CHAPTER 3

Vector Differentiation

1. Introduction

Having studied algebra of vectors in the previous chapter, we shall go to the next logical step i.e. calculus of vectors. Before considering how to differentiate a vector we shall first learn how a curve in space is represented in vector form. After defining the derivative of a vector we shall learn how it represents a tangent to a curve in space.

We shall then prove the rules of differentiation of vectors. In article 10 we shall discuss point functions. After defining a new operator ∇ we shall learn to find gradient, divergence and curl of a vector. Lastly we shall consider second order differential operators and an important operator ∇^2 called Laplacian. We shall then learn the technique of operating by these operators on different functions.

2. Vector Function of a Scalar Quantity

If t is a scalar quantity and if to each value of t in an interval there corresponds a vector $\bar{r}$ then we say that $\bar{r}$ is a vector function of a scalar variable t and we denote it as

$$\bar{r} = \bar{f}(t)$$

If c is a particular value of the scalar t, then we denote the corresponding vector $\bar{c}$ under $\bar{f}$ by

$$\bar{c} = \bar{f}(c)$$

Illustration : Suppose a particle P moves along a curve and t denotes time and $\bar{r}$ denotes its position vector. Then clearly $\bar{r}$ is a function of t. Similarly, the velocity and acceleration of the moving particle P are also vector functions of time t.

3. Decomposition of a Vector Function

If we can write $\bar{r} = \bar{f}(t)$ as $\boxed{\bar{f}(t) = f_1(t)\,i + f_2(t)\,j + f_3(t)\,k}$

where $f_1(t)$, $f_2(t)$, $f_3(t)$, are scalar functions of t then it is called the **decomposition** of the vector function $\bar{f}(t)$.

Illustrations : If t is a parameter then we know that the equations of circle, ellipse, hyperbola and parabola in parametric form are $x = a\cos t$, $y = a\sin t$; $x = a\cos t$, $y = b\sin t$; $x = a\sec t$, $y = b\tan t$; $x = at^2$, $y = 2at$.

If $\bar{r}$ is the position vector of a point P on a curve then clearly,

$$\bar{r} = xi + yj + zk$$

if (x, y, z) are the coordinates of P.

Hence, using the above parametric forms of the curves the position vector of a point on circle, ellipse, hyperbola and parabola can be given by

$$\bar{r} = a\cos t\, i + a\sin t\, j + ok$$

$$\bar{r} = a\cos t\, i + b\sin t\, j + ok$$

$$\bar{r} = a\sec t\, i + b\tan t\, j + ok$$

$$\bar{r} = at^2 i + 2at\, j + ok.$$

Thus, above equations give us vector decomposition of a position vector of a point on these curves.

Since the above equations give us the position vector of a point on the curves they can also be called as the equations of these curves in vector form.

4. Curves in Space

If $\bar{r}$ is a position vector of a point $P(x, y, z)$ on a curve $x = f_1(t), y = f_2(t), z = f_3(t)$, then

$$\bar{r} = xi + yj + zk$$

where $x = f_1(t), y = f_2(t), z = f_3(t)$ is the vector equation of the curve in space.

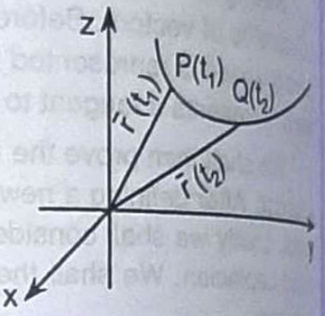

Fig. 3.1

For a fixed value of t, say t_1, $x_1 = f_1(t_1)$, $y_1 = f_2(t_1)$, $z_1 = f_3(t_1)$ are constants and $\bar{r} = x_1 i + y_1 j + z_1 k$ is a fixed point P on the curve. As t changes from t_1 to t_2, we move on from a point P to another point Q on the curve.

The following are the equations of some more curves in space,

$$\bar{r} = 3ti + 3t^2 j + 2t^3 k,$$

$$\bar{r} = e^t\cos t\, i + e^t \sin t\, j + e^t k$$

$$\bar{r} = a\cos t\, i + b\sin t\, j + tk$$

$$\bar{r} = t\cos t\, i + t\sin t\, j + atk \quad \text{etc.}$$

5. Point Functions

(a) Scalar Valued Point Functions

Consider any region R of space and suppose that to each point P of R there corresponds by some law a scalar quantity denoted by $\Phi(P)$. Then Φ is called a scalar point function defined for the region R.

Illustrations : Consider a material body occupying some region R. If $\Phi(P)$ denotes the density of the material at P, temperature at P or charge at P then Φ is a scalar point function over R.

(b) Vector Valued Point Functions

Consider any region R of space and suppose that to each point P of R there corresponds by some law a vector quantity $\bar{f}(P)$. Then $\bar{f}$ is called a vector point function defined for the region R.

Illustrations : Consider a fluid in motion. At any time t if $\bar{f}(P)$ denotes velocity at a point P which varies from point to point or acceleration at a point P which varies from point to point then $\bar{f}$ is a vector point function.

6. Vector Operator Del ∇

We define a new operation ∇ (read as 'del' or 'nabla') by

$$\boxed{\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}}$$

where i, j, k have usual meanings.

7. Gradient

If Φ is a scalar point function then the vector function $\nabla\Phi$ is called the gradient of Φ.

Thus,

$$\boxed{\text{grad } \Phi = \nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}}$$

Remark ....

We also denote grad Φ as,

$$\boxed{\text{grad } \Phi = \nabla\Phi = \sum i\frac{\partial \Phi}{\partial x} = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}}$$

8. Standard Results

The following results can be proved very easily by using the definition of ∇.

(i) $$\nabla(\Phi \pm \Psi) = \nabla\Phi \pm \nabla\Psi$$

(ii) $$\nabla(\Phi\Psi) = \Phi(\nabla\Psi) + (\nabla\Phi)\Psi$$

(iii) $$\nabla f(u) = i\frac{\partial}{\partial x}f(u) + j\frac{\partial}{\partial y}f(u) + k\frac{\partial}{\partial z}f(u) = f'(u)\nabla u$$

Example 1 : If $\Phi = xz^2 - 5yz + xz$, find $\nabla\Phi$ at $(1, -1, 2)$.

Sol. : By definition $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$$\therefore \nabla\Phi = i(z^2 + z) + j(-5z) + k(2xz - 5y + x)$$
$$= i(4 + 2) + j(-10) + k(4 + 5 + 1)$$
$$= 6i - 10j + 10k.$$

Example 2 : If $\Phi = \log(x^2 + y^2 + z^2)$, find $\nabla\Phi$.

Sol. : By definition $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$$\therefore \nabla\Phi = i\frac{1}{x^2 + y^2 + z^2}\cdot 2x + j\frac{1}{x^2 + y^2 + z^2}\cdot 2y + k\frac{1}{x^2 + y^2 + z^2}\cdot 2z$$
$$= \frac{2}{x^2 + y^2 + z^2}(xi + yj + zk)$$
$$= \frac{2\bar{r}}{r^2} \quad \text{where } \bar{r} = xi + yj + zk.$$

Example 3 : If $\Phi = (x^2+y^2+z^2)\cdot e^{-\sqrt{x^2+y^2+z^2}}$, find $\nabla\Phi$.

Sol. : $$\frac{\partial \Phi}{\partial x} = (x^2+y^2+z^2)\cdot e^{-\sqrt{x^2+y^2+z^2}} \times \frac{-1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x + e^{-\sqrt{x^2+y^2+z^2}}\cdot(2x)$$

where, $r = \sqrt{x^2+y^2+z^2}$.

$$\therefore \quad \frac{\partial \Phi}{\partial x} = -r\cdot e^{-r}x + e^{-r}\cdot 2x = e^{-r}(2x - xr)$$

Similarly, $\frac{\partial \Phi}{\partial y} = e^{-r}(2y - yr)$, $\quad \frac{\partial \Phi}{\partial z} = e^{-r}(2z - zr)$

Hence, $$\nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}$$
$$= e^{-r}(2-r)[xi + yj + zk] = e^{-r}(2-r)\bar{r}.$$

Example 4 : If $\Phi = x^2+y^2+z^2$, $\Psi = x^2y^2+y^2z^2+z^2x^2$, find $\nabla[\nabla\Phi\cdot\nabla\Psi]$.

Sol. : $$\nabla\Phi = 2xi + 2yj + 2zk$$
$$\nabla\Psi = (2xy^2+2xz^2)i + (2yx^2+2yz^2)j + (2zx^2+2zy^2)k$$
$$\therefore \quad \nabla\Phi\cdot\nabla\Psi = 4x^2(y^2+z^2) + 4y^2(x^2+z^2) + 4z^2(x^2+y^2)$$
$$= 8(x^2y^2+y^2z^2+z^2x^2)$$
$$\therefore \quad \nabla(\nabla\Phi\cdot\nabla\Psi) = 16x(y^2+z^2)i + 16y(z^2+x^2)j + 16z(x^2+y^2)k$$

Example 5 : If $u = x+y+z$, $v = x+y$, $w = -2xz-2yz-z^2$, show that $\nabla u\cdot[\nabla v\times\nabla w] = 0$.

Sol. : $$\nabla u = i+j+k, \ \nabla v = i+j$$
$$\nabla w = -2zi - 2zj - (2x+2y+2z)k$$
$$\nabla u\cdot[\nabla v\times\nabla w] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -2z & -2z & -2x-2y-2z \end{vmatrix} = 0 \qquad \text{[Scalar Triple Product]}$$

(Since first two columns are identical.)

Example 6 : Prove that $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$ and hence, find f if $\nabla f = 2r^4\,\bar{r}$. **(M.U. 1996, 2008)**

Sol. : We have, $\nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} + k\frac{\partial \Phi}{\partial z}$

Here, $\Phi = f(r)$ and f is a function of r and r is a function of (x, y, z).

$$\therefore \quad \nabla f(r) = i\frac{df}{dr}\frac{\partial r}{\partial x} + j\frac{df}{dr}\frac{\partial r}{\partial y} + k\frac{df}{dr}\frac{\partial r}{\partial z}$$

But $r^2 = x^2+y^2+z^2 \quad \therefore \quad 2r\frac{\partial r}{\partial x} = 2x$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla f(r) = \frac{f'(r)}{r}[xi + yj + zk] = \frac{f'(r)}{r}\bar{r}$$

Comparing this with the given expression. *i.e.*, comparing

$$\nabla f(r) = f'(r)\frac{\bar{r}}{r} \text{ with } \nabla f(r) = 2r^4\bar{r} = 2r^5\frac{\bar{r}}{r}$$ we see that $f'(r) = 2r^5$.

Here, by integration $f(r) = \dfrac{2r^6}{6} + C = \dfrac{r^6}{3} + C.$

Example 7 : Prove that $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\bar{r}}{r^3}$.

Sol. : Here, $f(r) = \dfrac{1}{r}$ $\therefore$ $f'(r) = -\dfrac{1}{r^2}$

But $\boxed{\nabla f(r) = f'(r)\cdot\dfrac{\bar{r}}{r}}$ [By Ex. 6 above]

(This can be used as a standard result e.g. $\nabla r^2 = 2r\cdot\dfrac{\bar{r}}{r}$, $\nabla \log r = \dfrac{1}{r}\cdot\dfrac{\bar{r}}{r}$)

$$\therefore\ \nabla\left(\frac{1}{r}\right) = f'(r)\frac{\bar{r}}{r} = -\frac{1}{r^2}\frac{\bar{r}}{r} = -\frac{\bar{r}}{r^3}.$$

Alternatively, we have $\nabla\Phi = \dfrac{\partial\Phi}{\partial x}i + \dfrac{\partial\Phi}{\partial y}j + \dfrac{\partial\Phi}{\partial z}k.$

Here, $\Phi = \dfrac{1}{r}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore\ \frac{d\Phi}{dr} = -\frac{1}{r^2} \quad \text{and} \quad \therefore\ \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore\ \nabla\Phi = \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial x}i + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial y}j + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial z}k$$

$$= -\frac{1}{r^2}\cdot\frac{x}{r}i - \frac{1}{r^2}\cdot\frac{y}{r}j - \frac{1}{r^2}\cdot\frac{z}{r}k$$

$$= -\frac{1}{r^3}(xi + yj + zk) = -\frac{\bar{r}}{r^3}.$$

Example 8 : Prove that $\nabla r^n = n\,r^{n-2}\,\bar{r}.$ **(M.U. 1994, 2006)**

Sol. : As proved in Ex. 6 above $\nabla f(r) = f'(r)\dfrac{\bar{r}}{r}$

Here, $f(r) = r^n$ $\therefore$ $f'(r) = n\,r^{n-1}$ $\therefore$ $\nabla f(r) = n\,r^{n-1}\dfrac{\bar{r}}{r} = n\,r^{n-2}\,\bar{r}$

Alternatively we have $\nabla\Phi = \dfrac{\partial\Phi}{\partial x}i + \dfrac{\partial\Phi}{\partial y}j + \dfrac{\partial\Phi}{\partial z}k$

Here, $\Phi = r^n$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore\ \frac{d\Phi}{dr} = n\,r^{n-1} \quad \text{and} \quad \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore\ \nabla\Phi = \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial x}i + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial y}j + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial z}k$$

$$= n\,r^{n-1}\left(\frac{x}{r}\right)i + n\,r^{n-1}\left(\frac{y}{r}\right)j + n\,r^{n-1}\left(\frac{z}{r}\right)k$$

$$= n\,r^{n-2}(xi + yj + zk)$$

$$\therefore \quad \boxed{\nabla\Phi = \nabla r^n = n\,r^{n-2}\,\bar{r}}$$

Example 9 : Find $\nabla(e^{r^2})$.

(M.U. 2004)

Sol. : Here $f(r) = e^{r^2} \quad \therefore \quad f'(r) = e^{r^2} \cdot 2r$

$$\therefore \quad \nabla f(r) = f'(r) \cdot \frac{\bar{r}}{r} = e^{r^2} 2r \cdot \frac{\bar{r}}{r} = 2\,e^{r^2}\,\bar{r}$$

Alternatively we have $\quad \nabla\Phi = \dfrac{\partial\Phi}{\partial x} i + \dfrac{\partial\Phi}{\partial y} j + \dfrac{\partial\Phi}{\partial z} k$

Here, $\Phi = e^{r^2}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{d\Phi}{dr} = e^{r^2}\,2r \quad \text{and} \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla\Phi = \frac{d\Phi}{dr} \cdot \frac{\partial r}{\partial x} i + \frac{d\Phi}{dr} \cdot \frac{\partial r}{\partial y} j + \frac{d\Phi}{dr} \cdot \frac{\partial r}{\partial z} k$$

$$= e^{r^2}\,2r\left(\frac{x}{r}\right) i + e^{r^2}\,2r\left(\frac{y}{r}\right) j + e^{r^2}\left(\frac{z}{r}\right) k$$

$$= 2e^{r^2} r \cdot \frac{(xi + yj + zk)}{r} = 2e^{r^2} \cdot \bar{r}.$$

Example 10 : Show that $\nabla\left[\dfrac{(\bar{a} \cdot \bar{r})}{r^n}\right] = \dfrac{\bar{a}}{r^n} - \dfrac{n(\bar{a} \cdot \bar{r})\,\bar{r}}{r^{n+2}}$. **(M.U. 1995, 98, 2005)**

Sol. : We have $\quad \dfrac{\bar{a} \cdot \bar{r}}{r^n} = \dfrac{(a_1 i + a_2 j + a_3 k) \cdot (xi + yj + zk)}{r^n} = \dfrac{a_1 x + a_2 y + a_3 z}{r^n}$

Let $\quad \Phi = \dfrac{\bar{a} \cdot \bar{r}}{r^n} = \dfrac{a_1 x + a_2 y + a_3 z}{r^n}$

$$\therefore \quad \frac{\partial\Phi}{\partial x} = \frac{r^n \cdot a_1 - (a_1 x + a_2 y + a_3 z)\,n\,r^{n-1}(\partial r / \partial x)}{r^{2n}}$$

But $r^2 = x^2 + y^2 + z^2 \quad \therefore \quad 2r\dfrac{\partial r}{\partial x} = 2x \quad \therefore \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$$\therefore \quad \frac{\partial\Phi}{\partial x} = \frac{a_1 r^n - (a_1 x + a_2 y + a_3 z) \cdot n\,r^{n-2} \cdot x}{r^{2n}} = \frac{a_1}{r^n} - \frac{n(a_1 x + a_2 y + a_3 z)\,x}{r^{n+2}}$$

Similarly, $\quad \dfrac{\partial\Phi}{\partial y} = \dfrac{a_2}{r^n} - \dfrac{n(a_1 x + a_2 y + a_3 z)\,y}{r^{n+2}}$

and $\quad \dfrac{\partial\Phi}{\partial z} = \dfrac{a_3}{r^n} - \dfrac{n(a_1 x + a_2 y + a_3 z)\,z}{r^{n+2}}$

$$\therefore \quad \nabla\Phi = \frac{\partial\Phi}{\partial x} i + \frac{\partial\Phi}{\partial y} j + \frac{\partial\Phi}{\partial z} k$$

$$= \frac{1}{r^n}(a_1 i + a_2 j + a_3 k) - \frac{n}{r^{n+2}}\left[(a_1 x + a_2 y + a_3 z)(xi + yj + zk)\right]$$

But $\bar{a}\cdot\bar{r} = (a_1 i + a_2 j + a_3 k)\cdot(xi + yj + zk) = a_1 x + a_2 y + a_3 z$

$$\therefore \nabla\Phi = \frac{\bar{a}}{r^n} - \frac{n}{r^{n+2}}(\bar{a}\cdot\bar{r})\bar{r}.$$

Example 11 : Prove that $\nabla\left[\frac{\bar{a}\cdot\bar{r}}{r^3}\right] = \frac{\bar{a}}{r^3} - \frac{3(\bar{a}\cdot\bar{r})\bar{r}}{r^5}$.

Sol. : Putting $n = 3$ in the above Ex. 10, we get the required result.

Example 12 : If $\bar{r} = xi + yj + zk$ and $\bar{a}, \bar{b}$ are constant vectors, prove that

$$\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5} - \frac{\bar{a}\cdot\bar{b}}{r^3}.$$ **(M.U. 1999, 2007)**

Sol. : As in the Ex. 7, we have

$$\nabla\frac{1}{r} = -\frac{1}{r^2}\cdot\frac{\bar{r}}{r} = -\frac{\bar{r}}{r^3} = -\frac{1}{r^3}(xi + yj + zk)$$

$$\therefore \bar{b}\cdot\nabla\left(\frac{1}{r}\right) = (b_1 i + b_2 j + b_3 k)\cdot\left(-\frac{1}{r^3}(xi + yj + zk)\right)$$

$$= -\frac{1}{r^3}(b_1 x + b_2 y + b_3 z) = \Phi \text{ say}$$

$$\therefore \nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} \qquad \text{.................... (1)}$$

Now, $$\frac{\partial\Phi}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{b_1 x + b_2 y + b_3 z}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= -\frac{(x^2 + y^2 + z^2)^{3/2}\cdot b_1 - (b_1 x + b_2 y + b_3 z)\cdot(3/2)(x^2 + y^2 + z^2)^{1/2}\, 2x}{(x^2 + y^2 + z^2)^3}$$

$$= -\frac{b_1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3(b_1 x + b_2 y + b_3 z)\, x}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= -\frac{b_1}{r^3} + \frac{3(b_1 x + b_2 y + b_3 z)\, x}{r^5}$$

Similarly, $$\frac{\partial\Phi}{\partial y} = -\frac{b_2}{r^3} + \frac{3(b_1 x + b_2 y + b_3 z)\, y}{r^5}; \qquad \frac{\partial\Phi}{\partial z} = -\frac{b_3}{r^3} + \frac{3(b_1 x + b_2 y + b_3 z)\, z}{r^5}$$

Hence, from (1)

$$\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = -\frac{1}{r^3}(b_1 i + b_2 j + b_3 k) + \frac{3}{r^5}(b_1 x + b_2 y + b_3 z)(xi + yj + zk)$$

(But $\bar{b}\cdot\bar{r} = (b_1 i + b_2 j + b_3 k)\cdot(xi + yj + zk) = b_1 x + b_2 y + b_3 z$)

$$\therefore \nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = -\frac{\bar{b}}{r^3} + \frac{3(\bar{b}\cdot\bar{r})\bar{r}}{r^5}$$

$$\therefore \bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5} - \frac{\bar{a}\cdot\bar{b}}{r^3}.$$

Example 13 : Find $\Phi(r)$ such that $\nabla\Phi = -\dfrac{\bar{r}}{r^5}$ and $\Phi(2) = 3$. (M.U. 1988, ...)

Sol. : We have

$$\nabla\Phi = -\bar{r}(x^2+y^2+z^2)^{-5/2} \qquad [\because r = \sqrt{x^2+y^2+z^2}\,]$$

$$= -(x^2+y^2+z^2)^{-5/2}(xi+yj+zk) \qquad \ldots\ldots(1)$$

But $$\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} \qquad \ldots\ldots(2)$$

Comparing (1) and (2), we get,

$$\frac{\partial\Phi}{\partial x} = -x(x^2+y^2+z^2)^{-5/2}, \qquad \frac{\partial\Phi}{\partial y} = -y(x^2+y^2+z^2)^{-5/2},$$

$$\frac{\partial\Phi}{\partial z} = -z(x^2+y^2+z^2)^{-5/2}.$$

But $$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = -(x^2+y^2+z^2)^{-5/2}(xdx+ydy+zdz)$$

Now let $x^2+y^2+z^2 = t$

$$\therefore\ 2(x\,dx+y\,dy+z\,dz) = dt \qquad \therefore\ d\Phi = -t^{-5/2}\cdot\frac{dt}{2}$$

Integrating, $$\Phi = -\frac{1}{2}\frac{t^{-3/2}}{-3/2} + C = \frac{t^{-3/2}}{3} + C$$

Now resubstituting $t = x^2+y^2+z^2$,

$$\therefore\ \Phi = \frac{1}{3}(x^2+y^2+z^2)^{-3/2} + C = \frac{1}{3}\cdot\frac{1}{r^3} + C$$

But by data $\Phi(r) = 3$ when $r = 2$

$$\therefore\ 3 = \frac{1}{3}\cdot\frac{1}{8} + C \quad \therefore\ C = \frac{71}{24} \qquad \therefore\ \Phi = \frac{1}{3}\cdot\frac{1}{r^3} + \frac{71}{24} = \frac{1}{3}\left(\frac{1}{r^3} + \frac{71}{8}\right).$$

EXERCISE - I

1. If $\Phi = 2xz^2 - 3xy - 4x$, find $\nabla\Phi$ at $(1, -1, 2)$. [**Ans. :** $7i - 3j + 8k$]
2. If $\Phi = 5x^2y - 3y^2z^2$, find $\nabla\Phi$ at $(1, -2, 1)$. [**Ans. :** $-20i + 17j - 24k$]
3. If $\Phi = 2xz^4 - x^2y$, find $\nabla\Phi$ and $|\nabla\Phi|$ at $(2, -2, 1)$. [**Ans. :** $10i - 4j + 16k$, $2\sqrt{93}$]
4. If $\Phi = x^2+y^2+z^2$ and $\Psi = x^2y^2 + y^2z^2 + z^2x^2$, find $\nabla[\nabla\Phi\cdot\nabla\Psi]$.
   [**Ans. :** $16x(y^2+z^2) + 16y(z^2+x^2) + 16z(x^2+y^2)$]
5. If $u = x+y+z$, $v = x+y$, $w = -2xz - 2yz - z^2$, show that $\nabla u[\nabla v \times \nabla w] = 0$.
6. If $\bar{F} = 2x^2i - 3yzj + xz^2k$ and $\Phi = 2z - x^3y$, find $\bar{F}\cdot\nabla\Phi$, at $(1, -1, 1)$. [**Ans. :** 5] (M.U. 1991)
7. Find $\nabla\Phi$ if (i) $\Phi = \sqrt{x^2+y^2+z^2}$ (ii) $\Phi = \log(x^2+y^2+z^2)$
   (iii) $\Phi = e^{r^2}$ where $r^2 = x^2+y^2+z^2$. (iv) $\Phi = 3x^2y - y^3z^2$ at $(1, -2, 1)$. (M.U. 1992)

[**Ans. :** (i) $(x^2+y^2+z^2)^{-1/2}(xi+yj+zk)$ (ii) $2(x^2+y^2+z^2)^{-1}(xi+yj+zk)$
(iii) $2e^{r^2}(xi+yj+zk)$ (iv) $-12i - 9j + 16k$.]

8. (a) Find $\Phi(r)$ such that $\nabla\Phi = -\frac{\bar{r}}{r^5}$ and $\Phi(1) = 0$.

[Ans. : $\Phi(r) = \frac{1}{3}\left(\frac{1}{r^3} - 1\right)$]

(M.U. 1994, 96)

(b) If $\nabla u = 2r^4\bar{r}$, find u. (M.U. 1996, 2008) [Ans. : $u = \frac{(x^2+y^2+z^2)^3}{3} + c$]

9. If $\bar{a}$ is a constant vector and r and $\bar{r}$ have usual meanings, prove that

(i) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$ (M.U. 2000) (ii) $\nabla\left(\bar{a}\cdot\nabla\frac{1}{r}\right) = -\frac{\bar{a}}{r^3} + \frac{3(\bar{a}\cdot\bar{r})\bar{r}}{r^5}$ (M.U. 2000, 01)

10. If $\Phi_1 = x + y + z$, $\Phi_2 = x + y + z^2$, $\Phi_3 = 2xz + 2yz + z^3$, prove that

$$\nabla\Phi_1 \cdot [\nabla\Phi_2 \times \nabla\Phi_3] = 0.$$

11. If $\Phi = 3x^2y$, $\Psi = xz^2 - 2y$, show that $\nabla(\nabla\Phi\cdot\nabla\Psi) = (6yz^2 - 12x)\,i + 6xz^2\,j + 12xyzk$.

12. If $\Phi(x, y, z) = x^2yz$, $\bar{u} = 3x^2yi + yz^2j - xy^2k$, find $\frac{\partial^2}{\partial y\,\partial z}(\Phi\bar{u})$ at $(1, -2, 1)$.

[Ans. : $-12(i + j + k)$]

9. Geometrical Meaning of Grad Φ

Consider a scalar point function and let

$$\bar{r} = xi + yi + zk$$

be the position vector of a point P on the surface $\Phi(x, y, z) = c$.

Such a surface for which the value of the function is constant is called a **level surface.**

Now, $\overline{dr} = dx\,i + dy\,j + dz\,k$ and it lies in the plane tangential to the surface $\Phi(x, y, z) = c$.

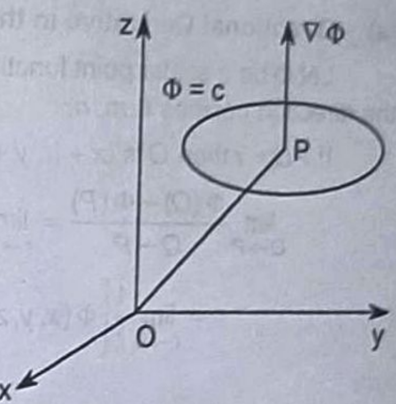

Fig. 3.2

Also $d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz.$

Since $\Phi(x, y, z) = c$, $d\Phi = 0$

$$\therefore \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = 0$$

Hence, $\nabla\Phi\cdot\overline{dr} = \left(i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}\right)\cdot(dx\,i + dy\,j + dz\,k)$

$$= \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = 0$$

$\nabla\Phi$ is a vector perpendicular to $\overline{dr}$. But since $\overline{dr}$ lies in the tangent plane, $\nabla\Phi$ *is a vector* **perpendicular to the tangent plane to the surface.** $\Phi(x, y, z) = c$. (See Ex. of (A) on page 3-12)

10. Directional Derivative

Let Φ be a scalar point function and let $\Phi(P)$ and $\Phi(Q)$ be the values of Φ at two neighbouring points P and Q in the field. Then,

$$\lim_{Q\to P}\frac{\Phi(Q) - \Phi(P)}{Q - P}$$

If it exists is called the **directional derivative** of Φ in the direction of PQ.

If $\bar{f}$ is a vector point function then $\lim\limits_{Q \to P} \dfrac{\bar{f}(Q) - \bar{f}(P)}{Q - P}$

If it exists is called the **directional derivative of** $\bar{f}$ in the direction of PQ.

Fig. 3.3

If we take the direction of PQ along the coordinate axes then we get the directional derivatives along the coordinate axes.

Taking PQ along the x-axis

$$\lim_{Q \to P} \frac{\Phi(Q) - \Phi(P)}{Q - P} = \lim_{\delta x \to 0} \frac{\Phi(x + \delta x, y, z) - \Phi(x, y, z)}{\partial x} = \frac{\partial \Phi}{\partial x}$$

$\therefore$ $\dfrac{\partial \Phi}{\partial x}, \dfrac{\partial \Phi}{\partial y}, \dfrac{\partial \Phi}{\partial z}$ **are the directional derivatives of** Φ **in the direction of the coordinate axes at** P.

Similarly, it can be seen that

$\dfrac{\partial \bar{f}}{\partial x}, \dfrac{\partial \bar{f}}{\partial y}, \dfrac{\partial \bar{f}}{\partial z}$ **are the directional derivatives of** $\bar{f}$ **in the direction of the coordinate axes.**

(a) Directional Derivative in the given direction

Let Φ be a scalar point function in a scalar field. Let P be (x, y, z). Let a line segment PQ have the direction cosines l, m, n.

If $PQ = r$ then Q is $(x + lr, y + mr, z + nr)$.

$$\therefore \lim_{Q \to P} \frac{\Phi(Q) - \Phi(P)}{Q - P} = \lim_{r \to 0} \frac{\Phi(x + lr,\ y + mr,\ z + nr) - \Phi(x, y, z)}{r}$$

$$= \lim_{r \to 0} \frac{1}{r}\left[\Phi(x, y, z) + lr\frac{\partial \Phi}{\partial x}(x, y, z) + mr\frac{\partial \Phi}{\partial y}(x, y, z) + nr\frac{\partial \Phi}{\partial z}(x, y, z) - \Phi(x, y, z)\right]$$

[By Taylor's Theorem]

$$= l\frac{\partial \Phi}{\partial x} + m\frac{\partial \Phi}{\partial y} + n\frac{\partial \Phi}{\partial z}$$

This is the directional derivative of a scalar function Φ in the direction of a line whose directions are l, m, n.

The directional derivative of Φ **in the direction** $l, m, n,$ $\Big\} = l\dfrac{\partial \Phi}{\partial x} + m\dfrac{\partial \Phi}{\partial y} + n\dfrac{\partial \Phi}{\partial z}$ [See Ex. 3, page 3-12]

If $\bar{f}$ is a vector point function then the directional derivative of $\bar{f}$ in the direction of the line whose direction cosines are l, m, n is

$$\boxed{l\frac{\partial \bar{f}}{\partial x} + m\frac{\partial \bar{f}}{\partial y} + n\frac{\partial \bar{f}}{\partial z}}$$

(b) Directional Derivative in the direction of a vector $\bar{a}$

Since $\nabla\Phi$ is a vector quantity its **component** (or resolved part) in the direction of a vector is $\frac{\nabla\Phi \cdot \bar{a}}{|\bar{a}|}$. This component is called the **directional derivative of Φ in the direction of $\bar{a}$**.

$$\boxed{\text{Thus, the directional derivative of } \Phi \text{ in the direction of } \bar{a} = \frac{\nabla\Phi \cdot \bar{a}}{|\bar{a}|}}$$

[See Ex. 4, page 3-13]

(c) Maximum Directional Derivative

Since the resolved part of a vector is maximum in its own direction, the directional derivative is maximum in the direction $\nabla\Phi$. Since $\nabla\Phi$ is normal to the surface, we can also say that $\nabla\Phi$ is maximum in the direction of the normal to the surface and the maximum directional derivative is $|\nabla\Phi|$.

Remark ....

Though Φ is a scalar point function, grad Φ is a vector point function whose components are $\partial\Phi/\partial x$, $\partial\Phi/\partial y$ and $\partial\Phi/\partial z$.

[See Ex. 7, page 3-13]

(d) Angle between Two Surfaces

We know that $\nabla\Phi$ is perpendicular to the tangent plane to the surface $\Phi(x, y, z) = c$. Hence, if $\Phi(x, y, z) = c_1$ and $\Psi(x, y, z) = c_2$ are two surfaces the angle between the two surfaces is equal to the angle between the normals *i.e.*, the angle between $\nabla\Phi$ and $\nabla\Psi$. [See Ex. 1, 2, page 3-14]

(e) Gradient of a constant

If Φ is a constant then $\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial z} = 0 \quad \therefore \text{ grad } \Phi = \bar{0}$.

(f) Differential $d\Phi$ where Φ is a scalar point function

We have $$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz \qquad \text{.......... (1)}$$

Also $$d\bar{r} = i\,dx + j\,dy + k\,dz \quad \text{and} \quad \nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$$

$$\therefore \nabla\Phi \cdot d\bar{r} = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz \qquad \text{.......... (2)}$$

Hence, from (1) and (2), we get $\boxed{d\Phi = \nabla\Phi \cdot d\bar{r}}$

(g) Differential $d\bar{f}$ where $\bar{f}$ is a vector point function

We have, $$d\bar{f} = \frac{\partial\bar{f}}{\partial x}dx + \frac{\partial\bar{f}}{\partial y}dy + \frac{\partial\bar{f}}{\partial z}dz = dx\frac{\partial\bar{f}}{\partial x} + dy\frac{\partial\bar{f}}{\partial y} + dz\frac{\partial\bar{f}}{\partial z} \qquad \text{.......... (1)}$$

Also $$d\bar{r} = i\,dx + j\,dy + k\,dz \quad \text{and} \quad \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\therefore d\bar{r} \cdot \nabla = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} \qquad \text{.......... (2)}$$

From (1) and (2) we get $\boxed{d\bar{f} = (d\bar{r} \cdot \nabla)\bar{f}}$

(A) To Find Unit Normal

Example : Find the unit vector normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$. **(M.U. 2008)**

Sol. : We know that $\nabla\Phi$ is the vector normal to the surface $\Phi(x, y, z) = c$ at P.

$$\text{Now, } \nabla\Phi = i\frac{\partial}{\partial x}(xy^3z^2) + j\frac{\partial}{\partial y}(xy^3z^2) + k\frac{\partial}{\partial z}(xy^3z^2)$$

$$= y^3z^2 i + 3xy^2z^2 j + 2xy^3 zk$$

$$= -4i - 12j + 4k \text{ at } (-1, -1, 2)$$

unit vector normal to the surface at $(-1, -1, 2)$.

$$= \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{-4i - 12j + 4k}{\sqrt{16 + 144 + 16}} = -\frac{1}{\sqrt{11}}(i + 3j - k)$$

(B) To Find The Directional Derivative

Example 1 : Find the directional derivative of $\Phi = x^4 + y^4 + z^4$ at point $A(1, -2, 1)$ in the direction of AB where B is $(2, 6, -1)$. **(M.U. 1993, 2005, 09, 14)**

Sol. :

$$\nabla\Phi = i\frac{\partial}{\partial x}(x^4 + y^4 + z^4) + j\frac{\partial}{\partial y}(x^4 + y^4 + z^4) + k\frac{\partial}{\partial z}(x^4 + y^4 + z^4)$$

$$\therefore \quad \nabla\Phi = 4(x^3 i + y^3 j + z^3 k) = 4(i - 8j + k) \text{ at } (1, -2, 1)$$

$$\overline{AB} = \overline{0B} - \overline{0A} = (2-1)i + (6+2)j + (-1-1)k = i + 8j - 2k$$

$\therefore$ Directional derivative at A in the direction of $\overline{AB}$

$$= 4(i - 8j + k)\cdot\frac{(i + 8j - 2k)}{\sqrt{1 + 64 + 4}} = \frac{4(1 - 64 - 2)}{\sqrt{69}} = -\frac{260}{\sqrt{69}}$$

Example 2 : Find the directional derivative of $\Phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at $(1, 2, 3)$. **(M.U. 1996, 2007)**

Sol. :

$$\nabla\Phi = i\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + j\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + k\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= 2(xi + yi + zk) = 2(i + 2j + 3k) \text{ at } (1, 2, 3)$$

Given direction $= 3i + 4j + 5k$.

Directional derivative in the given direction

$$= \nabla\Phi\cdot\frac{\overline{a}}{|\overline{a}|} = 2(i + 2j + 3k)\cdot\frac{(3i + 4j + 5k)}{\sqrt{9 + 16 + 25}} = \frac{2(3 + 8 + 15)}{5\sqrt{2}} = \frac{26}{5}\sqrt{2}.$$

Example 3 : Find the directional derivative of $\Phi(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $i + 2j + 2k$. **(M.U. 2005)**

Sol. :

$$\nabla\Phi = i\frac{\partial}{\partial x}(xy^2 + yz^3) + j\frac{\partial}{\partial y}(xy^2 + yz^3) + k\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= y^2 i + (2xy + z^3)j + 3yz^2 k$$

$$= i - 3j - 3k \text{ at } (2, -1, 1).$$

Directional derivative in the direction of $(i + 2j + 2k)$

$$= (i - 3j - 3k) \cdot \frac{(i + 2j + 2k)}{\sqrt{1+4+4}} = \frac{1}{3}(1 - 6 - 6) = -\frac{11}{3}.$$

Example 4 : Find the directional derivative of $\Phi = x^2 y \cos z$ at $(1, 2, \pi/2)$ in the direction of $\bar{a} = 2i + 3j + 2k$. **(M.U. 2013)**

Sol. :

$$\nabla\Phi = i\frac{\partial}{\partial x}(x^2 y \cos z) + j\frac{\partial}{\partial y}(x^2 y \cos z) + k\frac{\partial}{\partial z}(x^2 y \cos z)$$

$$= 2xy \cos z\, i + x^2 \cos z\, j - x^2 y \sin z\, k$$

At $(1, 2, \pi/2)$, $\nabla\Phi = 0\,i + 0\,j - 2\,k.$

Directional derivative in the direction of $2i + 3j + 2k$

$$= (0i + 0j - 2k) \cdot \frac{2i + 3j + 2k}{\sqrt{4+9+4}} = -\frac{4}{\sqrt{17}}.$$

Example 5 : Find the directional derivative of $\Phi = \dfrac{y}{x^2 + y^2}$ at $(0, 1)$ in the direction making are angle of $30°$ with positive x-axis.

Sol. : Directional derivative

$$\nabla\Phi = i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial y} = \left[-\frac{y}{(x^2+y^2)^2} \cdot 2x\right] i + \left[\frac{(x^2+y^2) \cdot 1 - y \cdot 2y}{(x^2+y^2)^2}\right] j$$

$$= \frac{-2xy}{(x^2+y^2)^2}\, i + \frac{(x^2 - y^2)}{(x^2+y^2)^2}\, j = 0i - j \text{ at } (0, 1)$$

Unit vector making an angle of $30°$ with the x-axis

$$= \cos 30^0\, i + \sin 30^0\, j = \frac{\sqrt{3}}{2} i + \frac{1}{2} j$$

$\therefore$ Required directional derivative $= (0i - j) \cdot \left(\dfrac{\sqrt{3}}{2} i + \dfrac{1}{2} j\right) = -\dfrac{1}{2}.$

Example 6 : In what direction from the point $(2, 1, -1)$ is the directional derivative of $\Phi = x^2 y z^3$ maximum ? What is its magnitude ?

Sol. : $$\nabla\Phi = \nabla x^2 y z^3 = i\frac{\partial}{\partial x}(x^2 y z^3) + j\frac{\partial}{\partial y}(x^2 y z^3) + k\frac{\partial}{\partial z}(x^2 y z^3)$$

$$= 2xyz^3 i + x^2 z^3 j + 3x^2 y z^2 k$$

$$= -4i - 4j + 12k \text{ at } (2, 1, -1)$$

Directional derivative is maximum in the direction of $\nabla\Phi$. Hence, directional derivative is maximum in the direction of $-4i - 4j + 12k$.

Its magnitude $= \sqrt{16 + 16 + 144} = 4\sqrt{11}.$

Example 7 : Find the maximum directional derivative of $\Phi = (4x - y + 2z)^2$ at $(1, 2, 1)$. **(M.U. 2001)**

Sol. : Directional derivative of Φ is given by

$$\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$$

$$= 2(4x - y + 2z)\,4i + 2(4x - y + 2z)(-1)\,j + 2(4x - y + 2z)\cdot 2k$$

$$= 2(4x - y + 2z)(4i - j + 2k)$$

At $(1, 2, 1)$, $\nabla\Phi = 8(4i - j + 2k)$

Maximum directional derivative $= |\nabla\Phi| = 8\sqrt{16 + 1 + 4} = 8\sqrt{21}.$

(C) To Find The Angle Between The Normals

Example 1 : Find the angle between the normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ and $(-3, -3, 3)$. **(M.U. 1998, 2005)**

Sol. : Let $\Phi = xy - z^2$

$$\therefore \quad \nabla\Phi = i\frac{\partial}{\partial x}(xy - z^2) + j\frac{\partial}{\partial y}(xy - z^2) + k\frac{\partial}{\partial z}(xy - z^2)$$

$$= yi + xj - 2zk$$

$$\nabla\Phi = 4i + j - 4k \text{ at } (1, 4, 2)$$

and $\nabla\Phi = -3i - 3j - 6k$ at $(-3, -3, 3)$

But these are the normals to the surface at the given points.

Angle between two vectors $\bar{a}, \bar{b}$ is given by $\bar{a}\cdot\bar{b} = |a|\,|b|\cos\theta.$

If θ is the angle between them,

$$(4i + j - 4k)\cdot(-3i - 3j - 6k) = |4i + j - 4k|\cdot|-3i - 3j - 6k|\cos\theta$$

$$\therefore \quad -12 - 3 + 24 = 9 = \sqrt{33}\,\sqrt{54}\cos\theta$$

$$\therefore \quad \cos\theta = \frac{9}{\sqrt{3}\,\sqrt{11}\,\sqrt{27}\,\sqrt{2}} = \frac{9}{9\sqrt{22}} = \frac{1}{\sqrt{22}}.$$

Example 2 : Find the angle between the surfaces $x\log z + 1 - y^2 = 0$, $x^2y + z = 2$ at $(1, 1, 1)$. **(M.U. 2003, 05)**

Sol. : Let $\Phi = x\log z + 1 - y^2.$

$$\therefore \quad \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} = \log z\,i - 2yj + \frac{x}{z}k$$

$$= 0i - 2j + k \text{ at } (1, 1, 1).$$

Unit normal vector at $(1, 1, 1) = \dfrac{0i - 2j + k}{\sqrt{5}}$

Let $\Psi = x^2y + z - 2$

$$\nabla\Psi = i\frac{\partial\Psi}{\partial x} + j\frac{\partial\Psi}{\partial y} + k\frac{\partial\Psi}{\partial z} = 2xy\,i + x^2j + k = 2i + j + k$$

Unit normal vector at $(1, 1, 1) = \dfrac{2i + j + k}{\sqrt{6}}$

$$\cos\theta = \frac{(0 - 2j + k)}{\sqrt{5}}\cdot\frac{(2i + j + k)}{\sqrt{6}} = -\frac{1}{30}.$$

(D) **To Find The Constants**

Example 1 : Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1, -1, 2)$. **(M.U. 2000, 06)**

Sol. : Let $u = ax^2 - byz - (a+2)x$ and $v = 4x^2y + z^3 - 4$.

$$\therefore \quad \nabla u = (2ax - a - 2)\,i + (-bz)\,j + (-by)\,k$$

$$= (a-2)\,i - 2bj + bk \text{ at } (1, -1, 2)$$

The direction ratios of the normal to this surface at $(1, -1, 2)$ are $a-2, -2b, b$.

And $\nabla v = 8xyi + 4x^2 j + 3z^2 k = -8i + 4j + 12k$ at $(1, -1, 2)$

The direction ratios of the normal to this surface at $(1, -1, 2)$ are $-8, 4, 12$ *i.e.* $-2, 1, 3$.

Since, the surfaces are orthogonal, normals are perpendicular to each other.

$$\therefore (-2)(a-2) + (1)(-2b) + (3)(b) = 0 \quad i.e. \quad -2a + b = -4 \qquad \text{.................... (1)}$$

Since $(1, -1, 2)$ lies on the surface

$ax^2 - byz - (a+2)x = 0$, we have $a + 2b - a - 2 = 0$.

i.e. $b = 1$. .... (2)

Then from (1) we get $a = 5/2$. Hence, $a = 5/2$ and $b = 1$.

Example 2 : Find the values of a, b, c if the directional derivative of $\Phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has maximum magnitude 64 in the direction parallel to the z axis. **(M.U. 2003, 04)**

Sol. : We have $\Phi = axy^2 + byz + cz^2x^3$

$$\therefore \quad \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} = (ay^2 + 3cx^2z^2)\,i + (2axy + bz)\,j + (by + 2czx^3)\,k$$

$$= (4a + 3c)\,i + (4a - b)\,j + (2b - 2c)\,k \text{ at } (1, 2, -1) \qquad \text{.................... (1)}$$

The directional derivative is maximum in the direction of $\nabla\Phi$ *i.e.* in the direction of $(4a + 3c)\,i + (4a - b)\,j + (2b - 2c)\,k$.

But by data the directional derivative is maximum in the direction of the z-axis *i.e.* in the direction of $0i + 0j + k$.

$$\therefore \frac{4a + 3c}{0} = \frac{4a - b}{0} = \frac{2b - 2c}{1} \qquad \therefore \quad 4a + 3c = 0 \text{ and } 4a - b = 0$$

Hence, from (1), $\nabla\Phi = (2b - 2c)\,k \qquad \therefore \quad |\nabla\Phi| = |2b - 2c|$.

But directional derivative is maximum in the direction of $\nabla\Phi$ and is given to be 64,

$\therefore \; 2b - 2c = 64 \qquad \therefore \; b - c = 32$.

Solving $4a + 3c = 0$, $4a - b = 0$ and $b - c = 32$, we get $a = 6$, $b = 24$, $c = -8$.

EXERCISE - II

(A) **To Find The Unit Normal**

1. Find the unit normal to the surface $2x^2 + 4yz - 5z^2 = -10$ at $(3, -1, 2)$.

[**Ans. :** $\dfrac{3i + 2j + 6k}{7}$]

2. Find the unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $(a/\sqrt{3}, a/\sqrt{3}, a/\sqrt{3})$.

[**Ans. :** $(i + j + k)/\pm\sqrt{3}$]

3. Find unit vector normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2).
[**Ans. :** $(8i + j + 8k)/\sqrt{129}$]

(B) To Find The Directional Derivative

1. Find the directional derivative of $\Phi = xy + yz + zx$ at (1, 2, 3) in the direction of $3i + 4j + 5k$.
[**Ans. :** $46/5\sqrt{2}$]

2. Find the directional derivative of $\Phi = x^2y + y^2z + z^2x^2$ at P(1, 2, 1) in the direction of the normal to the surface $x^2 + y^2 - z^2x = 1$ at Q(1, 1, 1). **(M.U. 2000)** [**Ans. :** 4 / 3]

3. Find the directional derivative of $4xz^2 + x^2yz$ at (1, –2, –1) in the direction of $2i - j - 2k$.
(M.U. 1999) [**Ans. :** 37 / 3]

4. Find the directional derivative of $\Phi = x/(x^2 + y^2)$ at (0, 1) in the direction making an angle of 30° with positive x-axis. [**Ans. :** $\sqrt{3}/2$]

5. Find the directional derivative of $\Phi = 2x^3y - 3y^2z$ at P(1, 2, –1) in the direction towards Q (3, –1, 5). In what direction from P is the directional derivative maximum ? Find the magnitude of maximum directional derivative. **(M.U. 1993)** [**Ans. :** – 90 / 7 ; $12i + 14j - 12k$; 22]

6. Find the directional derivative of the function $\Phi = x^2 - y^2 + 2z^2$ at the point P(1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4). In what direction will it be maximum ? Find also the magnitude of the maximum. **(M.U. 1998)** [**Ans. :** $4\sqrt{7}/\sqrt{3}$; $\nabla\Phi = 2i - 4j + k$; $|\nabla\Phi| = 2\sqrt{41}$]

7. Find the maximum directional derivative of $xy(x - y + 2z)$ at (1, 1, 0). **(M.U. 2000)**
[**Ans. :** $\sqrt{6}$]

8. Find the maximum directional derivative of $\Phi = x(x - y) + y(y + z)$ at P(1, 2, 1).
(M.U. 1997) [**Ans. :** $2\sqrt{5}$]

9. Find the directional derivative of $\Phi = 4xz^3 - 3x^2y^2z$ at (2, –1, 2) in the direction from this point towards the point (4, – 4, 8). [**Ans. :** 376 / 7]

10. Find the directional derivative of $\Phi = xy^2 + yz^2$ at (2, –1, 1) in the direction of the vector $i + 2j + 2k$.
[**Ans. :** – 3]

11. In what direction is the directional derivative of $\Phi = x^2y^2z^4$ at (3, –1, – 2) maximum ? Find its magnitude.
[**Ans. :** 418·45]

12. In what direction is the directional derivative of $\Phi = 2xz - y^2$ at (1, 3, 2) maximum ? Find its magnitude.
[**Ans. :** $2\sqrt{14}$]

13. Find the directional derivative of $\Phi = x^2y + y^2z + z^2x$ at (2, 2, 2) in the direction of the normal to the surface $4x^2y + 2z^2 = 2$ at the point (2, –1, 3). [**Ans. :** $36/\sqrt{41}$]

14. Find the directional derivative of $\Phi = xy^2 + yz^3$ at (2, –1, 1) in the direction of the normal to $x\log z - y^2 + 4 = 0$ at (–1, 2, 1). **(M.U. 1999, 2003, 13)** [**Ans. :** $-15/\sqrt{17}$]

15. Find the directional derivative of $\Phi = xy^2 + yz^2$ at (2, –1, 1) in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at (1, 1, 1). [**Ans. :** $-13/\sqrt{29}$]

16. Find the rate of change of $\Phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1). **(M.U. 1998, 2005)** [**Ans. :** $9/\sqrt{29}$]

17. Find the rate of change of $\Phi = xy + yz + zx$ in the direction of the normal to the surface $x^2 + y^2 = z + 4$ at (1, 1, – 2).
[**Ans. :** – 2]

(C) **To Find the Angle Between The Normals**

1. Find the angle between the normals to the surfaces $x^2y + 2xz = 4$ at $(2, -2, 3)$ and to $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$. [**Ans. :** $\cos\theta = 11/3\sqrt{14}$]

2. Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 9$, $z = x^2 + y^2 - 3$ at $(2, -1, 2)$. **(M.U. 2003, 14)** [**Ans. :** $\cos\theta = 8/3\sqrt{21}$]

3. Find the angle between the normals to the surface $xy = z^2$ at $P(1, 1, 1)$ and $Q(4, 1, 2)$. **(M.U. 2003)** [**Ans. :** $\cos\theta = 13/\sqrt{198}$]

4. Find the angle between the normals to the surfaces $x^2y + z = 3$ and $x \log z - y^2 + 4 = 0$ at $(-1, 2, 1)$. [**Ans. :** $\cos\theta = -5/3\sqrt{34}$]

5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 12$ and $x^2 + y^2 - z = 6$ at $(2, -2, 2)$. [**Ans. :** $\cos\theta = 7/3\sqrt{11}$]

6. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 7$ at the point $(2, 1, -2)$. **(M.U. 1995)** [**Ans. :** $\cos\theta = 4/\sqrt{21}$]

•7. Find the angle between the two surfaces $x^2 + y^2 + az^2 = 6$ and $z = 4 - y^2 + bxy$ at $P(1, 1, 2)$. [**Ans. :** $(1, 1, 2)$ lies on both surfaces $\therefore a = 1, b = -1, \cos\theta = \sqrt{6/11}$]

•8. Find the angle between the surfaces $ax^2 + y^2 + z^2 - xy = 1$ and $bx^2y + y^2z + z = 1$ at $(1, 1, 0)$. **(M.U. 2000, 06)** [**Ans. :** $(1, 1, 0)$ lies on both the surfaces $\therefore a = 1, b = 1, \cos\theta = 1/\sqrt{2}, \theta = 45^\circ$]

(D) **To Find The Constants**

1. Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ cut orthogonally at $(1, -1, 2)$. **(M.U. 2004)** [**Ans. :** $a = 4, b = 1$]

2. Find the constants a, b such that the surface $ax^2 - bxy + xz = 10$ is orthogonal to the surface $x^2 + y^2 = 4 + xz$ at $(1, 2, 1)$. **(M.U. 2002)** [**Ans. :** $a = 27$, $b = 9$]

3. Find the constants a, b, c if the normal to the surface $ax^2 + bxz + z^2y = c$ at $P(-1, 1, 2)$ is parallel to the normal to the surface $x^2 - y^2 + 2z = 2$ at $Q(1, 1, 1)$. **(M.U. 2001)** [**Ans. :** $a = 10$, $b = 8$, $c = -2$]

4. Find the constants a and b such that the surface $ax^2 - 2byz = (a + 4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1, -1, 2)$. **(M.U. 1996)** [**Ans. :** $a = 5$, $b = 1$]

5. Find the constants a, b, c if the normal to the surface $ax^2 + yz + bxz^3 = c$ at $P(1, 2, 1)$ is parallel to the normal to the surface $y^2 + xz = 61$ at $(10, 1, 6)$. **(M.U. 1997, 2010)** [**Ans. :** $a = 1$, $b = 1$, $c = 4$]

6. Find the constants a, b if the angle between the surfaces $x^2 + axz + byz = 2$ and $x^2z + xy + y + 1 = z$ at $(0, 1, 2)$ is $\cos^{-1}(1/\sqrt{3})$. **(M.U. 2000)** [**Ans. :** $a = 1$, $b = 1$]

7. If the directional derivative of $\Phi = ax^2 + by + 2z$ at $(1, 1, 1)$ is maximum in the direction of $i + j + k$, find a and b. **(M.U. 2001)** [**Ans. :** 1, 2]

11. Divergence and Curl

We have seen in the previous article that if we operate upon a scalar point function by the operator ∇ we get a vector point function called grad denoted by ∇Φ. In this article we shall see that by operating upon a vector point function $\bar{f}$ by the operator ∇ *scalarly* (*i.e.* by taking dot product) we get *divergence* $\bar{f}$ denoted by *div* $\bar{f} = (\nabla \cdot \bar{f})$ and by operating on $\bar{f}$ *vectorially* (*i.e.* by taking cross product) we get *curl* $\bar{f}$ denoted by *curl* $\bar{f} = (\nabla \times \bar{f})$.

From the results (i) and (ii) of § 8, page 3-3, it is clear that ∇ is a differential operator just as d/dx is in differential calculus. We can apply ordinary rules of differential calculus to the differential operator ∇ keeping in mind its vector character.

(a) Definition : Let $\bar{f} = f_1 i + f_2 j + f_3 k$ then

$$\nabla \cdot \bar{f} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (f_1 i + f_2 j + f_3 k) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

is called the **divergence** of $\bar{f}$. Thus,

$$\boxed{\textbf{div } \bar{f} = \nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}} \quad \text{.................... (1)}$$

Note that divergence $\bar{f}$ is a scalar point function. (M.U. 2002)

Remark ....

If $\bar{f}$ is a vector point function such that $\nabla \cdot \bar{f} = 0$ then $\bar{f}$ is called **solenoidal.**

Note ....

Writing $\nabla \cdot \bar{f}$ as $\bar{f} \cdot \nabla$ is wrong. $\nabla \cdot \bar{f}$ is a scalar while $\bar{f} \cdot \nabla$ is an operator

$$\left(f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z}\right).$$

We can also write $\nabla \cdot \bar{f}$ as,

$$\boxed{\textbf{div } \bar{f} = \nabla \cdot \bar{f} = i \cdot \frac{\partial \bar{f}}{\partial x} + j \cdot \frac{\partial \bar{f}}{\partial y} + k \cdot \frac{\partial \bar{f}}{\partial z}} \quad \text{.................... (2)}$$

$$\boxed{\nabla \cdot \bar{f} = \sum i \cdot \frac{\partial \bar{f}}{\partial x}} \quad \text{.................. (2A)}$$

The equivalence between (1) and (2) can be proved as follows.

$$\nabla \cdot \bar{f} = i \cdot \frac{\partial \bar{f}}{\partial x} + j \cdot \frac{\partial \bar{f}}{\partial y} + k \cdot \frac{\partial \bar{f}}{\partial z} \qquad \text{[By (2)]}$$

$$= i \cdot \left(i \frac{\partial f_1}{\partial x} + j \frac{\partial f_2}{\partial x} + k \frac{\partial f_3}{\partial x}\right) + j \cdot \left(i \frac{\partial f_1}{\partial y} + j \frac{\partial f_2}{\partial y} + k \frac{\partial f_3}{\partial y}\right) + k \cdot \left(i \frac{\partial f_1}{\partial z} + j \frac{\partial f_2}{\partial z} + k \frac{\partial f_3}{\partial z}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Hence, the equality between (1) and (2).

(b) Definition: If $\bar{f} = f_1 i + f_2 j + f_3 k$ then

$$\nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$ is called the **curl of** $\bar{f}$. Thus,

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} \qquad \text{.................... (3)}$$

Note that curl $\bar{f}$ is a vector point function.

Remark ....

If $\bar{f}$ is a vector point function such that curl $\bar{f} = 0$ then $\bar{f}$ is called **irrotational** or **conservative.** (The reason for this nomenclature will be made clear later.)

We can also write curl $\bar{f}$ as

$$\boxed{\nabla \times \bar{f} = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + k \times \frac{\partial \bar{f}}{\partial z}} \qquad \text{.................... (4)}$$

$$\boxed{\nabla \times \bar{f} = \sum i \times \frac{\partial \bar{f}}{\partial x}}$$

The equivalence between (1) and (2) can be proved as follows.

$$\nabla \times \bar{f} = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + k \times \frac{\partial \bar{f}}{\partial z} \qquad [\text{ By (4) }]$$

$$= i \times \left(i \frac{\partial f_1}{\partial x} + j \frac{\partial f_2}{\partial x} + k \frac{\partial f_3}{\partial x} \right) + j \times \left(i \frac{\partial f_1}{\partial y} + j \frac{\partial f_2}{\partial y} + k \frac{\partial f_3}{\partial y} \right) + k \times \left(i \frac{\partial f_1}{\partial z} + j \frac{\partial f_2}{\partial z} + k \frac{\partial f_3}{\partial z} \right)$$

$$= k \frac{\partial f_2}{\partial x} - j \frac{\partial f_3}{\partial x} - k \frac{\partial f_1}{\partial y} + i \frac{\partial f_3}{\partial y} + j \frac{\partial f_1}{\partial z} - i \frac{\partial f_2}{\partial z}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Hence, the equality between (3) and (4).

We denote cut $\bar{f}$ also as curl $\bar{f} = \nabla \times \bar{f} = \sum i \times \dfrac{\partial \bar{f}}{\partial x}$.

(c) Standard Results : The following results can be proved very easily by using the above finition.

1. $\text{div}(\bar{f} \pm \bar{g}) = \text{div } \bar{f} \pm \text{div } \bar{g}$ or $\nabla \cdot (\bar{f} \pm \bar{g}) = \nabla \cdot \bar{f} \pm \nabla \cdot \bar{g}$
2. curl $(\bar{f} \pm \bar{g}) =$ curl $\bar{f} \pm$ curl $\bar{g}$ or $\nabla \times (\bar{f} \pm \bar{g}) = \nabla \times \bar{f} \pm \nabla \times \bar{g}$ **(M.U. 2003)**
3. $\nabla \cdot (\bar{f} \times \bar{g}) = \bar{g} \cdot (\nabla \times \bar{f}) - \bar{f} \cdot (\nabla \times \bar{g})$
4. $\nabla \cdot (\Phi \bar{f}) = \Phi (\nabla \cdot \bar{f}) + \nabla \Phi \cdot \bar{f}$

Type I : To Find div and curl of $\bar{F}$

Example 1 : If $\bar{F} = x^2 z\, i - 2y^3 z^3\, j + xy^2 z^2\, k$ find div $\bar{f}$ and curl $\bar{f}$ at (1, −1, 1).

Sol. : By definition

$$\operatorname{div} \bar{F} = \nabla \cdot \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^3) + \frac{\partial}{\partial z}(xy^2 z^2)$$

$$= 2xz - 6y^2 z^3 + 2xy^2 z$$

$$\therefore \quad \operatorname{div} \bar{F} = (2 - 6 + 2) = -2 \text{ at } (1, -1, 1)$$

$$\therefore \quad \operatorname{curl} \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & -2y^3 z^3 & xy^2 z^2 \end{vmatrix} = i(2xyz^2 + 6y^3 z^2) - j(y^2 z^2 - x^2) + k(0 - 0)$$

$$= i(-2 - 6) - j(1 - 1) + k(0) = -8\, i \text{ at } (1, -1, 1).$$

Example 2 : If $\bar{F} = xy\, e^{2z}\, i + xy^2 \cos z\, j + x^2 \cos xy\, k$ find div $\bar{F}$ and curl $\bar{F}$. **(M.U. 2013)**

Sol. :

$$\operatorname{div} \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = y\, e^{2z} + 2xy \cos z + 0\, k$$

$$\operatorname{curl} \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy\, e^{2z} & xy^2 \cos z & x^2 \cos xy \end{vmatrix}$$

$$= i(-x^3 \sin xy + xy^2 \sin z) - j(-x^2 y \sin xy + 2x \cos xy - 2xy\, e^{2z}) + k(y^2 \cos z - x\, e^{2z})$$

Example 3 : If $\bar{a}$ is a constant vector find div $\bar{a}$ and curl $\bar{a}$.

Sol. : Let $\bar{a} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are constants

$$\therefore \quad \nabla \cdot \bar{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = 0$$

$$\nabla \times \bar{a} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \end{vmatrix} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) + j\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial z}\right)$$

$$\therefore \quad \nabla \times \bar{a} = 0i + 0j + 0k = \bar{0}$$

$$\therefore \quad \boxed{\operatorname{div} \bar{a} = 0 \text{ and } \operatorname{curl} \bar{a} = \bar{0}.}$$

Example 4 : If $\bar{r} = xi + yj + zk$ find (i) grad r, (ii) div $\bar{r}$ and (iii) curl $\bar{r}$.

Sol. : (i) grad $r = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\sqrt{x^2 + y^2 + z^2}$

Now, $\frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{r} \quad \left[\because r = \sqrt{x^2 + y^2 + z^2}\right]$

$$\therefore \text{ grad } r = \frac{x}{r} i + \frac{y}{r} j + \frac{z}{r} k = \frac{1}{r}(xi + yj + zk) = \frac{1}{r}\bar{r}$$

(ii) $$\text{div } \bar{r} = \left(\frac{\partial}{\partial x} x\right) + \left(\frac{\partial}{\partial y} y\right) + \left(\frac{\partial}{\partial z} z\right) = 1 + 1 + 1 = 3.$$

(iii) $$\text{curl } \bar{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0i + 0j + 0k = \bar{0}$$

Thus, $$\boxed{\text{grad } r = \frac{\bar{r}}{r}, \quad \text{div } \bar{r} = 3, \quad \text{crul } \bar{r} = \bar{0}}$$

Example 5 : If $\bar{a}$ is a constant vector and $\bar{r} = xi + yj + zk$, prove that

(1) $\nabla(\bar{a} \cdot \bar{r}) = \bar{a}$ (2) $\text{div}(\bar{a} \times \bar{r}) = 0$ (3) $\text{div}(\bar{a} \cdot \bar{r})\bar{a} = a^2$

(4) $\text{curl}(\bar{a} \times \bar{r}) = 2\bar{a}$ (5) $\text{div}[\bar{a} \times (\bar{r} \times \bar{a})] = 2a^2$ **(M.U. 1999, 2005, 08)**

Sol. : (1) We have

$$\bar{a} \cdot \bar{r} = (a_1 i + a_2 j + a_3 k) \cdot (xi + yj + zk) = a_1 x + a_2 y + a_3 z$$

$$\therefore \nabla(\bar{a} \cdot \bar{r}) = i\frac{\partial}{\partial x}(a_1 x + a_2 y + a_3 z) + j\frac{\partial}{\partial y}(\ldots\ldots) + k\frac{\partial}{\partial z}(\ldots\ldots)$$

$$= a_1 i + a_2 j + a_3 k = \bar{a}$$

(2) We have

$$\bar{a} \times \bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y) i - (a_1 z - a_3 x) j + (a_1 y - a_2 x) k$$

$$\text{div}(\bar{a} \times \bar{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) = 0$$

(3) We have, as above

$$\bar{a} \cdot \bar{r} = a_1 x + a_2 y + a_3 z$$

$$(\bar{a} \cdot \bar{r})\bar{a} = (a_1 x + a_2 y + a_3 z) a_1 i + (\ldots.) a_2 j + (\ldots.) a_3 k$$

$$\text{div}(\bar{a} \cdot \bar{r})\bar{a} = \frac{\partial}{\partial x}[(a_1 x + a_2 y + a_3 z) a_1] + \frac{\partial}{\partial y}[(\ldots.) a_2] + \frac{\partial}{\partial z}[(\ldots.) a_3]$$

$$= a_1^2 + a_2^2 + a_3^2 = a^2$$

(4) We have as in (2)

$$\bar{a} \times \bar{r} = (a_2 z - a_3 y) i + (a_3 x - a_1 z) j + (a_1 y - a_2 x) k$$

$$\therefore \text{curl}(\bar{a} \times \bar{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$\therefore \text{curl}(\bar{a} \times \bar{r}) = i(a_1 + a_1) + j(a_2 + a_2) + k(a_3 + a_3)$$

$$= 2(a_1 i + a_2 j + a_3 k) = 2\bar{a}.$$

(5) We have as in (2)

$$\bar{a} \times \bar{r} = (a_2 z - a_3 y) i + (a_3 x - a_1 z) j + (a_1 y - a_2 x) k$$

$$\therefore\quad \bar{r}\times\bar{a} = -(\bar{a}\times\bar{r}) = (a_3\,y - a_2\,z)\,i + (a_1\,z - a_3\,x)\,j + (a_2\,x - a_1\,y)\,k$$

$$\therefore\quad \bar{a}\times(\bar{r}\times\bar{a}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_3\,y - a_2\,z & a_1\,z - a_3\,x & a_2\,x - a_1\,y \end{vmatrix}$$

$$= [\,a_2\,(a_2\,x - a_1\,y) - a_3\,(a_1\,z - a_3\,x)\,]\,i + [\,\ldots..\,]\,j + [\,\ldots..\,]\,k$$

$$\text{div}\,[\,\bar{a}\times(\bar{r}\times\bar{a})] = \frac{\partial}{\partial x}[\,a_2\,(a_2\,x - a_1\,y) - a_3\,(a_1\,z - a_3\,x)] + \frac{\partial}{\partial y}[\ldots..] + \frac{\partial}{\partial z}[\ldots..]$$

$$= (a_2^2 + a_3^2) + (a_3^2 + a_1^2) + (a_1^2 + a_2^2)$$

$$= 2(a_1^2 + a_2^2 + a_3^2) = 2\,\bar{a}.$$

List of Formulae

1. If $\bar{a}$ is constant, then $\boxed{\nabla\cdot\bar{a} = 0 \quad\text{and}\quad \nabla\times\bar{a} = \bar{0}}$
2. If $\bar{r} = xi + yj + zk$, then $\boxed{\nabla\cdot\bar{r} = 3 \quad\text{and}\quad \nabla\times\bar{r} = \bar{0}}$
3. If $\bar{a}$ is constant and $\bar{r}$ is a position vector then

$$\boxed{\nabla(\bar{a}\cdot\bar{r}) = \bar{a} \quad\text{and}\quad \nabla\cdot(\bar{a}\times\bar{r}) = 0 \quad\text{and}\quad \nabla\times(\bar{a}\times\bar{r}) = 2\,\bar{a}}$$

4. $\boxed{\nabla\, r^n = n\,r^{n-2}\,\bar{r}}$ [Ex. 8, page 3-5]

Example 6 : If $\Phi = x^3 + y^3 + z^3 - 3xyz$, find (i) $\bar{r}\cdot\nabla\Phi$, (ii) div $\bar{F}$ and curl $\bar{F}$ where $\bar{F} = \nabla\Phi$. **(M.U. 2003)**

Sol. : (a) $\nabla\Phi = i\dfrac{\partial\Phi}{\partial x} + j\dfrac{\partial\Phi}{\partial y} + k\dfrac{\partial\Phi}{\partial z}$

$$\therefore\quad \bar{F} = \nabla\Phi = i(3x^2 - 3yz) + j(3y^2 - 3xz) + k(3z^2 - 3xy)$$

But $\bar{r} = xi + yj + 2k$

$$\therefore\quad \bar{r}\cdot\nabla\Phi = \bar{r}\cdot\bar{F} = x(3x^2 - 3yz) + y(3y^2 - 3xz) + z(3z^2 - 3xy)$$

$$= 3(x^3 + y^3 + z^3 - 3xyz) = 3\Phi$$

(b) div $\bar{F} = \nabla\cdot\bar{F}$

$$\therefore\quad \text{div}\,\bar{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z = 6(x + y + z)$$

(c) curl $\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3(x^2 - yz) & 3(y^2 - xz) & 3(z^2 - xy) \end{vmatrix}$

$$= i(-3x + 3x) + j(3y - 3y) + k(-3z + 3z)$$

$$= 0\,i + 0\,j + 0\,k = \bar{0}$$

Example 7 : If $\bar{f} = (x + y + 1)\,i + j - (x + y)\,k$, prove that $\bar{f}\cdot\text{curl}\,\bar{f} = 0$. **(M.U. 2004)**

Sol. : curl $\bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + y + 1 & 1 & -x - y \end{vmatrix} = i(-1) - j(-1) + k(-1) = -i + j - k$

$$\therefore \quad \bar{f} \cdot \operatorname{curl} \bar{f} = [(x+y+1)i + j - (x+y)k] \cdot [-i + j - k]$$
$$= -(x+y+1) + 1 + (x+y) = 0$$

Example 8 : Prove that $\nabla \cdot (\nabla \times \bar{F}) = 0$ where $\bar{F}$ is a vector point function. **(M.U. 2000)**

Sol. : Let $\bar{F} = F_1 i + F_2 j + F_3 k$

$$\text{Then } \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + j\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\therefore \quad \nabla \cdot (\nabla \times \bar{F}) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y \partial z} = 0.$$

Example 9 : If $u\bar{F} = \nabla v$ where u and v are scalar fields and $\bar{F}$ is a vector field, prove that $\bar{F} \cdot \operatorname{curl} \bar{F} = 0$.

Sol. : Let $\bar{F} = F_1 i + F_2 j + F_3 k$. By data $u\bar{F} = \nabla v$

$$\therefore \quad uF_1 i + uF_2 j + uF_3 k = i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}$$

$$\therefore \quad uF_1 = \frac{\partial v}{\partial x}, \quad uF_2 = \frac{\partial v}{\partial y}, \quad uF_3 = \frac{\partial v}{\partial z}$$

$$\therefore \quad F_1 = \frac{1}{u}\frac{\partial v}{\partial x}, \quad F_2 = \frac{1}{u}\frac{\partial v}{\partial y}, \quad F_3 = \frac{1}{u}\frac{\partial v}{\partial z} \qquad \text{.................... (1)}$$

$$\operatorname{curl} \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= i\left[\left\{-\frac{1}{u^2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{1}{u}\frac{\partial^2 v}{\partial y \partial z}\right\} - \left\{-\frac{1}{u^2}\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} + \frac{1}{u}\frac{\partial^2 v}{\partial y \partial z}\right\}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\frac{1}{u^2}\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + j\left(\ldots\ldots\right) + k\left(\ldots\ldots\right)$$

Now, using (1), we get

$$\therefore \quad \bar{F} \cdot \operatorname{curl} \bar{F} = \frac{1}{u^3}\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\ldots\ldots\right) + \left(\ldots\ldots\right) = 0$$

(The terms are cancelled because of symmetry.)

Type II : Div and Curl of Functions of $\bar{r}$

Example 1 : Prove that $\nabla\left\{\nabla \cdot \frac{\bar{r}}{r}\right\} = -\frac{2}{r^3}\bar{r}$. **(M.U. 2005)**

Sol. : We have $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2 + y^2 + z^2}$

Now, let $\bar{f} = \frac{\bar{r}}{r} = \frac{xi}{\sqrt{x^2+y^2+z^2}} + \frac{yj}{\sqrt{x^2+y^2+z^2}} + \frac{zk}{\sqrt{x^2+y^2+z^2}}$

$= f_1 i + f_2 j + f_3 k$

$$\therefore \nabla \cdot \frac{\bar{r}}{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Now, $\frac{\partial f_1}{\partial x} = \frac{\sqrt{x^2+y^2+z^2} - x \cdot x \Big/ \sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)} = \frac{r^2 - x^2}{(r^2)^{3/2}} = \frac{r^2-x^2}{r^3}$

Similarly, we get two more results.

$$\therefore \nabla \cdot \frac{\bar{r}}{r} = \frac{(r^2-x^2)+(r^2-y^2)+(r^2-z^2)}{r^3} = \frac{3r^2-(x^2+y^2+z^2)}{r^3}$$

$$= \frac{2r^2}{r^3} = \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \nabla\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right) = 2\left[i \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + j(\ldots\ldots) + k(\ldots\ldots)\right]$$

$$= 2\left[i \cdot \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot 2x + j(\ldots\ldots) + k(\ldots\ldots)\right]$$

$$= -2\left[\frac{xi}{r^3} + \frac{yj}{r^3} + \frac{zk}{r^3}\right] = \frac{-2}{r^3}\bar{r}$$

Example 2 : Prove that $\nabla \cdot \left(r \nabla \frac{1}{r^n}\right) = \frac{n(n-2)}{r^{n+1}}$. **(M.U. 2006, 09)**

Sol. : We have $r^{-n} = (x^2+y^2+z^2)^{-n/2}$

$$\therefore \nabla\left(\frac{1}{r^n}\right) = \left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-\frac{n}{2}-1} \cdot (2x)\, i + (\ldots\ldots)\, j + (\ldots\ldots)\, k$$

$$= -nr^{-n-2}[xi + yj + zk]$$

$$\therefore r\nabla\left(\frac{1}{r^n}\right) = -nr^{-n-1}xi - nr^{-n-1}yj - nr^{-n-1}zk$$

$$\therefore \nabla \cdot \left[r\nabla\left(\frac{1}{r^n}\right)\right] = -n\frac{\partial}{\partial x}\left[r^{-n-1}x\right] - [\ \ldots\ldots\ ] - [\ \ldots\ldots\ ]$$

$$= -n\left[(-n-1)x \cdot r^{-n-2}\frac{\partial r}{\partial x} + r^{-n-1} \cdot 1\right] - \left[\ldots\ldots\right] - \left[\ldots\ldots\right]$$

$$= -n\left[(-n-1)\frac{x^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right] - \left[\ldots\ldots\right] - \left[\ldots\ldots\right]$$

$$= -n\left[\left\{(-n-1)\frac{x^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right\} + \left\{(-n-1)\frac{y^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right\} + \left\{(-n-1)\frac{z^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right\}\right]$$

$$\therefore \quad \nabla \cdot \left[r \nabla \left(\frac{1}{r^n} \right) \right] = -n \left[-\frac{(n+1)}{r^{n+3}} (x^2 + y^2 + z^2) + \frac{3}{r^{n+1}} \right]$$

$$= -n \left[-\frac{n+1}{r^{n+1}} + \frac{3}{r^{n+1}} \right] = \frac{n(n-2)}{r^{n+1}}$$

Example 3 : Prove that $\nabla \cdot \left(r \nabla \frac{1}{r^3} \right) = \frac{3}{r^4}$. **(M.U. 1999, 2006)**

Sol. : Putting $r = 3$ in the above example, we get the result.

Example 4 : Prove that div grad $r^n = n(n+1) r^{n-2}$. **(M.U. 2004)**

Sol. : As proved in Ex. 8, page 3-5

$$\text{grad } r^n = n r^{n-2} \bar{r}.$$

$$\therefore \quad \text{div grad } r^n = \nabla \cdot \left[n r^{n-2} (xi + yj + zk) \right]$$

$$= n \left[\frac{\partial}{\partial x} (r^{n-2} \cdot x) + \frac{\partial}{\partial y} (r^{n-2} \cdot y) + \frac{\partial}{\partial z} (r^{n-2} \cdot z) \right]$$

$$= n \left[r^{n-2} + x \cdot (n-2) \cdot r^{n-3} \frac{\partial r}{\partial x} + \ldots\ldots + \ldots\ldots \right]$$

$$= n \left[\left\{ r^{n-2} + x \cdot (n-2) \cdot r^{n-3} \frac{x}{r} \right\} + \{ \ldots\ldots \} + \{ \ldots\ldots \} \right]$$

$$= n \left[\left\{ r^{n-2} + (n-2) r^{n-4} \cdot x^2 \right\} + \left\{ r^{n-2} + (n-2) r^{n-4} \cdot y^2 \right\} + \left\{ r^{n-2} + (n-2) r^{n-4} \cdot z^2 \right\} \right]$$

$$= n \left[3r^{n-2} + (n-2) r^{n-4} \cdot (x^2 + y^2 + z^2) \right]$$

$$= n \left[3r^{n-2} + (n-2) r^{n-2} \right]$$

$$= n(n+1) r^{n-2}$$

(The term r^{n-2} comes from three terms.)

Example 5 : Prove that **(i)** $\nabla \cdot (\bar{a} \times \bar{r}) = 0$, **(ii)** $\nabla \cdot \left(\frac{\bar{a} \times \bar{r}}{r} \right) = 0$. **(M.U. 2004)**

Sol. : Assuming the result that $\nabla \cdot (\bar{a} \times \bar{r})$ can be looked upon as a scalar triple product treating ∇ as a vector we have [Scalar Triple Product]

$$\text{(i)} \quad \nabla \cdot (\bar{a} \times \bar{r}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \frac{\partial}{\partial x} (a_2 z - a_3 y) + \frac{\partial}{\partial y} (a_3 x - a_1 z) + \frac{\partial}{\partial z} (a_1 y - a_2 x) = 0$$

(ii) In the same way

$$\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r}\right)=\nabla\cdot\left(\bar{a}\times\frac{\bar{r}}{r}\right)=\begin{vmatrix}\partial/\partial x & \partial/\partial y & \partial/\partial z\\ a_1 & a_2 & a_3\\ x/r & y/r & z/r\end{vmatrix}$$

$$=\frac{\partial}{\partial x}\left(\frac{a_2z-a_3y}{r}\right)+\frac{\partial}{\partial y}\left(\frac{a_3x-a_1z}{r}\right)+\frac{\partial}{\partial z}\left(\frac{a_1y-a_2x}{r}\right)$$

$$=-\frac{1}{r^2}(a_2z-a_3y)\frac{\partial r}{\partial x}+\ldots\ldots+\ldots\ldots$$

$$=-(a_2z-a_3y)\frac{x}{r^3}+\ldots\ldots+\ldots\ldots$$

$$=-\frac{1}{r^3}\left[(a_2z-a_3y)\,x+(a_3x-a_1z)\,y+(a_1y-a_2x)\,z\right]$$

$$=0$$

Example 6 : Prove that $\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right)=\frac{(2-n)\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}}$. **(M.U. 1996, 2002, 03, 05, 09)**

Sol. : We have $\frac{\bar{a}\times\bar{r}}{r^n}=\frac{1}{r^n}(\bar{a}\times\bar{r})=\frac{1}{r^n}\begin{vmatrix}i & j & k\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}$

$$=\frac{1}{r^n}(a_2z-a_3y)\,i+\frac{1}{r^n}(a_3x-a_1z)\,j+\frac{1}{r^n}(a_1y-a_2x)\,k$$

$$\therefore\quad \nabla\times\frac{(\bar{a}\times\bar{r})}{r^n}=\begin{vmatrix}i & j & k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \dfrac{a_2z-a_3y}{r^n} & \dfrac{a_3x-a_1z}{r^n} & \dfrac{a_1y-a_2x}{r^n}\end{vmatrix}$$

$$=i\left[\frac{\partial}{\partial y}\left(\frac{a_1y-a_2x}{r^n}\right)-\frac{\partial}{\partial z}\left(\frac{a_3x-a_1z}{r^n}\right)\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

$$\because\ r^2=x^2+y^2+z^2\quad\therefore\ 2r\frac{\partial r}{\partial x}=2x$$

$$\therefore\ \frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$=i\left[\left\{-n\,r^{-n-1}\cdot\left(\frac{y}{r}\right)(a_1y-a_2x)+\frac{1}{r^n}a_1\right\}-\left\{-n\,r^{-n-1}\left(\frac{z}{r}\right)(a_3x-a_1z)+\frac{1}{r^n}(-a_1)\right\}\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

$$=i\left[-\frac{n}{r^{n+2}}(a_1y^2-a_2xy)+\frac{a_1}{r^n}+\frac{n}{r^{n+2}}(a_3xz-a_1z^2)+\frac{a_1}{r^n}\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

$$=i\left[\frac{2a_1}{r^n}-\frac{n}{r^{n+2}}a_1(y^2+z^2)\frac{n}{r^{n+2}}(a_2xy+a_3xz)\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

Adding $\frac{n}{r^{n+2}} a_1 x^2$ to the third term and subtracting it from the second term

$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}(x^2+y^2+z^2) + \frac{n}{r^{n+2}}(a_1x^2 + a_2xy + a_3xz)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}} r^2 + \frac{n}{r^{n+2}} x(a_1x + a_2y + a_3z)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}} r^2 + \frac{n}{r^{n+2}} x(a_1x + a_2y + a_3z)\right]$$
$$+ j\left[\frac{2a_2}{r^n} - \frac{na_2}{r^{n+2}} r^2 + \frac{n}{r^{n+2}} y(a_1x + a_2y + a_3z)\right]$$
$$+ k\left[\frac{2a_3}{r^n} - \frac{na_3}{r^{n+2}} r^2 + \frac{n}{r^{n+2}} z(a_1x + a_2y + a_3z)\right]$$

$$= \frac{(2-n)}{r^n}(a_1 i + a_2 j + a_3 k) + \frac{n}{r^{n+2}}(a_1x + a_2y + a_3z)(xi + yj + zk)$$

$$= \frac{(2-n)}{r^n}\bar{a} + \frac{n}{r^{n+2}}(\bar{a}\cdot\bar{r})\bar{r}$$

Example 7 : Prove that $\nabla \times (\bar{a} \times \bar{r})\, r^n = (n+2)\, r^n \bar{a} - n\, r^{n-2}(\bar{a}\cdot\bar{r})\bar{r}$. **(M.U. 1993)**

Sol. : Change the sign of n in the above Ex. 6 or try independently on the same lines.

Example 8 : Prove that $\nabla \times \left(\frac{\bar{a}\times\bar{r}}{r}\right) = \frac{\bar{a}}{r} + \frac{\bar{a}\cdot\bar{r}}{r^3}\bar{r}$.

Sol. : Put $n = 1$ in the above Ex. 6 or try independently.

Example 9 : Prove that $\nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{a}{r^3} + \frac{3(\bar{a}\cdot\bar{r})\,r}{r^5}$. **(M.U. 1998, 2006)**

Sol. : Put $n = 3$ in Ex. 6 or try independently.

Example 10 : Prove that $\nabla \log r = \frac{\bar{r}}{r^2}$ and hence, show that

$$\nabla \times (\bar{a} \times \nabla \log r) = 2\frac{(\bar{a}\cdot\bar{r})\bar{r}}{r^4}$$ where $\bar{a}$ is a constant vector. **(M.U. 2000)**

Sol. : $\log r = \frac{1}{2}\log(x^2+y^2+z^2)$ $\quad \therefore \quad \frac{\partial}{\partial x}(\log r) = \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{x}{r^2}$

Similarly, $\frac{\partial}{\partial y}(\log r) = \frac{y}{r^2}$, $\frac{\partial}{\partial z}(\log r) = \frac{z}{r^2}$.

$$\therefore \quad \nabla \log r = i\frac{x}{r^2} + j\cdot\frac{y}{r^2} + k\cdot\frac{z}{r^2} = \frac{1}{r^2}(xi + yj + zk) = \frac{\bar{r}}{r^2}$$

Now, $\bar{a} \times \nabla \log r = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix} = \frac{(a_2z - a_3y)}{r^2} i + \frac{(a_3x - a_1z)}{r^2} j + \frac{(a_1y - a_2x)}{r^2} k$

$$\therefore\ \nabla\times(\bar{a}\times\nabla\log r)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{(a_2z-a_3y)}{r^2} & \frac{(a_3x-a_1z)}{r^2} & \frac{(a_1y-a_2x)}{r^2}\end{vmatrix}$$

$$=i\left[\frac{\partial}{\partial y}\left\{\left(\frac{a_1y-a_2x}{r^2}\right)\right\}-\frac{\partial}{\partial z}\left\{\left(\frac{a_3x-a_1z}{r^2}\right)\right\}\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

$$\because\ r^2=x^2+y^2+z^2,\ 2r\frac{\partial r}{\partial x}=2x,$$

$$\therefore\ \frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$=i\left[\left\{\frac{r^2(a_1)-(a_1y-a_2x)\,2r\,(y/r)}{r^4}\right\}-\left\{\frac{r^2(-a_1)-(a_3x-a_1z)\,2r\,(z/r)}{r^4}\right\}\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

$$=\frac{2i}{r^4}\left[a_1r^2-a_1(y^2+z^2)+(a_2xy+a_3xz)\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

[By adding a_1x^2 to the second term and sebtracting a_1x^2 from the third term.]

$$=\frac{2i}{r^4}\left[a_1r^2-a_1(x^2+y^2+z^2)+(a_1x^2+a_2xy+a_3xz)\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

$$=\frac{2i}{r^4}\left[a_1r^2-a_1r^2+x(a_1x+a_2y+a_3z)\right]+j\left[\ldots\ldots\right]+k\left[\ldots\ldots\right]$$

$$=\frac{2}{r^4}(a_1x+a_2y+a_3z)(xi+yj+zk)=\frac{2}{r^4}(\bar{a}\cdot\bar{r})\,\bar{r}$$

Example 11 : Prove that $\nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\}=\frac{1}{r^2}\frac{d}{dr}\left[r^2f(r)\right]$. **(M.U. 2002, 08)**

Hence, or otherwise prove that $\operatorname{div}(r^n\,\bar{r})=(n+3)\,r^n$.

Sol. : $\frac{f(r)}{r}\bar{r}=\frac{f(r)}{r}(xi+yj+zk)$

$$\therefore\ \nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\}=\frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right]+\frac{\partial}{\partial y}\left[\frac{f(r)}{r}y\right]+\frac{\partial}{\partial z}\left[\frac{f(r)}{r}z\right]$$

$$=\frac{f'(r)}{r}x\frac{\partial r}{\partial x}-\frac{f(r)}{r^2}x\frac{\partial r}{\partial x}+\frac{f(r)}{r}+\ldots\ldots+\ldots\ldots$$

But $r^2=x^2+y^2+z^2$ $\therefore\ \frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$

$$\therefore\ \nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\}=\frac{f'(r)}{r}\frac{x^2}{r}-\frac{f(r)}{r^2}\frac{x^2}{r}+\frac{f(r)}{r}+[\ldots\ldots]+[\ldots\ldots]$$

$$\therefore \nabla \cdot \left\{\frac{f(r)}{r}\bar{r}\right\} = f'(r)\frac{x^2}{r^2} - \frac{f(r)}{r}\frac{x^2}{r^2} + \frac{f(r)}{r} + [.....] + [.....]$$

$$= f'(r)\frac{(x^2+y^2+z^2)}{r^2} - \frac{f(r)}{r}\frac{(x^2+y^2+z^2)}{r^2} + 3\frac{f(r)}{r}$$

$$= f'(r) + 2\frac{f(r)}{r} \qquad .... (1)$$

[$\because$ The term $\frac{f(r)}{r}$ will come from each bracket.]

Now, $\frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right] = \frac{1}{r^2}\left[r^2 f'(r) + 2r f(r)\right] = f'(r) + \frac{2f(r)}{r}$ .... (2)

From (1) and (2) we get the required result.

$$\left\{ e.g. \quad \nabla \cdot \left[\frac{e^r}{r}\bar{r}\right] = e^r + \frac{2e^r}{r} \right\}$$

Now put $\frac{f(r)}{r} = r^n$ i.e. $f(r) = r^{n+1}$

$$\therefore \nabla \cdot \left\{r^n \bar{r}\right\} = \frac{1}{r^2}\frac{d}{dr}\left\{(r^{n+1})r^2\right\} = \frac{1}{r^2}\frac{d}{dr}\left\{r^{n+3}\right\}$$

$$= \frac{1}{r^2}(n+3)r^{n+2} = (n+3)r^n.$$

(Also see the next example.)

Example 12 : Prove that $\nabla \cdot \left[\frac{\log r}{r}\bar{r}\right] = \frac{1}{r}[1 + 2\log r]$.

Sol. : Putting $f(r) = \log r$ in the result (1) of the above Ex. 11, we get

$$\nabla \cdot \left[\frac{\log r}{r}\bar{r}\right] = \frac{1}{r^2} + \frac{2\log r}{r} \qquad [\text{ Because } f(r) = \log r]$$

$$= \frac{1}{r}[1 + 2\log r]$$

EXERCISE - III

State true or false with proper justification.

(i) Maximum value of the directional derivative of $\Phi = x^3 + 2xy + 3z$ cannot be less than 3. **(M.U. 1997)**

(ii) Grad r^3 is an irrotational vector. **(M.U. 1997)**

(iii) Div $\bar{r} = 3$, curl $\bar{r} = 0$.

(iv) If $\bar{a}$ is a constant vector then $\bar{a} \times \bar{r}$ is irrotational.

(v) If $\bar{a}$ is a constant vector then $\bar{a}$ is solenoidal.

(vi) Grad r^n is an irrotational vector.

(vii) If $\bar{a}$ is a constant vector then curl $(\bar{a} \times \bar{r}) = 0$.

(viii) If $\bar{a}$ is a constant vector then curl $(\bar{a} \cdot \bar{r}) = 0$.

(ix) If $\bar{a}$ is a constant vector then div $(\bar{a} \cdot \bar{r})\bar{a} = a^2$.

[**Ans. :** (i) True, (ii) True, (iii) True, (iv) False, (iv) True, (v) True,
(vi) True, (vii) False, (viii) False, (ix) True.]

EXERCISE - IV

1. If $\bar{f} = x^2zi - 2y^3z^2j + xy^2zk$, find $\nabla \cdot \bar{f}$ at $(1, -1, 1)$. **(M.U. 1993)** [**Ans. :** -3]

2. If $\bar{f} = 3x^2i + 5xyj + xyz^3k$, find div $\bar{f}$ and curl $\bar{f}$ at (1, 2, 3).
[**Ans. :** (i) 65, (ii) $27i - 18j + 10k$]

3. Find div $\bar{F}$ and curl $\bar{F}$ where
(a) $\bar{F} = (x^2 + yz)\,i + (y^2 + zx)\,j + (z^2 + xy)\,k$.
(b) $\bar{F} = (x^2 - y^2)\,i + 2xyj + (y^2 - xy)\,k$.
(c) $\dfrac{xi - yj}{x^2 + y^2}$ **(M.U. 2000, 10)**

[**Ans. :** (a) (i) $2(x + y + z)$, (ii) $\bar{0}$; (b) (i) $4x$, (ii) $(2y - x)\,i + yj + 4yk$
(c) (i) $-\dfrac{2(x^2 - y^2)}{(x^2 + y^2)^2}$, (ii) $\dfrac{4xy}{(x^2 + y^2)^2}$.]

4. If $\bar{A} = \nabla(xy + yz + zx)$, find $\nabla \cdot \bar{A}$ and $\nabla \times \bar{A}$. **(M.U. 2004, 14)** [**Ans. :** 0, $\bar{0}$]

5. If $\bar{F} = x^2i + xzj + yzk$ and $\bar{r} = xi + yj + zk$, find div $(\bar{F} \times \bar{r})$ and curl $(\bar{F} \times \bar{r})$. **(M.U. 2001)**
[**Ans. :** (i) $z^2 + xz - x^2$, (ii) $(2x^2 - xy)\,i + (4xz - 2xy - y^2)\,j + (3yz - 2xz)\,k$]

6. If $\bar{u} = x^2yi + y^2x^3j - 3x^2z^2k$ and $\bar{v} = 2xz^2i - yzj + x^2y^3k$, find $\nabla \cdot (\bar{u} \times \bar{v})$ at (1, 2, 1).
[**Ans. :** 96]

7. If $\bar{u} = x^2yi + y^2x^3j - 3x^2z^2k$ and $\Phi = x^2yz$, find $\nabla \cdot (\Phi\,\bar{u})$ at (1, 2, 1). [**Ans. :** 10]

8. If $\bar{A} = \dfrac{x}{r}i + \dfrac{y}{r}j + \dfrac{z}{r}k$ and $r = \sqrt{x^2 + y^2 + z^2}$, find div $\bar{A}$. [**Ans. :** $\dfrac{2}{r}$]

9. If $\bar{F} = (\bar{a} \cdot \bar{r})\,\bar{r}$ where $\bar{a}$ is a constant vector, find curl $\bar{F}$ and prove that it is perpendicular to $\bar{a}$.
(M.U. 2002)

10. Prove that $\nabla \cdot \left(\dfrac{\bar{r}}{r^3}\right) = 0$. **(M.U. 1992)**

11. If $u = x^2 + y^2 + z^2$ prove that curl grad $u = \bar{0}$.

12. If $\bar{A} = yz^2i + zx^2j + xy^2k$, prove that $\bar{A} \cdot \text{curl}\,\bar{A} = xyz(xy + yz + zx)$.

12. Physical Interpretation of Divergence

Consider a region of space filled with a fluid which moves so that its velocity vector at any point $P(x, y, z)$ is $\bar{f}(x, y, z)$. With usual notation let, $\bar{f} = f_1i + f_2j + f_3k$, so that f_1, f_2, f_3 are scalar functions of x, y, z and are the components of velocity parallel to the axes.

Now construct a parallelepiped having centre at $P(x, y, z)$ and edges parallel to the coordinate axes and having magnitudes δx, δy, δz respectively.

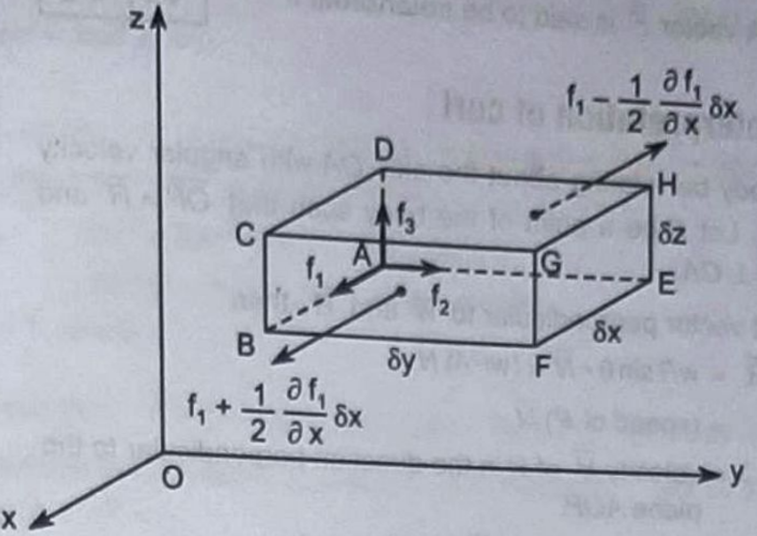

Fig. 3.4

Since the component of $\bar{f}$ normal to any face is responsible for the flow through that face,

The amount of fluid, leaving the face $CBFG$ in time δt

$$= \text{velo. comp. normal to } CBFG \times \text{area} \times \text{time}$$

$$= \left(f_1 + \frac{1}{2}\frac{\partial f_1}{\partial x}\delta x\right)\delta y\,\delta z\,\delta t.$$

Similarly, amount of fluid entering the face $DAEH$

$$= \text{velo. como. normal to } DAEH \times \text{area} \times \text{time}$$

$$= \left(f_1 - \frac{1}{2}\frac{\partial f_1}{\partial x}\delta x\right)\delta y\,\delta z\,\delta t.$$

$\therefore$ Gain of fluid in the parallelepiped in the direction of the x-axis.

$$= \frac{\partial f_1}{\partial x}\delta x\,\delta y\,\delta z\,\delta t.$$

Similarly, we can calculate the gain of fluid in the parallelepiped in the directions of the y and z-axis.

$\therefore$ Total gain of fluid in the parallelepiped

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)\delta x\,\delta y\,\delta z\,\delta t.$$

But $\delta x\,\delta y\,\delta z$ is the volume of the parallelepiped.

$\therefore$ Total gain of fluid in the parallelepiped per unit volume per unit time

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)\left(\frac{\delta x\,\delta y\,\delta z}{\delta x\,\delta y\,\delta z}\right)\left(\frac{\delta t}{\delta t}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \text{div}\,\bar{f} = \nabla \cdot \bar{f}$$

Definition : A vector $\bar{F}$ whose divergence $\bar{F}$ is zero is called **solenoidal**. For such a vector there is no loss or gain of fluid.

Thus, we have another definition.

Definition : A vector $\overline{F}$ is said to be **solenoidal** if $\boxed{\nabla \cdot \overline{F} = 0}$

13. Physical Interpretation of curl

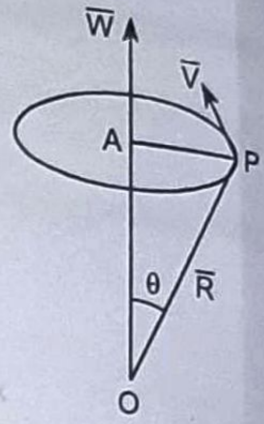

Fig. 3.5

Let a rigid body be rotating about the axis OA with angular velocity w radians per sec. Let P be a point of the body such that $\overline{OP} = \overline{R}$ and $\angle AOP = \theta$. Let $PA \perp OA$.

If $\overline{N}$ is a unit vector perpendicular to $\overline{w}$ and $\overline{R}$, then

$$\overline{w} \times \overline{R} = wR\sin\theta \cdot \overline{N} = (wPA)\,\overline{N}$$
$$= (\text{speed of } P)\,\overline{N}$$
$= \text{velocity } \overline{V}$ of P in the direction perpendicular to the plane AOP.

If $\overline{w} = w_1 i + w_2 j + w_3 k$ and $\overline{R} = xi + yj + zk$

Then

$$\overline{V} = \overline{w} \times \overline{R} = \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = (w_2 z - w_3 y)\, i + (w_3 x - w_1 z)\, j + (w_1 y - w_2 x)\, k$$

$$\therefore \quad \text{curl } \overline{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix} = (w_1 + w_1)\, i + (w_2 + w_2)\, j + (w_3 + w_3)\, k$$
$$= 2w_1 i + 2w_2 j + 2w_3 k = 2\overline{w}$$

$$\therefore \quad \overline{w} = \frac{1}{2}\,\text{curl } \overline{V}$$

Thus, the angular velocity of rotation at any point is equal to half the curl of the velocity vector.

Definition : Any motion in which the curl of the velocity vector is zero *i.e.* if curl $\overline{v} = 0$ then $\overline{w} = 0$ *i.e.* angular velocity is zero, the motion is said to be **irrotational**.

Note ....

In view of this interpretation of curl, curl $\overline{F}$ is also called the **rotation** of $\overline{F}$ and is **sometimes denoted by rot** $\overline{F}$.

Definition : A vector $\overline{F}$ is said to be **irrotational** if $\boxed{\text{curl } \overline{F} = \overline{0}}$

(A) To Show That A Vector Is Solenoidal or Irrotational

Example 1 : If $\overline{F} = (x + 3y)\, i + (y - 2z)\, j + (az + x)\, k$ is solenoidal, find the value of a. **(M.U. 1998)**

Sol. : We know that $\overline{F}$ is solenoidal if div $\overline{F} = \nabla \cdot \overline{F} = 0$.

Now, $$\nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(az + x)$$
$$= 1 + 1 + a = 2 + a$$

$\overline{F}$ is solenoidal if divergence $\overline{F} = 0$. $\quad \therefore\ 2 + a = 0 \quad \therefore\ a = -2.$

Example 2 : Find a, b, c if $\overline{F} = (axy + bz^3)i + (3x^2 - cz)j + (3xz^2 - y)k$ is irrotational. (M.U. 1999, 2005, 2014)

Sol. : $\overline{F}$ is irrotational if curl $\overline{F} = 0$.

$$\therefore \text{curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy + bz^3 & 3x^2 - cz & 3xz^2 - y \end{vmatrix} = i(-1+c) - j(3z^2 - 3bz^2) + k(6x - ax)$$

$$\therefore c - 1 = 0, \quad 3z^2 - 3bz^2 = 0, \quad 6x - ax = 0$$

$$\therefore c = 1, \quad 3z^2(1-b) = 0, \quad x(6-a) = 0$$

$$\therefore c = 1, \; b = 1, \; a = 6.$$

Example 3 : Prove that $\overline{F} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is solenoidal and determine the constants a, b, c if $\overline{F}$ is irrotational. (M.U. 1995, 2000, 04, 14)

Sol. : $\overline{F}$ is solenoidal if $\nabla \cdot \overline{F} = 0$.

$$\text{Now, } \nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 - 3 + 2 = 0$$

Hence, for all values of a, b, c, $\overline{F}$ is solenoidal. $\overline{F}$ is irrotational if curl $\overline{F} = \overline{0}$.

$$\text{Now, curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

and $F_1 = x + 2y + az, \; F_2 = bx - 3y - z, \; F_3 = 4x + cy + 2z$

$$\therefore \text{curl } \overline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k$$

$$= (c+1)\,i + (a-4)\,j + (b-2)\,k = 0i + 0j + 0k$$

$$\therefore c + 1 = 0, \quad a - 4 = 0, \quad b - 2 = 0$$

$$\therefore a = 4, \; b = 2, \; c = -1.$$

Example 4 : Is $\overline{F} = \dfrac{\overline{a} \times \overline{r}}{r^n} a$, solenoidal vector ? ($\overline{a}$ is a constant vector). (M.U. 1997)

Sol. : By data

$$\overline{F} = \frac{\overline{a} \times \overline{r}}{r^n} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x/r^n & y/r^n & z/r^n \end{vmatrix}$$

$$\therefore \overline{F} = i\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) + j\left(\frac{a_3 x}{r^n} - \frac{a_1 z}{r^n}\right) + k\left(\frac{a_1 y}{r^n} - \frac{a_2 x}{r^n}\right)$$

$$\text{Now, } \nabla \cdot \overline{F} = \frac{\partial}{\partial x}\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) + \frac{\partial}{\partial y}\left(\frac{a_3 x}{r^n} - \frac{a_1 z}{r^n}\right) + \frac{\partial}{\partial z}\left(\frac{a_1 y}{r^n} - \frac{a_2 x}{r^n}\right)$$

$$\text{Now, } \frac{\partial}{\partial x}\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) = (a_2 z - a_3 y)\frac{\partial}{\partial x} r^{-n} = (a_2 z - a_3 y)\left(-n r^{-n-1}\frac{\partial r}{\partial x}\right)$$

$$= (a_2 z - a_3 y)\left(-n r^{-n-1}\frac{x}{r}\right) = (a_2 z - a_3 y)\left(\frac{-nx}{r^{n+2}}\right)$$

$$\left(\because r^2 = x^2 + y^2 + z^2 \quad \therefore 2r\frac{\partial r}{\partial x} = 2x\right)$$

By symmetry, we get two more expressions.

$$\therefore \quad \nabla \cdot \bar{F} = \frac{n}{r^{n+2}}(a_3xy - a_2xz + a_1zy - a_3xy + a_2xz - a_1yz) = 0$$

Hence, $\bar{F}$ is solenoidal.

Example 5 : If $\bar{r}$ is the position vector of a point (x, y, z) and r is the modulus of $\bar{r}$ then prove that $r^n\bar{r}$ is an irrotational vector for any value of n but is solenoidal only if $n = -3$.

(M.U. 2002, 03, 06)

Sol. : (a) By definition

$$\text{curl } r^n\bar{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

$$= i\left[z n r^{n-1}\frac{\partial r}{\partial y} - y n r^{n-1}\frac{\partial r}{\partial z}\right] + j\left[\ldots\ldots\right] + k\left[\ldots\ldots\right]$$

Now, $r^2 = x^2 + y^2 + z^2 \quad \therefore \; 2r\frac{\partial r}{\partial x} = 2x \quad \therefore \; \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \quad \text{curl } r^n\bar{r} = i[n r^{n-2} zy - n r^{n-2} zy] + j[\ldots] + k[\ldots] = \bar{0}$$

Hence, $r^n\bar{r}$ is irrotational for any value of n.

(b) $\text{div}(r^n\bar{r}) = \nabla \cdot (r^n xi + r^n yj + r^n zk)$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= \left[r^n + xnr^{n-1}\frac{\partial r}{\partial x}\right] + \left[\ldots\right] + \left[\ldots\right]$$

$$= \left[r^n + nxr^{n-1}\frac{x}{r}\right] + \left[r^n + nyr^{n-1}\frac{y}{r}\right] + \left[r^n + nzr^{n-1}\frac{z}{r}\right]$$

$$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{n-2} \cdot r^2 + nr^n = (n+3)r^r.$$

Hence, $\text{div}(r^n\bar{r}) = 0$ if $n = -3$.

Example 6 : Find $f(r)$, so that the vector $f(r)\bar{r}$ is both solenoidal and irrotational.

(M.U. 1996, 2003, 05, 09)

Sol. : (a) We have $f(r)\bar{r} = f(r)xi + f(r)yj + f(r)zk$

$$\text{div}[f(r)\bar{r}] = \nabla \cdot f(r)\bar{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot [f(r)xi + f(r)yj + f(r)zk]$$

$$= \frac{\partial}{\partial x}[f(r)x] + \frac{\partial}{\partial y}[f(r)y] + \frac{\partial}{\partial z}[f(r)z]$$

Now, $\dfrac{\partial f(r)}{\partial x} = f'(r)\dfrac{\partial r}{\partial x} = f'(r)\dfrac{x}{r}$

Similarly, $\dfrac{\partial}{\partial y} f(r) = f'(r)\dfrac{y}{r}$, $\dfrac{\partial f(r)}{\partial z} = f'(r)\dfrac{z}{r}$

$\therefore \dfrac{\partial}{\partial x}[f(r)\, x] = x\dfrac{\partial}{\partial x} f(r) + f(r) = x\dfrac{f'(r)}{r}\cdot x + f(r)$

$\therefore$ div $[f(r)\,\bar{r}] = f'(r)\dfrac{x}{r}\cdot x + f(r) + f'(r)\dfrac{y}{r}\cdot y + f(r) + f'(r)\dfrac{z}{r}\cdot z + f(r)$

$$= 3f(r) + f'(r)\cdot\frac{1}{r}[x^2 + y^2 + z^2] = 3f(r) + f'(r)\, r$$

If $f(r)\bar{r}$ is solenoidal

div $[f(r)\,\bar{r}] = 3f(r) + f'(r)\, r = 0 \qquad \therefore \dfrac{f'(r)}{f(r)} = -\dfrac{3}{r}$

Integrating $\log f(r) = -3\log r + \log c$

$\therefore \log f(r) = \log\dfrac{c}{r^3} \qquad \therefore f(r) = \dfrac{c}{r^3}$

Thus, $f(r)\,\bar{r}$ is solenoidal if $f(r) = \dfrac{c}{r^3}$.

(b) Now, curl $[f(r)\,\bar{r}] = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y} z f(r) - \frac{\partial}{\partial z} y f(r)\right] + j[\,.....\,] + k[\,.....\,]$$

$$= i\left[z f'(r)\cdot\frac{y}{r} - y\cdot f'(r)\frac{z}{r}\right] + j[\,.....\,] + k[\,.....\,]$$

$$= f'(r)\, i\left[\frac{zy}{r} - \frac{zy}{r}\right] + j[\,.....\,] + k[\,.....\,] = 0$$

Thus, $f(r)\,\bar{r}$ is irrotational if for any $f(r)$.

Hence, $f(r)\,\bar{r} = \dfrac{c}{r^3}\bar{r}$ is both solenoidal and irrotational.

Note ....

Note the similarity and difference between the above two examples 5 and 6. Also compare them with the following Example 7.

Example 7 : Prove that $\bar{F} = \dfrac{\bar{r}}{r^3}$ is both irrotational and solenoidal. **(M.U. 2002, 04, 14)**

Sol. : As above.

Example 8 : Show that $\bar{F} = \dfrac{\bar{r}}{r^2}$ is irrotational.

Find F such that $\bar{F} = -\nabla\Phi$ where $\bar{r} = xi + yj + zk$. **(M.U. 2004)**

Sol. : We have $\text{curl}\ \dfrac{\bar{r}}{r^2} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{r^2} & \dfrac{y}{r^2} & \dfrac{z}{r^2} \end{vmatrix}$

$$\therefore \quad \text{curl}\ \frac{\bar{r}}{r^2} = \left[\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right)\right] i + \left[\ldots..\right] j + \left[\ldots..\right] k$$

$$= \left(-\frac{2}{r^3}\frac{yz}{r} + \frac{2}{r^3}\frac{z}{r}\cdot y\right) i + \left(\ldots..\right) j + \left(\ldots..\right) k$$

$$= 0i + 0j + 0k = \bar{0} \quad \therefore \ \bar{F} \text{ is irrotational.}$$

Now, $\bar{F} = -\nabla\Phi$ gives $\dfrac{\bar{r}}{r^2} = \dfrac{xi + yj + zk}{r^2} = -\left[\dfrac{\partial \Phi}{\partial x} i + \dfrac{\partial \Phi}{\partial y} j + \dfrac{\partial \Phi}{\partial z} k\right]$.

$$\therefore \quad \frac{\partial \Phi}{\partial x} = -\frac{x}{r^2} = -\frac{x}{x^2 + y^2 + z^2}; \quad \frac{\partial \Phi}{\partial y} = -\frac{y}{r^2} = -\frac{y}{x^2 + y^2 + z^2};$$

$$\frac{\partial \Phi}{\partial z} = -\frac{z}{r^2} = -\frac{z}{x^2 + y^2 + z^2}.$$

But $d\Phi = \dfrac{\partial \Phi}{\partial x} dx + \dfrac{\partial \Phi}{\partial y} dy + \dfrac{\partial \Phi}{\partial z} dz = -\dfrac{xdx + ydy + zdz}{x^2 + y^2 + z^2}$

By integration, $\Phi = -\dfrac{1}{2}\log(x^2 + y^2 + z^2)$.

(B) To Find The Scalar Potential

Example 1 : A vector field is given by $\bar{F} = (x^2 + xy^2)\, i + (y^2 + x^2 y)\, j$. Show that $\bar{F}$ is irrotational and find its scalar potential. **(M.U. 2004, 17)**

Sol. : We have $\text{curl}\ \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2 + x^2 y)\right] i - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 + xy^2)\right] j + \left[\frac{\partial}{\partial x}(y^2 + x^2 y) - \frac{\partial}{\partial y}(x^2 + xy^2)\right] k$$

$$= 0\,i + 0\,j + (2xy - 2xy)\,k = 0\,i + 0\,j + 0\,k$$

Hence, $\bar{F}$ is irrotational.

If Φ is the scalar potential then $\bar{F} = \nabla\Phi$.

$$\therefore \quad (x^2 + xy^2)\, i + (y^2 + x^2 y)\, j + 0\, k = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial y} k$$

$$\therefore \quad \frac{\partial \Phi}{\partial x} = x^2 + xy^2 \quad \ldots\ldots\ldots. (1) \qquad \frac{\partial \Phi}{\partial y} = y^2 + x^2 y \quad \ldots\ldots\ldots. (2)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \ldots\ldots\ldots. (3)$$

But $d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$

$$= \left[x^2 + xy^2\right] dx + \left[y^2 + x^2 y\right] dy + 0\, dz$$

$$= x^2\, dx + y^2\, dy + (xy^2\, dx + x^2 y\, dy)$$

By integration $\Phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{1}{2} x^2 y^2 + c$ where c is the constant of integration.

Aliter : Integrating (1), (2), (3) w.r.t. x, y, z respectively treating the other variables constant, we get,

$$\Phi = \frac{x^3}{3} + \frac{x^2 y^2}{2} + \Psi_1(y, z); \quad \Phi = \frac{y^3}{3} + \frac{x^2 y^2}{2} + \Psi_2(x, z); \quad \Phi = \Psi_3(x, y).$$

Comparing these equations, we find that

$$\Psi_1(y, z) = \frac{y^3}{3}, \quad \Psi_2(x, z) = \frac{x^3}{3}, \quad \Psi_3(x, y) = 0.$$

$$\therefore \ \Phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2 y^2}{2} + c.$$

Example 2 : A vector field $\overline{F}$ is given by

$$\overline{F} = (y \sin z - \sin x)\, i + (x \sin z + 2yz)\, j + (xy \cos z + y^2)\, k$$

Prove that it is irrotational and hence, find its scalar potential. **(M.U. 2013)**

Sol. : We have

$$\text{curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy \cos z + y^2) - \frac{\partial}{\partial z}(x \sin z + 2yz)\right] i$$

$$+ \left[\frac{\partial}{\partial z}(y \sin z - \sin x) - \frac{\partial}{\partial x}(xy \cos z + y^2)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(x \sin z + 2yz) - \frac{\partial}{\partial y}(y \sin z - \sin x)\right] k$$

$$= [x \cos z + 2y - x \cos z - 2y]\, i + [y \cos z - y \cos z]\, j + [\sin z - \sin z]\, k$$

$$= 0\, i + 0\, j + 0\, k = \overline{0}$$

Hence, $\overline{F}$ is irrotational.

If Φ is the scalar potential then $\overline{F} = \nabla \Phi$.

$$\therefore \ (y \sin z - \sin x)\, i + (x \sin z + 2yz)\, j + (xy \cos z + y^2)\, k = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k$$

$$\therefore \ \frac{\partial \Phi}{\partial x} = y \sin z - \sin x \quad \text{.......... (1)} \qquad \frac{\partial \Phi}{\partial y} = x \sin z + 2yz \quad \text{.......... (2)}$$

$$\frac{\partial \Phi}{\partial z} = xy \cos z + y^2 \quad \text{.......... (3)}$$

But $d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$

$= (y \sin z - \sin x) dx + (x \sin z + 2yz) dy + (xy \cos z + y^2) dz$

$= [y \sin z\, dx + x \sin z dy + xy \cos z dz] + (-\sin x) dx + (2yz dy + y^2 dz)$

By integration, $\Phi = xy \sin z + \cos x + y^2 z + c$ where c is the constant of integration.

Aliter : Integrating (1), (2) and (3) partially w.r.t. x, y, z treating other variables constant, we get,

$$\Phi = (xy \sin z + \cos x) + \Psi_1(y, z) \quad \text{.................... (4)}$$

$$\Phi = (xy \sin z + y^2 z) + \Psi_2(x, z) \quad \text{.................... (5)}$$

$$\Phi = (xy \sin z + y^2 z) + \Psi_3(x, y) \quad \text{.................... (6)}$$

Comparing (4), (5) and (6), we find that

$$\Psi_1(y, z) = y^2 z,\ \Psi_2(x, z) = \cos x,\ \Psi_3(x, y) = \cos x$$

$$\therefore\ \Phi = xy \sin z + \cos x + y^2 z + c.$$

Example 3 : Prove that $\overline{F} = (z^2 + 2x + 3y)\, i + (3x + 2y + z)\, j + (y + 2zx)\, k$ is irrotational and find its scalar potential Φ if $\Phi(1, 1, 0) = 4$. **(M.U. 2016)**

Sol. : We have

$$\text{curl } \overline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 + 2x + 3y & 3x + 2y + z & y + 2xz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(y + 2zx) - \frac{\partial}{\partial z}(3x + 2y + z)\right] i - \left[\frac{\partial}{\partial x}(y + 2zx) - \frac{\partial}{\partial z}(z^2 + 2x + 3y)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(3x + 2y + z) - \frac{\partial}{\partial y}(z^2 + 2x + 3y)\right] k$$

$$= (1 - 1)\, i - (2z - 2z)\, j + (3 - 3)\, k$$

$$= 0i + 0j + 0k = \overline{0}$$

Hence, $\overline{F}$ is irrotational.

If Φ is the scalar potential then $\overline{F} = \nabla\Phi$.

$$\therefore\ (z^2 + 2x + 3y)\, i + (3x + 2y + z)\, j + (y + 2zx)\, k = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k$$

$$\therefore\ \frac{\partial \Phi}{\partial x} = z^2 + 2x + 3y;\quad \frac{\partial \Phi}{\partial y} = 3x + 2y + z;\quad \frac{\partial \Phi}{\partial z} = y + 2zx.$$

But $d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$

$= (z^2 + 2x + 3y)\, dx + (3x + 2y + z)\, dy + (y + 2zx)\, dz$

$= z^2 dx + 2x\, dx + 3y\, dx + 3x\, dy + 2y\, dy + z\, dy + y\, dz + 2zx\, dz$

$= 2x\, dx + 2y\, dy + (z^2 dx + 2zx\, dz) + 3(y\, dx + x\, dy) + (z\, dy + y\, dz)$

By Integration, we get

$\therefore\ \Phi = x^2 + y^2 + z^2 x + 3xy + zy + c$ where c is the constant of integration.

Now, $\Phi(1, 1, 0) = 4$. Hence, putting $x = 1$, $y = 1$ and $z = 0$, we get

$\Phi(1, 1, 0) = 1 + 1 + 0 + 3 + 0 + c = 4 \quad \therefore\ 5 + c = 4 \quad \therefore\ c = -1.$

$\therefore\ \Phi = x^2 + y^2 + z^2 x + 3xy + zy - 1.$

EXERCISE - V

(A) To Show That A Vector is Solenoidal or Irrotational

1. Show that the vector $\overline{F} = (z + \sin y)\, i + (x \cos y - z)\, j + (x - y)\, k$ is irrotational.
2. Show that the vector $\overline{F} = (y + z)\, i + (z + x)\, j + (x + y)\, k$ is solenoidal.
3. Show that the vector $\overline{F} = \dfrac{xi + yj}{x^2 + y^2}$ is solenoidal.
4. Determine the constant a, so that the vector $\overline{F} = (x + 3y^2)\, i + (2y + 2z^2)\, j + (x^2 + az)\, k$ is solenoidal. [**Ans. :** $a = -3$]
5. Show that the vector $\overline{F} = yzi + zxj + xyk$ is solenoidal.
6. Show that the vector $\overline{F} = \dfrac{-yi + xj}{x^2 + y^2}$ is irrotational.
7. If the vector $(ax + 3y + 4z)\, i + (x - 2y + 3z)\, j + (3x + 2y - z)\, k$ is solenoidal, determine the constant a. [**Ans. :** $a = 3$]
8. Show that if $(xyz)^b (x^a i + y^a j + z^a k)$ is an irrotational vector then either $b = 0$ or $a = -1$. **(M.U. 1995)**

(B) To Find The Scalar Potential

1. Show that the velocity given by $\overline{V} = (y + z)\, i + (z + x)\, j + (x + y)\, k$ is irrotational and find its scalar potential. [**Ans. :** $\Phi = yz + zx + xy$]
2. Is the above motion possible for an incompressible fluid ? [**Ans. :** Fluid motion is possible because $\nabla \cdot \overline{V} = 0$.]
3. Show that the vector, $\overline{F} = (x^2 - yz)\, i + (y^2 - zx)\, j + (z^2 - xy)\, k$ is irrotational and hence, find Φ such that $\overline{F} = \nabla\Phi$. [**Ans. :** $\Phi = \dfrac{x^3}{3} + \dfrac{y^3}{3} + \dfrac{z^3}{3} - xyz$]
4. Show that the vector $\overline{F} = (6xy + z^3)\, i + (3x^2 - z)\, j + (3xz^2 - y)\, k$ is irrotational. Find the function Φ such that $\overline{F} = \nabla\Phi$. [**Ans. :** $\Phi = 3x^2 y + z^3 x - yz$]
5. Show that the vector $\overline{F} = (y^2 \cos x + z^3)\, i + (2y \sin x - 4)\, j + (3xz^2 + 2)\, k$ is irrotational and find its scalar potential. [**Ans. :** $\Phi = y^2 \sin x + z^3 x - 4y + 2z$]
6. Show that the vector field given by $\overline{F} = 2xyz^3\, i + x^2 z^3\, j + 3x^2 yz\, k$ is irrotational and find its scalar potential. [**Ans. :** $\Phi = x^2 yz^3$]
7. Prove that $\overline{F} = (z^2 + 2x + 3y)\, i + (3x + 2y + z)\, j + (y + 2zx)\, k$ is irrotational and find scalar potential function Φ such that $\overline{F} = \nabla\Phi$ and $\Phi(1, 1, 0) = 4$. Hence, find the workdone by $\overline{F}$ in moving a particle from $A(0, 1, 1)$ to $B(3, 0, 2)$. **(M.U. 2002, 03, 16)** [**Ans. :** $\Phi = x^2 + y^2 + z^2 x + 3xy + zy - 1$; 19]

8. Show that $\bar{F} = (y^2 - z^2 + 3yz - 2x)\, i + (3xz + 2xy)\, j + (3xy - 2xz + 2z)\, k$ is both solenoidal and irrotational. (M.U. 2005)

9. If $\nabla\Phi = (y^2 - 2xyz^3)\, i + (3 + 2xy - x^2z^3)\, j + (6z^3 - 3x^2yz^2)\, k$, find Φ where $\Phi(1, 0, 1) = 8$. (M.U. 2005) [**Ans. :** $xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 - \frac{13}{2}$]

EXERCISE - VI

Theory

1. Define div. and curl of a vector point function. (M.U. 2002)

2. Define the gradient of a scalar point function Φ and prove that the directional derivative of Φ is maximum in the direction of grad Φ. (M.U. 2001)

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