12311410 陈恺

Numerical Analysis. HW 1.

$\boxed{\text{Sec } 1.2}$

1. a. $|p-p^*| = 0.126 \times 10^{-2}$.

$$\frac{|p-p^*|}{|p|} = 0.402 \times 10^{-3}.$$

c. $|p-p^*| = 0.282 \times 10^{-3}$.

$$\frac{|p-p^*|}{|p|} = 0.104 \times 10^{-3}.$$

3. a. $\frac{|p-p^*|}{|p|} \le 10^{-3}$. $\frac{|150-p^*|}{150} \le 10^{-3}$.

$|150-p^*| \le 0.15 \Rightarrow 149.85 \le p^* \le 150.15$.

c. $\frac{|p-p^*|}{|p|} \le 10^{-3}$. $\frac{|1500-p^*|}{1500} \le 10^{-3}$.

$|1500-p^*| \le 1.5 \Rightarrow 1498.5 \le p^* \le 1501.5$

5. a. (i) $\frac{4}{5}+\frac{1}{3} = \frac{12+5}{15} = \frac{17}{15}$.

(ii) $\frac{4}{5} \oplus \frac{1}{3} = fl(fl(\frac{4}{5}) + fl(\frac{1}{3})) = fl(0.800\times 10^0 + 0.333\times 10^0)$

$= fl(0.1133\times 10^1) = 0.113\times 10^1$

(iii) $\frac{4}{5} \oplus \frac{1}{3} = fl(fl(\frac{4}{5}) + fl(\frac{1}{3})) = fl(0.800\times 10^0 + 0.333\times 10^0)$

$= fl(0.1133\times 10^1) = 0.113\times 10^1$.

(iv). chopping: $\frac{|\frac{17}{15} - 0.113\times 10^1|}{|\frac{17}{15}|} = 0.294\times 10^{-2}$.

rounding: $\frac{|\frac{17}{15} - 0.113\times 10^1|}{|\frac{17}{15}|} = 0.294\times 10^{-2}$.

c. (i) $(\frac{1}{3} - \frac{3}{11}) + \frac{3}{20} = \frac{2}{33} + \frac{3}{20} = \frac{40+99}{660} = \frac{139}{660}$.

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(ii) $\left(\frac{1}{3} \ominus \frac{3}{11}\right) \oplus \frac{3}{20} = fl\left(fl\left(fl\left(\frac{1}{3}\right) - fl\left(\frac{3}{11}\right)\right) + fl\left(\frac{3}{20}\right)\right)$

$= fl\left[fl\left(0.333 \times 10^0 - 0.272 \times 10^0\right) + 0.150 \times 10^0\right]$

$= fl\left(0.061 \times 10^0 + 0.150 \times 10^0\right)$

$= 0.211 \times 10^0.$

(iii) $\left(\frac{1}{3} \ominus \frac{3}{11}\right) \oplus \frac{3}{20} = fl\left(fl\left(fl\left(\frac{1}{3}\right) - fl\left(\frac{3}{11}\right)\right) + fl\left(\frac{3}{20}\right)\right).$

$= fl\left[fl\left(0.333 \times 10^0 - 0.273 \times 10^0\right) + 0.150 \times 10^0\right]$

$= fl\left(0.060 \times 10^0 + 0.150 \times 10^0\right) = 0.210 \times 10^0.$

(iv): chopping: $\dfrac{\left|\frac{139}{660} - 0.211 \times 10^0\right|}{\left|\frac{139}{660}\right|} = 0.187 \times 10^{-2}.$

rounding: $\dfrac{\left|\frac{139}{660} - 0.210 \times 10^0\right|}{\left|\frac{139}{660}\right|} = 0.288 \times 10^{-2}.$

14. a. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{x} = \lim\limits_{x \to 0} \dfrac{e^x + e^{-x}}{1} = 2.$

b. $f(0.1) = fl\left(\dfrac{fl\left[fl\left(e^{0.1}\right) - fl\left(e^{-0.1}\right)\right]}{fl(0.1)}\right)$

$= fl\left(\dfrac{fl\left(\left[0.111 \times 10^1 - 0.905 \times 10^0\right]\right)}{0.100 \times 10^0}\right)$

$= fl\left(\dfrac{0.205 \times 10^0}{0.100 \times 10^0}\right) = 0.205 \times 10^1.$

c. $f(x) \approx \dfrac{\cancel{1} + x + \cancel{\frac{1}{2}x^2} + \frac{1}{6}x^3 - \left(\cancel{1} - x + \cancel{\frac{1}{2}x^2} - \frac{1}{6}x^3\right)}{x} = \dfrac{2x + \frac{1}{3}x^3}{x}$

$= \frac{1}{3}x^2 + 2.$

$$f(0.1) = fl\{ fl[fl(\tfrac{1}{3}) \cdot fl(0.1) \cdot fl(0.1)] + fl(2) \}$$

$$= fl\{ fl[0.333 \times 10^0 \cdot 0.100 \times 10^0 \cdot 0.100 \times 10^0] + 0.200 \times 10^1 \}$$

$$= fl\{ 0.333 \times 10^{-2} + 0.200 \times 10^1 \} = 0.200 \times 10^1.$$

d. (b): $$\frac{|f(0.1) - 0.205 \times 10^1|}{|f(0.1)|} = 0.233 \times 10^{-1}.$$

(c): $$\frac{|f(0.1) - 0.200 \times 10^1|}{|f(0.1)|} = 0.166 \times 10^{-2}.$$

Rk: We could also see we need to avoid $x \ominus y$ while $y \ll x$ here.

15. a. $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$

by (1.1): $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{(\frac{123}{4}) + \sqrt{(\frac{123}{4}) \times (\frac{123}{4}) - 4 \times \frac{1}{3} \times \frac{1}{6}}}{2 \times \frac{1}{3}}$$

$$= \frac{0.3075 \times 10^2 + \sqrt{0.9456 \times 10^3 - 0.2222 \times 10^0}}{0.6666 \times 10^0}.$$

$$= \frac{0.3075 \times 10^2 + \sqrt{0.9454 \times 10^3}}{0.6666 \times 10^0}$$

$$= \frac{0.3075 \times 10^2 + 0.3075 \times 10^2}{0.6666 \times 10^0}$$

$$= \frac{0.6150 \times 10^2}{0.6666 \times 10^0} = 0.9226 \times 10^2.$$

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this is inaccurate

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{0.3075\times10^2 - 0.3075\times10^2}{0.6666} = 0$$

by (1.2), $x_1 = \frac{-2c}{b+\sqrt{b^2-4ac}} = \frac{-2\times(\frac{1}{6})}{(-\frac{123}{4})+0.3075\times10^2} = \frac{-0.3334}{0}$

error!!

by (1.3), $x_2 = \frac{-2c}{b-\sqrt{b^2-4ac}} = \frac{-0.3334}{-0.6150\times10^2} = 0.5421\times10^{-2}$.

Thus, $x_1^* = 0.9226\times10^2$, $x_2^* = 0.5421\times10^{-2}$.

(these are all approximations).

We use calculator to find the exact value of x_1 and x_2, then:

$|x_1-x_1^*| = 0.1542\times10^{-1}$. $|x_2-x_2^*| = 0.6273\times10^{-6}$

$\frac{|x_1-x_1^*|}{|x_1|} = 0.1672\times10^{-3}$. $\frac{|x_2-x_2^*|}{|x_2|} = 0.1157\times10^{-3}$.

19. a. $s=0 \Rightarrow$ it's positive.

~~c~~ $1000000\ 1010 \Rightarrow c = 2^1+2^3+2^{10} = 1034$

~~f~~ $100100110 0\cdots\cdots0 \Rightarrow f = (\frac{1}{2})^1+(\frac{1}{2})^4+(\frac{1}{2})^7+(\frac{1}{2})^8$.

$$(-1)^s\cdot 2^{c-1023}\cdot(1+f)$$

$$= +1\cdot 2^{1034-1023}\cdot\left(1+\frac{1}{2}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256}\right)$$

$$= 3224.$$

20. a. the next largest one:

$$+1\cdot 2^{1034-1023}\cdot\left(1+\frac{1}{2}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256}+\frac{1}{2^{52}}\right)$$

$$= 3224+2^{-41}.$$

the next smallest one:

$$+1\cdot 2^{1034-1023}\cdot\left(1+\frac{1}{2}+\frac{1}{16}+\frac{1}{128}+\frac{1}{256}-\frac{1}{2^{52}}\right)$$

$$= 3224-2^{-41}.$$

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25. a. $f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^{x} - 1.99$

$$= (((1.01e^{x} - 4.62)e^{x} - 3.11)e^{x} + 12.2)e^{x} - 1.99$$

28. <u>Proof</u>: ① If $d_{k+1} < 5$, then $fl(y) = 0.d_1d_2\cdots d_k \times 10^n$.

$$\Rightarrow \frac{|y - fl(y)|}{|y|} \leq \frac{|+0.\underbrace{00\cdots0}_{k\text{ times}}5 \times 10^n|}{0.1 \times 10^n} = 0.5 \times 10^{-k+1}.$$

② If $d_{k+1} \geq 5$, then $fl(y) = 0.d_1d_2\cdots d_k \times 10^n + 10^{n-k}$.

$$\Rightarrow \frac{|y - fl(y)|}{|y|} \leq \frac{|-0.\underbrace{00\cdots0}_{k\text{ times}}5 \times 10^n|}{0.1 \times 10^n} = 0.5 \times 10^{-k+1}. \quad \square$$

29. a. $|f(x_0) - \tilde{f}(x_0)| \underset{\text{MVT}}{=} |f'(\xi)| \cdot \varepsilon$, where $x_0 < \xi < x_0 + \varepsilon$.

$$\frac{|f(x_0) - \tilde{f}(x_0)|}{|f(x_0)|} = \frac{|f'(\xi)| \cdot \varepsilon}{|f(x_0)|}.$$

b. i. $f(x) = e^x$. $f'(x) = e^x$. $f(x_0) = f(1) = e$.

$1 < \xi < 1 + 5 \times 10^{-6} \Rightarrow e < f'(\xi) < 0.27182954 2 \times 10^{1}$.

$\Rightarrow 0.135914 \times 10^{-4} < |f(x_0) - \tilde{f}(x_0)| < 0.135915 \times 10^{-4}$

$$0.499997 \times 10^{-5} < \frac{|f(x_0) - \tilde{f}(x_0)|}{|f(x_0)|} < 0.50000 25 \times 10^{-5}$$

ii. $f(x) = \sin x$. $f'(x) = \cos x$. $f(x_0) = f(1) = \sin 1$.

$0.5402981 < f'(\xi) < 0.5403023$

$\Rightarrow 0.270149 \times 10^{-5} < |f(x_0) - \tilde{f}(x_0)| < 0.270151 \times 10^{-5}$.

bound: $\approx 2.70 \times 10^{-6}$.

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$\Rightarrow 0.32104337\times10^{-5} < \dfrac{|f(x_0)-\tilde{f}(x_0)|}{|f(x_0)|} < 0.32104163\times10^{-5}$.

bound: $\approx 3.21\times10^{-6}$.

c. i. $\varepsilon = 5\times10^{-5}$. $x_0=10$

$f(x)=e^x$. $f'(x)=e^x$. $f(x_0)=e^{10}$.

$10<\xi<10+5\times10^{-5}$.

$\Rightarrow 0.1101323\times10^{1} < |f(x_0)-\tilde{f}(x_0)| < 0.1101378\times10^{1}$.

bound: ≈ 1.10

$0.49999987\times10^{-4} < \dfrac{|f(x_0)-\tilde{f}(x_0)|}{|f(x_0)|} < 0.5000025\times10^{-4}$.

bound: $\approx 5\times10^{-5}$.

ii. $f(x)=\sin x$. $f'(x)=\cos x$.

$\Rightarrow$

~~$0.4195358\times10^{-4} < |f(x_0)-\tilde{f}(x_0)| < 0.4195222\times10^{-4}$~~.

$0.4195222\times10^{-4} < |f(x_0)-\tilde{f}(x_0)| < 0.4195358\times10^{-4}$.

bound: $\approx 4.20\times10^{-5}$

$0.77115059\times10^{-4} < \dfrac{|f(x_0)-\tilde{f}(x_0)|}{|f(x_0)|} < 0.77117559\times10^{-4}$.

bound: $\approx 7.71\times10^{-5}$.

$\boxed{\text{Sec 2.1}}$

1. $f(x)=\sqrt{x}-\cos x$. on $[0,1]$.

$a_1, b_1 = 0, 1. \Rightarrow p_1 = 0.5$.

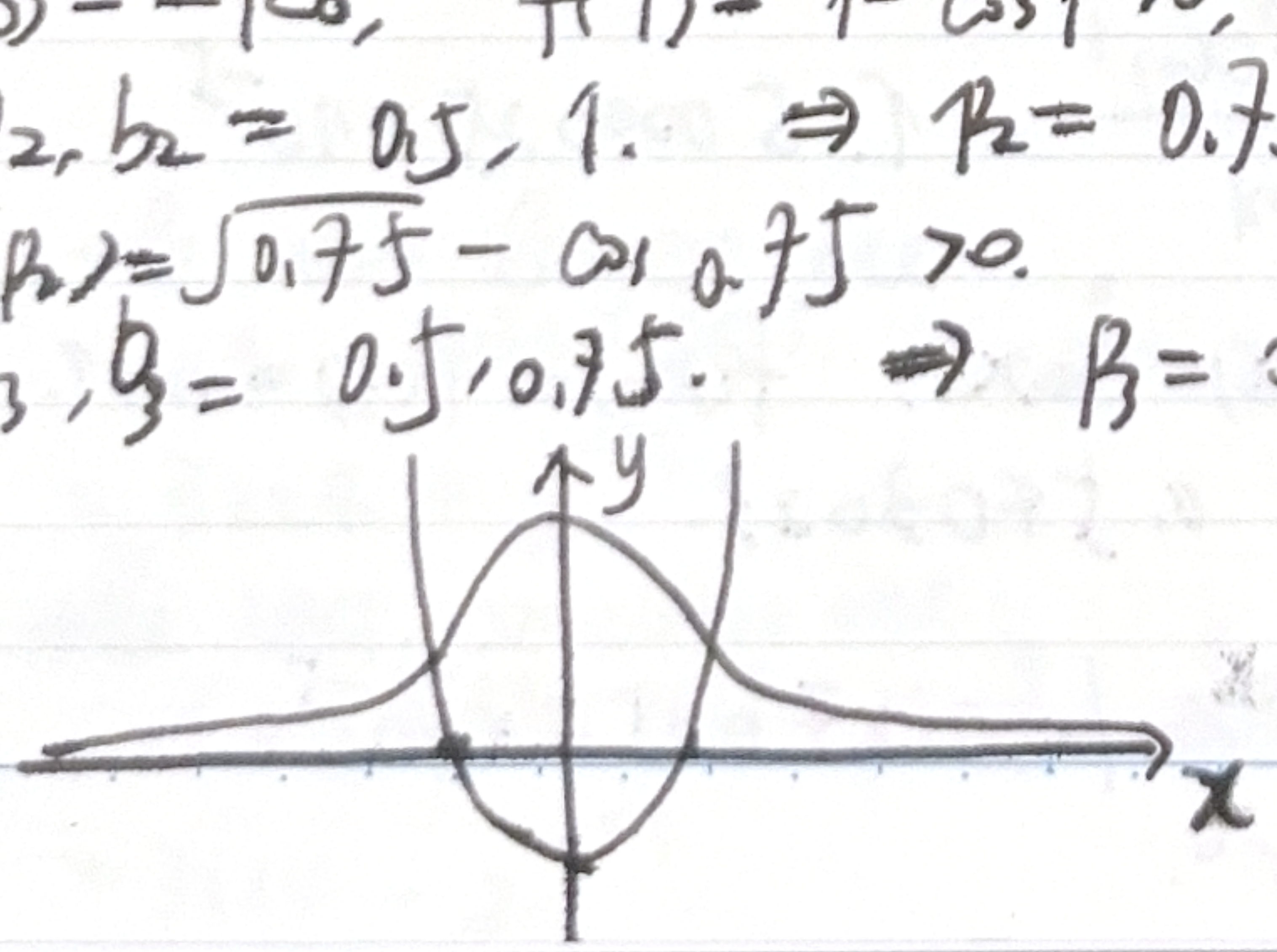
$f(0)=-1<0$, $f(1)=1-\cos 1>0$, $f(p_1)=\sqrt{0.5}-\cos\frac{1}{2}<0$.

$\Rightarrow a_2, b_2 = 0.5, 1. \Rightarrow p_2 = 0.75$.

$f(p_2)=\sqrt{0.75}-\cos 0.75>0$.

$\Rightarrow a_3, b_3 = 0.5, 0.75. \Rightarrow p_3 = 0.625$.

10. a.



b. $x^2-1=e^{1-x^2}$.

$f(x)=x^2-1-e^{1-x^2}$

$|p_n-p^*|\le \frac{b-a}{2^n}=\frac{2}{2^n}=\frac{1}{2^{n-1}}<10^{-3}$. $\Rightarrow n\ge 11$.

We'll do 11 iterations:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	0	-1	-1
2	-2	-1	-1.5	0.9635
3	-1.5	-1	-1.25	-0.0073
4	-1.5	-1.25	-1.375	0.4802
5	-1.375	-1.25	-1.3125	0.2372
6	-1.3125	-1.25	-1.28125	0.1152
7	-1.28125	-1.25	-1.265625	0.0540
8	-1.265625	-1.25	-1.2578125	0.0234
9	-1.2578125	-1.25	-1.25390625	0.0081
10	-1.25390625	-1.25	-1.25195313	0.0040
11	-1.25195313	-1.25	-1.25097656	-0.0035.

$\Rightarrow$ An approximation is $p_{11}\approx -1.25097656$.

≈ -1.251.

~~a.~~ $\cancel{f(-3)=(-1)(-2)(-3)(-4)^3(-5)<0}$

$\cancel{f(2.5)=(0.5)(3.5)(2.5)(1.5)^3(-0.5)}<0$.

$\cancel{f(-0.25)}=$

11.

	$x<-2$	$-2<x<-1$	$-1<x<0$	$0<x<1$	$1<x<2$	$x>2$
$(x+2)$	$-$	$+$	$+$	$+$	$+$	$+$
$(x+1)$	$-$	$-$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$-$	$-$	$+$
sign of f	$-$	$+$	$-$	$+$	$-$	$+$

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a. $[-3, 2.5]$.

$f(-3)<0$. $f(2.5)>0$

$f(-0.25)<0$ $f(1.125)<0$ $\Rightarrow$ converges to 2

b. $[-2.5, 3]$

$f(-2.5)<0$ $f(3)>0$

$f(0.25)>0$ $f(-1.125)>0$ $\Rightarrow$ Converges to -2.

c. $[-1.75, 1.5]$.

$f(-1.75)>0$. $f(1.5)<0$

$f(-0.125)<0$ $\Rightarrow$ converges to -1.

d. $[-1.5, +1.75]$

$f(-1.5)>0$. $f(1.75)<0$

$f(0.125)>0$ $\Rightarrow$ converges to $+1$.

13. $f(x)= x^3-25. =0.$

$f(2)<0$. $f(3)>0$. $\Rightarrow$ start with $[2,3]$.

$a_1, b_1 = 2, 3$.

$|p_n-p^*| \le \frac{1}{2^n}$ $\frac{1}{2^n} \le 10^{-4}$ $\Rightarrow n \ge 14$.

$\Rightarrow$ need 14 iterations.

~~14.~~ $\cancel{f(x)=x^2-3}$.

$\cancel{f(1)<0}$. $\cancel{f(2)>0}$. ~~$\Rightarrow$ start with $[1,2]$~~.

$\cancel{|p_n-p^*| \le \frac{1}{2^n}}$. $\cancel{\frac{1}{2^n} \le 10^{-4} \Rightarrow n \ge 14}$.

~~$\Rightarrow$ also need 14 iterations.~~

17. $|p_n-p^*| \le \frac{1}{2^n}$ $\frac{1}{2^n} \le 10^{-4}$ $\Rightarrow$ $n \ge 14$.

$\Rightarrow$ need 14 iterations.

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$\boxed{\text{Sec 2.2}}$

2\. a. $g_1(x) = (3+x-2x^2)^{\frac{1}{4}}$.

$p_0 = 1$. $\quad p_1 = g_1(1) = 1.1892$

$p_2 = g_1(p_1) = 1.08005775$.

$g_2(x) = (\frac{1}{2}x + \frac{3}{2} - \frac{1}{2}x^4)^{\frac{1}{2}}$.

$p_0 = 1$. $\quad p_1 = g_2(1) = 1.2247$

$p_2 = g_2(p_1) = 0.993666159$.

$g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{\frac{1}{2}}$.

$p_0 = 1$. $\quad p_1 = g_3(1) = 1.1547$

$p_2 = g_3(p_1) = 1.11642741$.

$g_4(x) = \frac{3x^4+2x^2+3}{4x^3+4x-1}$.

$p_0 = 1$. $\quad p_1 = g_4(1) = 1.142857$

$p_2 = g_4(p_1) = 1.12448$.

b. $g_4(x)$.

5\. a. $p_n = \frac{20p_{n-1} + 21/p_{n-1}^2}{21} \Rightarrow g(x) = \frac{20x + \frac{21}{x^2}}{21} = \frac{20}{21}x + \frac{1}{x^2}$.

$g'(x) = \frac{20}{21} - \frac{2}{x^3}$. ~~$g'(p_0) = \frac{20}{21}$~~

$p = 21^{\frac{1}{3}}$

$g'(p) = \frac{6}{7} \approx 0.857 \Rightarrow |g'(p)| < 1$

$\Rightarrow$ It can converge.

b. $p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}$.

This is actually the Newton's method, which gets $g'(p) = 0$. It converges and is the fastest.

c. $p_n = p_{n-1} - \frac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$. $\Rightarrow g(x) = x - \frac{x^4 - 21x}{x^2 - 21}$.

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$$g'(x) = 1 - \frac{(4x^3-21)(x^2-21)-(x^4-21x)(2x)}{(x^2-21)^2}.$$

$g'(p) \approx 5.7 \Rightarrow |g'(p)| > 1.$

$\Rightarrow$ It diverges.

In fact $p_0 = 1. \Rightarrow p_1 = 0$ and $p_n = 0$ for $n > 1$.

d. $p_n = \left(\frac{21}{p_{n-1}}\right)^{\frac{1}{2}}. \Rightarrow g(x) = \sqrt{\frac{21}{x}}.$

$g'(x) = -\frac{\sqrt{21}}{2x^{\frac{3}{2}}} \Rightarrow g'(p) = -\frac{1}{2}.$

$|g'(p)| = 0.5 < 1. \Rightarrow$ It converges.

Since $0.5 < 0.857$.

$\Rightarrow$ From fastest to slowest : b, d, a.

And c : not converge.

13. a. $x = \frac{2-e^x+x^2}{3}.$ $g(x) = \frac{2-e^x+x^2}{3}.$ $g'(x) = -\frac{1}{3}e^x + \frac{2}{3}x.$

$g''(x) = -\frac{1}{3}e^x + \frac{2}{3}.$

$g(0) = \frac{1}{3}$ $g(1) = \frac{3-e}{3} \in (0,1).$

Thus $g \in C[0,1]$. $g'(x) < 0$ on $[0,1] \Rightarrow g(x) \downarrow$

$\Rightarrow g([0,1]) \subset [0,1]$

$g'(x)$ exists on $(0,1)$ and

$g'(0) = -\frac{1}{3}$, $g'(1) = \frac{2-e}{3}.$

$\Rightarrow |g'(x)| \leq \frac{1}{3} < 1$, $\forall x \in (0,1).$

$\Rightarrow$ fixed-point iteration will converge on $[0,1]$.

b. $x = \frac{5}{x^2} + 2.$ $g(x) = \frac{5}{x^2} + 2.$ $g'(x) < 0.$ $g'(x) = -\frac{10}{x^3}.$

$g''(x) = \frac{30}{x^4} > 0.$

$g(3) = \frac{5}{9} + 2 < 3.$ $g(4) = 2 + \frac{5}{16} < 3.$

$g(2.5) = 2.8 < 3$

$\Rightarrow g([2.5,3]) \subset [2.5,3]$

$g \in C[2.5, 3]$ $\quad g([2.5, 3]) \subset [2.5, 3]$.
$g'(x)$ exists on $[2.5, 3]$ and
$|g'(x)| \le |g'(2.5)| = \left|\frac{10}{2.5^3}\right| < 1$.

$\Rightarrow$ fixed-point iteration'll converge on $[2.5, 3]$.

14. a. $x = 2 + \sin x$. $\quad g(x) = \sin x + 2$. $\quad g'(x) = \cos x$. $\quad g''(x) = -\sin x$.
$g \in C[2, 3]$. $\quad g([2, 3]) \subset [2, 3]$.
$g'(x)$ exists on $[2, 3]$ and
$|g'(x)| \le |g'(3)| = |\cos 3| < 1$.
$\Rightarrow$ fixed-point iteration'll converge on $[2, 3]$.

b. $x = \frac{1}{2}x^3 - \frac{5}{2}$. $\quad \cancel{g(x) = \frac{1}{2}x^3 - \frac{5}{2} \quad g'(x) = \frac{3}{2}x^2 > 0 \quad g(x) \nearrow}$
$\cancel{g''(x) = 3x > 0}$.
$\cancel{g \in [2, 3]. \quad g([2, 3])}$
$x^3 = 2x + 5$. $\quad x = \sqrt[3]{2x+5}$.
$g(x) = \sqrt[3]{2x+5}$. $\quad g'(x) = \frac{2}{3}(2x+5)^{-\frac{2}{3}}$. $\quad g''(x) < 0$.
$g(2) = \sqrt[3]{9} > 2$. $\quad g(3) = \sqrt[3]{11} < 3$.
$g \in C[2, 3]$, $\quad g([2, 3]) \subset [2, 3]$.
$g'(x)$ exists on $[2, 3]$ and $|g'(x)| \le |g'(2)| = \frac{2}{3}(9)^{-\frac{2}{3}} < 1 \times 1 = 1$.
$\Rightarrow$ fixed-point iteration will converge on $[2, 3]$

c. $3x^2 - e^x = 0$ $\quad x = \sqrt{\frac{1}{3}e^x}$. ——— a root in $[0, 1]$
$g(x) = \sqrt{\frac{1}{3}e^x}$. $\quad g'(x) = \frac{1}{2}\sqrt{\frac{1}{3}}\sqrt{e^x} > 0$. $\quad g''(x) > 0$.
$g(0) = \sqrt{\frac{1}{3}} \in (0, 1)$ $\quad g(1) = \sqrt{\frac{e}{3}} \in (0, 1)$.
$g \in C[0, 1]$. $\quad g([0, 1]) \subset [0, 1]$.
$g'(x)$ exists on $(0, 1)$ and $|g'(x)| \le |g'(1)| = \frac{1}{2}\sqrt{\frac{e}{3}} < 1$,
$\forall x \in (0, 1)$.
$\Rightarrow$ fixed-point iteration will converge on $[0, 1]$.

d. $x - \cos x = 0$. ——— a root in $[0, 1]$
$g(x) = \cos x$ $\quad g'(x) = \sin x$.
$g([0, 1]) \subset [0, 1]$. $\quad g \in C[0, 1]$.

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$g'(x)$ exists on $(0,1)$ and $|g'(x)| < 1$. $\forall x \in (0,1)$

$\Rightarrow$ fixed-point iteration will converge.

21. $g(x) = -\ln(x+\frac{1}{2})+1 \qquad g'(x) = -\frac{1}{x+\frac{1}{2}}$.

✗

① $g(0) = -\ln\frac{1}{2}+1 = 1+\ln 2 > 1$.

② $|g'(\frac{1}{2})| = 1$. ✗.

but $g(x)=x$ has a unique solution on $[0,1]$.

let $f(x) = g(x) - x$. $\quad f'(x) = -\frac{1}{x+\frac{1}{2}} - 1 < 0$ on $[0,1]$

$f(x)\downarrow$. $\quad f(0) = -\ln\frac{1}{2}+1 > 0$

$f(1) = -\ln\frac{3}{2}+1-1 = -\ln\frac{3}{2} < 0$.

$\Rightarrow \exists\, x_0 \in (0,1)$ s.t. $f(x_0) = 0$. ✓.

23. a. Pf: $x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}$.

$g(x) = \frac{1}{2}x + \frac{1}{x}$. $\quad g(\sqrt2) = \frac{\sqrt2}{2}+\frac{\sqrt2}{2} = \sqrt2$. $\Rightarrow \sqrt2$ is a fixed pt!

$(p=\sqrt2)$

$g'(x) = \frac{1}{2} - \frac{1}{x^2}$. $\quad g''(x) > 0$. $\quad \psi_{\sqrt2}$

$g(1) = \frac{3}{2}$. $\quad g(2) = \frac{3}{2}$. $\quad g(\sqrt2) = \sqrt2$

$g \in C[1,2]$. $\quad g([1,2]) \subset [1,2]$

And: $g'(x)$ exists on $(1,2)$ and $|g'(x)| \le \frac{1}{2} < 1$. $\forall x \in (1,2)$

$\Rightarrow$ Since $x_n = g(x_{n-1})$

$\Rightarrow x_n \to p\,(=\sqrt2)$. $(n\to\infty)$, which is the unique fixed pt on $[1,2]$, which is $\sqrt2$.

$(\forall x_0 \in [1,2].)$

23. a. Proof. $\forall x_0 > \sqrt2$. Consider $[\sqrt2, x_0]$.

$g(x) = \frac{1}{2}x + \frac{1}{x}$. $\quad g'(x) = \frac{1}{2} - \frac{1}{x^2} > 0$. $\quad g''(x) > 0$.

$g \in C[\sqrt2, x_0]$. $\quad g(x) = \frac{1}{2}x + \frac{1}{x} < \frac{1}{2}x + 1 < x$. $\quad \forall x \in [\sqrt2, x_0]$

$\Rightarrow g([\sqrt2, x_0]) \subset [\sqrt2, x_0]$.

(since $g(x)\uparrow$, $g(\sqrt2) = \sqrt2$, $g(x) < x$).

$g'(x) = \frac{1}{2} - \frac{1}{x^2}$ exists on $(\sqrt{2}, x_0)$ and

$|g'(x)| \le \frac{1}{2} - \frac{1}{x_0^2} < 1$
$\uparrow x_0 > \sqrt{2}$.

Moreover, $x_n = g(x_{n-1})$. and ~~$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}$~~ ~~$\frac{1}{2}x_{n-1} + 1 < x_{n-1}$~~

$x_0 \in [\sqrt{2}, x_0]$

Thus, $x_n \to \sqrt{2}$ $(n \to \infty)$, which is the ~~x_0~~ unique fixed point of $g(x)$ on $[\sqrt{2}, x_0]$. $\square$.

b. Proof. $x_1 = \frac{1}{2}x_0 + \frac{1}{x_0}$

$$= \frac{\frac{1}{2}x_0^2 + 1}{x_0} = \frac{\frac{1}{2}(x_0^2 - 2\sqrt{2}x_0 + 2) + 1}{x_0} + \sqrt{2}$$

$$= \frac{\frac{1}{2}(x_0 - \sqrt{2})^2 + 1}{x_0} + \sqrt{2} > 0 + \sqrt{2} = \sqrt{2}. \quad \square.$$

$\uparrow 0 < x_0 < \sqrt{2}$, $(x_0 - \sqrt{2})^2 > 0$

c. Proof: We only need to show it holds for $\forall x_0 \in (0, \sqrt{2}]$
(since we have shown it holds for $\forall x_0 > \sqrt{2}$).

~~$g(x) = \frac{1}{2}x + \frac{1}{x}$. $g'(x) = \frac{1}{2} - \frac{1}{x^2}$. $g''(x) > 0$.~~

by b: $x_0 \in (0, \sqrt{2}) \Rightarrow x_1 > \sqrt{2}$.

And it holds naturally by viewing x_1 as a new "x_0".

$\Rightarrow$ it holds for $x_0 \in (0, \sqrt{2})$.

For $x_0 = \sqrt{2}$, it's the fixed-point itself!

$\Rightarrow x_n = \sqrt{2}$ for $n \ge 1$. $\Rightarrow$ ✓.

Thus, it holds for $x_0 \in (0, \sqrt{2}]$ $\square$.

26. Since g is cts. diff. on (c, d), $|g'(p)| < 1$. $\Rightarrow \exists (c', d') \subset (c, d)$, $\ni p$, s.t. $|g'(x)| < 1$ $\forall x \in (c', d')$

Proof: Claim: there exists $\delta > 0$, s.t. $[p-\delta, p+\delta] \subset (c', d')$ and $g([p-\delta, p+\delta]) \subset [p-\delta, p+\delta]$.

Proof of claim: Suppose rather.

WLOG, suppose $\exists\, s \in [p-\delta, p+\delta]$, s.t. $g(s) > p+\delta$.

$\Rightarrow |g(s) - g(p)| = |g(s) - p| > \delta$

but $|g(s) - g(p)| \underset{\uparrow \text{MVT.}}{=} |g'(\xi)| \cdot |s - p| < 1 \cdot |s-p| \le 1 \cdot \delta = \delta$.

$\Rightarrow$ a contradiction. ↯

NO.

DATE

Thus, now, by our claim, the δ in the claim is the "δ" that we are looking for here in our problem.

More detailedly, for this δ (that exists),

① $g \in C[p-\delta, p+\delta]$. $g([p-\delta, p+\delta]) \subset [p-\delta, p+\delta]$

② $g'(x)$ is defined on $(p-\delta, p+\delta)$ and $|g'(x)| < 1$. $\forall x \in (p-\delta, p+\delta) \subset (c', d')$.

③ $\forall p_0 \in [p-\delta, p+\delta]$, $p_n = g(p_{n-1})$.

$\Rightarrow$ The fixed-point iteration converges. $\square$