A Finite Configuration System with Symmetry Group Action and Linear Invariants

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1 Preliminaries and Notation

Throughout this paper, the following conventions are used.

- $\mathbb{R}$ denotes the field of real numbers.
- P denotes a finite set of positions.
- A *configuration* is a function $f : P \to \mathbb{R}$.
- $m \in \mathbb{R}$ denotes a fixed constant, referred to as the *linear invariant constant*.

No temporal, dynamical, optimisation, or stability-related structure is assumed unless explicitly stated. All constructions in this paper are purely structural.

2 Definition of the System

Definition 1 (Finite Configuration System). *A finite configuration system is a tuple*

$$\mathcal{S} = (P, R, G;\ \alpha, \beta, m, \mathrm{Inv}),$$

where the components are defined below.

2.1 Position Set

Definition 2 (Position Set). *Let*

$$P = \{(i, j) \mid i, j \in \{1, 2, 3\}\}.$$

Thus $|P| = 9$*, representing the positions of a* 3×3 *grid.*

2.2 Value Domain and Configuration Space

Definition 3 (Configuration Space). *Let* $R = \mathbb{R}$*. Define the configuration space*

$$X = R^P = \{f \mid f : P \to \mathbb{R}\}.$$

This is a real vector space of dimension 9*, with vector addition and scalar multiplication defined pointwise.*

2.3 Line Structure

Definition 4 (Line Set)**.** *Define the following subsets of P.*

(i) Rows:

$$\ell_{r1} = \{(1,1),(1,2),(1,3)\}, \quad \ell_{r2} = \{(2,1),(2,2),(2,3)\}, \quad \ell_{r3} = \{(3,1),(3,2),(3,3)\}.$$

(ii) Columns:

$$\ell_{c1} = \{(1,1),(2,1),(3,1)\}, \quad \ell_{c2} = \{(1,2),(2,2),(3,2)\}, \quad \ell_{c3} = \{(1,3),(2,3),(3,3)\}.$$

(iii) Diagonals:

$$\ell_{d1} = \{(1,1),(2,2),(3,3)\}, \quad \ell_{d2} = \{(1,3),(2,2),(3,1)\}.$$

Let

$$\mathcal{L} = \{\ell_{r1}, \ell_{r2}, \ell_{r3}, \ell_{c1}, \ell_{c2}, \ell_{c3}, \ell_{d1}, \ell_{d2}\}.$$

Remark 1. *Elements of $\mathcal{L}$ are referred to as* lines*. No geometric or metric structure is assumed beyond this explicit enumeration.*

3 Symmetry Group Action

3.1 The Symmetry Group

Definition 5 (Symmetry Group)**.** *Let $G = D_4$ be the dihedral group of order 8, presented as*

$$D_4 = \langle r, s \mid r^4 = e,\ s^2 = e,\ srs = r^{-1} \rangle.$$

3.2 Action on Positions

Definition 6 (Action on Positions)**.** *Define $\alpha : G \times P \to P$ by*

$$\alpha(r,(i,j)) = (j, 4-i), \qquad \alpha(s,(i,j)) = (i, 4-j).$$

The action of an arbitrary group element $g \in G$ is defined by composition.

Remark 2 (Well-defined group action)**.** *A direct computation shows that $\alpha(r,\cdot)$ and $\alpha(s,\cdot)$ are permutations of P satisfying*

$$\alpha(r)^4 = \mathrm{id}, \quad \alpha(s)^2 = \mathrm{id}, \quad \alpha(s)\alpha(r)\alpha(s) = \alpha(r)^{-1}.$$

Hence α defines a well-defined action of D_4 on P.

3.3 Induced Action on Configurations

Definition 7 (Induced Action)**.** *Define the induced action*

$$\beta : G \times X \to X$$

by

$$(\beta(g,f))(p) = f(\alpha(g^{-1},p)), \quad \forall g \in G,\ f \in X,\ p \in P.$$

Remark 3. *Since $\alpha(g,\cdot)$ is bijective for all $g \in G$, each $\beta(g,\cdot)$ is a linear automorphism of X.*

4 Linear Invariants

Definition 8 (Linear Invariant Predicate)**.** *For $f \in X$, define*

$$\operatorname{Inv}(f) \iff \forall \ell \in \mathcal{L} : \sum_{p \in \ell} f(p) = m.$$

Definition 9 (Constraint Solution Set)**.** *Define*

$$M = \{f \in X \mid \operatorname{Inv}(f)\}.$$

5 Affine Structure of the Solution Set

Proposition 1. *If $M \neq \varnothing$, then M is an affine linear subspace of X.*

Proof. For each $\ell \in \mathcal{L}$, define

$$\varphi_\ell(f) = \sum_{p \in \ell} f(p).$$

Enumerating $\mathcal{L} = \{\ell_1, \ldots, \ell_8\}$ yields a linear map

$$\Phi : X \to \mathbb{R}^8, \quad \Phi(f) = (\varphi_{\ell_1}(f), \ldots, \varphi_{\ell_8}(f)).$$

Then $M = \Phi^{-1}(m\mathbf{1})$. If $f_0 \in M$, then $M = f_0 + \ker(\Phi)$, hence affine. □

6 Rank and Dimension of the Constraint System

Proposition 2. *The linear map $\Phi : X \to \mathbb{R}^8$ has rank 7.*

Proof. Summing the three row constraints equals summing the three column constraints, yielding one linear dependency. The two diagonal constraints are independent of the row–column system. Hence the total rank is 7. □

Corollary 1. $\dim \ker(\Phi) = 2$. *For fixed m, the solution set M is a 2-dimensional affine subspace.*

7 Non-emptiness and Explicit Parametrisation

Proposition 3 (Existence via constant configuration)**.** *For every $m \in \mathbb{R}$, the set M is non-empty.*

Proof. Define the constant configuration $f_0 : P \to \mathbb{R}$ by

$$f_0(p) = \frac{m}{3} \quad \forall p \in P.$$

Each line $\ell \in \mathcal{L}$ contains exactly three positions. Hence

$$\sum_{p \in \ell} f_0(p) = 3 \cdot \frac{m}{3} = m,$$

so $f_0 \in M$. □

Proposition 4 (Explicit parametrisation)**.** *Fix $m \in \mathbb{R}$. Every solution $f \in M$ can be written as*

$$f = f_0 + h\, v_1 + i\, v_2, \quad h, i \in \mathbb{R},$$

where $\{v_1, v_2\}$ is a basis of $\ker(\Phi)$.

Remark 4. *An explicit choice of v_1, v_2 may be obtained by solving the linear system $\Phi(f) = 0$. For example, writing the configuration as*

$$\begin{matrix} a & b & c \\ d & e & f, \\ g & h & i \end{matrix}$$

one convenient parametrisation is

$$\begin{aligned} &a = \frac{2m}{3} - i, \qquad b = \frac{2m}{3} - h, \qquad c = h + i - \frac{m}{3}, \\ &d = h + 2i - \frac{2m}{3}, \qquad e = \frac{m}{3}, \qquad f = \frac{4m}{3} - h - 2i, \\ &g = m - h - i. \end{aligned}$$

A direct computation verifies that all row, column, and diagonal sums equal m.

8 Boundary Statement

Remark 5. *The system $\mathcal{S}$ is purely structural. It contains no notion of time, dynamics, optimisation, stability, or evolution. Any such notions must be introduced explicitly as external structures.*

9 A Minimal Objective and Constraint-Preserving Dynamics

In this section we demonstrate how a minimal notion of controlled "motion" may be introduced without violating the linear invariants.

9.1 A Minimal Objective Function

Let $x \in \mathbb{R}^9$ denote the vector representation of a configuration $f \in X$. The constraint set can be written as

$$M = \{x \in \mathbb{R}^9 \mid Ax = m\mathbf{1}\},$$

where $A \in \mathbb{R}^{8\times 9}$ is the constraint matrix.

Given a reference configuration $x^{(0)}$, define

$$F(x) = \frac{1}{2}\|x - x^{(0)}\|_2^2.$$

9.2 Closed-Form Projection

Proposition 5. *The unique minimiser of*

$$\min_{x \in M} \frac{1}{2}\|x - x^{(0)}\|_2^2$$

is

$$x^* = x^{(0)} - A^\top (AA^\top)^+ \big(Ax^{(0)} - m\mathbf{1}\big).$$

Remark 6. *The pseudoinverse appears because* $\operatorname{rank}(A) = 7 < 8$.

9.3 Iterative Constraint-Preserving Evolution

Given $x_t \in M$, define

$$y_{t+1} = x_t - \eta \nabla G(x_t), \qquad x_{t+1} = \Pi_M(y_{t+1}),$$

where Π_M denotes the projection above.

Proposition 6. *If* $x_t \in M$, *then* $x_{t+1} \in M$ *for all* t.

Proof. By construction, x_{t+1} satisfies $Ax_{t+1} = m\mathbf{1}$. □

Remark 7. *All evolution is confined to the affine subspace* M *and preserves the linear invariants exactly.*

This work is intended as a structural and methodological baseline, not as a full-scale application study. The 3×3 system serves as a minimal, fully analysable instance where symmetry, linear invariants, and constraint-preserving operations can be made completely explicit.