Homework 6 (Due November 7)

Grade Distribution (Total=10+8+8+4+8+8+10=56).

Please simply answer as much as possible.

1. If the joint probability mass function of X, Y is given by

X Y	-1	0	1
-1	a	0	0.2
0	0.1	b	0.1
1	0	0.2	c

and
$$P(X \cdot Y \neq 0) = 0.4$$
, $P(X \le 0 | Y \le 0) = \frac{2}{3}$.

- (a) (3 points) Find the values of a, b, c.
- (b) (4 points) Compute the marginal probability mass function of X and Y.
- (c) (3 points) Find the probability mass function of X + Y.
- 2. The joint probability density function of (X, Y) is given by

$$f(x,y) = \begin{cases} cx^4y, & \text{if } x^4 < y < 1; \\ 0, & \text{otherwise,} \end{cases}$$
 (0.1)

where c > 0 is some constant.

- (a) (4 points) Find the marginal probability density functions f_X and f_Y .
- (b) (4 points) Calculate EX and EY.
- 3. You spend the night in a teepee shaped as a right circular cone whose base is a disk of radius r centered at the origin and the height at the apex is h. A fly is buzzing around the teepee at night. At some time point the fly dies in mid-flight and falls directly on the floor of the teepee at a random location (X,Y). Assume that the position of the fly at the moment of its death was uniformly random in the volume of the teepee.
 - (a)(4 points) Derive the joint probability density function $f_{XY}(x,y)$ of the point (X,Y) where you find the dead fly in the morning.
 - (b)(4 points) Let Z be the height from which the dead fly fell to the floor. Find the probability density function $f_Z(z)$ of Z.

4. (4 points) If X is exponential with rate λ , find

$$P([X] = n, X - [X] \le x)$$

for $n \in \mathbb{Z}$ with $n \geq 0$ and $x \in \mathbb{R}$ with x > 0. Here [x] is defined as the largest integer less than or equal to x.

- 5. If X and Y are independent exponential random variables with parameters λ_1, λ_2 , express the density function of
 - (a) (4 points) Z = X/Y.
 - (b) (4 points) Z = XY.
- 6. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over (0, L/2) and Y is uniformly distributed over (L/2, L).]
 - (a) (4 points) Let Z = |X Y| be the distance between the two points. Find the probability that Z is greater than L/3.
 - (b) (4 points) Compute EZ.
- 7. Let X, Y have a bivariate normal distribution $\mathcal{N}(\mu_x, \mu_y; \sigma_x^2, \sigma_y^2; \rho)$.
 - (a) (3 points) Show that

$$E[(X - \mu_x)(Y - \mu_y)] = \rho \sigma_x \sigma_y.$$

(b) (7 points) Let

$$G(\lambda) = \left\{ (x,y) \in \mathbb{R}^2 : \left(\frac{x - \mu_x}{\sigma_x} \right)^2 + \left(\frac{y - \mu_y}{\sigma_y} \right)^2 - 2\rho \frac{(x - \mu_x)(y - \mu_y)}{\sigma_x \sigma_y} \le \lambda^2 \right\}.$$

Calculate $P((X,Y) \in G(\lambda))$. [Hint: Use Substitutions in Multiple Integrals.]