

# HW3

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In what follows,  $\Omega \subset \mathbb{R}^d$  will be a bounded domain,  $T > 0$ , and the parabolic interior  $\Omega_T$  and boundary  $\partial_p \Omega_T$  are given by

$$\Omega_T = (0, T] \times \Omega, \quad \partial_p \Omega_T = ([0, T] \times \partial\Omega) \cup (\{0\} \times \Omega).$$

**Exercise 1** Consider the differential operator

$$(\mathcal{L}u)(t, x) = -\Delta u(t, x) + c(x)u(x),$$

where  $c : \Omega \rightarrow [-M, +\infty)$  is continuous,  $M \geq 0$ . The goal is to establish the following weak maximum principle: if  $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$  and

$$\boxed{(\partial_t + \mathcal{L})u \geq 0 \text{ in } \Omega_T}, \quad \min_{\partial_p \Omega_T} u \geq 0 \implies \min_{\overline{\Omega_T}} u \geq 0. \quad (0.1)$$

1. Prove (0.1) under the condition  $\mathcal{L}u > 0$  in  $\Omega_T$  and  $M = 0$ .

2. Prove (0.1) under the condition  $\mathcal{L}u \geq 0$  in  $\Omega_T$  and  $M = 0$ .

*Hint: consider  $u_\varepsilon(t, x) = u(t, x) - t\varepsilon$ .*

3. Prove (0.1) under the condition  $\mathcal{L}u \geq 0$  in  $\Omega_T$  and  $M > 0$ .

*Hint: consider  $v(t, x) = e^{\lambda t}u(t, x)$  for an appropriate  $\lambda$ . (0, 1)*

**Exercise 2** Let  $\Omega = (0, \ell)$ .

1. Let  $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{1,0}(\partial_p \Omega_T)$  satisfy

$$\begin{cases} u_t - u_{xx} \geq 0, & (t, x) \in \Omega_T, \\ u|_{t=0} \geq 0, & x \in \Omega, \\ u(t, 0) \geq 0, & t > 0, \\ u_x(t, \ell) \geq 0, & t > 0. \end{cases}$$

Show that  $u \geq 0$  on  $\overline{\Omega_T}$ .

*Hint: you may consider  $u_\varepsilon(t, x) = u(t, x) + \varepsilon x$ .*

2. Let  $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{1,0}(\partial_p \Omega_T)$  satisfy

$$\begin{cases} u_t - u_{xx} = f, & (t, x) \in \Omega_T, \\ u|_{t=0} = \varphi, & x \in \Omega, \\ u(t, 0) = 0, & t > 0, \\ u_x(t, \ell) = g(t), & t > 0, \end{cases}$$

where  $f, \varphi, g$  are bounded, continuous functions in their domains. Show that

$$\max_{\Omega_T} |u| \leq C(|T| + 1)(F + G + \Phi)$$

for some constant  $C$  depending only on  $\ell$ , where  $F = \sup |f|$ ,  $G = \sup |g|$  and  $\Phi = \sup |\varphi|$ .

*Hint: consider  $v(t, x) = tF + Gx + \Phi \pm u(t, x)$  and use part 1.*

**Exercise 3** Let  $\Omega = (0, \ell)$ . Suppose that  $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$  solves

$$\begin{cases} u_t - u_{xx} = f(t, x), & (t, x) \in \Omega_T, \\ u(0, x) = 0, & x \in [0, \ell], \\ -u_x + \alpha u = 0, & t > 0, x = 0, \\ u_x + \beta u = 0, & t > 0, x = \ell, \end{cases}$$

where  $\alpha, \beta \geq 0$  are constants. Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t, x) dx + \int_0^T \int_0^\ell u_x^2(t, x) dx dt \leq C \int_0^T \int_0^\ell f^2(t, x) dx dt,$$

for some constant  $C$  depending only on  $T$ .

*Hint: multiply the equation by  $u$  on both sides, perform suitable integration by parts in  $x$ , then integrate in  $t$ ; use  $|2ab| \leq a^2 + b^2$  and Gronwall at some point.*

**Exercise 4** Let  $\Omega = (0, \ell)$  and  $b, c \in \mathcal{C}(\overline{\Omega_T})$ . Suppose that  $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$  solves

$$\begin{cases} u_t - u_{xx} + b(t, x)u_x + c(t, x)u = 0, & (t, x) \in \Omega_T, \\ u(0, x) = \varphi(x), & x \in [0, \ell], \\ u(t, 0) = u(t, \ell) = 0, & t \in [0, T]. \end{cases}$$

Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t, x) dx + \int_0^T \int_0^\ell u_x^2(t, x) dx dt \leq C \int_0^\ell \varphi^2(x) dx,$$

for some constant  $C$  depending only on  $T$ ,  $\beta$  and  $\gamma$ , where

$$\beta = \sup_{\overline{\Omega_T}} |b(t, x)|, \quad \gamma = \sup_{\overline{\Omega_T}} |c(t, x)|.$$