Discrete Mathematics for Computer Science

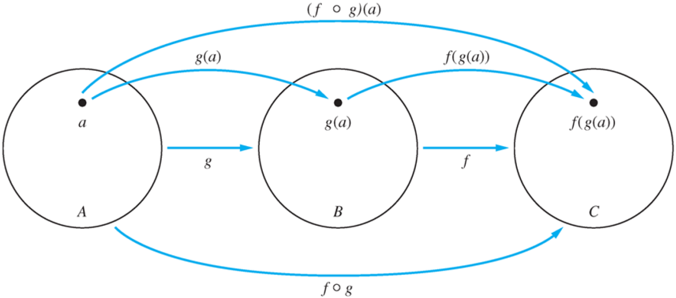
Lecture 5: Set and Function

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

Composition of Functions

Let f be a function from B to C and let g be a function from A to B. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.



Composition of Functions

- **Example 1**:

  Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$.

  $g : A \to A$ $f : A \to B$

  $1 \mapsto 3$ $1 \mapsto b$

  $2 \mapsto 1$ $2 \mapsto a$

  $3 \mapsto 2$ $3 \mapsto d$

  What is $f \circ g$?

- **Example 1**:

  Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$.

Composition of Functions

- **Example 2**:

  Let $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$, where $f(x) = 2x$ and $g(x) = x^2$.

  What are $g \circ f$ and $f \circ g$?

- **Example 2**:

  Let $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$, where $f(x) = 2x$ and $g(x) = x^2$.

  What are $g \circ f$ and $f \circ g$?

Composition of Functions

- Suppose that f is a bijection from A to B. Then $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$, Since

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$
$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b,$$

  where I_A, I_B denote the *identity functions* on the sets A and B, respectively.

Note: Identity function is sometimes denoted by $\iota_A(\cdot)$:

$$\iota_A(x) = x$$

Floor and Ceiling Functions

- The floor function assigns a real number x the largest integer that is $\leq x$, denoted by $\lfloor x \rfloor$. E.g., $\lfloor 3.5 \rfloor = 3$.
- The ceiling function assigns a real number x the smallest integer that is $\geq x$, denoted by $\lceil x \rceil$. E.g., $\lceil 3.5 \rceil = 4$.

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x+1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Note: n is an integer, x is a real number.

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Floor and Ceiling Functions: Example 1

Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Proof: Let $x = n + \epsilon$, where n is an integer and $0 \leq \epsilon < 1$.

- $0 \leq \epsilon < \frac{1}{2}$: In this case, $2x = 2n + 2\epsilon$. Since $0 \leq 2\epsilon < 1$, we have $\lfloor 2x \rfloor = 2n$. Similarly, $x + \frac{1}{2} = n + \frac{1}{2} + \epsilon$. Since $0 \leq \frac{1}{2} + \epsilon < 1$, we have $\lfloor x + \frac{1}{2} \rfloor = n$. Thus, $\lfloor 2x \rfloor = 2n$, and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = 2n$.
- $\frac{1}{2} \leq \epsilon < 1$: In this case, $2x = 2n + 2\epsilon = (2n+1) + (2\epsilon - 1)$. Since $0 \leq 2\epsilon - 1 < 1$, we have $\lfloor 2x \rfloor = 2n + 1$. ....

Floor and Ceiling Functions: Example 2

Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Proof: This statement is false. Consider a counterexample $x = \frac{1}{2}$ and $y = \frac{1}{2}$. We can find that $\lceil x + y \rceil = 1$, but $\lceil x \rceil + \lceil y \rceil = 2$.

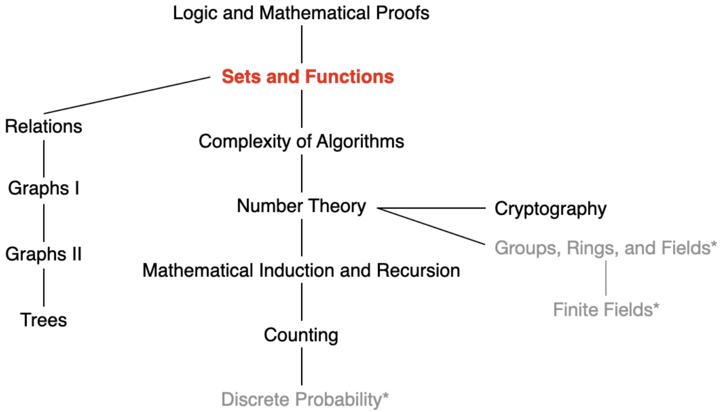
Factorial Function

The factorial function $f : \mathbf{N} \to \mathbf{Z}^+$ is the product of the first n positive integers when n is a nonnegative integer, denoted by $f(n) = n!$.

Summary of Function

- Function $f : A \to B$: an assignment of exactly one element of B to each element of A
- One-to-one function
- Onto function
- One-to-one correspondence: one-to-one function and onto
- Inverse function
- Floor function, ceiling function, factorial function

This Lecture

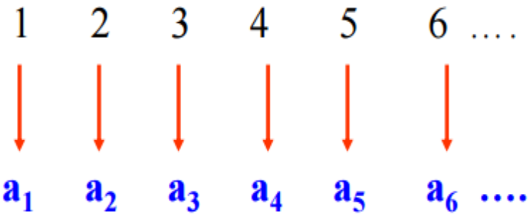


Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Sequences

A sequence is a function from a subset of the set of integers (typically the set $\{0, 1, 2, ...\}$ or $\{1, 2, 3, ...\}$) to a set S.

We use the notation a_n to denote the image of the integer n. $\{a_n\}$ represents the ordered list $\{a_1, a_2, a_3, ...\}$



Sequences

Examples:

- $a_n = n^2$, where $n = 1, 2, 3, \ldots$
- $a_n = (-1)^n$, where $n = 1, 2, 3, \ldots$
- $a_n = 2^n$, where $n = 1, 2, 3, \ldots$

Geometric Progression

A geometric progression is a sequence of the form

$$a, ar, ar^2, ..., ar^n, ...$$

where the initial term a and the common ratio r are real numbers.

Example: $a_n = 3 \times (\frac{1}{2})^n$, where $n = 0, 1, 2, 3, ...$

Arithmetic Progression

An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, a+3d, ..., a+nd, ...$$

where the initial term a and common difference d are real numbers.

Example: $a_n = -1 + 4n$, where $n = 0, 1, 2, 3, ...$

Recursively Defined Sequences

1 Providing explicit formulas, e.g., $a_n = -1 + 4n$, where $n = 0, 1, 2, 3, \ldots$

2 Recursively Defined Sequences: provide

- one or more initial terms
- a rule for determining subsequent terms from those that precede them.

The n-th element of the sequence $\{a_n\}$ is defined recursively in terms of the previous elements of the sequence and the initial elements of the sequence.

Examples:

- $a_0 = 1$, $a_n = a_{n-1} + 2$ for $n = 1, 2, 3, \ldots$.
- $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$ (Fibonacci sequence)

Summations

The summation of the terms of a sequence is

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

- j: the index of summation; the choice of the letter is arbitrary
- m: the lower limit of the summation
- n: the upper limit of the summation

$$\sum_{j=1}^{n}(ax_j + by_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j$$

Summations

The sum of the first n terms of the arithmetic progression:

$$S_n = \sum_{j=0}^{n} (a + jd) = (n+1)a + d\sum_{j=0}^{n} j = (n+1)a + d\frac{n(n+1)}{2}$$

The sum of the first n terms of the geometric progression:

- $r \neq 1$

$$S_n = \sum_{j=0}^{n} (ar^j) = a\sum_{j=0}^{n} r^j = \frac{ar^{n+1} - a}{r - 1}$$

- $r = 1$

$$S_n = \sum_{j=0}^{n} (ar^j) = (n+1)a$$

Summations: Example

- **Examples**:
  - $S = \sum_{j=1}^{5}(2+3j)$
  - $S = \sum_{j=3}^{5}(2+3j)$
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)$
  - $S = \sum_{j=0}^{3} 2(5)^j$
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{3} ij$
- **Examples**:
  - $S = \sum_{j=1}^{5}(2+3j)$ 55
  - $S = \sum_{j=3}^{5}(2+3j)$ 42
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)$ 28
  - $S = \sum_{j=0}^{3} 2(5)^j$ 312

Infinite Series

Infinite geometric series can be computed in the closed form for $|x| < 1$.

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{x^{n+1}-1}{x-1} = \frac{1}{1-x}$$

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Some Useful Summation Formulas

$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

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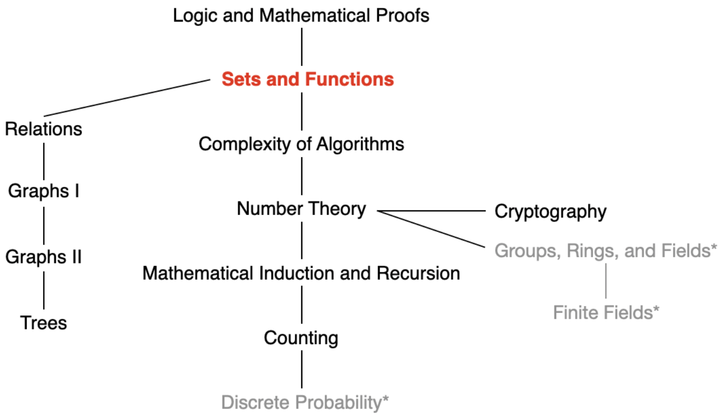


Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Cardinality of Sets

Recall: the cardinality of a finite set is defined by the number of the elements in the set.

The sets A and B have the same cardinality if there is a one-to-one correspondence between elements in A and B.

If there is a one-to-one function from A to B, the cardinality of A is less than or equal to the cardinality of B, denoted by $|A| \leq |B|$.

Moreover, when $|A| \leq |B|$ and A and B have different cardinalities, we say that the cardinality of A is less than the cardinality of B, denoted by $|A| < |B|$.

Countable Sets

A set that is either finite or has the same cardinality as the set of positive integers $\mathbf{Z}^+$ is called countable. A set that is not countable is called uncountable.

Why are these called countable?

The elements of the set can be enumerated and listed.

Hilbert's Paradox: Grand Hotel

The Grand Hotel has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel.

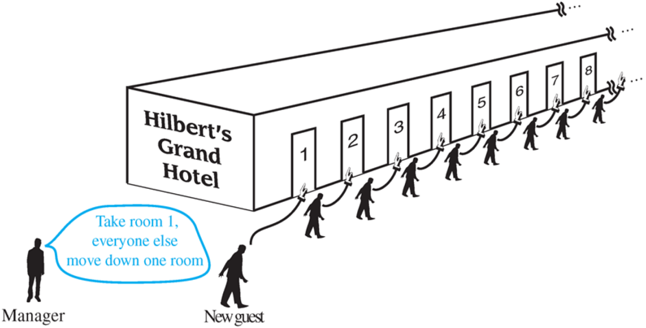


FIGURE 2 **A New Guest Arrives at Hilbert's Grand Hotel.**

Finitely many room: "All rooms are occupied" is equivalent to "no new guests can be accommodated".
Infinitely many room: This equivalence no longer holds.

Countable Sets: Example 1

The set of odd positive integers: $A = \{1, 3, 5, 7, ...\}$. Is it countable?

Proof: Using the definition: If there is a one-to-one correspondence from the set of positive integers $\mathbf{Z}^+$ to this set A?

Consider the function

$$f(n) = 2n - 1$$

- One-to-one: Suppose $f(n) = f(m)$. Then, $2n - 1 = 2m - 1$, which leads to $n = m$.
- Onto: For any arbitrary element in $t \in A$, we have an $n = (t + 1)/2 \in \mathbf{Z}^+$ such that $f(n) = t$.

Countable Sets: Example 2

Theorem: The set of integers $\mathbf{Z}$ is countable.

Proof: We can list the set of integers into a sequence:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

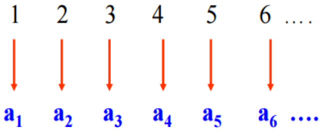
Thus, it is countable.

Theorem: An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers):

- All elements are listed

Why?

A sequence is a function from a subset of the set of integers to a set S.



Countable Sets: Example 2

Theorem: The set of integers $\mathbf{Z}$ is countable.

Proof: We can list the set of integers into a sequence:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

Thus, it is countable.

Alternatively, show there is a one-to-one correspondence from $\mathbf{Z}^+$ to $\mathbf{Z}$:

- when n is even: $f(n) = n/2$
- when n is odd: $f(n) = -(n-1)/2$

Thus, it is countable.

Do $\mathbf{Z}^+$ and $\mathbf{Z}$ have the same cardinality? Yes, because there is a one-to-one correspondence between $\mathbf{Z}^+$ and $\mathbf{Z}$.
Hilbert's Paradox: Grand Hotel

Countable Sets: Example 3

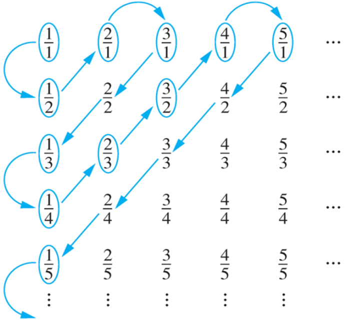
Theorem: The set of positive rational numbers is countable.

Hint: prove by showing that the set of positive rational numbers can be listed in a sequence: specifying the initial term and rule

Solution:

Constructing the list: first list p/q with $p+q=2$, next list p/q with $p+q=3$, and so on.

$1, 1/2, 2, 3, 1/3, 1/4, 2/3, \ldots$



Countable Sets: Example 4

Theorem: The set of finite strings S over a finite alphabet A is countably infinite. (Assume an alphabetical ordering of symbols in A)

For example, let $A = \{$'a', 'b', 'c'$\}$. Then, set $S = \{$'', 'a', 'b', 'c', 'ab' ..., 'aaaaa', ...$\}$

Solution:

We show that the strings can be listed in a sequence. First list

(i) all the strings of length 0 in alphabetical order.

(ii) then all the strings of length 1 in lexicographic order.

(iii) and so on.

This implies a bijection from $\mathbf{Z}^+$ to S.

Countable Sets: Example 5

The set of all Java programs is countable.

Solution:

Let S be the set of strings constructed from the characters which may appear in a Java program. Use the ordering from the previous example. Take each string in turn

– feed the string into a Java compiler

– if the complier says YES, this is a syntactically correct Java program, we add this program to the list

– we move on to the next string

In this way, we construct a bijection from $\mathbf{Z}^+$ to the set of Java programs.

Countable Sets: Example 6

Theorem: Any subset of a countable set is countable.

Proof: Consider a countable set A and its subset $B \subseteq A$.

- A is a finite set: $|B| \leq |A| < \infty$. Thus, $|B|$ is a finite set and hence countable.
- A is not a finite set: Since A is countable, the elements of A can be listed in a sequence. By removing the elements in the list that are not in B, we can obtain a list for B. Thus, B is countable

Theorem: If A and B are countable sets, then $A \cup B$ is also countable.

Uncountable Sets: Example 1

A set that is not countable is called uncountable.

Theorem: The set of real numbers $\mathbf{R}$ is uncountable.

Proof by Contradiction: Suppose $\mathbf{R}$ is countable. Then, the interval from 0 to 1 is countable. This implies that the elements of this set can be listed as $r_1, r_2, r_3, \ldots$, where

- $r_1 = 0.d_{11}d_{12}d_{13}d_{14}$
- $r_2 = 0.d_{21}d_{22}d_{23}d_{24}$
- $r_3 = 0.d_{31}d_{32}d_{33}d_{34}$

where all $d_{ij} \in \{0, 1, 2, \ldots, 9\}$.

We want to show that not all real numbers in the interval between 0 and 1 are in this list. Form a new number called $r = 0.d_1d_2d_3d_4$, where $d_i = 2$ if $d_{ii} \neq 2$, and $d_i = 3$ if $d_{ii} = 2$.

Example: suppose	r1 = 0.75243...	d1 = 2
	r2 = 0.524310...	d2 = 3
	r3 = 0.131257...	d3 = 2
	r4 = 0.9363633...	d4 = 2

Uncountable Sets: Example 2

Theorem: The set $\mathcal{P}(\mathbf{N})$ is uncountable.

Proof by contradiction:

Assume that $\mathcal{P}(\mathbb{N})$ is countable. This implies that the elements of this set can be listed as $S_0, S_1, S_2, \ldots$, where $S_i \subseteq \mathbb{N}$, and each S_i can be represented uniquely by the bit string $b_{i0}b_{i1}b_{i2}\ldots$, where $b_{ij} = 1$ if $j \in S_i$ and $b_{ij} = 0$ if $j \notin S_i$

- $S_0 = b_{00}b_{01}b_{02}b_{03}\cdots$
- $S_1 = b_{10}b_{11}b_{12}b_{13}\cdots$
- $S_2 = b_{20}b_{21}b_{22}b_{23}\cdots$

$\vdots$

all $b_{ij} \in \{0, 1\}$.

Form a new set called $R = b_0b_1b_2b_3\ldots$, where $b_i = 0$ if $b_{ii} = 1$, and $b_i = 1$ if $b_{ii} = 0$. R is different from each set in the list. Each bit string is unique, and R and S_i differ in the i-th bit for all i.

Schroder-Bernstein Theorem

Theorem: If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B, and hence $|A| = |B|$.

Example: Show that $|(0,1)| = |(0,1]|$

$f(x) = x$, $g(x) = x/2$

Computable vs Uncomputable

Definition: We say that a function is computable if there is a computer program in some programming language that finds the values of this function. If a function is not computable, we say it is uncomputable.

Theorem: There are functions that are not computable.

- The set of all programs is countable.
- There are uncountably many different functions from a particular countably infinite set to itself (omitted).

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