HW7

1. Prove that if G is a simple group with order 60 , then $G \simeq A_5$.

Proof. See here. □

2. Prove that:

- **(1). Prove there exists no such group G satisfying $G' \simeq S_3$;**
- **(2). Prove there exists no such group G satisfying $G' \simeq S_4$.**

Proof. i) Assume that $G' \simeq S_3$, then $G'' \simeq S_3' \simeq C_3$ and by NC lemma

$$G/C_G(G'') = N_G(G'')/C_G(G'') \lesssim \mathrm{Aut}(G'') = C_2.$$

It follows that $C_G(G'') \geqslant G'$ and so $C_{G'}(G'') = G'$. Hence, all elements in S_3 commute with elements in $S_3' = \langle (123) \rangle$, which is impossible.

ii) Similarly, $S_4' \simeq A_4$ and $A_4' = \{1, (12)(34), (14)(23), (13)(24)\} = V$. Note that $G'/G''' = S_3$ and $(G/G''')' = G'/G''' \simeq S_3$, which contradicts with i). □

3. Prove that all 3-cycles in group $A_n (n \geq 5)$ can be represented by a commutator in A_n. Then prove $A_n' = A_n$.

Proof. Note that $[(ij), (ik)] = (ij)(ij)^{(ik)} = (ij)(jk) = (ikj)$. □

4. Let p be a prime number, $F = \mathbb{Z}/p\mathbb{Z}, G = \mathrm{GL}_n(F)$. Write a specific Sylow p-subgroup of G.

Proof. Note that $|G| = p^{\frac{n(n-1)}{2}} m$ with $(m, p) = 1$. Thus

$$H = \left\{ \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & 1 \end{bmatrix} : u_{i,j} \in F \text{ for all } i < j \right\}$$

is a Sylow p-subgroup of G. □

5. Let G be a group, $H \trianglelefteq G, K \trianglelefteq G$ and $H \cap K = 1$, prove that $\forall h \in H$ and $k \in K$, $hk = kh$.

Proof. Since $hkh^{-1}k^{-1} = (k^h)k^{-1} \in K$ and $hkh^{-1}k^{-1} = h(h^{-1})^k \in H$, we have that $hkh^{-1}k^{-1} \in H \cap K = \{1\}$ and so $hk = kh$. □

6. Let G be a finite group. Prove the minimal normal subgroup of G is a direct product of several (maybe 1) isomorphic simple groups.

Proof. See here. $\square$

7. Prove for any prime p, there exists exact 2 different types of non-abelian groups of order p^3 up to isomorphic.

Proof. Note that $C_p{:}C_{p^2}$ and $C_{p^2}{:}C_p$ are non-isomorphic.

Assume that $C_p{:}C_{p^2} = \langle\lambda\rangle{:}\langle\mu\rangle$, then $\lambda^\mu = \lambda^k$ for some $k \in \mathbb{Z}$ and $\lambda^{\mu^{p^2}} = \lambda^{k^{p^2}} = \lambda$. It holds for all $(k, p) = 1$. Choose any $k \not\equiv 1 \pmod p$ and $C_p{:}C_{p^2}$ is non-abelian.

Assume that $C_{p^2}{:}C_p = \langle\lambda\rangle{:}\langle\mu\rangle$, then $\lambda^\mu = \lambda^k$ for some $k \in \mathbb{Z}$ and $\lambda^{\mu^p} = \lambda^{k^p} = \lambda$. It holds if $k^p \equiv 1 \pmod{p^2}$. Choose $k = p + 1$ and $C_p{:}C_{p^2}$ is non-abelian.

It remains to show these two groups are not isomorphic. Assume that $\varphi : C_{p^2}{:}C_p \to C_p{:}C_{p^2}$ is an isomorphism. Since $C_{p^2}{:}C_p = \langle\lambda\rangle{:}\langle\mu\rangle$ has a normal subgroup $\langle\lambda^p\rangle \simeq C_p$, we have that $\varphi(\langle\lambda^p\rangle) \lhd C_p{:}C_{p^2}$. Hence $C_p{:}C_{p^2}$ has a normal subgroup $N \simeq C_p$ such that $C_p{:}C_{p^2}/N \simeq C_p \times C_p$.

Assume that $C_p{:}C_{p^2} = H{:}K = \langle\lambda_1\rangle{:}\langle\mu_1\rangle$. Then $N \cap H = \{1\}$ otherwise $C_p{:}C_{p^2}/N \simeq K \not\simeq C_p \times C_p$. It follows that $N \times H \lhd H{:}K$ and there exists $\lambda_1^i\mu_1^j$ commute with λ_1. Then $\mu_1^j\lambda_1 = \lambda_1\mu_1^j$ yields that $\lambda^\mu = \lambda$, which is impossible. $\square$

8. Let G be of order p^3q with $p < q, p, q$ are different primes. Prove G is not simple.

Proof. By Burnside's theorem. $\square$

9. Let $G = A_4$. Write G into a semidirect product of 2 subgroups.

Proof. Notice that $G = V_4{:}C_3$ where $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$ and $C_3 = \langle(123)\rangle$. $\square$

10. Let $G = S_4$. Write G into a semidirect product of 2 subgroups.

Proof. Notice that $G = V_4{:}S_3$ where $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$ and S_3 acts on V_4 by $S_3 \overset{\varphi}{\simeq} \mathrm{Aut}(V_4)$. $\square$