Southern University of Science and Technology
Department of Statistics and Data Science

MA204: Mathematical Statistics | **Final Examination (Paper A)**

Date: 18 June 2025 | **Time: 7:00 p.m. – 10:00 p.m.**

(I) Acronyms:

CI	confidence interval
iid	independently and identically distributed
$I(\cdot)$	indicator function
LRT	likelihood ratio test
mgf	moment generating function
MLE	maximum likelihood estimator
MPT	most powerful test
pdf/pmf	probability density/mass function
$\mathbb{R}$	real line, $(-\infty, \infty)$
r.v.	random variable
UMVUE	uniformly minimum variance unbiased estimator
z_α, $t(\alpha, \nu)$, $\chi^2(\alpha, \nu)$	upper α-th quantile of $N(0,1)$, $t(0,1,\nu)$ and $\chi^2(\nu)$
$f(\alpha, \nu_1, \nu_2)$	upper α-th quantile of $F(\nu_1, \nu_2)$

(II) Commonly used pdfs or pmfs:

- Gamma/Exponential distribution. The pdf of $X \sim \text{Gamma}(\alpha, \beta)$ is

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

  Its expectation and variance are $E(X) = \alpha/\beta$ and $\text{Var}(X) = \alpha/\beta^2$, respectively. In particular, $\text{Exponential}(\beta) = \text{Gamma}(1, \beta)$.

- Inverse gamma distribution. The pdf of $Y \sim \text{IGamma}(\alpha, \beta)$ is

$$\frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} \mathrm{e}^{-\beta/y}, \quad y > 0,\ \alpha > 0,\ \beta > 0.$$

- Laplace distribution. The pdf of $X \sim \text{Laplace}(\mu, \sigma)$ is

$$\frac{1}{2\sigma} \exp\left(-\frac{|x-\mu|}{\sigma}\right), \quad x \in \mathbb{R},\ \mu \in \mathbb{R},\ \sigma > 0.$$

- Geometric distribution. The pmf of $X \sim \text{Geometric}(\theta)$ is

$$\Pr(X = x) = \theta(1-\theta)^{x-1}, \quad x = 1, 2, \ldots, \infty, \quad \theta \in (0, 1).$$

Its expectation and variance are $E(X) = 1/\theta$ and $\text{Var}(X) = (1-\theta)/\theta^2$, respectively.

Answer ALL 6 questions. Marks are shown in square brackets

1. [**Total: 40 ms**]. Directly give your answers to the following questions:

1.1 Let two discrete r.v.'s $X, Y \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$ with $p = 0.5$ and define $Z = X + Y - 2XY$. The support of Z is ______ and the value of $\Pr(Z = 1)$ is ______. [**2 ms**]

1.2 Let $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2$ and $X_1 \perp\!\!\!\perp X_2$. Define $Y \triangleq a_1X_1 + a_2X_2$, where a_1, a_2 are two constants, then the distribution of Y is ______.

[**2 ms**]

1.3 (a) If X is a discrete r.v. , what is the definition of the median of X, denoted by $\text{med}(X)$? [**1 mk**]

(b) The pmf of X is defined by $p_i = \Pr(X = i)$ for $i = 1, 2, 3, 4, 5, 6$, where $p_1 = 0.20$, $p_2 = 0.15$, $p_3 = 0.10$, $p_4 = 0.05$, $p_5 = 0.10$ and $p_6 = 0.40$. The median $\text{med}(X)$ of X is ______. [**2 ms**]

1.4 Let $X \sim \text{Exponential}(\beta)$, then $\Pr(X > t + s | X > s) =$ ______, where t and s are two positive real numbers. [**2 ms**]

1.5 Given two conditional densities $f_{(X|Y)}(x|y)$ and $f_{(Y|X)}(y|x)$, if the joint support is a product space, i.e., $\mathcal{S}_{(X,Y)} = \mathcal{S}_X \times \mathcal{S}_Y$, then the sampling-wise formula for the marginal density $f_X(x)$ is ____. [**2 ms**]

1.6 (a) Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} U(0, \theta)$ (i.e., uniform distribution on $(0, 1)$), where $\theta \in (0, 1)$. Does the MLE of θ exist? [**1 mk**]

(b) Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} U(0, \theta]$, where $\theta \in (0, 1)$. Then a sufficient statistic of θ is ______ and the MLE of θ is ______. [**2 ms**]

1.7 Assume that we want to find the unique MLE $\hat{\theta}$ of the concave log-likelihood function $\ell(\theta)$ for $\theta \in \Theta$. What is Newton's method to iteratively calculate the MLE $\hat{\theta}$? [**2 ms**]

1.8 Let $X|(Y=y) \sim \text{Poisson}(y)$ and $Y \sim \text{Gamma}(\alpha, \beta)$ with known $\alpha\,(>0)$ and $\beta\,(>0)$. Then $E(X) =$ ______ and $\text{Var}(X) =$ ______. **[2 ms]**

1.9 State the Lehmann–Scheffé theorem. **[2 ms]**

1.10 State the definition of a pivotal quantity. **[2 ms]**

1.11 Let $\mathbb{C}$ be the critical region of a test for testing H_0: $\theta \in \Theta_0$ against H_1: $\theta \in \Theta_1$. What are the definitions of the Type II error function $\beta(\theta)$ and the power function $p(\theta)$? **[2 ms]**

1.12 How to compare two given tests T_1 and T_2? In other words, under what kind of conditions, we say that T_1 is better than T_2. **[2 ms]**

1.13 State the Neyman–Pearson Lemma. **[2 ms]**

1.14 Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma_0^2)$ with unknown μ and known σ_0^2. Suppose that we want to test the null hypothesis H_0: $\mu = \mu_0$ against H_1: $\mu \neq \mu_0$.

(a) The pivotal quantity $Z =$ ______ and the test statistic $Z_0 =$ ______.

(b) Under H_0, the distribution of Z_0 is ______.

(c) The critical region of size α for the test is ______.

(d) The corresponding p-value is ______. **[4 ms]**

1.15 Let $3.3, -0.3, -0.6, -0.9$ be a random sample from $N(\mu, \sigma^2)$.

(a) If $\sigma = 3$, The 90% CI of μ. **[2 ms]**

(b) What would be the CI of μ if σ were unknown? **[2 ms]**

[Note: $z_{0.05} = 1.645$, $t(0.05, 3) = 2.3534$]

1.16 Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Suppose that $n = 12$, $\bar{x} = 66.3$ and $s = 8.4$. Construct a 95% CI for μ, where $t(0.025, 11) = 2.201$. **[2 ms]**

1.17 Given the null hypothesis H_0, the test statistic T and its observed value t_{obs}, what is the definition of the p-value for testing H_0? **[2 ms]**

2. [**Total: 10 ms**]. Let $X_1, \ldots, X_{n_1} \sim N(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2} \sim N(\mu_2, \sigma_2^2)$, where $n_1 = 18$, $\bar{x} = 13.5$, $s_1 = 5$; $n_2 = 12$, $\bar{y} = 9.5$, $s_2 = 6$; and $f(0.025, 11, 17) = 2.8696$, $f(0.025, 17, 11) = 3.2816$.

2.1 Show that $f(1-\alpha/2, \nu_1, \nu_2) = f^{-1}(\alpha/2, \nu_2, \nu_1)$. **[6 ms]**

2.2 Construct and compute a 95% CI for σ_1/σ_2. **[4 ms]**

3. [**Total: 18 ms**]. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Geometric(θ) and define $\boldsymbol{X} \triangleq (X_1, \ldots, X_n)^\top$.

3.1 Show that $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ is a sufficient statistic of θ. [**3 ms**]

3.2 Let the prior distribution of θ be $U(0, 1)$. Find the posterior distribution of θ and the Bayesian estimator of θ. [**5 ms**]

3.3 Suppose that among 100 people who win in a certain lottery, the number of tickets each person purchases up to and including the first winning ticket shows the following frequency distribution:

Number of tickets purchased (i)	1	2	3	4	5	6	7	Total
Frequency of people (N_i)	46	27	14	6	5	1	1	100

Does the number of tickets purchased by each person follow a geometric distribution at the significance level of 0.05? [Note: $\chi^2(0.05, 3) = 7.81$, $\chi^2(0.05, 4) = 9.49$, $\chi^2(0.05, 5) = 11.07$]. [**10 ms**]

4. [**Total: 20 ms**]. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Exponential(λ).

4.1 Show that $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ is an efficient estimator of λ^{-1}. [**5 ms**]

4.2 Find a $100(1-\alpha)\%$ equal-tail CI for λ. [**5 ms**]

4.3 Find the MPT of size α for testing H_0: $\lambda = \lambda_0$ vs H_1: $\lambda = \lambda_1\,(> \lambda_0)$. [**10 ms**]

5. [**Total: 12 ms**]. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Exponential(λ). Find the LRT of size α for testing H_0: $\lambda \leqslant \lambda_0$ versus H_1: $\lambda > \lambda_0$.

6. [Bonus question, **Total: 5 ms**]. Let $Z \sim N(0, \sigma^2)$, $\tau \sim$ IGamma(α, β) with $\alpha = 1$ and $\beta = 1/2$, and $Z \perp\!\!\!\perp \tau$. Define

$$X = \mu + \frac{Z}{\sqrt{\tau}},$$

find the distribution of X.

********** **END OF THE PAPER A** **********

1. Suggested Solutions.

1.1 The support of Z is $\{0,1\}$. $Z \sim$ Bernoulli(0.5).

Solution: Note that X, Y take values 0 and 1, then

$$Z = Y \cdot I(X=0) + (1-Y) \cdot I(X=1)$$

only takes 0 and 1. Since

$$\begin{aligned}
\Pr(Z=1) &= \Pr(X+Y-2XY=1) = \Pr(X=1, Y=0) + \Pr(X=0, Y=1) \\
&= \Pr(X=1) \times \Pr(Y=0) + \Pr(X=0) \times \Pr(Y=1) \\
&= 0.5 \times 0.5 + 0.5 \times 0.5 = 0.5,
\end{aligned}$$

indicating that $Z \sim \text{Bernoulli}(0.5)$. ‖

1.2 $Y \sim N(a_1\mu_1 + a_2\mu_2, a_1^2\sigma_1^2 + a_2^2\sigma_2^2)$. ‖

1.3 Solution: (a) The median of X satisfies

$$\Pr\{X \leqslant \text{med}(X)\} \geqslant 0.5 \quad \textbf{and} \quad \Pr\{X \geqslant \text{med}(X)\} \geqslant 0.5.$$

See page 18 of the Textbook.

(b) We have $\text{med}(X) = 4$ because

$$\begin{aligned}
\Pr(X \leqslant 4) &= 0.20 + 0.15 + 0.10 + 0.05 = 0.50 \geqslant 0.5 \quad \text{and} \\
\Pr(X \geqslant 4) &= 0.05 + 0.10 + 0.40 = 0.55 \geqslant 0.5.
\end{aligned}$$

Alternatively, We have $\text{med}(X) = 5$ because

$$\begin{aligned}
\Pr(X \leqslant 5) &= 0.20 + 0.15 + 0.10 + 0.05 + 0.10 = 0.60 \geqslant 0.5 \quad \text{and} \\
\Pr(X \geqslant 5) &= 0.10 + 0.40 = 0.50 \geqslant 0.5.
\end{aligned}$$

Similar to **Q1.16 in Assignment 1**. ‖

1.4 Solution: Let $X \sim \text{Exponential}(\beta)$, then the cdf of X is $F_X(x) = 1 - \exp(-\beta x)$ so that the survival function $S_X(x) = \Pr(X > x) = \mathrm{e}^{-\beta x}$. Thus

$$\begin{aligned}
\Pr(X > t+s | X > s) &= \frac{\Pr(X > t+s)}{\Pr(X > s)} \\
&= \frac{S_X(t+s)}{S_X(s)} \\
&= \frac{\mathrm{e}^{-\beta(t+s)}}{\mathrm{e}^{-\beta s}} \\
&= \mathrm{e}^{-\beta t}.
\end{aligned}$$

1.5 Solution: See 42.5• **The sampling-wise formula** on page 42. The sampling-wise formula is

$$f_X(x) \propto \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)},$$

for all $x \in \mathcal{S}_X$ and an arbitrarily fixed $y_0 \in \mathcal{S}_Y$. ‖

1.6 Solution: (a) The MLE of θ **does not exist**.

(b) The likelihood function of θ is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 < x_i \leqslant \theta) = \frac{1}{\theta^n} I(0 < x_{(n)} \leqslant \theta) \cdot 1,$$

According to the factorization theorem, we known that $X_{(n)} = \max(X_1, \ldots, X_n)$ is a sufficient estimator of θ. Note that $L(\theta)$ is an decreasing function of θ over the interval $[x_{(n)}, \infty)$, thus, $X_{(n)}$ **is the MLE of** θ.

See **Example 3.3** on pages 108–109 of the textbook. ‖

1.7 Solution: Newton's method to iteratively calculate the MLE $\hat{\theta}$ of the equation $\ell'(x) = 0$ is

$$\theta^{(t+1)} = \theta^{(t)} - \frac{\ell'(\theta^{(t)})}{\ell''(\theta^{(t)})}, \quad t = 0, 1, 2, \ldots, \infty. \qquad \|$$

1.8 **Solution**: (i) The expectation and variance of X are given by

$$\begin{aligned} E(X) &= E[E(X|Y)] = E(Y) = \alpha/\beta \triangleq \mu, \quad \text{and} \\ \mathrm{Var}(X) &= E[\mathrm{Var}(X|Y)] + \mathrm{Var}[E(X|Y)] \\ &= E(Y) + \mathrm{Var}(Y) = \alpha/\beta + \alpha/\beta^2 = \mu(1 + 1/\beta). \end{aligned}$$

1.9 **Solution**: See **Theorem 3.7** on page 149.
Let $T(\mathbf{x})$ be a complete sufficient statistic for θ. If $g(T)$ is an unbiased estimator of $\tau(\theta)$, then $g(T)$ is the unique UMVUE for $\tau(\theta)$.

1.10 **Solution**: See **Definition 4.1** on page 164.
Assume that $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x;\theta)$ and $T = T(X_1, \ldots, X_n)$ is a sufficient statistic of θ. Let $P = P(T, \theta)$ be a function of T and θ. If the distribution of P does not depend on θ, then P is called a *pivotal quantity*.

1.11 **Solution**: (i) The Type II error function is defined by

$$\begin{aligned} \beta(\theta) = \Pr(\text{Type II error}) &= \Pr(\text{accepting } H_0 | H_0 \text{ is false}) \\ &= \Pr(\mathbf{x} \in \mathbb{C}' | \theta \in \Theta_1), \end{aligned}$$

which is a function of θ defined in Θ_1, where $\mathbf{x} = (X_1, \ldots, X_n)^\top$.

(ii) The power function is defined by

$$p(\theta) = \Pr(\text{rejecting } H_0 | \theta) = \Pr(\mathbf{x} \in \mathbb{C} | \theta). \qquad \|$$

1.12 If $\alpha_{T_1}(\theta), \alpha_{T_2}(\theta) \leqslant \alpha^*$ **and** $\beta_{T_1}(\theta) \leqslant \beta_{T_2}(\theta)$, then T_1 is better than T_2, where α^* $(0 < \alpha^* < 1)$ is a preassigned (small) level.

Alternatively, if $p_{T_1}(\theta) \geqslant p_{T_2}(\theta)$, then T_1 is better than T_2. $\|$

1.13 **Solution**: **See Lemma 5.1 on pages 194** of the Textbook.

Assume that $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x;\theta)$. Let the likelihood function be $L(\theta) = L(\theta; \boldsymbol{x})$. Then a test φ with critical region

$$\mathbb{C} = \left\{ \boldsymbol{x} = (x_1, \ldots, x_n)^{\top} \colon \frac{L(\theta_0)}{L(\theta_1)} \leqslant k \right\}$$

and size α is the most powerful test of size α for testing H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$, where k is a value determined by the size α. ||

1.14 **Solution**: (i) The pivotal quantity is

$$Z \triangleq \frac{\bar{X} - \mu}{\sigma_0/\sqrt{n}} \sim N(0,1),$$

where $\bar{X} = (1/n)\sum_{i=1}^n X_i$. The test statistic is

$$Z_0 \triangleq \frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}}.$$

(ii) When H_0 is true, we obtain $Z_0 \sim N(0,1)$.

(iii) The critical regions of size α for the test is $\mathbb{C} = \{\boldsymbol{x}\text{:}\ |z_0| \geqslant z_{\alpha/2}\}$, where

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}} \quad \text{with } \bar{x} = \frac{\sum_{i=1}^n x_i}{n}.$$

and z_α denotes the upper α-th quantile of $N(0,1)$.

(iv) The corresponding p-value can be calculated by

$$p\textbf{-value} = 2\Pr(Z \geqslant |z_0|). \qquad ||$$

1.15 **Solution**: (a) When $\sigma = \sigma_0$ is known, from (4.4) of Chapter 4, we know that

$$\left[\bar{X} - z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}},\ \bar{X} + z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right] = [-2.0925,\ 2.8425]$$

is a $100(1-\alpha)\%$ CI for the mean μ, where $n=4$, $\alpha=0.1$, $z_{\alpha/2}=z_{0.05}=1.645$, $\sigma_0=3$, and

$$\bar{X}=\frac{3.3-0.3-0.6-0.9}{4}=0.375.$$

(b) When σ is unknown, from (4.6) of Chapter 4, we know that

$$\left[\bar{X}-t\Big(\frac{\alpha}{2},n-1\Big)\frac{S}{\sqrt{n}},\ \bar{X}+t\Big(\frac{\alpha}{2},n-1\Big)\frac{S}{\sqrt{n}}\right]=[-1.937, 2.687]$$

is a $100(1-\alpha)\%$ CI for the mean μ, where $\bar{X}=0.375$, $n=4$, $t(\alpha/2,n-1)=t(0.05,3)=2.3534$, and

$$S=\sqrt{\frac{\sum_{i=1}^n(X_i-\bar{X})^2}{n-1}}=\sqrt{3.863}=1.965. \qquad \|$$

1.16 **<u>Solution</u>**: Since $(\bar{X},S^2)$ are a pair of joint sufficient statistics for (μ,σ^2), we have a pivotal quantity

$$P=\frac{\sqrt{n}(\bar{X}-\mu)}{S}\sim t(n-1),$$

$$\Rightarrow\ \Pr\left\{-t(\alpha/2,n-1)\leqslant\frac{\sqrt{n}(\bar{X}-\mu)}{S}\leqslant t(\alpha/2,n-1)\right\}=1-\alpha,$$

$$\Rightarrow\ \Pr\left(\bar{X}-t(\alpha/2,n-1)\frac{S}{\sqrt{n}}\leqslant\mu\leqslant\bar{X}+t(\alpha/2,n-1)\frac{S}{\sqrt{n}}\right)=1-\alpha,$$

$$\Rightarrow\ \Pr\left(\bar{X}-t(0.025,n-1)\frac{S}{\sqrt{n}}\leqslant\mu\leqslant\bar{X}+t(0.025,n-1)\frac{S}{\sqrt{n}}\right)=0.95.$$

Therefore, a 95% CI for μ is given by

$$\left[66.3-2.201\frac{8.4}{\sqrt{12}},\ 66.3+2.201\frac{8.4}{\sqrt{12}}\right]=[61.0,\ 71.6]\,. \qquad \|$$

1.17 **<u>Solution</u>**: The p-value (or probability value) is defined as the probability, under the null hypothesis H_0, the test statistic T is equal to or more extreme than the observed value t_{obs}; i.e.,

$$\Pr(T \text{ is equal to or more extreme than the observed value } t_{\text{obs}}|H_0). \qquad \|$$

2. Solution.

2.1 See 10.1• on pages 173–174. Note that

$$\frac{1}{F(\nu_2,\nu_1)} \overset{\mathrm{d}}{=} \frac{1}{\dfrac{\chi^2(\nu_2)/\nu_2}{\chi^2(\nu_1)/\nu_1}} = \frac{\chi^2(\nu_1)/\nu_1}{\chi^2(\nu_2)/\nu_2} \overset{\mathrm{d}}{=} F(\nu_1,\nu_2). \tag{1}$$

— On the one hand, the definition of $f(\alpha/2,\nu_2,\nu_1)$ indicates that

$$\frac{\alpha}{2} = \Pr\{F(\nu_2,\nu_1) > f(\alpha/2,\nu_2,\nu_1)\},$$

we obtain

$$\begin{aligned} 1-\frac{\alpha}{2} &= \Pr\{F(\nu_2,\nu_1) \leqslant f(\alpha/2,\nu_2,\nu_1)\} \\ &= \Pr\left\{\frac{1}{F(\nu_2,\nu_1)} \geqslant f^{-1}(\alpha/2,\nu_2,\nu_1)\right\} \\ &\overset{(1)}{=} \Pr\left\{F(\nu_1,\nu_2) \geqslant f^{-1}(\alpha/2,\nu_2,\nu_1)\right\}. \end{aligned} \tag{2}$$

— On the other hand, the definition of $f(1-\alpha/2,\nu_1,\nu_2)$ means that

$$1-\frac{\alpha}{2} = \Pr\{F(\nu_1,\nu_2) > f(1-\alpha/2,\nu_1,\nu_2)\}. \tag{3}$$

— By comparing (3) with (2), we immediately obtain the needed result.

2.2 Define $\nu_i = n_i - 1$, $i = 1, 2$. Since $f(1-\alpha/2,\nu_1,\nu_2) = f^{-1}(\alpha/2,\nu_2,\nu_1)$ and

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(\nu_1,\nu_2),$$

$$\Rightarrow \Pr\left\{f(1-\alpha/2,\nu_1,\nu_2) \leqslant \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \leqslant f(\alpha/2,\nu_1,\nu_2)\right\} = 1-\alpha$$

$$\Rightarrow \Pr\left\{\frac{S_1^2}{S_2^2}\cdot f^{-1}(\alpha/2,\nu_1,\nu_2) \leqslant \frac{\sigma_1^2}{\sigma_2^2} \leqslant \frac{S_1^2}{S_2^2}\cdot f^{-1}(1-\alpha/2,\nu_1,\nu_2)\right\} = 1-\alpha$$

$$\Rightarrow \Pr\left\{\frac{S_1^2}{S_2^2}\cdot f^{-1}(\alpha/2,\nu_1,\nu_2) \leqslant \frac{\sigma_1^2}{\sigma_2^2} \leqslant \frac{S_1^2}{S_2^2}\cdot f(\alpha/2,\nu_2,\nu_1)\right\} = 1-\alpha$$

$$\Rightarrow \Pr\left\{\sqrt{\frac{S_1^2}{S_2^2}\cdot f^{-1}(0.025,\nu_1,\nu_2)} \leqslant \frac{\sigma_1}{\sigma_2} \leqslant \sqrt{\frac{S_1^2}{S_2^2}\cdot f(0.025,\nu_2,\nu_1)}\right\} = 0.95.$$

Therefore, a 95% CI for σ_1/σ_2 is given by

$$\left[\sqrt{\frac{s_1^2}{s_2^2}\cdot f^{-1}(0.025,17,11)},\ \sqrt{\frac{s_1^2}{s_2^2}\cdot f(0.025,11,17)}\ \right]$$

$$= \left[\sqrt{\frac{5^2}{6^2}\cdot 3.2816^{-1}},\ \sqrt{\frac{5^2}{6^2}\cdot 2.8696}\ \right] = [0.4600,\ 1.4117].$$

3. Solution.

3.1 The joint density of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n; \theta) = \theta^n (1-\theta)^{n\bar{x}-n} \times 1, \quad x_i = 1, 2, \ldots, \infty,$$

so that $\bar{X}$ is a sufficient statistic of θ based on the factorization theorem.

3.2 Since $\theta \sim U(0,1)$, the posterior density of θ is

$$p(\theta|\boldsymbol{x}) \propto \theta^n (1-\theta)^{n\bar{x}-n},$$

so that $\theta|\boldsymbol{x} \sim \text{Beta}(n+1, n\bar{x}-n+1)$, where $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$. Therefore,

$$E(\theta|\boldsymbol{x}) = \frac{n+1}{n\bar{x}+2}$$

is the Bayesian estimate of θ, and $(n+1)/(n\bar{X}+2)$ is the Bayesian estimator of θ.

3.3 We wish to test

$H_0:$ The distribution is Geometric(θ) against

$H_1:$ The distribution is not Geometric(θ).

Under H_0, the maximum likelihood estimate of θ is

$$\hat{\theta} = \frac{1}{\bar{x}} = \frac{100}{204} \approx 0.49.$$

Now

$$\hat{p}_{i0} = p_{i0}(\hat{\theta}) = \hat{\theta}(1-\hat{\theta})^{i-1}, \quad i = 1, 2, \ldots, 6, \qquad \hat{p}_{7,0} = 1 - \sum_{i=1}^{6} \hat{p}_{i0},$$

and $n = 100$, we obtain

i	1	2	3	4	5	6	$7(\geq 7)$
N_i	46	27	14	6	5	1	1
$\hat{p}_{i0}$	0.4902	0.2499	0.1274	0.0650	0.0331	0.0169	0.0176
$n\hat{p}_{i0}$	49.0196	24.9904	12.7402	6.4950	3.3112	1.6881	1.7556

Those classes with expected frequencies less than 5 should be combined with the adjacent class. Therefore, we combine the last 3 classes, and the revised table is

i	1	2	3	4	$5(\geq 5)$
N_i	46	27	14	6	7
$\hat{p}_{i0}$	0.4902	0.2499	0.1274	0.0650	0.0675
$n\hat{p}_{i0}$	49.0196	24.9904	12.7402	6.4950	6.7548

So we have

$$\hat{Q}_{100} = \sum_{i=1}^{5} \frac{(N_i - n\hat{p}_{i0})^2}{n\hat{p}_{i0}} = 0.1739 < \chi^2(0.05, 5-1-1) = 7.81.$$

Thus, we cannot reject H_0 when the significance level is 0.05.

4. Solution.

4.1 Let $f(x;\lambda) = \lambda \mathrm{e}^{-\lambda x}$ and $\tau(\lambda) = 1/\lambda$. The Fisher information is

$$I_n(\lambda) = nE\left\{\frac{\mathrm{d}\log f(X;\lambda)}{\mathrm{d}\lambda}\right\}^2 = nE\left(\frac{1}{\lambda} - X\right)^2 = n\mathrm{Var}(X) = \frac{n}{\lambda^2}.$$

The CR lower bound is

$$\frac{[\tau'(\lambda)]^2}{I_n(\lambda)} = \frac{1}{n\lambda^2}.$$

Since

$$E(\bar{X}) = E(X_1) = \frac{1}{\lambda} = \tau(\lambda) \quad \text{and} \quad \mathrm{Var}(\bar{X}) = \frac{1}{n\lambda^2}$$

i.e., $\bar{X}$ is an unbiased estimator of $\tau(\lambda)$ and its variance reaches the CR lower bound. Hence, $\bar{X}$ is an efficient estimator of λ^{-1}.

4.2 See **Example 4.1 on page 164** of Textbook Chapter 4. From $X \sim$ Exponential(λ), we have $n\bar{X} = \sum_{i=1}^n X_i \sim \mathrm{Gamma}(n, \lambda)$, and

$$2\lambda n\bar{X} \sim \mathrm{Gamma}(2n/2, 1/2) = \chi^2(2n),$$

so that $2\lambda n\bar{X}$ **is a pivotal quantity**. Thus, using the equal-probability (or equal-tail) method, we have

$$\begin{aligned} 1-\alpha &= \Pr\left\{\chi^2(1-\alpha/2, 2n) \leqslant 2\lambda n\bar{X} \leqslant \chi^2(\alpha/2, 2n)\right\} \\ &= \Pr\left\{\frac{\chi^2(1-\alpha/2, 2n)}{2n\bar{X}} \leqslant \lambda \leqslant \frac{\chi^2(\alpha/2, 2n)}{2n\bar{X}}\right\}; \end{aligned}$$

that is,

$$[L_p, U_p] = \left[\frac{\chi^2(1-\alpha/2, 2n)}{2n\bar{X}}, \frac{\chi^2(\alpha/2, 2n)}{2n\bar{X}}\right]$$

is a $100(1-\alpha)\%$ exact CI for λ.

4.3 We consider a test of size α for testing H_0: $\lambda = \lambda_0$ versus H_1: $\lambda = \lambda_1 > \lambda_0$. The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} \lambda \exp(-\lambda x_i) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right).$$

Then

$$\frac{L(\lambda_0)}{L(\lambda_1)} = \frac{\lambda_0^n \exp(-\lambda_0 \sum_{i=1}^n x_i)}{\lambda_1^n \exp(-\lambda_1 \sum_{i=1}^n x_i)} = \left(\frac{\lambda_0}{\lambda_1}\right)^n \exp\left\{(\lambda_1 - \lambda_0)\sum_{i=1}^{n} x_i\right\} \leqslant k$$

is equivalent to

$$\bar{x} \leqslant \frac{\log(k)}{n(\lambda_1 - \lambda_0)} + \frac{\log(\lambda_1/\lambda_0)}{\lambda_1 - \lambda_0} \ \hat{=}\ c,$$

when $\lambda_1 > \lambda_0$. To determine c, we noted that

$$\begin{aligned} X_i &\overset{\text{iid}}{\sim} \text{Exponential}(\lambda) \\ \Rightarrow n\bar{X} = \sum_{i=1}^{n} X_i &\sim \text{Gamma}(n, \lambda) \\ \Rightarrow \qquad 2\lambda n\bar{X} &\sim \text{Gamma}\left(n, \frac{1}{2}\right) = \chi^2(2n), \end{aligned}$$

and

$$\begin{aligned} \alpha &= \Pr(\bar{X} \leq c \mid \lambda = \lambda_0) = \Pr(2\lambda n\bar{X} \leq 2\lambda nc \mid \lambda = \lambda_0) \\ &= \Pr(\chi^2(2n) \leq 2\lambda_0 nc) = 1 - \Pr(\chi^2(2n) \geq 2\lambda_0 nc) \\ \Rightarrow\ 1 - \alpha &= \Pr(\chi^2(2n) \geq 2\lambda_0 nc) \\ \Rightarrow\ 2\lambda_0 nc &= \chi^2(1 - \alpha, 2n) \\ \Rightarrow\ c &= \frac{\chi^2(1-\alpha, 2n)}{2\lambda_0 n}. \end{aligned}$$

By the Neyman–Pearson Lemma, a test φ with critical region

$$\mathbb{C} = \left\{\boldsymbol{x}\colon\ \bar{x} \leqslant \frac{\chi^2(1-\alpha, 2n)}{2\lambda_0 n}\right\}$$

is the MPT of size α for testing H_0: $\lambda = \lambda_0$ versus H_1: $\lambda = \lambda_1 > \lambda_0$. $\|$

5. **Solution**. Note that $\Theta_0 = (0, \theta_0]$ and $\Theta_1 = (\theta_0, \infty)$, then $\Theta = (0, \infty) = \Theta^*$.

<u>*Step 1: Calculate* $\lambda(\boldsymbol{x})$</u>. The likelihood function is given by

$$L(\theta) = \prod_{i=1}^{n} \theta \mathrm{e}^{-\theta x_i} = \theta^n \mathrm{e}^{-\theta n \bar{x}}$$

so that $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ is a sufficient statistic of θ. The log-likelihood function is $\ell(\theta) = n \log \theta - \theta n \bar{x}$. We have

$$\ell'(\theta) = \frac{n}{\theta} - n\bar{x} \quad \text{and} \quad \ell''(\theta) = -n\theta^{-2} < 0 \quad \forall \theta \in \Theta.$$

Hence, $\ell(\theta)$ is strictly concave and has the maximum at $\hat{\theta} = 1/\bar{x}$. On the other hand, the restricted MLE $\hat{\theta}^R$ of θ in Θ_0 is given by

$$\hat{\theta}^R = \begin{cases} 1/\bar{x}, & \text{if } \theta_0 \geqslant 1/\bar{x}, \\ \theta_0, & \text{if } \theta_0 < 1/\bar{x}. \end{cases}$$

Therefore,

$$\max_{\theta \in \Theta} L(\theta) = L(\hat{\theta}) = (1/\bar{x})^n \mathrm{e}^{-n},$$

and

$$\max_{\theta \leqslant \theta_0} L(\theta) = L(\hat{\theta}^R) = \begin{cases} L(\hat{\theta}) = (1/\bar{x})^n \mathrm{e}^{-n}, & \text{if } \theta_0 \geqslant 1/\bar{x}, \\ L(\theta_0) = \theta_0^n \mathrm{e}^{-\theta_0 n \bar{x}}, & \text{if } \theta_0 < 1/\bar{x}, \end{cases}$$

so that

$$\lambda(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \theta_0 \geqslant 1/\bar{x}, \\ \dfrac{\theta_0^n \mathrm{e}^{-\theta_0 n \bar{x}}}{(1/\bar{x})^n \mathrm{e}^{-n}}, & \text{if } \theta_0 < 1/\bar{x}. \end{cases}$$

<u>*Step 2: Find the critical region* $\mathbb{C}$</u>. The LRT of size α has the critical region

$$\begin{aligned} \mathbb{C} &= \{\boldsymbol{x}\colon \theta_0 < 1/\bar{x} \quad \text{and} \quad (\theta_0 \bar{x})^n \mathrm{e}^{-\theta_0 n \bar{x} + n} \leqslant \lambda_\alpha\} \\ &= \{\boldsymbol{x}\colon \theta_0 \bar{x} < 1 \quad \text{and} \quad (\theta_0 \bar{x})^n \mathrm{e}^{-n(\theta_0 \bar{x} - 1)} \leqslant \lambda_\alpha\} \\ &= \{\boldsymbol{x}\colon 0 < y < 1 \quad \text{and} \quad y^n \mathrm{e}^{-n(y-1)} \leqslant \lambda_\alpha\} \\ &= \{\boldsymbol{x}\colon 0 < y < 1 \quad \text{and} \quad h(y) \leqslant \lambda_\alpha\}, \end{aligned} \tag{4}$$

where $y \;\hat{=}\; \theta_0 \bar{x} > 0$ and $h(y) \;\hat{=}\; y^n \mathrm{e}^{-n(y-1)}$.

Step 2(a): Check if or not $h(y)$ is log-concave. Define

$$H(y) \ \hat{=} \ \log\{h(y)\} = n\log(y) - n(y-1).$$

Letting $H'(y) = n/y - n = 0$, we obtain $y = 1$. In addition

$$H''(y) = -ny^{-2} < 0.$$

Therefore, $h(y)$ is strictly log-concave and has a maximum at $y = 1$.

Step 2(b): Find an equivalent $\mathbb{C}$ involving $\bar{X}$ and k. Hence $0 < y < 1$ and $h(y) \leqslant \lambda_\alpha$ if and only if $y \leqslant k$, where k is a constant satisfying $0 < k < 1$. Thus, (4) becomes

$$\mathbb{C} = \{\boldsymbol{x}\colon\ \theta_0 \bar{x} \leqslant k\},$$

where $0 < k < 1$.

Step 2(c): Find the constant k. We know that

$$2\theta n\bar{X} \sim \chi^2(2n).$$

Then k can be determined by the size

$$\begin{aligned} \alpha &= \sup_{\theta\in\Theta_0} \Pr(\mathbf{x}\in\mathbb{C}|\theta) \\ &= \sup_{\theta\leqslant\theta_0} \Pr(\theta_0\bar{X} \leqslant k|\theta) \\ &= \sup_{\theta\leqslant\theta_0} \Pr(2\theta n\bar{X} \leqslant 2\theta nk/\theta_0|\theta) \\ &= \max_{\theta\leqslant\theta_0} \Pr\{\chi^2(2n) \leqslant 2\theta nk/\theta_0|\theta\} \\ &= \Pr\{\chi^2(2n) \leqslant 2nk\}, \end{aligned}$$

or equivalently

$$1 - \alpha = \Pr\{\chi^2(2n) > 2nk\} = \Pr\{\chi^2(2n) > \chi^2(1-\alpha, 2n)\},$$

we obtain $2nk = \chi^2(1-\alpha, 2n)$ or $k = \chi^2(1-\alpha, 2n)/2n$. The null hypothesis H_0 is rejected when $\bar{X} \leqslant \chi^2(1-\alpha, 2n)/(2n\theta_0)$. $\|$

6. **Solution**. From $X = \mu + \tau^{-1/2} Z$, we obtain $X|\tau \sim N(\mu, \tau^{-1}\sigma^2)$. Thus, the pdf of X is

$$
\begin{aligned}
f_X(x) &= \int_0^\infty f_\tau(\tau) \cdot f_{X|\tau}(x|\tau)\, \mathrm{d}\tau \\
&= \int_0^\infty \mathrm{IGamma}(\tau|1, 1/2) \cdot N(\tau|\mu, \tau^{-1}\sigma^2)\, \mathrm{d}\tau \\
&= \int_0^\infty \frac{1}{2}\tau^{-2}\mathrm{e}^{-1/(2\tau)} \cdot \frac{\sqrt{\tau}}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\tau(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}\tau \\
&= \frac{1}{2\sqrt{2\pi}\sigma} \int_0^\infty \tau^{-3/2} \exp\left[-\frac{1}{2\tau} - \frac{\tau(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}\tau \qquad (5) \\
&= \frac{1}{2\sigma} \exp\left(-\frac{|x-\mu|}{\sigma}\right), \quad x \in \mathbb{R},\ \mu \in \mathbb{R},\ \sigma > 0, \qquad (6)
\end{aligned}
$$

indicating that $X \sim \mathrm{Laplace}(\mu, \sigma^2)$, where we only need to prove the following integral identity: For any $z \in \mathbb{R}$,

$$
F(z) \triangleq \int_0^\infty \tau^{-3/2} \exp\left(-\frac{1}{2\tau} - \frac{\tau z^2}{2}\right) \mathrm{d}\tau = \sqrt{2\pi}\mathrm{e}^{-|z|}. \tag{7}
$$

Case 1: $z = 0$. The identity (7) reduces to

$$
F(0) \triangleq \int_0^\infty \tau^{-3/2}\mathrm{e}^{-1/(2\tau)}\, \mathrm{d}\tau = \sqrt{2\pi}. \tag{8}
$$

Proof. From

$$
1 = \int_0^\infty \mathrm{IGamma}(\tau|\alpha, \beta)\, \mathrm{d}\tau = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{-(\alpha+1)}\mathrm{e}^{-\beta/\tau}\, \mathrm{d}\tau,
$$

we have

$$
\int_0^\infty \tau^{-(\alpha+1)}\mathrm{e}^{-\beta/\tau}\, \mathrm{d}\tau = \frac{\Gamma(\alpha)}{\beta^\alpha}. \tag{9}
$$

By setting $\alpha = \beta = 1/2$, we obtain

$$
F(0) = \int_0^\infty \tau^{-3/2}\mathrm{e}^{-1/(2\tau)}\, \mathrm{d}\tau \overset{(9)}{=} \frac{\Gamma(1/2)}{\sqrt{1/2}} = \frac{\sqrt{\pi}}{\sqrt{1/2}} = \sqrt{2\pi}. \qquad \|
$$

<u>**Case 2:** $z \geqslant 0$</u>. The identity (7) reduces to

$$F(z) = \int_0^\infty \tau^{-3/2} \exp\left(-\frac{1}{2\tau} - \frac{\tau z^2}{2}\right) \mathrm{d}\tau = \sqrt{2\pi}\mathbf{e}^{-z}. \tag{10}$$

Proof. We have

$$\begin{aligned}
F'(z) \quad &\triangleq \quad \frac{\mathrm{d}F(z)}{\mathrm{d}z} = \int_0^\infty \tau^{-3/2} \exp\left(-\frac{1}{2\tau} - \frac{\tau z^2}{2}\right) \cdot \frac{-\tau \times 2z}{2} \,\mathrm{d}\tau \\
&= \quad -z \int_0^\infty \tau^{-1/2} \exp\left(-\frac{1}{2\tau} - \frac{\tau z^2}{2}\right) \mathrm{d}\tau \qquad \left[\text{Let } y = \frac{1}{\tau z^2}\right] \\
&= \quad -z \int_\infty^0 (yz^2)^{1/2} \exp\left(-\frac{yz^2}{2} - \frac{1}{2y}\right) \cdot z^{-2}(-1)y^{-2} \,\mathrm{d}y \\
&= \quad -\int_0^\infty y^{-3/2} \exp\left(-\frac{1}{2y} - \frac{yz^2}{2}\right) \mathrm{d}y \\
&= \quad -F(z),
\end{aligned}$$

so that

$$\frac{F'(z)}{F(z)} = \frac{\mathrm{d}\log F(z)}{\mathrm{d}z} = [\log F(z)]' = -1.$$

Thus, there exists a consant c_0 such that

$$\log F(z) = -z + c_0 \quad \text{or} \quad F(z) = \mathrm{e}^{-z+c_0}. \tag{11}$$

From (8), we obtain

$$\sqrt{2\pi} = F(0) \overset{(11)}{=} \mathrm{e}^{c_0} \Rightarrow F(z) = \sqrt{2\pi}\mathrm{e}^{-z}. \qquad \|$$

<u>**Case 3:** $z < 0$</u>. The proof is similar. $\|$