HOMEWORK 1

1. Describe analytically and illustrate the image of the given set E under the map $w = f(z)$:
(a) $E = \{0 \le \operatorname{Re} z \le 1,\ 0 \le \operatorname{Im} z \le 1\}$, $f(z) = z^2$.
(b) $E = \{\operatorname{Im} z = 2\operatorname{Re} z + 3\}$, $f(z) = 1/(z+i)$.
(c) $E = \mathbb{C} \setminus (-\infty, 1]$, and $f(z)$ is the branch of $\sqrt{z}$ determined by $f(i) = -(1+i)/\sqrt{2}$.
(d) $E = \{0 < \operatorname{Re} z < \pi/4\}$, and $f(z)$ is the cotangent function $f(z) = \cot z$.
(e) $E = \{\operatorname{Im} w > (\operatorname{Re} z)^2 + 10\}$, $f(z) = e^z$.
(f) $E = \Pi^+ \setminus B_1(0)$, and $f(z)$ is the branch of $\operatorname{Ln} z$ in Π^+ with $f(i) = -3\pi i/2$.

2. Construct a conformal map of the following domains onto the upper half plane Π^+:
(a) $D = \{0 < \operatorname{Re} z < 1,\ \operatorname{Im} z > 0\}$
(b) $D = G \setminus [0, i/2]$, where G is the upper unit half disc
(c) $D = G \setminus [-2, -1]$, where $G = \{|z| > 1\}$
(d) $D = \Pi^+ \setminus \big([0, i] \cup [2i, \infty)\big)$

3. Prove that the group of conformal automorphisms of the upper half plane consists of linear-fractional mappings $f(z) = \frac{az+b}{cz+d}$ for which the respective matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a positive determinant and real entries.

4. The number

$$\frac{z_2 - z_3}{z_2 - z_4} : \frac{z_1 - z_3}{z_1 - z_4}$$

is called the *cross-ratio* of four points in complex plane. Prove that the cross-ratio is invariant under linear-fractional mappings.

5. Prove that four points z_1, z_2, z_3, z_4 belong to the same generalized circle iff their cross-ratio

$$\frac{z_2 - z_3}{z_2 - z_4} : \frac{z_1 - z_3}{z_1 - z_4}$$

is real.