Homework-9

November 17, 2024

1. (1). Give an example of integral domain s.t. dose not satisfy the factor chain condition.
(2). Let R be a integral domain satisfying factor chain condition. Prove R is a UFD iff any two elements in R have greatest common divisor.

2. (1). Let R be a UFD, S is a multiplicatively closed set of R, $0 \notin S$. Prove that the ring of fractions $S^{-1}R$ is a UFD.
(2). Give an example to show that the subring of a UFD may not a UFD.
(3). Let R be a UFD, P is a prime ideal of R. Give an example to show that the quotient ring R/P may not a UFD.

3. (1). Prove that any principle ideal of $\mathbb{Z}[x]$ is not a maximal ideal.
(2). Prove that all non-zero prime ideals in PID are maximal ideals.

4. Let K be a field. The formal power series $\sum_{i=0}^{\infty} a_i x^i$ $(a_i \in K)$ form a ring under the usual addition and multiplication, which is called the ring of formal power series in one variable over K, denoted by $K[[x]]$.
(1). Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in K[[x]]$, prove that $f(x)$ is invertible iff $a_0 \neq 0$.
(2). Prove $K[[x]]$ is a PID.

5. Let R be a PID, $a, b, d \in R$, then $(a, b) = (d)$ (as ideals) iff d is the greatest common divisor of a and b.

6. Let D,R be PIDs, $R \subseteq D$. $a, b, d \in R$, d is the greatest common divisor of a and b in R. Prove that d is also the greatest common divisor of a and b in D.

7. Let K be a algebraic number field. We call $\alpha \in K$ an algebraic integer if α is a root of a monoic polynomial with integer coefficients. Let d be integer with no square factors. i.e. $\sqrt{d}$ is not a integer. Let $K = \mathbb{Q}(\sqrt{d})$.
(1). If $d \equiv 2, 3 \pmod 4$, prove that all algebraic integer in K is a set:

$$\{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$$

(2). If $d \equiv 1 \pmod 4$, prove that all algebraic integer in K is a set:

$$\{a + b\frac{1+\sqrt{d}}{2} \mid a, b \in \mathbb{Z}\}$$

Therefore all algebraic integers in K form a ring, called the algebraic integer ring of K.

8. (1). Prove the algebraic integer ring of $\mathbb{Q}(\sqrt{-3})$ is a ED.
(2). Prove the algebraic integer ring of $\mathbb{Q}(\sqrt{2})$ is a ED.
(3). Prove the algebraic integer ring of $\mathbb{Q}(\sqrt{5})$ is a ED.

9. Prove the invertible elements in $\mathbb{Z}[i]$ where $i = \sqrt{-1}$ is $\{\pm 1, \pm i\}$.

10. Let p be a prime, if $p \equiv 1 \pmod 4$, prove there exist $a, b \in \mathbb{Z}$ s.t. $p = a^2 + b^2$.