1. Let A, B be two nonintersecting sets. Let A be infinite and B at most countable. Prove that the union of A and B has the same cardinality as A does.

2. Prove that the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ has the cardinality 2^C, where C denotes the continual cardinality.

HINT: determine first the cardinality of $\mathbb{R}^2$.

3. Prove that an at most countable union of continual sets is again continual.

4. Let X be a metric space, and for two balls it holds that $B_r(x) \subset B_R(y)$. Does it imply $r \leq R$?

5. Let A be a subset in $\mathbb{R}^n$, and A' is the set of all its accumulation points. Prove that $A \setminus A'$ is at most countable.

6. Let G be an open dense subset in a metric space X. Prove that its complement is nowhere dense in X.

7. Let $G_1, G_2, ...$ be a sequence of open dense subsets in a complete metric space X. Prove that $G_1 \cap G_2 \cap \cdots$ is also a dense subset in X.

8. Let E be a closed subset in a metric space X. Prove that for any $a \in X$, the infimum

$$\tau_E(a) := \inf_{x \in E} d(x, a)$$

(where d is the distance function) is attained if either:

(i) E is in addition compact;

(ii) E is merely closed, but $X = \mathbb{R}^n$.

$\tau_E(a)$ is called *the distance from a to E.*

9. Prove that the function $\tau_E(a)$ in Problem 8 is continuous in a on X.

10. Let K be a compact in $\mathbb{R}^n$, and f a continuous real-valued function on K. Prove that its graph

$$G_f := \{(x, f(x)\}_{x \in K}$$

is a compact subset in $\mathbb{R}^{n+1}$.

11*. Prove that the space $C(K)$ (where K is a compact) is always separable.

HINT: employ functions $\tau_E(x)$ for one point sets E and the Stone-Weierstrass Theorem.

12*. Deduce from the Stone-Weierstrass Theorem its complex generalization: if X is the (complex) algebra of complex-valued continuous functions on a compact K and $A \ni 1$ is a subalgebra in X which separates points and satisfies $\lambda(A) \subset A$, where λ is the involutive map

$$f(x) \mapsto \overline{f(x)}, \quad f \in X,$$

then $\bar{A} = X$.

13. Let $A = \{f \in C[0,1] : \ |f(x)| \leq x^2\}$. Is A precompact?

14*. Let the functions $f_a(x)$ be defined as $x^a \ln x$ for $x > 0$ and $f_a(0) = 0$. Is the family $\{f_a(x)\}_{a>0}$ precompact in $C[0,1]$?

15. Let B be a closed ball in $\mathbb{R}^n$ and $E \subset C(B)$ be the subset of differentiable in $\mathbb{R}^n$ functions with $|f(x)| \leq 1$, $||\nabla f|| \leq 1$. Prove that E is precompact.

16*. Endow the set of natural numbers $\mathbb{N}$ with a metric which equals $1 + 1/(m+n)$ for m, n distinct, and 0 otherwise. Prove that: (i) the resulting $\mathbb{N}$ is a complete metric space; (ii) the nested balls principle fails for it.

*problems count for 2 points, while regular problems for 1 point.