MA204: Mathematical Statistics

Assignment 2

You have a total of 15 questions in Assignment 2.

Submit your solutions to 7 questions randomly chosen from Q2.1–Q2.10 in Exercise 2 on pages 99–101 of the Textbook "Mathematical Statistics", plus 3 questions chosen from the following Q2.11–Q2.15, plus the last 5 questions (i.e., Q2.16–Q2.20).

2.11 Let $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$ and $X_1 \perp\!\!\!\perp X_2$, where $\lambda_1 > 0$ and $\lambda_2 > 0$.

(a) Find the pmf of $Y = X_2 - X_1$.

(b) Calculate $E(Y)$ and $\text{Var}(Y)$.

2.12 Let $X_1, X_2 \overset{\text{iid}}{\sim} \text{Exponential}(\lambda)$ with pdf $\lambda \exp(-\lambda x)$, $x \geqslant 0$.

(a) Find the joint pdf of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

(b) Find the marginal pdf of Y_1.

(c) Find the marginal pdf of Y_2.

2.13 Let $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics of a random sample $X_1, \ldots, X_n \overset{\text{iid}}{\sim} F(x)$, where $F(x)$ is the population cdf and $f(x)$ is the population pdf. The *sample range* and *sample mid-range* are respectively defined to be

$$R \hat{=} X_{(n)} - X_{(1)} \quad \text{and} \quad T \hat{=} \frac{X_{(1)} + X_{(n)}}{2}.$$

(a) Find the joint pdf of R and T.

(b) Find the marginal pdf of R.

(c) Find the marginal pdf of T.

2.14 (a) Find the cdf of the Bernoulli(p) distribution with $p \in (0, 1)$. (b) Let the mgf of the random variable X is $M_X(t) = \exp(c\,t)$, where c is a constant, what is the distribution of X?

2.15 Let Y be a positive continuous random variable cdf $F_Y(y)$ or pdf $f_Y(y)$. Define a discrete random variable $X = \lceil Y \rceil$, which denotes the largest integer less than or equal to Y. Find the pmf of X for the following cdf or pdf of Y.

(a) Half logistic distribution:

$$F_Y(y) = 1 - \frac{2}{1 + \exp(y/\sigma)}, \quad y > 0,\ \sigma > 0.$$

(b) Gamma distribution:

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \mathrm{e}^{-\beta y}, \quad y > 0, \quad \alpha > 0,\ \beta > 0.$$

(c) Lindley distribution:

$$F_Y(y) = 1 - \frac{(1 + \theta + \theta y)\exp(-\theta y)}{1 + \theta}, \quad y > 0,\ \theta > 0.$$

2.16 Let $X \sim N(0, 1)$. Use the *moment generating function* (mgf) technique to find the distribution of $Y = X^2$.

2.17 Let $X_1, X_2 \overset{\text{iid}}{\sim} U(0, 1)$. Use the mixture technique,

(a) Find the pdf of $Y = X_1 X_2$.

(b) Find the pdf of $Z = X_1/X_2$.

2.18 In Example 2.10 on page 69 of the Textbook, we rewrite (2.5) in the following vector form

$$\mathbf{x} = \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix} = \boldsymbol{\mu} + \frac{\mathbf{y}}{\sqrt{Z/\nu}}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)^\top$, $\mathbf{y} = (Y_1, \ldots, Y_d)^\top \sim N_d(\mathbf{0}, \boldsymbol{\Sigma})$, $Z \sim \chi^2(\nu)$ and $Z \perp\!\!\!\perp \mathbf{y}$. Use the mixture technique, find the joint pdf of $\mathbf{x}$.

2.19 Let $F(\cdot)$ and $f(\cdot)$ be a cdf and pdf of a continuous random variable. Prove that

(a) For any $\alpha\,(> 0)$, $\alpha f(x)[F(x)]^{\alpha-1}$ is a pdf of some random variable, X say.

(b) Find the *stochastic representation* (SR) of X via the uniform random variable $U \sim U(0, 1)$.

2.20 Let $X \sim t(\mu, \sigma^2, \nu)$, then X can be stochastically represented by

$$X \stackrel{\mathrm{d}}{=} \mu + \frac{Z}{\sqrt{\xi/\nu}},$$

where $Z \sim N(0, \sigma^2)$, $\xi \sim \chi^2(\nu)$, and $Z \perp\!\!\!\perp \xi$. Now assume that

$$Y \stackrel{\mathrm{d}}{=} \mu + \frac{Z}{\sqrt{\tau}},$$

where $Z \sim N(0, \sigma^2)$, $\tau \sim \mathrm{Gamma}(\alpha, \beta)$ and $Z \perp\!\!\!\perp \tau$. Use the SR technique to prove that $Y \sim t(\mu, \sigma_*^2, \nu_*)$ and find the expressions of σ_*^2 and ν_*.

[Hint: (1) $\mathrm{Gamma}(\nu/2, 1/2) = \chi^2(\nu)$. (2) If the r.v. $\xi \sim \mathrm{Gamma}(\alpha, \beta)$ and $c\,(> 0)$ is a constant, then $\eta \triangleq c\xi \sim \mathrm{Gamma}(\alpha, \beta/c)$.]