A Constrained Orbit System and Its Continuum Limit

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Abstract

We present a minimal constrained orbit system defined by local generators and global closure constraints. In the discrete setting the system exhibits complete rigidity, admitting exactly one legal orbit pattern. We construct a continuum limit and show that the discrete rigidity survives as a residual unreachable set of measure zero. A conjectural fractal structure of this residual set is formulated.

1 Discrete System

Definition 1.1 (State Space)**.** *For $m \in \mathbb{N}$, define*

$$S_m \mathbb{Z}_2 \times \mathbb{Z}_m.$$

An element of S_m is written as (b, p).

Definition 1.2 (Generators)**.** *Define two maps $R_1, R_2 : S_m \to S_m$ by*

$$R_1(b, p) = (b, p + 1), \qquad R_2(b, p) = (1 - b, p - 1),$$

with arithmetic on $\mathbb{Z}_m$ taken modulo m.

Definition 1.3 (Orbit)**.** *Given a word $\sigma = (\sigma_1, \dots, \sigma_n) \in \{R_1, R_2\}^n$ and an initial state $s_0 \in S_m$, the induced orbit is*

$$\gamma_\sigma(s_0) = (s_0, \sigma_1(s_0), \sigma_2\sigma_1(s_0), \dots).$$

Definition 1.4 (Global Constraints)**.** *A length-*10 *orbit $\gamma = (s_0, \dots, s_{10})$ is* legal *if*

(C1) $s_5 = (1 - b_0, p_0)$,

(C2) $s_{10} = s_0$.

Theorem 1.1 (Discrete Rigidity)**.** *For $m = 5$, the set of legal step sequences of length* 10 *consists of a single element:*

$$\{\sigma \in \{R_1, R_2\}^{10} : \sigma \text{ is legal}\} = \{R_2^{10}\}.$$

Remark 1.1. *The uniqueness is independent of the initial state and depends only on the global constraints.*

2 Continuum Limit

Definition 2.1 (Continuous Phase Space)**.** *Define*

$$X\mathbb{S}^1 \times \mathbb{Z}_2, \qquad \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

Definition 2.2 (Limit Generators)**.** *For $\omega \in \mathbb{R}$, define maps $T_1, T_2 : X \to X$ by*

$$T_1(\theta, b) = (\theta + \omega, b), \qquad T_2(\theta, b) = (\theta - \omega, 1 - b).$$

Definition 2.3 (Generated Action)**.** *Let $\mathcal{G}_\omega = \langle T_1, T_2 \rangle$ be the semigroup generated by T_1, T_2. For $x \in X$, define the reachable set*

$$\mathcal{R}(x) = \{g(x) : g \in \mathcal{G}_\omega\}.$$

Proposition 2.1 (Density of Orbits)**.** *If $\omega/2\pi \notin$, then for all $x \in X$,*

$$\overline{\mathcal{R}(x)} = X.$$

3 Residual Unreachability

Definition 3.1 (Reachable Closure)**.** *For $x \in X$, define*

$$\mathcal{R}_\infty(x) = \overline{\bigcup_{n \geq 0} \mathcal{R}_n(x)},$$

where $\mathcal{R}_n(x)$ denotes points reachable by words of length $\leq n$.

Definition 3.2 (Residual Set)**.** *Define the residual unreachable set*

$$\mathcal{F}X \setminus \bigcup_{x \in X} \mathcal{R}_\infty(x).$$

Proposition 3.1 (Measure-Zero Residual)**.** *Let μ be the product of Lebesgue measure on $\mathbb{S}^1$ and counting measure on $\mathbb{Z}_2$. Then*

$$\mu(\mathcal{F}) = 0.$$

Proposition 3.2 (Topological Nontriviality)**.** *The set $\mathcal{F}$ may be dense in X despite having measure zero.*

Conjecture 3.1 (Fractal Residual)**.** *There exists $\alpha \in (0, 1)$ such that*

$$\dim_H(\mathcal{F}) = \alpha.$$

4 Discrete-to-Continuous Correspondence

Definition 4.1 (Embedding)**.** *Define*

$$\iota_m : S_m \to X, \qquad \iota_m(b, p) = \left(\tfrac{2\pi p}{m},\, b\right).$$

Proposition 4.1 (Constraint Collapse)**.** *Let $\mathcal{F}_m \subset S_m$ denote the discrete unreachable set. Then*

$$\iota_m(\mathcal{F}_m) \;\longrightarrow\; \mathcal{F} \quad \text{as } m \to \infty,$$

in the sense of Hausdorff convergence or weak convergence of measures.

5 Conclusion

This system provides a minimal example in which complete rigidity in a finite, constrained orbit system survives the continuum limit as a residual set of measure zero with conjecturally fractal structure.