MA204: Mathematical Statistics

Assignment 3

You have a total of 15 questions in Assignment 3.

Submit your solutions to 10 questions randomly chosen from Q3.1–Q3.19 in Exercise 3 on pages 156–161 of the Textbook "Mathematical Statistics", plus 5 questions chosen from the following six questions.

3.20 Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} f(x;\sigma)$, where

$$f(x;\sigma) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0,\ \sigma > 0.$$

(a) Show that $X_1^2 \sim \text{Exponential}(\beta)$ with $\beta = 1/(2\sigma^2)$.

(b) Find the C-R lower bound of σ.

(c) Find the C-R lower bound of σ^2.

3.21 Let $X_1, X_2, X_3 \stackrel{\text{iid}}{\sim} \text{Bernoulli}(\theta)$, where $\theta \in (0,1)$.

(a) Find the pmf of $T = T(X_1, X_2, X_3) = X_1X_2 + X_3$.

(b) Show that T is not a sufficient statistic for θ.

3.22 Let $X_1, \ldots, X_n, X \stackrel{\text{iid}}{\sim} f(x;\theta)$, where X is the population random variable,

$$f(x;\theta) = \frac{\log\theta}{\theta - 1}\theta^x, \quad 0 \leqslant x \leqslant 1,\ \theta > 1.$$

(a) Prove that $f(x;\theta)$ is a density function.

(b) Show that $T \triangleq \sum_{i=1}^n X_i$ is sufficient for θ.

(c) Show that the *moment generating function* (mgf) of the population random variable X is

$$M_X(t) = \frac{(\theta e^t - 1)\log\theta}{(\theta - 1)(\log\theta + t)}.$$

(d) Based on the mgf of X, prove that

$$\begin{aligned} E(X) &= \frac{\theta}{\theta - 1} - \frac{1}{\log\theta} \triangleq \tau(\theta), \\ E(X^2) &= \frac{\theta(\log\theta)^2 - 2\theta(\log\theta - 1) - 2}{(\theta - 1)(\log\theta)^2}, \\ \mathrm{Var}(X) &= \frac{(\theta - 1)^2 - \theta(\log\theta)^2}{(\theta - 1)^2(\log\theta)^2}, \end{aligned}$$

and prove that $\bar{X} = T/n$ is an unbiased estimator of $\tau(\theta)$.

(e) Show that the Fisher information $I_n(\theta)$ is given by

$$I_n(\theta) = nI(\theta) = n\frac{(\theta - 1)^2 - \theta(\log\theta)^2}{\theta^2(\theta - 1)^2(\log\theta)^2}.$$

(f) Show that $\bar{X}$ is the efficient estimator of $\tau(\theta)$.

3.23 A r.v. X is said to follow a Conway–Maxwell–Poisson (CoM-Poisson) distribution with parameters $\lambda > 0$ and $\nu \geqslant 0$, denoted by $X \sim \mathrm{CoMP}(\lambda, \nu)$, if its pmf is

$$\mathrm{CoMP}(x|\lambda, \nu) = \frac{1}{Z(\lambda, \nu)} \cdot \frac{\lambda^x}{(x!)^\nu}, \quad x = 0, 1, \ldots, \infty,$$

where

$$Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu}$$

is the normalized constant. Let $\{X_i\}_{i=1}^n \overset{\text{iid}}{\sim} \mathrm{CoMP}(\lambda, \nu)$ and $Y_{\text{obs}} = \{x_i\}_{i=1}^n$ denote the observed counts. Show that $\{T_1, T_2\}$ are joint sufficient statistics for $\{\lambda, \nu\}$, where

$$T_1 \triangleq \sum_{i=1}^n X_i \quad \text{and} \quad T_2 \triangleq \sum_{i=1}^n \log(X_i!).$$

3.24 Let $X_1, \ldots, X_n$ be independent random variables and X_i have the following pdf

$$f_{X_i}(x;\theta) = \begin{cases} \dfrac{1}{2i\theta}, & -i(\theta-1) < x < i(\theta+1), \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, n$, where $\theta > 0$. Find a sufficient statistic of θ.

3.25 Let $X_1, \ldots, X_n$ be a random sample from an unknown population with mean μ and variance $\sigma^2 < +\infty$.

(a) If $\sum_{i=1}^n a_i = 1$, show that the estimator $\varphi(\mathbf{x}) \triangleq \sum_{i=1}^n a_i X_i$ is an unbiased estimator of μ, where $\mathbf{x} = (X_1, \ldots, X_n)^\top$, and $\{a_1, \ldots, a_n\}$ are known constants.

(b) Among all unbiased estimators of this form (called *linear unbiased estimators*), find the one with minimum variance, and calculate the variance.