Solutions to Problems 81-90 on Pearson's Chisq Test for Goodness of Fit, Duality between Confidence Interval and Hypothesis Testing, and Re-enforcements

81. Four coins were tossed 160 times and 0,1,2,3, or 4 heads showed, respectively, 19, 54, 58, 23, and 6 times. Use the 0.05 level of significance to test whether it is reasonable to suppose that the coins are balanced and randomly tossed.

Solution. This is a hypothesis testing problem. The original null hypothesis is a verbal statement "coin is fair and sample is i.i.d." But we test an its immediate consequence which can be expressed in a distributional statement:

$$H_0: Data\ (\#Heads) \overset{iid}{\sim} Binomial(4, 0.5).$$

We first tabulate (i.e., make histograms) the data to streamline the calculation:

#Heads	Frequency	Expected Frequency
0	19	10
1	54	40
2	58	60
3	23	40
4	6	10

The third column "Expected Frequency" = $160 \times p_i$, where 160 is the size of the Binomial(4,0.5) sample and $p_i = \binom{4}{i} 0.5^4$. The Pearson's Chisq test for Goodness-of-fit employs the following statistic

$$T = \sum_{i=1}^{r} \frac{(O_i - E_i)^2}{E_i} \overset{n\to\infty}{\sim} \chi^2(r-1-d)$$

whose large sample distribution was deduced to be a Chisq distribution with $r-1-d$ degrees of freedom where r is the number of the items in the summation, which is the same as the number of rows in the table and as the number of bars in the histogram; d is the number of incompletely fixed scalar parameters (e.g., there are two parameters of the null distribution between them only the probability is unknown by the experiment—the alternative hypothesis will change its value.)

For the data here, $T = 21.89$, $r = 5$, $d = 1$, the large-sample p-value of T is $1 - ChisqCDF(21{,}89; \mathrm{df} = 3) = 6.9 \times 10^{-5}$ ➔ Reject H_0 that the data follows $Binomial(4{,}0.5)$ at 5% significance level.

82. It is desired to test whether the number of gamma rays emitted per second by a certain radio-active substance is a random variable having the Poisson distribution with $\lambda = 2.4$. Use the following data obtained for 300 1-second intervals to test this null hypothesis at the 0.05 level of significance:

Number of gamma rays	Frequency	Expected Frequency
0	19	
1	48	
2	66	
3	74	
4	44	
5	35	
6	10	
7 or more	4	

Solution. The null hypothesis is

$$H_0: Data \overset{iid}{\sim} Poisson(2.4).$$

Sample size is 300, which is large enough. Expected frequency = $300 \times p_i$, where 300 is the size of Poisson(2.4) sample and $p_i = e^{-2.4} \frac{2.4^i}{i!}$ (for the last row "7 or more", we simply took it as 7.)

Number of gamma rays	Frequency	Expected Frequency
0	19	27.22
1	48	65.32
2	66	78.38
3	74	62.70
4	44	37.62
5	35	18.06
6	10	7.22
7 or more	4	2.48

The Pearson's Chisq test statistic is

$$T = \sum_{i=1}^{r} \frac{(O_i - E_i)^2}{E_i} \overset{n\to\infty}{\sim} \chi^2(r-1-d)$$

which realizes to $t = 30.04$, and $r = 8$ is the number of bars in the histogram, $d = 1$ is the number of unfixed parameters. The large sample p-value for $t = 30.04$ is $1 - ChisqCDF(30.04; \mathrm{df} = 6) = 3.9 \times 10^{-5}$ ➔ Reject H_0 that the data follows $Poisson(2.4)$ at 5% significance level.

83. The following is the distribution of the readings obtained with a Geiger counter of the numbers of particles emitted by a radioactive substance in 100 successive 40-second intervals:

Number of particles	Frequency	Expected Frequency
5-9	1	
10-14	10	
15-19	37	
20-24	36	
25-29	13	
30-34	2	
35-39	1	

Test at the 0.05 level of significance whether the data may be looked upon as a random sample from a normal population.

Solution. The null hypothesis is

$$H_0: Data \overset{iid}{\sim} N(\mu, \sigma^2).$$

We need to plug in the two parameters with estimates. We will use their MLEs

$$\left(\hat{\mu}, \widehat{\sigma^2}\right) = \left(\bar{X}, \frac{(n-1)S^2}{n}\right)$$

The raw data are not provided, we decide to use the midpoints of each bin as the observed value for all observations falling in that bin. The mid points are 7.5, 12.5, 17.5, 22.5, 27.5, 32.5, 37.5. Thus the MLEs are

$$\left(\hat{\mu}, \widehat{\sigma^2}\right) = \left(\bar{X}, \frac{(n-1)S^2}{n}\right) = (20.5,25).$$

Thus the modified null hypothesis is

$$H_0^*: Data \overset{iid}{\sim} N(20.5, 5^2).$$

The Expected frequency is calculated as $100 \times p_i$, where 100 is the size of the $N(20.5, 5^2)$ sample and $p_i = \text{NormalProbability}\{\text{Bin}_i;\ \mu = 20.5, \sigma^2 = 5^2\}$.

Number of particles	Frequency	Expected Frequency
5-9	1	1.69
10-14	10	11.78
15-19	37	32.45
20-24	36	35.58
25-29	13	15.53
30-34	2	2.69
35-39	1	0.18

The Pearson's Chisq test statistic is

$$T = \sum_{i=1}^{r} \frac{(O_i - E_i)^2}{E_i} \overset{n\to\infty}{\sim} \chi^2(r-1-d)$$

which realizes to $t = 2.755$, and $r = 7, d = 2$ (as both mean and variance are plugged in by their MLEs). The large-sample p-value for $t = 2.755$ is $1 - \mathrm{ChisqCDF}(2.755; \mathrm{df} = 4) = 0.60$ ➔ Accept both H_0^* and H_0 at 5% significance level.

84. Suppose that for each value $\theta_0 \in \Theta$ there is a test at level α of the hypothesis $H_0: \theta = \theta_0$. Denote the acceptance region of the test by $A(\theta_0)$. Then the set $B(\mathbf{X}) = \{\theta: \mathbf{X} \in A(\theta)\}$ is a $100(1-\alpha)\%$ confidence region for θ.

Proof. We have to show that $\mathbb{P}\{B(\mathbf{X}) \ni \theta_0\} \geq 1 - \alpha$. We start the proof with the following clarification: $A(\theta_0)$ is a subset of the joint sample space S of the data $\mathbf{X}$ whereas $B(\mathbf{X})$ is a subset of the parameter space Θ of the parameter θ. By definition of the significance level of a test: $1 - \alpha \leq \mathbb{P}_{\theta_0}\{\mathbf{X} \in A(\theta_0)\}$. By definition of $B(\mathbf{X})$: $\mathbf{X} \in A(\theta_0) \Rightarrow \theta_0 \in B(\mathbf{X})$, therefore $\mathbb{P}_{\theta_0}\{X \in A(\theta_0)\} \leq \mathbb{P}_{\theta_0}\{B(\mathbf{X}) \ni \theta_0\}$.

85. Suppose that $B(\mathbf{X})$ is a $100(1-\alpha)\%$ confidence region for θ; that is, for each θ_0, $\mathbb{P}\{\theta_0 \in B(\mathbf{X}) | \theta = \theta_0\} = 1 - \alpha$. Then an acceptance region for a test at level α of the hypothesis $H: \theta = \theta_0$ is $A(\theta_0) = \{\mathbf{X}: \theta_0 \in B(\mathbf{X})\}$.

Proof. We have to show that $\mathbb{P}_{\theta_0}\{\mathbf{X} \in A(\theta_0)\} \geq 1 - \alpha$. We start the proof with the following clarification: $A(\theta_0)$ is a subset of the joint sample space S of the data $\mathbf{X}$ whereas $B(\mathbf{X})$ is a subset of the parameter space Θ of the parameter θ. By definition of $A(\theta_0)$: $\theta_0 \in B(\mathbf{X}) \Rightarrow \mathbf{X} \in A(\theta_0)$, therefore $\mathbb{P}_{\theta_0}\{X \in A(\theta_0)\} \geq \mathbb{P}_{\theta_0}\{B(\mathbf{X}) \ni \theta_0\} = 1 - \alpha$.

86. Let $X_1, \dots, X_n$ be a random sample from $N(\mu_0, \sigma^2)$, where μ_0 is known and σ^2 is unknown. (a) Using the Neyman-Pearson Lemma, find a most powerful test (MPT) of size α for testing the simple null hypothesis $H_0: \sigma^2 = \sigma_0^2$ against the simple alternative hypothesis $H_1: \sigma^2 = \sigma_1^2$, where $\sigma_1^2 > \sigma_0^2$. (b) Find a uniformly most powerful test (UMPT) of size α for testing the null hypothesis $H_1: \sigma^2 < \sigma_0^2$ against the alternative hypothesis $H_1: \sigma^2 > \sigma_0^2$.

Solution.

Part (a)

This is testing simple-vs-simple hypotheses → the Neyman-Pearson Lemma is in effect. The MP critical region is $\left\{\mathbf{X}: \frac{L_0}{L_1} \leq k\right\}$. We proceed to expand its form to a more explicit one.

$$L_0 = \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\mu_0)^2} \qquad L_1 = \left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^n e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(X_i-\mu_0)^2}$$

$$k \geq \frac{L_0}{L_1} = e^{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n}(X_i-\mu_0)^2}$$

$$k_1 \geq -\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n}(X_i-\mu_0)^2$$

$$k_2 \leq \sum_{i=1}^{n}(X_i-\mu_0)^2$$

$$k_3 \leq \sum_{i=1}^{n}\frac{(X_i-\mu_0)^2}{\sigma_0^2} \sim \chi^2(n)$$

Thus we can use $T = \sum_{i=1}^{n}\frac{(X_i-\mu_0)^2}{\sigma_0^2}$ as the test statistic for its null sampling distribution is the familiar Chisq distribution. The size-α MP critical region is $\{T \geq \chi_\alpha^2(n)\}$ where $\chi_\alpha^2(n) = k_3$ is the α-upper quantile of the Chisq distribution with n degrees of freedom.

Part (b):

This part is testing a composite-vs-composite hypotheses pair. The Neyman-Pearson Lemma is *not* in direct effect. We are required to find the UMP critical region. To do this, we make logical bridges that will help us extrapolate the effect of Neyman-Pearson Lemma to this particular pair of c-v-c hypotheses. There are two bridges to build.

The first bridges the s-v-s "$\sigma^2 = \sigma_0^2$ vs $\sigma^2 = \sigma_1^2 \geq \sigma_0^2$" to the s-v-c "$\sigma^2 = \sigma_0^2$ vs $\sigma^2 \geq \sigma_0^2$". This is readily extendable because the s-v-s critical region does not depend on the value of σ_1^2, in other words, the s-v-s critical region is MP for any value of σ_1^2—the s-v-s critical region is UMP for the s-v-c pair of hypotheses.

The second bridge establishes that the s-v-c critical region (which is the same as the s-v-s Neyman-Pearson MP critical region) is UMP for the c-v-c hypotheses "$\sigma^2 < \sigma_0^2$ vs $\sigma^2 = \sigma_1^2 \geq \sigma_0^2$". This requires that the new composite null space is still under size α. But since, $\forall \sigma^2 < \sigma_0^2$, we have

$$\sum_{i=1}^{n} \frac{(X_i - \mu_0)^2}{\sigma^2} \geq \sum_{i=1}^{n} \frac{(X_i - \mu_0)^2}{\sigma_0^2}$$

where $\sum_{i=1}^{n} \frac{(X_i-\mu_0)^2}{\sigma^2}$ is now following $\chi^2(n)$. This means if $\sum_{i=1}^{n} \frac{(X_i-\mu_0)^2}{\sigma_0^2} \geq \chi_\alpha^2(n)$, so is $\sum_{i=1}^{n} \frac{(X_i-\mu_0)^2}{\sigma^2}$, or the lower bound of the critical region on the range of $\sum_{i=1}^{n} \frac{(X_i-\mu_0)^2}{\sigma^2}$ is above $\chi_\alpha^2(n)$ ➔ type 1 error probability $\leqslant$ α ➔ the new composite null space is having size α.

With the above two logical bridges, we have found the UMP critical region for the test of "$\sigma^2 < \sigma_0^2$ vs $\sigma^2 = \sigma_1^2 \geq \sigma_0^2$" being the same as the MP critical region of "$\sigma^2 = \sigma_0^2$ vs $\sigma^2 = \sigma_1^2 \geq \sigma_0^2$".

87. Let $X_1, \dots, X_n$ be a random sample from the following density $f(x;\lambda) = \lambda(1-x)^{\lambda-1}\,\mathbb{I}(0 < x < 1)$ where $\lambda > 0$. (a) Let $Y_i = -\ln(1 - X_i)$ and $Q(X) = \prod_{i=1}^{n}(1 - X_i)$. Prove that Y_i follows an exponential distribution with mean parameter $1/\lambda$ and $-2\lambda \ln Q(X) \sim \chi^2(2n)$. [Hint: $\chi^2(2n) = Gamma\left(n, \frac{1}{2}\right)$ and the density of $Gamma(w; m, \lambda) = \frac{\lambda^m}{\Gamma(m)} w^{m-1} e^{-\lambda w}$]. (b) Find the likelihood ratio test (LRT) for testing $H_0: \lambda = 1$ against $H_1: \lambda \neq 1$.

Solution.

(a) **Proof**. $f(y_i)dy_i = f(x_i)dx_i \Rightarrow f(y_i) = f(x_i)\frac{dx_i}{dy_i} = \lambda(1-x_i)^{\lambda-1}(1-x_i) = \lambda(1-x_i)^{\lambda} = \lambda e^{-\lambda y_i}$ $(y > 0)$, which is the exponential density with mean $1/\lambda$. The other result is shown as $-2\lambda \ln Q = 2\lambda \sum_{i=1}^{n}\left(-\ln(1 - X_i)\right) \sim 2\lambda \cdot \Gamma(n, \lambda) = \Gamma(n, \frac{1}{2}) = \chi^2(2n)$.

(b) The (generalized) likelihood ratio statistic is defined as $\Lambda = \frac{\sup L_0}{\sup L}$ where $\sup L_0$ is the likelihood maximized over the null space for the parameter and $\sup L$ is the likelihood maximized over the entire parameter space. Here the null space contains only the point 1 while the entire space of the parameter λ is $[0, +\infty)$. Hence, here, $\Lambda = \frac{L(\lambda=1)}{L(\hat{\lambda})}$ where $\hat{\lambda}$ is the MLE for λ over the entire parameter space. $L(1) = 1$. Hence

$$\ln \Lambda = -\ln L(\hat{\lambda})$$

For $\hat{\lambda}$:

$$L(\lambda)=\lambda^n\prod_{i=1}^{n}(1-X_i)^{\lambda-1}$$

$$\ln L(\lambda)=n\ln\lambda+\sum_{i=1}^{n}\lambda\ln(1-X_i)-\sum_{i=1}^{n}\ln(1-X_i)=n\ln\lambda+(\lambda-1)\ln Q\sim n\ln\lambda+\Gamma\left(n,\frac{\lambda}{\lambda-1}\right)$$

$$0=\frac{\partial\ln L}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln(1-X_i)\Rightarrow\hat{\lambda}=\frac{n}{-\sum_{i=1}^{n}\ln(1-X_i)}$$

Hence

$$T=\frac{-2\ln\Lambda-2n\ln\hat{\lambda}}{\frac{\hat{\lambda}-1}{\hat{\lambda}}}\sim\chi^2(2n)$$

The critical region is $\{\Lambda>k\}$ or equivalently $\{T\leq\chi^2_{1-\alpha}(2n)\}$ where $\chi^2_{1-\alpha}(2n)$ is the α-lower quantile or equivalently the (1-α)-upper quantile of the chisq distribution with $2n$ degrees of freedom.

88. Let $X_1,\ldots,X_n$ be a random sample from the Bernoulli distribution with parameter $\theta=\mathbb{P}(X=1)$. Define $\hat{\theta}=\bar{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i$. Based on the convergence results $\frac{\sqrt{n}(\bar{X}_n-\theta)}{\sqrt{\hat{\theta}(1-\hat{\theta})}}\xrightarrow{L}N(0,1)$ and $\frac{\sqrt{n}(\bar{X}_n-\theta)}{\sqrt{\theta(1-\theta)}}\xrightarrow{L}N(0,1)$, (a) Find two approximate $100(1-\alpha)\%$ confidence intervals for θ, denoted by CI_1 and CI_2. (b) Let $n=10$, $(x_1,\ldots,x_{10})=(1,1,1,1,0,1,1,0,0,1)$, $\alpha=0.05$ and $z_{0.025}=1.96$. Calculate CI_1 and CI_2 and compare their widths.

Solution.

(a) From convergence 1, clearly, $\mathbb{P}\left\{-z_{\alpha/2}\leq\frac{\sqrt{n}(\bar{X}_n-\theta)}{\sqrt{\hat{\theta}(1-\hat{\theta})}}\leq z_{\alpha/2}\right\}\xrightarrow{n\to\infty}1-\alpha$. Therefore:

$$-z_{\alpha/2}\leq\frac{\sqrt{n}(\bar{X}_n-\theta)}{\sqrt{\hat{\theta}(1-\hat{\theta})}}\leq z_{\alpha/2}\Leftrightarrow\bar{X}_n-z_{\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}\leq\theta\leq\bar{X}_n+z_{\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$$

is a 100(1-α)% large-sample CI for θ.

From convergence 2, clearly, $\mathbb{P}\left\{\left|\frac{\sqrt{n}(\bar{X}_n-\theta)}{\sqrt{\theta(1-\theta)}}\right| \le z_{\alpha/2}\right\} \xrightarrow{n\to\infty} 1-\alpha$. Therefore

$$\left|\frac{\sqrt{n}(\bar{X}_n-\theta)}{\sqrt{\theta(1-\theta)}}\right| \le z_{\alpha/2} \Leftrightarrow \frac{2n\bar{X}_n + z_{\alpha/2}^2 - z_{\alpha/2}\sqrt{4n\bar{X}_n - 4n\bar{X}_n^2 + nz_{\alpha/2}^2}}{2n+2z_{\alpha/2}^2} \le \theta \le \frac{2n\bar{X}_n + z_{\alpha/2}^2 + z_{\alpha/2}\sqrt{4n\bar{X}_n - 4n\bar{X}_n^2 + nz_{\alpha/2}^2}}{2n+2z_{\alpha/2}^2}$$

is a 100(1-α)% large-sample CI for θ.

(b)

CI1=[0.416, 0.984], width = 0.568

CI2=[0.397, 0.892], width=0.495

CI2 is shorter.

89. Let $X_1, \dots, X_n$ be a random sample from a Poisson distribution with probability mass function $Poisson(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$ $(x = 0,1,2,\dots)$. Prove that the sample mean $\bar{X}$ is the unique uniformly minimum variance unbiased estimator (UMVUE) of λ.

Proof. It is clear that $\mathbb{E}(\bar{X}) = \mathbb{E}(X_1) = \lambda$ and $\mathbb{V}(\bar{X}) = \frac{1}{n}\mathbb{V}(X_1) = \frac{\lambda}{n}$. Then $\bar{X}$ is unbiased for λ and it remains to prove that $\bar{X}$ is a function of some complete sufficient statistic T and thereby it will satisfy the condition of the Lehman-Scheffé Theorem which states that *any unbiased estimator that is a function of a complete sufficient statistic must be the UMVUE, which is essentially unique*.

The candidate for the complete sufficient statistic is the sample sum: $T = \sum_i^n X_i$ of which $\bar{X}$ is a function and $T \sim Poisson(n\lambda)$. From the likelihood:

$$L(\lambda; \mathbf{X}) = \left(\prod_{i=1}^{n} \frac{1}{X_i!}\right)\left(e^{-n\lambda}\lambda^{\sum_{i=1}^{n} X_i}\right) = h(\mathbf{X})g\left(\sum_{i=1}^{n} X_i, \lambda\right)$$

we know from the Neyman's factorization theorem that T is sufficient for λ. We proceed to check its completeness. Suppose that an arbitrary function f of T has expectation k:

$$\mathbb{E}\left(f\big(T(X)\big)\right) = k \Leftrightarrow \mathbb{E}\big(f\big(T(X)\big) - k\big) = 0 \Leftrightarrow \sum_{t=0}^{\infty} [f(t) - k] p_T(T = t; n\lambda) = 0$$

As long as f has an expectation, the infinite sum must converge to 0, for whichever λ in the parameter space. For this question, the parameter space has an (uncountably) infinite number of points, which means the only possibility for the above infinite sum to converge to 0 is that $f(t) - k = 0$ with probability 1. This entails that T is complete.

90. Let $\theta = (\theta_1, \theta_2, \theta_3)^\intercal$ be an unknown parameter vector, where $0 < \theta_i < 1$ for $i = 1, 2, 3$ and $\sum_{i=1}^{3} \theta_i = 1$. Let x_1, x_2, x_3, y_1 and y_2 be observed values of random variables X_1, X_2, X_3, Y_1 and Y_2, respectively. Assume that the likelihood function of θ is $L(\theta) = \left(\prod_{i=1}^{3} \theta_i^{x_i}\right) \theta_1^{y_1} (\theta_1 + \theta_2)^{y_2}$. (a) Let $x_i > 0$ for $i = 1,2,3$ and $y_1 = y_2 = 0$. Find the maximum likelihood estimates (mles) of θ_i for $i = 1,2,3$ subject to the equality constraint $\sum_{i=1}^{3} \theta_i = 1$. (b) Let $x_i > 0$ for $i = 1,2,3$ and $y_j > 0$ for $j = 1,2$. Find the mles of θ_i with explicit expressions for $i = 1,2,3$ subject to the equality constraint $\sum_{i=1}^{3} \theta_i = 1$. (c) Let $x_1 = 100$, $x_2 = 50$, $x_3 = 20$, $y_1 = 10$ and $y_2 = 30$. Calculate the mles of θ_i for $i = 1,2,3$.

Solution.

(a) $L(\theta) = \prod_{i=1}^{3} \theta_i^{x_i}$,

$$\ln L(\theta) = \sum_{i=1}^{3} x_i \ln\theta_i \equiv \sum_{i=1}^{3} x_i \ln\theta_i + \lambda\left(\sum_{i=1}^{3} \theta_i - 1\right)$$

$$\left.\begin{aligned} 0 &= \frac{\partial \ln L(\theta)}{\partial \theta_i} = \frac{x_i}{\theta_i} + \lambda \Rightarrow \hat{\theta}_i = -\frac{x_i}{\lambda} \\ 1 &= \sum_{i=1}^{3} \theta_i \Rightarrow \lambda = -\sum_{i=1}^{3} x_i \end{aligned}\right\} \Rightarrow \hat{\theta}_i = \frac{x_i}{\sum_{i=1}^{3} x_i}$$

(b) $L(\theta) = \left(\prod_{i=1}^{3} \theta_i^{x_i}\right) \theta_1^{y_1} (\theta_1 + \theta_2)^{y_2}$

$$\ln L(\theta) = \sum_{i=1}^{3} x_i \ln\theta_i + y_i \ln\theta_1 + y_2 \ln(\theta_1 + \theta_2) + \lambda\left(\sum_{i=1}^{3} x_i - 1\right)$$

$$0=\frac{\partial \ln L(\theta)}{\partial \theta_i}=\frac{x_i}{\theta_i}+\frac{y_i}{\theta_1}\mathbb{I}(i=1)+\frac{y_2}{\theta_1+\theta_2}\mathbb{I}(i=1,2)+\lambda \Rightarrow \begin{cases} -\lambda\theta_1 = x_1+y_1+\dfrac{(x_1+y_1)y_2}{x_1+x_2+y_1} \\ -\lambda\theta_2 = x_2+\dfrac{x_2y_2}{x_1+x_2+y_1} \\ -\lambda\theta_3 = x_3 \end{cases}$$

where

$$-\lambda=-\lambda\sum_{i=1}^{3}\theta_i = x_1+x_2+x_3+y_1+y_2$$

(c)

$$\begin{cases} -\lambda\theta_1 = x_1+y_1+\dfrac{(x_1+y_1)y_2}{x_1+x_2+y_1}=\dfrac{1045}{8} \\ -\lambda\theta_2 = x_2+\dfrac{x_2y_2}{x_1+x_2+y_1}=\dfrac{475}{8} \\ -\lambda\theta_3 = x_3 = 20 \end{cases}$$

$$-\lambda = 210 \Rightarrow \begin{cases} \theta_1=\dfrac{209}{336} \\ \theta_2=\dfrac{95}{336} \\ \theta_3=\dfrac{2}{21} \end{cases}$$