The KFX Patriarch: A Structural Classification of Residual Unreachability

Kaifan XIE
ORCID: 0009-0005-6911-7295

February 7, 2026

1 Motivation and Scope

This document formulates and consolidates the *KFX family* of deterministic counterexamples exhibiting *structural residual unreachability* under coarse-grained observation.

Rather than presenting a collection of isolated constructions, we elevate the analysis to a *patriarch-level formulation*: a single structural framework from which all known KFX-type residual constraints arise as direct and necessary consequences.

The guiding question is deliberately elementary in appearance but subtle in content:

> In a finite, fully deterministic system, which coarse-grained residual configurations are structurally reachable, and which are forbidden, independently of noise, randomness, or asymptotic limits?

The answer developed here shows that residual unreachability can arise purely from arithmetic coupling between modular phase evolution and parity constraints. No stochastic assumptions, thermodynamic limits, or probabilistic reasoning are invoked. All statements are exact, finite, and combinatorial in nature.

2 The KFX-R System

We consider a deterministic system with state variables

$$(\theta_t,\, b_t) \in \mathbb{Z}_m \times \mathbb{Z}_2,$$

where θ_t is a phase variable modulo m and b_t is a binary spin.

The system evolves in discrete time by selecting one of two deterministic update rules:

- R_1: $\theta \mapsto \theta + 1 \pmod m$, with b unchanged;
- R_2: $\theta \mapsto \theta - 1 \pmod m$, with $b \mapsto b + 1 \pmod 2$.

A full trajectory of length $2k$ is generated by choosing a sequence

$$R_{i_1}, R_{i_2}, \ldots, R_{i_{2k}}, \qquad i_t \in \{1, 2\}.$$

We define the observable residuals as follows:

$$\varepsilon_p^{(k)} := \theta_k - \theta_0 \pmod{m},$$
$$\varepsilon_p^{(2k)} := \theta_{2k} - \theta_0 \pmod{m},$$
$$\varepsilon_b^{(2k)} := b_{2k} - b_0 \pmod{2}.$$

The central object of study is the residual triple

$$\big(\varepsilon_p^{(k)}, \varepsilon_p^{(2k)}, \varepsilon_b^{(2k)}\big),$$

and the question of which such triples are realizable by some deterministic rule sequence of length $2k$.

3 Counting Variables and Generative Relations

3.1 Counting Variables

Let

- a be the number of applications of R_2 in the first k steps,
- b be the number of applications of R_2 in the second k steps,
- $s := a + b$ be the total number of applications of R_2 over $2k$ steps.

By construction,

$$0 \le a \le k, \qquad 0 \le b \le k, \qquad 0 \le s \le 2k.$$

3.2 Phase and Spin Contributions

Each application of R_1 contributes $+1$ to the phase and leaves the spin unchanged. Each application of R_2 contributes -1 to the phase and flips the spin.

Consequently, the residual observables satisfy the exact relations

$$\varepsilon_p^{(k)} \equiv k - 2a \pmod{m}, \tag{1}$$
$$\varepsilon_p^{(2k)} \equiv 2k - 2s \pmod{m}, \tag{2}$$
$$\varepsilon_b^{(2k)} \equiv s \pmod{2}. \tag{3}$$

Equations (1)–(3) form the generative skeleton of the entire KFX family.

4 Interval Feasibility and Congruence Structure

Since m is assumed odd, the map $x \mapsto 2x \pmod{m}$ is invertible. Let 2^{-1} denote the multiplicative inverse of 2 modulo m.

Equations (1) and (2) are equivalent to the congruences

$$a \equiv a_0 := 2^{-1}(k - \varepsilon_p^{(k)}) \pmod{m},$$
$$s \equiv s_0 := 2^{-1}(2k - \varepsilon_p^{(2k)}) \pmod{m}.$$

Residual reachability therefore requires integers a and s satisfying these congruences together with the finite constraints

$$0 \le a \le k, \qquad 0 \le s \le 2k, \qquad 0 \le s - a \le k.$$

Equivalently,

$$a \in I(s) := [0, k] \cap [s - k,\ s].$$

Theorem 4.1 (Interval Congruence Feasibility). *Let $x \equiv x_0 \pmod{m}$. There exists an integer x with $L \le x \le U$ satisfying the congruence if and only if*

$$\left\lceil \frac{L - x_0}{m} \right\rceil \le \left\lfloor \frac{U - x_0}{m} \right\rfloor .$$

5 Lift Structure of the Total Flip Count

All admissible values of s are of the form

$$s = s_0 + um, \qquad 0 \le s \le 2k.$$

The admissible lift set is therefore

$$\mathcal{S} = \{ s_0 + um \mid u \in \mathbb{Z},\ 0 \le s_0 + um \le 2k \} .$$

Because m is odd, successive lifts alternate in parity. As a result, the spin residual $\varepsilon_b^{(2k)}$ effectively selects a parity class of lifts.

Residual unreachability occurs precisely when all lifts of one parity are eliminated by the interval feasibility constraint.

6 Critical Band Analysis

We define the *critical band* by the inequality

$$m \le 2k < 2m.$$

In this regime, there are exactly two admissible lifts:

$$s \in \{s_0,\ s_0 + m\}.$$

These two values necessarily have opposite parity. If the feasibility condition excludes exactly one of them, then one spin parity becomes structurally forbidden, independently of any dynamical or probabilistic considerations.

7 Classification Theorem

Theorem 7.1 (KFX Patriarch Classification). *Assume m is odd and $m \le 2k < 2m$. A residual triple*

$$(\varepsilon_p^{(k)},\, \varepsilon_p^{(2k)},\, \varepsilon_b^{(2k)})$$

is reachable if and only if there exists

$$s \in \{s_0,\ s_0 + m\}$$

such that:

- $0 \leq s \leq 2k$,
- $s \equiv \varepsilon_b^{(2k)} \pmod 2$,
- *there exists* $a \equiv a_0 \pmod m$ *with* $a \in I(s)$.

8 Discussion

The KFX family demonstrates that residual unreachability can arise in finite, fully deterministic systems without invoking randomness, noise, or asymptotic limits.

The obstruction is not dynamical but structural: it is produced by the interaction between modular arithmetic, parity constraints, and finite observation windows.

The patriarch formulation presented here isolates this mechanism in its most compressed and transferable form. Any extension that preserves these structural ingredients necessarily inherits the same class of residual constraints.