Homework-11

December 5, 2024

1. (1). Let p be a prime number, show the splitting field of $x^{p^n}-1$ over $GF(p)$;
(2). Show the splitting field of x^6+2x^3+2 over $GF(3)$.
(3). Show the splitting field of x^4-2 over $\mathbb{Q}(i)$ and its Galois group.

2. Let F be a field, K be the splitting field of $f(x) \in F[x]$ over F, E be an intermediate field of K/F. Prove that K is also the splitting field of $f(x)$ over E.

3. Let K/F be a finite normal extension, E is a intermediate field. Prove that E/F is a normal extension iff E is stable of K/F. i.e. For any F-automorphism σ of K, $\sigma(E)=E$.

4. Let E, K be two intermediate fields of finite extension L/F. Prove: if E/F and K/F are normal extension, then $E \cap K/F$ and EK/F are normal extension.

5. Let E, K be two intermediate fields of finite extension L/F. Prove: if K/F is a normal extension, then EK/E is a normal extension.

6. If K/E and E/F are normal extension, is K/F a normal extension? Prove it or give a counterexample.

7. Let $p_1, \ldots, p_m$ be distinct prime numbers. $K=\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_m})$, show $Gal(K/\mathbb{Q})$.

8. Give the Galois groups of the splitting fields of following polynomials over $\mathbb{Q}$, and show all those subgroups and fixed fields.
(1). x^3-3x-1;
(2). x^3-x-1.

9. Give all subgroups of $Gal(GF(p^n)/GF(p))$ and their fixed field.

10. Let p be an odd prime number, K be the splitting field of $x^{p^n}-1$ over $\mathbb{Q}$.
(1). Prove $[K:\mathbb{Q}]=p^{n-1}(p-1)$;
(2). Prove $Gal(K/\mathbb{Q})$ is a cyclic group.