MA204: Mathematical Statistics

Suggested Solutions to Assignment 5

5.1 Solution. In Example 5.5, let $\mathbb{C} = \{1, 7, 3, 8, 4\}$. (a) The Type I error rate is

$$\begin{aligned}
\alpha(0) &= \Pr(X \in \mathbb{C}|\theta = 0) \\
&= \Pr(X = 1|0) + \Pr(X = 7|0) + \Pr(X = 3|0) \\
&\quad + \Pr(X = 8|0) + \Pr(X = 4|0) \\
&= 0 + 0.01 + 0.02 + 0.07 + 0.05 \\
&= 0.15.
\end{aligned}$$

(b) The acceptance region is $\mathbb{C}' = \{5, 9, 10, 6, 2\}$ so that the Type II error rate is given by

$$\begin{aligned}
\beta(1) &= \Pr(X \in \mathbb{C}'|\theta = 1) \\
&= \Pr(X = 5|1) + \Pr(X = 9|1) + \Pr(X = 10|1) \\
&\quad + \Pr(X = 6|1) + \Pr(X = 2|1) \\
&= 0.03 + 0.02 + 0.04 + 0.01 + 0 \\
&= 0.1.
\end{aligned}$$

5.2 Solution. Since $Y \sim \text{Binomial}(n, \theta)$, we have

$$E(Y) = n\theta \quad \text{and} \quad \text{Var}(Y) = n\theta(1 - \theta).$$

Let $Z \sim N(0,1)$. By the normal approximation, the power function is

$$\begin{aligned} p(\theta) &= \Pr(\text{reject } H_0|\theta) \\ &= \Pr(Y \geqslant c|\theta) \\ &= \Pr\left\{ \frac{Y-n\theta}{\sqrt{n\theta(1-\theta)}} \geqslant \frac{c-n\theta}{\sqrt{n\theta(1-\theta)}} \middle| \theta \right\} \\ &\doteq \Pr\left\{ Z \geqslant \frac{c-n\theta}{\sqrt{n\theta(1-\theta)}} \middle| \theta \right\}. \end{aligned}$$

Since $0.1 = p(0.5)$, we obtain

$$0.1 \doteq \Pr\left(Z \geqslant \frac{2c-n}{\sqrt{n}} \middle| \theta \right)$$

so that

$$\frac{2c-n}{\sqrt{n}} = z_{0.10} = 1.2816. \tag{SA5.1}$$

On the other hand, $0.95 = p(2/3)$, we obtain

$$0.95 \doteq \Pr\left(Z \geqslant \frac{3c-2n}{\sqrt{2n}} \middle| \theta \right)$$

so that

$$\frac{3c-2n}{\sqrt{2n}} = z_{0.95} = -1.645. \tag{SA5.2}$$

Solving equations (SA5.1) and (SA5.2), we get

$$1.5(n + 1.2816\sqrt{n}\,) - 2n = -1.645\sqrt{2n},$$

i.e.,

$$\sqrt{n} = 8.49756 \quad \text{or} \quad n = 72.2086.$$

Then

$$c = \frac{n + 1.2816\sqrt{n}}{2} = 41.5495.$$

Approximately, $n = 72$ and $c = 42$.

5.3 Solution. (a) The distribution of $Y = \sum_{i=1}^n X_i$ is Gamma$(2n, \theta)$.

(b) The likelihood function is

$$L(\theta) = \prod_{i=1}^n \frac{\theta^2}{\Gamma(2)} x_i \mathrm{e}^{-\theta x_i} = \theta^{2n} (\textstyle\prod_{i=1}^n x_i) \cdot \mathrm{e}^{-\theta \sum_{i=1}^n x_i}.$$

Since $\theta_1 > 1$, the ratio

$$\frac{L(1)}{L(\theta_1)} = \theta_1^{-2n} \mathrm{e}^{(\theta_1 - 1)\sum_{i=1}^n x_i} \leqslant k$$

is equivalent to

$$\sum_{i=1}^n x_i \leqslant c.$$

By Neymann–Pearson Lemma, a test φ of size α with critical region

$$\mathbb{C} = \{\boldsymbol{x}\colon\ L(1)/L(\theta_1) \leqslant k\} = \left\{\boldsymbol{x}\colon\ \sum_{i=1}^n x_i \leqslant c\right\}$$

is the most powerful test for testing H_0 against H_1.

How to determine the c? Under H_0 (i.e., $\theta = 1$),

$$Y = \sum_{i=1}^n X_i \sim \text{Gamma}(2n, 1).$$

Using Property 1 in Appendix A.2.4, under H_0, we have

$$2Y \sim \chi^2(4n).$$

Hence

$$\begin{aligned}
\alpha &= \Pr(\mathbf{x} \in \mathbb{C} | H_0) \\
&= \Pr(\textstyle\sum_{i=1}^n X_i \leqslant c | H_0) \\
&= \Pr(Y \leqslant c | H_0) \\
&= \Pr(2Y \leqslant 2c | H_0)
\end{aligned}$$

or

$$1 - \alpha = \Pr(2Y > 2c)$$

so that $2c = \chi^2(1-\alpha, 4n)$, i.e.,

$$c = \frac{1}{2}\chi^2(1-\alpha, 4n).$$

(c) The power function, for $\theta \geqslant 1$,

$$\begin{aligned} p(\theta) &= \Pr(\mathbf{x} \in \mathbb{C}|\theta) \\ &= \Pr(\textstyle\sum_{i=1}^n X_i \leqslant c|\theta) \\ &= \Pr(Y \leqslant c|\theta) \qquad \text{by } Y \sim \text{Gamma}(2n, \theta) \\ &= \int_0^c \frac{\theta^{2n}}{\Gamma(2n)} y^{2n-1} \mathrm{e}^{-\theta y}\, \mathrm{d}y. \end{aligned}$$

5.4 Solution. (a) Let $\mathbf{x} = (X_1, \ldots, X_n)^\top$, $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$,

$$Q(\mathbf{x}) \mathrel{\hat=} \prod_{i=1}^n (1 - X_i) \quad \text{and} \quad Q(\boldsymbol{x}) \mathrel{\hat=} \prod_{i=1}^n (1 - x_i),$$

then, the likelihood function is

$$L(\theta) = \prod_{i=1}^n \theta(1 - x_i)^{\theta-1} = \theta^n [Q(\boldsymbol{x})]^{\theta-1}. \tag{SA5.3}$$

Note that

$$\frac{L(1)}{L(\theta_1)} = \frac{1}{\theta_1^n [Q(\boldsymbol{x})]^{\theta_1 - 1}} \leqslant k$$

is equivalent to

$$\log Q(\boldsymbol{x}) \geqslant c.$$

By Neymann–Pearson Lemma, a test φ of size α with critical region

$$\mathbb{C} = \{\boldsymbol{x}\colon\ L(1)/L(\theta_1) \leqslant k\} = \{\boldsymbol{x}\colon\ \log Q(\boldsymbol{x}) \geqslant c\}$$

is the most powerful test for testing H_0 against H_1. To determine the c, we note that

$$Y_i = -\log(1 - X_i) \sim \text{Exponential}(\theta) = \text{Gamma}(1, \theta),$$

thus

$$-\log Q(\mathbf{x}) = -\sum_{i=1}^{n} \log(1 - X_i) = \sum_{i=1}^{n} Y_i \sim \text{Gamma}(n, \theta).$$

Under H_0, $-\log Q(\mathbf{x}) \sim \text{Gamma}(n, 1)$. Using Property 1 in Appendix A.2.4, under H_0, we have

$$-2\log Q(\mathbf{x}) \sim \chi^2(2n). \tag{SA5.4}$$

Hence

$$\begin{aligned} \alpha &= \Pr(\mathbf{x} \in \mathbb{C} | H_0) \\ &= \Pr\{\log Q(\mathbf{x}) \geqslant c | H_0\} \\ &= \Pr\{-2\log Q(\mathbf{x}) \leqslant -2c | H_0\} \end{aligned}$$

or

$$1 - \alpha = \Pr\{-2\log Q(\mathbf{x}) > -2c\}$$

so that $-2c = \chi^2(1 - \alpha, 2n)$, i.e., $c = -\chi^2(1 - \alpha, 2n)/2$.

(b) Now $\Theta_0 = \{1\}$ and $\Theta = (0, \infty)$. To derive the likelihood ratio statistic (LRS), we first need to find the MLE of θ. From (SA5.3), we have

$$\begin{aligned} \log L(\theta) &= n\log\theta + (\theta - 1)\log Q(\boldsymbol{x}), \\ \frac{\mathrm{d}\log L(\theta)}{\mathrm{d}\theta} &= \frac{n}{\theta} + \log Q(\boldsymbol{x}). \end{aligned}$$

Thus, the MLE is

$$\hat{\theta} = -n/\log Q(\mathbf{x})$$

with $Q(\mathbf{x})$ as a sufficient statistic, and the LRS is

$$\lambda(\mathbf{x}) = \frac{L(1)}{L(\hat{\theta})} = \frac{1}{\hat{\theta}^n [Q(\mathbf{x})]^{\hat{\theta}-1}}.$$

Denoting $Q(\boldsymbol{x})$ by Q, we have

$$\lambda(\boldsymbol{x}) = \frac{1}{(-n/\log Q)^n \cdot Q^{-n/\log Q - 1}}.$$

Thus, based on (5.29) in Chapter 5 (page 99), the critical region that the H_0 is rejected is

$$\mathbb{C} = \{\boldsymbol{x}\colon\ \lambda(x) \leqslant \lambda_\alpha\} = \{\boldsymbol{x}\colon\ (-n/\log Q)^n \cdot Q^{-n/\log Q - 1} \geqslant c\}. \tag{SA5.5}$$

To determine c, we let

$$h(Q) = (-n/\log Q)^n \cdot Q^{-n/\log Q - 1},$$

then

$$\log h(Q) = n \log n - n \log(-\log Q) - n - \log Q.$$

We have

$$\frac{\mathrm{d} \log h(Q)}{\mathrm{d}Q} = \frac{n}{Q(-\log Q)} - \frac{1}{Q} = \frac{n + \log Q}{Q(-\log Q)}.$$

Setting $\frac{\mathrm{d} \log h(Q)}{\mathrm{d}Q} = 0$, we obtain $Q = \mathrm{e}^{-n}$. Note that $0 < Q < 1$, then

$$\frac{\mathrm{d} \log h(Q)}{\mathrm{d}Q} = \frac{n + \log Q}{Q(-\log Q)} < 0, \quad \text{when } Q < \mathrm{e}^{-n},$$

and

$$\frac{\mathrm{d} \log h(Q)}{\mathrm{d}Q} = \frac{n + \log Q}{Q(-\log Q)} > 0, \quad \text{when } Q > \mathrm{e}^{-n},$$

Hence, $Q = \mathrm{e}^{-n}$ is the minimum of $h(Q)$, and $h(Q)$ is decreasing when $Q < \mathrm{e}^{-n}$ and increasing when $Q < \mathrm{e}^{-n}$. Therefore, (SA5.5) is equivalent to

$$\mathbb{C} = \{\boldsymbol{x}\colon\ Q \leqslant c_1 \quad \text{or} \quad Q \geqslant c_2\}. \tag{SA5.6}$$

Namely, determining c is equivalent to determining c_1 and c_2.

How to determine c_1 and c_2? Based on the Type I error rate

$$\begin{aligned} \alpha &= \Pr\{Q(\mathbf{x}) \leqslant c_1 \quad \text{or} \quad Q(\mathbf{x}) \geqslant c_2 | H_0\} \\ &= \Pr\{Q(\mathbf{x}) \leqslant c_1 | H_0\} + \Pr\{Q(\mathbf{x}) \geqslant c_2 | H_0\}, \end{aligned}$$

we use the equal-tail approach, i.e.,

$$\alpha/2 = \Pr\{Q(\mathbf{x}) \leqslant c_1 | H_0\} \tag{SA5.7}$$

and

$$\alpha/2 = \Pr\{Q(\mathbf{x}) \geqslant c_2 | H_0\}. \tag{SA5.8}$$

Recall (SA5.4), from (SA5.7), we have

$$\begin{aligned} \alpha/2 &= \Pr\{\log Q(\mathbf{x}) \leqslant \log c_1 | H_0\} \\ &= \Pr\{-2\log Q(\mathbf{x}) \geqslant -2\log c_1 | H_0\} \end{aligned}$$

so that $-2\log c_1 = \chi^2(\alpha/2, 2n)$, i.e.,

$$c_1 = \exp\{-0.5\chi^2(\alpha/2, 2n)\}. \tag{SA5.9}$$

Similarly, from (SA5.8), we have

$$\begin{aligned} 1 - \alpha/2 &= \Pr\{Q(\mathbf{x}) < c_2 | H_0\} \\ &= \Pr\{\log Q(\mathbf{x}) \leqslant \log c_2 | H_0\} \\ &= \Pr\{-2\log Q(\mathbf{x}) \geqslant -2\log c_2 | H_0\} \end{aligned}$$

so that $-2\log c_2 = \chi^2(1 - \alpha/2, 2n)$, i.e.,

$$c_2 = \exp\{-0.5\chi^2(1 - \alpha/2, 2n)\}. \tag{SA5.10}$$

5.5 Solution. We first consider to test

$$H_0\colon\ \theta = \theta_0 \quad \text{against} \quad H_1\colon\ \theta = \theta_1\,(< \theta_0).$$

Let φ be a test with critical region satisfying (5.17). The likelihood function is given by

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x_i-\theta)^2\right\} \\ &= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}. \end{aligned}$$

Then

$$\begin{aligned} \frac{L(\theta_0)}{L(\theta_1)} &= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}[(x_i-\theta_0)^2-(x_i-\theta_1)^2]\right\} \\ &= \exp\left\{(\theta_0-\theta_1)\sum_{i=1}^{n}x_i - n(\theta_0^2-\theta_1^2)/2\right\} \leqslant k \end{aligned}$$

is equivalent to (by noting that $\theta_0 - \theta_1 > 0$)

$$\bar{x} \leqslant \frac{\log k}{n(\theta_0-\theta_1)} + \frac{\theta_0+\theta_1}{2} \mathrel{\hat{=}} c.$$

To determine c, we consider the size

$$\begin{aligned} \alpha &= \Pr(\bar{X} \leqslant c|\theta = \theta_0) \\ &= \Pr\{\sqrt{n}(\bar{X}-\theta_0) \leqslant \sqrt{n}(c-\theta_0)\} \\ &= \Pr\{Z \leqslant \sqrt{n}(c-\theta_0)\} \\ &= \Pr(Z \leqslant -z_\alpha). \end{aligned}$$

Then, $\sqrt{n}(c-\theta_0) = -z_\alpha$ or $c = \theta_0 - z_\alpha/\sqrt{n}$. Therefore, the test with critical region

$$\mathbb{C} = \{\boldsymbol{x}\colon\ \bar{x} \leqslant \theta_0 - z_\alpha/\sqrt{n}\,\}$$

is a most powerful test (MPT) of size α. Since the $\mathbb{C}$ depends only on n, θ_0, α and the fact $\theta_1 < \theta_0$, but not on the value of θ_1, the test φ is also a UMPT of size α for testing

$$H_0\colon\ \theta = \theta_0 \quad \text{against} \quad H_1\colon\ \theta < \theta_0.$$

On the other hand, the power function is given by

$$\begin{aligned} p_\varphi(\theta) &= \Pr(\mathbf{x} \in \mathbb{C}|\theta) \\ &= \Pr(\bar{X} \leqslant \theta_0 - z_\alpha/\sqrt{n}|\theta) \\ &= \Pr\{\sqrt{n}(\bar{X} - \theta) \leqslant -z_\alpha + \sqrt{n}(\theta_0 - \theta)|\theta\} \\ &= \Pr\{Z \leqslant -z_\alpha + \sqrt{n}(\theta_0 - \theta)\} \\ &= \Phi(-z_\alpha + \sqrt{n}(\theta_0 - \theta)) \end{aligned}$$

so that

$$\begin{aligned} \sup_{\theta \in \Theta_0} p_\varphi(\theta) &= \sup_{\theta \geqslant \theta_0} \Phi(-z_\alpha + \sqrt{n}(\theta_0 - \theta)) \\ &= \max_{\theta \geqslant \theta_0} \Phi(-z_\alpha + \sqrt{n}(\theta_0 - \theta)) \\ &= \Phi(-z_\alpha) \\ &= \alpha = p_\varphi(\theta_0). \end{aligned}$$

Then, the test φ is also a UMPT of size α for testing

$$H_0\colon\ \theta \geqslant \theta_0 \quad \text{against} \quad H_1\colon\ \theta < \theta_0.$$

5.6 Solution. (a) We first consider

$$H_0\colon\ \sigma^2 = \sigma_0^2 \quad \text{against} \quad H_1\colon\ \sigma^2 \neq \sigma_0^2.$$

Note that $\mathbf{\Theta}_0 = \{\mu\colon\ -\infty < \mu < +\infty\}$ and the whole parameter space $\mathbf{\Theta}^* = \mathbf{\Theta} = \{(\mu, \sigma^2)\colon\ -\infty < \mu < +\infty,\ \sigma^2 > 0\}$. The likelihood function is given by

$$L(\mu, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right\}.$$

so that

$$\log L(\mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

Hence, the unrestricted maximum likelihood estimates of μ and σ^2 are given by

$$\hat{\mu} = \bar{x} = \frac{\sum_{i=1}^n x_i}{n}, \quad \text{and} \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} = \frac{(n-1)s^2}{n}.$$

Under H_0, the restricted maximum likelihood estimate of μ is $\hat{\mu}^{\mathrm{R}} = \bar{x}$. Thus, the likelihood ratio is

$$\lambda(\boldsymbol{x}) = \frac{L(\hat{\mu}^{\mathrm{R}}, \sigma_0^2)}{L(\hat{\theta}, \hat{\sigma}^2)} = \left[\frac{(n-1)s^2}{n\sigma_0^2}\right]^{n/2} \exp\left\{-\frac{(n-1)s^2}{2\sigma_0^2} + \frac{n}{2}\right\}.$$

Define $f(s^2) = \log \lambda(\boldsymbol{x})$, then

$$\begin{aligned} \frac{\mathrm{d}f(s^2)}{\mathrm{d}s^2} = f'(s^2) &= \frac{n}{2s^2} - \frac{n-1}{2\sigma_0^2}, \\ f''(s^2) &= -\frac{n}{2s^4} < 0, \end{aligned}$$

i.e., $f(s^2)$ has a maximum at $s^2 = n\sigma_0^2/(n-1)$. Since $\lambda(\boldsymbol{x}) \leqslant \lambda_\alpha$ is equivalent to

$$\log \lambda(\boldsymbol{x}) = f(s^2) \leqslant \log \lambda_\alpha,$$

the critical region is given by

$$\mathbb{C}_1 = \{\boldsymbol{x}\colon\ s^2 \leqslant c_1 \quad \text{or} \quad s^2 \geqslant c_2\}.$$

Namely, determining λ_α is equivalent to determining c_1 and c_2.

How to determine c_1 and c_2? Based on the Type I error rate

$$\begin{aligned} \alpha &= \Pr(S^2 \leqslant c_1 \quad \text{or} \quad S^2 \geqslant c_2 | H_0) \\ &= \Pr(S^2 \leqslant c_1 | H_0) + \Pr(S^2 \geqslant c_2 | H_0), \end{aligned}$$

we use the equal-tail approach, i.e.,

$$\alpha/2 = \Pr(S^2 \leqslant c_1 | H_0) \tag{SA5.11}$$

and

$$\alpha/2 = \Pr(S^2 \geqslant c_2 | H_0). \tag{SA5.12}$$

Recall that $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$, from (SA5.11), we have

$$\begin{aligned}
1-\alpha/2 &= \Pr(S^2 > c_1 | H_0) \\
&= \Pr\{(n-1)S^2/\sigma^2 > (n-1)c_1/\sigma^2 | \sigma^2 = \sigma_0^2\} \\
&= \Pr\{(n-1)S^2/\sigma_0^2 > (n-1)c_1/\sigma_0^2\}
\end{aligned}$$

so that $(n-1)c_1/\sigma_0^2 = \chi^2(1-\alpha/2, n-1)$, i.e.,

$$c_1 = \frac{\sigma_0^2 \chi^2(1-\alpha/2, n-1)}{n-1}.$$

Similarly, from (SA5.12), we have

$$\begin{aligned}
\alpha/2 &= \Pr(S^2 \geqslant c_2 | H_0) \\
&= \Pr\{(n-1)S^2/\sigma^2 \geqslant (n-1)c_2/\sigma^2 | \sigma^2 = \sigma_0^2\} \\
&= \Pr\{(n-1)S^2/\sigma_0^2 \geqslant (n-1)c_2/\sigma_0^2\}
\end{aligned}$$

so that $(n-1)c_2/\sigma_0^2 = \chi^2(\alpha/2, n-1)$, i.e.,

$$c_2 = \frac{\sigma_0^2 \chi^2(\alpha/2, n-1)}{n-1}.$$

Therefore, the critical region is

$$\begin{aligned}
\mathbb{C}_1 &= \left\{ \boldsymbol{x}\colon\ s^2 \leqslant \frac{\sigma_0^2 \chi^2(1-\alpha/2, n-1)}{n-1} \right. \\
&\qquad\qquad \left. \text{or} \quad s^2 \geqslant \frac{\sigma_0^2 \chi^2(\alpha/2, n-1)}{n-1} \right\}.
\end{aligned}$$

(b) We then consider

$$H_0\colon\ \sigma^2 = \sigma_0^2 \quad \text{against} \quad H_1\colon\ \sigma^2 > \sigma_0^2.$$

The critical region is

$$\mathbb{C}_2 = \left\{ \boldsymbol{x}\colon\ s^2 \geqslant \frac{\sigma_0^2 \chi^2(1-\alpha, n-1)}{n-1} \right\}.$$

(c) We finally consider

$$H_0\colon\ \sigma^2 = \sigma_0^2 \quad \text{against} \quad H_1\colon\ \sigma^2 < \sigma_0^2.$$

The critical region is

$$\mathbb{C}_3 = \left\{ \boldsymbol{x}\colon\ s^2 \leqslant \frac{\sigma_0^2 \chi^2(1-\alpha, n-1)}{n-1} \right\}.$$

5.7 Solution. Let $\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma^2)^\top$, $\boldsymbol{\Theta}_0 = \{(0, 0, \sigma^2)^\top\colon\ \sigma^2 > 0\}$ and $\boldsymbol{\Theta} = \{\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma^2)^\top\colon\ -\infty < \mu_1, \mu_2 < +\infty, \sigma^2 > 0\}$, then

$$H_0\colon\ \boldsymbol{\theta} \in \boldsymbol{\Theta}_0 \quad \text{against} \quad H_1\colon\ \boldsymbol{\theta} \in \boldsymbol{\Theta}_1 = \boldsymbol{\Theta} - \boldsymbol{\Theta}_0.$$

The likelihood function is

$$L(\boldsymbol{\theta}) = (2\pi\sigma^2)^{-n} \exp\left[-\frac{\sum_{i=1}^n (x_i - \mu_1)^2 + \sum_{i=1}^n (y_i - \mu_2)^2}{2\sigma^2} \right]$$

so that

$$\log L(\boldsymbol{\theta}) = -n\log(2\pi) - n\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n [(x_i - \mu_1)^2 + (y_i - \mu_2)^2].$$

Hence, the unrestricted maximum likelihood estimates of μ_1, μ_2 and σ^2 are given by

$$\begin{aligned}
\hat{\mu}_1 &= \frac{\sum_{i=1}^n x_i}{n} = \bar{x}, \\
\hat{\mu}_2 &= \frac{\sum_{i=1}^n y_i}{n} = \bar{y}, \quad \text{and} \\
\hat{\sigma}^2 &= \frac{\sum_{i=1}^n [(x_i - \bar{x})^2 + (y_i - \bar{y})^2]}{2n} = \frac{(n-1)(s_1^2 + s_2^2)}{2n}.
\end{aligned}$$

Under H_0 (i.e., $\mu_1 = \mu_2 = 0$), the restricted maximum likelihood estimate of σ^2 is

$$\begin{aligned}\hat{\sigma}^{2\mathrm{R}} &= \frac{\sum_{i=1}^n (x_i^2 + y_i^2)}{2n} \\ &= \frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n\bar{x}^2 + \sum_{i=1}^n (y_i - \bar{y})^2 + n\bar{y}^2}{2n} \\ &= \frac{(n-1)(s_1^2 + s_2^2) + n\bar{x}^2 + n\bar{y}^2}{2n}.\end{aligned}$$

Thus, the likelihood ratio is

$$\begin{aligned}\lambda(\boldsymbol{x}, \boldsymbol{y}) &= \frac{L(0, 0, \hat{\sigma}^{2\mathrm{R}})}{L(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2)} = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}^{2\mathrm{R}}}\right)^n \\ &= \left[\frac{(n-1)(s_1^2 + s_2^2)}{(n-1)(s_1^2 + s_2^2) + n\bar{x}^2 + n\bar{y}^2}\right]^n \\ &= \left(\frac{1}{1+F}\right)^n,\end{aligned}$$

where

$$F = \frac{n\bar{x}^2 + n\bar{y}^2}{(n-1)(s_1^2 + s_2^2)}.$$

Since $\lambda(\boldsymbol{x}, \boldsymbol{y}) \leqslant \lambda_\alpha$ is equivalent to $F \geqslant c$, the critical region is given by

$$\mathbb{C} = \{(\boldsymbol{x}, \boldsymbol{y}) \colon\ F \geqslant c\}.$$

<u>How to determine</u> c? Note that

$$\frac{\sqrt{n}(\bar{X} - \mu_1)}{\sigma} \sim N(0, 1), \quad \frac{\sqrt{n}(\bar{Y} - \mu_2)}{\sigma} \sim N(0, 1),$$

$(n-1)S_i^2/\sigma^2 \sim \chi^2(n-1)$, $i = 1, 2$, and they are independent. Thus, under H_0,

$$\frac{n(\bar{X}^2 + \bar{Y}^2)}{\sigma^2} \sim \chi^2(2) \quad \text{and} \quad \frac{(n-1)(S_1^2 + S_2^2)}{\sigma^2} \sim \chi^2(2n-2)$$

so that

$$(n-1)F = \frac{\frac{n(\bar{X}^2 + \bar{Y}^2)}{\sigma^2}/2}{\frac{(n-1)(S_1^2 + S_2^2)}{\sigma^2}/2(n-1)} \sim F(2, 2n-2).$$

Based on the Type I error rate

$$\begin{aligned}\alpha &= \Pr(F \geqslant c|H_0) \\ &= \Pr\{(n-1)F \geqslant (n-1)c\},\end{aligned}$$

we have $(n-1)c = f(\alpha, 2, 2n-2)$, i.e.,

$$c = \frac{f(\alpha, 2, 2n-2)}{n-1}.$$

5.8 Solution. (a) We first consider

$$H_0\colon\ \sigma_1^2 = \sigma_2^2 = \sigma^2 \quad \text{against} \quad H_1\colon\ \sigma_1^2 \neq \sigma_2^2.$$

Let $\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)^\top$. The likelihood function is given by

$$\begin{aligned}L(\boldsymbol{\theta}) &= (2\pi\sigma_1^2)^{-n_1/2} \exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n_1}(x_i - \mu_1)^2\right\} \\ &\quad \times (2\pi\sigma_2^2)^{-n_2/2} \exp\left\{-\frac{1}{2\sigma_2^2}\sum_{j=1}^{n_2}(y_j - \mu_2)^2\right\}\end{aligned}$$

so that

$$\begin{aligned}\log L(\boldsymbol{\theta}) &= -\frac{n_1}{2}\log(2\pi) - \frac{n_1}{2}\log(\sigma_1^2) - \frac{1}{2\sigma_1^2}\sum_{i=1}^{n_1}(x_i - \mu_1)^2 \\ &\quad -\frac{n_2}{2}\log(2\pi) - \frac{n_2}{2}\log(\sigma_2^2) - \frac{1}{2\sigma_2^2}\sum_{j=1}^{n_2}(y_j - \mu_2)^2.\end{aligned}$$

Hence, the unrestricted maximum likelihood estimates of the 4 parameters are given by

$$\hat{\mu}_1 = \bar{x}, \quad \hat{\mu}_2 = \bar{y}, \quad \text{and} \quad \hat{\sigma}_k^2 = \frac{(n_k - 1)s_k^2}{n_k},\ k = 1, 2.$$

Under H_0, the restricted maximum likelihood estimates of the 3 parameters are

$$\hat{\mu}_1^{\mathrm{R}} = \bar{x}, \quad \hat{\mu}_2^{\mathrm{R}} = \bar{y}, \quad \text{and} \quad \hat{\sigma}^{2\mathrm{R}} = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2}.$$

Thus, the likelihood ratio is

$$\lambda(\boldsymbol{x},\boldsymbol{y}) = \frac{L(\hat\mu_1^{\mathrm{R}},\hat\mu_2^{\mathrm{R}},\hat\sigma^{2\mathrm{R}},\hat\sigma^{2\mathrm{R}})}{L(\hat\mu_1^{\mathrm{R}},\hat\mu_2^{\mathrm{R}},\hat\sigma_1^{2\mathrm{R}},\hat\sigma_2^{2\mathrm{R}})} = \frac{(\hat\sigma_1^2)^{n_1/2}(\hat\sigma_2^2)^{n_2/2}}{(\hat\sigma^{2\mathrm{R}})^{(n_1+n_2)/2}}$$
$$= \left[\frac{(n_1+n_2)(n_1-1)/n_1}{n_1-1+(n_2-1)s_2^2/s_1^2}\right]^{\frac{n_1}{2}}\left[\frac{(n_1+n_2)(n_2-1)[s_2^2/s_1^2]/n_2}{n_1-1+(n_2-1)s_2^2/s_1^2}\right]^{\frac{n_2}{2}}.$$

Define $t = s_2^2/s_1^2$ and

$$f(t) = \frac{t^{n_2/2}}{[n_1-1+(n_2-1)t]^{\frac{n_1+n_2}{2}}},$$

then

$$\frac{\mathrm{d}\log f(t)}{\mathrm{d}t} = \frac{n_2}{2t} - \frac{n_1+n_2}{2}\cdot\frac{n_2-1}{n_1-1+(n_2-1)t}$$

Setting $\frac{\mathrm{d}\log f(t)}{\mathrm{d}t} = 0$, we obtain $t = n_2(n_1-1)/[n_1(n_2-1)] \mathrel{\hat=} t_0$. Note that

$$\frac{\mathrm{d}\log f(t)}{\mathrm{d}t} = \frac{n_2(n_1-1)-n_1(n_2-1)t}{2t[n_1-1+(n_2-1)t]} < 0, \quad \text{when } t > t_0,$$

and

$$\frac{\mathrm{d}\log f(t)}{\mathrm{d}t} = \frac{n_2(n_1-1)-n_1(n_2-1)t}{2t[n_1-1+(n_2-1)t]} > 0, \quad \text{when } t < t_0.$$

Hence, $t = t_0$ is the maximum of $f(t)$, and $f(t)$ is increasing when $t < t_0$ and decreasing when $t > t_0$. Therefore, $\lambda(\boldsymbol{x},\boldsymbol{y}) \leqslant \lambda_\alpha$ is equivalent to $f(t) \leqslant c$, resulting in the following critical region

$$\mathbb{C}_1 = \{(\boldsymbol{x},\boldsymbol{y})\colon\ s_2^2/s_1^2 \leqslant c_1 \quad \text{or} \quad s_2^2/s_1^2 \geqslant c_2\}.$$

How to determine c_1 and c_2? Based on the Type I error rate

$$\begin{aligned}\alpha &= \Pr(S_2^2/S_1^2 \leqslant c_1 \quad \text{or} \quad S_2^2/S_1^2 \geqslant c_2|H_0)\\ &= \Pr(S_2^2/S_1^2 \leqslant c_1|H_0) + \Pr(S_2^2/S_1^2 \geqslant c_2|H_0),\end{aligned}$$

we use the equal-tail approach, i.e.,

$$\alpha/2 = \Pr(S_2^2/S_1^2 \leqslant c_1 | H_0)$$

and

$$\alpha/2 = \Pr(S_2^2/S_1^2 \geqslant c_2 | H_0).$$

Note that under H_0, $S_2^2/S_1^2 \sim F(n_2 - 1, n_1 - 1)$, we have

$$c_1 = f(1-\alpha/2, n_2-1, n_1-1) \quad \text{and} \quad c_2 = f(\alpha/2, n_2-1, n_1-1).$$

(b) We then consider

$$H_0\colon\ \sigma_1^2 = \sigma_2^2 \quad \text{against} \quad H_1\colon\ \sigma_1^2 > \sigma_2^2.$$

The critical region is

$$\mathbb{C}_2 = \{(\boldsymbol{x}, \boldsymbol{y})\colon\ s_2^2/s_1^2 \leqslant f(1-\alpha, n_2-1, n_1-1)\}.$$

(c) We finally consider

$$H_0\colon\ \sigma_1^2 = \sigma_2^2 \quad \text{against} \quad H_1\colon\ \sigma_1^2 < \sigma_2^2.$$

The critical region is

$$\mathbb{C}_3 = \{(\boldsymbol{x}, \boldsymbol{y})\colon\ s_2^2/s_1^2 \geqslant f(\alpha, n_2-1, n_1-1)\}.$$

5.9 Solution. We consider

$$H_0\colon\ \theta = 0.5 \quad \text{against} \quad H_1\colon\ \theta \neq 0.5.$$

Let $X \sim \text{Binomial}(n, \theta)$, the likelihood function is given by

$$L(\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

so that

$$\log L(\theta) = \log \binom{n}{x} + x \log \theta + (n-x)\log(1-\theta).$$

Hence, the maximum likelihood estimate of θ is $\hat{\theta} = x/n$.

(a) The likelihood ratio is

$$\lambda(x) = \frac{L(0.5)}{L(\hat{\theta})} = \frac{(0.5n)^n}{x^x(n-x)^{n-x}}$$

so that the LR statistic is

$$\lambda(X) = \frac{(0.5n)^n}{X^X(n-X)^{n-X}}.$$

(b) Note that $\lambda(x) \leqslant \lambda_\alpha$ is equivalent to $x^x(n-x)^{n-x} \geqslant k$, or

$$x\log(x) + (n-x)\log(n-x) \geqslant c.$$

(c) Define

$$f(x) = x\log(x) + (n-x)\log(n-x),$$

then

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \log(x) - \log(n-x) = \log\frac{x}{n-x}.$$

Setting $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0$, we obtain $x = n/2$. Note that

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \log\frac{x}{n-x} < 0, \quad \text{when } x < n/2,$$

and

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \log\frac{x}{n-x} > 0, \quad \text{when } x > n/2.$$

Hence, $x = n/2$ is the minimum of $f(x)$, and $f(x)$ is decreasing when $x < n/2$ and increasing when $x > n/2$. Therefore, $f(x) \geqslant c^*$, resulting in the following critical region

$$\mathbb{C} = \{x\colon\ 0 \leqslant x \leqslant c_1 \quad \text{or} \quad c_2 \leqslant x \leqslant n\},$$

where $c_1 < n/2 < c_2$.

Note that

$$\begin{aligned}
f(0) &= f(n) = n\log(n), \\
f(1) &= f(n-1) = (n-1)\log(n-1), \\
f(2) &= f(n-2) = 2\log(2) + (n-2)\log(n-2), \\
&\vdots \\
f(n/2) &= n\log(n/2),
\end{aligned}$$

i.e., the function $f(x)$ is symmetrical about $x = n/2$. Therefore, the critical region $\mathbb{C}$ can be written as

$$\mathbb{C} = \{x\colon\ |x - n/2| \geqslant c\}.$$

5.10 Solution. Now $n = 556$, $np_{10} = 556 \times 9/16 = 312.75$, $np_{20} = 556 \times 3/16 = 104.25$, $np_{30} = 556 \times 3/16 = 104.25$, and $np_{40} = 556 \times 1/16 = 34.75$. According to (5.40) and (5.42), we have

$$\begin{aligned}
Q_n &= \sum_{i=1}^{4} \frac{(N_i - np_{i0})^2}{np_{i0}} \\
&= \frac{(315 - 312.75)^2}{312.75} + \frac{(108 - 104.25)^2}{104.25} \\
&\quad + \frac{(101 - 104.25)^2}{104.25} + \frac{(32 - 34.75)^2}{34.75} \\
&= 0.470 < \chi^2(0.05, 3) = 7.8147,
\end{aligned}$$

we cannot reject H_0, so there is a good agreement with null hypothesis or there is a good fit of the data to the model.

5.11 Solution. The null hypothesis is H_0: $p_1 = \cdots = p_6 = 1/6$. Now $n = 300$, $np_{i0} = 300 \times 1/6 = 50$, $i = 1, \ldots 6$. According to (5.40) and (5.42), we have

$$\begin{aligned} Q_n &= \sum_{i=1}^{6} \frac{(N_i - np_{i0})^2}{np_{i0}} \\ &= \frac{(43-50)^2}{50} + \frac{(49-50)^2}{50} + \frac{(56-50)^2}{50} \\ &\quad + \frac{(45-50)^2}{50} + \frac{(66-50)^2}{50} + \frac{(41-50)^2}{50} \\ &= 8.96 < \chi^2(0.05, 5) = 11.07, \end{aligned}$$

we cannot reject H_0.

5.12 Solution. (a) We first consider

$$H_0\colon\ \theta_1 = \theta_2 (= \theta) \quad \text{against} \quad H_1\colon\ \theta_1 \neq \theta_2.$$

Note that $\mathbf{\Theta}_0 = \{\theta\colon\ \mu > 0\}$ and the whole parameter space $\mathbf{\Theta}^* = \mathbf{\Theta} = \{(\theta_1, \theta_2)\colon\ \theta_1 > 0,\ \theta_2 > 0\}$. The likelihood function of (θ_1, θ_2) is given by

$$L(\theta_1, \theta_2) = \left(\prod_{i=1}^{m} \theta_1 \mathrm{e}^{-\theta_1 x_i}\right)\left(\prod_{j=1}^{n} \theta_2 \mathrm{e}^{-\theta_2 y_j}\right),$$

so that

$$\log L(\theta_1, \theta_2) = m \log(\theta_1) - \theta_1 m\bar{x} + n \log(\theta_2) - \theta_2 n\bar{y},$$

where $\bar{x} = (1/m)\sum_{i=1}^{m} x_i$ and $\bar{y} = (1/n)\sum_{j=1}^{n} y_j$. Hence, the unrestricted MLEs of θ_1 and θ_2 are given by

$$\hat{\theta}_1 = 1/\bar{x} \quad \text{and} \quad \hat{\theta}_2 = 1/\bar{y}.$$

Under H_0, the log-likelihood function of θ is

$$\log L(\theta, \theta) = (m+n)\log(\theta) - \theta(m\bar{x} + n\bar{y}),$$

so that the restricted maximum likelihood estimate of θ is

$$\hat{\theta}^{\mathrm{R}} = \frac{m+n}{m\bar{x}+n\bar{y}}.$$

Thus, the likelihood ratio is

$$\begin{aligned}
\lambda(\boldsymbol{x},\boldsymbol{y}) &= \frac{L(\hat{\theta}^{\mathrm{R}},\hat{\theta}^{\mathrm{R}})}{L(\hat{\theta}_1,\hat{\theta}_2)} = \frac{(\hat{\theta}^{\mathrm{R}})^{m+n}\mathrm{e}^{-(m+n)}}{\bar{x}^{-m}\bar{y}^{-n}\mathrm{e}^{-(m+n)}} \\
&= \left(\frac{m+n}{m\bar{x}+n\bar{y}}\right)^{m+n}\bar{x}^m\bar{y}^n,
\end{aligned}$$

so that the likelihood ratio statistic is

$$\lambda(\mathbf{x},\mathbf{y}) = \left(\frac{m+n}{m\bar{X}+n\bar{Y}}\right)^{m+n}\bar{X}^m\bar{Y}^n,$$

where $\bar{X} = (1/m)\sum_{i=1}^m X_i$ and $\bar{Y} = (1/n)\sum_{j=1}^n Y_j$.

(b) Since $m\bar{X} = \sum_{i=1}^m X_i \sim \text{Gamma}(m,\theta_1)$, we have

$$\theta_1 m\bar{X} \sim \text{Gamma}(m,1).$$

Similarly, we have $\theta_2 m\bar{Y} \sim \text{Gamma}(n,1)$. In addition $\bar{X} \perp\!\!\!\perp \bar{Y}$. Under H_0, i.e., $\theta_1 = \theta_2 = \theta$, we obtain

$$\begin{aligned}
T &= T(\mathbf{x},\mathbf{y}) = \frac{\sum_{i=1}^m X_i}{\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j} = \frac{m\bar{X}}{m\bar{X}+n\bar{Y}} \\
&= \frac{\theta_1 m\bar{X}}{\theta_1 m\bar{X} + th_2 n\bar{Y}} = \frac{\text{Gamma}(m,1)}{\text{Gamma}(m,1)+\text{Gamma}(n,1)} \\
&\sim \text{Beta}(m,n).
\end{aligned}$$

(c) Let $t = T(\boldsymbol{x},\boldsymbol{y})$ be the value of T. Then

$$\begin{aligned}
\lambda(\boldsymbol{x},\boldsymbol{y}) &= \left(\frac{m+n}{m\bar{x}+n\bar{y}}\right)^{m+n}\bar{x}^m\bar{y}^n \\
&= \frac{(m+n)^{m+n}}{m^m n^n} t^m(1-t)^n \leqslant \lambda_\alpha
\end{aligned}$$

if and only if

$$\log[\lambda(\boldsymbol{x}, \boldsymbol{y})] = \text{constant} + f(t) \leqslant \log(\lambda_\alpha),$$

where $f(t) \hat{=} m \log t + n \log(1-t)$. Since

$$f'(t) = \frac{m}{t} - \frac{n}{1-t} \quad \text{and} \quad f''(t) = -\left[\frac{m}{t^2} + \frac{n}{(1-t)^2}\right] < 0,$$

i.e., $f(t)$ has a maximum at $t = m/(m+n)$, the critical region $\mathbb{C}$ of the LRT with size α has the form

$$\mathbb{C} = \{(\boldsymbol{x}, \boldsymbol{y})\colon\ t = T(\boldsymbol{x}, \boldsymbol{y}) \leqslant k_1 \quad \text{or} \quad t = T(\boldsymbol{x}, \boldsymbol{y}) \geqslant k_2\},$$

where $k_1 < \frac{m}{m+n} < k_2$, and k_1, k_2 are two constants. Namely, determining λ_α is equivalent to determining k_1 and k_2.

How to determine k_1 and k_2? Based on the Type I error rate

$$\begin{aligned} \alpha &= \Pr(T \leqslant k_1 \quad \text{or} \quad T \geqslant k_2 | H_0) \\ &= \Pr(T \leqslant k_1 | H_0) + \Pr(T \geqslant k_2 | H_0), \end{aligned}$$

we use the equal-tail approach, i.e.,

$$\alpha/2 = \Pr(T \leqslant c_1 | H_0) \tag{SA5.13}$$

and

$$\alpha/2 = \Pr(T \geqslant k_2 | H_0). \tag{SA5.14}$$

Recall that $T \sim \text{Beta}(m, n)$ under H_0, from (SA5.13), we have

$$1 - \alpha/2 = \Pr(T > k_1 | H_0) = \Pr\{T > \text{Beta}(1-\alpha/2; m, n)\}$$

so that $k_1 = \text{Beta}(1-\alpha/2; m, n)$, which is the upper $(1-\alpha/2)$-th quantile of the $\text{Beta}(m, n)$ distribution. Similarly, from (SA5.14), we have $k_2 = \text{Beta}(\alpha/2; m, n)$.