Lecture 35

Review for Final Examination

Part I: Chapters 1 – 4 & Appendix C
Part II: Chapter 5

(1) 2025 final examination

- **Time and Date**
  19:00 – 22:00, 3 hours
  18 June 2025
- **Venue**
  The Third Teaching Building,
  Rooms 206 & 207 for Class I & Class II
- **Range**
  Chapters 1–5, Appendix C
- **Assessment**
  Final Examination (50%)

(2) Please bring your calculators

- **Candidates taking examinations that permit the use of calculators may use any calculator which fulfils the following criteria:**
  - **(a) it should be self-contained, silent, battery-operated and pocket-sized.**
  - **(b) it should have numeral-display facilities only.**
  - **(c) it should be used only for the purposes of calculation.**

(2) Please bring your calculators (cont'd)

- **It is the candidate's responsibility to ensure that the calculator operates satisfactorily.**
- **The candidate must record the name and type of the calculator on the front page of the examination scripts.**

(3) Please bring two pens/pencils

- **When one pen does not work, you can use another one.**
- **You can prepare anything on two sides of an A4 paper and bring it with you to the Final Examination venue.**
- **You are not allowed to bring any other material (including iPhone/iPad) to the Final Examination venue.**

(4) The range of the examination

- Chapter 1 excluding §1.11–§1.12
- Chapter 2
  excluding Theorem 2.1 on p.74, §2.5.2, §2.5.3 and §2.5.4
- Chapters 3
- Appendix C
  excluding C.3
- Chapter 4
- Chapter 5 excluding §5.2.2

(5) The distribution and marks of the questions in the Exam of 2025

- There are six questions in the Final Examination with a total of 100+5 marks.
- Q1 has 20 sub-questions with 2 marks per sub-question, ranging from Chapter 1 to Chapter 5; Directly giving your answers.
- Q2 in Chapter 1 has a total of 10 Marks.

(5) The distribution and marks of the questions in the Exam of 2025 (cont's)

- **Q3** (including MLE, unbiased estimator, Fisher information, sufficient statistics, UMVUE) in Ch. **3** with **18** Marks.
- There is **1** sub-question in Ch. **4** with a total of **2** Marks.
- There are **2** questions (**Q4–Q5**) in Chapter **5** with a total of **30** Marks, where MPT (10), LRT (10), goodness-of-fit test (10).
- **Q6** is a bonus question with **5** Marks.

8) Key points for Chapter 5

31. Type I error function

$$\begin{aligned} \alpha(\theta) &= \Pr(\textbf{Type I error}) \\ &= \Pr(\textbf{rejecting } H_0 | H_0 \textbf{ is true}) \\ &= \Pr(\mathbf{x} \in \mathbb{C} | \theta \in \Theta_0), \end{aligned} \tag{5.3}$$

where $\mathbb{C}$ is the critical region.

32. Type II error function

$$\begin{aligned}\beta(\theta) &= \Pr(\textbf{Type II error}) \\ &= \Pr(\textbf{accepting } H_0 | H_0 \textbf{ is false}) \\ &= \Pr(\mathbf{x} \in \mathbb{C}' | \theta \in \Theta_1), \end{aligned} \tag{5.4}$$

where $\mathbb{C}'$ is the acceptance region.

33. Power function and the relationship with $\alpha(\theta)$ and $\beta(\theta)$

$$p(\theta) = \Pr(\textbf{rejecting } H_0|\theta) = \Pr(\mathbf{x} \in \mathbb{C}|\theta) \tag{5.6}$$

— **If** $\theta \in \Theta_0$**,**

$$p(\theta) = \Pr(\mathbf{x} \in \mathbb{C}|\theta \in \Theta_0) = \alpha(\theta). \tag{5.7}$$

— **If** $\theta \in \Theta_1$**,**

$$p(\theta) = \Pr(\mathbf{x} \in \mathbb{C}|\theta \in \Theta_1) = 1 - \beta(\theta).$$

34. Neyman–Pearson Lemma

Let $X_1, \ldots, X_n$ be a random sample from a population with the pdf (or pmf) $f(x;\theta)$. Let the likelihood function be $L(\theta)$. Then a test φ with critical region

$$\mathbb{C} = \left\{ \boldsymbol{x} = (x_1, \ldots, x_n)^{\top} \colon \frac{L(\theta_0)}{L(\theta_1)} \leqslant k \right\} \tag{5.12}$$

and size α is the most powerful test of size α for testing $H_0\colon \theta = \theta_0$ against $H_1\colon \theta = \theta_1$, where k is a value determined by the size α.

35. Uniformly most powerful test (not test)

- Definition 5.3 on page 199
- Examples 5.7 and 5.8
- Questions 5.1–5.12 in Assignment 5.

36. How to find a UMPT (not test)

(a) For the given two composite hypotheses H_0: $\theta \in \Theta_0$ **and** H_1: $\theta \in \Theta_1$, **first consider two simple hypotheses** H_{0s}: $\theta = \theta_0 \in \Theta_0$ **and** H_{1s}: $\theta = \theta_1 \in \Theta_1$.

(b) **By using the Neyman–Pearson lemma, to find a most powerful test** φ **of size** α **with critical region** $\mathbb{C}$.

(c) If $\mathbb{C}$ is free from θ_1, then φ is the UMPT of size α for testing H_{0s}: $\theta = \theta_0 \in \Theta_0$ against H_1: $\theta \in \Theta_1$.

(d) If $\sup_{\theta \in \Theta_0} p_\varphi(\theta) = \alpha = p_\varphi(\theta_0)$, then φ is the UMPT of size α for testing H_0: $\theta \in \Theta_0$ against H_1: $\theta \in \Theta_1$.

37. Likelihood ratio (LR) test

- **Step 1: Find $\hat{\theta}^{\mathrm{R}}$, the restricted MLE of θ in Θ_0, and $\hat{\theta}$, the (un)restricted MLE of θ in Θ.**
- **Step 2: Calculate the LR statistic**

$$\lambda(\mathbf{x}) = L(\hat{\theta}^{\mathrm{R}})/L(\hat{\theta}). \qquad (5.23)$$

- **Step 3: Determine λ_α or c in**

$$\mathbb{C} = \{\boldsymbol{x}\colon\ \lambda(\boldsymbol{x}) \leqslant \lambda_\alpha\} = \{\boldsymbol{x}\colon\ \log\lambda(\boldsymbol{x}) \leqslant c\}$$

based on

$$\alpha = \Pr\{\log\lambda(\mathbf{x}) \leqslant c | H_0 \textbf{ is true}\}$$

38. How to determine the c in LR Test

- **Step I.** Express $\lambda(\boldsymbol{x})$ or $\log \lambda(\boldsymbol{x})$ as a function of a statistic Q, e.g., $\lambda(\boldsymbol{x}) = h(Q)$, such that under H_0, Q follows a given distribution (e.g., Normal, Gamma, Chi-squares, t, F).
- **Step II.** Prove that $h(Q)$ is concave with a maximum or convex with a minimum.

- **<u>Step III</u>. If $h(Q)$ is concave, then**

$$h(Q) \leqslant c$$

is equivalent to

$$Q \leqslant c_1 \quad \textbf{or} \quad Q \geqslant c_2.$$

- **<u>Step IV</u>. If $h(Q)$ is convex, then $h(Q) \geqslant c$ is equivalent to**

$$Q \leqslant c_1 \quad \textbf{or} \quad Q \geqslant c_2.$$

- **Step V.** Use the **equal-tail approach,** let

$$\alpha/2 = \Pr(Q \leqslant c_1 | H_0\}$$

and

$$\alpha/2 = \Pr(Q \geqslant c_2 | H_0\}$$

to determine the c_1 and c_2.

39. Tests on normal means

- **One-sample normal test when variance is known (§5.4.1)**
- **One-sample t test (§5.4.2)**
- **Two-sample t test (§5.4.3)**

40. The critical region approach and the p-value approach

- **If you are asked to use the critical region approach, you must obey.**
- **If you are asked to use the p-value approach, you must obey.**
- **You could use either approach if the approach is not be specified in questions.**

41. Goodness of fit test

Please review §5.5.1 – §5.5.3

42. All questions in Assignments 1–5.

43. All questions in Tutorials.

End of the Review

GOD Bless You!