Solutions to Problems 61-70

61. As a statistician, you want to determine whether a coin is fair. You tossed the coin n times and record the 0-1 outcomes as $\{Y_i\}_{i=1}^{n}$ and modeled the population distribution as $Y_1 \sim$Bernoulli(θ) where $\theta = \mathbb{P}(Y_1 = 1)$. You started with the belief that the coin *is* fair. In other words, your hypothesis is

$$H_0: \theta = 0.5$$

Then you selected a sufficient statistic

$$T = \sum_{i=1}^{n} Y_i$$

and you know that T follows the distribution $Binomial(n, \theta)$. Specifically under H_0, T follows the distribution: $Binomial(n, 0.5)$. Finally, you compute the p-value of T's realization as

$$pvalue = \sum_{k \in C} \binom{n}{k} 0.5^n$$

where

$$C = \left\{k: \binom{n}{k} \leq \binom{n}{t}\right\}.$$

In an alternative approach you specified the critical region with size 5% as

$$\mathbb{C}(T, H_0, 5\%) = \{0,1, \dots, \lfloor BinomialQuantile(2.5\%; n, 0.5) \rfloor\} \cup \{\lceil BinomialQuantile(97.5\%; n, 0.5) \rceil, \dots, n-1, n\}$$

and the critical region's power function is expressed as

$$p(\theta, \mathbb{C}) = \sum_{t=0}^{q_1} \binom{n}{t} \theta^t (1-\theta)^{n-t} + \sum_{t=q_2}^{n} \binom{n}{t} \theta^t (1-\theta)^{n-t}$$

where the integers $q_1 = \lfloor BinomialQuantile(2.5\%; n, 0.5) \rfloor$ and $q_2 = \lceil BinomialQuantile(97.5\%; n, 0.5) \rceil$.

62. As a scientist you collected the following data recording the 7 repetitions of measurements of the same quantity:

$$\{y_i\} = \{22.1, \quad 20.3, \quad 19.7, \quad 21.1, \quad 18.9, \quad 19.8, \quad 21.2\}$$

You believe that Y_1 follows a normal distribution with mean μ and variance σ^2. You started with the hypothesis that the mean is 20, that is,

$$H_0: \mu = 20.$$

Then you selected a sufficient statistic $m = \bar{Y}$ and you know that a simple algebraic transformation of m into a pivot

$$T(\mu) = \frac{\bar{Y} - \mu}{S/\sqrt{7}} \sim t(6) \text{ where } S = 1.09.$$

Specifically under H_0, the pivot becomes a statistic,

$$T(\mu = 20) = \frac{\bar{Y} - 20}{1.09/\sqrt{7}} \sim t(6).$$

Finally, you compute the T's realization (under H_0) =1.076 and its

$$\text{pvalue} = 0.323$$

which leads to the `acceptance` of H_0 at 5% significance level. Alternatively, you specified the critical region with size 5% as

$$\mathbb{C}(T, H_0, 5\%) = (-\infty, -2.447] \cup [+2.447, +\infty)$$

which `excludes` the realization of T, leading to the `acceptance` of H_0 at 5% significance level.

63. As a financial analyst you collected the following monthly log-return data of the past year on the same index:

$$\{y_i\} = \{0.41, \quad 0.21, \quad -0.04, \quad 0.03, \quad -0.54, \quad 0.19, \quad 0.31, \quad -0.08, \quad 0.35, \quad 0.04, \quad 0.68, \quad -0.20\}.$$

You assume that Y_1 follows a normal distribution with mean μ and variance σ^2, where the standard deviation σ is an important indicator of the volatility of the market. You started with the hypothesis that the standard deviation is 20%, that is, $H_0: \sigma = 0.2$. Then you selected a pivot

$$T(\sigma) = \frac{11S^2}{\sigma^2} \sim \chi^2(11).$$

Specifically under H_0, the pivot becomes a statistic

$$T(\sigma = 0.2) = \frac{11S^2}{0.04} \sim \chi^2(11).$$

Finally, you compute T's realization (under H_0) = 27.93 and its

$$pvalue = 0.00367$$

leads to the `rejection` of H_0 at 5% significance level. Alternatively, you specified the critical region with size 5% as $\mathbb{C}(T, H_0, 5\%) = [0,\ 3.816] \cup [21.920, \infty)$ which `includes` the realization of T, leading to the `rejection` of H_0 at 5% significance level.

64. Two samples of data are collected: $\{x_i\}$ ={178.6, 185.3, 179.5, 175.1, 189.7} and $\{y_j\}$ ={160.8, 165.2, 168.3, 170.2, 177.5, 162.9, 164.5, 167.2, 178.1}. As a statistician, you modeled them as both normal: $X_i \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$ and $Y_i \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$. You believe that the two populations have equal variance and you this belief: $H_0: \sigma_1^2 = \sigma_2^2$. You then select a pivot $T(\sigma_1^2, \sigma_2^2) = \frac{S_1^2 \sigma_2^2}{S_2^2 \sigma_1^2} \sim F(4,8)$. Specifically under H_0, $T(\sigma_1^2 = \sigma_2^2) = \frac{S_1^2}{S_2^2} \sim F(4,8)$. Finally, you compute T's realization (under H_0) = 0.915 and its p-value=0.5402 which leads to the `acceptance` of H_0 at 5% significance level. Alternatively, you specified the critical region with size 5% as $\mathbb{C}(T, H_0, 5\%) = [0, 0.111] \cup [5.053, \infty)$ which `excludes` the realization of T, leading to the `acceptance` of H_0 at 5% significance level.

65. With the same data and model as in 61, construct a 95% confidence interval for θ.

Solution. We shall use the same sufficient statistic $T = \sum_{i=1}^{n} Y_i$ to construct the pivot. For binomial(n,θ), the normal approximation exists with good quality. Therefore we can safely write $T \sim N(n\theta, n\theta(1-\theta))$ with $n > 30$. The pivot is thus $\frac{T-n\theta}{\sqrt{n\theta(1-\theta)}} \sim N(0,1)$. The pivot inequality for constructing the 95% confidence interval for θ is $\left|\frac{T-n\theta}{\sqrt{n\theta(1-\theta)}}\right| \leq 1.96$ ➔ $(n^2 + 1.96^2 n)\theta^2 - (2nT + 1.96^2 n)\theta + T^2 \leq 0$; The solution set of this quadratic inequality is the 95% confidence interval for θ.

66. With the same data and model as in 62, construct a 95% confidence interval for μ.

Solution. We shall use the same pivot $T = \frac{\bar{Y}-\mu}{S/\sqrt{7}} \sim t(6)$. The pivot inequality for constructing 95% CI for μ is $\left|\frac{\bar{Y}-\mu}{S/\sqrt{7}}\right| \leq 2.447$ with $\bar{Y} = 1.09$ and $S = 1.09$ ➔ $19.44 \leq \mu \leq 21.45$.

67. With the same data and model as in 63, construct a 95% confidence interval for σ.

Solution. We shall use the same pivot $T = \frac{11S^2}{\sigma^2} \sim \chi^2(11)$. The pivot inequality for constructing 95% CI for σ is $3.816 \leq \left|\frac{11S^2}{\sigma^2}\right| \leq 21.92$ with $S^2 = 0.1016$ ➔ $0.051 \leq \sigma^2 \leq 0.293$ ➔ $0.2258 \leq \sigma \leq 0.5412$.

68. With the same data and model as in 64, construct a 95% confidence interval for σ_1^2/σ_2^2 .

Solution. We shall use the same pivot $T = \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} \sim F(4,8)$. The pivot inequality for constructing 95% CI for σ_1^2/σ_2^2 is $0.1114 \leq \left|\frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2}\right| \leq 5.053$ with $\frac{S_1^2}{S_2^2} = 0.915$ ➔ $0.1217 \leq 0.181 \leq \frac{\sigma_1^2}{\sigma_2^2} \leq 8.213$.

69. An examination was given to two classes consisting of 40 and 50 students, respectively. In the first class the mean grade was 74 with a standard deviation of 8, while in the second class the mean grade was 78 with a standard deviation of 7. Is there a significant difference between the performance of the two classes at a level of significance of (a) 0.05, (b) 0.01? What is the p-value of the test?

Solution. Suppose two classes come from two populations having the respective means μ_1 and μ_2. The H_0: $\mu_1 = \mu_2$. Since the examination scores are essentially binominal with large number of trails, we can use the normal approximation. The two class sizes are large enough to allow for a simple z-test to be used. The test statistic is clearly the difference of the sample means: $T = \bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right) = N(\mu_1 - \mu_2, 2.58)$. Specifically under H_0, $T \sim N(0,2.48)$. The realization of T is $74 - 78 = -4$, standardized to $z = \frac{74-78}{\sqrt{2.58}} = -2.49$.

For (a): significance=0.05; the acceptance interval for the standardized T is (-1.96, 1.96) and -2.49 is not in the acceptance interval, hence the test is rejected at 0.05 level.

For (b): significance=0.01; the acceptance interval for the standardized T is (-2.58, 2.58) and -2.49 is inside the acceptance interval, hence the test is accepted at 0.01 level.

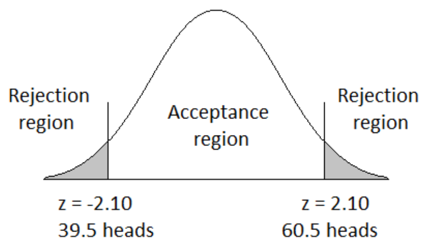
The p-value for T is the same as that of the standardized T, which is equal to $\Phi(-2.49) * 2 = 0.013$ ➔ accept at 0.01 significance level but reject at 0.05.

70. To test the hypothesis that a coin is fair, the following decision rules are adopted: Accept the hypothesis if the number of heads in a single sample of 100 tosses is between 40 and 60 inclusive and reject the hypothesis otherwise. (1) Find the probability of rejecting the hypothesis when it is actually correct; (2) Interpret graphically the decision rule and the result of part (1); (3) What conclusions would you draw if the sample of 100 tosses yielded 53 head? 60 head? (4) Could you be wrong in your conclusions to (c)? Explain; (5) What is the probability of accepting the hypothesis that the coin is fair when the actual probability of heads is $p = 0.7$? 0.6? 0.8? 0.9? (6) Construct the graph of the power function for the test.

Solution.

(1) The sample size is large enough to use normal approximation to binomial. $\mu = np = 50$, $\sigma^2 = npq = 25$. Therefore the approximating normal distribution is $N(50,25)$. Thus the type-1 error is approximately $\Phi\left(\frac{39.5-50}{5}\right) * 2 = 0.0357$.

(2) The graphical illustration is the following:



(3) Both 53 and 60 are within the acceptance region [40,60], hence we accept the hypothesis.

(4) Yes. We could accept the hypothesis when it actually should be rejected, as would be the case, for example, when the probability of heads is actually 0.7 instead of 0.5. This is an instance of the type-2 error.

(5) If $p = 0.7$, then the distribution of heads in 100 tosses is approximated by $N(70, 21)$. The type-2 error is approximately $\Phi\left(\frac{60.5-70}{\sqrt{21}}\right) - \Phi\left(\frac{39.5-70}{\sqrt{21}}\right) \approx \Phi\left(\frac{60.5-70}{\sqrt{21}}\right) = 0.019$. Therefore, with the given decision rule there is very little chance of accepting the hypothesis that the coin is fair when actually $p = 0.7$.

If $p = 0.6$, the approximating normal distribution is $N(60, 24)$. The type-2 error is approximately $\Phi\left(\frac{60.5-60}{\sqrt{24}}\right) = 0.541$ → accept

If $p = 0.8$, the approximating normal distribution is $N(80,16)$. The type-2 error is approximately $\Phi\left(\frac{60.5-80}{4}\right) = 0.000$ → reject

If $p = 0.9$, The type-2 error is approximately 0.000 → reject.

(6) The type-2 error as a function of p is symmetric about $p = 0.5$. The power as a function of p is equal to 1 minus the type-2 error function, thus also symmetric about $p = 0.5$.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$power\ at\ p$	1	1	0.9808	0.459	0.357	0.459	0.9808	1	1