The KFX Minimal Core: Rigidity under a Fixed Admissibility Scale

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Abstract

We introduce a finite-state transition system in which locally permissive rules, when combined with simple global admissibility constraints, induce complete rigidity: only a single admissible trajectory exists. We prove that this phenomenon is minimal under a fixed admissibility scale, and that any non-degenerate system exhibiting the same rigidity must contain the KFX system as a core factor. We further formalize the notion of residual structure, capturing how admissible behavior may be released in extensions of the core.

1 The KFX Minimal Core

Let

$$S := \{0,1\} \times \mathbb{Z}_5$$

be the state space, where the first component is a binary flag and the second component is a cyclic phase variable.

We define two local transition rules on S:

$$\begin{aligned} R_1 &: (b,p) \longrightarrow (b,p+1), \\ R_2 &: (b,p) \longrightarrow (1-b,p-1), \end{aligned}$$

where arithmetic on the second component is taken modulo 5.

We consider admissibility relative to an arbitrary initial state $s_0 \in S$.

A trajectory

$$\gamma = (s_0, s_1, \ldots, s_{10})$$

is said to be *admissible* if it satisfies the following global constraints:

(C1) (*Midpoint state constraint*) If $s_0 = (b,p)$, then $s_5 = (1-b,p)$.

(C2) (*Closure*) $s_{10} = s_0$.

Remark 1. *Constraint (C1) fixes the state at time* 5 *and does not prescribe when intermediate flips may occur; multiple flips before time* 5 *are permitted.*

Theorem 1 (Rigidity of the KFX core)**.** *There exists exactly one admissible step-type sequence of length* 10*, namely*

$$R_2^{10}.$$

Equivalently, every admissible trajectory is uniquely determined by its initial state.

Proof. Let m denote the number of applications of R_1 among the first five steps of an admissible trajectory. Then the number of R_2 steps in this segment is $5 - m$.

Each application of R_1 contributes a phase displacement of $+1$, while each application of R_2 contributes -1. Thus the net phase displacement after five steps is

$$\Delta p = m - (5 - m) = 2m - 5.$$

By constraint (C1), the phase must return to its initial value at time 5, hence

$$2m - 5 \equiv 0 \pmod 5.$$

This congruence forces $m \equiv 0 \pmod 5$, hence $m = 0$ or $m = 5$. Constraint (C1) also requires the binary component to differ from its initial value at time 5, which implies that the number of R_2 steps in the first five moves, namely $5 - m$, must be odd. This excludes $m = 5$, leaving $m = 0$ as the unique possibility.

Thus the first five steps must all be of type R_2. By constraint (C2), the same reasoning applies to the last five steps. Therefore the unique admissible step-type sequence is R_2^{10}. □

2 Minimality Under Fixed Admissibility Scale

We now show that the above construction is minimal among all non-degenerate systems satisfying the same admissibility template.

Consider the family of systems with state space

$$S_n := \{0, 1\} \times \mathbb{Z}_n,$$

equipped with the same local rules R_1, R_2 and the same admissibility constraints (C1),(C2) imposed at steps 5 and 10.

Theorem 2 (Minimality of the phase space)**.** *Assume that a system with state space S_n admits at least one admissible trajectory and exhibits rigidity in the sense of a unique admissible step-type sequence. Then $n = 5$.*

Proof. As before, let m be the number of R_1 steps among the first five moves. Constraint (C1) yields

$$2m - 5 \equiv 0 \pmod n. \tag{1}$$

Since $m \in \{0, 1, 2, 3, 4, 5\}$, the quantity $2m - 5$ belongs to

$$\{-5, -3, -1, 1, 3, 5\}.$$

For $n > 1$, congruence (1) can only hold if $n \in \{3, 5\}$.

Constraint (C1) also requires $5 - m$ to be odd.

If $n = 5$, congruence (1) implies $m = 0$ or $m = 5$, and the parity condition excludes $m = 5$. Thus $m = 0$ is uniquely determined, yielding rigidity.

If $n = 3$, congruence (1) admits values such as $m = 4$, which also satisfy the parity condition. Hence admissible trajectories with mixed step types exist, and rigidity fails.

For $n = 1$, the phase space is trivial and multiple step-type sequences satisfy both (C1) and (C2), again violating uniqueness.

Therefore the only non-degenerate case admitting rigidity is $n = 5$. □

Corollary 1. *Among all non-degenerate systems obeying the admissibility template (C1),(C2), the KFX core has minimal state cardinality*

$$|S| = 10.$$

3 The KFX Family via Core Projection

We now formalize what it means for a more general system to belong to the KFX family.

Definition 1 (KFX-family system)**.** *A finite transition system* $(X, \to_X, \mathcal{C}_X)$ *belongs to the KFX family if there exists a surjective map*

$$\pi : X \longrightarrow S$$

such that:

1. (Step homomorphism) *For every transition* $x \to_X x'$*, the projected states* $\pi(x) \to \pi(x')$ *follow either* R_1 *or* R_2*.*
2. (Constraint compatibility) *For every admissible trajectory* γ *in* X*, the projected trajectory* $\pi(\gamma)$ *satisfies (C1),(C2).*
3. (Core realization) *Every admissible KFX trajectory admits at least one admissible lift in* X*.*

Theorem 3 (Inheritance of rigidity)**.** *For any KFX-family system, the projection of every admissible trajectory follows the unique KFX step-type sequence* R_2^{10}*.*

4 Residual Structure

The core projection π identifies the KFX system as a factor of a larger system. Any remaining degrees of freedom are captured by the residual structure.

Define an equivalence relation on X by

$$x \sim y \iff \pi(x) = \pi(y).$$

The equivalence classes

$$F_s := \pi^{-1}(s), \quad s \in S,$$

are called *residual fibres.*

For each core transition $s \to s'$, the local dynamics induces a residual transition relation

$$\delta_{s \to s'} \subseteq F_s \times F_{s'}.$$

Definition 2 (Residual lift)**.** *Let* $\bar{\gamma}$ *denote the unique admissible KFX trajectory. The residual lift set is defined as*

$$\mathrm{Lift}(\bar{\gamma}) := \{\gamma \in \mathcal{C}_X \mid \pi(\gamma) = \bar{\gamma}\}.$$

Definition 3 (Residual rank)**.** *The residual rank of a KFX-family system is defined by*

$$\rho(X) := |\mathrm{Lift}(\bar{\gamma})|.$$

Remark 2. *Systems with* $\rho(X) = 1$ *exhibit zero residual freedom, while systems with* $\rho(X) > 1$ *quantify controlled releases of admissible behavior beyond the KFX core.*