HW7: Review

October 30, 2025

Exercise 1 Consider the following initial value problem for Burgers equation.

$$\partial_t u(t,x) + u(t,x) \cdot \partial_x u(t,x) = 0, \quad u(0,x) = \varphi(x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

1. Let
$$\varphi(x) = \begin{cases} 0, & x \le 0, \\ x, & x > 0. \end{cases}$$
Find a solution $u(t,x) \in \mathcal{C}^{1,1}\big((0,\infty)\times\mathbb{R}\big) \cap \mathcal{C}\big([0,\infty)\times\mathbb{R}\big)$.

2. Let
$$\varphi(x) = \begin{cases} 1, & x \le 0, \\ 1 - \frac{1}{2}x, & 0 < x \le 2, \\ 0, & x > 2. \end{cases}$$
Find the largest time t_s such that all characteristics do not intersect.

3. Find the expression of $u(t,x)$ for $t < t_s$.

Exercise 2

1. Find all the eigenvalues and eigenfunctions of the Sturm–Liouville problem:
$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X'(\pi) = 0.$$

2. Use Separation of Variables to solve the homogeneous heat equation:
$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & x \in (0,\pi),\ t > 0, \\ u(0,x) = \sin\frac{x}{2}, & x \in [0,\pi], \\ u(t,0) = \partial_x u(t,\pi) = 0, & t \ge 0. \end{cases}$$

3. Use Duhamel's principle to solve the inhomogeneous heat equation:
$$\begin{cases} \partial_t u - \partial_{xx} u = f, & x \in (0,\pi),\ t > 0, \\ u(0,x) = \sin\frac{x}{2}, & x \in [0,\pi], \\ u(t,0) = \partial_x u(t,\pi) = 0, & t \ge 0. \end{cases}$$

Exercise 3 Let $\Omega = B_1(0) \subset \mathbb{R}^2$ and $\Omega_T = (0,T] \times \Omega$.

1. Prove the following comparison principle: if $v \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ satisfies

$$\begin{cases} \partial_t v - \Delta v \geq 0, & (t,x) \in \Omega_T, \\ v(0,x) \geq 0, & x \in \Omega, \\ \dfrac{\partial v}{\partial n}(t,x) > 0, & t > 0,\ x \in \partial\Omega, \end{cases}$$

then $v(t,x) \geq 0$ on $\overline{\Omega_T}$.

2. Assume instead that $\dfrac{\partial}{\partial n} v(t,x) \geq 0$ for all $t > 0$ and $x \in \partial\Omega$. Show that $v(t,x) \geq 0$ on $\overline{\Omega_T}$.

   Hint: consider $v_\varepsilon(t,x) = v(t,x) + \varepsilon(4t + x_1^2 + x_2^2 - 1)$.

3. Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ solve

$$\begin{cases} \partial_t u - \Delta u = f(t,x), & (t,x) \in \Omega_T, \\ u(0,x) = \varphi(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial n}(t,x) = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

   Show that there exists a constant $C = C(T)$ such that

$$\sup_{\overline{\Omega_T}} |u| \leq C\big(\sup_{\Omega_T} |f| + \sup_{\Omega} |\varphi|\big).$$

Exercise 4 Let $K(x,y) : \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}$ be defined by

$$K(x,y) = \frac{1}{\pi} \frac{x_1}{(x_2 - y)^2 + x_1^2}.$$

1. Show that $\displaystyle\int_{\mathbb{R}} K(x,y)\, dy = 1$, and for every $\varepsilon > 0$ and $z \in \mathbb{R}$,

$$\lim_{h \to 0+} \int_{z-\varepsilon}^{z+\varepsilon} K\big((h,z),y\big)\, dy = 1.$$

2. Let $g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $|g| \leq 1$. Let

$$g_h(z) = \int_{\mathbb{R}} K\big((h,z),y\big) g(y)\, dy, \quad h > 0.$$

   Show that $g_h \to g$ uniformly on $\mathbb{R}$.