Homework-8

November 12, 2024

1. (1). Let $G = A_4 \rtimes Z_2$, find (1234) by this semidirect product.
(2). Identify $Z_{17} \rtimes Z_{16}$.
(3). Construct $G = Z_3^2 \rtimes Q_8$ s.t. $Z(G) = 1$.

2. (1). Prove $(\mathbb{Z}, +)$ has no composition series.
(2). Write two different composition series of Z_6.
(3). Write the composition series of S_3 and S_4 respectively.
(4). Let $F = \mathbb{Z}/2\mathbb{Z}$, write one composition series of $\mathrm{GL}_2(F)$.

3. (1). Prove $S_4 \simeq \langle a, b | a^2 = b^3 = e, (ab)^4 = e \rangle$.
(2). Prove $A_4 \simeq \langle a, b | a^2 = b^3 = e, (ab)^3 = e \rangle$.
(3). Prove $Q_8 \simeq \langle a, b | a^4 = b^4 = e, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$.

4. (Universal property) Let F be a free group, G, H be groups. Let $\alpha : F \to G$ be a group homomorphism, $\beta : H \to G$ be a group epimorphism. Prove that there exsits homomorphism $\gamma : F \to H$ s.t. $\alpha = \beta\gamma$.

From now we assume over ring R is a commutative ring with identity.

5. (1). Let I, J be ideals of ring R, I, J are coprime. Prove $IJ = I \cap J$.
(2). Let $I_1, I_2, \ldots, I_n$ be ideals of ring $I_1, I_2, \ldots, I_n$ are coprime. Prove
$I_1 \cap I_2 \cap \cdots \cap I_n = I_1 I_2 \cdots I_n$.
(3). Let I, J, K be ideals of R, $IJ \subseteq K$ and I, K are coprime. Prove $J \subseteq K$.
(4). Let I, J, K be ideals of R, $I, J \supseteq K$ and I, J are coprime. Prove $IJ \supseteq K$.

6. Let p be a prime number, n be a positive integer and $n > 1$. Let $R = \mathbb{Z}/(p^n)$, Prove:
(1). If for $r \in R$ where r is not a unit, then r must be a nilpotent element.
(2). R has only one prime ideal.
(3). We denote this prime ideal as P, then the quotient ring R/P is a field.

7. (1). Let $\varphi : R \to R_1$ be a ring homomorphism s.t. $\varphi(1_R) = 1_{R_1}$. prove that if Q is a prime ideal of R_1 then $P = \varphi^{-1}(Q)$ is a prime ideal of R.
(2). If Q is a maximal ideal of R_1, is $\varphi^{-1}(Q)$ must a maximal ideal of R?

8. (1). Let P be a prime ideal of R which contains a intersection of finitely many ideals $I_i (1 \leq i \leq n)$, prove that there exist some i s.t. $I_i \subseteq P$.
(2). Let I be an ideal which contained in the union of finitely many prime ideals $P_i (1 \leq i \leq n)$, prove that there exist some i s.t. $I \subseteq P_i$.
(3). Prove that a prime ideal of a finite ring R is maximal ideal.

9. (1). Let p be a prime number, write the ring of fractions $\mathbb{Z}_{(P)}$ (as a subset of $\mathbb{Q}$).

(2). Let $m \in \mathbb{Z}$, $m \neq 0$, write the ring of fractions $m^{-1}\mathbb{Z}$ (as a subset of $\mathbb{Q}$).

10. Let P be a prime ideal of R, then R can be regarded as a subring of R_P.
(1). For any ideal I of R, prove IR_P is an ideal of R_P.
(2). Let Q be a prime ideal of R. Prove QR_P is a prime ideal of R_P or $QR_P = (1)$.
(3). Prove PR_P is the unique maximal ideal of R_P.
(4). Prove there is a one to one and onto correspondence between prime ideals of R which contained in P and prime ideals in R_P given by $Q \mapsto QR_P$.