Sheet 6
Sabastian & Kai
(Die Zusatzaufgabe Sheet 4 letzter Woche knnte in der Klausur dran kommen.)

1a
See the markdown file on LiMo Homepage. The example ist only for expanation of Bayer inference. The procedure is often not possible because the posterior is often unknown/not calculatable.

2a

$$P(\tau) = \frac{\delta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} \exp(-\delta\tau)$$

$$\text{where} \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$$

$$\begin{aligned}
P(\tau) &= \int_0^\infty \frac{\delta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} \exp(-\delta\tau) d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \delta^\alpha \tau^{\alpha-1} \exp(-\delta\tau) d\tau \\
&\propto \int_0^\infty \delta^\alpha \tau^{\alpha-1} \exp(-\delta\tau) d\tau \\
&= \int_0^\infty (\delta\tau)^\alpha \tau^{-1} \exp(-\delta\tau) d\tau && \mu = \tau\delta; \quad \frac{d\tau}{d\mu} = \frac{1}{\delta} \\
&= \int_0^\infty \mu^\alpha \left(\frac{\mu}{\delta}\right)^{-1} exp(-\mu) \frac{1}{\delta} d\mu \\
&= \int_0^\infty \mu^\alpha \mu^{-1} \exp(-\mu) d\mu \\
&= \int_0^\infty \mu^{\alpha-1} \exp(-\mu) d\mu && \text{Dies ist die Gamma Funktion.} \\
&\text{Mit dem zuvor ausgelassenem Term ergibt sich also:} \\
&= \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) \\
&= 1
\end{aligned}$$

Funktioniert, da $\tau \geq 0$, $\delta > 0$, $\alpha > 0$ and $\Gamma(\alpha) > 0 \rightarrow p(\tau) \geq 0$

2b

$$
\begin{aligned}
E(\tau) &= \int_0^\infty p(\tau) d\tau \\
&= \int_0^\infty \frac{\delta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} \exp(-\delta\tau) d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty (\delta\tau)^\alpha \exp(-\delta\tau) d\tau \quad \mu = \tau\delta \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty (\mu)^\alpha \exp(-\mu) \frac{1}{\delta} d\mu \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\delta} \int_0^\infty (\mu)^\alpha \exp(-\mu) d\mu \quad \text{dies ist wieder so aehnlich wie die Gamma Funktion (fehlt ein -)} \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\delta} \Gamma(\alpha+1) \\
&= \frac{\alpha}{\delta}
\end{aligned}
$$

$$
\begin{aligned}
Var(\tau) &= \int_0^\infty \tau^2 \frac{\delta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} exp(-\delta\tau) - (\frac{\alpha}{\delta})^2 && \text{Verschiebungssatz} \\
&= \frac{\delta^\alpha}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha+1} exp(-\delta\tau) - (\frac{\alpha}{\delta})^2 && \text{Schritt unklar, Gamma Funktion??} \\
&= \frac{\delta^\alpha}{\Gamma(\alpha)} \delta^{-(\alpha+2)} \Gamma(\alpha+2) - (\frac{\alpha}{\delta})^2 \\
&= \frac{\delta^{\alpha-2}}{\Gamma(\alpha)} \Gamma(\alpha+2) - (\frac{\alpha^2}{\delta^2}) && \Gamma(\alpha+1) = \alpha\Gamma(\alpha); \quad \Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha) \\
&= \frac{(\alpha+1)\alpha}{\delta^2} - \frac{\alpha^2}{\gamma^2} \\
&= \frac{\alpha}{\delta^2}
\end{aligned}
$$

3

$$
\begin{aligned}
y &= X\beta + \epsilon \\
\epsilon &\sim N_p(0, \sigma^2 I) \\
\beta &= (\beta_1, \ldots, \beta_p)' \\
\sigma^2 &\sim IG(a, b)
\end{aligned}
$$

a) posterior disribution of σ^2

$$p(\sigma^2|y,X,\beta) = f(y|X,\beta,\sigma^2)p(\sigma^2)$$

$$\begin{aligned}
f(y|X,\beta,\sigma^2) &= \frac{1}{(2\pi)^{p/2}det(I\sigma^2)^{1/2}}\exp(-\frac{1}{2}(y-X\beta)'(I\sigma^2)^{-1}(y-X\beta)) \\
&\propto \frac{1}{\sqrt{(\sigma^2)^n det(I}}\exp(-\frac{1}{2}(y-X\beta)'(I\sigma^2)^{-1}(y-X\beta)) \\
&\propto \frac{1}{(\sigma^2)^{n/2}}\exp(-\frac{1}{2}(y-X\beta)'(I\sigma^2)^{-1}(y-X\beta))
\end{aligned}$$

Prior

$$\begin{aligned}
p(\sigma^2) &= \frac{b^a}{\Gamma(a)}(\sigma^2)^{-(a+1)}\exp(-b/\sigma^2) \\
&\propto (\sigma^2)^{-(a+1)}\exp(-b/\sigma^2)
\end{aligned}$$

$$\begin{aligned}
p(\sigma^2|y,X,\beta) &\propto \frac{1}{(\sigma^2)^{n/2}}\exp(-\frac{1}{2}(y-X\beta)'(I\sigma^2)^{-1}(y-X\beta))(\sigma^2)^{-(a+1)}\exp(-b/\sigma^2) \\
&= (\sigma^2)^{-(a+1+n/2)}\exp(-\frac{1}{\sigma^2}(\frac{1}{2}(y-X\beta)'(y-X\beta)+b))
\end{aligned}$$

$$\sigma^2|y,X,\beta \sim IG(a+n/2, \frac{1}{2}(y-X\beta)'(y-X\beta)+b)$$

b flat prior = non informative prior we get flat priors if the IG distribution is constant. Thus the prior is flat for the values
$b = 0, a = -1 p(\sigma^2) = const$

$$p(\sigma^2) \propto (\sigma^2)^{-(a+1)}\exp(-b/\sigma^2)$$

c

$$\hat{\sigma}^2 = \frac{b_{post}}{a_{post}+1} = \frac{\frac{1}{2}(y-X\beta)'(y-X\beta)+b}{a+n/2+1}$$

fuer $b = 0, a = -1$

$$\hat{\sigma}^2 = \frac{\frac{1}{2}(y-X\beta)'(y-X\beta)}{n/2} = \frac{1}{n}(y-X\beta)'(y-X\beta) = \hat{\sigma}^2_{ML}$$

4

$$\begin{aligned}
y &\sim Po(\lambda) \\
f(y|\lambda) &= \frac{\lambda_i^y}{y_i!}\exp(-\lambda)
\end{aligned}$$

Likelihood

$$\begin{aligned} L(y|\lambda) &= \prod_i f(y_i|\lambda) = \prod_i \frac{\lambda_i^y}{y_i!} \exp(-\lambda) \\ &= \frac{1}{\prod_i y_i!} \lambda^{\sum_i y_i} \exp(-n\lambda) \end{aligned}$$

Prior (Gamma Verteilung)

$$\begin{aligned} p(\lambda) &= \frac{b^a}{\Gamma(a)} (\lambda^2)^{-(a+1)} \exp(-b/\lambda) \\ &\propto (\sigma^2)^{a-1} \exp(-b\sigma^2) \end{aligned}$$

Posterior

$$\begin{aligned} p(\lambda|y) &\propto f(y|\lambda) p(\lambda) \\ &\propto \lambda \sum_i y_i \exp(-n\lambda) \lambda^{a-1} \exp(-b\lambda) \\ &= \lambda a + \sum_i y_i - 1 \exp(-(b+n)\lambda) \end{aligned}$$

$$\lambda|y \sim Ga(a + \sum_i y_i, b+n)$$

Konjugierte Verteilung, da der Kern der Post.-Verteilung wieder einer Gamma Vertielung entspricht.
$p(\lambda|y)$ hat die Form einer Gamma Verteilung mit $a = a + \sum y_i 4$ und $b = b + n$, also $\lambda(y \sim Ga(a + \sum y_i, b+n))$

5 R. There was a mistake in an early version of the sheet in b.

6 see page 284

$$u(\tau|y) \rightarrow y(\tau,\beta|y) = cP(\tau,\beta)L(\tau,\beta|y)$$
$$P(\tau,\beta) = P_1\tau|\beta)P_2(\tau)$$

where $\quad \beta|\tau \sim N_{p+1}(\psi\tau^{-1}V) \quad$ and $\quad \tau \sim G(\alpha,\sigma)$

$$\begin{aligned} L(y|\beta,\tau) &= \prod_i \frac{1}{(2\pi)^{p/2} det(\tau^{-1}I)} \frac{1}{2} \exp(-\frac{1}{2}(y-X\beta)'(\tau I)(y-X\beta)) \\ &\propto (\tau^n)^{1/2} \exp(-\frac{\tau}{2}(y-X\mu)'(y-X\beta)) \end{aligned}$$

$$\begin{aligned} P_2(\tau) &= \frac{\sigma^\alpha}{\Gamma(\alpha)} \tau^{(\alpha-1)} \exp(-\sigma\tau) \\ &\propto \tau^{(\alpha-1)} \exp(-\sigma\tau) \end{aligned}$$

$$\begin{aligned} P_1(\beta|\tau) &= \frac{1}{(2\pi)^{\frac{k+1}{2}} det(\Sigma)^{1/2}} \exp(\tau(\beta-\psi)'V^{-1}\frac{(\beta-\psi)}{2}) \\ &\propto \tau^{\frac{k+1}{2}} \exp(-\tau(\beta-\psi)'V^{-1}\frac{(\beta-\psi)}{2}) \end{aligned}$$

$$\begin{aligned} P(\beta,\tau) &\propto P_1 P_2 \\ &\propto \tau^{\frac{k+1}{2}} + \frac{2(\alpha-1)}{2} \exp(\frac{-\tau(\beta-\psi)'V^{-1}(\beta-\psi)}{2} - \frac{2\sigma\tau}{2}) \end{aligned}$$

$$\begin{aligned} y(\tau,\beta|y) &= P(\beta,\tau)L(\tau,\beta|y) \\ &\propto \tau^{\frac{k+1+\alpha+n}{2}} exp(-\frac{\tau}{2}[\beta\psi'V^{-1}(\beta-\psi) + 2\sigma + (y-X\beta)'(y-X\beta)]) \end{aligned}$$

$\rightarrow$ post. Verteilung $G(\beta,\tau|y)$

c_1 [in (11.9) auf Seite 281] $= \frac{1}{2\pi^{\frac{k+1}{2}}}$

c_2 [in(11.10)] $= c_1 \frac{1}{2\pi^{\frac{n}{2}}}$

$$\delta_*[\text{in}(11.14)] = 2\delta$$
$$\alpha_{**}[\text{in}(11.14)] = 2\alpha - 2 + n$$
$$\delta_{**}[\text{in}(11.14)] = -\psi_*' V_*^{-1} + \psi V^{-1} \psi \ldots$$

Die Konstandten die zuvor weggelassen wurden muessen jetzt multiplikativ hinzugefgt werden, da das Integral der Dicht ansonsten nicht 1 ergeben wuerde. c_5 [in(11.14)] $= c_2 |V_*|^{1/2} * (2\pi)^{\frac{k+1}{2}}$

5b Change of Variable technique

$$\begin{aligned}
(\sigma^2|y) &= v(\tau|y)|\frac{d\tau}{d\sigma^2}| \\
&= v(\frac{1}{\sigma^2}|y) * \frac{1}{\sigma^4} \\
&= c_5(\frac{1}{\sigma^2})^{\frac{\alpha+n}{2}-1} \frac{1}{\sigma^4} - exp\left(-\left[\left(\frac{-\psi_*' V_*^{-1} \psi_* + \psi' V^{-1} \psi + y'y + 2\psi}{2}\right)\right] \frac{1}{\sigma^2}\right) \\
&= c_5(\sigma^2)^{-\left(\frac{\alpha+n}{2}-1\right)-2} exp\left(\ldots\right) \\
&= c_5(\sigma^2)^{-\frac{\alpha+n}{2}+1-2} exp\left(\ldots\right) \\
&= c_5(\sigma^2)^{-\frac{\alpha+n}{2-1}} exp\left(\ldots\right)
\end{aligned}$$