Discrete Mathematics for Computer Science

Lecture 19: Graph

Dr. Ming Tang

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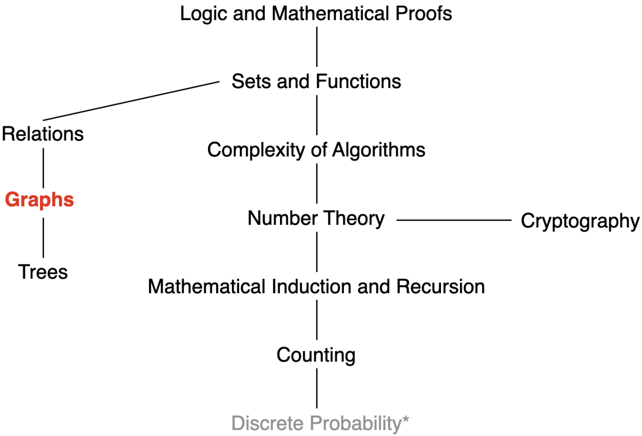
This Lecture



Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamilton path, ...

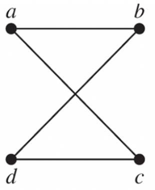
Counting Paths between Vertices

Theorem: Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)-th entry of $\mathbf{A}^r$.

Note: with directed or undirected edges, multiple edges and loops allowed

Counting Paths between Vertices:

How many paths of length 4 are there from a to d in the graph G?



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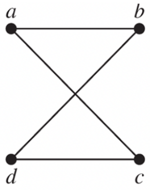

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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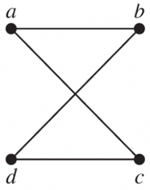

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

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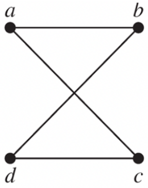

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a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d;
a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; a, c, d, c, d

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Proof (by induction):

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- Inductive Step: $\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$. The (i, j)-th entry of $\mathbf{A}^{r+1}$ equals

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{ik}a_{kj} + \cdots b_{in}a_{nj}.$$

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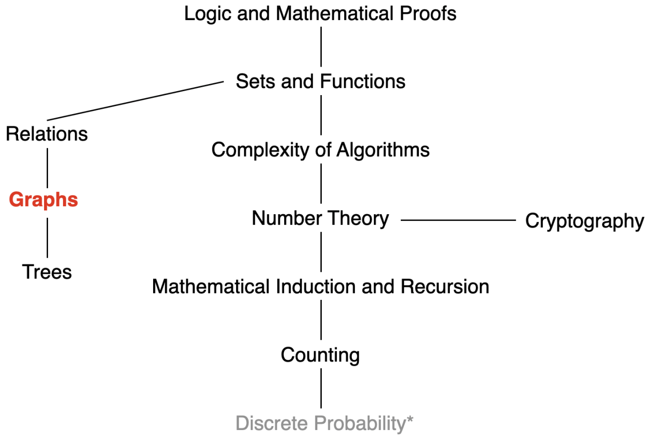
  - $b_{ik}a_{kj}$: the number of paths from i to j with k as the interior point of length $r + 1$.

This Lecture



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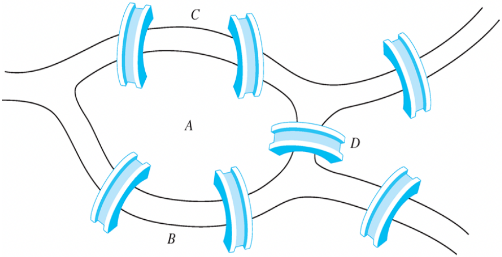
Euler Paths

Königsberg seven-bridge problem: People wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



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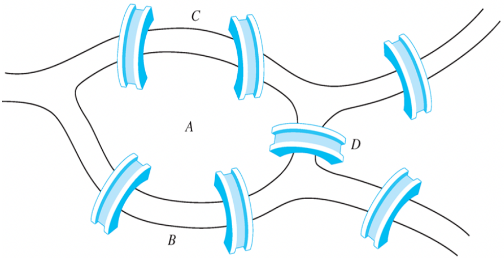

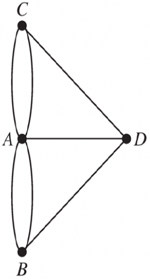

Euler Paths and Circuits

Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

Recall that a path or circuit is simple if it does not contain the same edge more than once.

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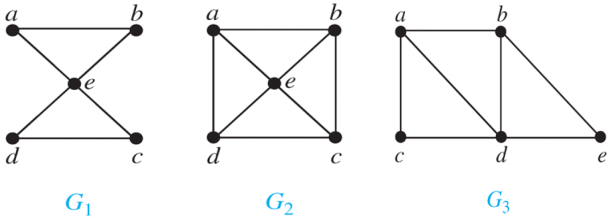

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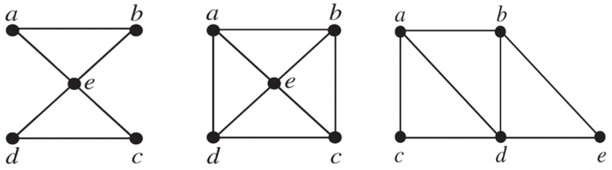

G_1 G_2 G_3

G_1: an Euler circuit, e.g., a, e, c, d, e, b, a;
G_2: neither; G_3: an Euler path, e.g., a, c, d, e, b, d, a, b

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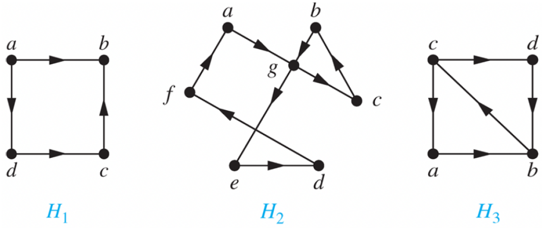

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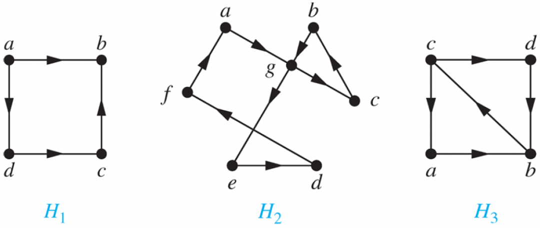

H_1: neither; H_2: an Euler circuit, e.g., a, g, c, b, g, e, d, f, a; H_3: an Euler path, e.g., c, a, b, c, d, b

Necessary Conditions for Euler Circuits and Paths

Consider undirected graph:

Euler Circuit $\Rightarrow$ The degree of every vertex must be even

- Each time the circuit passes through a vertex, it contributes two to the vertex's degree.
- The circuit starts with a vertex a and ends at a, then contributes two to $deg(a)$.

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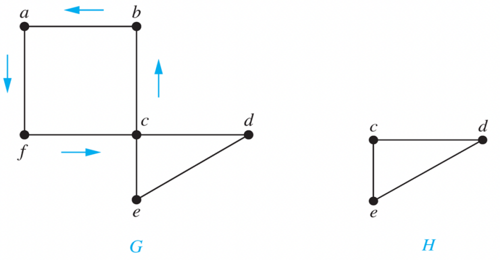
Are these conditions also sufficient?

Sufficient Conditions for Euler Circuits and Paths

G is a connected multigraph with ≥ 2 vertices, all of even degree.



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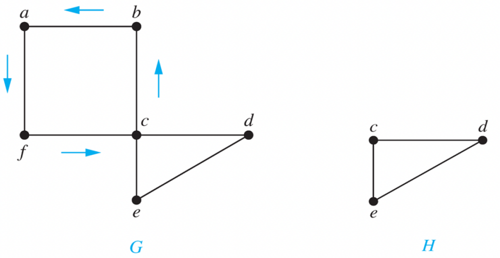

We will form a simple circuit that begins at an arbitrary vertex a of G, building it edge by edge.

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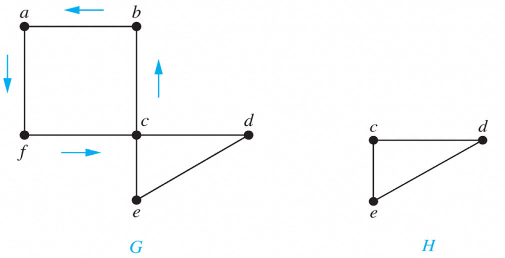

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The path begins at a, and it must terminate at a. This is because every time we enter a vertex other than a, we can leave it.

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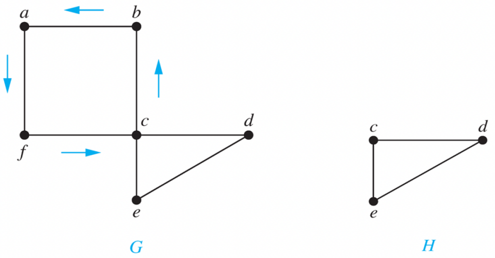

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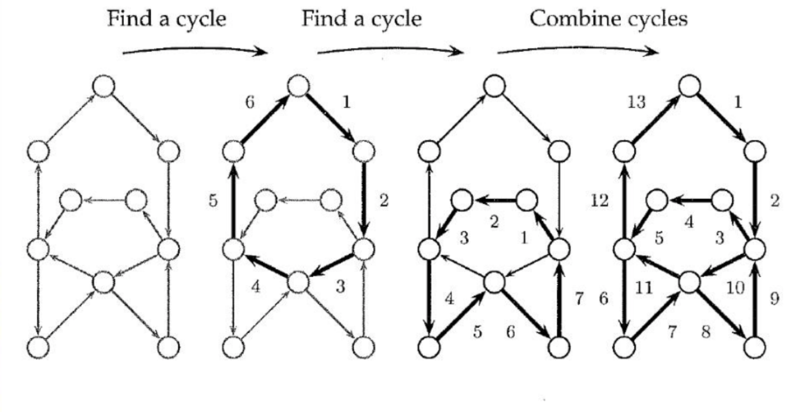
An Euler circuit has been constructed if all the edges have been used. Otherwise, consider the subgraph H obtained from G by deleting the edges already used. Every vertex in H has even degree ...

Algorithm for Constructing an Euler Circuit

ALGORITHM 1 Constructing Euler Circuits.

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
H := G with the edges of this circuit removed
while H has edges
    subcircuit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge of *circuit*
    H := H with edges of *subcircuit* and all isolated vertices removed
    circuit := *circuit* with *subcircuit* inserted at the appropriate vertex
return *circuit* {*circuit* is an Euler circuit}

Algorithm for Constructing an Euler Circuit



Euler Circuits and Paths

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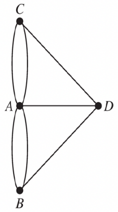

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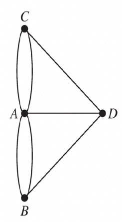
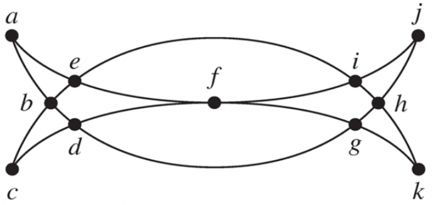


No Euler circuit, no Euler path

Euler Circuits and Paths: Example



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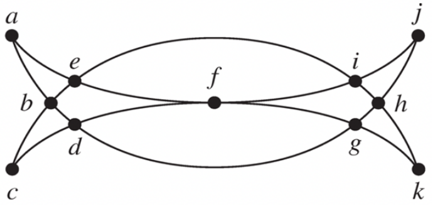

It has such a circuit because all its vertices have even degree.

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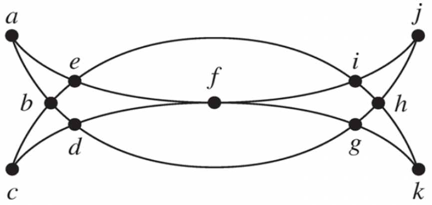

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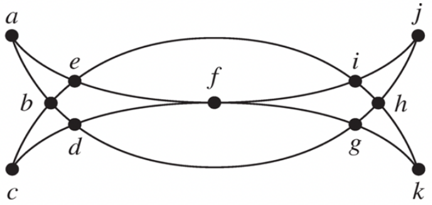

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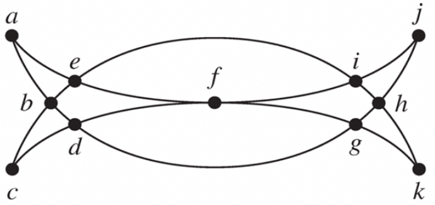

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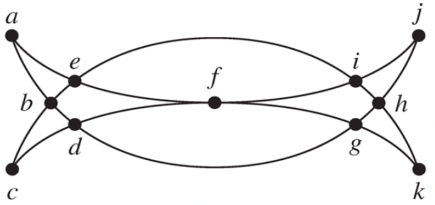

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- Form the circuit d, g, h, j, i, h, k, g, f, d in *H*;

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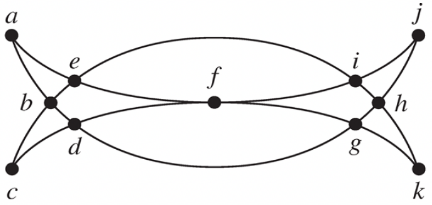

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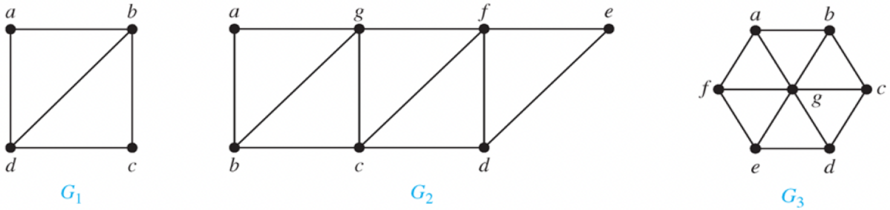
- Form the circuit a, b, d, c, b, e, i, f, e, a;
- Obtain the subgraph H by deleting the edges in this circuit and all vertices that become isolated;
- Form the circuit d, g, h, j, i, h, k, g, f, d in H;
- Splice this new circuit into the first circuit at the appropriate place produces the Euler circuit
  a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a.

Euler Circuits and Paths: Example



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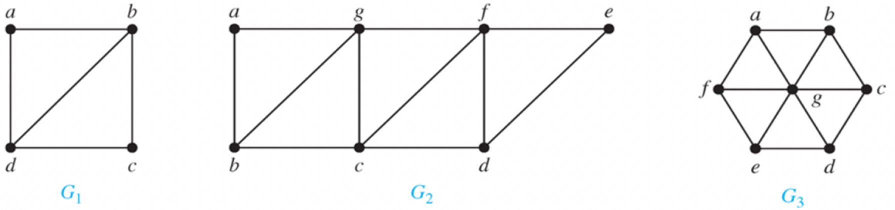

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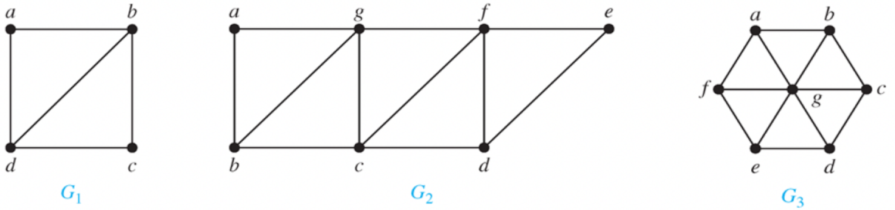

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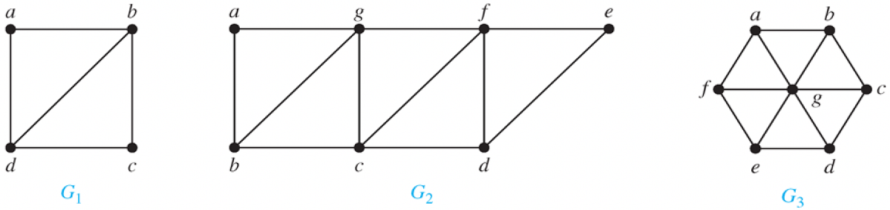

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- G_2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints.
- G_3 has no Euler path because it has six vertices of odd degree.

Applications of Euler Paths and Circuits

Finding a path or circuit that traverses each

- street in a neighborhood
- road in a transportation network
- link in a communication network
- ...

Hamilton Paths and Circuits

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What about containing every vertex exactly once?

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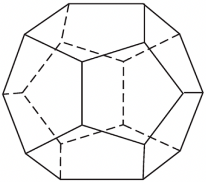
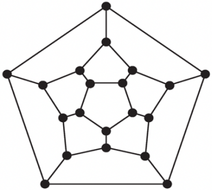


(a)

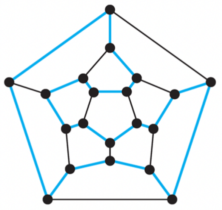


(b)

Hamilton Paths and Circuits

Euler paths and circuits contained every edge only once.

What about containing every vertex exactly once?



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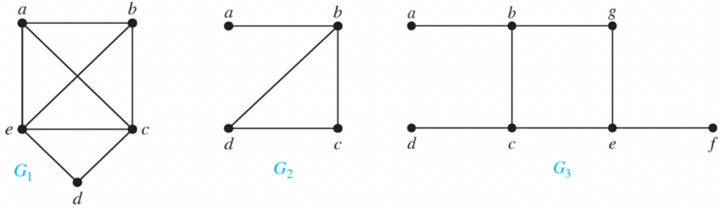

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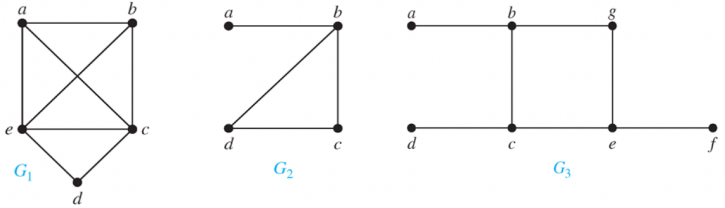

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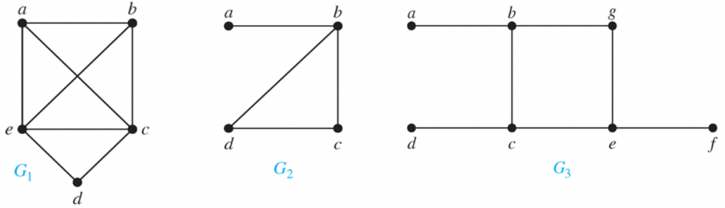

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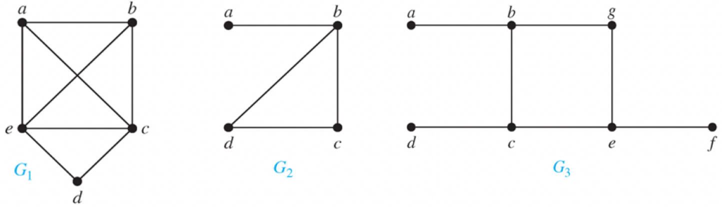

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- G_3 has neither, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

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Hamilton path problem $\in$ NP-Complete

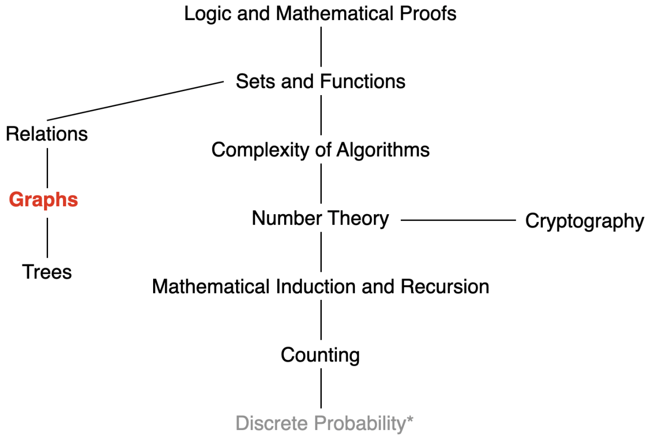
Applications of Hamilton Paths and Circuits

A path or a circuit that visits each city, or each node in a communication network exactly once, can be solved by finding a Hamilton path.

Traveling Salesperson Problem (TSP) asks for the shortest route a traveling salesperson should take to visit a set of cities.

the decision version of the TSP $\in$ NP-Complete

This Lecture



Graph and terminologies, representing graphs and graph isomorphism, connectivity, Euler and Hamiliton path, shortest-path problem ...

Shortest Path Problems

Using graphs with weights assigned to their edges

Such graphs are called weighted graphs and can model lots of questions involving distance, time consuming, fares, etc.

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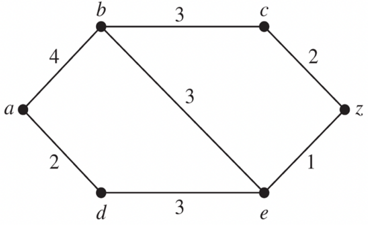

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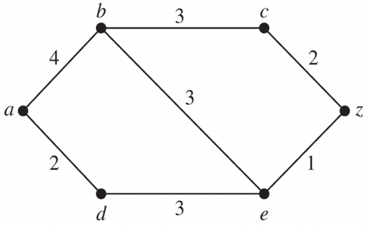


What is the length of a shortest path between a and z?

Dijkstra's Algorithm

S: a distinguished set of vertices;
$L(v)$: the length of a shortest path from a to v that contains only the vertices in S as the interior vertices.

(i) Set $L(a) = 0$ and $L(v) = \infty$ for all v, $S = \emptyset$

(ii) While $z \notin S$

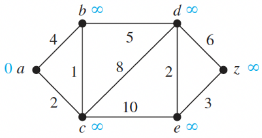
$u :=$ a vertex not in S with $L(u)$ minimal

$S := S \cup \{u\}$

For all vertices v not in S

$$L(v) := \min\{L(u) + w(u,v), L(v)\}$$

Dijkstra's Algorithm



$S = \emptyset$

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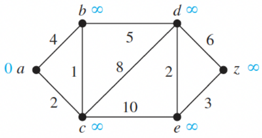
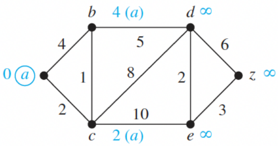
Dijkstra's Algorithm

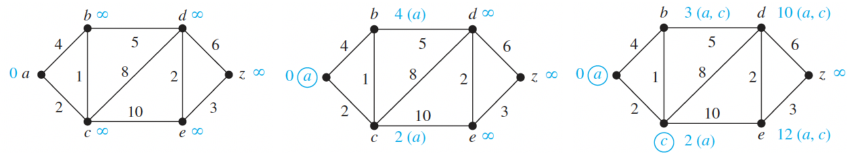




$S = \{a\}$

$L(a) = 0$, $L(b) = 4$, $L(c) = 2$, $L(d) = \infty$, $L(e) = \infty$, $L(z) = \infty$

Dijkstra's Algorithm



$S = \{a, c\}$
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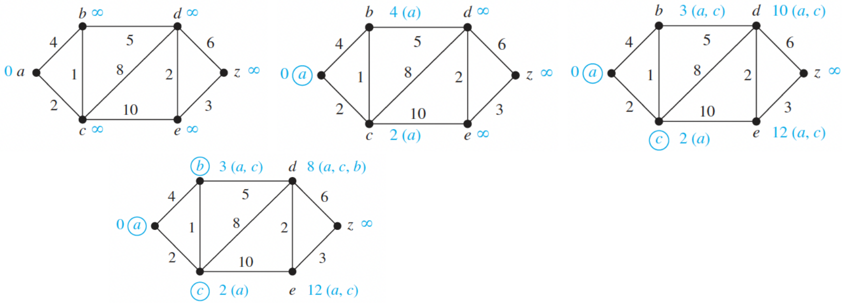

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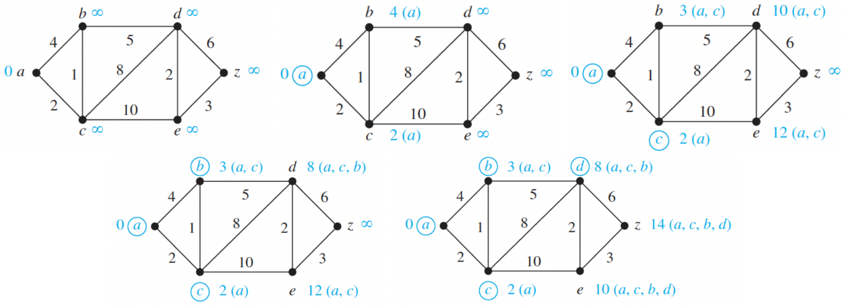

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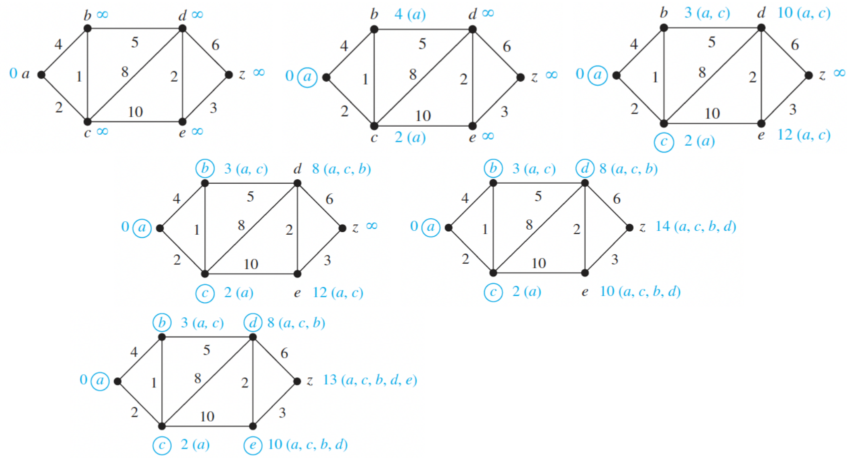

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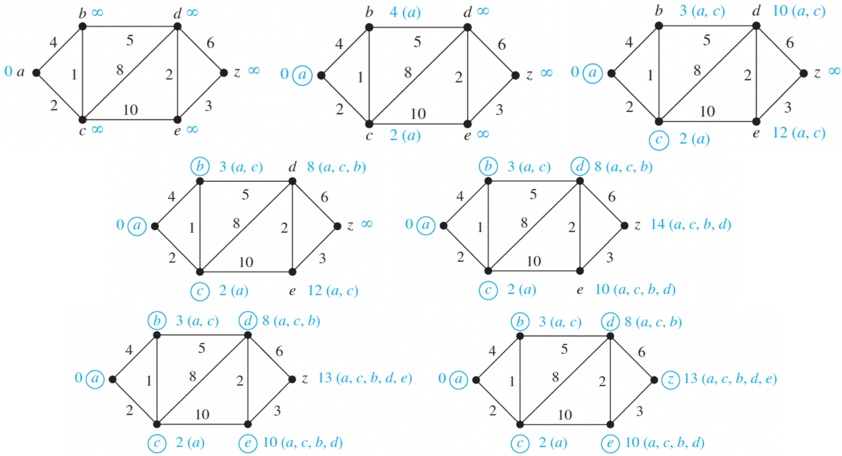

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Theorem: Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

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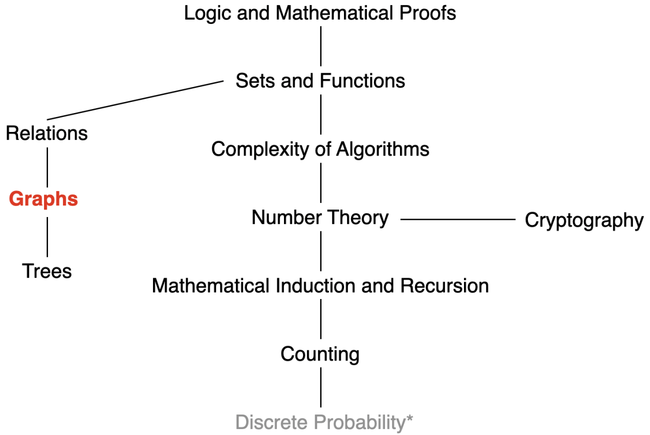
Proof by induction ... (P713 on textbook)

This Lecture



..., Euler and Hamiliton path, shortest-path problem, Planar Graphs, ...

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Join three houses to each of three separate utilities.

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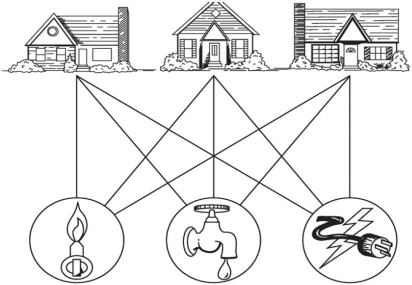

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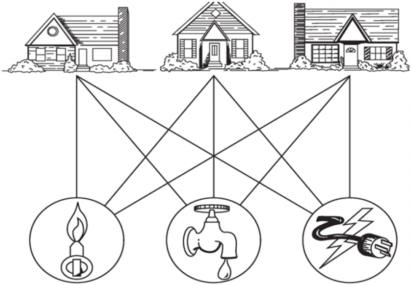

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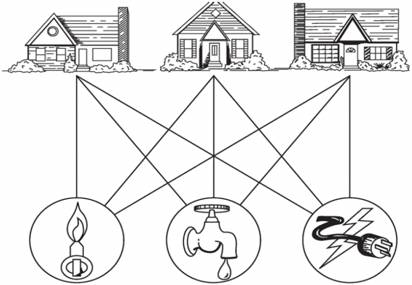

Can this graph be drawn in the plane such that no two of its edges cross?
Complete bipartite graph $K_{3,3}$

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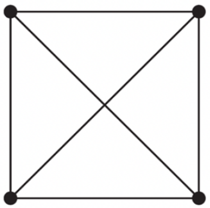

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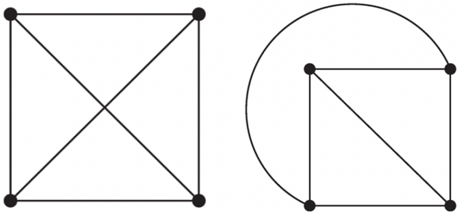
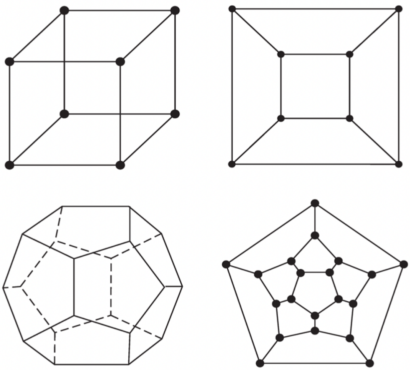


Planar Graphs: Example



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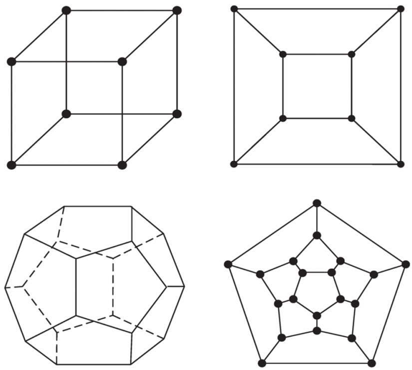

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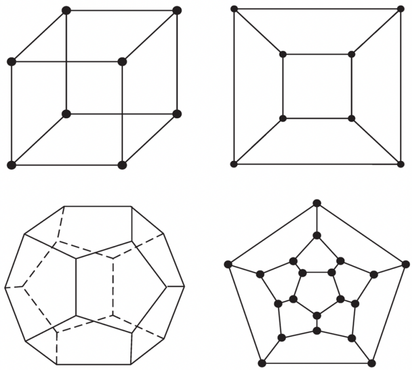

- We can show that a graph is planar by displaying a planar representation.
- It is harder to show that a graph is nonplanar.

Planar Graphs: Example

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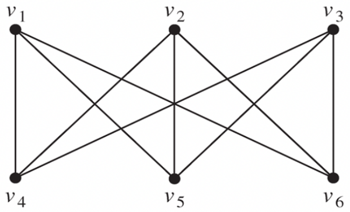

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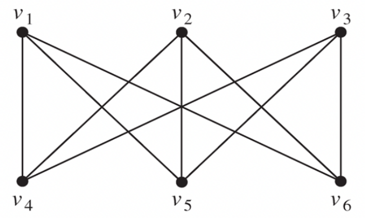

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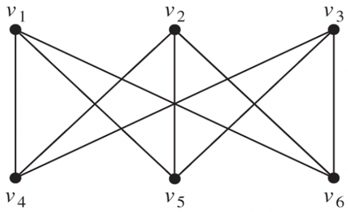

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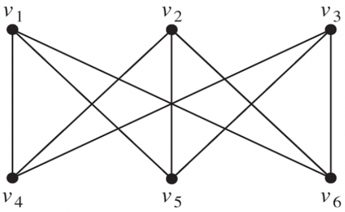

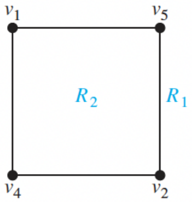

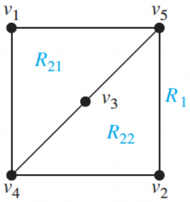

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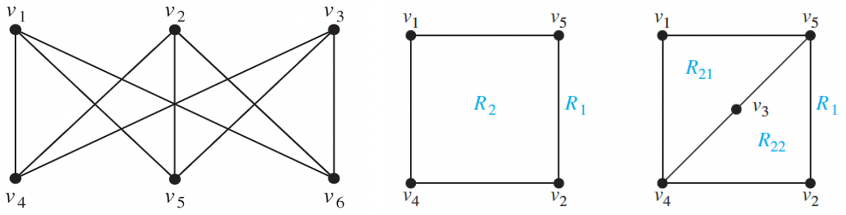

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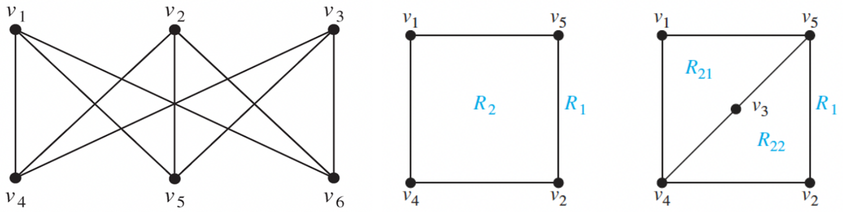

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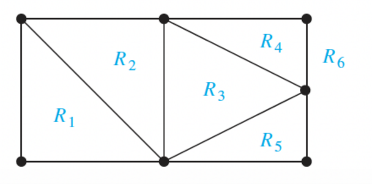
- These four edges form a closed curve that splits the plane into two regions, R_1 and R_2.
- The vertex v_3 is in either R_1 or R_2. Suppose v_3 is in R_2, there is no way to place the final vertex v6 without forcing a crossing. ....

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.



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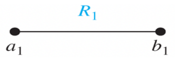

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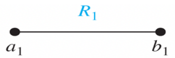

- Inductive Hypothesis: $r_k = e_k - v_k + 2$

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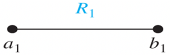
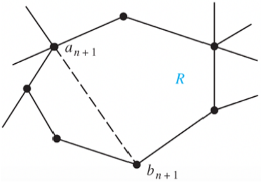
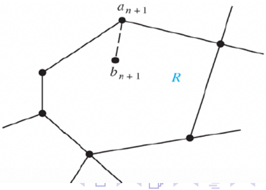


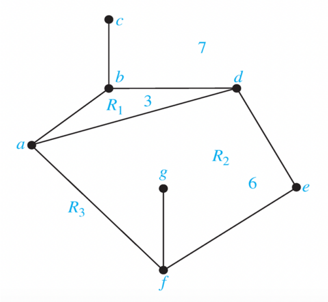
- Inductive Hypothesis: $r_k = e_k - v_k + 2$
- Inductive step: Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1}.





The Degree of Regions

Definition: The degree of a region is defined to be the number of edges on the boundary of this region. When an edge occurs twice on the boundary, it contributes two to the degree.



Corollaries

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Hence, $(2/3)e \geq r$. By Euler's formula (i.e., $r = e - v + 2$), $e \leq 3v - 6$.

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Corollary 3: In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

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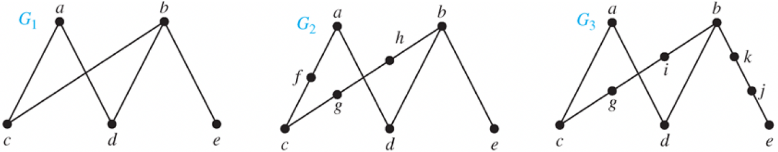
Kuratowski's Theorem

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.

The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.



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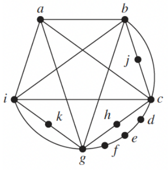

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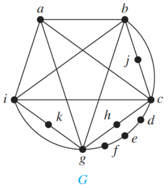

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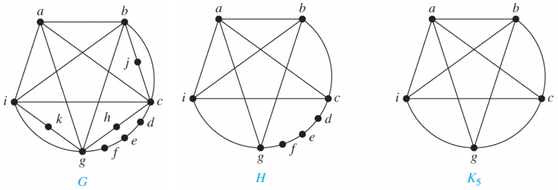

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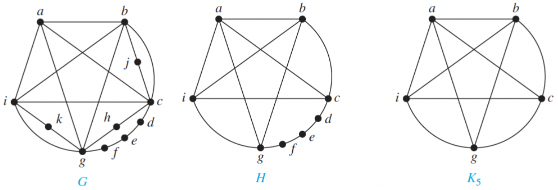

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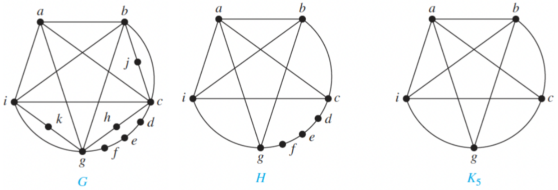

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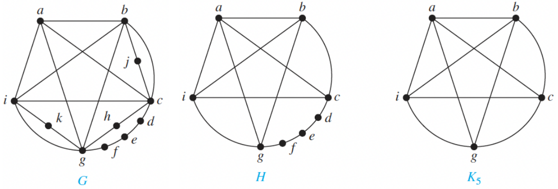

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Hence, G is nonplanar.

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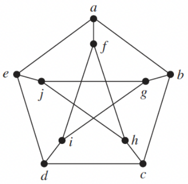

(a)

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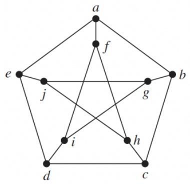

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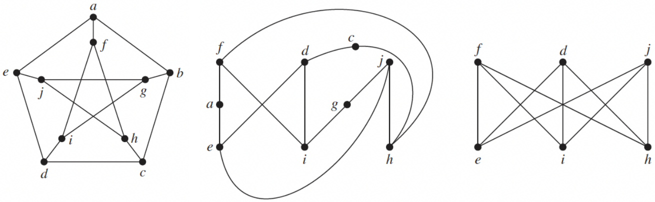

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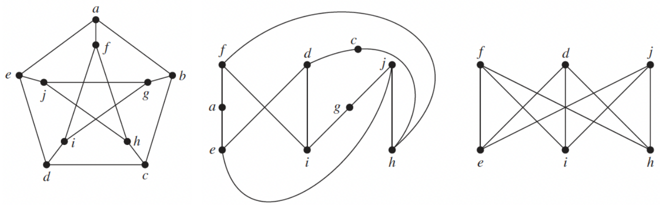

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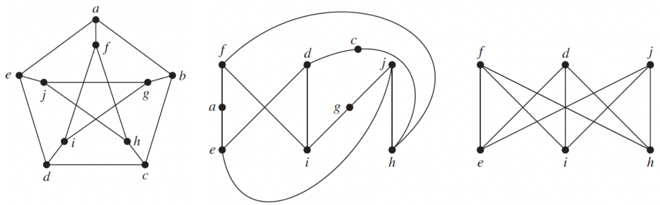

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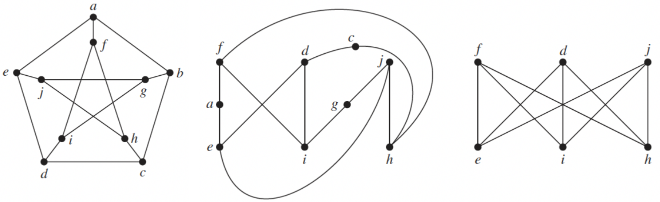

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