Residual Reachability Constraints in a Finite Deterministic Spin–Phase System

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Abstract

We investigate a minimal deterministic dynamical system combining a binary spin variable and a cyclic phase variable. Despite its simplicity and complete determinism, the system exhibits nontrivial residual reachability constraints under coarse-grained observation. For a fixed observation window, we define residuals capturing phase and spin deviations over time and introduce a compressed residual representation. We prove that, starting from a given initial state, not all compressed residuals are reachable. In particular, for a system defined on $\mathbb{Z}_2 \times \mathbb{Z}_5$ and a 10-step observation window, exactly 14 out of 16 compressed residuals are reachable. The remaining two are excluded by a number-theoretic coupling between modular arithmetic in $\mathbb{Z}_5$ and parity constraints in $\mathbb{Z}_2$. Our results demonstrate that strict reachability constraints can arise in finite, deterministic systems without stochasticity, noise, or asymptotic limits.

1 Introduction

Markovian and coarse-grained descriptions play a central role in the modeling of dynamical systems across physics, computer science, and applied mathematics. Such descriptions often assume that macroscopic observables evolve without hidden constraints, or that any apparent non-Markovian behavior can be eliminated by augmenting the state space.

In this work, we present a minimal counterexample to this intuition. We construct a finite, deterministic system in which coarse-grained residual observables exhibit strict reachability constraints that cannot be removed by local rule selection. Our system contains no stochastic elements, no continuous limits, and no asymptotic approximations. All results are exact and can be exhaustively verified.

The goal of this paper is not to propose a universal model, but to demonstrate that even extremely simple deterministic systems may possess nontrivial residual structure imposed by modular arithmetic and parity constraints.

2 Definition of the KFX-R System

Definition 1 (State Space)**.** *The state space is defined as*

$$\mathcal{S} = \mathbb{Z}_2 \times \mathbb{Z}_5,$$

where $b \in \mathbb{Z}_2$ represents a binary spin variable and $p \in \mathbb{Z}_5$ represents a cyclic phase variable.

Definition 2 (Update Rules)**.** *We define two deterministic update rules:*

$$R_1(b, p) = (b, p + 1),$$
$$R_2(b, p) = (1 - b, p - 1),$$

where arithmetic on p is taken modulo 5.

A trajectory of length n is generated by applying a sequence $(r_1, \ldots, r_n)$ with $r_i \in \{R_1, R_2\}$ to an initial state (b_0, p_0).

3 Residual Observables

Let (b_t, p_t) denote the system state at time t.

Definition 3 (Residuals)**.** *Given an initial state (b_0, p_0), define*

$$\varepsilon_5^p = p_5 - p_0 \pmod 5,$$
$$\varepsilon_{10}^p = p_{10} - p_0 \pmod 5,$$
$$\varepsilon_{10}^b = b_{10} - b_0 \pmod 2.$$

To facilitate coarse-grained analysis, we introduce a compressed residual representation that records only whether each residual component is zero or nonzero. This yields a binary vector in $\{0, 1\}^4$.

4 Main Result

We now state the central result of this paper.

Theorem 1 (Residual Unreachability)**.** *Consider the KFX-R system starting from the initial state $(0, 0)$. If*

$$\varepsilon_5^p \equiv 1 \pmod 5 \quad \textit{and} \quad \varepsilon_{10}^p \equiv 0 \pmod 5,$$

then necessarily

$$\varepsilon_{10}^b \equiv 1 \pmod 2.$$

Consequently, compressed residuals of the form $(*, 1, 0, 0)$ are unreachable.

Proof. Each application of R_1 contributes $+1$ to the phase and leaves the spin unchanged, while each application of R_2 contributes -1 to the phase and flips the spin.

Let a be the number of applications of R_2 in the first 5 steps, and b the number in the remaining 5 steps.

The phase residual after 5 steps is

$$\varepsilon_5^p \equiv 5 - 2a \pmod 5.$$

The condition $\varepsilon_5^p \equiv 1$ implies $a \equiv 2 \pmod 5$, hence $a = 2$.

The total phase residual after 10 steps is

$$\varepsilon_{10}^p \equiv 10 - 2(a + b) \pmod 5.$$

The condition $\varepsilon_{10}^p \equiv 0$ implies $a + b \equiv 0 \pmod 5$, hence $b = 3$.

The total number of applications of R_2 is $a + b = 5$, which is odd. Since each R_2 flips the spin, we conclude that

$$\varepsilon_{10}^{b} \equiv 1 \pmod 2.$$

This contradicts $\varepsilon_{10}^{b} \equiv 0$, establishing the result. □

Corollary 1 (Exact Reachability Count)**.** *Out of the* 16 *possible compressed residuals, exactly* 14 *are reachable from the initial state* $(0, 0)$ *in* 10 *steps.*

5 Generalizations and Extensions

The reachability constraint proven above arises from a number-theoretic coupling between modular arithmetic in $\mathbb{Z}_5$ and parity in $\mathbb{Z}_2$. This section outlines natural generalizations of the system and associated conjectures.

5.1 General Phase Modulus

Let $m \geq 3$ be an odd integer. Define the generalized state space

$$\mathcal{S}_m = \mathbb{Z}_2 \times \mathbb{Z}_m,$$

with update rules identical in form to those of the KFX-R system.

The same reasoning shows that residual reachability is governed entirely by congruence constraints on the number of applications of R_2 in each segment of the trajectory.

5.2 General Window Length

Let $n = 2k$ be an even observation window. Define residuals

$$\varepsilon_k^p = p_k - p_0 \pmod m, \quad \varepsilon_{2k}^p = p_{2k} - p_0 \pmod m, \quad \varepsilon_{2k}^b = b_{2k} - b_0 \pmod 2.$$

Let a and b denote the number of applications of R_2 in the first and second halves of the trajectory. Then

$$\begin{aligned}
\varepsilon_k^p &\equiv k - 2a \pmod m, \\
\varepsilon_{2k}^p &\equiv 2k - 2(a + b) \pmod m, \\
\varepsilon_{2k}^b &\equiv a + b \pmod 2.
\end{aligned}$$

5.3 Parity–Phase Coupling

Conjecture 1 (Parity–Phase Coupling Constraint)**.** *Let m be odd and $n = 2k$. Suppose a trajectory satisfies*

$$\varepsilon_k^p \equiv \delta \pmod m, \quad \varepsilon_{2k}^p \equiv 0 \pmod m.$$

Then the parity of ε_{2k}^b is uniquely determined by δ. In particular, for certain values of δ, residuals with $\varepsilon_{2k}^b \equiv 0$ are unreachable.

Theorem 1 corresponds to the special case $m = 5$, $k = 5$, and $\delta = 1$.

6 Discussion

The KFX-R system demonstrates that strict reachability constraints may arise in finite, deterministic systems purely from modular arithmetic and parity considerations. No randomness, noise, or asymptotic arguments are involved.

Although physical analogies such as spin–phase coupling or topological constraints may provide intuition, the results of this paper are entirely combinatorial in nature.

7 Conclusion

We have presented a minimal deterministic system exhibiting nontrivial residual reachability constraints. By combining exact enumeration with modular arithmetic analysis, we proved that certain coarse-grained residual configurations are structurally forbidden.

This work highlights how simple deterministic rules can give rise to hidden constraints at the residual level, suggesting new directions for the study of coarse-grained dynamics in finite systems.

References