ISTA 421 + INFO 521
Machine Learning

Probability Review

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References for probability

Recommend:
- (lvl 1) Doing Bayesian Data Analysis (**DBDA**)
  - Ch 2, 4, 5
- (lvl 2) First Course in Machine Learning (**FCML**)
  - Ch 2.2 (foundations),
  - Ch 2.3 (Discrete),
  - Ch 2.4-2.5 (Continuous)
  - Ch 2.6-2.7 (Expectation and Maximum Likelihood)
  - Ch 3 (Bayesian)
- (lvl 3) Pattern Recognition and Machine Learning (**PRML**)
  - Ch 1.2 (foundations),
  - Ch 2.1-2.2 (Discrete),
  - Ch 2.3 (Continuous)

Google (and WikiPedia) for unfamiliar terms and alternative explanations.

Wisdom from tea dipper handle

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Probability semantics

Two broad interpretations of probability
(variants exist for both)

1) Representation of expected frequency (“frequentist”)

2) Degree of belief (“Bayesian”)

There is a 20% chance of rain tomorrow.

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Basic terminology and rules

Sample Space of *outcomes* (often denoted by Ω)

{H, T}

{1, 2, 3, 4, 5, 6}

An outcome is just ONE element of the sample space
A “generic” outcome is often denoted by ω
and we can say things like, e.g., “for each $\omega \in \Omega$…”

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Event (subset of Ω) …does or does not contain (is true or false for) a particular outcome

odd {1, 3, 5}, even {2, 4, 6}, prime {2, 3, 5}

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Semantics of Set Operations

Equivalence between "set" and "proposition" representations.

1. Set E: outcomes s.t. proposition E is true.
2. Union, $E \cup F$: logical OR between propositions E and F.
3. Intersection, $E \cap F$: logical AND
4. Complement, E^C: logical negation

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Denote the **collection of measurable events** (ones we want to assign probabilities to) by S.

S must include $\varnothing$ and Ω

These special events represent the cases where "nothing" among all the choices happens (impossible), and "something" happens (certain).

Reason for being technical: It is important to be tuned into **what** a particular probability is **about** (precisely!).

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(ones we want to assign probabilities to) by S.

S must include $\varnothing$ and Ω

S is *closed* under set operations …aka: σ-algebra

$$\alpha,\beta \in S \;\Rightarrow\; \alpha \cup \beta \in S\ ,\ \alpha \cap \beta \in S\ ,\ \alpha^{C} = \Omega - \alpha \in S\ , \text{etc.}$$

Translation: We need to be able to deal with concepts such as "either A or B" happens, or "both A and B" happen.

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E.g., I'll accept either an even or prime number

E.g., If I roll a 3, it is both odd and prime

Basic terminology and rules

Probability Space

A **probability space** is a sample space Ω augmented with a function, P, that assigns a **probability** to each event, $E \subset S$.

Kolmogorov Axioms

1. $0 \le P(E) \le 1$ for all $E \subset S$.
2. $P(\Omega) = 1$.
3. If $E \cap F = \varnothing$ then $P(E \cup F) = P(E) + P(F)$.

Important Consequences

1. $P(\varnothing) = 0$.
2. $P(E^{C}) = 1 - P(E)$
3. In general, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

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Random Variables

Random variables

Defined by **functions** mapping **outcomes** (ω) to **values**

A random variable is a way of reporting an attribute of an outcome
Typically r.v. are denoted by uppercase letters (e.g., X)
Generic values are corresponding lower case letters (e.g., x)
Shorthand: P(x) = P(X=x)
Value “type” is arbitrary (typically categorical or real)

Example (from K&F)

Outcomes are student grades (A,B,C)
Random variable G=f_{GRADE}(student)

$$P('A') = P(G = 'A') = P(\{w \in \Omega : f_{GRADE}(w) = 'A'\})$$

We sometimes use sets, but usually R.Vs.: $P(\overbrace{A \cap B \cap C}^{\text{Sets}}) \equiv \overbrace{P(A, B, C)}^{\text{R. Vs.}}$

Random Variables

Random Variable

- Formally, a **random variable** is a function, X that assigns a number to each outcome in S (e.g., dead $\to 0$, alive $\to 1$).
- Key consequence: a random variable divides the sample space into **equivalence classes**: sets of outcomes that share some property (differ only in ways irrelevant to X)

Example

- Let S = all sequences of 3 coin tosses.
- We can define a r.v. X that counts number of heads.
- Then HHT and HTH are equivalent in the eyes of X:

$$X(HHT) = X(HTH) = 2$$

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Random Variables

Distribution of a Random Variable

- The expression $P(X = x)$ refers to the probability of the event $E = \{\omega \in S : X(\omega) = x\}$.
- Sometimes we can obtain it by breaking it down into simpler, mutually exclusive events and adding their probabilities (Kolmogorov axiom 3)

Example

- $S =$ all sequences of 3 coin tosses.
- $X(\omega) = \#$ of heads in ω.

$$\begin{aligned}\{X = 2\} &= \{HHT\} \cup \{HTH\} \cup \{THH\} \\ P(X = 2) &= P(HHT) + P(HTH) + P(THH) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\end{aligned}$$

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Random Variables

Distribution of a Random Variable

- Similarly, $P(X < x)$ is the probability of the event $E = \{\omega \in S : X(\omega) < x\}$.
- Can sometimes obtain it the same way as we did above.

Example

- $S =$ all sequences of 3 coin tosses.
- $X(\omega) = \#$ of heads in ω.

$$\begin{aligned}\{X < 2\} &= \{TTT\} \cup \{TTH\} \cup \{THT\} \cup \{HTT\} \\ P(X < 2) &= P(TTT) + P(TTH) + P(THT) + P(HTT) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\end{aligned}$$

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Random Variables

Distribution of a Random Variable

Example, continued

- Notice that in this example we could also have written

$$\{X < 2\} = \{X = 0\} \cup \{X = 1\}$$
$$P(X < 2) = P(X = 0) + P(X = 1)$$

which is useful if we have already calculated $P(X = x)$ for each value of x.

- This always works if X is always an integer.

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Joint Probability

Joint Probability

- We have already seen the concept of *intersecting events*: $A \cap B$ is the event that occurs when *both* A and B are true *at the same time*.
- $P(A \cap B)$ is called the **joint probability** of A and B.
- If A is $\{X = x\}$ and B is $\{Y = y\}$, then $A \cap B$ means $X = x$ and $Y = y$ *at the same time*.
- If X and Y are discrete, $P(X = x, Y = y)$, for different combinations of x and y, characterize the **joint distribution** of X and Y.

We write $P(x,y)$ for $P(\{w \in \Omega : X(w) = x \text{ and } Y(w) = y\})$

Alternatively, $P((X = x) \cap (Y = y))$

Note that the comma in the usual form, $P(x,y)$, is read as "and".

Here events are being defined by assignments of random variables

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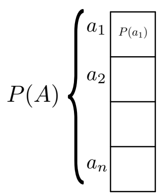
Joint Probability



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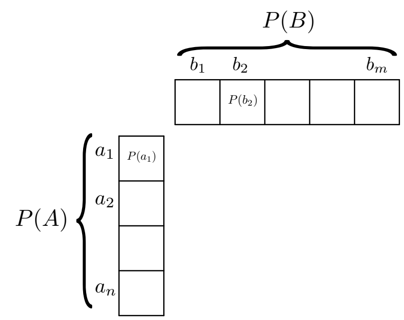
Joint Probability



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Joint Probability

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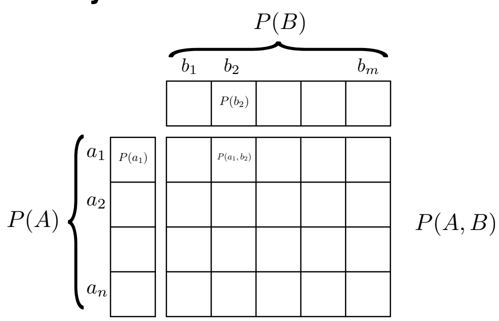

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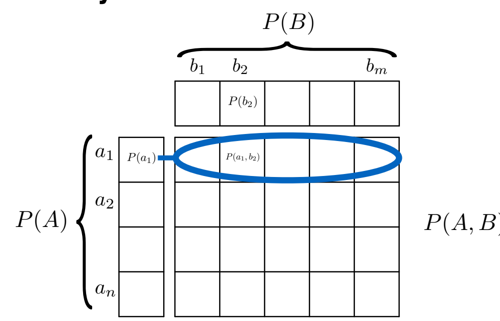
Joint Probability



Marginalization: $P(A) = \sum_{b \in B} P(A, B)$ *

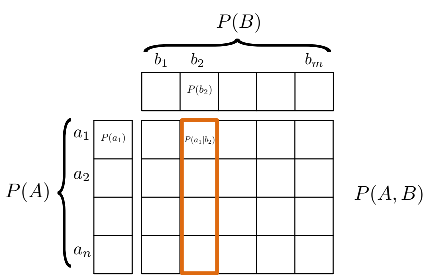
Formulas that you should be comfortable with are marked by * .

Conditional Probability

"probability in context"

Conditional probability (definition)

$$P\left(A\middle|B\right) \equiv \frac{P(A \cap B)}{P(B)}$$ *



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Conditional Probability

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Conditional probability (definition)

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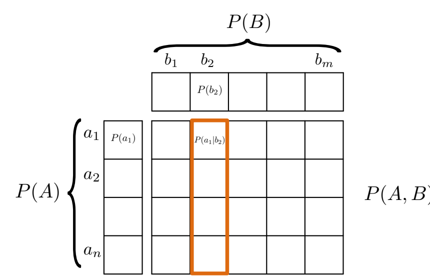
Example: what is the probability that you roll 2 (on a six sided die), given that you know you have rolled a prime number?



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Conditional probability from constraints on belief update

- Conditional probability can be viewed as following from reasonable constraints on updating beliefs given evidence (Darwiche 2009, p.31).
- Think in terms of updating beliefs from joint probability, where you know evidence, β, is true:
  1. All worlds where evidence is true should have probability that sums to 1 (across worlds): $\sum_{\omega \models \beta} P(\omega|\beta) = 1$
  2. All worlds where evidence is false should have probability 0 (they aren't possible): $P(\omega|\beta) = 0 \;\; \forall\omega \; P(\omega) = 0$
  3. For all pairs of world in which evidence is true (and where the probabilities of those worlds are > 0), the ratios of the probabilities of the pair should be the same before as after: $\frac{P(\omega)}{P(\omega')} = \frac{P(\omega|\beta)}{P(\omega'|\beta)}, \;\; \forall\omega,\omega' \models \beta, P(\omega) > 0, P(\omega') > 0$
- These three constraints leave us with only one option for the new beliefs in the worlds that satisfy the evidence β:

$$P(\omega|\beta) = \frac{P(\omega)}{P(\beta)} \;\; \forall\omega \models \beta$$

Product Rule

"probability in context"

Conditional probability (definition)

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}$$ *

Applying a bit of algebra,

$$P(A \cap B) = P(B)P(A|B)$$

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Chain (Product) Rule

"probability in context"

Conditional probability (definition)

$$P\left(A|B\right) \equiv \frac{P(A\cap B)}{P(B)} \quad *$$

Applying a bit of algebra,

$$P(A\cap B) = P(B)P(A|B)$$

In general, we have the **chain** (**product**) rule:

Product $\quad P\left(A_1\cap A_2\right) = P(A_1)P(A_2|A_1)$

Chain $\quad P\left(A_1\cap A_2\cap \;\dots\; A_N\right) = P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2)\;\dots\; P(A_N|A_1\cap A_2\cap \;\dots\; A_{N-1}) \quad *$

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Bayes Rule

Going back to the definition of conditional probability

$$P\left(A|B\right) \equiv \frac{P(A\cap B)}{P(B)}$$

Applying a little bit more algebra,

$$P\left(A\cap B\right) = P(A)P(B|A)$$

and $\quad P\left(A\cap B\right) = P(B)P(A|B)$

and thus $\quad P(B)P(A|B) = P(A)P(B|A)$

and we get $\quad P(A|B) = \dfrac{P(A)P(B|A)}{P(B)} \qquad$ **Bayes rule** *

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Bayes Rule

Going back to the definition of conditional probability

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}$$

Applying a little bit more algebra,

$$P(A \cap B) = P(A)P(B|A)$$

and $P(A \cap B) = P(B)P(A|B)$

and thus $P(B)P(A|B) = P(A)P(B|A)$

and we get $P(A|B) = \dfrac{P(A)P(B|A)}{P(B)}$

Pro tip!: Common to represent denominator as marginalization of numerator:

$$P(B) = \sum_{a \in A} P(A, B) = \sum_{a \in A} P(A)P(B|A)$$

Bayes rule *

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- STOP HERE — better to move to ML-lec-07, which is more consistent…

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Expectation

The **expected value** of a function of a random variable *X* that is distributed according to P(X) is:

$$\mathbf{E}_{P(x)}\left\{f(X)\right\} \quad = \quad \sum_x f(x)P(x)$$

The expected value of a (function of a) random variable is the **weighted (by probability) average** of all possible values of that variable (through that function).

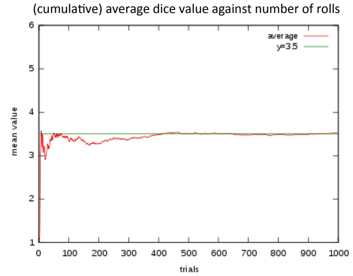
The expected value of the random variable *X* itself: the **mean**

$$\mathbf{E}_{P(x)}\left\{X\right\} \quad = \quad \sum_x xP(x)$$

What is the relationship of the *arithmetic mean* to the expected value?

$$= \frac{1}{N}\sum_{i=1}^{N} x_i$$

(cumulative) average dice value against number of rolls



Expectation

$$\mathbf{E}_{P(x)}\left\{f(X)\right\} \quad = \quad \sum_x f(x)P(x)$$

The expectation of the value of *X* if *X* is a fair die:

$$\mathbf{E}_{P(x)}\{X\} = \sum_x x\frac{1}{6} = \frac{1}{6} + \frac{2}{6} + ... + \frac{6}{6} = \frac{21}{6} = (3.5)^2 = 12.25$$

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