HW2

September 18, 2025

Exercise 1 Let Ω be a bounded $\mathcal{C}^1$-domain. Show that there is at most one solution $u \in \mathcal{C}^{1,2}((0,\infty)\times\Omega) \cap \mathcal{C}^{0,1}([0,\infty)\times\bar{\Omega})$ that solves

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + f(t,x), & t>0,\ x\in\Omega, \\ \dfrac{\partial u(t,x)}{\partial n} = g(x), & x \in \partial\Omega, \\ u(0,x) = h(x), & x\in\Omega. \end{cases}$$

Hint: for two solutions u_1, u_2, compute $\phi'(t)$ where $\phi(t) = \int_\Omega |\nabla(u_1-u_2)|^2\,dx$.

Exercise 2 Compute the Fourier transform of the following functions defined on $\mathbb{R}$.

1. $$f_1(x) = \begin{cases} 1, & |x|\le A, \\ 0, & |x|>A, \end{cases} \quad A>0.$$

2. $$f_2(x) = \begin{cases} e^{-ax}, & x>0, \\ 0, & x<0, \end{cases} \quad a>0.$$

3. $f_3(x) = e^{-a|x|}$, $a>0$.

4. $f_4(x) = \frac{1}{a^2+x^2}$, $a>0$.

Exercise 3 Let $f\in L^1(\mathbb{R}^d)$. Use Fourier transform to solve the equation

$$-\Delta u(x) + u(x) = f(x), \quad x\in\mathbb{R}^d.$$

Exercise 4

1. Show that $$\Big[\frac{1}{2}\mathbb{1}_{(-t,t)}(x)\Big]^\wedge = \frac{\sin(2\pi\xi t)}{2\pi\xi}.$$

2. Show that $$\big[\delta(x-t)+\delta(x+t)\big]^\wedge = 2\cos(2\pi\xi t).$$
   You can treat $\delta(x)$ as a function such that $\int \delta(x)f(x)\,dx = f(0)$ for any continuous f.

3. Use Fourier transform to solve the wave equation in $\mathbb{R}^1$:
   $$\begin{cases} \partial_{tt}u = \partial_{xx}u, & t>0,\ x\in\mathbb{R}, \\ u(0,x) = \phi(x), & x\in\mathbb{R}, \\ \partial_t(0,x) = \psi(x), & x\in\mathbb{R}. \end{cases}$$

Ex 1. 证：若 u_1, u_2 均为原方程的解，则 $u = u_1 - u_2$ 即为 $\begin{cases} \partial_t u = \Delta u, & t>0,\ x\in\Omega \\ \frac{\partial u}{\partial n} = 0, & x\in\partial\Omega \\ u(0,x) = 0, & x\in\Omega \end{cases}$ 的解

令 $\phi(t) = \int_\Omega |\nabla u^2| dx$

因 Ω 有界，故 $\phi(t) < \infty$ 且 well-defined.

又 $\phi'(t) = 2\int_\Omega \nabla(\partial_t u)\cdot\nabla u\, dx$

$$= 2\int_\Omega (-\partial_t u)\Delta u + \int_{\partial\Omega} \partial_t u \cdot \frac{\partial u}{\partial n} dS = -2\int_\Omega |\partial_t u|^2 \leq 0$$

且 $\phi(0) = 0$，$\phi(t) \geq 0$

故 $\phi(t) \equiv 0$ for $t \geq 0$，从而 $\nabla u(t,x) \equiv 0$ for $(t,x) \in \mathbb{R}_+ \times \bar{\Omega}$

因 Ω connected，故 $u(t,\cdot)$ 在 Ω 上为常数

又 $u(t,\cdot) \in C(\bar{\Omega})$ 且 $u|_{\partial\Omega} = 0$，故 $u \equiv 0$，即得唯一性.

Ex 2. (1) $\hat{f}_1(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f_1(x)\,dx$

$$= \int_{-A}^{A} e^{-2\pi i \xi x} dx = \frac{e^{-2\pi i\xi A} - e^{2\pi i \xi A}}{-2\pi i \xi} = \frac{\sin(2\pi A\xi)}{\pi\xi}$$

(2) $\hat{f}_2(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f_2(x)\,dx$

$$= \int_0^\infty e^{-(2\pi i\xi + a)x} dx = \frac{1}{a + 2\pi\xi i}$$

(3) $\hat{f}_3(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f_3(x)\,dx$

$$= \int_{-\infty}^0 e^{-(2\pi i\xi - a)x} dx + \int_0^\infty e^{-(2\pi i\xi + a)x} dx = \frac{2a}{a^2 + 4\pi^2\xi^2}$$

(4) 由 (3) 知，$(e^{-2a\pi|x|})^\wedge = \frac{a}{\pi(a^2+\xi^2)} := g(\xi)$

若记 $h(x) := e^{-2a\pi|x|}$，则上式即为 $\hat{h}(\xi) = g(\xi) = \hat{h}(-\xi) = (h(x))^\vee$

从而 $\hat{g}(\xi) = h(\xi)$

故 $\hat{f}_4(\xi) = \left(\frac{\pi}{a} g(\xi)\right)^\wedge = \frac{\pi}{a}\hat{g}(\xi) = \frac{\pi}{a} h(\xi) = \frac{\pi}{a} e^{-2a\pi|\xi|}$

Ex 3. 考虑 $\hat{u}(\xi) = \mathcal{F}[u(\cdot)]$

则 $\hat{u}$ s.t. $4\pi^2|\xi|^2\hat{u}(\xi) + \hat{u}(\xi) = \hat{f}(\xi)$

求解得 $\hat{u}(\xi) = \dfrac{\hat{f}(\xi)}{1+4\pi^2|\xi|^2}$

故解为 $u(x) = f(x) * \left(\dfrac{1}{1+4\pi^2|\xi|^2}\right)^{\vee}(x) = f(x) * \left(\frac{1}{2}e^{-|x|}\right)$

Ex 4. (1) $\left[\frac{1}{2}1_{(-t,t)}(x)\right]^{\wedge} = \frac{1}{2}\int_{-t}^{t} e^{-2\pi i\xi x}\,dx = \dfrac{e^{-2\pi i\xi t} - e^{2\pi i\xi t}}{-4\pi i\xi} = \dfrac{\sin(2\pi\xi t)}{2\pi\xi}$

(2) $$\begin{aligned}\left[\delta(x-t)+\delta(x+t)\right]^{\wedge} &= \int_{\mathbb{R}} e^{-2\pi i\xi x}\left[\delta(x-t)+\delta(x+t)\right]dx \\ &= e^{-2\pi i\xi t} + e^{2\pi i\xi t} \\ &= 2\cos(2\pi\xi t)\end{aligned}$$

(3) 考虑 $\hat{u}(t,\xi) = \mathcal{F}[u(t,\cdot)]$

则 $\hat{u}$ s.t. $\begin{cases}\partial_{tt}\hat{u}(t,\xi) = -4\pi^2\xi^2\hat{u}(t,\xi) \\ \hat{u}(0,\xi) = \hat{\phi}(\xi) \\ \partial_t\hat{u}(0,\xi) = \hat{\psi}(\xi)\end{cases}$

求解得 $\hat{u}(t,\xi) = \cos(2\pi\xi t)\cdot\hat{\phi}(\xi) + \dfrac{\sin(2\pi\xi t)}{2\pi\xi}\cdot\hat{\psi}(\xi)$

故解为 $$\begin{aligned}u(t,x) &= \phi(x) * (\cos(2\pi\xi t))^{\vee}(x) + \psi(x) * \left(\frac{\sin(2\pi\xi t)}{2\pi\xi}\right)^{\vee}(x) \\ &= \phi(x) * \frac{\delta(x-t)+\delta(x+t)}{2} + \psi(x) * \left(\frac{1}{2}1_{(-t,t)}(x)\right) \\ &= \frac{\phi(x+t)+\phi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t}\psi(y)\,dy\end{aligned}$$