Discrete Mathematics for Computer Science

Lecture 3: Nested Quantifier, Mathematical Proofs

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

Questions from Students: Limitations of Proposition

- p: Every computer in Room 101 is functioning properly.
- q: Computer MATH3 is in Room 101.

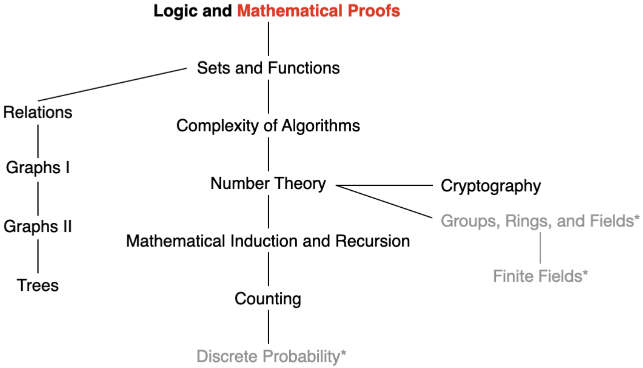
Can we conclude r: "MATH3 is functioning properly" using the rules of propositional logic? NO! Cannot infer r from p and q.

With predicate and quantifier:

- $C(x)$: Computer x is in Room 101.
- $D(x)$: Computer x is functioning properly.

- $\forall x(C(x) \to D(x))$ within the domain of computers: Every computer in Room 101 is functioning properly.
- $C(MATH3)$: Computer MATH3 is in Room 101.
- $D(MATH3)$: MATH3 is functioning properly.

This Lecture



Mathematical Proofs: Rules of inference, introduction to proofs

Argument

Argument: A sequence of propositions that end with a conclusion.

Premises:

“If you have a current password, then you can log onto the network.”

“You have a current password.”

Therefore,**Conclusion:**

“You can log onto the network.”

An argument is valid if the truth of all its premises implies that the conclusion is true.

Argument Form

Premises:

“If you have a current password, then you can log onto the network.”

“You have a current password.”

Conclusion: “You can log onto the network.”

An argument form in propositional logic is a sequence of compound propositions involving propositional variables.

- p: “You have a current password”
- q: “You can log onto the network”

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

The validity of an argument follows from the validity of its argument form.

Validity

Validity of Argument Form: The argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q \text{ is a tautology.}$$

Note: According to the definition of $p \to q$, we do not worry about the case where $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ is false.

Thus, equivalently, an argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Is the following argument form valid?

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

Is $(p \to q) \wedge p \to q$ a tautology?

Validity of Argument: The validity of an argument follows from the

Validity

Is the following argument valid?

- If you do every problem in this book, then you will learn discrete mathematics.
- You learned discrete mathematics.
- Therefore, you did every problem in this book.

No! $((p \to q) \land q) \to p$ is not a tautology.

Validity

Is the following argument valid?

- If you do every problem in this book, then you will learn discrete mathematics.
- You did not do every problem in this book.
- Therefore, you did not learn discrete mathematics.

No! $((p \to q) \land \neg p) \to \neg p$ is not a tautology.

Rules of Inference for Propositional Logic

To see the validity of $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$, we need to draw a table with 2^{n+1} row. A tedious approach!

Construct complicated valid argument forms using the validity of some relatively simple argument forms, called rules of inference.

- **modus ponens** (*law of detachment*) 肯定前件式

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

corresponding tautology:
$(p \wedge (p \rightarrow q)) \rightarrow q$

Rules of Inference for Propositional Logic

- **modus tollens** 否定后件式

$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

corresponding tautology:
$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

- **hypothetical syllogism** 假言三段论

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

corresponding tautology:
$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Rules of Inference for Propositional Logic

- **disjunctive syllogism** 选言三段论

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

corresponding tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

- **Addition**

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

corresponding tautology:
$p \rightarrow (p \vee q)$

- **Simplication**

$$\begin{array}{l} p \wedge q \\ \hline \therefore q \end{array}$$

corresponding tautology:
$(p \wedge q) \rightarrow p$

Rules of Inference for Propositional Logic

- **Conjunction**

$$\begin{array}{l} p \\ \underline{q \qquad\quad} \\ \therefore p \land q \end{array}$$

corresponding tautology:
$((p) \land (q)) \rightarrow (p \land q)$

- **Resolution**

$$\begin{array}{l} \neg p \lor r \\ \underline{p \lor q \qquad} \\ \therefore q \lor r \end{array}$$

corresponding tautology:
$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$

Example

Show that the premises (i) $(p \wedge q) \vee r$ and (ii) $r \rightarrow s$ imply the conclusion $p \vee s$.

1.	$(p \vee r) \wedge (q \vee r)$	Premise (i), Distributive Law
2.	$p \vee r$	Simplification from step 1
3.	$\neg r \vee s$	Premise (ii), Useful Law
4.	$p \vee s$	Resolution

Using Rules of Inference to Build Arguments

- "It is not sunny this afternoon and it is colder than yesterday."

  $\neg p \wedge q$

- "We will go swimming only if it is sunny this afternoon."

  $r \rightarrow p$

- "If we do not go swimming then we will take a canoe trip."

  $\neg r \rightarrow s$

- "If we take a canoe trip, then we will be home by sunset."

  $s \rightarrow t$

- Show the conclusion that "we will be home by sunset."

  t

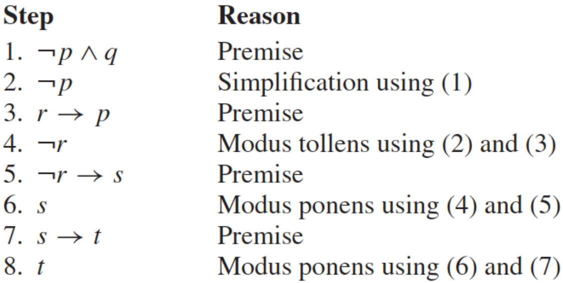
- p: It is sunny this afternoon.
- q: It is colder than yesterday.
- r: We will go swimming.
- s: We will take a canoe trip.
- t: We will be home by sunset.

Using Rules of Inference to Build Arguments

- p: It is sunny this afternoon.
- q: It is colder than yesterday.
- r: We will go swimming.
- s: We will take a canoe trip.
- t: We will be home by sunset.

Premises: $\neg p \land q$, $r \to p$, $\neg r \to s$, $s \to t$

Conclusion: t



Step	Reason
1. $\neg p \land q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Example 2

In a small town, theft(s) have occurred, and the police have narrowed down the suspects to three people: Alice, Bob, and Charlie. Based on their investigation, the police have gathered the following clues:

- Clue 1: If Alice is a suspect, then Bob is also a suspect.
- Clue 2: If Alice is not a suspect, then Charlie is a suspect.
- Clue 3: Charlie is not a suspect.

The police's goal is to identify the real suspect(s).

Rules of Inference for Quantified Statements

- **Universal Instantiation** (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

- **Universal Generalization** (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

- **Existential Instantiation** (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

- **Existential Generalization** (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Applying Rules of Inference for Quantified Statements

- "A student in this class has not read the book."

  $\exists x(C(x) \land \neg B(x))$
- "Everyone in this class passed the first exam."

  $\forall x(C(x) \rightarrow P(x))$
- Show the conclusion that "Someone who passed the first exam has not read the book."

  $\exists x(P(x) \land \neg B(x))$

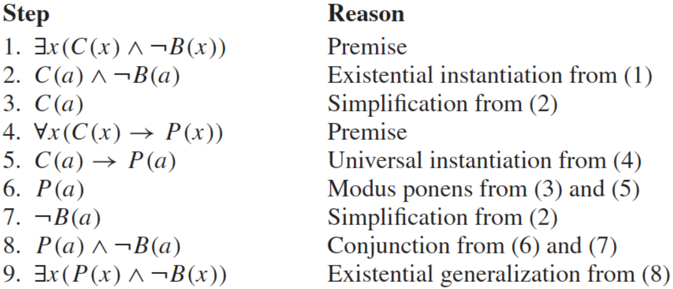
- $C(x)$: x is in this class.
- $B(x)$: x has read the book.
- $P(x)$: x passed the first exam.
- Domain of x: all students

Applying Rules of Inference for Quantified Statements

- $C(x)$: x is in this class.
- $B(x)$: x has read the book.
- $P(x)$: x passed the first exam.
- Domain of x: all students

Premises: $\exists x(C(x) \wedge \neg B(x))$, $\forall x(C(x) \rightarrow P(x))$

Conclusion: $\exists x(P(x) \wedge \neg B(x))$



Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

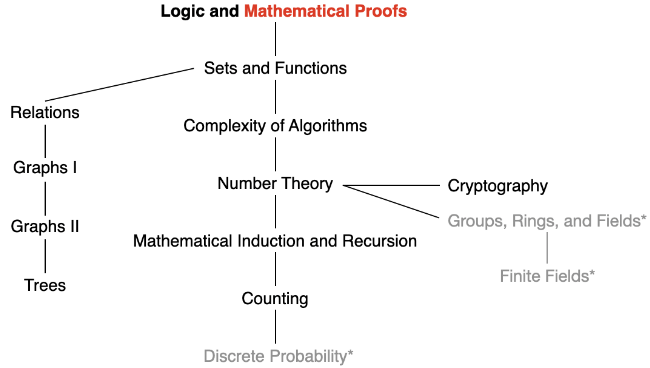
Example: Game of Logic

- “Logic is difficult or not many students like logic.”
- “If mathematics is easy, then logic is not difficult.”

Which of the followings are valid conclusions?

1. That mathematics is not easy, if many students like logic.
2. That not many students like logic, if mathematics is not easy.
3. That mathematics is not easy or logic is difficult.
4. That logic is not difficult or mathematics is not easy.
5. That if not many students like logic, then either mathematics is not easy or logic is not difficult.

This Lecture



Mathematical Proofs: Rules of inference, introduction to proofs

Proofs

A proof is a valid argument that establishes the truth of a mathematical statement. (Note: the truth of all its premises implies that the conclusion is true.)

Premises:

- hypotheses of the theorem
- axioms assumed to be true
- previously proven theorems or lemmas

Conclusion:

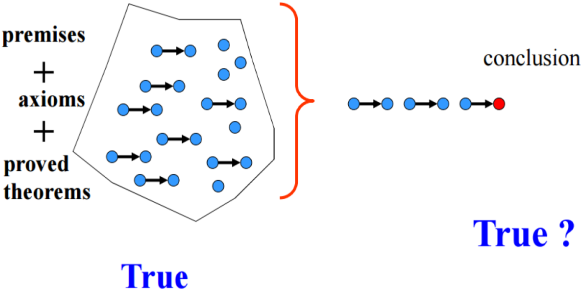
- the truth of the statement

Using rules of inference

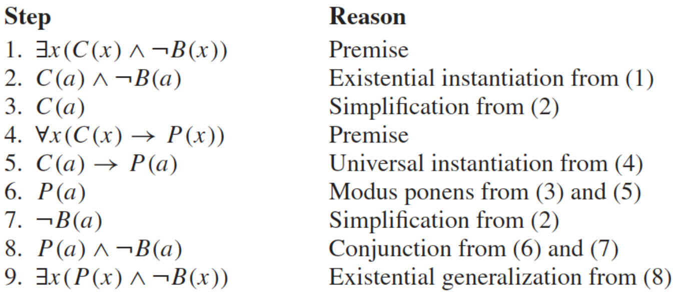
- Axiom: a statement or proposition which is regarded as being established.
- Theorem: a statement that can be shown to be true.
- Lemma: a statement that can be proved to be true, and is used in proving a theorem or proposition.

Formal Proofs

Formal proofs: steps follow logically from the set of premises, axioms, lemmas, and other theorems.



Formal Proof and Informal Proof



Step	Reason
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

In practice, **informal proofs:** steps are not expressed in any formal language of logic; steps may be skipped; the axioms being assumed and the rules of inference used are not explicitly stated; ...

Methods of Proving Theorems

Premises: p; Conclusion: q

- **Direct proof**

  If p is true, then q follows
- **Proof by contrapositive**

  Show the contrapositive: If $\neg q$ is true, then $\neg p$ follows
- **Proof by contradiction**

  show that $(p \wedge \neg q)$ contradicts the assumptions
- **Proof by cases**

  give proofs for all possible cases
- **Proof of equivalence**

  If p is true, then q follows; If q is true, then p follows

A proof is a valid argument. We work on propositions in proofs.

Direct Proof

If p is true, then q follows.

Example: Prove that "if n is odd, then n^2 is odd"

Proof:

Assume that (the hypothesis is true, i.e., n is odd)
$n = 2k + 1$ where k is an integer.
Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Therefore, n^2 is odd.

Proof by Contrapositive

If $\neg q$ is true, then $\neg p$ follows.

Example: Prove that "if $3n+2$ is odd, then n is odd"

Proof:

Assume that n is even, i.e., $n = 2k$, where k is an integer. Then

$3n + 2 = 3(2k) + 2 = 2(3k + 1)$.

Therefore, $3n + 2$ is even.

Proof by Contrapositive

Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof:

- Assume that $a > \sqrt{n}$ and $b > \sqrt{n}$.
- Then, $ab > n$, that is $ab \neq n$.

Proof by Contradiction

Assume that p is true but q is false (i.e., $p \wedge \neg q$). Then show a contradiction to p, or $\neg q$, or other settled results.

Example: Prove that "if $3n + 2$ is odd, then n is odd"

Proof:

Assume that $3n + 2$ is odd and n is even, i.e., $n = 2k$, where k is an integer. Then

$$3n + 2 = 3(2k) + 2 = 2(3k + 1).$$

Thus, $3n + 2$ is even. This is a contradiction to the assumption that $3n + 2$ is odd. Therefore, n is odd.

Proof by Contradiction

Assume that p is true but q is false (i.e., $p \wedge \neg q$). Then show a contradiction to p, or $\neg q$, or other settled results.

Example: Suppose we choose any 22 days. At least four of them fall on the same day of the week.

- Suppose we choose any 22 days; and at most three of them fall on the same day of the week.
- There are seven days of the week.
- Thus, at most 21 days could have been chosen. Contradiction!

Proof by Contradiction

Prove that $\sqrt{2}$ is irrational. (Rational numbers are those of the form $\frac{m}{n}$, where m and n are integers.)

Proof: Suppose that $\sqrt{2}$ is rational. Then, there exist integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors (so that the fraction a/b is in lowest terms.)

Since $\sqrt{2} = a/b$, it follows that $2b^2 = a^2$. By the definition of an even integer, it follows that a^2 is even, so a is even (see Exercise 16).

Since a is even, $a = 2k$ for some integer k. Thus, $b^2 = 2k^2$. This implies that b^2 is even, so b is even.

As a result, a and b have a common factor 2, which contradicts our assumption.

Proof by Cases

We want to show

- Premises: $p_1 \vee p_2 \vee \ldots \vee p_n$; Conclusion: q

This is equivalent to showing

- Premise 1: p_1; conclusion: q and
- Premise 2: p_2; conclusion: q and
- ... and
- Premise n: p_n; conclusion: q

Why?

$$
\begin{aligned}
& (p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q \\
\equiv\ & \neg(p_1 \vee p_2 \vee \ldots \vee p_n) \vee q \\
\equiv\ & (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n) \vee q \\
\equiv\ & (\neg p_1 \vee q) \wedge (\neg p_2 \vee q) \wedge \ldots \wedge (\neg p_n \vee q) \\
\equiv\ & (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)
\end{aligned}
$$

Proof by Cases

We want to show $(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$. This is equivalent to $(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge ... \wedge (p_n \rightarrow q)$. Why?

Example: Prove that "$|x||y| = |xy|$ for real numbers x, y"

Proof: Four cases:

- ◇ $x \geq 0, \; y \geq 0$
- ◇ $x \geq 0, \; y < 0$
- ◇ $x < 0, \; y \geq 0$
- ◇ $x < 0, \; y < 0$

Proof by Cases

Show that there exist irrational numbers x and y such that x^y is rational.

Proof: We know that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$.

Case 1: If $\sqrt{2}^{\sqrt{2}}$ is rational, then we have two irrational numbers $x = \sqrt{2}$ and $y = \sqrt{2}$ with $x^y = \sqrt{2}^{\sqrt{2}}$ rational.

Case 2: If $\sqrt{2}^{\sqrt{2}}$ is irrational, then we let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. We have $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ is rational.

Note that although we do not know which case works, we know that one of the two cases has the desired property.

Proof of Equivalences

If p is true, then q follows; If q is true, then p follows

Example: Prove that "An integer n is odd if and only if n^2 is odd"

Proof:

- ◇ proof of $p \to q$: direct proof
- ◇ proof of $q \to p$: proof by contrapositive

Vacuous Proof

Premise: p; Conclusion: q

Show that p (the hypothesis) is always false.

Example:

- $P(n)$: if $n > 1$, then $n^2 > n$.
- Domain: integers

Show $P(0)$ is true.

Proof: Since the premise $0 > 1$ is always false. Thus $P(0)$ is true.

Vacuous proofs are often used to establish special cases of theorems that state that a conditional statement is true for all positive integers.

Trivial Proof

Premise: p; Conclusion: q

Show that q (the conclusion) is always true.

Example: $P(n)$: if $a \geq b$, then $a^n \geq b^n$. Show $P(0)$ is true.

Proof: Since the conclusion $a^0 \geq b^0$ is always true for any value of a and b. Thus $P(0)$ is true.

Proofs with Quantifiers

Universal quantified statements

- Prove the property holds for all examples
  - proof by cases to divide the proof into different parts

- Disprove universal statements
  - existential quantified statements
  - counterexamples

Proofs with Quantifiers

Existential quantified statements

- Constructive
  - find a specific example to show the statement holds
- Nonconstructive
  - any method other than the constructive method
  - e.g., proof by contradiction

- Disprove: there does not exist any ...
  - universal quantified statements

Proofs with Quantifiers

Uniqueness proofs: assert the existence of a unique element with a particular property.

- Existence: We show that an element x with the desired property exists.
- Uniqueness: We show that if $y \neq x$, then y does not have the desired property. Or, if y has the desired property, then $y = x$.

Example

Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Solution:

- Existence: The real number $r = -b/a$ is a solution of $ar + b = 0$. Consequently, a real number r exists for which $ar + b = 0$.
- Uniqueness: Suppose that s is a real number such that $as + b = 0$. Then, $ar + b = as + b$, where $r = -b/a$. Dividing both sides of this last equation by a, which is nonzero, we see that $r = s$.

Proof Exercise

Prove that there are infinitely many prime numbers.

Proof: Suppose that there are only a finite number of primes. Then, there exists a prime number p that is the largest of all the prime numbers. Also, we can list the prime numbers in ascending order: $2, 3, 5, 7, 11, ..., p$

Let $n = (2 \times 3 \times 5 \times \cdots \times p) + 1$. Then, $n > 1$, and n cannot be divided by any prime number in the list above (to be proven in number theory section). This means that n is also a prime.

Clearly, n is larger than all the primes in the list above. This is contrary to the assumption that all primes are in the list.

Proof Exercise

Prove that there are infinitely many solutions in positive integers x, y, and z to the equation $x^2 + y^2 = z^2$.

Proof:

- $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$
- m and n are integers

Proof Exercise

Prove that if x is irrational and $x \geq 0$, then $\sqrt{x}$ is irrational.

Proof Exercise

- Write the numbers $1, 2, 3, ..., 2n$ on a blackboard, where n is an odd integer.
- Pick any two of the numbers, j and k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board.

Prove that this integer must be odd.

Proof Exercise

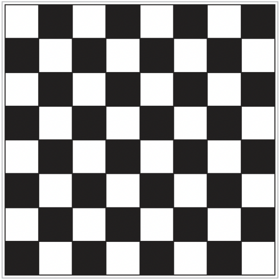


FIGURE 2 The Standard Checkerboard.

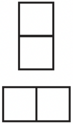


FIGURE 3 Two Dominoes.

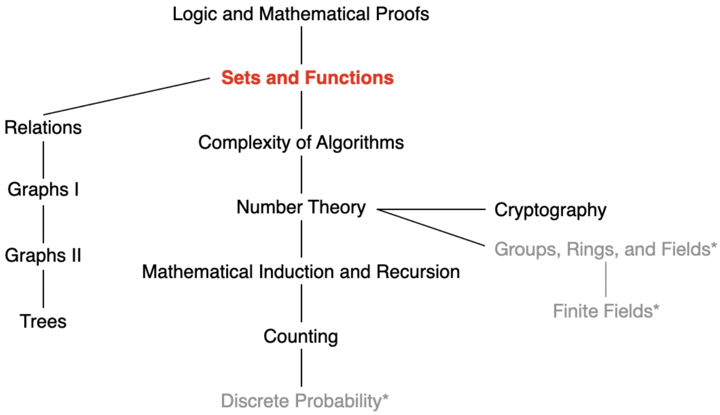
We say that a board is **tiled** by dominoes when all its squares are covered with no overlapping dominoes and no dominoes overhanging the board.

Q1 Can we tile the standard checkerboard using dominoes?

Q2 What if we remove the upper left corner square?

Q3 What if we remove the upper left and lower right corner squares?

Next Lecture


Logic and Mathematical Proofs
Sets and Functions
Relations
Complexity of Algorithms
Graphs I
Number Theory
Cryptography
Graphs II
Groups, Rings, and Fields*
Mathematical Induction and Recursion
Finite Fields*
Trees
Counting
Discrete Probability*