12311410 陈悦 PDE(H) HW7.

Exercise 1.

1. Sol: Let $w(t) = u(t, \eta(t))$. $\Rightarrow \dot{w}(t) = \partial_t u + \eta'(t)\, \partial_x u(t, \eta(t))$.

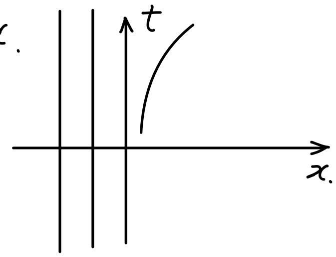
Let $\eta'(t) = u(t, \eta(t)) = w(t)$. $\dot{w}(t) = 0$. $\Rightarrow w(t) = \text{Const}$.

$\eta(t) = w(0)\cdot t + x_0$ with $x_0 = \eta(0)$.

Since $u(0,x) = \varphi(x)$, $w(0) = u(0, x_0) = \varphi(x_0)$. $\Rightarrow \eta(t) = \varphi(x_0)\cdot t + x_0$.

① $x_0 \le 0$. $w(t) = w(0) = 0$. $\eta(t) = x_0$. $\Rightarrow x_0 = x$.

$\Rightarrow u(t,x) = w(t) = 0$. for $x \le 0$.



② $x_0 > 0$. $w(t) = w(0) = x_0$. $\eta(t) = x_0(t+1)$.

$\Rightarrow x_0 = \dfrac{x}{1+t}$.

$\Rightarrow u(t,x) = w(t) = \dfrac{x}{1+t}$. for $x > 0$.

$$\Rightarrow u(t,x) = \begin{cases} 0, & x \le 0. \\ \dfrac{x}{1+t}, & x > 0. \end{cases}$$

2. Sol: We have deduced that $x = \varphi(x_0)\cdot t + x_0$.

$$\frac{\partial x}{\partial x_0} = 1 + \varphi'(x_0)\cdot t = \begin{cases} 1, & x_0 \le 0 \\ 1 - \frac{1}{2}t, & 0 < x_0 \le 2 \\ 1, & x_0 > 2. \end{cases}$$

$\Rightarrow$ key changes for $0 < x_0 \le 2$: let $1 - \frac{1}{2}t = 0$ $\Rightarrow t_s = 2$.

3. Sol: $0 \le t < 2$:

① $x_0 \le 0$. $x = \varphi(x_0)t + x_0 = t + x_0$.

$\Rightarrow u(t,x) = w(t) = \varphi(x_0) = 1$. $(x \le t)$.

② $0 < x_0 \le 2$. $x = (1 - \frac{1}{2}x_0)t + x_0 = (1 - \frac{1}{2}t)x_0 + t \Rightarrow x_0 = \dfrac{x-t}{1-\frac{1}{2}t}$.

$$\Rightarrow u(t,x) = w(t) = \varphi(x_0) = 1 - \frac{1}{2}x_0 = 1 - \frac{x-t}{2-t} = \frac{2-t-x+t}{2-t} = \frac{2-x}{2-t}.$$

$(t < x \le 2)$.

③ $x_0 > 2$. $x = \varphi(x_0)t + x_0 = x_0$.

$\Rightarrow u(t,x) = w(t) = \varphi(x_0) = 0. \quad (x>2).$

$$\Rightarrow u(t,x) = \begin{cases} 1, & x \le t. \\ \frac{2-x}{2-t}, & t < x \le 2. \\ 0, & x > 2. \end{cases} \qquad (0 \le t < 2).$$

Exercise 2

1. Sol: $X''(x) = -\lambda X(x)$.

① $\lambda = 0$. $X''(x) = 0$. $\Rightarrow X(x) = Ax + B$. $\quad X(0) = 0 \Rightarrow B = 0$. $\Rightarrow X(x) = 0$

$X'(\pi) = 0 \Rightarrow A = 0$.

② $\lambda > 0$. $r^2 = -\lambda \Rightarrow r = \pm\sqrt{\lambda} i$. $\Rightarrow X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$.

$X(0) = 0 \Rightarrow A = 0$.

$X'(\pi) = 0 \Rightarrow B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda}\pi = \frac{\pi}{2} + k\pi. \Rightarrow \lambda = (\frac{1}{2}+k)^2$.

$k = 0, 1, 2, \cdots\cdots$

$\Rightarrow \lambda_k = (\frac{1}{2}+k)^2, \quad X_k(x) = \sin((\frac{1}{2}+k)x)$.

③ $\lambda < 0$. $r^2 = \lambda$. $\Rightarrow r = \sqrt{\lambda}$. $\Rightarrow X(x) = Ae^{\sqrt{\lambda}x} + B\cdot e^{-\sqrt{\lambda}x}$.

$$\left.\begin{aligned} &X(0) = 0 \Rightarrow A + B = 0. \\ &X'(\pi) = 0 \Rightarrow \sqrt{\lambda}e^{\pi\sqrt{\lambda}}A - \sqrt{\lambda}e^{-\pi\sqrt{\lambda}}B = 0. \end{aligned}\right\} \Rightarrow A = B = 0. \Rightarrow X(x) = 0$$

Thus, eigen-pairs are: $\lambda_k = (\frac{1}{2}+k)^2$. $X_k(x) = \sin((\frac{1}{2}+k)x)$. $k = 0, 1, 2, \cdots$

2. Sol: Look for $u(t,x) = T(t)X(x)$.

$$0 = \partial_t u - \partial_{xx} u = T'(t)X(x) - T(t)X''(x) \Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \underset{\text{let}}{=} -\lambda.$$

$$\Rightarrow \begin{cases} X''(x) = -\lambda X(x) \\ X(0) = X'(\pi) = 0. \end{cases}$$

From 1, we know: $\lambda_k = (\frac{1}{2}+k)^2$. $X_k(x) = \sin((\frac{1}{2}+k)x)$. $\quad k = 0, 1, 2, \cdots\cdots$

$T'(t) = -\lambda T(t) \Rightarrow T(t) = T(0)\cdot e^{-\lambda_k t} = T(0)e^{-(\frac{1}{2}+k)^2 t}$.

$\Rightarrow u_k(t,x) = e^{-(\frac{1}{2}+k)^2 t} \sin((\frac{1}{2}+k)x), \quad k=0, 1, 2, \cdots\cdots$

Now, solve the HE with b.c.

Want to find $\{C_k\}$, s.t. $\quad u(t,x) = \sum_{k=0}^{\infty} C_k \cdot u_k(t,x)$.

$$u(0,x) = \sum_{k=0}^{\infty} C_k \sin((\tfrac{1}{2}+k)x) = \sin\frac{x}{2}.$$

Fourier series.

$$\int_0^{\pi} \phi(x) \sin((\tfrac{1}{2}+n)x) = C_n \int_0^{\pi} \sin^2((\tfrac{1}{2}+n)x)\, dx \Rightarrow C_0 = 1, \quad C_k = 0, \ k=1,2,\cdots\cdots$$

$$\Rightarrow u(t,x) = e^{-\frac{1}{4}t} \sin\frac{x}{2}.$$

3. Sol: $u(t,x) = \sum_{n=0}^{\infty} T_n(t) \sin((\frac{1}{2}+n)x)$

$$f(t,x) = \sum_{n=0}^{\infty} f_n(t) \sin((\tfrac{1}{2}+n)x). \qquad f_n(t) = \frac{2}{\pi}\int_0^{\pi} f(t,y) \sin((\tfrac{1}{2}+n)y)\, dy.$$

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n \sin((\tfrac{1}{2}+n)x). \qquad \phi_n = \frac{2}{\pi}\int_0^{\pi} \phi(y) \sin((\tfrac{1}{2}+n)y)\, dy.$$

$\phi(x) = \sin\frac{x}{2}$.

$$\Rightarrow T_n(t) \text{ solves: } \begin{cases} T_n'(t) + (\frac{1}{2}+n)^2 T_n(t) = f_n(t) \\ T_n(0) = \phi_n. \end{cases}$$

$\Rightarrow$ By Duhamel's Principle, $T_n(t) = e^{-(\frac{1}{2}+n)^2 t}\phi_n + \int_0^t e^{-(\frac{1}{2}+n)^2(t-s)} f_n(s)\, ds.$

And $u(t,x) = \sum_{n=0}^{\infty} T_n(t) \sin((\frac{1}{2}+n)x)$.

Exercise 3.

1. Proof: Suppose rather and denote $(t_0, x_0) = \arg\min_{\overline{\Omega}_T} v(t,x)$, $\quad v(t_0,x_0) < 0$.

Since $v(0,x) \geq 0$, we have $t_0 > 0$.

If $x_0 \in \partial\Omega$, then $\frac{\partial v}{\partial n}(t_0, x_0) \leq 0$. — a contradiction.

If $x_0 \in \Omega$, then $\partial_t v(t_0,x_0) \leq 0$, $\nabla u(t_0,x_0) = 0$, $\Delta u(t_0,x_0) \geq 0$

$\Rightarrow (\partial_t - \Delta) v(t_0,x_0) \leq 0$. But $(\partial_t - \Delta) v(t_0,x_0) \geq 0$

$\Rightarrow (\partial_t - \Delta) v(t_0,x_0) = 0.$

By the Strong Maximum Principle, $v(t,x) \equiv \text{Const}$.

$\Rightarrow v(0,x) = v(t_0,x_0) < 0$ — a contradiction.

Thus, all lead to contradictions. $\Rightarrow v(t,x) \geq 0$ on $\overline{\Omega}_T$. $\square$.

2. Proof: Consider $v_\varepsilon(t,x) = v(t,x) + \varepsilon(4t + x_1^2 + x_2^2 - 1)$. $\varepsilon > 0$.

$$\left.\begin{aligned} \partial_t v_\varepsilon(t,x) &= \partial_t v + 4\varepsilon. \\ \Delta v_\varepsilon(t,x) &= \Delta v + 4\varepsilon. \end{aligned}\right\} \Rightarrow \partial_t v_\varepsilon - \Delta v_\varepsilon \leq 0.$$

$v_\varepsilon(0,x) = v(0,x) + \varepsilon(x_1^2 + x_2^2 - 1) \geq -\varepsilon$.

On $\partial\Omega$: $x_1^2 + x_2^2 = 1$. $\dfrac{\partial v_\varepsilon}{\partial n}(t,x) = \dfrac{\partial v}{\partial n}(t,x) + \varepsilon \cdot (2x_1, 2x_2) \cdot (x_1, x_2)$

$$= \frac{\partial v}{\partial n}(t,x) + 2\varepsilon > 0.$$

$\Rightarrow$ By 1. we know that $\min\limits_{\overline{\Omega}_T} v_\varepsilon(t,x) = \min\limits_{\overline{\Omega}} v_\varepsilon(0,x) \geq -\varepsilon$.

$\Rightarrow v + \varepsilon(4t + x_1^2 + x_2^2 - 1) \geq -\varepsilon$

$v \geq -\varepsilon(4t + x_1^2 + x_2^2)$

Since $x_1^2 + x_2^2$ is bounded and $t \leq T$,

let $\varepsilon \to 0^+$, $v \geq 0$. $\square$.

3. Proof: Let $w_\pm(t,x) = \pm u + t \cdot \sup\limits_{\Omega_T} |f| + \sup\limits_{\Omega} |\varphi|$.

Then,
$$\begin{cases} \partial_t w_\pm - \Delta u = \pm f + \sup\limits_{|\Omega_T|} |f| \geq 0. \\ w_\pm(0,x) = \pm u(0,x) + \sup\limits_{\Omega} |\varphi| \geq 0. \\ \dfrac{\partial w_\pm}{\partial n}(t,x) = 0 \Rightarrow \dfrac{\partial w_\pm}{\partial n}(t,x) \geq 0. \end{cases}$$

By 2, we have $w_\pm(t,x) \geq 0$ on $\overline{\Omega}_T$.

$\Rightarrow -t \cdot \sup\limits_{\Omega_T} |f| - \sup\limits_{\Omega} |\varphi| \leq u \leq t \cdot \sup\limits_{\Omega_T} |f| + \sup\limits_{\Omega} |\varphi|$.

$$\Rightarrow \sup_{\overline{\Omega}_T} |u| \leq T \sup_{\Omega_T} |f| + \sup_{\Omega} |\varphi|$$

$$\leq \max\{T, 1\} \left(\sup_{\Omega_T} |f| + \sup_{\Omega} |\varphi| \right). \quad \square.$$

Exercise 4

1. Proof: $K(x,y) = \frac{1}{\pi} \frac{x_1}{(x_2-y)^2 + x_1^2}$.

$$\int_R K(x,y)\, dy \underset{t=\frac{x_2-y}{x_1}}{=} \int_{\infty}^{-\infty} \frac{1}{\pi} \frac{1}{x_1} \frac{1}{1+t^2} (-x_1)\, dt = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+t^2}\, dt.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2}\, dt$$

$$= \frac{1}{\pi} \cdot 2\pi i \cdot \lim_{z\to i} (z-i) \frac{1}{1+z^2}$$

$$= \frac{1}{\pi} \cdot 2\pi i \cdot \frac{1}{2i} = 1.$$

$$\int_{z-\varepsilon}^{z+\varepsilon} K((h,z),y)\, dy = \int_{z-\varepsilon}^{z+\varepsilon} \frac{1}{\pi} \frac{h}{(z-y)^2 + h^2}\, dy$$

$$\underset{t=\frac{z-y}{h}.}{=} \int_{\frac{\varepsilon}{h}}^{-\frac{\varepsilon}{h}} \frac{1}{\pi} \frac{1}{h} \frac{1}{t^2+1} (-h)\, dt = \int_{-\frac{\varepsilon}{h}}^{\frac{\varepsilon}{h}} \frac{1}{\pi} \frac{1}{1+t^2}\, dt.$$

$$\xrightarrow{h\to 0^+} \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+t^2}\, dt = 1. \qquad \square$$

2. Proof: Since g is uniformly continous,

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t $\forall |x-y| < \delta$. $|g(x) - g(y)| < \varepsilon$.

$$|g_h(z) - g(z)| = \left| \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{h}{(z-y)^2+h^2} g(y)\, dy - g(z) \right|$$

$$= \left| \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{h}{(z-y)^2+h^2} (g(y) - g(z))\, dy \right|$$

$$\le \left| \int_{z-\varepsilon}^{z+\varepsilon} \frac{1}{\pi} \frac{h}{(z-y)^2+h^2} (g(y) - g(z))\, dy \right| + \left| \int_{|y-z|>\varepsilon} \frac{1}{\pi} \frac{h}{(z-y)^2+h^2} (g(y) - g(z))\, dy \right|$$

$$\le \varepsilon \cdot \left| \int_{z-\varepsilon}^{z+\varepsilon} \frac{1}{\pi} \frac{h}{(z-y)^2+h^2}\, dy \right| + \frac{1}{\pi} \int_{|t|>\frac{\varepsilon}{h}} \frac{1}{1+t^2}\, dt \cdot 2\sup|g|$$

$$\le \varepsilon \cdot 1 + \frac{1}{\pi} \int_{|t|>\frac{\varepsilon}{h}} \frac{1}{1+t^2}\, dt \cdot 2.$$

$$= \varepsilon + \frac{2}{\pi} \int_{|t| > \frac{\varepsilon}{h}} \frac{1}{1+t^2}\, dt. \xrightarrow[h \searrow 0]{} \varepsilon \quad \text{uniformly.}$$

(independent of z)

Let $\varepsilon \to 0$, we have $\underset{(h\searrow 0)}{g_h \to g}$ uniformly on $\mathbb{R}$. $\square$.