Group extention and semidirect product.

Def. a short exact sequence of groups is a sequence of groups and group homomorphisms of the form

$$1 \xrightarrow{d_1} K \xrightarrow{d_2} G \xrightarrow{d_3} H \xrightarrow{d_4} 1$$

K, G, H are groups, 1 means trivial group.

Exactness means for all d_i and given groups. we have

$\operatorname{im} d_{i-1} = \ker d_i$

i.e. $\ker d_2 = \operatorname{im} d_1 = 1$ so d_1 is $\begin{matrix} 1 \longrightarrow k \\ e \longmapsto e \end{matrix}$ embedding.

$\ker d_3 = \operatorname{im} d_2$ d_2 is monomorphism.

Thus d_2 is an embedding $K \hookrightarrow G$, $\operatorname{im} d_2 \cong K$

$\ker d_4 = \operatorname{im} d_3$

Since d_4 is $H \longrightarrow 1$ trivial homo.

$\ker d_4 = H$

So d_3 is epimorphism.

Thus we have $H \cong G/K$

Any homomorphism, write as $\varphi: G_1 \longrightarrow G_2$ induces a SES. that is $1 \longrightarrow \ker\varphi \longrightarrow G_1 \longrightarrow \operatorname{im}\varphi \longrightarrow 1$

i.e $\operatorname{im}\varphi \cong G/\ker\varphi$, just 1st isomorphism. theorem.

e.g.:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1.$$

$$1 \to \mathbb{Z}_n \to D_{2n} \to \mathbb{Z}_2 \to 1.$$

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1.$$

$$1 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 1.$$

warning. in our class, if $G = K \rtimes H$, $1 \to K \to G \to H \to 1$ we sometimes say K by H.

Def When we have a short exact sequence as

$$1 \to K \to G \to H \to 1.$$

We say G is a group extension of H by K.

Def An extension G' of K by H is said to be isomorphic if there exist an iso. $\varphi: G \to G'$ s.t.

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "id"'] & G \arrow[r] \arrow[d, "\varphi"] & H \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & K \arrow[r] & G' \arrow[r] & H \arrow[r] & 1
\end{tikzcd}$$

commute.

As you can see. give K. H there may be different G

i.e. Let $K = H = \mathbb{Z}_2$.

$1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1$ is an extension

$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$ is another extension

and $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Sometimes. identify all extension types up to iso is hard.

Def. Let $1 \to K \to G \underset{h}{\overset{\pi}{\rightleftarrows}} H \to 1$ is an extension of H by K

a section of G is a map $h: H \to G$ s.t

$$\pi \circ h = 1_H$$

If h is further a homo. then

h is called a splitting of G

If such splitting h exists. we say G split or this

extension is a split extension.

$h(H)$ is a subgroup of G which $h(H) \cap K = 1$ and

$G = K\,h(H)$

$\pi: G \to H$ gives isomorphism $h(H) \xrightarrow{\sim} H$

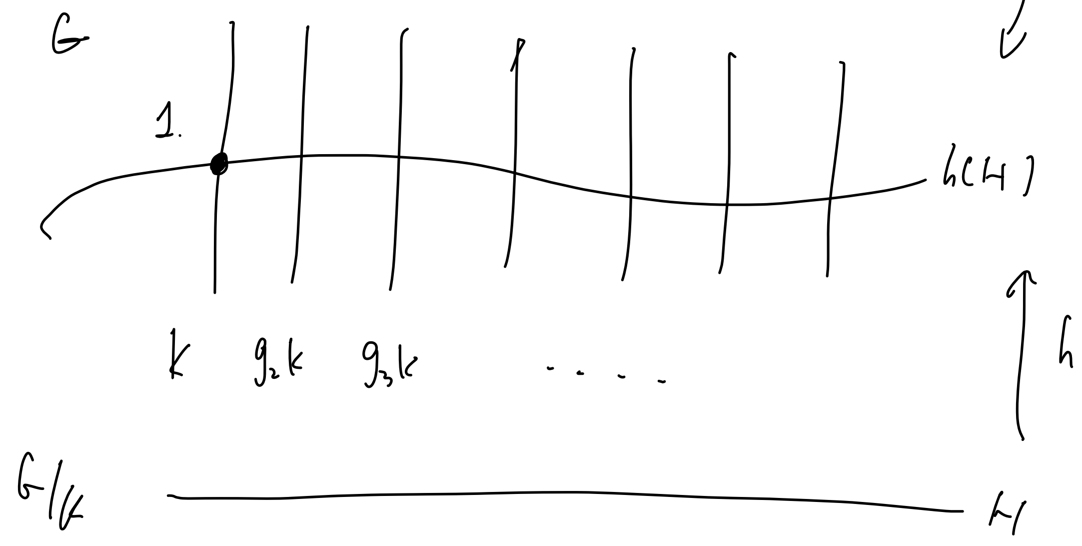
$h(H)$ is called a lift of H in G

G is called semidirect product of H and K

$$G = K \rtimes H$$

for $a \in K$, $b \in H$, $\quad \underbrace{\underset{K}{a_1}\,\underset{H}{b_1}\cdot\underset{K}{a_2}\,\underset{H}{b_2} = \underbrace{a_1\,a_2 b_1 a_2^{-1}}_{K}\cdot\underbrace{a_2 b_2}_{H}}_{\text{formally well-defined.}}$

Split extension: every element in H is exactly forms a complete repre. set of G/K.



For a split extension we write $G = K \rtimes_\varphi H$

To classification different $K \rtimes H$ is the same thing to identify

$\varphi: H \to \mathrm{Aut}(K)$

Given a split extension we let $H \curvearrowright K$ by conjugation.

This means:

$$\begin{aligned} \varphi: H &\to \mathrm{Aut}(K) \\ h &\mapsto \varphi_h: K \to K \\ &\qquad\quad k \mapsto hkh^{-1} \end{aligned}$$

Remark: This is the only possible way

Since the existence of h let us identify H with h(H) a subgroup of G

$\boxed{\text{E.g}}$. trivial split extension

consider $G = K \times H$

$$1 \to K \to K \times H \underset{s}{\rightleftarrows} H \to 1 \quad \text{is splitting.}$$

$$S: H \to K \times H$$
$$h \mapsto (1, h)$$

$$\varphi: H \to \mathrm{Aut}(K)$$
$$h \mapsto \varphi_h = id: K \to K$$
$$k \mapsto h^{-1}kh = k$$

The split extension of $G = K \rtimes H$ s.t $H \trianglelefteq G$

iff $G = K \times H$

$\boxed{\text{E.g}}$ Dihedral group (fin).

D_{2n} is a split extension of $\mathbb{Z}_n$ and $\mathbb{Z}_2$.

$$1 \to \mathbb{Z}_n \to D_{2n} \underset{h}{\rightleftarrows} \mathbb{Z}_2 \to 1$$

$h(\mathbb{Z}_2) = \langle b \rangle$ for $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$

$D_{2n} = \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ $\quad \varphi: \mathbb{Z}_2 \to \mathrm{Aut}\,\mathbb{Z}_n$.

$$h \mapsto \sigma: \mathbb{Z}_n \to \mathbb{Z}_n$$
$$a \mapsto a^{-1}$$

$\boxed{\text{Eg}}$ Symmetric group.

$$1 \longrightarrow A_n \longrightarrow S_n \underset{h}{\overset{\text{sgn}}{\rightleftarrows}} \mathbb{Z}_2 \longrightarrow 1.$$

lift of $\mathbb{Z}_2 = \langle a \rangle$, a must be a odd order 2 permu.

$\boxed{\text{Def}}$ Let $1 \to K \to G \to H \to 1$ be a short exact sequence and Let $\varphi : H \to \operatorname{Aut} K$ be a homomorphism

We say G is a semidirect product of K and H realizing φ

If there exists a splitting map $s : H \to G$ s.t the action of H on K by conjugation conincides with φ

write $G = K \rtimes_\varphi H$ and $(k_1, h_1)(k_2, h_2) = (k_1 k_2^{h_1^{-1}}, h_1 h_2)$

i.e. s.t $\varphi_{h_1} : K \longrightarrow K$, $k \longmapsto h_1 k h_1^{-1}$

coincide with $k^{h_1^{-1}}$

$\boxed{\text{Thm}}$. Given any groups K and H and any homomorphism $\varphi : H \to \operatorname{Aut} K$, there exists a semidirect product of K and H realizing φ. denoted by $G = K \rtimes_\varphi H$. Moreover any semidirect product G of K and H realising φ is iso to $K \rtimes_\varphi H$

Proof: As a set $G = K \times H$, we define product in G as follows:

$$(k,h)(k',h') := (k\varphi_h(k'), hh')$$

1°. Check this make G into a group

identity : $(1_k, 1_h)$

closed : ✓

associative :

On one hand: $(k_1,h_1)(k_2,h_2)(k_3,h_3)$

$$= (k_1\varphi_{h_1}(k_2), h_1h_2)(k_3,h_3)$$

$$= (k_1\varphi_{h_1}(k_2)\varphi_{h_1h_2}(k_3), h_1h_2h_3)$$

On the other hand: $(k_1,h_1)(k_2,h_2)(k_3,h_3)$

$$= (k_1,h_1)(k_2\varphi_{h_2}(k_3), h_2h_3)$$

$$= (k_1\varphi_{h_1}(k_2\varphi_{h_2}(k_3)), h_1h_2h_3)$$

$$= (k_1\varphi_{h_1}(k_2)\varphi_{h_1}\varphi_{h_2}(k_3), h_1h_2h_3)$$

Only need $\varphi_{h_1}\varphi_{h_2} = \varphi_{h_1h_2}$ this hold because $\varphi: H \to Aut(K)$ is a homomorphism.

inverse : for any $(k,h) \in G$

observe: $(k.h)^{-1} = (\varphi_{h^{-1}}(k^{-1}), h^{-1})$

check. $(k.h)(\varphi_{h^{-1}}(k^{-1}), h^{-1})$

$= (k\,\varphi_h(\varphi_{h^{-1}}(k^{-1})), 1)$

$= (1.1)$.

This group $K \rtimes_\varphi H$ is a semidirect product of K and H for we have the following split short. exact sequence:

$$1 \to K \xrightarrow{i} K \rtimes_\varphi H \underset{\underset{s}{\leftarrow\!--}}{\xrightarrow{\pi}} H \to 1$$

S.t i: $K \to K \rtimes_\varphi H$ $\quad$ π: $K \rtimes_\varphi H \to H$

$k \qquad (k,1) \qquad\qquad (k,h) \mapsto h$

s: $H \to K \rtimes_\varphi H$

$h \mapsto (1,h) = s_h$

$\mathrm{Con}(s_h)$

$\downarrow$

And the cojugation homomorphism iduced by $s(h)$ is.

$$\mathrm{Con}(s_h)(k,1) = (1,h)(k,1)(1,h)^{-1} = (\varphi_h(k), 1)$$

so $\mathrm{Con}(s_h)$ is a Aut of K s.t $\mathrm{Con}(s_h): K \to K$

$k \mapsto \varphi_h(k)$

i.e. $\mathrm{Con}(s_h) = \varphi_h$ $\quad$ i.e

this φ is consides with con. action of H on K.

For the uniqueness part:

Let G be any semidirect product of K and H realizing φ

By definition $\exists\ s: H \to G$ s.t. identify $s(H) \leq G$

the action φ corresponding to conjugation authom

$Con(s_h)$

Let $\psi: \ K \rtimes_{\varphi} H \longrightarrow G$
$(k, h) \longmapsto kh$

$$\begin{aligned}
\psi((k_1,h_1)(k_2,h_2)) &= \psi(k_1\varphi_{h_1}(k_2), h_1h_2) \\
&= k_1\varphi_{h_1}(k_2)\, h_1h_2 \\
&= k_1\, h_1k_2h_1^{-1}h_1h_2 \\
&= k_1h_1k_2h_2 \\
&= \psi(k_1,h_1)\psi(k_2,h_2).
\end{aligned}$$

$\Rightarrow \psi$ is group homo and ψ is a bijection

since $\forall g \in G$. $g = kh$ is unique by defi. of splitting extension.

$\boxed{\text{Corollary}}$ Characterization of direct products:

Let $1 \longrightarrow K \xrightarrow{m} G \xrightarrow{f} H \longrightarrow 1.$

be a SES of groups. Then the following equivalence:

(1). $G \simeq K \times H$

(2). The SES split and H is normal in G with respect to the splitting.

(3). G is a semidirect product of K and H realizing the trivial homomorphism $\varphi: H \to \mathrm{Aut}(K)$ s.t $\varphi(H) = id$.

(4). There exist a retraction $r: G \to K$ s.t $r \circ m = 1_K$

proof: (1) → (2) (1) → (3) (1) → (4) trivial.

(2) → (1). SES $\Rightarrow$ $K \trianglelefteq G$, $K \cap H = 1$ $H \trianglelefteq G$ $\Rightarrow$ $G \simeq H \times K$.

(3) → (1). $\forall k \in K$, $h \in H$ $[k, h] = 1$ $\Rightarrow$ $H \trianglelefteq G$

$K \trianglelefteq G$, $K \cap H = 1$ $\Rightarrow$ $G \simeq K \times H$

(4) → (1). Let $r: G \to K$ be retraction. We have

$\psi: G \to K \times H$ s.t $g \mapsto (r(g), f(g))$

Claim: ψ is iso.

inj.: $g \in \ker \psi \Rightarrow r(g) = f(g) = 1$.

by exactness $f(g) = 1 \Rightarrow \exists k \in K$ s.t $g = m(k)$

r retraction $\Rightarrow k = r(m(k)) = r(g) = 1 \Rightarrow g = m(1) = 1$

sur: $\forall (k, h) \in K \times H$

let $g \in G$ be preimage of h by f (since f sur).

then take $m(kr(g^{-1})g)$ s.t.

$$\psi(m(kr(g^{-1})g)) = \big(r(m(kr(g^{-1})g)),\ f(m(kr(g^{-1})g))\big)$$

$$= (kr(g^{-1})r(g),\ f(g)) = (k, h)$$