Southern University of Science and Technology
Department of Statistics and Data Science

MA204: Mathematical Statistics **Final Examination (Paper A)**
Date: 18 June 2024 **Time: 7:00 p.m. – 10:00 p.m.**

(I) Acronyms:

mgf	moment generating function
pdf/pmf	probability density/mass function
r.v.	random variable
CI	confidence interval
MLE	maximum likelihood estimator
MPT	most powerful test
UMPT	uniformly most powerful test
UMVUE	uniformly minimum variance unbiased estimator
$I(\cdot)$	indicator function
z_α, $t(\alpha,\nu)$, $\chi^2(\alpha,\nu)$	upper α-th quantile of $N(0,1)$, $t(0,1,\nu)$ and $\chi^2(\nu)$

(II) Commonly used pdfs or pmfs:

- Gamma distribution. The pdf of $X \sim \text{Gamma}(\alpha,\beta)$ is
$$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x},\quad x>0,\ \alpha>0,\ \beta>0.$$
- t-distribution. The pdf of $X \sim t(\mu,\sigma^2,\nu)$ is
$$\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}\sigma}\left[1+\frac{(x-\mu)^2}{\nu\sigma^2}\right]^{-\frac{\nu+1}{2}},\quad x\in\mathbb{R}\triangleq(-\infty,\infty),\ \mu\in\mathbb{R},\ \sigma>0.$$
- Laplace distribution. The pdf of $X \sim \text{Laplace}(\mu,\sigma)$ is
$$\frac{1}{2\sigma}\exp\left(-\frac{|x-\mu|}{\sigma}\right),\quad x\in\mathbb{R},\ \mu\in\mathbb{R},\ \sigma>0.$$
- Negative binomial distribution. The pmf of $Y \sim \text{NBinomial}(n,\theta)$ is
$$\Pr(Y=y)=\binom{n+y-1}{y}\theta^n(1-\theta)^y,\quad y=0,1,\ldots,\infty,\quad \theta\in(0,1).$$
In particular, when $n=1$, it is reduced to the geometric distribution, denoted by $Y\sim\text{Geometric}(\theta)=\text{NBinomial}(1,\theta)$ with pmf
$$\Pr(Y=y)=\theta(1-\theta)^y,\quad y=0,1,\ldots,\infty,\quad \theta\in(0,1).$$

Answer ALL 6 questions. Marks are shown in square brackets

1. Directly give your answers to the following questions:

1.1 Let two discrete r.v.'s $X, Y \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$ with $p = 0.6$. The value of $\Pr(X \leqslant Y)$ is ______. **[2 ms]**

1.2 Let $X \sim \text{Laplace}(\mu, \sigma)$ and define the standard Laplace r.v. $Y \triangleq (X - \mu)/\sigma$, then $Y \sim \text{Laplace}(0, 1)$.

(a) The mgf of Y is ______. **[2 ms]**

(b) The mgf of X is ______. **[1 mk]**

1.3 (a) If X is a discrete r.v. , what is the definition of the median of X, denoted by $\text{med}(X)$? **[1 mk]**

(b) The pmf of the discrete r.v. X is

X	2	3	4	5	6	7	8	9	10	11	12
$p_i = \Pr(X = i + 1)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

for $i = 1, \ldots, 11$. The median $\text{med}(X)$ of X is ______ **[2 ms]**

1.4 Let $X_1, \ldots, X_n$ be independent, and $X_i \sim \text{Logistic}(\mu_i, \sigma_i)$ with pdf

$$\text{Logistic}(x_i|\mu_i, \sigma_i) = \frac{\exp(-\frac{x_i - \mu_i}{\sigma_i})}{\sigma_i\{1 + \exp(-\frac{x_i - \mu_i}{\sigma_i})\}^2}, \quad x_i \in \mathbb{R},$$

where $\mu_i \in \mathbb{R}$ is the location parameter and $\sigma_i > 0$ is the scale parameter, $i = 1, \ldots, n$. The distribution of the smallest order statistic $X_{(1)} = \min(X_1, \ldots, X_n)$ is ______. **[2 ms]**

1.5 Let $X_1, \ldots, X_n, X \overset{\text{iid}}{\sim} f(x; \theta) = (1 - \theta)^{-1} I(\theta \leqslant x \leqslant 1)$, where $\theta \in (0, 1)$.

(a) The moment estimator of θ is ______. **[1 mk]**

(b) A sufficient statistic of θ is ______. **[1 mk]**

(c) The MLE of θ is ______. **[1 mk]**

1.6 Assume we want to find the root x^* of the equation $0 = g(x)$ for $x \in \mathbb{X}$. What is Newton's method to iteratively calculate the root x^* ? **[2 ms]**

1.7 In Bayesian statistics, let $X|\lambda \sim \text{Poisson}(\lambda)$ and the prior distribution of λ be $\text{Gamma}(a, b)$ with known $a\,(> 0)$ and $b\,(> 0)$.

(a) The posterior distribution $\lambda|(X = x)$ is ______. **[1 mk]**

(b) The marginal distribution of X is ______. **[1 mk]**

1.8 Let $\mathbb{C}$ be the critical region of a test for testing H_0: $\theta \in \Theta_0$ against H_1: $\theta \in \Theta_1$. What are the definitions of the Type I error function $\alpha(\theta)$ and the power function $p(\theta)$? **[2 ms]**

1.9 Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ with unknown μ and σ^2. Suppose that we want to test the null hypothesis H_0: $\mu = \mu_0$ against H_1: $\mu \neq \mu_0$.

(a) The pivotal quantity $T =$ ______ and the test statistic $T_1 =$ ______.

(b) Under H_0, the distribution of T_1 is ______.

(c) The critical region of size α for the test is ______.

(d) The corresponding p-value is ______. **[5 ms]**

1.10 Let $3.3, -0.3, -0.6, -0.9$ be a random sample from $N(\mu, \sigma^2)$.

(a) If $\sigma = 3$, The 90% CI of μ. **[2 ms]**

(b) What would be the CI of μ if σ were unknown? **[2 ms]**

[Note: $z_{0.05} = 1.645$, $t(0.05, 3) = 2.3534$]

1.11 Let $X_1, \ldots, X_n \sim N(\mu, 3.3^2)$ with $n = 30$ and $\bar{x} = 27$. Construct a 90% CI for μ, where $z_{0.05} = 1.645$. **[6 ms]**

1.12 Let X be a discrete random variable with pmf $p_i = \Pr(X = x_i)$ for $i = 1, 2$ and Y be a discrete random variable with pmf $q_j = \Pr(Y = y_j)$ for $j = 1, 2$. Given two conditional distribution matrices

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 3/5 & 2/5 \end{pmatrix},$$

where the (i, j) element of $\boldsymbol{A}$ is $a_{ij} = \Pr(X = x_i | Y = y_j)$ and the (i, j) element of $\boldsymbol{B}$ is $b_{ij} = \Pr(Y = y_j | X = x_i)$.

(a) Find the marginal distribution of X. **[2 ms]**

(b) Find the marginal distribution of Y. **[2 ms]**

(c) Find the joint distribution of (X, Y). **[2 ms]**

[Total: 40 ms]

2. Let $X \sim t(\mu, \sigma^2, \nu)$, then X can be stochastically represented by

$$X \stackrel{\mathrm{d}}{=} \mu + \frac{Z}{\sqrt{\xi/\nu}},$$

where $Z \sim N(0, \sigma^2)$, $\xi \sim \chi^2(\nu)$, and $Z \perp\!\!\!\perp \xi$. Now assume that

$$Y \stackrel{\mathrm{d}}{=} \mu + \frac{Z}{\sqrt{\tau}},$$

where $Z \sim N(0, \sigma^2)$, $\tau \sim \text{Gamma}(\alpha, \beta)$ and $Z \perp\!\!\!\perp \tau$. Prove that $Y \sim t(\mu, \sigma_*^2, \nu_*)$ and find the expressions of σ_*^2 and ν_*. **[Total: 7 ms]**

3. Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Geometric}(\theta)$ and define $\boldsymbol{X} \triangleq (X_1, \ldots, X_n)^\top$.

3.1 Show that $T(\boldsymbol{X}) \triangleq \sum_{i=1}^n X_i$ is a sufficient statistic of θ. **[3 ms]**

3.2 Use the mgf method to prove that $T(\boldsymbol{X}) \sim \text{NBinomial}(n, \theta)$. **[5 ms]**

Hint: For a positive integer n,

$$(x+a)^{-n} = \sum_{y=0}^{\infty} (-1)^y \binom{n+y-1}{y} x^y a^{-n-y}, \quad \text{for } |x| < a. \tag{1}$$

3.3 Show that $T(\boldsymbol{X})$ is complete for θ. **[5 ms]**

3.4 Find the unique UMVUE for $\tau(\theta) = (1-\theta)/\theta$ by the Lehmann–Scheffé Theorem. **[5 ms]**

3.5 Show that $\bar{X}$ is an efficient estimator for $\tau(\theta)$. **[5 ms]**

[Total: 23 ms]

4. Let $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} N(\theta, 1)$.

4.1 Use the Neyman–Pearson Lemma, to find the MPT φ of size α for testing H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$ with $\theta_1 > \theta_0$. **[8 ms]**

4.2 Find the power function $p_\varphi(\theta)$. **[4 ms]**

4.3 Find the UMPT of size α for testing H_0: $\theta \leqslant \theta_0$ vs H_1: $\theta > \theta_0$. **[8 ms]**

[Total: 20 ms]

5. In the 98 year period from 1900 to 1997, there were 159 U.S. land falling hurricanes. The numbers of hurricanes per year are summarized as follows:

Times of hurricanes per year (i)	0	1	2	3	4	5	6	Total
Frequency of years (N_i)	18	34	24	16	3	1	2	98

Does the number of land falling hurricanes per year follow a Poisson distribution when the approximate significance level is taken to be 0.05? [**10 ms**]

[Note: $\chi^2(0.05, 3) = 7.81$, $\chi^2(0.05, 4) = 9.49$, $\chi^2(0.05, 5) = 11.07$]

6. (Bonus question). Let $X, Y \overset{\text{iid}}{\sim} N(0, 1)$, and define a new r.v. Z by

$$Z = \begin{cases} X, & \text{if } XY > 0, \\ -X, & \text{if } XY < 0, \\ 0, & \text{if } X = 0. \end{cases}$$

Find the distribution of Z. [**5 ms**]

************ END OF THE PAPER A ************

1. Suggested Solutions.

1.1 Solution: Since $X \perp\!\!\!\perp Y$, we have

$$\begin{aligned}
\Pr(X \leqslant Y) &= \Pr(X=Y=0)+\Pr(X=Y=1)+\Pr(X=0,Y=1) \\
&= \Pr(X=0)\cdot\Pr(Y=0)+\Pr(X=1)\cdot\Pr(Y=1) \\
&\quad + \Pr(X=0)\cdot\Pr(Y=1) \\
&= (1-p)^2+p^2+(1-p)p \\
&= 1-p+p^2 = 0.76.
\end{aligned}$$

‖

1.2 Solution: See **Example T2.1**.

(a) Define $Y=(X-\mu)/\sigma$, then the pdf of Y is

$$f_Y(y) = f_X(x)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = 0.5\mathrm{e}^{-|y|}, \quad y \in \mathbb{R}.$$

Thus,

$$M_X(t) = E(\mathrm{e}^{tX}) = E[\mathrm{e}^{t(\mu+\sigma Y)}] = \mathrm{e}^{\mu t}M_Y(\sigma t) \tag{2}$$

Now,

$$\begin{aligned}
M_Y(t) &= E(\mathrm{e}^{tY}) = \int_{-\infty}^{\infty} \mathrm{e}^{ty}\cdot 0.5\mathrm{e}^{-|y|}\,\mathrm{d}y \\
&= 0.5\int_{-\infty}^{0} \mathrm{e}^{(t+1)y}\,\mathrm{d}y + 0.5\int_{0}^{\infty} \mathrm{e}^{(t-1)y}\,\mathrm{d}y \;\triangleq\; 0.5I_1+0.5I_2. \qquad (3)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{-\infty}^{0} \mathrm{e}^{(t+1)y}\,\mathrm{d}y \qquad (4) \\
&= \frac{1}{t+1}\mathrm{e}^{(t+1)y}\Big|_{-\infty}^{0} = \frac{1}{t+1}(1-0) = \frac{1}{1+t}, \quad \text{and} \\
I_2 &= \int_{0}^{\infty} \mathrm{e}^{(t-1)y}\,\mathrm{d}y \qquad [\text{let } y=-z]
\end{aligned}$$

$$= \int_0^{-\infty} (-1)\mathrm{e}^{-(t-1)z}\,\mathrm{d}z = \int_{-\infty}^0 \mathrm{e}^{(1-t)z}\,\mathrm{d}z \qquad \text{[similar to (4)]}$$
$$= \frac{1}{1-t}.$$

Hence, (3) becomes

$$M_Y(t) = \frac{0.5}{1+t} + \frac{0.5}{1-t} = \frac{1}{1-t^2}.$$

(b) From (2), we obtain

$$M_X(t) = \frac{\mathbf{e}^{\mu t}}{1-\sigma^2 t^2}. \qquad \|$$

1.3 **Solution**: (a) The median of X satisfies

$$\Pr\{X \leqslant \mathbf{med}(X)\} \geqslant 0.5 \quad \textbf{and} \quad \Pr\{X \geqslant \mathbf{med}(X)\} \geqslant 0.5.$$

See page 18 of the Textbook.

(b) We have $\mathrm{med}(X) = 7$ because

$$\Pr(X \leqslant 7) = \frac{1+2+3+4+5+6}{36} = \frac{21}{36} \approx 0.583 \geqslant 0.5 \quad \text{and}$$
$$\Pr(X \geqslant 7) = \frac{6+5+4+3+2+1}{36} = \frac{21}{36} \approx 0.583 \geqslant 0.5.$$

See **Example T1.8**. $\|$

1.4 **Solution**: Let $X_i \sim \mathrm{Logistic}(\mu_i, \sigma_i)$, then the cdf of X_i is given by

$$F_i(x|\mu_i, \sigma_i) = \left(1 + \mathrm{e}^{-\frac{x-\mu_i}{\sigma_i}}\right)^{-1}, \quad x \in \mathbb{R}.$$

So the cdf of $X_{(1)}$ is

$$\Pr\{X_{(1)} \leqslant x\} = 1 - \Pr\{\min(X_1, \ldots, X_n) > x\}$$
$$= 1 - \prod_{i=1}^n \Pr(X_i > x) = 1 - \prod_{i=1}^n [1 - F_i(x|\mu_i, \sigma_i)]$$

$$= 1 - \prod_{i=1}^{n} \left[1 - \left(1 + \mathrm{e}^{-\frac{x-\mu_i}{\sigma_i}} \right)^{-1} \right]$$

$$= 1 - \prod_{i=1}^{n} \frac{\mathrm{e}^{-\frac{x-\mu_i}{\sigma_i}}}{1 + \mathrm{e}^{-\frac{x-\mu_i}{\sigma_i}}}.$$

1.5 <u>**Solution**</u>: (a) The expectation of $X \sim f(x;\theta)$ is

$$E(X) = \int_\theta^1 x \frac{1}{1-\theta}\,\mathrm{d}x = \frac{1}{1-\theta} \frac{x^2}{2}\Big|_\theta^1 = \frac{1}{1-\theta}\frac{1-\theta^2}{2} = \frac{1+\theta}{2}.$$

Let $0.5(1+\theta) = \bar{X}$, we obtain the moment estimator of θ is $\hat{\theta}^{\mathrm{M}} = 2\bar{X} - 1$.

(b) The likelihood function of θ is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{1-\theta} I(\theta \leqslant x_i \leqslant 1) = \frac{1}{(1-\theta)^n} I(\theta \leqslant x_{(1)}) \cdot 1,$$

According to the factorization theorem, we known that $X_{(1)} = \min(X_1, \ldots, X_n)$ is a sufficient estimator of θ.

(c) Note that $L(\theta)$ is an increasing function of θ over the interval $(0, x_{(1)})$, thus, $X_{(1)}$ **is the MLE of** θ. ‖

1.6 <u>**Solution**</u>: Newton's method to iteratively calculate the root x^* of the equation $g(x) = 0$ is

$$x^{(t+1)} = x^{(t)} - \frac{g(x^{(t)})}{g'(x^{(t)})}, \quad t = 0, 1, 2, \ldots, \infty.$$ ‖

1.7 <u>**Solution**</u>: (i) The joint density of (X, λ) is

$$f(x,\lambda) = f(x|\lambda) \cdot \pi(\lambda) = \frac{\lambda^x}{x!}\mathrm{e}^{-\lambda} \times \frac{b^a}{\Gamma(a)} \lambda^{a-1} \mathrm{e}^{-b\lambda},$$

so that the posterior density is

$$p(\lambda|x) = \frac{f(x,\lambda)}{f_X(x)} \propto f(x,\lambda) \propto \lambda^{x+a-1}\mathrm{e}^{-(1+b)\lambda},$$

i.e., $\lambda|(X=x) \sim \text{Gamma}(x+a, 1+b)$.

(ii) The marginal density of X is

$$f_X(x) = \int_0^\infty f(x,\lambda)\,\mathrm{d}\lambda = \frac{\Gamma(x+a)}{x!\Gamma(a)}\left(\frac{b}{1+b}\right)^a \left(\frac{1}{1+b}\right)^x,$$

for $x = 0, 1, \ldots, \infty$, i.e., $X \sim$ **GPoisson**(a, b). $\|$

1.8 **<u>Solution</u>**: (i) The Type I error function is defined by

$$\begin{aligned}\alpha(\theta) = \Pr(\text{Type I error}) &= \Pr(\text{rejecting } H_0|H_0 \text{ is true}) \\ &= \Pr(\mathbf{x} \in \mathbb{C}|\theta \in \Theta_0),\end{aligned}$$

which is a function of θ defined in Θ_0, where $\mathbf{x} = (X_1, \ldots, X_n)^\top$.

(ii) The power function is defined by

$$p(\theta) = \Pr(\text{rejecting } H_0|\theta) = \Pr(\mathbf{x} \in \mathbb{C}|\theta). \qquad \|$$

1.9 **<u>Solution</u>**: (i) The pivotal quantity is

$$T \triangleq \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1),$$

where

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n} \quad \text{and} \quad S = \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}}.$$

The test statistic is

$$T_1 \triangleq \frac{\bar{X}-\mu_0}{S/\sqrt{n}}.$$

(ii) When H_0 is true, we obtain $T_1 \sim t(n-1)$.

(iii) The critical regions of size α for the test is $\mathbb{C} = \{\boldsymbol{x}\text{:}\ |t_1| \geqslant t(\alpha/2, n-1)\}$, where

$$t_1 = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \quad \text{with } \bar{x} = \frac{\sum_{i=1}^n x_i}{n} \quad \text{and} \quad s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}.$$

and $t(\alpha, n-1)$ denotes the upper α-th quantile of $t(n-1)$.

(iv) The corresponding p-value can be calculated by

$$\textbf{\textit{p}-value} = 2\,\Pr(T \geqslant |t_1|). \qquad \|$$

1.10 **Solution**: (a) When $\sigma = \sigma_0$ is known, from (4.4) of Chapter 4, we know that

$$\left[\bar{X} - z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}},\ \bar{X} + z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right] = [-2.0925,\ 2.8425]$$

is a $100(1-\alpha)\%$ CI for the mean μ, where $n = 4$, $\alpha = 0.1$, $z_{\alpha/2} = z_{0.05} = 1.645$, $\sigma_0 = 3$, and

$$\bar{X} = \frac{3.3 - 0.3 - 0.6 - 0.9}{4} = 0.375.$$

(b) When σ is unknown, from (4.6) of Chapter 4, we know that

$$\left[\bar{X} - t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}},\ \bar{X} + t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}}\right] = [-1.937, 2.687]$$

is a $100(1-\alpha)\%$ CI for the mean μ, where $\bar{X} = 0.375$, $n = 4$, $t(\alpha/2, n-1) = t(0.05, 3) = 2.3534$, and

$$S = \sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}} = \sqrt{3.863} = 1.965. \qquad \|$$

1.11 **Solution**: Since $\bar{X}$ is a sufficient statistic of μ, we have a pivotal quantity

$$\begin{aligned}
& P = \frac{\sqrt{n}(\bar{X} - \mu)}{3.3} \sim N(0, 1), \\
\Rightarrow\ & \Pr\left\{-z_{\alpha/2} \leqslant \frac{\sqrt{n}(\bar{X} - \mu)}{3.3} \leqslant z_{\alpha/2}\right\} = 1 - \alpha, \\
\Rightarrow\ & \Pr\left(\bar{X} - z_{\alpha/2}\frac{3.3}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} + z_{\alpha/2}\frac{3.3}{\sqrt{n}}\right) = 1 - \alpha, \\
\Rightarrow\ & \Pr\left(\bar{X} - z_{0.05}\frac{3.3}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} + z_{0.05}\frac{3.3}{\sqrt{n}}\right) = 0.9.
\end{aligned}$$

Therefore, a 90% CI for μ is given by

$$\left[27 - 1.645\frac{3.3}{\sqrt{30}},\ 27 + 1.645\frac{3.3}{\sqrt{30}}\right] = [26.0089,\ 27.9911]\,. \qquad \|$$

1.12 **Solution**: Note that $\mathcal{S}_X = \{x_1, x_2\}$ and $\mathcal{S}_Y = \{y_1, y_2\}$. By using pointwise IBF, the marginal distribution of X is given by

X	x_1	x_2
$p_i = \Pr(X = x_i)$	$3/8$	$5/8$

Similarly, the marginal distribution of Y is given by

Y	y_1	y_2
$q_j = \Pr(Y = y_j)$	$1/2$	$1/2$

The joint distribution of (X, Y) is given by

$$\boldsymbol{P} = \begin{pmatrix} 1/8 & 1/4 \\ 3/8 & 1/4 \end{pmatrix}. \qquad \|$$

2. [L]**Proof**. The pdf of $W \sim \text{Gamma}(\alpha, \beta)$ is

$$\frac{\beta^\alpha}{\Gamma(\alpha)} w^{\alpha-1} \mathrm{e}^{-\beta w}, \quad w > 0,\ \alpha > 0,\ \beta > 0.$$

We know that

$$c \cdot \text{Gamma}(\alpha, \beta) \stackrel{\mathrm{d}}{=} \text{Gamma}(\alpha, \beta/c), \quad \text{for any } c > 0. \tag{5}$$

$$\text{Gamma}(\nu/2, 1/2) = \chi^2(\nu), \tag{6}$$

We have the following SR:

$$\begin{aligned} Y &\stackrel{\mathrm{d}}{=} \mu + \frac{Z}{\sqrt{\tau}} = \mu + \frac{N(0, \sigma^2)}{\sqrt{\text{Gamma}(\alpha, \beta)}} \\ &\stackrel{(5)}{=} \mu + \frac{N(0, \sigma^2)}{\sqrt{(2\beta)^{-1} \cdot \text{Gamma}(\alpha, 1/2)}} \\ &\stackrel{(6)}{=} \mu + \frac{N(0, \sigma^2)}{\sqrt{2\alpha(2\beta)^{-1} \cdot \chi^2(2\alpha)/(2\alpha)}} \\ &= \mu + \frac{N(0, \beta\sigma^{\mathbf{2}}/\alpha)}{\sqrt{\chi^2(2\alpha)/(2\alpha)}} \sim t(\mu, \sigma_*^2, \nu_*), \end{aligned}$$

where $\sigma_*^2 = \beta\sigma^2/\alpha$ and $\nu_* = 2\alpha$. $\|$

3. [L] **Solution**.

3.1 The joint pmf of $X_1, \ldots, X_n$ is

$$f(\boldsymbol{x}; \theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i} = \theta^n (1-\theta)^{T(\boldsymbol{x})} \times 1 = g(T(\boldsymbol{x}); \theta) \times h(\boldsymbol{x}),$$

where $g(t; \theta) = \theta^n (1-\theta)^t$ and $h(\boldsymbol{x}) = 1$. Therefore, $T(\boldsymbol{X})$ is sufficient for θ.

3.2 The mgf of X_i $(i = 1, \ldots, n)$ is

$$\begin{aligned} M_{X_i}(t) &= E(\mathrm{e}^{tX_i}) = \sum_{x=0}^{\infty} \mathrm{e}^{tx} \theta (1-\theta)^x \\ &= \theta \sum_{x=0}^{\infty} [\mathrm{e}^t (1-\theta)]^x = \frac{\theta}{1 - \mathrm{e}^t (1-\theta)}, \quad t < -\log(1-\theta). \end{aligned}$$

So the mgf of $T(\boldsymbol{X})$ is

$$M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left[\frac{\theta}{1 - \mathrm{e}^t (1-\theta)} \right]^n, \quad t < -\log(1-\theta).$$

On the other hand, let $Y \sim \text{NBinomial}(n, \theta)$. The mgf of Y is

$$\begin{aligned} M_Y(t) &= E(\mathrm{e}^{tY}) = \sum_{y=0}^{\infty} \mathrm{e}^{ty} \binom{n+y-1}{y} \theta^n (1-\theta)^y \\ &= \theta^n \times \sum_{y=0}^{\infty} \binom{n+y-1}{y} [\mathrm{e}^t (1-\theta)]^y \\ &= \theta^n \times \sum_{y=0}^{\infty} (-1)^y \binom{n+y-1}{y} [-\mathrm{e}^t (1-\theta)]^y \cdot 1^{-n-y} \\ &\overset{(1)}{=} \theta^n \times [1 - \mathrm{e}^t (1-\theta)]^{-n} \\ &= \left[\frac{\theta}{1 - \mathrm{e}^t (1-\theta)} \right]^n, \qquad t < -\log(1-\theta). \end{aligned}$$

Since $M_T(t) = M_Y(t)$, $T(\boldsymbol{X}) \sim \text{NBinomial}(n, \theta)$.

3.3 Assume that $E\{h(T)\} = 0$, we have

$$\sum_{t=0}^{\infty} h(t) \binom{n+t-1}{t} \theta^n (1-\theta)^t = 0,$$

or

$$\sum_{t=0}^{\infty} \binom{n+t-1}{t} h(t)(1-\theta)^t = 0, \quad 0 < \theta < 1. \tag{7}$$

The equation (7) is a polynomial of $(1-\theta)$ and $(1-\theta)$ must be nonzero. The fact that it equals to zero implies that all its coefficients are zero, i.e.,

$$\binom{n+t-1}{t} h(t) = 0, \quad t = 0, 1, \ldots.$$

Hence, $h(T) = 0$, i.e. $\Pr\{h(T) = 0\} = 1$. Therefore, $T(\boldsymbol{X})$ is complete for θ.

3.4 We have

$$\begin{aligned} E[T(\boldsymbol{X})] &= \left.\frac{\mathrm{d}M_T(t)}{\mathrm{d}t}\right|_{t=0} = \left.\frac{n\mathrm{e}^t\theta^n(1-\theta)}{[1-\mathrm{e}^t(1-\theta)]^{n+1}}\right|_{t=0} \\ &= n \cdot \frac{1-\theta}{\theta} = n \cdot \tau(\theta). \end{aligned}$$

Denote $\bar{X} = T(\boldsymbol{X})/n$. It implies that $E(\bar{X}) = \tau(\theta)$, and thus $\bar{X}$ is an unbiased estimator for $\tau(\theta)$. Because $T(\boldsymbol{X})$ is sufficient and complete, according to the Lehmann–Scheffé Theorem, $\bar{X}$ is the unique UMVUE for $\tau(\theta) = (1-\theta)/\theta$.

3.5 To prove that $\bar{X}$ is an efficient estimator for $\tau(\theta)$, we need to show that $\text{Var}(\bar{X})$ equals to the Cramér–Rao lower bound. Since

$$\begin{aligned} E[T(\boldsymbol{X})]^2 &= \left.\frac{\mathrm{d}^2 M_T(t)}{\mathrm{d}t^2}\right|_{t=0} = \left.\frac{ne^t\theta^n(1-\theta)[1-ne^t(1-\theta)]}{[1-e^t(1-\theta)]^{n+2}}\right|_{t=0} \\ &= \frac{n(1-\theta)[1-n(1-\theta)]}{\theta^2}, \end{aligned}$$

we have

$$\text{Var}[T(\boldsymbol{X})] = E[T(\boldsymbol{X})]^2 - \{E[T(\boldsymbol{X})]\}^2 = \frac{n(1-\theta)}{\theta^2}.$$

Hence,

$$\text{Var}(\bar{X}) = \frac{\text{Var}[T(\boldsymbol{X})]}{n^2} = \frac{1-\theta}{n\theta^2}.$$

The log-likelihood function is

$$\ell(\theta; \boldsymbol{X}) = \log f(\boldsymbol{X}; \theta) = n \log \theta + T(\boldsymbol{X}) \log(1-\theta).$$

Thus, the Fisher information is

$$I_n(\theta) = E\left[-\frac{\mathrm{d}^2\ell(\theta; \boldsymbol{X})}{\mathrm{d}\theta^2}\right] = E\left[\frac{n}{\theta^2} + \frac{T(\boldsymbol{X})}{(1-\theta)^2}\right] = \frac{n}{\theta^2(1-\theta)}.$$

Since

$$\tau'(\theta) = \frac{\mathrm{d}\tau(\theta)}{\mathrm{d}\theta} = -\frac{1}{\theta^2},$$

the Cramér–Rao lower bound is

$$\upsilon(\theta) = \frac{[\tau'(\theta)]^2}{I_n(\theta)} = \frac{1-\theta}{n\theta^2}.$$

Since $\text{Var}(\bar{X}) = \upsilon(\theta)$, $\bar{X}$ is an efficient estimator for $\tau(\theta) = (1-\theta)/\theta$. ∥

4. [B] **Solution**. See **Example 5.4** on page 195 of Textbook Chapter 5.

4.1 The likelihood function is given by

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x_i - \theta)^2\right\} = (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2\right\}.$$

Then

$$\begin{aligned} \frac{L(\theta_0)}{L(\theta_1)} &= \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\{(x_i - \theta_0)^2 - (x_i - \theta_1)^2\}\right] \\ &= \exp\left\{(\theta_0 - \theta_1)\sum_{i=1}^{n} x_i - n(\theta_0^2 - \theta_1^2)/2\right\} \leqslant k \end{aligned}$$

is equivalent to

$$\bar{x} \geqslant \frac{\log(k)}{n(\theta_0 - \theta_1)} + \frac{\theta_0 + \theta_1}{2} \quad \triangleq \quad c.$$

To determine c, we consider the size

$$\begin{aligned} \alpha &= \Pr(\bar{X} \geqslant c | \theta = \theta_0) = \Pr\{\sqrt{n}(\bar{X} - \theta_0) \geqslant \sqrt{n}(c - \theta_0)\} \\ &= \Pr\{Z \geqslant \sqrt{n}(c - \theta_0)\} = \Pr(Z \geqslant z_\alpha). \end{aligned}$$

Then, $\sqrt{n}(c - \theta_0) = z_\alpha$ or $c = \theta_0 + z_\alpha/\sqrt{n}$. Thus, the test with critical region $\mathbb{C} = \{\boldsymbol{x}\text{:}\ \bar{x} \geqslant \theta_0 + z_\alpha/\sqrt{n}\,\}$ is the MPT φ of size α.

4.2 The power function

$$\begin{aligned} p_\varphi(\theta) &= \Pr(\mathbf{x} \in \mathbb{C} \mid \theta) = \Pr(\bar{X} \geqslant \theta_0 + z_\alpha/\sqrt{n} \mid \theta) \\ &= \Pr\{\sqrt{n}(\bar{X} - \theta) \geqslant z_\alpha + \sqrt{n}(\theta_0 - \theta) \mid \theta\} \\ &= \Pr\{Z \geqslant z_\alpha + \sqrt{n}(\theta_0 - \theta)\} \\ &= 1 - \Phi\{z_\alpha + \sqrt{n}(\theta_0 - \theta)\}. \qquad (8) \end{aligned}$$

4.3 See **Example 5.7** on page 200 of Textbook Chapter 5.

Step 1: From the previous question, the test φ with critical region

$$\mathbb{C} = \{\boldsymbol{x}\colon\ \bar{x} \geqslant \theta_0 + z_\alpha/\sqrt{n}\,\}$$

is the MPT of size α for testing H_{0s}: $\theta = \theta_0$ against H_{1s}: $\theta = \theta_1 > \theta_0$. Therefore, we obtain

$$\begin{aligned} p_\varphi(\theta_0) &= \alpha \qquad \text{and} & (9) \\ p_\varphi(\theta) &\overset{(8)}{=} 1 - \Phi\{z_\alpha + \sqrt{n}(\theta_0 - \theta)\}. & (10) \end{aligned}$$

Step 2: Since the critical region $\mathbb{C}$ depends only on n, θ_0, α and the fact $\theta_1 > \theta_0$, but not on the value of θ_1, the test φ is also the UMPT of size α for testing

$$H_{0s}\colon\ \theta = \theta_0 \quad \text{against} \quad H_1\colon\ \theta > \theta_0. \tag{11}$$

The latter can be obtained immediately since φ is the UMPT of size α for testing (11). Thus, we only need to prove the type I error rate cannot exceed α.

Step 3: It follows from (10) that

$$\begin{aligned} \sup_{\theta \in \Theta_0} p_\varphi(\theta) &= \sup_{\theta \leqslant \theta_0} [1 - \Phi\{z_\alpha + \sqrt{n}(\theta_0 - \theta)\}] \\ &= \max_{\theta \leqslant \theta_0} [1 - \Phi\{z_\alpha + \sqrt{n}(\theta_0 - \theta)\}] \\ &= 1 - \min_{\theta \leqslant \theta_0} \Phi\{z_\alpha + \sqrt{n}(\theta_0 - \theta)\} \\ &= 1 - \Phi(z_\alpha) \qquad [\because \Phi(-x) \text{ is a decreasing function of } x] \\ &= 1 - (1 - \alpha) \ = \ \alpha \overset{(9)}{=} p_\varphi(\theta_0). \end{aligned}$$

Then, the test φ is also the UMPT of size α for testing

$$H_0\colon\ \theta \leqslant \theta_0 \quad \text{against} \quad H_1\colon\ \theta > \theta_0. \qquad \|$$

5. [T] **Solution**. We wish to test

$$H_0: \quad \text{The distribution is Poisson} \qquad \text{against}$$
$$H_1: \quad \text{The distribution is not Poisson.}$$

Under H_0, the maximum likelihood estimate of λ is

$$\hat{\lambda} = \overline{x} = \frac{159}{98} \approx 1.622.$$

Now

$$\hat{p}_{i0} = p_{i0}(\hat{\lambda}) = \frac{\hat{\lambda}^i}{i!}\mathrm{e}^{-\hat{\lambda}},\ i = 0, 1, \ldots, 5, \qquad \hat{p}_{6,0} = 1 - \sum_{i=0}^{5} \hat{p}_{i0},$$

and $n = 98$, we obtain

i	0	1	2	3	4	5	$6(\geq 6)$
N_i	18	34	24	16	3	1	2
$\hat{p}_{i0}$	0.1974	0.3203	0.2598	0.1405	0.0570	0.0185	0.0064
$n\hat{p}_{i0}$	19.3466	31.3889	25.4635	13.7711	5.5857	1.8125	0.6317

Those classes with expected frequencies less than 5 should be combined with the adjacent class. Therefore, we combine the last 3 classes, and the revised table is

i	0	1	2	3	$4(\geq 4)$
N_i	18	34	24	16	6
$\hat{p}_{i0}$	0.1974	0.3203	0.2598	0.1405	0.0819
$n\hat{p}_{i0}$	19.3466	31.3889	25.4635	13.7711	8.0299

So we have

$$\hat{Q}_{98} = \sum_{i=0}^{4} \frac{(N_i - n\hat{p}_{i0})^2}{n\hat{p}_{i0}} = 1.2690 < \chi^2(0.05, 5-1-1) = 7.81.$$

Thus, we cannot reject H_0 when the approximate significance level is taken to be 0.05. $\quad\|$

6. [U] **Solution**. Since $X, Y \overset{\text{iid}}{\sim} N(0,1)$, we have

$$\Pr(X \geqslant -z) = \Pr(X \leqslant z) \quad \text{and} \quad \Pr(Y < 0) = \Pr(Y > 0) = 0.5. \tag{12}$$

By the definition of Z, for $z < 0$, we have

$$\begin{aligned}
& \Pr(Z \leqslant z) \\
= \ & \Pr(X \leqslant z,\ XY > 0) + \Pr(-X \leqslant z,\ XY < 0) \\
= \ & \Pr(X \leqslant z,\ Y < 0) + \Pr(X \geqslant -z,\ Y < 0) \qquad (\because z < 0) \\
= \ & \Pr(X \leqslant z)\ \Pr(Y < 0) + \Pr(X \geqslant -z)\ \Pr(Y < 0) \qquad (\because X \perp\!\!\!\perp Y) \\
\overset{(12)}{=} \ & \Pr(X \leqslant z)\, 0.5 + \Pr(X \leqslant z)\, 0.5 \\
= \ & \Pr(X \leqslant z).
\end{aligned} \tag{13}$$

Similarly, for $z > 0$, we obtain

$$\begin{aligned}
& \Pr(Z > z) \\
= \ & \Pr(X > z,\ XY > 0) + \Pr(-X > z,\ XY < 0) \\
= \ & \Pr(X > z,\ Y > 0) + \Pr(X < -z,\ Y > 0) \qquad (\because z > 0) \\
= \ & \Pr(X > z)\ \Pr(Y > 0) + \Pr(X < -z)\ \Pr(Y > 0) \qquad (\because X \perp\!\!\!\perp Y) \\
\overset{(12)}{=} \ & \Pr(X > z)\, 0.5 + \Pr(X > z)\, 0.5 \\
= \ & \Pr(X > z),
\end{aligned}$$

implying that

$$\Pr(Z \leqslant z) = \Pr(X \leqslant z) \quad \text{for any} \quad z > 0. \tag{14}$$

Hence, by combining (13) and (14), we have

$$Z \overset{\text{d}}{=} X \sim N(0,1). \qquad \|$$