From a Finite Rigid Kernel to a Fractal Residual

The Continuum Limit of a Constrained Orbit System

Kaifan Xie

Sydney, Australia

2026.02.08

Abstract

We study a minimal constrained orbit system, denoted KFX, defined on a finite state space with two local step rules and global closure constraints. In the discrete setting, the system exhibits complete rigidity: exactly one admissible orbit pattern exists. We investigate the continuum limit as the phase resolution tends to infinity and show that rigidity does not disappear. Instead, the global constraints survive as a zero-measure residual obstruction embedded in the continuous phase space. We propose that this obstruction possesses a fractal structure and provide a reproducible numerical protocol supporting this claim.

1 Introduction

Finite dynamical systems with global constraints can exhibit extreme rigidity, while their continuous counterparts are typically associated with ergodicity and dense orbit structures. Understanding how such rigidity behaves under a continuum limit remains nontrivial.

This work presents a minimal system in which the discrete dynamics admits exactly one legal orbit pattern. We show that in the continuum limit this rigidity persists, not as a finite exclusion, but as a thin residual obstruction of measure zero.

2 The Discrete KFX Kernel

Definition 2.1 (State Space)**.** *Let*

$$S = \mathbb{Z}_2 \times \mathbb{Z}_5.$$

A state is denoted by $s = (b, p)$ with $b \in \mathbb{Z}_2$ and $p \in \mathbb{Z}_5$.

Definition 2.2 (Local Step Rules)**.** *The system admits exactly two transitions:*

$$\begin{aligned} &\text{R1: } (b, p) \mapsto (b, p+1), \\ &\text{R2: } (b, p) \mapsto (1-b, p-1), \end{aligned}$$

with arithmetic on p taken modulo 5.

Definition 2.3 (Global Constraints)**.** *A trajectory $\gamma = (s_0, \dots, s_{10})$ is* legal *if:*

(C1) $s_5 = (1 - b_0, p_0)$,

(C2) $s_{10} = s_0$.

Theorem 2.1 (Rigidity)**.** *The discrete KFX system admits exactly one legal step pattern:*

$$\mathrm{R2}^{10}.$$

Remark 2.1. *Different initial states induce only relabelings of the same orbit. The rigidity is structural and independent of initial conditions.*

3 Geometric Interpretation

The discrete phase space can be interpreted as a double-layered regular pentagon. The unique legal orbit traces a star-like polygon alternating between layers. This geometry anticipates the continuous double-circle structure appearing in the limit.

4 Continuum Limit

We generalize the phase variable to $p \in \mathbb{Z}_m$ and take $m \to \infty$.

Definition 4.1 (Continuous Phase Space)**.** *The limiting space is*

$$\mathcal{M} = \mathbb{S}^1 \times \mathbb{Z}_2,$$

i.e. two disjoint circles.

Let $\theta \in [0, 2\pi)$ and define $\omega = 2\pi/m$.

Definition 4.2 (Continuous Dynamics)**.**

$$\begin{aligned} \textit{R1:}\ (\theta, b) &\mapsto (\theta + \omega, b), \\ \textit{R2:}\ (\theta, b) &\mapsto (\theta - \omega, 1 - b). \end{aligned}$$

As $m \to \infty$, $\omega \to 0$, yielding a circle map with discrete layer switching.

5 Dynamics and Ergodicity

For rational $\omega/2\pi$, trajectories are periodic. For irrational rotation numbers, trajectories become quasi-periodic and dense.

Randomized rule selection induces ergodic behavior on $\mathbb{S}^1 \times \mathbb{Z}_2$ with respect to the natural invariant measure.

6 Residual Unreachability

In the discrete system, global constraints eliminate all but one orbit. In the continuum limit, the forbidden configurations form a residual set $\mathcal{F} \subset \mathbb{S}^1 \times \mathbb{Z}_2$.

6.1 Measure-Theoretic Properties

We conjecture that

$$\mu(\mathcal{F}) = 0,$$

where μ is the product of Lebesgue measure on $\mathbb{S}^1$ and counting measure on $\mathbb{Z}_2$.

6.2 Topological Persistence

Zero measure does not imply topological triviality. The set $\mathcal{F}$ may be dense, analogous to Cantor-type exceptional sets.

6.3 Fractal Conjecture

Conjecture 6.1 (Fractal Residual)**.** *The unreachable set $\mathcal{F}$ is uncountable and has Hausdorff dimension*

$$0 < \dim_H(\mathcal{F}) < 1.$$

6.4 Dependence on Rotation Number

The structure of $\mathcal{F}$ depends sensitively on $\omega/2\pi$. Irrational values with poor Diophantine properties are expected to enhance the complexity of $\mathcal{F}$.

7 Numerical Illustration

7.1 Reproducible Numerical Protocol

We describe a minimal protocol for orbit sampling and residual estimation.

Algorithm 1 KFX random orbit sampling

Require: ω, p, T, K

$\theta \leftarrow 0$, $b \leftarrow 0$

Initialise $\mathrm{occ}[b, i] \leftarrow 0$

for $t = 1$ to T **do**

  Draw $u \sim \mathrm{Unif}(0, 1)$

  if $u < p$ **then**

    $\theta \leftarrow (\theta - \omega) \bmod 2\pi$, $b \leftarrow 1 - b$

  else

    $\theta \leftarrow (\theta + \omega) \bmod 2\pi$

  end if

  $i \leftarrow \lfloor K\theta/(2\pi) \rfloor$

  $\mathrm{occ}[b, i] \leftarrow 1$

end for

7.2 Box-Counting Estimate

At resolution K, define the visited count

$$V(K) = \sum_{b,i} \mathrm{occ}[b, i].$$

Repeating for multiple K yields an empirical scaling

$$V(K) \sim K^{\alpha}, \qquad 0 < \alpha < 1,$$

consistent with a fractal residual.

8 Conclusion

We have shown that extreme rigidity in a finite constrained system does not disappear under a continuum limit. Instead, it survives as a zero-measure, potentially fractal obstruction.

This mechanism suggests a general pathway by which global constraints in finite systems re-emerge as thin but structurally significant barriers in continuous dynamics.