抽象代数H 2024.9.19.

Def ① $H \lhd G$ if H is a subgroup of G and ② $g^{-1}Hg \subseteq H$ for every $g \in G$

or ③ $g^{-1}hg \in H, \forall h \in H$

or ④ $gH = Hg$

Rmk: ①, ②, ③④ are equivalent

eg. $SL_n(\mathbb{F}) \lhd GL_n(\mathbb{F})$ (b/c $\det(g^{-1}hg) = \det(g^{-1}) \cdot \det(h) \cdot \det(g)$
$= \det(h) = 1$ for $h \in SL_n(\mathbb{F})$, $g \in GL_n(\mathbb{F})$)

Def. Let $N \lhd G$, and let $G/N := \{gN \mid g \in G\}$,

define $\cdot : G/N \times G/N \to G/N$ by $(g_1N) \cdot (g_2N) = (g_1g_2N)$.

Prop. Then $(G/N, \cdot)$ is a group, called $\boxed{\text{factor group}}$ or $\boxed{\text{quotient group}}$

Proof: "$\cdot$" is well-defined (Change the representive element by $gN = ghN$ for $h \in N$)

$((g_1N) \cdot (g_2N)) \cdot (g_3N) \overset{?}{=} (g_1N) \cdot ((g_2N) \cdot (g_3N))$

N is the identity for $(G/N, \cdot)$

(gN)'s inverse is $(g^{-1}N)$ □

eg. $|GL_n(\mathbb{F}_p)/SL_n(\mathbb{F}_p)| = p-1$, $GL_n(\mathbb{F}_p)/SL_n(\mathbb{F}_p) \cong C_{p-1}$ for prime p.

for $g \in GL_n(\mathbb{F}_p)$, $g = g_1 h$, $\det(h) = 1$, $g_1 = \begin{pmatrix} a & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$, $a = \det(g) \neq 0$.

Easily, $\left\{ \begin{bmatrix} a & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix} \middle| a \in \mathbb{F}_p^{\times} \right\} < GL_n(\mathbb{F}_p)$.

Question: how to define "essentially the same"

$V = F^3$ $\quad \{(a,0,0) \mid a \in F\}$

$\{(0,b,0) \mid b \in F\}$ $\quad \big\updownarrow$ 1-1 ϕ

$\{(a,a,0) \mid a \in F\}$

Def. Two groups G and H are said to be isomorphic if there exists a bijection $\phi: G \to H$ s.t. $\phi(g_1 \cdot g_2) = \phi(g_1) * \phi(g_2)$ for every $g_1, g_2 \in G$, ϕ is a $\boxed{\text{homomorphism}}$

eg. 1. $G := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in F^\times \right\}$, $H := \left\{ \begin{pmatrix} b & & \\ & b & \\ & & b \end{pmatrix} \mid b \in F^\times \right\}$

G, H are isomorphic: let $\phi: G \to H$

$$\begin{pmatrix} a & \\ & a \end{pmatrix} \mapsto \begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix}$$

2. $G := \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{F}_p \right\}$, $H = (\mathbb{F}_p, +)$

G, H are isomorphic: let $\phi: G \to H$

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mapsto c \quad \left(\text{b/c } \begin{pmatrix} 1 & c_1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & c_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_1+c_2 \\ & 1 \end{pmatrix}\right)$$

Def. G is homomorphic to H if a mapping $\phi: G \to H$ s.t.

$$\phi: g_1 g_2 \to g_1^\phi g_2^\phi \quad \text{or } \phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$

eg. let $G = \left\{ \begin{bmatrix} a & \\ & b \end{bmatrix} \mid a, b \in \mathbb{F} \setminus \{0\} \right\}$, $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{F} \setminus \{0\} \right\}$

$$\phi: \begin{bmatrix} a & \\ & b \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$ homomorphism.

let $N \triangleleft G$, Then G is homomorphic to G/N

$\phi: G \to G/N$ by $g \mapsto gN$, $\forall g \in G$

Def. for G, H and homomorphism ϕ, let $K = \{g \in G \mid g^{\phi} = 1\}$

Then ① $K \triangleleft G$ ② $G/K \cong H$ (only for ϕ is a surjection!)

proof: $K \leq G$ as K is closed under $\cdot$

For $x \in G$ and $h \in K$.

$x^{-1}hx \in K$ $\phi(x^{-1}hx) = (x^{-1})^{\phi} \cdot x^{\phi} = (x^{\phi})^{-1} x^{\phi} = 1.$

define $\Phi: G/K \to H$, Φ is isomorphism

$gK \mapsto g^{\phi}$

① Φ is homomorphism

② Φ is injection } bijection.

③ Φ is surjection

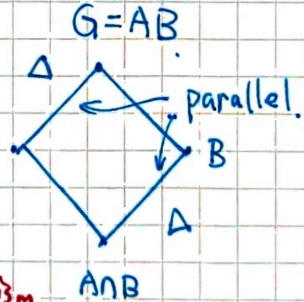
$\square$

Rmk: Thm above is called "First Theorem of Isomorphism"!

Let $G = AB = \{ab \mid a \in A, b \in B\}$

Theorem 2. $A \triangleleft G$, Then $A \cap B \triangleleft B$ and $G/A \cong B/A\cap B$.

$AB \leq G$



Proof: ① Take $x \in A \cap B$, and $g \in B$

Then $g^{-1}xg \in A, B \Rightarrow A \cap B \triangleleft B$

② Def $\varphi: B \to G/A$, φ is homomorphism.

$b \mapsto bA$.

use Thm 1 of Isomorphism: $\operatorname{Im}\varphi \cong B/\operatorname{Ker}\varphi$

$\Rightarrow G/A \cong B/A\cap B$ $\square$