Introduction to Mathematical Logic

For CS Students

CS104

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Course Info

Welcome!

- **Lectures**: Week 1-16
  - Tuesday 5-6, 一教 111
- **No Lab**
- **Instructor**
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- **Teaching Assistants**
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Course Info

Welcome!

- **Course Site**: Blackboard
- **Course Group**: see announcement on Blackboard course site

Course Evaluation

Welcome!

- Class participation, Quiz: 10%
- Assignments: 40%
- Final Exam: 50%

Table of Contents

A Logic Puzzle

Zhang San, Li Si, and Wang Wu each have cat(s).
Among them, two people have white cats, and two people have black cats.
The person who has black cat(s) always tells lies, and the person who only has a white cat may not always tell the truth.
Zhang San says, “Li Si has a white cat.”
Li Si says, “Wang Wu has a white cat.”

Question: Who has what cat(s)?

Source: 微信公众号：数理逻辑与哲学逻辑

What is Logic?

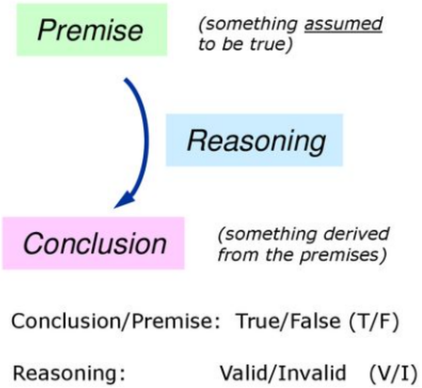
1 Warm Up

- **Intuition (直觉)**: Have you ever said to someone, "be logical"? Whatever your intuition was, that's logic.
- **Reasoning (推理)**: Logic is the study of reasoning: the process of using reasons and critical thinking to analyze ideas, find connections, and draw conclusions.
- **Proof (证明)**: conclusion(s) can be drawn with **absolute certainty** from a particular set of premises.

What is Logic?

1 Warm Up

- What entails what?
- What follows from what?
- How do we know whether a reasoning is valid?
  (We can't always rely on our intuition.)



What is Mathematical Logic?

- Mathematization/Formalization of the intuition is Mathematical Logic
- Using Mathematical logic, we could formally prove that a reasoning is valid, like we did with Math.
- Using Mathematical logic, even machines could perform reasoning, and validating reasoning.

Table of Contents

Traditional Logic

- "Logic" originates from ancient greek
- The study of principles of reasoning; a part of philosophy
- Explicit analyses of the principles of reasoning was developed in three cultures: China, India, and Greece.
- Modern treatment descends from the Greek tradition, particularly Aristotelian Logic, core of which includes Reductio ad absurdum (归谬法，Reduction to absurdity) and syllogism (三段论).

Reduction to absurdity

亚里士多德，《形而上学》

同一个东西同时且在同一方面既属于又不属于同一个东西，这是不可能的。

如果你从一个主张推出了矛盾，你就要否定这个主张 (A method of proving the falsity of a statement by assuming its truth and then deriving a contradiction.)。

Syllogism (三段论)

- 大前提（Major premise）：一般性原则，描述一个大范围的情况或关系 (A broad statement or generalization)。
- 小前提（Minor premise）：特殊陈述，描述一个较小范围的情况或关系 (A specific statement related to or within the scope of the major premise)。
- 结论（Conclusion）：基于主前提和副前提，推导出一个逻辑上的结论 (A statement that logically follows from the combination of the major and minor premises)。

Syllogism (三段论)

Example I

- Major premise: All humans are mortal
- Minor premise: Socrates is a human
- Conclusion: Socrates is mortal

Example II

- Major premise: Metal conducts electricity.
- Minor premise: Copper is a metal.
- Conclusion: Copper conducts electricity.

Example III

- Major premise: All humans are mortal
- Minor premise: Socrates is mortal
- Conclusion: Socrates is human

This argument has all true premises (and a true conclusion), but it it is invalid.

Example III

- Major premise: All humans are mortal
- Minor premise: Socrates is mortal
- Conclusion: Socrates is human

Example IV

- Major premise: All humans are mortal
- Minor premise: My friend’s dog is mortal.
- Conclusion: My friend’s dog is human.

The Validity of Reasoning

Truth for statements, Validity for arguments/reasoning. They are not the same.

- Valid reasoning does not guarantee a true conclusion.
- Invalid reasoning does not guarantee a false conclusion.
- A false conclusion does not guarantee a invalidity.
- True premises and a true conclusion together do not guarantee a validity.

Ambiguity of Natural Languages

Example V

- (Major Premise) A student from Class A is the captain of the soccer team .
- (Minor Premise) x knows a student from Class A .
- (Conclusion) x knows the captain of the soccer team.

Example VI

- (Major Premise) Nothing is better than happiness.
- (Minor Premise) Junk food is better than nothing.
- (Conclusion) Junk food is better than happiness.

Ambiguity of Natural Languages

- The similarity in natural language forms (e.g., “a student from Class A” in Example V, “nothing” in Example VI) does not guarantee that the logical forms are identical.
- Different natural language forms (Example I and II, Example III and IV) may have the same logical form.

Natural language is NOT reliable for logical reasoning.

Early Phase

- 1710 年，莱布尼茨在《神正论》中提出建立一种普遍语言的设想，“这种语言是一种用来代替自然语言的人工语言，它通过字母和符号进行逻辑分析与综合，把一般逻辑推理的规则改变为演算规则，以便更精确更敏捷地进行推理”。
- In the early 18th century, Leibniz outlined his characteristica universalis, an artificial language in which grammatical and logical structure would coincide, allowing reasoning to be reduced to calculation.

Early Phase

2 History

"When there are disputes among persons, we can simply say, 'Let us calculate,' and without further ado, see who is right." —Gottfried Wilhelm Leibniz

Leibniz 将这种语言称为 "人类思想的字母表"(alphabet of human thought)：在这种语言中，一切理性真理都会被还原为一种演算，所有推理的错误都只成为计算的错误。

Early Phase

Although Leibniz didn’t succeed in realizing it, this idea becomes the essence of mathematical logic. The origins of modern mathematical logic are often attributed to Leibniz.

- Constructing a “universal language” (通用语言， a universally applicable, precise scientific language).
- Constructing “calculus of reasoning” (推理演算， similar to a computational calculus, capable of proving correctness or incorrectness).

Development

2 History

- George Boole and A. De Morgan presented systematic mathematical treatments of propositional logic
- Gottlob Frege and Charles Sanders Peirce independently developed the foundation of first-order logic
- Great Logicians: Bertrand Russell (罗素), David Hilbert(希尔伯特), Gerhard Gentzen(甘岑), Kurt Gödel(哥德尔), and Alfred Tarski(塔斯基)

Table of Contents

3 An Informal Introduction to Reasoning

A Motivating Example

《韩非子 · 难一》

楚人有鬻盾与矛者，誉之曰：“吾盾之坚，物莫能陷也。”又誉其矛曰：“吾矛之利，于物无不陷也。”或曰：“以子之矛陷子之盾，何如？”其人弗能应也。

”In the state of Chu, there was a person selling shields and spears. He praised the shields, saying, ’My shield is sturdy, and nothing can penetrate it.’ He also praised his spear, saying, ’My spear is so sharp that it can pierce through anything.’ Someone asked, ’What if you use your spear to pierce through your own shield?’ The person could not provide an answer.” - Translated by AI

What is Reasoning?

- A reasoning consists of (a finite number of) declarative sentences, where some sentences infer another sentence (indicated by the word “therefore”).
- Declarative sentences: Used for declarations or statements, characterized by being true or false (not sentences posing questions, giving commands, or expressing emotions), which is referred to as propositions(命题).

1.1 No spear can pierce through this shield. Therefore, this spear cannot pierce through this shield.

1.2 If this spear can pierce through all shields, then it can pierce through this shield. This spear cannot pierce through this shield. Therefore, it is not the case that this spear can pierce through all shields.

What is Reasoning?

Reasoning is a set of propositions. One of them (called the conclusion) is deduced/derived from others (called premises). We use rules for deduction/derivation.

1.1 No spear can pierce through this shield. Therefore, this spear cannot pierce through this shield.

1.2 If this spear can pierce through all shields, then it can pierce through this shield. This spear cannot pierce through this shield. Therefore, it is not the case that this spear can pierce through all shields.

What are Rules?

1.3 All spears cannot pierce through this shield. Therefore, this spear cannot pierce through this shield.

1.4 Everyone reasons. Han Fei is a person. Therefore, Han Fei also reasons.

1.5 Everything undergoes change. A table is something. Therefore, the table undergoes change.

1.6 All Greeks are philosophers. Aristotle is a Greek. Therefore, Aristotle is a philosopher.

What are Rules?

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Rule: All X is Y. a is X. Therefore, a is Y.

1.3 All spears cannot pierce through this shield. Therefore, this spear cannot pierce through this shield.

1.4 Everyone reasons. Han Fei is a person. Therefore, Han Fei also reasons.

1.5 Everything undergoes change. A table is something. Therefore, the table undergoes change.

1.6 All Greeks are philosophers. Aristotle is a Greek. Therefore, Aristotle is a philosopher.

Takeaway: Logic studies forms of reasoning. The above reasoning have the same form. The content doesn't matter; the reasoning process should be applicable to any subject: water purity, mathematics, cooking, nuclear physics, ethics, or whatever.

Is the Rule Valid?

If this spear can pierce through all shields, then this spear can pierce through this shield.

This spear can pierce through all shields.

Therefore, this spear can pierce through this shield.

Valid Rule: If P then Q; P; Therefore Q

Is the Rule Valid?

If this spear can pierce through all shields, then this spear can pierce through this shield.

This spear cannot pierce through this shield.

Therefore, it is not the case that this spear can pierce through all shields.

Valid Rule: If P then Q; Not Q; Therefore not P.

Is the Rule Valid?

If this spear can pierce through all shields, then this spear can pierce through this shield.

It is not the case that this spear can pierce through all shields.

Therefore, this spear cannot pierce through this shield.

Invalid Rule: If P then Q; Not P; Therefore not Q

What We'll Learn in This Course

- How do we express ambiguous natural language reasoning formally?
- How do we prove that a reasoning is valid?

What We'll Learn in This Course

- How do we express ambiguous natural language reasoning formally?
  - **Formal languages** (形式语言): a precisely defined set of symbols and syntax rules. Using symbols and apply syntax rules to form formulas, where formulas represent propositions
- How do we prove that a reasoning is valid?

Formal Languages

1. (Digital sequence understood by computer)

   0010101010000010111101000

2. (Programme Language, eg. Java or C)

   ```
   s = 1; i = n; while (i > 0) { s *= a; i--; }
   ```

3. (Propositional Logic)

   $(\neg((p \vee q) \to p))$

4. (First-Order Logic)

   $\forall \epsilon \exists \delta \forall x (|x - a| < \delta \to |f(x) - c| < \epsilon)$

5. (Modal Logic)

   $\neg(\Diamond p) \leftrightarrow \Box(\neg p)$

What We'll Learn in This Course

- How do we express ambiguous natural language reasoning formally?
  - **Formal languages** (形式语言): a precisely defined set of symbols and syntax rules. Using symbols and apply syntax rules to form formulas, where formulas represent propositions
- How do we prove that a reasoning is valid?
  - **Formal Deduction System** (形式推演系统): logical truth or falsity of propositions is expressed through deductive calculations based on the formulas and rules.

Formal Deduction System - Propositional Logic

3 An Informal Introduction to Reasoning

1.	$((p \to q) \wedge (\neg r \to \neg q))$	Supposition
2.	p	Supposition
3.	$((p \to q) \wedge (\neg r \to \neg q))$	1 Reiterate
4.	$(p \to q)$	3 Simplification
5.	q	2, 4 Modus Ponens
6.	$(\neg r \to \neg q)$	3 Simplification
7.	$\neg r$	Supposition
8.	$(\neg r \to \neg q)$	6 Reiterate
9.	$\neg q$	7, 8 Modus Ponens
10.	q	5 Reiterate
11.	r	7–10 Reductio ad Absurdum
12.	$p \to r$	2-11 Conditionalization
13.	$(((p \to q) \wedge (\neg r \to \neg q)) \to (p \to r))$	1–12 Conditionalization

1	$\forall x \exists y (Px \wedge Qy)$	ass.
2	$\exists y (Pa \wedge Qy)$	1, $(\forall E)$
3	$\quad Pa \wedge Qb$	ass.
4	$\quad Pa$	3, $(\wedge E)$
5	Pa	3–4, $(\exists E)$
6	$\forall x Px$	5, $(\forall I)$
7	$\quad Pa \wedge Qc$	ass.
8	$\quad Qc$	7, $(\wedge E)$
9	$\quad \exists x Qx$	8, $(\exists I)$
10	$\exists x Qx$	7–9, $(\exists E)$
11	$\forall x Px \wedge \exists x Qx$	6, 10, $(\wedge I)$
12	$\forall x \exists y (Px \wedge Qy) \rightarrow \forall x Px \wedge \exists x Qx$	1–11, $(\rightarrow I)$

What We'll Learn in This Course

For propositional logic and first-order logic:

- (Modeling) How do we use this logic (formal language) to model real world phenomenon (natural language)?
- (Syntax) How do we formulate well-formed formulas in this formal language?
- (Semantics) How do we interpret the formulas written in this formal language?
- (Deduction) How do we perform reasoning and inference using this formal language?

Table of Contents

4 Applications

Logic is fun!

The Barber Paradox: a barber who shaves all and only those men in the town who do not shave themselves. Then, who shaves the barber?

Logic is fun!

The Liar Paradox: This sentence is false.

Logic is fun!

Find the true statement and get a A+ for this course!

The statement in the right box is true.	The statement in the left box is false.

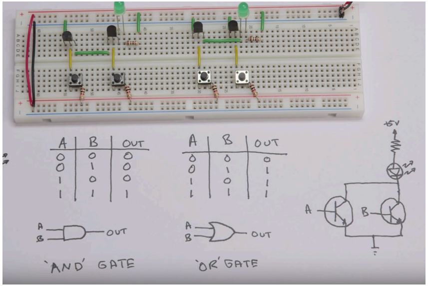
Why should we study logic?

- Logic is fun!
- Logic improves one's ability to think analytically and to communicate precisely
- Logic has many applications in Computer Science.

Circuit Design

4 Applications

- Digital circuits are the basic building blocks of an electronic computer
- We can represent the inputs and outputs of the logic gate with symbols and boolean algebra (propositional logic). Subsequently, we verify the correctness of the circuit through methods such as truth tables.



Databases

4 Applications

- First-order logic is one of the theoretical foundations of Relational Databases.
- A relational database implements a set of predicates to represent relations:
  - $Major(pid, cs)$: the student with PID pid has declared the major CS.
  - $DeansList(pid, sem)$: the student with PID pid made the Dean's List in semester sem.
- Properties of the database can be expressed in predicate logic
  - $\exists p(DeansList(p, 23Spring))$: Some student made the Dean's list in 23Spring.
  - $\forall p(DeansList(p, 23Spring) \to \exists m(Major(p, m)))$: Every student who made the Dean's list in 23Spring has declared some major.

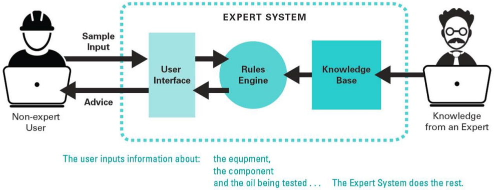
Formal Verification

4 Applications

- Bugs can be costly and dangerous in real life
  - (Hardware bug) Intel's Pentium FDIV bug (1994) cost them half a billion dollars.
  - (Software bug) Cancer patients died due to severe overdose of radiation
- Formal verification can be used to prove that a program is bug-free.

Knowledge representation and reasoning

- AI systems (e.g., the expert system) utilize logic-based formalisms such as first-order logic and ontologies to represent and organize knowledge.
- These formal representations enable AI algorithms to reason about the relationships between entities, properties, and events, facilitating intelligent decision-making



Example: an expert system to assist in diagnosing user illnesses: the system represent the relationships between different symptoms and diseases through a set of symbols and rules based on the knowledge of doctors. This allows reasoning and diagnosis based on the symptoms provided by the patients.

Automated Discovery in Science

- Reasoning and inference systems have also been used to prove previously unproven results, rather than just simply check proofs produced by a human. One of the most impressive feats of machine reasoning occurred in 1996, when a first-order logic reasoner was able to prove an open math conjecture that was unsolved for 60 years.
- In 2007, a group in Wales and England created a system named Adam (Automated Discovery and Analysis Machine). Adam can automatically form scientific hypotheses, perform experiments to test hypotheses, and record results of experiments. Adam was the first automated system to discover non-trivial scientific information. It successfully identified the function of some genes in yeast.

- PROLOG (PROgramming with LOGic) has roots in first-order logic.
- Prolog can be used for theorem proving, expert systems, and natural language processing
- Example: On February 14-16, 2011, IBM Watson won the Jeopardy Man vs. Machine Challenge by defeating two former grand champions. Watson relied on natural language processing technology to analyze the vast amount of unstructured text (encyclopedias, dictionaries, news articles, etc.). Prolog was the ideal language for this due to its simplicity and expressiveness.

It was only in the early 1970's that the idea emerged to use the formal language of logic as a programming language. An example is PROLOG, which stands for PROgramming in LOGic. A logic program is simply a set of formulas (of a particular form) in the language of predicate logic. The formulas below constitute a logic program for kinship relations. The objects are people and there are two binary predicates 'parent of' (p), and 'grandparent of' (g).

- A_1: $p(\text{art}, \text{bob})$.
- A_2: $p(\text{art}, \text{bud})$.
- A_3: $p(\text{bob}, \text{cap})$.
- A_4: $p(\text{bud}, \text{coe})$.
- A_5: $g(x,z)$:- $p(x,y)$, $p(y,z)$.

'art', 'bob', 'bud', 'cap' and 'coe' are individual constants and A_5 stands for $p(x,y) \wedge p(y,z) \rightarrow g(x,z)$.

Now if we ask the question

$$?\text{- } g(\text{art}, \text{cap})$$

the answer will be 'yes', corresponding with the fact that $g(\text{art}, \text{cap})$ can be logically deduced from the premisses or data $A_1, \ldots, A_5$.

But if we ask the question

$$?\text{- } g(\text{art}, \text{amy})$$

the answer will be 'no', corresponding with the fact that $g(\text{art}, \text{amy})$ cannot be logically deduced from $A_1, \ldots, A_5$. Note that this does not mean that $\neg g(\text{art}, \text{amy})$ logically follows from $A_1, \ldots, A_5$.

Table of Contents

Main Topics Covered (Negotiable)

- Preliminaries (预备知识)
- Propositional Logic (经典命题逻辑)
- First-order Logic (经典一阶逻辑)
- Program Verification (程序验证)

Expected Output

- Thinking and communicating precisely
- Problem solving
- Critical thinking
- Creative thinking

Textbooks & References

- Text1:《面向计算机科学的数理逻辑》（科学出版社）陆钟万
- Text2:《数理逻辑》（北京大学出版社）邢滔滔
- Text3: 《数理逻辑证明及其限度》（复旦大学出版社）郝兆宽等

Textbooks & References

- TextA: Introduction to Logic. 14ed. Copi Cohen. 2011
- TextB: Logic in Computer Science. Huth Ryan. 2004
- TextC: Philosophical and Mathematical Logic. H. Swart. 2018
- TextD: Logic and Proof. Jeremy Avigad, Robert Y. Lewis, and Floris van Doorn
- TextE: Mathematical Logic. Slaman Woodin. 2019.
- TextF: Mathematical Logic for Computer Science. Mordechai Ben-Ari
- TextG: Logic for Computer Scientists. Uwe Schöning
- TextH: A Mathematical Introduction to Logic. Herbert B. Enderton. 2001.
- TextI: A First Course in Logic. Shawn Hedman.

Readings

Finished in 2 weeks

- TextA: Chapter 1
- TextF: Chapter 1

Academic Integrity

- Although discussion with others is allowed, assignments and quizzes should be done individually.
- If you get an idea for a solution from others or online resources, you must acknowledge the source in your submission.
- It is strictly prohibited to directly copy AI-generated answers for assignment submissions or to extensively plagiarize AI-generated content. Such actions will be considered plagiarism and will be handled in accordance with academic integrity regulations.

Academic Integrity

- If an undergraduate assignment is found to be plagiarized, the first time the score of the assignment will be 0.
- The second time the score of the course will be 0.
- If a student does not sign the Assignment Declaration Form or cheats in the course, including regular assignments, midterms, final exams, etc., in addition to the grade penalty, the student will not be allowed to enroll in the two CS majors through 1+3, and cannot receive any recommendation for postgraduate admission exam exemption and all other academic awards.

Your First Assignment

Sign this form and upload it to Blackboard (Assignments -> Declaration Form).

南方科技大学 SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY | 计算机科学与工程系 Department of Computer Science and Engineering

Undergraduate Students Declaration Form

This is____________(student ID:____________, who has enrolled in ________ course of the Department of Computer Science and Engineering. I have read and understood the regulations on courses according to "Regulations on Academic Misconduct in courses for Undergraduate Students in the SUSTech Department of Computer Science and Engineering". I promise that I will follow these regulations during the study of this course.

Introduction to Mathematical Logic

Thank you for listening!

Any questions?