# COMPLEX ANALYSIS (H) COURSE, FINAL EXAM

**1. (5 points)** Let $f(z) \in \mathcal{O}(\bar{D})$, $D = B_1(0)$, and $|f(z)| = 1$ for $|z| = 1$. Prove that $f(D) = D$, unless $f$ is constant.

**2. (5 points)** Let $P_n$ be a sequence of polynomials converging uniformly on the circle $C = \{|z| = 1\}$ to a function $f$. Prove that $f$ extends to the disc $D = \{|z| < 1\}$ as a function holomorphic in $D$ and continuous up to the boundary.

**3. (7 points)** Show that any meromorphic function in a domain $D \subset \mathbb{C}$ can be approximated in it by rational functions (in the sense that for any compact $K$ in the domain and any $\epsilon > 0$ one can find a rational function $R(z)$ such that $|f(z) - R(z)| < \epsilon$ for all $z \in K$ including the possible poles of $f$).

**4. (10 points)** Prove that there exists a Weierstrass analytic function $F(z)$ in $\mathbb{C} \setminus \{0\}$ such that

$$\operatorname{Re} F(x+iy) = \frac{x^2 - y^2}{(x^2+y^2)^2} - \ln(x^2+y^2), \quad x, y \in \mathbb{R}.$$

Determine and characterize all the branch points of $F(z)$ in $\overline{\mathbb{C}}$, and provide its Riemann surface diagram.

**5. (10 points)** Let $\mathcal{M}$ be the space of holomorphic maps of $B_1(0)$ into itself with $f(0) = 1/2$. Find $\sup |f'(0)|$, $f \in \mathcal{M}$. **Hint.** Use the Schwarz Lemma.

**6. (10 points)** Prove that if $P_n(z)$ is a sequence of polynomials converging uniformly on compacts in $\mathbb{C}$ to a function $f(z)$, and such that all the roots of all $P_n(z)$ lie on the real line, then all the zeroes of $f(z)$ lie on the real line. **Additional question (5 points):** prove that all the zeroes of $f'(z)$ lie on the real line as well.

**7. (10 points)** Prove the "Inverse Caratheodory Theorem": Let $D, D'$ be two Jordan domains, $f \in \mathcal{O}(\bar{D})$, $f(D) \subset D'$, and assume that $f$ performs a homeomorphism of $\partial D$ onto $\partial D'$. Prove that $f$ is then a conformal map of $D$ onto $D'$. **Hint**. Use the argument principle.

**8. (10 points)** Let $f \in \mathcal{O}(D) \cap C(\bar{D})$, where $D$ is the upper half plane. Assume $f(R) \subset \mathbb{R}$ and $|f(z)| = O(|z|^N), N > 0$. Prove that $f$ is a polynomial. **Hint.** Inspect the proof of the Schwarz Reflection Principle.

**9. (10 points)** Prove the following mean value properties of holomorphic functions: if $f(z)$ is holomorphic in a domain $D \subset \mathbb{C}$ and the disc $B_r(a)$ together with its closure belongs to $D$, then

(i)

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt$$

(Mean Value Property on a circle);

(ii)

$$f(a) = \frac{1}{\pi r^2} \iint_{B_r(a)} f(z) dx dy$$

(Mean Value Property on a disc).

*In the following 2 problems, you are allowed to use the outcome of Problem 9.*

**10. (10 points)** Let $f, g$ be two holomorphic functions in a domain $D \subset \mathbb{C}$ at least one of which is nonconstant. Show that the maximum principle holds for the function $|f(z)| + |g(z)|$ (i.e. the latter function can't have a local maximum at a point $a \in D$).

**11. (15 points)** Let us introduce, for a bounded domain $D \subset \mathbb{C}$, the linear space

$$\mathcal{A}_1(D) := \mathcal{O}(D) \cap L_1(D)$$

(w.r.t the Lebegues measure in $D$). Prove that $\mathcal{A}_1(D)$, endowed with the standard induced $L_1(D)$ norm, is a closed subspace in $L_1(D)$.

**12. (15 points)** By considering integrals over appropriate circles from the function

$$f(\xi) = \cot \xi \left( \frac{1}{\xi - z} - \frac{1}{\xi} \right),$$

prove for all $z \in \mathbb{C}$ the following expansion of the cotangence function:

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z - \pi n} + \frac{1}{z + \pi n} \right).$$

**13. (10 points)** Deduce from the outcome of Problem 12 the Weierstrass Factorization for $\sin z$:

$$\sin z = z \cdot \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{\pi^2 n^2} \right).$$

**14. (15 points)** Let $f_n$ be a sequence of holomorphic functions in a domain $D \subset \mathbb{C}$, and let all $f_n$ miss in $D$ two fixed values $a, b \in \mathbb{C}$, $a \neq b$. Prove that the sequence $f_n$ contains a subsequence normally convergent in $D$, assuming $D$ is *simply-connected*. **Additional question (5 points):** prove the desired fact without assuming the simple-connectness.

**15. (10 points)** Formulate and prove the theorem on the description of the conformal automorphism group of the complex place $\mathbb{C}$.

**16. (20 points)** Prove that there is no norm on the linear space $\mathcal{O}(D)$ such that the convergence in it coincides with the uniform convergence of holomorphic functions on compacts. **Hint**. Arguing by contradiction, construct a linear operator in $\mathcal{O}(D)$ with an unbounded spectrum.