Discrete Mathematics for Computer Science

Lecture 4: Set and Function

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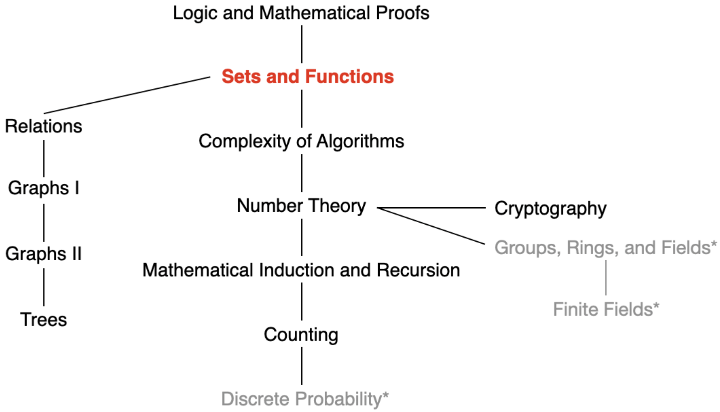
This Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Sets

A set is an unordered collection of objects. These objects are called elements or members.

- $A = \{1, 2, 3, 4\}$
- $B = \{a, b, c, d\}$
- $C = \{a, 2, 1, \text{Mary}\}$

Many discrete structures are built with sets:

- combinations
- relations
- graphs

Set Representation

Examples:

- $A = \{2, 3, 5, 7\}$
- $B = \{1, 2, 3, ..., 100\}$
- $C = \{a \mid a \geq 2, a \text{ is a prime}\}$
- $D = \{2n \mid n = 0, 1, 2, ..., \}$

Representing a set by:

- listing (enumerating) the elements
- if enumeration is hard, use ellipses (...)
- definition by property, using the set builder

$$\{x \mid x \text{ has property } P \text{ or property } P(x))\}$$

Notation:

- $a \in A$: a is an element of set A
- $a \notin A$: a is not an element of set A

Important sets

- Natural numbers:
  - $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$
- Integers:
  - $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Positive integers:
  - $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$
- Rational numbers:
  - $\mathbf{Q} = \{\frac{p}{q} \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$
- Real numbers:
  - $\mathbf{R}$
- Complex numbers:
  - $\mathbf{C}$

Important sets

- $[a, b] = \{x \mid a \leq x \leq b\}$
  $[a, b) = \{x \mid a \leq x < b\}$
  $(a, b] = \{x \mid a < x \leq b\}$
  $(a, b) = \{x \mid a < x < b\}$

- Two sets A, B are *equal* if and only if
  $\forall x \, (x \in A \leftrightarrow x \in B)$.

Are sets $\{1, 2, 5\}$ and $\{2, 5, 1\}$ equal? Yes

Are sets $\{1, 2, 2, 2, 5\}$ and $\{2, 5, 1\}$ equal? Yes

Universal and Empty Set

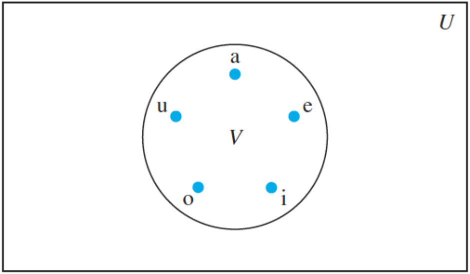
The universal set is the set of all objects under consideration, denoted by U.

The empty set is the set of no object, denoted by $\emptyset$ or $\{\}$.

- Are $\emptyset$ and $\{\emptyset\}$ equal? No

Venn Diagrams

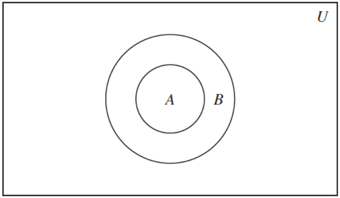
A set can be visualized using Venn diagrams



Subset

The set A is a subset of B if and only if every element of A is also an element of B, i.e., $\forall x(x \in A \rightarrow x \in B)$, denoted by $A \subseteq B$.

If $A \subseteq B$, but $A \neq B$, then we say A is a proper subset of B, i.e., $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$, denoted by $A \subset B$.



Proof of Subset

Proof:

- Showing $A \subseteq B$: if x belongs to A, then x also belongs to B.
- Showing $A \nsubseteq B$: find a single $x \in A$ such that $x \notin B$.

Theorems

Prove that $\emptyset \subseteq S$.

Proof:

By definition, we need to prove $\forall x(x \in \emptyset \to x \in S)$. Since the empty set does not contain any element, $x \in \emptyset$ is always false. Then the implication is always true.

Prove that $S \subseteq S$.

Proof:

By definition, we need to prove $\forall x(x \in S \to x \in S)$. This is obviously true.

Note: two sets are equal if and only if each is a subset of the other:

$$\forall x(x \in A \leftrightarrow x \in B)$$

The Size of a Set – Cardinality

Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say that S is a finite set and n is the cardinality of S, denoted by $|S|$.

A set is said to be infinite if it is not finite.
Examples:

- $A = \{1, 2, 3, ..., 20\}$, where $|A| = 20$
- $B = \{1, 2, 3, ...\}$, which is infinite
- $|\emptyset| = 0$
- $|\{\emptyset\}| = 1$

Power Set

Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations,

Given a set S, the power set of S is the set of all subsets of the set S, denoted by $\mathcal{P}(S)$.

Example: What is the power set of the set $\{0, 1, 2\}$?

$$\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

If S is a set with $|S| = n$, then $|\mathcal{P}(S)| = 2^n$. Why?

Power Set

What is the power set of $\emptyset$?

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

What is the power set of the set $\{\emptyset\}$?

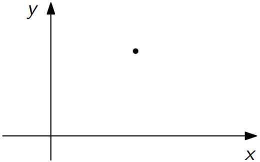
$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

Tuples

The ordered n-tuple $(a_1, a_2, \dots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on.

Ordered 2-tuples are called ordered pairs

Example:



coordinates of a point in the 2-D plane

Two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. That is, $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \dots, n$.

Cartesian Product

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Example:

- $A = \{1, 2\}$, $B = \{a, b, c\}$
- $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Are $A \times B$ and $B \times A$ equal? No, $A \times B \neq B \times A$

What is the cardinality $|A \times B|$? $|A \times B| = |A| \times |B|$

Cartesian Product

The Cartesian product of the sets $A_1, A_2, ..., A_n$, denoted by $A_1 \times A_2 \times ... \times A_n$, is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$ where $a_i \in A_i$ for $i = 1, ..., n$:

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) \mid a_i \in A_i \text{ for } i = 1, 2, ..., n\}$$

Example:

$A = \{0, 1\},\ B = \{1, 2\},\ C = \{0, 1, 2\}$

$A \times B \times C =$

$$\{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2),$$
$$(1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,2,2)\}$$

Let A be a set. A^n denotes $A \times A \times ... \times A$ with n sets:

$$A^n = \{(a_1, a_2, ..., a_n) \mid a_i \in A \text{ for } i = 1, 2, ..., n\}$$

Relation

A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B.

Example: What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently,

$$R = \{(0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

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Summary of Set

- Set: unordered collection of objects
- Subset $A \subseteq B$
- Cardinality: size of set
- Power of set $\mathcal{P}(A)$
- Tuple: (a, b)
- Cartesian Product $A \times B$
- Relation: a subset of $A \times B$

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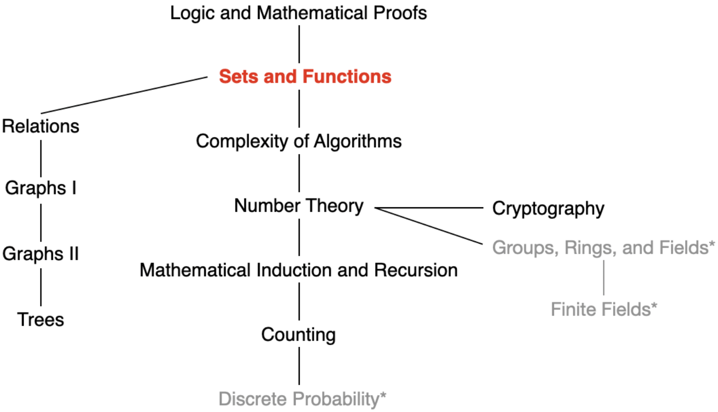
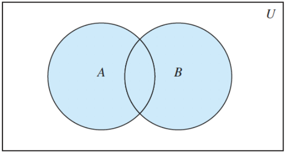


Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

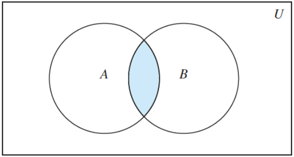
Set Operations

Union: Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set $\{x \mid x \in A \vee x \in B\}$.



$A \cup B$ is shaded.

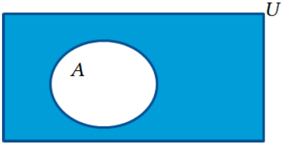
Intersection: The intersection of the sets A and B, denoted by $A \cap B$, is the set $\{x \mid x \in A \wedge x \in B\}$.



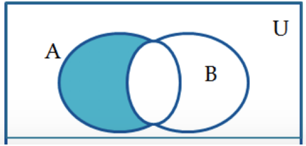
$A \cap B$ is shaded.

Set Operations

Complement: If A is a set, then the complement of the set A (with respect to U), denoted by $\bar{A}$ is the set $U - A$, $\bar{A} = \{x \in U \mid x \notin A\}$



Difference: Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.
$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$.



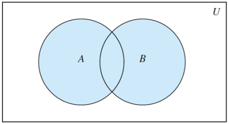
Disjoint Sets

Two sets A and B are called disjoint if their intersection is empty, i.e., $A \cap B = \emptyset$.

Example: $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ are disjoint, because $A \cap B = \emptyset$.

Cardinality of the Union

What is the cardinality of $A \cup B$?



$A \cup B$ is shaded.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The generalization of this result to unions of an arbitrary number of sets is called the principle of inclusion–exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

THE PRINCIPLE OF INCLUSION–EXCLUSION Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Exercises

- $U = \{0, 1, 2, \ldots, 10\}, A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7, 8\}$

1. $A \cup B$
2. $A \cap B$
3. $\bar{A}$
4. $\bar{B}$
5. $A - B$
6. $B - A$

- $U = \{0, 1, 2, \ldots, 10\}, A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7, 8\}$

1. $A \cup B$ $\{1,2,3,4,5,6,7,8\}$

Set Identities

The properties and laws of sets that help us demonstrate and prove set operations, subsets and equivalence.

- Identity laws
  - $A \cup \emptyset = A$
  - $A \cap U = A$
- Domination laws
  - $A \cup U = U$
  - $A \cap \emptyset = \emptyset$
- Idempotent laws
  - $A \cup A = A$
  - $A \cap A = A$
- Complementation laws
  - $\bar{\bar{A}} = A$

Set Identities

- **Commutative laws**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative laws**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive laws**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **De Morgan's laws**
  - $\overline{A \cap B} = \bar{A} \cup \bar{B}$
  - $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Set Identities

- Absorbtion laws
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- Complement laws
  - $A \cup \bar{A} = U$
  - $A \cap \bar{A} = \emptyset$

Proof of Set Identities

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof 1: Using membership tables. Consider an arbitrary element x: 1, x is in A; 0, x is not in A.

A	B	$\overline{A}$	$\overline{B}$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	0	0
1	0	0	1	1	1
0	1	1	0	1	1
0	0	1	1	1	1

Proof 2: by showing that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

- $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$:
  - Suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. Using the definition of intersection, $\neg((x \in A) \wedge (x \in B))$ is true.
  - By applying De Morgan's law, $\neg(x \in A) \vee \neg(x \in B))$. Thus, $x \notin A$ or $x \notin B$. Using the definition of the complement of a set, $x \in \bar{A}$ or $x \in \bar{B}$.
  - By the definition of union, we see that $x \in \bar{A} \cup \bar{B}$. Thus, $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$.
- $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Proof of Set Identities

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Proof 1: using membership tables.
Proof 2: by showing that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$
Proof 3: Using set builder and logical equivalences

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	by definition of complement
$= \{x \mid \neg(x \in (A \cap B))\}$	by definition of does not belong symbol
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by definition of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by the first De Morgan law for logical equivalences
$= \{x \mid x \notin A \vee x \notin B\}$	by definition of does not belong symbol
$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	by definition of complement
$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	by definition of union
$= \overline{A} \cup \overline{B}$	by meaning of set builder notation

Generalized Unions and Intersections

- The *union of a collection of sets* is the set that contains those elements that are members of at least one set in the collection $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$.

- The *intersection of a collection of sets* is the set that contains those elements that are members of all sets in the collection $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$.

Computer Representation of Sets

Question: How to represent sets in a computer?

- One solution: explicitly store the elements in a list
  - Computing the union, intersection, or difference operations would be time-consuming, because of the needs for searching elements.
- A better solution: assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is in the set.
  - Universal set U is finite and with n elements
  - Represent a subset A of U with n bits, where the i-th bit is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Computer Representation of Sets

Example: $U = \{1, 2, 3, 4, 5\}$
$A = \{2, 5\}$. Thus, A is represented by 01001
$B = \{1; 5\}$. Thus, B is represented by 10001

- Union: $A \vee B = 11001$, i.e., $\{1, 2, 5\}$
- Intersection: $A \wedge B = 00001$, i.e., $\{5\}$
- Complement: $\bar{A} = 10110$, i.e., $\{1, 3, 4\}$

Summary of Set Operations

- Union $A \cup B$, cardinality (principle of inclusion-exclusion)
- Intersection $A \cap B$
- Complement $\bar{A}$
- Difference $A - B$
- Disjoint set
- Set identities
- Proof of set identities
  - membership table, subset, set build and logical equivalences
- Computer representations

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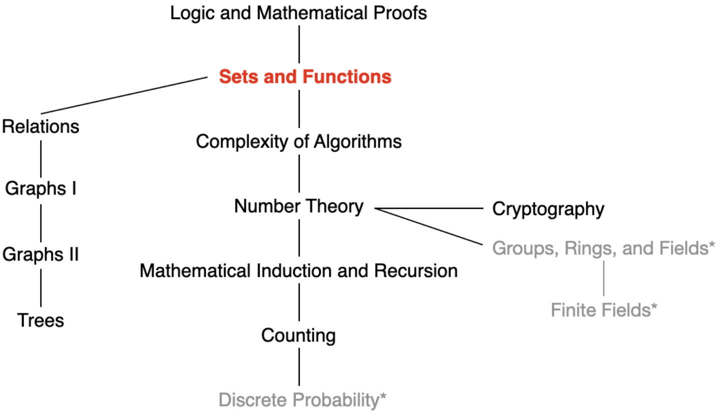
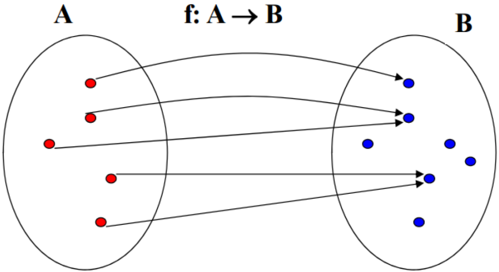


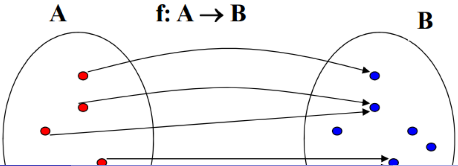
Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets

Function

Let A and B be two sets. A function from A to B, denoted by $f : A \to B$, is an assignment of exactly one element of B to each element of A.

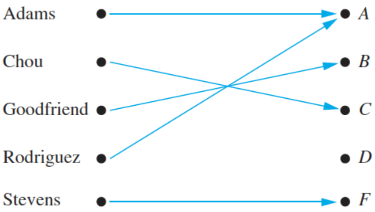
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.





Representing Functions

1 Explicitly state the assignments between elements of the two sets



Note: Admas $\mapsto A$, Chou $\mapsto C$, ...

2 By a formula: $\mathrm{f(x)} = \mathrm{x} + 1$

3 By a relation from A to B: (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22).

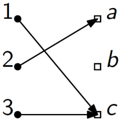
Important Sets of Functions

Let f be a function from A to B.

- A is the domain of f; B is the codomain of f
- If $f(a) = b$, b is called the image of a and a is a preimage of b.
- The range of f is the set of all images of elements of A, denoted by $f(A)$.
- We also say f maps A to B.

Example:

$A = \{1, 2, 3\}$, $B = \{a, b, c\}$



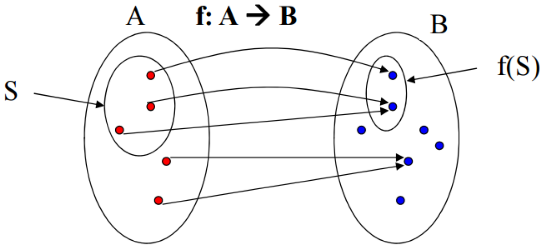
Example:

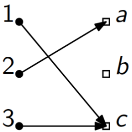
$A = \{1, 2, 3\}$, $B = \{a, b, c\}$

– c is the image of 1

Image of a Subset

For a function $f : A \to B$ and $S \subseteq A$, the image of S is a subset of B that consists of the images of the elements of S, denoted by $f(S)$, where $f(S) = \{f(s) | s \in S\}$





Let $S = \{1, 3\}$, what is $f(S)$?

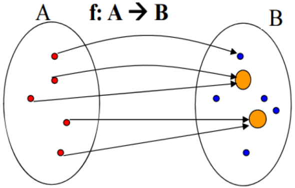
One-to-One and Onto Functions

- **One-to-one function**
  - never assign the same value to two different domain elements.
- **Onto function**
  - every member of the codomain is the image of some element of the domain.
- **One-to-one correspondence**
  - One-to-one and onto

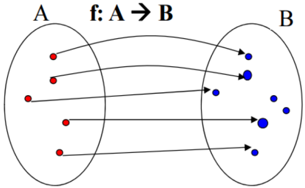
One-to-One (Injective) Function

A function f is called one-to-one or injective if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f. Also called an injunction.

Alternatively: A function is one-to-one if and only if $x \neq y$ implies $f(x) \neq f(y)$. (contrapositive!)



Not injective



Injective function

How about:

- $f(x) \neq f(y)$ implies $x \neq y$?
- $x = y$ implies $f(x) = f(y)$?

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One-to-One (Injective) Function

Example 1:
Whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one? Yes.

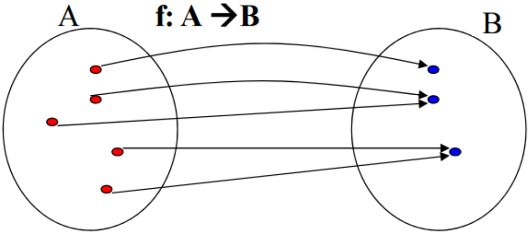
Example 2:
Whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one? No, $f(-1) = f(1)$

What if it is from the set of positive integers to the set of integers? Yes.

Onto (Surjective) Function

A function f is called onto or surjective if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$. Also called a surjection.

Alternatively: A function is onto if and only if all codomain elements are covered, i.e., $f(A) = B$.



Onto (Surjective) Function: Example

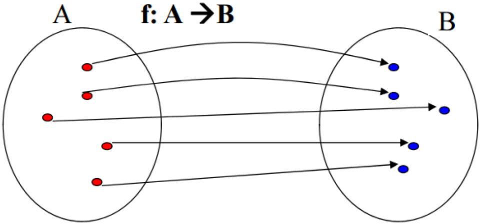
Example 1:
Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function? Yes.

What if the codomain were $\{1, 2, 3, 4\}$? No.

Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto? No, as there is no integer x with $x^2 = -1$.

One-to-One Correspondence (Bijective Function)

A function f is called one-to-one correspondence or bijective, if and only if it is both one-to-one and onto. Also called bijection.

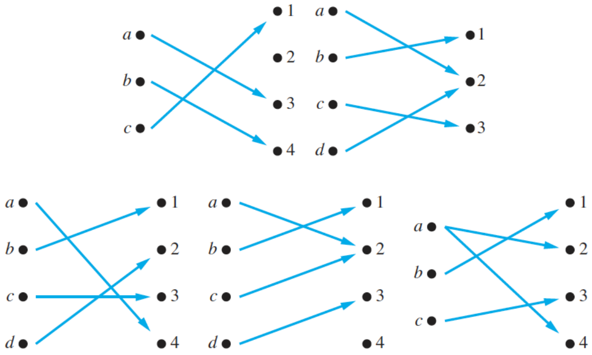


One-to-One Correspondence: Example

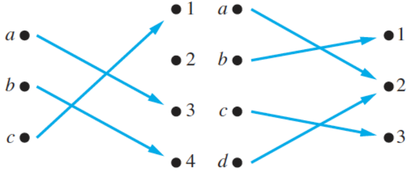
Example 1:
Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a one-to-one correspondence? Yes.

Example 2: Consider an identity function on A, i.e., $\iota : A \to A$, where $\iota_A(x) = x$. Is this function a one-to-one correspondence? Yes.

Are These Functions Injective, Surjective, Bijective?



(a) One-to-one, not onto (b) Onto, not one-to-one



Proof for One-to-One and Onto

To show that f is *injective*	Show that if $f(x) = f(y)$ for all $x, y \in A$, then $x = y$
To show that f is not *injective*	Find specific elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$
To show that f is *surjective*	Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$
To show that f is not *surjective*	Find a specific element $y \in B$ such that $f(x) \neq y$ for all $x \in A$

Example

$f : \boldsymbol{Z} \to \boldsymbol{Z}$, where $f(x) = x + 1$. Is f injective? Surjective? Bijective?

Proof:

- Injective (one-to-one function): If $f(x) = f(x')$ for any arbitrary x and x', then $x = x'$.
- Surjective (onto function): For every integer y, these exists an integer x such that $f(x) = y$.
- Bijective (one-to-one correspondence): injective and surjective

One-to-One and Onto

Prove that "for a function $f : A \rightarrow B$ with $|A| = |B| = n$, f is one-to-one if and only if f is onto."

Proof: Since $|A| = n$, let $\{x_1, x_2, ..., x_n\}$ be elements of A.

- If f is one-to-one, then f is onto (direct proof): Suppose that f is one-to-one. According to the definition of one-to-one function, $f(x_i) \neq f(x_j)$ for any $i \neq j$. Thus, $|f(A)| = |\{f(x_1), ..., f(x_n)\}| = n$. Since $|B| = n$ and $f(A) \subseteq B$, we have $f(A) = B$.
- If f is onto, then f is one-to-one (contradiction): Suppose that f is onto. Suppose that f is not one-to-one. Thus, $f(x_i) = f(x_j)$ for some $i \neq j$. Then, $|\{f(x_1), ..., f(x_n)\}| \leq n - 1$. Note that $|f(A)| = |B| = n$, which leads to a contradiction.

One-to-One and Onto

Consider an infinite set A and a function from A to A. Consider the statement "For any arbitrary $f : A \to A$, f is one-to-one if and only if f is onto". Is this statement true?

Proof (Counterexample): Consider the following $f : \boldsymbol{Z} \to \boldsymbol{Z}$, where $f(x) = 2x$. f is one-to-one but not onto:

- $f(1) = 2$
- $f(2) = 4$
- $f(3) = 6$
- ...

We can prove that 3 has no preimage.

Two Functions on Real Numbers

Let f_1 and f_2 be functions from A to $\boldsymbol{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\boldsymbol{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Example:

$f_1 = x - 1$ and $f_2 = x^3 + 1$

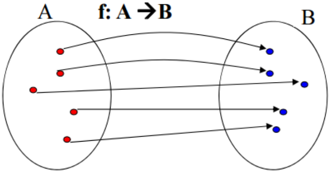
Then

$$(f_1 + f_2)(x) = x^3 + x$$
$$(f_1 f_2)(x) = x^4 - x^3 + x - 1$$

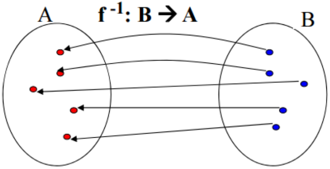
Inverse Functions

Let f be a one-to-one correspondence (bijection) from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.

The inverse function of f is denoted by f^{-1}.
Hence, $f^{-1}(b) = a$ when $f(a) = b$.



f is bijective



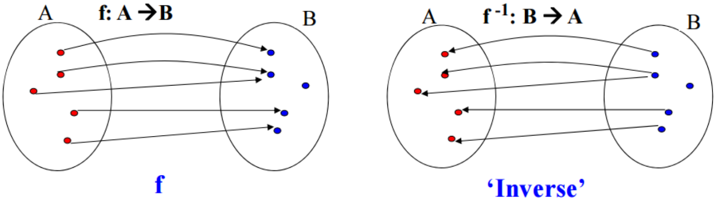
Inverse of f

A bijection is called invertible.

Inverse Functions

If is not a one-to-one correspondence (bijection), it is impossible to define the inverse function of f. Why?

Assume f is not one-to-one (injective):

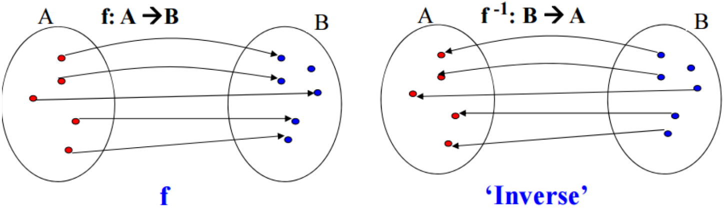


The inverse is not a function: one element of B is mapped to two different elements of A.

Inverse Functions

If is not a one-to-one correspondence (bijection), it is impossible to define the inverse function of f. Why?

Assume f is not onto (surjective):



The inverse is not a function: one element of B is not assigned an element of A.

Proof for Inverse Function

1. Prove function f is a bijection: injective, surjective

To show that f is *injective*	Show that if $f(x) = f(y)$ for all $x, y \in A$, then $x = y$
To show that f is not *injective*	Find specific elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$
To show that f is *surjective*	Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$
To show that f is not *surjective*	Find a specific element $y \in B$ such that $f(x) \neq y$ for all $x \in A$

2. If f is a bijection, then it is invertible
3. Determine the inverse function

Inverse Functions: Example 1

$f : \boldsymbol{Z} \to \boldsymbol{Z}$, where $f(x) = x + 1$. Is f invertible? If yes, then what is the inverse function f^{-1}?

Proof: f is invertible, as it is a bijection (one-to-one correspondence):

- Injective (one-to-one function): If $f(x) = f(x')$ for any arbitrary x and x', then $x = x'$.
- Surjective (onto): For every integer y, these exists an integer $x = y - 1$ such that $f(x) = y$.

To reverse the function, suppose that y is the image of x, so that $y = x + 1$. Then, $x = y - 1$. This means that $y - 1$ is the unique element of $\boldsymbol{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

Inverse Functions: Example 2

Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

Proof: No, f is not invertible. This is because f is not injective, as $f(-2) = f(2)$.

What if we restrict function $f(x) = x^2$ to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers?

Proof: It is invertible, as it is a bijection:

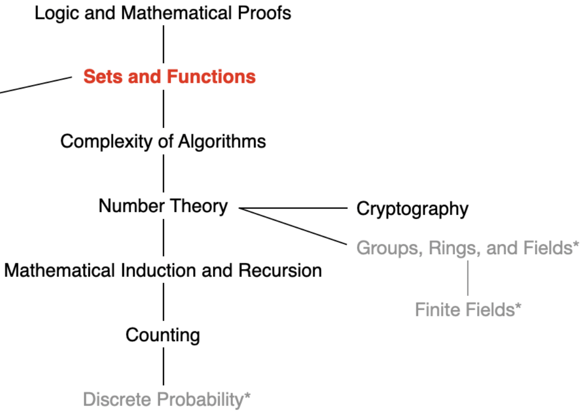
- Injective: Consider x and x'. If $f(x) = f(x')$ (i.e., $x^2 = (x')^2$), then we have $x^2 - (x')^2 = (x + x')(x - x') = 0$. Since we consider the set of all nonnegative real numbers, we must have $x = x'$.
- Surjective: Consider an arbitrary nonnegative real number y. There exists a nonnegative real number $x = \sqrt{y}$ such that $f(x) = y$.

To reverse the function, suppose that y is the image of x, so that $y = x^2$. Then, $x = \sqrt{y}$. Consequently, $f^{-1}(y) = \sqrt{y}$.

Summary of Function

- Function $f : A \to B$: an assignment of exactly one element of B to each element of A
- Domain, codedomain, image, preimage, range
- One-to-one function
  - also called an injunction or injective function
- Onto function
  - also called a surjection or surjective function
- One-to-one correspondence
  - one-to-one and onto
  - also called a bijection or bijective function
- Inverse function
  - One-to-one correspondence

Next Lecture



Set and Functions: set, set operations, functions, sequences and summation, cardinality of sets