陈澄 12311410 Complex Analysis (H) Homework 2.

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$\boxed{1}$ 6

Solution:

(a) Zeros: all come from $(1-\cos z^6)$, requiring $(\sin z)^5 \neq 0$, $e^{\frac{1}{\pi-z}}$ finite.

Case 1: $z=0$.

$$1-\cos z^6 = \frac{(z^6)^2}{2!} - \frac{(z^6)^4}{4!} + \cdots = \frac{z^{12}}{2} - \frac{z^{24}}{24} + \cdots$$

ord: 12 at $z=0$.

$$(\sin z)^5 = \left(z - \frac{z^3}{3!} + \cdots\right)^5 = z^5\left(1-\frac{z^2}{3!}+\cdots\right)^5$$

ord: 5 at $z=0$.

$\Rightarrow \text{ord}_0 f = 12-5 = 7$.

$\frac{1}{(\sin z)^5}$ has a pole of ord 5 at $z=0$ (but the more powerful part is $1-\cos z^6$)

Case 2: $z \neq 0$.

$\Rightarrow \cos z^6 = 1 \Rightarrow z^6 = 2k\pi$, $k\in\mathbb{Z}$, $k\neq 0$.

Let $g(z) = 1-\cos z^6$. $g(z_0) = 1-\cos z_0^6 = 0$.

$g'(z_0) = 6z_0^5 \sin(2k\pi) = 0$.

$g''(z_0) = 30 z_0^4 \sin(2k\pi) + 36 z_0^{10}\cos 2k\pi = 36 z_0^{10} \neq 0 \Rightarrow$ order 2.

for $z^6 = 2k\pi \neq 0$.

since $2k\pi \in \mathbb{R}$, $\sqrt[6]{\cdot}$ gives 6 different roots.

Note that this results follow for each $k\in\mathbb{Z}$, $k\neq 0$.

To sum up, the zeros of f are:

① $z=0$, $\text{ord}_0 f = 7$.

② $\forall k\in\mathbb{Z}$, $k\neq 0$, $\sqrt[6]{\cdot}$ gives 6 different roots: $(2k\pi)^{\frac{1}{6}}$. all of order 2.

(There are infinitely many zeros here)

(Rk: ∞ is not a zero here.)

Singularities: all come from $\frac{1}{(\sin z)^5}$ or $e^{\frac{1}{\pi-z}}$. (Now, we don't consider ∞)

Note that: for $e^{\frac{1}{\pi-z}}$, singularity is $z=\pi$, which is also a singularity for $\frac{1}{(\sin z)^5}$.

Case 1: $z=\pi$.

for $\lim\limits_{z\to\pi} e^{\frac{1}{\pi-z}}$ in $\mathbb{C}$, $\begin{cases} z\in\mathbb{R},\ z<\pi,\ z\to\pi, & \text{limit} = \infty \\ z\in\mathbb{R},\ z>\pi,\ z\to\pi, & \text{limit} = 0. \end{cases}$

$\Rightarrow \nexists \lim\limits_{z\to\pi} e^{\frac{1}{\pi-z}}$, it's an essential singularity of $e^{\frac{1}{\pi-z}}$.

$\Rightarrow$ essential singularity of f.

Case 2: $z\neq\pi$. ($z\neq 0$ → zero.)

$(\sin z)^5=0. \Rightarrow z=n\pi,\ n\neq 0,1.$ $\qquad \sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$

$\lim\limits_{z\to n\pi} f(z)=\infty. \Rightarrow$ pole.

$f(z)=\left[e^{\frac{1}{\pi-z}}\cdot(1-\cos z^6)\cdot\frac{1}{\left(1-\frac{(z-n\pi)^2}{3!}+\cdots\right)}\right]\times\frac{1}{(z-n\pi)^5}$

$\Rightarrow z=n\pi$ is a pole of order 5.

To sum up. the singularities of f are:

① $z=\pi$, essential.

② $z=n\pi$, $n\in\mathbb{Z}$, $n\neq 0,1$. pole of order 5.
(infinitely many).

The point ∞:

Note that $\{2\pi, 3\pi, \cdots, n\pi, \cdots\}$ is a series of poles.

$n\pi \xrightarrow{n\to\infty} \infty.$

Consider $\nexists \lim\limits_{z\to\infty} f(z)$, ∞ is a non-isolated singularity.

(b). Zeros: all come from $\sqrt{z}$ and $\sin\frac{1}{\sqrt{z^7}}$.

Case 1: $z=0$.

However, $f(z)=\sqrt{z}\,\frac{1}{\left(z^{\frac{7}{2}}-\frac{z^{\frac{21}{2}}}{3!}+\cdots\right)} = \frac{1}{z^3\left(1-\frac{z^7}{3!}+\cdots\right)}$

$z=0$ is a singularity point. (Since $\nexists \lim\limits_{z\to 0} f(z)$ because $\sin\frac{1}{\sqrt{z^7}}$ occillates, essential sing.)

Case 2: $\frac{1}{\sqrt{z^7}} = n\pi,\ n\in\mathbb{Z}.\ n\neq 0$ (not considering $z=\infty$ now)

$\sqrt{z^7} = \frac{1}{n\pi}. \Rightarrow z = \left(\frac{1}{n^2\pi^2}\right)^{\frac{1}{7}}.$

$f(z_0) = 0 \qquad f'(z) = \frac{1}{2\sqrt{z}} \sin\left(\frac{1}{\sqrt{z^7}}\right) + \sqrt{z}\cos\frac{1}{\sqrt{z^7}}\left(-\frac{7}{2}z^{-\frac{9}{2}}\right)$

$f'(z_0) = -\frac{7}{2}(-1)^k z_k^{-4} \neq 0. \quad \Rightarrow$ order : 1.

Case 3: $z=\infty$.

let $w=\frac{1}{z}$. $z\to\infty,\ w\to 0.$ $f(z) = f(\frac{1}{w}) = w^{-\frac{1}{2}}\sin(w^{\frac{7}{2}})$

$= w^{-\frac{1}{2}}(w^{\frac{7}{2}} - \frac{w^{\frac{21}{2}}}{3!} + \cdots)$

$= w^3(1 - \frac{w^7}{3!} + \cdots) \quad \Rightarrow$ order 3.

To sum up, zeros are

① $z = \left(\frac{1}{n^2\pi^2}\right)^{\frac{1}{7}}$, $n\in\mathbb{Z},\ n\neq 0$, order : 1

(infinitely many)

② $z=\infty$, order: 3.

Singularities: all come from $\sin(\frac{1}{\sqrt{z^7}})$

The only possible one is $z=0$.

$f(z) = z^{\frac{1}{2}}\left(z^{-\frac{7}{2}} - \frac{z^{-\frac{21}{2}}}{3!} + \cdots\right) = z^{-3} - \frac{z^{-10}}{3!} + \cdots$

$\Rightarrow$ essential singularity. infinite principle part.

So, the only isolated singularity is $z=0$, which is an essential one.

Why hol. outside isolated singularities:

$\sqrt{z}$: hol at its each branch.

Same as $\sqrt{z^7}$.

$\sin z$: hol. entire

$\frac{1}{z}$: hol. on $\mathbb{C}\setminus\{0\}$

$\Rightarrow$ composition : $\sin\frac{1}{\sqrt{z^7}}$ hol on $\mathbb{C}\setminus\{0\}$.

hol. outside iso. sing.

$\Rightarrow$ composition: $\sqrt{z}\sin\frac{1}{\sqrt{z}}$ hol outside iso. sing.

$\boxed{2.}$ 10

Solution: $f(z) = \dfrac{z^3}{(z-1)(z-2)} = z^3\left(\dfrac{1}{z-2} - \dfrac{1}{z-1}\right)$ $(z \neq 0)$.

Region 1: $|z|<1$. (<2)

$$f(z) = z^3\left(\frac{1}{-2(1-\frac{z}{2})} + \frac{1}{1-z}\right)$$

$$= z^3\left(-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} + \sum_{n=0}^{\infty} z^n\right) = z^3\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n$$

$$= \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^{n+3}. \quad \text{(Taylor Series)}.$$

Singularity: $z=0$. no principle part $\Rightarrow$ removable. zero of order 3.

Region 2: $1<|z|<2$.

$$f(z) = z^3\left(\frac{1}{-2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}\right)$$

$$= z^3\left(-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}\right)$$

$$= z^3\left(-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \sum_{n=-1}^{-\infty} z^n\right) = -\sum_{n=-1}^{-\infty} z^{n+3} - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^{n+3}. \quad \text{(Laurent Series)}.$$

Region 3: $|z|>2$.

$$f(z) = z^3\left(\frac{1}{z(1-\frac{2}{z})} - \frac{1}{z(1-\frac{1}{z})}\right)$$

$$= z^3\left(\frac{1}{z}\sum_{n=0}^{\infty}2^n z^{-n} - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n}\right)$$

$$= \sum_{n=0}^{\infty}(2^n-1)\,z^{2-n}. \quad \text{(Laurent Series)}.$$

Singularity: $z=\infty$

let $w=\frac{1}{z}$. $g(w) = f(\frac{1}{w}) = \frac{1}{w} + 3 + 7w + 15w^2 + \cdots$ $\Rightarrow$ $z=\infty$ is a pole of ord=1.

$\boxed{3.}$ 10

Solution: $\displaystyle\lim_{R\to+\infty}\int_{|z|=R} z^{100}e^{\frac{1}{z}}$.

$f(z) = z^{100} e^{\frac{1}{z}}$: the only singl is $z=0$.

$$\int_{|z|=R} f(z)\,dz \underset{\substack{\downarrow \\ \text{Residue} \\ \text{Thm.}}}{=} 2\pi i \cdot \operatorname{res}_0 f$$

$$f(z) = z^{100} \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} n!\, z^{100-n}. \Rightarrow \operatorname{res}_0 f = C_1 = \frac{1}{101!}$$

$$\Rightarrow \int_{|z|=R} f(z)\,dz = 2\pi i \cdot \frac{1}{101!}. \quad \Rightarrow \lim_{R\to+\infty} \int_{|z|=R} f(z)\,dz = \frac{2\pi i}{101!}.$$

$\boxed{4.}$ 10

<u>Proof</u>: Suppose all the finite ones among $\{a_1, \cdots, a_s\}$ are $\{a_1, \cdots, a_k\}$.

<u>Note</u>: If all $a_1 \sim a_s$ are finite, $k=s$, and $\operatorname{Res}_\infty f = 0$.

Otherwise, $k = s-1$. #.

Let R: large s.t. $a_1, \cdots, a_k \in B_R(0)$.

By the Residue Thm we learned from class, $\int_{\partial B_R(0)} f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{res}_{a_j} f$. (✱1)

for $|z|>R$, $f(z) = \sum_{n=-\infty}^{\infty} C_n z^n$.

$$\int_{\partial B_R(0)} f(z)\,dz = \int_{\partial B_R(0)} \sum_{n=-\infty}^{\infty} C_n z^n \underset{\substack{\uparrow \\ \text{normally} \\ \text{conv.}}}{=} \sum_{n=-\infty}^{\infty} C_n \int_{\partial B_R(0)} z^n\,dz = 2\pi i\, C_{-1}. \qquad (✱2)$$

$\Rightarrow$ (✱1) and (✱2) give us: $\sum_{j=1}^{k} \operatorname{res}_{a_j} f = C_{-1}$.

$$\Rightarrow \sum_{j=1}^{s} \operatorname{res}_{a_j} f = C_{-1} + \operatorname{res}_\infty f = 0. \quad \square$$

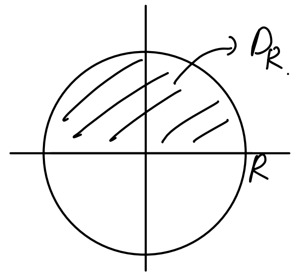
$\boxed{5.}$ 10

<u>Solution</u>: $\int_{-\infty}^{+\infty} \frac{(\cos(2x))^2}{x^2+2x+2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos 4x}{x^2+2x+2}\,dx + \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+2}\,dx.$

$$\int_{-\infty}^{+\infty} \frac{\cos 4x}{x^2+2x+2}\,dx = \operatorname{Re}\left(\int_{-\infty}^{+\infty} \frac{e^{i4x}}{x^2+2x+2}\,dx\right)$$

$$= \mathrm{Re}\left(\lim_{R\to+\infty} \int_{-R}^{R} \frac{e^{i4x}}{x^2+2x+2}\,dx \right)$$

$$= \mathrm{Re}\left[\lim_{R\to+\infty} \left(\int_{\partial D_R} \frac{e^{i4z}}{z^2+2z+2}\,dz - \int_{\substack{|z|=R \\ \mathrm{Im}\, z\geq 0}} \frac{e^{i4z}}{z^2+2z+2}\,dz \right) \right].$$



$$= \mathrm{Re}\left(\lim_{R\to+\infty} \int_{\partial D_R} \frac{e^{i4z}}{z^2+2z+2}\,dz - \lim_{R\to+\infty} \int_{\substack{|z|=R \\ \mathrm{Im}\, z\geq 0}} \frac{e^{i4z}}{z^2+2z+2}\,dz \right)$$

$\partial D_R \to R$: large! $\qquad \lim_{R\to+\infty} \int_{\substack{|z|=R \\ \mathrm{Im}\, z\geq 0}} \frac{e^{i4z}}{z^2+2z+2}\,dz = 0 \to$ Jordan's Lemma.

$$= \mathrm{Re}\left(\lim_{R\to+\infty} \left(2\pi i \operatorname*{Res}_{-1+i} \frac{e^{i4z}}{z^2+2z+2} \right) \right)$$

$\downarrow$ Residue Thm

$$= \mathrm{Re}\left(2\pi i \lim_{z\to -1+i} (z+1-i) \frac{e^{i4z}}{z^2+2z+2} \right) = \mathrm{Re}\left(2\pi i \cdot \frac{e^{i4(-1+i)}}{2i} \right)$$

$$= \mathrm{Re}\left(\pi e^{-4-4i} \right) = \pi e^{-4} \cos 4.$$

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+2}\,dx = \lim_{R\to+\infty} \int_{\partial D_R} \frac{1}{z^2+2z+2}\,dz - \lim_{R\to+\infty} \int_{\substack{|z|=R \\ \mathrm{Im}\, z\geq 0}} \frac{1}{z^2+2z+2}\,dx$$

$\partial D_R \to R$: large!

$$= \lim_{R\to+\infty} 2\pi i \cdot \operatorname*{Res}_{-1+i} f - 0$$

$$= 2\pi i \cdot \lim_{z\to -1+i} (z+1-i) \frac{1}{z^2+2z+2} = 2\pi i \cdot \frac{1}{2i} = \pi.$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{(\cos(2x))^2}{x^2+2x+2}\,dx = \frac{1}{2}\pi\left(1 + e^{-4}\cos 4\right).$$

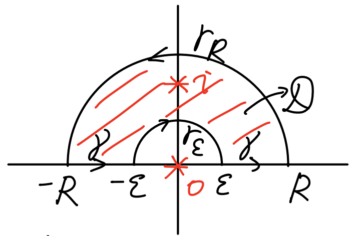
$\boxed{6.}$ 10

Solution: $\displaystyle \int_{-\infty}^{+\infty} \frac{\sin 3x}{x(x^2+1)}\,dx = \mathrm{Im} \int_{-\infty}^{+\infty} \frac{e^{i3z}}{z(z^2+1)}\,dz.$

$$= \mathrm{Im} \lim_{\varepsilon\to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \frac{e^{i3z}}{z(z^2+1)}\,dz.$$

$$= \mathrm{Im} \lim_{\varepsilon\to 0} \lim_{R\to+\infty} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{e^{i3z}}{z(z^2+1)}\,dz.$$

$$= \operatorname{Im} \lim_{\varepsilon\to 0} \lim_{R\to+\infty} \left(\int_{\partial D} - \int_{\gamma_R} - \int_{\gamma} \right) \frac{e^{i3z}}{z(z^2+1)}\, dz$$



$$= \operatorname{Im} \lim_{\varepsilon\to 0} \lim_{R\to+\infty} \left(2\pi i \cdot \operatorname{res}_i f - \underbrace{\int_{\gamma_R} \frac{e^{i3z}}{z(z^2+1)}\, dz}_{\downarrow R\to+\infty \ 0} - \int_{\gamma_\varepsilon} \frac{e^{i3z}}{z(z^2+1)}\, dz \right)$$

(Jordan's lemma)

$$\frac{1+i3z+\cdots}{z(z^2+1)} = \frac{1}{z(z^2+1)} + \frac{i3+\cdots}{z^2+1}$$

hol. at $z=0$

$$= \operatorname{Im} \left(2\pi i \cdot \operatorname{res}_i f - 0 - \lim_{\varepsilon\to 0} \int_0^{\pi} \frac{1}{e^{it}(e^{2it}+1)}\, dt + \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{e^{i3z}}{z(z^2+1)}\, dz \right)$$

$= 0.$ $\quad \le \pi\cdot\varepsilon\cdot C \to 0. \ (\varepsilon\to 0)$

$$= \operatorname{Im} \left(2\pi i \cdot \left(-\frac{e^{-3}}{2}\right) + \pi i \right) = \pi(1-e^{-3}).$$

$\boxed{7.}$

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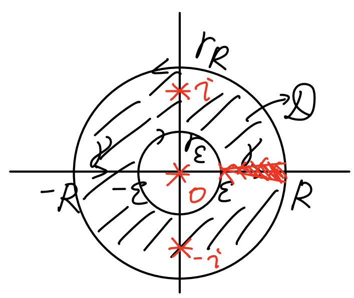
Solution:

$$I = \int_0^{+\infty} \frac{x+1}{\sqrt[3]{x}\,(x^2+1)}\, dx. \qquad f(z) = \frac{z+1}{\sqrt[3]{z}\,(z^2+1)} = \frac{z+1}{z^{\frac13}(z+i)(z-i)}.$$

$$\operatorname{res}_i f = \lim_{z\to i} \frac{z+1}{z^{\frac13}(z+i)} = \frac{1+i}{2e^{i\frac{2\pi}{3}}} = \frac{\sqrt3-1}{4} - i\frac{\sqrt3+1}{4}.$$



$$\operatorname{res}_{-i} f = \lim_{z\to -i} \frac{z+1}{z^{\frac13}(z-i)} = \frac{-i+1}{(-2i)\,e^{i\frac{\pi}{2}}} = \frac{1-i}{2}.$$

$$\sum \operatorname{res} f = \frac{\sqrt3+1}{4} - i\frac{\sqrt3+3}{4}.$$

$$\left| \int_{\gamma_R} f\, dz \right| \le 2\pi R \cdot \frac{R+1}{R^{\frac13}\cdot(R^2+1)} \xrightarrow{R\to+\infty} 0.$$

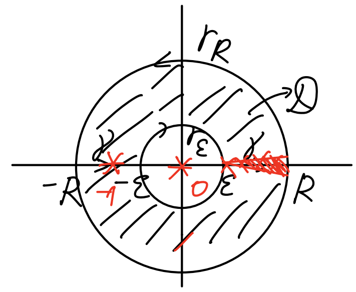
$$\left| \int_{\gamma_\varepsilon} f\, dz \right| \le 2\pi\varepsilon\cdot(\varepsilon+1)\cdot O(1) \xrightarrow{\varepsilon\to 0} 0.$$

$$\Rightarrow \int_D f\, dz = 2\pi i \sum \operatorname{res} f = I - e^{-i\frac{2\pi}{3}} I \quad \Rightarrow I = \pi\left(1+\frac{\sqrt3}{3}\right).$$

$\boxed{8.}$ 10

Solution:

$$I=\int_0^{+\infty}\overbrace{\frac{\ln x}{\sqrt[4]{x}(x^2-1)}}^{g(z)}dx. \qquad \operatorname{res}_{-1} g=\lim_{z\to-1}(z+1)g(z)=\frac{i\pi}{e^{\frac{i\pi}{4}}(-2)}=\frac{\pi\sqrt{2}(1+i)}{4}.$$



$$J(a)\triangleq\int_0^{+\infty}\underbrace{\frac{x^a}{x^2-1}}_{f(z)}dx. \Rightarrow I=\frac{d}{da}J(a)\Big|_{a=-\frac{1}{4}}.$$

$$J(a)=\frac{\pi}{2}\tan\left(\frac{\pi a}{2}\right) \qquad I=\frac{\pi}{2}\sec^2\left(\frac{\pi a}{2}\right)\cdot\frac{\pi}{2}\Big|_{a=-\frac{1}{4}}=\frac{\pi^2}{4}\sec^2\left(\frac{\pi a}{2}\right)\Big|_{a=-\frac{1}{4}}$$

$$=\frac{\pi^2}{4}\sec^2\left(\frac{\pi}{8}\right)=\pi^2\left(1-\frac{\sqrt{2}}{2}\right).$$

$\boxed{9.}$ 10

Solution: $f(z)=\frac{1}{z+1}\cos g(z)) \qquad g(z)=\frac{1}{(z-1)\cdots(z-100)}.$

singularities: pole: $z=-1$.

essential: $z=1,2,\cdots,100$.

Contour $C:\{|z|=101\}$.

$$\operatorname{res}_\infty f=-\operatorname{res}_0 \underbrace{\frac{1}{w^2}f\left(\frac{1}{w}\right)}_{\triangleq}.$$

$$h(w)=\frac{1}{w^2}\frac{w}{1+w}\cos\left(\frac{w^{100}}{(1-w)(1-2w)\cdots(1-100w)}\right).$$

$$=\left(\frac{1}{w}-(1-w+w^2-\cdots)\right)\left(1-O(w^{200})\right)$$

$$=\underset{\uparrow}{\frac{1}{w}}-1+w-\cdots.$$

$$\Rightarrow \operatorname{res}_0 h=1. \quad\Rightarrow \operatorname{res}_\infty f=-1.$$

$$\Rightarrow \int_{|z|=101}f(z)\,dz \underset{\substack{\uparrow\\ \text{residue}\\ \text{thm}}}{=} 2\pi i\sum_{k\in\{-1,1,\cdots,100\}}\operatorname{res}_{z_k}f=2\pi i\left(-\operatorname{res}_\infty f\right)=2\pi i.$$

$\boxed{10.}$ 10

Solution: Consider $f(z) = \dfrac{1}{z^2+1}$.

poles: $z = \pm i$. let $g(z) = \pi \cot(\pi z)\, f(z)$, $\longrightarrow$ poles: $n \in \mathbb{Z}$, $\pm i$.

$$\operatorname{res}_i g = \lim_{z \to i} (z-i)\, g(z) = -\frac{\pi}{2} \coth(\pi).$$

$$\operatorname{res}_n g = \frac{1}{n^2+1}. \quad n \in \mathbb{Z}$$

$$\operatorname{res}_{-i} g = \lim_{z \to -i} (z+i)\, g(z) = -\frac{\pi}{2} \coth(\pi).$$

$$\Rightarrow 1 + 2\sum_{n=1}^{+\infty} \frac{1}{n^2+1} = \frac{1}{0^2+1} + \sum_{n=1}^{+\infty} \frac{1}{n^2+1} + \sum_{n=-\infty}^{-1} \frac{1}{n^2+1}.$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$$

$$= \sum_{n=-\infty}^{\infty} \operatorname{res}_n g = -\left(\operatorname{res}_i g + \operatorname{res}_{-i} g\right) = \pi \coth(\pi).$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2+1} = \frac{1}{2}\left(\pi \coth(\pi) - 1\right).$$