1.

Step 1: Extension to the Boundary

Since D is a Jordan domain, it is bounded and simply connected with a boundary that is a simple closed curve. By the Riemann mapping theorem, there exists a conformal map $\phi : D \to \mathbb{D}$, where $\mathbb{D}$ is the unit disk. By Carathéodory's theorem, since D is a Jordan domain, ϕ extends to a homeomorphism $\bar{\phi} : \overline{D} \to \overline{\mathbb{D}}$, and its inverse $\phi^{-1} : \mathbb{D} \to D$ extends to a homeomorphism $\overline{\phi^{-1}} : \overline{\mathbb{D}} \to \overline{D}$.

Let $f : D \to D$ be a conformal automorphism. Consider the composition:

$$g = \phi \circ f \circ \phi^{-1}.$$

Since f and ϕ are biholomorphic, $g : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of the unit disk. The conformal automorphisms of $\mathbb{D}$ are Möbius transformations of the form:

$$g(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}, \quad |a| < 1, \quad \theta \in \mathbb{R}.$$

Such maps extend continuously to $\overline{\mathbb{D}}$ and are homeomorphisms of $\overline{\mathbb{D}}$ to itself.

Now express f in terms of g:

$$f = \phi^{-1} \circ g \circ \phi.$$

Since $\bar{\phi} : \overline{D} \to \overline{\mathbb{D}}$, $\overline{\phi^{-1}} : \overline{\mathbb{D}} \to \overline{D}$, and $g : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ are all continuous, their composition is continuous. Thus, $f : \overline{D} \to \overline{D}$ is continuous, and $f|_D$ is holomorphic. Therefore, f extends continuously to ∂D.

Step 2: Uniquely Determined by Images of Three Boundary Points

Let $a, b, c \in \partial D$ be three distinct boundary points. Suppose f_1 and f_2 are conformal automorphisms of D such that:

$$f_1(a) = f_2(a), \quad f_1(b) = f_2(b), \quad f_1(c) = f_2(c).$$

We need to show that $f_1 = f_2$.

Fix a conformal map $\phi : D \to \mathbb{D}$ extending to a homeomorphism $\bar{\phi} : \overline{D} \to \overline{\mathbb{D}}$ as before. Define:

$$\alpha = \bar{\phi}(a), \quad \beta = \bar{\phi}(b), \quad \gamma = \bar{\phi}(c).$$

Since $\bar{\phi}$ is a homeomorphism, $\alpha, \beta, \gamma \in \partial\mathbb{D}$ are distinct.

For each $i = 1, 2$, define:

$$g_i = \phi \circ f_i \circ \phi^{-1}.$$

Then $g_i : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism extending to $\overline{\mathbb{D}}$.

Now compute $g_i(\alpha)$:

$$g_i(\alpha) = \phi(f_i(\phi^{-1}(\alpha))) = \phi(f_i(a)),$$

since $\phi^{-1}(\alpha) = a$. Similarly:

$$g_i(\beta) = \phi(f_i(b)), \quad g_i(\gamma) = \phi(f_i(c)).$$

Given that $f_1(a) = f_2(a)$, $f_1(b) = f_2(b)$, and $f_1(c) = f_2(c)$, it follows that:

$$g_1(\alpha) = g_2(\alpha), \quad g_1(\beta) = g_2(\beta), \quad g_1(\gamma) = g_2(\gamma).$$

Thus, g_1 and g_2 agree on the three distinct boundary points $\alpha, \beta, \gamma \in \partial\mathbb{D}$.

Conformal automorphisms of $\mathbb{D}$ are Möbius transformations and are uniquely determined by their values at three distinct boundary points because:

- any linear-fractional mapping $L(z) = \frac{az+b}{cz+d}$ is uniquely determined by three points
- proof: Recall Problem 4 in HW1, the cross-ratio of four points $\frac{z_2 - z_3}{z_2 - z_4} : \frac{z_1 - z_3}{z_1 - z_4}$ is invariant under linear-fractional mapping.

If $w_i = L(z_i)$ for $i = 1, 2, 3$, let $w = L(z)$,

let $F(z) = \frac{z_2 - z_3}{z_2 - z} : \frac{z_1 - z_3}{z_1 - z}$, $G(w) = \frac{w_2 - w_3}{w_2 - w} : \frac{w_1 - w_3}{w_1 - w}$

Then $F(z) = G(w)$, then we can solve a, b, c, d $(ad - bc = 1)$ uniquely from $F(z) = G(w)$.

2. Solution: Since $f(z) = z^3 \cdot e^{\frac{1}{z}}$, hence $z=0$ is an essential singularity for the Taylor expansion near 0 is: $f(z) = z^3 \cdot (1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \dots)$

$$= z^3 + z^2 + \frac{z}{2!} + \dots$$

By the Big Picard Theorem, for any small $\varepsilon > 0$,

$f(B_\varepsilon^*(0)) = \overline{\mathbb{C}} \setminus E$ where $E = \phi$ or $\{a\}$ or $\{a, b\}$.

Hence: $f(\mathbb{C} \setminus \{0\}) \subset \overline{\mathbb{C}} \setminus E$. Since $e^{\frac{1}{z}} \neq 0$ for any $z \neq 0$,

hence: $f(z) = 0$ iff $z = 0$. Hence: $0 \notin f(\mathbb{C} \setminus \{0\})$.

Hence $E \neq \phi$. For $f(z) = z^3 e^{\frac{1}{z}}$, then: $f(\overline{z}) = (\overline{z})^3 \cdot e^{\frac{1}{\overline{z}}}$

$$= (\overline{z^3}) \cdot (\overline{e^{\frac{1}{z}}}) = \overline{(z^3 \cdot e^{\frac{1}{z}})} = \overline{f(z)}.$$

Hence: if $z \notin f(\mathbb{C} \setminus \{0\})$, then $\overline{z} \notin f(\mathbb{C} \setminus \{0\})$.

Hence there doesn't exist $z \neq 0$, s.t: $z \in f(\mathbb{C} \setminus \{0\})$

for E has at most 2 elements. Hence $E = \{0\}$

Hence: $f(\mathbb{C} \setminus \{0\}) = \overline{\mathbb{C}} \setminus \{0\}$. $\boxed{///}$.

3. $f(z) = \ln(1+\sqrt{z^2+1})$

① If we choose the branch of $\sqrt{z^2+1}$ such that $\sqrt{1}=1$, let $w = 1+\sqrt{z^2+1}$

choose the branch of $\ln w$ such that $\ln 1 = 0$

The branch points of $\sqrt{z^2+1}$ are $\pm i$ and ∞,

branch points of $\ln w$ are 0 and ∞, but $1+\sqrt{z^2+1} \neq 0$.

So branch points of $\ln(1+\sqrt{z^2+1})$: $\pm i$, ∞.

At i, $\ln(1+\sqrt{z^2+1}) = \ln(1+\sqrt{z-i}\sqrt{z+i})$, $z=i$ is a branch point of $\sqrt{z-i}$ with order 2, so $z=i$ is a branch point of $f(z)$ with order 2, similarly, $z=-i$ is also a branch point of $f(z)$ with order 2. $\pm i$ are algebraic branch points of $f(z)$.

For ∞, ∞ is a logarithmic branch point of $\ln w$, so also is a logarithmic branch point of $f(z)$.

So the domain of $f(z)$: a simply conneted domain in $\overline{\mathbb{C}}$ without $\pm i$, ∞, such as $\mathbb{C} \setminus (-\infty,-1)i \cup (1,+\infty)i$

Puiseux series: (notice that $\sqrt{2i} = 1+i$)

At $z=i$:

$\sqrt{z^2+1} = \sqrt{z-i}\sqrt{(z-i)+2i}$, Let $t = \sqrt{z-i}$, then $\sqrt{z^2+1} = t\sqrt{t^2+2i}$

$\sqrt{t^2+2i} = \sqrt{2i} + \dfrac{t^2}{2\sqrt{2i}} + o(t^2)$. let $w = \sqrt{z^2+1}$

$\ln(1+w) = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 + o(w^3)$

$$= (1+i)(z-i)^{1/2} - i(z-i) + \frac{-5+5i}{12}(z-i)^{3/2} + o\left((z-i)^{3/2}\right)$$

Similarly, at $z=-i$, $\sqrt{-2i} = 1-i$

$$f(z)=(1-i)(z+i)^{\frac{1}{2}}+i(z+i)+\frac{-5-5i}{12}(z+i)^{\frac{3}{2}}+o\left((z+i)^{\frac{3}{2}}\right)$$

(2) If we choose the branch of $\sqrt{z^2+1}$ such that $\sqrt{1}=-1$, let $w=1+\sqrt{z^2+1}$

choose the branch of $\ln w$ such that $\ln 1=0$

The branch points of $\sqrt{z^2+1}$ are $\pm i$ and ∞,

branch points of $\ln w$ are 0 and ∞, the when $w=0$, $z=0$

So branch points of $\ln(1+\sqrt{z^2+1})$: $\pm i$, ∞, 0.

Also, similary, $z=\pm i$ are branch points of $f(z)$ with order 2.

$z=\infty$ is a logarithmic branch point of $f(z)$. $w=0$ is a a logarithmic branch point of $\ln w$, so also a logarithmic branch point of $f(z)$.

So the domain of $f(z)$: a simply conneted domain in $\bar{\mathbb{C}}$ without $\pm i$, ∞, 0, such as $\mathbb{C}\setminus(-\infty,-1)i\cup(1,+\infty)i\cup(-\infty,0)$

Puiseux series: (notice that $\sqrt{2i}=-1-i$)

At $z=i$:

$\sqrt{z^2+1}=\sqrt{z-i}\sqrt{(z-i)+2i}$, Let $t=\sqrt{z-i}$, then $\sqrt{z^2+1}=t\sqrt{t^2+2i}$

$\sqrt{t^2+2i}=\sqrt{2i}+\frac{t^2}{2\sqrt{2i}}+o(t^2)$. let $w=\sqrt{z^2+1}$

$\ln(1+w)=w-\frac{1}{2}w^2+\frac{1}{3}w^3+o(w^3)$

$$=(-1-i)(z-i)^{1/2}-i(z-i)+\frac{5-5i}{12}(z-i)^{3/2}+o\left((z-i)^{\frac{3}{2}}\right)$$

Similarly at $z=-i$, $\sqrt{-2i}=-1+i$

$$f(z)=(1+i)(z+i)^{\frac{1}{2}}+i(z+i)+\frac{5+5i}{12}(z+i)^{\frac{3}{2}}+o\left((z+i)^{\frac{3}{2}}\right)$$