Automorphism

Let G be a group. $Aut(G)$ is the automorphism group of G

<u>Def 1</u> $\psi: G \longrightarrow Aut(G)$

$$g \longmapsto \sigma_g : G \longrightarrow G$$
$$h \longmapsto g^{-1}hg$$

σ_g is called the inner automorphism induced by g

$Im(\psi)$ is a subgroup (normal) of $Aut(G)$

called the Inner automorphism group of G. denoted by $Inn(G)$

$\ker\psi = Z(G)$, which means that, $Inn(G) \cong G/Z(G)$

<u>Def 2</u>. $\forall \tau \in Aut(G) \setminus Inn(G)$, τ is called outer automorphism of G

$Aut(G)/Inn(G)$ is called outer automorphism group of G

denoted by $Out(G)$

Schreier Conjecture: Let G be a finite simple group. then $Out(G)$ is solvable.

<u>Thm 3</u> (N-C Lemma) Let $H \le G$ then $N_G(H)/C_G(H) \lesssim Aut(H)$

proof: Let π : $N_G(H) \longrightarrow Aut(H)$

$$g \longmapsto \varphi_g : H \longrightarrow H$$
$$h \longmapsto g^{-1}hg$$

$\ker\pi = C_G(H) \Rightarrow N_G(H)/C_G(H) \lesssim Aut(H)$

<u>Def 4</u> Let $H \le G$, if $\forall \alpha \in Aut(G)$ $H^{\alpha} \subseteq H$

Then H is called the characteristic subgroup of G. denoted by

H char G

Obviously $H \trianglelefteq G$ since H is stable under $\mathrm{Inn}(G)$

If $\forall \phi \in \mathrm{End}(G)$ $\phi(H) \subseteq H$, then H is called the full invariant subgrp of G

E.g. $\{e\}$, G are full invariant.

$Z(G)$ is char, but not full inva.

prop5. If K char H and H char G then K char G

(Remark. in general, $K \trianglelefteq H$ and $H \trianglelefteq G$ $\not\Rightarrow$ $K \trianglelefteq G$.

E.g. $C_2 = \{1, (12)(34)\} \trianglelefteq V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$.

$V_4 \trianglelefteq S_4$ but $C_2 \ntrianglelefteq S_4$

$\forall \sigma \in \mathrm{Aut}G$ $\sigma(H) = H \Rightarrow \sigma|_H : H \to H$ is a aut of H

$\Rightarrow \sigma(K) = K \Rightarrow K$ char G.

prop6 K char H. $H \trianglelefteq G \Rightarrow K \trianglelefteq G$ Consider $\mathrm{Inn}(G)|_H$

Def7. If G has no non-trivial characteristic subgrp

G is called characteristic simple group.

Thm8 Finite characteristic simple grp is a product of isomorphic simple grps. (In par. $M \trianglelefteq_{min} G$. then M ✓)

Let N be a minimal normal subgrp of G.

i.e. for a normal subgrp partial set $\{1, N, \cdots N_t\}$

there is no N_i s.t $1 \subsetneq N_i \subsetneq N$.

Then $\forall \alpha \in \mathrm{Aut}(G)$. N^α is also a minimal normal subgrp of G

Let $M \leq G$ be the maximal element of product of N^α

i.e. $M = N_1 \times \cdots \times N_s$ where $N_i = N^{\alpha_i}$, $\alpha_i \in Aut(G)$, $i = 1 \cdots s$.

Then $M \trianglelefteq G$ (e.g. $\forall (g_1, g_2) \in N_1 \times N_2$ then $(g_1, g_2)^g = (g_1^g, g_2^g) \in N_1 \times N_2$)

Claim. $\forall \alpha \in Aut G$, $N^\alpha \leq M$.

If $\exists \beta \in Aut G$, $N^\beta \not\leq M$ since $N^\beta \trianglelefteq G \Rightarrow N^\beta \cap M \trianglelefteq G$

Since N^β is minimal $\Rightarrow N^\beta \cap M = 1$.

$\Rightarrow \langle M, N^\beta \rangle = M \times N^\beta = N_1 \times \cdots \times N_s \times N^\beta$ ↯ M is maximal.

$\Rightarrow M = \langle N^\alpha \mid \alpha \in Aut G \rangle \Rightarrow M$ char G

Since G is characteristic simple.

$\Rightarrow G = M = N_1 \times \cdots \times N_s$

Claim N_i are simple

If not. $\exists K_i \ntrianglelefteq N_i$

Then $K_i \trianglelefteq G = N_1 \times \cdots \times N_s$ and $K_i \subsetneq N_i$

↯ to N_i are minimal normal.

Thm. The direct product of isomorphic simple groups is characteristic simple

Char simple $\Longleftrightarrow$ direct product of iso simple group.

Step 1. elementary abelian p groups are char simple groups.

Let $G = (\mathbb{Z}_p)^n$, the $G \cong (F^n, +)$ where $F = \mathbb{F}_p$

Then $Aut(G) \cong GL_n(F)$

and char G is $GL_n(F)$ invariant subspace of $(F^n, +)$

$\Rightarrow$ char$G$ is 0 space $\Rightarrow$ G is char simple.

Step 2. Let $G = N_1 \times \cdots \times N_s$, N_i are iso. non-abelian simple group.

Claim. Any non-trivial normal subgroup $K \triangleleft G$ has the following form:

$$K = N_{i_1} \times \cdots \times N_{i_t}, \quad 1 \leq i_1 < \cdots < i_t \leq s$$

Let $K \triangleleft G$, K non-trivial.

$\forall g \in K$, since $G = N_1 \times \cdots \times N_s$

we have $g = g_1 g_2 \cdots g_s$ where $g_i \in N_i$ (regard as inner dir. prod.)

since K non-trivial, for $g \neq 1$, $\exists j$ s.t. $g_j \neq 1$.

Then we consider the normal closure of g in G.

$$g^G = \langle \{ h^{-1} g h \mid h \in G \} \rangle$$

Since $\forall x \in N_j$, $g^{-1} g^x \in g^G$

i.e. $(g_1 \cdots g_s)^{-1} (g_1 \cdots g_s)^x = g_j^{-1} g_j^x = g_j^{-1} x^{-1} g_j x = [g_j, x] \in g^G$

Since N_j is simple $\Rightarrow$ $Z(N_j) = 1$

$\Rightarrow \exists x \in N_j$ s.t. $g^{-1} g^x \neq 1$.

and $g^{-1} g^x \in N_j$ Let $h_j = g^{-1} g^x = [g_j, x] \neq 1$ be such an element. then since N_j simple. And $h_j^{N_j} \leq g^G$, obviously. the normal closure of h_j in N_j is N_j itself

Thus $K \geq g^G \geq N_j$ (normal closure is the minimal normal subgroup of G containing g. and we know $g \in K$, $K \triangleleft G$)

In a word, if $\exists\, g \in K$, $g \neq 1$ $\Rightarrow$ $\exists$ $g_j \neq 1$ where $g = g_1 \cdots g_s$

$\Rightarrow$ $K \geq N_j$ since $K \trianglelefteq G$, $G = N_1 \times \cdots \times N_s$

$\Rightarrow$ K must has the form $N_{i_1} \times \cdots \times N_{i_t}$

Step 3. Suppose $\exists$ $1 \neq K$ char G, then $K \trianglelefteq G$

WLOG, Let $K = N_1 \times \cdots \times N_{t-1}$ $1 < t \leq s$

Let α : $N_1 \xrightarrow{\sim} N_t$
$g_1 \longmapsto g_t$

Let β : $G \longrightarrow G$

$g = g_1 \cdots g_s \longmapsto g_1^{\alpha} g_2 \cdots g_{t-1} g_t^{\alpha^{-1}} g_{t+1} \cdots g_s$

Then $\beta \in Aut(G)$

But $K^{\beta} = N_t N_2 \cdots N_{t-1} \neq K$, thus K is not char of G

Contradiction!

Q1. prove $Z(A \times B) = Z(A) \times Z(B)$ just need to check as a set.

Q2 If G has finite index subgrp. then G also has finite index normal subgrp.

Let G be a grp. $H \leq G$, $[G:H] = n$

Let $\Omega = \{Hg \mid g \in G\}$.

Consider ψ : $G \longrightarrow Sym(\Omega)$
$x \longmapsto \rho_x : \Omega \longrightarrow \Omega$
$Hg \longmapsto Hgx$

Then $\ker\varphi = \bigcap_{g\in G} g^{-1}Hg \trianglelefteq G$.

This group is denoted by $Core_G(H)$

Q3: $\forall g\in G$, $g^{-1}g^{\alpha}\in Z(G)$ then α is called central automorphism

$\alpha\in \operatorname{Aut} G$

of G. prove if $\operatorname{Aut}(G)$ abelian, then $\forall \beta\in \operatorname{Aut} G$. β is a central automorphism.

①. $I \trianglelefteq J \trianglelefteq R$ means $I \trianglelefteq J$ and $J \trianglelefteq R$.

So the home work should be

I, J ideal of R and $I \trianglelefteq J$, but not

② the hint of one homework (also in quiz)

should be. $G/_{H\cap K} \lesssim G/_H \oplus G/_K$

$G \to G/_H \oplus G/_K$

$g \mapsto (g+H, g+K)$ it may not be epi:

So is equal iff. $G = HK$.

③ product of 2 ideal is $\{\sum_{i=1}^{n} a_i b_i \mid a_i\in I, b_i\in J\}$.

④. maximal subgroup may not unique (actually in General)

$\mathbb{Z}_6 = G = \mathbb{Z}_2\times\mathbb{Z}_3$, $\mathbb{Z}_2, \mathbb{Z}_3$ are all maximal.

⑤ Integral domain : 1. commutative.

2. identity

3. no non-zero divisor

i.e if $a \neq 0$, $\forall b \in R$ $ab \neq 0$.

⑥ subring substructure of R structure

$(+ , \cdot)$

abelian semigroup.

sub sub.

S

S may not has $1 \in R$

But this S may also has a $1'$

there means a subsemigroup of a monoid can be a monoid but not submonoid.