MA204: Mathematical Statistics

Tutorial 8

T8.1 Type I/II Error and Power Function

— Consider a test for testing H_0: $\theta \in \Theta_0$ against H_1: $\theta \in \Theta_1$ with *critical region* $\mathbb{C}$ and *acceptance region* $\mathbb{C}'$ (the complement of $\mathbb{C}$).

— Rejection of the null hypothesis H_0 when it is true is called *Type I error*. The probability of making a Type I error is called the *Type I error function*:

$$\alpha(\theta) = \Pr(\text{rejecting } H_0 \mid H_0 \text{ is true}) = \Pr(\mathbf{x} \in \mathbb{C} \mid \theta \in \Theta_0),$$

where $\mathbf{x} = (X_1, \ldots, X_n)^\top$.

— Acceptance of the null hypothesis H_0 when it is false is called *Type II error*. The probability of making a Type II error is called the *Type II error function*:

$$\beta(\theta) = \Pr(\text{accepting } H_0 \mid H_0 \text{ is false}) = \Pr(\mathbf{x} \in \mathbb{C}' \mid \theta \in \Theta_1).$$

— The *power function* of a test for testing H_0: $\theta \in \Theta_0$ against H_1: $\theta \in \Theta_1$ is

$$p(\theta) = \Pr(\text{rejecting } H_0 \mid \theta) = \Pr(\mathbf{x} \in \mathbb{C}|\theta) = \begin{cases} \alpha(\theta), & \theta \in \Theta_0, \\ 1 - \beta(\theta), & \theta \in \Theta_1. \end{cases}$$

T8.2 Size of a Test

A test φ with critical region $\mathbb{C}$ is said to have size α if

$$\sup_{\theta \in \Theta_0} p_\varphi(\theta) = \sup_{\theta \in \Theta_0} \alpha_\varphi(\theta) = \alpha.$$

T8.3 Most Powerful Test

A test φ with critical region $\mathbb{C}$ is said to be a *most powerful test* with size α for testing $H_0\colon\ \theta = \theta_0$ against $H_1\colon\ \theta = \theta_1$, if

(i) $p_\varphi(\theta_0) = \alpha$,

(ii) $p_\varphi(\theta_1) \geqslant p_\psi(\theta_1)$ for any other test ψ with $p_\psi(\theta_0) \leqslant \alpha$.

T8.4 Neyman–Pearson Lemma

The test φ with size α and critical region

$$\mathbb{C} = \left\{ \boldsymbol{x} = (x_1, \ldots, x_n)^\top\colon\ \frac{L(\theta_0)}{L(\theta_1)} \leqslant k \right\}$$

is the most powerful test of size α for testing $H_0\colon\ \theta = \theta_0$ against $H_1\colon\ \theta = \theta_1$, where k is a value determined by the size α and $L(\theta)$ denotes the likelihood function.

Example T8.1 (Exponential distribution). Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exponential}(\lambda)$, we want to test $H_0\colon\ \lambda = \lambda_0 = 0.01$ versus $H_1\colon\ \lambda = \lambda_1 = 0.04$. If $n = 8$, find the most powerful test of size 0.1.

Solution: The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} \lambda \mathrm{e}^{-\lambda x_i} = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right).$$

Then

$$\frac{L(\lambda_0)}{L(\lambda_1)} = \frac{\lambda_0^n \exp\left(-\lambda_0 \sum_{i=1}^{n} x_i\right)}{\lambda_1^n \exp\left(-\lambda_1 \sum_{i=1}^{n} x_i\right)} = \left(\frac{\lambda_0}{\lambda_1}\right)^n \exp\left[(\lambda_1 - \lambda_0) \sum_{i=1}^{n} x_i\right] \leqslant k$$

is equivalent to

$$\bar{x} \leqslant \frac{\log(k)}{n(\lambda_1 - \lambda_0)} + \frac{\log(\lambda_1/\lambda_0)}{\lambda_1 - \lambda_0} \ \hat{=}\ c,$$

when $\lambda_1 > \lambda_0$. To determine c, we noted that

$$\begin{aligned} & X_i \sim \text{Exponential}(\lambda), \\ \Rightarrow\quad & n\bar{X} = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda), \\ \Rightarrow\quad & 2\lambda n\bar{X} \sim \text{Gamma}\left(n, \frac{1}{2}\right) \equiv \chi^2(2n), \end{aligned}$$

and

$$\begin{aligned} \alpha &= \Pr(\bar{X} \leqslant c \mid \lambda = \lambda_0) = \Pr(2\lambda n\bar{X} \leqslant 2\lambda nc \mid \lambda = \lambda_0) \\ &= \Pr(\chi^2(2n) \leqslant 2\lambda_0 nc) = 1 - \Pr(\chi^2(2n) \geqslant 2\lambda_0 nc), \end{aligned}$$

i.e., $1 - \alpha = \Pr(\chi^2(2n) \geqslant 2\lambda_0 nc)$, so that $2\lambda_0 nc = \chi^2(1 - \alpha, 2n)$. Thus,

$$c = \frac{\chi^2(1-\alpha, 2n)}{2\lambda_0 n} = \frac{\chi^2(0.9, 16)}{2 \times 0.01 \times 8} = \frac{9.312236}{0.16} = 58.2.$$

Therefore, by the Neyman–Pearson Lemma, the critical region of the most powerful test with size 0.1 is

$$\mathbb{C} = \{\boldsymbol{x}\colon\ \bar{x} \leqslant 58.2\}. \qquad \|$$

Example T8.2 (A discrete distribution). Let X be an r.v. with pmf under H_0 and H_1 being given in the following table:

x	1	2	3	4	5	6	7
$f(x \mid H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x \mid H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Given a single observation, find

(a) The most powerful test for testing H_0 versus H_1 with size $\alpha = 0.04$.

(b) The Type II error rate for this test.

Solution: (a) First, we calculate $f(x|H_0)/f(x|H_1)$ for all x, i.e.,

x	1	2	3	4	5	6	7
$f(x\|H_0)/f(x\|H_1)$	0.17	0.2	0.25	0.33	0.5	1	1.19

The likelihood ratios are increasing with the values of x.

By the Neyman–Pearson Lemma, a test of size α with critical region

$$\mathbb{C} = \left\{ x\colon \ \frac{f(x|H_0)}{f(x|H_1)} \leqslant k \right\}$$

is the most powerful test of size α. Now $\alpha = 0.04$, we have

$$\begin{aligned} \alpha &= \Pr\{X \in C \mid H_0\} \\ &= f(1|H_0) + f(2|H_0) + f(3|H_0) + f(4|H_0) \\ &= 0.01 + 0.01 + 0.01 + 0.01 = 0.04. \end{aligned}$$

Therefore, the most powerful test of size 0.04 is that with critical region

$$\mathbb{C} = \left\{ x\colon \ \frac{f(x|H_0)}{f(x|H_1)} \leqslant k \right\} = \{1, 2, 3, 4\},$$

where $k \in [0.33, 0.5)$.

(b) The acceptance region is $\mathbb{C}' = \{5, 6, 7\}$, so the Type II error rate is

$$\begin{aligned} \beta &= \Pr\{X \in \mathbb{C}' \mid H_1\} \\ &= \Pr\{X \in \{5, 6, 7\} \mid H_1\} \\ &= f(5|H_1) + f(6|H_1) + f(7|H_1) \\ &= 0.02 + 0.01 + 0.79 \\ &= 0.82. \end{aligned}$$

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