Discrete Mathematics for Computer Science

Lecture 2: Propositional and Predicate Logic

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

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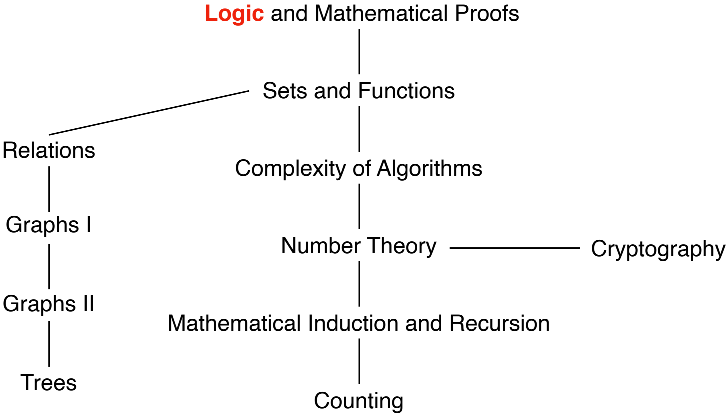

Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers

Tautology and Contradiction

- Tautology: A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.
- Contradiction: A compound proposition that is always false.
- Contingency: A compound proposition that is neither a tautology nor a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

The compound propositions p and q are called logically equivalent, denoted by $p \equiv q$ or $p \Leftrightarrow q$, if $p \leftrightarrow q$ is a tautology.

That is, two compound propositions are equivalent if they always have the same truth value.

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

De Morgan's Laws

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

  $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Important Logical Equivalences

- *Identity laws*
  - $p \wedge T \equiv p$
  - $p \vee F \equiv p$
- *Domination laws*
  - $p \vee T \equiv T$
  - $p \wedge F \equiv F$
- *Idempotent laws*
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$

Important Logical Equivalences

- *Double negation laws*
  - $\neg(\neg p) \equiv p$
- *Commutative laws*
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- *Associative laws*
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

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Important Logical Equivalences

- *Distributive laws*
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- *De Morgan's laws*
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- *Others*
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

    Absorption laws
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

    Negation laws
  - $p \rightarrow q \equiv \neg p \vee q$

    Useful law

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Using Logical Equivalences

Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

Example: Show that $(p \land q) \rightarrow p$ is a tautology.

Proof:
$$
\begin{aligned}
(p \land q) \rightarrow p &\equiv \neg(p \land q) \lor p && \text{Useful} \\
&\equiv (\neg p \lor \neg q) \lor p && \text{De Morgan's} \\
&\equiv (\neg q \lor \neg p) \lor p && \text{Commutative} \\
&\equiv \neg q \lor (\neg p \lor p) && \text{Associative} \\
&\equiv \neg q \lor T && \text{Negation} \\
&\equiv T && \text{Domination}
\end{aligned}
$$

Proof (alternatively):

p	q	$p \land q$	$(p \land q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**

Using Logical Equivalences

Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

Example: Show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$

Proof:

$$\begin{aligned} \neg q \rightarrow \neg p &\equiv \neg(\neg q) \vee (\neg p) && \text{Useful} \\ &\equiv q \vee (\neg p) && \text{Double negation} \\ &\equiv (\neg p) \vee q && \text{Communtative} \\ &\equiv p \rightarrow q && \text{Useful} \end{aligned}$$

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Limitations of Propositional Logic

Propositional logic: describe the world in terms of elementary propositions and their logical combinations.

Example 1: $1^2 \geq 0$

However, we also have

- $2^2 \geq 0$, $3^2 \geq 0$, ...
- $(-1)^2 \geq 0$, $(-2)^2 \geq 0$, ...

What is a more natural solution to express the knowledge?

Include variables!

- Predicates: $P(x)$: $x^2 \geq 0$
- Quantifiers: For all integer x, we have $x^2 \geq 0$.

Limitations of Propositional Logic

Example 2:

- Every computer in Room 101 is functioning properly.
- MATH3 is a computer in Room 101.

Can we conclude "MATH3 is functioning properly" using the rules of propositional logic?

NO!

Solution: Predicates and Quantifiers

- $P(x)$: Computer x is functioning properly.
- $\forall x P(x)$: $P(x)$ holds for all computer x in Room 101.
- Universal quantifier, existential quantifier

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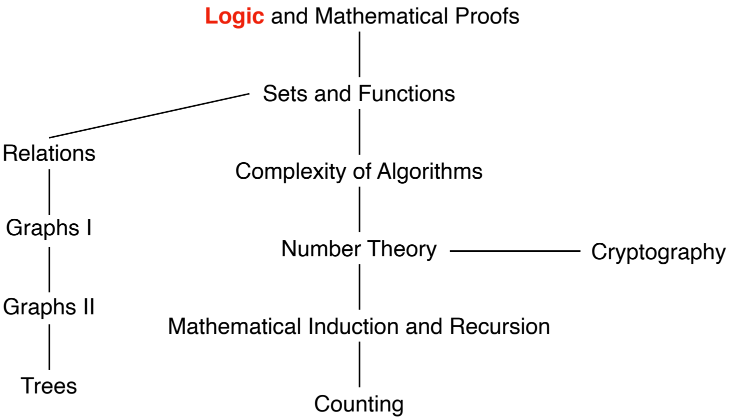

Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers

Predicate Logic

Predicate Logic: make statements with variables

Example: x is greater than 3

- Variable x
- Predicate P: "is greater than 3"
- Propositional function $P(x)$: the truth value of P at x

Predicate Logic

A propositional function $P(x)$ assigns a value T or F to each x depending on whether the property holds or not for x

Example: $P(x)$ denote the statement "$x > 3$":

- $P(2)$ is F
- $P(4)$ is T

Is $P(x)$ a proposition? No!

Is $P(2)$ a proposition? Yes!

Predicates

- A predicate is a statement $P(x_1, x_2, ..., x_n)$ that contains n variables x_1, x_2, ... x_n. It becomes a proposition when specific values are substituted for the variables x_1, x_2, ... x_n.
- The domain (universe) D of the predicate variables x_1, x_2, ... x_n is the set of all values that may be substituted in place of the variables.
- The truth set of $P(x_1, x_2, ..., x_n)$ is the set of all values of the predicate variables $(x_1, x_2, ..., x_n)$ such that the proposition $P(x_1, x_2, ..., x_n)$ is true.

Predicates: Example 1

Let $P(x)$ be the predicate "$x^2 > x$" with domain of the real numbers.

1. What are the truth values of $P(2)$ and $P(1)$?

   $P(2) = \text{T}$, $P(1) = \text{F}$

2. What is the truth set of $P(x)$?

   $x > 1$ or $x < 0$

Predicates: Example 2

Let $Q(x, y)$ be the predicate "$x = y + 3$" with domain of the real numbers.

1. What are the truth values of $Q(1, 2)$ and $Q(3, 0)$?
   $Q(1, 2) = \mathrm{F}$, $Q(3, 0) = \mathrm{T}$
2. What is the truth set of $Q(x, y)$?
   $(a, a - 3)$ for all real numbers a

Compound Statements in Predicate Logic

Compound statements are obtained via logical connectives.

$P(x)$: x is a prime
$Q(x)$: x is an integer

- $P(2) \wedge P(3)$: Both 2 and 3 are primes. (T)
- $P(2) \wedge Q(2)$: 2 is a prime or an integer. (T)
- $Q(x) \rightarrow P(x)$: If x is an integer, then x is a prime. (Not a proposition!)

How to make it a proposition?

Note: Researchers may use Prime(x) to refer to "x is a prime", Integer(x) to refer to "x is an integer", and others. It is only a way of notation. If you use such notations, please define it clearly beforehand.

Quantified Statements

Propositional function $P(x) \stackrel{\text{specify } x}{\Longrightarrow}$ Proposition

An alternative way to obtain proposition:

Propositional function $P(x) \stackrel{\text{for all/some } x \text{ in domain}}{\Longrightarrow}$ Proposition

Predicate logic permits quantified statement where variables are substituted for statements about the group of objects.

Quantified Statements

Two types of quantified statements:

- Universal quantifier $\forall x P(x)$
  - All CS-major graduates have to pass CS201.
  - (This is true for all CS-major graduates.)
- Existential quantifier $\exists x P(x)$
  - Some CS-major students graduate with honor.
  - (This is true for some students.)

Universal Quantifier

The universal quantification of $P(x)$ is the statement

$P(x)$ for all values of x in the domain.

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$."

Universal Quantifier: Example

$P(x)$: $|x| \leq x$

What is the truth value of $\forall x P(x)$?

- Assuming the domain to be all positive real numbers? True
- All real numbers? False

The domain must always be specified!

Universal Quantifier: Questions

The universal quantification of $P(x)$ is the statement

$$P(x) \text{ for all values of } x \text{ in the domain.}$$

Question 1: Is $\forall x P(x)$ a proposition?

Yes. Its truth value?

- True if $P(x)$ is true for all x in the domain.
- False if there is an x in the domain such that $P(x)$ is false. (counterexample)

Question 2: What is the truth value of $\forall x P(x)$ when the domain is empty?

Proposition $\forall x P(x)$ is true for every propositional function $P(x)$.

Existential Quantifier

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Example: $P(x)$: $x > 0$

What is the truth value of $\exists x P(x)$?

- What if assuming the domain to be all real numbers? True
- What if all negative real numbers? False

The domain must always be specified!

Existential Quantifier: Questions

The existential quantification of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

Question 1: Is $\exists x P(x)$ a proposition?

Yes. Its truth value?

- True if there is an x in the domain such that $P(x)$ is true. (an example)
- False if $P(x)$ is false for all x in the domain.

Question 2: What is the truth value of $\exists x P(x)$ when the domain is empty?

Proposition $\exists x P(x)$ is false for every propositional function $P(x)$.

Summary of Quantified Statements

Statement	When true?	When false?
$\forall x\, P(x)$	P(x) true for all x	There is an x where P(x) is false.
$\exists x\, P(x)$	There is some x for which P(x) is true.	P(x) is false for all x.

Suppose that the elements in the domain can be enumerated as $x_1, x_2, ..., x_n$ then:

- $\forall x P(x)$ is true whenever $P(x_1) \wedge P(x_2) \wedge ... \wedge P(x_n)$ is true.
- $\exists x P(x)$ is true whenever $P(x_1) \vee P(x_2) \vee ... \vee P(x_n)$ is true.

Properties of Quantifiers

The truth values of $\forall x P(x)$ and $\exists x P(x)$ depend on both the propositional function $P(x)$ and the domain.

Example: $P(x)$: $x < 2$

- domain: the positive integers
  $\forall x P(x)$: F, $\exists x P(x)$: T
- domain: the negative integers
  $\forall x P(x)$: T, $\exists x P(x)$: T
- domain: $\{3, 4, 5\}$
  $\forall x P(x)$: F, $\exists x P(x)$: F

Precedence of Proposition and Quantifiers

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

- $\neg p \wedge q$ means $(\neg p) \wedge q$ rather than $\neg(p \wedge q)$
- $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

- $\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x (P(x) \vee Q(x))$

Translation with Quantifiers

Every student in this class has studied algebra.

Logic Expression 1:

- $A(x)$: "x has studied algebra".
- Domain: the students in the class
- $\forall x A(x)$

Logic Expression 2:

- $A(x)$: "x has studied algebra".
- $C(x)$: "x is in this class"
- Domain: all students
- $\forall x(C(x) \rightarrow A(x))$

Note: Implication $p \rightarrow q$.

How about $\forall x(C(x) \wedge A(x))$? All students are in this class and has studied algebra.

Logic Expression 3:

- $A(x)$: "x has studied algebra".

Translation with Quantifiers

Some student in this class has visited Mexico.

Logic Expression 1:

- $M(x)$: "x has visited Mexico".
- Domain: the students in the class
- $\exists x M(x)$

Logic Expression 2:

- $M(x)$: "x has visited Mexico".
- $C(x)$: "x is a student in this class."
- Domain: all people
- $\exists x(C(x) \land M(x))$

How about $\exists x(C(x) \rightarrow A(x))$? No! This is even true when there is some people not in the class.

Translation with Quantifiers

- p: Every computer in Room 101 is functioning properly.
- q: Computer MATH3 is in Room 101.

Can we conclude r: "MATH3 is functioning properly" using the rules of propositional logic? NO! Cannot infer r from p and q.

With predicate and quantifier:

- $C(x)$: Computer x is in Room 101.
- $D(x)$: Computer x is functioning properly.

- $\forall x(C(x) \rightarrow D(x))$ within the domain of computers: Every computer in Room 101 is functioning properly.
- $C(MATH3)$: Computer MATH3 is in Room 101.
- $D(MATH3)$: MATH3 is functioning properly.

Negation of Quantifiers

Every student in this class has taken a course in calculus.

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\forall x P(x)$

The negation of this statement: It is not the case that every student in this class has taken a course in calculus.

- $\neg(\forall x P(x))$
- $\exists x(\neg P(x))$

$$\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$$

Negation of Quantifiers

There is a student in this class who has taken a course in calculus."

- $P(x)$: x has taken a course in calculus
- Domain: All students in this class
- $\exists x P(x)$

The negation of this statement: It is not the case that there is a student in this class has taken a course in calculus.

- $\neg(\exists x P(x))$
- $\forall x(\neg P(x))$

$$\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$$

Negation of Quantified Statements

A.k.a, De Morgan laws for quantifiers

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x \ P(x)$	$\forall x \ \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x \ P(x)$	$\exists x \ \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

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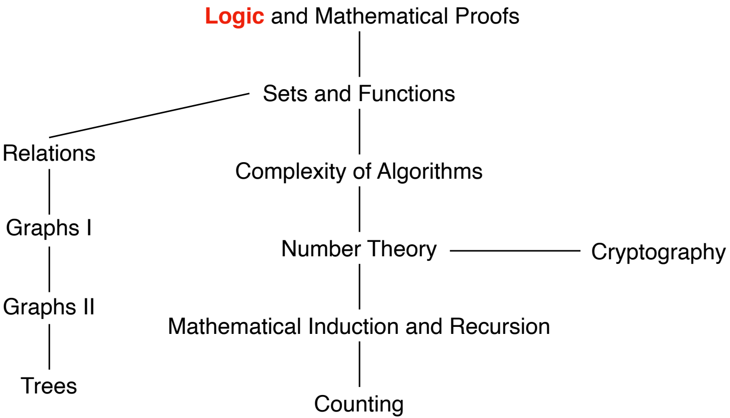

Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers
Mathematical Proofs: Rules of inference, introduction to proofs,

Nested Quantifiers

More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example 1: For every real number, there is a real number such that their summation is equal to zero.

- $P(x, y)$: $x + y = 0$
- Domain of x and y: all real number
- $\forall x \exists y P(x, y)$

Nested Quantifiers

More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example 2: There is a real number such that it is larger than all negative real numbers.

- $P(x, y)$: $x > y$
- Domain of x: all real number
- Domain of y: all negative real numbers
- $\exists x \forall y P(x, y)$

Does the order matter?

Order of Quantifiers

The order of nested quantifiers matters if quantifiers are of different type.

Example:

- $P(x, y)$: $x + y = 0$
- Domain of x: all real number
- Domain of y: all negative real numbers

$\forall x \exists y P(x, y)$ is not equivalent to $\exists y \forall x P(x, y)$

- $\forall x \exists y P(x, y)$: for every x, there exists a y such that ...
- $\exists y \forall x P(x, y)$: exists a y such that for every x ...

Order of Quantifiers

The order of nested quantifiers does no matter if quantifiers are of the same type.

Example:

- $P(x, y)$: $x + y = y + x$
- Domain of x: all real number
- Domain of y: all negative real numbers

$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$:

- $\exists x \exists y P(x, y)$: exists an x such that there exists a y ...
- $\exists y \exists x P(x, y)$: exists a y such that there exists an x ...

Exist a pair x, y for which $P(x, y)$ is true.

$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$:

- $\forall x \forall y P(x, y)$: for every x, for every y, ...
- $\forall y \forall x P(x, y)$: for every y, for every x, ...

For every pair x, y, $P(x, y)$ is true.

Nest Quantifier with Two Variables

Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

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Try to Translate

1. The sum of two positive integers is always positive.
   - Domain of x and y: all integers
   - $P(x, y)$: $(x > 0) \wedge (y > 0)$
   - $Q(x, y)$: $x + y > 0$
   - $\forall x \forall y (P(x, y) \rightarrow Q(x, y))$
   - Or, we can write it as $\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow x + y > 0)$
2. Every real number except zero has a multiplicative inverse.
   - Domain of x and y: all real numbers
   - $\forall x ((x \neq 0) \rightarrow \exists y (xy = 1))$

Negating Nested Quantifiers

For every real number x, there exists a real number y such that $xy = 1$.

$$\forall x \exists y (xy = 1)$$

$$\begin{aligned} & \neg \forall x \exists y \ (xy = 1) \\ \equiv \ & \exists x \neg \exists y \ (xy = 1) \\ \equiv \ & \exists x \forall y \ \neg (xy = 1) \\ \equiv \ & \exists x \forall y \ (xy \neq 1) \end{aligned}$$

Note: $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$, $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

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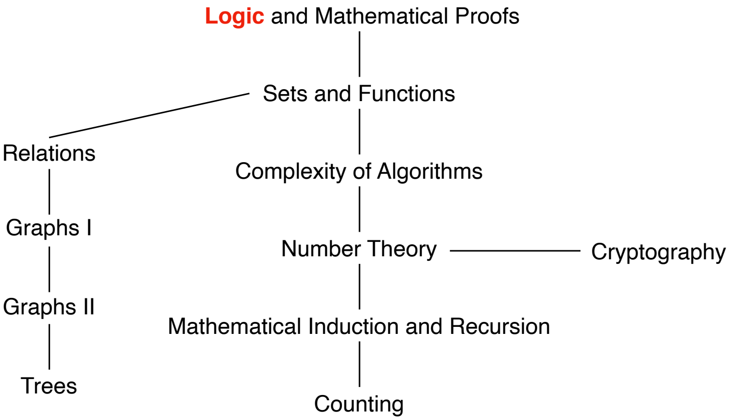

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Argument

Argument: A sequence of propositions that end with a conclusion.

Premises:

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore, **Conclusion:**

"You can log onto the network."

An argument is valid if <u>the truth of all its premises</u> implies that the conclusion is true.

Argument Form

Premises:

"If you have a current password, then you can log onto the network."

"You have a current password."

Conclusion: "You can log onto the network."

An argument form in propositional logic is a sequence of compound propositions involving propositional variables.

- p: "You have a current password"
- q: "You can log onto the network" or "You can change your grade"

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

The validity of an argument follows from the validity of its argument form.

Validity

Validity of Argument Form: The argument form with premises $p_1, p_2, ..., p_n$ and conclusion q is valid, if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q \text{ is a tautology.}$$

Note: According to the definition of $p \rightarrow q$, we do not worry about the case where $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ is false.

Thus, equivalently, an argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Is the following argument form valid?

$$\begin{array}{l} \quad p \rightarrow q \\ \quad p \\ \hline \therefore q \end{array}$$

Is $(p \rightarrow q) \wedge p \rightarrow q$ a tautology?

Validity of Argument: The validity of an argument follows from the

Validity

Is the following argument valid?

- If you do every problem in this book, then you will learn discrete mathematics.
- You learned discrete mathematics.
- Therefore, you did every problem in this book.

No! $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology.

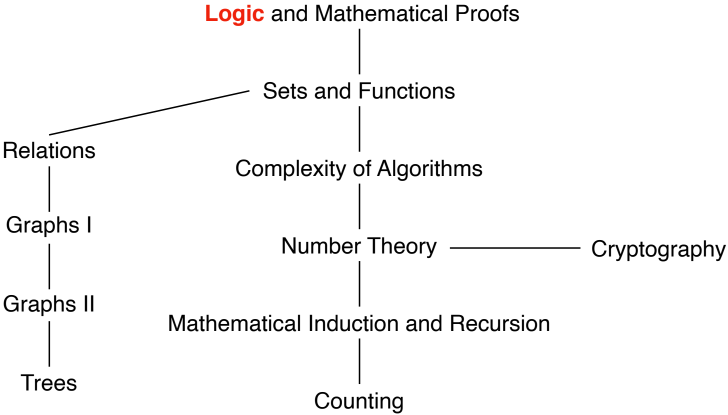
Validity

Is the following argument valid?

- If you do every problem in this book, then you will learn discrete mathematics.
- You did not do every problem in this book.
- Therefore, you did not learn discrete mathematics.

No! $((p \to q) \land \neg p) \to \neg p$ is not a tautology.

Next Lecture



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