陈隽 12311410 概率统计 Assignment 3

Part 1: 10 questions randomly chosen from Q3.1 ~ Q3.19

Q3.1

Solution: $f(x_i;\theta_1,\theta_2)=\frac{1}{\theta_2-\theta_1}\mathbb{1}(\theta_1\le x_i\le\theta_2)$. $\vec{\theta}=(\theta_1,\theta_2)^T$.

$$L(\vec{\theta})=\frac{1}{(\theta_2-\theta_1)^n}\cdot\prod_{i=1}^{n}\mathbb{1}(\theta_1\le x_i\le x_2)=\begin{cases}\frac{1}{(\theta_2-\theta_1)^n}, & \theta_1\le x_{(1)},\ \theta_2\ge x_{(n)}\\ 0, & \text{otherwise}\end{cases}$$

By partially differentiating $L(\theta_1,\theta_2)$ w.r.t $\theta_1$:

$$\frac{\partial L(\vec{\theta})}{\partial\theta_1}=-n\cdot\frac{1}{(\theta_2-\theta_1)^{n+1}}\cdot(-1)=n\cdot\frac{1}{(\theta_2-\theta_1)^{n+1}}>0 \quad \text{since } \theta_2>\theta_1.$$

$\Rightarrow L(\vec{\theta})$ is strictly increasing in $\theta_1$ when $\theta_2$ is fixed.

Since $\theta_1\le x_{(1)}$, $L(\vec{\theta})$ is maximized at $\theta_1=x_{(1)}$.

By partially differentiating $L(\theta_1,\theta_2)$ w.r.t. $\theta_2$:

$$\frac{\partial L(\vec{\theta})}{\partial\theta_2}=-n\cdot\frac{1}{(\theta_2-\theta_1)^{n+1}}<0 \quad \text{since } \theta_2>\theta_1.$$

$\Rightarrow L(\vec{\theta})$ is strictly decreasing in $\theta_2$ when $\theta_1$ is fixed.

Since $\theta_2\ge x_{(n)}$, $L(\vec{\theta})$ is maximized at $\theta_2=x_{(n)}$.

We obtain:

$$\hat{\theta}_1=X_{(1)},\ \hat{\theta}_2=X_{(n)} \text{ are the MLEs of } \theta_1 \text{ and } \theta_2.$$

Q3.2

Solution: let $X_1,\dots,X_{n_1}$; $Y_1,\dots,Y_{n_2}$ denote the sample r.v.s

(a) $f(x_i;\mu_1,\sigma_1^2)=\frac{1}{\sqrt{2\pi\sigma_1^2}}\cdot e^{-\frac{(x_i-\mu_1)^2}{2\sigma_1^2}}$, $g(y_i;\mu_2,\sigma_2^2)=\frac{1}{\sqrt{2\pi\sigma_2^2}}\cdot e^{-\frac{(y_i-\mu_2)^2}{2\sigma_2^2}}$

$$L(\vec{\theta})=\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^{n_1}\left(\frac{1}{\sqrt{2\pi\sigma_2^2}}\right)^{n_2}e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n_1}(x_i-\mu_1)^2-\frac{1}{2\sigma_2^2}\sum_{j=1}^{n_2}(y_j-\mu_2)^2}$$

$\vec{\theta}=(\mu_1,\mu_2)^T$

Solving the following system of equations:

$$\frac{\partial L(\vec{\theta})}{\partial \mu_1} \propto e^{-\frac{1}{2a_1^2}\sum_{i=1}^{n_1}(x_i-\mu_1)^2} \times \left[-\frac{1}{2a_1^2}\cdot 2\sum_{i=1}^{n_1}(\mu_1-x_i)\right] = 0.$$

$$\frac{\partial L(\vec{\theta})}{\partial \mu_2} \propto e^{-\frac{1}{2a_2^2}\sum_{j=1}^{n_2}(y_j-\mu_2)^2} \times \left[-\frac{1}{2a_2^2}\cdot 2\sum_{j=1}^{n_2}(\mu_2-y_j)\right] = 0.$$

we obtain: $\mu_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i. \Rightarrow \hat{\mu}_1 = \bar{X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$

$\mu_2 = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j \Rightarrow \hat{\mu}_2 = \bar{Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j.$

which are the MLEs of $\mu_1$ and $\mu_2$.

By extension thm, $\hat{\theta} = \hat{\mu}_1 - \hat{\mu}_2 = \bar{X} - \bar{Y}$ is the MLE for $\mu_1 - \mu_2$.

(b) $Var(\hat{\theta}) = Var\left(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i + \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j\right)$

$= \frac{1}{n_1^2}\sum_{i=1}^{n_1} Var(X_i) + \frac{1}{n_2^2}\sum_{j=1}^{n_2} Var(Y_j)$

$= \frac{1}{n_1^2}\cdot n_1 \cdot a_1^2 + \frac{1}{n_2^2}\cdot n_2 \cdot a_2^2$

$= \frac{a_1^2}{n_1} + \frac{a_2^2}{n_2} = \frac{a_1^2}{n_1} + \frac{a_2^2}{n-n_1}$

let $\frac{\partial Var(\hat{\theta})}{\partial n_1} = -\frac{a_1^2}{n_1^2} + \frac{a_2^2}{(n-n_1)^2} = 0. \Rightarrow a_1^2(n-n_1)^2 = a_2^2 n_1^2.$

i.e. $a_1^2 n_2^2 = a_2^2 n_1^2 \Leftrightarrow \frac{n_1}{n_2} = \frac{a_1}{n_2}. \Leftrightarrow n_1 = \frac{a_1}{a_1+a_2} n,\ n_2 = \frac{a_2}{a_1+a_2} n.$

Thus. n should be devided as:

$$n_1 = \frac{a_1}{a_1+a_2} n,\quad n_2 = \frac{a_2}{a_1+a_2} n.$$

Q3.3.

Solution: $L(\vec{\theta}) = \binom{n}{n_1 \cdots n_4} (\alpha\beta)^{n_1} [\alpha(1-\beta)]^{n_2} [(1-\alpha)\beta]^{n_3} [(1-\alpha)(1-\beta)]^{n_4}$

$$= \binom{n}{n_1, \cdots, n_4} \alpha^{n_1+n_2} (1-\alpha)^{n_3+n_4} \beta^{n_1+n_3} (1-\beta)^{n_2+n_4}$$

Solving the following system of equations:

$$\frac{\partial L(\vec{\theta})}{\partial \alpha} \propto \left[ (n_1+n_2)\alpha^{n_1+n_2-1}(1-\alpha)^{n_3+n_4} - \alpha^{n_1+n_2}(n_3+n_4)(1-\alpha)^{n_3+n_4-1} \right] = 0.$$

$$\frac{\partial L(\vec{\theta})}{\partial \beta} \propto \left[ (n_1+n_3)\beta^{n_1+n_3-1}(1-\beta)^{n_2+n_4} - \beta^{n_1+n_3}(n_2+n_4)(1-\beta)^{n_2+n_4-1} \right] = 0.$$

we obtain: $\begin{cases} (n_1+n_2)(1-\alpha) = (n_3+n_4)\alpha \\ (n_1+n_3)(1-\beta) = (n_2+n_4)\beta \end{cases} \Rightarrow \begin{cases} \alpha = \dfrac{n_1+n_2}{n_1+n_2+n_3+n_4} \\ \beta = \dfrac{n_1+n_3}{n_1+n_2+n_3+n_4} \end{cases}$

$\Rightarrow$ the MLEs of $\alpha, \beta$ are

$$\hat{\alpha} = \frac{N_1+N_2}{N_1+N_2+N_3+N_4}. \qquad \hat{\beta} = \frac{N_1+N_3}{N_1+N_2+N_3+N_4}$$

## Q3.4

Solution: $f(x_{11}, \cdots, x_{in}; a, b, c, \alpha^2)$

$$= \left(\frac{1}{\sqrt{2\pi\alpha^2}}\right)^{4n} \exp\left\{-\frac{1}{2a^2}\sum_{j=1}^{4}\sum_{i=1}^{n}(x_{ij}-\mu_j)^2\right\}.$$

Let:

$$\frac{\partial f}{\partial a} \propto \sum_{i=1}^{n}(a+b+c-x_{1i} + a+b-c-x_{2i} + a-b+c-x_{3i} + a-b-c-x_{4i})$$

$$= 4na - \sum_{i=1}^{n}(x_{1i}+x_{2i}+x_{3i}+x_{4i}) = 0$$

$$\frac{\partial f}{\partial b} \propto \sum_{i=1}^{n}(b+a+c-x_{1i} + b+a-c-x_{2i} + b-c-a+x_{3i} + b-a+c+x_{4i})$$

$$= 4nb - \sum_{i=1}^{n}(x_{1i}+x_{2i}-x_{3i}-x_{4i}) = 0.$$

$$\frac{\partial f}{\partial c} \propto \sum_{i=1}^{n}(c+a+b-x_{1i} + c-a-b+x_{2i} + c+a-b-x_{3i} + c-a+b+x_{4i})$$

$$= 4nc - \sum_{i=1}^{n}(x_{1i}-x_{2i}+x_{3i}-x_{4i})$$

$$\frac{\partial f}{\partial a^2} \propto \frac{4n\cdot\left(\frac{1}{\sqrt{2\pi a^2}}\right)^{4n-1}\frac{1}{\sqrt{2\pi}}\left(-\frac{1}{2}\right)(a^2)^{-\frac{3}{2}}}{\left(\frac{1}{\sqrt{2\pi a^2}}\right)^{4n}} + \frac{1}{2}(a^2)^{-2}\sum_{j=1}^{4}\sum_{i=1}^{n}(x_{ij}-\mu_j)^2$$

$$= -2n\frac{1}{a^2} + \frac{1}{2}\frac{1}{a^4}\sum_{j=1}^{n}\sum_{i=1}^{n}(x_{ij}-\mu_j)^2 = 0.$$

$\Rightarrow \hat{a} = \frac{1}{4}(\bar{X_1}+\bar{X_2}+\bar{X_3}+\bar{X_4})$. $\hat{b} = \frac{1}{4}(\bar{X_1}+\bar{X_2}-\bar{X_3}-\bar{X_4})$.

$\hat{c} = \frac{1}{4}(\bar{X_1}-\bar{X_2}+\bar{X_3}-\bar{X_4})$, $\quad \bar{X_j} = \frac{1}{n}\sum_{i=1}^{n}X_{ij}$. $j=1,2,3,4$.

$\hat{a^2} = \frac{1}{4}\frac{1}{n}\sum_{j=1}^{4}\sum_{i=1}^{n}(X_{ij}-\bar{\mu_j})^2$, where $\bar{\mu_1} = \frac{3}{4}\bar{X_1}+\frac{1}{4}\bar{X_2}+\frac{1}{4}\bar{X_3}-\frac{1}{4}\bar{X_4}$,

$\bar{\mu_2} = \frac{3}{4}\bar{X_2}+\frac{1}{4}\bar{X_1}-\frac{1}{4}\bar{X_3}+\frac{1}{4}\bar{X_4}$, $\bar{\mu_3} = \frac{3}{4}\bar{X_3}+\frac{1}{4}\bar{X_1}-\frac{1}{4}\bar{X_2}+\frac{1}{4}\bar{X_4}$, $\bar{\mu_4} = \frac{3}{4}\bar{X_4}-\frac{1}{4}\bar{X_1}+\frac{1}{4}\bar{X_2}+\frac{1}{4}\bar{X_3}$.

## Q3.5

Solution:

(a) $f(x_i;\mu, a) = \frac{1}{2\sqrt{3}a}\mathbb{1}(\mu-\sqrt{3}a \le x_i \le \mu+\sqrt{3}a)$.

$$L(\vec{\theta}) = \left(\frac{1}{2\sqrt{3}a}\right)^n \cdot \prod_{i=1}^{n}\mathbb{1}(\mu-\sqrt{3}a \le x_i \le \mu+\sqrt{3}a).$$

$\vec{\theta} = (\mu, a)^T$

$$= \begin{cases} \left(\frac{1}{2\sqrt{3}a}\right)^n, & \mu-\sqrt{3}a \le x_{(1)},\ \mu+\sqrt{3}a \ge x_{(n)} \\ 0. & \text{otherwise} \end{cases}$$

By partially differentiating $L(\vec{\theta})$ w.r.t $a$,

$$\frac{\partial L(\vec{\theta})}{\partial a} = \frac{1}{(2\sqrt{3})^n}(-n)\frac{1}{a^{n+1}} < 0. \text{ since } a>0.$$

Thus, $L(\vec{\theta})$ is strictly decreasing in $a$ when $\mu$ is fixed.

$\Rightarrow$ we are supposed to find minimum of $a$ of all possible $\mu$.

$$\begin{cases} \mu-\sqrt{3}a \le x_{(1)} \\ \mu+\sqrt{3}a \ge x_{(n)} \end{cases} \Rightarrow \begin{cases} \sqrt{3}a \ge \mu - x_{(1)} \\ \sqrt{3}a \ge x_{(n)}-\mu \end{cases} \Rightarrow 2\sqrt{3}a \ge x_{(n)}-x_{(1)}$$

$$a \ge \frac{x_{(n)}-x_{(1)}}{2\sqrt{3}}$$

"=" holds for $\mu = \frac{x_{(n)}+x_{(1)}}{2}$.

Thus, the MLEs for $\mu$ and $\alpha$ are

$$\hat{\mu} = \frac{X_{(n)} + X_{(1)}}{2}, \quad \hat{\alpha} = \frac{X_{(n)} - X_{(1)}}{2\sqrt{3}}.$$

(b) We know that $E(X) = \mu$, $Var(X) = \frac{(2\sqrt{3}\alpha)^2}{12} = \alpha^2$.

(by Appendix A.2.1)

$\Rightarrow$ the moment estimators of $\mu$ and $\alpha$ are:

$$\hat{\mu}^M = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\alpha}^M = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2}.$$

Q3.6

Solution:

(a) $f(x_i; \theta) = e^{-x_i + \theta}$ $(x \geq \theta)$.

$$L(\theta) = e^{n\theta - \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} 1(x_i \geq \theta).$$

$$= \begin{cases} e^{n\theta - \sum_{i=1}^{n} x_i}, & \theta \leq x_{(1)} \\ 0. & \text{otherwise.} \end{cases}$$

$\frac{\partial L(\theta)}{\partial \theta} = e^{-\sum_{i=1}^{n} x_i} \cdot e^{n\theta} \cdot n > 0 \quad \forall \theta \in \mathbb{R}$. $\Rightarrow L(\theta)$ is strictly increasing w.r.t. $\theta$. $\Rightarrow \arg\max_{\theta} L(\theta) = x_{(1)}$.

$\Rightarrow \hat{\theta} = X_{(1)}$. i.e. the MLE of $\theta$ is $X_{(1)}$.

(b) $f(x) = e^{-x+\theta}$. $x \geq \theta$. $E(X) = \int_{\theta}^{+\infty} x \cdot e^{-x+\theta} dx = \int_{\theta}^{+\infty} e^{\theta} x (-1)\, d(e^{-x})$

$= -e^{\theta} x e^{-x} \Big|_{x=\theta}^{x=+\infty} + e^{\theta}\int_{\theta}^{+\infty} e^{-x} dx = \theta + 1.$

$\Rightarrow \hat{\theta}^M = \bar{X} - 1 = \frac{1}{n}\sum_{i=1}^{n} X_i - 1.$

(c) $\pi(\theta) = e^{-\theta} I_{(0,+\infty)}(\theta)$.

$$f(\vec{x}, \theta) = f(\vec{x}|\theta) \times \pi(\theta)$$

$$= e^{-\sum_{i=1}^{n} x_i + n\theta} \times e^{-\theta} \quad \theta > 0.$$

$$P(\theta|\vec{x}) \propto e^{(n-1)\theta} \times e^{-\sum_{i=1}^{n} x_i}.$$

$$\propto e^{(n-1)\theta}$$

$$\Rightarrow p(\theta|\vec{x}) = \frac{e^{(n-1)\theta}}{\int_0^{x_{(1)}} e^{(n-1)t} dt} 1(0<\theta\leq x_{(1)}).$$

Case 1: $n=1$.

$$E(\theta|X_1)$$

$$= \int_0^{x_{(1)}} \theta \cdot \frac{1}{x_{(1)}} d\theta = \frac{x_{(1)}}{2}. \Rightarrow \hat{\theta}_{Bayes} = \frac{X_{(1)}}{2}.$$

Case 2: $n>1$.

$$\int_0^{x_{(1)}} e^{(n-1)t} dt = \frac{1}{n-1} e^{(n-1)t} \Big|_{t=0}^{t=x_{(1)}} = \frac{e^{(n-1)x_{(1)}}-1}{n-1}.$$

$$E(\theta|\vec{X}) = \int_0^{x_{(1)}} \theta \frac{e^{(n-1)\theta}}{\int_0^{x_{(1)}} e^{(n-1)t} dt} d\theta$$

$$= \int_0^{x_{(1)}} \theta \frac{e^{(n-1)\theta}}{e^{(n-1)x_{(1)}}-1}(n-1) d\theta = \frac{n-1}{e^{(n-1)x_{(1)}}-1} \int_0^{x_{(1)}} \theta e^{(n-1)\theta} d\theta$$

$$\int_0^{x_{(1)}} \theta e^{(n-1)\theta} d\theta = \int_0^{x_{(1)}} \theta \frac{1}{n-1} d(e^{(n-1)\theta}) = \frac{\theta}{n-1} e^{(n-1)\theta} \Big|_{\theta=0}^{\theta=x_{(1)}} - \int_0^{x_{(1)}} \frac{1}{n-1} e^{(n-1)\theta} d\theta.$$

$$= \frac{x_{(1)}}{n-1} e^{(n-1)x_{(1)}} - \frac{1}{(n-1)^2} e^{(n-1)\theta} \Big|_{\theta=0}^{\theta=x_{(1)}}$$

$$= \frac{x_{(1)}}{n-1} e^{(n-1)x_{(1)}} - \frac{1}{(n-1)^2} e^{(n-1)x_{(1)}} + \frac{1}{(n-1)^2}.$$

$$E(\theta|\vec{X}) = \frac{n-1}{e^{(n-1)x_{(1)}}-1} \left( \frac{x_{(1)}}{n-1} e^{(n-1)x_{(1)}} - \frac{e^{(n-1)x_{(1)}}-1}{(n-1)^2} \right)$$

$$= \frac{x_{(1)} e^{(n-1)x_{(1)}}}{e^{(n-1)x_{(1)}}-1} - \frac{1}{n-1}. \Rightarrow \hat{\theta}_{Bayes} = \frac{X_{(1)} e^{(n-1)X_{(1)}}}{e^{(n-1)X_{(1)}}-1} - \frac{1}{n-1}$$

To conclude,

$$\hat{\theta}_{Bayes} = \begin{cases} \dfrac{X_{(1)}}{2} & n=1. \\ \dfrac{X_{(1)} e^{(n-1)X_{(1)}}}{e^{(n-1)X_{(1)}}-1} - \dfrac{1}{n-1}, & n>1. \end{cases}$$

Q 3.7.

Solution:

(a) $E(t_1(X)) = E(X) = \theta \Rightarrow t_1(X)$ is unbiased.

$E(t_2(X)) = E(\frac{1}{2}) = \frac{1}{2}$. $\begin{cases} \text{If } \theta = \frac{1}{2}, \ \frac{1}{2} = E(X) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \\ \qquad \Rightarrow t_2(X) \text{ is unbiased.} \\ \text{If } \theta \neq \frac{1}{2}, \ \frac{1}{2} \neq E(X) = 0 \times (1-\theta) + 1 \times \theta = \theta \\ \qquad \Rightarrow t_2(X) \text{ is biased.} \end{cases}$

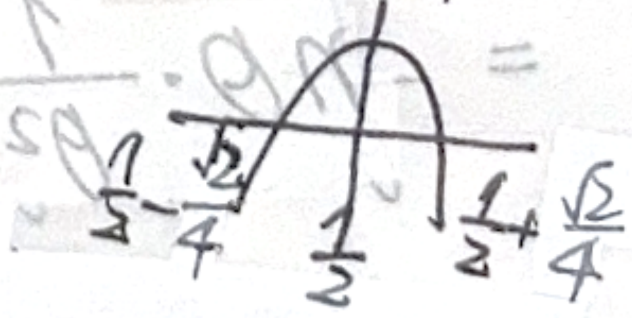
(b) $MSE_1 = E\{t_1(X) - \theta\}^2 = E\{X - \theta\}^2 = \theta(1-\theta)^2 + (1-\theta)\theta^2 = \theta(1-\theta) = \theta - \theta^2.$

$MSE_2 = E\{t_2(X) - \theta\}^2 = E\{\frac{1}{2} - \theta\}^2$

$= \frac{1}{4} - \theta + \theta^2.$

Since $(\theta - \theta^2) - (\frac{1}{4} - \theta + \theta^2) = \theta - \theta^2 - \frac{1}{4} + \theta - \theta^2 = -2\theta^2 + 2\theta - \frac{1}{4} = -2(\theta - \frac{1}{2})^2 + \frac{1}{4}$

$\Rightarrow \frac{1}{2} - \frac{\sqrt{2}}{4} < \theta < \frac{1}{2} + \frac{\sqrt{2}}{4}$: $MSE_1 > MSE_2$




$\theta = \frac{1}{2} - \frac{\sqrt{2}}{4}$ or $\theta = \frac{1}{2} + \frac{\sqrt{2}}{4}$: $MSE_1 = MSE_2$

$0 \leq \theta < \frac{1}{2} - \frac{\sqrt{2}}{4}$ or $\frac{1}{2} + \frac{\sqrt{2}}{4} < \theta \leq 1$: $MSE_1 < MSE_2$.

Q 3.15.

Solution:

(a) $T'(\theta) = 1 - \theta - \theta = 1 - 2\theta.$

$$L(\theta)=\prod_{i=1}^{n}\theta^{x_i}(1-\theta)^{1-x_i}=\theta^{\sum_{i=1}^{n}x_i}(1-\theta)^{n-\sum_{i=1}^{n}x_i}.$$

$$L'(\theta)=\left(\sum_{i=1}^{n}x_i\right)\theta^{\sum_{i=1}^{n}x_i-1}(1-\theta)^{n-\sum_{i=1}^{n}x_i}+\left(\sum_{i=1}^{n}x_i-n\right)\theta^{\sum_{i=1}^{n}x_i}(1-\theta)^{n-\sum_{i=1}^{n}x_i-1}.$$

$$=\theta^{\sum_{i=1}^{n}x_i-1}(1-\theta)^{n-\sum_{i=1}^{n}x_i-1}\left[\left(\sum_{i=1}^{n}x_i\right)\cdot(1-\theta)+\left(\sum_{i=1}^{n}x_i-n\right)\cdot\theta\right].$$

$$S(\theta)=\frac{L'(\theta)}{L(\theta)}=\frac{\left(\sum_{i=1}^{n}x_i\right)\cdot(1-\theta)+\left(\sum_{i=1}^{n}x_i-n\right)\cdot\theta}{\theta(1-\theta)}=\frac{\sum_{i=1}^{n}x_i-n\theta}{\theta(1-\theta)}$$

$f(x;\theta)=\theta^{x}(1-\theta)^{1-x}$; support: $\{0,1\}$.

Since the support of the population density $f(x;\theta)$ doesn't depend on $\theta$,

$\Rightarrow E\{S(\theta)\}=0$. (From $S(\theta)=\frac{\sum_{i=1}^{n}x_i-n\theta}{\theta(1-\theta)}$, we also obtain $E\{S(\theta)\}=0$).

$$\Rightarrow I(\theta)=E\left\{-\frac{d^2\log f(X;\theta)}{d\theta^2}\right\}$$

$$=E\left\{-\frac{d^2}{d\theta^2}\left(x\cdot\log\theta+(1-x)\cdot\log(1-\theta)\right)\right\}$$

$$=E\left\{-\frac{d}{d\theta}\left[x\cdot\frac{1}{\theta}+(1-x)\cdot\frac{-1}{1-\theta}\right]\right\}.$$

$$=E\left\{x\cdot\frac{1}{\theta^2}+(1-x)\cdot\frac{1}{(1-\theta)^2}\right\}$$

$$=\frac{1}{\theta}+\frac{1}{(1-\theta)}=\frac{1}{\theta(1-\theta)}.$$

$$\Rightarrow I_n(\theta)=n\,I(\theta)=\frac{n}{\theta(1-\theta)}$$

$\Rightarrow$ The CR lower bound is: $\frac{\{T'(\theta)\}^2}{I_n(\theta)}=\frac{1}{n}(1-2\theta)^2\theta(1-\theta).$

(b) Consider the sample variance:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad \text{where } \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - \frac{2}{n}\left(\sum_{i=1}^{n} X_i\right)^2 + \left(\sum_{i=1}^{n} X_i\right)^2 \cdot \frac{1}{n}\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i - \frac{2}{n}\left(\sum_{i=1}^{n} X_i\right)^2 + \left(\sum_{i=1}^{n} X_i\right)^2 \cdot \frac{1}{n}\right]$$

Since $X_1, \cdots, X_n \sim \text{Bernoulli}(\theta)$, $\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i$.

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i\right)\left(1 - \frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{\left(\sum_{i=1}^{n} X_i\right)\left(n - \sum_{i=1}^{n} X_i\right)}{(n-1)\,n}.$$

We know: $E(S^2) = Var(X) = \theta(1-\theta)$ (By (2.9) from book: prop. of sample variance).

Thus, $S^2$ is an unbiased estimator of $\tau(\theta)$.

let $T \triangleq \sum_{i=1}^{n} X_i$.

(c)

$$f(x_1, \cdots, x_n; \theta) = \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n - \sum_{i=1}^{n} x_i} = \theta^{t}(1-\theta)^{n-t}$$

By Thm 3.5, $T = \sum_{i=1}^{n} X_i$ is sufficient for $\theta$ and $T \sim \text{Binomial}(n, \theta)$.

Now, assume that a function $h(T)$ satisfies

$$E(h(T)) = \sum_{i=1}^{n} h(t) \Pr(T=t) = \sum_{i=1}^{n} h(t)\binom{n}{t}\theta^{t}(1-\theta)^{n-t} = 0, \quad 0<\theta<1.$$

let $y = \frac{\theta}{1-\theta}$ $(y>0)$ $\Rightarrow \sum_{i=1}^{n} h(t)\binom{n}{t} y^t \cdot (1-\theta)^n = 0$

$$\Rightarrow \sum_{i=1}^{n} h(t)\binom{n}{t} y^t = 0 \quad \forall y>0$$

$$\Rightarrow h(t)\binom{n}{t} = 0 \quad \forall t = 0, 1, \cdots, n.$$

$\Rightarrow h(T) \equiv 0 \Rightarrow$ By def, $T$ is also complete.

Thus, $T$ is a complete sufficient statistic for $\theta$.

Since $S^2 = g(T)$ is an unbiased estimator of $\tau(\theta)$,

by Lehmann-Scheffé theorem, $S^2$ is the unique UMVUE for $\tau(\theta)$.

Thus the UMVUE of $\tau(\theta)$ (also unique) is

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \text{ where } \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

$$= \frac{\left(\sum_{i=1}^{n} X_i\right)\left(n - \sum_{i=1}^{n} X_i\right)}{n(n-1)}.$$

Q3.18

Solution:

(a) $M_X(t) = E(e^{tX}) = \int_0^{+\infty} \sqrt{\frac{\lambda}{2\pi}}\, x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2} \cdot e^{tx}\, dx$

$$= \sqrt{\frac{\lambda}{2\pi}} \int_0^{+\infty} x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2}x + \frac{\lambda}{\mu} - \frac{\lambda}{2x} + tx}\, dx.$$

$$= \sqrt{\frac{\lambda}{2\pi}} \int_0^{+\infty} x^{-\frac{3}{2}} e^{\frac{\lambda}{\mu}\left(1 - \frac{1}{2\mu}x - \frac{\mu}{2x} + \frac{\mu}{\lambda}tx\right)}\, dx$$

$$= e^{\frac{\lambda}{\mu}} \sqrt{\frac{\lambda}{2\pi}} \int_0^{+\infty} x^{-\frac{3}{2}} e^{\frac{\lambda}{\mu}\left(-\frac{1}{2\mu}\right)\left[\left(1 - \frac{2\mu^2}{\lambda}t\right)x + \frac{\mu^2}{x} - 2\mu\sqrt{1-\frac{2\mu^2}{\lambda}t} + 2\mu\sqrt{1-\frac{2\mu^2}{\lambda}t}\right]}\, dx$$

$$= e^{\frac{\lambda}{\mu}} \sqrt{\frac{\lambda}{2\pi}} \int_0^{+\infty} x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2}\left(\sqrt{1-\frac{2\mu^2}{\lambda}t}\cdot\sqrt{x} - \frac{\mu}{\sqrt{x}}\right)^2 - \frac{\lambda}{\mu}\sqrt{1-\frac{2\mu^2}{\lambda}t}}\, dx.$$

$$= e^{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)} \int_0^{+\infty} \sqrt{\frac{\lambda}{2\pi}}\,(-2)\, e^{-\frac{\lambda}{2\mu^2}\left(\sqrt{1-\frac{2\mu^2}{\lambda}t}\cdot\sqrt{x} - \frac{\mu}{\sqrt{x}}\right)^2} d\left(x^{-\frac{1}{2}}\right),$$

$$= e^{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)} \int_0^{+\infty} \sqrt{\frac{\lambda}{2\pi}}\, x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2 x}\left(\sqrt{1-\frac{2\mu^2}{\lambda}t}\, x - \mu\right)^2}\, dx.$$

$$= e^{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)} \sqrt{\frac{\lambda}{2\pi}} \int_0^{+\infty} x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\frac{\mu^2}{1-\frac{2\mu^2}{\lambda}t}x}\left(x - \frac{\mu}{\sqrt{1-\frac{2\mu^2}{\lambda}t}}\right)^2}\, dx$$

Note that: $f_X\left(x \,\middle|\, \frac{\mu}{\sqrt{1-\frac{2\mu^2}{\lambda}t}}, \lambda\right) = \sqrt{\frac{\lambda}{2\pi}}\, x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\frac{\mu^2}{1-\frac{2\mu^2}{\lambda}t}x}\left(x - \frac{\mu}{\sqrt{1-\frac{2\mu^2}{\lambda}t}}\right)^2}.$

which is the p.d.f of $IG\left(\frac{\mu}{\sqrt{1-\frac{2\mu^2}{\lambda}t}}, \frac{\mu^3}{(1-\frac{2\mu^2}{\lambda}t)^{\frac{3}{2}}\lambda}\right)$.

$$\Rightarrow \int_0^{+\infty} x^{-\frac{3}{2}} e^{-\frac{\lambda}{2\frac{\mu^2}{1-\frac{2\mu^2}{\lambda}t}x}\left(x-\frac{\mu}{\sqrt{1-\frac{2\mu^2}{\lambda}t}}\right)^2} dx = \sqrt{\frac{\lambda}{2\pi}},$$

$$\Rightarrow M_X(t) = e^{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)}.$$

(b) $$E(X) = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left. -\frac{\lambda}{\mu}\frac{1}{2}\frac{-\frac{2\mu^2}{\lambda}}{\sqrt{1-\frac{2\mu^2}{\lambda}t}} e^{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)}\right|_{t=0} = \mu.$$

$$E(X^2) = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \left.\left[\left(\frac{\mu}{\sqrt{1-\frac{2\mu^2}{\lambda}t}}\right)^2 + \left(-\frac{1}{2}\right)\cdot\mu\cdot\left(1-\frac{2\mu^2}{\lambda}t\right)^{-\frac{3}{2}}\cdot\left(-\frac{2\mu^2}{\lambda}\right)\right] e^{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)}\right|_{t=0}$$

$$= \mu^2 + \frac{\mu^3}{\lambda}.$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{\mu^3}{\lambda}.$$

(c)
$$M_{\sum_{i=1}^n X_i}(t) = E\left(e^{t\sum_{i=1}^n X_i}\right) \underset{X_1,\cdots,X_n:\, i.i.d.}{=} \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n e^{\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)}$$

$$= e^{n\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2}{\lambda}t}\right)} = e^{\frac{n^2\lambda}{n\mu}\left(1-\sqrt{1-\frac{2n^2\mu^2}{n^2\lambda}t}\right)}$$

$$\Rightarrow \sum_{i=1}^n X_i \sim IG\left(n\mu, \frac{n\mu^3}{\lambda}\right). \qquad (\lambda' = n^2\lambda,\ \mu' = n\mu).$$

(d)
We know that $E(X)=\mu$, $Var(X) = \frac{\mu^3}{\lambda}$.

Thus, $\hat{\mu}^M = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}$, $\hat{\lambda}^M = \frac{\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^3}{\sum_{i=1}^n (X_i-\bar{X})^2} = \frac{\bar{X}^3}{\sum_{i=1}^n (X_i-\bar{X})^2}$.

$$L(\vec{\theta}) = \prod_{i=1}^{n} f_{X_i}(x_i|\mu,\lambda)$$

$$\vec{\theta}=(\mu,\lambda) \quad = \left(\sqrt{\frac{\lambda}{2\pi}}\right)^n \prod_{i=1}^{n} x_i^{-\frac{3}{2}} e^{\sum_{i=1}^{n}\left[-\frac{\lambda}{2\mu^2 x_i}(x_i-\mu)^2\right]}$$

$$l(\vec{\theta}) = \frac{n}{2}\log\frac{\lambda}{2\pi} - \frac{3}{2}\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\left[-\frac{\lambda}{2\mu^2 x_i}(x_i-\mu)^2\right].$$

$$\begin{cases} \frac{\partial l(\vec{\theta})}{\partial \lambda} = \frac{n}{2\lambda} + \sum_{i=1}^{n} -\frac{(x_i-\mu)^2}{2\mu^2 x_i} = 0. \quad ① \\ \frac{\partial l(\vec{\theta})}{\partial \mu} = \sum_{i=1}^{n}\left(\frac{\lambda x_i}{\mu^3} - \frac{\lambda}{\mu^2}\right) = 0. \quad ② \end{cases}$$

$$② \Rightarrow \sum_{i=1}^{n}\left(\frac{x_i}{\mu}-1\right)=0 \Rightarrow \frac{\sum_{i=1}^{n} x_i}{\mu} = n \Rightarrow \mu = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

$$① \Rightarrow \frac{n}{2\lambda} - \frac{\sum_{i=1}^{n} x_i}{2\mu^2} + \frac{n}{\mu} - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i} = 0.$$

$$\Rightarrow \frac{n}{2\lambda} = \frac{\sum_{i=1}^{n} x_i}{\frac{2}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{n^2}{\sum_{i=1}^{n} x_i} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i} = -\frac{n}{2\sum_{i=1}^{n} x_i} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i}.$$

$$\Rightarrow \lambda = \frac{n}{2}\,\frac{1}{\frac{-n}{2\sum_{i=1}^{n} x_i} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i}} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\bar{X}}}.$$

Thus, the MLEs for $\mu$ and $\lambda$ are.

$$\hat{\mu} = \bar{X}, \quad \hat{\lambda} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\bar{X}}}.$$

Q3.19

Solution:

(a) For $x \notin \mathbb{Z}$ or $x<0$, $f(x;\theta)=0$. let $a(\theta)$ or $b(x)$ be $0$ is fine.

For $x\in\mathbb{Z}$ & $x\geq 0$, $f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!} = e^{-\theta}\frac{1}{x!}e^{(\log\theta)\cdot x}$

let $a(\theta)=e^{-\theta}$, $b(x)=\frac{1}{x!}$, $c(\theta)=\log\theta$, $d(x)=x$.

Thus, the p.m.f. of the Poisson distribution with mean $\theta$ belongs to the one-parameter exponential family.

(b) For $x<0$, $f(x;\theta)=0$. letting $a(\theta)$ or $b(x)$ to be 0 is fine.

For $x>0$, $f(x;\theta)=\theta\cdot e^{-\theta x}$. let $a(\theta)=\theta$, $b(x)=1$, $c(\theta)=-\theta$, $d(x)=x$.

Thus, the p.d.f. of the exponential distribution with mean $1/\theta$ belongs to the one-parameter exponential family.

(c) $f(x;\theta)=a(\theta)\,b(x)\exp[c(\theta)d(x)]$, $x\in\mathbb{R}$.

$f(x_1,\cdots,x_n;\theta)=f(\vec{x};\theta)$

$=\prod_{i=1}^{n} f(x_i;\theta)=[a(\theta)]^n\left(\prod_{i=1}^{n} b(x_i)\right)\cdot\exp\left[c(\theta)\sum_{i=1}^{n} d(x_i)\right]$

$=[a(\theta)]^n\cdot\exp\left\{c(\theta)\sum_{i=1}^{n} d(x_i)\right\}\times\left(\prod_{i=1}^{n} b(x_i)\right)$

Since $[a(\theta)]^n\cdot\exp\{c(\theta)\sum_{i=1}^{n} d(x_i)\}$ is a function of both $\theta$ and $\sum_{i=1}^{n} d(x_i)$, and $\prod_{i=1}^{n} b(x_i)$ does not depend on $\theta$,

by the Factorization Theorem, $\sum_{i=1}^{n} d(X_i)$ is a sufficient statistic of $\theta$.

Part 2: 1 question chosen from Q3.20 ~ Q3.25.

Q3.20.

Solution:

(a) $f(x;\theta)=\frac{x}{a^2}e^{-\frac{x^2}{2a^2}}$

$g(u;\theta)\underset{u=x^2}{=}f(x;\theta)\times\left|\frac{dx}{du}\right|=\frac{1}{2a^2}e^{-\frac{u}{2a^2}}$.

Thus, $X_1^2\sim\text{Exponential}\left(\frac{1}{2a^2}\right)$.

(b) Since the support of $f(x;\theta)$ is $(0,+\infty)$ which is independent of $\alpha$ $\Rightarrow E\{S(\alpha)\}=0$.

$T(\alpha) = \alpha$, $T'(\alpha) = 1$.

$$I(\alpha) = E\left\{-\frac{d^2 \log f(X;\alpha)}{d\alpha^2}\right\}.$$

$$f(x;\alpha) = \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}$$

$$\frac{d \log f(X;\alpha)}{d\alpha} = \frac{d}{d\alpha}\left\{\log x - 2\log \alpha - \frac{1}{2\alpha^2}\cdot x^2\right\}.$$

$$= -\frac{2}{\alpha} + \frac{1}{\alpha^3} x^2.$$

$$\frac{d^2 \log f(X;\lambda)}{d\alpha^2} = \frac{2}{\alpha^2} - \frac{3}{\alpha^4} x^2.$$

$$I(\alpha) = E\left\{-\frac{d^2 \log f(X;\alpha)}{d\alpha^2}\right\} = E\left\{-\frac{2}{\alpha^2} + \frac{3}{\alpha^4} X^2\right\}$$

$$= -\frac{2}{\alpha^2} + \frac{3}{\alpha^4} E(X^2)$$

$$= -\frac{2}{\alpha^2} + \frac{3}{\alpha^4} 2\alpha^2. = \frac{4}{\alpha^2}.$$

$\Rightarrow I_n(\alpha) = n I(\alpha) = \frac{4n}{\alpha^2}$.

$\Rightarrow$ the CR-lower bound of $\alpha$ is

$$\frac{\{T'(\alpha)\}^2}{I_n(\alpha)} = \frac{\alpha^2}{4n}.$$

(C) Denote $\alpha^2$ as $\lambda$.

$Y \triangleq X_1^2 \sim \text{Exponential}(\frac{1}{2\lambda})$. $f(y;\frac{1}{2\lambda}) = \frac{1}{2\lambda} e^{-\frac{y}{2\lambda}}$. $y>0$.

$T(\lambda) = \lambda$. $T'(\lambda) = 1$.

Since the support of $f(y;\frac{1}{2\lambda})$ doesn't depend on $\lambda$, $\Rightarrow E\{S(\lambda)\} = 0$.

$$\Rightarrow I(\lambda) = E\left\{-\frac{d^2 \log f(y;\frac{1}{2\lambda})}{d\lambda^2}\right\}$$

$$\frac{d \log f(y;\frac{1}{2\lambda})}{d\lambda} = \frac{d}{d\lambda}\left\{-\log 2 - \log \lambda - \frac{y}{2\lambda}\right\} = -\frac{1}{\lambda} + \frac{y}{2\lambda^2}.$$

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$$\frac{d^2 \log f(y; \frac{1}{2\lambda})}{d\lambda^2} = \frac{1}{\lambda^2} - \frac{y}{\lambda^3}.$$

$$I(\lambda) = E\left\{-\frac{d^2 \log f(y; \frac{1}{2\lambda})}{d\lambda^2}\right\} = E\left(-\frac{1}{\lambda^2} + \frac{y}{\lambda^3}\right)$$

$$= \frac{-1}{\lambda^2} + \frac{2\lambda}{\lambda^3} = \frac{1}{\lambda^2}.$$

$$\Rightarrow I_n(\lambda) = nI(\lambda) = \frac{n}{\lambda^2}.$$

$\Rightarrow$ the CR-lower bound of $\alpha^2$ is

$$\frac{\{T'(\lambda)\}^2}{I_n(\lambda)} = \frac{\lambda^2}{n} = \frac{\alpha^4}{n}.$$

**Q 3.21**

**Solution:**

(a) For any nonnegative and measurable $g(\cdot)$,

$$E\{g(T)\} = E\{g(X_1X_2+X_3)\}$$

$$= \sum_{x_1}\sum_{x_2}\sum_{x_3} g(x_1x_2+x_3)\cdot p_{(X_1,X_2,X_3)}(x_1,x_2,x_3)$$

$$= \sum_{x_1}\sum_{x_2}\sum_{x_3} g(x_1x_2+x_3)\cdot \theta^{x_1}(1-\theta)^{1-x_1}\,\theta^{x_2}(1-\theta)^{1-x_2}\,\theta^{x_3}(1-\theta)^{1-x_3}$$

$$P_r(T=2) = P_r(X_1=X_2=X_3=1) = P_r(X_1=1)\,P_r(X_2=1)\,P_r(X_3=1) = \theta^3.$$

$$P_r(T=0) = P_r(X_1X_2=X_3=0) = P_r(X_1X_2=0)\,P_r(X_3=0)$$

$$= [P_r(X_1=0) + P_r(X_2=0) - P_r(X_1=X_2=0)]\,P_r(X_3=0)$$

$$= [1-\theta + 1-\theta - (1-\theta)^2](1-\theta)$$

$$= (1+\theta)(1-\theta)^2 = (1+\theta)(1-2\theta+\theta^2) = 1-2\theta+\theta^2+\theta-2\theta^2+\theta^3$$

$$= 1-\theta-\theta^2+\theta^3.$$

$Pr(T=1) = 1 - Pr(T=2) - Pr(T=0)$

$= 1-\theta^3 - 1+\theta+\theta^2-\theta^3 = \theta+\theta^2-2\theta^3$.

Thus, the p.m.f of $T$ is

| $T$ | 0 | 1 | 2 |
|---|---|---|---|
| Prob. | $1-\theta-\theta^2+\theta^3$ | $\theta+\theta^2-2\theta^3$ | $\theta^3$. |

(b) Suppose otherwise. On one hand, by factorization theorem, the joint p.m.f can be written as

$$f(x_1, x_2, x_3; \theta) = g(x_1 x_2 + x_3; \theta) \times h(x_1, x_2, x_3)$$

On the other hand $f(x_1, x_2, x_3; \theta) = \theta^{x_1+x_2+x_3}(1-\theta)^{3-x_1-x_2-x_3}$.

Thus, $\theta^{x_1+x_2+x_3}(1-\theta)^{3-x_1-x_2-x_3} = g(x_1x_2+x_3; \theta) \times h(x_1, x_2, x_3)$.

Consider 2 set of assignment of $(x_1, x_2, x_3)$:

① $(x_1, x_2, x_3) = (0, 0, 1)$

$\Rightarrow \theta(1-\theta)^2 = g(1; \theta) \times h(0, 0, 1)$ ———— $(*1)$

② $(x_1, x_2, x_3) = (1, 1, 0)$

$\Rightarrow \theta^2(1-\theta)^1 = g(1, \theta) \times h(1, 1, 0)$. ———— $(*2)$

Dividing both sides of $(*2)$ by both sides of $(*1)$ separately,

we obtain: $\frac{\theta}{1-\theta} = \frac{h(1, 1, 0)}{h(0, 0, 1)}$.

The left hand does depend on $\theta$, while the right hand doesn't.

$\Rightarrow$ A Contradiction!

1b Thus, $T$ is not a sufficient statistic for $\theta$.

Q 3.22

Solution:

(a) $\int_0^1 f(x;\theta)\,dx = \int_0^1 \frac{\log\theta}{\theta-1}\,\theta^x\,dx = \frac{1}{\theta-1}\,\theta^x\Big|_{x=0}^{x=1} = 1.$

Thus, $f(x;\theta)$ is a density function.

(b). $f(x_1 \cdots x_n;\theta) = \prod_{i=1}^n f(x_i;\theta) = \frac{(\log\theta)^n}{(\theta-1)^n}\cdot\theta^{\sum_{i=1}^n x_i}.$

$= \left\{\frac{(\log\theta)^n}{(\theta-1)^n}\cdot\theta^{\sum_{i=1}^n x_i}\right\} \times 1.$

Thus, $T \triangleq \sum_{i=1}^n X_i$ is sufficient for $\theta$.

(c) $M_X(t) = E(e^{tX}) = \int_0^1 \frac{\log\theta}{\theta-1}(\theta e^t)^x\,dx = \frac{\log\theta}{\theta-1}\,\frac{1}{t+\log\theta}\,(\theta e^t)^x\Big|_{x=0}^{x=1}$

$= \frac{\log\theta}{\theta-1}\cdot\frac{\theta e^t-1}{t+\log\theta}.$

(d) $E(X) = \frac{dM_X(t)}{dt}\Big|_{t=0} = \frac{\log\theta}{\theta-1}\,\frac{\theta e^t(t+\log\theta)-\theta e^t+1}{(t+\log\theta)^2}\Big|_{t=0}$

$= \frac{\log\theta}{\theta-1}\,\frac{\theta\log\theta-\theta+1}{(\log\theta)^2} = \frac{\theta}{\theta-1} - \frac{1}{\log\theta} = \frac{\theta\log\theta-\theta+1}{(\theta-1)\log\theta} \triangleq T(\theta)$

$E(X^2) = \frac{d^2M_X(t)}{dt^2}\Big|_{t=0} = \frac{d}{dt}\left\{\frac{\log\theta}{\theta-1}\left[\frac{\theta e^t}{t+\log\theta} + \frac{1-\theta e^t}{(t+\log\theta)^2}\right]\right\}\Big|_{t=0}$

$= \frac{\log\theta}{\theta-1}\left\{\frac{\theta e^t(t+\log\theta)-\theta e^t}{(t+\log\theta)^2} + \frac{-\theta e^t(t+\log\theta)^2-(1-\theta e^t)\cdot 2(t+\log\theta)}{(t+\log\theta)^4}\right\}\Big|_{t=0}$

$$= \frac{\log\theta}{\theta-1}\left\{\frac{\theta\log\theta-\theta}{(\log\theta)^2}+\frac{-\theta(\log\theta)^2-2(1-\theta)\log\theta}{(\log\theta)^4}\right\}.$$

$$= \frac{\theta(\log\theta)^2-\theta\log\theta-\theta\log\theta-2+2\theta}{(\theta-1)(\log\theta)^2}.$$

$$= \frac{\theta(\log\theta)^2-2\theta(\log\theta-1)-2}{(\theta-1)(\log\theta)^2},$$

$$Var(X)=E(X^2)-[E(X)]^2$$

$$= \frac{\theta(\log\theta)^2-2\theta(\log\theta-1)-2}{(\theta-1)(\log\theta)^2}-\frac{\theta^2(\log\theta)^2+\theta^2+1-2\theta^2\log\theta+2\theta\log\theta-2\theta}{(\theta-1)^2(\log\theta)^2}$$

$$= \frac{(\theta^2-\theta)(\log\theta)^2-\cancel{2(\theta^2-\theta)\log\theta}+2\theta^2-2\theta-\cancel{2\theta}+2-\cancel{\theta^2(\log\theta)^2}+\cancel{2(\theta^2-\theta)\log\theta}-\theta^2-1+\cancel{2\theta}}{(\theta-1)^2(\log\theta)^2}$$

$$= \frac{(\theta-1)^2-\theta(\log\theta)^2}{(\theta-1)^2(\log\theta)^2},$$

$$E(\bar{X})=E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\frac{1}{n}\sum_{i=1}^{n}E(X_i)=E(X)=\tau(\theta).$$

Thus, $\bar{X}=\frac{1}{n}T$ is an unbiased estimator of $\tau(\theta)$.

(e) Since the support of $f(x;\theta)$ is $[0,1]$, which is independent of $\theta$.

$\Rightarrow E\{S(\theta)\}=0$

$$\Rightarrow I(\theta)=E\left\{-\frac{d^2\log f(x;\theta)}{d\theta^2}\right\}.$$

$$\frac{d \log f(x;\theta)}{d\theta} = \frac{d}{d\theta}\left\{\log(\log\theta) - \log(\theta-1) + x\log\theta\right\}.$$

$$= \frac{1}{\log\theta}\cdot\frac{1}{\theta} - \frac{1}{\theta-1} + \frac{x}{\theta}.$$

$$\frac{d^2 \log f(x;\theta)}{d\theta^2} = \frac{-1}{(\log\theta)^2}\cdot\frac{1}{\theta}\cdot\frac{1}{\theta} + \frac{-1}{\theta^2}\frac{1}{\log\theta} + \frac{1}{(\theta-1)^2} - \frac{x}{\theta^2}.$$

$$= \frac{-1}{\theta^2(\log\theta)^2} - \frac{1}{\theta^2\log\theta} + \frac{1}{(\theta-1)^2} - \frac{x}{\theta^2}.$$

$$\Rightarrow I(\theta) = E\left\{-\frac{d^2\log f(x;\theta)}{d\theta^2}\right\} = \frac{1}{\theta^2(\log\theta)^2} + \cancel{\frac{1}{\theta^2\log\theta}} - \frac{1}{(\theta-1)^2} + \frac{1}{\theta(\theta-1)} - \cancel{\frac{1}{\theta^2\log\theta}}$$

$$= \frac{(\theta-1)^2 - \theta^2(\log\theta)^2 + \theta(\theta-1)(\log\theta)^2}{\theta^2(\theta-1)^2(\log\theta)^2}.$$

$$= \frac{(\theta-1)^2 - \theta(\log\theta)^2}{\theta^2(\theta-1)^2(\log\theta)^2}.$$

$$\Rightarrow I_n(\theta) = n\, I(\theta) = n\frac{(\theta-1)^2 - \theta(\log\theta)^2}{\theta^2(\theta-1)^2(\log\theta)^2},$$

(f) the CR-lower bound of $\tau(\theta)$ is

$$\frac{\{\tau'(\theta)\}^2}{I_n(\theta)} = \frac{\dfrac{[(\log\theta+1-1)(\theta-1)\log\theta - (\log\theta+1-\frac{1}{\theta})(\theta\log\theta-\theta+1)]^2}{(\theta-1)^4(\log\theta)^4}}{n\dfrac{(\theta-1)^2-\theta(\log\theta)^2}{\theta^2(\theta-1)^2(\log\theta)^2}}$$

$$= \frac{\theta^2\{\cancel{\theta}-1)(\log\theta)^2 - \cancel{\theta(\log\theta)^2} - \theta\log\theta + \cancel{\log\theta} + \cancel{\theta\log\theta} + \theta - 1 - \cancel{\log\theta} - 1 + \frac{1}{\theta}\}^2}{n(\theta-1)^2(\log\theta)^2[(\theta-1)^2 - \theta(\log\theta)^2]}.$$

$$= \frac{\{(\theta-1)^2 - \theta(\log\theta)^2\}^2}{n(\theta-1)^2(\log\theta)^2[(\theta-1)^2 - \theta(\log\theta)^2]} = \frac{(\theta-1)^2 - \theta(\log\theta)^2}{n(\theta-1)^2(\log\theta)^2}.$$

Meanwhile,

$$Var(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n} Var(X) = \frac{(\theta-1)^2 - \theta(\log\theta)^2}{n(\theta-1)^2(\log\theta)^2};$$

Thus. $\dfrac{\{\tau'(\theta)\}^2}{I_n(\theta)} = Var(\bar{X})$.

Since $\bar{X}$ is also an unbiased estimator of $\tau(\theta)$,

by def, $\bar{X}$ is the efficient estimator of $\tau(\theta)$.

Q3.23.

Solution:

$$f(x_1,\cdots,x_n;\lambda,\nu) = \frac{1}{(Z(\lambda,\nu))^n}\frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}(x_i!)^\nu}$$

$$= \frac{\frac{1}{(Z(\lambda,\nu))^n}\cdot\lambda^{\sum_{i=1}^{n} x_i}}{e^{\nu\cdot\log\prod_{i=1}^{n}(x_i!)}} = \frac{\frac{1}{(Z(\lambda,\nu))^n}\cdot\lambda^{\sum_{i=1}^{n} x_i}}{e^{\nu\cdot\sum_{i=1}^{n}\log(x_i!)}} = \frac{\frac{1}{(Z(\lambda,\nu))^n}\lambda^{t_1}}{e^{\nu\cdot t_2}} \times 1.$$

By Theorem 3.6, $\{T_1, T_2\}$ are joint sufficient statistics for $\{\lambda, \nu\}$.

Q3.24.

Solution:

$$f(x_1,\cdots,x_n;\theta) = \frac{1}{(2\theta)^n}\prod_{i=1}^{n}\frac{1}{i}\cdot\prod_{i=1}^{n} 1(-i(\theta-1) < x_i < i(\theta+1))$$

$$= \frac{1}{(2\theta)^n}\prod_{i=1}^{n}\frac{1}{i}\cdot\prod_{i=1}^{n} 1\left(\theta > 1-\frac{x_i}{i} \;\&\; \theta > \frac{x_i}{i}-1\right)$$

$$= \frac{1}{(2\theta)^n}\prod_{i=1}^{n}\frac{1}{i}\cdot 1\left(\theta > \max_i\left|\frac{x_i}{i}-1\right|\right).$$

$$= \left\{\frac{1}{(2\theta)^n}\cdot\prod_{i=1}^{n}\frac{1}{i}\cdot 1\left(\theta > \max_i\left|\frac{x_i}{i}-1\right|\right)\right\} \times 1.$$

By Factorization theorem, $T = \max_i\left|\frac{x_i}{i}-1\right|$ is a sufficient statistic.