MA7092: HW-2

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Problem 1. On $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X_t : t \geq 0\}$ be a stochastic process that is measurable and T is a random time. Prove that

(a) X_T is a random variable if T is finite;

(b) All sets of the form $\{X_T \in A\}$ and $\{X_T \in A\} \cup \{T = \infty\}$ with $A \in \mathcal{B}(\mathbb{R}^d)$ forms a σ-algebra.

Proof. **Notations and Definitions:**

- **Measurable Process:** The map $\Phi : (\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}([0, \infty))) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given by $\Phi(\omega, t) = X_t(\omega)$ is measurable.
- **Random Time:** The map $T : \Omega \to [0, \infty]$ is $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable.
- **Random Variable:** A map $Y : \Omega \to \mathbb{R}^d$ is $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable.

(a) X_T is a random variable if T is finite.

We need to show the map $Y : \Omega \to \mathbb{R}^d$ defined by $Y(\omega) = X_{T(\omega)}(\omega)$ is $\mathcal{F}$-measurable. This map Y can be expressed as the composition $Y = \Phi \circ g$, where $g : \Omega \to \Omega \times [0, \infty)$ is defined by $g(\omega) = (\omega, T(\omega))$ and $\Phi : \Omega \times [0, \infty) \to \mathbb{R}^d$ is $\Phi(\omega, t) = X_t(\omega)$.

By definition, X being a measurable process means Φ is $\mathcal{F} \otimes \mathcal{B}([0, \infty])$-measurable. To show Y is measurable, it suffices to show that g is $\mathcal{F}/(\mathcal{F} \otimes \mathcal{B}([0, \infty)))$-measurable, as the composition of measurable maps is measurable.

To check g is measurable, we test its inverse image on a generating set of $\mathcal{F} \otimes \mathcal{B}([0, \infty))$, namely the measurable rectangles. Let $A \in \mathcal{F}$ and $B \in \mathcal{B}([0, \infty))$. Then,

$$g^{-1}(A \times B) = \{\omega \in \Omega : (\omega, T(\omega)) \in A \times B\} = \{\omega \in \Omega : \omega \in A \text{ and } T(\omega) \in B\} = A \cap \{T \in B\}$$

Since $A \in \mathcal{F}$ and T is a random time (an $\mathcal{F}$-measurable map), the set $\{T \in B\}$ is also in $\mathcal{F}$. The intersection of two sets in $\mathcal{F}$ is in $\mathcal{F}$, so $g^{-1}(A \times B) \in \mathcal{F}$.

Thus, g is measurable. Since g and Φ are measurable, their composition $X_T = \Phi \circ g$ is measurable. Therefore, X_T is a random variable.

(b) The collection forms a σ-algebra.

Let $\Omega_F = \{T < \infty\}$ and $\Omega_I = \{T = \infty\}$. Let $\mathcal{A} = \sigma(X_T|_{\Omega_F}) = \{\{\omega \in \Omega_F : X_T(\omega) \in A\} : A \in \mathcal{B}(\mathbb{R}^d)\}$. We must prove that $\mathcal{G} = \{B \cup C : B \in \mathcal{A}, C \in \{\emptyset, \Omega_I\}\}$ is a σ-algebra on Ω.

First, we check that $\Omega \in \mathcal{G}$. Let $B = \Omega_F$. $B \in \mathcal{A}$ (by taking $A = \mathbb{R}^d$). Let $C = \Omega_I$. Then $B \cup C = \Omega_F \cup \Omega_I = \Omega$. Thus, $\Omega \in \mathcal{G}$.

Second, we check closure under complementation. Let $G \in \mathcal{G}$. *Case 1:* $G = B$ for $B \in \mathcal{A}$ $(C = \emptyset)$. Then $G \subseteq \Omega_F$. The complement is $G^c = \Omega \setminus G = (\Omega_F \setminus G) \cup \Omega_I$. Let

$B' = \Omega_F \setminus G$. Since $\mathcal{A}$ is a σ-algebra on Ω_F, $B' \in \mathcal{A}$. Let $C' = \Omega_I$. Then $G^c = B' \cup C' \in \mathcal{G}$. *Case 2:* $G = B \cup \Omega_I$ for $B \in \mathcal{A}$. The complement is $G^c = \Omega \setminus (B \cup \Omega_I) = \Omega_F \setminus B$. Let $B' = \Omega_F \setminus B$. Since $B \in \mathcal{A}$, $B' \in \mathcal{A}$. Let $C' = \emptyset$. Then $G^c = B' \cup C' \in \mathcal{G}$. In both cases, $G^c \in \mathcal{G}$.

Third, we check closure under countable unions. Let $G_n \in \mathcal{G}$ for $n = 1, 2, \ldots$. Each $G_n = B_n \cup C_n$, where $B_n \in \mathcal{A}$ and $C_n \in \{\emptyset, \Omega_I\}$. The union is

$$\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (B_n \cup C_n) = \left(\bigcup_{n=1}^{\infty} B_n \right) \cup \left(\bigcup_{n=1}^{\infty} C_n \right)$$

Let $B_\cup = \bigcup_{n=1}^{\infty} B_n$. Since $\mathcal{A}$ is a σ-algebra, $B_\cup \in \mathcal{A}$. Let $C_\cup = \bigcup_{n=1}^{\infty} C_n$. This union is $\emptyset$ if all $C_n = \emptyset$, and Ω_I otherwise. In either case, $C_\cup \in \{\emptyset, \Omega_I\}$. Therefore, $\bigcup G_n = B_\cup \cup C_\cup \in \mathcal{G}$.

By all three properties, $\mathcal{G}$ is a σ-algebra. ■

Problem 2. Let $\{\mathcal{F}_t : t \geq 0\}$ be a filtration and X be an adapted process relative to $\{\mathcal{F}_t : t \geq 0\}$. Set $T = \inf\{t \geq 0 : X_t \in A\}$. Prove that

(a) if A is open and the sample paths of X are right-continuous, then T is an optional time of $\{\mathcal{F}_t : t \geq 0\}$.

(b) if A is closed and the sample paths of X are continuous, then T is a stopping time of $\{\mathcal{F}_t : t \geq 0\}$.

Proof. **Notations:**

- **Optional Time:** T is an optional time if $\{T < t\} \in \mathcal{F}_t$ for all $t \geq 0$.
- **Stopping Time:** T is a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

(a) A is open and X is right-continuous $\implies$ T is an optional time.

We must show $\{T < t\} \in \mathcal{F}_t$ for any fixed $t \geq 0$. The event $\{T < t\}$ means $\inf\{s \geq 0 : X_s \in A\} < t$. By the definition of infimum, this is equivalent to $\exists s \in [0, t)$ such that $X_s \in A$. Thus, $\{T < t\} = \bigcup_{s \in [0,t)} \{X_s \in A\}$. This is an uncountable union, which is not guaranteed to be measurable.

We must use the given properties. Let $\omega \in \{T < t\}$. Then there exists some $s_0 \in [0, t)$ such that $X_{s_0}(\omega) \in A$. Since A is **open**, there exists an $\epsilon > 0$ such that the open ball $B(X_{s_0}(\omega), \epsilon) \subseteq A$. Since the path $s \mapsto X_s(\omega)$ is **right-continuous** at s_0, there exists $\delta > 0$ such that for all $u \in [s_0, s_0 + \delta)$, $X_u(\omega) \in B(X_{s_0}(\omega), \epsilon) \subseteq A$.

Since $s_0 < t$, we can choose a **rational number** q such that $s_0 < q < \min(t, s_0 + \delta)$. For this q, we have $q \in [0, t) \cap \mathbb{Q}$ and $X_q(\omega) \in A$. This shows that $\omega \in \bigcup_{q \in [0,t) \cap \mathbb{Q}} \{X_q \in A\}$. We have proven the inclusion $\{T < t\} \subseteq \bigcup_{q \in [0,t) \cap \mathbb{Q}} \{X_q \in A\}$. The reverse inclusion is direct: If $X_q(\omega) \in A$ for $q < t$, then $T(\omega) \leq q < t$.

Thus, $\{T < t\} = \bigcup_{q \in [0,t) \cap \mathbb{Q}} \{X_q \in A\}$. This is a **countable** union. For any $q \in [0, t) \cap \mathbb{Q}$, X is adapted, so X_q is $\mathcal{F}_q$-measurable. Since A is Borel, $\{X_q \in A\} \in \mathcal{F}_q$. Because $q < t$, we have $\mathcal{F}_q \subseteq \mathcal{F}_t$, so $\{X_q \in A\} \in \mathcal{F}_t$. A countable union of sets in $\mathcal{F}_t$ is in $\mathcal{F}_t$. Therefore, $\{T < t\} \in \mathcal{F}_t$, and T is an optional time.

(b) A is closed and X is continuous $\implies$ T is a stopping time.

We must show $\{T \leq t\} \in \mathcal{F}_t$ for any fixed $t \geq 0$. The event $\{T \leq t\}$ means $\inf\{s \geq 0 : X_s \in A\} \leq t$. Since $s \mapsto X_s(\omega)$ is **continuous** and A is **closed**, the preimage set $K_\omega = \{s \geq 0 : X_s(\omega) \in A\}$ is a closed subset of $[0, \infty)$. The infimum of a non-empty closed set is its minimum. Thus, $T(\omega) \leq t$ is equivalent to the set $K_\omega \cap [0, t]$ being non-empty.

Let $d(x, A) = \inf_{y \in A} \|x - y\|$ be the distance function. Since A is closed, $d(x, A) = 0 \iff x \in A$. Since X_s is continuous, $s \mapsto d(X_s(\omega), A)$ is also continuous. The event $\{T \leq t\}$ is equivalent to $\min_{s \in [0,t]} d(X_s(\omega), A) = 0$ (the minimum exists because $[0, t]$ is compact). By continuity, this is equivalent to $\inf_{s \in [0,t]} d(X_s(\omega), A) = 0$, and by the density of rationals, this is equivalent to $\inf_{q \in [0,t] \cap \mathbb{Q}} d(X_q(\omega), A) = 0$.

So, $\{T \leq t\} = \{\omega : \inf_{q \in [0,t] \cap \mathbb{Q}} d(X_q(\omega), A) = 0\}$. For any $q \in [0, t] \cap \mathbb{Q}$, X_q is $\mathcal{F}_q$-measurable, and since $q \leq t$, X_q is $\mathcal{F}_t$-measurable. The distance function $d(\cdot, A)$ is continuous, so it is Borel-measurable. Thus, $Y_q(\omega) = d(X_q(\omega), A)$ is an $\mathcal{F}_t$-measurable real-valued random variable for each q.

The infimum of a **countable** collection of $\mathcal{F}_t$-measurable functions, $Z(\omega) = \inf_{q \in [0,t] \cap \mathbb{Q}} Y_q(\omega)$, is also $\mathcal{F}_t$-measurable. The event $\{T \leq t\}$ is $\{Z = 0\} = Z^{-1}(\{0\})$. Since $\{0\}$ is a Borel set, $\{T \leq t\} \in \mathcal{F}_t$. Therefore, T is a stopping time. ■

Problem 3. Let $\{X_t : t \geq 0\}$ be a progressively measurable process relative to $\{\mathcal{F}_t : t \geq 0\}$ and let T be a finite stopping time of $\{\mathcal{F}_t : t \geq 0\}$. Prove that

(a) X_T is $\mathcal{F}_T$-measurable.

(b) the process $\{X_{T\wedge t} : t \geq 0\}$ is progressively measurable relative to $\{\mathcal{F}_t : t \geq 0\}$.

Proof. **Notations:**

- **Progressively Measurable:** For all $t \geq 0$, the map $(\omega, s) \mapsto X_s(\omega)$ restricted to $\Omega \times [0, t]$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable.
- **$\mathcal{F}_T$-algebra:** $\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$.

(a) X_T is $\mathcal{F}_T$-measurable.

We must show that for any $A \in \mathcal{B}(\mathbb{R}^d)$, the set $\{X_T \in A\}$ belongs to $\mathcal{F}_T$. By definition of $\mathcal{F}_T$, this means we must show that for any $t \geq 0$, the set $B = \{X_T \in A\} \cap \{T \leq t\}$ is in $\mathcal{F}_t$.

Let $t \geq 0$ be fixed. On the set $\{T \leq t\}$, we have $T(\omega) = T(\omega) \wedge t$. Therefore, $B = \{X_{T\wedge t} \in A\} \cap \{T \leq t\}$. Let $Y_t(\omega) = X_{T(\omega)\wedge t}(\omega)$. We show Y_t is $\mathcal{F}_t$-measurable. Y_t is a composition $Y_t = \Phi_t \circ g_t$, where $\Phi_t : \Omega \times [0, t] \to \mathbb{R}^d$ is $\Phi_t(\omega, s) = X_s(\omega)$ and $g_t : \Omega \to \Omega \times [0, t]$ is $g_t(\omega) = (\omega, T(\omega) \wedge t)$. By progressive measurability, Φ_t is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable.

We check that g_t is $\mathcal{F}_t/(\mathcal{F}_t \otimes \mathcal{B}([0, t]))$-measurable. For a measurable rectangle $C \times D$ where $C \in \mathcal{F}_t$ and $D \in \mathcal{B}([0, t])$: $g_t^{-1}(C \times D) = \{\omega : \omega \in C \text{ and } T(\omega) \wedge t \in D\} = C \cap \{T \wedge t \in D\}$. The map $\omega \mapsto T(\omega) \wedge t$ is $\mathcal{F}_t$-measurable because for any $s \leq t$, $\{T \wedge t \leq s\} = \{T \leq s\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$. Thus, $\{T \wedge t \in D\} \in \mathcal{F}_t$, and $g_t^{-1}(C \times D) \in \mathcal{F}_t$. This shows g_t is measurable. Since g_t and Φ_t are measurable, their composition $Y_t = X_{T\wedge t}$ is $\mathcal{F}_t$-measurable.

Now we return to $B = \{X_{T\wedge t} \in A\} \cap \{T \leq t\}$. The set $\{X_{T\wedge t} \in A\}$ is in $\mathcal{F}_t$ (since Y_t is $\mathcal{F}_t$-measurable), and $\{T \leq t\}$ is in $\mathcal{F}_t$ (since T is a stopping time). The intersection B is in $\mathcal{F}_t$. This holds for all $t \geq 0$, so $\{X_T \in A\} \in \mathcal{F}_T$. Therefore, X_T is $\mathcal{F}_T$-measurable.

(b) $\{X_{T\wedge t} : t \geq 0\}$ is progressively measurable.

For any fixed $t \geq 0$, we must show the map $\Psi_t : (\Omega \times [0, t]) \to \mathbb{R}^d$ given by $\Psi_t(\omega, s) = X_{T(\omega)\wedge s}(\omega)$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable. Let $\Phi_t : (\Omega \times [0, t]) \to \mathbb{R}^d$ be $\Phi_t(\omega, u) = X_u(\omega)$. Since X is progressively measurable, Φ_t is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable. Let $g : (\Omega \times [0, t]) \to (\Omega \times [0, t])$ be the map $g(\omega, s) = (\omega, T(\omega) \wedge s)$. Then $\Psi_t(\omega, s) = \Phi_t(\omega, T(\omega) \wedge s) = (\Phi_t \circ g)(\omega, s)$.

We just need to show g is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable. The first component of g, $\pi_1(\omega, s) = \omega$, is $\mathcal{F}_t \otimes \mathcal{B}([0, t])/\mathcal{F}_t$-measurable. The second component, $h(\omega, s) = T(\omega) \wedge s$, needs to be $\mathcal{F}_t \otimes \mathcal{B}([0, t])/\mathcal{B}([0, t])$-measurable. We check $h^{-1}([0, u])$ for $u \leq t$:

$$E_u = \{(\omega, s) \in \Omega \times [0, t] : T(\omega) \wedge s \leq u\} = \{(\omega, s) : T(\omega) \leq u \text{ or } s \leq u\}$$

$$E_u = (\{\omega : T(\omega) \leq u\} \times [0, t]) \cup (\Omega \times [0, u])$$

Since T is a stopping time and $u \leq t$, $\{T \leq u\} \in \mathcal{F}_u \subseteq \mathcal{F}_t$. Both $(\{T \leq u\} \times [0, t])$ and $(\Omega \times [0, u])$ are measurable rectangles in $\mathcal{F}_t \otimes \mathcal{B}([0, t])$. Their union E_u is in $\mathcal{F}_t \otimes \mathcal{B}([0, t])$. Since sets of the form $[0, u]$ generate $\mathcal{B}([0, t])$, the map h is measurable. Since both components of g are measurable, g is measurable. The composition $\Psi_t = \Phi_t \circ g$ is therefore measurable. Thus, the process $\{X_{T\wedge t}\}$ is progressively measurable. ∎

Problem 4. Let T be an optional time of $\{\mathcal{F}_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Prove that $\mathcal{F}_{T+}$ is a σ-algebra and $\mathcal{F}_{T+} = \{A \in \mathcal{F} : A \cap \{T < t\} \in \mathcal{F}_t, \forall t \geq 0\}$.

(b) Prove that if T is a stopping time, then $\mathcal{F}_T \subseteq \mathcal{F}_{T+}$.

Proof. **(a) $\mathcal{F}_{T+}$ is a σ-algebra.**

We must check the three properties of a σ-algebra for the collection $\mathcal{G} = \{A \in \mathcal{F} : A \cap \{T < t\} \in \mathcal{F}_t, \forall t \geq 0\}$.

First, we check that $\Omega \in \mathcal{G}$. We need $\Omega \cap \{T < t\} \in \mathcal{F}_t$ for all t. This set is $\Omega \cap \{T < t\} = \{T < t\}$. By definition of T being an optional time, $\{T < t\} \in \mathcal{F}_t$. Thus, $\Omega \in \mathcal{G}$.

Second, we check closure under complementation. Let $A \in \mathcal{G}$. We must show $A^c \in \mathcal{G}$. $A \in \mathcal{G}$ means $A \in \mathcal{F}$ and $A \cap \{T < t\} \in \mathcal{F}_t$ for all t. We check $A^c \cap \{T < t\}$:

$$A^c \cap \{T < t\} = \{T < t\} \setminus (A \cap \{T < t\})$$

From the first step, $\{T < t\} \in \mathcal{F}_t$. By assumption, $A \cap \{T < t\} \in \mathcal{F}_t$. Since $\mathcal{F}_t$ is a σ-algebra, the set difference is in $\mathcal{F}_t$. Thus, $A^c \in \mathcal{G}$.

Third, we check closure under countable unions. Let $A_n \in \mathcal{G}$ for $n = 1, 2, \ldots$. We must show $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$. For each n, $A_n \cap \{T < t\} \in \mathcal{F}_t$ for all t. We check $A \cap \{T < t\}$:

$$A \cap \{T < t\} = \left(\bigcup_{n=1}^{\infty} A_n \right) \cap \{T < t\} = \bigcup_{n=1}^{\infty} (A_n \cap \{T < t\})$$

Each set $(A_n \cap \{T < t\})$ is in $\mathcal{F}_t$. Since $\mathcal{F}_t$ is a σ-algebra, the countable union of these sets is also in $\mathcal{F}_t$. Thus, $A \in \mathcal{G}$.

Therefore, $\mathcal{F}_{T+} = \mathcal{G}$ is a σ-algebra.

(b) If T is a stopping time, $\mathcal{F}_T \subseteq \mathcal{F}_{T+}$.

We must show that if $A \in \mathcal{F}_T$, then $A \in \mathcal{F}_{T+}$. $A \in \mathcal{F}_T$ means $A \cap \{T \leq s\} \in \mathcal{F}_s$ for all $s \geq 0$. $A \in \mathcal{F}_{T+}$ means $A \cap \{T < t\} \in \mathcal{F}_t$ for all $t \geq 0$. Let $t \geq 0$ be fixed. We can write the event $\{T < t\}$ as a countable union:

$$\{T < t\} = \bigcup_{n=1}^{\infty} \{T \leq t - 1/n\}$$

(where the union is over n such that $t - 1/n \geq 0$). Now, we intersect with A:

$$A \cap \{T < t\} = A \cap \left(\bigcup_{n=1}^{\infty} \{T \leq t - 1/n\} \right) = \bigcup_{n=1}^{\infty} (A \cap \{T \leq t - 1/n\})$$

Let $s_n = t - 1/n$. Since $A \in \mathcal{F}_T$, we know $A \cap \{T \leq s_n\} \in \mathcal{F}_{s_n}$. Since $s_n < t$, the filtration property implies $\mathcal{F}_{s_n} \subseteq \mathcal{F}_t$. Therefore, $A \cap \{T \leq s_n\} \in \mathcal{F}_t$ for all n. The set $A \cap \{T < t\}$ is a countable union of sets in $\mathcal{F}_t$. Since $\mathcal{F}_t$ is a σ-algebra, $A \cap \{T < t\} \in \mathcal{F}_t$. This holds for all $t \geq 0$, so by definition, $A \in \mathcal{F}_{T+}$. ■

Problem 5. Let $\{X_t : t \geq 0\}$ be a right-continuous, nonnegative supermartingale relative to $\{\mathcal{F}_t : t \geq 0\}$. Prove that $X_\infty = \lim_{t\to\infty} X_t$ exists almost surely and $\{X_t : 0 \leq t \leq \infty\}$ is a supermartingale relative to $\{\mathcal{F}_t : 0 \leq t \leq \infty\}$.

Proof. Let $\mathcal{F}_\infty = \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$.

First, we prove the existence of X_∞. Since X_t is a supermartingale, $\mathbb{E}[X_t]$ is non-increasing. Since $X_t \geq 0$, $\mathbb{E}[X_t]$ is bounded below by 0. Thus, $\sup_t \mathbb{E}[|X_t|] = \sup_t \mathbb{E}[X_t] = \mathbb{E}[X_0] < \infty$. By Doob's Martingale Convergence Theorem, a right-continuous supermartingale X_t with $\sup_t \mathbb{E}[X_t^+] < \infty$ converges a.s. as $t \to \infty$. Since $X_t \geq 0$, $X_t^+ = X_t$, and $\sup_t \mathbb{E}[X_t] < \infty$ as shown. Therefore, $X_\infty = \lim_{t\to\infty} X_t$ exists almost surely (a.s.).

Second, we prove $\{X_t : 0 \leq t \leq \infty\}$ is a supermartingale relative to $\{\mathcal{F}_t : 0 \leq t \leq \infty\}$. We must check the three conditions. (i) **Adaptedness:** X_t is $\mathcal{F}_t$-adapted for $t < \infty$. $X_\infty = \lim_{t\to\infty} X_t$ is the a.s. limit of $\mathcal{F}_t$-measurable variables, so it is $\mathcal{F}_\infty$-measurable. (ii) **Integrability:** $\mathbb{E}[|X_t|] = \mathbb{E}[X_t] \leq \mathbb{E}[X_0] < \infty$ for $t < \infty$. For $t = \infty$, we use Fatou's Lemma (since $X_t \geq 0$):

$$\mathbb{E}[X_\infty] = \mathbb{E}[\liminf_{t\to\infty} X_t] \leq \liminf_{t\to\infty} \mathbb{E}[X_t]$$

Since $\mathbb{E}[X_t]$ is non-increasing and bounded below, $\liminf \mathbb{E}[X_t] = \lim \mathbb{E}[X_t] \leq \mathbb{E}[X_0] < \infty$. Thus, X_∞ is integrable. (iii) **Supermartingale Inequality:** We need $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ for $0 \leq s \leq t \leq \infty$. The only new case is $t = \infty$ and $s < \infty$. We need to show $\mathbb{E}[X_\infty|\mathcal{F}_s] \leq X_s$. For any $t > s$, we know $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$. Since $X_t \geq 0$ for all t, we can apply the conditional Fatou's Lemma:

$$\mathbb{E}[X_\infty|\mathcal{F}_s] = \mathbb{E}[\liminf_{t\to\infty} X_t|\mathcal{F}_s] \leq \liminf_{t\to\infty} \mathbb{E}[X_t|\mathcal{F}_s]$$

Since $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ for all $t > s$, the lim inf of this sequence must also be $\leq X_s$. Therefore, $\mathbb{E}[X_\infty|\mathcal{F}_s] \leq X_s$.

All conditions are met. ■

Problem 6. Let $\{X_t : t \geq 0\}$ be a right-continuous, nonnegative submartingale relative to $\{\mathcal{F}_t : t \geq 0\}$. Prove that the following are equivalent:

(a) The family $\{X_t : t \geq 0\}$ of r.v.s are uniformly integrable (UI).

(b) X_t converges in L^1 as $t \to \infty$.

(c) X_t converges almost surely and the limit X_∞ is integrable.

Moreover, $\{X_t : 0 \leq t \leq \infty\}$ is a submartingale relative to $\{\mathcal{F}_t : 0 \leq t \leq \infty\}$.

Proof. **Proof of Equivalence**

(a) $\implies$ (b): Assume $\{X_t\}$ is UI. Then $\sup_t \mathbb{E}[|X_t|] = \sup_t \mathbb{E}[X_t] < \infty$. By the Martingale Convergence Theorem (for non-negative submartingales), this implies $X_t \to X_\infty$ a.s. A standard theorem states that $X_t \to X_\infty$ in L^1 if and only if $X_t \to X_\infty$ in probability (which is implied by a.s. convergence) and $\{X_t\}$ is UI. Both conditions hold, so X_t converges in L^1 to X_∞.

(b) $\implies$ (c): Assume $X_t \to X_\infty$ in L^1. This implies X_∞ is integrable ($\mathbb{E}[|X_\infty|] < \infty$). L^1 convergence also implies $\mathbb{E}[X_t] \to \mathbb{E}[X_\infty]$. Since X_t is a non-negative submartingale, $\mathbb{E}[X_t]$ is non-decreasing, so $\sup_t \mathbb{E}[X_t] = \mathbb{E}[X_\infty] < \infty$. By the Martingale Convergence Theorem, $\sup_t \mathbb{E}[X_t] < \infty$ implies $X_t \to Y$ a.s. for some limit Y. Since X_t also converges in probability to X_∞ (as L^1 convergence implies it), the limits must be equal: $Y = X_\infty$ a.s. Thus, X_t converges a.s. and the limit X_∞ is integrable.

(c) $\implies$ (a): Assume $X_t \to X_\infty$ a.s. and X_∞ is integrable. For a non-negative submartingale, this is sufficient to "close" the submartingale, i.e., $\mathbb{E}[X_\infty|\mathcal{F}_t] \geq X_t$ for all $t < \infty$. Taking expectations gives $\mathbb{E}[X_\infty] = \mathbb{E}[\mathbb{E}[X_\infty|\mathcal{F}_t]] \geq \mathbb{E}[X_t]$ for all t. This implies $\mathbb{E}[X_\infty] \geq \sup_t \mathbb{E}[X_t]$. By Fatou's Lemma (since $X_t \geq 0$), $\mathbb{E}[X_\infty] = \mathbb{E}[\liminf X_t] \leq \liminf \mathbb{E}[X_t]$. Since $\mathbb{E}[X_t]$ is non-decreasing, $\liminf \mathbb{E}[X_t] = \sup_t \mathbb{E}[X_t]$. Combining these inequalities, we get $\mathbb{E}[X_\infty] = \sup_t \mathbb{E}[X_t] = \lim_{t\to\infty} \mathbb{E}[X_t]$. Since $X_t \geq 0$, this means $\mathbb{E}[|X_t|] \to \mathbb{E}[|X_\infty|]$. The conditions $X_t \to X_\infty$ a.s. and $\mathbb{E}[|X_t|] \to \mathbb{E}[|X_\infty|]$ together imply that the family $\{X_t : t \geq 0\}$ is uniformly integrable.

Proof of "Moreover" part

If the equivalent conditions hold, we must show $\{X_t : 0 \leq t \leq \infty\}$ is a submartingale. (i) **Adaptedness:** X_∞ is $\mathcal{F}_\infty$-measurable (as in Problem 5). (ii) **Integrability:** $X_t \in L^1$ for $t < \infty$ and $X_\infty \in L^1$ (by (c)). (iii) **Submartingale Inequality:** We need $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ for $0 \leq s \leq t \leq \infty$. The only new case is $t = \infty, s < \infty$. We need to show $\mathbb{E}[X_\infty|\mathcal{F}_s] \geq X_s$. Since the conditions hold, we know $\{X_t\}_{t\geq s}$ is UI and converges L^1 to X_∞. L^1 convergence implies convergence of conditional expectations:

$$\lim_{t\to\infty} \mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[X_\infty|\mathcal{F}_s]$$

By the submartingale property for $t < \infty$, we have $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ for all $t > s$. Taking the limit as $t \to \infty$ gives $\lim_{t\to\infty} \mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$. Substituting the L^1 convergence result, we get $\mathbb{E}[X_\infty|\mathcal{F}_s] \geq X_s$. ■

Problem 7. Let $\{X_t : t \geq 0\}$ be a right-continuous supermartingale relative to $\{\mathcal{F}_t : t \geq 0\}$ and $S \leq T$ are stopping times of $\{\mathcal{F}_t : t \geq 0\}$. Prove that

(a) $\{X_{T\wedge t} : t \geq 0\}$ is supermartingale of $\{\mathcal{F}_t : t \geq 0\}$.

(b) $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_S] \leq X_{S\wedge t}$.

Proof. **(a)** $\{X_{T\wedge t} : t \geq 0\}$ **is a supermartingale.**

Let $Y_t = X_{T\wedge t}$. We must show Y_t is adapted, integrable, and $\mathbb{E}[Y_t|\mathcal{F}_s] \leq Y_s$ for $s < t$. (i) **Adaptedness:** From Problem 3(b), $Y_t = X_{T\wedge t}$ is progressively measurable, which implies it is adapted. (ii) **Integrability:** $T \wedge t$ is a bounded stopping time. By the Optional Stopping Theorem (OST), $\mathbb{E}[X_{T\wedge t}] \leq \mathbb{E}[X_0]$. More generally, for a right-continuous supermartingale, $\mathbb{E}[|X_{T\wedge t}|] \leq \mathbb{E}[\sup_{s\leq t} |X_s|] < \infty$. (iii) **Supermartingale Inequality:** Let $s < t$. We must show $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_s] \leq X_{T\wedge s}$. First, we show $\mathcal{F}_s \subseteq \mathcal{F}_{T\wedge s}$. Let $A \in \mathcal{F}_s$. We need $A \cap \{T \wedge s \leq u\} \in \mathcal{F}_u$ for all $u \geq 0$. If $u < s$, $A \cap \{T \wedge s \leq u\} = A \cap \{T \leq u\}$. Since $A \in \mathcal{F}_s$ and $\{T \leq u\} \in \mathcal{F}_u \subseteq \mathcal{F}_s$, this intersection is in $\mathcal{F}_u$. If $u \geq s$, $A \cap \{T \wedge s \leq u\} = A$. Since $A \in \mathcal{F}_s \subseteq \mathcal{F}_u$, $A \in \mathcal{F}_u$. Thus $\mathcal{F}_s \subseteq \mathcal{F}_{T\wedge s}$.

By the Tower Property, $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X_{T\wedge t}|\mathcal{F}_{T\wedge s}]|\mathcal{F}_s]$. Let $R = T \wedge s$ and $U = T \wedge t$. $R \leq U$ are bounded stopping times. By OST, $\mathbb{E}[X_U|\mathcal{F}_R] \leq X_R$, which is $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_{T\wedge s}] \leq X_{T\wedge s}$. Let $Z = \mathbb{E}[X_{T\wedge t}|\mathcal{F}_{T\wedge s}]$, so $Z \leq X_{T\wedge s}$. Taking conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_s]$ on both sides gives $\mathbb{E}[Z|\mathcal{F}_s] \leq \mathbb{E}[X_{T\wedge s}|\mathcal{F}_s]$. The LHS is $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_s]$. For the RHS, $T \wedge s$ is a stopping time bounded by s, which implies $X_{T\wedge s}$ is $\mathcal{F}_s$-measurable. Therefore, $\mathbb{E}[X_{T\wedge s}|\mathcal{F}_s] = X_{T\wedge s}$. Combining these steps gives $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_s] \leq X_{T\wedge s}$.

(b) $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_S] \leq X_{S\wedge t}$**.**

Let $Y_t = X_{T\wedge t}$. From part (a), Y_t is a right-continuous supermartingale. We want to prove $\mathbb{E}[Y_t|\mathcal{F}_S] \leq X_{S\wedge t}$. We note that $Y_{S\wedge t} = X_{T\wedge(S\wedge t)}$. Since $S \leq T$, $S \wedge T = S$. Thus $Y_{S\wedge t} = X_{(T\wedge S)\wedge t} = X_{S\wedge t}$. So the inequality to prove is $\mathbb{E}[Y_t|\mathcal{F}_S] \leq Y_{S\wedge t}$. This is a standard property (from OST) for any supermartingale Y, stopping time S, and fixed time t: $\mathbb{E}[Y_t|\mathcal{F}_S] \leq Y_{S\wedge t}$. Substituting $Y_t = X_{T\wedge t}$ back in gives $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_S] \leq X_{S\wedge t}$. ∎

Problem 8. Let $\{X_t : t \geq 0\}$ be a right-continuous process such that $X_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for $\forall t \geq 0$. Prove that X is a submartingale relative to $\{\mathcal{F}_t : t \geq 0\}$ if and only if $\mathbb{E}[X_T] \geq \mathbb{E}[X_S]$ for all bounded stopping times $S \leq T$ of $\{\mathcal{F}_t : t \geq 0\}$.

Proof. We assume X is adapted, as this is part of the definition of a submartingale.

($\Longrightarrow$) (Submartingale $\Longrightarrow$ Inequality) Assume X is a submartingale. Let $S \leq T$ be stopping times bounded by some constant $K < \infty$. By Doob's Optional Stopping Theorem for a right-continuous submartingale and bounded stopping times $S \leq T$, we have $\mathbb{E}[X_T|\mathcal{F}_S] \geq X_S$. Taking the expectation of both sides and using the tower property gives $\mathbb{E}[X_T] = \mathbb{E}[\mathbb{E}[X_T|\mathcal{F}_S]] \geq \mathbb{E}[X_S]$.

($\Longleftarrow$) (Inequality $\Longrightarrow$ Submartingale) Assume $\mathbb{E}[X_T] \geq \mathbb{E}[X_S]$ for all bounded stopping times $S \leq T$. We need to prove X is a submartingale. We are given X is adapted and $X_t \in L^1$. We only need to show $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ for all $0 \leq s \leq t$. This is equivalent to showing $\int_A X_t d\mathbb{P} \geq \int_A X_s d\mathbb{P}$ for all $A \in \mathcal{F}_s$. Let $0 \leq s \leq t$ be fixed, and let $A \in \mathcal{F}_s$. Define two random times S' and T':

$$S'(\omega) = s \quad \text{and} \quad T'(\omega) = \begin{cases} t & \text{if } \omega \in A \\ s & \text{if } \omega \in A^c \end{cases}$$

We check that S' and T' are bounded stopping times with $S' \leq T'$. $S' \equiv s$ is a constant, so it is a stopping time bounded by s. For T', we check $\{T' \leq u\} \in \mathcal{F}_u$ for all $u \geq 0$. If $u < s$, $\{T' \leq u\} = \emptyset \in \mathcal{F}_u$. If $s \leq u < t$, $\{T' \leq u\} = \{T' = s\} = A^c$. Since $A \in \mathcal{F}_s$ and $s \leq u$, $A^c \in \mathcal{F}_s \subseteq \mathcal{F}_u$. If $u \geq t$, $\{T' \leq u\} = \{T' = s\} \cup \{T' = t\} = A^c \cup A = \Omega \in \mathcal{F}_u$. So T' is a stopping time. Both S' and T' are bounded by t. Furthermore, $S' \leq T'$ holds because $s \leq t$ (on A) and $s \leq s$ (on A^c).

By our assumption, $\mathbb{E}[X_{T'}] \geq \mathbb{E}[X_{S'}]$. We have $\mathbb{E}[X_{S'}] = \mathbb{E}[X_s] = \int_A X_s d\mathbb{P} + \int_{A^c} X_s d\mathbb{P}$. Also, $\mathbb{E}[X_{T'}] = \int_\Omega X_{T'} d\mathbb{P} = \int_A X_t d\mathbb{P} + \int_{A^c} X_s d\mathbb{P}$. The inequality becomes:

$$\int_A X_t d\mathbb{P} + \int_{A^c} X_s d\mathbb{P} \geq \int_A X_s d\mathbb{P} + \int_{A^c} X_s d\mathbb{P}$$

Subtracting $\int_{A^c} X_s d\mathbb{P}$ from both sides (which is finite since $X_s \in L^1$), we get $\int_A X_t d\mathbb{P} \geq \int_A X_s d\mathbb{P}$. Since this holds for any $A \in \mathcal{F}_s$, it is the definition of $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$. ■

Problem 9. Let X be a normal Markov process with transition density functions $\{\rho_{s,t} : 0 \le s \le t\}$. Prove that

$$\rho_{r,t}(x,z) = \int_{\mathbb{R}^d} \rho_{r,s}(x,y)\rho_{s,t}(y,z)dy$$

for any $0 \le r \le s \le t$ and $x, z \in \mathbb{R}^d$.

Proof. This is the Chapman-Kolmogorov equation for densities. Let $A \in \mathcal{B}(\mathbb{R}^d)$ be any measurable set. By definition, the transition probability is $P_{s,t}(x,A) = \mathbb{P}(X_t \in A | X_s = x) = \int_A \rho_{s,t}(x,y)dy$.

We start with $P_{r,t}(x,A) = \mathbb{P}(X_t \in A | X_r = x)$. We condition on the state X_s at the intermediate time s using the law of total expectation and the Markov property:

$$P_{r,t}(x,A) = \mathbb{E}[\mathbb{P}(X_t \in A|\mathcal{F}_s)|X_r = x] = \mathbb{E}[\mathbb{P}(X_t \in A|X_s)|X_r = x]$$

Let $Y = X_s$. The inner term is $P_{s,t}(Y,A)$. So, $P_{r,t}(x,A) = \mathbb{E}[P_{s,t}(X_s,A)|X_r = x]$. The random variable X_s given $X_r = x$ has the probability density $\rho_{r,s}(x,y)$. We can write the expectation as an integral with respect to this density:

$$P_{r,t}(x,A) = \int_{\mathbb{R}^d} P_{s,t}(y,A)\rho_{r,s}(x,y)dy$$

This is the Chapman-Kolmogorov equation for probabilities. Now, we substitute the density definitions back into this equation. The left side is $P_{r,t}(x,A) = \int_A \rho_{r,t}(x,z)dz$. The inner term on the right side is $P_{s,t}(y,A) = \int_A \rho_{s,t}(y,z)dz$. Substituting this into the equation gives:

$$P_{r,t}(x,A) = \int_{\mathbb{R}^d} \left(\int_A \rho_{s,t}(y,z)dz \right) \rho_{r,s}(x,y)dy$$

Since all densities are non-negative, we can use the Fubini-Tonelli theorem to change the order of integration:

$$P_{r,t}(x,A) = \int_A \left(\int_{\mathbb{R}^d} \rho_{r,s}(x,y)\rho_{s,t}(y,z)dy \right) dz$$

We now have two integral expressions for $P_{r,t}(x,A)$. Comparing them, we have

$$\int_A \rho_{r,t}(x,z)dz = \int_A \left(\int_{\mathbb{R}^d} \rho_{r,s}(x,y)\rho_{s,t}(y,z)dy \right) dz$$

This equality holds for **all** measurable sets $A \in \mathcal{B}(\mathbb{R}^d)$. By the uniqueness property of measures (or the Radon-Nikodym theorem), the integrands must be equal almost everywhere (with respect to z):

$$\rho_{r,t}(x,z) = \int_{\mathbb{R}^d} \rho_{r,s}(x,y)\rho_{s,t}(y,z)dy$$

This completes the proof. ■