COMPLEX ANALYSIS (H) COURSE, FINAL EXAM

1. (5 points) If f is nonconstant, by the Openness Principle, $f(D)$ is also a domain. Since $|f| = 1$ for $|z| = 1$, we conclude that the boundary of the domain $f(D)$ lies inside the unit circle. This means that $f(D)$ is simply D itself.

2. (5 points) A corollary from the Maximum Principle states that if a sequence of holomorphic functions (continuous up to the boundary of a bounded domain) converges uniformly on the boundary, then it converges uniformly in the closed domain. We conclude from here and from the Weierstrass convergence theorem that the limit of P_n in the closed domain is a function F which is continuous in the cloased domain and holomorphic inside. F is the continuation of f, and we obtain the desired claim.

3. (7 points) Fix a compact K in D. Since the set of poles of f is discrete in D, by the definition of meromorphic, there are only finitely many such poles inside K. Let $Q(z)$ be the sum of the principal parts of the Laurent series for f at all such points. Then Q is a rational function, and $g := f - Q$ is also meromorphic in D, and additionally has no singularities on K. By the Runge Theorem, for any $\epsilon > 0$ there exists a rational function $T(z)$ with poles outside K such that $|g(z) - T(z)| = |f - Q - T < \epsilon$ on K. Then $R = Q + T$ is the desired approximating rational function for f.

4. (10 points) Note that

$$\frac{x^2 - y^2}{(x^2 + y^2)^2} = \mathrm{Re}\,(z^2)/|z|^4 = 0.5(z^2 + \bar{z}^2)/(z^2\bar{z}^2) = 0.5(1/z^2 + 1/\bar{z}^2) = \mathrm{Re}\,(1/z^2).$$

Next,

$$\ln(x^2 + y^2) = \ln(|z|^2) = 2\ln|z| = 2\mathrm{Re}\,(\mathrm{Ln}\,z).$$

Hence we may choose

$$F(z) = 1/z^2 - 2\mathrm{Ln}\,z.$$

Any other choice of F differs by an imaginary constant.

The branch points of $F(z)$ in $\overline{\mathbb{C}}$ are 0 and ∞. Since $1/z^2$ is single-valued, the branching at this points is identical to that for (Ln z: both 0 and ∞ are logarithmic branch points.

5. (10 points) Consider a linear-fractional automorphism ϕ of the disc mapping $1/2$ to 0. Specifically, we can choose $\phi(z) = \frac{z-1/2}{1-z/2}$. Note that $\phi^{-1}(w) = \frac{w+1/2}{1+w/2}$. Now fix $f \in \mathcal{M}$. Then $g := \phi \circ f$ is a map of the disc into itself with $g(0) = 0$. We also have $f = \phi^{-1} \circ g$. By the Schwarz Lemma, we have $|g(z)| \leq |z|$ for all z in the disc. Dividing by $|z|$ and taking the limit at 0, this gives $|g'(0)| \leq 1$. But $|f'(0)| = |(\phi^{-1})'(0)| \cdot |g'(0)|$. We easily compute that $(\phi^{-1})'(0) = 3/4$, so that finally $|f'(0)| \leq 3/4$.

At the same time, putting $f := \phi^{-1}$, we have $|f'(0)| = 3/4$, so that the desired supremum is realized and equals $3/4$.

6. (10 points) Assume, buy contradiction, f vanishes at a point $a \notin \mathbb{R}$. Choose a circle C centered at a and not intersecting $\mathbb{R}$, such that f is nonvanishing on C (this is possible since a is an isolated singularity: f doesn't vanish identically). Let $\epsilon > 0$ be the minimum of f on C. Choose, using the uniform convergence on C, such P_n that $|f(z) - P_n(z)| < \epsilon$ holds on C. We then have $|P_n - f| < |f|$ on C, and by the Rouche theorem we conclude that $f + (P_n - f) = P_n$ has the same number of zeroes inside the circle C as f has. But since P_n may vanish only on $\mathbb{R}$, the latter number of zeroes equals 0, which contradicts $f(a) = 0$.

Additional question (5 points): By Weierstrass Theorem, P_n' converges uniformly on compacts to f'. So, we may apply the previous statement, if proving that all the zeroes of P_n' lie on the real line. To see this, Let us factorize

$$P_n = A(z - x_1)\cdots(z - x_k),$$

where all x_j are real by the conditions of the problem. Without loss of generality, assume $x_1 \leq x_2 \leq ... \leq x_k$. Then either by Lagrange theorem (from Calculus) in case $x_j < x_{j+1}$ or for trivial reason in case $x_j = x_{j+1}$, we find points $y_j \in [x_j, x_{j+1}]$ with $P_n'(y_j) = 0$ ($j = 1, ..., k-1$). This means each P_n' has $(k-1)$ real roots counting with multiplicities, so all its roots are real, as desired. (Note that f' is not identical zero since f is nonconstant).

7. (10 points) To show that f maps conformally D onto D', it is sufficient to show that any point $b \in D'$ has exactly one preimage (then f is holomorphic and bijective onto D', hence conformal). Indeed, since f is a homeomorphism of ∂D onto $\partial D' = f(D)$, we first apply the Openness Principle and conclude that f is nonconstant and that $f(D)$ is a domain with the boundary $\partial D'$. Hence $f(D) = D'$. So, every $b \in D'$ has at least one preimage in D. Next, by the Argument Principle, the number of preimages of b under f (which is the number of zeroes of $f(z) - b$ in D) equals the index w.r..t. 0 of $(f(z) - b)(\partial D)$ with respect to 0. Since f is a homeomorphism on ∂D and D is a Jordan domain, the latter curve $(f(z) - b)(\partial D)$ has index 0 or 1. Since at least one preimage exists, this index equals 1, and we have exactly one preimage, as desired.

8. (10 points) We extend f to the closed lower half plane by the formula $f(z) := \overline{f(\bar{z})}$. The outcome is holomorphic in the open lower half plane (since the operator $f \mapsto \overline{f(\bar{z})}$ maps a complex power series convergent in a disc in upper half plane into such in lower half plane). Further, $f(\mathbb{R}) \subset \mathbb{R}$ means that the two functions in upper and lower half planes agree on the real line. Hence the naturally defined union of this to functions is a function F continuous in $\mathbb{C}$ and holomorphic oputside the real line. By a Lemma proved before the Schwarz Reflection Principle, F is holomorphic in C (and is a continuation to $\mathbb{C}$ of the original f, by construction).

It remain to prove that F is a polynomial. Note that, from the way F is constructed, it also satisfies $|F(z)| = O(|z|^N)$, i.e. $|f(z)|/|z|^N \leq M$ in $\mathbb{C}$ for a constant $M > 0$. Now we apply in a disc $B_R(0)$ the Cauchy inequalities for the Taylor series $\sum_{n\geq 0} c_n z^n$ of F at 0 and get $|c_n| \leq MR^N/R^n$. For $n > N$, letting R tend to ∞, we conclude that $c_n = 0$. This means that F has a finite Taylor seires, i.e. is a polynomial, as desired.

9. (10 points) To Prove the MVP on a circle, we first use the Cauchy Integral Formula and get:

$$f(a) = \frac{1}{2\pi i}\int_{|z-a|=r} f(\xi)/(\xi - a).$$

Reducing via the parameterization $\xi = a + re^{it}$, we get:

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a + re^{it})ire^{it}}{re^{it}}dt = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt,$$

as needed.

To prove the area MVP, we consider the respective integral and subject it to the polar substitution $z = x + iy = a + \rho e^{i\theta}$:

$$\iint_{B_r(a)} f(z)dxdy = \int_0^r d\rho \int_0^{2\pi} f(a + \rho e^{it})\rho d\theta.$$

Applying now (within the repeated integral) in each $B_\rho(a)$, $\rho \leq r$, the circle MVP and applying elementary integration, we obtain:

$$\iint_{B_r(a)} f(z)dxdy = 2\pi \int_0^r \rho f(a)d\rho = \pi r^2 f(a),$$

as desired.

In the following 2 problems, you are allowed to use the outcome of Problem 9.

10. (10 points) Assume, for example, that f is nonconstant. Let $u(z) := |f(z)| + |g(z)|$ achieve its local maximum at a point $a \in D$. Note that g is then also nonconstant (otherwise we would simply apply the Maximum Principle for f), and that $f(a)$ and $g(a)$ can't vanish at a (if, for example, f vanishes, that a is a local max mod for f). Choose a closed disc B with center a and radius r contained in D, where, in addition, a is a maximum point for $u(z)$ and f, g are nonvanishing. Then applying, for example, the Area MVP, we have:

$$u(a) = |f(a)| + |g(a)| = \frac{1}{\pi r^2}\left(\left|\int_B f(z)dxdy\right| + \left|\int_B g(z)dxdy\right|\right) \leq$$
$$\frac{1}{\pi r^2}\left(\int_B |f(z)|dxdy + \int_B |g(z)|dxdy\right) \leq \frac{1}{\pi r^2}\left(\int_B |f(a)|dxdy + \int_B |g(a)|dxdy\right) \leq$$
$$|f(a)| + |g(a)| = u(a).$$

Thus we conclude that all the above inequalities are in turn equalities. The latter means that $u(z) = const$. This means $|f| + |g| = const$. To "play" with this identity, we appeal to the Cauchy-Riemann equations $f_z = f'$, $f_{\bar z} = 0$, and rewrite it as

$$\sqrt{f\bar f} + \sqrt{g\bar g} = const.$$

Applying $\frac{\partial}{\partial z}$ (which "sees" conjugated holomorphic functions as constants) gives

$$f'\sqrt{\bar f}/\sqrt{f} + g'\sqrt{\bar g}/\sqrt{g} = 0.$$

Applying then $\frac{\partial}{\partial \bar z}$ (which sees holomorphic functions as constants) gives, after simplifications:

$$|f'|^2/|f| + |g'|^2/|g| = 0,$$

which implies $f' = g' = 0$, which is a contradictions since both f, g are nonconstant.

11. (15 points) Clearly, $\mathcal{A}_1(D)$ is a linear space. Hence it remains to prove that it is closed, i.e. the limit f of an $L_1(D)$ convergent sequence f_n is actually holomorphic in D. To see that, let us first prove that f_n converges uniformly on compact in D. Fix a compact K in D, and denote by $2r$ its distance to the boundary. Then all closed r-balls with centers $z \in K$ lie in D. We shall show that the sequence f_n is a (uniformly) Cauchy sequence on K. Indeed, we apply the Area MVP in discs $B_r(z)$ for $z \in K$ and have:

$$|f_n(z) - f_m(z)| = \frac{1}{\pi r^2}\left|\int_{B_r(z)} (f_n(z)dxdy - f_m(z))dxdy\right| \leq \frac{1}{\pi r^2}||f_n - f_m||_{L_1},$$

and this implies the desired fact. Hence f_n converges uniformly on compacts in D to a function $F \in \mathcal{O}(D)$. Cleally, f almost everywhere coincides with F: recall (from Real Analysis) that an L_1 convergent sequence converges in measure, hence contains an almost everywhere convergent subsequence, hence F coincides with f almost everywhere. Hence f can be chosen holomorphic, as required.

12. (15 points) As was shown in the lectures, $\cot z$ is uniformly bounded by a constant M, for example, on the union of circle $C_R(0)$, $R = \pi(n + 1/2)\, n \in \mathbb{N}$. This means, in view of

$$\frac{1}{\xi - z} - \frac{1}{\xi} = \frac{z}{\xi(\xi - z)} = O(1/|\xi|^2),$$

that for each fixed z outside the poles of $\cot z$, the integrals of $f(\xi) := \left(\frac{1}{\xi - z} - \frac{1}{\xi}\right) \cot \xi$ over $C_R(0)$ are bounded (by modulus) by C/R for a constant C and thus tend to 0 as $R = \pi(N + 1/2)$ tends to ∞.

On the other hand, each such integral by the Residue Theorem equals

$$2\pi i(\mathrm{res}_0 f + \mathrm{res}_z f + \sum_{j=1}^{N} (\mathrm{res}_{\pi j} f + \mathrm{res}_{-\pi j} f)).$$

The points $\xi = z$ and $\xi = \pi j$ are simple poles for f, while $\xi = 0$ is a pole of order 2. We then calculate the residues in the standard manner and get:

$$\mathrm{res}_0 f = -\frac{1}{z},\ \mathrm{res}_z f = \cot z,\ \mathrm{res}_{\pi j} f = \frac{1}{\pi j - z} - \frac{1}{\pi j}.$$

Substituting this into the Residue Theorem, dividing by $2\pi i$, and letting N tend to ∞ gives the desired identity for each fixed z (for z which is a pole of $\cot z$ the identity is trivial since both sides equal ∞).

13. (10 points) Deduce from the outcome of Problem 12 the Weierstrass Factorization for $\sin z$:

$$\sin z = z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

The idea is taking the Log derivatives f'/f of both sides and reducing to the outcome of Problem 12. First, make sure the infinite product on the right is convergent normally in $\mathbb{C}$: for this, denote by $\ln z$ the branch of Log in $D = \{\mathrm{Re}\, z > 0\}$ with $\ln 1 = 0$, and note that for any compact K and some natural N, for all $n \geq N$ the terms of the product belong to D (actually even to its compact subset P). Its easy to see that all ln of terms in the product with $n \geq N$ admit the uniform bound M/n^2 for all $n \geq N$ and $z \in K$ for some constant $M > 0$. Hence the series of Logs of elements in the product converges uniformly on K, and by ecponentiating we conclude that so does the product (in the sense that its partial products converge uniformly on K). Now computing the Log derivatives of both sides of the desired identity (we do it accurately by applying to partial products and passing to the limit thanks to Weierstrass Theorem) outside the zeroes of $\sin z$, we obtain

$$\cot z = 1/z + \sum_{n=1}^{\infty} \frac{-2z}{\pi^2 n^2 - z^2},$$

which is equivalent to the outcome of Problem 12.

Next, let's prove that if the Log derivatives of two holomorphic functions f, g coincide in a domain (here f, g are the left and the right sides of the desired identity), then f, g differ by a constant factor. Indeed, $f'/f = g'/g$ implies $(f/g)' = 0$, and we get the claim. So, $\sin z$ is proportional to the RHS of our identity with a constant C. However, comparing e.g. the derivatives at 0 we get $C = 1$. This implies the desired identity.

Finally, at the zeroes of $\sin z$ the identity is obvious.

14. (15 points) switching if necessary to the sequence $(f_n(z) - a)/(b - a)$, we may assume that precisely 0 and 1 are the missing values for all the f_n. Thus all f_n are valued in $G := \mathbb{C} \setminus \{0, 1\}$.

Recall that the modular function $\mu(z)$ maps holomorphically $B_1(0)$ onto G, while μ^{-1} is a (Weierstrass) AF in G valued in $B_1(0)$, as was proved in the lectures. This allows us to consider the composition $F_n := \mu^{-1} \circ f_n$. All F_n then are AFs defined in D, but D is simply-connected, hence all the F_n are simply holomorphic in D by the Monodromy Theorem. In addition, they all are valued in $B_1(0)$ (because μ^{-1} is), i.e. all have modulud bounded by 1. Now the Compactness Principle is applicable, and F_n is precompact, i.e. contains a normally convergent subsequence. Hence $f_n = \mu \circ F_n$ contains a normally convergent subsequence.

Additional question (5 points): If not assuming D to be simply connected, we repeat the argument used in the prove of the Compactness Principle itself. Namely, we first consider any disc B contained in D (with its closure) and note that f_n contains a convergent subsequence on B by the above argument. Next, any compact K in D is contained in a finite union of discs as above (by compactness, we choose such a disc for each point in K, then their union covers K, and then use the finite subcovering). Hence taking convergent subsequences for each of the discs one-by-one, we obtain a sunsequence converging on K. Finally, we use a compact exhausion $D = \cup_k K_j$, take a convergent subsequence on K_1, then its convergent subsequence on K_2, then its convergent subsequence on K_3, etc. Then the "diagonal" subsequence is the needed one (details were explained in the lectures; also search for "Cantor Diagonal Procedure").

15. (10 points) See the recommended books of Shabat or Alfors.

16. (20 points) Assume, by contradiction, that there exists a norm on $\mathcal{O}(D)$ the convergence in which coincides with the normal convergence. Consider then the linear operators:

$$A(f) := zf(z), \quad B(f) := f'.$$

Since A and B transform (normally) convergent sequences to (normally) convergent sequences, they are continuous operators (Weierstrass Theorem!), hence are bounded in the norm. Hence the operator $C := A \circ B$ is bounded. However, we may consider the sequence $f_n := z^n \in \mathcal{O}(D)$. We easily compute that $Cf_n = nf_n$. Thus C rescales the norm of each f_n by n, hence is unbounded, which is a contradiction.