Week 3, Tuesday (Quiz 30 mins)

△ Background: in physics, experiments collect many discrete datas. we try to find the continuum formula to indicate the routine.

x_j : independent variable
$f(x_j)$: dependent variable
} ⇒ continuum approximation $p(x)$ of f

$\boxed{\text{Thm}}$ (Weierstrass approximation thm) $\forall f \in C[a,b]$, $\forall \varepsilon > 0$, $\exists$ a polynomial $p(x)$ s.t. $|f(x)-p(x)|<\varepsilon$, $\forall x \in C[a,b]$.

RK: if $f(x)$ is analytical in I, then $\forall x_0 \in I$

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \text{Taylor expansion.}$$

$$\therefore P_N(x) = \sum_{j=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \sim f(x).$$

RK: besides approximation, we need more: $\boxed{\text{interpolation}}$

i.e. try to find $p(x) \sim f(x)$ and $p(x_j) = f(x_j)$ $j = 0, 1, \cdots, n$.

△ suppose we have $\{x_j, f(x_j)\}_{j=0}^{n}$, we can determine a polynomial at most degree of n, i.e. find: $a_0, \cdots, a_n$, s.t.

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad p(x_j) = f(x_j)$$

$$\begin{cases} a_0 + a_1 x_0 + \cdots + a_n x_0^n = f_0 \\ a_0 + a_1 x_1 + \cdots + a_n x_1^n = f_1 \\ \qquad \vdots \\ a_0 + a_1 x_n + \cdots + a_n x_n^n = f_n \end{cases} \Rightarrow D\vec{a} = \vec{f}, \quad \vec{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ & \vdots & & \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}$$

Vandermonde

$$|D| = \prod_{\substack{i,j=0 \\ i>j}}^{n} (x_i - x_j),$$ inventible! if $x_i \neq x_j$

$\boxed{\text{Thm}}$ $\vec{a}$ exists and is unique, i.e., the interpolation polynomial $\exists!$.

RK: Solve $D\vec{a}=\vec{f}$ is doable but not practical: $\begin{cases}\text{large computations}\\ \text{large rounding error.}\end{cases}$

Solution: Choose a good basis!

$\boxed{\text{Def}}$ Lagrange interpolation polynomials: $x_0, x_1, \cdots, x_n$.

$$L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} := \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i}$$

Key property: (i) each $L_k(x)$ is a polynomial of degree n.

(ii) $L_k(x_j) = \begin{cases} 0 & k\neq j \\ 1 & k=j \end{cases} := \delta_{kj}.$

RK: $P(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$

(i) $P(x)$ is a polynomial of degree $\boxed{\text{at most}}$ n.

(ii) $P(x_j) = f(x_j)$ → interpolation. (elegant formula!)

$\boxed{\text{Thm}}$ if $x_0, x_1, \cdots, x_n$ are $n+1$ distinct numbers, then a unique polynomial $p(x)$ of degree at most n exists with $P(x_j) = f(x_j)$, $j=0,1,2,\cdots,n$, which is given by:
$P(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$.

Proof: it is trival to verify the existence.

if we assume there exists two polynomials $P(x)$ and $Q(x)$ of degree at most n satisfying $P(x_j)=Q(x_j)=f(x_j)$, $j=0,1,\cdots,n$.

Define: $R(x) = P(x)-Q(x)$, which is also a polynomial of degree at most n. But $R(x)$ has $n+1$ roots, $x_0, \cdots, x_n$.

$\therefore R(x)\equiv 0$, i.e., $P(x) = Q(x)$

Lemma: polynomials of degree n have at most n real roots.

Q: why degree at most n?

e.g. $f(x)=\frac{1}{x}$, $x_0=2$, $x_1=2.75$, $x_2=4$.

$$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-2.75)(x-4)}{0.75\cdot 2}=\frac{2}{3}(x-2.75)(x-4),\quad f(x_0)=\frac{1}{2}$$

$$L_1(x)=\frac{(x-2)(x-4)}{(2.75-2)(2.75-4)}=-\frac{16}{15}(x-2)(x-4)\qquad f(x_1)=\frac{4}{11}$$

$$L_2(x)=\frac{(x-2)(x-2.75)}{(4-2)(4-2.75)}=\frac{2}{5}(x-2)(x-2.75)\qquad f(x_2)=\frac{1}{4}.$$

$$\Rightarrow P(x)=\frac{1}{2}L_0(x)+\frac{4}{11}L_1(x)+\frac{1}{4}L_2(x)=\frac{1}{22}x^2-\frac{35}{88}x+\frac{49}{44}$$

use $P(x)\sim f(x)$, take $x=3$

$P(\frac{1}{3})=\frac{29}{88}\approx 0.329555$, $f(3)=\frac{1}{3}$. error $=0.0038$.

Q: error estimate.

Lemma: (Rolle's thm) $f\in C[a,b]$, f' $\exists$ on (a,b), $f(a)=f(b)$, then $\exists\, c\in(a,b)$ s.t. $f'(c)=0$.

(Generalization) $f\in C[a,b]$, $f^{(n)}$ $\exists$ on (a,b), $\exists$ $x_0,x_1,\cdots,x_n$ s.t. $f(x_0)=\cdots=f(x_n)$, then $\exists\,\xi\in(a,b)$, s.t. $f^{(n)}(\xi)=0$.

$\boxed{\text{Thm}}$ $f\in C^{n+1}[a,b]$, $x_0<x_1<\cdots<x_n$, $\forall\, x\in[a,b]$, $\exists\,\xi(x)$ s.t.

$$f(x)-p(x)=\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n),\ \text{where}$$

$P(x)$ is the interpolation polynomial.

proof: let: $g(t)=f(t)-p(t)-(f(x)-p(x))\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}$, then

$g(x)=0$ and $g(x_k)=0$, $k=0,1,\cdots,n$ (n+2 roots)

(general Rolle's thm) $\Rightarrow$ $\exists\,\xi(x)$ s.t. $g^{(n+1)}(\xi(x))=0$,

as $g^{(n+1)}(t)=f^{(n+1)}(t)-\underbrace{p^{(n+1)}(t)}_{0\ \text{(degree of n)}}-(f(x)-p(x))\frac{d^{n+1}}{dt^{n+1}}\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}$

$$= f^{(n+1)}(t) - (f(x)-p(x))\frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

$$\Rightarrow f^{(n+1)}(\xi(x)) - (f(x)-p(x))\frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)} = 0$$

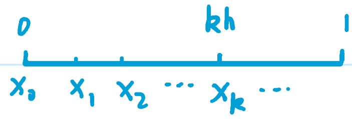
i.e. $f(x) = p(x) + \boxed{\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{i=0}^{n}(x-x_i)} := R(x)$.

e.g. $f(x) = \frac{1}{x}$, $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$.

$$R(x) = \frac{f^{(3)}(\xi(x))}{3!}(x-2)(x-2.75)(x-4) = -(\xi(x))^{-4}(x-2)(x-2.75)(x-4)$$

$\therefore \forall x \in [2,4]$, $|R(x)| \leq \frac{1}{16}\left|(x-2)(x-2.75)(x-4)\right| \leq \frac{9}{256} \approx 0.00586$

e.g. $f(x) = e^x$, table in $[0,1]$



$[0,1] = \bigcup_{k=1}^{n}[x_{k-1}, x_k]$, $\forall x \in (x_{k-1}, x_k)$

$$f(x) \approx f(x_{k-1})\frac{x-x_k}{x_{k-1}-x_k} + f(x_k)\frac{x-x_{k-1}}{x_k-x_{k-1}} := P_1(x)$$

Q: Find h s.t. err $\leq 10^{-6}$.

$$|f(x)-P_1(x)| = \left|\frac{f^{(2)}(\xi)}{2!}(x-x_{k-1})(x-x_k)\right| = \frac{e^{\xi}}{2}\left|(x-(k-1)h)(x-kh)\right|$$

$$\leq \frac{e}{2}\cdot\frac{h^2}{4} = \frac{e}{8}h^2 \leq 10^{-6} \Rightarrow h \leq 1.72\times 10^{-3}$$

HW3-1: Sec 3.1 1 b 2d 3b 4d 9 10 15 21 23

① Suppose we have (x_0, x_0^2), (x_1, x_1^2), (x_2, x_2^2) distinct.

write down the Lagrange polynomial agrees at x_0, x_1, x_2

② We have $n+1$ distinct points $x_0, x_1, \cdots, x_n$, $(n\geq 1)$, show that

$\sum_{k=0}^{n} L_k(x) \equiv 1$, where $L_k(x)$ is the Lagrange interpolating polynomials

show that $\forall$ integer $0\leq m\leq n$. $\sum_{k=0}^{n} x_k^m L_k(x) \equiv x^m$