Week 3, Thursday

△ Review: interpolation (插值) given $\{x_j, f(x_j)\}_{j=0}^{n}$, $\exists!$ a polynomial $P(x)$ degree of at most n, s.t. $P(x_j) = f(x_j)$

Lagrange interpolating polynomials: $L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}$

Key Property: $L_j(x_k) = \delta_{jk} = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}$

Then the interpolation is $P(x) = \sum_{j=0}^{n} f(x_j) L_j(x)$

The reminder:

$$R(x) = f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Cond: $f \in C^{n+1}[a, b]$, $\xi(x) \in [a, b]$,

Q: what if we try to add one more point to do the new interpolation? should we start over from the beginning since: $L_k(x)$ changed.

A: Neville's method

Def: $x_0, x_1, \cdots, x_n$ are $n+1$ points, $x_{m_1}, x_{m_2}, \cdots, x_{m_k}$ are k distinct points among $\{x_j\}_{j=0}^{n}$, the polynomial that agree with f at $x_{m_1}, x_{m_2}, \cdots, x_{m_k}$ is denoted $P_{m_1, m_2 \cdots m_k}(x)$.

e.g. $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 6$, $f(x) = e^x$.

$$P_{1,2,4} = \frac{(x-3)(x-6)}{(2-3)(2-6)} e^2 + \frac{(x-2)(x-6)}{(3-2)(3-6)} e^3 + \frac{(x-2)(x-3)}{(6-2)(6-4)} e^6$$

$\boxed{\text{Thm}}$ let f be defined at $x_0, x_1, \cdots, x_n$, and x_i and x_j are distinct then the Lagrange polynomial agrees f at $x_0, x_1, \cdots, x_k$

$$P(x) = \frac{(x - x_j) P_{0,1,\cdots,j-1,j+1,\cdots,k}(x) - (x - x_i) P_{0,1,\cdots,i-1,i+1,\cdots,k}(x)}{x_i - x_j}.$$

Proof: $\forall x_m \neq x_i$, $x_m \neq x_j$

$$p(x_m) = \frac{(x_m - x_j) f(x_m) - (x_m - x_i) f(x_m)}{x_i - x_j} = f(x_m)$$

$$x_m = x_i: \quad p(x_i) = \frac{(x_i - x_j) f(x_i) - 0}{x_i - x_j} = f(x_i)$$

$$x_m = x_j: \quad p(x_j) = \frac{0 - (x_j - x_i) f(x_j)}{x_i - x_j} = f(x_j)$$

Obviously: $p(x)$ is a polynomial of degree at most k. Done!

Rk: $P_k(x) \equiv f(x_k)$.

$$P_{0,1} = \frac{(x-x_0)P_1 - (x-x_1)P_0}{x_1 - x_0}, \quad P_{1,2} = \frac{(x-x_1)P_2 - (x-x_2)P_1}{x_2 - x_1}$$

$$\Rightarrow P_{0,1,2} = \frac{(x-x_0)P_{1,2} - (x-x_2)P_{0,1}}{x_2 - x_0}$$

Rk:

$$\begin{array}{llllll}
 & x_0 & P_0 & & & \\
 & x_1 & P_1 & P_{01} & & \\
 & x_2 & P_2 & P_{12} & P_{0,1,2} & \\
 & x_3 & P_3 & P_{23} & P_{1,2,3} & P_{0,1,2,3} \\
\text{add:} & x_4 & P_4 & P_{34} & P_{2,3,4} & P_{1,2,3,4} \quad P_{0,1,2,3,4}
\end{array}$$

Rk: a interpolating polynomial can be get by the combination of interpolating polynomials on the subset of nodes.

advantage: easy to implement for adding more points.

Δ look at the interpolation with another basis:

$\{1, x, x^2, \cdots, x^n\}$ natural basis $\qquad \{L_0(x), L_1(x), \cdots, L_n(x)\}$ Lagrange.

Q: other basis?

A: $1,\ x - x_0,\ (x-x_0)(x-x_1),\ \cdots,\ (x-x_0)(x-x_1)\cdots(x-x_{n-1})$ $\quad$ dim $= n+1$

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

determine coefficients a_j:

$$p(x_0) = a_0 = f(x_0)$$

$$P(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1) \Rightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Next introduce divided-difference notation (均差) (d-d)

idea: recursion formula.

zeroth d-d: $f[x_i] = f(x_i)$

first d-d: $$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

kth d-d: $$f[x_i, x_{i+1}, \cdots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \cdots, x_{i+k}] - f[x_i, x_{i+1}, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$\begin{tikzcd}
 & \text{0th} & \text{1st} & \text{2ed} & \text{3rd} \\
x_0 & f[x_0] \arrow[dr] & & & \\
x_1 & f[x_1] \arrow[r] \arrow[dr] & f[x_0,x_1] \arrow[dr] & & \\
x_2 & f[x_2] \arrow[r] \arrow[dr] & f[x_1,x_2] \arrow[r] \arrow[dr] & f[x_0,x_1,x_2] \arrow[dr] & \\
x_3 & f[x_3] \arrow[r] & f[x_2,x_3] \arrow[r] & f[x_1,x_2,x_3] \arrow[r] & f[x_0,x_1,x_2,x_3]
\end{tikzcd}$$

revisit $P(x) = a_0 + a_1(x - x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)\cdots(x-x_{n-1})$

$$a_0 = f[x_0]$$

$$a_1 = f[x_0, x_1]$$

$$f(x_2) = f[x_0] + f[x_0,x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\Leftrightarrow f[x_2] - f[x_1] \cancel{+ f[x_1] - f[x_0]} = \underline{f[x_0, x_1](x_2 - x_0)} + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\cancel{f[x_0,x_1](x_1 - x_0)} + f[x_0,x_1](x_2 - x_1)$$

$$\Leftrightarrow f[x_2] - f[x_1] - (x_2 - x_1) f[x_0, x_1] = a_2(x_2 - x_0)(x_2 - x_1)$$

$$\Leftrightarrow f[x_1, x_2] - f[x_0, x_1] = a_2(x_2 - x_0) \Leftrightarrow a_2 = f[x_0, x_1, x_2]$$

Newton's divided-difference formula for the interpolation:

$$\boxed{P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k](x - x_0)(x - x_1)\cdots(x - x_{k-1})}$$

Q: how to show above is the interpolation, i.e. $P(x_j) = f(x_j)$

Proof: fix $x \in [a, b)$ as one point:

by the definition of d-d formula:

$$f(x) = f(x_0) + f[x, x_0](x - x_0)$$

$$f[x, x_0] = f[x_0, x_1] + f[x, x_0, x_1](x - x_1)$$

$$\vdots$$

$$f[x, x_0, \cdots, x_{n-2}] = f[x_0, x_1, \cdots, x_{n-1}] + f[x, x_0, \cdots, x_{n-1}](x - x_{n-1})$$

$$f[x, x_0, \cdots, x_{n-1}] = f[x_0, x_1, \cdots, x_n] + f[x, x_0, \cdots, x_n](x - x_n)$$

$\Rightarrow$ $f(x) = P(x) + R_n(x)$, where: d-d formula for the reminder

$$\boxed{R_n(x) = f[x, x_0, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_n)}$$

$$\therefore f(x_j) = P(x_j) + R_n(x_j) = P(x_j) + \underbrace{R_n(x_j)}_{=0 \text{ for } j \le n} = P(x_j) \text{ for } \underline{j \le n}$$

advantages ① just add one more basis if we add one more point

② $\phi_j(x_i) = 0$, for $i < j$, if add one more points,

the coefficients does not change for the old basis,

since $\phi_{n+1}(x_j) = 0$, $j = 0, 1, \cdots, n$, $P_{new}(x_j) = P_{old}(x_j)$

	0th	1st	2ed	3rd	4th
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_2, x_3, x_4, x_5]$	$f[x_0, x_1, x_2, x_3, x_4]$

P1: $$f[x_0, x_1, \cdots, x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{(x_j - x_0)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_n)}$$

Proof: the coefficient of x^n

Proof: the coefficient of x^n

in Lagrange: $P(x)=\sum_{j=0}^{n} f(x_j)L_j(x)$ (RHS)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

in Newton's d-d: $P(x)=\sum_{j=0}^{n} f[x_0,\cdots,x_j]\omega_j(x)$ (LHS)

RK: d-d is independent of the order of nodal points.

P2: $f\in C^n[a,b]$, $x_0,x_1,\cdots,x_n$ distinct $\in[a,b]$, $\exists\,\xi\in(a,b)$ s.t.

$$f[x_0,x_1,\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!}$$

Proof: as $P(x_j)=f(x_j)$ $j=0,1,\cdots,n+1$. By generalized Rolle's thm

$\exists\,\xi\in(a,b)$ s.t. $f^{(n)}(\xi)=p^{(n)}(\xi)$

$p^{(n)}(\xi)=n!\,a_n=n!\,f[x_0,x_1,\cdots,x_n]$.

RK: for $f\in C^n[a,b]$, $\{x_i\}\subset[a,b]$, $\exists\,\xi\in(a,b)$ s.t.

$$\sum_{j=0}^{n}\frac{f(x_j)}{\prod_{k\neq j}(x_j-x_k)}=\frac{f^{(n)}(\xi)}{n!}$$

RK: coefficient of x^n in $P_n(x)$

RK: the reminder: $f[x,x_0,\cdots,x_n]\,\omega_n(x)=\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\omega_{n+1}(x)$

$\triangle$ equal spacing: $h=x_{i+1}-x_i$, $x_i=x_0+ih$, $i=1,2,\cdots,n$.

let $x=x_0+sh$ $\Rightarrow$ $x-x_i=(s-i)h$

$$P(x)=f[x_0]+\sum_{k=1}^{n} f[x_0,x_1,\cdots,x_k](x-x_0)(x-x_1)\cdots(x-x_{k-1}).$$

$$=f[x_0]+\sum_{k=1}^{n} s(s-1)\cdots(s-k+1)\,h^k f[x_0,x_1,\cdots,x_k]$$

Def: $\binom{s}{k}=\frac{s(s-1)\cdots(s-k+1)}{k!}$ binomial - coefficient (二项系数)

$$\Rightarrow P(x)=P(x_0+sh)=f[x_0]+\sum_{k=1}^{n}\binom{s}{k}k!\,h^k f[x_0,x_1,\cdots,x_k]$$

Def: forward difference formula: $\Delta f(x_k)=f(x_{k+1})-f(x_k)$

$$f[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{1}{h}\Delta f(x_0)$$

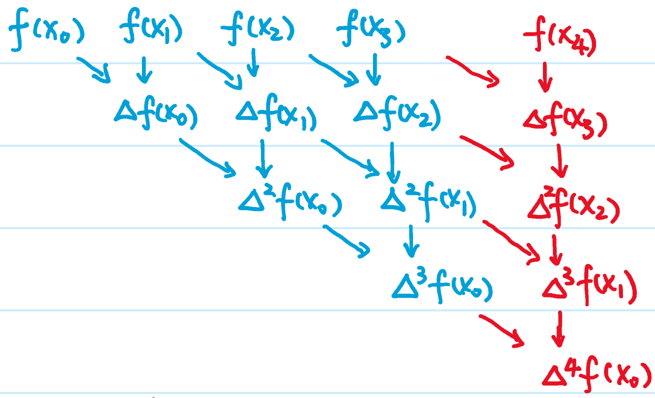
$$f[x_0,x_1,x_2] = \frac{f[x_2,x_1]-f[x_1,x_0]}{x_2-x_0} = \frac{\frac{1}{h}[\Delta f(x_1)-\Delta f(x_0)]}{2h} = \frac{1}{2h^2}\Delta^2 f(x_0)$$

$$\vdots$$

$$f[x_0,x_1,\cdots,x_k] = \frac{1}{h^k k!}\Delta^k f(x_0) \qquad \binom{s}{0}=1,\ \Delta^0 f(x_k) = If(x_k) = f(x_k)$$

Hence, Newton forward-difference formula (equal spacing)

$$\boxed{P(x) = f(x_0) + \sum_{k=1}^{n}\binom{s}{k}\Delta^k f(x_0) := \sum_{k=0}^{n}\binom{s}{k}\Delta^k f(x_0)}$$



$$P_4(x) = P_3(x) + \binom{s}{4}\Delta^4 f(x_0), \quad x = x_0 + sh.$$

Similarly: def: backward difference formula: $\nabla f(x_k) = f(x_k) - f(x_{k-1})$

$$\boxed{P(x) = \sum_{k=0}^{n}(-1)^k\binom{-s}{k}\nabla^k f(x_n)}$$

Δ Hermite interpolation: $f \in C^1[a,b]$, $x_0, x_1, \cdots, x_n \in [a,b]$ distinct

Def: Hermite polynomial: $H(x)$ agrees with f and f' at $x_0, x_1, \cdots, x_n$.

i.e. $H(x_j) = f(x_j)$, $H'(x_j) = f'(x_j)$ $j=0,1,\cdots,n$. ($2n+2$ conditions)

degree of at most $2n+1$.

Thm The unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \cdots, x_n$ is the Hermite polynomial of degree at most $2n+1$ given by:

$$H(x) = \sum_{j=0}^{n} f(x_j)H_j(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_j(x), \text{ where}$$

$$H_j(x) = [1-2(x-x_j)L_j'(x_j)]L_j^2(x), \quad \hat{H}_j(x) = (x-x_j)L_j^2(x)$$

and $L_j(x)$ is the jth Lagrange interpolating polynomial

Proof: to construct the Hermite polynomial, we need to find:

(i) $H_j(x_i) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ $\quad \hat{H}_j(x_i) = 0 \Rightarrow H(x_i) = f(x_i)$

(ii) $H_j'(x_i) = 0$, $\quad \hat{H}_j'(x_i) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \Rightarrow H'(x_i) = f'(x_i)$

i.e. ① x_i $(i\neq j)$ is a double root of $H_j(x)$: $H_j(x) = (ax+b)L_j^2(x)$

② x_i $(i\neq j)$ is a double root of $\hat{H}_j(x)$ and x_j is the single root

$$\hat{H}_j(x) = C(x-x_j)L_j^2(x)$$

Furthermore: $\begin{cases} H_j(x_j) = (ax_j+b)L_j^2(x_j) = 1 \\ H_j'(x_j) = aL_j^2(x_j) + 2(ax_j+b)L_j'(x_j)L_j(x_j) = 0 \end{cases}$

i.e. $\begin{cases} ax_j+b = 1 \\ a+2(ax_j+b)L_j'(x_j) = 0 \end{cases} \Rightarrow \begin{cases} b = 1+2x_jL_j'(x_j) \\ a = -2L_j'(x_j) \end{cases}$

$$\therefore H_j(x) = [1-2(x-x_j)L_j'(x_j)]L_j^2(x)$$

$$\hat{H}_j'(x_j) = CL_j^2(x_j) = 1 \Rightarrow C = 1$$

$$\therefore \hat{H}_j(x) = (x-x_j)L_j^2(x)$$

error bound: $f \in C^{2n+2}[a,b]$, $\exists \xi \in (a,b)$, s.t.

$$f(x) = H(x) + \frac{(x-x_0)^2(x-x_1)^2\cdots(x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

HW 3-2: Sec 3.2: 5, 8.

Sec 3.3: 6, 9 ab, 11 14 15 16 20 22 23

extra: use the definition of divided-difference.

define $P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k](x-x_0)(x-x_1)\cdots(x-x_{k-1})$

Show that $P(x_j) = f(x_j)$

Due: 2025.10.16