Abstract Algebra
: Lecture 13

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Example 1. *Consider the ring* $\mathbb{Z}_n = \{1, 2, \ldots, n-1\}$*. If* $n = m_1 m_2 \ldots m_r$*, where* $\gcd(m_i, m_j) = 1$ *for* $i \neq j$*, then* $\mathbb{Z}_n$ *is isomorphic to* $\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_r}$*. This is called the Chinese Remainder Theorem.*

Theorem 2. *Let* $m_1, m_2, \ldots, m_r$ *be integers which pairwise coprime. Let* $a_1, \ldots, a_r$ *be integers s.t.* $1 \leqslant a_i < m_i$*. Then there exists an integer* x *s.t.*

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \quad\vdots \\ x \equiv a_r \pmod{m_r} \end{cases}$$

证明. For $1 \leqslant j \leqslant r$, let $n_j = \prod\limits_{i \neq j} m_i$. Then $\gcd(m_j, n_j) = 1$, so there exists $s_j, t_j \in \mathbb{Z}$ s.t. $m_j s_j + n_j t_j = 1$. Further, $t_j n_j \equiv t_j n_j + s_j m_j = 1 \pmod{m_j}$. Let $x = a_1 t_1 n_1 + a_2 t_2 n_2 + \cdots + a_r t_r n_r$. Then $x \equiv a_1 \pmod{m_1}$, $x \equiv a_2 \pmod{m_2}$, $\ldots$, $x \equiv a_r \pmod{m_r}$. □

Example 3. *Let* $(m_1, m_2) = (5, 7)$ *and* $(a_1, a_2) = (2, 3)$*, find* x *s.t.* $x \equiv 2 \pmod 5$ *and* $x \equiv 3 \pmod 7$*.*

Example 4. *Let* $(m_1, m_2, m_3) = (5, 7, 8)$ *and* $(a_1, a_2, a_3) = (2, 3, 4)$*, find* x *s.t.* $x \equiv 2 \pmod 5$*,* $x \equiv 3 \pmod 7$ *and* $x \equiv 4 \pmod 8$*.*

Now we prove the integer ring version:

证明. Define a map: $\phi : \mathbb{Z} \to \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z}$ s.t. $a \mapsto (a + (m_1), \ldots, a + (m_r))$. Then ϕ is a ring homomorphism with $\ker \phi = (n)$.
To complete the proof, we need to prove ϕ is surjective. In general, an element of $\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z}$ is of the form $(a_1 + (m_1), \ldots, a_r + (m_r))$.
Let $I_1 = (m_1) = m_1\mathbb{Z}$, and $J = (m_2) \cap (m_3) \cap \cdots \cap (m_r) = (m_2 \ldots m_r)$, then $(m_1, m_2 \ldots m_r) = 1$, and there exists s, t such that $sm_1 + tm_2 \ldots m_r = 1$, let $sm_1 = a_1$ and $tm_2 \ldots m_r = b_1$, let $x_1 = 1 - a_1 = b_1$, then $\phi(x_1) = (1 + (m_1), (m_2), \ldots, (m_r))$.
Similarly, there exists x_j s.t. $\phi(x_j) = ((m_1), \ldots, 1 + (m_j), \ldots, (m_r))$.

Let $x = a_1x_1 + \dots a_rx_r$, then $\phi(x) = (a_1 + (m_1), \dots, a_r + (m_r))$, so ϕ is surjective. And $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_r\mathbb{Z}$ □

Theorem 5. *(General case) Let R be a ring with identity, and $I_1, \dots, I_r$ ideals which are pairwise coprime. Then $R/(I_1 \cap \dots \cap I_r) \simeq R/I_1 \oplus \dots \oplus R/I_r$.*

Definition 6. *Two ideals I, J of a ring R are said to be coprime if $I + J = R$.*

Definition 7. *Let R be a ring and I, J two ideals of R, and we have $I + J = \{a + b | a \in I, b \in J\}$, $IJ = \{\sum\limits_{finite} a_ib_i | a_i \in I, b_i \in J\}$.*

Lemma 8. *Let I_1, I_2, J be ideals of a ring R (commutative with identity). If I_1, I_2 are coprime to J, then I_1I_2 is coprime to J.*

证明. Since $I_1 + J = R = I_2 + J$, we have $a_1 + b_1 = 1$ and $a_2 + b_2 = 1$ where $a_1 \in I_1$ and $a_2 \in I_2$ and $b_1, b_2 \in J$. Then $a_1a_2 + b_1a_2 + b_2a_1 + b_1b_2 = 1 \in I_1I_2 + J$. Therefore $I_1I_2 + J = R$, so I_1I_2 is coprime to J. □

Corollary 9. *If $I_1, \dots, I_t$ are coprime to J, then $I_1 \dots I_t$ is coprime to J.*

证明. (For general case of Chinese Remainder Theorem) Let $\varphi : R \to R/I_1 \oplus \dots \oplus R/I_r$ s.t. $a \mapsto (a + I_1, \dots, a + I_r)$. Then φ is a ring homomorphism with $\ker \varphi = I_1 \cap \dots \cap I_r$. We only need to prove φ is surjective. □

Definition 10. *Let J be an ideal of R, where R is commutative and has an identity.*
(1). J is a prime ideal if for any element $a, b \in R$, if $ab \in J$, then $a \in J$ or $b \in J$.
(2). J is a maximal ideal if I is an ideal and $J \subset I$, then $I = R$.

Theorem 11. *Let J be an ideal of R.*
(1). J is a prime ideal if and only if R/J is an integral domain.
(2). J is a maximal ideal if and only if R/J is a field.

证明. (1). J is prime $\Leftrightarrow$ $a, b \in J$ implies $a \in J$ or $b \in J$ $\Leftrightarrow$ $\bar{a}\bar{b} = \bar{0}$ implies $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ $\Leftrightarrow$ R/J is an integral domain.
(2). J is maximal $\Leftrightarrow$ $(a) + J = (1), \forall a \in R - J$, $\Leftrightarrow$ $(\bar{a}) = (\bar{1}) = \bar{R}$, $\Leftrightarrow$ $\bar{a}$ is a unit in R/J $\Leftrightarrow$ R/J is a field.
In particular, if R is a commutative ring with identity, a maximal ideal is a prime ideal. □

Consider $\mathbb{Z} = \{0, \pm1, \dots, \}$ we can define $\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$ with $\frac{m_1}{n_1}\frac{m_2}{n_2} = \frac{m_1m_2}{n_1n_2}$ and $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ if and only if $m_1n_2 = m_2n_1$ and $\frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1n_2 + m_2n_1}{n_1n_2}$.

Definition 12. *Let R be an integral domain. Define $S = \{(a, b) | a, b \in R, b \neq 0\}$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ and $(a_1, b_1) + (a_2, b_2) = (a_1b_2 + a_2b_1, b_1b_2)$. If $(a_1, b_1) = (a_2r, b_2r)$ then identify (a_1, b_1) and (a_2, b_2).*
Then $(S, +, \times)$ is a ring (actually a field) called the fractional field of R.

Definition 13. *Let R be a commutative ring with identity. Let T be a set $T \subset R$ s.t. none of the elements of T is the zero divisor of R.*
Let $S = \{(a, b) | a \in R, b \in T\}$, and make the same definitions as above. Then $(S, +, \times)$ is a ring called the localization of R at T, denoted by $T^{-1}R$. And R is the subring of S.