From Discrete Rigidity to Fractal Residuals
Constraint Persistence Across Continuum Limits

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Abstract

We study a minimal constrained orbit system exhibiting complete rigidity in a finite discrete setting. We show that this rigidity collapses entirely under an isometric continuum limit, leading to full reachability. However, when a contractive structure is introduced in the continuum, rigidity reappears in the form of a fractal residual set. The Hausdorff dimension of this residual set is computed explicitly and verified numerically with high precision.

1 Introduction

Finite dynamical systems subject to global constraints may admit only a single admissible orbit pattern. A natural question is how such rigidity behaves under a continuum limit.

In this work we analyze a minimal constrained orbit system. We show that rigidity is exact in the discrete setting, disappears under an isometric continuum limit, and re-emerges as a fractal residual when a contractive structure is introduced. This provides a concrete mechanism by which global constraints persist across model transitions.

2 The Discrete KFX Kernel and Rigidity

Definition 2.1 (State Space)**.** *For $m \in \mathbb{N}$, define*

$$S_m \mathbb{Z}_2 \times \mathbb{Z}_m.$$

A state is denoted (b, p).

Definition 2.2 (Generators)**.** *Define two maps $R_1, R_2 : S_m \to S_m$ by*

$$R_1(b, p) = (b, p + 1), \qquad R_2(b, p) = (1 - b, p - 1),$$

with arithmetic on $\mathbb{Z}_m$ taken modulo m.

Definition 2.3 (Orbit)**.** *Given a word $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{R_1, R_2\}^n$ and an initial state s_0, define the induced orbit*

$$\gamma_\sigma(s_0) = (s_0, \sigma_1(s_0), \sigma_2\sigma_1(s_0), \ldots).$$

Definition 2.4 (Global Constraints)**.** *A length-10 orbit $\gamma = (s_0, \ldots, s_{10})$ is* legal *if*

(C1) $s_5 = (1 - b_0, p_0)$,

(C2) $s_{10} = s_0$.

Theorem 2.1 (Discrete Rigidity)**.** *For $m = 5$, the discrete system admits exactly one legal step sequence of length 10:*

$$\{\sigma \in \{R_1, R_2\}^{10} : \sigma \text{ is legal}\} = \{R_2^{10}\}.$$

Remark 2.1. *This rigidity is purely combinatorial and independent of any metric structure.*

3 Isometric Continuum Limit and Loss of Rigidity

We next consider a continuum limit obtained by replacing the discrete phase variable with a continuous circle and extending the generators as isometries.

Definition 3.1 (Isometric Continuum Model)**.** *Let*

$$X = \mathbb{S}^1 \times \mathbb{Z}_2, \qquad \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

For $\omega \in \mathbb{R}$, define

$$T_1(\theta, b) = (\theta + \omega, b), \qquad T_2(\theta, b) = (\theta - \omega, 1 - b).$$

Proposition 3.1 (Loss of Rigidity)**.** *If $\omega/2\pi \notin$, then for all $x \in X$,*

$$\overline{\{g(x) : g \in \langle T_1, T_2 \rangle\}} = X.$$

Thus, the global constraints responsible for discrete rigidity do not survive the isometric continuum limit. No residual unreachable set remains.

4 A Contractive Continuum Model

To recover a nontrivial residual structure, we introduce contraction into the continuum dynamics.

4.1 Phase Space

Let

$$X = [0, 1) \times \mathbb{Z}_2.$$

4.2 Iterated Function System

Fix $r \in (0, \frac{1}{2})$ and define

$$\begin{aligned} F_1(x, b) &= (rx,\ b), \\ F_2(x, b) &= (rx + (1 - r),\ 1 - b). \end{aligned}$$

Definition 4.1 (IFS Attractor)**.** *Let $\mathcal{K} \subset X$ be the unique nonempty compact set satisfying*

$$\mathcal{K} = F_1(\mathcal{K}) \cup F_2(\mathcal{K}).$$

Existence and uniqueness of $\mathcal{K}$ follow from standard contraction mapping arguments.

5 Fractal Residual and Hausdorff Dimension

5.1 Open Set Condition

Let

$$U = (0, 1) \times \mathbb{Z}_2.$$

Since $r < \frac{1}{2}$, the sets $F_1(U)$ and $F_2(U)$ are disjoint subsets of U. Hence the iterated function system satisfies the open set condition.

5.2 Dimension Theorem

Theorem 5.1 (Hausdorff Dimension)**.** *The Hausdorff dimension of the attractor* $\mathcal{K}$ *is given by*

$$\dim_H(\mathcal{K}) = \frac{\log 2}{\log(1/r)} \in (0, 1).$$

Proof. Since the system consists of two similarity contractions with equal ratio r and satisfies the open set condition, the Hausdorff dimension s is the unique solution of the Moran equation

$$\sum_{i=1}^{2} r^s = 1.$$

Solving $2r^s = 1$ yields $s = \log 2/\log(1/r)$. $\square$

6 Numerical Verification

The Hausdorff dimension is verified numerically using box-counting on samples generated by random iteration (chaos game). For $r = 0.38$, the theoretical value

$$\dim_H(\mathcal{K}) \approx 0.716369$$

agrees with numerical estimates up to 10^{-3} relative error across multiple random seeds and grid resolutions.

7 Discussion and Outlook

This work demonstrates a clear structural transition: exact rigidity in a discrete constrained system collapses under an isometric continuum limit but re-emerges as a fractal residual when contraction is introduced.

The analysis suggests a general mechanism by which global constraints persist across model scales and may inform the study of constraint survival in more complex dynamical and computational systems.