Solutions to Problems 71-80

71. Decide in each case whether the hypothesis is simple [S] or composite [C]:

[S] The hypothesis that a random variable has a gamma distribution with α=3 and β=2.

[C] The hypothesis that a random variable has a gamma distribution with α=3 and β≠2.

[C] The hypothesis that a random variable has an exponential density.

[C] The hypothesis that a random variable has a beta distribution with the mean μ=0.50.

[S] The hypothesis that a random variable has a Poisson distribution with λ=1.25.

[C] The hypothesis that a random variable has a normal distribution with λ>1.25.

[C] The hypothesis that a random variable has a negative binomial distribution with k=3 and θ<0.60.

72. Fill in the blanks.

(a) A single observation of a random variable having a hypergeometric distribution with $N=7$ and $n=2$ is used to test the null hypothesis $k=2$ against the alternative hypothesis $k=4$. The null hypothesis is rejected if and only if the value of the random variable is 2. The sample space of the test statistic is $\{0,1,2\}$. On the sample space of the test statistic, the critical region is $\mathbb{C}=\{2\}$. The power function is $\wp(k,\mathbb{C})=\frac{\binom{k}{2}\binom{7-k}{0}}{\binom{7}{2}}=\frac{k(k-1)}{42}$, the Type 1 error probability is $\alpha=\wp(2,\mathbb{C})=\frac{1}{21}$, the Type 2 error probability is $\beta=1-\wp(4,\mathbb{C})=5/7$.

(b) A single observation of a random variable having a geometric distribution (p.m.f. $(1-\theta)^x\theta$) is used to test the null hypothesis $\theta=\theta_0$ against the alternative hypothesis $\theta=\theta_1>\theta_0$. The null hypothesis is rejected if and only if the observed value of the random variable is greater than or equal to the positive integer k. The sample space of the test statistic is $\{0,\ 1,\ 2,\ 3,\ \ldots\}$. On the sample space of the test statistic, the critical region is $\mathbb{C}=\{k,\ k+1,\ k+2,\ldots\}$. The power function is $\wp(\theta,\mathbb{C})=\theta\sum_{x=k}^{\infty}(1-\theta)^x$, the Type 1 error probability is $\theta_0\sum_{x=k}^{\infty}(1-\theta_0)^x$, the Type 2 error probability is $1-\theta_1\sum_{x=k}^{\infty}(1-\theta_1)^x$.

(c) A single observation of a random variable having an exponential distribution is used to test the null hypothesis that the mean of the distribution is $\theta = 2$ against the alternative that it is $\theta = 5$. The null hypothesis is accepted if and only if the observed value of the random variable is less than 3. The sample space of the test statistic is $(0, +\infty)$. On the sample space of the test statistic, the critical region is $\mathbb{C} = [3, \infty)$. The power function is $\wp(\theta, \mathbb{C}) = e^{-3/\theta}$, the Type 1 error probability is 0.2231, the Type 2 error probability is 0.4512.

(d) A single observation of a random variable having a uniform density with $\alpha = 0$ is used to test the null hypothesis $\beta = \beta_0$ against the alternative hypothesis $\beta = \beta_0 + 2$. The null hypothesis is rejected if and only if the random variable takes on a value greater than $\beta_0 + 1$. The sample space of the test statistic is $[0, \infty)$. On the sample space of the test statistic, the critical region is $\mathbb{C} = (\beta_0 + 1, \infty)$. The power function is $\wp(\beta, \mathbb{C}) = \frac{1}{\beta}\mathbb{I}(\beta \geq \beta_0 + 1)$, the Type 1 error probability is 0, the Type 2 error probability is $\frac{\beta_0+1}{\beta_0+2}$.

73. Filling the blanks.

(a) Let X_1 and X_2 constitute a random sample from a normal population with $\sigma^2 = 1$. If the null hypothesis $\mu = \mu_0$ is to be rejected in favor of the alternative hypothesis $\mu = \mu_1 > \mu_0$ when $\bar{x} > \mu_0 + 1$, then the size of the test is $\alpha = 1 - \Phi(\sqrt{2}) = 0.07865$.

(b) Let X_1 and X_2 constitute a random sample of size 2 from the population given by $f(x; \theta) = \theta x^{\theta-1}\mathbb{I}(0 < x < 1)$. If the critical region $x_1 x_2 \geq \frac{3}{4}$ is used to test the null hypothesis $\theta = 1$ against the alternative hypothesis $\theta = 2$, then the power of this test at $\theta = 2$ is $\wp(\theta = 2) = \mathbb{P}\left(X_1 X_2 \geq \frac{3}{4}; \theta = 2\right) = \int_{t=\frac{3}{4}}^{1} \mathbb{P}\left(X_1 \geq \frac{3}{4t} | X_2 = t; \theta = 2\right) \mathbb{P}(X_2 = t; \theta = 2)\, dt = \int_{t=\frac{3}{4}}^{1} \left[\int_{\frac{3}{4t}}^{1} 2x dx\right] 2t dt = \frac{7}{16} + \frac{9}{8}\ln\frac{3}{4} = 0.1139$.

74. With i.i.d. data

$$\mathbf{X} = \{X_i\}_{i=1}^{n}$$

we would like to test hypotheses about a parameter θ of the statistical distribution of X_1:

$$H_0: \theta = \theta_0 \text{ vs. } H_1: \theta = \theta_1.$$

Natrually, we may directly compare their likelihoods:

$$L_0 \equiv L(\theta_0; \{X_i\}_{i=1}^n) \quad \text{vs.} \quad L_1 \equiv L(\theta_1; \{X_i\}_{i=1}^n)$$

via the likelihood ratio

$$\frac{L_0}{L_1}$$

whose only variable argument is the data vector

$$\mathbf{X} = \{X_i\}_{i=1}^n$$

as both θ_0 and θ_1 are fixed constants. This means the likelihood ratio is a statistic. We use it as the test statistic to specify the following critical region

$$\mathbb{C} := \left\{\mathbf{X}: \frac{L_0}{L_1} \le k\right\}.$$

where k is a constant to be specified according to the size α of the test, that is,

$$\int_{\mathbb{C}} L_0(\mathbf{x}) d\mathbf{x} = \alpha.$$

Suppose a different test for the same pair of hypotheses uses the critical region $\mathbb{D}$ with the same size α, that is,

$$\int_{\mathbb{D}} L_0(\mathbf{x}) d\mathbf{x} = \alpha.$$

The power of the $\mathbb{C}$-test at θ_1 is

$$\wp(\theta_1, \mathbb{C}) = \int_{\mathbb{C}} L_1(\mathbf{x}) d\mathbf{x}.$$

The power of the $\mathbb{D}$-test at θ_1 is

$$\wp(\theta_1, \mathbb{D}) = \int_{\mathbb{D}} L_1(\mathbf{x}) d\mathbf{x}.$$

Show that $\wp(\theta_1, \mathbb{C}) - \wp(\theta_1, \mathbb{D}) \ge 0$. **Proof** $\wp(\theta_1, \mathbb{C}) - \wp(\theta_1, \mathbb{D}) = \int_{\mathbb{C}} L_1 dx - \int_{\mathbb{D}} L_1 dx \overset{!}{=} \int_{\mathbb{C}\setminus\mathbb{D}} L_1 dx - \int_{\mathbb{D}\setminus\mathbb{C}} L_1 dx \overset{!!}{\ge} \int_{\mathbb{C}\setminus\mathbb{D}} \frac{1}{k} L_0 dx - \int_{\mathbb{D}\setminus\mathbb{C}} \frac{1}{k} L_0 dx \overset{!}{=} 0$. **End of Proof.**

Therefore, any simple-vs-simple test based on $\mathbb{C} = \left\{\mathbf{X}: \frac{L_0}{L_1} \le k\right\}$ is the most powerful test among all tests with size α.

75. Explain why a statistical hypothesis test is <u>identified</u> by its critical region (on the data's joint sample space), not by its test statistic. Explain why the power function <u>characterizes</u> a statistical hypothesis test.

Answer. It is clear that underlying the data is the joint sample space of n iid random variables and their joint distribution. Any statistic defined on the data, hence taken the data's joint sample space as domain, will only bring about a simplified sample space (the range of the statistic) and a sampling distribution thereon (on the range) commeasurable with the data's joint distribution (on the domain of the statistic). Any critical region specified on the statistic's range automatically specifies an unambiguous critical region on the data's joint sample space (the domain the statistic) because the latter is the pre-image of the former under the statistic. As the null hypothesis is judged on the range of the statistic by observing whether the realization of the statistic falls in the critical region here, an equivalent judgment is simultaneously carried out on the joint sample space of the data by observing whether the realization of the data vector falls in the pre-image critical region. An important difference between the two critical regions is concerned with their sophistication: the data's joint sample space is at least as sophisticated as any statistic's range. Therefore, not all critical regions on the data's joint sample space is *identifiable* from the statistic's range—some sophisticated structure is just lost in the statistic's simplification effort. In the extreme case, a barbarous statistic may manipulate the distribution on its range by concentrate those low-density regions of the domain to artificially create a high density region on the range and spread those high-density regions widely on its domain to artificially create a vast low-density region on its range.

For this question, we note that the judgment of the null hypothesis is equivalent to the indication of the critical region, regardless of whether the judgment is carried out on the statistic's range or directly on the data's joint sample space. And since all possible critical regions on the range of any statistic is always identifiable on the data's joint sample space, whereas the converse is not true, we prefer to say the test is identifiable from the critical region on the data's joint sample space, and this critical region exists independently of the construction of any statistic.

The power function gives the probability measure of the critical region for each parameter in the parameter space. It describes the interaction of the test (critical region) and the parameter using a probability, which can be interpreted as the likelihood of the parameter given the indicator function of the critical region (as a binary random variable). This likelihood holds all information about the parameter given the test's critical region.

76. A size-n random sample from a normal population with $\sigma^2 = 1$ is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative hypothesis $\mu = \mu_1$, where $\mu_1 < \mu_0$. Show that the most powerful size-α critical region does not depend on the value of μ_1.

Solution. This is a simple-vs-simple test. The Neyman-Pearson Lemma is in effect. The null likelihood is $L_0 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$; The alternative likelihood is $L_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_1)^2}$. Likelihood ratio of Null-over-Alternative:

$$k_1 \geq \frac{L_0}{L_1} = e^{-\frac{1}{2}\sum_{i=1}^{n}[(x_i-\mu_0)^2-(x_i-\mu_1)^2]} = e^{-\frac{1}{2}\sum_{i=1}^{n}[(2x_i-\mu_0-\mu_1)(\mu_1-\mu_0)]} = e^{\frac{\mu_0-\mu_1}{2}\sum_{i=1}^{n}(2x_i-\mu_0-\mu_1)} = e^{\frac{n}{2}(\mu_1^2-\mu_0^2)} e^{(\mu_0-\mu_1)\sum_{i=1}^{n}x_i}$$

$$\Rightarrow k_2 \geq (\mu_0-\mu_1)\sum_{i=1}^{n}x_i$$

$$\overset{\mu_0>\mu_1}{\Rightarrow} k_3 \geq \sqrt{n}(\bar{x}-\mu_0) \overset{H_0}{\sim} N(0,1)$$

where $k_{1\sim3}$ are constants linked to the size of the test. Thus if the sample mean $\bar{X}$ is used as a test statistic, the size-α critical region on the range of $\bar{X}$ is the half-line $\left\{\bar{X} \le \mu_0 - \frac{z_\alpha}{\sqrt{n}}\right\}$. (for the normal upper α-quantile z_α, e.g., $\alpha = 5\% \to z_\alpha = 1.645$.) Clearly this does *not* depend on μ_1 and this is the most powerful critical region by the Neyman-Pearson Lemma. The independence of μ_1 means that the critical region is the *uniformly most powerful* if the alternative hypothesis has been changed to $\mu < \mu_0$.

77. A size-n random sample from an exponential population (use the density form $\lambda e^{-\lambda x}$ where the parameter λ is reciprocal of the mean) is used to test the null hypothesis $\lambda = \lambda_0$ against the alternative hypothesis $\lambda = \lambda_1 < \lambda_0$. Construct the most powerful size-α critical region for this pair of hypotheses and argue that it is the uniformly most powerful critical region when the alternative hypothesis has been changed to the composite hypothesis $\lambda < \lambda_0$.

Solution. This is a simple-vs-simple test. The Neyman-Pearson Lemma is in effect. $L_0 = \lambda_0^n e^{-\lambda_0 \sum_{i=1}^n x_i}$; $L_1 = \lambda_1^n e^{-\lambda_1 \sum_{i=1}^n x_i}$.

$$k_1 \ge \frac{L_0}{L_1} = \frac{\lambda_0^n}{\lambda_1^n} e^{-(\lambda_0-\lambda_1)\sum_{i=1}^n x_i} \Rightarrow k_2 \overset{\lambda_0>\lambda_1}{\le} \sum_{i=1}^n x_i \overset{H_0}{\sim} \Gamma\left(n, \frac{1}{\lambda_0}\right)$$

where $k_{1\sim2}$ are constants linked to the size of the test. The Gamma distribution result is due to the fact that iid sum of exponentials has a Gamma distribution with $\alpha = n$ and β =the exponential's mean ($=\frac{1}{\lambda_0}$ here) . Thus if the sample sum $\sum_{i=1}^n X_i$ is used as a test statistic, the size-α critical region on the range of $\sum_{i=1}^n X_i$ is the half-line $\left\{\sum_{i=1}^n X_i \ge \Gamma_\alpha\left(n, \frac{1}{\lambda_0}\right)\right\}$, where $\Gamma_\alpha\left(n, \frac{1}{\lambda_0}\right)$ is the upper-α quantile of the distribution. Clearly this does not depend on λ_1 and this is the most powerful critical region by the Neyman-Pearson Lemma. The independence of λ_1 means that the critical region is the *uniformly most powerful* if the alternative hypothesis has been changed to $\lambda < \lambda_0$.

78. A size-n random sample from a normal population with $\mu = 0$ is used to test the null hypothesis $\sigma = \sigma_0$ against the alternative hypothesis $\sigma > \sigma_0$. Find the uniformly most powerful size-α critical region.

Solution. We start with the simple alternative hypothesis $H_1^*: \sigma = \sigma_1 > \sigma_0$ so that the Neyman-Pearson is in effect. Then we only need to show that the MP critical region is independent of σ_1 meaning the test of the original simpe-vs-composite hypotheses is UMP over H_1. Now → test $H_0: \sigma = \sigma_0$ vs $H_1^*: \sigma = \sigma_1 > \sigma_0$: $L_0 = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{\sum_{i=1}^n x_i^2}{2\sigma_0^2}}$;

$L_1 = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{\sum_{i=1}^n x_i^2}{2\sigma_1^2}}$. $k_1 \ge \frac{L_0}{L_1} = \sqrt{\frac{\sigma_1^2}{\sigma_0^2}} e^{-\frac{1}{2}\sum_{i=1}^n x_i^2\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} \Rightarrow k_2 \le \frac{1}{\sigma_0^2} \sum_{i=1}^n x_i^2 \overset{H_0}{\sim} \chi^2(n)$ where $k_{1\sim2}$ are constants linked to the size of the test. Thus, if the sum of squared and standardized sample $\sum_i^n \frac{X_i^2}{\sigma_0^2}$ is used as the test statistic, the size-α critical region on the range of $\sum_i^n \frac{X_i^2}{\sigma_0^2}$ is the half-line $\left\{\sum_i^n \frac{X_i^2}{\sigma_0^2} \ge \chi_\alpha^2(n)\right\}$ where $\chi_\alpha^2(n)$ is the upper α-quantile of the χ^2 distribution with n degrees of freedom. The critical region does not depend on σ_1 meaning the MP critical region for the particular simple-vs-simple hypotheses is the UMP critical region for our original simple-vs-composite hypotheses.

79. When we test a simple null hypothesis against a composite alternative, a critical region is said to be unbiased if the corresponding power function takes on its minimum value at the value of the parameter assumed under the null hypothesis. In other words, a critical region is unbiased if the probability of rejecting the null hypothesis is least when the null hypothesis is true. Given a single observation of the random variable X having the density

$$f(x) = 1 + \theta^2 \left(\frac{1}{2} - x\right) \mathbb{I}\{x \in (0,1)\}$$

where $-1 \leq \theta \leq 1$, show that the critical region $x \leq \alpha$ provides an unbiased critical region of size α for testing the null hypothesis $\theta = 0$ against the alternative hypothesis $\theta \neq 0$.

Solution. The power function is $\wp(\theta, \mathbb{C}) = \int_{-\infty}^{\alpha} f(x)dx = \int_0^{\alpha} 1 + \theta^2 \left(\frac{1}{2} - x\right) dx = \alpha + \frac{1}{2}\alpha(1-\alpha)\theta^2$ which attains the minimum value α iff $\theta = 0$. Therefore the critical region $(-\infty, \alpha]$ is unbiased.

80. A size-n random sample from a normal population with unknown mean and variance is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative $\mu \neq \mu_0$. Show that the (generalized) likelihood ratio statistic can be written in the form

$$\Lambda = \left(1 + \frac{T^2}{n-1}\right)^{-\frac{n}{2}}$$

where $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$. Then show that $-2\ln\Lambda \xrightarrow{n\to\infty} T^2$.

Solution. The generalized likelihood ratio is generalized from the simple likelihood ratio $\frac{L_0}{L_1}$ for testing composite-vs-composite hypotheses. The GLR is defined as $\Lambda = \frac{\max L_0}{\max L}$ where L_0 is the null likelihood as usual, while L is the unconstrained likelihood as we use in an unconstrained estimation problem. Unfortunately, the Neyman-Pearson Lemma is dropped during the generalization. The Wilk's Theorem gives the asymptotic distribution of a slight 1-1 transformation on Λ: $-2\ln\Lambda \overset{n\to\infty}{\sim} \chi^2(\nu - \nu_0)$ where ν is the degrees of freedom of the full parameter space and ν_0 is that of the null space. But here we do not cite the Wilk's theorem.

$$L_0 = \left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_0)^2}$$

$$\max L_0 = \max L_0 = \left[\frac{2\pi e}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2\right]^{-\frac{n}{2}}$$

$$\max L = \left[\frac{2\pi e}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right]^{-\frac{n}{2}}$$

$$\Lambda = \frac{\max L_0}{\max L} = \left(\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right)^{-\frac{n}{2}}$$

Fact:

$$\sum_{i=1}^{n}(X_i-\bar{X})^2 + (\bar{X}-\mu_0)^2 = \sum_{i=1}^{n} X_i^2 - 2\bar{X}X_i + 2\bar{X}^2 - 2\mu_0\bar{X} + \mu_0^2 = \sum_{i=1}^{n} X_i^2 - 2\mu_0\bar{X} + \mu_0^2 = \sum_{i=1}^{n}(X_i-\mu_0)^2$$

Therefore

$$\Lambda = \left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{i=1}^{n}(\bar{X}-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right)^{-\frac{n}{2}} = \left(1+\frac{n(\bar{X}-\mu_0)^2}{(n-1)S^2}\right)^{-\frac{n}{2}} = \left(1+\frac{T^2}{n-1}\right)^{-\frac{n}{2}}$$

and

$$-2\ln\Lambda = n\ln\left(1+\frac{T^2}{n-1}\right) \xrightarrow{n\to\infty} \frac{nT^2}{n-1} \xrightarrow{n\to\infty} T^2$$