陈愷 12311410 Complex Analysis (H) Homework 4.

1.

Proof: Let D be the Jordan dom.

By Riemann Mapping Thm, there exists a conf. map $\phi: B_1(0) \to D$.

By Carathéodory Thm, ϕ can be extend to a homeomorphism

$\tilde{\phi}: \overline{B_1(0)} \longrightarrow \bar{D}$.

Note that: a homeomorphism is a continuous bijection with continuous inverse.

$\Rightarrow \tilde{\phi}^{-1}$ exists.

Let f be the conf. automorphism of D. namely: $f: D \to D$. Let $g = \phi^{-1} \circ f \circ \phi$. $(B_1(0) \to B_1(0))$

$\Longrightarrow g(w) = e^{i\theta} \cdot \dfrac{w-\alpha}{1-\bar{\alpha} w}$ for some $\theta \in \mathbb{R}$, $\alpha \in B_1(0)$.

notation change of center. $(w \in B_1(0))$

$g(w)$ is continuous on $\overline{B_1(0)}$. $\Rightarrow$ can be extended to $\tilde{g}: \overline{B_1(0)} \to \overline{B_1(0)}$.

$\Rightarrow \tilde{f} = \tilde{\phi} \circ \tilde{g} \circ \tilde{\phi}^{-1}$.

Check: $\forall z \in D$, $\tilde{f}(z) = \tilde{\phi} \circ \tilde{g} \circ \tilde{\phi}^{-1}(z) = \phi(g(\phi^{-1}(z))) = \phi(\phi^{-1} \circ f \circ \phi(\phi^{-1}(z))) = f(z)$

$\Rightarrow \tilde{f}$ is a extension of f to $\bar{D}$. ✓

Uniqueness:

Let f_1, f_2 be two conf. map (automorphisms of D).

Suppose $f_1(a) = f_2(a)$, $f_1(b) = f_2(b)$, $f_1(c) = f_2(c)$.

By argument above. we have f_1, f_2 can be extended to $\bar{D}$.

Then, still use $\tilde{\phi}: \overline{B_1(0)} \to \bar{D}$. $\Rightarrow$ define $\tilde{g}_i = \tilde{\phi}^{-1} \circ \tilde{f}_i \circ \tilde{\phi}$, $i = 1, 2$.

Consider the map $g = \tilde{g}_2^{-1} \circ \tilde{g}_1$. Let $w_a = \tilde{\phi}^{-1}(a)$, $w_b = \tilde{\phi}^{-1}(b)$, $w_c = \tilde{\phi}^{-1}(c)$.

Then $g(w_x) = \tilde{g}_2^{-1} \circ \tilde{g}_1(w_x) = \tilde{\phi}^{-1} \circ \tilde{f}_2^{-1} \circ \tilde{\phi} \circ \tilde{\phi}^{-1} \circ \tilde{f}_1 \circ \tilde{\phi}(\tilde{\phi}^{-1}(x)) = \tilde{\phi}^{-1}(x) = w_x$

for $x = a, b, c$.

Since $\tilde{g} = \tilde{g}_2^{-1} \circ \tilde{g}_1$. $\tilde{g}_2, \tilde{g}_1 \in Aut(\overline{B_1(0)})$

$\tilde{g} \in Aut(\overline{B_1(0)})$

$\Rightarrow g = e^{i\theta} \cdot \frac{z-\alpha}{1-\bar{\alpha} z}$ for some $\theta \in \mathbb{R}$, $\alpha \in B_1(0)$.

It is a linear-fractional map.

Note that a linear-frac. map is uniquely determined by 3 distinct pts. (form a generalized circle)

$\Rightarrow g$ is uniquely determined by a, b, c.

$\Rightarrow \tilde{g}$ — — — | | — — — —

$\Rightarrow f$ — — — | | — — — —. $\square$

2.

Solution: $f(z) = z^3 \cdot e^{\frac{1}{z}}$. We want to find its image of $\mathbb{C} \setminus \{0\}$.

Step 1. Analyse f's sing.s.

f only has one sing. $z=0$.

Around $z=0$: $f(z) = z^3 \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^n}{n!} = z^3 \sum_{n=0}^{\infty} \frac{1}{n! \cdot z^n} = z^3 + z^2 + \frac{1}{2} z + \frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{(n+3)! z^n}$

$\nexists \lim_{z \to 0} f(z) \Rightarrow z=0$ is an essential sing.

Step 2: By the Big Picard Thm,

we know that in every neighborhood of 0 (punctured), f's image has at most 2 exceptions. (in $\bar{\mathbb{C}}$)

Step 3: Check possible exceptions

① $z=0$.

suppose $z^3 e^{\frac{1}{z}}=0$.

since $e^{\frac{1}{z}} \neq 0$, $\forall z$, $\Rightarrow$ $z^3=0$ $\Rightarrow$ $z=0$. ↯.

$\Rightarrow$ 0 is an exception.

② $z=\infty$.

However, we find $z^3 e^{\frac{1}{z}} \to \infty$. $z\to\infty$.

Here, we cannot determine whether there is another exception pt.

We find an alternative.

We prove a stronger Big Picard Thm. (We note as "Bigger Picard Thm.")

<u>Thm</u>: Suppose $f \in \mathcal{O}(\mathbb{C}\setminus\{z_0\})$, z_0 - essential sing of f.

Then $f(B^*_\varepsilon(z_0)) = \mathbb{C}\setminus E$, $E \begin{cases} \emptyset \\ \{a\} \end{cases}$, $\forall$ small ε.

<u>Pf by contrad.</u>:

Suppose $E=\{a,b\}$, $a\neq b$. a,b finite.

Define $g(z)=\dfrac{f(z)-a}{f(z)-b}$ $\Rightarrow$ $g(z)\neq 0$, $\forall z\in\mathbb{C}$. $\Leftarrow f(z)\neq a$

and $g(z)\neq\infty$, $\forall z\in\mathbb{C}$. $\Leftarrow f(z)\neq b$

and $g(z)\neq 1$, otherwise $f(z)-a=f(z)-b$, $a=b$ ↯.

$\Rightarrow$ By our Big Picard Thm in class, we have

g has no more than 2 exceptions.

(Note that z_0 is also an essential sing for g)

Contrad! $\square$.

Thus, for the f in the problem,

f has at most 1 exception in $\mathbb{C}$. that is $z=0$, as stated before.

Also, we have f can attain ∞ (iff $z \to \infty$).

$\Rightarrow f(\mathbb{C}\setminus\{0\}) = \mathbb{C}\setminus\{0\}$. (Moreover $f(\bar{\mathbb{C}}\setminus\{0\}) = \mathbb{C}\setminus\{0\}$!)

3. Analysis for $F(z) = \ln(1+\sqrt{1+z^2})$

Sol: $w(z) = \sqrt{1+z^2}$ has branch pts $z = \pm i$.

Its Riemann surface consists of two sheet. (sheet I & sheet II)

connected along a branch cut.

Sheet I: the sheet with $w(0) := +1$.

For large $|z|$, $w(z) \approx +z$

Sheet II: the sheet with $w(0) := -1$.

For large $|z|$, $w(z) \approx -z$.

To show/illustrate the domain of $F(z)$, we define a tuple representing its elt.s. (And the set of tuples will correspond to $F(z)$'s Riemann Surface diagram).

Define a pt as (z, w, k), where z is the location in $\mathbb{C}$, $w = \sqrt{1+z^2}$ implies which sheet (sheet I or II) the pt. is.

and k will denote which floor it is (corresponds to $\ln$'s domain's representation).

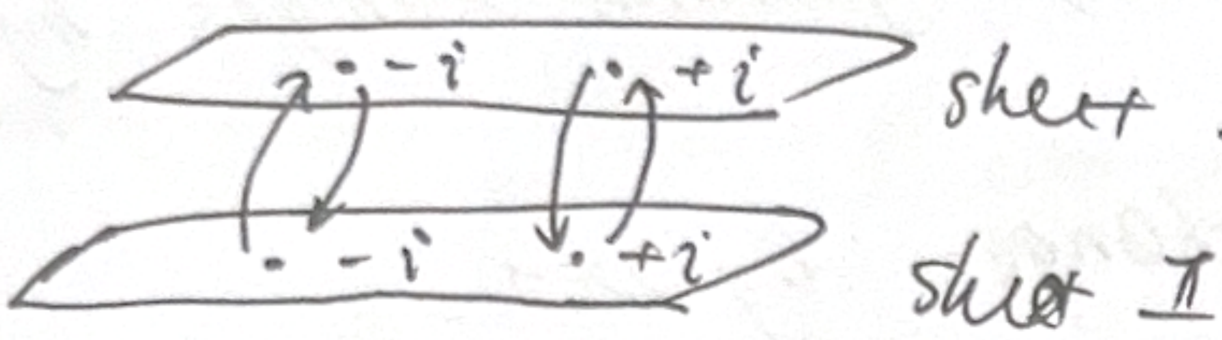
Then the domain is $\{(z, w, k) \in \mathbb{C}^2 \times \mathbb{Z} \mid w^2 = 1+z^2\}$.

Now, branch pts analysis:

① $z = \pm i$. (Algebraic branch pts)

since $F(\pm i) = \ln(1+0) = \ln(1) = 2\pi k i$. $k \in \mathbb{Z}$

all logarithm branches on both sheet connect at these pts.



these are algebraic branch pts of order 2.

② $z = 0$, (Logarithm branch pt.)

On sheet I: $w(0) = +1$.

near $z=0$ $\quad 1 + w(z) \approx 1 + (1 + \frac{1}{2}z^2) = 2 + \frac{1}{2}z^2$.

As z circles the origin, $1+w(z)$ stays (far) away from logarithm's branch pt. $z=0$

$\Rightarrow z=0$ is regular point. on sheet I

On sheet II: $w(0) = -1$.

near $z=0$; $\quad 1 + w(z) \approx 1 + (-(1+\frac{1}{2}z^2)) = -\frac{1}{2}z^2$

As z circles the origin, $1+w(z) = -\frac{1}{2}z^2$ makes two full circles around the origin.

$\Rightarrow z=0$ is a logarithm branch pt, but not on every sheet of its Riemann surface.

More precisely, every time z circle the origin once. logarithm changes by $4\pi i$. $\Rightarrow$ half of the sheets, skip one sheet upon a time.

③ $z=\infty$. (Logarithm branch pt.)

On sheet I: for large z, $w(z) \approx z$. $F(z) \approx \ln(1+z)$.

A large circuit around the origin causes $1+z$ to circle its origin once. $\Rightarrow$ logarithm changes by $2\pi i$.

On sheet II: similarly, logarithm changes by $2\pi i$.

$\Rightarrow$ $z=\infty$ is a logarithm branch pt. on all sheets of the Riemann surface diagram.

Now, for the Puiseux series:

Around $z=i$.

Let $t=\sqrt{z-i}$.

$$\sqrt{1+z^2} = \sqrt{1+(i+t^2)^2} = t\sqrt{2i}\sqrt{1-\frac{i}{2}t^2}.$$

$$= t(1+i)\left(1-\frac{i}{4}t^2-\frac{1}{32}t^4+\dots\right)$$

$$= (1+i)t+\frac{1-i}{4}t^3+\dots$$

$\ln(1+u) \approx u-\frac{1}{2}u^2+\frac{1}{3}u^3$. $\Rightarrow F(z) \approx \left[(1+i)t+\frac{1-i}{4}t^3\right]-\frac{1}{2}((1+i)t)^2+\frac{1}{3}((1+i)t)^3+\dots$

$$= (1+i)(z-i)^{\frac{1}{2}} - i(z-i) - \frac{5}{12}(1-i)(z-i)^{\frac{3}{2}}+\dots$$

Around $z=-i$:

Let $t=\sqrt{z+i}$

Symmetrically, $F(z) = (1-i)(z+i)^{\frac{1}{2}} + i(z+i) - \frac{5}{12}(1+i)(z+i)^{\frac{3}{2}}+\dots$