# Abstract Algebra
# : Lecture 9

Leo

2024.10.22

**Theorem 1.** *$A_n$ is a simple group when $n \geqslant 5$.*

**Remark 2.** *Let $G = \mathrm{Sym}(\Omega) = S_n$, if $|\Omega| = n$.*

- *each element of $G$ is a product of transpositions.* ✓.
- *each even permutation of $\Omega$ is a product of some 3-cycles.*

证明. Let $N \lhd A_n$, s.t. $N \neq \{e\}$. We aim to prove $N = A_n$.

Claim 1: $N$ cotains a 3-cycle. 拥有最多的不动点 $x(w) = w.$

Let $e \neq g \in N$ s.t. $|\mathrm{Fix}(g)| \geqslant |\mathrm{Fix}(x)|$ for all $x \in A_n$. In which $\mathrm{Fix}(x) = \{\omega \in \Omega | \omega^x = \omega,\ x \in S_n\}$ called the set of fixed points of $x$.

**Remark 3.** *Since $g$ is a even permutation, $|\mathrm{Fix}(g)| \leqslant n - 3$.*

If $|\mathrm{Fix}(g)| = n - 3$ then $g$ is a 3-cycle as claimed.

To complete the proof of the claim, we assume that $|\mathrm{Fix}(g)| < n - 3$. Then relabling if necessary, we may write $g = \begin{cases} (12345\dots)\dots \\ (1234)(5\dots)\dots \\ (123)(45\dots)\dots \\ (12)(34)(5\dots)\dots \end{cases}$ or $g = (12)(34)$ which means that we write $g$ into a product of disjoint cycles and put the longest cylce in the first place, and relabling if necessary. ✓. $g^{-1} 1 g.$

If $\mathrm{Fix}(g) \cap \{1, 2, 3, 4, 5\} = \varnothing$ and so $\mathrm{Fix}(g) \subseteq \{6, 7, \dots, n\}$. In particular, $1^g = 2$. $g(1) = 2$.

Let $\sigma = (345)$ and $h = [\sigma, g]$. $\sigma^{-1} g^{-1} \sigma g.$ Then $h \in N$ and $1^\sigma = 2^{g^{-1}} = 1$. So $1 \in \mathrm{Fix}(h)$ and $h$ fixes each point in $\mathrm{Fix}(g)$. Thus $|\mathrm{Fix}(h)| \geqslant |\mathrm{Fix}(g)| + 1$, and $h = 1$ by our assumption. Then $[\sigma, g] = 1 \Rightarrow \sigma g = g\sigma$ it is obviously wrong, just need to consider the result of conjugacy action of $\sigma$.

This is a contradiction, so $g$ is not a product of two disjoint cycles. It is just a 3-cycle as we claimed.

(or: contra. with $g$ has the most # of fixed points in $G$.)

**Remark 4.** *If $g = (12)(34)$ then $h = (354)$ also has more fixed points.*

**Lemma 5.** *Let $g = (ijk\dots)$, $\sigma \in S_n$, $g^\sigma = (i^\sigma j^\sigma k^\sigma \dots)$.* ✓

$\cancel{\alpha(i_1 i_2 \cdots i_k)\alpha^{-1} = (\alpha(i_1)\ \alpha(i_2) \cdots \alpha(i_t))}$.

$\alpha^{-1}(i_1\, i_2 \cdots i_t)\alpha = (\alpha(i_1)\ \alpha(i_2) \cdots \alpha(i_t))$

$\alpha^{-1}(i_1\, i_2 \cdots i_t)\alpha$ group action.

$(\alpha(i_j))$ $= (i_j)^{(i_1 \cdots i_t)\alpha} = (i_{j+1})^{\alpha} = \alpha(i_{j+1})$

**Remark 6.** *1.* $|\mathrm{Fix}(g)| = n - 3 \Leftrightarrow g$ *is a 3-cycle.*

*2. As $g$ is even, if $g$ is not a 3-cycle, then either $g$ moves at least 5 points or $g$ is a product of two disjoint transpositions.*

for $u \notin \{\alpha(i_1), \cdots, \alpha(i_t)\}$

$\alpha^{-1}(u) \notin \{i_1, \cdots, i_t\} \Rightarrow u^{\alpha^{-1}(i_1 \cdots i_t)\alpha} = \alpha^{-1}(u)^{(i_1 \cdots i_t)\alpha} = \alpha^{-1}(u)^{\alpha} = u.$ $\Rightarrow u$ is a fixed point.

Claim 2: $N$ contains all 3-cycles.

**Remark 7.** *each 3-cycle is conjugate to (123) by even permutations. Let $(ijk)$ be an arbitrary 3-cycle. Then there exists $x \in S_n$ such that $(ijk)^x = (123)$.*

*1. If $\{i, j, k\} \cap \{1, 2, 3\} = \emptyset$, then let $x = (1i)(2k)(3j)(23)$. Then $(ijk)^x = (123)$.*

Or.

*2. Assume $|\{i, j, k\} \cap \{1, 2, 3\}| = 1$. WLOG, $i = 1$ and $\{j, k\} \cap \{2, 3\} = \emptyset$. Let $x = (2j)(3k)$. Then $(ijk)^x = (123)$.*

*3. Assume $|\{i, j, k\} \cap \{1, 2, 3\}| = 2$. WLOG, $i = 1$ , $j = 2$ and $k = 4$. Let $x = (k35)$.*

That means all 3-cycles are in $N$ since $N \lhd A_n$, and each element fo $A_n$ is a product of 3-cycles so $N = A_n$.

**Remark 8.** *If $G$ is simple then for all $e \neq g \in G$, the normal closure $g^G = G$ where $g^G = \langle g^x | x \in G \rangle$.*

Permutation Group

由双射组成的group.

**Definition 9.** *For a set $\Omega$, each subgroup of* $\mathrm{Sym}(\Omega)$ *is called a permutation group on $\Omega$.*

**Theorem 10.** *(Cayley) Every finite group is isomorphic to ~~a permutation group~~.* subgp of

因为置换就是：一個双射！

证明. Let $G$ be a group, $G = \{g_1, g_2, \ldots g_n\} = \Omega$. For each element $x \in G$ define a permutation on $\Omega$ by $\hat{x} : g_i \mapsto g_i x$. Then $\hat{x} \in \mathrm{Sym}(\Omega)$ (the proof of $\hat{x}$ is a bijection is easy to check). $\to G \lesssim \mathrm{Sym}(\Omega)$

Let $\hat{G} = \{\hat{x} | x \in G\}$ then $\hat{G}$ is a group which isomorphic to $G$ since $\hat{x}\hat{y} = \widehat{xy}$. Also $\hat{G} \leqslant \mathrm{Sym}(\Omega)$ □

$\hat{x}\hat{y}: g_i \mapsto g_i x y$ (act as a group action?)

$\widehat{xy}: g_i \mapsto g_i xy.$

For any $g_i, g_j \in G$ there exists $\hat{x} \in \hat{G}$ s.t. $g_i^{\hat{x}} = g_j$. So $\hat{G}$ is transitive on $\Omega$. And such $\hat{x}$ is unique. So $\hat{G}$ is regular on $\Omega$ which means only the identity permutation fixes some point of $\Omega$.

下面证明 $H$ 含有所有的 3 轮换. 无妨设~~上述的~~ $\tau = (1\ 2\ 3)$. 对于任一 $(i\ j\ k) \in A_n$, 取

$$\nu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ i & j & k & x_4 & x_5 & \cdots & x_n \end{pmatrix},$$

其中 $(x_4\ x_5\ \cdots\ x_n)$ 是 $\{1, 2, \cdots, n\} \setminus \{i, j, k\}$ 的一个适当的排列, 使得 $\nu \in A_n$. 这样适当的排列是可以取到的, 原因是: 如果 $\nu$ 是奇置换, 将 $\nu$ 的表达式中的 $x_4$ 和 $x_5$ 对调, 即令

$$\nu' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ i & j & k & x_5 & x_4 & \cdots & x_n \end{pmatrix},$$

则 $\nu' = (x_4\ x_5)\nu$ 是偶置换. 由引理 1.10 即知 $(i\ j\ k) = \nu\tau\nu^{-1} \in H$.

这就完成了定理的证明. □