Def G: group.

subnormal chain. 次正规序列.

$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_i \rhd G_{i+1} \rhd \cdots \rhd \{1\}$ where $G_{i+1} \trianglelefteq G_i$ $\forall i$.

$G_2 \lhd G_1 \lhd G$. G_2 is a <u>subnormal subgroup</u> of G.
次正规子群.

G_i is a <u>subnormal subgroup</u> of G. denoted by $G_i \lhd\lhd G$.
次正规子群.

If G_i/G_{i+1} is simple for all i, then the chain is called a <u>composition chain</u> (合成序列). And each factor group G_i/G_{i+1} is called a <u>composition factor</u> 合成因子.

If G_i/G_{i+1} is not simple,

$G_0 = S_4 \underset{\rhd A_4}{\rhd} \mathbb{Z}_2\times\mathbb{Z}_2 \underset{\rhd \mathbb{Z}_2}{\rhd} \{1\}$.

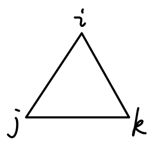
eg. ① Let $G = S_4$.

$G = G_0 \rhd \underset{G_1}{A_4} \rhd \underset{G_2}{\mathbb{Z}_2\times\mathbb{Z}_2} \rhd \underset{G_3}{\mathbb{Z}_2} \rhd \underset{G_4}{\{1\}}$. ($\mathbb{Z}_2\times\mathbb{Z}_2 \cong V_4$)

$G_0/G_1 \cong \mathbb{Z}_2$. $G_1/G_2 \cong \mathbb{Z}_3$, $G_2/G_3 \cong \mathbb{Z}_2$. $G_3/G_4 \cong \mathbb{Z}_2$.

$A_4 \rhd V_4$

$V_4 = \{\underset{1}{e}, \underset{i}{(12)(34)}, \underset{j}{(13)(24)}, \underset{k}{(14)(23)}\}$.



$i^2 = j^2 = k^2 = 1$.

② Let $G = S_5$.

$G = G_0 = S_5 \rhd \underset{G_1}{A_5} \rhd \underset{G_2}{\{1\}}$.
G_3.

$G_0/G_1 \cong \mathbb{Z}_2$. $G_2/G_3 \cong A_5$.

Thm. The number and the set of composition factors of a finite group is <u>uniquely</u> determined by G.

eg. Let $G = S_4 \times S_5$.

$G = G_0 = S_4\times S_5 \rhd \underset{G_1}{A_4\times S_5} \rhd \underset{G_2}{A_4\times A_5} \rhd \underset{G_3}{A_4} \rhd \underset{G_4}{\mathbb{Z}_2\times\mathbb{Z}_2} \rhd \underset{G_5}{\mathbb{Z}_2} \rhd \underset{G_6}{\{1\}}$.

$G_0/G_1 \cong \mathbb{Z}_2$ $G_1/G_2 \cong \mathbb{Z}_2$ $G_2/G_3 \cong A_5$ $G_3/G_4 \cong \mathbb{Z}_3$ $G_4/G_5 \cong \mathbb{Z}_2$ $G_5/G_6 = \mathbb{Z}_2$.

set of composition factors $= \{\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, A_5\}$.

$G = G_0 = S_4\times S_5 \rhd \underset{G_1}{S_4\times A_5} \rhd \underset{G_2}{S_4} \rhd \underset{G_3}{A_4} \rhd \underset{G_4}{\mathbb{Z}_2\times\mathbb{Z}_2} \rhd \underset{G_5}{\mathbb{Z}_2} \rhd \underset{G_6}{\{1\}}$.

$G_0/G_1 \cong \mathbb{Z}_2$ $G_1/G_2 \cong A_5$ $G_2/G_3 \cong \mathbb{Z}_2$ $G_3/G_4 \cong \mathbb{Z}_3$ $G_4/G_5 \cong \mathbb{Z}_2$ $G_5/G_6 \cong \mathbb{Z}_2$.

eg. Let $G = S_4 = \overset{\text{even permutation}}{A_4} \rtimes \mathbb{Z}_2$ $\leftarrow \langle (12) \rangle$.

$(23) = (123)(12)$
$(34) = (123)(143)(12)$

$(1234) = (234)(12)$.

$1\ 2\ 3\ 4$

$\mathbb{Z}_2 < S_4$, $A_4 \lhd S_4$ and $A_4 \cap \mathbb{Z}_2 = \{1\}$

$\langle H, K\rangle = HK = H \rtimes K = H : K$

eg. Let G be a group. $A = Aut(G)$. $\begin{matrix} 1 & 3 & 4 & 2 \\ 2 & 3 & 4 & 1 \end{matrix}$

Let $X = \{(g, \alpha) \mid g \in G, \alpha \in A\}$. $|X| = |G| \cdot |Aut(G)|$.

$(g_1, \alpha_1) \cdot (g_2, \alpha_2) = (g_1 g_2^{\alpha_1^{-1}}, \alpha_1 \alpha_2)$.

$$\boxed{\begin{aligned} & g_1\alpha_1 \, g_2\alpha_2 \\ &= g_1\alpha_1 \, \underline{g_2 \, \alpha_1^{-1}} \alpha_1 \alpha_2 \\ &= (g_1 g_2^{\alpha_1^{-1}}, \alpha_1\alpha_2) \end{aligned}}$$

Then $(X, \cdot)$ is a group.

called the <u>holomorph</u> of G,

denoted by $Hol(G)$.

- $G \cong G \times \{1\} = \{(g, 1) \mid g \in G\} \lhd Hol(G)$.
- $A \cong \{1\} \times A = \{(1, \alpha) \mid \alpha \in A\} < Hol(G)$.
- $Hol(G) = G \rtimes A$.

$G \rtimes \mathbb{Z}_2 = \mathbb{Z}_{15} \rtimes \mathbb{Z}_2 = (\mathbb{Z}_3 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_2$

$Aut(G) \cong Aut(\underset{\langle a \rangle \times \langle b \rangle}{\mathbb{Z}_3 \times \mathbb{Z}_5}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

eg. ① Let $G = \langle a \rangle \times \langle b \rangle = \mathbb{Z}_3 \times \mathbb{Z}_5$.

Let $\alpha \in Aut(G)$ s.t. $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$.

Define $X = G \rtimes \langle \alpha \rangle < G \rtimes Aut(G)$

$= \langle ab \rangle \rtimes \langle \alpha \rangle = \underline{\mathbb{Z}_{15} \rtimes \mathbb{Z}_2}$.

$= \langle ab, \alpha \mid (ab)^\alpha = (ab)^{-1} \rangle$

$= D_{30}$.

② Let $\tau \in Aut(G)$ s.t. $a^\tau = a^{-1}$, $b^\tau = b$.

Then $Y = G \rtimes \langle \tau \rangle = \underline{\mathbb{Z}_{15} \rtimes \mathbb{Z}_2}$

$= D_6 \times \mathbb{Z}_5$

$= \langle a, \tau \rangle \times \langle b \rangle$.

③ Let $\rho \in Aut(G)$, s.t. $a^\rho = a$, $b^\rho = b^{-1}$.

Then $Z = \underbrace{G \rtimes \langle \rho \rangle}_{= \mathbb{Z}_{15} \rtimes \mathbb{Z}_2} < Hol(G)$

$= \langle a \rangle \times \langle b, \rho \rangle$

$= \mathbb{Z}_3 \times D_{10}$.

eg. Let $G = \underset{= \langle a \rangle}{\mathbb{Z}_5}$. Let $\alpha, \tau \in Aut(G) = \mathbb{Z}_4$. $\underline{Aut(\mathbb{Z}_p) = \mathbb{Z}_{p-1}}$.

s.t. $a^\alpha = a^{-1}$. $\Rightarrow$ $G \rtimes \langle \alpha \rangle = D_{10}$

$a^\tau = a^2$.

$a^{\tau^2} = a^4 = a^{-1}$ $\Rightarrow$ $G \rtimes \langle \tau \rangle = \underset{\langle a, \tau \rangle}{\mathbb{Z}_5 \rtimes \mathbb{Z}_4}$

$$\boxed{\begin{aligned} & a^{\tau'} = a^3 \quad G \rtimes \langle \tau' \rangle = G \rtimes \langle \tau^3 \rangle \\ & \tau' = \tau^3 = \tau^{-1} \end{aligned}}$$ generators: a, τ.

$x^5 = 1$. y

$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

$Aut(\mathbb{Z}_5) = \mathbb{Z}_4$.

$G = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

$\langle x \rangle \rtimes \langle y \rangle$.

① $x^y = x$. $\Rightarrow$ $\langle x \rangle \times \langle y \rangle$. ($= \mathbb{Z}_{20}$.)

② $x^y = x^{-1}$. $\Rightarrow$ $Z(G) = \langle y^2 \rangle$.

③ $x^y = x^2$

$$\boxed{\begin{aligned} & \text{HW: } \mathbb{Z}_{17} \rtimes \mathbb{Z}_{16} = \langle x \rangle \rtimes \langle y \rangle \\ & \text{Construct} \\ & G = \mathbb{Z}_3^2 \rtimes Q_8 \text{ s.t. } Z(G) = 1. \end{aligned}}$$

Let $X = \{x_1, x_2, \cdots, x_r\}$

$X' = \{x_1', x_2', \cdots, x_r'\}$.

$Y = X \cup X' = \{x_1, x_2, \cdots, x_r, x_1', x_2', \cdots, x_r'\}$.

$\{\text{words on } Y\} = \text{word}(Y)$.

Let $u, v \in \text{word}(Y)$.

If $u = v x_i x_i'$ (or $v x_i' x_i$), then u, v are equivalent.
equivalence denoted by $u \sim v$.

If $u = u_1 x_i x_i' x_j x_j' u_2$, then $u \sim u_1 u_2$ equivalent.
(similarly, exchange position ✓)

"$\sim$" is an equivalence relation on word(Y).

Let $\bar{w}$ be the equivalent class of $w \in \text{word}(Y)$.

Define $G = \{\bar{w} \mid w \in \text{word}(Y)\}$.

For $\overline{w_1}, \overline{w_2} \in G$, let $\overline{w_1} \cdot \overline{w_2} = \overline{w_1 w_2}$. then $(G, \cdot)$ is a group. called a <u>free group</u> (finite generated) of rank r.

$$G = \langle x_1, \cdots, x_r, x_1', \cdots, x_r' \rangle.$$

$G = \langle a, \alpha \rangle = D_{30}$. $a^5 = 1$, $\alpha^2 = 1$, $a^{\alpha} = a^{-1}$. $\alpha a \alpha = a^{-1}$. $\alpha a \alpha a = 1$.

There is a free group F of rank 2 s.t. G is a homomorphism image of F.

$F = \langle x_1, x_2, x_1', x_2' \rangle$

$$\phi \begin{cases} x_1 \mapsto a \\ x_2 \mapsto \alpha \\ x_1' \mapsto a^{-1} \\ x_2' \mapsto \alpha^{-1}. \\ \phi \mapsto 1 \end{cases}$$

$\ker(\phi) = \langle x_1^5, x_2^2, x_1 x_2 x_1 x_2 \rangle$.

i.e. $F/\ker(\phi) \cong G$.