HW1

September 11, 2025

Exercise 1 Solve $\partial_t u + \partial_x u + u = e^{x+2t}$ with initial condition $u(0,x) = 0$.

Exercise 2 Consider the following initial value problem for Burgers equation

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(0,x) = \phi(x) = \begin{cases} 1, & x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & x > 1. \end{cases} \end{cases}$$

1. Find the largest time t_s such that all characteristics do not intersect.
2. Find an expression of $u(t,x)$ for $t < t_s$.

Exercise 3 Use the method of characteristics to solve the following PDEs.

1. $x_1 \partial_{x_1} u + x_2 u_{x_2} = 2u$, $u(x_1, 1) = g(x_1)$.
2. $u\partial_{x_1} u + \partial_{x_2} u = 1$, $u(x_1, x_1) = \frac{1}{2}x_1$.

Exercise 4 Suppose that u is smooth and solves $u_t - \Delta u = 0$ in $(0,\infty) \times \mathbb{R}^d$.

1. Show that $u_\lambda(t,x) := u(\lambda^2 t, \lambda x)$ solves the heat equation for every $\lambda \in \mathbb{R}$.
2. Use the above to derive that $v(t,x) := x \cdot \nabla u(t,x) + 2tu_t(t,x)$ also solves the heat equation. *Hint: take derivative in λ.*

Ex 1. 令 $U(t) := u(t, \eta(t))$

故特征线 s.t. $\dot{\eta}(t) = 1$，即 $\eta(t) = t + c$

对应的ODE为 $\dot{U}(t) = e^{\eta(t)+2t} - U(t) = e^{3t+c} - U(t)$

$$U(0) = u(0, \eta(0)) = u(0, c) = 0$$

求解得 $U(t) = \frac{1}{4}(e^{3t+c} - e^{c-t}) = u(t, t+c)$

取 $x = c+t$ 得 $c = x - t$

从而解为 $u(t,x) = \frac{1}{4}(e^{x+2t} - e^{x-2t})$

Ex 2. Recall: 特征线 $\eta(t) = \phi(c)t + c$

① 设2条特征线 $\phi(x_1)t + x_1$ 与 $\phi(x_2)t + x_2$ 在 t_0 相交（∵线性 ∴相交点只有一个）

即 $\phi(x_1)t_0 + x_1 = \phi(x_2)t_0 + x_2 \Rightarrow t_0 = -\frac{x_1 - x_2}{\phi(x_1) - \phi(x_2)} > 1$ 或不存在

故 $t_s = 1$.（画图也可得）

② 取 $x = \eta(t)$，注意到我们只需对 $\forall t < t_s = 1$ 讨论.

故得 $c = \begin{cases} x-t, & c \le 0 \\ \frac{x-t}{1-t}, & 0 < c \le 1 \\ x, & c > 1 \end{cases}$，代回 $\phi(c)$ 得 $u(t,x) = \begin{cases} 1, & x \le t \\ \frac{1-x}{1-t}, & t < x \le 1 \\ 0, & x > 1. \end{cases}$

Ex 3. ① 令 $U(s) = u(\eta(s), w(s))$，$\eta(s) = x_1$，$w(s) = x_2$

故特征线 s.t. $\begin{cases} \dot{\eta}(s) = \eta(s) \text{ with } \eta(0) = c \\ \dot{w}(s) = w(s) \text{ with } w(0) = 1 \end{cases}$，即 $\begin{cases} \eta(s) = c \cdot e^s \\ w(s) = e^s \end{cases} \Rightarrow c = \frac{x_1}{x_2}$

对应的ODE为 $\dot{U}(s) = 2U(s)$ with $U(0) = g(c)$

求解得 $U(s) = g(c) \cdot e^{2s}$

从而解为 $u(x_1, x_2) = g\left(\frac{x_1}{x_2}\right) x_2^2 \quad (x_2 \neq 0)$

② 令 $U(s) = u(\eta(s), w(s))$，$\eta(s) = x_1$，$w(s) = x_2$

故特征线 s.t. $\begin{cases} \dot{\eta}(s) = u \text{ with } \eta(0) = c \\ \dot{w}(s) = 1 \text{ with } w(0) = c \Rightarrow w(s) = c + s \end{cases}$

对应的ODE为 $\dot{U}(s) = 1$ with $U(0) = \frac{1}{2}c$

求解得 $U(s) = \frac{1}{2}c + s$，进而 $\dot{\eta}(s) = \frac{1}{2}c + s \xrightarrow{\eta(0)=c} \eta(s) = \frac{1}{2}s^2 + \frac{1}{2}cs + c$

从而由 $J(s)$, $w(s)$ 表达式得 $u(x_1, x_2) = \dfrac{x_2^2 - 4x_2 + 2x_1}{2(x_2 - 2)}$, $(x_2 \neq 2)$

Ex 4. ① 证: 令 $y = \lambda^2 t$, $z = \lambda x$, 则 $u = u(y, z)$

故 $\partial_t u_\lambda - \Delta u_\lambda = \lambda^2 \partial_y u_\lambda - \lambda^2 \partial_z u_\lambda = 0$, 即 u_λ s.t. heat equation.

② 证: 对上问中 $u_\lambda = u(\lambda^2 t, \lambda x) = u(y, z)$ 关于 λ 求导得

$\partial_\lambda u_\lambda = 2\lambda t \cdot \partial_y u_\lambda + x \cdot \nabla_z u$

取 $\lambda = 1$ 得 $\partial_\lambda u_\lambda \big|_{\lambda=1} = 2t \cdot \partial_y u_\lambda + x \cdot \nabla_z u = v$

因 $u_\lambda = u(\lambda, t, x)$ 光滑 on $\mathbb{R} \times (0, \infty) \times \mathbb{R}^d$

故 $0 = \partial_\lambda (\partial_t u_\lambda - \Delta u_\lambda) = \partial_t (\partial_\lambda u_\lambda) - \Delta (\partial_\lambda u_\lambda)$

$= \partial_t v - \Delta v$, 即 v s.t. heat equation.