MA204: Mathematical Statistics

Tutorial 2

T2.1 Negative skewness and positive skewness

Q: **What is the difference between negative skewness and positive skewness?**
A: The question is very interesting. To answer this question, I have read the related context in the following three textbooks.

[1] Mood, A. M., Graybill, F. A. and Boes, D. C. (1974). *Introduction to the Theory of Statistics (3rd Edition)*. McGraw-Hill Book Company, New York. **Page 75–76**.

[2] Miller, I. and Miller, M. (2003). *John E. Freund's Mathematical Statistics with Applications (7th Edition)*. Prentice-Hall, New Jersey. **Page 119**.

[3] Hogg, R. V., McKean, J. W. and Craig, A. T. (2005). *Introduction to Mathematical Statistics (6th Edition)*. Prentice-Hall, New Jersey. **Page 66**.

In page 75 of [1], they stated that a third moment $\mu_3 = E(X-\mu)^3$ about the mean $\mu = E(X)$ is sometimes called a measure of asymmetry, or **skewness**. In Figure T2.1, use R, I plot three curves of density:

(i) The standard normal distribution $X \sim N(0,1)$. We have $\mu_3 = E(X^3) = 0$. A curve like $\phi(y)$ in Figure T2.1(i) is said to be symmetrical.

(ii) Beta distribution $Y \sim \text{Beta}(20,3)$. We obtain $E(Y) = 20/23$ and $\mu_3 = E(Y - 20/23)^3 = -0.000288537 < 0$. A curve like $\text{Beta}(y|20,3)$ in Figure T2.1(ii) is said to be **skewed to the left** and can be shown to have a **negetive** μ_3.

(iii) Gamma distribution $Z \sim \text{Gamma}(2,1)$. We have $E(Z) = 2$ and $\mu_3 = E(Z-2)^3 = 3.9744 > 0$. A curve like $\text{Gamma}(z|2,1)$ in Figure T2.1(iii) is called **skewed to the right** and can be shown to have a **positive** μ_3.

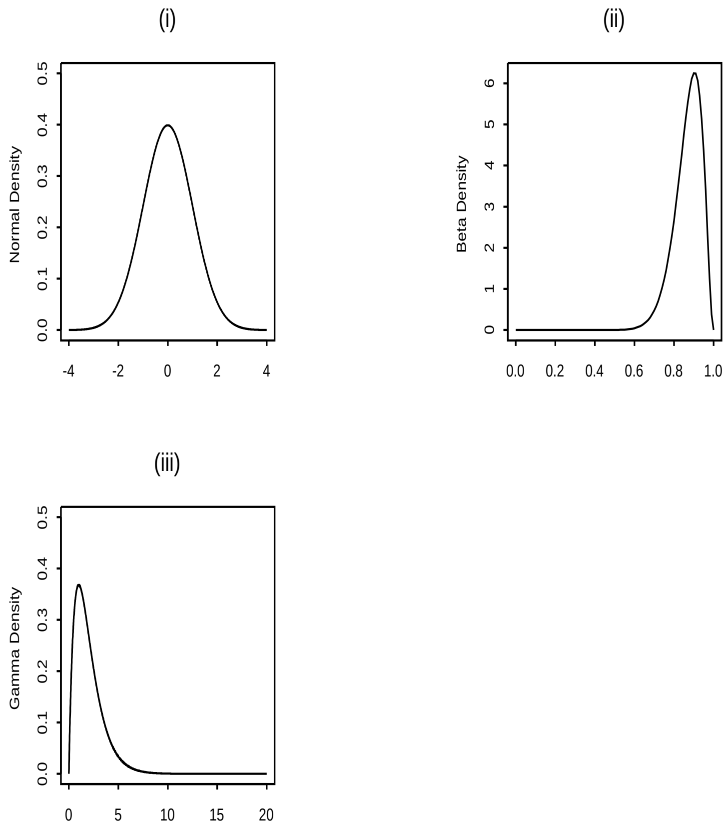

Figure T2.1 Three density functions. (i) $N(0, 1)$; (ii) Beta$(20, 3)$; (iii) Gamma$(2, 1)$.

In page 119 of [2], they stated that data having histograms with a long tail on the left or on the right are said to be **skewed**. A histogram exhibiting a **long left-hand tail** arises when the data have **negative skewness**. Likewise, if the histogram exhibiting a **long right-hand tail**, the data are said to have **positive skewness**.

Note the equivalence between the histogram and density, we can see that the two books have the same definition on the skewness.

Finally, I provide the R code for your reference.

```
function(My, Mz){
  # Function name: question1(My=100000, Mz=150000)
```

```
  par(pty = "s")
  par(mfrow = c(2, 2))
  x <- seq(-4, 4, 0.01)
  plot(x, dnorm(x, 0, 1), type = "l", lty = 1, main = "(i)", xlab = " ",
       ylab = "Normal Density", xlim = c(-4, 4), ylim = c(0, 0.5))
  y <- seq(0, 1, 0.01)
  plot(y, dbeta(y, 20, 3), type = "l", lty = 1, main = "(ii)",
       xlab = " ", ylab = "Beta Density", xlim = c(0, 1))
  Y <- rbeta(My, 20, 3)
  Ymu3 <- mean((Y - 20/23)^3)
  z <- seq(0, 20, 0.01)
  plot(z, dgamma(z, 2, 1), type = "l", lty = 1, main = "(iii)", xlab = " ",
       ylab = "Gamma Density", xlim = c(0, 20), ylim = c(0, 0.5))
  Z <- rgamma(Mz, 2, 1)
  Zmu3 <- mean((Z - 2)^3)
  return(Ymu3, Zmu3)
}
```

T2.2 Find $E(X^r)$ from the MGF of X

The mgf of the random variable X is defined as $M_X(t) = E(\mathrm{e}^{tX})$, and we have

$$\left.\frac{\mathrm{d}^r M_X(t)}{\mathrm{d}\,t^r}\right|_{t=0} = E(X^r). \qquad \blacksquare$$

Example T2.1 (Laplace distribution). Let $X \sim \text{Laplace}(\mu, \sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$. The density of X is

$$f(x) = \frac{1}{2\sigma} \exp\left\{-\frac{|x-\mu|}{\sigma}\right\}, \quad -\infty < x < \infty.$$

(a) Find the *moment generating function* (mgf) of the standard Laplace r.v. $Y = (X - \mu)/\sigma \sim \text{Laplace}(0, 1)$.

(b) Find the mgf of X.

(c) Find $E(X)$ and $\mathrm{Var}(X)$ from the moment generating function.

Solution: (a) Define $Y = (X - \mu)/\sigma$, then the pdf of Y is

$$f_Y(y) = f_X(x)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = 0.5\mathrm{e}^{-|y|}, \quad y \in \mathbb{R}.$$

Thus,

$$M_X(t) = E(\mathrm{e}^{tX}) = E[\mathrm{e}^{t(\mu+\sigma Y)}] = \mathrm{e}^{\mu t}M_Y(\sigma t) \tag{2.1}$$

Now,

$$\begin{aligned} M_Y(t) &= E(\mathrm{e}^{tY}) = \int_{-\infty}^{\infty} \mathrm{e}^{ty}\cdot 0.5\mathrm{e}^{-|y|}\mathrm{d}y \\ &= 0.5\int_{-\infty}^{0} \mathrm{e}^{(t+1)y}\mathrm{d}y + 0.5\int_{0}^{\infty} \mathrm{e}^{(t-1)y}\mathrm{d}y \ \triangleq \ 0.5I_1 + 0.5I_2. \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{0} \mathrm{e}^{(t+1)y}\mathrm{d}y && (2.3) \\ &= \frac{1}{t+1}\mathrm{e}^{(t+1)y}\Big|_{-\infty}^{0} = \frac{1}{t+1}(1-0) = \frac{1}{1+t}, \quad \text{and} \\ I_2 &= \int_{0}^{\infty} \mathrm{e}^{(t-1)y}\mathrm{d}y \qquad [\text{let } y = -z] \\ &= \int_{0}^{-\infty} (-1)\mathrm{e}^{-(t-1)z}\mathrm{d}z = \int_{-\infty}^{0} \mathrm{e}^{(1-t)z}\mathrm{d}z \qquad [\text{similar to (2.3)}] \\ &= \frac{1}{1-t}. \end{aligned}$$

Hence, (2.2) becomes

$$M_Y(t) = \frac{0.5}{1+t} + \frac{0.5}{1-t} = \frac{1}{1-t^2}.$$

(b) From (2.1), we have

$$M_X(t) = \frac{e^{\mu t}}{1-\sigma^2 t^2}. \qquad \|$$

(c) We have

$$\begin{aligned}
E(X) &= \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{(\mu + 2\sigma^2 t - \mu\sigma^2 t^2)e^{\mu t}}{(1-\sigma^2 t^2)^2}\right|_{t=0} = \mu, \\
E(X^2) &= \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} \\
&= \left.\frac{e^{\mu t}[(1-\sigma^2 t^2)(\mu^2 + 2\sigma^2 - \mu^2\sigma^2 t^2) + 4\sigma^2 t(\mu + 2\sigma^2 t - \mu\sigma^2 t^2)]}{(1-\sigma^2 t^2)^3}\right|_{t=0} \\
&= \mu^2 + 2\sigma^2, \\
\mathrm{Var}(X) &= E(X^2) - [E(X)]^2 = \mu^2 + 2\sigma^2 - \mu^2 = 2\sigma^2.
\end{aligned}$$

$\|$

Example T2.2 (Cauchy distribution). Let X follow the standard Cauchy distribution with density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

(a) What is the difference among "**exist**", "**not exist**" and "**undefined**"?

(b) Prove that $E(|X|)$ does not exist.

(c) Prove that $E(X)$ is undefined.

(d) Show that the mgf of X, $E(e^{tX})$, is undefined when $t \neq 0$.

Proof: (a) Given a number A,

(a1) If $A < +\infty$, we say that A exists. Usually, we may denote $+\infty$ by ∞.

(a2) If $A = +\infty$, we say that A does not exist.

(a3) If $A = \infty - \infty$, we say that A is undefined.

Can we claim that $\infty - \infty = 0$? We have the following result:

Let $B < \infty$ be an arbitrary real number. If $\infty - \infty = 0$, then we can prove that $B = 0$. Proof: $B = B + 0 = B + (\infty - \infty) = (B + \infty) - \infty = \infty - \infty = 0$. ■

Can we claim that $\infty - \infty = \infty$? No. ■

(b) We have

$$\begin{aligned} E(|X|) &= \int_{-\infty}^{+\infty} |x| f(x) \mathrm{d}x = 2 \int_0^{+\infty} \frac{x}{\pi(1+x^2)} \mathrm{d}x \\ &= \frac{1}{\pi} \int_0^{+\infty} \frac{\mathrm{d}(1+x^2)}{1+x^2} = \frac{1}{\pi} [\log(1+x^2)] \Big|_0^{+\infty} = +\infty, \end{aligned}$$

indicating that $E(|X|)$ does not exist. As a by-product, we obtain

$$\int_0^{+\infty} \frac{x}{\pi(1+x^2)} \mathrm{d}x = +\infty. \tag{T2.1}$$

(c) Recall that the expectation of a random variable X is defined as $E(X) = \int x f(x) \mathrm{d}x$, provided that $E(|X|)$ exists, i.e., $E(|X|) < +\infty$. Now

$$\begin{aligned} & E(X) = \int_{-\infty}^{+\infty} x f(x) \mathrm{d}x \\ = & \int_0^{+\infty} \frac{x}{\pi(1+x^2)} \mathrm{d}x + \int_{-\infty}^0 \frac{x}{\pi(1+x^2)} \mathrm{d}x \qquad [\text{let } y = -x \text{ in the second integral}] \\ = & \int_0^{+\infty} \frac{x}{\pi(1+x^2)} \mathrm{d}x - \int_0^{+\infty} \frac{y}{\pi(1+y^2)} \mathrm{d}y \overset{(\mathrm{T2.1})}{=} \infty - \infty, \end{aligned}$$

indicating that $E(X)$ is undefined.

Remark: Note that $f(-x) = f(x)$, i.e., the pdf of X is an even function, so $xf(x)$ is an odd function. Can we claim that

$$E(X) = \int_{-\infty}^{+\infty} xf(x)\mathrm{d}x = 0? \qquad \blacksquare$$

(d) Note that $M_X(t) = E(\mathrm{e}^{tX})$. When $t = 0$, we have $M_X(0) = 1$. In the follows, we consider the case of $t \neq 0$. By using the second-order Taylor expansion of e^y around 0, we have

$$\mathrm{e}^y = 1 + y + \frac{1}{2}y^2\mathrm{e}^\xi \geqslant 1 + y > y, \tag{T2.2}$$

where ξ is a point between 0 and y. Thus

$$\begin{aligned}
& M_X(t) = E(\mathrm{e}^{tX}) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{tx}}{1+x^2}\mathrm{d}x \\
= \quad & \frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{e}^{tx}}{1+x^2}\mathrm{d}x + \frac{1}{\pi}\int_{0}^{+\infty} \frac{\mathrm{e}^{tx}}{1+x^2}\mathrm{d}x \qquad [\text{let } y = -x \text{ in the first integral}] \\
= \quad & \frac{1}{\pi}\int_{+\infty}^{0} \frac{\mathrm{e}^{-ty}}{1+y^2}(-1)\mathrm{d}y + \frac{1}{\pi}\int_{0}^{+\infty} \frac{\mathrm{e}^{tx}}{1+x^2}\mathrm{d}x \\
\overset{\text{(T2.2)}}{=} \quad & \frac{1}{\pi}\int_{0}^{+\infty} \frac{\mathrm{e}^{-tx}}{1+x^2}\mathrm{d}x + \frac{1}{\pi}\int_{0}^{+\infty} \frac{\mathrm{e}^{tx}}{1+x^2}\mathrm{d}x \\
\overset{\text{(T2.1)}}{>} \quad & \frac{1}{\pi}\int_{0}^{+\infty} \frac{-tx}{1+x^2}\mathrm{d}x + \frac{1}{\pi}\int_{0}^{+\infty} \frac{tx}{1+x^2}\mathrm{d}x \\
= \quad & -t(+\infty) + t(+\infty),
\end{aligned}$$

implying that $M_X(t)$ is undefined when $t \neq 0$. $\square$

T2.3 Inverse Bayes Formulae under $\mathcal{S}_{(X,Y)} = \mathcal{S}_X \times \mathcal{S}_Y$

2.3.1 Continuous random variables, for any $x \in \mathcal{S}_X$

$$
\begin{aligned}
f_X(x) &= \left\{ \int_{\mathcal{S}_Y} \frac{f_{(Y|X)}(y|x)}{f_{(X|Y)}(x|y)} \, \mathrm{d}y \right\}^{-1} \\
&= \left\{ \int_{\mathcal{S}_X} \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)} \, \mathrm{d}x \right\}^{-1} \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)} \\
&\propto \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)}, \qquad \text{for an arbitrarily fixed } y_0 \in \mathcal{S}_Y.
\end{aligned}
$$

■

Example T2.3 (Quadratic distribution on the unit interval). Let two conditional distributions be quadratic and linear restricted to the unit interval $(0,1)$ and the interval $(0,2)$, respectively; that is,

$$
\begin{aligned}
f_{(X|Y)}(x|y) &= \frac{6x(y+x)}{3y+2}, \quad 0 < x < 1, \\
f_{(Y|X)}(y|x) &= \frac{y+x}{2(1+x)}, \quad 0 < y < 2.
\end{aligned}
$$

Find the marginal distribution of X.

Solution: Note that $\mathcal{S}_X = (0,1)$ and $\mathcal{S}_Y = (0,2)$. Let $y_0 = 1 \in \mathcal{S}_Y = (0,2)$, we have

$$
f_X(x) \propto \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)} = \frac{\dfrac{6x(y_0+x)}{3y_0+2}}{\dfrac{y_0+x}{2(1+x)}} \propto x + x^2,
$$

so that $f_X(x) = K^{-1} \cdot (x + x^2) \cdot I(0 < x < 1)$. From $1 = \int_0^1 f_X(x)\,\mathrm{d}x$, we obtain

$$
K = \int_0^1 (x + x^2)\,\mathrm{d}x = \frac{x^2}{2}\bigg|_0^1 + \frac{x^3}{3}\bigg|_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.
$$

Thus,

$$f_X(x) = \frac{6}{5} \cdot (x + x^2) \cdot I(0 < x < 1),$$

which is a quadratic pdf on the unit interval. ‖

2.3.2 Discrete random variables, for any $x_i \in \mathcal{S}_X$

$$\begin{aligned}
\Pr(X = x_i) &= \left\{ \sum_j \frac{\Pr(Y = y_j | X = x_i)}{\Pr(X = x_i | Y = y_j)} \right\}^{-1} \\
&= \left\{ \sum_k \frac{\Pr(X = x_k | Y = y_{j0})}{\Pr(Y = y_{j0} | X = x_k)} \right\}^{-1} \frac{\Pr(X = x_i | Y = y_{j0})}{\Pr(Y = y_{j0} | X = x_i)} \\
&\propto \frac{\Pr(X = x_i | Y = y_{j0})}{\Pr(Y = y_{j0} | X = x_i)}, \qquad \text{for an arbitrarily fixed } y_{j0} \in \mathcal{S}_Y.
\end{aligned}$$

■

Example T2.4 (Discrete conditional distributions). Let X be a discrete random variable with pmf $p_i = \Pr(X = x_i)$ for $i = 1, 2$ and Y be a discrete random variable with pmf $q_j = \Pr(Y = y_j)$ for $j = 1, 2$. Given two conditional distribution matrices

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 3/5 & 2/5 \end{pmatrix},$$

where the (i, j) element of $\boldsymbol{A}$ is $a_{ij} = \Pr(X = x_i | Y = y_j)$ and the (i, j) element of $\boldsymbol{B}$ is $b_{ij} = \Pr(Y = y_j | X = x_i)$.

(a) Find the marginal distribution of X.

(b) Find the marginal distribution of Y.

(c) Find the joint distribution of (X, Y).

Solution: Note that $\mathcal{S}_X = \{x_1, x_2\}$ and $\mathcal{S}_Y = \{y_1, y_2\}$. By using point-wise IBF, the marginal distribution of X is given by

X	x_1	x_2
$p_i = \Pr(X = x_i)$	$3/8$	$5/8$

Similarly, the marginal distribution of Y is given by

Y	y_1	y_2
$q_j = \Pr(Y = y_j)$	$1/2$	$1/2$

The joint distribution of (X, Y) is given by

$$\boldsymbol{P} = \begin{pmatrix} 1/8 & 1/4 \\ 3/8 & 1/4 \end{pmatrix}.$$

‖

T2.4 Distribution of the Function of Random Variables

Let a set of r.v.'s $X_1, \ldots, X_n$ have the joint cdf $F(x_1, \ldots, x_n)$ or the joint pdf $f(x_1, \ldots, x_n)$. We seek the distribution of $Y = h(X_1, \ldots, X_n)$ for some function $h(\cdot)$.

2.4.1 Cumulative distribution function technique

Step 1: Find the cdf of Y: $F(y) = \Pr\{h(X_1, \ldots, X_n) \leqslant y\}$;

Step 2: Find the pdf of Y: $f(y) = F'(y)$. ■

Example T2.5 (Chi-square distribution). Let X be a standard normal random variable. Using the cdf technique, find the pdf of $Y = X^2$.

Solution: Let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \quad -\infty < x < \infty$$

denote the density of $N(0, 1)$. The cdf of Y is

$$\begin{aligned} F(y) &= \Pr(X^2 \leqslant y) = \Pr(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \phi(x)\,\mathrm{d}x = 2\int_0^{\sqrt{y}} \phi(x)\,\mathrm{d}x. \end{aligned}$$

Therefore, the density of Y is

$$\begin{aligned} f(y) &= \frac{\mathrm{d}F(y)}{\mathrm{d}y} = \frac{\mathrm{d}F(y)}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}y} \qquad [\text{let } z = \sqrt{y}] \\ &= \begin{cases} \frac{2}{\sqrt{2\pi}}\mathrm{e}^{-y/2} \cdot \frac{y^{-1/2}}{2} = \frac{1}{\sqrt{2\pi}} y^{-1/2}\mathrm{e}^{-y/2}, & \text{if } 0 < y < \infty, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

That is, $Y \sim \chi^2(1)$. ‖

2.4.2 Monotone transformation technique

Case 1: Univariate case:

$$g(y) = f(h^{-1}(y)) \times \left| \frac{\mathrm{d}h^{-1}(y)}{\mathrm{d}y} \right|. \qquad \blacksquare$$

Case 2: Bivariate case:

$$g(y_1, y_2) = f(h_1^{-1}(y_1, y_2), h_2^{-1}(y_1, y_2))|J(x_1, x_2 \to y_1, y_2)|. \qquad \blacksquare$$

Example T2.6 (Two independent exponential distributions). Let X_1 and X_2 be two independently exponentially distributed r.v.'s with a rate parameter λ.

(a) Find the joint pdf of $Y_1 = X_1/X_2$ and $Y_2 = X_1 + X_2$.

(b) Find the marginal pdf's of Y_1 and Y_2.

NOTE: Let $Z_i \sim \text{Gamma}(\alpha_i, \beta)$ and $Z_1 \perp\!\!\!\perp Z_2$, then $Y = Z_1/Z_2$ is said to follow an inverted beta distribution with parameters α_1 and α_2. Its density is

$$f(y) = \frac{1}{B(\alpha_1, \alpha_2)} \cdot \frac{y^{\alpha_1 - 1}}{(1+y)^{\alpha_1+\alpha_2}}, \quad y > 0. \qquad \blacksquare$$

Solution: (a) The transformation is $y_1 = x_1/x_2$ and $y_2 = x_1 + x_2$. Hence, $y_1 > 0$ and $y_2 > 0$. The corresponding inverse transformation is

$$x_1 = \frac{y_1 y_2}{1+y_1} \quad \text{and} \quad x_2 = \frac{y_2}{1+y_1}.$$

Hence, the Jacobian determinant is

$$J = J(x_1, x_2 \to y_1, y_2) = \det \begin{pmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{pmatrix} = \det \begin{pmatrix} \dfrac{y_2}{(1+y_1)^2} & \dfrac{y_1}{1+y_1} \\ \dfrac{-y_2}{(1+y_1)^2} & \dfrac{1}{1+y_1} \end{pmatrix} = \frac{y_2}{(1+y_1)^2}.$$

The joint pdf of (Y_1, Y_2) is

$$\begin{aligned} f(y_1, y_2) &= f(x_1, x_2) \times |J| = f(x_1) f(x_2) |J| \\ &= \lambda \exp\left\{-\lambda \left(\frac{y_1 y_2}{1+y_1}\right)\right\} \cdot \lambda \exp\left\{-\lambda \left(\frac{y_2}{1+y_1}\right)\right\} |J| \\ &= \frac{\lambda^2 y_2}{(1+y_1)^2} \mathrm{e}^{-\lambda y_2}, \qquad y_1 > 0,\ y_2 > 0. \end{aligned}$$

(b) The marginal density of Y_1 is

$$\begin{aligned} f(y_1) &= \int_0^\infty \frac{\lambda^2 y_2}{(1+y_1)^2} \mathrm{e}^{-\lambda y_2}\, \mathrm{d}y_2 = \frac{\lambda}{(1+y_1)^2} \int_0^\infty y_2 \lambda \mathrm{e}^{-\lambda y_2}\, \mathrm{d}y_2 \\ &= \frac{\lambda}{(1+y_1)^2} E(Y_2) = \frac{1}{(1+y_1)^2}, \quad y_1 > 0. \end{aligned}$$

Hence, Y_1 follows the inverted beta distribution with parameters 1 and 1. On the other hand,

$$\begin{aligned} f(y_2) &= \int_0^\infty \frac{\lambda^2 y_2}{(1+y_1)^2} \mathrm{e}^{-\lambda y_2} \mathrm{d}y_1 = \lambda^2 y_2 \mathrm{e}^{-\lambda y_2} \int_0^\infty \frac{\mathrm{d}y_1}{(1+y_1)^2} \\ &= \left. \frac{-1}{1+y_1} \right|_0^\infty = \lambda^2 y_2 \mathrm{e}^{-\lambda y_2}, \quad y_2 > 0, \end{aligned}$$

i.e., $Y_2 \sim \text{Gamma}(2, \lambda)$. ‖

Example T2.7 (Distribution in the simplex $\mathbb{V}_2$). Let the joint density of $(X, Y)^\top$ be

$$\begin{aligned} f(x, y) &= K \cdot (x + y) I_{(0,1)}(x) I_{(0,1)}(y) I_{(0,1)}(x + y) \\ &= K \cdot (x + y), \quad (x, y)^\top \in \mathbb{V}_2, \end{aligned}$$

where K is a positive constant, $I_{(0,1)}(x)$ is the indicator function and

$$\mathbb{V}_n = \{(x_1, \ldots, x_n)^\top \colon \; x_i > 0, \; i = 1, \ldots, n, \; \textstyle\sum_{i=1}^n x_i < 1\}.$$

is the simplex in the n-dimensional Euclidean space.

(a) Find the marginal density of X.

(b) Find the joint pdf of $X + Y$ and $Y - X$.

(c) Find the marginal pdf's of $X + Y$ and $Y - X$.

Solution: (a) The marginal density of X is given by

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} K \cdot (x + y) I_{(0,1)}(x) I_{(0,1)}(y) I_{(0,1)}(x + y) \, \mathrm{d}y \\ &= K \cdot I_{(0,1)}(x) \int_0^{1-x} (x + y) \, \mathrm{d}y \\ &= K \cdot I_{(0,1)}(x) \left[xy + \frac{y^2}{2} \right] \Bigg|_0^{1-x} = \frac{K(1 - x^2)}{2} I_{(0,1)}(x). \end{aligned}$$

(b) The transformation is $u = x + y$ and $v = y - x$. The corresponding inverse transformation is

$$x = \frac{1}{2}(u - v) \quad \text{and} \quad y = \frac{1}{2}(u + v).$$

Hence, the Jacobian determinant is

$$J(x, y \to u, v) = \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix} = \frac{1}{2}.$$

The joint pdf of (U, V) is

$$\begin{aligned} f(u,v) &= \frac{K}{2}(x+y)I_{(0,1)}(x)I_{(0,1)}(y)I_{(0,1)}(x+y) \\ &= \frac{K}{2}uI_{(0,1)}\left[\frac{1}{2}(u-v)\right]I_{(0,1)}\left[\frac{1}{2}(u+v)\right]I_{(0,1)}(u). \end{aligned}$$

(c) The marginal pdf of U is

$$f(u) = \int_{-u}^{u} \frac{Ku}{2}\,\mathrm{d}v = Ku^2, \quad 0 < u < 1,$$

i.e., $U \sim \text{Beta}(3,1)$, so that

$$K = \frac{1}{B(3,1)} = \frac{\Gamma(4)}{\Gamma(3)\Gamma(1)} = \frac{3!}{2! \cdot 1} = 3.$$

The marginal density of V is

$$\begin{aligned} f(v) &= \begin{cases} \displaystyle\int_{v}^{1} \frac{Ku}{2}\,\mathrm{d}u = \frac{K(1-v^2)}{4}, & 0 \leqslant v < 1, \\ \displaystyle\int_{-v}^{1} \frac{Ku}{2}\,\mathrm{d}u = \frac{K(1-v^2)}{4}, & -1 < v \leqslant 0 \end{cases} \\ &= \frac{K}{4}(1-v^2), \quad -1 < v < 1. \end{aligned}$$

$\|$

2.4.3 Moment generating function technique

Let $Y = \sum_{i=1}^{n} X_i$. If $\{X_i\}_{i=1}^{n}$ are independent r.v.'s, then

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

We can find the moment generating function of $Y = h(X_1, \ldots, X_n)$ and match it with those of some standard distributions. ■

Example T2.8 (Independent gamma distributions). If $X_1, \ldots, X_n$ are independent gamma r.v.'s with shape parameters α_i, $i = 1, \ldots, n$ and a common rate parameter β. By using the moment generating function technique, find the distribution of $Y = \sum_{i=1}^n X_i$.

Solution: (a) Since

$$f(x_i) = \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-\beta x_i}, \quad 0 < x_i < \infty, \quad i = 1, \ldots, n,$$

we obtain

$$\begin{aligned} M_{X_i}(t) &= E(\mathrm{e}^{tX_i}) = \int_0^\infty \mathrm{e}^{tx_i} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-\beta x_i}\,\mathrm{d}x_i \\ &= \int_0^\infty \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-(\beta - t)x_i}\,\mathrm{d}x_i \\ &= \left(\frac{\beta}{\beta - t}\right)^{\alpha_i} \int_0^\infty \frac{(\beta - t)^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-(\beta - t)x_i}\,\mathrm{d}x_i \\ &= \left(\frac{\beta}{\beta - t}\right)^{\alpha_i}. \end{aligned}$$

(b) In fact,

$$M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n \left(\frac{\beta}{\beta - t}\right)^{\alpha_i} = \left(\frac{\beta}{\beta - t}\right)^{\sum_{i=1}^n \alpha_i}.$$

So $Y \sim \text{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$. ‖

T2.5 Expectation Technique

2.5.1 Continuous random variables

(a) The one-dimensional case

- Let $X \sim f_X(x)$ and $Y = h(X)$. The aim is to find the pdf of Y.
- For any nonnegative and measurable $g(\cdot)$, if

$$E[g(Y)] = E[g(h(X))] = \int g(h(x)) \cdot f_X(x)\,\mathrm{d}x = \int g(y) \cdot f_Y(y)\,\mathrm{d}y, \tag{T2.3}$$

  then we can claim that $f_Y(y)$ is the pdf of Y. ■

Example T2.9 (Absolute value of a continuous r.v. X defined in the real line). Let $X \sim f_X(x)$, $x \in \mathbb{R}$, find the distribution of $Y = |X|$.

Solution: For any nonnegative and measurable $g(\cdot)$, from (T2.3), we have

$$
\begin{aligned}
E[g(Y)] &= E[g(|X|)] = \int_{-\infty}^{\infty} g(|x|) \cdot f_X(x)\,\mathrm{d}x \\
&= \underbrace{\int_{-\infty}^{0} g(-x) \cdot f_X(x)\,\mathrm{d}x}_{\text{let } y=-x} + \int_{0}^{\infty} g(x) \cdot f_X(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} g(y) \cdot f_X(-y)\,\mathrm{d}y + \int_{0}^{\infty} g(y) \cdot f_X(y)\,\mathrm{d}y \\
&= \int_{0}^{\infty} g(y) \cdot [f_X(-y) + f_X(y)]\,\mathrm{d}y,
\end{aligned}
$$

so that the pdf of Y is $f_Y(y) = [f_X(-y) + f_X(y)] \cdot I(y \geqslant 0)$. ‖

Example T2.10 (Uniform distribution and Cauchy distribution). Let $X \sim U(-\pi/2, \pi/2)$, find the distribution of $Y = \sigma \cdot \tan(X)$, where $\sigma\,(> 0)$ is a known constant.

NOTE: The pdf of the general Cauchy distribution is defined by

$$
f_Y(y|\mu, \sigma^2) = \frac{1}{\pi} \cdot \frac{\sigma}{\sigma^2 + (y-\mu)^2}, \quad y \in (-\infty, \infty),\ \mu \in \mathbb{R},\ \sigma > 0,
$$

denoted by $Y \sim$ **Cauchy**(μ, σ^2). Especially, Cauchy$(0, 1)$ is called the standard Cauchy distribution. ■

Solution: Let $y = \sigma \cdot \tan(x) = \sigma \sin(x)/\cos(x)$, we have

$$
\begin{aligned}
y^2 &= \sigma^2 \cdot \frac{\sin^2(x)}{\cos^2(x)} = \sigma^2 \cdot \frac{1-\cos^2(x)}{\cos^2(x)} = \frac{\sigma^2}{\cos^2(x)} - \sigma^2, \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sigma \cdot \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sigma \cdot \frac{1}{\cos^2(x)} = \frac{y^2 + \sigma^2}{\sigma},
\end{aligned}
$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\sigma}{\sigma^2 + y^2}. \tag{T2.4}$$

For any nonnegative and measurable $g(\cdot)$, from (T2.3), we have

$$\begin{aligned} E[g(Y)] &= E[g(\sigma \cdot \tan(X))] = \underbrace{\int_{-\pi/2}^{\pi/2} g\big(\sigma \cdot \tan(x)\big) \cdot \frac{1}{\pi}\,\mathrm{d}x}_{\text{let } y=\sigma\cdot\tan(x)} \\ &\overset{\text{(T2.4)}}{=} \int_{-\infty}^{\infty} g(y) \cdot \left(\frac{1}{\pi} \cdot \frac{\sigma}{\sigma^2 + y^2}\right) \mathrm{d}y, \end{aligned}$$

so that the pdf of Y is

$$f_{_Y}(y|0,\sigma^2) = \frac{1}{\pi} \cdot \frac{\sigma}{\sigma^2 + y^2}, \qquad -\infty < y < \infty,$$

i.e., $Y \sim \text{Cauchy}(0, \sigma^2)$. ‖

(b) The two-dimensional case

- Let $(X_1, X_2)^\top \sim f_{_{(X_1,X_2)}}(x_1, x_2)$ and $Y = h(X_1, X_2)$. The aim is to find the pdf of Y.

- For any nonnegative and measurable $g(\cdot)$, if

$$\begin{aligned} E[g(Y)] = E[g(h(X_1, X_2))] &= \int\!\!\int g(h(x_1, x_2)) \cdot f_{_{(X_1,X_2)}}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \int g(y) \cdot f_{_Y}(y)\,\mathrm{d}y, \end{aligned} \tag{T2.5}$$

  then we can claim that $f_{_Y}(y)$ is the pdf of Y. ■

Example T2.11 (Cauchy distribution). Let $X_1, X_2 \overset{\text{iid}}{\sim} \text{Cauchy}(0, 1)$, find the distribution of $Y = X_1 + X_2$.

Solution: For any nonnegative and measurable $g(\cdot)$, from (T2.5), we have

$$\begin{aligned}
E[g(Y)] &= E[g(X_1+X_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1+x_2)\cdot f_{(X_1,X_2)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1+x_2)\cdot\frac{1}{(1+x_1^2)(1+x_2^2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{1+x_2^2}\underbrace{\left[\int_{-\infty}^{\infty} g(x_1+x_2)\cdot\frac{1}{1+x_1^2}\,\mathrm{d}x_1\right]}_{\text{let } y=x_1+x_2}\mathrm{d}x_2 \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{1+x_2^2}\left[\int_{-\infty}^{\infty} g(y)\cdot\frac{1}{1+(y-x_2)^2}\,\mathrm{d}y\right]\mathrm{d}x_2 \quad \text{[exchange } y \text{ and } x_2\text{]} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} g(y)\left\{\int_{-\infty}^{\infty}\frac{1}{1+x_2^2}\cdot\frac{1}{\pi[1+(x_2-y)^2]}\,\mathrm{d}x_2\right\}\mathrm{d}y \\
&\overset{\text{(T2.6)}}{=} \int_{-\infty}^{\infty} g(y)\cdot\left(\frac{1}{\pi}\cdot\frac{2}{4+y^2}\right)\mathrm{d}y,
\end{aligned}$$

indicating that $Y \sim \text{Cauchy}(0,2)$. ||

NOTE: Let $X \sim \text{Cauchy}(\mu,1)$, show that

$$E\left(\frac{1}{1+X^2}\right) = \int_{-\infty}^{\infty}\frac{1}{1+x^2}\cdot\frac{1}{\pi[1+(x-\mu)^2]}\,\mathrm{d}x = \frac{2}{4+\mu^2}. \tag{T2.6}$$

Proof: By writing

$$\begin{aligned}
&\frac{1}{1+x^2}\cdot\frac{1}{1+(x-\mu)^2} = \frac{ax+b}{1+x^2}+\frac{cx+d}{1+(x-\mu)^2} \\
=\; &\frac{(ax+b)[1+(x-\mu)^2]+(cx+d)(1+x^2)}{(1+x^2)[1+(x-\mu)^2]} \\
=\; &\frac{(a+c)x^3+(-2a\mu+b+d)x^2+[a(\mu^2+1)-2b\mu+c]x+b(\mu^2+1)+d}{(1+x^2)[1+(x-\mu)^2]},
\end{aligned}$$

and setting

$$\begin{aligned}0 &= a+c,\\ 0 &= -2a\mu+b+d,\\ 0 &= a(\mu^2+1)-2b\mu+c,\\ 1 &= b(\mu^2+1)+d,\end{aligned}$$

we obtain

$$a=\frac{2}{\mu(4+\mu^2)},\quad b=\frac{1}{4+\mu^2},\quad c=\frac{-2}{\mu(4+\mu^2)}=-a,\quad d=\frac{3}{4+\mu^2}=3b.$$

Thus,

$$\begin{aligned}\pi\cdot E\left(\frac{1}{1+X^2}\right) &= \underbrace{\int_{-\infty}^{\infty}\frac{ax}{1+x^2}\,\mathrm{d}x}_{I_1}+\underbrace{\int_{-\infty}^{\infty}\frac{b}{1+x^2}\,\mathrm{d}x}_{I_2}\\ &\quad+\underbrace{\int_{-\infty}^{\infty}\frac{c(x-\mu)}{1+(x-\mu)^2}\,\mathrm{d}x}_{I_3\text{: Let } y=x-\mu}+\underbrace{\int_{-\infty}^{\infty}\frac{c\mu+d}{1+(x-\mu)^2}\,\mathrm{d}x}_{I_4},\end{aligned}$$

where

$$\begin{aligned}I_1+I_3 &= \int_{-\infty}^{\infty}\frac{ax}{1+x^2}\,\mathrm{d}x+\int_{-\infty}^{\infty}\frac{cy}{1+y^2}\,\mathrm{d}y=0,\quad [\text{because } c=-a]\\ I_2 &= \int_{-\infty}^{\infty}\frac{b}{1+x^2}\,\mathrm{d}x=b\pi,\quad \left[\text{because } \int_{-\infty}^{\infty}\frac{1}{\pi(1+x^2)}\,\mathrm{d}x=1\right]\\ I_4 &= \int_{-\infty}^{\infty}\frac{c\mu+d}{1+(x-\mu)^2}\,\mathrm{d}(x-\mu)=\int_{-\infty}^{\infty}\frac{1}{\pi(1+y^2)}\,\mathrm{d}y=(c\mu+d)\pi.\end{aligned}$$

Hence,

$$E\left(\frac{1}{1+X^2}\right)=\frac{I_2+I_4}{\pi}=b+c\mu+d=4b+c\mu=\frac{2}{4+\mu^2},$$

indicating (T2.6). □

2.5.2 Discrete random variables

(a) The one-dimensional case

- Let $X \sim p_X(x)$ and $Y = h(X)$. The aim is to find the pmf of Y.
- For any nonnegative and measurable $g(\cdot)$, if

$$E[g(Y)] = E[g(h(X))] = \sum_x g(h(x)) \cdot p_X(x) = \sum_y g(y) \cdot p_Y(y), \tag{T2.7}$$

then we can claim that $p_Y(y)$ is the pmf of Y. ∎

Example T2.12 (Binomial distribution). Let $X \sim \text{Binomial}(n, \theta)$ with $n = 3$ and $\theta = 1/3$, find the distribution of $Y = X/(1 + X)$.

Solution: For any nonnegative and measurable $g(\cdot)$, from (T2.7), we have

$$\begin{aligned} E[g(Y)] &= E\left[g\left(\frac{X}{1+X}\right)\right] \\ &= \sum_{x=0}^{3} g\left(\frac{x}{1+x}\right) \cdot p_X(x) \\ &= g(0) \cdot p_X(0) + g(1/2) \cdot p_X(1) + g(2/3) \cdot p_X(2) + g(3/4) \cdot p_X(3) \\ &= \sum_{y \in \mathcal{S}_Y} g(y) \cdot p_Y(y), \end{aligned}$$

where $\mathcal{S}_Y = \{0, 1/2, 2/3, 3/4\}$ is the support of Y. Then, the pmf of Y, $\Pr(Y = y)$, is given by $p_X(0) = 8/27$, $p_X(1) = 4/9 = 12/27$, $p_X(2) = 2/9 = 6/27$ and $p_X(3) = 1/27$, respectively. We summarize them as follows:

X	0	1	2	3
$\Pr(X = x)$	$p_X(0)$	$p_X(1)$	$p_X(2)$	$p_X(3)$
$Y = X/(1 + X)$	0	1/2	2/3	3/4
$\Pr(Y = y)$	8/27	12/27	6/27	1/27

$\|$

(b) The two-dimensional case

- Let $(X_1, X_2)^\top \sim p_{(X_1,X_2)}(x_1, x_2)$ and $Y = h(X_1, X_2)$. The aim is to find the pmf of Y.

- For any nonnegative and measurable $g(\cdot)$, if

$$E[g(Y)] = E[g(h(X_1, X_2))] = \sum_{x_1}\sum_{x_2} g(h(x_1, x_2)) \cdot p_{(X_1,X_2)}(x_1, x_2)$$

$$= \sum_y g(y) \cdot p_Y(y), \tag{T2.8}$$

then we can claim that $p_Y(y)$ is the pmf of Y. ■

Example T2.13 (Two-dimensional discrete distribution). Define $\mathbf{x} = (X_1, X_2)^\top$ and let $\mathbf{x} \sim p_{\mathbf{x}}(x_1, x_2)$, where

$$p_{\mathbf{x}}(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{36}, & \text{for } x_1 = 1, 2, 3 \quad \text{and} \quad x_2 = 1, 2, 3, \\ 0, & \text{otherwise,} \end{cases}$$

find the distribution of $Y = X_1 + X_2$.

Solution: For any nonnegative and measurable $g(\cdot)$, from (T2.8), we have

$$\begin{aligned} E[g(Y)] &= E[g(X_1 + X_2)] = \sum_{x_1}\sum_{x_2} g(x_1 + x_2) \cdot p_{\mathbf{x}}(x_1, x_2) \\ &= g(2) \cdot p_{\mathbf{x}}(1, 1) + g(3) \cdot p_{\mathbf{x}}(1, 2) + g(4) \cdot p_{\mathbf{x}}(1, 3) \\ &\quad + g(3) \cdot p_{\mathbf{x}}(2, 1) + g(4) \cdot p_{\mathbf{x}}(2, 2) + g(5) \cdot p_{\mathbf{x}}(2, 3) \\ &\quad + g(4) \cdot p_{\mathbf{x}}(3, 1) + g(5) \cdot p_{\mathbf{x}}(3, 2) + g(6) \cdot p_{\mathbf{x}}(3, 3) \end{aligned}$$

$$
\begin{aligned}
&= g(2) \cdot p_{\mathbf{x}}(1,1) + g(3) \cdot [p_{\mathbf{x}}(1,2) + p_{\mathbf{x}}(2,1)] \\
&\quad + g(4) \cdot [p_{\mathbf{x}}(1,3) + p_{\mathbf{x}}(2,2) + p_{\mathbf{x}}(3,1)] \\
&\quad + g(5) \cdot [p_{\mathbf{x}}(2,3) + p_{\mathbf{x}}(3,2)] + g(6) \cdot p_{\mathbf{x}}(3,3) \\
&= g(2) \cdot \frac{1}{36} + g(3) \cdot \frac{4}{36} + g(4) \cdot \frac{10}{36} + g(5) \cdot \frac{12}{36} + g(6) \cdot \frac{9}{36} \\
&= \sum_{y \in \mathcal{S}_Y} g(y) \cdot p_{Y}(y),
\end{aligned}
$$

where $\mathcal{S}_Y = \{2, 3, 4, 5, 6\}$ is the support of Y. Then, the pmf of Y is given by

Y	2	3	4	5	6
$\Pr(Y = y)$	$1/36$	$4/36$	$10/36$	$12/36$	$9/36$

$\|$