The Limit Structure of Goal-Free Evolutionary Formal Systems: From Minimal Single-Orbit Cores to the Inevitability of Targets

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1 Motivation

This paper studies a class of finite-state formal systems in which evolution is generated by local step rules, while legality is not defined by target states, attractors, or optimality criteria. Instead, legality is determined by structural constraints imposed on complete orbits or paths.

We refer to such systems as **goal-free evolutionary systems**.

The central question is:

> Without introducing any target states, how structurally complex can a finite evolutionary system become?

This work presents a stepwise relaxation of structural constraints and proves that **goal-free evolution admits a strict upper limit in allowable structural freedom**.

2 Basic Paradigm: Orbit-Constrained Formal Systems

Definition 1 (Orbit-Constrained Formal System). *An orbit-constrained formal system consists of:*

1. *A finite state set S;*
2. *A collection of local one-step rules $\mathcal{R} \subseteq S \times S$;*
3. *A global legality criterion defined on finite-length paths.*

The world *of the system is the set of all paths satisfying the legality constraints.*

Throughout this paper, **legality is not permitted to reference any designated target state or subset of states**. Only structural properties of entire paths may be used.

3 KFX-β: The Minimal Non-Unique Single-Orbit Core

3.1 Structural Definition

Definition 2 (KFX-β)**.** *Let*

$$S = \{0, 1\} \times \mathbb{Z}_4,$$

and define the phase-advance operator

$$\sigma(b, p) = (b, p + 1 \bmod 4).$$

Two local rules are allowed:

$$\begin{aligned} R_0 &: (b, p) \mapsto (b, p + 1), \\ R_1 &: (b, p) \mapsto (1 - b, p + 1). \end{aligned}$$

*A length-*4 *orbit* $\gamma = (s_0, \ldots, s_4)$ *is defined to be legal if and only if:*

1. $s_4 = s_0$*;*

2. *The number of applications of* R_1 *is even.*

3.2 Properties

Theorem 1. *The number of legal step patterns in KFX-*β *is* 8*, and the system contains no target states or attractor structures.*

This system is:

- Non-unique;
- Goal-free;
- Minimal in state cardinality under the imposed structural constraints.

4 KFX-δ: Generalization via Finite Group Covers

4.1 Structural Generalization

Definition 3 (Group-Cover Goal-Free Core)**.** *Let* G *be a finite group and* $\Gamma \subseteq G$ *a generating set. Define a finite set* S *together with maps*

$$\pi : S \to G, \quad \sigma : S \to S,$$

such that:

1. σ *generates a phase cycle:* $\sigma^n = \mathrm{id}$*;*

2. *For each* $g \in \Gamma$*, there exists a local rule* R_g *satisfying*

$$(s, t) \in R_g \Rightarrow \begin{cases} t = \sigma(s), \\ \pi(t) = \pi(s) \cdot g. \end{cases}$$

4.2 Legality

A length-n orbit is legal if and only if

$$\prod_{i=1}^{n} g_i = e_G.$$

Theorem 2. *The number of legal step patterns is*

$$|\Omega| = |\Gamma|^{n-1}.$$

This result is completely independent of the concrete representation of S and depends only on the group-cover structure.

5 KFX-ε: Direct Sums of Multiple Phase Orbits

5.1 Orbit Decomposition

Definition 4. *Let*

$$S = \bigsqcup_{j=1}^{m} S_j,$$

where each S_j is a disjoint σ-orbit with period n_j.

Local rules and group-cover structure act independently within each orbit.

5.2 World Decomposition

Theorem 3. *The world of the system decomposes strictly as a direct sum:*

$$\mathcal{W} = \bigsqcup_{j=1}^{m} \mathcal{W}_j.$$

The number of legal paths satisfies:

$$|\Omega| = \sum_{j=1}^{m} |\Gamma|^{n_j - 1}.$$

6 KFX-ζ: Sparse Cross-Orbit Transitions

6.1 Jump Rules

Definition 5. *Allow cross-orbit jump rules $J \subseteq S \times S$ satisfying:*

1. $\sigma(t) = \sigma(s)$;
2. $\pi(t) = \pi(s)$;
3. *s and t belong to different orbits.*

Such jumps preserve phase and group value, altering only the orbit label.

6.2 Consequences

Theorem 4. *If nontrivial jump rules exist, then:*

- *Orbit labels cease to be invariants;*
- *The world is no longer decomposable;*
- *The system remains goal-free.*

7 KFX-ω: The Limit of Goal-Free Evolution

7.1 Three Equivalent Structural Violations

Consider any one of the following modifications:

1. Jumps alter the group-cover value;
2. Jumps alter the phase;
3. Different local rules induce different phase advances.

7.2 Inevitability of Targets

Theorem 5 (Inevitability of Targets)**.** *In a finite-state system, if:*

1. *Paths are generated by local rules;*
2. *Legality requires filtering;*
3. *Any of the above structural violations is permitted;*

then there exists an irreducible subset $T \subseteq S$ such that legal paths are structurally biased to pass through or converge toward T.

Such a subset T is formally equivalent to a target state or attractor.

8 Conclusion

Goal-free evolution is not the absence of structure, but the presence of a highly constrained and fragile structural regime. This paper characterizes the complete trajectory from the minimal non-unique core to the precise point at which goal structures become unavoidable.