[Power] can be thought of as the probability that the test will "detect" that the alternative θ holds. –Bickel and Docksum

Problems 71-80 on Composite Alternative Hypotheses, Power function, Likelihood Ratio Test, and the Chisq Test for Goodness of Fit. (STAT2802 Statistical Models Tutorial notes for the week of 26-NOV-2012)

Power is a probability. It is the probability of the test's critical region $\mathbb{C}$, which is specified under $\mathbb{P}(T|H_0)$ with the only formal constraint that $\mathbb{P}(\mathbb{C}|H_0) =$ the prescribed *size of the test*. Power is a bunch of probabilities indexed by the parameter—Power $\wp$ is a function $\wp(\theta)$ on the parameter space Θ to the unit interval $[0,1]$. A good test desires more power after its size is given. This is exactly the idea of designing a test that minimizes its Type 2 error probability, controlling Type 1 error.

At its core, a statistical test is the indication of the critical region $\mathbb{C}$, or, equivalently, a partition of the sample space of the test statistic, or, equivalently, a partition of the sample space for the data vector. A statistical test is characterized by its power function: $\wp: \Theta \ni \theta \mapsto \mathbb{P}(\mathbb{C}|\theta) \in [0,1]$. If we plot the graph of the power function, with horizontal axis listing all possible points of Θ and the vertical axis listing all points on [0,1], we often see a continuous curve. This curve characterizes the underlying test. When we plot different tests' curves, one curve may be higher everywhere than every other test and the test being characterized by this highest curve is the Uniformly Most Powerful test.

Problem 71-80.

71. Decide in each case whether the hypothesis is simple [S] or composite [C]:

[S | C] The hypothesis that a random variable has a gamma distribution with α=3 and β=2.

[S | C] The hypothesis that a random variable has a gamma distribution with α=3 and β≠2.

[S | C] The hypothesis that a random variable has an exponential density.

[S | C] The hypothesis that a random variable has a beta distribution with the mean μ=0.50.

[S | C] The hypothesis that a random variable has a Poisson distribution with λ=1.25.

[S | C] The hypothesis that a random variable has a normal distribution with λ>1.25.

[S | C] The hypothesis that a random variable has a negative binomial distribution with k=3 and θ<0.60.

72. Fill in the blanks.

(a) A single observation of a random variable having a hypergeometric distribution with $N = 7$ and $n = 2$ is used to test the null hypothesis $k = 2$ against the alternative hypothesis $k = 4$. The null hypothesis is rejected if and only if the value of the random variable is 2. The sample space of the test statistic is ___________. On the sample space of the test statistic, the critical region is $\mathbb{C} =$____________. The power function is $\wp(k, \mathbb{C}) =$________________________, the Type 1 error probability is __________, the Type 2 error probability is _________.

(b) A single observation of a random variable having a geometric distribution is used to test the null hypothesis $\theta = \theta_0$ against the alternative hypothesis $\theta = \theta_1 > \theta_0$. The null hypothesis is rejected if and only if the observed value of the random variable is greater than or equal to the positive integer k. The sample space of the test statistic is _____________. On the sample space of the test statistic, the critical region is $\mathbb{C} =$_____________. The power function is $\wp(\theta, \mathbb{C}) =$__________________________, the Type 1 error probability is __________, the Type 2 error probability is _________.

(c) A single observation of a random variable having an exponential distribution is used to test the null hypothesis that the mean of the distribution is $\theta = 2$ against the alternative that it is $\theta = 5$. The null hypothesis is accepted if and only if the observed value of the random variable is less than 3. The sample space of the test statistic is ___________. On the sample space of the test statistic, the critical region is $\mathbb{C} =$____________. The power function is $\wp(\theta, \mathbb{C}) =$________________________, the Type 1 error probability is __________, the Type 2 error probability is _________.

(d) A single observation of a random variable having a uniform density with $\alpha = 0$ is used to test the null hypothesis $\beta = \beta_0$ against the alternative hypothesis $\beta = \beta_0 + 2$. The null hypothesis is rejected if and only if the random variable takes on a value greater than $\beta_0 + 1$. The sample space of the test statistic is ___________. On the sample space of the test statistic, the critical region is $\mathbb{C} =$____________. The power function is $\wp(\beta, \mathbb{C}) =$________________________, the Type 1 error probability is __________, the Type 2 error probability is _________.

73. Filling the blanks.

(a) Let X_1 and X_2 constitute a random sample from a normal population with $\sigma^2 = 1$. If the null hypothesis $\mu = \mu_0$ is to be rejected in favor of the alternative hypothesis $\mu = \mu_1 > \mu_0$ when $\bar{x} > \mu_0 + 1$, then the size of the test is $\alpha =$___________.

(b) Let X_1 and X_2 constitute a random sample of size 2 from the population given by $f(x;\theta) = \theta x^{\theta-1}\mathbb{I}(0 < x < 1)$. If the critical region $x_1 x_2 \geq \frac{3}{4}$ is used to test the null hypothesis $\theta = 1$ against the alternative hypothesis $\theta = 2$, then the power of this test at $\theta = 2$ is $\wp(\theta = 2) =$____________.

74. With i.i.d. data

$$\mathbf{X} = \{X_i\}_{i=1}^n$$

we would like to test hypotheses about a parameter θ of the statistical distribution of X_1:

$$H_0: \theta = \theta_0 \text{ vs. } H_1: \theta = \theta_1.$$

Natrually, we may directly compare their likelihoods:

$$L_0 \equiv L(\theta_0; \{X_i\}_{i=1}^n) \quad \text{vs.} \quad L_1 \equiv L(\theta_1; \{X_i\}_{i=1}^n)$$

via the likelihood ratio

$$\frac{L_0}{L_1}$$

whose only variable argument is the data vector

$$\mathbf{X} = \{X_i\}_{i=1}^n$$

Because both θ_0 and θ_1 are fixed constants on the numerator and the denominator. This means the likelihood ratio is a statistic. We use it as the test statistic to construct the following critical region

$$\mathbb{C} := \left\{\mathbf{X}: \frac{L_0}{L_1} \_\_\_\_\_ k\right\}.$$

where k is a constant to be specified according to the size α of the test, that is,

$$\int_{\_\_\_\_} L_0(\mathbf{x}) d\mathbf{x} = \_\_\_\_\_.$$

Suppose a different test for the same hypotheses uses the critical region $\mathbb{D}$ with the same size α, that is,

$$\int_{\_\_\_\_} \_\_\_\_\_\_\_\_ d\mathbf{x} = \_\_\_\_\_.$$

The power of the $\mathbb{C}$-test at θ_1 is

$$\wp(\theta_1, \mathbb{C}) = \int_{\_\_\_\_} \_\_\_\_\_\_\_\_ d\mathbf{x}.$$

The power of the $\mathbb{D}$-test at θ_1 is

$$\wp(\theta_1, \mathbb{D}) = \int_{\_\_\_\_} \_\_\_\_\_\_\_\_ d\mathbf{x}.$$

Show that $\wp(\theta_1, \mathbb{C}) - \wp(\theta_1, \mathbb{D}) \geq 0$. Therefore, any ____________________ test based on $\mathbb{C} =$_________________ is the most powerful test among all tests with size ____.

75. Explain why a statistical hypothesis test is identified by its critical region(on the data's joint sample space), not by its test statistic. Explain why the power function characterizes a statistical hypothesis test.

76. A size-n random sample from a normal population with $\sigma^2 = 1$ is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative hypothesis $\mu = \mu_1$, where $\mu_1 < \mu_0$. Show that the most powerful size-α critical region does not depend on the value of μ_1.

77. A size-n random sample from an exponential population (use the density form $\lambda e^{-\lambda x}$ where the parameter λ is reciprocal of the mean) is used to test the null hypothesis $\lambda = \lambda_0$ against the alternative hypothesis $\lambda = \lambda_1 < \lambda_0$. Construct the most powerful size-α critical region for this pair of hypotheses and argue that it is the uniformly most powerful critical region when the alternative hypothesis has been changed to the composite hypothesis $\lambda < \lambda_0$.

78. A size-n random sample from a normal population with $\mu = 0$ is used to test the null hypothesis $\sigma = \sigma_0$ against the alternative hypothesis $\sigma > \sigma_0$. Find the uniformly most powerful size-α critical region.

79. When we test a simple null hypothesis against a composite alternative, a critical region is said to be unbiased if the corresponding power function takes on its minimum value at the value of the parameter assumed under the null hypothesis. In other words, a critical region is unbiased if the probability of rejecting the null hypothesis is least when the null hypothesis is true. Given a single observation of the random variable X having the density

$$f(x) = 1 + \theta^2 \left(\frac{1}{2} - x\right) \mathbb{I}\{x \in (0,1)\}$$

where $-1 \leq \theta \leq 1$, show that the critical region $x \leq \alpha$ provides an unbiased critical region of size α for testing the null hypothesis $\theta = 0$ against the alternative hypothesis $\theta \neq 0$.

80. A size-n random sample from a normal population with unknown mean and variance is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative $\mu \neq \mu_0$. Show that the (generalized) likelihood ratio statistic can be written in the form

$$\Lambda = \left(1 + \frac{T^2}{n-1}\right)^{-\frac{n}{2}}$$

where $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$. Then show that $-2\ln\Lambda \xrightarrow{n\to\infty} T^2$.