightarrow} \infty$.

homeomophic of $\overline{\mathbb{C}}$ onto itself.

Now consider $\Omega \subset \overline{\mathbb{C}}$.

dom.

Let $z_0 \in \Omega$. $w_0 = f(z_0)$.

We want to define $\mathbb{C}$-diff / holomophic if some of $\{w_0, z_0\}$ is ∞.

<u>Def</u>: f is $\mathbb{C}$-diff at z_0

if it is $\mathbb{C}$-diff in the usual sense "modulo φ".

<u>Meaning of "modulo φ":</u>

$$\begin{cases} z_0=\infty,\ w_0\neq\infty \Rightarrow f(\varphi(z)) \text{ is } \mathbb{C}\text{-diff at } 0. \\ z_0\neq\infty,\ w_0=\infty \Rightarrow \varphi(f(z)) \text{ is } \mathbb{C}\text{-diff at } z_0. \\ z_0=w_0=\infty \Rightarrow \varphi(f(\varphi(z))) \text{ is } \mathbb{C}\text{-diff at } 0 \end{cases} \quad \begin{array}{|l} f\circ\varphi = f(\frac{1}{z}) \\ \varphi\circ f = \dfrac{1}{f(z)} \\ \varphi\circ f\circ\varphi = \dfrac{1}{f(\frac{1}{z})}. \end{array}$$

<u>Ex</u>. $f(z)=z^m$, $m\in\mathbb{N}$. is $\mathbb{C}$-diff at ∞.

$f(\infty)=\infty$. $\varphi\circ f\circ\varphi = z^m$

$\mathbb{C}$-diff at 0.

<u>Ex</u>. $f(z)=\frac{1}{z^m}$, $m\in\mathbb{N}$ is $\mathbb{C}$-diff at both $\{0,\infty\}$.

$f=$ const same.

Now. for $\Omega\in\overline{\mathbb{C}}$, f valued in $\overline{\mathbb{C}}$.

$f \in \mathcal{O}(\Omega) \Leftrightarrow f$ is $\mathbb{C}$-diff $\forall z_0 \in \Omega$.

f is conformal in $\overline{\mathbb{C}}$: $\begin{cases} f \text{ is bijective} \\ f \in \mathcal{O}(\Omega) \\ f \text{ is conformal } \forall \text{ p.t "modulo } \varphi \text{"} \end{cases}$

All $z^m, m \in \mathbb{Z}$ are hol. in $\overline{\mathbb{C}}$.
(as $\mathbb{C}$-valued func.s).

$m = \pm 1$. actually conformal.

e^z: not hol in $\overline{\mathbb{C}}$ ($\nexists\, e^\infty$ making it hol at ∞).

$\underbrace{\text{Elementary func.s (continued)}}$

8) $\ast(z) = \frac{1}{2}(z + \frac{1}{z})$ —— Zhukovsky func.

$\ast \in \mathcal{O}(\overline{\mathbb{C}})$ as a $\overline{\mathbb{C}}$-valued func.

$\ast(0) = \infty$. $\ast(\infty) = 0$.

$\ast \in \mathcal{O}(\mathbb{C} \setminus \{0\})$ as a $\mathbb{C}$-valued func.

Consider $\ast$ so far just on $\mathbb{C} \setminus \{0\}$.

Q: where is it conf. ?

$\ast'(z) = (1 - \frac{1}{z^2})\frac{1}{2}$; $\ast'(z) = 0 \Leftarrow z = \pm 1$.

Equil.: $\ast(z) = \ast(w) \Leftrightarrow z + \frac{1}{z} = w + \frac{1}{w}$. $\Leftrightarrow z - w = \frac{z-w}{zw}$

$\Leftrightarrow \begin{cases} z = w. \\ \underbrace{zw = 1}. \rightarrow \text{the observation for inj.} \end{cases}$

$*$ is bij. in $\Omega \iff \nexists\, z, w \in \Omega$, $z \neq w$. s.t $zw = 1$.

Option 1: $\Omega = \{ |z| > 1 \}$.

Option 2: $\Omega = \{ 0 < |z| < 1 \}$.

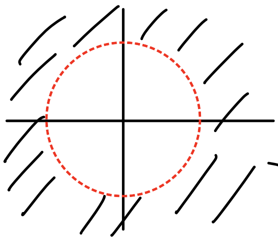
Option 3: $\Omega = \Pi^+$. ~ ($\mathrm{Arg}\, z$, $\mathrm{Arg}\, w \in (0, \pi)$

Option 4: $\Omega = \Pi^-$. $\mathrm{Arg}\, zw \in (0, 2\pi)$, but $\mathrm{Arg}\, 1 = 0$ or 2π).

$1 \overset{\varphi}{\longleftrightarrow} 2$, $3 \overset{\varphi}{\longleftrightarrow} 4$. $* \circ \varphi \sim *$

$\frac{1}{2}(z + \frac{1}{z})$ $\frac{1}{2}(z + \frac{1}{z})$

Option 1:



~ Ω.

$*(\Omega) = ?$

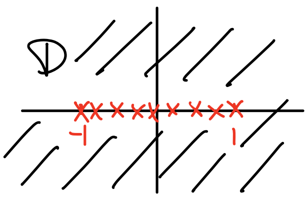
$*|_\Omega$ - conf.

$*(\partial\Omega) = \{ *(e^{it}) \} = \{ \frac{1}{2}(e^{it} + e^{-it}) \} = \{ \cos t \}_{t \in [0, 2\pi]}$

$= [-1, 1]$.

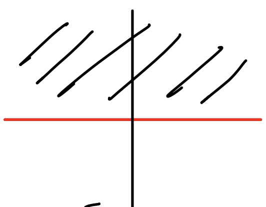
$*(\Omega) = \mathbb{C} \setminus [-1, 1]$.

$*$ is a homeomorphism of Ω (conf), so $\partial(*(\Omega)) = [-1, 1]$.



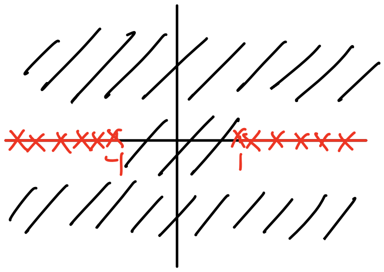
$\Omega \overset{*}{\longleftrightarrow} \mathbb{D}$

Option 3: Ω:



$*(\partial\Omega) = *(R) = \{ \frac{1}{2}(e^t + e^{-t}) \}_{t \in \mathbb{R}} = \{ \mathrm{ch}\, t \}_{t \in \mathbb{R}}$

$= (-\infty, -1] \cup [1, +\infty)$.

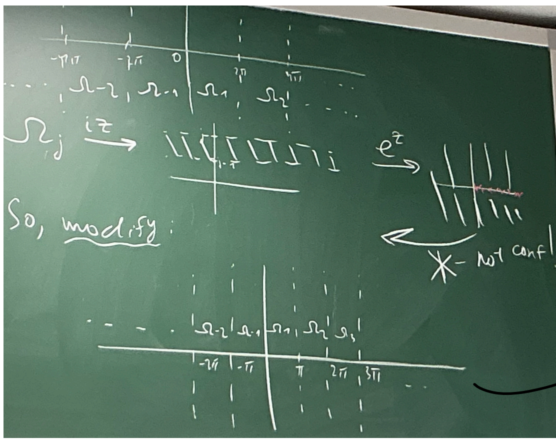


8) $f(z) = \cos z := \frac{1}{2}(e^{iz} + e^{-iz})$ $\quad \left(\begin{array}{l} e^{iz} = \cos z + i \sin z \\ e^{-iz} = \cos z - i \sin z \end{array} \right)$

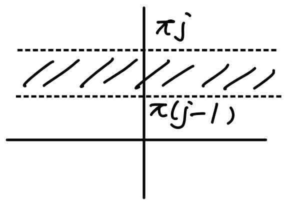
$\ast$ $\cos z$ is again 2π-periodic.

$\ast$ $\cos z = \ast \circ e^{iz}$.

Now, find domains of conformality.



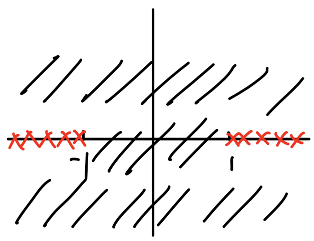
$\Omega_j \xrightarrow{iz}$



$\xrightarrow{e^z}$

Π^+ (j odd)

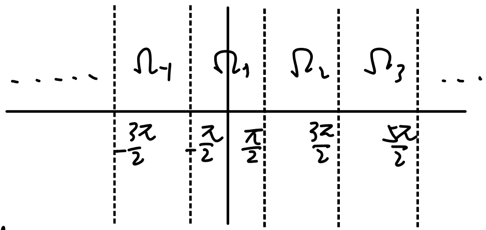
Π^- (j even) $\xrightarrow{\ast}$



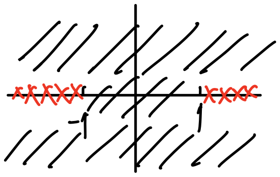
9). $\sin z := \frac{1}{2i}(e^{iz} - e^{-iz})$. $\quad \left(\begin{array}{l} e^{iz} = \cos z + i \sin z \\ e^{-iz} = \cos z - i \sin z \end{array} \right)$

Its properties follows from $\sin z = \cos(z - \frac{\pi}{2})$.

We conclude :



$\sin z$: $\Omega_j \xrightarrow{\text{conf}}$



Linear-fractional maps

Def: A linear-fractional map is a map $f(z): \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$f(z) = \frac{az+b}{cz+d}, \quad \text{with } \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0, \quad a, b, c, d \in \mathbb{C}.$$

$$f\left(-\frac{d}{c}\right) = \infty, \quad f(\infty) = \frac{a}{c}, \quad c \neq 0.$$

$$f(\infty) = \infty, \quad c = 0.$$

Properties:

* if $c = 0$, then f is an affine map.
* if $c \neq 0$, then we do division of poly.s and get:

$$f(z) = \frac{A}{z+B} + C, \quad A \neq 0.$$

So, $\boxed{f = \underbrace{\lambda}_{\text{affine map}} \circ \underbrace{\varphi}_{z \mapsto \frac{1}{z}} \circ \underbrace{\mu}_{\text{shift}}}$

$\lambda: z \mapsto Az + C$

$\mu: z \mapsto z + B$

So, $\boxed{f \text{ is conf in } \overline{\mathbb{C}}}$, as a map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

$\overline{\mathbb{C}} \xrightarrow[\text{conf.}]{f} \overline{\mathbb{C}}$ —— true since it's true for all the 3 maps λ, φ and μ.

* $\forall$ linear-frac. map is a conf. auto. of $\overline{\mathbb{C}}$.

* linear-frac. maps form a group:

$$f = \frac{az+b}{cz+d}, \quad g = \frac{\alpha z + \beta}{\gamma z + \delta}$$

$$f \circ g = \frac{a\frac{\alpha z+\beta}{\gamma z+\delta} + b}{c\frac{\alpha z+\beta}{\gamma z+\delta} + d} = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}$$

a new lin-frac map corresponds to the mat. prod.:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Rightarrow$ we have group homo:

$$\underbrace{GL(2, \mathbb{C})}_{\text{invert. mat. } 2\times 2} \xrightarrow[\text{onto}]{\text{homo}} \underbrace{Aut(\overline{\mathbb{C}})}_{\text{lin. frac.}}$$

In partic. $\exists$ f^{-1} — the inverse lin-frac. map

correspond to the inverse mat.

lin-frac maps $\leftrightsquigarrow$ 2×2 invert. complex mat.

considered up to $A \to k \cdot A$, $k \neq 0$, $k \in \mathbb{C}$

* "Three pt. prop.":

$\forall$ 3 distinct pts. $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ and distinct $w_1, w_2, w_3 \in \overline{\mathbb{C}}$.

$\exists!$ lin-frac. map f: $f(z_j) = w_j$, $j = 1, 2, 3$.

Pf: first, consider the special case $(w_1, w_2, w_3) = (0, 1, \infty)$

$$\boxed{\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}}$$ works if all $z_j \neq \infty$.

If some $z_j = \infty$, first apply $g = \frac{1}{z - p}$. $p \neq$ any z_j.

$\Rightarrow$ the new triple $\not\ni \infty$.

Uniqueness: follows from the same argument.

General case:

$$\begin{tikzcd}
(z_1, z_2, z_3) \arrow[dr, "f"] & \\
& (0, 1, \infty) \\
(w_1, w_2, w_3) \arrow[ur, "g"'] &
\end{tikzcd}$$

the desired map: $h = g^{-1} \circ f$.

Uniqueness also follows.

* Circle prop.

lin.-frac maps transform generalized circles into generalized circles.

Pf: $\forall$ lin.-frac map is built of:

$\Big\{$ aff-maps: $z \mapsto az + b$ — built of $\begin{cases} \text{rotation} \\ \text{shift} \\ \text{homopety.} \end{cases}$

| inversion.

all of those map $\begin{cases} \text{lines} \to \text{lines} \\ \text{circles} \to \text{circles.} \end{cases}$

So it's sufficient to consider the special case: $f(z) = \frac{1}{z}$.

Case 1: (closed) line. $\gamma = \{\alpha x + \beta y + \delta = 0\}$.

$x = \frac{z+\bar{z}}{2}$. $y = \frac{z-\bar{z}}{2i}$. $\Rightarrow$ $\gamma = \{az + \bar{a}\bar{z} + \delta = 0\}$.

$z \mapsto \frac{1}{z}$. $\Rightarrow$ $\frac{a}{z} + \frac{\bar{a}}{\bar{z}} + \delta = 0$.

$$\delta z\bar{z} + a\bar{z} + \bar{a}z = 0.$$

$\delta = 0 \Rightarrow$ line.

$\delta \neq 0 \Rightarrow$ $(z + \frac{a}{\delta})(\bar{z} + \frac{\bar{a}}{\delta}) = \underbrace{|z + \frac{a}{\delta}|^2 = \text{const}.}_{\text{circle.}}$

Case 2: circle: $\gamma = \{|z-a|^2 = R^2\}$.

$z \mapsto \frac{1}{z}$ $\Rightarrow$ $|\frac{1}{z} - a|^2 = R^2$.

$$|1-az|^2 = R^2|z|^2.$$

$$(1-az)(1-\bar{a}\bar{z}) = 1 + |a|^2|z|^2 - az - \bar{a}\bar{z}.$$

$$|z|^2 (R^2 - |a|^2) = 1 - az - \bar{a}\bar{z}$$

$$\Rightarrow z\bar{z} + bz + \bar{b}\bar{z} = C \in \mathbb{R}$$

$$\Rightarrow |z+b|^2 = \tilde{C} \in \mathbb{R}.$$ circle.

Def: Let γ a generalized circle.

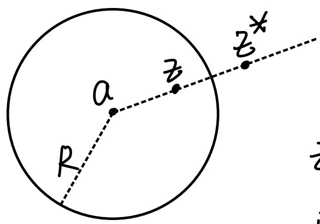
Case 1: γ — line.

choose $\forall z \in \mathbb{C}$, then the mirror reflected pt z^* is called **dual** to z w.r.t. γ.

(we put $\infty^* = \infty$)

Case 2: γ — circle.

Reflection across a circle $|z|=r$ is the inversion $z \mapsto \frac{r^2}{\bar{z}}$.



z^*: the dual of z w.r.t. γ.

is a pt. with 2 props:

(i) z^* lies on the ray $[a, z]$.

(ii) $|z-a|\,|z^*-a| = R^2$.

we put: $a^* = \infty$.

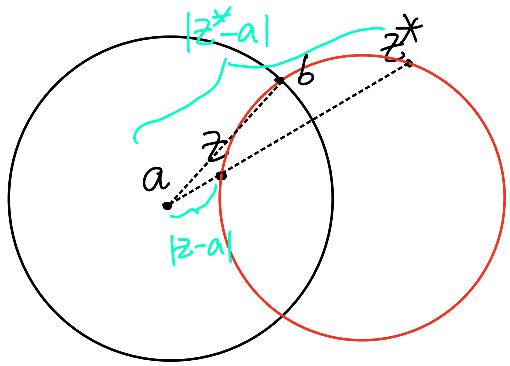
$\infty^* = a$.

(sometimes dual pt. is called symmetric pt or reflected pt.)

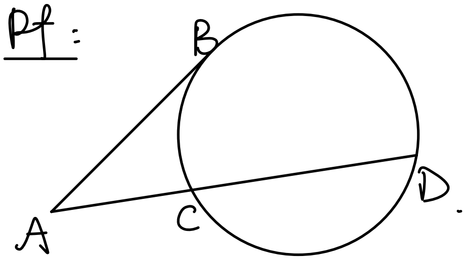
easy fact: $z = z^* \Leftrightarrow z \in \gamma$.

Lemma: Let γ be a generalized circle.

$z \in \gamma$, then $w = z^*$ $\Leftrightarrow$ $\forall$ generalized circle passing through z, w is orthogonal to γ.



Pf:



$AB^2 = AC \cdot AD$

Compare to the definition of dual: $R^2 = |z-a||z^*-a|$.

$AD \sim |z^*-a|$, $AC \sim |z-a|$.

$\Rightarrow$ $[a, b]$ is a segment tangent to the 2rd circle!

$\Rightarrow$ the circles are ortho.

Works in both directions.

If γ is a line. — even easier. . . .

$*$ lin-frac. maps send dual to dual.

if $w = f(z) \Rightarrow w^* = f(z^*)$ w.r.t $\forall$ gener. circle γ.

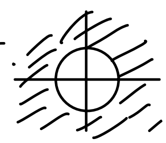
Pf: we know f sends gener. circ $\to$ gener. circ.

we know f is conf. $\Rightarrow$ f preserves angles.

$\Rightarrow$ ortho. curves are sent to ortho. curves.

Now the lemma implies the proof. $\square$.

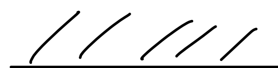
Def: A generalized disc is a domain $D \subset \overline{\mathbb{C}}$ s.t. ∂D is a generalized circle.

In fact (i) disc. (ii) extension of disc $\cup \{\infty\}$. 

(iii) half plane.

$*$ $\forall$ two generalized discs $D, \tilde{D}$, $\exists$ a lin-frac map sending them into each other.

Pf: Choose distinct $z_1, z_2, z_3 \in \partial D$.

First consider $\tilde{D} = \Pi^+$ 

Choose $f: (z_1, z_2, z_3) \longrightarrow (0, 1, \infty)$.

$\Rightarrow f(\partial D)$: generalized circle $\ni (0, 1, \infty)$

$\Rightarrow f(\partial D) = \mathbb{R} \cup \{\infty\} = \bar{\mathbb{R}}.$

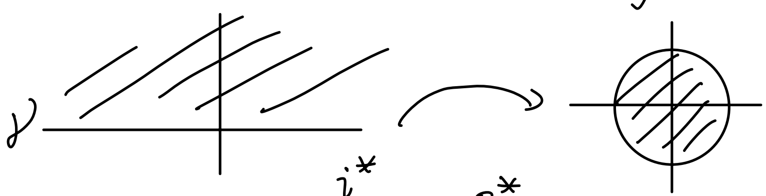
$\Rightarrow f(D) = \begin{cases} \Pi^+ - \text{ done.} \\ \Pi^- - \text{ taken } g = \varphi \circ f \end{cases}$

inversion.

$$\begin{array}{ccc} \mathbb{D} & \searrow^{f} & \\ & & \Pi^+ \\ \tilde{\mathbb{D}} & \nearrow_{g} & \end{array} \Rightarrow h = g^{-1} \circ f.$$

$\Box$.

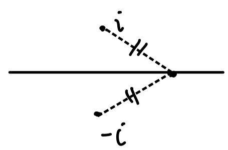
Ex. build a lin-frac map. transforming Π^+ onto $B_1(0)$.



Let $i \xrightarrow{f} 0$, $\underset{-i}{\overset{i^*}{\shortparallel}} \xrightarrow{f} \underset{\infty}{\overset{0^*}{\shortparallel}}$.

Try the map: $f(z) = \dfrac{z-i}{z+i}$

Find $f(\mathbb{R})$: $\left|\dfrac{z-i}{z+i}\right| = 1$ ($z \in \mathbb{R}$) $\Rightarrow f(\bar{\mathbb{R}}) = \partial B_1(0)$.



$\Rightarrow f(\Pi^+) = \begin{cases} B_1(0) \\ \bar{\mathbb{C}} \setminus B_1(0). \end{cases}$

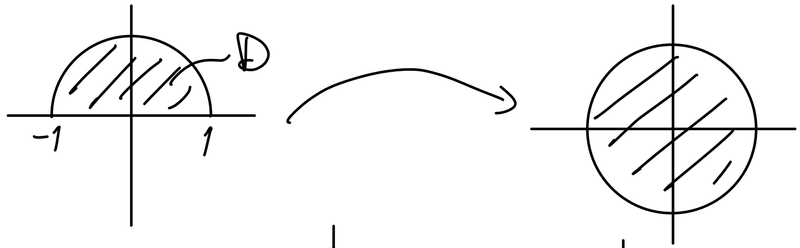
$f(\Pi^+) \ni 0. \Rightarrow f(\Pi^+) = B_1(0).$

Find f^{-1}: $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} = \dfrac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$

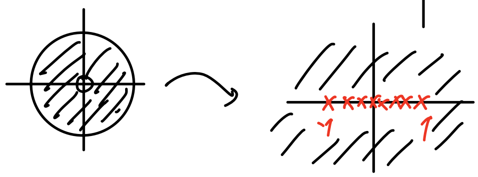
$\Rightarrow f^{-1}(w) = \dfrac{w+1}{iw-i} = -i\,\dfrac{w+1}{w-1}.$ $\quad f^{-1}(B_1(0)) \rightarrow \Pi^+.$

Construct conf. maps between domains

Ex. find a conf map transforming upper unit half-disc onto the unit disc.



Reminder: Ж:



$$\text{Ж}\big((-1,0)\cup(0,1)\big) = (-\infty,-1)\cup(1,\infty).$$

$$\text{Ж}(\partial B_1(0)) = [-1,1] \Rightarrow \text{Ж}(\partial D) = \bar{\mathbb{R}}.$$

$$\Rightarrow \text{Ж}(D) = \Pi^-. \quad \left(\text{Ж}\left(\tfrac{i}{2}\right) = -\tfrac{3}{2}i\right).$$

$$\Rightarrow \text{use } \underset{\substack{\parallel \\ g(z)}}{-\text{Ж}(z)} : D \to \Pi^+.$$

Then take $h(z) = \dfrac{z-i}{z+i}$, $\Pi^+ \to B_1(0)$.

$$f = h \circ g.$$

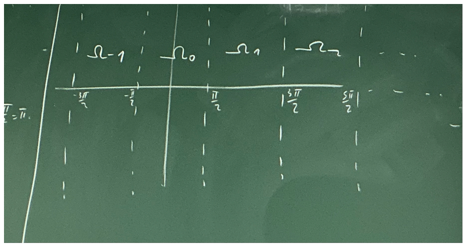
Ex. find (max) domains of conformality for the func. $f(z) = \tan z$.

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \cdot \frac{1}{i} = -i\,\frac{e^{2iz} - 1}{e^{2iz} + 1} = L \circ e^{2iz}.$$

$$L(w) = -i\,\frac{w-1}{w+1}.$$

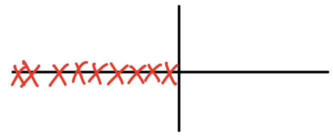
we need vertical strips with width $\frac{2\pi}{2} = \pi$.

$\tan z$ is periodic.

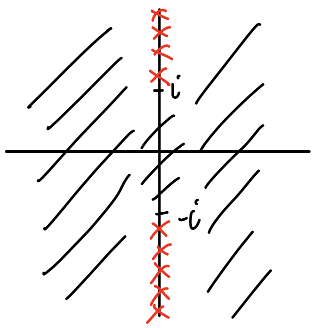


find $f(\Omega_0)$:

$\Omega_0 \xrightarrow{e^{2iz}} [\exp \text{ in } -\pi < \mathrm{Re}\, z < \pi] =$

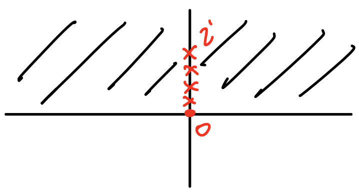


apply L:



Conformal maps (continued)

Ex.

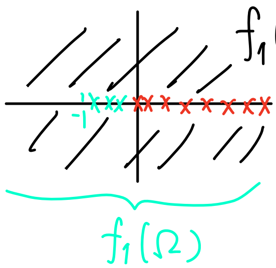


$\Omega = \Pi^+ \setminus [0, i]$.

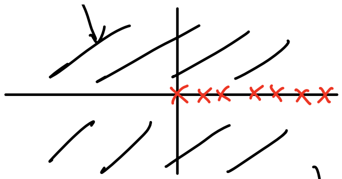
Find $f: \Omega \xrightarrow{conf} \Pi^+$.

Step 1: apply $f_1(z) = z^2$.

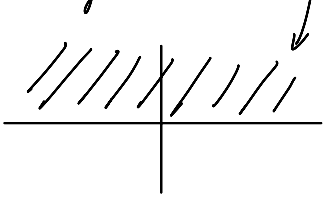
$f_1(\Pi^+) = \mathbb{C} \setminus [0, +\infty)$.



Step 2: $f_2(z) = z+1$.



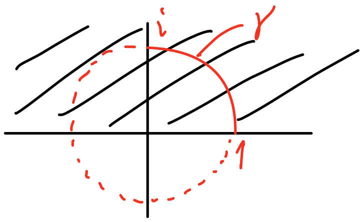
Step 3: $f_3(z) = \sqrt{z}$. with $\operatorname{Arg} z \in (0, 2\pi)$



Finally, $f = f_3 \circ f_2 \circ f_1$

$$f(z) = \sqrt{z^2+1}. \quad \text{branch with } \operatorname{Arg} z \in (0, 2\pi).$$

Ex.



γ: $\frac{1}{4}$ of the unit circle.

$$\Omega = \Pi^+ \setminus \gamma.$$

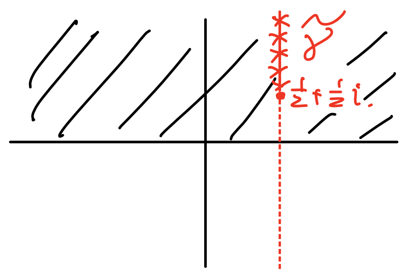
Find $f: \Omega \xrightarrow{\text{conf.}} \Pi^+$.

Since γ is an arc of a generalized circle, we may try using linear-fractional maps.

Step 1: $1 \to \infty$, take $f_1(z) = \dfrac{1}{1-z}$.
keep Π^+.

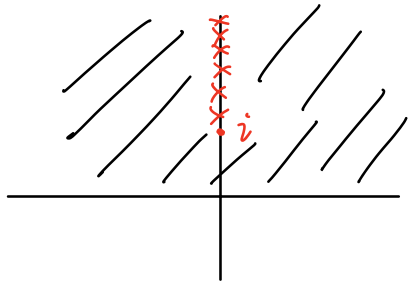
$f_1(1) = \infty$. $f_1(-1) = \frac{1}{2}$. $f_1(i) = \frac{1}{2} + \frac{1}{2} i$.

So, the image of $\partial B_1(0)$ is a line.

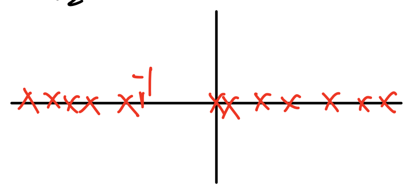


$$f_1(\Omega) = \Pi^+ \setminus \tilde{\gamma}$$

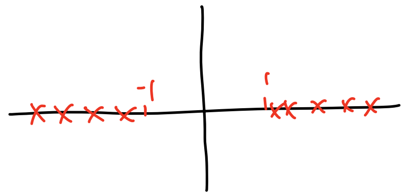
Step 2: $f_2(z) = 2(z - \frac{1}{2})$.



Step 3: $f_2(z) = z^2$.



Step 4: $f_4(z) = 2(z + \frac{1}{2})$.



Step 5: $f_5(z) = K^{-1}(z)$

branch mapping onto Π^+.

Integration in Complex Plane

Def: a (parameterized) curve in $\mathbb{R}^n$ is a continuous map

$$\gamma(t) : [\alpha, \beta] \longrightarrow \mathbb{R}^n.$$

$n = 2 \leadsto \mathbb{C}$. $\quad z = z(t)$, $t \in [\alpha, \beta]$. $\quad z(t) = x(t) + i\,y(t)$

continuous.

One has to distinguish between a curve and its support.

(support $[\gamma] :=$ the image of $[\alpha, \beta]$ under $\gamma(t)$).

(same support can have different curves).

Eg. $[\gamma] = \partial B_1(0)$.

$\gamma_n(t) = e^{int}$, $t \in [0, 2\pi]$ $\quad n \in \mathbb{Z},\ n \neq 0.$

$[\gamma_n] = \partial B_1(0).$ $\quad \operatorname{Arg}(\gamma_n(t)) = nt$. $\quad |e^{int}| = 1.$

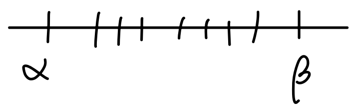
n: the winding number.

$n>0$: walk through the circle n times counter-clockwise.

$n<0$: ———————— // —— $(-n)$ ———— clockwise

<u>Smooth curve</u>: $\exists\ z'(t) = x'(t) + i y'(t) \in C[\alpha, \beta].$

<u>Smooth by parts</u>:

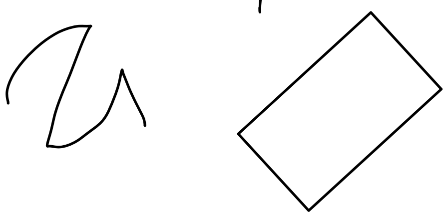


segments: $I_1, \cdots, I_n$

γ - smooth on $\forall\ I_1, \cdots, I_n.$

(and $\gamma(t) \in C[\alpha, \beta]$).

Eg.



<u>Equivalent curves</u>: $\gamma(t): [\alpha, \beta] \to \mathbb{C}.$

$\tilde{\gamma}(t): [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{C}.$

$\exists$ strictly increasing $\varphi(t): [\alpha\ \beta] \xrightarrow{\text{onto}} [\tilde{\alpha}, \tilde{\beta}]$

$(\varphi(\alpha) = \tilde{\alpha},\ \varphi(\beta) = \tilde{\beta}) \quad \Rightarrow \boxed{\tilde{\gamma}(\varphi(t)) = \gamma(t)}$

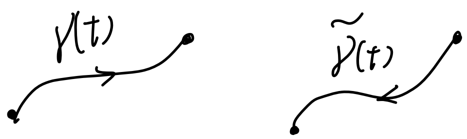
In short, $\tilde{\gamma}$ is obtained from γ by a <u>reparameterization.</u>

can require φ - smooth / smooth by parts.

If, on the other hand, $\varphi(t)$ is strictly decreasing and

$\varphi(\alpha)=\tilde{\beta}$, $\varphi(\beta)=\tilde{\alpha}$. and $\boxed{\tilde{\gamma}(\varphi(t))=\gamma(t)}$

then we called γ and $\tilde{\gamma}$ <u>oppositely parameterized.</u>



<u>Ex.</u> $[\alpha,\beta]=[\tilde{\alpha},\tilde{\beta}]=[0,1]$

$\varphi(t)=1-t$.

$\tilde{\gamma}(t):=\gamma(1-t)$. then $\tilde{\gamma}$ is opp. param-d.

<u>Ex:</u> $[0,1]$; $\gamma(t)=e^{2\pi i t}$. — $n=1$.

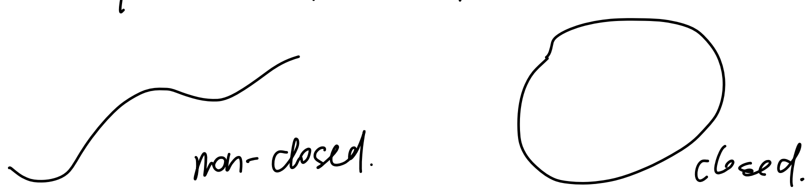
$\tilde{\gamma}(t)=e^{2\pi i(1-t)}=e^{-2\pi i t}$. — $n=-1$.

<u>Simple curve</u>: such that $\gamma(t)$ is <u>injective</u> on $[\alpha,\beta]$.

accept possible $\gamma(\alpha)=\gamma(\beta)$.

(no self-intersections. no self-overloops)

$\gamma(\alpha)=\gamma(\beta)$: closed curves.



$[\gamma]$ defines a simple curve uniquely (up to equi.) - up to changing orientation. ($\gamma \longleftrightarrow$ opp. param-d).

<u>Reminder</u>: line integral. $\vec{F}=(P, Q)$ vector field.

$$\int_\gamma \vec{F}\cdot d\vec{r}. \qquad \vec{r}=\gamma(t).$$

$$= \int_\gamma P\,dx + Q\,dy. \qquad \left|\int_\gamma \vec{F}\cdot d\vec{r}\right| \le \max|\vec{F}|\cdot \underbrace{|\gamma|}_{\text{length}}.$$

$$|\gamma| = \int_\alpha^\beta \|\gamma'(t)\|\,dt.$$

<u>Def</u>: Let $\gamma = \{z = z(t)\}$ — a smooth (by parts) curve in $\mathbb{C}$.

Let $f(x,y) = u(x,y) + iv(x,y)$ be a complex func. continuous on $[\gamma]$. Then, the integral of f along γ is:

$$\boxed{\int_\gamma f\,dz := \int (u+iv)(dx+i\,dy)}$$

line integral.

$$\int_\gamma f\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx).$$

Directly from the def. and the prop.s of the line integral.

we get

<u>Prop.</u> 0) Existence from the theory of line integral.

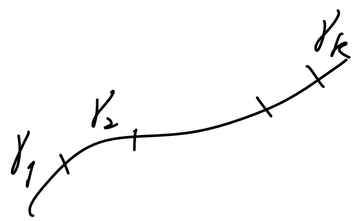
1) $\int_\gamma (af+bg)\,dz = a\int_\gamma f\,dz + b\int_\gamma g\,dz.$ (linearity) $a,b\in\mathbb{C}.$

2) Additivity:

$$\begin{array}{ccccc} & I_1 & I_2 & \cdots & I_k \\ \alpha & | & | & | \cdots & | \ \beta \end{array}$$

using the same param-d map $\gamma(t)$. (or equi.)

Then: $\int_\gamma f\,dz = \int_{\gamma_1} f\,dz + \cdots + \int_{\gamma_k} f\,dz.$



3). <u>Reduction formula</u>:

$$\boxed{\int_\gamma f\,dz = \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt}$$

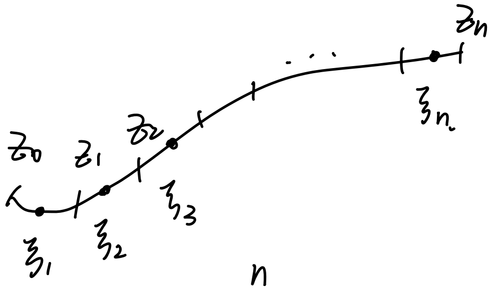
4). <u>Basic bound</u>:

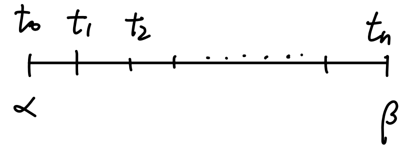
$$\left|\int_\gamma f\,dz\right| \le M\cdot|\gamma|,$$ where $M = \max_{[\gamma]}|f|$, $|\gamma|$ — the length of γ.

5). The integral is <u>invariant</u> under switching to an equi.

curve, and the integral changes sign when switching to the opposite param.-n.

<u>Remark:</u> $\exists$ a way to define the integral using Riemann sums.





$$\sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

$$\int_\gamma f dz = \lim_{\max_j |t_j - t_{j-1}| \to 0} (\text{integral sums})$$ exists as long as:

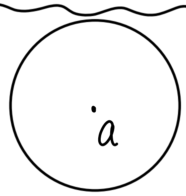
(i) f is continuous on $[\gamma]$.

(ii) $|\gamma| < +\infty$.

rectifiable curve.

$$|\gamma| := \sup_{\text{all partitions}} \sum_{j=1}^{n} |z_j - z_{j-1}|.$$

Main calculation of the course



$$\partial B_R(a) =: \gamma$$

considered as a simple counter-clockwise oriented curve.

$z(t) = a + R \cdot e^{it}$. $\quad 0 \le t \le 2\pi$.

Evaluate $\displaystyle\int_\gamma (z-a)^n \, dz$

$n \in \mathbb{Z}$. $\qquad$ if f is hol. then

$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\, \gamma'(t)$.

Use reduction formula:

$$= \int_0^{2\pi} (Re^{it})^n \cdot R \cdot i \cdot e^{it} \, dt.$$

$$= i R^{n+1} \int_0^{2\pi} e^{it(n+1)} \, dt.$$

$$= \begin{cases} (n \neq -1.) & i R^{n+1} \frac{1}{n+1} e^{it(n+1)} \Big|_0^{2\pi} = 0. \\ (n = -1) & i \int_0^{2\pi} dt = 2\pi i. \end{cases}$$

Summary: $\displaystyle\int_{\partial B_R(a)} (z-a)^n \, dz = \begin{cases} 0, & n \neq -1. \\ 2\pi i, & n = -1. \end{cases}$

Def: Let $f(z)$ be a function in a domain $\Omega \subset \mathbb{C}$, then

$\phi(z)$ is called an anti-derivative of f in Ω, if

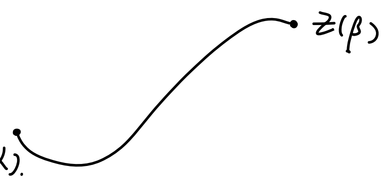
$\phi \in \mathcal{O}(\Omega)$, and $\phi'(z) = f(z)$.

Prop. (Newton - Lebnitz formular)

curve smooth (by parts).

Let $\Omega \supset [\gamma]$, $f \in C(\Omega)$, and $\exists\ \phi$ - an anti- deriv. of f in Ω. Then:

$$\int_\gamma f\, dz = \phi(z(\beta)) - \phi(z(\alpha)).$$



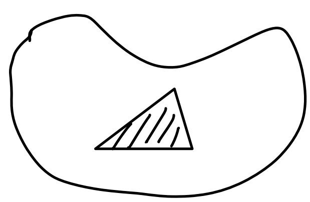
Pf: $\frac{d}{dt}(\phi(z(t))) = \phi'(z(t))\, z'(t) = f(z(t))\, z'(t)$.

$$\int_\gamma f dz \underset{\substack{\uparrow \\ \text{Reduction f-lar}}}{=} \int_\alpha^\beta f(z(t))\, z'(t) dt = \int_\alpha^\beta \frac{d}{dt}(\phi(z(t)) dt \underset{\substack{\uparrow \\ \text{usual N-L}}}{=} \phi(z(\beta)) - \phi(z(\alpha))$$

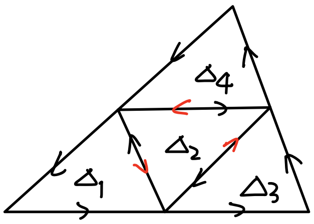
Lemma (Morera)

Let $f \in \mathcal{O}(\Omega)$. Let $\bar{\Delta}$ be a closed triangle. $\bar{\Delta} \subset \Omega$.

Then $$\int_{\partial\Delta} f(z)\, dz = 0.$$



Proof: Assume, by contradiction, that $I := \left| \int_{\partial\Delta^+} f(z)\, dz \right| \neq 0$. (>0).

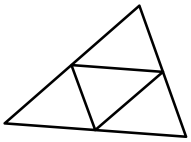


Note: $\displaystyle\int_{\partial\Delta^+} f\,dz = \sum_{i=1}^{4} \int_{\partial\Delta_i^+} f(z)\,dz.$

Clearly, $\exists j \in [1,4]$, $\displaystyle\left|\int_{\partial\Delta_j^+} f(z)\,dz\right| \geqslant \frac{1}{4} I.$

(by triangle inequality).

Pick this Δ_j:



build $\Delta_{j,1}, \Delta_{j,2}, \Delta_{j,3}, \Delta_{j,4}$.

Argue similarly, and find $\Delta_{j,j1}$: $\displaystyle\left|\int_{\partial\Delta_{j,j1}^+} f\,dz\right| \geqslant \frac{I}{4^2}.$

continue ……

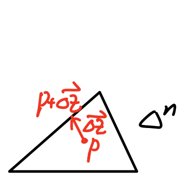
So, $\forall n \in \mathbb{N}^*$, get a triangle Δ^n: $\displaystyle\left|\int_{\partial\Delta^{n+}} f\,dz\right| \geqslant \frac{I}{4^n}$

diam $\Delta^n \to 0$ $(n\to\infty)$. $\displaystyle\bigcap_{n=1}^{\infty} \Delta^n = \{p\} \in \Omega.$

Since these are compacts in $\mathbb{R}^2$ with $\Delta^1 \supset \Delta^2 \supset \cdots \supset \Delta^n \supset \cdots$

Then, there exists $f'(p)$:

$f(p+\Delta z) = f(p) + f'(p)\Delta z + \bar{\bar{o}}(\Delta z)$. $\Delta z = z - p$.



$$\left|\int_{\partial\Delta^{n+}} f\,dz\right| = \left|\underbrace{\int_{\partial\Delta^{n+}} [f(p) + f'(p)(z-p)]\,dz}_{\substack{\| \\ 0}} + \int_{\partial\Delta^{n+}} \bar{\bar{o}}(z-p)\,dz\right|$$

follows from $\int (z-p)^n\,dz = 0$, $n \geqslant 0$.

$$= \left| \int_{\partial \Delta^{n^*}} \bar{\bar{o}}(z-p)\,dz \right| \overset{\text{basic}}{\underset{\text{bound}}{\leq}} \underbrace{\frac{\rho}{2^n}}_{\downarrow} \cdot \max|z-p| \cdot \max|\bar{\bar{o}}|$$

(perimeter of Δ^n.) $\qquad \lim_{z\to p} = 0$

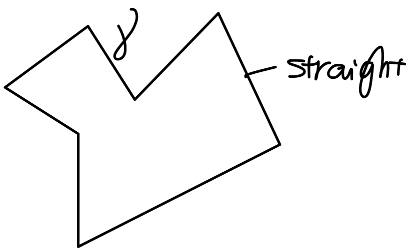
(ρ = the perimeter of Δ).

$$\leq \frac{\rho}{2^n} \cdot \frac{\rho}{2^n} \cdot \alpha_n \quad ; \qquad \text{So, } \frac{\rho^2}{\cancel{4^n}} \cdot \alpha_n \geq \frac{I}{\cancel{4^n}} > 0.$$

$\alpha_n \xrightarrow{n\to\infty} 0$; $\quad \alpha_n \xrightarrow{n\to\infty} 0$

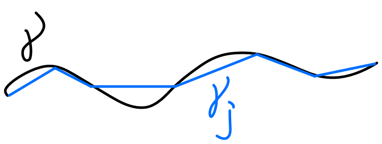
a contradiction. $\square$.

<u>Corollary 1</u>: Let $f \in \mathcal{O}(\Omega)$. Let γ a simple closed broken line in Ω. And the interior of $\gamma \subset \Omega$. Then $\int_\gamma f dz = 0$.



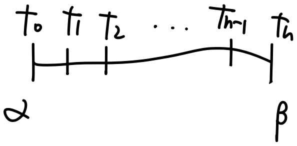
<u>Proof</u>: Break γ into triangles. It follows from Morera Lemma. $\square$.

<u>Lemma (Approximation)</u> Let Ω: domain, $f \in C(\Omega)$.
γ: a smooth by-parts curve in Ω. Then $\exists\, \gamma_j$: a sequence of broken lines inscribed in γ, s.t. $\int_\gamma f\,dz = \lim_{j\to\infty} \int_{\gamma_j} f\,dz$.



Proof: enough to prove $\forall$ smooth part of γ.

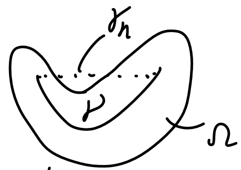
(then can take union of broken line $\forall$ piece).

Take a partition: 

$\gamma(t): [\alpha, \beta] \to \mathbb{C}$.

$\max_i |t_i - t_{i-1}| \to 0 \ (n \to \infty)$.

Just take γ_n: spanned by $\gamma(t_0), \gamma(t_1), \cdots, \gamma(t_n)$. if n is large,

then $\gamma_n \subset \Omega$. (avoiding 

).

Compare:

$$\int_\gamma f dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\, dt. \quad (1)$$

$$\int_{\gamma_n} f dz = \int_\alpha^\beta f(\gamma_n(t))\,\gamma_n'(t)\, dt. \quad (2)$$

$f(\gamma_n(t)) \longrightarrow f(\gamma(t)) \ (n \to \infty)$, since $f \in C(\Omega)$.

γ_n' is close to γ', $\forall$ smooth part of γ_n'.

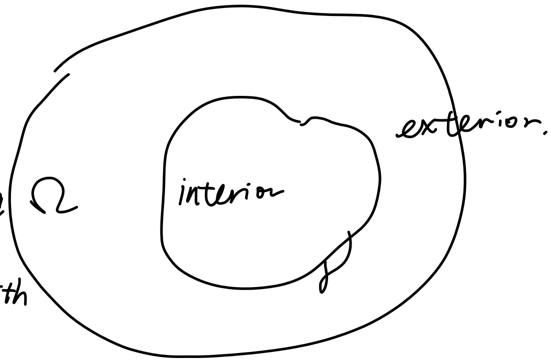
$$\left|(1) - (2)\right| \le \underset{\propto \max|f| \cdot \max|\gamma'|}{\text{Const}} \cdot \overset{\text{small}}{\varepsilon} \cdot \underbrace{\sum |t_i - t_{i-1}|}_{= \beta - \alpha}.$$

$\square$

Theorem (Cauchy Integral)

Let $\Omega \subset \mathbb{C}$: domain, $f \in \mathcal{O}(\Omega)$, and γ: a simple closed smooth by-parts curve in Ω, lying in Ω together with its interior.

Then $\int_{\gamma^{\pm}} f(z)dz = 0.$



Proof: By approximation lemma,

$\exists\ \gamma_n$, $\gamma_n \in \Omega$ together with its interior.
(inscribed in γ, simple broken lines.)

$\int_{\gamma_n} f dz \longrightarrow \int_{\gamma} f dz \quad (n \to \infty)$. by corollary 1, $\int_{\gamma_n} f\, dz = 0. \Rightarrow \int_{\gamma} f dz = 0$

□.

Remark: If one has $f \in C^1(\Omega)$ $\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in C(\Omega)\right]$.

then: $\int_{\gamma} f dz = \int_{\gamma} (u+iv)(dx+idy)$.

$$= \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (u dy + v dx)$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy.$$

Green's formula.

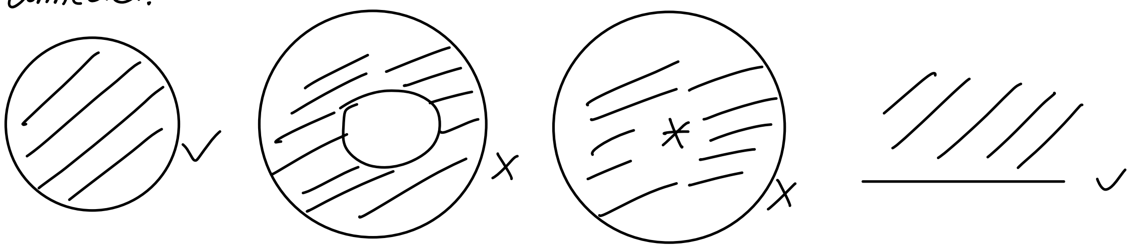
$$\left(\int_{\gamma} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \right)$$

$$= 0 + 0 = 0$$

Cauchy-Riemann conditions.

Def: a bounded domain $\Omega \in \mathbb{C}$ is called <u>simply-connected</u> if $\partial\Omega$ is connected.

E.g.



(No proof) Equivalent conditions:

(i) $\forall\, a, b \in \Omega$, $\forall$ curves γ_0, γ_1 connecting a, b are homotopic.

E.g. If $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$, $\exists\ \gamma(t,s) : [0,1] \times [0,1] \to \mathbb{C}$, s.t.

$$\gamma(0,s) = a,\ \gamma(1,s) = b,\ \gamma(t,0) = \gamma_0(t),\ \gamma(t,1) = \gamma_1(t).$$

($\gamma_s(t) := \gamma(t,s)$ is the "s-intermediate curve").

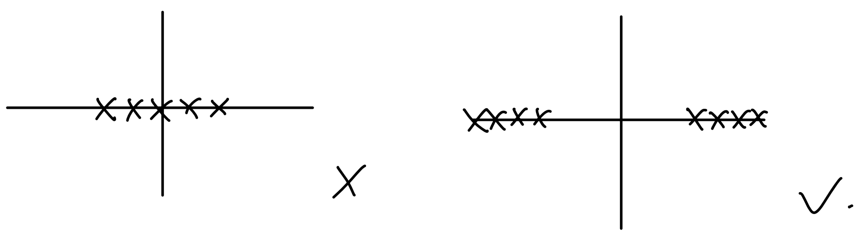
(ii) $\forall$ closed path $\gamma_0 \in \Omega$ is homotopic to $\{p\}$.
simple pt. curve.

$$\gamma(t,0) = \gamma_0(t),\ \gamma(t,1) \equiv p,\ \gamma(0,s) = \gamma(1,s).$$

simply-connected $\Longleftrightarrow$ "no holes".

(iii) If γ_0: any simple curve in Ω. then its interior $\subset \Omega$.

Using any of (i), (ii), (iii), we extend the notion <u>simply-connected</u> to unbounded domains:



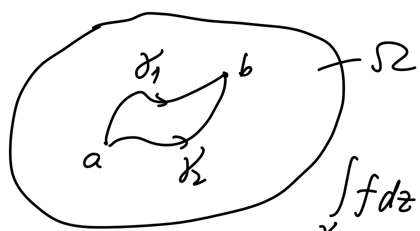
Corollary (of Cauchy Thm):

If $\Omega \subset \mathbb{C}$: simply-connected. γ: simple closed curve in Ω (smooth by parts).

then, $\int_\gamma f dz = 0$. $\forall f \in \mathcal{O}(\Omega)$.

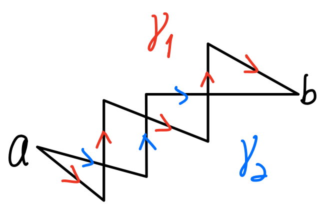
Theorem: Let Ω: simply-connected dom. in $\mathbb{C}$, $f \in \mathcal{O}(\Omega)$; $a, b \in \Omega$.

then $\int_\gamma f dz$, where γ is simple, smooth-by-parts. connecting a and b, does NOT depend on γ.

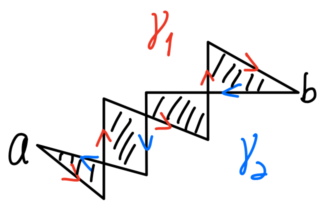


$$\int_{\gamma_1} f dz = \int_{\gamma_2} f dz.$$

Proof: Because of approximation lemma, it's sufficient to consider the case: γ_1, γ_2 are broken lines.



$$\int_{\gamma_1} f dz - \int_{\gamma_2} f dz$$



$$= \int_{\gamma_1^+} f dz + \int_{\gamma_2^-} f dz = \sum_{j=1}^{N} \int_{\partial P_j^\pm} f dz = 0$$

γ_1^+: $a \to b$ $\quad$ γ_2^-: $b \to a$ $\quad$ P_j: polygons $\quad$ $= 0$: by the corollary.

$$\Rightarrow \int_{\gamma_1} f dz = \int_{\gamma_2} f dz. \quad \square$$

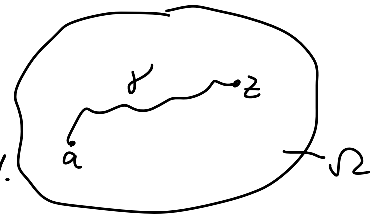
So, hol. func.s have the path-independent property for cx. integration.

Theorem: If $\Omega \subset \mathbb{C}$ is a simply-connected dom, then $\forall f \in \mathcal{O}(\Omega)$,

f has an anti-deriv.

Proof: Set $\phi(z) := \int_a^z f(\xi)\, d\xi$ = integral along $\forall$ simply broken line,

connecting a and z.



From the previous thm, ϕ is well-defined.

So, we just need to prove: $\phi'(z) = f(z)$, $\forall z \in \Omega$.

By def. $z+\Delta z$ is a pt. nearby z.

Let's arrive from a to $z+\Delta z$ by taking $\tilde{\gamma} := \gamma \cup [z, z+\Delta z]$.

$$\phi(z+\Delta z) - \phi(z) - f(z)\Delta z$$

$$= \int_a^{z+\Delta z} f(\xi)\, d\xi - \int_a^{z} f(\xi)\, d\xi - \int_z^{z+\Delta z} f(z)\, d\xi$$

$$= \int_z^{z+\Delta z} [f(\xi) - f(z)]\, d\xi.$$

$$\left| \int_z^{z+\Delta z} [f(\xi) - f(z)]\, d\xi \right| \le |\Delta z| \cdot \max_{\xi \in [z, z+\Delta z]} |f(z) - f(\xi)| \to 0 \quad (\Delta z \to 0).$$

$\Rightarrow$ By def: $\exists\ \phi'(z) = f(z)$ $\square$.

Questions for HW:

(i) Does $\int_\gamma f\, dz = 0$ hold for any γ: simple closed, smooth-by-parts curve,

$\forall$ domain ?

(ii) Does the anti-deriv. thm hold in any dom. ?

1) $\Omega = \mathbb{C} \setminus \{0\}$, $\gamma = \{|z| = 1\}$, $f(z) = \frac{1}{z}$. $\int_\gamma f(z)\, dz = 2\pi i \neq 0$.

2) if $\exists \phi$, $\phi' = \frac{1}{z}$ $\Rightarrow \int_\gamma \frac{1}{z} dz = \phi(1) - \phi(1) = 0 \neq 2\pi i$.

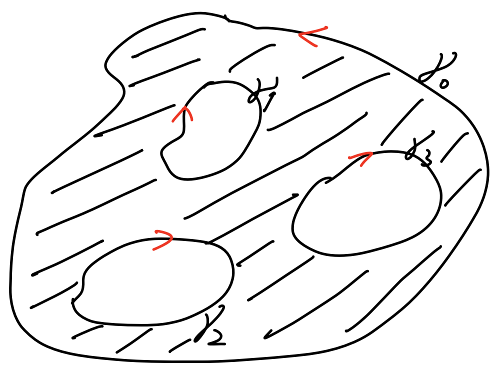
Remark: If, instead, choose $f = \frac{1}{z^2}$, then $\int_\gamma f dz = 0$. $\exists \phi = -\frac{1}{z}$.

Def: If $E \subset \mathbb{C}$, then "$f \in \mathcal{O}(\bar{E})$" means $\exists \Omega$-dom. $\Omega \supset \bar{E}$, $f \in \mathcal{O}(\Omega)$
set.

Def: A domain $D \subset \mathbb{C}$ is called an admissible domain, if $\partial D = \gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_s$. ($s$ can be 0), where all γ_j - closed simple curves, and $\gamma_1, \cdots, \gamma_s$ are lie in the interior of γ_0.

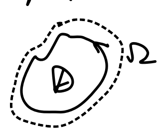
($S \geq 1$ — connected)

($s = 0$ — simply connected)



Cauchy Thm implies:

If D - admissible, simply connected, ∂D: SBP (smooth by parts) and $f \in \mathcal{O}(\bar{D}) \Rightarrow \int_{\partial D} f dz = 0$.
$= \int_\gamma f dz = \int_{\partial D^+} f dz$.



Good: for all admissible dom.

Def: Let D - adm. dom. ∂D - SBP. $f \in C(\partial D)$.

Then $\int_{\partial D} f\,dz := \int_{\gamma_0^+} f\,dz + \int_{\gamma_1^-} f\,dz + \cdots + \int_{\gamma_s^-} f\,dz$.

$$= \int_{\gamma_0^+} f\,dz - \int_{\gamma_1^+} f\,dz - \cdots - \int_{\gamma_s^+} f\,dz.$$

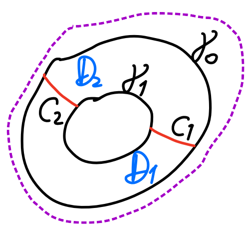
Thm. (Cauchy Thm for multi-conn. dom.)

Let D - adm. dom. ∂D - SBP. $f \in \mathcal{O}(\bar{D})$.

Then $\int_{\partial D} f\,dz = 0$.

Proof: $S=0$: proved above.

$S=1$:



C_1, C_2 - two simple SBP curves connecting γ_0, γ_1.

Ω: between the "purple"

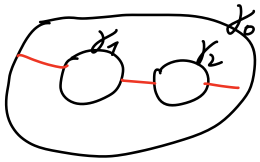
Now: $\bar{D} = \bar{D}_1 \cup \bar{D}_2$.

By the $S=0$ case: $\int_{\partial D_1} f\,dz = \int_{\partial D_2} f\,dz = 0$.

But $\int_{\partial D} f\,dz = \int_{\partial D_1} f\,dz + \int_{\partial D_2} f\,dz = 0$.

($\int_{C_{1,2}^{\pm}} f\,dz$ canceled!)

$\underline{S>1}$: Analog.

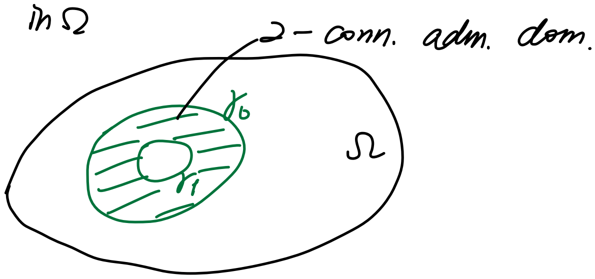


$\square$.

Corol: Let $f \in \mathcal{O}(\Omega)$; γ_0, γ_1 – two simple closed curves,

$\gamma_1 \subset$ interior of γ_0. $\gamma_0 \underset{\text{in } \Omega}{\overset{\text{homot.}}{\sim}} \gamma_1$.

Then $\displaystyle\int_{\gamma_0^+} f dz = \int_{\gamma_1^+} f dz$.



Cauchy Integral Formula

Reminder: integ. depending on a param.

$$I(\alpha) := \int_a^b f(x,\alpha)\,dx. \quad \alpha \in \mathbb{R}^m. \quad \alpha \in G - \text{dom}.$$

$*$ If $f \in C([a,b] \times G) \Rightarrow I(\alpha) \in C(G)$.

$*$ If additionaly $\exists\ \frac{\partial f}{\partial \alpha} \in C([\alpha, \beta] \times G)$

$$\frac{\partial f}{\partial \alpha} = \left(\frac{\partial f}{\partial \alpha_1}, \dots, \frac{\partial f}{\partial \alpha_m}\right).$$

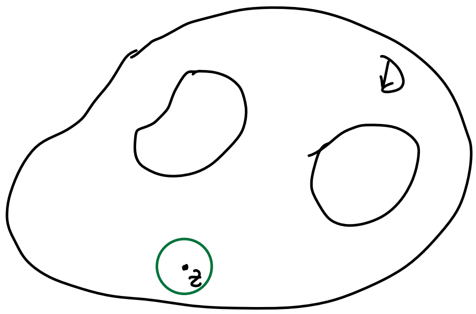
$\Rightarrow I(\alpha) \in C^1(G)$. and $\displaystyle\frac{\partial}{\partial \alpha_j} I(\alpha) = \int_a^b \frac{\partial f}{\partial \alpha_j}\,dx.$

Using reduction to Riemann intergral, same holds for f on $\gamma \times G$
curve.
SBP.

Thm. (Cauchy Intergral f-la).

Let D - adm. dom. ∂D - SBP. $f \in \mathcal{O}(\bar{D})$, Then, $\forall z \in D$, it holds:

$$\boxed{f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta.}$$



Proof: choose δ, s.t. $\overline{B_\delta(z)} \subset D$.

Consider $\tilde{D} := D \setminus \overline{B_\delta(z)}$.

$\tilde{D}$ - new adm. dom. with the extra

$\gamma_{s+1} = \partial B_\delta(z)$

Then by Cauchy Intergral Thm: $\int_{\partial \tilde{D}} f(\zeta) d\zeta = 0$.

also: $\int_{\partial \tilde{D}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0$. $\quad \in \mathcal{O}(\tilde{D})$

$$\Rightarrow \int_{\partial D} \frac{f(\zeta) d\zeta}{\zeta - z} = \underbrace{\int_{\partial \tilde{D}}}_{\text{by def}} + \int_{\partial B_\delta(z)} \frac{f(\zeta) d\zeta}{\zeta - z}, \ \forall \delta > 0! \text{ (small enough)}.$$

$$= \int_{\partial B_\delta(z)} \frac{f(\zeta)}{\zeta - z} d\zeta. \overset{\zeta = z + e^{it} \cdot \delta,\ t \in [0, 2\pi]}{=} \int_0^{2\pi} \frac{f(z + \delta \cdot e^{it})}{\delta \cdot e^{it}} \cdot \delta \cdot i e^{it} dt.$$

$$= i \int_0^{2\pi} f(z + \delta \cdot e^{it}) dt \xrightarrow[\delta \to 0.]{\text{by conti. of } f} i \int_0^{2\pi} f(z) dt = 2\pi i \cdot f(z).$$

Now, here take $\lim_{\delta \to 0}$: $\boxed{\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i \cdot f(z)}$. $\square$.

Thm. (Cauchy f-la for derivatives)

Let $f \in \mathcal{O}(\bar{D})$ D – adm. dom. ∂D – SBP.

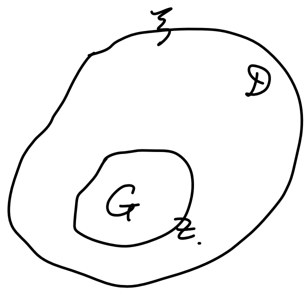
Then $\exists f', f'', \dots, f^{(k)}, \dots$ all $\in \mathcal{O}(\bar{D})$, and

$$\forall k: \quad f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(\xi)}{(\xi - z)^{k+1}} \, d\xi.$$

Proof: Apply Cauchy Intergral f-la. + props of interg. with param.

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \underbrace{\frac{f(\xi)}{\xi - z}}_{\varphi(\xi, z)} \, d\xi.$$

Let z vary in a subdomain G: $\bar{G} \subset D$.



then $\varphi \in C(\partial D \times G)$. $(|\xi - z| \geq \delta > 0)$.

$$\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \xrightarrow{\text{instead}} \frac{\partial \varphi}{\partial z}, \frac{\partial \varphi}{\partial \bar{z}}. \qquad z = x + iy.$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial}{\partial z}\left(\frac{f(\xi)}{\xi - z}\right) = \frac{f(\xi)}{(\xi - z)^2}.$$

$\dfrac{\partial \varphi}{\partial \bar{z}} = 0$ – because $\dfrac{f(\xi)}{\xi - z}$ is hol in z.

$\Rightarrow$ apply differ-n. in the param. z on f:

get $\frac{\partial f}{\partial \bar{z}} = 0$. $\frac{\partial f}{\partial z} = f' = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$.

Argueing similarly: $\frac{\partial f'}{\partial \bar{z}} = 0$. $(\Rightarrow f' \in \mathcal{O}(D))$.

and $\frac{\partial f'}{\partial z} = f'' = \frac{2}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$.

analog, all $f^{(k)}$ exist and are hol in D.

Show hol in $\bar{D}$: increase D "a bit", set $\hat{D}$: $\bar{D} \subset \Omega$, argue $f^{(k)} \in \mathcal{O}(\hat{D})$

$\Rightarrow f^{(k)} \in \mathcal{O}(\bar{D})$. $\blacksquare$

Corol: $\forall$ dom Ω, if $f \in \mathcal{O}(\Omega)$, then $\exists f', f'', \ldots, f^{(k)} \in \mathcal{O}(\Omega)$, $\forall k \in \mathbb{N}$.

Proof. the statement is local $\Rightarrow$ enough to prove $\forall$ disc $B_\delta(a)$: $\overline{B_\delta(a)} \subset \Omega$

Then apply the prev. Thm in $D = B_\delta(a)$

Get, $f', f'', \ldots, f^{(k)} \in \mathcal{O}(D)$. $\blacksquare$

Ex. $f(x) = |x|^{k+\frac{1}{2}}$. $x \in \mathbb{R}$. $k \in \mathbb{N}$.

$\exists f', f'', \ldots, f^{(k)}$ but $\nexists f^{(k+1)}(0)$.

Remark:

* CIF recovers a hol func f from its boundary values.

If $\exists$ a hol conti. of a func f from ∂D to D. then it's unique

conti.

* CIF for $f^{(k)}$: remarkable. since allows to estimate $f^{(k)}$ in terms

of func. !

$$|f^{(k)}(z)| \le \frac{|\partial D|}{2\pi} \cdot \max_{\partial D} |f| \cdot \underbrace{\frac{1}{\operatorname{dist}(z, \partial D)^{k+1}}}_{\le C_0 \text{ in a subdom.}} \cdot k!$$

$$\sim \text{Const} \cdot k! \cdot \max_{\partial D} |f| \quad \text{in a subdomain.}$$

HW: Compute the Cauchy Integ. for $f(z) = \frac{1}{z}$, $D = B_1(0)$.

Inverse Thm: D - adm. dom. ∂D: SBP. $f \in C(\partial D)$ satisfying the CIF. Then $f \in \mathcal{O}(D)$.

Proof: just follows as before from differ-n. in the param. z.

Thm: Let Ω - dom. $f \in C(\Omega)$. satisfying the

triangle prop. / Morera prop. : $\forall \overline{\Delta} \subset \Omega$, $\int_{\partial \Delta} f(\xi)\, d\xi = 0$.

closed triangle.

Then $f \in \mathcal{O}(\Omega)$.

(integral prop. implies differ. prop.).

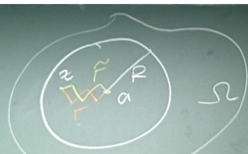
Proof: (Out lines).

Step 1: $\forall \gamma \in \Omega$, simply-closed SBP with interior in Ω,

$\int_\gamma f\, dz = 0$. (polygons estimation: $\gamma_n \to \gamma$, $n \to \infty$).

Step 2: $\forall a \in \Omega$. $\overline{B_\delta(a)} \subset \Omega$. $\exists \phi \in \mathcal{O}(\overline{B_\delta(a)})$:

$\phi' = f$. — why?



Set $\Phi(z) := \int_\Gamma f(\xi)\,d\xi$

Γ - broken line, conn-g a, z.

Def of Φ is indep of Γ. Similarly to how we argued before, $\int_\Gamma f\,dz - \int_{\tilde\Gamma} f\,dz = \sum_{j=1}^{m} \int_{\partial\Gamma_j} f\,dz =$ (polygon)

$= \sum 0 = 0$ – by Step I.

$\Rightarrow$ We show $\Phi'(z) = f(z)$ word-by-word as in the Thm on $\exists$ anti-deriv of hol func.

(We switch $\Omega \to B_R(a)$ since $B_R(a)$ - simply-conn!)

<u>Step 3:</u> $\forall\, \overline{B_R(a)} \subset \Omega,\ \exists\, \phi : \phi' = f$ in $B_R(a)$.

$\Rightarrow f \in \mathcal{O}(B_R(a))$ as $f = \phi'$. $\square$

<u>Thm (Inverse CIF)</u>

Let $f \in C(\Omega)$, and $\forall\, \bar\Delta \subset \Omega$, Cauchy f-la holds:

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(\xi)}{\xi - z}\,d\xi \quad \forall z \in \Omega \Rightarrow f \in \mathcal{O}(\Omega).$$

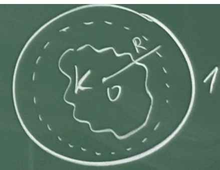
<u>Proof:</u> $f \in \mathcal{O}(\Delta)$ an integ. (by the inverse thm.) $\Rightarrow$ by the above thm $\Downarrow$ ✓

hol-ly depending on param. z. $\Rightarrow f \in \mathcal{O}(\Omega)$ $\square$

<u>Def:</u> Let $f_n \in \mathcal{O}(\Omega)$. We say that $f_n \xrightarrow{\Omega} f$ (normally converges, or converges unif-ly on compacts), if

$\forall \underset{\text{cpt}}{K} \subset \Omega,\ f_n \underset{\text{unif.}}{\overset{K}{\rightrightarrows}} f$

$\forall K \underset{\text{cpt}}{\subset} \Omega$, $f_n \xrightarrow[\text{unif.}]{K} f$



Ex: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, $\{|z|<1\}$ (geom progr.)
$=: \Omega$; $\forall K \subset \Omega$, $K \subset B_R(0)$, $R<1$.

$S_n(z) = \frac{1-z^{n+1}}{1-z}$, $|z^{n+1}| \le R^{n+1}$ on $K \Rightarrow$

$z^{n+1} \xrightarrow{K} 0 \Rightarrow S_n(z) = \frac{1-z^{n+1}}{1-z} \xrightarrow{K} \frac{1}{1-z} \Rightarrow \sum z^n \xrightarrow{\Omega} \frac{1}{1-z}$

$(R<1)$

设 $\sum_{n=0}^{\infty} f_n(z)$ 为函数级数, 其中函数 f_n 定义于某个集合 $M \subset \mathbb{C}$. 称其在 M 上一致连续是说, 如果它在每个点 $z \in M$ 收敛, 且对于任意 $\varepsilon > 0$ 可以找到序号 $N = N(\varepsilon)$ 使得对于所有 $n \geqslant N$ 及所有 $z \in M$, 级数的余部 $\left|\sum_{k=n+1}^{\infty} f_k(z)\right| < \varepsilon$.

But $\sum z^n \underset{B_1(0)}{\not\rightrightarrows} \frac{1}{1-z}$ since $z^{n+1} \underset{\text{in } B_1(0)}{\not\rightrightarrows} 0$ (stand. exercise)

Remark: normal conv can be also tested by Cauchy criter. (fundam. seq.)

Remark: instead of considering $\forall K \subset \Omega$ it's enough to consider just the cpt exhaustion: $\{K_j\}_{j=1}^{\infty}$ $K_1 \subset K_2 \subset \dots$

$\bigcup_{j=1}^{\infty} K_j = \Omega$, because

$\forall K \subset K_n$ for n suff. large

Why exists? For ex.: $K_j = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \ge \frac{1}{j}\} \cap \{|z| \le j\}$

Thm: (Weierstrass Thm) Let $f_n \in \mathcal{O}(\Omega)$, $f_n \xrightarrow{\Omega} f$, then $f \in \mathcal{O}(\Omega)$

Remark: entirely wrong in real anal.! $\forall f \in C[0,1]$, $P_n(x) \rightrightarrows f$

Furthermore, $f_n^{(k)} \xrightarrow{\Omega} f^{(k)}$ $\forall$ fixed $k \in \mathbb{N}$

$\frac{\sin nx}{n} \xrightarrow{\mathbb{R}} 0$, but $\left(\frac{\sin nx}{n}\right)' = \cos nx \not\to 0$

Proof: take $\forall\, \overline{\Delta} \subset \Omega$; $f_n \overset{\overline{\Delta}}{\rightrightarrows} f$

$$\Rightarrow \int_{\partial\Delta} f\,dz = \lim_{n\to\infty} \underbrace{\int_{\partial\Delta} f_n\,dz}_{=0} = 0$$

(since $f_n \in \mathcal{O}(\Omega)$)

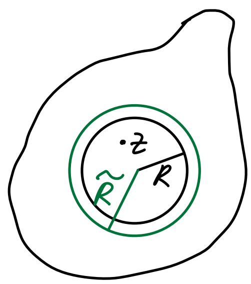
$\Rightarrow$ by the Inverse CIT: $f \in \mathcal{O}(\Omega)$

Now let's prove $f_n^{(k)} \overset{\Omega}{\rightrightarrows} f^{(k)}$

Enough to prove $f_n^{(k)} \underset{K}{\rightrightarrows} f^{(k)}$,

$K = \overline{B_R(a)} \subset \Omega$ — because $\forall K \subset \bigcup_{j=1}^{n} B_{R_j}(a) \subset$
$\subset \bigcup_{j=1}^{n} \overline{B_{R_j}(a_j)}$

So, fix $\overline{B_R(a)} \subset \Omega$, choose $\overline{B_R(a)} \subset \overline{B_{\tilde{R}}(a)} \subset \Omega$.



$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int \frac{f_n(\xi)}{(\xi - z)^{k+1}}\,d\xi, \quad \forall\, k \in \mathbb{N}.$$

$$|\xi - z| \geq \tilde{R} - R \Rightarrow \left| \frac{1}{(\xi - z)^{k+1}} \right| \leq \text{const.}$$

$$\Rightarrow \frac{f_n(\xi)}{(\xi - z)^{k+1}} \underset{\substack{z \in \overline{B_R(a)} \\ \xi \in \partial B_{\tilde{R}}(a)}}{\overset{n\to\infty}{\rightrightarrows}} \frac{f(\xi)}{(\xi - z)^{k+1}} \Rightarrow \text{integrate.}$$

$$\frac{1}{2\pi i} k! \int_{\partial B_{\tilde{R}}(a)} \frac{f_n(\xi)}{(\xi - z)^{k+1}}\,d\xi \underset{z \in \overline{B_R(a)}}{\overset{n\to\infty}{\rightrightarrows}} \frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi - z)^{k+1}}\,d\xi.$$

$$\begin{matrix} \| \\ f_n^{(k)}(z) \underset{z \in \overline{B_R(a)}}{\overset{n\to\infty}{\rightrightarrows}} f^{(k)}(z). \end{matrix}$$

$\square$.

Space $\mathcal{O}(\Omega)$.

Linear space (proved earlier).

Take $\Omega = \bigcup_{j=1}^{\infty} K_j$ – cpt exhausion.

Define semi-norms: $p_j(f) := \max_{K_j} |f|$.

Metric: $d(f,g) := \sum_{j=1}^{\infty} \frac{p_j(f-g)}{1+p_j(f-g)} \cdot \frac{1}{2^j}$.

Fact: $d(f,g)$ is a metric, convergence $\longleftrightarrow$ normal converg.

From Weiers. Thm: $\mathcal{O}(\Omega)$ is a complete metric space.

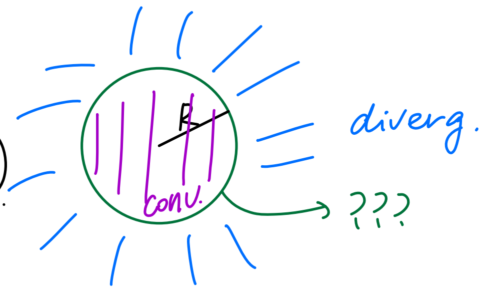
Power series expansion of hol func.

Review of power series: repeating word-by-word the args in the real case, one gets:

Cauchy-Adamar Thm: Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a cx power series, then $\exists\, R$ (called the radius of the conv.) s.t. the following holds:

1) The series is conv. in $B_R(a)$ and diverg. in $\mathbb{C} \setminus \overline{B_R(a)}$.
disc. of conv.

2) $\forall\, \varepsilon < R$, the series is conv. unif in $\overline{B_\varepsilon(a)}$ ($\Rightarrow$ normally conv. in $B_R(a)$).



3). $\frac{1}{R} = \varlimsup\limits_{n\to\infty} \sqrt[n]{|C_n|}$. (Cauchy-Adamar f-la).

Ex. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^2}$, abs. unif. conv. in $|z| \le 1$.

$R = 1$. $B_1(0)$.

$\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$, $B_1(0)$ conv. conditionaly for $z \neq 1$ on $\partial B_1(0)$.

Corollary. (Cauchy-Adamar + Weierstrass).

1) $f(z) = \sum\limits_{n=0}^{\infty} C_n (z-a)^n$ is hol in the disc. of convergence $B_R(a)$

2) the identity $f(z) = \sum\limits_{n=0}^{\infty} C_n (z-a)^n$ can be differentiated k times.

$\forall k \in \mathbb{N}$. (keeping the unif. conv.)

In part., the Taylor f-las hold: $\boxed{C_k = \frac{f^{(k)}(a)}{k!}}$.

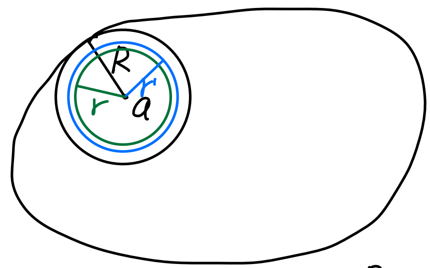
$\left(\frac{d}{dz^k}\Big|_{z=a} f^{(k)}(a) = 0 + C_k \cdot k! + 0 \right)$.

Inverse Thm (power series expansion of hol. func.).

Let $f \in \mathcal{O}(\Omega)$. Let $a \in \Omega$. $R = \operatorname{dist}(a, \partial\Omega)$.

Then $\exists!$ cx. power series $\sum_{n=0}^{\infty} c_n (z-a)^n$, s.t.

$\forall z \in B_R(a)$, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$.



Furthermore,

1) The conv. of the series to $f(z)$ is normal in $B_R(a)$.

(In part., $\forall r < R$, $\sum_{n=0}^{\infty} c_n (z-a)^n \underset{\overline{B_r(a)}}{\rightrightarrows} f(z)$).

2) Cauchy f-las hold for coef-s:

$$c_n = \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta, \quad \forall r \in (0, R).$$

3) Cauchy estimates hold: $|c_n| \le \frac{M}{R^n}$.

Proof: We prove (1), (2), (3) at once.

Choose $r < r' < R$.

For $z \in \overline{B_r(a)}$, apply CIF in the dom. $B_{r'}(a)$:

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_{r'}(0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

$z \in \overline{B_r(a)}, \ \zeta \in \partial B_{r'}(a) \Rightarrow |\zeta - z| \geq r' - r$

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\zeta - a| = r'} \frac{f(\zeta)\, d\zeta}{(\zeta - a) - (z - a)}$$

$$= \frac{1}{2\pi i} \int\limits_{|\zeta-a|=r'} \frac{1}{\zeta - a} f(\zeta)\, d\zeta \cdot \frac{1}{1 - \underbrace{\frac{z-a}{\zeta-a}}_{\substack{\parallel \\ w:\ |w| \leq \frac{r}{r'} < 1.}}}$$

$$= \left\{ \begin{matrix} \text{Geom.} \\ \text{progression} \end{matrix} \right\} = \frac{1}{2\pi i} \int\limits_{|\zeta-a|=r'} \frac{1}{\zeta - a} f(\zeta)\, d\zeta \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta-a} \right)^n.$$

$$= \left\{ \begin{matrix} \text{the converg. of series} \\ \text{is uniform as } z \in \overline{B_r(a)}, \\ \zeta \in \partial B_{r'}(a) \end{matrix} \right\} = \{ \text{integrate} \}$$

$$= \sum_{n=0}^{\infty} c_n (z-a)^n. \quad \text{where } c_n = \frac{1}{2\pi i} \int\limits_{|\zeta-a|=r'} \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{n+1}}.$$

The converg. is uniform in $z \in \overline{B_r(a)}$.

Now, if we tend $r' \to r$: $c_n = \frac{1}{2\pi i} \int\limits_{|\zeta-a|=r} \frac{f(\zeta)\, d\zeta}{(\zeta-a)^{n+1}}.$ — as desired.

So, (3) and (2) are proved.

Since $B_R(a) = \bigcup\limits_{r<R} \overline{B_r(a)} \Rightarrow$ get expansion in $B_R(a) \Rightarrow$ get (1).

Finally, $|c_n| = \left| \frac{1}{2\pi i} \int\limits_{|\zeta-a|=r} \frac{f(\zeta)\, d\zeta}{(\zeta-a)^{n+1}} \right| \leq \frac{1}{2\pi} 2\pi r \frac{M(r)}{r^{n+1}}.$

So, $|C_n| \le \dfrac{M(r)}{r^n}$. $M(r) = \max\limits_{|z-a|=r} |f|$.

if let $r \to R$, $|C_n| \le \dfrac{M}{R^n}$, $M = \sup\limits_{B_R(a)} |f|$. $\square$.

<u>Remark</u>: $f(z) = \sum\limits_{n=0}^{\infty} (z-a)^n$ can be differentiated k times in $B_R(a)$

(by Weierstrass Thm). $\forall k \in \mathbb{N}$.

<u>Corollary</u>: Let $f \in \mathcal{O}(B_R(a))$, assume that $f(a) = f'(a) = f''(a) = \cdots = 0$;

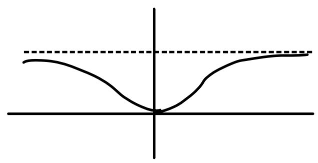
$f^{(k)}(a) = 0$, $\forall k \in \mathbb{N}$. Then $f \equiv 0$.

<u>Proof</u>: By the prev. thm, $f(z) = \sum\limits_{n=0}^{\infty} C_n (z-a)^n$.

$\forall z \in B_r(a)$. Taylor F-las: $C_n = \dfrac{f^{(n)}(a)}{n!} = 0 \Rightarrow f(z) = 0$. $\forall z$. $\square$.

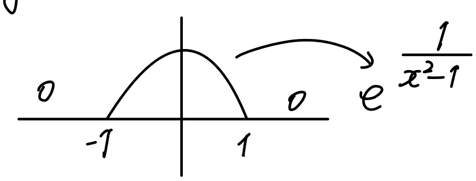
<u>Very big difference from Real Analysis!</u>

<u>E.g.</u> $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$



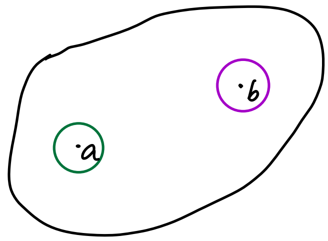
$f^{(n)}(0) = 0$. $\forall n \in \mathbb{N}$. $f \in C^{\infty}(\mathbb{R})$ but $f \not\equiv 0$.

Part of unity: cut off funcs.



Inverse Thm: Let f- func in Ω; $\forall a \in \Omega$, $\exists B_r(a)$ where

$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$. then $f \in \mathcal{O}(\Omega)$



Proof: By the prev. thms, $f \in \mathcal{O}(B_r(a))$

a- arbitr. in $\Omega \Rightarrow f \in \mathcal{O}(\Omega)$

(Big) Theorem (Equiv. cond-s of holom-ty).

Let $\Omega \subset \mathbb{C}$- domain. TFAE:

(1) $f \in \mathcal{O}(\Omega)$ $\left(\Leftrightarrow \forall z \in \Omega, \exists f'(z) \Leftrightarrow df\big|_z \text{ is } \mathbb{C}\text{- linear}\right)$.

(2) $\forall a \in \Omega$, $\exists B_r(a)$, f admits an anti-deriv. in $B_r(a)$

$(\Leftrightarrow \forall B_r(a) \subset \Omega$, - - - - .$)$

(3) $f \in C(\Omega)$ and $\forall$ simple, SBP, closed $\gamma \in \Omega$ with inter. $\subset \Omega$, CIT holds. $\left(\int_\gamma f \, d\zeta = 0\right)$.

(3') same with $\gamma = \partial \triangle$.

(4) $f \in C(\Omega)$, $\forall \gamma$ as in (3), CIF holds: $f(z) = \frac{1}{2\pi i} \int_{\gamma^+} \frac{f(\zeta)}{\zeta - z} d\zeta$.

(4') same with $\gamma = \partial \triangle$.

(5) $\forall a \in \Omega$, $\exists B_r(a) \subset \Omega$: there, $f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$.

(5') same but $\forall B_r(a) \subset \Omega$.

Def: $\forall f \in \mathcal{O}(\mathbb{C})$ is called an entire func.

Thm (Liouville): if f - entire and bounded $\Rightarrow f = $ const.

Remark: no functions like $f(x) = \sin x$ $(x \in \mathbb{R})$, $f(x) = \frac{1}{1+x^2}$.

Proof: Apply the p.s. expansion thm for $\Omega = \mathbb{C}$.

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in B_r(0). \qquad \forall n \geq 0, \ \forall r > 0,$$

$$|c_n| \leq \frac{M(r)}{r^n} \leq \frac{C}{r^n} \text{ (where } |f| \leq C \text{ in } \mathbb{C})$$

Fix $n > 0$, let $r \to +\infty$. $|c_n| \leq 0 \Rightarrow c_n = 0$.

$\Rightarrow f(z) = c_0 = $ const. $(\forall z \in B_r(a) \Rightarrow \forall z \in \mathbb{C})$.

Reminder: for $E \subset \mathbb{C}$, a point $a \in \mathbb{C}$ is called an

accumulation pt. for E, if $\exists a_n \in E$, $a_n \neq a$, s.t. $a_n \xrightarrow{n \to \infty} a$.

$(\Leftrightarrow \exists \delta > 0, \ \exists b \in E, \ b \neq a, \ b \in B_\delta(a))$.

Ex 1: $E = \mathbb{Z} \subset \mathbb{C} \Rightarrow$ no accum. pts. in $\mathbb{C}$.

Ex 2: $E = \mathbb{R} \subset \mathbb{C} \Rightarrow \mathbb{R}$ is the set of accum. pts.

Ex 3: $E = B_r(a) \subset \Rightarrow \overline{B_r(a)}$ - the set of acc. pts.

Thm (Uniqueness Thm)

can be $\{a\}$, a is an accum. pt.

Let $f \in \mathcal{O}(\Omega)$, $E \subset \Omega$ - a subset with at least one accum. pt. $a \in \Omega$. Then if $f|_E \equiv 0 \Rightarrow f \equiv 0$ in Ω.

Corollary: $f, g \in \mathcal{O}(\Omega)$. $E \subset \Omega$ - a subset with accum. pt. $a \in \Omega$. and $f \equiv g|_E \Rightarrow f \equiv g$ in Ω.

Proof of coro: Apply the uniqueness thm to $h = f - g$.

Proof of thm: take $a_n \in E$, $a_n \to a$. $a_n \neq a$. we have $f(a_n) = 0$.

$\exists \overline{B_r(a)}$. there, $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$. - unif. converg.

Plug in $z = a_n$: $0 = f(a_n) = \sum_{k=0}^{\infty} c_k (a_n - a)^k$.

take $\lim_{n\to\infty}$: $0 = \lim_{n\to\infty} \sum_{k=0}^{\infty} c_k (a_n - a)^k$

$$= \{\text{unif. converg.}\} = \sum_{k=0}^{\infty} \lim_{n\to\infty} c_k (a_n - a)^k$$

$$= c_0.$$

So, $f(z) = c_1 (z-a) + c_2 (z-a)^2 + \cdots\cdots$

Plug in $z = a_n$: $0 = f(a_n) = c_1 (a_n - a) + c_2 (a_n - a)^2 + \cdots$

$(a_n \neq a!)$ $\quad 0 = c_1 + c_2 (a_n - a) + c_3 (a_n - a)^2 + \cdots$

take $\lim_{n\to\infty}$: $0 = c_1$. and so on $\cdots\cdots$

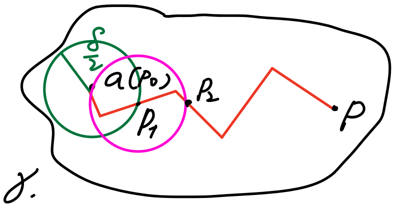
$c_k = 0$, $\forall k \geq 0$. $\Rightarrow f(z) \equiv 0$ in $B_r(a)$.

So, we have proved: $\forall$ accum. pt. $b \in \Omega$ of the zero-set of f. $\forall B_R(b) \subset \Omega$, $f \equiv 0$ in $B_R(b)$.

Now, we shall prove: $f(p) = 0$, $\forall p \in \Omega$.

Let's connect a, p by a broken line $\gamma \subset \Omega$. $(|\gamma| < +\infty)$

Fix $\delta := \text{dist}(\gamma, \partial\Omega) > 0$.



This guarantees $\overline{B_{\frac{\delta}{2}}(z)} \subset \Omega$, $\forall z \in \gamma$.

$\partial B_{\frac{\delta}{2}}(a)$ intersects γ at some P_1 (set $a = P_0$).

P_1 - accum. pt. of the zero-set of f.

$\Rightarrow$ By the above, $f \equiv 0$ in $B_{\frac{\delta}{2}}(P_1)$.

So, we have moved along γ with the distance $\geq \frac{\delta}{2}$.

Similarly, obtain P_2 and again distance $\geq \frac{\delta}{2}$.

...

So, in finitely many steps (thanks to $|\gamma| < +\infty$),

we arrive to some $\overline{B_{\frac{\delta}{2}}(P_m)} \ni P$, where $f \equiv 0$

$\Rightarrow f(p) = 0$. $\Rightarrow f \equiv 0$ in Ω. $\square$

<u>Applications</u>

Use E = curve, open set.

* Trigonometric f-las.

e.g. $\sin 2z = 2 \sin z \cos z$ - why holds for $z \in \mathbb{C}$?

holds for $z \in E = \{z \in \mathbb{R}\} \subset \mathbb{C}$.

By the uniquess thm or the corollary: holds $\forall z \in \mathbb{C}$!

e.g. $\sin(z+w) = \sin z \cos w + \cos z \sin w$. $z, w \in \mathbb{C}$. why?

(i) Case $\underset{\text{fixed}}{z \in \mathbb{R}}$, $\underset{\text{variable}}{w \in \mathbb{C}}$.

$f(w) = g(w)$ holds for $w \in \mathbb{R}$

$\Rightarrow f(w) = g(w)$ for $w \in \mathbb{C}$.

(ii) Case $z, w \in \mathbb{C}$

fix $w \in \mathbb{C}$, $\lambda(\underset{\text{variable}}{z}) = \mu(z)$.

by (i), $\lambda(z) = \mu(z)$ holds for $z \in \mathbb{R} \subset \mathbb{C}$.

$\Rightarrow \lambda(z) = \mu(z)$ for $z \in \mathbb{C}$.

Thus, LHS = RHS.

* $e^{z+w} = e^z \cdot e^w$, etc.

* Taylor expansion: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ why holds for $z \in \mathbb{C}$?

Because it holds for $z \in \mathbb{R}$!

Zeroes of a hol. func.

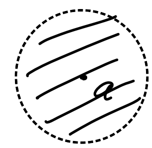
Def: Let $f \in \mathcal{O}(\Omega)$, $f \not\equiv 0$. Then $\forall a \in \Omega$ with $f(a) = 0$ is called a zero of f.

__Main fact:__ By uniquenes thm, all the zeroes are __isolated__.

i.e. $\forall\, a$ - zero of f, $\exists\, B_\delta(a)$: $f(z) \neq a$ for $z \neq a$.

So, take a zero pt. a for $f(z)$. fix such $B_\delta(a)$.

Power series expansion:



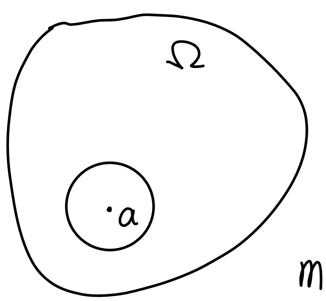
$$f(z) = \sum_{n=1}^{\infty} c_n (z-a)^n \quad (c_0 = f(a) = 0).$$

Take min. m : $c_m \neq 0 \Rightarrow f(z) = \sum_{n=m}^{\infty} c_n (z-a)^n$.

__Def__: the integer m is called the order of zero for $f(z)$ at a. noted as $m = \mathrm{ord}_a f$.

__Prop.__ (from the Taylor f-las)

$$m = \mathrm{ord}_a f \iff f(a) = f(a) = \cdots = f^{(m-1)}(a) = 0, \text{ but } f^{(m)}(a) \neq 0.$$



$f \in \mathcal{O}(\Omega)$ $\left(\Rightarrow f \in \mathcal{O}(B_r(a))\right)$

a - isolated zero of f.

$m = \mathrm{ord}_a f$.

$$f(z) = c_m (z-a)^m + c_{m+1}(z-a)^{m+1} + \cdots\cdots, \quad c_m \neq 0.$$

__Equil. def ①:__ $f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$, while $f^{(m)}(a) \neq 0$.

__Equil. def ②:__ $f(z) = c_m (z-a)^m + c_{m+1}(z-a)^{m+1} + \cdots\cdots$

$$= (z-a)^m \left(C_m + C_{m+1}(z-a) + \cdots\cdots \right)$$

$$= \boxed{(z-a)^m \cdot g(z)} \quad \text{where } g \in \mathcal{O}(B_r(a)),\ g(a) \neq 0.$$

local factorization at a zero pt.

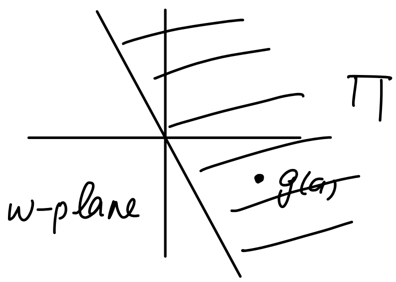
<u>Equiv. def ③</u>: Let $m = \mathrm{ord}_a f$. Choose a factor.

$$f(z) = (z-a)^m g(z),\ g(a) \neq 0.$$

By continuity, we may shrink $B_r(a) \longrightarrow B_\delta(a)$.

$g(B_\delta(a)) \subset \Pi$, where Π — half plane, $\partial \Pi \ni 0$.

$\Pi \ni g(a) \neq 0$.



In Π, $\exists$ a contin. branch of Arg w

$\Rightarrow \exists$ hol. branch of $\mathrm{Ln}\, w = \ln|w| + i \mathrm{Arg}\, w$.

$\Rightarrow \exists$ hol. branch of $\sqrt[m]{w} := e^{\frac{1}{m}\mathrm{Ln}\, w} = \varphi(w)$

$\left((\varphi(w))^m = e^{\mathrm{Ln}\, w} = w \right) \Rightarrow$ we may consider $\tilde{h}(z) := \varphi(g(z))$

$\Rightarrow \boxed{\tilde{h}^m = g}$; finally set $h := (z-a)\tilde{h}$. $\in \mathcal{O}(B_\delta(a))$.

$f = h^m$, $h \in \mathcal{O}(B_\delta(a))$, $h(a) = 0$ but $h'(a) \neq 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\tilde{h}(a)}$ (since $\tilde{h}(a) \neq 0$)

$\mathrm{ord}_a h = 1$.

Conversely, if $f = h^m$ like before,

$\Rightarrow f = (p_1(z-a) + p_2(z-a)^2 + \cdots\cdots)^m$

$= |\text{multiply}| = p_1^m (z-a)^m + \cdots\cdots$. $\quad p_1 \neq 0 \Rightarrow \text{ord}_a f = m$.

<u>Thm. (Criterion of local injectivity).</u>

Let $f \in \mathcal{O}(\Omega)$, $a \in \Omega$, then f is locally injective at a (that is, $\exists B_\delta(a)$: $f|_{B_\delta(a)}$ is inj.) $\Leftrightarrow f'(a) \neq 0$.

<u>Rk</u>: fails in Real Analysis.

E.g. $f(x) = x^3$. $f'(0) = 0$. but $f(x) = x^3$ is bij. $(\mathbb{R} \to \mathbb{R})$

<u>Proof</u>: if f is const. trivial.

Suppose $f \neq$ const. $f(z) - f(a)$ has zero of order $m \geq 1$ at $a \Rightarrow f(z) = f(a) + h(z)^m$, $h \in \mathcal{O}(B_\delta(a))$, $h(a) = 0$, $h'(a) \neq 0$

Now, if $m = 1$ $(\Leftrightarrow f'(a) = h'(a) \neq 0)$.

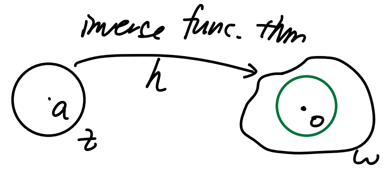
then f is locally inj. by the (real) inverse func thm in $B_\delta(a)$.

$(J_f(a) = |f'(a)|^2 \neq 0)$.

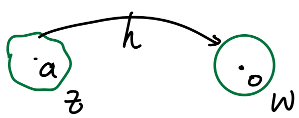
If $m \geq 2$ $(\Leftrightarrow f'(a) = 0)$. $f'(a) = m h'(a) \cdot h(a)^{m-1} = 0$.

we shall prove that f is not inj.

$h'(a) \neq 0 \Rightarrow h$ is inj. in $B_\delta(a) \Rightarrow$



$\Rightarrow$



w^m is not inj.

all $w_j^m = b$, $\forall j$. $\qquad w = h(z)$.

Take $z_j' \in B_\delta(a)$. $h(z_j') = w_j$.

$f(z_j) = f(a) + b$. $\forall j = 1, \cdots, m$.

$\Rightarrow f$ is not locally inj. (and choosing a smaller δ will give the same picture), as desired. $\square$.

Corollary: a map $f: \underset{\mathbb{C}}{\underset{\cap}{\Omega_1}} \xrightarrow{\text{onto}} \underset{\mathbb{C}}{\underset{\cap}{\Omega_2}}$ is conformal

$\Leftrightarrow$ (1) $f \in \mathcal{O}(\Omega_1)$ (2) $f' \neq 0$ in Ω_1.

(3) f is bijective.

Computing Order.

1) $f(z) = \sin z$. $a = 0$.

$$= z - \frac{z^3}{3!} + \cdots \Rightarrow \operatorname{ord}_a f = 1.$$

2) $f(z) = \tan z - z$, $a = 0$.

$f(0) = 0$. $f'(0) = \left(\frac{1}{\cos^2 z} - 1\right)\Big|_{z=0} = 0$.

$f''(0) = \frac{2\sin z}{\cos^3 z}\Big|_{z=0} = 0$. $f'''(0) = \left(2\cos z \frac{1}{\cos^3 z} + \sin z(\cdots)\right)\Big|_{z=0} = 2 \neq 0$.

$\Rightarrow \operatorname{ord}_a f = 3$.

Maximum modulus principle.

Thm: Let $f \in \mathcal{O}(\Omega)$; $a \in \Omega$; $\max\limits_{B_\delta(a)} |f| = |f(a)|$. $B_\delta(a) \subset \Omega$.

(f attains its local max. mod. at $z = a \in \Omega$)

Then $f = \text{Const}$!

Rk: Real Analysis: $y = 1 - x^2$. $a = 0$.

Proof: Assume $f \neq$ const. Then by uniqueness thm, $f \neq$ const in

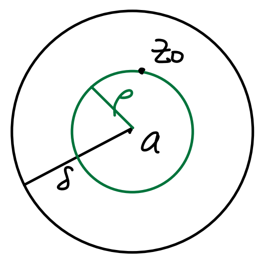
$B_\delta(a)$. $\Rightarrow$ take $m := \mathrm{ord}_a(f(z) - f(a)) \in \mathbb{N}$.

We have in $B_\delta(a)$:

$$f(z) = f(a) + C_m (z-a)^m + C_{m+1}(z-a)^{m+1} + \cdots, \quad C_m \neq 0.$$

we have: $\max\limits_{|z-a|<\delta} |f| = |f(a)|$. $|f(a)| \neq 0$ since $f \neq$ const.

$$f(z) = f(a) + (z-a)^m (C_m + \underbrace{C_{m+1}(z-a) + \cdots\cdots}_{=g(z)})$$



$\lim\limits_{z \to a} g(z) = g(a) = 0$. $\Rightarrow \exists\, 0 < \rho < \delta$: $|g(z)| < \frac{|C_m|}{2}$

for $|z-a| \leq \rho$.

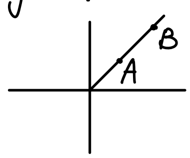
Then choose z_0: (1) $|z_0 - a| = \rho$.

(2) $\mathrm{Arg}\left[(z_0-a)^m \cdot C_m\right] = \mathrm{Arg}\, f(a)$.

($\Leftrightarrow$ $m \cdot \mathrm{Arg}(z_0 - a) + \mathrm{Arg}\, C_m = \mathrm{Arg}\, f(a)$).

Observe: $A, B \in \mathbb{C}$, $A, B \neq 0$, $\mathrm{Arg}\, A = \mathrm{Arg}\, B$.

Then $|A+B| = |A| + |B|$.



$\Rightarrow$ now, $|f(z_0)| \geq |f(a) + C_m(z_0-a)^m| - |z_0-a|^m |g(z_0)|$

$= |f(a)| + |C_m| \cdot |z_0-a|^m - |z_0-a|^m \cdot |g(z_0)|$

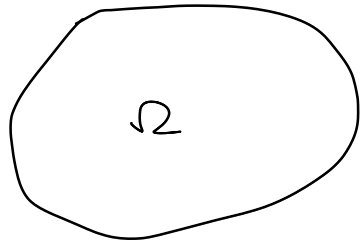
$> |f(a)| + |C_m| \cdot |z_0-a|^m - |z_0-a|^m \cdot \frac{|C_m|}{2}$.

$> |f(a)|$. Contradiction $\square$.

<u>Corollary</u>: Let $\Omega \subset \mathbb{C}$ – bounded dom

$f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$

Then $\max\limits_{\overline{\Omega}} |f| = \max\limits_{\partial\Omega} |f|$.



So, $\max |f|$ in $\overline{\Omega}$ is attained on $\partial\Omega$.

<u>Proof</u>: if $f = c =$ const in $\Omega \Rightarrow f \equiv c$ in $\overline{\Omega}$. trivial

if $f \not\equiv$ const. $\Rightarrow$ take $a \in \overline{\Omega}$: $\max\limits_{\overline{\Omega}} |f| = |f(a)|$.

then $a \notin \Omega$ by the Max Princ. $\Rightarrow a \in \partial\Omega$.

$\Rightarrow \max\limits_{\overline{\Omega}} |f| = \max\limits_{\partial\Omega} |f|$. $\square$.

<u>Ex.</u> Let $f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$, Ω – bdd. dom. and $f|_{\partial\Omega} = 0$.

then $f \equiv 0$ in $\overline{\Omega}$. (since $\max\limits_{\overline{\Omega}} |f| = 0$).

<u>Ex.</u> Ω – bdd dom. $f_n \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$, s.t. $f_n \overset{\partial\Omega}{\rightrightarrows} \varphi$,

then $f_n \overset{\overline{\Omega}}{\rightrightarrows} f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$.

<u>Proof</u>: Apply Cauchy Criterion, $\max\limits_{\overline{\Omega}} |f_n - f_m| = \max\limits_{\partial\Omega} |f_n - f_m| \xrightarrow{n,m\to\infty} 0$.

$\Rightarrow f_n \overset{\overline{\Omega}}{\rightrightarrows} f$, $f \in C(\overline{\Omega})$.

$f \in \mathcal{O}(\Omega)$ by Weierstrass Thm. $\square$.

<u>Ex.</u> (Fundemental Thm of Algebra)

$\forall$ poly. $p(z) \not\equiv$ const, p has a cx root.

Proof: Assume by contradiction. $p(z) \neq 0$, $\forall z \in \mathbb{C}$.

Set $f(z) := \dfrac{1}{P(z)} \in \mathcal{O}(\mathbb{C})$.

$$\max_{|z|\le R} |f| \underset{\text{max princ.}}{=} \max_{|z|=R} |f| \xrightarrow{R \to +\infty} 0 \quad \text{because}$$

$$|p(z)|\Big|_{|z|=R} = |a_0 + a_1 z + \cdots a_n z^n|\Big|_{|z|=R} = R^n \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n \right|\Bigg|_{|z|=R}$$

$$\longrightarrow +\infty \quad (R \to +\infty)$$

$\Rightarrow |f| \xrightarrow{R \to +\infty} 0 \Big|_{|z|=R}$. $\Rightarrow f \equiv 0$. — contrad. $\lightning$. $\square$.

Rk: can also be proved using Liouville's Thm.

Laurent Series

Def: a func. series $\sum_{n=-\infty}^{+\infty} c_n (z-a)^n$ is called a Laurent Series with center a.

Convergence: by def, means the convergence of both

$$\underbrace{\sum_{n=0}^{+\infty} c_n (z-a)^n}_{\text{regular part.}} \quad \text{and} \quad \underbrace{\sum_{n=-1}^{-\infty} c_n (z-a)^n}_{\text{principle part.}}.$$

regular part: $\begin{cases} \text{conv. only at } a. \\ \text{disc of conv. } |z-a| < R \text{. } R \in (0, +\infty]. \end{cases}$

principle part: Set $w := \dfrac{1}{z-a}$.

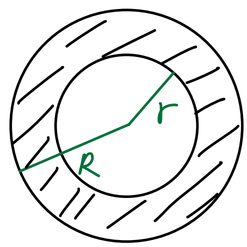
$\sum_{k=1}^{\infty} C_{-k} w^k \Rightarrow$ reduced to Taylor series with center 0

Dom of conv. $\begin{cases} w=0 \Leftrightarrow z \in \phi. \\ |w| < \varepsilon \Leftrightarrow |z-a| > \frac{1}{\varepsilon}. \end{cases}$ $\varepsilon \in (0, +\infty]$.

exterior of a disc. or a pt.

<u>Conclusion</u>: the dom. of conv. of a Laurent series (i.e. the interior of the conv. set) is either:

$$\begin{cases} \phi \\ r < |z-a| < R \text{ — annulus.} \\ |z-a| < R \text{ (no principle part).} \end{cases} \qquad 0 \le r < R \le +\infty$$



<u>Ex.</u> $f(z) = \frac{1}{z^2-1}$. find all possible Tayl/Laur. expansion with center 0.

Sol: <u>Case 1</u>: $|z| < 1 \Rightarrow f(z) = -\frac{1}{1-z^2} = -\sum_{n=0}^{\infty} z^{2n}$. Tayl. series.

Geom. progr.

$f \in \mathcal{O} \in \mathbb{C} \setminus \{\pm 1\}$.

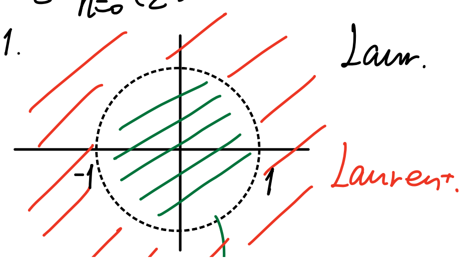
<u>Case 2</u>: $|z| > 1$.

$$f(z) = \frac{1}{z^2} \frac{1}{1-\frac{1}{z^2}} = \frac{1}{z^2} \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n = z^{-2} + z^{-4} + z^{-6} + \cdots\cdots$$

Geom. progr.

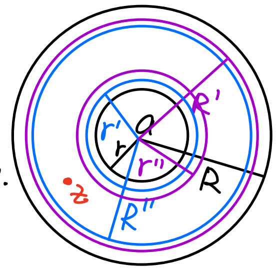
$|w| < 1$.

Laur. series.



Thm Let $f \in \mathcal{O}(\Omega)$, where   Taylor.

$\Omega = \{ r < |z-a| < R \}$, $0 \le r < R \le +\infty$.



Then, f admits in Ω a Laurent expan.

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n, \quad \forall z \in \Omega.$$

Furthermore,

(1) The series conv. (abs) unif. on compacts in Ω.

(2) $c_n = \displaystyle\int_{|z-a|=\rho} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi$. $\forall\, r < \rho < R$. — (Cauchy f-la.)

(the integrals are indep. of ρ). in partc, the Laurent expan. is unique.

(3) Cauchy integral holds:

$$|c_n| \le \frac{M(\rho)}{\rho^n}, \quad M(\rho) := \max_{|\xi-a|=\rho} |f|. \quad \forall \rho \in (r, R).$$

Proof: Shrink twice the orig. annulus by fixing

$$r < r' < r'' < R'' < R' < R$$

Consider z: $r'' \le |z-a| \le R''$.

Apply CIF in $\Omega' = \{ r' < |z-a| < R' \}$

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{f(\xi)}{\xi - z} d\xi \qquad |\xi - a| = \begin{cases} R' \\ r' \end{cases}.$$

$$r'' \le |z-a| \le R'' \Rightarrow |z-\xi| \ge \underbrace{\delta}_{\min\{R'-r'',\ r''-r'\}} > 0.$$

Develop: $f(z) = \displaystyle\frac{1}{2\pi i} \int_{|\xi-a|=R'} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{|\xi-a|=r'} \frac{f(\xi)}{\xi - z} d\xi.$

$$f_1(z) \qquad\qquad f_2(z).$$

<u>Work out $f_1(z)$:</u>

$$f_1(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=R'} \frac{f(\zeta)}{(\zeta-a)-(z-a)}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta-a|=R'} \frac{f(\zeta)}{\zeta-a}\cdot\frac{1}{1-\frac{z-a}{\zeta-a}}\,d\zeta.$$

$$= \{\text{Geom. progr.}\}$$

$$\sum_{n=0}^{+\infty}\frac{(z-a)^n}{(\zeta-a)^n}$$

$$\frac{|z-a|}{|\zeta-a|} \le \frac{r''}{R'} < 1.$$

$$= \left\{\begin{array}{l}\text{using unif. conv.}\\ \text{in both } \zeta, z \text{ due}\\ \text{to } |z-\zeta|\geq\delta\end{array}\right\} = \{\text{integrate}\} = \sum_{n=0}^{\infty} C_n(z-a)^n.$$

$$C_n := \frac{1}{2\pi i}\int_{|\zeta-a|=R'}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta.$$

<u>Work out $f_2(z)$:</u>

$$f_2(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r'}\frac{f(\zeta)\,d\zeta}{(\zeta-a)-(z-a)}$$

$$= -\frac{1}{2\pi i}\int_{|\zeta-a|=r'}\frac{f(\zeta)}{z-a}\;\frac{1}{1-\underbrace{\frac{\zeta-a}{z-a}}_{\text{mod}<1}}\,d\zeta$$

$$= \{\text{Geom. progr.}\} \qquad \frac{|z-a|}{|\zeta-a|} \geq \frac{r''}{r'} > 1.$$

$$= \left\{\begin{array}{l}\text{unif. conv.}\\ \text{due to}\\ |z-\zeta|\geq\delta\end{array}\right\} = \{\text{integrate term by term}\}$$

$$= -\sum_{k=0}^{\infty} b_k (z-a)^{-k-1}, \quad b_k = \frac{1}{2\pi i}\int_{|\zeta-a|=r'} f(\zeta)(\zeta-a)^k\,d\zeta.$$

$$= -\sum_{n=-\infty}^{-1} c_n (z-a)^n, \quad c_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r'} f(\zeta)(\zeta-a)^n\,d\zeta.$$

Now, $f = f_1 - f_2$ gives desired expan.

(1) unif. and abs. conv. : follow from above since r', r'', R', R'' are arbitrary.

($\forall$ cpt $K \subset \{r'' < |z-a| < R''\}$).

(2) Since r', R'' are arbitrary, Cauchy f-las follow from above. (or: the integrals $\int_{|z-a|=\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta$ are independent of ρ by CIT in an annulus).

Uniqueness of Laurent expan:

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n.$$

Apply $\int_{|z-a|=\rho}$. $r<\rho<R$.

$\int_{|z-a|=\rho} f(z)\,dz = 2\pi i\, c_{-1}$ So c_{-1} is unique.

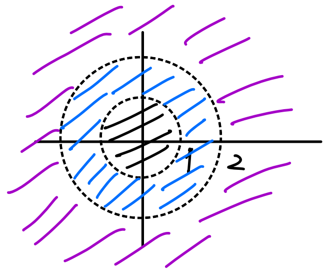
$$(z-a)f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^{n+1}$$

$\int_{|z-a|=\rho} (z-a)f(z)\,dz = 2\pi i \cdot c_{-2} \Rightarrow c_{-2}$ is unique etc. $\Rightarrow$ uniqueness.

(3) Just apply basic boundary:

$$|C_n| = \frac{1}{2\pi}\left|\int_{|\zeta-a|=\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right| \le \frac{1}{2\pi}\cdot 2\pi\rho\cdot\frac{M(\rho)}{\rho^{n+1}} = \frac{M(\rho)}{\rho^n}. \quad \square.$$

<u>Ex.</u> $f(z) = \frac{1}{(z-1)(z-2)}$ $\quad a=0.$



(i) $|z|<1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}.$$

$$= -\frac{1}{2}\frac{1}{1-\frac{z}{2}} + \frac{1}{1-z}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} C_n z^n. \qquad C_n = 1-\frac{1}{2^{n+1}}.$$

(ii) $1<|z|<2$

$$f(z) = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \sum_{n=-\infty}^{-1} z^n. \qquad C_n = \cdots\cdots$$

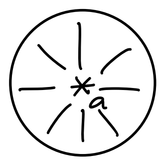
(See: difference arises from requirement for Geom. progr.)

(iii) $|z|>2$

$$f(z) = \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}} - \frac{1}{1-\frac{1}{z}}\right) = \{|z|>2\}$$

$$= \sum_{n=-\infty}^{-1} z^n 2^{-n-1} - \sum_{n=-\infty}^{-1} z^n.$$

<u>Isolated singularities of hol. func.</u>



<u>Def</u>: Let $f \in \mathcal{O}(B_\delta^*(a))$, $B_\delta^*(a) = B_\delta(a) \setminus \{a\}$.

Then we say $z=a$ is an isolated singularity for f.
(a itself is a singularity for f).

Eg. $f(z) = \frac{1}{z}$. $a=0$.

By the Laurent. expan. Thm: $f(z) = \sum_{n=-\infty}^{+\infty} C_n (z-a)^n$. ($C_n$ - unique)

Case 1: all $C_n = 0$, $n<0$. (zero principle part.)

Then a is called a removable singularity.

Case 2: $C_n = 0$, $n < -m$, $m \in \mathbb{N}$. $C_{-m} \neq 0$. (principle part. is finite)

Then a is called a pole of order $=m$. rational func.

$m := \mathrm{ord}_a f$. $\begin{cases} \mathrm{ord} > 0: \text{iso. zero} \\ \mathrm{ord} < 0: \text{pole.} \\ \mathrm{ord} = 0: f \in \mathcal{O}(B_\delta(a)).\ f(a) \neq 0. \end{cases}$

$$f(z) = \frac{C_{-m}}{(z-a)^m} + \frac{C_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{C_{-1}}{z-a} + \sum_{n=0}^{\infty} (z-a)^n.$$

$C_{-m} \neq 0$. $f(z) = \frac{g(z)}{(z-a)^m}$, $g \in \mathcal{O}(B_\delta(m))$, $g(a) \neq 0$.

$$g(z) = C_{-m} + C_{-m+1}(z-a) + \cdots + C_{-1}(z-a)^{m-1} + \sum_{n=0}^{\infty} (z-a)^{n+m} \cdot C_n.$$

The series for f conv. $\forall z \in B_\delta^*(a) \Rightarrow$

By Cauchy-Adamar Thm, $B_\delta(a) \subset \{\text{disc of conv.}\}$

$\Rightarrow g \in \mathcal{O}(B_\delta(a))$.

Case 3: the principle part is infinite (not 1, not 2).

then a is called an essential singularity.

Ex. (1) $f(z) = \frac{\sin z}{z}$, $a=0$.

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{4!} - \cdots$$

no neg. terms $\Rightarrow$ removable

(2) $f(z) = z^2 e^{\frac{1}{z}}$, $a=0$.

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots\right) = z^2 + z + \frac{1}{2!} + \frac{1}{3!\,z} + \cdots$$ (∞ terms) $\Rightarrow$ essen.

(3) $f(z) = \cot z - \frac{1}{z}$, $a=0$.

$$f(z) = \frac{z\cos z - \sin z}{z \cdot \sin z} = \frac{z\left(1 - \frac{z^2}{2!} + \cdots\right) - \left(z - \frac{z^3}{3!} + \cdots\right)}{z\left(z - \frac{z^3}{3!} + \cdots\right)}$$

$$= \frac{-\frac{1}{3}z^3 + \cdots}{z^2\left(1 - \frac{z}{3!} + \cdots\right)} = z \cdot \text{hol}$$ $\Rightarrow$ removable.

(4) $f(z) = \cot z + \frac{1}{z}$.

$$f(z) = \frac{2z + O(z^2)}{z^2(1 - \cdots)} = \frac{1}{z} \cdot g(z)$$

hol near $z=0$ as ratio.

$g(0) = 2 \neq 0 \Rightarrow z=0$ — pole of order 1.

(5) $f(z) = \dfrac{1}{\cos\frac{1}{z}}$. $a=0$.

$\cos\frac{1}{z}$ vanishes for $z = \dfrac{1}{\frac{\pi}{2} + k\pi} \xrightarrow{k\to\infty} 0$

$\Rightarrow$ $z=0$ is not an iso. sing. !

景墙面，注意保护，严禁书写！

Theorem (classif of isol. sing-s).

Let $z=a$ be an isol. sing. for $f(z)$. Then:

(1) $z=a$ is removable $\Leftrightarrow$ f extends hol. to $z=a$

$\Leftrightarrow \exists \lim\limits_{z\to a} f(z) \Leftrightarrow f$ is bounded in some $B^*_\delta(a)$ (Riemann Thm).

$\boxed{f(x)=\sin\frac{1}{x};\ \text{bdd, but } \nexists \lim\limits_{x\to 0} \text{ in Real Analysis}}$

(2) $z=a$ a pole of order $m \Leftrightarrow f(z)=\dfrac{g(z)}{(z-a)^m}$, $g \in \mathcal{O}(B_\delta(a))$, $g(a)\neq 0$

$\Leftrightarrow \lim\limits_{z\to a} f(z)=\infty$

(3) $z=a$ is an essential sing $\Leftrightarrow \nexists \lim\limits_{z\to a} f(z)$ in $\overline{\mathbb{C}}$

$\Leftrightarrow \forall A\in\overline{\mathbb{C}},\ \exists z_k\to a:\ f(z_k)\longrightarrow A$ (!)

$\left(\Leftrightarrow \overline{f(B^*_\delta(a))}=\overline{\mathbb{C}}\right)$.

Sokhotsky-Weierstrass Thm.

<u>Proof</u>: (1) If $z=a$ is removable $\Rightarrow f(z)=\sum\limits_{n=0}^{+\infty} c_n(z-a)^n$. $\forall z\in B^*_\delta(a)$.

$\Rightarrow$ By Cauchy-Adamar Thm, $B_\delta(a)\subset\{$disc. of conv$\}$.

$\Rightarrow$ the f-la $\sum\limits_{n=0}^{+\infty} c_n(z-a)^n$ gives a hol. exten. of

f to $B_\delta(a)$ $\quad (f(a):=c_0)$.

$\exists$ lim, boundness – obvious. then

To prove (1), it remains to prove

f bounded $\Rightarrow a$ is a remov. sing

$f(z)=\sum\limits_{n=-\infty}^{+\infty} c_n(z-a)^n$. $\quad c_n=\dfrac{1}{2\pi i}\displaystyle\int_{|z-a|=\rho}\dfrac{f(z)}{(z-a)^{n+1}}\,dz$. $\forall\, 0<\rho<\delta$. $|c_n|\le\dfrac{M(\rho)}{\rho^n}$.

f is bdd $\Rightarrow |f|\le M_0$ – const $\Rightarrow |c_n|\le\dfrac{M_0}{\rho^n}$; now, if $n<0$.

then $\frac{1}{\rho^n} \xrightarrow{\rho\to 0} 0 \Rightarrow |C_n| \le 0. \Rightarrow C_n = 0 \quad \forall n<0.$

(2) If $z-a$ pole of ord $= m \Rightarrow f(z) = \frac{g(z)}{(z-a)^m} \Rightarrow \lim_{z\to a} f(z) = \frac{g(a)^{\neq 0}}{0} = \infty.$

Conversely, let $\lim_{z\to a} f(z) = \infty$. Then in some $B^*_\delta(a)$: $f(z)\neq 0$

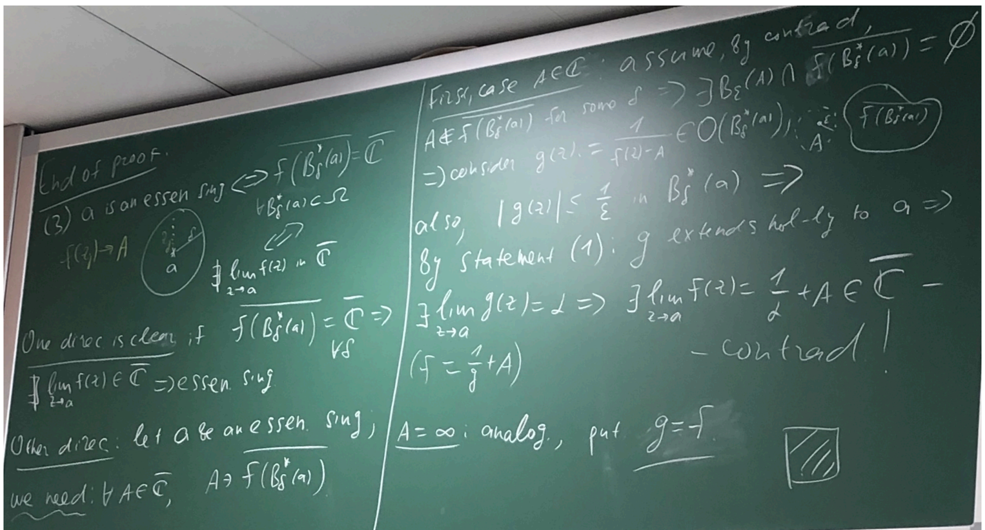
$\Rightarrow$ consider $h(z) := \frac{1}{f(z)} \in \mathcal{O}(B^*_\delta(a)) \Rightarrow z=a$ is an iso. sing. for $h(z)$.

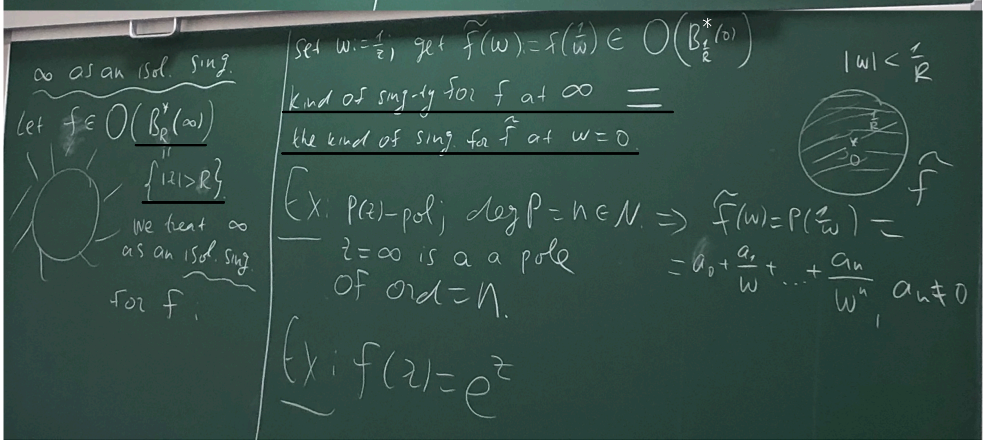
and $\lim_{z\to a} h(z) = \frac{1}{\infty} = 0 \Rightarrow z=a$ – remov. sing. for h.

$\Rightarrow h$ extends hol to $\mathcal{O}(B_\delta(a)) \Rightarrow h = (z-a)^m \cdot \lambda(a)$, $\lambda(a)\neq 0$.

(let $h(a)=0$).

$f = \frac{g(z)}{(z-a)^m}$. $g := \frac{1}{\lambda} \in \mathcal{O}(B_\delta(a))$.





$z=\infty$: essen. sing.

$\left(f(w)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{w^n}\right)$

Ex: $f(z)=\frac{1+z}{1+z^2}$

$\lim_{z\to\infty} f(z)=0 \Rightarrow z=\infty$ is removable

$\boxed{\text{The class-n of isol. sing. applies to } \infty \text{ too}}$

Laurent series at ∞: the Laur. series for $f(w)$

substituted $w=\frac{1}{z}$.

$P(z)=a_0+a_1z+\dots a_nz^n$

$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ $(\forall z\in\mathbb{C})$

overall, for $f\in\mathcal{O}(\mathbb{C})$ (entire func) Laur at ∞ = = Taylor at 0, because of uniq of Laur. expan!

$\frac{1+z}{1+z^2}=\frac{\frac{1}{z}+\frac{1}{z^2}}{\frac{1}{z^2}+1}=$

$=\left(\frac{1}{z}+\frac{1}{z^2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}}=\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}+\dots$ (!).

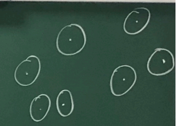
positive powers of z: principal part

nonpositive powers: regular part

$f=e^z$: $\left[\begin{array}{ll} 1 & \text{reg. part} \\ \sum_{n=1}^{\infty}\frac{z^n}{n!} & \text{princ. part}\end{array}\right.$

Meromorphic functions

Def: Let Ω-dom. in $\overline{\mathbb{C}}$! then a func. $f(z)$ is called merom. in Ω, if it's hol in $\Omega\setminus E$, E - a discr. set in Ω (discrete set = all its points are isolated) and $\forall a\in E$ is a remov. sing. for f or a pole.



This means we can extend f to whole Ω by allowing values in $\overline{\mathbb{C}}$.

$f(a):=\lim_{z\to a}f(z)$

$\boxed{f\in Mer(\Omega)}$

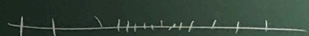
Ex: $f(z)=\frac{1}{\sin\frac{1}{z}}$

$E=\{0,\frac{1}{\pi k}\}_{k\in\mathbb{Z}}$, 0 - not isol.

$\Rightarrow f\notin Mer(\mathbb{C})$

But $f\in Mer(\mathbb{C}\setminus\{0\})$

$f(\frac{1}{\pi k})=\infty$



0

$f\in Mer(\overline{\mathbb{C}}\setminus\{0\})$!

$f(\infty)=\infty$

Ex: $f(z)=e^z\in Mer(\mathbb{C})$

$f\notin Mer(\overline{\mathbb{C}})$

$E=\{\infty\}$

Ex: $\forall$ pol $P(z)\in Mer(\overline{\mathbb{C}})$

$\forall$ rat. $R(z)\in Mer(\overline{\mathbb{C}})$

$E=\{b_1,\dots,b_m,\infty\}$

$\exists$lim everywhere

Thm $\forall f\in Mer(\overline{\mathbb{C}})$ is actually rational

Proof: $E=\{b_1,b_2,\dots,b_m,\infty\}$ (can be $m=0$)

because $\overline{\mathbb{C}}$ is compact $\Rightarrow$ every ∞ seq has an accum pt

So, E is finite ✓

$\forall b_j$, take R_j = the princ. part of the Laur. ser of f at b_j)

($R_j=0$ if b_j is remov.)

b_j - a pole $\Rightarrow R_j$ - rational

finite Laur ser

∞ - pole $\Rightarrow$ its princ part P - polyn

Now, set $g:=f-\sum_{j=1}^{m}R_j-P$

(so, we substruct all the princ. parts!)

Now, $\forall a\in\overline{\mathbb{C}}$, $\exists\lim_{z\to a} g(z)$ finite $\Rightarrow g$ extends to $g\in\mathcal{O}(\overline{\mathbb{C}})$

$\Rightarrow g$ - a bounded entire func. $\Rightarrow g=C$ = const

So, $f=\sum_{j=1}^{m}R_j+P+C$ - ration ✓. ▨

Residues 留数.

<u>Def</u>: Let a - an isol sing for $f \in \mathcal{O}(B^*_\delta(a))$. Let $f = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n$ be the Laur. expan of f in $B^*_\delta(a)$. Then the coef c_{-1} in this expan is called the residue of f at a.

$$\boxed{\operatorname{res}_a f} = c_{-1} = \frac{1}{2\pi i} \int_{|\xi - a| = \rho} f(\xi)\, d\xi.$$

By Cauchy f-las: $\boxed{c_{-1} = \frac{1}{2\pi i} \int_{|\xi-a|=\rho} f(\xi)\, d\xi}$; $0 < \rho < \delta$ $\forall$

<u>Ex</u>. a - remov. sing $\Rightarrow$ $\boxed{\operatorname{res}_a f = 0}$

<u>Ex</u>. $f = e^{\frac{1}{z}}$, $a = 0$ $\quad \operatorname{res}_0 e^{\frac{1}{z}} = 1$

$f = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \dots$

<u>Ex</u>: Let a - a pole of ord $= m$;

How to compute $\operatorname{res}_a f$?

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \frac{c_{-m+1}}{(z-a)^{m-1}} + \dots + \frac{c_{-1}}{z-a} + c_0 + \dots$$

$c_{-1}(z-a)^{m-1}$

<u>$m = 1$</u>: $\operatorname{res}_a f = c_{-1} = \lim_{z \to a} \left[f(z)(z-a) \right]$

<u>$m \geq 2$</u>: $\operatorname{res}_a f = \frac{1}{(m-1)!} \lim_{z \to a} \left[\left(f(z)(z-a)^m \right)^{(m-1)} \right]$ derivative

<u>Ex</u>: $\operatorname{res}_0 \frac{\operatorname{cotan} z}{z}$

$f(z) = \frac{\cos z}{z \sin z}$; $\operatorname{ord}_0 f = 0 - 1 - 1 = -2$.

$$\frac{1 - \frac{z^2}{2} + \dots}{z\left(z - \frac{z^3}{3!} + \dots\right)} = \frac{1}{z^2} \cdot \frac{1 + \dots}{1 + \dots}$$

ord of pole $= 2$.

$$\operatorname{res}_0 f = \frac{1}{1!} \lim_{z \to 0} (z \operatorname{cotan} z)' = \lim_{z \to 0} \left(\frac{\cos z}{\sin z} - \frac{z}{\sin^2 z} \right) =$$

$$= \lim_{z \to 0} \frac{\sin z \cos z - z}{\sin^2 z} = \lim_{z \to 0} \frac{\left(z - \frac{z^3}{6} + \dots\right)\left(1 - \frac{z^2}{2} + \dots\right) - z}{z^2 \left(1 - \frac{z^2}{6} + \dots\right)^2} =$$

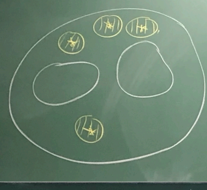
$$= \lim_{z \to 0} \frac{O(z^3)}{z^2 (1 + \dots)} = 0$$

<u>Notation</u>: $f \in Mer(\overline{\Omega})$ if $f \in Mer(D)$, $D \supset \overline{\Omega}$.

$\overline{\Omega}$ - closed $\Rightarrow$ if Ω is bounded, $\overline{\Omega}$ - cpt $\Rightarrow$ only fin many sing in $\overline{\Omega}$.

otherwise the infinite sequence must have an accumulation pt.

Residue Thm: Let Ω - admissible dom, $\partial\Omega$ - SBP. Let $f \in Mer(\overline{\Omega})$, not having sing on $\partial\Omega$.



<u>Then</u>: $\boxed{\int_{\partial\Omega} f(z)\, dz = 2\pi i \left(\sum_{j=1}^{m} \operatorname{res}_{a_j} f \right)}$, where $E = \{a_1, \dots, a_m\}$ is the sing locus of f - the set of sing-s.

<u>Proof</u>. take $B_{\varepsilon_j}(a_j) \subset \Omega$ and having $\emptyset$ pairwise intersec-s $\Rightarrow$ consider $\Omega' := \Omega \setminus \bigcup_{j=1}^{m} B_{\varepsilon_j}(a_j)$ - again an adm dom; $f \in \mathcal{O}(\overline{\Omega'})$ $\Rightarrow$ By CIT:

$$\boxed{\int_{\partial\Omega'} f(z)\, dz = 0} \Rightarrow$$

$$\int_{\partial\Omega} f(z)\, dz - \sum_{j=1}^{m} \int_{\partial B^+_{\varepsilon_j}(a_j)} f(z)\, dz = 0 \Rightarrow$$

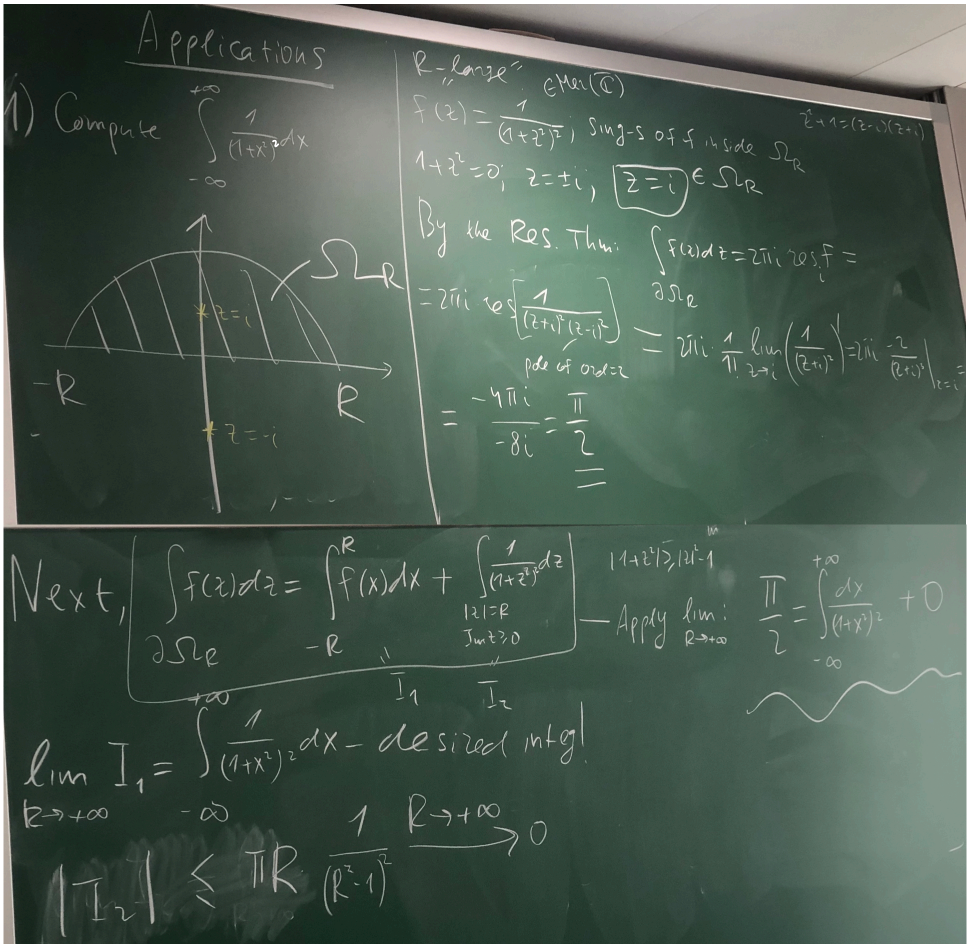
$$\int_{\partial\Omega} f(z)\, dz = \sum_{j=1}^{m} \int_{\partial B_{\varepsilon_j}(a_j)} f\, dz = \sum_{j=1}^{m} 2\pi i \operatorname{res}_{a_j} f.$$

Remark: the Residue Thm is applicable for

Ω - admis. dom with $\partial\Omega$ - SBP, $f \in \mathcal{O}(D \setminus E)$, $D \supset \overline{\Omega}$; E - discrete set, $E \cap \partial\Omega$ (But pts in E need not be poles!)

$$\int_{\partial\Omega} f\, dz = 2\pi i \sum_{a \in E \cap \Omega} \operatorname{res}_a f$$

(Same proof)



Jordan's Lemma. Let $f \in C(\overline{\Pi^+})$, $\lambda > 0$: constant.

Then $\lim\limits_{R\to+\infty} \int\limits_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} f(z)\, e^{i\lambda z} dz = 0$, as long as $\lim\limits_{\substack{z\to\infty \\ z\in\overline{\Pi^+}}} f(z) = 0$.

Proof: $I(R) = \int\limits_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} f(z)\, e^{i\lambda z} dz = \{z = Re^{it},\ 0 \le t \le \pi\} = iR \int_0^{\pi} f(Re^{it})\, e^{i\lambda Re^{it}} e^{it} dt$

$|e^w| = e^{\operatorname{Re} w}$

$|e^{i\lambda Re^{it}}| = e^{-\lambda R \sin t}$

$\Rightarrow |I(R)| \le R \cdot \int_0^{\pi} |f(Re^{it})| \cdot e^{-\lambda R \sin t} dt$

Fact: $\varphi(t) = \frac{\sin t}{t} \in C[0, \frac{\pi}{2}]$, $\varphi(t) > 0 \Rightarrow$

$\exists \min\limits_{0 \le t \le \frac{\pi}{2}} \varphi(t) = m > 0 \Rightarrow \sin t \ge mt$, $t \in [0, \frac{\pi}{2}]$.

$= R \left[\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}\right] |f(Re^{it})| \cdot e^{-\lambda R \sin t} dt.$

$$= R(I_1 + I_2).$$

$$R I_1 \le R \max_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} |f(z)| \int_0^{\frac{\pi}{2}} e^{-\lambda R m t}\, dt = R \cdot \max_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} |f| \cdot \left(-\frac{1}{\lambda R m} e^{-\lambda R m t} \Big|_{t=0}^{t=\frac{\pi}{2}} \right)$$

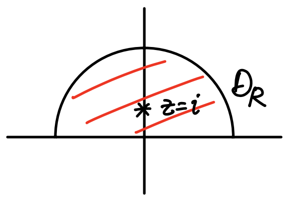
$$= \underbrace{\max_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} |f|}_{\to 0 \ (R\to\infty) \text{ as } \lim\limits_{\substack{R\to\infty \\ z \in \Gamma^+}} f(z) = 0.} \cdot \frac{1}{\lambda m} \left(1 - e^{-\lambda R m \frac{\pi}{2}} \right) \xrightarrow{R\to\infty} 0.$$

$$R I_2 \le R \max_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} |f(z)| \int_{\frac{\pi}{2}}^{\pi} e^{-\lambda R \sin t}\, dt = \{ t = \pi - s \} = R \max_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} |f(z)| \int_0^{\frac{\pi}{2}} e^{-\lambda R \sin s}\, ds$$

$$\xrightarrow[\text{analog.}]{R\to+\infty} 0$$

$$\Rightarrow I(R) \xrightarrow{R\to+\infty} 0. \qquad \square.$$

<u>Ex.</u> $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\, dx = \operatorname{Re} \underbrace{\int_{-\infty}^{+\infty} \frac{e^{ix}}{1+x^2}\, dx.}_{:= I.}$



R-large.

$$\int_{\partial D_R} \frac{e^{iz}}{1+z^2}\, dz = 2\pi i \operatorname*{res}_{i} \frac{e^{iz}}{1+z^2}$$

$$= 2\pi i \lim_{z\to i} \frac{e^{iz}}{(z+i)(z-i)} (z-i)$$

$$= \frac{\pi}{e}.$$

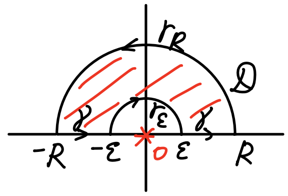
$$\int_{\partial D_R} \frac{e^{iz}}{1+z^2}\, dz = \underbrace{\int_{\substack{|z|=R \\ \operatorname{Im} z \ge 0}} f(z)\, dz}_{\xrightarrow{R\to+\infty} 0 \text{ by Jordan's lemma.}} + \int_{-R}^{R} f(x)\, dx \xrightarrow{R\to+\infty} I$$

$$\Rightarrow I = \frac{\pi}{e}. \Rightarrow \int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}.$$

<u>Ex.</u> Dirichlet Integral.

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = \lim_{\varepsilon\to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \frac{\sin x}{x}\, dx$$

$$= \underbrace{\operatorname{Im} \lim_{\varepsilon\to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \frac{e^{ix}}{x}\,dx}_{= I.}$$



$\int_{\mathcal{D}} \frac{e^{iz}}{z}\,dz = 0$ by CIT (no singularities in $\mathcal{D}$!)

$\int_{r_R} \frac{e^{iz}}{z}\,dz \xrightarrow{R\to+\infty} 0$ by Jordan's Lemma.

$\int_{\gamma} \frac{e^{ix}}{x}\,dx \xrightarrow[\varepsilon\to 0]{R\to+\infty} I.$

$$\lim_{\varepsilon\to 0}\left(\int_{\gamma_\varepsilon^-} \frac{e^{iz}}{z}\,dz \right) = \left\{ \begin{array}{l} e^{iz} = 1 + iz + \cdots \\ \frac{e^{iz}}{z} = \frac{1}{z} + \underbrace{g(z)}_{\text{hol. at } 0} \end{array} \right\} = -\lim_{\varepsilon\to 0}\left(\int_{\gamma_\varepsilon} \frac{1}{z}\,dz + \int_{\gamma_\varepsilon} g(z)\,dz \right)$$

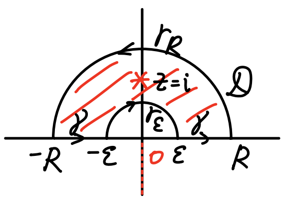
$$= \left\{ \begin{array}{c} z = \varepsilon e^{it} \\ 0 \le t \le \pi \end{array} \right\} = -\underbrace{\lim_{\varepsilon\to 0} \int_0^{\pi} \frac{1}{\varepsilon e^{it}} \varepsilon \cdot i \cdot e^{it}\,dt}_{= -\pi i.} - \underbrace{\lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} g(z)\,dz}_{= 0}$$

since $|\text{integral}| \le \pi \cdot \varepsilon \cdot C \to 0$ $(\varepsilon\to 0)$

Finally, if taking $\lim\limits_{\substack{R\to+\infty \\ \varepsilon\to 0}}$:

$$0 = 0 + I - \pi i \Rightarrow I = \pi i \Rightarrow \int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \operatorname{Im} I = \pi.$$

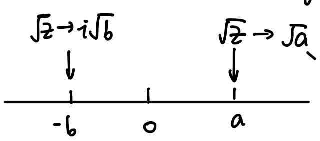
Ex. $\int_0^{+\infty} \frac{\sqrt{x}\,dx}{1+x^2}.$



R - large. ε - small.

$f(x)$ we extend to Π^+ by $f(z) = \frac{\sqrt{z}}{1+z^2}$, where we consider the branch of $\sqrt{z}$: $0 < \operatorname{Arg} z < \pi$.

Then,



$\sqrt{z} = \sqrt{r}\, e^{\frac{i\varphi}{2}}.$

Actually, better even to extend to $\mathbb{C} \setminus \{\operatorname{Arg} z = -\frac{\pi}{2}\}$. $-\frac{\pi}{2} < \operatorname{Arg} z < \frac{3}{2}\pi.$

$f(z)$ is meromorphic here.

Now, $f \in \mathcal{M}(\overline{\mathcal{D}})$. can apply then the Residue Thm.

$$\int_{\partial D} f(z)\,dz = 2\pi i \cdot \operatorname{res}_{i} f(z) = 2\pi i \cdot \lim_{z\to i} \frac{\sqrt{z}}{(z+i)(z-i)}(z-i)$$

$$= 2\pi i \cdot \frac{\sqrt{i}}{2i} = \pi \cdot e^{\frac{\pi i}{4}} = \pi\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$i = e^{\frac{\pi}{2} i}$.

$$\int_{\partial D} = \int_{\gamma_R} - \int_{\gamma_\varepsilon} + \int_{\gamma}.$$

$$\left|\int_{\gamma_R} f\,dz\right| \le \pi R \cdot \frac{\sqrt{R}}{R^2-1} \xrightarrow{R\to+\infty} 0 \quad \Rightarrow \int_{\gamma_R} \to 0.$$

$$\left|\int_{\gamma_\varepsilon} f\,dz\right| \le \pi \varepsilon \cdot \sqrt{\varepsilon}\, O(1) \xrightarrow{\varepsilon\to 0} 0 \quad \Rightarrow \int_{\gamma_\varepsilon} \to 0.$$

$|\sqrt{z}| = |z|^{\frac{1}{2}}$.

$$\int_{\gamma} f\,dz = \int_{\varepsilon}^{R} \frac{\sqrt{x}}{1+x^2}\,dx + \int_{-R}^{-\varepsilon} \frac{\sqrt{z}}{1+z^2}\,dz. = \{ z=-y,\ y\in[\varepsilon,R] \}$$

$$= \int_{\varepsilon}^{R} \frac{\sqrt{x}}{1+x^2}\,dx + \int_{\varepsilon}^{R} \frac{i\sqrt{y}}{1+y^2}\,dy$$

$$= (1+i)\int_{\varepsilon}^{R} \frac{\sqrt{x}}{1+x^2}\,dx.$$

Take $\lim_{\substack{\varepsilon\to 0\\ R\to+\infty}}$: $\quad \pi\frac{\sqrt{2}}{2}(\cancel{1+i}) = (\cancel{1+i})\int_0^{+\infty} \frac{\sqrt{x}}{1+x^2}\,dx.$

$$\Rightarrow \int_0^{+\infty} \frac{\sqrt{x}}{1+x^2}\,dx = \frac{\sqrt{2}}{2}\pi.$$

Application to summation of series

Ex. using residues, compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$\cot \pi z$: has poles of order 1 at all $n\in\mathbb{Z}$.

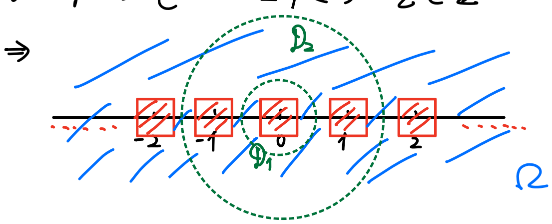
$$\frac{\cos\pi z}{\sin\pi z} = i\,\frac{e^{iz\pi}+e^{-iz\pi}}{e^{iz\pi}-e^{-iz\pi}} = i\,\frac{e^{2iz\pi}+1}{e^{2iz\pi}-1} = L\circ e^{2\pi i z}. \quad L(w) = i\frac{w+1}{w-1}.$$

$\Rightarrow$ $\cot \pi z$ is unbounded whenever $w \to 1$.

Hence, if w is away from $w=1$, $\cot\pi z$ is away from ∞.

$w = 1 \Leftrightarrow e^{2\pi i z} = 1 \Leftrightarrow z \in \mathbb{Z}$

$\Rightarrow$



In Ω: $|\cot \pi z| \le M = \text{const}$.

In partc.: $|\cot \pi z| \le M$ for $|z| = N + \frac{1}{2}$.

So, consider $f(z) := \frac{1}{z^2} \cot(\pi z)$

$\mathcal{D}_N = \{ |z| < N + \frac{1}{2} \}$.

$$\int_{\partial \mathcal{D}_N} f(z)\, dz = \{\text{Residue Thm}\} = 2\pi i \sum_{z \in \{-N, \dots, N\}} \operatorname{res} f(z)$$

$$= 2\pi i \Big(\operatorname{res}_0 f + \sum_{\substack{j=-N \\ j \neq 0}}^{N} \operatorname{res}_j f \Big)$$

$f(z)$ — odd func. $\Rightarrow$ $\operatorname{res}_{-j} f = \operatorname{res}_j f$.

$$\Rightarrow \int_{\partial \mathcal{D}_N} f(z) dz = 2\pi i \Big(\operatorname{res}_0 f + 2 \sum_{j=1}^{N} \operatorname{res}_j f \Big)$$

$$\operatorname{res}_j f = \{\text{pole of ord} = 1\} = \lim_{z \to j} \frac{\cot(\pi z)}{z^2} (z - j)$$

$$= \{ z - j = w \} = \lim_{w \to 0} \frac{\cot \pi w}{(w+j)^2} \cdot w = \lim_{w \to 0} \frac{\cos \pi w}{\sin \pi w} \pi w \cdot \frac{1}{\pi} \cdot \frac{1}{(w+j)^2} = \frac{1}{j^2} \cdot \frac{1}{\pi}.$$

$$\sum_{j=1}^{N} \operatorname{res}_j f = \frac{1}{\pi} \sum_{j=1}^{N} \frac{1}{j^2}.$$

$$\operatorname{res}_0 \frac{\cot(\pi z)}{z^2} = \{\text{pole of ord} = 3\} = \frac{1}{(3-1)!} \lim_{z \to 0} \left(f(z) \cdot z^3 \right)''$$

$$= \frac{1}{2}\lim_{z\to 0}\left(z\,\mathrm{cotan}(\pi z)\right)'' = \frac{1}{2}\lim_{z\to 0}\left(\frac{-2\pi}{(\sin\pi z)^2} + \frac{2\pi^2 z\cos\pi z}{(\sin\pi z)^3}\right)$$

$$= \pi\lim_{z\to 0}\left(\frac{-1}{(\sin\pi z)^2} + \frac{\pi z\cos\pi z}{(\sin\pi z)^3}\right) = \pi\lim_{z\to 0}\frac{\pi z\cos\pi z - \sin\pi z}{(\sin\pi z)^3}$$

$$= \pi\lim_{z\to 0}\frac{\pi z\left(1 - \frac{\pi^2 z^2}{2} + \dots\right) - \left(\pi z - \frac{\pi^3 z^3}{6} + \dots\right)}{\pi^3 z^3(1 + \dots)}$$

$$= \pi\cdot\frac{1}{\pi^3}\left(-\frac{1}{3}\right)\pi^3 = -\frac{\pi}{3}.$$

$$\left|\int_{\partial D} f(z)\,dz\right| \leq 2\pi\left(N+\frac{1}{2}\right)\cdot M\frac{1}{\left(N+\frac{1}{2}\right)^2} \xrightarrow{N\to\infty} 0.$$

Applying $\lim_{N\to\infty}$: $\quad 0 = 2\pi i\left(-\frac{\pi}{3} + 2\sum_{j=1}^{\infty}\frac{1}{j^2}\cdot\frac{1}{\pi}\right) \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$

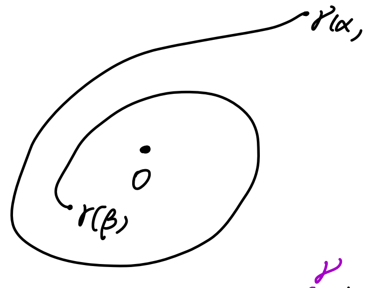
Argument Principle

Def: Let $\gamma(t): [\alpha, \beta] \to \mathbb{C}$ be a curve, and $\gamma(t) \neq 0$ ($0 \notin [\gamma] = \mathrm{Im}\,\gamma$).

Let's choose a continuous branch $\varphi(t)$ of the multi-valued func: $\mathrm{Arg}(\gamma(t))$.

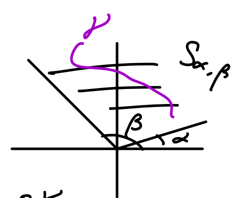
Then, the delta-argument of γ ($\Delta_\gamma \mathrm{Arg}$)

is the difference: $\varphi(\beta) - \varphi(\alpha)$.



Q1: why $\varphi(t) \in C[\alpha, \beta]$ exists at all?

Q2: why $\Delta_\gamma \mathrm{Arg}$ is indep. of φ?



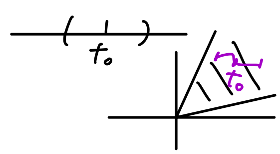
Proof: Q1: Note that if $[\gamma] \subset S_{\alpha,\beta}$, then $\exists\,\varphi$ is obvious.

You use a global branch of $\mathrm{Arg}\,z$ in $S_{\alpha,\beta}$. $\varphi = \mathrm{Arg}\circ\gamma$.

Then the general case is reduced to:

$\forall t_0$, $\exists\, B_\delta(t_0) \subset [\alpha, \beta]$, where $[\gamma]\big|_{B_\delta(t_0)} \subset S_{\alpha,\beta}$.

(By conti. of γ).



Get an open covering of $[\alpha,\beta]$

Choose a finite subcovering: $\vdash\!\!\stackrel{\varphi_1}{-}\!\!+\!\!\stackrel{\varphi_2}{-}\!\!+\!\!\stackrel{\cdots}{-\!-\!-}\!\!\dashv$

On those intersections: φ_j may differ by $2\pi k$.

$\Rightarrow$ we repair each φ_j to obtain a conti. func. over all.

Q2: follows from the construction in Q1 that all possible φ different by $\varphi \to \varphi + 2\pi k$ $\Rightarrow$ $\Delta_\gamma \operatorname{Arg}$ doesn't change.

<u>Proposition</u>: If γ - SBP. closed $\Rightarrow$ $\Delta_\gamma \operatorname{Arg} = \frac{1}{i}\int_\gamma \frac{1}{z}\,dz$.

<u>Proof</u>: $\forall\, \varphi_j$ corresponds to a conti. branch of $\operatorname{Arg} z$. $z = \gamma(t) \in S_{\alpha,\beta}$.

<u>recall</u>: $\operatorname{Ln} z := \ln|z| + i \operatorname{Arg} z$. $\Rightarrow$ In $S_{\alpha,\beta} \supset [\gamma_j]$, $\exists$ a hol. branch of

$\operatorname{Ln} z = \phi_j(z)$ $\Rightarrow$ By Newton-Lebnitz: $\int_{\gamma_j} \frac{1}{z}\,dz = \phi_j(z)\Big|_{\gamma(t_j)}^{\gamma(t_{j+1})} = i\Delta_{\gamma_j}\operatorname{Arg} + \Delta_{\gamma_j} \ln|z|$.

$$\Rightarrow \int_\gamma \frac{1}{z}\,dz = \sum_j \int_{\gamma_j} \frac{1}{z}\,dz \underset{\text{by def.}}{=} i\cdot\Delta_\gamma \operatorname{Arg} + \underbrace{\Delta \ln|z|}_{= \ln|\gamma(\beta)| - \ln|\gamma(\alpha)| = 0} \qquad \square.$$

<u>Remark</u>: $\Delta_\gamma \operatorname{Arg}$ does depend on the parameterization!

<u>Def</u>: Let $\gamma: [\alpha,\beta] \to \mathbb{C}$ be a curve, $p \notin [\gamma]$, and γ closed.

Then the index (or winding number) of γ w.r.t the pt. p is

$$\operatorname{ind}_p \gamma := \frac{1}{2\pi} \Delta_{\{\gamma - p\}} \operatorname{Arg}.$$

<u>Ex. 1</u> $\gamma(t) = e^{int}$, $0 \le t \le 2\pi$, $n \in \mathbb{Z}$.

$[\gamma] = \{|z| = 1\}$.

$\Delta_\gamma \operatorname{Arg} = 2\pi n$

$\operatorname{ind}_0 \gamma = n$.

Ex.2 $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, $\Gamma(t) = f \circ \gamma$, $f(z) := z^2 + 2$.

$$[\Gamma] = f(\partial B_1(0)) = \partial B_1(2) \subset S_{-\frac{\pi}{2}, \frac{\pi}{2}} \quad - \exists \operatorname{Arg} z - \text{continu.}$$

$\downarrow$ $\operatorname{Re} \Gamma \geq 1$.

$$\Delta_{\Gamma} \operatorname{Arg} = 0 \quad \Rightarrow \operatorname{ind}_0 \Gamma = 0.$$

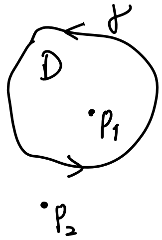
Ex.3 Same but $f(z) = e^z$.

$$= e^{\cos t + i \sin t}$$
$$= e^{\cos t}[\cos(\sin t) + i \sin(\sin t)]$$

$$\operatorname{Re} e^z = e^{\cos t} \cdot \underbrace{\cos(\sin t)}_{>0} \geq e^{-1} \cdot c. > 0$$

$$\Rightarrow \operatorname{ind}_0 \Gamma = 0.$$

Ex.4 Let γ – a closed simple curve. Then (Jordan's thm) $[\gamma] = \partial D$, D simply-conn. bounded.



Then $\operatorname{ind}_p \gamma^+ = \begin{cases} 1, & p \in D \\ 0, & p \in \mathbb{C} \setminus \overline{D}. \end{cases}$

(No proof.)

Thm (Argument Principle)

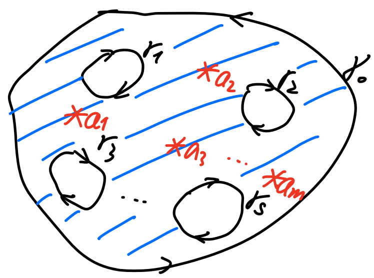
Let D be an admissible dom. with a SBP boundary $\partial D = \underset{\text{outer}}{\gamma_0} \cup \underbrace{\gamma_1 \cup \cdots \cup \gamma_s}_{\text{inner}}$.

Let $f \in \mathcal{O}(\Omega \setminus \{a_1, \cdots, a_m\})$, where $\underset{\text{dom}}{\Omega} \supset \overline{D}$.

all $a_j \notin \partial D$, all a_j – poles.

Assume further that $f \neq 0$ on ∂D.

($\Rightarrow$ By Uniqueness Thm. $\underset{\text{zero set}}{Z_f}$ in D is $\{b_1, \cdots, b_n\}$).



Then, define $\operatorname{ind}_0 f(\partial D) := \operatorname{ind}_0 f \circ \gamma_0^+ - \operatorname{ind}_0 f \circ \gamma_1^+ - \cdots - \operatorname{ind}_0 f \circ \gamma_s^+$.

Then, the following holds:

$$\boxed{\operatorname{ind}_0 f(\partial D) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'}{f} \, dz = \# \text{ zeros } (f) - \# \text{ poles } f.}$$

where zeros and poles are counted with their multiplicities (orders).

Proof: Consider the integral

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'}{f}dz = \frac{1}{2\pi i}\int_{\gamma_0}\frac{f'}{f}dz - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f'}{f}dz - \cdots\cdots - \frac{1}{2\pi i}\int_{\gamma_s}\frac{f'}{f}dz$$

Compare with $\operatorname{ind}_0 f(\partial D) = \frac{1}{2\pi}\Delta_{f(\gamma_0)}\operatorname{Arg} - \frac{1}{2\pi}\Delta_{f(\gamma_1)}\operatorname{Arg} - \cdots - \frac{1}{2\pi i}\Delta_{f(\gamma_s)}\operatorname{Arg}.$

By proposition. $\frac{1}{2\pi}\Delta_{f(\gamma_j)}\operatorname{Arg} = \frac{1}{2\pi i}\int_{f(\gamma_j)}\frac{1}{w}dw \underset{w=f(\gamma_j(t))}{\uparrow}= \frac{1}{2\pi i}\int_{\alpha_j}^{\beta_j}\frac{1}{f(\gamma_j(t))}f'(\gamma_j(t))\,\gamma_j'(t)\,dt.$

$$= \frac{1}{2\pi i}\int_{\gamma_j}\frac{f'}{f}dz.$$

So, the two things coincide. $\Rightarrow$ the first identity is proved.

On the other hand, let's apply the residue thm.

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'}{f}dz = \left\{\begin{array}{l}\text{singularities come from zeros}\\ \text{and singularities of } f\end{array}\right\}.$$

$$= \sum_{\{a_j, b_k\}}\operatorname{res}\frac{f'}{f}.$$

Consider $z \to b_k$: $\underset{\text{locally}}{f(z) \overset{\downarrow}{=}} (z-b_k)^{l_k} g(z)$

$$\Rightarrow \frac{f'}{f} = \frac{l_k(z-b_k)^{l_k-1}g + (z-b_k)^{l_k}g'}{(z-b_k)^{b_k}\cdot g}.$$

$$= \underbrace{\frac{l_k}{z-b_k}}_{\substack{\text{pole of order 1.}\\ \text{res} = l_k.}} + \underbrace{\frac{g'}{g}}_{\text{hol.}} \qquad \Rightarrow \operatorname{res}_{b_k}\frac{f'}{f} = l_k.$$

Similarly, at $z\to a$, $f = \frac{g}{(z-a_j)^{\beta_j}} \Rightarrow \frac{f'}{f} = -\frac{\beta_j}{z-a_j} + \frac{g'}{g} \Rightarrow \operatorname{res}_{a_j}\frac{f'}{f} = -\beta_j.$

$$\Rightarrow \frac{1}{2\pi i}\int_{\partial D}\frac{f'}{f}dz = \sum_k l_k - \sum_j \beta_j. \qquad \square.$$

<u>Rouche Thm.</u>

Let $f, g \in \mathcal{O}(\overline{D})$, where D - an admissible dom. with ∂D - SBP.

Assume that it holds $|g|<|f|$ $\forall z\in\partial D$. Then $\underline{\#\text{ zeroes}(f+g) = \#\text{ zeroes}(f)}$ in D (counted with multiplicities). And actually, $f, f+g\neq 0$ on ∂D.

Notation: $N_f = N_{f+g}$.

<u>Proof</u>: $|f|>|g|\geq 0$, $|f+g|\geq |f|-|g|>0$ $\Rightarrow$ $f, f+g$ don't vanish on ∂D.

By the Argument Principle:

$$\#\text{ zeroes}(f) = \text{ind}_0\, f(\partial D) = \frac{1}{2\pi i}\int_{\partial D}\frac{f'}{f}\,dz.$$

$$\#\text{ zeroes}(f+g) = \text{ind}_0 (f+g)(\partial D) = \frac{1}{2\pi i}\int_{\partial D}\frac{(f+g)'}{f+g}\,dz.$$

Note that $f+g = f(1+\frac{g}{f})$ on ∂D.

$\text{Re}\,(1+\frac{g}{f})>0$, since $\text{Re}\,(1+\frac{g}{f})\geq 1-\left|\frac{g}{f}\right|>0$ on ∂D.

$\Rightarrow \text{ind}_0 (1+\frac{g}{f})(\partial D) = 0.$

So, $\#\text{ zeroes}(f+g) = \#\text{ zeroes}\left(f(1+\frac{g}{f})\right) = \frac{1}{2\pi i}\int_{\partial D}\left[\ln\left(f(1+\frac{g}{f})\right)\right]'dz$

$= \frac{1}{2\pi i}\int_{\partial D}(\ln f)' + \left[\ln(1+\frac{g}{f})\right]'\,dz = \#\text{ zeroes}(f) + 0.$ $\square$

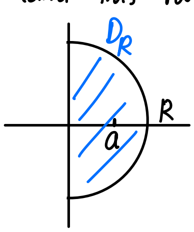
by the Argument Principle.

<u>Ex.</u> How many roots for $z^8-3z+1=0$ in $B_1(0)$?

$f=-3z$, $g=z^8+1$ on $\partial B_1(0)$: $|g|\leq 2<3=|f|$.

$\Rightarrow$ By Rouche Theorem: $\#\text{zeroes}(f+g) = \#\text{ zeroes}(-3z) = 1.$ $\Rightarrow$ 1 root in $B_1(0)$

<u>Ex.</u> Show that the eq. $z-e^{-z}=a$ has exactly one root in $\{\text{Re}\,z\geq 0\}$ for $a>1$ (and this root is real.)



Choose large $R>0$, $R\to+\infty$

$f = z-a$, $g=-e^{-z}$. $\Rightarrow f+g=0$.

$N_f(D_R) = 1$ $(z=a)$.

on ∂D_R: (1) $|z|=R$: $|g| = e^{-\text{Re}\,z}\leq 1.$

$|f| \geq R-a > 1$ for $R > a+1$. $\Rightarrow R \to +\infty$, $|g| < |f|$.

(ii) $z \in [-iR, iR]$: $|g| = 1$.

$|f| = |z-a| > 1$ since $a > 1$ $\Rightarrow |g| < |f|$.

$\Rightarrow$ finally, $N_{f+g}(D_R) = N_f(D_R) = 1$.

$R \to +\infty$, $\{\mathrm{Re}\, z \geq 0\} = \bigcup_{R>0} \overline{D_R}$. $\Rightarrow N_{f+g}(\mathrm{Re}\, z \geq 0) = 1$.

Q1: why the root is real?

A1: we can find a real root: $x - a = e^{-x}$, $x \in \mathbb{R}$.

A2: if the root $z_0 \notin \mathbb{R} \Rightarrow \bar{z}_0 \neq z_0$, $\bar{z}_0 - e^{-\bar{z}_0} = a \Rightarrow \bar{z}_0$: another root. — contradiction.

Theorem (Openness Principle)

Let $f \in \mathcal{O}(\Omega)$, $f \not\equiv$ const.

Then $f(\Omega)$ is also a domain!

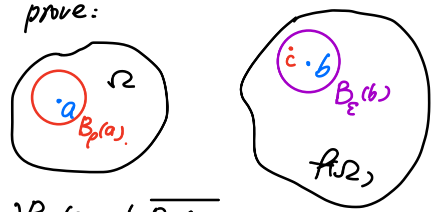
Rk: Fails for $\mathbb{R}$!

$y = x^2$, $\Omega = \mathbb{R}$. $\Rightarrow y(\mathbb{R}) = [0, +\infty)$ — not open!

Proof: $f(\Omega)$ is connected (as continuous image of connected set Ω).

$\Rightarrow$ need openness. Take $a \in \Omega$, $b = f(a)$, need to prove:

$\exists\, B_\varepsilon(b) \subset f(B_\delta(a))$.



Take $f(z) - b$ has $m = \mathrm{ord}_a(f(z) - b) \geq 1$.

a - isolated zero $\Rightarrow \exists\, B_\rho(a)$: $f(z) - b \neq 0$ on $\partial B_\rho(a)$ ($\overline{B_\rho(a)} \subset \Omega$).

$\varepsilon := \min_{|z-a|=\rho} (f(z) - b)$, take $\forall\, c \in B_\varepsilon(b)$.

We deal with the eq: $f(z) - c = 0$ in $B_\rho(a)$.

$$f(z) - c = \underbrace{(f(z) - b)}_{F(z)} + \underbrace{(b - c)}_{G(z)}$$

Take on $\partial B_\rho(a)$: $|G| < \varepsilon = \min |F|$.

$\Rightarrow$ By Rouche Thm: $N_{F+G}(B_\rho(a)) = N_F(B_\rho(a)) = m \geq 1$

$\Rightarrow f(z) - c = 0$ has ≥ 1 roots in $B_\rho(a)$.

So, $B_\varepsilon(b) \subset f(B_\rho(a))$. $\square$.

Theorem (Huzwitz)

Let f_n be a seq. of conf. maps of a dom Ω ($\mathbb{C}$-valued).

$f_n : \Omega \xrightarrow{\text{conf.}} \Omega_n \subset \mathbb{C}$. Let $f_n \xrightarrow[\text{in } \Omega]{\text{norm.}} f$.

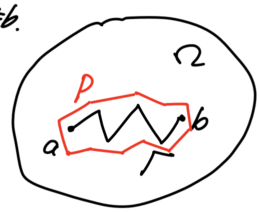
Then, $\begin{cases} f = \text{const} \\ \quad \text{or} \\ f \text{ is a conf. map} \end{cases}$

Proof: assume $f \neq$ const. assume, by contrad. f is not conf.

$\Rightarrow f$ is not injective. $\Rightarrow \exists\, a, b \in \Omega$, s.t $f(a) = f(b)$, $a \neq b$.

Connect a, b by a broken line $\Gamma \subset \Omega$.

Take closed simple polygon $\bar{P} \supset \Gamma$, $\bar{P} \subset \Omega$.



We will apply Rouche Thm in Ω.

Perturbing P if needed, we may assume:

$f(z) - f(a) \neq 0$ on ∂P (possible by uniqueness Thm).

Fix $\varepsilon := \min\limits_{\partial P} |f(z) - f(a)| > 0$.

$f(z) - f(a) = (f(z) - f_n(z)) + (f_n(z) - f_n(a)) + (f_n(a) - f(a))$

Since $f_n(z) \overset{\bar{P}\text{-cpt.}}{\rightrightarrows} f(z) \Rightarrow N: \forall n \geq N$, we have:

(i) $\min\limits_{\bar{P}} |f_n(z) - f_n(a)| > \frac{\varepsilon}{2}$

(ii) $|f_n(a) - f(a)| < \frac{\varepsilon}{4}$

(iii) $\min\limits_{\bar{P}} |f_n(z) - f(z)| < \frac{\varepsilon}{4}$.

$f(z) - f(a) = F(z) + G(z)$,

$F(z) = f_n(z) - f_n(a)$, $G(z) = (f(z) - f_n(z)) + (f_n(a) - f(a))$

on ∂P: $|F| > \frac{\varepsilon}{2}$, $|G| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$.

$\Rightarrow |G| < |F| \Rightarrow$ by Rouche: $N_{F+G}(P) = N_F(P) = 1$ since f_n is injective.

but this contradicts $f(a) = f(b)$. $F+G$ vanishes on both pts. $\square$

Come back to Conf Maps.

3 model simple connected dom-s in $\overline{\mathbb{C}}$:

(1) $\Omega = \overline{\mathbb{C}}$

(2) $\Omega = \mathbb{C}$

(3) $\Omega = B_1(0)$

Why pairwise conf inequil.?

(1) - cpt. (2)(3) - non-cpt. $\Rightarrow$ (1) $\not\sim$ (2) (3).

(2) $\not\sim$ (3) by Liouville ($f: \mathbb{C} \to B_1(0) \Rightarrow f$ const.)

Riemann Mapping Thm

$\forall$ simply-connected dom $D \subset \overline{\mathbb{C}}$ is equil. to (1), (2) or (3).

First Goal: find Aut groups of model domains

Case (2):

Thm. $Aut(\mathbb{C}) = \{az+b;\ a,b \in \mathbb{C},\ a \neq 0\}$ - cx affine maps.

Proof: We'll prove a stronger claim:

Any inj. hol. func. in $\mathbb{C}$ is affine.

Let's consider ∞ as an iso. sing for $f(z)$.

If $z=\infty$ is removable. then $f \in C(\infty)$ bounded $\Rightarrow f$ is overall bounded in $\mathbb{C}$

$\Rightarrow f = $ Const. Contrad.

If $z=\infty$ is a pole. then since $f \in \mathcal{O}(\mathbb{C})$.

$f(z)=\sum_{n=0}^{\infty} c_n z^n$ — both Taylor and Laurent.

Since $z=\infty$ a pole. $\Rightarrow f=\sum_{n=0}^{m} c_n z^n$ — a poly.

if $\deg \geq 2 \Rightarrow f'$ has zeros $\Rightarrow$ not injective — contrad.

So $\deg = 1 \Rightarrow f = c_0 + c_1 z$.

Finally, if $z=\infty$ is essential $f: \mathbb{C} \xrightarrow{\text{conf.}} \Omega$ — dom. $\exists f^{-1}$.

Take $\forall \overline{B_\varepsilon(b)} \subset \Omega$.

$K = f^{-1}(\overline{B_\varepsilon(b)})$ — cpt.

But this contradicts the Sokholsky–Weierstrass:

$\exists z_n \to \infty$, $f(z_n) \to b$ (z_n has to be in K for n large)

Contrad. $\square$.

<u>Case (1):</u>

<u>Thm.</u> $\mathrm{Aut}(\overline{\mathbb{C}}) = \{$ linear fractional maps $\}$.

<u>Proof:</u> Take $f: \overline{\mathbb{C}} \xrightarrow{\text{conf.}} \overline{\mathbb{C}}$.

Let $A = f(\infty)$, take $\varphi(w) = \dfrac{1}{w-A}$. $\Rightarrow \underbrace{\varphi \circ f}_{= g}: \infty \to \infty \Rightarrow g|_{\mathbb{C}}$: a conf. map $\mathbb{C} \to \mathbb{C}$.

$\Rightarrow g$ affine (in partic. linear-frac.)

$\Rightarrow f = \varphi^{-1} \circ g$ is also linear-frac. $\square$.

<u>Case (3):</u>

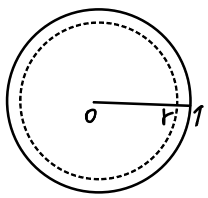
<u>Lemma (Schwarz)</u>

Let $f \in \mathcal{O}(B_1(0))$, $|f| \leq 1$, $f(0)=0$. Then it actually holds that $|f(z)| \leq |z|$, and if equility here holds for $z = z_0 \neq 0$, then in fact $f = e^{i\theta} z$, $\theta \in \mathbb{R}$.

Proof: $f(0)=0 \Rightarrow f(z)=z^m h(z)$ $(m\geq 2)$ near 0.

$\Rightarrow$ the func. $g(z):=\frac{f(z)}{z} \in \mathcal{O}(B_1(0))$.

Apply to $g \in \overline{B_r(0)}$ $(r<1)$ the Max Principle:



$$\max_{|z|\leq r} |g(z)| = \max_{|z|=r} |g(z)|.$$

$$\underbrace{\max_{|z|\leq r} \left|\frac{f(z)}{z}\right|}_{\text{non decreasing in } r.} = \frac{1}{r} \max_{|z|=r} |f(z)| \leq \frac{1}{r}\cdot 1$$

take $\lim_{r\to 1}$: $\sup_{|z|\leq 1} \left|\frac{f(z)}{z}\right| \leq 1 \Rightarrow |f(z)|\leq |z|$ in $B_1(0)$.

If now $\exists\ z_0\neq 0$, $|f(z_0)|=|z_0|$. $\Rightarrow |g(z_0)|=1$

but $|g(z)|\leq 1$ in $B_1(0) \Rightarrow$ by the Max Princ.

in $B_r(0)$ with $r>|z_0|$, we have $g=$ const. $|g(z_0)|=1$

$\Rightarrow f=e^{i\theta}\cdot z$. $\square$

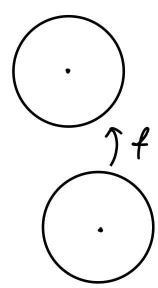
Thm. $\mathrm{Aut}(B_1(0)) = \left\{ \frac{z-a}{1-\bar{a}z} e^{i\theta} \right\}_{|a|<1,\ \theta\in\mathbb{R}}$ - the group of Linear-frac aut-s of $B_1(0)$.

In partic., all aut-s are linear-frac.

Proof:

Part 1: find the linear-frac aut-s of $B_1(0)$.

Let $f: B_1(0) \xrightarrow[\text{linear-frac.}]{} B_1(0)$



Let $a=f^{-1}(0)$, $f(a)=0$.

assume first $a\neq 0$.

$a\to 0$

$a^*\to 0^*=\infty$.

$a^* = \frac{R^2}{\bar a} = \frac{1}{\bar a}. \Rightarrow f(z) = \frac{z-a}{z-\frac{1}{\bar a}}\cdot k$. $|f(1)| = 1 \Rightarrow f(1) = e^{i\theta}$.

$\Rightarrow \frac{1-a}{1-\frac{1}{\bar a}}\cdot k = e^{i\theta} \Rightarrow \frac{(1-a)\bar a}{\bar a - 1}\cdot k = e^{i\theta}$. $b \triangleq \frac{1-a}{\bar a - 1}$. $|b| = \frac{|1-a|}{|1-\bar a|} = 1$.

$\Rightarrow \bar a k = e^{i\theta}/b = e^{i\tilde\theta} \Rightarrow k = \frac{1}{\bar a} e^{i\tilde\theta}$.

$\Rightarrow f(z) = \frac{z-a}{1-\bar a z}(-e^{i\tilde\theta}) = \frac{z-a}{1-\bar a z} e^{i\theta'}$.

So, any f is as desired.

Opposite direction: any $f = \frac{z-a}{1-\bar a z} e^{i\theta}$ maps $B_1(0) \xrightarrow{conf} B_1(0)$

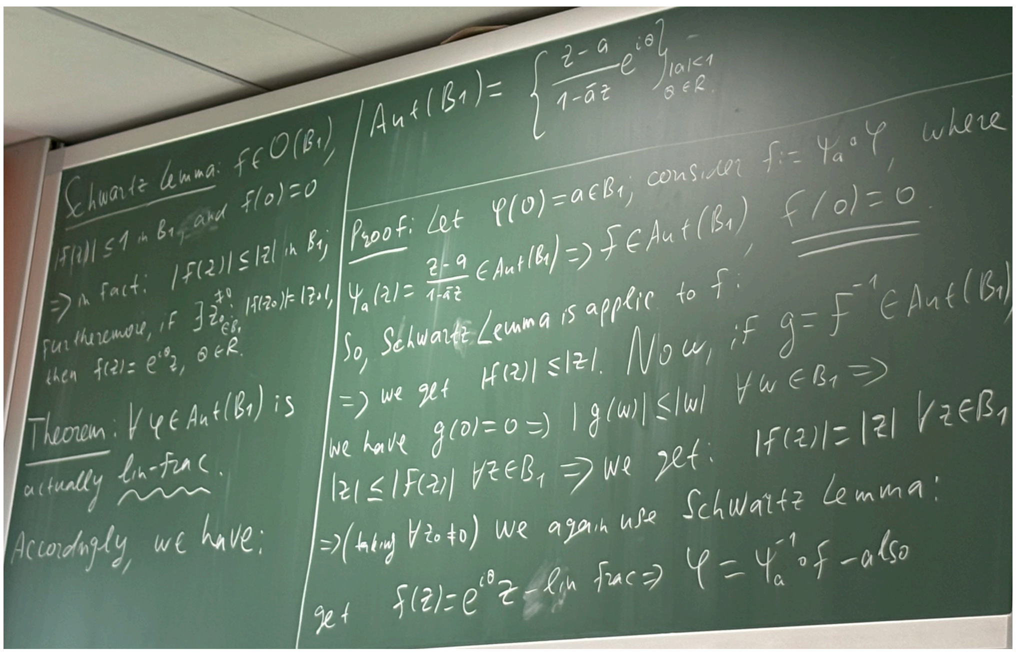
Why? $a \mapsto 0$

$\frac{1}{\bar a} \mapsto 0^* = \infty \Rightarrow$ the image of $\partial B_1(0)$ is such a curve that

$\frac{1}{\bar a} = a^*$ it's $\{|z| = R\}$

since $|f(1)| = 1$, we get $R = 1$.

$f(a) = 0 \in B_1(0) \Rightarrow f(B_1(0)) = B_1(0)$. So Part 1 is done.



lin-frac

$\mathrm{Aut}(\overline{\mathbb{C}}) = \left\{ \frac{az+b}{cz+d} \right\}_{ad-bc \neq 0}$ $\boxed{\dim_{\mathbb{R}} = 6}$

8-2

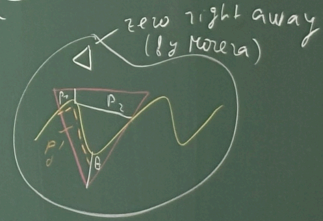
$= x+yi$

$\mathrm{Aut}(\mathbb{C}) = \{ az+b \}_{a \neq 0},$ $\boxed{\dim_{\mathbb{R}} = 4}$

$\mathrm{Aut}(B_1) = \left\{ \frac{z-a}{1-\bar{a}z} e^{i\theta} \right\}_{|a|<1,\ \theta \in \mathbb{R}},$ $\boxed{\dim_{\mathbb{R}} = 3}$

$\sim SU(1,1)$

<u>Lemma</u>: Let Ω - domain, $\gamma \subset \Omega$ - broken line, $f \in C(\Omega) \cap \mathcal{O}(\Omega \setminus \gamma)$. Then $f \in \mathcal{O}(\Omega)$.



<u>Outline of proof</u>:

Use the inverse Morera Thm

Will be proving: $\int_{\partial\Delta} f dz = 0, \ \forall \bar{\Delta} \subset \Omega$.

$\int_{\partial\Delta} f dz = \sum_{j=1}^{5} \int_{\partial P_j} f dz$, $P_j \in \Omega \setminus \gamma$, ∂P_j may overlap with γ.

$\overline{P_j'} \in \Omega \setminus \gamma \Rightarrow \int_{\partial P_j'} f dz = 0 \Rightarrow$

approx. P_j by a seq of such $P_j' \Rightarrow$

conclude $\int_{\partial P_j} f dz = 0. \Rightarrow$

Finally $\int_{\partial\Delta} f dz = 0.$ ▨

<u>Remark</u>: actually true $\forall \gamma$ - curve of regularity C^1.

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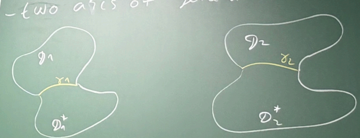
Thm (Schwartz (Riemann) Reflection Principle)

Let $\mathcal{D}_1, \mathcal{D}_2$ - domains in $\overline{\mathbb{C}}$, let $\gamma_1 \subset \partial\mathcal{D}_1$, $\gamma_2 \subset \partial\mathcal{D}_2$ - two arcs of generalized circles.



Let $\mathcal{D}_1^*, \mathcal{D}_2^*$ be the symmetric images of resp. $\mathcal{D}_1, \mathcal{D}_2$ w.r.t. γ_1, γ_2.

Assume that $\mathcal{D}_j \cap \mathcal{D}_j^* = \emptyset$, $j = 1,2$, and $\Omega_j := \mathcal{D}_j \cup \gamma_j \cup \mathcal{D}_j^*$ are again domains, $j=1,2$. Finally, let $f: \mathcal{D}_1 \longleftrightarrow \mathcal{D}_2$ be a conf. map

Assume f extends to a homeom of $\mathcal{D}_1 \cup \gamma_1$ onto $\mathcal{D}_2 \cup \gamma_2$.

Then f extends to a conf. map $F: \Omega_1 \longleftrightarrow \Omega_2$.

(extension by reflection)

<u>Proof</u>: Let φ_1, φ_2 be two lin-frac maps, mapping γ_1 and γ_2 resp. onto arcs of $\mathbb{R} \subset \mathbb{C}$.

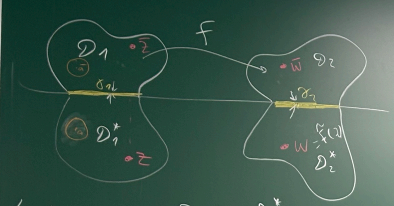
(exist since all gener. circles are lin-frac equiv.)

Then, modifying accordingly the domains $\mathcal{D}_1, \mathcal{D}_2$, and setting $g := \varphi_2 \circ f \circ \varphi_1^{-1}$, we arrive to an analog. setting as originally, <u>but</u> having γ_1, γ_2 - arcs of $\mathbb{R}$

(Based on:
- lin-frac are homeom-s of $\overline{\mathbb{C}}$
- lin-frac are hol in $\overline{\mathbb{C}}$
- lin-frac map symm pt to symm pt)

(we „switch to more convenient coord-s“).

$z^* = \bar{z}$



Let's extend f to D_1^* as follows:

$$\boxed{\hat{f}(z) = \overline{f(\bar{z})}} \quad z \in D_1^*$$

$$F(z) := \begin{cases} f(z), & z \in D_1 \cup \gamma_1 \\ \hat{f}(z), & z \in D_1^* \end{cases}$$

Clearly, F is a homeom of Ω_1 onto Ω_2.
(just need to check: $\hat{f} = f$ on γ_1 – but this follows from $f(\gamma_1) = \gamma_2$).

Next, $\hat{f} \in O(D_1^*)$: take any $B_r(a) \subset D_1^* \Rightarrow$ take $(B_r(a))^* = B_r(\bar{a}) \subset D_1 \Rightarrow$

$$f(z) = \sum_{n=0}^{\infty} c_n (z - \bar{a})^n \text{ in } B_r(\bar a) \Rightarrow$$

$$\hat{f}(z) = \overline{f(\bar{z})} = \sum_{n=0}^{\infty} \bar{c}_n (z-a)^n \in O(B_r(a))$$

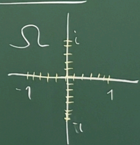
So, $\hat{f} \in O(D_1^*)$

So, $F \in O(\Omega_1 \setminus \gamma_1)$, $F \in C(\Omega_1) \Rightarrow$
By the Lemma: $\underline{F \in O(\Omega_1)}$.
And since F is a bijection, we conclude: $F : \Omega_1 \xleftrightarrow{\text{conf.}} \Omega_2$

$\blacksquare$

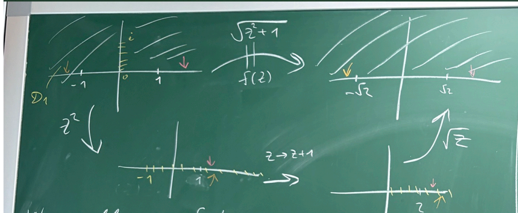
Remark: the Refl. Princ. can be generalized to the situation of $f \in O(D_1)$, $f(D_1) \subset D_2$, f extends contin. to $D_1 \cup \gamma_1$ and $f(\gamma_1) \subset \gamma_2$ (and the conclusion is the hol. exten. of f from D_1 to Ω_1)

Example: construct a conf map of



onto $\{|z| > 1\}$.

Step I: recall a simpler problem.



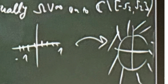
We will use Schwartz Refl with this $D_1 = \Pi^+ \setminus [0, i]$, and the map f
(so $D_2 = \Pi^+$)

$\gamma_1 = \{ z \in \bar{\mathbb{R}} : |z| > 1 \}$

$\gamma_2 = \{ z \in \bar{\mathbb{R}} : |z| > \sqrt{2} \}$

From the „arrows“ considered, f extends to a homeom of $D_1 \cup \gamma_1$ onto $D_2 \cup \gamma_2$
$\Rightarrow$ By Refl. Princ, f extends to a conf. map $F : \underset{\text{the orig. dom}}{\Omega} \longrightarrow \mathbb{C} \setminus [-\sqrt{2}, \sqrt{2}]$
(actually onto $\bar{\mathbb{C}} \setminus [-\sqrt{2}, \sqrt{2}]$)

Step 2: $F_2(z) = \dfrac{z}{\sqrt{2}}$



$F_3(z) = \mathcal{K}^{-1}(z)$

the needed map: $\boxed{F_3 \circ F_2 \circ F}$

↑投影背景墙面，注意保护，严禁书写！↑

Thm (Montel Thm). Let $f_n \in O(\Omega)$, and let it be unif bdd $\forall$ cpt: $\forall \underset{\text{cpt}}{K} \subset \Omega$, $\exists C = C(K)$: $|f_n(z)| \le C(K)$ $\forall n \in \mathbb{N}, \forall z \in K$.
Then f_n contains a subseq. $f_{n_j} \underset{\text{normally}}{\overset{\Omega}{\rightrightarrows}} f$

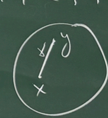
Proof: recall the Arcela-Ascolli Thm
($E \subset C(K)$, then E is precompact $\Leftrightarrow$ E is unif. bounded and equicontinuous)

Equicontin: $\forall \varepsilon > 0$, $\exists \delta > 0$: $\forall x, y \in K$ with $d(x,y) < \delta$ and $\forall f \in E$, it holds $|f(x) - f(y)| < \varepsilon$.

Suff. cond. (real case): if $K = [a, b]$, and $|f'| \le M$ $\forall f \in E \Rightarrow E$ is equicontin.
$|f(x) - f(y)| = |f'(\xi)(y-x)| \le M|y-x| \Rightarrow \delta < \dfrac{\varepsilon}{M}$.

Complex case - analog.: $K = \overline{B_r(a)}$,
$E \subset C(K)$, $\forall f \in O(\overline{B_r(a)})$, $|f'| \le M$ (with $f \in E$)

$$\Rightarrow |f(y) - f(x)| \underset{\text{Newton-Leibn}}{=} \left| \int_x^y f'(z)\,dz \right| \le$$



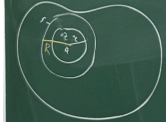
$\leq M \cdot |y-x|$

Now, back to the Thm: take $K = \overline{B_r(a)} \subset \Omega$.

$E = \{f_n\} \subset C(K)$. Use $\overline{B_r(a)} \subset \overline{B_R(a)} \subset \Omega$ $(R > r)$; $\forall z \in K$, $\forall f \in E$, $f'(z) = \frac{1}{2\pi i} \int_{|\xi - a| = R} \frac{f(\xi)}{(\xi - z)^2} d\xi$

$\Rightarrow |f'(z)| \leq \frac{1}{2\pi} 2\pi R \cdot C \cdot \frac{1}{(R-r)^2}$ $\Rightarrow$

$C = C(\partial B_R(a))$



the family $\{f_n'\}$ is also unif bounded on K!

$\Rightarrow E$ is equicontin. on K and unif bdd $\Rightarrow$ by Arcela-Ascolli:

E is precompact in $C(K) \Rightarrow$

$\boxed{\exists f_{n_k} \xrightarrow[\text{unif.}]{K} f}$

Arcela-Ascolli Thm:

The most common statement of the theorem for real-valued continuous functions on a closed and bounded interval $[a, b] \subset \mathbb{R}$ is as follows:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued continuous functions defined on the closed and bounded interval $[a, b]$. If this sequence is **uniformly bounded** and **equicontinuous,** then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges uniformly on $[a, b]$.

Let's break down the key conditions:

- **Uniformly Bounded:** A sequence of functions $\{f_n\}$ is uniformly bounded on $[a, b]$ if there exists a single positive constant M such that $|f_n(x)| \leq M$ for all n and for all $x \in [a, b]$. This means that the graphs of all the functions in the sequence are contained within a horizontal strip $[-M, M]$.
- **Equicontinuous:** A sequence of functions $\{f_n\}$ is equicontinuous on $[a, b]$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all n and for all $x, y \in [a, b]$ with $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \epsilon$. The key here is that the same δ works simultaneously for all functions in the sequence. This implies that the functions in the sequence have a uniform degree of continuity.

Now take $\forall$ cpt $T \subset \Omega$;

$T \subset \bigcup_{j=1}^{n} B_{r_j}(a_j)$, $\overline{B_{r_j}(a_j)} \subset \Omega$.

(to get it: take some covering by open discs $\Rightarrow$ finite subcovering...)

$\forall B_{r_j}(a_j)$ $\exists$ a conv. subseq of f_n; after n steps: get a conv. subseq. $\{f_{n_k}\}$ on $\bigcup B_{r_j}(a_j) \supset T$.

$\Rightarrow \forall T \subset \Omega$ cpt, $\exists$ a unif. converg. subseq.

Finally, $\Omega = \bigcup_{j=1}^{\infty} K_j$, $K_1 \subset K_2 \subset \ldots$ cpt exhausion

f_{11} f_{12} f_{13} ...

f_{21} f_{22} f_{23} ...

f_{31} f_{32} f_{33} ...

...

- the conv. subs on K_1
- its conv. subseq. on K_2
- the conv. subseq. of $\{f_{22}, f_{22}, \ldots\}$ on K_3

Then take $\{f_{jj}\}_{j=1}^{\infty}$ unif conv. $\forall K_j$ $\Rightarrow$ norm. conv. in Ω.

$\{f_{jj}\}$ is a subseq. of $\forall$ of the „rows" $\Rightarrow$ it is conv. unif on the respective cpt.

(„Cantor diagonal process/trick")

Montel Thm. if $f_n \in \mathcal{O}(\Omega)$, f_n is unif bdd $\forall \underset{\text{cpt}}{K} \subset \Omega \Rightarrow$ $\exists f_{n_k} \underset{\text{norm}}{\overset{\Omega}{\rightrightarrows}} f$

Remark on compact exhausions:

$K_1 \subset K_2 \subset K_3 \subset \dots,\ \bigcup_{j=1}^{\infty} K_j = \Omega,$

$\forall K_j \subset \mathring{K}_{j+1}$ interior

↑ 投影背景墙面，注意保护，严禁书写！↑

Remind: $\mathcal{O}(\Omega)$ - a (linear) metric space

$d(f,g) = \frac{1}{2^j} \frac{\rho_j(f,g)}{1+\rho_j(f,g)}$, where $\rho_j(f,g) := \max_{K_j} |f-g|$

converg. in $d \Longleftrightarrow$ normal conv. in Ω

Corollary (Compactness Principle)

Let $E \subset \mathcal{O}(\Omega)$ which is unif bdd on cpts.

$\forall \underset{\text{cpt}}{K} \subset \Omega,\ \exists C = C(K) : \max_K |f| \le C,\ \forall f \in E$

Then E is precompact (i.e. $\bar{E}$ is cpt)

Proof: just say that, in a metric space, a sequencially cpt set is cpt,

$\forall$ seq. contains conv subseq

and then apply Montel Thm $\square$

Propos. Let $\Omega \overset{\circ}{\subset} \mathbb{C}$ - a simply-conn dom. Then it admits hol branches of $\operatorname{Ln} z$, $\sqrt[n]{z}$ $(n \in \mathbb{N})$.

Proof: pick $\forall p \in \Omega$, and set

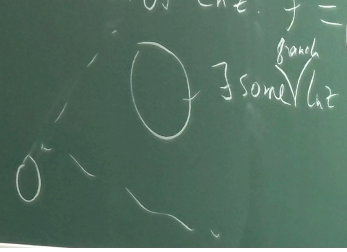
$f(z) := \int_\gamma \frac{1}{\xi} d\xi$, where γ : broken line $(\gamma \subset \Omega)$

Since $\frac{1}{z} \in \mathcal{O}(\Omega)$ and Ω is simply-conn, the def of $f(z)$ is correct (path-independ)

As proved, $f'(z) = \frac{1}{z}$

(actually can simply say: f - anti-deriv of $\frac{1}{z}$).

Then $\forall B_\varepsilon(a) \subset \Omega$, f is a hol branch of $\operatorname{Ln} z$: $f' = (\underset{\text{branch}}{\operatorname{Ln} z})'$

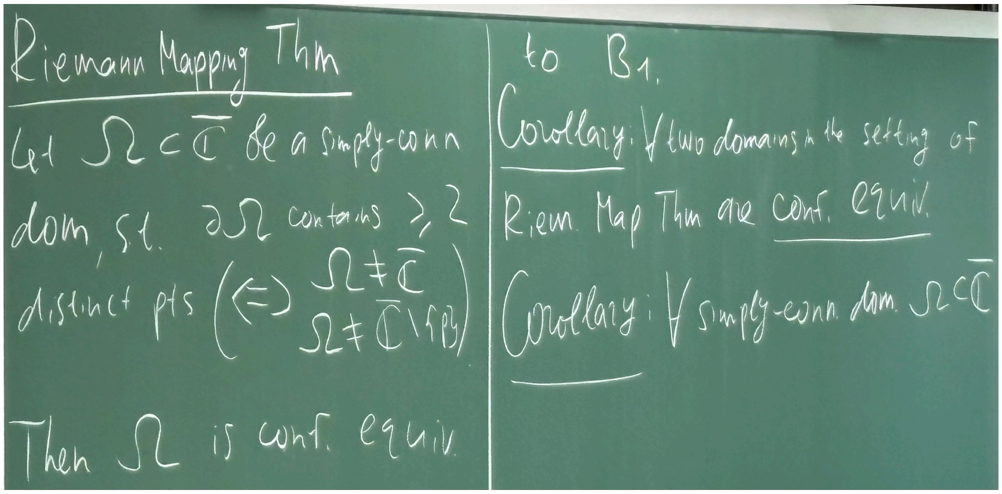


$\Rightarrow f(z) = \underset{\text{branch}}{\operatorname{Ln} z} + C$ in $B_\varepsilon(a)$, since $f \in \mathcal{O}(\Omega)$, C is a const in $\Omega \Rightarrow$

$f - C$ is a branch of $\operatorname{Ln} z$ in Ω.

To find a hol branch of $\sqrt[n]{z}$: choose $e^{\frac{1}{n}\underset{\text{branch}}{\operatorname{Ln} z}} = g(z)$

$g(z) \in \mathcal{O}(\Omega)$, $g^n(z) = z \Rightarrow g$ is a hol branch of $\sqrt[n]{z}$. $\square$



is conf. equiv. to $\begin{cases} \overline{\mathbb{C}}, & f = Id \quad \partial\Omega = \emptyset \\ \mathbb{C}, & f = \dfrac{1}{z-p} \quad \partial\Omega = \{p\} \\ B_1(0), & f \text{ is the Riemann Map.} \end{cases}$

Proof of Riem. Map. Thm:

The desired map f will be a composition of several maps.

Step 1: $\exists\, \alpha, \beta \in \Omega$, $\alpha \neq \beta$, apply the map $z \mapsto \dfrac{z-\alpha}{z-\beta}$.

Then the image of Ω:

* simply-connected.
* $\not\ni \{0, \infty\}$ (i.e. the new $\Omega \subset \mathbb{C}$) $\quad \overset{\not\ni 0}{}$

Step 2: by the Prop. $\exists\, \varphi(z) \in \mathcal{O}(\Omega)$. — a branch of $\sqrt{z}$.

$\left.\begin{array}{l} \Omega_1 = \varphi(\Omega) \\ \Omega_2 = -\varphi(\Omega) \end{array}\right|$ — two branches of $\sqrt{z}$.

Then $\Omega_1 \cap \Omega_2 = \emptyset$, because if $\exists\, w \in \Omega_1 \cap \Omega_2 \Rightarrow w = \underset{\text{one branch}}{\sqrt{z_1}} = \underset{\text{another branch}}{\sqrt{z_2}}$

$\Rightarrow$ apply w^2: $z_1 = z_2$

$w = \varphi(z_1) = -\varphi(z_2) \Rightarrow \varphi(z_1) = w = -\varphi(z_1)$

$\Rightarrow \varphi(z_1) = 0$ — contradiction. $z_1 = 0 \notin \Omega$.

$\Rightarrow$ take $\forall B_R(a) \subset \Omega_2 \Rightarrow \Omega_1 \subset \mathbb{C} \setminus \underbrace{B_R(a)}_{\text{a generalized disc.}}$

$\Rightarrow$ apply a lin-frac map $L: \bar{\mathbb{C}} \setminus B_R(a) \to B_1$.

$\Rightarrow L(\Omega_1) \subset B_1$.

Finally, taking some $p \in \Omega$, and using $\psi(z) = \frac{z-p}{1-\bar{p}^2} \in Aut(B_1)$

we get a new $\Omega \subset B_1$, $\Omega \ni 0$.

<u>Step 3</u>: Consider

$$E = \{ f \in O(\Omega),\ f(0)=0,\ |f'(0)| \geq 1,\ |f| \leq 1,\ f \text{ is Bijective} \} \subset O(\Omega)$$

* E is unif. bounded. (in particular, on cpt. s).
* E is closed. (if $f_n \overset{\Omega}{\rightrightarrows} f \Rightarrow \underset{\uparrow \text{ Hurwitz Thm.}}{f} = \begin{cases} \text{bij.} \\ \text{const.} \end{cases}$ but $|f'(0)| \geq 1 \Rightarrow f$- bijective).
* $E \neq \emptyset$. since $z \in E$.

Hence, by the Compactness Principle, $E \subset O(\Omega)$ is cpt.

Consider now the functional $J(f) := |f'(0)|$ on E.

By weierstrass's thm, J is continuous. $\Rightarrow \exists$ a max pt $h \in E$.

(i.e. $|h'(0)| \geq |f'(0)|$, $\forall f \in E$).

We claim that h is the desired conf. map!

That is, we need: $h(\Omega) = B_1$.

Assume, by contradiction, $h(\Omega) \neq B_1$.

fix $b \in B_1 \setminus h(\Omega)$. $(h(z) \neq b,\ \forall z \in \Omega)$ $|b| < 1$.

Then consider $g(z) := \frac{h(z) - b}{1 - \bar{b} h(z)} = \underset{\underset{Aut(B_1)}{\cap}}{T} \circ h$.

Then $g(\Omega) \subset B_1$, $\underset{\text{simply connected.}}{\overset{\text{conf.}}{g(\Omega)}} \not\ni 0$.

Hence, $\exists$ a branch of $\sqrt{z}$ is $g(\Omega)$.

$\Rightarrow$ consider $f(z) := \sqrt{g(z)}$ branch.

$$f'(0) = \frac{1}{2\sqrt{g(0)}} \cdot g'(0) = \frac{1}{2\sqrt{-b}} \cdot h'(0) \cdot \frac{1-|b|^2}{1} \Rightarrow |f'(0)| = |h'(0)| \cdot \frac{1-|b|^2}{2\sqrt{|b|}}$$ but $f \notin E$ since $f(0) \neq 0$.

Finally, take $F(z) := \dfrac{f(z) - f(0)}{1 - \overline{f(0)}\, f(z)} = \underset{\underset{\mathrm{Aut}(B_1)}{\cap}}{\lambda} \circ f$. $\quad F(0) = 0 \Rightarrow F \in E$.

Compute $|F'(0)|$

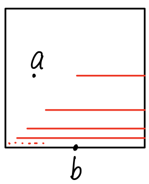
$|F'(0)| = |f'(0)| \cdot |\lambda'(f(0))|$

$$= |h'(0)| \cdot \frac{1 - |b|^2}{2\sqrt{|b|}} \cdot \frac{1 - |f(0)|^2}{(1 - |f(0)|^2)^2} = |h'(0)| \cdot \frac{1 - |b|^2}{2\sqrt{|b|}} \cdot \frac{1}{1 - |b|} > |h'(0)|$$

(since $|b| < 1$).

— a contradiction with the Max prop. of h. $\boxed{\text{Q}}$

<u>Remark</u>: it's a surprising thm, since very exotic shape can be mapped onto B_1 (conformally)!



Ω = square \ $\cup$ deleted segment

Ω is simply connected!

$\exists$ no $\underset{\underset{\Omega}{\cap}}{\gamma}$ connecting a, b.

<u>Remark</u>: the Riemann mapping is <u>unique</u>, up to $\varphi \in \mathrm{Aut}(B_1)$

$f: \Omega \to B_1$.

$\tilde{f} = \varphi \circ f$. because $\underset{\underset{\varphi}{\parallel}}{\tilde{f} \circ f^{-1}} \in \mathrm{Aut}(B_1)$.

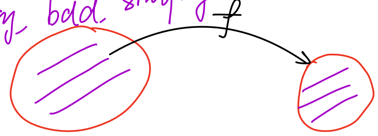
So, to fix a Riem. map $\Leftrightarrow$ to fix $\varphi \in \mathrm{Aut}(B_1)$

<u>Exercise</u>: $\forall\, \varphi \in \mathrm{Aut}(B_1)$ is uniquely determined by $\{\underset{\underset{B_1}{\cap}}{\varphi(a)}, \underset{\underset{\mathbb{R} \bmod 2\pi n}{\cap}}{\mathrm{Arg}\, \varphi'(a)}\}$, $a \in B_1$ fixed.

In this way, a Riem. map $f: \Omega \to B_1$ is uniquely fixed by a pair $\{\underset{\underset{B_1}{\cap}}{f(a)}, \underset{\underset{\mathbb{R} \bmod 2\pi n}{\cap}}{\mathrm{Arg}\, f'(a)}\}$.

<u>Caratheodory Thm</u>

a non-empty, bdd, simply-connected open subset whose boundary is a Jordan curve
↓
a simple closed curve.
↓
no intersections



Let $D \subset \mathbb{C}$ be a <u>Jordan dom</u>.

Let $f: D \to B$ be a Riem. map. Then f extends to a homeomorphism of $\overline{D}$ onto $\overline{B}$.

(no proof).

<u>Exercise:</u> $\forall\, \varphi \in \mathrm{Aut}(B_1(0))$ is uniquely determined by the images of 3 boundary pts.

$\{a_1, a_2, a_3\} \in \partial B_1$.

<u>This implies:</u> For a Jordan dom D (after applying Carath. Thm),

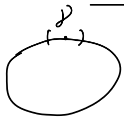
$\forall$ Riem. map f is uniquely determined by

$\{f(a_1), f(a_2), f(a_3)\}$ $\quad a_1, a_2, a_3 \in \partial D$

$\cap$ $\quad$ distinct.

∂B_1

<u>Remark:</u> For the case when ∂D is <u>real-analytic</u>.

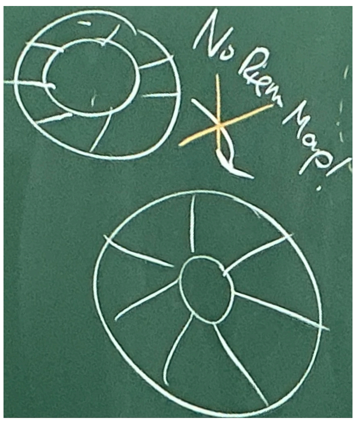


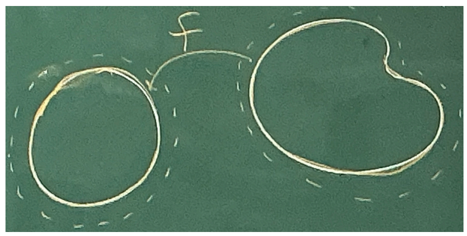
$\gamma(t) = \sum_{j=0}^{\infty} c_j (t-t_0)^j, \quad t \in [t_0-\delta, t_0+\delta]$.

<u>Ex.</u> γ: $y = f(x)$, $x \in [\alpha, \beta]$, $f(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k$.

$\gamma(t) = t + i f(t)$

the Riem. map extends across the boundary! ($f \in \mathcal{O}(\overline{D})$).

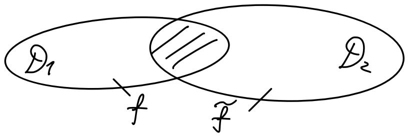




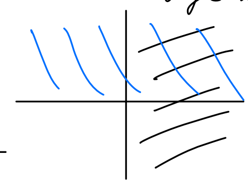
<u>Multiple-valued Analytic Functions.</u>

<u>Def:</u> Let $D_1, D_2 \subset \mathbb{C}$ — two domains. $D_1 \cap D_2 \neq \emptyset$, $f \in \mathcal{O}(D_1)$.

Then a function $\tilde{f} \in \mathcal{O}(D_2)$ is called the <u>analytical continuation</u> of f to D_2, if $f = \tilde{f}$ on $D_1 \cap D_2$.



Ex. $f(z) = \operatorname{Ln} z \big|_{-\frac{\pi}{2} < \operatorname{Arg} z < \frac{\pi}{2}}$ in $D_1 = \{\operatorname{Re} z > 0\}$ extends analytically to $\tilde{f} = \operatorname{Ln} z \big|_{0 < \operatorname{Arg} z < \pi}$ in Π^+.



Def: Let $D_1, D_2, \cdots, D_n$ be a finite sequence of domains and $f \in \mathcal{O}(D_1)$. Then a func. $\tilde{f}$ is called an analytical contin. of f along the chain $D_1, \cdots, D_n$ (finite) if $\exists\ f_1 = f, f_2, \cdots, f_{n-1}, f_n = \tilde{f}$, all $f_j \in \mathcal{O}(D_j)$ st. $f_{j+1} = f_j$ in $D_j \cap D_{j+1}$ (assume $\neq \emptyset$). ($j = 1 \cdots n-1$.)

Ex. $f = \operatorname{Ln} z \big|_{\operatorname{Arg} z \in (-\frac{\pi}{2}, \frac{\pi}{2})}$

$D_k = \left\{ -\frac{\pi}{2} + \frac{\pi}{2}k < \operatorname{Arg} z < \frac{\pi}{2} + \frac{\pi}{2}k \right\}_{k=-\infty}^{+\infty}$

$f_k = \operatorname{Ln} z \big|_{z \in D_k} \in \mathcal{O}(D_k)$

Then, $\forall f_k$ is a chain contin. of $\forall f_j$.

All values (and even single-valued hol. branches of $\operatorname{Ln} z$) can be described in this way!

Def 1 of Analytic func. in the sense of Weierstrass:

Fix a pair (D, f): $D \subset \mathbb{C}$ – dom. $f \in \mathcal{O}(D)$

analytic elt.

Then the entire collection of all possible anal. elt. obtained via chain contin. of (D, f), is called a complete analytic func. (in the sense of Weierstrass).

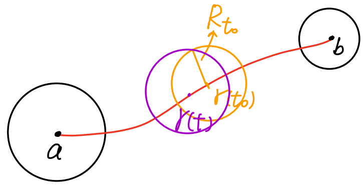
$\operatorname{Ln} z$, $\sqrt[n]{z}$ can be considered like that.

Ex. if $f \in \mathcal{O}(\mathbb{C})$, then the AF built out of (D, f) $D \in \mathbb{C}$, can be identified with the original f. (by uniqueness thm)

Def 2. of AF.

Let $(B_R(a), f)$ be a canon anal. elt. (canonically) (means, $f \in \mathcal{O}(B_R(a))$, R – the radius of conv. of f at $z = a$, i.e. $R = \max \cdots$) Next, let γ be a (contin.) curve, $\gamma(t) : [0,1] \to \mathbb{C}$, $a = \gamma(0)$,

$b=\gamma(1)$. Then a canon anal. elt. $(B_{\tilde R}(b), \tilde f)$ is called the anal contin. of $(B_R(a), f)$ along γ, if $\exists$ a family of canon. elt.s $(B_{R_t}(\gamma(t)), f_t)$, s.t. for $t=0, 1$, we get resp. $(B_R(a), f)$ and $(B_{\tilde R}(b), \tilde f)$, and the following holds:



$\hookrightarrow$ infinite.

$\forall \varepsilon>0$, $\forall t_0\in[0,1]$, $\exists \delta>0$: $\forall t$ with $|t-t_0|<\delta$ we have $|\gamma(t)-\gamma(t_0)|<\varepsilon$, and $(B_{R_t}(\gamma(t)), f_t)$ is the direct anal. contin. of $(B_{R_{t_0}}(\gamma(t_0)), f_{t_0})$.

Finally, an AF is the collection of all canon. elt.s, obtained from a given one $(B_R(a), f)$ by anal contin. along all possible γs' as above.

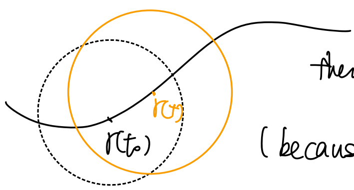
AF in Ω: $\Omega \supset B_R(a)$, and we were able to extend $(B_R(a), f)$ along all possible γs' as above with $[\gamma]\subset\Omega$.

Lemma. In the def of contin of f, the func $t\longrightarrow R_t$ is contin. on $[0,1]$.

In partic., $\exists\ \varepsilon := \min\limits_{[0,1]} R_t > 0$.

Proof: fix $t_0\in[0,1]$. $\forall\varepsilon>0$, $\exists\delta>0$: $\forall t$, $|t-t_0|<\delta$, $|\gamma(t)-\gamma(t_0)|<\varepsilon$. and direct anal. cont. holds.

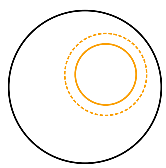
Pick $\varepsilon = R_{t_0}$, then $\gamma(t)\in B_{R_{t_0}}(\gamma(t_0))$.



then $\partial B_{R_{t_0}}(\gamma(t_0)) \cap \partial B_{R_t}(\gamma(t)) \neq \emptyset$.

(because otherwise $B_{R_t}\supset B_{R_{t_0}}$ or vice versa, and this contradicts with the max. of R_t, R_{t_0})

Now, triangle ineq. gives: $|R_t - R_{t_0}| \leq |\gamma(t)-\gamma(t_0)| \Rightarrow R_t \in C[0,1]$.



Thm 1. (Uniqueness)

The anal. contin. of a canon. elt. $(B_R(a), f)$ along a path γ is unique. $\to$ indep. of the family along γ.

Proof: Let $\{B_{R_t}, f_t\}$, $\{B_{\tilde R_t}, \tilde f_t\}$ be two families.

Let $E=\{t\in[0,1] : f_t=\tilde f_t$ in $B_{R_t}\cap B_{\tilde R_t}\}$

We need: $E \ni 1$.

So, first $E \neq \emptyset$, since $0 \in E$. ($f_0 = \tilde{f}_0$: given!)

Next, E is open: indeed, let $t_0 \in E$, take $\varepsilon = \min_{t \in [0,1]} \{R_t, \tilde{R}_t\}$. $\exists \delta > 0$, s.t.

$\forall t$, $|t - t_0| < \delta$, $|\gamma(t) - \gamma(t_0)| < \varepsilon$. and the anal. contin. prop holds for both families

Here ($t = t_0 \in E$): $f_t = f_{t_0}$, $\tilde{f}_t = \tilde{f}_{t_0}$, and $f_{t_0} = \tilde{f}_{t_0}$ by $t_0 \in E$.

$\Rightarrow f_t = \tilde{f}_t$ here $\Rightarrow$ By uniqueness thm: $f_t = \tilde{f}_t$ in $B_{R_t} \cap \tilde{B}_{R_t}$

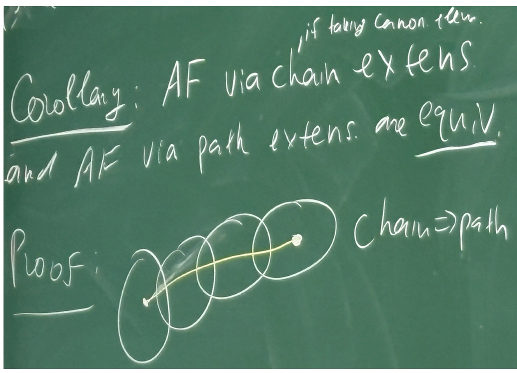
$\Rightarrow E$ is open at $t_0 \Rightarrow E$ is open.

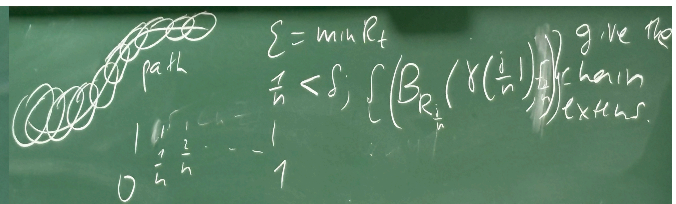
Analog: E is closed: if $t_j \in E$, $t_j \to t_0 \Rightarrow t_0 \in E$.

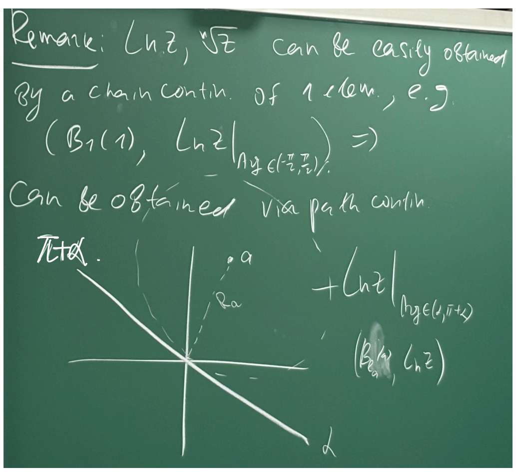
do the same arrangement, get the same picture.

$\Rightarrow$ arrive to the analog. conclus: $f_{t_0} = \tilde{f}_{t_0} \Rightarrow t_0 \in E$.

$\Rightarrow E \subset [0,1]$: open & closed $\Rightarrow E = [0,1]$. (since $[0,1]$ is connected) $\Rightarrow 1 \in E$. $\square$



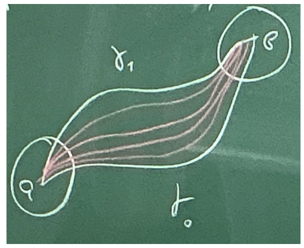




$$\operatorname{Ln} z \sim \bigcup_{a,n} \{ B_{R_a}(a), f_a^{\,n} \}.$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ | Ln.

Thm (Homotopy Thm). Let F be an AF in Ω. Let $(B_R(a), f)$ be its elt.
Let $(B_\xi(b), g)$ _ $(B_\xi(b), \tilde{g})$ be two of its anal. contin. s along homotopic in Ω paths γ_0, γ_1. Then $g = \tilde{g}$.



Proof: Let $\underset{\gamma_s(t)}{\overset{\shortparallel}{\gamma(t,s)}} : [0,1] \times [0,1] \xrightarrow{\text{contin.}} \mathbb{C}$ be the homotopy.

$\gamma(t,0) = \gamma_0(t)$, $\gamma(t,1) = \gamma_1(t)$.

and $\gamma_s(0) = a$, $\gamma_s(1) = b$. $\forall s$.

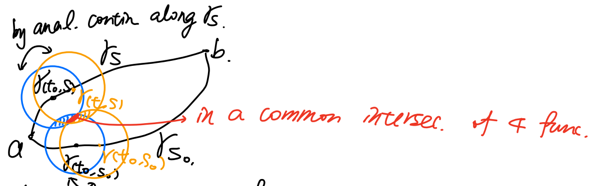
$E = \{ s \in [0,1] :$ extentions of $(B_R(a), f)$ along γ_s coincide with $g \}$.

We need $1 \in E$. (will prove $E = [0,1]$)

First, $E \neq \emptyset$, $0 \in E$.

Second, E is open: indeed, fix $s_0 \in E$, and prove that E is open at s_0.

Fix $\varepsilon = \min\limits_{t \in [0,1]} R_{t,s_0}$, pick δ to fulfill the uniform contin. of $\gamma(t,s)$ on $[0,1] \times [0,1]$ with $\frac{\varepsilon}{2}$.



Now, take $s : |s - s_0| < \delta$

Take $A = \{ t \in [0,1] ; f_{t,s}$ is the direct contin of $f_{t,s_0} \}$.

$A \neq \emptyset$ since $A \ni 0$. $(f_{0,s} = f_{0,s_0} = f)$.

Next, A is open at $\forall t_0 \in A$: take t, $|t - t_0| < \delta$.

By uniqueness thm: all 4 funcs coincide in the intersec!

$\Rightarrow t \in A \Rightarrow A$ is open.

$f_{1,s} = f_{1,s_0} \Rightarrow s \in E \Rightarrow E$ is open at s_0.

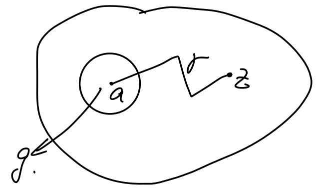
Analog. E is closed. $\Rightarrow E = [0,1]$. $\square$.

Corollary (Monodromy Theorem)

Let Ω – a simply-conn. dom. F be an AF in Ω.
Then in fact F represents a (single-valued) $f \in \mathcal{O}(\Omega)$,
i.e. $\forall$ elt. $(B_R(a), g) \in F$, $g = F|_{B_R(a)}$.

Proof: choose $\forall (B_R(a), g) \in F$. Now $\forall z \in \Omega$, find γ (e.g a broken line)

$\gamma \subset \Omega$ and connects a and z.



Let $(B_\xi(z), h)$ be the cont. of $(B_R(a), g)$ along γ.

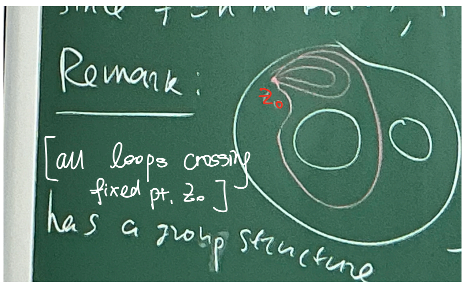
Then we set $f(z) := h(z)$

Ω is simply-conn. $\Rightarrow \forall$ other γ' as above,

$\gamma' \overset{hom.}{\sim} \gamma \Rightarrow$ by Homotopy Thm, h is defined uniq.

$\Rightarrow f(z)$ is unique. (single-valued)

Since $f = h$ in $B_R(a)$, $f \in \mathcal{O}(B_R(a)) \Rightarrow f \in \mathcal{O}(\Omega)$



$\gamma \sim \gamma'$ if γ is homotopic to γ'.

$\Pi_1(\Omega)$ – the fundemental group

– the space of equiv. classes.

If Ω is simply-connected. $\Pi_1(\Omega) = \{e\}$.

$\Omega = \{r < |z| < R\}$. $\Pi_1(\Omega) = \mathbb{Z}$.

Take $\gamma_1, \gamma_2, \cdots, \gamma_s, \cdots$ be generators.

Claim: if F is an AF in Ω, $(B_R(z_0), f) \in F$; $(B_R(z_0), f) \xrightarrow{\gamma_j} (B_R(z_0), f)$ $\forall j$.

$\Rightarrow F$ represents $\varphi \in \mathcal{O}(\Omega)$

Thm (Poincaré – Wolterra Thm)

Let F be an AF in Ω.

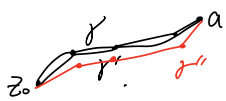
Let $A = \{(B_R(a), f)\}$ be the collection of all the elt. of F at a.

Then the set $\{f(a)\}$ is at most countable.

In fact. the coll. of distinct elt.s is $\leq$ countable.

Proof: $\forall$ elt. E_a is obtained: $E_{z_0} \overset{\gamma}{\rightsquigarrow} E_a$, z_0: fixed.

Then choose a broken line $\gamma' \sim \gamma$ connecting z_0, a.



Finally, find another broken line γ'' connecting z_0, a, and having rational vertices (besides z_0, a) $\gamma'' \sim \gamma'$.

$\mathbb{Q}^2 = \underset{\substack{\downarrow \\ x}}{\mathbb{Q}} \times \underset{\substack{\downarrow \\ y}}{\mathbb{Q}}$ vertices is countable. $\Rightarrow \exists$ countably many $\{\gamma''\}$, $\gamma'' \sim \gamma$

$\Rightarrow$ anal. contin. again gives E_a. $\Rightarrow$ we have $\leq$ countable $\{E_a\}$.

<u>(Arythmetric) operations with AF and beyond.</u>

1) $F \pm G$, $F \cdot G$, F/G.

For ex., $F+G$: consider all poss. choices

$$\{(\underset{\substack{\cap \\ F}}{B_R(z_0), f}), (\underset{\substack{\cap \\ G}}{B_\zeta(z_0), g})\}. \qquad (z_0 \text{ - fixed base pt.})$$

Then, we extend $f+g$ defined in $B_R(z_0) \cap B_\zeta(z_0)$ along $\forall$ path in Ω starting at z_0.

We end up with (1 or more) AF.

$\leq$ count. many.

<u>Ex.</u> $\sqrt{z} + \sqrt{z} = \begin{cases} 2\sqrt{z} & \text{choose base pt. choice. } f, g \text{ belong to the same branch} \\ 0. & \text{diff. branch.} \end{cases}$

$\mathrm{Ln}\, z - \mathrm{Ln}\, z = \{2\pi k\}_{k \in \mathbb{Z}}$.

<u>Rk</u>: if either F or $G \in \mathcal{O}(\Omega)$, then the outcome is a (single) AF.

$\sin z \cdot \sqrt{z}$.

2) Compositions.

F is an AF in Ω, and the values of el.s of F lie in D. and G is an AF in D

Then $G \circ F$ — the comp. of AF, obtained by extending $\{\overset{\in G}{g} \circ \overset{\in F}{f}\}$ along γ's in Ω.

<u>Special case:</u> if $G \in \mathcal{O}(\mathbb{D})$, then $G \circ F$ is a single AF. (Ex. $\sin\sqrt{z}$)

<u>Ex.</u> $\sqrt{z^4} \longrightarrow \begin{cases} z^2 & g(1) = 1. \\ -z^2 & g(1) = -1. \end{cases}$ $\quad F = z^4.\ G = \sqrt{z}.$

$\Omega = \mathbb{C} \setminus \{0\}.$

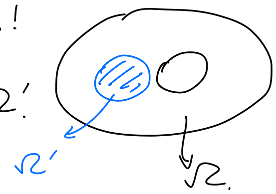
3) <u>Restricting onto subdomain.</u>

Let F be an AF in Ω; $\Omega' \subset \Omega$.

Then considering elt.s of F centered at pt.s $a \in \Omega'$, and squeezing $B_R(a)$ to achieve $B_R(a) \subset \Omega'$, we consider the resulting elt.s.

This gives generally a coll-n (at most countable) of AFs!

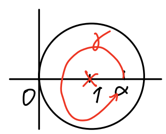
$\gamma \subset \Omega'$.



<u>Ex.</u> $\sqrt{1+\sqrt{z}}$. one can prove: a (single) AF.

$\Omega = \mathbb{C} \setminus \{0, 1\}.$

But when restricted onto $\Omega' = B_1(1) \setminus \{1\}$.



if choosing f: $f(\alpha) \in \mathbb{R}^+ . \Rightarrow f(\gamma) \in \{\mathrm{Re}\, z > 0\}$.

we arrive to two AFs in Ω' (depending on the choice of $g|_{\mathbb{R}^+}$) | two 1-valued func.s.

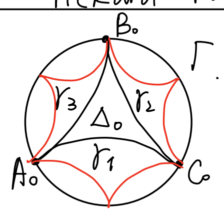
if choosing $f(\alpha) \in \mathbb{R}^- \Rightarrow$ a 2-valued AF.

<u>Application of AFs: Pickard Thm.</u>

<u>Modular function:</u>

$\Gamma = \partial B_1.$

$\gamma_j \perp \Gamma,\ j = 1, 2, 3.$



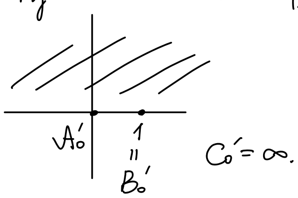
$\varphi_0(A_0) = 0.$

$\varphi_0(B_0) = 1.$

$\varphi_0(C_0) = \infty.$

By Riemann Mapping Thm, $\exists$ a conf. $\varphi_0 : \Delta_0 \longrightarrow \Pi^+$.

By Carath Thm, φ_0 extends to a homeom: $\bar{\Delta}_0 \longrightarrow \overline{\Pi^+}$.



$C_0' = \infty.$

By Schwarz Refl. Princ, φ extends to 3 dom-s $\Delta_1, \Delta_2, \Delta_3$ — symm. images of Δ_0 w.r.t. $\gamma_1, \gamma_2, \gamma_3$

Get $\varphi_1, \varphi_2, \varphi_3$ mapping resp. $\Delta_1, \Delta_2, \Delta_3$ to Π^+.

Note: $\varphi_0 \cup \varphi_1 \cup \varphi_2 \cup \varphi_3$ is not a conf. map!

Then, we proceed similarly and reflect the maps $\varphi_1, \varphi_2, \varphi_3, \ldots\ldots$

Continue infinitely,

We end up with a hol. map μ: $B_1 \longrightarrow \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$.

<u>Def</u>: $\mu(z)$ is called the <u>modula function</u>.

<u>Remark</u>: μ^{-1} is an AF in $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. all valued in B_1

because it's obtained from chain extensions of the conf. $\varphi_0^{-1} \in \mathcal{O}(\Pi^+)$

<u>Thm (Small Pickard Thm)</u>

If a func $f \in \mathcal{O}(\mathbb{C})$ has two exception values a, b. (i.e. $f(\mathbb{C}) \subset \mathbb{C} \setminus \{a, b\}$) $a \neq b$. then $f = \text{const}$.

<u>Proof</u>: Let $f \neq$ const. Let $L \in \mathrm{Aff}(\mathbb{C})$, mapping $\{a, b\} \longrightarrow \{0, 1\}$.

Take $g = L \circ f$, $g \in \mathcal{O}(\mathbb{C})$, $g(z) \neq \{0, 1\}$. $g \neq$ const.

So $g(\mathbb{C}) \subset \overline{\mathbb{C}} \setminus \{0, 1, \infty\} \Rightarrow$ we may consider the composition of AFs:

$H := \mu^{-1} \circ g$; it gives (a coll of) AFs in $\mathbb{C}$;

but $\mathbb{C}$ is simply-conn. $\Rightarrow H$ is a (coll. of) hol. func (by Monod Thm)

Take a representative h of H; $h \in \mathcal{O}(\mathbb{C})$, $|h| \leq 1$. (since μ^{-1} is valued in B_1)

$\Rightarrow h =$ const (by Liouville).

— a contradiction. since $\mu^{-1} \circ g \neq$ Const.

<u>Corollary</u>. (Small Picard Thm for Merom. func.s)

Let $f \in \mathrm{Mer}(\mathbb{C})$, f has excep. values $\{a, b, c\} \in \overline{\mathbb{C}}$ (all distinct). then $f =$ const.

Proof: choose a lin. frac. $L:\{a,b,c\}\to\{0,1,\infty\}$.

Then $g:=L\circ f$ is Merom. in $\mathbb{C}$, and $g\neq\infty$ $\Rightarrow$ $g\in\mathcal{O}(\mathbb{C})$, $g\neq\{0,1\}$.

$\Rightarrow$ By the hol. Thm: $g=$ const. $\Rightarrow$ $f=$ const.

Ex. 1. $f=e^z\in\mathcal{O}(\mathbb{C})$, then $f\neq 0$.

Ex 2. $f=\tan z\in \mathrm{Mer}(\mathbb{C})$, then $f\neq\pm i$.

So, the Small Picard Thm is sharp.

Upshot: Small Picard Thm:

$$f\in\mathcal{O}(\mathbb{C})\Rightarrow f(\mathbb{C})=\mathbb{C}\setminus E,\quad E=\begin{cases}\emptyset\\ \{p\}.\end{cases}$$

$$f\in \mathrm{Mer}(\mathbb{C})\Rightarrow f(\mathbb{C})=\overline{\mathbb{C}}\setminus E,\quad E=\begin{cases}\emptyset\\ \{a\}\\ \{a,b\}\end{cases}$$

$f=\tan z$: $\{\pm i\}$. $\tan z=i$

$$\Leftrightarrow \frac{1}{i}\,\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}=i\Rightarrow 2e^{iz}=0.\quad \text{no sol.}$$

Ex. prove: $f=z+e^z$ takes $\forall$ value $a\in\mathbb{C}$.

$f\big|_{\mathbb{R}}$: $f(+\infty)=+\infty$. $f(-\infty)=-\infty$, $f\in C(\mathbb{R})\Rightarrow f(\mathbb{R})=\mathbb{R}$.

$\Rightarrow$ If a — missing value, then $a\notin\mathbb{R}$.

Also, $\bar a$: also missing, since if $z+e^z=\bar a\Rightarrow \bar z+e^{\bar z}=a$. — Contrad.

$\Rightarrow$ 2 missing values. a & $\bar a$. $\Rightarrow f=$ const. — Contrad.

Development: $ze^z=a$. $f(\mathbb{R})\neq\mathbb{R}$.

e.g. no "$-\infty$".

$f(0)=0\Rightarrow 0$ is not missing

For $z\neq 0$: $z=e^{w}$ $(\exists w)$.

Trick: $g(w) = e^w e^{e^w} = e^{w+e^w}$ — func. above.
$w+e^w = h(w)$.

$h(\mathbb{C}) = \mathbb{C} \Rightarrow g(\mathbb{C}) = \mathbb{C}\setminus\{0\} \Rightarrow f(\mathbb{C}) = \mathbb{C}$.

Ex. $f, g \in \mathcal{O}(\mathbb{C})$, $f^3+g^3=1$.

Prove: $f, g =$ const.

$$g^3\left(\left(\frac{f}{g}\right)^3+1\right)=1. \qquad h \in Mer(\mathbb{C}).$$

$\frac{f}{g} = h$. $\qquad g^3(h^3+1)=1$.

Claim: h misses all the 3: $\{\sqrt[3]{-1}\}$.

indeed, if $h(a) \in \sqrt[3]{-1}$, $g(a)\neq 0$ — contrad.

$h(a)\in\sqrt[3]{-1}$, $g(a)=0$ — still contrad.

$\Rightarrow h = $ const. $\Rightarrow f = cg \Rightarrow g^3(c^3+1)=1 \Rightarrow g=$ const $\Rightarrow f =$ const.

Remark: Big Picard Theorem.

If a – isol. sing. for f, a-essential,

then $f(B^*_\varepsilon(a)) = \overline{\mathbb{C}}\setminus E$, $E = \begin{cases}\varnothing \\ \{a\} \\ \{a,b\}\end{cases}$. $\forall$ small ε!

Come back to AFs.

Another application of Monod Thm:

If Ω – simply-conn, $f\in\mathcal{O}(\Omega)$, $f\neq 0 \Rightarrow \exists\, g\in\mathcal{O}(\Omega)$: $f = e^g$ in Ω.

Idea: Consider $\ln f := g$ — AF in Ω. thanks to $f\neq 0$.

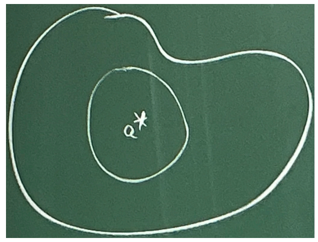
Ω – simply-conn. $\Rightarrow$ by Monod Thm: $g\in\mathcal{O}(\Omega)$.

Another proof: $g :=$ anti-der. of $\frac{f'}{f}$.

Isolated sing.s (branch pts) of AFs

Def: Let F be an AF in Ω, let $\Omega \supset B^*_\varepsilon(a)$ $\quad(\Omega \not\ni a)$

Then we call $\underset{\triangle}{a}$: a branch pt. for F.



More precisely: consider $F\big|_{B^*_\varepsilon(a)}$ — this gives a collection $\{F_\alpha\}_{\alpha\in A}$ ($\#A \le \aleph_0$) of AFs.

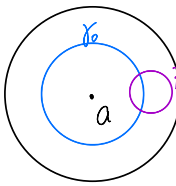
Then $\underset{\triangle}{a}$ becomes a <u>collection</u> of branch pts. for respective F_α.

We call that $\underset{\triangle}{a}$ <u>contains</u> branch pts. for $\{F_\alpha\}$.

For the moment, let $\Omega = B^*_\varepsilon(a)$.

Then $\exists$ 3 senarios for the branch pt. a.

1)



γ_0: simple loop.

F_i - elt. of F.

$F_0 \xrightarrow{\gamma_0} F_0 \Rightarrow \underset{\triangle}{a}$ is called single valued branch pt.

in fact, $F \in \mathcal{O}(\Omega)$.

Ex. $F = \dfrac{\sin(\sqrt{z})}{\sqrt{z}}$. can take diff. branches. $\Omega = \mathbb{C}\setminus\{0\}$.

two analy. funcs.

$F = \{F_1, F_2\}$ $\qquad F_1, F_2 \in \mathcal{O}(\Omega)$

$$F_1 = \frac{\sqrt{z} - \frac{(\sqrt{z})^3}{6} + \cdots}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \cdots \qquad F_2 = -F_1.$$

Here $z=0$ contains two single-valued branch pts.

2). $F_0 \xrightarrow{\gamma_0} F_1 \xrightarrow{\gamma_0} F_2 \xrightarrow{\gamma_0} \cdots \xrightarrow{\gamma_0} F_m = F_0$.

Here, $\underset{\triangle}{a}$ is called an algebraic branch pt.

m is called <u>order</u> of branching

(F is in fact $\le m$-valued)

Ex. $F = \sqrt[m]{z}$. $a=0$. $\sqrt[m]{z} = \sqrt[m]{|z|}\cdot e^{i\frac{\operatorname{Arg} z}{m}}$.

3). $F_0 \xrightarrow{\gamma_0} F_1 \xrightarrow{\gamma_1} F_2 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_{k-1}} F_k \xrightarrow{\gamma_k} F_{k+1} \xrightarrow{\gamma_{k+1}} \cdots$ $\qquad$ never $F_i = F_j$, $i\ne j$!

then $\underset{\Delta}{a}$ is called a logarithmic branch pt.

Ex. $F = \operatorname{Ln} z$. $z=0$, $z=\infty$. Remark: $a=\infty$ is possible!

$$F_k = \operatorname{Ln} z + 2\pi k i.$$

org.

Puiseux Series.

Def: a Puiseux Series centered at $z=a$ is a series of the kind:

$$f = \sum_{n=-\infty}^{+\infty} c_n (z-a)^{\frac{n}{m}}, \quad m \in \mathbb{N},\ n \in \mathbb{Z}.$$

order

fixed.

$$= \sum_{n=-\infty}^{+\infty} c_n \left(\sqrt[m]{z-a}\right)^n.$$

Thm: Let F be an AF in $\Omega = B^*_\varepsilon(a)$.

Let $z=a$ be an alg. branch pt. for F of order $=m \geq 2$.

Then F is described in Ω by a (converg.) Puiseux series:

$$F(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^{\frac{n}{m}}$$ (the branches of $\sqrt[m]{z-a}$ are all the terms agree)

In partic., $F = \varphi \circ G$, $G = \sqrt[m]{z-a}$. $\varphi \in \mathcal{O}(B^*_\delta(0))$

Proof: Consider $\chi(w) := a + w^m$, $\chi(B_\delta(0)) = B_\varepsilon(a)$. ($\delta^m = \varepsilon$)

Then consider $H(w) := F \circ \chi$ – an AF in $B^*_\delta(0)$

Take a simple loop $\Gamma_0 \subset B^*_\delta(0)$.

$\chi(\Gamma_0) = (\gamma_0)^m$, γ_0 – loop around a.

$$\begin{tikzcd} \Gamma_0 \subset B^*_\delta(0) \arrow[d, "\chi"] \arrow[dr, "H"] & \\ (\gamma_0)^m \subset B^*_\varepsilon(a) \arrow[r, "F"] & {} \end{tikzcd}$$

$\Rightarrow$ by the def of alg. branch pt. $H_0 \xrightarrow{\Gamma_0} H_0$

elt. of H $\quad \int$

$$F_0 \xrightarrow{\gamma_0^m} F_0$$

$\Rightarrow$ $w=0$ is a simple-valued branch pt. for H.

$\Rightarrow$ $H \in \mathcal{O}(B^*_\delta(0))$ $\Rightarrow$ since $H = F \circ \chi$. $F = H \circ \chi^{-1}$, $\chi^{-1}(z) = \sqrt{z-a}$.

By Laur. expan Thm:

$$H(w) = \sum_{n=-\infty}^{+\infty} C_n w^n \; ; \quad w = \sqrt[m]{z-a}.$$

$\Rightarrow$ We get the Puiseux expan.

<u>Ex.</u> $F = \sqrt{\operatorname{Ln} z}$.

branch pt.s in $\overline{\mathbb{C}}$: $z=0$, $z=1$, $z=\infty$.

<u>$z=1$:</u> in $B_\varepsilon(1)$:

$$\underset{\substack{| \\ f(1)=0.}}{\overset{\circ}{\operatorname{Ln}}} z = (z-1) - \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} - \cdots$$

$$\Rightarrow \sqrt{\overset{\circ}{\operatorname{Ln}} z} = \sqrt{z-1} \cdot \underbrace{\sqrt{1 - \frac{(z-1)}{2!} + \frac{(z-1)^2}{3!} - \cdots}}_{\text{single-valued.}}$$

taylor series

fix $\forall$ of the two roots.

developing this, one gets the Puiseux ser.

$z=1$: contains a branch pt of order 2,

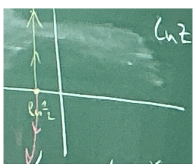
Now other $\overset{(k)}{\operatorname{Ln}} z$: $f(1) = 2\pi k i$, $k \neq 0$.

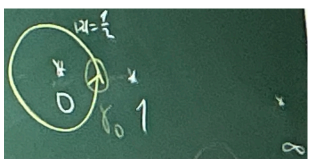
$\forall k$, get ∂ single-valued branch pts.

$\Rightarrow$ overall, $z=1$ contains an alg. branch pt of ord $=2$, and ∞ single-valued branch pts.

<u>$z=0$:</u> Since $z=0$ is log branch pt. already for $\operatorname{Ln} z$,

it will be such for $\sqrt{\operatorname{Ln} z}$ too.





Applying $\operatorname{Ln} z$ to γ_0, we remain in $\{\operatorname{Re} w < 0\}$. where $\sqrt{z}$ doesn't branch.

$\Rightarrow$ choosing φ_1, φ_2 — the two hol. branches of $\sqrt{w}$, our $F|_{B_1(0)}$ splits into

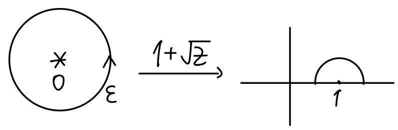
$\{F_1, F_2\}$, $F_1 = \varphi_1 \circ \operatorname{Ln} z$, $F_2 = \varphi_2 \circ \operatorname{Ln} z$.

$\Rightarrow$ <u>$z=0$ contains two log branch pts.</u>

$z=\infty$: $\ln z = -\ln \frac{1}{z} \Rightarrow 0 \Leftrightarrow \infty$. $\Rightarrow$ same picture with $z=\infty$.

Ex. $F(z) = \sqrt{1+\sqrt{z}}$. $z=0$, $z=1$, $z=\infty$.

$z=0$:



$\{\mathrm{Re}\, z > 0\}$.

φ_1, φ_2: two branches of $\sqrt{z}$ in $\{\mathrm{Re}\, z > 0\}$.

$G(z) := 1+\sqrt{z}$.

$\Rightarrow F_1 = \varphi_1 \circ G$, $F_2 = \varphi_2 \circ G$.

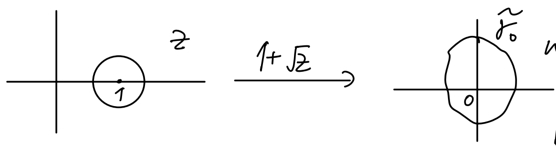
$z=0$: contains two branch pts. of ord $=2$.

$z=1$: if around $z=1$ we choose $\sqrt{z}$ branch, $f(1+\varepsilon)>0$

$\Rightarrow$ image of γ_0 lies in $\{\mathrm{Re}\, w > 0\}$.

$\Rightarrow$ get 2 single-valued branch pts.

but, if choosing $\sqrt{z}$ branch with $f(1+\varepsilon)<0$.



branch pt of ord $=2$.

So, $z=1$ contains $\begin{cases} 2 & \text{single-valued branch pts} \\ 1 & \text{branch pt. with ord}=2. \end{cases}$

$z=\infty$: 1 branch pt. of ord $=4$.

$$F(z) = \sqrt[4]{z} \cdot \underbrace{\sqrt{\frac{1}{\sqrt{z}}+1}}_{\text{Hol. branch}}.$$

Riemman Surface diagram of an AF.

Global characterization of AF.

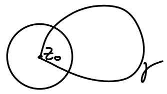
Let $D \subset \overline{\mathbb{C}}$ – simply-conn. dom, $E \subset D$ – discrete set.
branching locus.

$\Omega = D \setminus E$. F – AF in Ω.

* We consider a collection of „copies" of Ω (which we usually display by line intervals)
corresponding to all the el't.s of an AF near z_0.

* We mark all the branch pts. $a \in E$.

* We fix a regular (base) pt. $z_0 \in \Omega$.



and then $\forall \gamma \in \pi_1(\Omega)$, we study the permutation of el't.s of F near z_0.

el't. of $\begin{cases} S_n \\ S_\infty \end{cases}$ $F_0 \xrightarrow{\gamma} F_1$.

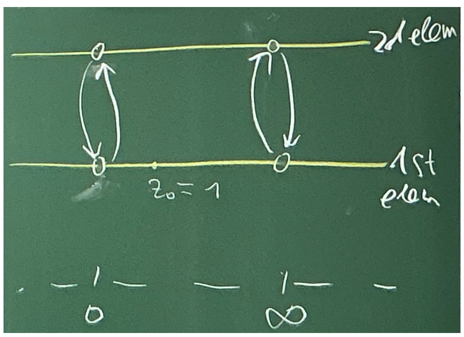
(AF is associated to a representation of $\pi_1(\Omega)$ in S_n or S_∞)

We fix the generators of Ω: loops $\gamma_1, \gamma_2, \cdots$ around $\underbrace{a_0, a_1, \cdots}_{E}$

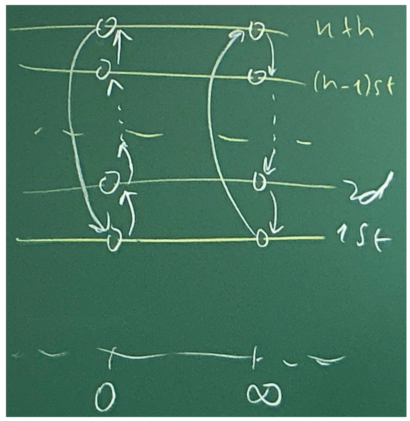
and get a coll of permutations: $T_1, T_2, \cdots\cdots$

* Finally, we display this permutations on the diagram.

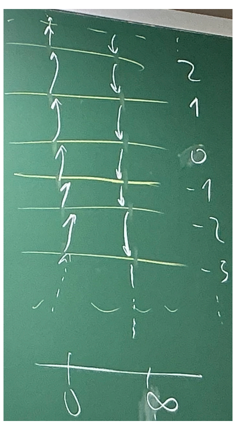
Ex. $F(z) = \sqrt{z}$. $\Omega = \overline{\mathbb{C}} \setminus \{0, \infty\}$.

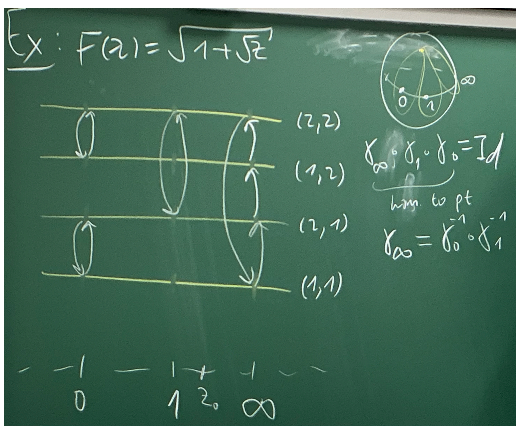


Ex. $F(z) = \sqrt[n]{z}$.



Ex. $F(z) = \operatorname{Ln} z$.





Ex: $F(z) = \operatorname{Arc\,sin} z$

– will send.

Additional Topics

Weierstrass Factorization Thm:

Let $\Omega \subset \mathbb{C}$ - dom ; $E \subset \Omega$ - a discrete subset. $E = \{a_1, a_2, a_3, \dots\}$.

Consider $\{k_1, k_2, k_3, \dots\} \subseteq \mathbb{N}$ - coll. of orders.

Then, $\exists\ f \in \mathcal{O}(\Omega)$ - s.t.

1) $\{f=0\} = E$.
2) $\operatorname{ord}_{a_j} f = k_j$.

$$f = \prod_{j=1}^{\infty} (z-a_j)^{k_j} e^{g_j}$$

Application 1: $\forall\ f \in \mathrm{Mer}(\Omega)$ is a rotation of hol:

$$\exists\ g, h \in \mathcal{O}(\Omega),\ f = \frac{g}{h}.$$

Proof: Let $E = \{$ the set of poles for $f\}$

discrete.

$E = \{a_1, a_2, \dots\}$. Let k_j - ord. of pole at a_j.

Build $h : \{h=0\} = E$, $\operatorname{ord}_{a_j} h = k_j$.

Set $g := f \cdot h$, $g \in \mathcal{O}(\Omega \setminus E)$

but $\forall a_j$ - removable sing. for g ($(z-a_j)^{k_j}$ cancels).

$\Rightarrow g \in \mathcal{O}(\Omega) \Rightarrow f = \dfrac{g}{h}$.

Application 2: $\forall \Omega \in \mathbb{C}$, $\exists\ f \in \mathcal{O}(\Omega)$, s.t f can't be extended to $\tilde{\Omega} \supsetneq \Omega$ holomorphically.

Proof: Build a discrete $E \subset \Omega$: $\forall\ p \in \partial\Omega$, is an accum. pt. of E.

Then take $f \in \mathcal{O}(\Omega)$: $\{f=0\} = E$, $f \not\equiv 0$.

f can't extend hol. to p. (by Uniqueness Thm.)

Runge Thm.

Thm. Let $\Omega \in \mathbb{C}$, $K \subset \Omega$ (cpt.); $f \in \mathcal{O}(\Omega)$.

Then $\forall \varepsilon > 0$, $\exists R(z)$ – rational func. with poles in $\mathbb{C} \setminus K$.

s.t. $|f(z) - R(z)| < \varepsilon$, $z \in K$.

(rational approximation).

Proof: Take $\overline{D} \subset \Omega$ an admissible dom. $D \supset K$,

∂D – broken line. $\partial D \cap K = \emptyset$.

We have in $\overline{D}$ the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in K.$$

Consider „integ. sums" $R_n(z) = \sum_{j=1}^{n} \frac{f(\zeta_j)}{\zeta_j - z} (\zeta_j - \zeta_{j-1}) \frac{1}{2\pi i}$.

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n+1} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta. \quad R_n(z) = \frac{1}{2\pi i} \sum_{j=1}^{n+1} \int_{\gamma_j} \frac{f(\zeta_j)}{\zeta_j - z} \, d\zeta$$

$$|f(z) - R_n(z)| \leq \frac{1}{2\pi} \sum_{j=1}^{n+1} \left| \int_{\gamma_j} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta_j)}{\zeta_j - z} \right) d\zeta \right|$$

$$\leq \{\text{continous of } f, \text{ smallness of } \gamma_j\}$$

$$\leq \frac{1}{2\pi} \cdot \varepsilon \cdot M \cdot \sum_{j=1}^{n+1} \gamma_j = C \cdot \varepsilon.$$

Ex. $f(z) = \frac{1}{z}$ in $\{1 < |z| < 2\} = \Omega$ can't be approximated by polys.

$\int_{|z| = \frac{3}{2}} P_n(z) \, dz = 0$, while $\int \frac{1}{z} \, dz = 2\pi i$. $f \in \mathcal{O}(\Omega)$

Thm. (Polynomial Runge)

Let $\Omega \in \mathbb{C}$, $K \subset \Omega$, $\Omega \setminus K$ – conn.

Then, $\forall \varepsilon > 0$, $\exists P(z)$ – poly. s.t. $|f(z) - P(z)| < \varepsilon$, $z \in K$.

Proof: „approximable by poly. func" forms an algebra.

$\Rightarrow$ since we may approximate f by rationals and $\forall R(z) = \sum_{j=1}^{n} \frac{A_j}{(z - a_j)^{m_j}}$. $a_j \in \mathbb{C} \setminus K$.

$\Rightarrow$ it's enough to approximate $f = \frac{1}{z-a}$, $a \in \mathbb{C} \setminus K$.

$E = \{ a \in \mathbb{C} \setminus K : f_a = \frac{1}{z-a}$ can be approximated to $\forall \varepsilon$ by polynomials$\}$.

$*$ $E \neq \emptyset$: $K \subset B_R(0)$. if $|a| > R$, then $\frac{1}{z-a} = \frac{1}{a} \cdot \frac{1}{1-\frac{z}{a}} = -\frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$ unif. conveg. on K.

$\Rightarrow$ sums approx. $\frac{1}{z-a}$.

$*$ E is open: if $a' \in B_\delta(a)$, $\max_K \left| \frac{1}{z-a} - \frac{1}{z-a'} \right| \xrightarrow{\delta \to 0} 0$.

$\Rightarrow$ if $\frac{1}{z-a}$ admits appro. same holds for close a'.

$*$ E is closed: analog

$\Rightarrow$ since $\Omega \setminus K$ - connected $\Rightarrow$ $\mathbb{C} \setminus K$ - connected.

$\Rightarrow$ $E = \mathbb{C} \setminus K$ $\Rightarrow$ $\forall \frac{1}{z-a}$ can be approximated.

<u>Corollary</u>: if $\Omega \subset \mathbb{C}$ - simply connected.

then $\forall f \in \mathcal{O}(\Omega)$, $\exists \{P_n(z)\}$ - poly. $P_n \xrightarrow[\Omega]{} f$.

<u>Proof</u>: This follows from $\exists$ cpt. exhausion $K_1 \subset K_2 \subset \cdots$, with $\Omega \setminus K_j$ - conn.

<u>Harmonic functions</u>

<u>Def</u>: a func. u in $\Omega \in \mathbb{C}$ $(u(x,y) \in \mathbb{R})$ is called <u>harmonic</u>,

if it satisfies:

1) $u \in C^2(\Omega)$

2) $\Delta u = 0$, where Δ is the Laplace operator.

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Notation: $u \in H(\Omega)$

<u>Main Ex</u>: Let $f \in \mathcal{O}(\Omega)$, $f = u + iv$.

C-R equations: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ $\Rightarrow \begin{cases} u_{xx} + u_{yy} = 0. \\ v_{xx} + v_{yy} = 0. \end{cases}$ $\Rightarrow$ $\operatorname{Re} f, \operatorname{Im} f \in H(\Omega)$

Q: is the converse true?

Thm: $\forall$ simply-connected dom, $\forall u \in H(\Omega)$, $\exists f \in \mathcal{O}(\Omega)$: $u = \operatorname{Re} f$.

f is called the hol. generator for u.

v is called the harmonic conjugated func.

In partic., $\forall$ dom. $D \subset \mathbb{C}$, $\forall a \in D$, hol. generator exists in $\forall B_R(a) \subset \Omega$.

Proof: we need v: $\begin{cases} v_x = u_y = P \\ v_y = -u_x = Q \end{cases}$ grad $v = \nabla v = (P, Q)$.

when v exists?

Cauchy conditions: $P_y = Q_x$ → necessary

→ suff. for simply-conn. dom.

$u_{yy} = -u_{xx}$ holds!

Remark: in a multi-conn. dom, the hol generator f exists as an AF.

(it's multi-valued!)

Ex: $u = \ln|z| = \frac{1}{2}\ln(x^2+y^2)$.

$F = \operatorname{Ln} z$.

$v = \operatorname{Arg} z$.

special prop: $F_0 \xrightarrow{\gamma_0} F_1$, $F_1 = F_0 + \text{const}$.