Abstract Algebra
: special homework: 14

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In our class we didn't introduce a lot of the concepts about algebraic closure. But I think it's a very important concept to learn for all of you. So this homework is based on this. We begin with the definition of algebraically closed. All things based on the book of Birkhauser. Warning, this 'homework' is like a proof rush.

Definition 1. *A field K is called algebraically closed if every nonconstant polynomial f of $K[X]$ admits a zero in K, or in other words, if f decomposes in $K[X]$ into a product of linear factors. This means that f admits a product decomposition $f = c\prod_i (X - \alpha_i)$ with a constant $c \in K^*$ as well as zeros $\alpha_i \in K$.*

Example 2. $\mathbb{C}$ *is actually algebraically closed. Because every polynomial with complex coefficients has a complex root.*

Exercise 1. *A field K is algebraically closed if and only if every algebraic field extension L/K is trivial.*

证明. Assume that K is algebraically closed and that $K \subset L$ is an algebraic field extension. Furthermore, consider an element $\alpha \in L$ together with its minimal polynomial $f \in K[X]$. Then f decomposes over K into a product of linear factors and hence is linear, since it is irreducible. In particular, this shows that $\alpha \in K$ and therefore $L = K$. Conversely, assume that K does not admit any nontrivial algebraic field extension and consider a polynomial $f \in K[X]$ of degree ≥ 1. Using Kronecker's construction, there exists an algebraic field extension L/K such that f admits a zero in L. However, by our assumption we must have $L = K$, so that f has a zero in K. Consequently, K is algebraically closed. □

Theorem 3. *Every field K admits an extension field L that is algebraically closed.*

This theorem also tells us there exists an algebraic closure for all fields. There are many ways to prove this theorem, one of the most famous proof is by Artin using Zorn's Lemma. If you are interested in this proof you can find it in many abstract algebra textbooks. My favorite one is in GTM 167 by Patrick Morandi. In our course you can just admit that this theorem is true and use it as a tool.

Definition 4. *Let K be a field. Then there exists an algebraically closed field $\bar{K}$ extending K, where $\bar{K}$ is algebraic over K; such a field $\bar{K}$ is called an algebraic closure of K.*

Example 5. $\mathbb{C}$ *is an algebraic closure of* $\mathbb{R}$.

Example 6. *Define an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ by setting:*

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C}; \alpha \textit{ is algebraic over } \mathbb{Q}\}.$$

Exercise 2. *(1). Show that $\overline{\mathbb{Q}}$ is a field.*
(2). Show that $\overline{\mathbb{Q}}$ has countably infinite many elements.
(3). Show that $\overline{\mathbb{Q}[x]}$ is a subfield of $\mathbb{C}$.

证明. (1). $\forall \alpha, \beta \in \overline{\mathbb{Q}}$, take $\mathbb{Q}(\alpha, \beta)$ as an extension of $\mathbb{Q}$, since α and β are algebraic over $\mathbb{Q}$, $\mathbb{Q}(\alpha, \beta)$ is a finite extension of $\mathbb{Q}$. Thus since $\alpha+\beta, \alpha-\beta, \alpha\beta, \alpha/\beta \in \mathbb{Q}(\alpha, \beta)$, they are all algebraic over $\mathbb{Q}$. Therefore, all algebraic numbers over $\mathbb{Q}$ form a field.
(2). For every nonnegative integer N, define the set of polynomials

$$P_N := \{(a_n z^n + \cdots + a_1 z + a_0) \, | n + |a_0| + \cdots + |a_n| = N\}.$$

Then that P_N is finite. Now, each polynomial has finite number of roots (counting with multiplicity, a polynomial f has exactly $\deg(f)$ roots in $\mathbb{C}$).
Let R_N be the set of all roots of polynomials in P_N :

$$R_N := \{\alpha \in \mathbb{C} \mid \exists f \in P_N : f(\alpha) = 0\}$$

So, R_N is still finite. Finally $\{$ algebraic numbers $\} = \bigcup_{N \geq 0} R_N$.
(3). $\mathbb{Q}(\pi) \simeq \mathbb{Q}(x)$. And $\overline{\mathbb{Q}(x)} \simeq \overline{\mathbb{Q}(\pi)} \subset \mathbb{C}$. □

Exercise 3. *Let K be a field and $K' = K(\alpha)$ a simple algebraic field extension of K with attached minimal polynomial $f \in K[X]$ of α. Furthermore, let $\sigma : K \longrightarrow L$ be a field homomorphism.*
(i) If $\sigma' : K' \longrightarrow L$ is a field homomorphism extending σ, then $\sigma'(\alpha)$ is a zero of f^σ.
(ii) Conversely, for every zero $\beta \in L$ of $f^\sigma \in L[X]$, there is precisely one extension $\sigma' : K' \longrightarrow L$ of σ such that $\sigma'(\alpha) = \beta$.
In particular, the different extensions σ' of σ are in one-to-one correspondence with the distinct zeros of f^σ in L, and the number of these is $\leq \deg f$.

证明. For every extension $\sigma' : K' \longrightarrow L$ of σ we get from $f(\alpha) = 0$ necessarily $f^\sigma(\sigma'(\alpha)) = \sigma'(f(\alpha)) = 0$.
Think about it in this way: $\sigma = \sigma' |_K$. So $f^\sigma(\sigma'(\alpha)) = a_n^\sigma (\alpha^n)^{\sigma'} + \cdots + a_0^\sigma = (a_n \alpha^n + \cdots + a_0)^{\sigma'} = \sigma'(f(\alpha)) = 0$.
Moreover, since $K' = K[\alpha]$, every extension $\sigma' : K' \longrightarrow L$ of σ is uniquely determined by the image $\sigma'(\alpha)$ of α.

It remains to show for each zero $\beta \in L$ of f^σ that there is an extension $\sigma' : K' \longrightarrow L$ of σ satisfying $\sigma'(\alpha) = \beta$. To do this, consider the substitution homomorphisms:

$$\varphi : K[X] \to K[\alpha] : g \mapsto g(\alpha)$$

$$\psi : K[X] \to L, g \mapsto g^\sigma(\beta).$$

We have $(f) = \ker \varphi$ by, as well as $(f) \subset \ker \psi$, since $f^\sigma(\beta) = 0$. If $\pi : K[X] \longrightarrow K[X]/(f)$ denotes the canonical projection, we obtain a commutative diagram:

$$\begin{tikzcd}
 & K[X] \arrow[dl, "\varphi"'] \arrow[d, "\pi"] \arrow[dr, "\psi"] & \\
K[\alpha] & K[X]/(f) \arrow[l, "\bar{\varphi}"] \arrow[r, "\bar{\psi}"'] & L
\end{tikzcd}$$

with homomorphisms $\bar{\varphi}$ and $\bar{\psi}$ that are unique. Since $\bar{\varphi}$ is an isomorphism, we recognize $\sigma' := \bar{\psi} \circ \bar{\varphi}^{-1}$ as an extension of σ satisfying $\sigma'(\alpha) = \beta$. $\square$

Theorem 7. *Let $K \subset K'$ be an algebraic field extension and $\sigma : K \longrightarrow L$ a field homomorphism with image in an algebraically closed field L. Then σ admits an extension $\sigma' : K' \longrightarrow L$. In addition, if K' is algebraically closed and L algebraic over $\sigma(K)$, then every extension σ' of σ is an isomorphism.*

This proof needs Zorn's Lemma.

Theorem 8. *Let $\bar{K}_1$ and $\bar{K}_2$ be two algebraic closures of a field K. Then there exists an isomorphism $\bar{K}_1 \simeq \bar{K}_2$, noncanonical in general, that extends the identity map on K.*

This theorem shows that the algebraic closure of a given field is unique up to isomorphic. After that, we reformulate something about the concepts splitting field and normal extension.

Definition 9. *Let $\mathfrak{F} = (f_i)_{i \in I}, f_i \in K[X]$, be a family of nonconstant polynomials with coefficients in a field K. A splitting field (over K) of the family $\mathfrak{F}$ is a field L extending K such that:*
(i) Every polynomial f_i decomposes into a product of linear factors over L.
(ii) The extension L/K is generated by the zeros of the polynomials f_i.

Definition 10. *An algebraic field extension $K \subset L$ is called normal if it satisfies the equivalent conditions of the following exercise/theorem.*

Exercise 4. *The following conditions are equivalent for a field K and an algebraic extension $K \subset L$:*
(i) Every K-homomorphism $L \longrightarrow \bar{L}$ into an algebraic closure $\bar{L}$ of L restricts to an automorphism of L.
(ii) L is a splitting field of a family of polynomials in $K[X]$.
(iii) Every irreducible polynomial in $K[X]$ that admits a zero in L decomposes over L completely into linear factors.

If we only consider finite extension then "normal" equivalent to "a splitting field of one (not a family of) polynomial".

证明. We start with the implication from (i) to (iii) and consider an irreducible polynomial $f \in K[X]$ admitting a zero $a \in L$. If $b \in \bar{L}$ is another zero of f, we can conclude that there is a K-homomorphism $\sigma : K(a) \longrightarrow \bar{L}$ satisfying $\sigma(a) = b$. Furthermore, we can extend σ to a K-homomorphism $\sigma' : L \longrightarrow \bar{L}$. Now, if condition (i) is given, we obtain $\sigma'(L) = L$ and thereby see that $b = \sigma'(a) \in L$. Hence, all zeros of f are contained in L, and f decomposes over L into a product of linear factors.

Next we show that (iii) implies condition (ii). Let $(a_i)_{i \in I}$ be a family of elements in L such that the field extension L/K is generated by the elements a_i. Furthermore, let f_i be the minimal polynomial of a_i over K. Then, according to (iii), all f_i decompose over L into a product of linear factors, and we see that L is a splitting field of the family $\mathfrak{F} = (f_i)_{i \in I}$.

Finally, assume condition (ii), i.e., that L is a splitting field of a family $\mathfrak{F}$ of polynomials in $K[X]$. If $\sigma : L \longrightarrow \bar{L}$ is a K-homomorphism, it follows that $\sigma(L)$, just like L, is a splitting field of $\mathfrak{F}$. However, then we must have $\sigma(L) = L$, since both fields are subfields of $\bar{L}$. $\square$