

HM2

1 Continuous-time stochastic processes

1. On $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X_t : t \geq 0\}$ be a stochastic process that is measurable and T is a random time. Prove that
 - (a) X_T is a random variable if T is finite;
 - (b) All sets of the form $\{X_T \in A\}$ and $\{X_T \in A\} \cup \{T = \infty\}$ with $A \in \mathcal{B}(\mathbb{R}^d)$ forms a σ -algebra.
2. Let $\{\mathcal{F}_t : t \geq 0\}$ be a filtration and X be an adapted process relative to $\{\mathcal{F}_t : t \geq 0\}$. Set $T = \inf\{t \geq 0 : X_t \in A\}$. Prove that
 - (a) if A is open and the sample paths of X are right-continuous, then T is an optional time of $\{\mathcal{F}_t : t \geq 0\}$.
 - (b) if A is closed and the sample paths of X are continuous, then T is a stopping time of $\{\mathcal{F}_t : t \geq 0\}$.
3. Let $\{X_t : t \geq 0\}$ be a progressively measurable process relative to $\{\mathcal{F}_t : t \geq 0\}$ and let T be a finite stopping time of $\{\mathcal{F}_t : t \geq 0\}$. Prove that
 - (a) X_T is \mathcal{F}_T -measurable;
 - (b) the process $\{X_{T \wedge t} : t \geq 0\}$ is progressively measurable relative to $\{\mathcal{F}_t : t \geq 0\}$.
4. Let T be an optional time of $\{\mathcal{F}_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.
 - (a) Prove that \mathcal{F}_{T+} is a σ -algebra and $\mathcal{F}_{T+} = \{A \in \mathcal{F} : A \cap \{T < t\} \in \mathcal{F}_t, \forall t \geq 0\}$.
 - (b) Prove that if T is a stopping time, then $\mathcal{F}_T \subseteq \mathcal{F}_{T+}$.
5. Let $\{X_t : t \geq 0\}$ be a right-continuous, nonnegative supermartingale relative to $\{\mathcal{F}_t : t \geq 0\}$. Prove that $X_\infty = \lim_{t \rightarrow \infty} X_t$ exists almost surely and $\{X_t : 0 \leq t \leq \infty\}$ is a supermartingale relative to $\{\mathcal{F}_t : 0 \leq t \leq \infty\}$.
6. Let $\{X_t : t \geq 0\}$ be a right-continuous, nonnegative submartingale relative to $\{\mathcal{F}_t : t \geq 0\}$.

- (a) The family $\{X_t : t \geq 0\}$ of r.v.s are uniformly integrable.
- (b) X_t converges in L^1 as $t \rightarrow \infty$.
- (c) X_t converges almost surely and the limit X_∞ is integrable. Moreover, $\{X_t : 0 \leq t \leq \infty\}$ is a submartingale relative to $\{\mathcal{F}_t : 0 \leq t \leq \infty\}$.

Prove that (a), (b) and (c) are equivalent.

7. Let $\{X_t : t \geq 0\}$ be a right-continuous supermartingale relative to $\{\mathcal{F}_t : t \geq 0\}$ and $S \leq T$ are stopping times of $\{\mathcal{F}_t : t \geq 0\}$. Prove that
 - (a) $\{X_{T \wedge t} : t \geq 0\}$ is supermartingale of $\{\mathcal{F}_t : t \geq 0\}$;
 - (b) $\mathbb{E}[X_{T \wedge t} | \mathcal{F}_S] \leq X_{S \wedge t}$.
8. Let $\{X_t : t \geq 0\}$ be a right-continuous process such that $X_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for $\forall t \geq 0$. Prove that X is a submartingale relative to $\{\mathcal{F}_t : t \geq 0\}$ if and only if $\mathbb{E}[X_T] \geq \mathbb{E}[X_S]$ for all bounded stopping times $S \leq T$ of $\{\mathcal{F}_t : t \geq 0\}$.
9. Let X be a normal Markov process with transition density functions $\{\rho_{s,t} : 0 \leq s \leq t\}$. Prove that

$$\rho_{r,t}(x, z) = \int_{\mathbb{R}^d} \rho_{r,s}(x, y) \rho_{s,t}(y, z) dy$$

for any $0 \leq r \leq s \leq t$ and $x, z \in \mathbb{R}^d$.