HW4

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In what follows, $\Omega \subset \mathbb{R}^d$ will be a bounded domain, $T > 0$, and the parabolic interior Ω_T and boundary $\partial_p \Omega_T$ are given by

$$\Omega_T = (0, T] \times \Omega, \quad \partial_p \Omega_T = \big([0,T] \times \partial\Omega\big) \cup \big(\{0\} \times \Omega\big).$$

Exercise 1 Consider the differential operator

$$(\mathcal{L}u)(t,x) = -\Delta u(t,x) + c(x)u(t,x)\,,$$

where $c : \Omega \to [-M, +\infty)$ is continuous, $M \geq 0$. The goal is to establish the following weak maximum principle: if $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ and

$$(\partial_t + \mathcal{L})u \geq 0 \text{ in } \Omega_T, \; \min_{\partial_p \Omega_T} u \geq 0 \implies \min_{\overline{\Omega_T}} u \geq 0. \tag{0.1}$$

1. Prove (0.1) under the condition $(\partial t + \mathcal{L})u > 0$ in Ω_T and $M = 0$.
2. Prove (0.1) under the condition $(\partial t + \mathcal{L})u \geq 0$ in Ω_T and $M = 0$.

   Hint: consider $u_\varepsilon(t,x) = u(t,x) - t\varepsilon$.
3. Prove (0.1) under the condition $(\partial t + \mathcal{L})u \geq 0$ in Ω_T and $M > 0$.

   Hint: consider $v(t,x) = e^{\lambda t}u(t,x)$ *for an appropriate* λ.

Exercise 2 Let $\Omega = (0, \ell)$.

1. Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{1,0}(\partial_p \Omega_T)$ satisfy

$$\begin{cases} u_t - u_{xx} \geq 0, & (t,x) \in \Omega_T, \\ u\big|_{t=0} \geq 0, & x \in \Omega, \\ u(t,0) \geq 0, & t > 0, \\ u_x(t,\ell) \geq 0, & t > 0. \end{cases}$$

   Show that $u \geq 0$ on $\overline{\Omega_T}$.

   Hint: you may consider $u_\varepsilon(t,x) = u(t,x) + \varepsilon x$.
2. Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{1,0}(\partial_p \Omega_T)$ satisfy

$$\begin{cases} u_t - u_{xx} = f, & (t,x) \in \Omega_T, \\ u\big|_{t=0} = \varphi, & x \in \Omega, \\ u(t,0) = 0, & t > 0, \\ u_x(t,\ell) = g(t), & t > 0, \end{cases}$$

where f, φ, g are bounded, continuous functions in their domains. Show that

$$\max_{\overline{\Omega_T}} |u| \leq C(|T|+1)(F+G+\Phi)$$

for some constant C depending only on ℓ, where $F = \sup |f|$, $G = \sup |g|$ and $\Phi = \sup |\varphi|$.

Hint: consider $v(t,x) = tF + Gx + \Phi \pm u(t,x)$ and use part 1.

Exercise 3 Let $\Omega = (0, \ell)$. Suppose that $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ solves

$$\begin{cases} u_t - u_{xx} = f(t,x), & (t,x) \in \Omega_T, \\ u(0,x) = 0, & x \in [0,\ell], \\ -u_x + \alpha u = 0, & t > 0,\ x = 0, \\ u_x + \beta u = 0, & t > 0,\ x = \ell, \end{cases}$$

where $\alpha, \beta \geq 0$ are constants. Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t,x)\,dx + \int_0^T \int_0^\ell u_x^2(t,x)\,dxdt \leq C \int_0^T \int_0^\ell f^2(t,x)\,dxdt,$$

for some constant C depending only on T.

Hint: multiply the equation by u on both sides, perform suitable integration by parts in x, then integrate in t; use $|2ab| \leq a^2 + b^2$ and Gronwall at some point.

Exercise 4 Let $\Omega = (0, \ell)$ and $b, c \in \mathcal{C}(\overline{\Omega_T})$. Suppose that $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ solves

$$\begin{cases} u_t - u_{xx} + b(t,x)u_x + c(t,x)u = 0, & (t,x) \in \Omega_T, \\ u(0,x) = \varphi(x), & x \in [0,\ell], \\ u(t,0) = u(t,\ell) = 0, & t \in [0,T]. \end{cases}$$

Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t,x)\,dx + \int_0^T \int_0^\ell u_x^2(t,x)\,dxdt \leq C \int_0^\ell \varphi^2(x)\,dx,$$

for some constant C depending only on T, β and γ, where

$$\beta = \sup_{\overline{\Omega_T}} |b(t,x)|, \quad \gamma = \sup_{\overline{\Omega_T}} |c(t,x)|.$$

Ex1. (1) 反证. 假设 $\min\limits_{\overline{\Omega}_T} u<0$

因 $\min\limits_{\partial_p\Omega_T} u\geq 0$，故 $\exists (t^*,x^*)\in\Omega_T$ s.t. $\min\limits_{\overline{\Omega}_T} u = u(t^*,x^*)<0$

因对 $\forall x$, $u(t^*,x^*)\leq u(t^*,x)$

故 $u(t^*,\cdot)$ 在 $x=x^*$ 处 Hessian 矩阵半正定，从而 $\Delta u(t^*,x^*)\geq 0$

又由 $\forall t<t^*$, $u(t^*,x^*)\leq u(t,x^*)$ 知，$\partial_t u(t^*,x^*)\leq 0$

又 $c(x)\geq 0$，故 $c(x^*)u(t^*,x^*)\leq 0$

从而 $\partial_t u(t^*,x^*)+\mathcal{L}u(t^*,x^*)=\partial_t u(t^*,x^*)-\Delta u(t^*,x^*)+c(x^*)u(t^*,x^*)\leq 0$

与 $(\partial_t+\mathcal{L})u>0$ in Ω_T 矛盾

故即得结论

(2) 考虑 $u_\varepsilon(t,x):=u(t,x)+t\varepsilon$, $\varepsilon>0$

则 $(\partial_t+\mathcal{L})u_\varepsilon(t,x)=(\partial_t+\mathcal{L})u(t,x)+\varepsilon+c(x)t\varepsilon\geq\varepsilon>0$ in Ω_T

故类似上问 (1) 知，$\min\limits_{\partial_p\Omega_T} u_\varepsilon\geq\min\limits_{\partial_p\Omega_T} u\geq 0\Rightarrow\min\limits_{\overline{\Omega}_T} u_\varepsilon\geq 0$

令 $\varepsilon\to 0+$ 即得结论.

(3) 考虑 $v(t,x):=e^{-Mt}u(t,x)$

则 $(\partial_t+\mathcal{L})v(t,x)=e^{-Mt}(\partial_t+\mathcal{L})u(t,x)-Me^{-Mt}u(t,x)$

即 $(\partial_t+\tilde{\mathcal{L}})v(t,x)=e^{-Mt}(\partial_t+\mathcal{L})u(t,x)\geq 0$ in Ω_T，其中 $\tilde{\mathcal{L}}v=\mathcal{L}v+Mv$

又由 v 定义知 $\min\limits_{\partial_p\Omega_T} v\geq 0$，且 $c+M\in[0,\infty)$

故类似上问 (2) 知，$\min\limits_{\overline{\Omega}_T} v\geq 0$

从而 $\min\limits_{\overline{\Omega}_T} u\geq 0$，即得结论.

Ex2. (1) 考虑 $u_\varepsilon(t,x):=u(t,x)+\varepsilon x$, $\varepsilon>0$

则 $(\partial_t-\Delta)u_\varepsilon(t,x)=\partial_t u(t,x)-\Delta u(t,x)\geq 0$ in Ω_T

由 Weak maximum principle 知，$\min\limits_{\overline{\Omega}_T} u_\varepsilon=\min\limits_{\partial_p\Omega_T} u_\varepsilon$

假设 $\min\limits_{\overline{\Omega}_T} u_\varepsilon=u_\varepsilon(t^*,x^*)<0$, with $(t^*,x^*)\in\partial_p\Omega_T$

因 $u_\varepsilon|_{t=0}=u|_{t=0}+\varepsilon x>0$, $\forall x\in\Omega$ 且 $u_\varepsilon(t,0)=u(t,0)\geq 0$, $\forall t>0$

故 $x^*=l$

从而由 $u_\varepsilon(t^*,x^*)\geq u_\varepsilon(t^*,l)$, $\forall x\in\Omega$ 知 $\partial_x u_\varepsilon(t^*,l)\leq 0$

这与 $\partial_x u_\varepsilon(t^*,l)=\partial_x u(t^*,l)+\varepsilon>0$ 矛盾

故 $u_\varepsilon\geq 0$ on $\overline{\Omega}_T$

令 $\varepsilon\to 0+$ 即得 $u\geq 0$ on $\overline{\Omega}_T$

(2) 考虑 $v(t,x)=tF+Gx+\Phi\pm u(t,x)$

则 $\begin{cases}(\partial_t-\Delta)v=F\pm f\geq 0, & (t,x)\in\Omega_T\\ v|_{t=0}=Gx+\Phi\pm\varphi\geq 0, & x\in\Omega\\ v(t,0)=tF+\Phi\geq 0, & t>0\\ v_x(t,l)=G\pm g\geq 0, & t>0\end{cases}$

由上问(1)知，$v(t,x)\geq 0$ on $\overline{\Omega}_T$

从而 $\max\limits_{\overline{\Omega}_T}|u|\leq\max\limits_{\overline{\Omega}_T}(tF+Gx+\Phi)=|T|F+Gl+\Phi\leq C(|T|+1)(F+G+\Phi)$

其中 $C:=\max\{L,1\}$ 只依赖 l.

Ex3. 对 $\forall t\in(0,T]$，有 $\int_0^l(u\cdot u_t-u\cdot u_{xx})dx=\int_0^l u\cdot f\,dx$

分部积分得，$LHS=\int_0^l u\cdot u_t dx-u\cdot u_x\Big|_0^l+\int_0^l|u_x|^2dx$

$\geq\frac{1}{2}\frac{d}{dt}\int_0^l u^2dx+\int_0^l u_x^2dx$

又 $RHS\leq\frac{1}{2}\int_0^l u^2dx+\frac{1}{2}\int_0^l f^2dx$

两边关于变元 t 积分得 $\frac{1}{2}\int_0^l u^2(t,x)dx+\int_0^t ds\int_0^l u_x^2(s,x)dx\leq\frac{1}{2}\int_0^t ds\int_0^l u^2(s,x)dx+\frac{1}{2}\int_0^t ds\int_0^l f^2(s,x)d$

一方面，$\int_0^l u^2(t,x)dx\leq\int_0^t ds\int_0^l u^2(s,x)dx+\int_0^t ds\int_0^l f^2(s,x)dx$，$t\in[0,T]$

由Gronwall不等式知，$\int_0^l u^2(t,x)dx\leq\int_0^t ds\int_0^l f^2(s,x)dx+\int_0^t e^{t-s}\left(\int_0^s dr\int_0^l f^2(r,x)dx\right)ds$

$\leq[1+te^t]\int_0^t ds\int_0^l f^2(s,x)dx$

从而 $\sup\limits_{0\leq t\leq T}\int_0^l u^2(t,x)dx\leq[1+Te^T]\int_0^T dt\int_0^l f^2(t,x)dx$

另一方面，$\int_0^t ds\int_0^l u_x^2(s,x)dx\leq\frac{t}{2}\int_0^l u^2(t,x)dx+\frac{1}{2}\int_0^t ds\int_0^l f^2(s,x)dx$

$\leq\left[\frac{1}{2}+\frac{t}{2}+\frac{t}{2}e^t\right]\int_0^t ds\int_0^l f^2(s,x)dx$

从而 $\int_0^T ds\int_0^l u_x^2(s,x)dx\leq\left[\frac{1}{2}+\frac{T}{2}+\frac{T}{2}e^T\right]\int_0^T ds\int_0^l f^2(s,x)dx$

整理即得结论.

Ex4. (与Ex3.证明类似)

注意到，$\frac{1}{2}\frac{d}{dt}\int_0^l u^2dx+\int_0^l u_x^2dx\leq\frac{\beta^2}{2}\int_0^l u^2dx+\frac{1}{2}\int_0^l u_x^2dx+\gamma\int_0^l u^2dx$

两边关于变元 t 积分得 $\frac{1}{2}\int_0^l u^2(t,x)dx+\frac{1}{2}\int_0^t ds\int_0^l u_x^2(s,x)dx$

$\frac{1}{2}\int_0^l u^2(t,x)dx+\frac{1}{2}\int_0^t ds\int_0^l u_x^2(s,x)dx\leq\left(\frac{\beta^2}{2}+\gamma\right)\int_0^t ds\int_0^l u^2(s,x)dx+\frac{1}{2}\int_0^l\varphi^2(x)dx$

一方面，由Gronwall不等式知，$\int_0^l u^2(t,x)dx\leq\underbrace{\left[1+(\beta^2+2\gamma)te^{(\beta^2+2\gamma)t}\right]}_{\text{记为}\tilde{C}}\int_0^l\varphi^2(x)dx$

另一方面，$\int_0^t ds\int_0^l u_x^2(s,x)dx\leq(\beta^2+2\gamma)\int_0^t ds\int_0^l u^2(s,x)dx+\int_0^l\varphi^2(x)dx$

$\leq\left[(\beta^2+2\gamma)t\tilde{C}+1\right]\int_0^l\varphi^2(x)dx$

整理即得结论