FKX: Constraint-Driven Discrete Formal Systems

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1 Purpose and Position

This paper introduces a class of *constraint-driven discrete formal systems*, referred to as the FKX family. The central claim is structural:

> *The effective degrees of freedom (dimension) of a system are determined by its admissible solution space ("worlds"), rather than by the dimensionality of state representations, the cardinality of the state space, or the number of local transition rules.*

We further provide three *minimal counterexample templates*, showing that extremely small modifications suffice to break the zero-dimensional rigidity of FKX systems.

2 The Base FKX System: $\mathrm{FKX_H}$

2.1 System Quadruple

Definition 1 (FKX System)**.** *A discrete formal system is defined as a quadruple*

$$\mathrm{FKX_H} = \langle \Sigma, S, \mathcal{I}, \mathcal{R} \rangle,$$

where:

- Σ *is a symbol domain (elements, positions, relations);*
- S *is the state space;*
- $\mathcal{I} = \{I_1, I_2, \dots\}$ *is a set of invariants (constraints);*
- $\mathcal{R}$ *is a transition relation (or a set of transition rules).*

2.2 A Concrete Instance: 3×3 Bijection States

Definition 2 (Symbols and Adjacency)**.** *Let the element set be* $\Sigma_e = \{1, 2, \dots, 9\}$ *and the position set*

$$\Sigma_p = \{(x, y) \mid x, y \in \{-1, 0, 1\}\}.$$

Adjacency is defined by

$$(x_1, y_1) \sim (x_2, y_2) \iff |x_1 - x_2| + |y_1 - y_2| = 1.$$

Definition 3 (State Space)**.** *A state is a bijection*

$$s : \Sigma_p \to \Sigma_e,$$

such that for each $e \in \Sigma_e$ *there exists exactly one* $p \in \Sigma_p$ *with* $s(p) = e$*. All such bijections constitute the state space* S*.*

Definition 4 (Invariant Set $\mathcal{I}$ (Example))**.** *Consider the invariant set* $\mathcal{I} = \{I_1, I_2, I_3\}$*:*

(I1) ***Line-Sum Invariant:*** *Let L be the set of all horizontal, vertical, and diagonal lines of length 3. Then*

$$\forall \ell \in L, \quad \sum_{p\in\ell} s(p) = 15.$$

(I2) ***Center Anchor:*** *$s(0,0) = 5$.*

(I3) ***Parity Symmetry:***

$$\forall p \in \Sigma_p \setminus \{(0,0)\}, \quad \mathrm{parity}(s(p)) \neq \mathrm{parity}(s(-p)).$$

Definition 5 (Transition Relation $\mathcal{R}$ (Constraint-Driven))**.** *A transition $s \xrightarrow{r} s'$ is allowed if:*

(R1) There exists a finite subset $P \subset \Sigma_p$;

(R2) The transition is local: $\forall p \notin P,\ s'(p) = s(p)$;

(R3) ***Invariant preservation:*** *s' satisfies all constraints in $\mathcal{I}$.*

All such transitions form $\mathcal{R}$.

2.3 Legal Set, Reachability, and Worlds

Definition 6 (Legal State Set)**.** *The legal (constraint-satisfying) set is*

$$\mathcal{L} := \{s \in S \mid s \models \mathcal{I}\}.$$

Definition 7 (Reachability)**.** *Define reachability $\Rightarrow$ by*

$$s \Rightarrow s' \iff \exists r_1, \dots, r_n \in \mathcal{R} : s \xrightarrow{r_1} \cdots \xrightarrow{r_n} s'.$$

Definition 8 (Worlds)**.** *Two states $s, s' \in \mathcal{L}$ belong to the same world if they are mutually reachable. A* world *$\mathcal{W}$ is a connected component of $\mathcal{L}$ under $\Rightarrow$.*

3 Structural Result: Zero-Dimensional Rigidity

3.1 Closure

Lemma 1 (Closure of the Legal Set)**.** *If $s \in \mathcal{L}$ and $s \xrightarrow{r} s'$ is an allowed transition, then $s' \in \mathcal{L}$.*

Proof. By condition (R3), every allowed transition preserves all invariants in $\mathcal{I}$. Hence $s' \models \mathcal{I}$, and $s' \in \mathcal{L}$. □

3.2 Working Definition of Dimension

Definition 9 (FKX Working Dimension)**.** *The dimension of a world $\mathcal{W}$ is defined as the minimal number of independent parameters required to index all states in $\mathcal{W}$. If $\mathcal{W}$ is finite and admits no adjustable external parameters, we write*

$$\dim(\mathcal{W}) = 0.$$

Theorem 1 (Zero-Dimensional Rigidity of FKX)**.** *In* $\mathrm{FKX_H}$*, every world $\mathcal{W}$ satisfies* $\dim(\mathcal{W}) = 0$.

Proof. The state space S is finite, hence $\mathcal{L}$ and every $\mathcal{W} \subseteq \mathcal{L}$ are finite. The system introduces no external parameters (weights, probabilities, thresholds). Transitions only traverse the pre-existing finite solution cluster. Therefore no independent parameters are required to index $\mathcal{W}$, implying $\dim(\mathcal{W}) = 0$. □

Corollary 1 (Dynamics Does Not Imply Freedom)**.** *A system may exhibit nontrivial evolution while still having zero degrees of freedom.*

4 Minimal Counterexample Templates

We now present three minimal modifications, each breaking exactly one key condition and thereby destroying zero-dimensional rigidity.

4.1 Counterexample A: One-Bit World Parameter

Definition 10 (FKX_A: Constraint Mode Bit). *Introduce an external parameter* $c \in \{0, 1\}$ *and define*

$$\mathrm{FKX_A}(c) = \langle \Sigma, S, \mathcal{I}_c, \mathcal{R} \rangle,$$

where

$$\mathcal{I}_c = \begin{cases} \{I_1, I_2, I_3\}, & c = 0, \\ \{I_1, I_2\}, & c = 1. \end{cases}$$

Theorem 2 (Discrete Parameter Breaks Rigidity). *The family* $\mathrm{FKX_A}$ *has dimension at least* 1.

Proof. The parameter c cannot be inferred from internal states and selects distinct legal sets $\mathcal{L}(c)$. Thus at least one independent parameter is required to specify the world. □

4.2 Counterexample B: Leakage Transition

Definition 11 (FKX_B: Temporary Constraint Violation). *Extend* $\mathcal{R}$ *with a transition* r_ϵ *allowing*

$$s \xrightarrow{r_\epsilon} s'$$

such that s' *may violate at most one invariant once, provided the system can return to* $\mathcal{L}$ *within a fixed number of steps.*

Theorem 3 (Loss of Closure). *In* $\mathrm{FKX_B}$*, the legal set* $\mathcal{L}$ *is no longer closed under transitions.*

4.3 Counterexample C: Soft Constraints and Energy Landscape

Definition 12 (FKX_C: Energy-Based Transitions). *Define a violation cost*

$$E(s) = \sum_i w_i \, \mathrm{viol}(I_i, s), \quad w_i > 0,$$

and allow transitions

$$s \to s' \iff s' \text{ differs locally from } s \text{ and } E(s') \le E(s).$$

Theorem 4 (Parameter-Dependent World Structure). *The weight vector* $(w_1, w_2, \dots)$ *introduces external parameters, yielding a non-rigid, multi-basin structure.*

5 The FKX Family Perspective

The FKX family captures systems in which:

- Candidate spaces may be large, yet admissible worlds are constraint-determined;
- Enlarging state representations does not necessarily increase freedom;
- Degrees of freedom typically arise only through external parameters, loss of closure, or softening of constraints.

6 How to Use This Framework

1. To prove a system has no genuine freedom, verify FKX-style closure and absence of external parameters.
2. To demonstrate minimal emergence of freedom, apply one of the counterexample templates.
3. To test a model before investing computation, abstract it as $\langle \Sigma, S, \mathcal{I}, \mathcal{R} \rangle$ and check rigidity.