An Exponential Lower Bound for Markovian State Augmentation under Minimal Observation

Kaifan XIE

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Abstract

Markovian descriptions are often recovered by augmenting the state space when an observed process exhibits apparent non-Markovian behavior. In this short note, we present a minimal deterministic construction showing that such Markovian recovery may incur an unavoidable exponential cost.

We construct a family of finite deterministic systems together with a binary observation map such that any state augmentation rendering the observed process Markovian must have cardinality exponential in the underlying system size. The result provides a structural lower bound on the complexity of Markovian representations under coarse observation.

1 Introduction

The Markov property plays a central role in the modeling of dynamical systems. When an observed process fails to satisfy first-order Markov closure, a standard response is to enlarge the state space so as to restore Markovianity.

While this strategy is formally valid, its cost is rarely quantified. It is often implicitly assumed that a finite and manageable state augmentation always suffices.

In this note, we show that this assumption fails in a strong sense. We construct a simple class of deterministic finite systems for which any Markovian state augmentation consistent with a minimal observation must have exponentially many states. The result is purely structural and does not rely on stochasticity, noise, or approximation.

2 System Construction

2.1 Microstate Dynamics

Fix an integer $n \geq 1$. Let the microstate space be

$$X = \{0, 1\}^n,$$

with $|X| = 2^n$.

We define a deterministic autonomous dynamics

$$X_{t+1} = F(X_t),$$

where $F : X \to X$ is a permutation consisting of a single cycle of length 2^n. That is, the dynamics visits every state exactly once per period.

Such a permutation can be realized, for example, by identifying X with the set of length-n substrings of a binary de Bruijn sequence and letting F correspond to a one-step shift along the cycle.

2.2 Observation Map

Define a minimal observation

$$Y_t = \pi(X_t) \in \{0, 1\},$$

where $\pi(x)$ returns the last bit of x.

The observed process (Y_t) is thus a deterministic binary sequence induced by the underlying cycle.

3 Predictive Equivalence

Definition 1 (Predictive Equivalence)**.** *Two microstates $x, x' \in X$ are said to be* predictively equivalent *if the future observation sequences generated from them coincide:*

$$\pi(F^k(x)) = \pi(F^k(x')) \quad \textit{for all } k \geq 0.$$

In a deterministic system, predictive equivalence means that the two states are indistinguishable by any possible future observation.

Proposition 1. *For the system constructed above, no two distinct states in X are predictively equivalent.*

Proof. Under the de Bruijn-cycle realization, each state corresponds to a unique position on the cycle. The future observation sequence generated from a state is the infinite suffix of the de Bruijn sequence starting at that position.

Distinct positions on the cycle yield distinct infinite binary sequences. Hence, for $x \neq x'$, there exists $k \geq 0$ such that

$$\pi(F^k(x)) \neq \pi(F^k(x')).$$

□

Thus, the number of predictive equivalence classes is exactly 2^n.

4 Markovian State Augmentation

We now formalize the notion of Markovian augmentation.

Definition 2 (Markovian Augmentation)**.** *A process (S_t) is a Markovian augmentation of (Y_t) if:*

1. *$S_t = g(X_t)$ for some function $g : X \to S$,*
2. *Y_t is a function of S_t,*
3. *(S_t) is first-order Markov and is sufficient to predict the entire future observation sequence $(Y_{t+k})_{k \geq 0}$.*

Remark 1. *In deterministic systems, predictive sufficiency for the full future is equivalent to predictive sufficiency at all finite horizons.*

5 Exponential Lower Bound

Theorem 1 (Exponential Lower Bound)**.** *For the system constructed above, any Markovian augmentation* (S_t) *must satisfy*

$$|S| \geq 2^n.$$

Proof. Assume $|S| < 2^n$. By the pigeonhole principle, there exist distinct $x, x' \in X$ such that $g(x) = g(x')$.

Since (S_t) is sufficient to predict the entire future observation sequence, states with the same S_t value must generate identical future observations. This implies that x and x' are predictively equivalent.

This contradicts Proposition 1. Therefore, $|S| \geq 2^n$. □

6 Numerical Illustration

For small values of n, the construction can be verified explicitly. For each n, we enumerate all 2^n states, generate finite prefixes of the future observation sequence from each state, and count the number of distinct prefixes.

In all tested cases ($n \leq 12$), the number of distinct prefixes equals 2^n, confirming the theoretical lower bound.

7 Discussion

The result shows that while Markovian recovery via state augmentation is always possible in principle, its cost may be exponential in the size of the underlying system.

Importantly, the observed non-Markovian behavior does not reflect any fundamental indeterminism of the system itself. Rather, it reveals a provable lower bound on the complexity required to restore Markovianity under minimal observation.

8 Conclusion

We have presented a minimal deterministic construction demonstrating that Markovian state augmentation under coarse observation can require exponentially many states.

The result clarifies a fundamental limitation of Markovian modeling strategies and places a concrete lower bound on the complexity of state-space augmentation.