12311410 [illegible] Real Analysis (H) HW 1.

1. Proof: Warning: A is infinite, it might be impossible to list all its elements out explicitly.

Since $(A \cup B) \supset A$, $|A \cup B| \geq |A|$.

Since A is infinite, $\exists$ C – countable set, $C \subset A$.

$C = \{c_1, c_2, \dots\dots\}$ – list all elements out.

① $B = \{x_1, \cdots, x_n\}$ – finite.

Define $\varphi: A \cup B \to A$ $\quad \varphi(x) = \begin{cases} c_i, & \text{if } x = x_i,\ i \in [\![1, n]\!]. \\ c_{i+n}, & \text{if } x = c_i,\ i \in \mathbb{N} \setminus [\![1, n]\!] \\ x, & \text{otherwise} \end{cases}$

Clearly, φ is well-defined.

Suppose $\varphi(x) = \varphi(y)$.

$$\left.\begin{aligned} \Rightarrow\ & x = y = x_i \text{ for some } i \in [\![1, n]\!], \\ \text{or } & x = y = c_i \text{ for some } i \in \mathbb{N} \setminus [\![1, n]\!]. \\ \text{or } & x = y \text{ for } x, y \in (A \cup B) \setminus (B \cup C). \end{aligned}\right\} \Rightarrow x = y \Rightarrow \varphi: \text{injective.} \Rightarrow |A \cup B| \leq |A|.$$

② $B = \{x_1, x_2, \cdots\}$ – countable.

Define $\psi: A \cup B \to A$. $\quad \psi(x) = \begin{cases} c_{2i-1}, & \text{if } x = x_i \text{ for } i \in \mathbb{N}. \\ c_{2i}, & \text{if } x = c_i \text{ for } i \in \mathbb{N}. \\ x, & \text{otherwise} \end{cases}$

Also, ψ is well-defined.

Suppose $\psi(x) = \psi(y)$.

$$\left.\begin{aligned} \Rightarrow\ & x = y = x_i \text{ for some } i \in \mathbb{N} \\ \text{or } & x = y = c_i \text{ for some } i \in \mathbb{N} \\ \text{or } & x = y \text{ for } x, y \in (A \cup B) \setminus (B \cup C) \end{aligned}\right\} \Rightarrow x = y \Rightarrow \psi: \text{injective.} \Rightarrow |A \cup B| \leq |A|.$$

Combining ① and ②, $|A \cup B| \leq |A|$.

Since $A \cup B \supseteq A$ also, we have $|A \cup B| = |A|$. $\square$.

2. Proof: $|\mathbb{R}^2| = |\mathbb{R} \times \mathbb{R}| = |2^{\mathbb{N}} \times 2^{\mathbb{N}}| \underset{\uparrow}{=} |2^{\mathbb{N}}| = |\mathbb{R}|$. $\Rightarrow |2^{\mathbb{R}^2}| = |2^{\mathbb{R}}|$.

Define $\varphi: 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$

$A \times B \mapsto \{2n : n \in A\} \cup \{2n+1 : n \in B\}$. $\quad \varphi^{-1} \circ \varphi = I_d$.

Then $\varphi^{-1}: 2^{\mathbb{N}} \to 2^{\mathbb{N}} \times 2^{\mathbb{N}}$

$C \mapsto \{n : 2n \in C\} \times \{n : 2n+1 \in C\}$. $\quad \varphi \circ \varphi^{-1} = I_d$.

Since $f: \mathbb{R} \to \mathbb{R}$ is an element of $2^{\mathbb{R}^2}$, we have $|\mathbb{R}^{\mathbb{R}}| \leq |2^{\mathbb{R}^2}| = |2^{\mathbb{R}}|$.

Since $2^{\mathbb{R}}$ can be viewed as all the func. from $\mathbb{R}$ to $\{0,1\}$,

we have $|2^{\mathbb{R}}| \leq |\mathbb{R}^{\mathbb{R}}|$.

Thus, $|2^{\mathbb{R}}| = |\mathbb{R}^{\mathbb{R}}| = |\{\text{all func. from } \mathbb{R} \text{ to } \mathbb{R}\}|$. $\square$.

3. Proof: WTS: $\left|\bigcup_{a\in A} \mathbb{R}_a\right| = |\mathbb{R}|$, where A is an at most countable set, $\mathbb{R}_a$: countinual, $\forall a \in A$.

Step 1: We deduce it holds for A: finite. Enough, by induction, to show that it holds for 2 continual sets' union.

Claim: $\forall$ sets A, B, it holds $|A \cup B| \leq |A \times B|$ if $A \neq \emptyset$, $|B| \geq 2$.

Indeed consider
$$\varphi: A \cup B \to A \times B$$
$$x \mapsto \begin{cases} (x, b_1), & \text{if } x \in A \setminus B. \\ (a, x), & \text{if } x \in B \setminus A. \\ (x, b_2), & \text{if } x \in A \cap B. \end{cases}$$
where $a \in A$, $b_1, b_2 \in B$ are fixed ($b_1 \neq b_2$).

$\Rightarrow \varphi$ is injective since $A \cup B$ are mapped to 3 distinct „catagories": $(\cdot, b_1)$, $(a, \cdot)$ and $(\cdot, b_2)$.

Thus, $|\mathbb{R}_a \cup \mathbb{R}_b| \leq |\mathbb{R}_a \times \mathbb{R}_b| = |\mathbb{R} \times \mathbb{R}| \underset{\uparrow}{=} |\mathbb{R}|$

by prob. 2.

Also, $|\mathbb{R}_a \cup \mathbb{R}_b| \geq |\mathbb{R}_a| = |\mathbb{R}|$. $\Rightarrow |\mathbb{R}_a \cup \mathbb{R}_b| = |\mathbb{R}|$. ✓

Step 2: prove $\left|\bigcup_{n=1}^{\infty} \mathbb{R}_n\right| = |\mathbb{R}|$.

It's obvious that $\left|\bigcup_{n=1}^{\infty} \mathbb{R}_n\right| \geq |\mathbb{R}|$.

Switch from $\bigcup_{n=1}^{\infty}$ to $\bigsqcup_{n=1}^{\infty}$: let $A_1 = \mathbb{R}_1$, $A_2 = \mathbb{R}_2 \setminus \mathbb{R}_1$, $A_3 = \mathbb{R}_3 \setminus (\mathbb{R}_1 \cup \mathbb{R}_2)$, ...

$\Rightarrow$ Only need to show $\left|\bigsqcup_{n=1}^{\infty} A_n\right| \leq |\mathbb{R}|$, A_n: at most countable.

The proof for A_n: finite follows Step 1.

Since union of a countinual set and a finite set is still continual, we only care about $\left|\bigsqcup_{n=1}^{\infty} A_n\right| \overset{?}{\leq} |\mathbb{R}|$, A_n: all continual.

Since $|A_n| = |(0,1)|$, $\exists$ a bijection $\psi_n: A_n \to (0,1)$, $\forall n \in \mathbb{N}$.

Define $\Phi: \coprod_{n=1}^{\infty} A_n \to \mathbb{N} \times (0,1)$.

$\forall n: a \in A_n \mapsto (n, \psi_n(a))$

extract information.

Clearly, Φ is injective.

$\Rightarrow \left|\coprod_{n=1}^{\infty} A_n\right| \leq |\mathbb{N} \times (0,1)| \leq |\mathbb{R} \times (0,1)| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

by prob. 2.

Thus, $\left|\bigcup_{n=1}^{\infty} A_n\right| = |\mathbb{R}|$. $\square$.

4. NO!

A counter example: $\rho(x,y) = \begin{cases} 0, & x \neq y \\ 1, & x = y. \end{cases}$

Then, $B_{\frac{1}{2}}(x) \subset B_{\frac{1}{3}}(x)$ since the two balls degenerate to singletons.

BUT: $r = \frac{1}{2} > \frac{1}{3} = R$.

5. <u>Proof</u>: Define $\mathcal{B} := \{$all open balls in $\mathbb{R}^n$ with centre in $\mathbb{Q}^n$ and radius in $\mathbb{Q}^+\}$. It's a family of balls in $\mathbb{R}^n$, which is a base for $\mathbb{R}^n$,

i.e. $\forall x \in \mathbb{R}^n$, $\exists$ a rational ball B_x in $\mathcal{B}$ s.t. $x \in B$.

$|\mathbb{N}| \leq |\mathcal{B}| = |\mathbb{Q}^n \times \mathbb{Q}^+| = |\mathbb{N}^n \times \mathbb{N}| = |\mathbb{N}^{n+1}| = |\mathbb{N}|$. $\Rightarrow |\mathcal{B}| = |\mathbb{N}|$!

Since $|\mathbb{Q}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Let $x \in A \setminus A'$, x must be isolated.

i.e. $\exists$ ball $B'_x \ni x$, $B'_x \cap A = \{x\}$.

Otherwise x will be an accumulation pt.

More precisely, there exists a rational ball $U_x \subset B'_x$ s.t. $x \in U_x$.

Now, define: $\varphi: A \setminus A' \to \mathcal{B}$.

$x \mapsto U_x$.

φ is well-defined since the selection's possibility is guaranteed above.

φ is injective since $U_x \neq U_y$ for $x \neq y$ because $\begin{cases} x \in U_x \\ y \notin U_x \end{cases}$ and $\begin{cases} x \notin U_y \\ y \in U_y \end{cases}$.

$\Rightarrow |A \setminus A'| \leq |\mathcal{B}| = |\mathbb{N}|$. $\square$

6. <u>Proof</u>: Prove by contradiction.

Suppose $X \setminus G$ is not nowhere dense in X. i.e. $X \setminus G$ is somewhere dense. i.e. $\exists B_\delta(a)$, s.t. $(X \setminus G) \cap B_\delta(a)$ is dense in $B_\delta(a)$.

$\Rightarrow \forall B_\varepsilon(b) \subset B_\delta(a)$. $B_\varepsilon(b) \cap \left((X \setminus G) \cap B_\delta(a)\right) \neq \emptyset$. ······ $(*)$

Since G is dense, $B_\delta(a) \cap G \neq \emptyset$. let $x \in B_\delta(a) \cap G$.

Since G is open, $B_\delta(a) \cap G$ is still open.

$\Rightarrow \exists B_r(x)$, $B_r(x) \subset (B_\delta(a) \cap G)$.

Note that $B_r(x) \subset B_\delta(a)$. thus by $(*)$ $B_r(x) \cap (X \setminus G) \cap B_\delta(a) \neq \emptyset$.

$\Rightarrow B_r(x) \cap (X \setminus G) \neq \emptyset$

But $B_r(x) \subset G$. $\Rightarrow B_r(x) \cap (X \setminus G) = \emptyset$. $\Rightarrow$ a contradiction. $\square$

7. <u>Proof</u>: $\forall B_\varepsilon(a) \subset X$, Since G_1 is dense, $\exists x_1 \in B_\varepsilon(a) \cap G_1$.

Since G_1, $B_\varepsilon(a)$ open $\Rightarrow B_\varepsilon(a) \cap G_1$ is open.

$\Rightarrow \exists \overline{B_1} \subset B_\varepsilon(a) \cap G_1$, s.t. $B_1 \ni x$ and $\mathrm{diam}(B_1) < \frac{1}{2}$.

Since G_2 is dense, $B_1 \cap G_2 \neq \emptyset$.

$\Rightarrow$ Similarly, $\exists \overline{B_2} \subset (B_1 \cap G_2) \subset (B_\varepsilon(a) \cap G_1 \cap G_2)$ and $\mathrm{diam}(B_2) < \frac{1}{2^2}$.

Repeat this process, we get a sequence of nested balls:

$\overline{B_1} \supset \overline{B_2} \supset \overline{B_3} \supset \cdots\cdots$ s.t $\overline{B_n} \subset \left(B_\varepsilon(a) \cap \bigcap_{i=1}^{n} G_i\right)$ and $\mathrm{diam}(B_n) < \frac{1}{2^n}$.

Now, by the Cantor intersection property for nested balls, there exists unique $x \in \bigcap_{n=1}^{\infty} \overline{B_n}$. Note that, $x \in \bigcap_{n=1}^{\infty} G_n$ and $x \in B_\varepsilon(a)$. $\Rightarrow \bigcap_{n=1}^{\infty} G_n$ is dense. $\square$

8. Proof: To prove "can be attained", we need to find the $x^* \in E$ s.t we attain inf.

(i) if E is cpt.

Fix a. Define $f(x) := d(x,a)$.

Since $|d(x,a) - d(y,a)| \le d(x,y)$, $f(x)$ is 1-Lipschitz

$\Rightarrow f(x)$ is continuous on X.

Since E is cpt. f is cts. on E. $f \in C(E)$

$\Rightarrow \exists x^* \in E$ s.t $f(x^*) = \min_{x \in E} f(x) = \tau_E(a)$.

(ii) if E is merely closed, $X = \mathbb{R}^n$.

Now, $d(x,a) = \|x-a\|$. Define $f(x) := \|x-a\|$.

Since the set E is not "nice" enough, we define $K := E \cap \overline{B}_{(\tau_E(a)+1)}(a)$

$\Rightarrow K$ is closed and bdd.

Since $K \subset X = \mathbb{R}^n \Rightarrow K$ is cpt.

$\Rightarrow$ Similarly, following (i), $\exists x^* \in K$, s.t $f(x^*) = \min_{x \in K} f(x)$

For $x \in E \setminus K$, $x \notin B_{(\tau_E(a)+1)}(a) \Rightarrow f(x) \ge \tau_E(a)+1 > \tau_E(a)$.

But $x^* \in K \subset E \Rightarrow f(x) \le \tau_E(a) \Rightarrow f(x^*) = \min_{x \in E} f(x)$. □.

9. Proof: So now we fix $E \subset X$. $\bar{E} = E$

$\forall f, g: X \to \mathbb{R}$, $f(x) - g(x) \ge -|f(x) - g(x)|$.

$\Rightarrow \inf f - \inf g \ge \inf(-|f-g|) \Rightarrow \inf f - \inf g \ge -\sup|f-g|$.

Similarly, $\inf g - \inf f \ge -\sup|f-g|$

$\Rightarrow -\sup|f-g| \le \inf f - \inf g \le \sup|f-g|$

$\Rightarrow |\inf f - \inf g| \le \sup|f-g|$.

$\forall \varepsilon > 0$, $\exists \delta = \varepsilon$, s.t $\forall a, b \in X$, with $d(a,b) < \delta$.

$$\left|\tau_E(a) - \tau_E(b)\right| = \left|\inf_{x \in E} d(x,a) - \inf_{y \in E} d(y,b)\right|$$

$$\le \sup_{x\in E} |d(x,a)-d(x,b)| \le \sup_{x\in E} d(a,b) = d(a,b) < \varepsilon$$

$\Rightarrow T_E(a)$ is continuous in a on X. $\square$.

10. Proof: Since K is cpt. in $\mathbb{R}^n$, K is closed and bdd.

Since $f\in C(K)$, f is bdd on K. $\Rightarrow G_f$ is bdd.

Only need to show G_f is closed.

We prove G_f contains all its accumulation pts.

Let $(x_k, f(x_k)) \xrightarrow{k\to\infty} (x,y)$ with $x_k\in K$.

Then, by f's continuity, $y=f(x)$. $\Rightarrow (x,y)=(x,f(x))\in G_f$. $\square$.

11. Proof: Since K is cpt. $\Rightarrow K$ is separable.

Pick a countable dense subset $D=\{x_i\}_{i=1}^{\infty}\subset K$.

Define $T_{x_i}(a) := T_{\{x_i\}}(a) = \inf_{x\in\{x_i\}} d(x,a) = d(x_i,a)$.

Claim: $\{T_{x_i}(a)\}$ separates pts.

$\forall a\neq b\subset K$. $\exists$ subseq. $x_{i_k}\xrightarrow{k\to\infty} a$.

$\Rightarrow T_{x_{i_k}}(a) = d(x_{i_k},a)\xrightarrow{k\to\infty} 0$. but $T_{x_{i_k}}(b) = d(x_{i_k},b)\xrightarrow{k\to\infty} d(a,b)>0$

$\Rightarrow T_{x_{i_k}}(a)\neq T_{x_{i_k}}(b)$ for large k. ✓.

Let $\mathcal{A}$ be the (real) algebra generated by const func 1 and $\{T_{x_i}\}_{i=1}^{\infty}$.

By Stone-Weierstrass Thm, $\bar{\mathcal{A}} = C(K)$.

Let $\mathcal{A}_{\mathbb{Q}}$ be the subalgebra of $\mathcal{A}$ consisted of all polys with rational coefficients in finitely many generators from $\{T_{x_i}\}_{i=1}^{\infty}$ and 1.

$\Rightarrow \mathcal{A}_{\mathbb{Q}}$ is countable.

Since rational coefficients approximate real ones, $\overline{\mathcal{A}_{\mathbb{Q}}}$ is the same as $\bar{\mathcal{A}}$. $\Rightarrow \overline{\mathcal{A}_{\mathbb{Q}}} = C(K)$. $\Rightarrow C(K)$ is separable. $\square$.

12. Proof: $X = C(K, \mathbb{C})$.

$A \subset X$ is a complex algebra.

For $f = u + iv \in A$, with u, v being real-valued cts. func. $u, v \in A$.

$\bar{f} = u - iv \in A$. $\Rightarrow$ A is closed under conjugation.

Define $A_{\mathbb{R}} = A \cap C(K, \mathbb{R})$.

$\forall f \in A$, $\mathrm{Re}(f) = \frac{1}{2}(f + \bar{f}) \in A$, $\mathrm{Im}(f) = \frac{1}{2}(f - \bar{f}) \in A$.

$A_{\mathbb{R}}$ inherites its identity of an algebra from A.

$1 \in A$, $1 \in C(K, \mathbb{R}) \Rightarrow 1 \in A_{\mathbb{R}}$.

$\forall x \neq y$, $\exists f \in A$, $f(x) \neq f(y)$. $f(x) - f(y) \in \mathbb{C}$. write $f(x) - f(y) = r \cdot e^{i\theta}$. $r \neq 0$

$\Rightarrow (f(x) - f(y)) \cdot e^{-i\theta} = r \in \mathbb{R} \setminus \{0\}$.

$\Rightarrow u := \mathrm{Re}(e^{-i\theta} \cdot f) \in A_{\mathbb{R}}$. and $u(x) - u(y) = (f(x) - f(y)) \cdot e^{-i\theta} \neq 0$.

$\Rightarrow A_{\mathbb{R}}$ also separates pts.

By the real Stone-Weierstrass Thm, $\overline{A_{\mathbb{R}}} = C(K, \mathbb{R})$.

Now, $\forall f = u + iv \in C(K, \mathbb{C})$, with $u, v \in C(K, \mathbb{R})$.

$\exists\ u_n \overset{n\to\infty}{\rightrightarrows} u$, $v_n \overset{n\to\infty}{\rightrightarrows} v$ in $A_{\mathbb{R}}$. Define $f_n = u_n + iv_n \in A$.

$$\Rightarrow \sup_{x \in K} |f_n(x) - f(x)| \leq \sup_{x \in K} |u_n(x) - u(x)| + \sup_{x \in K} |v_n(x) - v(x)| \longrightarrow 0, \quad n \to \infty$$

$\Rightarrow A$ is dense in $C(K, \mathbb{C})$. i.e. $\bar{A} = C(K, \mathbb{C}) = X$. $\square$

13. NOT precompact!

Proof: By Arcela-Ascolli Thm:

$A \subset C([0,1])$ is precompact $\Leftrightarrow$ A is bdd and equiconti.

① bdd: $\forall f \in A$, $|f| \leq x^2 \leq 1$. ✓.

② equiconti: We propose a counter example:

Define $f_n(x) = x^2 \sin(nx)$. Apparently, $|f_n(x)| \leq x^2$, $\forall n \in \mathbb{R}$.

For fixed $\varepsilon = 1 > 0$, $\forall \delta > 0$, choose $N \in \mathbb{N}$ large enough. s.t $\frac{\pi}{N} < \delta$

Now, we evaluate $\{f_n(x),\ n \geq N,\ n \in \mathbb{R}\}$.

Choose $x=1$, $y=1-\frac{\pi}{n}$. $|x-y|=\frac{\pi}{n}<\delta$.

$$|f_n(x)-f_n(y)| = \left|\sin n - (1-\tfrac{\pi}{n})^2 \sin(n-\pi)\right|$$
$$= \left|\sin n + (1-\tfrac{\pi}{n})^2 \sin n\right|$$
$$= |\sin n| \cdot \left|2+\tfrac{\pi^2}{n^2}-\tfrac{2\pi}{n}\right| = |\sin n|\cdot\left|2+\tfrac{\pi}{n}(\tfrac{\pi}{n}-2)\right|$$

Since $\left|2+\frac{\pi^2}{n^2}-\frac{2\pi}{n}\right| \to 2$ $(n\to\infty)$, $\exists N'>0$, s.t. $\left|2+\frac{\pi^2}{n^2}-\frac{2\pi}{n}\right|>1$, $\forall n\geq N'$.

Now, for $\{f_n(x), n\geq \max\{N,N'\}, n\in\mathbb{R}\}$.

We further narrow down to $\{f_n(x), n=\frac{\pi}{2}+2k\pi, k\in\mathbb{N}, n\geq\max\{N,N'\}\}$.

$\Rightarrow |f_n(x)-f_n(y)| > |\sin n| = 1 = \varepsilon$.

$\Rightarrow \mathcal{A}$ is not equiconti.

Thus, $\mathcal{A}$ is not precompact. $\square$

14.

Proof: $f_a(x)=x^a \ln x$. $f_a'(x) = x^{a-1} + a x^{a-1}\ln x$
$= x^{a-1}(1+a\ln x)$.

$f_a'(x)=0 \Rightarrow x=e^{-\frac{1}{a}}$. $f_a(e^{-\frac{1}{a}}) = e^{-1}(-\frac{1}{a}) = -\frac{1}{ae}$.

$0<x<e^{-\frac{1}{a}}$: $f_a'(x)<0$. $f_a(x)\downarrow$

$e^{-\frac{1}{a}}<x<1$: $f_a'(x)>0$. $f_a(x)\uparrow$.

Note that $f_a(x)<0$ on $(0,1)$. and $f_a(0)=f_a(1)=0$

$$\Rightarrow \|f_a\| = \sup_{x\in[0,1]}|f_a(x)| = |f_a(e^{-\frac{1}{a}})| = \frac{1}{ae}.$$

By Arcela-Ascolli Thm:

$\{f_a(x)\}_{a>0}\subset C([0,1])$ is precompact $\Leftrightarrow$ $\{f_a(x)\}_{a>0}$ is bdd and equiconti.

BUT: as $a\downarrow 0$, $\|f_a\|$ begins unbounded. $\Rightarrow$ CANNOT be precompact. $\square$

Rmk: the subfamily $\{f_a(x): a\geq\alpha\}$ with $\alpha>0$ is relatively cpt.

15. Proof: By Arcela-Ascolli Thm:

$E\subset C([0,1])$ is precompact $\Leftrightarrow$ E is bdd and equiconti.

Since $|f(x)| \le 1$, E is bdd. Only need to show equicontinuity.
Since B is convex, we have: $\forall x, y \in B$, the line connecting x and y completely lie in B.

$\forall \varepsilon > 0$, $\exists \delta = \varepsilon > 0$, $\forall x, y$ with $d(x,y) < \delta = \varepsilon$,

Define $g(t) = f(x + t(y-x))$, $t \in [0,1] \Rightarrow g'(t) = \nabla f(x + t(y-x)) \cdot (y-x)$

$|f(x) - f(y)| = |g(0) - g(1)| \underset{\text{MVT}}{=} |g'(\xi)| \cdot |1-0| \le \|\nabla f\| \cdot d(x,y) \le \varepsilon$, $\forall f \in E$. $\square$

16. Proof: $d(m,n) = \begin{cases} 1 + \frac{1}{m+n}, & m \ne n. \\ 0, & m = n. \end{cases}$

(i) Consider a Cauchy seq. $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{N}$.

$\Rightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $d(a_m, a_n) < \varepsilon$, $\forall m, n > N$.

Specifically, for $0 < \varepsilon < 1$, $d(a_m, a_n) < \varepsilon < 1$ means $a_m = a_n$ for $m \ne n$.

$\Rightarrow \forall$ Cauchy seq. in $\mathbb{N}$, $\exists N \in \mathbb{N}$, s.t. $a_N = a_{N+1} = a_{N+2} = \cdots\cdots$,

Such a seq. clearly has a limit: just the value of a_N!

$\Rightarrow \mathbb{N}$ with distance $d(m,n)$ is a complete metric sp.

(ii) Note that $d(m,n) = \begin{cases} 1 + \frac{1}{m+n}, & m \ne n \\ 0, & m = n. \end{cases}$

When one of (m,n) tends to very large, m, n tends to be very "close" under such metric! (BUT they actually become far away from each other!)

Define $\overline{B_{r_k}}(k) = \{x : d(x,k) \le 1 + \frac{1}{2k}\}$ — $r_k = 1 + \frac{1}{2k}$.

$= \{x : 1 + \frac{1}{x+k} \le 1 + \frac{1}{2k}\}$

$= \{x : x \ge k\}$.

Then, $\overline{B_{r_k}}(k) = \{x : x \ge k\} \supset \{x : x \ge k+1\} = \overline{B_{r_{k+1}}}(k+1)$

$\Rightarrow \{\overline{B_{r_k}}(k)\}$ are actually nested balls!

Now, consider: $\bigcap_{k=1}^{\infty} \overline{B}_{r_k}(k) = \bigcap_{k=1}^{\infty} \{x : x \geq k\} = \emptyset$,

since no element in $\mathbb{N}$ is greater than any other elements!

$\Rightarrow$ The nested balls principle fails. $\square$