Real Analysis (H).

25% HW. 25% Midterm. 50% Final.

textbook: Kolomogorov.

Crash Course on Set Theory.

Elements of Set Theory

Set: undefinable notion.

A set consists of its elts: $a \in A$.

$\emptyset$: set with no elts. (i.e. $\forall x$, it holds: $x \notin \emptyset$.)

Def: $A \subset B$ (A is included in B) if

$\{x \in A\} \underset{\text{set}}{\Rightarrow} \{x \in B\}$.

Prop: $\emptyset \subset A$, $\forall A$

Q: why one should be careful with the notion of sets?

Ex. (Barber's Paradox).

In a city X, there are residents with one of them being a barber. It's known that the barber only shaves everyone who doesn't shave himself. Does the barber shave himself? This leads to a contrad. $\times$

$\Rightarrow E = \{$residents who don't shave themselves$\}$ doesn't exist.

Def: 2^A: the set of all subsets of A. (always exists)

Prop: $|2^A| = 2^{|A|}$ if A is finite.

Operations with sets.

1) Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

2) Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

3) Difference: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Remark: union and intersection can be similarly defined for any amount of sets:

(Let E - a set: $\forall \alpha \in E$, we associate it with a set A_α, then $\bigcup_{\alpha \in E} A_\alpha$, $\bigcap_{\alpha \in E} A_\alpha$.)

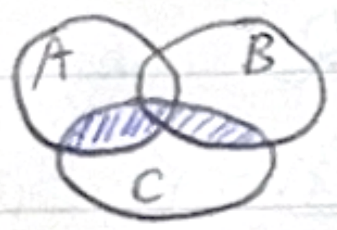
Notation: $\bigsqcup_{\alpha\in E} A_\alpha$ – the same as $\bigcup_{\alpha\in E} A_\alpha$, but in the special situation when all A_α are disjoint.

Ex. $\bigsqcup_{x\in\mathbb{R}} \{x\} = \mathbb{R}$.

Some Properties.

1). Commutativity: $A\cup B = B\cup A$; $A\cap B = B\cap A$.

2). Associativity: $(A\cup B)\cup C = A\cup(B\cup C)$.
$(A\cap B)\cap C = A\cap(B\cap C)$.



3). Distributivity: $(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$.
$(A\cap B)\cup C = (A\cup C)\cap(B\cup C)$.

Note: if „$\cup$" is $\oplus$, „$\cap$" is $\odot$, then we get axioms of a comm. ring in algebra.

4) Duality of „$\cup$" and „$\cap$":
$$\Omega\setminus\bigcup_{\alpha\in E} A_\alpha = \bigcap_{\alpha\in E}(\Omega\setminus A_\alpha).$$
$$\Omega\setminus\bigcap_{\alpha\in E} A_\alpha = \bigcup_{\alpha\in E}(\Omega\setminus A_\alpha).$$

Proof of the first identity:

$x\in LHS \Rightarrow x\in\Omega$ and $x\notin\bigcup_{\alpha\in E} A_\alpha \Rightarrow x\in\Omega$ and
$\forall\alpha\in E$, it holds: $x\notin A_\alpha$. $\Rightarrow$ by def. $\forall\alpha\in E$,
it holds: $x\in\Omega\setminus A_\alpha$. $\Rightarrow x\in\bigcap_{\alpha\in E}(\Omega\setminus A_\alpha)$.
$\Rightarrow x\in RHS$.
By similar logic, $x\in RHS \Rightarrow x\in LHS$.
Thus, $LHS = RHS$.

Cartesian Product: $\forall$ sets A, B, $\exists$ $A\times B = \{(x,y):$
$x\in A,\ y\in B\}$.

Ex. $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$. $\mathbb{R}^n = \underbrace{\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}}_{n \text{ times}}$.

Function from A to B.

$X\subset A\times B$, s.t.
(1) $\forall x\in X$, $\exists y: (x,y)\in X$.
(2) such y is unique for each $x\in X$.

Notation: $y = f(x)$. (function $\Leftrightarrow$ mapping).

* injective: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

* surjective: $\forall y\in B$, $\exists x: y = f(x)$.

* bijective: surj. + inj.

(bijection = equivalence = 1-to-1 mapping).

Def: Let A be a set, then an equivalence relation on A is a subset $X\subset A\times A$, with the following properties: (Notation: $(x,y)\in X$, we write $x\sim y$)
(1) $x\sim x$. (reflexivity). (2) $x\sim y \Rightarrow y\sim x$. (symmetry).
(3) $x\sim y$ and $y\sim z \Rightarrow x\sim z$ (transitivity).

Def: Having an equiv. relation, an equiv. class is the set of all $y\in A$ with $y\sim x$. (x is fixed).

Prop: two equiv. classes either coincide or don't intersect.

Thm: $\exists$ set $E\subset A$, s.t. $A = \bigsqcup_{\alpha\in E} X_\alpha$.
(A is a disjoint union of equiv. classes.)

Ex. $A = \mathbb{C}$ – α plane. $z\sim w$ if $|z| = |w|$.
$\mathbb{C} = \bigsqcup_{r\in[0,+\infty)} C_r$, $C_r = \{z\in\mathbb{C} : |z| = r\}$.

Def: two sets A, B are called equivalent if $\exists$ a bijection $f: A\to B$.
If X is a set of sets, this gives an equiv. relation on X. We obtain equiv. classes of sets. Each equiv. class is

called a cardinal (cardinal number).

Notation: $\#A$ – cardinal of the class generated by A.

Ex $A=\{a_1,\dots,a_n\}$. $B=\{b_1,\dots,b_m\}$. then $A\sim B \Leftrightarrow m=n$.

So, it's natural to write $\#A=n$.

Q: how to deal with infinite sets?

Ex: $\mathbb{N}$.

A set A is called countable if $A\sim\mathbb{N}$.

Fact: A is infinite $\Leftrightarrow$ $A\supset B$ – countable.

Proof: "$\Leftarrow$": clear.

"$\Rightarrow$": take $a_1\in A$. then take $A\setminus\{a_1\}$ – also infinite. so take $a_2\in A\setminus\{a_1\}$, then take $A\setminus\{a_1,a_2\}$ – also infinite. so take a_3, ...

Let $B=\{\underbrace{a_1,a_2,a_3,\dots}_{\text{all distinct}}\}$ $\Rightarrow A\supset B$. □

We want to show that not all infinite sets are countable.

Standard Equivalences:

1) $\mathbb{Z}\sim\mathbb{N}$: we'll now "list" all the elts. of $\mathbb{Z}$:

$$\mathbb{Z}=\{0,1,-1,2,-2,3,-3,\dots\dots\}.$$

$\mathbb{N}$: No1 No2 No3 ......

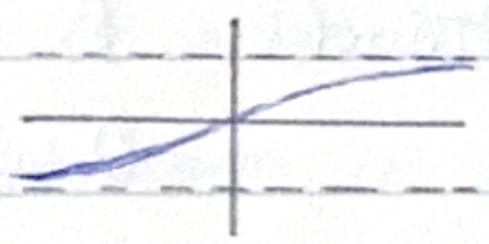
2) $\mathbb{Q}\sim\mathbb{N}$: $\forall x\in\mathbb{Q}$, $x=\frac{p}{q}$, $p\in\mathbb{Z}$, $q\in\mathbb{N}$.

denominator \ numerator	0	1	-1	2	-2	......
1	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	......
2	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{2}$	$\frac{-2}{2}$	......
3	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-2}{3}$	......
⋮	⋮	⋮	⋮	⋮	⋮	

$\forall x\in\mathbb{Q}$ is an elt. in the infin table. (Some may repeat).
So, all $x\in\mathbb{Q}$ are "listed".

3) $\mathbb{R}\sim(0,1)$:

* $(a,b)\sim(c,d)$: $y=\alpha x+\beta$.
* $f(x)=\arctan x$: $\mathbb{R}\to(-\frac{\pi}{2},\frac{\pi}{2})$.



4) $[a,b]\sim(a,b)$: choose E – a countable subset in $[a,b]$ with $a,b\in E$. Take $C=E\setminus\{a,b\}$ – also countable.

$[a,b]\setminus E=(a,b)\setminus C$ — on this set: $f(x)=x$.

$E\to C$: we just take a bijection between the two countable sets, which exists. At last, take "union".

$\Rightarrow$ Continual Sets

$\sim\mathbb{R}\sim[a,b]\sim(a,b)$.

Q: Is Contin. $\sim$ Count.?

Cantor's Thm: $\forall$ set E: $E\not\sim 2^E$.

Proof: Assume rather: $\exists$ a bijection $f: E\to 2^E$.

Consider $A=\{x\in E: x\notin \underbrace{f(x)}_{\in 2^E}\}\subset E$.

$\Rightarrow A\in 2^E$!

Since f is bij. there exists unique $a\in E$ s.t. $f(a)=A$.

Q: is $a\in A$?

If yes: $a\in A=f(a)$ but A only contains x satisfying $x\notin f(x)$. ↯

If no: $a\notin A=f(a)$ $\Rightarrow$ by def. $a\in A$. ↯

So, such f doesn't exist! □

Now, let's consider $2^{\mathbb{N}}$.

In fact $2^{\mathbb{N}}\sim\{$ the set of all sequences of $\{0,1\}\}$.

since $\forall A\in 2^{\mathbb{N}}$, $\forall x\in\mathbb{N}$, $x\in A\Rightarrow$ label: 1. $x\notin A\Rightarrow$ label: 0.

can be represented as

Remind: $\forall x \in [0,1]$, $x = 0.a_1 a_2 a_3 \cdots$, with $a_n \in \{0,1\}$ $\forall n$.

Method: | $\frac{1}{2}$ | $x \in [0, \frac{1}{2}) \Rightarrow a_1 = 0$

0 1 $x \in [\frac{1}{2}, 1] \Rightarrow a_1 = 1$. etc.

e.g. $1 = 0.1111 \cdots\cdots 11 \cdots$

$\Rightarrow 2^{\mathbb{N}} \sim [0,1]$ $2^{\mathbb{N}} \not\sim \mathbb{N}$

$\Rightarrow$ continual $\neq$ countable.

$\aleph_1$ $\aleph_0$.

Ordered Sets.

Def: Let E - a set. then an order (partial order) on E is a subset $X \subset E \times E$ with the following properties: (Notation: if $(a,b) \in X$, we write $a \leq b$)

(i) $a \leq a$;   (ii) if $a \leq b$ and $b \leq c$, then $a \leq c$.

(iii) if $a \leq b$ and $b \leq a$, then $a = b$.

Ex. $\mathbb{R}$ — natural ordering; $E \subset \mathbb{R}$;

$\mathbb{R}^n$ with the lexicographic order:

$\vec{x} = (x_1, \cdots, x_n)$, $\vec{y} = (y_1, \cdots, y_n)$, then

$\vec{x} \leq \vec{y}$ means $x_i \leq y_i$ $\forall i \in [\![1, n]\!]$.

Note: $(1,2)$ and $(2,1)$ are incomparable.

Def: Linearly ordered set:

we add the axiom (iv): $\forall a, b$ it holds $a \leq b$ or $b \leq a$.

(all are comparable).

Ex. $\mathbb{R}$; $E \subset \mathbb{R}$.

Def: Well ordered set:

we add to (i) - (iv) the axiom (v): $\forall$ subset $A \subset E$,

A has its least elt, i.e. $\exists a \in A$, s.t. $a \leq x$, $\forall x \in A$.

Ex. $\mathbb{N}$ with the usual order.

Q: A linearly ordered set but not a well orderd set?

E.g. $\mathbb{Z}$ with the usual order.

$\mathbb{Z}_{<0} \subset \mathbb{Z}$ but it doesn't have a least elt.

Def. For two cardinals C_1, C_2, we say that $C_1 \leq C_2$ if for some representative set E_1 in C_1; E_2 in C_2, it holds: $E_1 \sim E_2' \subset E_2$.

Ex. Natually, $\aleph_0 \leq \aleph_1$, since $\mathbb{N} \sim \mathbb{N} \subset \mathbb{R}$.

Thm 1. (Cermelo) $\forall$ cardinals C_1, C_2, it holds: either

$C_1 \leq C_2$ or $C_2 \leq C_1$.

(i.e. $\forall$ set A, B, either $A \sim B' \subset B$ or $B \sim A' \subset A$),

Thm 2. if $C_1 \leq C_2$ and $C_2 \leq C_1$, then $C_1 = C_2$.

(So, if $A \sim B' \subset B$ and $B \sim A' \subset A$, then $A \sim B$.

— Cantor-Bernshtein Thm)

Corollary: The set of all cardinal numbers is linearly ordered.

Further Thm (Cermelo): Cardinal numbers are actually well ordered.

So, we can natually order cardinals.

Cantor Thm means $\underset{\text{cardinal generated by set } E.}{\underbrace{[E]}} \lneqq [2^E]$.

$\Rightarrow \exists$ bigger and bigger cardinals...

e.g. $[2^{\mathbb{R}}] \gneqq [\mathbb{R}]$.

Famous open question (– Continuum Hypothesis)

We know that $\aleph_0 < \aleph_1$.

? $\exists \aleph$ s.t. $\aleph_0 < \aleph < \aleph_1$?

Thm: this question has no solution! (i.e. the existence of such $\aleph$ cannot be proved or disproved.)

(Illustration of Godel's Thm!)

A few exercises:

Ex1: prove $\mathbb{R}\setminus\mathbb{N} \sim \mathbb{R}$.

Proof: clearly $\mathbb{R}\setminus\mathbb{N} \le \mathbb{R}$ $(\mathbb{R}\setminus\mathbb{N} \subset \mathbb{R})$.

On the other hand $\mathbb{R}\setminus\mathbb{N} \supset (0,1) \sim \mathbb{R}$.

$\Rightarrow \mathbb{R} \le \mathbb{R}\setminus\mathbb{N}$. $\Rightarrow$ by Cantor-Bernshtein Thm:

$\mathbb{R}\setminus\mathbb{N} \sim \mathbb{R}$. $\square$

Thm: $\mathbb{R}\setminus\mathbb{N} \sim \mathbb{R}$.

Ex2. Prove that $\underbrace{C[0,1]}_{\text{space of contin. func-s on } [0,1]} \in \aleph_1$.

Reason: $\forall$ contin. funct. is uniquely determined by its values at $x \in \mathbb{Q}$! $\left(f(x) = \lim_{k\to\infty} f(x_k)^{\in\mathbb{Q}} \right)$.

$[C[0,1]] \le [\text{set of all } f: \mathbb{Q} \to \mathbb{R}]$

$= [\text{set of all } f: \mathbb{N} \to \mathbb{R}]$

$= [(a_1, a_2, a_3, \cdots) \text{ with } a_n \in \mathbb{R}, \forall n \in \mathbb{N}]$.

$= [(a_1, a_2, a_3, \dots) \text{ with } a_n \in [0,1], \forall n \in \mathbb{N}]$.

Let's analyze this cardinality.

$a_1: 0.\, d_{11}\, d_{12}\, d_{13} \cdots$

$a_2: 0.\, d_{21}\, d_{22}\, d_{23} \cdots$

$a_3: 0.\, d_{31}\, d_{32}\, d_{33} \cdots$

then use the „Cantor's Trick":

Take $d = 0.\, d_{11}\, d_{21}\, d_{12}\, d_{31}\, d_{22}\, d_{13} \cdots$

(d is a number in $[0,1]$!).

This number codes the whole sequence! $\Rightarrow$

$\forall$ sequence $(a_1, a_2, \cdots)$, it could be coded with a real number in $[0,1]$.

$[C[0,1]] \le \aleph_1$.

Since all const. funct. are in $C[0,1]$,

$\aleph_1 \le [C[0,1]]$. $\Rightarrow$ by C-B, $[C[0,1]] = \aleph_1$. $\square$

Thm $[C[0,1]] = \aleph_1$.

Axiom of choice:

Let $\{A_\alpha\}_{\alpha\in E}$ be a group of sets. (means $\exists$ a Bijection $\varphi: E$ onto some set and $\forall$ $\varphi(\alpha)$ is a set)

Let $\{A_\alpha\}_{\alpha \in E}$ be disjoint. i.e $A_\alpha \cap A_\beta = \emptyset$, for $\alpha \ne \beta$.

Then, $\exists$ a set X: it contains one and only one elt of each A_α. ($\Leftrightarrow$ $\exists$ a bijection f on E with $f(\alpha) \in A_\alpha$)

Rk: this is an axiom!

It has several equivalent formulations, and the most applicable one is as follows:

Zorn's Lemma: Let E be a (partially) ordered set.

Assume that the followings hold:

$\forall$ chain A in E (i.e. a subset $A \subset E$ where every 2 elts are comparable) has an upper bound. (i.e. $\exists a \in E$ s.t. $x \le a$, $\forall x \in A$.)

Then, E has a maximal elt. m. (i.e. $\forall x \in E$, $m \le x \Rightarrow x = m$).

KOKUYO

One application: $\exists$ of a basis in $\forall$ linear space (even ∞-dimentional).

Remind: a basis for a linear space V is such a system $\{e_\alpha^{\in V}\}$ that:

(i) $\exists$ no (finite) nontrivial linear combinations between elt.s of $\{e_\alpha\}$. i.e. $\{e_\alpha\}$ are linearly independent.

(ii). $\forall x \in V$, x is a (finite) lin. combination of elts of $\{e_\alpha\}$.

Rk: $Poly[x]$ — ∞-dim.

$C[0,1]$ — ∞-dim.

Thm: $\forall$ lin. space V admits a basis. (called the Gammel basis)

Proof: Let $E = \{$the set of all lin. indep. systems of vectors in $V\}$.

Let's introduce ordering on E:

$S_1 \le S_2$ if $S_1 \subset S_2$.

(easy to check that we've got a patial order)

Now, let's check that the conditions of Zorn's Lemma are satisfied:

Take a chain $\{S_\alpha\}_{\alpha \in A} \subset E$.

Then $\{S_\alpha\}$ has an upper bound: $S := \bigcup_{\alpha \in A} S_\alpha$.

Then S is also a lin. indep. system.

S is clearly an upper bound for $\{S_\alpha\}$.

$\Rightarrow$ by Zorn's Lemma, $\exists S_0$ - maximal lin. indep. syst.

Then, easy to deduce that S_0 is the desired basis.

Indeed, if some $u \in V$ is not a finite lin. combin. of elt.s of S_0, then $\widetilde{S_0} := S_0 \cup \{u\}$ is a strictly bigger lin. indep. syst. (by def.) — a contrad. $\not\downarrow$. $\square$

Remark: generally - no way to see explicitly the Gammel basis.

The Cantor - Schröder - Bernstein Theorem:

If $|X| \le |Y|$ and $|Y| \le |X|$, then $|X| = |Y|$.

Proof: Let $f: X \to Y$ and $g: Y \to X$ be injections.

Consider a point $x \in X$.

If $x \in g(Y)$, we form $g^{-1}(x) \in Y$.

If $g^{-1}(x) \in f(X)$, we form $f^{-1}(g^{-1}(x))$, and so forth.

Either this process can be continued indefinitely, or it terminates with an element of $X \setminus g(Y)$, or $Y \setminus f(X)$.

In these 3 cases we say that x is in X_∞, X_X or X_Y.

$\Rightarrow X = X_\infty \sqcup X_X \sqcup X_Y$.

In the same way, $Y = Y_\infty \sqcup Y_X \sqcup Y_Y$.

Clearly, $X_\infty \xrightarrow{f} Y_\infty$, $X_X \xrightarrow{f} Y_X$, $Y_Y \xrightarrow{g} X_Y$.

Therefore we define $h: X \to Y$ by

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_\infty \cup X_X \\ g^{-1}(x) & \text{if } x \in X_Y. \end{cases}$$

Then h is bijective. $\square$

KOKUYO

Metric Spaces.

Def: a metric space is a set X with a given distance function (metric) $\rho: X \times X \to \mathbb{R}_{\geq 0}$ on it with the following prop.:

1) $\rho(x,y) \geq 0$, and $\rho(x,y) = 0 \iff x = y$. (nondegeneracy).
2) $\rho(x,y) = \rho(y,x)$, $\forall x, y \in X$. (symmetry)
3) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$, $\forall x,y,z \in X$. (triangle inequality).

Examples: 1) $X = \mathbb{R}$, $\rho(x,y) = |x-y|$.

2) $\forall$ set E with $\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$

Prop. If X - a metric space then $\forall Y \subset X$, Y is also a metric space with the same ρ.

Ex $\mathbb{N} \subset \mathbb{R}$; $(a,b) \subset \mathbb{R}$; ...

Def. a normed space is a linear space X equipped with a func. $\|\cdot\|: X \to \mathbb{R}_{\geq 0}$ with the following prop.:

1) $\|x\| \geq 0$, and $\|x\| = 0 \iff x = 0$.
2) $\|\alpha x\| = |\alpha| \cdot \|x\|$, $\forall x \in X$, $\alpha \in \mathbb{R}$.
3) $\|x+y\| \leq \|x\| + \|y\|$ $\forall x, y \in X$. (indicating $\|x-y\| \geq \big| \|x\| - \|y\| \big|$)

Fact: $\forall$ normed space is a metric space with $\rho(x,y) := \|x-y\|$.

(for symmetry: $\|y-x\| = \|(-1)\cdot(x-y)\| = |-1| \cdot \|x-y\| = \|x-y\|$.)

Ex. $\mathbb{R}^n$, with $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. ρ: accordingly defined

scalar product in $\mathbb{R}^n$: $(x,y) = x_1 y_1 + \cdots + x_n y_n$.

then $\|x\| = \sqrt{(x,x)}$.

triang. ineq.: $\|x+y\| \leq \|x\| + \|y\|$

$\iff (x+y, x+y) \leq (x,x) + (y,y) + 2\sqrt{(x,x)\cdot(y,y)} \iff (x,y) \leq \sqrt{(x,x)\cdot(y,y)}$

— Cauchy-Schwarz Inequality.

Remark: a more general fact is that a space with scalar product (Euclidean space) is always a normed space.

So, $\mathbb{R}^n$ and all its subsets are metric spaces with $\rho(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$.

Ex. $C[a,b] = \{$ cts func. on $[a,b]\}$.

$\|f\| := \max_{x\in[a,b]} |f(x)|$. ρ: accordingly defined.

Ex. $\ell_1 = \{ (x_1, x_2, x_3, \cdots)$ with $\sum_{j=1}^{\infty} |x_j| < +\infty \}$.

$\|x\| := \sum_{j=1}^{\infty} |x_j|$.

Ex. $X = \{ C[a,b]$. but equipped with the norm $\|f\| := \int_a^b |f(x)|\,dx. \}$.

Def. an open ball in a metric space X with center $a \in X$ and radius $R > 0$ is $B_R(a) = \{x \in X : \rho(x,a) < R\}$.

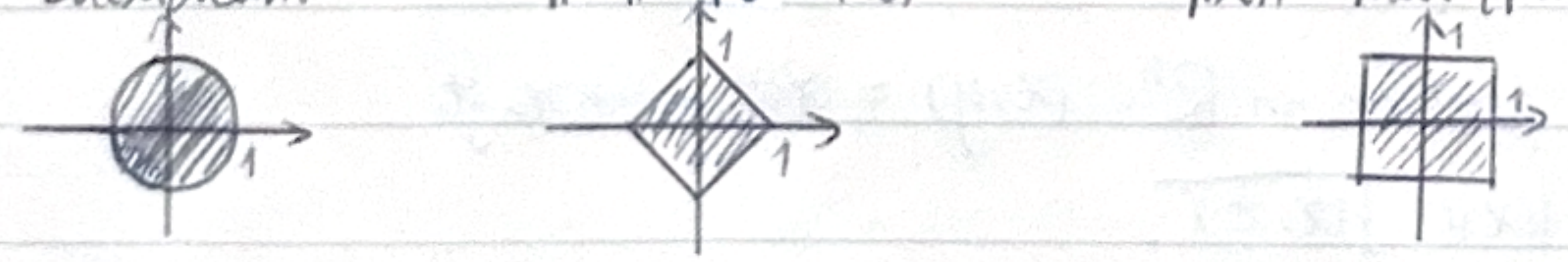
a closed ball: $\overline{B_R(a)} = \{x \in X : \rho(x,a) \le R\}$.

Ex. $\forall$ set E with $\rho(x,y) = \begin{cases} 0, & x=y \\ 1, & x \ne y. \end{cases}$

$B_{\frac{1}{2}}(a) = \{a\}$. $B_1(a) = E$.

Ex. $\mathbb{R}^2$. $B_1(0)$:

ρ: Euclidean. $\|x\| = |x_1| + |x_2|$ $\|x\| = \max\{|x_1|, |x_2|\}$.



From now on, set X denote a metric space.

Def. a set $G \subset X$ is called open, if $\forall a \in G$, $\exists \varepsilon > 0$, st. $B_\varepsilon(a) \subset G$.

A set $E \subset X$ is closed if $X \setminus E$ is open.

Ex. open sets in $\mathbb{R} = \bigsqcup$ open intervals.
could be made possible by selecting the intervals.

Prop. 1) $\bigcup_{\alpha\in A} G_\alpha$ is open if $\forall G_\alpha$ is open.

2) $\bigcap_{\alpha\in A} E_\alpha$ is closed if $\forall E_\alpha$ is closed.

(follows from 1) and $X \setminus \bigcap_{\alpha\in A} E_\alpha = \bigcup_{\alpha\in A} (X \setminus E_\alpha)$.)

3) G - open. E - closed. $G \setminus E$ is open.

Remark: if X - a metric space. $Y \subset X$ - a metric subspace, then $G \subset Y$ is open in Y. iff $G = Y \cap \tilde{G}$. with $\tilde{G}$: open in X. — subspace topology.

this is relatively open!

Ex. $X = \mathbb{R}$, $Y = [-1, 1]$.

$G = (0, 1]$ $\begin{cases} \text{not open in } X. \\ \text{is open in } Y. \text{ since } G = (0, +\infty) \cap Y. \end{cases}$

Remark. $B_R(a)$ - open. $\overline{B_R(a)}$ - closed. (in our metric space)

Proof: choose $b \in B_R(a)$, $\varepsilon < R - \rho(a,b) \Rightarrow B_\varepsilon(b) \subset B_R(a)$

Def: a seq. $\{a_k\}_{k=1}^{\infty} \subset X$ is convergent to $a \in X$ (notation: $\lim_{k\to\infty} a_k = a$), if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}^*$ st $a_k \in B_\varepsilon(a)$. $\forall k \ge N$.

(equivalently, we can say: "$\forall$ open $G \ni a$" instead of "$B_\varepsilon(a)$".)

open neighborhood of a. (nbd).

Def. Let $E \subset X$; an interior point $a \in E$ is a point such that $\exists \varepsilon > 0$, $B_\varepsilon(a) \subset E$.

$\mathring{E} :=$ the interior of E = {the set of all its interior points.}
open. by its def.

Def. An exterior point of E is an interior pt. of $X \setminus E$.
exterior of $E = (X \mathring{\setminus} E) =$ {all exterior points}.

Def. a pt. $a \in X$ is called a boundary pt. of E if it's neither interior nor exterior. ~ isolated pt. is bdry pt.
($\Leftrightarrow \forall B_\varepsilon(a)$ contains pts both from E and $X \setminus E$).
$\partial E :=$ the bdry of E = {all bdry pts}.

Prop. $\forall$ pt. $a \in E$ is either interior or bdry.

Def. a is a closure pt. for $E \subset X$, if it's either interior or bdry pt. of E.
($\Leftrightarrow \exists \{a_k\}_{k=1}^{\infty} \subset E$, $a_k \xrightarrow{k \to \infty} a$). ~ isolated pt. is closure pt.

$$\begin{cases} a \in \mathring{E} \Rightarrow \{a_k\} = \{a\}. \\ a \in \partial E \Rightarrow a_k \in B_{\frac{1}{k}}(a) \cap E. \end{cases}$$

$\bar{E} :=$ the closure of E = {all closure pts}.

Def. $a \in X$ is an accumulation pt. (limit pt.) for $E \subset X$, if $\forall B_\varepsilon(a)$ contains a pt. $b \in E$, $b \neq a$.
($\Leftrightarrow \exists \{a_k\} \subset E$, $a_k \neq a$, $a_k \xrightarrow{k \to \infty} a$).

Fact: $\bar{E}$ is the smallest closed set containing E.

Ex. let $E = \mathbb{Q} \subset \mathbb{R}$.
$\mathring{E} = \emptyset$. $(X \mathring{\setminus} E) = \emptyset$. $\partial E = \mathbb{R}$.
$\bar{E} = \mathbb{R}$. $\forall a \in \mathbb{R}$ is an accumulation pt.

Ex. let $E = (0, 1] \subset \mathbb{R}$.
$\mathring{E} = (0, 1)$. $(X \mathring{\setminus} E) = (-\infty, 0) \cup (1, +\infty)$. $\partial E = \{0, 1\}$.
$\bar{E} = [0, 1]$. {accumulation pts} = $[0, 1]$.

Fact. $\bar{E} = E \cup \partial E$.

Ex. $E = \mathbb{Z} \subset \mathbb{R}$.
$\mathring{E} = \emptyset$. $(X \mathring{\setminus} E) = \bigcup_{k \in \mathbb{Z}} (k, k+1)$. $\partial E = E$.
$\bar{E} = E$. {accumulation pts} $= \emptyset$.

Thm. (Characterization of closed sets).
Let X - a metric space. $E \subset X$. TFAE:
(i) E is closed.
(ii) $E = \bar{E}$. ($\Leftrightarrow E \supset \bar{E}$).
(iii) $E \supset \partial E$.
(iv) $E \supset$ {accumulation pts}.

Proof: left as homework.

Continuous Maps

Def. Let X, X' – two metric spaces separately equipped with ρ, ρ'. A map $f: X \to X'$ is called continuous at $a \in X$, if $\forall B'_\varepsilon(f(a)) \subset X'$, $\exists B_\delta(a) \subset X$ $(\varepsilon, \delta > 0)$, s.t. $f(B_\delta(a)) \subset B'_\varepsilon(f(a))$.

f is called continuous if it's cts. at $\forall a \in X$.

Remark: if $X' = \mathbb{R}$ with the standard metric, f is called a continuous function on X.

Proposition. (characterization of continuity).
$f: X \to X'$ is cts. $\Leftrightarrow$ $f^{-1}(G^{-1})$ is open in X $\forall$ open set $G' \subset X'$. ——— proof: left as homework.

Remark: algebraic properties of contin. func.s ($f \pm g$, $f \cdot g$, $\frac{f}{g}$, ...) persist with word-by-word the same proof.

Proposition: E is closed $\Leftrightarrow$ E contains all its accumulation points.

Proof: "$\Rightarrow$": Let E be closed. $X \setminus E$ – open.
Now, if a is an accumulation pt. and $a \in X \setminus E$.
$\Rightarrow$ by def. $\exists B_\varepsilon(a) \subset X \setminus E \Rightarrow B_\varepsilon(a)$ has no pts in E.
$\Rightarrow \nexists \{x_k\} \subset E$, $x_k \to a$ $(k \to \infty)$ $\Rightarrow a$ is not an accum. pt. ↯.

"$\Leftarrow$": Let E contain all its accum. pts.
Take $a \in X \setminus E$. $\Rightarrow a$ is not an accum. pts.
If $\nexists B_\varepsilon(a) \subset X \setminus E \Rightarrow \forall B_{\frac{1}{k}}(a) \ni x_{k \in E}$, $x_k \neq a$.
$\Rightarrow \rho(a, x_k) < \frac{1}{k}$. $\Rightarrow x_k \to a$ $(k \to \infty)$
$\Rightarrow a$ is an accum. pt. ↯. $\square$

Prop. (i) map is cts. $\Leftrightarrow$ $f^{-1}(G)$ is open for $\forall G$ – open.
(ii) algebraic operations $\alpha f \pm \beta g$, $f \cdot g$, $f/g_{(g \neq 0)}$ with cts functions again gives cts. func.
(iii) $X \xrightarrow{f} Y \xrightarrow{g} Z$. $\Rightarrow g \circ f: X \to Z$.
X, Y, Z – metric spaces. f, g – cts. $\Rightarrow g \circ f$: also cts.
$\left((g \circ f)^{-1}(G) = f^{-1} \circ \underbrace{g^{-1}(G)}_{\text{open}} = f^{-1}(\Omega) \text{ – open} \right)$.
(iv). $f: X \to Y$ is cts. $\Leftrightarrow$ f is sequentially cts. i.e. $\forall a \in X$. $\forall x_k \to a$ $(k \to \infty)$, we have $f(x_k) \to f(a)$.

Proof: "$\Rightarrow$": $\forall a \in X$. $\forall \underbrace{B_\varepsilon(f(a))}_{\varepsilon>0 \text{ small s.t.}} \subset Y$, $\exists B_\delta(a) \in X$. $f(B_\delta(a)) \subset B_\varepsilon(f(a))$. $\forall x_k \to a$ $(k \to \infty)$. $\exists K \in \mathbb{N}^*$ s.t. $\rho(x_k, a) < \delta$. $\forall k > K$. $\Rightarrow \rho'(f(x_k), f(a)) < \varepsilon$.

"$\Leftarrow$": $\forall a \in X$. suppose $\forall B_\varepsilon(f(a)) \subset Y$, we have: $\forall \underbrace{B_\delta(a)}_{\delta>0 \text{ small}} \subset X$. $f(B_\delta(a)) \not\subset B_\varepsilon(f(a))$.
$\Rightarrow f(B_\delta(a)) \setminus B_\varepsilon(f(a)) \neq \emptyset$.
Choose $\{y_k\}_{k=1}^\infty \subset f(B_\delta(a)) \setminus B_\varepsilon(f(a))$.
$\Rightarrow \{f^{-1}(y_k)\}_{k=1}^\infty \subset B_\delta(a)$.
For $B_{\frac{1}{k}}(a) \subset X$. ($k \in \mathbb{N}^*$, k may be large. $k \to \infty$).
$\exists y_k \in f(B_{\delta_k}(a))$. $x_k \in B_{\frac{1}{k}}(a)$.
$\Rightarrow x_k \xrightarrow{k \to \infty} a$ but $\rho(y_k, f(a)) > \varepsilon$. $\Rightarrow y_k = f(x_k) \not\to f(a)$ ↯. $(k \to \infty)$ $\square$

Def. A metric space X is called complete if $\forall$ Cauchy sequence $\{x_k\}$ in X.
(means: $\forall \varepsilon > 0$. $\exists N$: $\forall k, l > N$, it holds $\rho(x_k, x_l) < \varepsilon$)
is convergent to some $a \in X$.

Examples:
1). $\mathbb{R}$ with standard norm.
2). $\mathbb{R}^n$ with Euclidean metric.
3). $C[a, b]$.

Convergence in $C[a,b] \Leftrightarrow$ uniformly convergence.

$(\forall \varepsilon>0,\ \exists N: \forall k\geq N,\ \forall x\in[a,b]: |f_n(x)-f(x)|<\varepsilon.$

$\Leftrightarrow \underbrace{\max_{x\in[a,b]} |f_n(x)-f(x)|}_{=\rho(f_n,f)} < \varepsilon.)$

Cauchy sequence $\Leftrightarrow$ uniform Cauchy.

$f_n \rightrightarrows f \Rightarrow f\in C[a,b]$.

$\underset{C[a,b]}{\cap}$

4) Let X a complete metric sp. then

$\underset{X}{\underset{\cap}{E}}$ is complete $\Leftrightarrow$ E is closed.

Proof. "$\Rightarrow$": E is complete. then take a - a closure pt of E.

$\Rightarrow \exists x_k \xrightarrow{\in E} a\ (k\to\infty)$. ρ on E is induced by X.

x_k is Cauchy in X. $\Rightarrow$ Cauchy in E.

$\Rightarrow$ has a limit $\in E$. i.e. $a\in E$. ✓

"$\Leftarrow$": E is closed. take a Cauchy seq. in E. $\{x_k\}\subset E$.

X complete. $\Rightarrow x_k \to \underset{X}{\overset{\cap}{a}}\ (k\to\infty)$. $\Rightarrow$ a is a closure pt. of E $\Rightarrow$ Since E-closed. $a\in E$.

$\Rightarrow$ E-complete. ✓ □.

5). $B_\varepsilon(a)\subset \mathbb{R}^n$ is not complete.

$\overline{B_\varepsilon(a)}$ is complete.

6). $\mathbb{R}$ with $\{0,1\}$-valued metrices.

$\forall$ Cauchy seq. = const. $\Rightarrow$ has a limit ✓.

7). $E=\{P_n(x)\}_{n\geq 0} \subset C[0,1]$ —— not complete,

all polyn.s. since E is not closed!

$\forall \underset{C[0,1]}{\underset{\cap}{\varphi}} \Leftarrow \{P_n(x)\}$ exist. by Weierstrass Thm. $\Rightarrow$ e.g. $\exists P_n(x) \rightrightarrows \sin x$, cauchy. $\Rightarrow$ no limit in E.

Thm. Let $\{\overline{B_{r_k}(a_k)}\}$ be a nested seq. of closed balls.

$(\overline{B_{r_1}(a_1)} \supset \overline{B_{r_2}(a_2)} \supset \cdots \supset \cdots)$ in a complete metric sp. X. and $r_k \xrightarrow{k\to\infty} 0$.

Then $\exists!\ a\in \overline{B_{r_k}(a_k)}$, $\forall k$.

i.e. $\bigcap_{k=1}^{\infty} \overline{B_{r_k}(a_k)} = \{a\}$. (nested balls principle)

Proof: Consider $\{a_k\}$. let's show that $\{a_k\}$ is Cauchy

Indeed. if $k\geq l$, then $\rho(a_k,a_l)\leq r_l$,

since $a_k \in \overline{B_{r_l}(a_l)}$.

$\forall \varepsilon>0,\ \exists N: r_l<\varepsilon$, (since $r_k\to 0, k\to\infty$), $\forall l\geq N$.

$\Rightarrow \forall k,l\geq N,\ \rho(a_k,a_l)<\varepsilon$.

So $\{a_k\}$ — Cauchy. $\Rightarrow \exists a = \lim_{k\to\infty} a_k$.

From the nested prop. $\forall \overline{B_{r_k}(a_k)}$ all $a_l\in \overline{B_{r_l}(a_l)}$ for $l\geq k$. $\Rightarrow$ a is a closure pt. for all $\overline{B_{r_k}(a_k)}$.

Since $\overline{B_{r_k}(a_k)}$ - complete and closed, $a\in \overline{B_{r_k}(a_k)}$ $\forall k$

$\Rightarrow a\in \bigcap_{k=1}^{\infty} \overline{B_{r_k}(a_k)}$.

Uniqueness: if $\exists b\neq a$, $b\in \bigcap_{k=1}^{\infty} \overline{B_{r_k}(a_k)}$.

then $a,b\in \overline{B_{r_k}(a_k)}$ $\forall k$.

$\rho(a,b) \leq \rho(a,a_k)+\rho(a_k,b) \leq 2r_k \xrightarrow{k\to\infty} 0$.

$\Rightarrow \rho(a,b)=0 \Rightarrow a=b$. ↯. □

Def: a map $f: \underset{\text{metric sp.}}{X\to X}$ is call contracting if $\exists q\in[0,1)$

squeezing prop.

s.t. $\rho(f(x),f(y)) \leq q\cdot\rho(x,y)$, $\forall x,y\in X$.

Thm: Let $f: X\to X$ be contracting. and X is complete.

Then f has a unique fixed pt. (i.e. $f(x)=x$).

Proof: Take $\forall x_0 \in X$. then let $x_1 = f(x_0)$, $x_2 = f(x_1)$, ...

Observe that: $\rho(x_{k+m}, x_k) \le q\, \rho(x_{k+m-1}, x_{k-1})$

$\le q^2 \rho(x_{k+m-2}, x_{k-2}) \le \dots \le q^k \rho(x_m, x_0)$

$\le q^k(\rho(x_0,x_1) + \rho(x_1,x_2) + \dots + \rho(x_{m-1}, x_m))$

$\le q^k(1+q+\dots+q^{m-1})\, \rho(x_0,x_1)$

$\le q^k \rho(x_0,x_1) \frac{1}{1-q}$ $m\to\infty$

$\longrightarrow 0$ $(k\to\infty)$.

$\Rightarrow \{x_k\}$ - Cauchy seq. $\Rightarrow \exists$ a limit $x = \lim_{k\to\infty} x_k$.

$x = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} f(x_k)$ squeezing prop.

From the contracting prop. f is cts.

$\Rightarrow x = \lim_{k\to\infty} f(x_k) = f(\lim_{k\to\infty} x_k) = f(x)$ i.e. $x = f(x)$. ✓.

Uniqueness: if $\tilde{x}$ - another fixed pt. then

$\rho(f(x), f(\tilde{x})) \le q\rho(x,\tilde{x})$

$\quad \overset{\shortparallel}{x} \quad\quad \overset{\shortparallel}{\tilde{x}} \quad = q\, \rho(f(x), f(\tilde{x}))$.

$\Rightarrow \rho(x,\tilde{x}) = 0 \Rightarrow x = \tilde{x}$. □.

Examples:

1) $I_n = (0, \frac{1}{n}) = B_{\frac{1}{2n}}(\frac{1}{2n}) \subset \mathbb{R}$.

$\bigcap_{n=1}^{\infty} I_n = \emptyset$. $\Rightarrow$ So, nested balls principle fails for open balls.

2) $\mathbb{Q} \subset \mathbb{R}$. take $q_n \to \sqrt{2}$. $\overline{B_{\frac{1}{2n}}(q_n)} \cap \mathbb{Q}$ - balls in $\mathbb{Q}$.
digits.

$\bigcap_{n=1}^{\infty} (\overline{B_{\frac{1}{2n}}(q_n)} \cap \mathbb{Q}) = \sqrt{2} \notin \mathbb{Q}$.

$\Rightarrow$ nested balls principle fails for incomplete spaces like $\mathbb{Q}$.

3) $X = (0,1)$ - not complete.

$f(x) = \frac{x}{2}$. $X \xrightarrow{f} X$ no fixed pt.!

4). Show that the eq. $x = \frac{x^4}{5} + \frac{1}{2}$ has a solution on $[0,1]$.

$X = [0,1]$ - complete metric sp.

$f(x) = \frac{x^4}{5} + \frac{1}{2}$. $X \to X$.

Show f is squeezing: $|f(x) - f(y)| = \left\{ \begin{matrix} \text{Languang} \\ \text{Thm} \end{matrix} \right\} = |f'(c)\cdot|x-y||$

$f'(x) = \frac{4}{5}x^3$ $|f'(x)| \le \frac{4}{5}$.

$\Rightarrow |f(x) - f(y)| \le \frac{4}{5}|x-y|$. $\Rightarrow f$ is squeezing.

$\Rightarrow \exists!$ fixed pt.

5). Remind: Cauchy problem for ODEs.

$$\begin{cases} y' = f(x,y) \\ y(a) = b. \end{cases} \Bigg| \; y = y(x).$$

Main Thm: If $f \in C^1(\Omega)$, $(a,b) \in \mathring{\Omega}$. then in some $(a-\delta, a+\delta)\times(b-\varepsilon, b+\varepsilon)$. $\exists!$ solution $y \in C^1$.

Proof: the original prob. is equiv. to

$$y(x) = \underbrace{y(a) + \int_a^x f(t, y(t))\,dt}_{=: H(y(t))}.$$

complete metric sp.! X $\shortparallel$

H: a mapping of $C[a-\delta, a+\delta]$ into itself.

We need to solve: $y = H(y)$.

So we need a fixed pt. of H!

Let's show that for δ-small, H is contracting!

$\max_{x\in[a-\delta,a+\delta]} |H(y)(x) - H(\tilde{y})(x)| \le \max_x \left| \int_a^x [f(t,y(t)) - f(t,\tilde{y}(t))]\,dt \right|$

$\le \max_x \int_a^x |f(t,y(t)) - f(t,\tilde{y}(t))|\,dt = \left\{ \begin{matrix} \text{Lagrange} \\ \text{Thm} \end{matrix} \right\}$

$\le \delta \cdot \underbrace{\max_{\substack{x\in[a-\delta,a+\delta] \\ y\in[b-\varepsilon,b+\varepsilon]}} \left|\frac{\partial f}{\partial y}\right|}_{\le \text{Const } C} \cdot \max |y(x) - \tilde{y}(x)|$

you can take C first in Ω.
then take δ s.t. $C\delta<1$.

Now, if $C\cdot\delta = q < 1$. (δ small enough).

$\Rightarrow \rho(\mathcal{H}(y), \mathcal{H}(\tilde{y})) \leq q \cdot \rho(y, \tilde{y}) \Rightarrow \exists!$ fixed pt. $\square$.

Rmk: For numerical sol. of ODEs:

Peano Iterations: choose arbitrary y. then keep applying $\mathcal{H}$.
$\Rightarrow$ approach the solution.

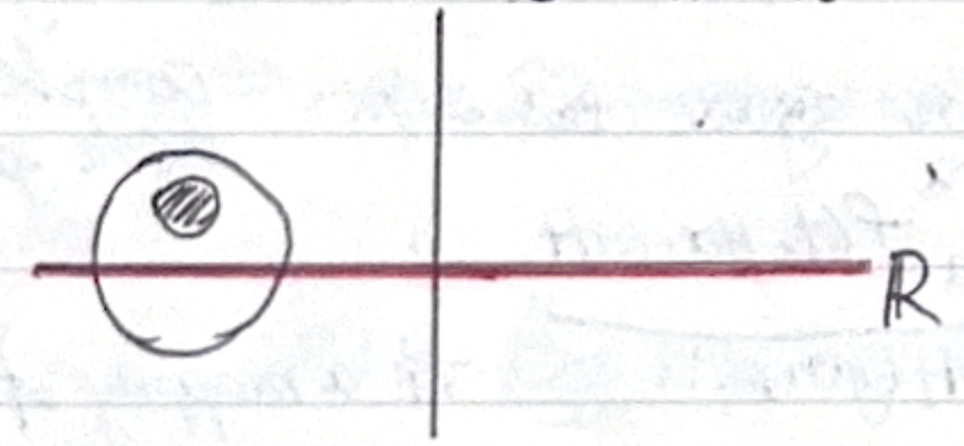
Def: Let X - a metric sp. A subset of X: E is called (everywhere) dense in X, if $\forall B_\varepsilon(a)$ in X contains a pt. $p \in E$.

Def: $E \subset X$ is called somewhere dense in X, if $\exists B_\delta(a)$ ($\Leftrightarrow \exists \overline{B_\delta(a)}$) s.t. $E \cap B_\delta(a)$ is dense in $B_\delta(a)$. ($\Leftrightarrow (\bar{E})^\circ \neq \phi$)

Def: $E \subset X$ is nowhere dense, if it's not somewhere dense.
($\Leftrightarrow \forall B_\delta(a) \subset X$, $\exists B_\varepsilon(b) \subset B_\delta(a)$: $E \cap B_\varepsilon(b) = \emptyset$).
($\Leftrightarrow (\bar{E})^\circ = \phi$).

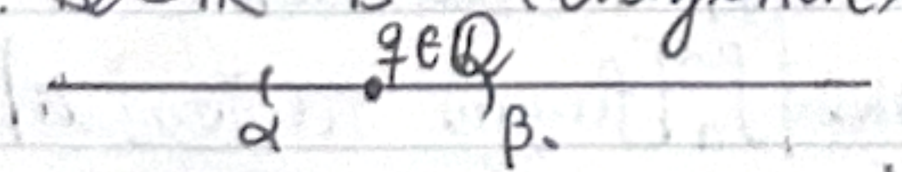
Examples:

1) $\mathbb{R} \subset \mathbb{C} \simeq \mathbb{R}^2$ is nowhere dense.



More generally, C^1 curves are nowhere dense in $\mathbb{R}^n$.
(Same for C^1 submanifolds.)

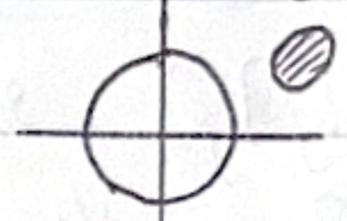
2). $\mathbb{Q} \subset \mathbb{R}$ is (everywhere) dense



3). $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense.
$\{(q_1, q_2, \dots, q_n)\}$.

4). $B_1(0) = E \subset \mathbb{R}^2$

E is somewhere dense (dense in $B_1(0)$), not everywhere.



Simple Facts:

1) E is dense in X. $\Rightarrow$ E is dense in Y: $E \subset Y \subset X$.
($\Leftrightarrow E \cap Y$ is dense in Y).

2) E is dense in $X \Leftrightarrow \bar{E} = X$.

In partic, a closed E is dense in $X \Leftrightarrow E = X$.

less trivial ex: $E \subset C[a,b]$. $\overset{?}{=} X$.

(i) $E = \{\text{polys}\}$. $\mathbb{R}[x]$.
follows from Weierstrass Approx. Thm: $\bar{E} = X$.

(ii) $E = \mathbb{Q}[x]$. (poly-s with rational coefficients)
countable. dense.

Def: a metric sp. is called separable, if $\exists$ E - a countable, dense subset; that is, $\exists \{x_n\}_{n=1}^\infty$ of elements of the sp s.t. every nonempty open subset of the space contains at least one element of the sequence. ($E = \{x_n\}_{n=1}^\infty$ - dense).

Ex: $\mathbb{R}^n$, $C[a,b]$. ✓.
$\cup$ $\cup$
$\mathbb{Q}^n$. $\mathbb{Q}[x]$.

$\mathbb{R}$ with $\{0,1\}$ metric is not separable, because: every
$\rho(x,y) = \begin{cases} 0, & x=y \\ 1, & x \neq y \end{cases}$

single pt. set $\{x\}$ is an open set. (choose $B_{\frac{1}{2}}(x)$.)
$\Rightarrow$ these open sets only contains its centre!

Important prop: separable spaces have a countable base, i.e. a countable system of open sets $\{V_j\}_{j=1}^{\infty}$, s.t. $\forall$ G - open in X, $G = \bigcup_{j \in T} V_j$, and $\forall x \in G$, $\exists V_k \in G$ s.t $x \in V_k$.

Why? Take $E = \{a_j\}_{j=1}^{\infty}$ - count, dense.

$\Rightarrow \{B_{r_k}(a_j)\}$. $r_k \in \mathbb{Q}$. $\to$ countable.

Ex: in $\mathbb{R}^n$: balls, rational centre, rational radius

$\Rightarrow$ countable system of open sets.

Baire Thm: Let X be a complete metric sp.

then $X \neq \bigcup_{j=1}^{\infty} E_j$, E_j - nowhere dense.

Proof: take $\forall a_1 \in X$. consider $\overline{B_1(a_1)}$.

E_1 - not dense in $\overline{B_1(a)} \Rightarrow \exists \overline{B_{r_2}(a_2)} \subset \overline{B_1(a_1)}$.

$r_2 < \frac{1}{2}$, $\overline{B_{r_2}(a_2)} \cap E_1 = \emptyset$.

Continue: $\exists \overline{B_{r_3}(a_3)} \subset \overline{B_{r_2}(a_2)}$, s.t $r_3 < \frac{1}{3}$.

$\overline{B_{r_3}(a_3)} \cap E_2 = \emptyset$.

$\Rightarrow \overline{B_{r_3}(a_3)} \cap E_1 = \emptyset$. and $\overline{B_{r_3}(a_3)} \cap E_2 = \emptyset$

$\cdots\cdots$

$\Rightarrow \overline{B_{r_k}(a_k)} : \overline{B_{r_k}(a_k)} \cap E_j = \emptyset$, $j = 1, 2, \dots k-1$.

Now, $\{\overline{B_{r_k}(a_k)}\}$: nested balls with $r_k \to 0$, $(k \to \infty)$.

$\Rightarrow \exists\, a = \bigcap_{k=1}^{\infty} \overline{B_{r_k}(a_k)}$.

$\Rightarrow a \notin E_j$, $\forall j \in \mathbb{N}$. $\Rightarrow X \neq \bigcup_{k=1}^{\infty} E_k$. - contrad. $\square$.

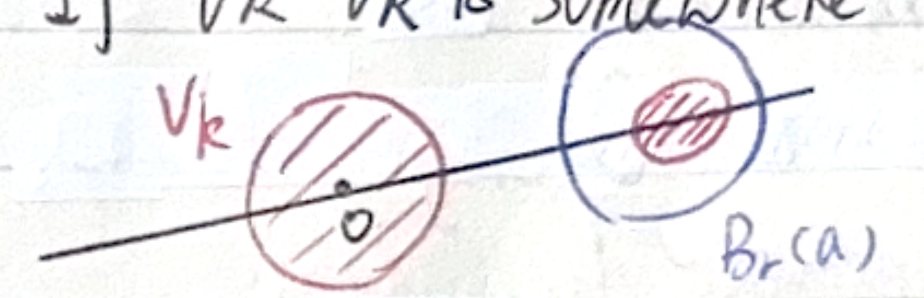
Application: An ∞-dim. $\underbrace{\text{Banach sp.}}_{}$ (complete normed.) has no countable Gammel basis.

Proof: Assume, by contrad., $\exists \{e_k\}_{k=1}^{\infty}$ - basis in X.

Let $V_k := \operatorname{span}\{e_1, e_2, \dots, e_k\}$. $\Rightarrow X = \bigcup_{k=1}^{\infty} V_k$.

We now show that $\forall V_k$ - nowhere dense.

If $\forall k$ V_k is somewhere dense (dense in $\overline{B_r(a)}$).



V_k - closed. 'like $\mathbb{R}^k$.

$\Rightarrow \overline{V_k} \cap \overline{B_r(a)} = \overline{B_r(a)}$. $\Rightarrow \overline{B_r(a)} \subset V_k$.

$\overline{V_k} = V_k$.

$\Rightarrow \exists\, \overline{B_R(b)} \subset \overline{B_r(a)} \subset V_k$, $b \in V_k$

$V_k - b = V_k$. $\overline{B_R(b)} - b = \overline{B_R(0)}$

$\Rightarrow V_k \supset \overline{B_R(0)} \Rightarrow V_k \supset X$. ↯. $\dim X = \infty$! $\square$.

Ex: $\nexists$ countable basis in $C[a,b]$.

Compactness.

Def. a metric sp. K is called compact, if $\forall \bigcup_{\alpha \in A} G_\alpha = K$ (an open covering of K). (G_α open)

$\exists\, G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}$: $K = \bigcup_{j=1}^{n} G_{\alpha_j}$. (finite covering).

Examples:

1). $K = [a,b] \subset \mathbb{R}$. - Heine-Borel Lemma. ✓.

2). $(a,b) \subset \mathbb{R}$ is not cpt. $(a,b) = \bigcup_{j=1}^{\infty} (a + \frac{1}{j}, b - \frac{1}{j})$

3). Any (closed) cell in $\mathbb{R}^n$:

$C = [a_1, b_1] \times \dots \times [a_n, b_n]$. ✓.

Properties:

1) A closed E in a cpt K is cpt itself.

Proof: $E = \bigcup \underset{\text{open}}{G_\alpha} \Rightarrow E \subset \bigcup \underset{\text{open in } K}{\widetilde{G_\alpha}}$. add $\underset{\text{open}}{\widetilde{G_0}} := K \setminus E$.

$K = (\bigcup \tilde{G}_\alpha) \cup \tilde{G}_0$ – open covering

$\Rightarrow \exists$ finite subcover.

Remove from it (if needed): $\tilde{G}_0$.

$\Rightarrow E \subset \bigcup_{j=1}^{n} \tilde{G}_{\alpha_j} \Rightarrow E = \bigcup_{j=1}^{n} G_{\alpha_j}$. □

Coro: a bounded, closed subset in $\mathbb{R}^n$ is cpt.

$(E \subset [-R \times R] \times \cdots \times [-R \times R])$

closed subset of a cell.

Rmk: metric spaces have the Hausdorff Property:

$\forall a, b \in X,\ a \neq b,\ \exists B_\varepsilon(a), B_\delta(b): B_\varepsilon(a) \cap B_\delta(b) = \emptyset.$

$\varepsilon, \delta < \frac{1}{2}\rho(a,b)$.

2). If X – a metric sp. $K \subset X$ – cpt subsp.

then K is closed.

Proof: Choose $a \in X \setminus K$. $\forall b \in K, \exists B_{R_a}(a), B_{r_b}(b)$:

$B_{R_a}(a) \cap B_{r_b}(b) = \emptyset$.

Trivially: $K \subset \bigcup_{a \in K} B_{r_a}(a)$. – open.

$\Rightarrow \exists$ a finite subcovering $K \subset \bigcup_{j=1}^{n} B_{r_{b_j}}(b_j)$.

$a \in G = \bigcap_{j=1}^{n} B_{R_j}(a)$ – open ball.

$G \cap K = \emptyset$. $\Rightarrow X \setminus K$ is open by def. □

3). X – metric sp. $K \subset X$ – cpt.

$\Rightarrow K$ is bounded, i.e. $K \subset B_R(a)$.

Proof: $K \subset \bigcup_{n=1}^{\infty} B_n(a)$. $\Rightarrow$ finite subcover $\Rightarrow K \subset B_N(a)$. □

So, cpt $\Rightarrow$ bdd & closed.

Coro: $K \subset \mathbb{R}^n$ is cpt $\Leftrightarrow$ K is bdd & closed.

Rmk: Very different for ∞-dim sp!

Def: Let $\{F_\alpha\}_{\alpha \in A}$ be a system of subsets in a metric sp X, then it's called centered if $\forall \{F_{\alpha_1}, \dots, F_{\alpha_n}\}$ has $\bigcap_{j=1}^{n} F_{\alpha_j} \neq \emptyset$.

$\forall \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{N}$.

Thm: K is cpt $\Leftrightarrow$ $\forall$ centered system of closed subsets of K has a nonempty intersection.

Proof: "$\Rightarrow$": Suppose a centered system of closed subsets of K: $\{A_i\}_{i=1}^{\infty}$, satisfies $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

$\Rightarrow K = \bigcup_{i=1}^{\infty} A_i^c \Rightarrow K = \bigcup_{j=1}^{N} A_{a_j}^c$

$\Rightarrow \bigcap_{j=1}^{N} A_{a_j} = \emptyset \Rightarrow \{A_i\}$ is not centered! ↯.

"$\Leftarrow$": Suppose $K = \bigcup_{i=1}^{\infty} U_i$. – U_i: closed subset of K.

$\Rightarrow \bigcap_{j=1}^{\infty} U_i = \emptyset$. $\Rightarrow \{U_i\}$ is not centered.

$\exists \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{N}$ s.t. $\bigcap_{i=1}^{n} U_{\alpha_i} = \emptyset$.

$\Rightarrow K = \bigcup_{i=1}^{n} U_{\alpha_i}$. $\Rightarrow K$ is cpt. □

Def: A metric sp. K is called sequencially cpt. if $\forall \{a_n\} \subset K$ contains a convergent subseq. $a_{n_k} \xrightarrow{k \to \infty} a$

($\Leftrightarrow$ $\forall$ infinite $E \subset K$ has an accum. pt.)

($\Leftrightarrow$ ∞ subset has a closure pt.)

Claim: a sequencially cpt. K is complete.

Proof: take a Cauchy seq. $\{a_n\} \subset K$.

* $\exists a_{n_k} \xrightarrow{k \to \infty} a$.

* $\forall \varepsilon > 0, \exists N \in \mathbb{N}: \forall k, l > N.\ \rho(a_k, a_l) < \varepsilon$.

So choose $\forall l > N$, $k : n_k > N$.

$$\rho(a_{n_k}, a) < \varepsilon$$

$$\Rightarrow \rho(a_l, a) \le \rho(a_l, a_{n_k}) + \rho(a_{n_k}, a) < 2\varepsilon. \quad \square$$

<u>Thm</u>: K is cpt. $\Rightarrow$ K is sequentially cpt.

$\boxed{K \text{ is sequencially cpt.}} \Rightarrow \boxed{\text{complete.}} \Rightarrow \boxed{\text{separable.}}$

$\Uparrow$

$\boxed{K \text{ is cpt.}} \Rightarrow \boxed{\text{closed in } \forall X \supset K.}$

$\boxed{K \text{ is cpt.}} \Rightarrow \boxed{\text{bounded in } \forall X \supset K.}$

$\Updownarrow$

$\boxed{\text{$\forall$ centered family of (closed) subsets of K has nonempty intersection.}}$

$\boxed{K: \text{cpt} \Rightarrow \forall \text{ closed } E \subset K \text{ is cpt.}}$

$\boxed{\text{In } \mathbb{R}^n: K: \text{cpt} \Leftrightarrow K: \text{bdd \& closed.}}$

<u>Counter Example for $\dim X = \infty$:</u>

$K = \overline{B_1(0)}$. $X = C[0, 2\pi]$

bdd. closed. Banach. (complete normed). L_2-norm.

$\boxed{f_n(x) = \sin nx} \in K$.

Consider $\displaystyle\int_0^{2\pi} (\sin nx - \sin mx)^2 \, dx$ (requiring $m \neq n$).

$$= \int_0^{2\pi} \Big[\sin^2 nx + \sin^2 mx - \underbrace{\cos(m+n)x - \cos(m-n)x}_{\text{integral}: 0} \Big] dx = 2\pi > 0.$$

$\Rightarrow \nexists$ subseq. converging within f_n.

(if $\exists f_{n_k} \rightrightarrows f \Rightarrow \displaystyle\int_0^{2\pi} (f_{n_k} - f_{n_l})^2 dx \xrightarrow{k, l \to \infty} 0$. ↯)

$\Rightarrow K$ is <u>not</u> cpt!

Def. A metric sp. K is called totally bounded,
if $\forall \varepsilon>0$, $\exists$ a finite ε-net for K, i.e.
a set $E=\{x_1, x_2, \cdots, x_m\}$ s.t. $\forall x\in K$, $\exists y\in E: \rho(x,y)<\varepsilon$.
$m=m(\varepsilon)<+\infty$
In other words: $K=\bigcup_{j=1}^{m} B_\varepsilon(x_j) \iff K=\bigcup_{j=1}^{m} \overline{B_\varepsilon(x_j)}$.

Prop: If K - totally bdd
then $\forall A\subset K$ is totally bdd.

Proof. $\forall \varepsilon>0$, $\exists$ a finite ε-net $E\subset K$.
$E=\{x_1,\cdots,x_m\}$. Now, if $B_\varepsilon(x_j)\cap A=\emptyset$,
remove it from others.
Choose $\forall y_j \in B_\varepsilon(x_j)\cap A$.
Then $E'=\{$all chosen $y_j\}\subset A$.
is a 2ε-net for A:
$\forall x\in A$. $\exists x_j\in E: \rho(x_j,x)<\varepsilon$.
x_j is removed above $\Rightarrow \rho(x,y_j)\le\rho(x_j,x)+\rho(x_j,y_j)<2\varepsilon$. □

Prop: $\forall$ tot. bdd K is separable.

Proof: let $E=\{$union of all the finite $\frac{1}{k}$-nets for $\forall k\in\mathbb{N}\}$.
$E\subset K$. E-countable dense subset by def. □

(**Coro:** tot. bdd K has a countable base.)

Prop: If K is seq. cpt. then K is tot. bdd.

Proof: Assume by contradiction: $\exists\ \varepsilon>0$,
$\nexists$ a finite ε-net for K.
Pick $\forall x_1\in K$. $E=\{x_1\}$ is not an ε-net.
$\Rightarrow \exists x_2\in K$. $\rho(x_1,x_2)\ge\varepsilon$.
$E=\{x_1,x_2\}$ is not an ε-net.
$\Rightarrow \exists x_3\in K$. $\rho(x_1,x_3)\ge\varepsilon$, $\rho(x_2,x_3)\ge\varepsilon$.
$\cdots\cdots \Rightarrow$ we could get a seq. $\{x_n\}$ s.t.
$\rho(x_k,x_l)\ge\varepsilon$. $\forall k\ne l$. $\Rightarrow \nexists$ convergent subseq. ↯ □

Coro: $\forall$ cpt & $\forall$ seq. cpt is: 1) Tot. bdd.
2) separable.
3) has a countable base.

Prop: If K is a space with a countable base,
then K is cpt.
$\iff \forall$ countable covering admits a finite subcovering.
$\iff \forall$ countable centered syst. of closed subsets has a nonempty intersection.

Proof: 1st "$\iff$":
"$\Rightarrow$": obvious.
"$\Leftarrow$": let $K=\bigcup_{\alpha\in A} G_\alpha$ $\forall G_\alpha$ - open. A — countable.
$\exists$ a countable base $\{E_j\}_{j\in\mathbb{N}}$.
Then $\forall G_\alpha$ is some union of $\{E_j\}$.
$\Rightarrow$ take those $\{E_j\}$: only those which are „necessary" for forming $\{G_\alpha\}_{\alpha\in A}$.
$\Rightarrow$ countable subcovering $\{E_{j_k}\}\subset\{E_j\}$.
Now, $\{E_{j_k}\}$ is a countable covering of K.
$\Rightarrow \exists$ a finite subcovering $E_{\alpha_1}, \cdots, E_{\alpha_n}$.
$\forall E_{\alpha_i} \subset G_{\alpha_j}$ for some $\alpha_j\in A$.
(since these E_{α_i} are those chosen by us previously
Based on the fact that the are part of
those E_j forming $\{G_\alpha\}$).
$\Rightarrow K\subset \bigcup G_{\alpha_j}$ finite covering of the original covering.
$\Rightarrow K$ is cpt.

2nd "$\Leftarrow$": for centered syst: word-by-word the same proof as for arbitrary centered syst. $\square$

<u>Prop</u>: If K is seq. cpt. then it's cpt.

<u>Proof</u>: By the above prop. K is seq. cpt.

$\Rightarrow K$ is complete $\Rightarrow K$ is separable

$\Rightarrow K$ has a countable base

$\Rightarrow$ by the last prop. to check the cptness of K, Only need to show:

$\forall$ countable centered syst. $\{F_j\}_{j=1}^{\infty}$ of closed subsets has $\bigcap_{j=1}^{\infty} F_j \neq \emptyset$

Choose $\forall x_1 \in F_1$. $\forall x_2 \in F_1 \cap F_2$, $\forall x_3 \in F_1 \cap F_2 \cap F_3$, ... $\forall x_k \in \bigcap_{j=1}^{k} F_j$.

$\{x_k\}$ - a seq in $K \Rightarrow \{x_k\}$ contains a convergent subseq. $x_{k_\ell} \xrightarrow{\ell \to \infty} x \in K$.

Since $\forall F_j$ contains "nearly" all the terms in $\{x_{k_\ell}\}$ (except for finite exceptions in the front of the seq), x is a closure pt. of F_j, $\forall j$.

Since $\forall F_j$ is closed, $x \in F_j$, $\forall j$.

$\Rightarrow x \in \bigcap_{j=1}^{\infty} F_j \Rightarrow \bigcap_{j=1}^{\infty} F_j \neq \emptyset$. $\square$

<u>Prop</u>: Let K be: (1) complete
(2) tot. bdd.

then K is seq. cpt.

<u>Proof</u>: we need to prove that $\forall \infty$ subset $E \subset K$ has an accumulation pt.

since there're only finite balls. Choose a finite $\frac{1}{2}$-net. $K = \bigcup_{j=1}^{N} \overline{B_{\frac{1}{2}}(x_j)}$.

$\Rightarrow$ One of the balls contains ∞ elts. of E.

Fix it and call it K_1.

K_1 - tot. bdd. since $K \subset K$ - tot. bdd.

$\Rightarrow$ Choose in K_1 a finite $\frac{1}{4}$-net. $K_1 = \bigcup_{\text{finite}} \overline{B_{\frac{1}{4}}(x_j')}$

$\Rightarrow$ One of the balls contains ∞ elts. in E.

Fix it and call it K_2

...... $\Rightarrow$ Get a seq. $\underset{\frac{1}{2}\text{-ball}}{K_1}, \underset{\frac{1}{4}\text{-ball}}{K_2}, \cdots, \underset{\frac{1}{2^n}\text{-ball}}{K_n}, \cdots$

$\forall K_j$ contains ∞ elt.s of E. (but K_n may $\not\supset K_{n+1}$!)

Let's "double" all these balls!

$\widetilde{K_n} \supset \widetilde{K_{n+1}}$ by triang. ineq. $\rho(x_{n+1}'' - a_n) \le \rho(x_{n+1}' - a_{n+1}) + \rho(a_{n+1} - a_n) \le \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}} = \text{radius}(\widetilde{K_n})$ ✓

Now by the nested ball principle:

$\bigcap_{j=1}^{\infty} \widetilde{K_j} = \{a\}$. a: a limit of an ∞ seq. in E.

($\forall \widetilde{K_j}$ contains ∞ elt.s of E). $\square$

Putting together all these claims:

<u>Thm</u>: K- metric sp. TFAE:

① K is cpt.

② $\forall$ centered syst. of closed subsets in E has a nonempty intersection.

③ K is seq. cpt.

④ K is tot. bdd & complete.

Properties of func-s contin. on comp-s.

1) Let $f \in C(K)$. K - cpt.

Then f is bdd on K.

Proof: Assume rather:

$\forall n \in \mathbb{N}, \exists x_n \in K: |f(x_n)| \geq n$.

$\{x_n\}$ has a conveg. subseq.

$\Rightarrow x_{n_k} \to a \ (k \to \infty)$.

f - contin. $\Rightarrow f(x_{n_k}) \to f(a)$.

But $|f(x_{n_k})| \geq n_k \to \infty$. ↯. □

2) Let $f \in C(K)$. K - cpt.

Then $\exists M = \max_{x \in K} f(x)$, $\exists m = \min_{x \in K} f(x)$,

Proof: f - bdd $\Rightarrow$ set $M := \sup_{x \in K} f(x)$.

Claim: $\exists x_0 \in K: f(x_0) = M$.

Indeed by contrad. let $f(x) < M, \forall x$.

Build $g(x) := \frac{1}{M - f(x)} \in C(K)$.

But since $M = \sup_{x \in K} f(x)$, $\exists \{x_j\}: f(x_j) \xrightarrow{j \to \infty} M$.

$\Rightarrow g(x)$ is unbounded. ↯.

Similarly, $\exists m = \min_{x \in K} f(x)$. □

Def: $f(x)$ is equicontinuous (uniformly continuous) on K (K - metric sp.)

if $\forall \varepsilon > 0, \exists \delta > 0: \forall x, y \in K$ with $\rho(x,y) < \delta$, it holds

$|f(x) - f(y)| < \varepsilon$.

Rmk: Can by analog. formulated for metric sp. of valued func.

Ex: $f(x) = \frac{1}{x}$ is not equicontin. on $(0,1)$.

Ex. Let K - an interval on $\mathbb{R}$.

$\exists f'(x)$ on K, $|f'(x)| \leq C$. Then f is equicontin. on K:

$|f(x) - f(y)| \underset{\text{Lagrange}}{=} |f'(\xi)||x - y| \leq C|x-y| \Rightarrow \delta = \frac{\varepsilon}{C}$ works!

3). Let $f \in C(K)$. K - cpt.

Then f is equicontin.

Proof: Assume, by contrad. $\exists \varepsilon: \forall \delta > 0$,

$\exists x_\delta, y_\delta: |f(x_\delta) - f(y_\delta)| \geq \varepsilon$.

Pick $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, then $\exists x_n, y_n \in K$:

$|f(x_n) - f(y_n)| \geq \varepsilon$. th. with $\rho(x_n, y_n) < \frac{1}{n}$.

But, there exists convergent seq. $\{x_{n_k}\}: x_{n_k} \xrightarrow{k \to \infty} x$.

$\rho(x_n, y_n) < \frac{1}{n} \Rightarrow y_{n_k} \xrightarrow{k \to \infty} x$.

$\Rightarrow |f(x_{n_k}) - f(y_{n_k})| \to |f(x) - f(x)| = 0. \ (k \to \infty)$

But $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon \ \forall k$. ↯. □

Rmk: Properties 1) & 3) identically hold for func-s valued in $\forall$ metric sp.

(For ex. for $\mathbb{C}$ - valued func-s).

$C(K)$: sp. $C(K) = \{\text{conti. } f: \overset{\text{cpt.}}{K} \to \mathbb{R}\}$.

Let's introduce $\|f\| := \max_{x \in K} |f(x)|$.

$C(K)$ now is a normed sp.

Prop: $C(K)$ is a Banach sp. (i.e. it's complete).

Proof: Obviously. $\boxed{f_n \xrightarrow{C(K)} f \Leftrightarrow f_n \overset{K}{\rightrightarrows} f}$

So, if f_n - a Cauchy seq. in $C(K)$

$\Rightarrow$ uniformly Cauchy. $\left(\max_{x \in K} |f_n(x) - f_m(x)| \xrightarrow{n, m \to \infty} 0\right)$ on K.

$\Rightarrow$ Like in Calculus: $f_n(x) \rightrightarrows f(x)$ on K.

KOKUYO

$f \in C(K)$ - same proof as in Calculas.

$\|f_n - f\| \to 0 \Rightarrow \lim_{n\to\infty} f_n \overset{C(K)}{=} f$. $\square$

Rmk: one may consider $C(\underset{\text{cpt}}{K}^{\rho}, \underset{\text{complete}}{X}^{\rho'}) = \{ \text{Contin. } f: K \to X \}$.

Then $C(K, X)$ is a complete metric space with

$\rho(f, g) := \max_{x\in K} \rho'(f(x), g(x))$.

(Similar proof).

Compactness in $C(K)$

Def: a set $E \subset C(K)$ is called equicontinuous,

if $\forall \varepsilon>0, \exists \delta>0, \forall x, y$ with $\rho(x,y)<\delta$, it holds

$|f(x) - f(y)| < \varepsilon$, $\forall f \in E$.

Ex: let $E = \{ f \in C[a,b], \exists f', |f'| < M \}$.

Then $|f(x) - f(y)| = \{ \text{Lagrange} \} = |f'(\xi)\cdot(x-y)| \le M \cdot |x-y|$, $\forall f$!

$\Rightarrow \forall \varepsilon>0$, take $\delta < \frac{\varepsilon}{M} \Rightarrow E$ - equicontin.

Ex: $E = \{x^n\}_{n=1}^{\infty} \subset C[0,1]$.

if E is equicontin., take in the def:

$x = 1, y < 1$.

$\forall \varepsilon>0$, need δ: $|1 - y^n| < \varepsilon$. $\underset{\forall f \in E}{\forall n}$ and $\forall y$: $\underset{1-y}{|1-y|} < \delta$.

$1-\delta < y$. $y^n > 1-\varepsilon \Rightarrow n < \log_y (1-\varepsilon)$. fix $y > 1-\delta$.

if n violates this, get contrad.

Def: a set E in a metric sp. X is called precompact

if $\bar{E}$ is cpt.

($\Leftrightarrow E \subset K$ for some cpt. K).

Ex: $B_R(a) \subset \mathbb{R}^n$.

Arcela - Ascolli Thm:

A set $E \subset C(K)$ is precompact

$\Leftrightarrow E$ is bounded and equicontin.

unif. bdd: $\|f\| < C$, $\forall f \in E$.

Accordingly, E is cpt $\Leftrightarrow E$ is closed, bdd, equicontin.

Prof: "$\Rightarrow$": let E be cpt. E is closed & bdd.

WTS: equicontin. use cpt.

Take $\forall \varepsilon>0$; take now a finite ε-net for E:

$\{f_1, f_2, \dots, f_N\}$ $\forall f_j$ is equicontin. of K

$\Rightarrow \exists \delta_j$: $\forall x, y$. $\rho(x,y) < \delta_j$, it holds $|f_j(x) - f_j(y)| < \varepsilon$.

Set $\delta = \min\{\delta_1, \dots, \delta_N\} > 0$.

Now, $\forall f \in E$, choose f_k: $\|f_k - f\| < \varepsilon$ (from the ε-net).

$\Rightarrow \forall \rho(x,y) < \delta$. $|f(x) - f(y)| \le |f(x) - f_k(x)| + \underset{<\varepsilon}{|f_k(x) - f_k(y)|}$

$+ |f_k(y) - f(y)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$. since $\delta \le \delta_k$

"$\Leftarrow$": Given E - closed, bdd, equicontin.

WTS: $\forall$ seq. $\{f_n\}$ contains a convergent subseq.

$f_{n_k} \to f$ ($f \in E$ since E-closed),

Fix $\{f_n\}$. $\forall j \in \mathbb{N}$: choose $A_j \subset K$ - finite $\frac{1}{j}$-net

$|f_n(x)| \le M$ (Boundness). $\forall x, \forall n$.

(*) $\Rightarrow \forall y \in A_j$. $\{f_n(y)\}_{n=1}^{\infty} \overset{\subset \mathbb{R}}{-}$ Bdd. seq.

$\Rightarrow$ contains a conver. subseq. $A_j = \{y_1, \dots, y_m\}$.

$f_n(y_1) \longrightarrow f_{n_k}(y_1)$; $f_{n_k}(y_2) \longrightarrow f_{n_{k_\ell}}(y_2)$, $\dots$

End up with a subseq. $\{f_{n_p}\}$ conv. on A_j.

i.e. $\{f_{n_p}(y_j)\}$ converges. $\forall j$.

Denote

$$\Rightarrow \begin{array}{llll} f_{11}, & f_{12}, & f_{13}, & \cdots \\ f_{21}, & f_{22}, & f_{23}, & \cdots \\ f_{31}, & f_{32}, & f_{33}, & \cdots \\ \cdots & & & \end{array}$$

get using the same way as (*).
- subseq. conv. on A_1.
- its subseq. conv. on A_2
- its subseq. conv. on A_3.

Now, „Cantor's diagonal trick":

Consider $\{f_{11}, f_{22}, f_{33}, \cdots\}$ - a subseq. of the orig. $\{f_n\}$. converges on $\forall A_j$. (satisfying from the j-th term, it becomes a subseq. of $f_{j1}, f_{j2}, \cdots$).

$\Rightarrow$ Get $\{f_{nn}\} \triangleq \{g_n\}$ - a subseq. of $\{f_n\}$, converging on $\bigcup_{j=1}^{\infty} A_j$ pointwisely.

Now let's prove that $\{g_n\}$ is the desired coverg. subseq. of $\{f_n\}$.

Take $\forall \varepsilon > 0$, by equicontin. of E, $\exists \delta > 0$, $\forall x, y$, $\rho(x,y) < \delta$, it holds $|g_n(x) - g_n(y)| < \varepsilon$.

Take j: $\frac{1}{j} < \delta$. $\{g_n\}$ is conveg. pointwise on A_j.

$\Rightarrow$ since A_j is finite, $\{g_n\}$ conveg. unif. on A_j.

$\Rightarrow \exists N, \forall m, n > N, |g_m(y) - g_n(y)| < \varepsilon, \forall y \in A_j$.

Now, $\forall x \in K$, take $y \in A_j$: $\rho(x,y) = \frac{1}{j} < \delta$.

$$\Rightarrow |g_n(x) - g_m(x)| \le |g_n(x) - g_n(y)| + |g_n(y) - g_m(y)| + |g_m(y) - g_m(x)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

$\Rightarrow \{g_n\}$ unif. conv. on K. $\square$

Rmk: Also holds for $\mathbb{C}$-valued funcs.

Ex of application:

$\forall f_n \in C[a,b]$, s.t. $|f_n| \le M$. $\exists f_n'$: $|f_n'| \le \tilde{M}$. contains a conv. subseq.

Density of C(K)

Weierstrass Approximation: $\forall f \in C[a,b]$, $\exists P_n(x)$ - seq. of polys. s.t. $P_n(x) \xrightarrow{C[a,b]} f(x)$.

$[a,b] = [0,1]$, can take $P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$.

Bernstein Polyn.-s.

Q: what about $K \subset \mathbb{R}^n$?
$f \in C(K)$?

Def: A subset $A \subset C(K)$ is called an algebra, if it's a lin. sp + closed under multiplication.

Ex. the algebra of poly-s in $x_1, \cdots, x_n$ for $K \subset \mathbb{R}^n$.

Simple Property: $\bar{A}$ is also an algebra in $C(K)$.

Q: which algebras are dense in $C(K)$?

Thm (Stone-Weierstrass).

Let $A \subset C(K)$ be an algebra, $A \ni 1$, which separates pts: $\forall x, y \in K$, $x \ne y$, $\exists f \in A$, $f(x) \ne f(y)$.

Then $\bar{A} = C(K)$, i.e. A: dense in $C(K)$.

Ex: $A = \{f \in C(K): f(p) = 0, p \in K. \text{ fixed}\}$.

Then $\bar{A} \ne C(K)$. $(\bar{A} = A)$. $A \not\ni 1$.

$B = \{f \in C(K) : f(p) = f(q), p \ne q. \text{ fixed}.\}$.

$\bar{B} = B$. B doesn't separate pts.

Coro: $\forall K \subset \mathbb{R}^n$ cpt. $\forall f \in C(K)$ can be uniformly on K approximated by poly-s in $x_1, \dots, x_n$.

Proof of Thm: Switching from $A \to \bar{A}$, we may assume A to be closed, and then need to prove: $A = C(K)$.

Step 1): Let's prove that if $f \in A$, then $|f| \in A$.

Consider $\alpha \in \mathbb{R}$: $\|\alpha f\| < 1$ $\left(\alpha < \frac{1}{\|f\|}\right)$.

Switch $f \to \alpha f$. so $\|f\| < 1 \Rightarrow f^2 - 1 \in [-1, 0)$

$$|f(x)| = \sqrt{f^2} = \sqrt{1 + f^2 - 1} = \left\{\text{Taylor series of } (1+t)^{\frac{1}{2}}\right\}$$
$$= 1 + \frac{1}{2}(f^2 - 1) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!}(f^2-1) + \dots$$

Conv. unif. on $[-1, 1]$.

So, $|f(x)|$ is a limit of pol-s in f

$\Rightarrow |f| \in A$.

Step 2): $\max(a, b) = \frac{|a-b| + a + b}{2}$,

$\min(a, b) = \frac{a + b - |a-b|}{2}$, for $a, b \in \mathbb{R}$.

$\Rightarrow \forall f, g \in A$, $\underset{\varphi(x)}{\underbrace{\max\{f(x), g(x)\}}} \in A$

$\psi(x) := \min\{f(x), g(x)\} \in A$.

Same for $f_1, f_2, \dots, f_N$ by induction.

Step 3): take $\forall f \in C(K)$. Let's prove that $f \in \bar{A} = A$.

Take $\forall \varepsilon > 0$.

Note: $\forall p, q \in K$, $p \neq q$. $\exists h \in A$. $h(p) \neq h(q)$.

By switching $h \to \tilde{h} = \alpha h + \beta \in A$, we may obtain $\tilde{h}(p) = A$, $\tilde{h}(q) = B$, $\forall A, B \in \mathbb{R}$, $A \neq B$.

Now, $\forall p, q \in K$, $p \neq q$, choose $g_{p,q} \in A$ satisfying $g_{p,q}(p) = f(p)$, $g_{p,q}(q) = f(q)$.

$\Rightarrow \exists U_{p,q}, V_{p,q}$ - neighbourhoods of p & q respectively.

In both nbds. we have $|f(x) - g_{p,q}(x)| < \varepsilon$.

Fix p, vary q. $\{V_{p,q}\}_{q \in K}$ - open covering of K.

$\Rightarrow \exists V_{p,q_1}, \dots, V_{p,q_n}$ - fin. subcovering.

Set $g_p := \max\{g_{p,q_1}, \dots, g_{p,q_n}\} \in A$.

$g_{p,q} - \varepsilon < f(x) < g_{p,q} + \varepsilon$.

Also, set $U_p := U_{p,q_1} \cap \dots \cap U_{p,q_n}$.

It follows from the above:

- on K: $f(x) < g_p(x) + \varepsilon$.
- on U_p: $g_p(x) - \varepsilon < f(x)$.

Finally $\{U_p\}_{p \in K}$ - open covering of K.

$\Rightarrow \exists$ finite subcovering $U_{p_1}, \dots, U_{p_m}$.

Set $g := \min\{g_{p_1}, \dots, g_{p_m}\} \in A$.

Directly follows: $g(x) - \varepsilon < f(x) < g(x) + \varepsilon$. $\forall x \in K$.

$\Rightarrow g$ is an ε approx. of $f \Rightarrow f \in A$. $\square$

In fact, $g = \min \max$