Homework-9

November 21, 2024

1. Prove that the irreducible elements in $\mathbb{Z}[i]$ (up to associative) has and only has the following 3 forms:
(1). 1+i;
(2). a+bi, where $a, b \in \mathbb{Z}$ satisfy $a^2 + b^2 \equiv 1 \pmod 4$, $a^2 + b^2$ is prime.
(3). prime p where $p \equiv 3 \pmod 4$.

2. Let R be a UFD, $K = \text{frac}(R)$, $f(x) \in R[x]$ and monic. If $g(x) \in K[x]$ s.t. $g(x)$ is monic and $g(x)|f(x)$, prove $g(x) \in R[x]$.

3. (1). Prove Eisenstein Criterion: Let R be a UFD,
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x]$. If there exists irreducible element $p \in R$ s.t. $p \nmid a_n$, $p|a_i$ $(\forall i < n)$, $p^2 \nmid a_0$. Prove $f(x)$ is irreducible in $F[x]$ where $F = \text{frac}(R)$.
(2). Determine whether the following polynomials in $\mathbb{Q}(i)[x]$ are reducible:
(i). $x^{p-1} + x^{p-2} + \cdots + 1$, p is a prime number in $\mathbb{Z}$;
(ii). $x^4 + (8 + i)x^3 + (3 - 4i)x + 5$.

4. Let E, F be subfields of K, prove $E \cup F$ is a field iff $E \subseteq F$ or $F \subseteq E$.

5. (1). Prove the field automorphism of $\mathbb{Q}$ is only dientity automorphism.
(2). Give all field embedding from $\mathbb{Q}(i)$ to $\mathbb{C}$.
(3). Prove there exists no field embedding from $\mathbb{Q}(i)$ to $\mathbb{Q}(\sqrt{2})$.

6. Let K be a field and α is a transcendental element over K. Prove there exists infinitely many field embeddings from $K(\alpha)$ to $K(\alpha)$.

7. Give minimal polynomial over $\mathbb{Q}$ for following elements.
(1). $a + bi$, $a, b \in \mathbb{Q}$, $b \neq 0$;
(2). $e^{\frac{2\pi i}{p}}$ where p is an odd prime.

8. Let K/F be a finite field extension, $[K : F]$ is a prime, $\alpha \in K - F$, prove $K = F(\alpha)$.

9. Find the basis of following field as a vector space over $\mathbb{Q}$:
(1). $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$;
(2). $K = \mathbb{Q}(\sqrt{3}, i, \omega)$ where $\omega = \frac{-1+\sqrt{-3}}{2}$;
(3). $K = \mathbb{Q}(e^{\frac{2\pi i}{p}})$ where p is an odd prime.

10. Let K/F be a finite field extension. Prove that any $F-$endomorphism of K is automorphism.