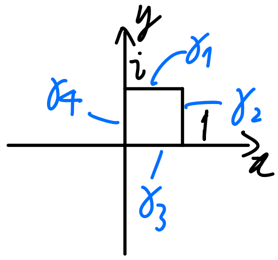
陈恺. 12311410 复变函数 (H) Homework 1.

$\boxed{1.}$ (a)



$z = x + iy$, $x, y \in [0,1]$

$f(z) = x^2 - y^2 + 2ixy.$

let $u = x^2 - y^2$, $v = 2xy$.

① $y = 1$, $u = x^2 - 1$, $v = 2x$. $\quad u = \frac{v^2}{4} - 1$ with $x \in [0,1]$, $v \in [0,2]$.

$\Rightarrow \gamma_1: u = \frac{v^2}{4} - 1 \quad v \in [0,2]$.

② $x = 1$, $u = 1 - y^2$, $v = 2y$. $\quad u = 1 - \frac{v^2}{4}$. with $y \in [0,1]$, $v \in [0,2]$.

$\Rightarrow \gamma_2: u = 1 - \frac{v^2}{4}$. $v \in [0,2]$.

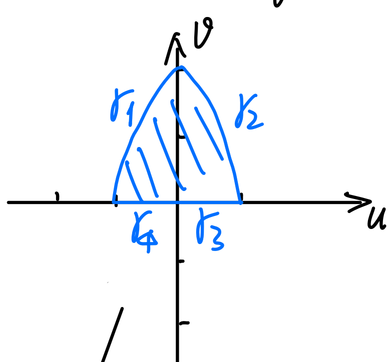
③ $y = 0$, $u = x^2 > 0$, $v = 0$. $\quad u = x^2 \in [0,1]$.

$\Rightarrow \gamma_3: u = x^2 \in [0,1]$.

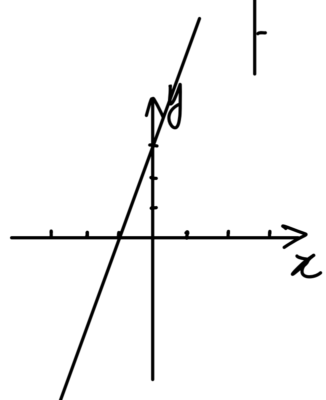
④ $x = 0$. $u = -y^2 < 0$. $v = 0$. $\quad u = -y^2 \in [-1, 0]$.

$\Rightarrow \gamma_4: u = -y^2 \in [-1, 0]$.

$\Rightarrow$



(b)



$y = 2x + 3$

$z = x + iy$.

$f(z) = \frac{1}{z + i}$ $\quad f(\infty) = 0$.

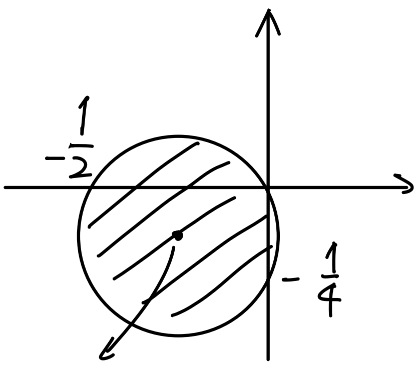
$f(3i) = -\frac{1}{4}i$.

$f(-2-i) = -\frac{1}{2}$.

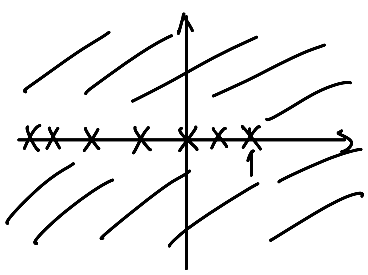
Since f is a linear-frac. map, f maps generalized cycles to generalized cycles. i.e. the image is a generalized cycle.

Now, we have got 0, $-\frac{1}{4}i$, $-\frac{1}{2}$ as image pts.

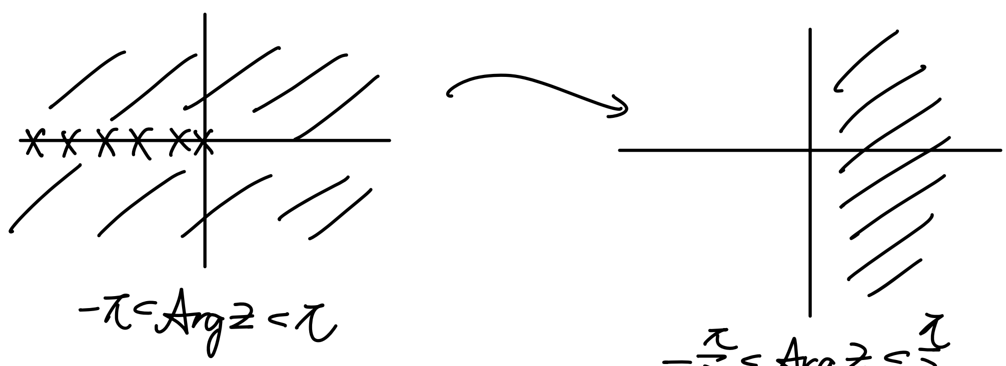
$\Rightarrow$ the center of the circle is $-\frac{1}{4}-\frac{1}{8}i$.



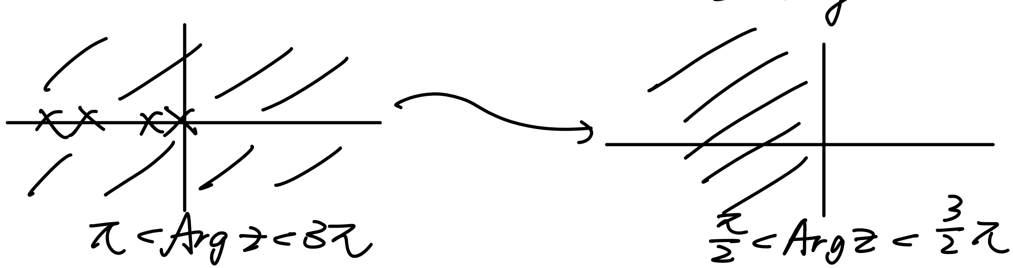
(C)



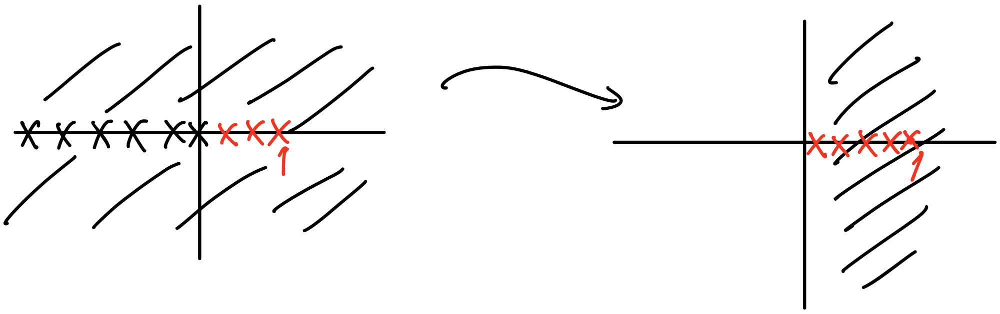
$\sqrt{z}^{(1)}$:



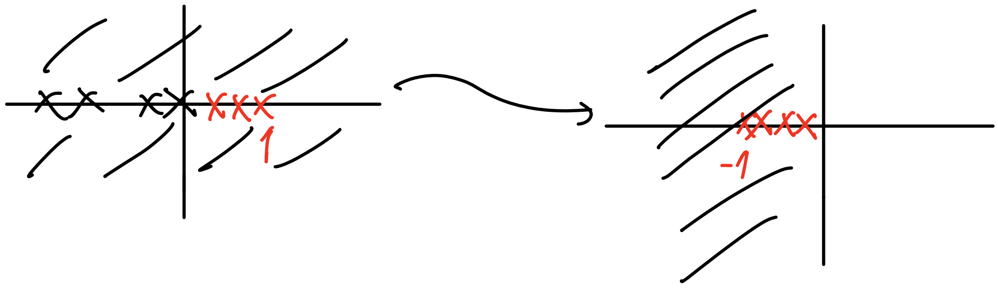
$\sqrt{z}^{(2)}$:



⇒ $\sqrt{z}^{(1)}$:

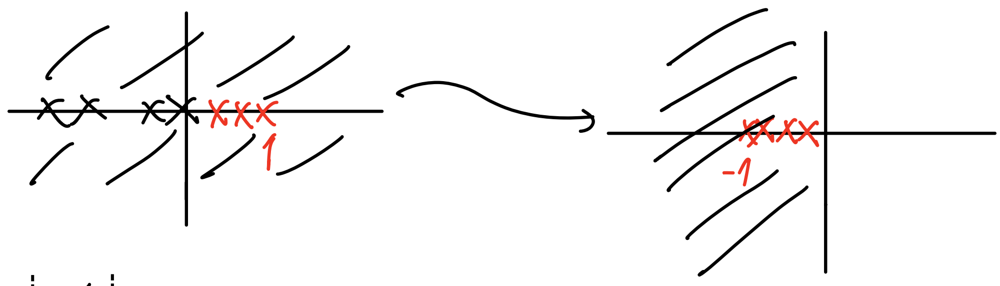


$\sqrt{z}^{(2)}$:

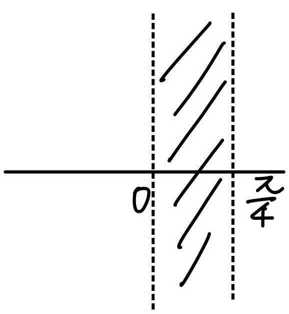


Here we chose the branch that maps i to $-\frac{1}{\sqrt{2}}(1+i)$

⇒ $\sqrt{z}^{(2)}$:



(d)

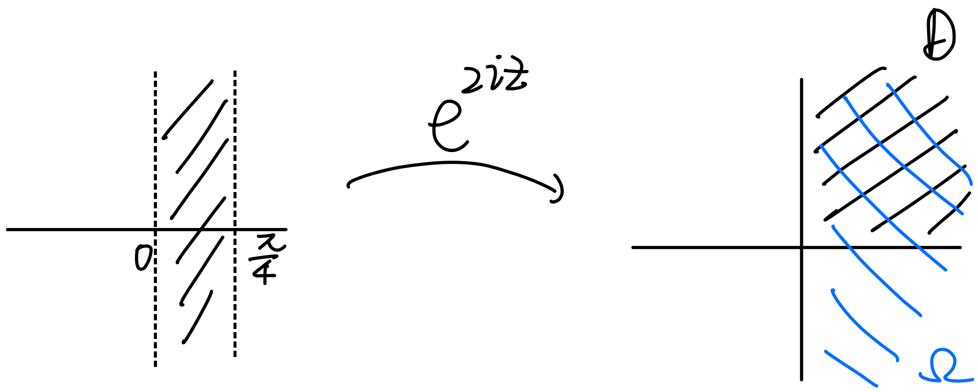


$$f(z)=\cot z = \frac{\cos z}{\sin z} = \frac{e^{iz}+e^{-iz}}{\frac{1}{i}(e^{iz}-e^{-iz})}$$

$$= i\,\frac{e^{2iz}+1}{e^{2iz}-1} = L\circ e^{2iz},$$ where $L(w)=i\frac{w+1}{w-1}$.

$z=x+iy$. $e^{2iz}=e^{2ix-2y}=e^{-2y}\cdot e^{i2x}$ $\quad 0<2x<\frac{\pi}{2}$.

$$\begin{cases} L(0)=-i \\ L(1)=\infty \\ L(-1)=0. \end{cases}$$



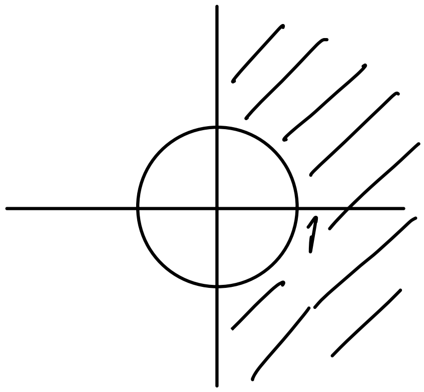
⇒ $L(\mathbb{R})=i\mathbb{R}$

$L(i\mathbb{R})$: $\left|i\frac{w+1}{w-1}\right|=1$

⇒ $L(\Omega)=\overline{\mathbb{C}}\setminus B_1(0)$

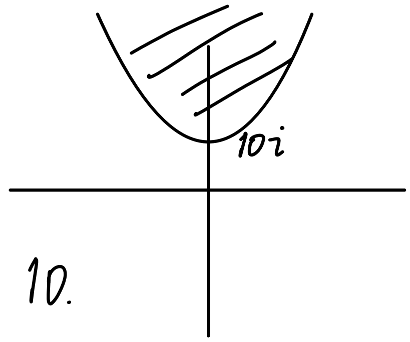
$L(1+i)=2+i$

$\xrightarrow{L(w)}$



$\left(\mathbb{C} \setminus \overline{B_1(0)}\right) \cap \{z \mid \operatorname{Re} z > 0\}$.

(e) $\operatorname{Im} z > (\operatorname{Re} z)^2 + 10$.



$\forall w \neq 0,\ w = re^{i\theta},\ \operatorname{Arg} w = \theta$.

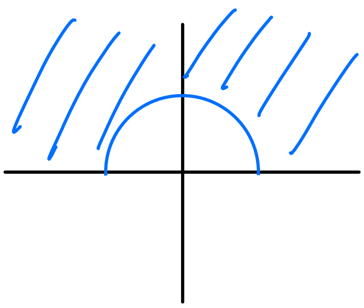
$\exists k \in \mathbb{Z}$, s.t. $\theta + 2k\pi > (\ln r)^2 + 10$.

$\Rightarrow f(\ln r + i(\theta + 2k\pi)) = w$.

Also, $|e^z| > 0,\ \forall z \Rightarrow w$ cannot be 0.

$\Rightarrow$ Image $= \mathbb{C} \setminus \{0\}$.

(f)



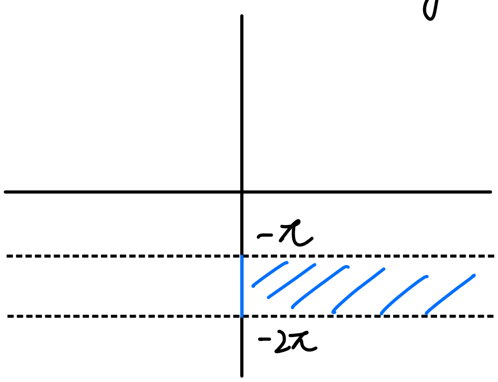
$\operatorname{Ln}(z) = \ln|z| + i\operatorname{Arg}(z)$

$\operatorname{Ln}(i) = 0 + i\left(\frac{\pi}{2} + 2k\pi\right) = -\frac{3\pi}{2} i$.

$\Rightarrow \frac{1}{2} + 2k = -\frac{3}{2}$ $\quad k = -1$

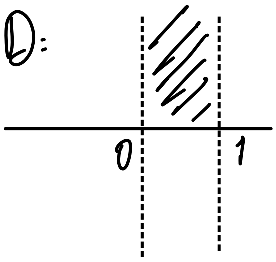
$\Rightarrow \ln|z| \geq 0, \quad \operatorname{Arg}(z) \in (-2\pi, -\pi)$

$\Rightarrow$



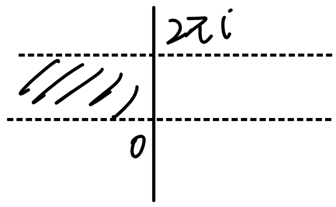
$\{x + iy \mid x \geq 0,\ -2\pi < y < -\pi\}$.

$\boxed{2.}$ (a) $\mathbb{D}$:



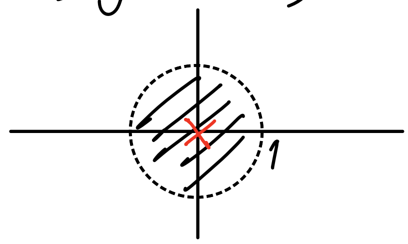
$\mathbb{D} \to \Pi^+$.

Step 1: $f_1(z) = 2\pi i z$

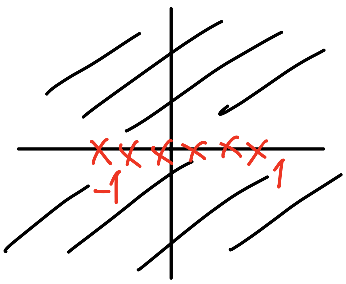


Step 2: $f_2(z) = e^z = e^{x+iy}$

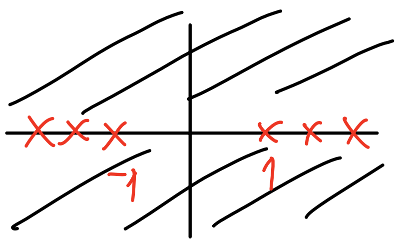
$x<0,\ y \in (0, 2\pi)$



Step 3: $f_3(z) = \ast(z)$.



Step 4: $f_4(z) = \frac{1}{z}$

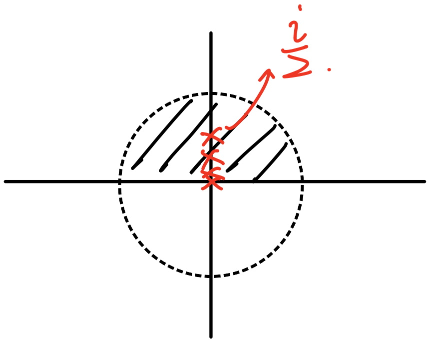


Step 5: $f_5(z) = \ast^{-1}(z)$

branch mapping onto Π^+!

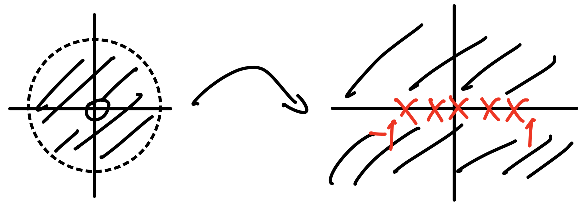
To conclude: $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$.

(b)



Step 1:

Recall: $\ast$:



$$\ast\big((-1,0)\cup(0,1)\big) = (-\infty,-1)\cup(1,+\infty).$$

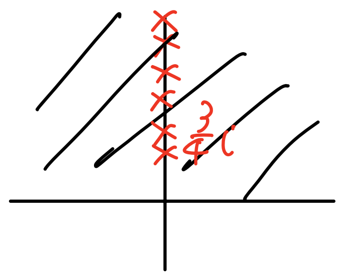
$$\ast(\partial B_1(0)) = [-1,1].$$

$$\ast\left(\left[0,\tfrac{i}{2}\right]\right) = \left(\infty, -\tfrac{3}{4}i\right]$$

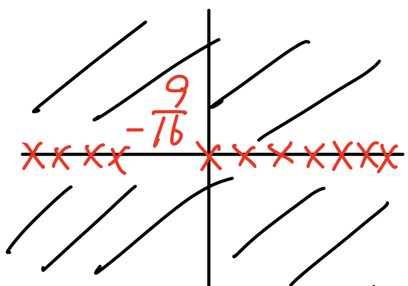
$$\Rightarrow -\ast(D) = \Pi^+ \setminus \left[\tfrac{3}{4}i, \infty\right)$$

$$\Rightarrow f_1(z) = -\ast(z)$$

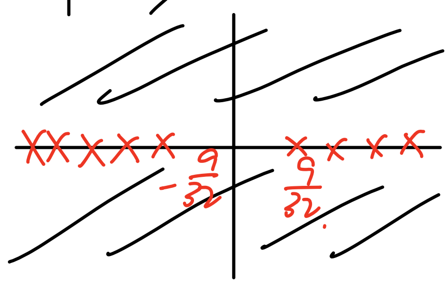
$1+\frac{1}{x^2}$



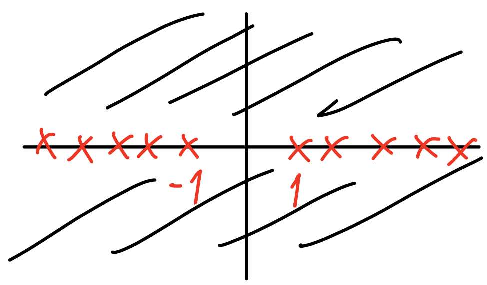
Step 2: $f_2(z) = z^2$



Step 3: $f_3(z) = z + \frac{9}{32}$

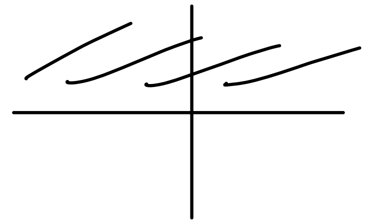


Step 4: $f_4(z) = \frac{32}{9}z$



Step 5: $f_5(z) = \ast^{-1}(z)$

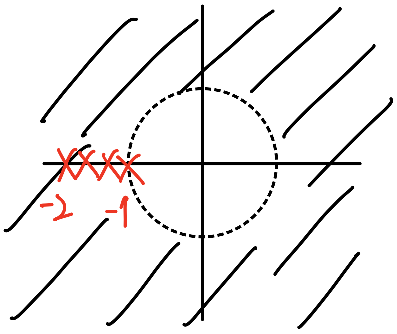
branch mapping onto Π^+.



Π^+.

To conclude: $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$.

(c)



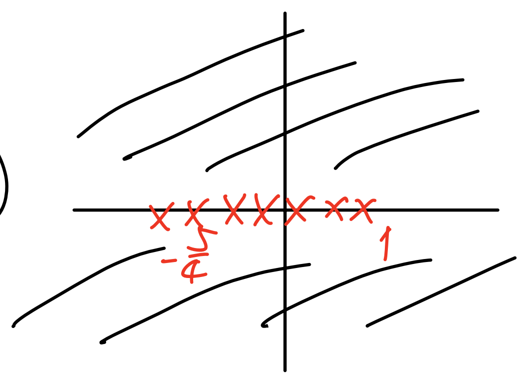
Step 1: $f_1(z) = \ast(z)$

$\ast(\{|z|>1\}) = \mathbb{C} \setminus [-1,1]$

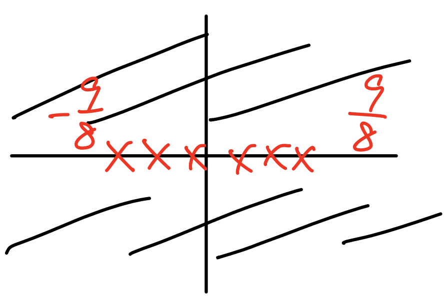
$\ast([-2,-1]) = [-\frac{5}{4}, -1]$

$\Rightarrow \ast(D) = \mathbb{C} \setminus ([-1,1] \cup [-\frac{5}{4}, -1])$

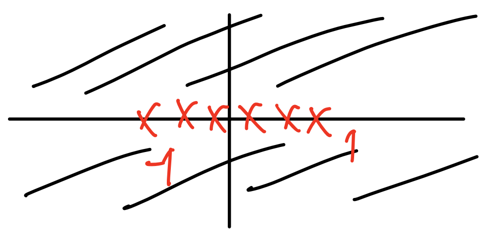
$= \mathbb{C} \setminus [-\frac{5}{4}, 1]$.



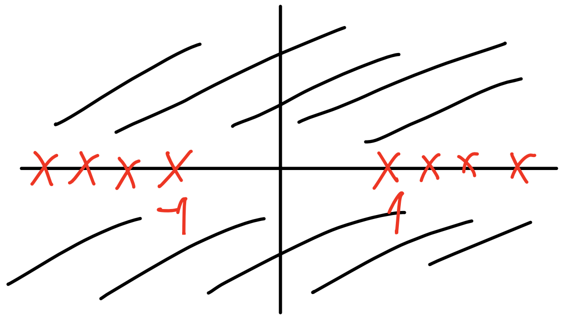
Step 2: $f_2(z) = z + \frac{1}{8}$



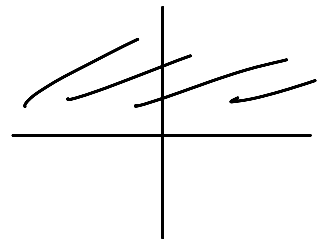
Step 3: $f_3(z) = \frac{8}{9} z$



Step 4: $f_4(z) = \frac{1}{z}$

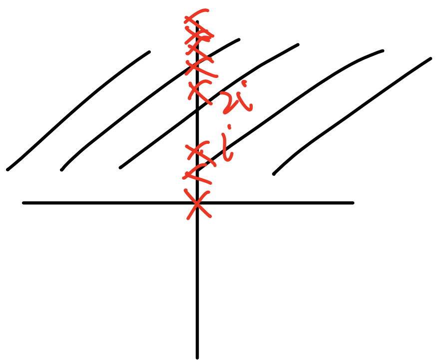


Step 5: $f_5(z) = \chi^{-1}(z)$ branch mapping onto Π^+.

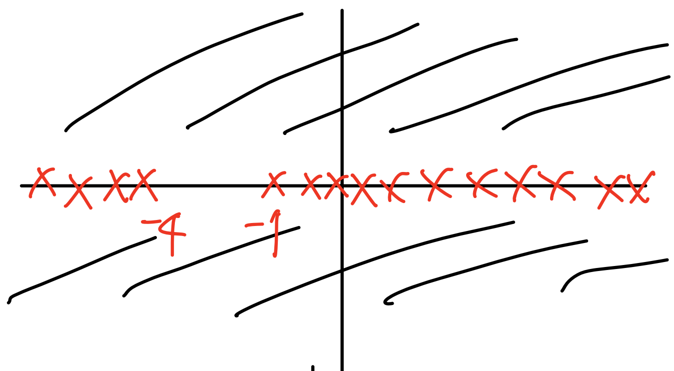


To conclude: $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$.

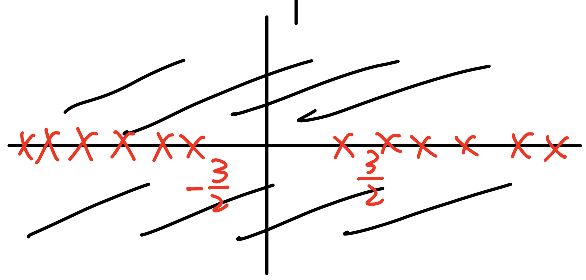
(d)



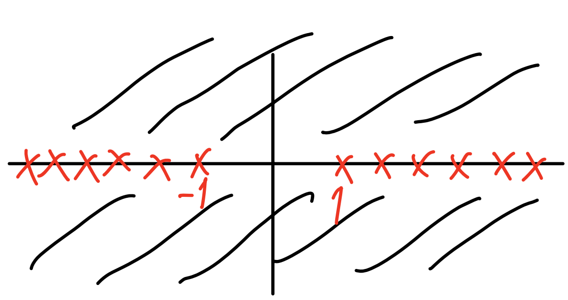
Step 1: $f_1(z) = z^2$.



Step 2: $f_2(z) = z + \frac{5}{2}$



Step 3: $f_3(z) = \frac{2}{3} z$



Step 4: $f_4(z) = \ast^{-1}(z)$

$\quad$ branch mapping onto Π^+.

To conclude: $f = f_4 \circ f_3 \circ f_2 \circ f_1$.

$\boxed{3.}$ Proof:

Step 1: If $M \in SL_2(\mathbb{R})$, then f_M maps Π^+ to itself.

Pf: Since $\operatorname{Im}(f_M(z)) = \dfrac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2} = \dfrac{\operatorname{Im}(z)}{|cz+d|^2} > 0, \ \forall z \in \Pi^+$.

(*1)

Step 2: If $M, M' \in SL_2(\mathbb{R})$, then $f_M \circ f_{M'} = f_{MM'}$. (proved in class)

Thus, $\forall$ linear fractional map f_M, $f_M \in \operatorname{Aut}(\Pi^+)$

Pf: $(f_M \circ f_{M^{-1}})(z) = f_{MM^{-1}}(z) = f_I(z) = z$

$\Rightarrow f_M$ has a simple inverse $f_{M^{-1}}$.

Step 3: $\forall z, w \in \Pi^+$, $\exists\, M \in SL_2(\mathbb{R})$ s.t. $f_M(z) = w$.

(Therefore, $SL_2(\mathbb{R})$ acts transitively on Π^+)

Pf: It's sufficient to show: we can map any $z \in \Pi^+$ to i.

Set $d=0$ in (*1) $\Rightarrow \operatorname{Im}(f_M(z)) = \dfrac{\operatorname{Im}(z)}{|cz|^2}$.

Choose $c \in \mathbb{R}$ s.t. $\operatorname{Im}(f_M(z)) = 1$.

Define $M_1 = \begin{pmatrix} 0 & c^{-1} \\ c & 0 \end{pmatrix}$. $\Rightarrow \operatorname{Im}(f_{M_1}(z)) = 1$.

Define $M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbb{R}$.

$M = M_2 M_1$. $\Rightarrow f_M(z) = i$.

Step 4: For $\theta \in \mathbb{R}$, the mat. $M_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SL_2(\mathbb{R})$.

Let $F(z) = \dfrac{z-i}{z+i}$, $F: \mathbb{H}^+ \to B_1(0)$.

$\Rightarrow F \circ f_{M_\theta} \circ F^{-1}$ denotes the rotation of angle -2θ in the disc,

since $F \circ f_{M_\theta} = e^{-2i\theta} F(z)$.

Step 5: Suppose $f \in \mathrm{Aut}(\mathbb{H}^+)$ with $f(\beta) = i$.

Consider $N \in SL_2(\mathbb{R})$ s.t. $f_N(i) = \beta$.

$\Rightarrow g := f \circ f_N$ satisfies $g(i) = i$.

$\Rightarrow F \circ g \circ F^{-1}$ is an automorphism of the disc that fixes the origin. $\Rightarrow F \circ g \circ F^{-1}$ is a rotation.

By step 4, there exists $\theta \in \mathbb{R}$ s.t.

$$F \circ g \circ F^{-1} = F \circ f_{M_\theta} \circ F^{-1}.$$

$\Rightarrow g = f_{M_\theta}$. $\Rightarrow f = f_{M_\theta N^{-1}}$.

i.e. $\forall f \in \mathrm{Aut}(\mathbb{H}^+)$, f is a linear fraction map. $\square$

$\boxed{4.}$ Proof: $\forall$ a linear fractional map $T(z)$.

Let $w_i = T(z_i)$, $i=2,3,4$.

Let $S(w) = \dfrac{w_2-w_3}{w_2-w_4} : \dfrac{w-w_3}{w-w_4}$, $\Rightarrow$ S is also a linear fractional map.

Let $L(z) = \dfrac{z_2-z_3}{z_2-z_4} : \dfrac{z-z_3}{z-z_4}$

Only need to show: $L(z_1) = S(T(z_1))$.

Since $S^{-1}\circ L$ is a linear fractional map,

$$S^{-1}\circ L(z_2) = S^{-1}(1) = w_2,$$

$$S^{-1}\circ L(z_3) = S^{-1}(0) = w_3,$$

$$S^{-1}\circ L(z_4) = S^{-1}(\infty) = w_4,$$

by "three pt. prop.", $S^{-1}\circ L = T$.

$\Rightarrow L = S\circ S^{-1}\circ L = S\circ T \Rightarrow L(z_1) = S(T(z_1))$. $\square$.

$\boxed{5.}$ Proof:

1° Suppose $z_1 \sim z_4$ belong to the same gene-d cycle.

Let $L(z) = \dfrac{z_2-z_3}{z_2-z_4} : \dfrac{z-z_3}{z-z_4}$. be a linear fractional map.

$\Rightarrow L(z_2) = 1 \quad L(z_4) = \infty \quad L(z_3) = 0$.

$\Rightarrow L$ maps the gener-d cycle determined by z_2, z_3, z_4 to $\mathbb{R}$.

Since $z_1 \sim z_4$ belong to the same gene-d cycle,

$L(z_1) \in \mathbb{R}$. $\Rightarrow$ $\dfrac{z_2 - z_3}{z_2 - z_4} : \dfrac{z_1 - z_3}{z_1 - z_4} \in \mathbb{R}$.

2° Suppose $\dfrac{z_2 - z_3}{z_2 - z_4} : \dfrac{z_1 - z_3}{z_1 - z_4} \in \mathbb{R}$. i.e $L(z_1) \in \mathbb{R}$.

Since $L(z_2), L(z_3), L(z_4) \in \mathbb{R}$.

$L(z_1) \sim L(z_4)$ belongs to the same gener-d cycle.
namely $\mathbb{R}$.

Since L^{-1} is also a linear fractional map.

by the circle prop. (linear frac. map maps

gener-d cycles to gener-d cycles).

$\Rightarrow$ $z_1 \sim z_4$ belong to the same gener-d cycle. $\square$.