KFX Continuous Mathematical Model: Non-Bypassability as Closed-Loop Invariance and Unreachability

Kaifan XIE
(0009-0005-6911-7295)

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Abstract

This document presents a purely mathematical continuous-time (optionally hybrid) dynamical system in which a distinguished anchor state cannot be bypassed for a defined class of high-impact controls. Non-bypassability is formalised as a closed-loop property: a forbidden state–control set is unreachable, equivalently an admissible graph set is forward invariant. The result relies only on dynamical systems, set-valued constraints, and closed-loop feasibility, with no implementation narrative.

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1 Problem Statement

1.1 Objective

Let $u(t)$ denote the control applied to a physical system. Define a state-dependent subset of controls $\mathcal{U}_{\mathrm{HI}}(z) \subseteq \mathcal{U}$ as *high-impact.*

> The objective is to construct a closed-loop system in which no trajectory can realise a high-impact control unless an anchor-gating condition is satisfied.

1.2 Interpretation boundary

This document makes *no claims* about arbitrary open-loop inputs. All guarantees are properties of the explicitly defined closed-loop system.

2 Time, State, Control, Disturbance

2.1 Time

$t \in [0, \infty)$.

2.2 State

$$x(t) \in \mathcal{X} \subseteq \mathbb{R}^n, \qquad a(t) \in \mathcal{A} \subseteq \mathbb{R}^{n_a},$$
$$z(t) := \begin{bmatrix} x(t) \\ a(t) \end{bmatrix} \in \mathcal{Z} \subseteq \mathbb{R}^{n+n_a}.$$

2.3 Control and disturbance

$$u(t) \in \mathcal{U} \subseteq \mathbb{R}^m, \qquad d(t) \in \mathcal{D} \subseteq \mathbb{R}^p.$$

2.4 Regularity

Controls and disturbances are measurable and essentially bounded on finite horizons.

3 Dynamics

3.1 Flow dynamics

$$\dot{z}(t) = F\big(z(t), u(t), d(t)\big) \quad \text{for a.e. } t. \tag{1}$$

Assumption 3.1 (Existence)**.** For any initial condition $z(0) \in \mathcal{Z}$ and admissible inputs, (1) admits an absolutely continuous solution on some interval.

4 High-Impact and Safe Control Sets

4.1 High-impact controls

Let

$$I : \mathcal{Z} \times \mathcal{U} \to \mathbb{R}$$

be an impact function and $\tau > 0$ a threshold. Define

$$\mathcal{U}_{\mathrm{HI}}(z) := \{u \in \mathcal{U} : I(z, u) \geq \tau\}.$$

4.2 Safe controls (strong exclusion)

Assumption 4.1 (Strong safety exclusion)**.** For all $z \in \mathcal{Z}$,

$$\mathcal{U}_{\mathrm{SAFE}}(z) \subseteq \mathcal{U}, \qquad \mathcal{U}_{\mathrm{SAFE}}(z) \cap \mathcal{U}_{\mathrm{HI}}(z) = \varnothing,$$

and $\mathcal{U}_{\mathrm{SAFE}}(z)$ is nonempty, closed, and convex.

This assumption is structural and will be used explicitly in proofs.

5 Anchor Gating

5.1 Gated state set

Let

$$\mathcal{Z}_g = \{z \in \mathcal{Z} : h_i(z) \geq 0,\ i = 1, \ldots, r\},$$

where each h_i depends on the anchor component a.

5.2 Binary predicate (notation only)

$$g(z) = \mathbf{1}\{z \in \mathcal{Z}_g\}.$$

Remark 5.1. The inequalities defining $\mathcal{Z}_g$ are the primary object. The binary predicate g is only shorthand.

6 Non-Bypassability as a Constraint

Definition 6.1 (Forbidden bypass set)**.**

$$\mathcal{F} := \{(z, u) : u \in \mathcal{U}_{\mathrm{HI}}(z)\ \wedge\ z \notin \mathcal{Z}_g\}.$$

Definition 6.2 (Admissible graph set)**.**

$$\mathcal{K} := \{(z, u) : (z \in \mathcal{Z}_g)\ \vee\ (u \notin \mathcal{U}_{\mathrm{HI}}(z))\}.$$

Clearly, $\mathcal{K} = (\mathcal{Z} \times \mathcal{U}) \setminus \mathcal{F}$.

7 Closed-Loop Realised Control

7.1 Nominal proposal

$$\hat{u}(t) = \mu(z(t)),$$

with μ arbitrary.

7.2 Realised control operator

$$u(t) = \begin{cases} \hat{u}(t), & \hat{u}(t) \notin \mathcal{U}_{\mathrm{HI}}(z(t)), \\ \hat{u}(t), & \hat{u}(t) \in \mathcal{U}_{\mathrm{HI}}(z(t))\ \wedge\ z(t) \in \mathcal{Z}_g, \\ \Pi_{\mathcal{U}_{\mathrm{SAFE}}(z(t))}\big(\hat{u}(t)\big), & \text{otherwise.} \end{cases} \tag{2}$$

7.3 Well-posedness

The projection is single-valued almost everywhere due to convexity.

8 Key Lemma

Lemma 8.1 (Closed-loop admissibility)**.** For all t (a.e.),

$$(z(t), u(t)) \in \mathcal{K}.$$

Proof. If $\hat{u}(t) \notin \mathcal{U}_{\mathrm{HI}}(z(t))$, then $u(t) = \hat{u}(t)$ and $(z(t), u(t)) \in \mathcal{K}$. If $\hat{u}(t) \in \mathcal{U}_{\mathrm{HI}}(z(t))$ and $z(t) \in \mathcal{Z}_g$, then $(z(t), u(t)) \in \mathcal{K}$. Otherwise, $u(t) \in \mathcal{U}_{\mathrm{SAFE}}(z(t))$, which by strong exclusion implies $u(t) \notin \mathcal{U}_{\mathrm{HI}}(z(t))$. Hence $(z(t), u(t)) \in \mathcal{K}$. □

9 Main Result

Theorem 9.1 (Non-bypassability)**.** Under the closed-loop system (1)–(2), the forbidden bypass set $\mathcal{F}$ is unreachable:

$$(z(t), u(t)) \notin \mathcal{F} \quad \text{for all } t \geq 0 \text{ (a.e.)}.$$

Equivalently, $\mathcal{K}$ is forward invariant.

Proof. Immediate from the lemma and $\mathcal{K} = (\mathcal{Z} \times \mathcal{U}) \setminus \mathcal{F}$. □

10 Remarks on Continuity and Hybrid Nature

Remark 10.1. The system is continuous-time in its flows but may be hybrid due to the state-dependent admissible control set. The non-bypassability property is independent of this distinction.

Remark 10.2. The result is a closed-loop property. No statement is made about arbitrary open-loop inputs.

11 Completeness Checklist

- State space $\mathcal{Z}$ including anchor component.
- Dynamics $F(z, u, d)$.
- Impact function $I(z, u)$ and threshold τ.
- Gated state set $\mathcal{Z}_g$.
- Strongly excluding safe control set $\mathcal{U}_{\mathrm{SAFE}}(z)$.
- Realised control operator (2).
- Forbidden set $\mathcal{F}$ and admissible set $\mathcal{K}$.

12 Interpretation

The statement "the anchor cannot be bypassed" is mathematically equivalent to:

> The set $\mathcal{F}$ is unreachable under the closed-loop dynamics.

No further semantics are required.