ADA Mini HW #6

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(1) Consider the 2 endpoints of the longest path, each of them must be either a root with only one child or a leaf, or we can extend the longest path by the endpoints' parent/child. And because there is only one root, we have 2 cases:

(a) one of the endpoints is a root with only one child, while the other one is a leaf, then the height is $x \geq \lceil \frac{x}{2} \rceil$ because any root-to-leaf path's length cannot exceed x or a contradiction happens.

(b) both of the endpoints are leaves, then we can split the longest path into 2 components where each of them belongs to exactly one root-to-leaf path. Let the lengths of them be ℓ and $x - \ell$, then we have at least one root-to-leaf path with length $\geq max(\ell, x - \ell)$, which $\geq \lceil \frac{x}{2} \rceil$ for all possible ℓ. Thus, the height is at least $\lceil \frac{x}{2} \rceil$.

(2) Proof by Contradiction:

Assume $v \notin S$, then the longest path lies in a subtree formed by one of the children of v, let this child node denoted as u. By the statement in question (1), the height of this subtree $h \geq \lceil \frac{x}{2} \rceil$. We can see that any root-to-leaf path which does not pass u should not have length $\geq \lfloor \frac{x}{2} \rfloor$ $(\leq \lfloor \frac{x}{2} \rfloor - 1)$, or we can construct a longest path($\lceil \frac{x}{2} \rceil + \lfloor \frac{x}{2} \rfloor = x$) which passes v by combining this path with the path from v to the deepest leaf in the subtree of u. By the above observation, the height of this tree(denoted by H) is determined by the height of the subtree contains the longest path $\implies H = h + 1 \geq \lceil \frac{x}{2} \rceil + 1$.

Now, we can construct a better rooted tree by changing the root from v to u. After changing, the height of the subtree which contains the longest path decreases by $1(h' = h - 1)$, while all the other subtrees' height increase by 1 but still less than or equal to $\lfloor \frac{x}{2} \rfloor$). The new tree's height $H' = h' + 1 = H - 1$, which is always less than H. $\implies$ *contradiction* $\implies v \in S$

(3) If v is the middle vertex of one of the longest simple paths, then v splits this path into 2 root-to-leaf paths whose lengths are $\lfloor \frac{x}{2} \rfloor$ and $\lceil \frac{x}{2} \rceil$. And for the other root-to-leaf paths, their length should less than or equal to $\lfloor \frac{x}{2} \rfloor$, or we can construct a longer path($> x$). Thus, the height of the tree $= \lceil \frac{x}{2} \rceil$, which is the lower bound proved in question (1), so it must be an answer to the problem.