Answers for Selected Exercises

Exercise Set 1.1 (Page 11)

1. For each part, $f \in C[a, b]$ on the given interval. Since $f(a)$ and $f(b)$ are of opposite sign, the Intermediate Value Theorem implies that a number c exists with $f(c) = 0$.

3. **a.** $[0, 1]$ contains a solution of $x - 2^{-x} = 0$
   b. $[-1, 0]$ contains a solution of $2x\cos(2x) - (x+1)^2 = 0$
   c. $[0, 1]$ contains a solution of $3x - e^x = 0$
   d. $[-\frac{3}{2}, -\frac{1}{2}]$ contains a solution of $x + 1 - 2\sin(\pi x) = 0$

5. The maximum value for $|f(x)|$ is given below.
   a. 0.4620981 **b.** 0.8 **c.** 5.164000 **d.** 1.582572

7. For each part, $f \in C[a, b]$, f' exists on (a, b) and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a, b) with $f'(c) = 0$. For part (d), we can use $[a, b] = [-1, 0]$ or $[a, b] = [0, 2]$.

9. **a.** $P_2(x) = 0$
   b. $R_2(0.5) = 0.125$; actual error $= 0.125$
   c. $P_2(x) = 1 + 3(x-1) + 3(x-1)^2$
   d. $R_2(0.5) = -0.125$; actual error $= -0.125$

11. Since

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^{\xi}(\sin\xi + \cos\xi)}{6}x^3$$

for some ξ between x and 0, we have the following:

   a. $P_2(0.5) = 1.5$ and $|f(0.5) - P_2(0.5)| \le 0.0932$;
   b. $|f(x) - P_2(x)| \le 1.252$;
   c. $\int_0^1 f(x)\,dx \approx 1.5$;
   d. $|\int_0^1 f(x)\,dx - \int_0^1 P_2(x)\,dx| \le \int_0^1 |R_2(x)|\,dx \le 0.313$, and the actual error is 0.122.

13. $P_3(x) = (x-1)^2 - \frac{1}{2}(x-1)^3$
   a. $P_3(0.5) = 0.312500$, $f(0.5) = 0.346574$. An error bound is $0.291\overline{6}$, and the actual error is 0.034074.
   b. $|f(x) - P_3(x)| \le 0.291\overline{6}$ on $[0.5, 1.5]$
   c. $\int_{0.5}^{1.5} P_3(x)\,dx = 0.08\overline{3}$, $\int_{0.5}^{1.5}(x-1)\ln x\,dx = 0.088020$
   d. An error bound is $0.058\overline{3}$, and the actual error is 4.687×10^{-3}.

15. $P_4(x) = x + x^3$
   a. $|f(x) - P_4(x)| \le 0.012405$
   b. $\int_0^{0.4} P_4(x)\,dx = 0.0864$, $\int_0^{0.4} xe^{x^2}\,dx = 0.086755$
   c. 8.27×10^{-4}
   d. $P_4'(0.2) = 1.12$, $f'(0.2) = 1.124076$. The actual error is 4.076×10^{-3}.

17. Since $42^\circ = 7\pi/30$ radians, use $x_0 = \pi/4$. Then

$$\left|R_n\left(\frac{7\pi}{30}\right)\right| \le \frac{\left(\frac{\pi}{4} - \frac{7\pi}{30}\right)^{n+1}}{(n+1)!} < \frac{(0.053)^{n+1}}{(n+1)!}.$$

For $|R_n(\frac{7\pi}{30})| < 10^{-6}$, it suffices to take $n = 3$. To 7 digits, $\cos 42^\circ = 0.7431448$ and $P_3(42^\circ) = P_3(\frac{7\pi}{30}) = 0.7431446$, so the actual error is 2×10^{-7}.

19. $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}x^k$, $n \ge 7$

21. A bound for the maximum error is 0.0026.

23. Since $R_2(1) = \frac{1}{6}e^{\xi}$, for some ξ in $(0, 1)$, we have $|E - R_2(1)| = \frac{1}{6}|1 - e^{\xi}| \le \frac{1}{6}(e - 1)$.

25. **a.** $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$. The shapes of P_n and f are the same at x_0.

b. $P_2(x) = 3 + 4(x-1) + 3(x-1)^2$.

27. First, observe that for $f(x) = x - \sin x$, we have $f'(x) = 1 - \cos x \geq 0$ because $-1 \leq \cos x \leq 1$ for all values of x.

a. The observation implies that $f(x)$ is nondecreasing for all values of x, and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence, for $x \geq 0$, we have $x \geq \sin x$, and $|\sin x| = \sin x \leq x = |x|$.

b. When $x < 0$, we have $-x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \leq (-x)$ implies that $|\sin x| = -\sin x \leq -x = |x|$.

As a consequence, for all real numbers x, we have $|\sin x| \leq |x|$.

29. **a.** The number $\frac{1}{2}(f(x_1) + f(x_2))$ is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of f. By the Intermediate Value Theorem 1.11, there exists a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

b. Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \leq f(x_1) \leq M$ and $m \leq f(x_2) \leq M$, so

$$c_1 m \leq c_1 f(x_1) \leq c_1 M \quad \text{and} \quad c_2 m \leq c_2 f(x_2) \leq c_2 M.$$

Thus,

$$(c_1 + c_2)m \leq c_1 f(x_1) + c_2 f(x_2) \leq (c_1 + c_2)M$$

and

$$m \leq \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \leq M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints x_1 and x_2, there exists a number ξ between x_1 and x_2 for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Let $f(x) = x^2 + 1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then for all values of x,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

Exercise Set 1.2 (Page 25)

1.	Absolute Error	Relative Error
a.	0.001264	4.025×10^{-4}
b.	7.346×10^{-6}	2.338×10^{-6}
c.	2.818×10^{-4}	1.037×10^{-4}
d.	2.136×10^{-4}	1.510×10^{-4}

3. The largest intervals are

a. (149.85,150.15) **b.** (899.1, 900.9) **c.** (1498.5, 1501.5) **d.** (89.91,90.09)

5. The calculations and their errors are:

a. (i) 17/15 (ii) 1.13 (iii) 1.13 (iv) both 3×10^{-3}

b. (i) 4/15 (ii) 0.266 (iii) 0.266 (iv) both 2.5×10^{-3}

c. (i) 139/660 (ii) 0.211 (iii) 0.210 (iv) 2×10^{-3}, 3×10^{-3}

d. (i) 301/660 (ii) 0.455 (iii) 0.456 (iv) 2×10^{-3}, 1×10^{-4}

7.

	Approximation	Absolute Error	Relative Error
a.	1.80	0.154	0.0786
b.	−15.1	0.0546	3.60×10^{-3}
c.	0.286	2.86×10^{-4}	10^{-3}
d.	23.9	0.058	2.42×10^{-3}

9.

	Approximation	Absolute Error	Relative Error
a.	3.55	1.60	0.817
b.	−15.2	0.054	0.0029
c.	0.284	0.00171	0.00600
d.	23.8	0.158	0.659×10^{-2}

11.

	Approximation	Absolute Error	Relative Error
a.	3.14557613	3.983×10^{-3}	1.268×10^{-3}
b.	3.14162103	2.838×10^{-5}	9.032×10^{-6}

13. a. $\lim_{x\to 0} \frac{x\cos x - \sin x}{x - \sin x} = \lim_{x\to 0} \frac{-x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{-\sin x - x\cos x}{\sin x} = \lim_{x\to 0} \frac{-2\cos x + x\sin x}{\cos x} = -2$

b. −1.941

c. $\dfrac{x(1-\frac{1}{2}x^2) - (x - \frac{1}{6}x^3)}{x - (x - \frac{1}{6}x^3)} = -2$

d. The relative error in part (b) is 0.029. The relative error in part (c) is 0.00050.

15.

	x_1	Absolute Error	Relative Error	x_2	Absolute Error	Relative Error
a.	92.26	0.01542	1.672×10^{-4}	0.005419	6.273×10^{-7}	1.157×10^{-4}
b.	0.005421	1.264×10^{-6}	2.333×10^{-4}	−92.26	4.580×10^{-3}	4.965×10^{-5}
c.	10.98	6.875×10^{-3}	6.257×10^{-4}	0.001149	7.566×10^{-8}	6.584×10^{-5}
d.	−0.001149	7.566×10^{-8}	6.584×10^{-5}	−10.98	6.875×10^{-3}	6.257×10^{-4}

17.

	Approximation for x_1	Absolute Error	Relative Error
a.	92.24	0.004580	4.965×10^{-5}
b.	0.005417	2.736×10^{-6}	5.048×10^{-4}
c.	10.98	6.875×10^{-3}	6.257×10^{-4}
d.	−0.001149	7.566×10^{-8}	6.584×10^{-5}

	Approximation for x_2	Absolute Error	Relative Error
a.	0.005418	2.373×10^{-6}	4.377×10^{-4}
b.	−92.25	5.420×10^{-3}	5.875×10^{-5}
c.	0.001149	7.566×10^{-8}	6.584×10^{-5}
d.	−10.98	6.875×10^{-3}	6.257×10^{-4}

19. The machine numbers are equivalent to

a. 3224 **b.** −3224 **c.** 1.32421875

d. 1.32421875000000002220446049250313080847263336181640625

21. b. The first formula gives −0.00658, and the second formula gives −0.0100. The true three-digit value is −0.0116.

23. The approximate solutions to the systems are

a. $x = 2.451$, $y = -1.635$ **b.** $x = 507.7$, $y = 82.00$

25. a. In nested form, we have $f(x) = (((1.01e^x - 4.62)e^x - 3.11)e^x + 12.2)e^x - 1.99$.

b. −6.79 **c.** −7.07

27. a. $m = 17$

b.

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k-1)(m-k)!}{k!(m-k)!}$$

$$= \left(\frac{m}{k}\right)\left(\frac{m-1}{k-1}\right)\cdots\left(\frac{m-k-1}{1}\right)$$

c. $m = 181707$ **d.** 2,597,000; actual error 1960; relative error 7.541×10^{-4}

29. a. The actual error is $|f'(\xi)\epsilon|$, and the relative error is $|f'(\xi)\epsilon| \cdot |f(x_0)|^{-1}$, where the number ξ is between x_0 and $x_0 + \epsilon$.
b. (i) 1.4×10^{-5}; 5.1×10^{-6} (ii) 2.7×10^{-6}; 3.2×10^{-6} **c.** (i) 1.2; 5.1×10^{-5} (ii) 4.2×10^{-5}; 7.8×10^{-5}

Exercise Set 1.3 (Page 35)

1. a. The approximate sums are 1.53 and 1.54, respectively. The actual value is 1.549. Significant round-off error occurs earlier with the first method.
b. The approximate sums are 1.16 and 1.19, respectively. The actual value is 1.197. Significant round-off error occurs earlier with the first method.

3. a. 2000 terms **b.** 20,000,000,000 terms

5. 3 terms

7. The rates of convergence are:
a. $O(h^2)$ **b.** $O(h)$ **c.** $O(h^2)$ **d.** $O(h)$

9. a. If $F(h) = L + O(h^p)$, there is a constant $k > 0$ such that

$$|F(h) - L| \leq kh^p,$$

for sufficiently small $h > 0$. If $0 < q < p$ and $0 < h < 1$, then $h^q > h^p$. Thus, $kh^p < kh^q$, so

$$|F(h) - L| \leq kh^q \quad \text{and} \quad F(h) = L + O(h^q).$$

b. For various powers of h, we have the entries in the following table.

h	h^2	h^3	h^4
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	10^{-8}
0.001	10^{-6}	10^{-9}	10^{-12}

The most rapid convergence rate is $O\left(h^4\right)$.

11. Since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$

we have

$$x = 1 + \frac{1}{x}, \quad \text{so} \quad x^2 - x - 1 = 0.$$

The quadratic formula implies that

$$x = \frac{1}{2}\left(1 + \sqrt{5}\right).$$

This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

13. $SUM = \sum_{i=1}^{N} x_i$. This saves one step since initialization is $SUM = x_1$ instead of $SUM = 0$. Problems may occur if $N = 0$.

15. (a) $n(n+1)/2$ multiplications; $(n+2)(n-1)/2$ additions.
(b) $\displaystyle\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j\right)$ requires n multiplications; $(n+2)(n-1)/2$ additions.

Exercise Set 2.1 (Page 53)

1. $p_3 = 0.625$

3. The Bisection method gives:
a. $p_7 = 0.5859$ **b.** $p_8 = 3.002$ **c.** $p_7 = 3.419$

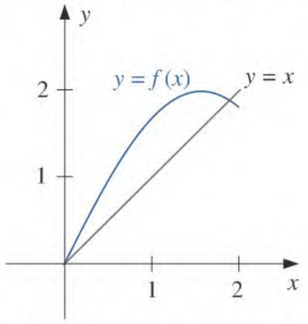
5. The Bisection method gives:

a. $p_{17} = 0.641182$ **b.** $p_{17} = 0.257530$

c. For the interval $[-3, -2]$, we have $p_{17} = -2.191307$, and for the interval $[-1, 0]$, we have $p_{17} = -0.798164$.

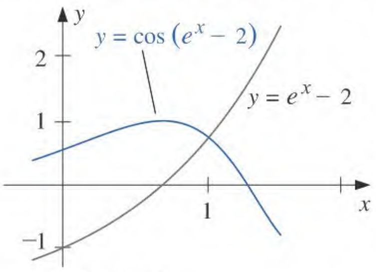
d. For the interval $[0.2, 0.3]$, we have $p_{14} = 0.297528$, and for the interval $[1.2, 1.3]$, we have $p_{14} = 1.256622$.

7. a.



b. Using $[1.5, 2]$ from part (a) gives $p_{16} = 1.89550018$.

9. a.



b. $p_{17} = 1.00762177$

11. a. 2 **b.** -2 **c.** -1 **d.** 1

13. The cubed root of 25 is approximately $p_{14} = 2.92401$, using $[2, 3]$.

15. The depth of the water is 0.838 ft.

17. A bound is $n \geq 14$, and $p_{14} = 1.32477$.

19. Since $\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty} 1/n = 0$, the difference in the terms goes to zero. However, p_n is the nth term of the divergent harmonic series, so $\lim_{n\to\infty} p_n = \infty$.

21. Since $-1 < a < 0$ and $2 < b < 3$, we have $1 < a + b < 3$ or $1/2 < 1/2(a + b) < 3/2$ in all cases. Further,

$$f(x) < 0, \quad \text{for } -1 < x < 0 \quad \text{and} \quad 1 < x < 2;$$

$$f(x) > 0, \quad \text{for } 0 < x < 1 \quad \text{and} \quad 2 < x < 3.$$

Thus, $a_1 = a$, $f(a_1) < 0$, $b_1 = b$, and $f(b_1) > 0$.

a. Since $a + b < 2$, we have $p_1 = \frac{a+b}{2}$ and $1/2 < p_1 < 1$. Thus, $f(p_1) > 0$. Hence, $a_2 = a_1 = a$ and $b_2 = p_1$. The only zero of f in $[a_2, b_2]$ is $p = 0$, so the convergence will be to 0.

b. Since $a + b > 2$, we have $p_1 = \frac{a+b}{2}$ and $1 < p_1 < 3/2$. Thus, $f(p_1) < 0$. Hence, $a_2 = p_1$ and $b_2 = b_1 = b$. The only zero of f in $[a_2, b_2]$ is $p = 2$, so the convergence will be to 2.

c. Since $a + b = 2$, we have $p_1 = \frac{a+b}{2} = 1$ and $f(p_1) = 0$. Thus, a zero of f has been found on the first iteration. The convergence is to $p = 1$.

Exercise Set 2.2 (Page 63)

1. For the value of x under consideration, we have

a. $x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$

b. $x = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$

c. $x = \left(\dfrac{x+3}{x^2+2}\right)^{1/2} \Leftrightarrow x^2(x^2+2) = x+3 \Leftrightarrow f(x) = 0$

d. $x = \dfrac{3x^4+2x^2+3}{4x^3+4x-1} \Leftrightarrow 4x^4+4x^2-x = 3x^4+2x^2+3 \Leftrightarrow f(x) = 0$

3. **a.** Solve for $2x$ then divide by 2. $p_1 = 0.5625$, $p_2 = 0.58898926$, $p_3 = 0.60216264$, $p_4 = 0.60917204$

b. Solve for x^3, divide by x^2. $p_1 = 0$, p_2 undefined

c. Solve for x^3, divide by x, then take positive square root. $p_1 = 0$, p_2 undefined

d. Solve for x^3, then take negative of the cubed root. $p_1 = 0$, $p_2 = -1$, $p_3 = -1.4422496$, $p_4 = -1.57197274$. Parts (a) and (b) seem promising.

5. The order in descending speed of convergence is (b), (d), (a). The sequence in (c) does not converge.

7. With $g(x) = (3x^2+3)^{1/4}$ and $p_0 = 1$, $p_6 = 1.94332$ is accurate to within 0.01.

9. Since $g'(x) = \frac{1}{4}\cos\frac{x}{2}$, g is continuous and g' exists on $[0, 2\pi]$. Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi \le g(x) = \le g(\pi) = \pi + \frac{1}{2}$ and $|g'(x)| \le \frac{1}{4}$, for $0 \le x \le 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0, 2\pi]$. With $k = \frac{1}{4}$ and $p_0 = \pi$, we have $p_1 = \pi + \frac{1}{2}$. Corollary 2.5 implies that

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0| = \frac{2}{3}\left(\frac{1}{4}\right)^n.$$

For the bound to be less than 0.1, we need $n \ge 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.

11. For $p_0 = 1.0$ and $g(x) = 0.5(x + \frac{3}{x})$, we have $\sqrt{3} \approx p_4 = 1.73205$.

13. **a.** With $[0, 1]$ and $p_0 = 0$, we have $p_9 = 0.257531$.

b. With $[2.5, 3.0]$ and $p_0 = 2.5$, we have $p_{17} = 2.690650$.

c. With $[0.25, 1]$ and $p_0 = 0.25$, we have $p_{14} = 0.909999$.

d. With $[0.3, 0.7]$ and $p_0 = 0.3$, we have $p_{39} = 0.469625$.

e. With $[0.3, 0.6]$ and $p_0 = 0.3$, we have $p_{48} = 0.448059$.

f. With $[0, 1]$ and $p_0 = 0$, we have $p_6 = 0.704812$.

15. For $g(x) = (2x^2 - 10\cos x)/(3x)$, we have the following:

$$p_0 = 3 \Rightarrow p_8 = 3.16193; \quad p_0 = -3 \Rightarrow p_8 = -3.16193.$$

For $g(x) = \arccos(-0.1x^2)$, we have the following:

$$p_0 = 1 \Rightarrow p_{11} = 1.96882; \quad p_0 = -1 \Rightarrow p_{11} = -1.96882.$$

17. With $g(x) = \frac{1}{\pi}\arcsin\left(-\frac{x}{2}\right) + 2$, we have $p_5 = 1.683855$.

19. Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \ge |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

21. One of many examples is $g(x) = \sqrt{2x - 1}$ on $\left[\frac{1}{2}, 1\right]$.

23. **a.** Suppose that $x_0 > \sqrt{2}$. Then

$$x_1 - \sqrt{2} = g(x_0) - g\left(\sqrt{2}\right) = g'(\xi)\left(x_0 - \sqrt{2}\right),$$

where $\sqrt{2} < \xi < x$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2}$$

and $\sqrt{2} < x_1 < x_0$. By an inductive argument,

$$\sqrt{2} < x_{m+1} < x_m < \ldots < x_0.$$

Thus, $\{x_m\}$ is a decreasing sequence which has a lower bound and must converge.

Suppose $p = \lim_{m\to\infty} x_m$. Then

$$p = \lim_{m\to\infty}\left(\frac{x_{m-1}}{2} + \frac{1}{x_{m-1}}\right) = \frac{p}{2} + \frac{1}{p}. \quad \text{Thus,} \quad p = \frac{p}{2} + \frac{1}{p},$$

which implies that $p = \pm\sqrt{2}$. Since $x_m > \sqrt{2}$ for all m, we have $\lim_{m\to\infty} x_m = \sqrt{2}$.

b. We have

$$0 < (x_0 - \sqrt{2})^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

so $2x_0\sqrt{2} < x_0^2 + 2$ and $\sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1$.

c. Case 1: $0 < x_0 < \sqrt{2}$, which implies that $\sqrt{2} < x_1$ by part (b). Thus,

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \ldots < x_1 \quad \text{and} \quad \lim_{m\to\infty} x_m = \sqrt{2}.$$

Case 2: $x_0 = \sqrt{2}$, which implies that $x_m = \sqrt{2}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{2}$.
Case 3: $x_0 > \sqrt{2}$, which by part (a) implies that $\lim_{m\to\infty} x_m = \sqrt{2}$.

25. Replace the second sentence in the proof with: "Since g satisfies a Lipschitz condition on $[a, b]$ with a Lipschitz constant $L < 1$, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| \leq L|p_{n-1} - p|."$$

The rest of the proof is the same, with k replaced by L.

Exercise Set 2.3 (Page 74)

1. $p_2 = 2.60714$

3. **a.** 2.45454 **b.** 2.44444 **c.** Part (b) is better.

5. **a.** For $p_0 = 2$, we have $p_5 = 2.69065$. **b.** For $p_0 = -3$, we have $p_3 = -2.87939$.
c. For $p_0 = 0$, we have $p_4 = 0.73909$. **d.** For $p_0 = 0$, we have $p_3 = 0.96434$.

7. Using the endpoints of the intervals as p_0 and p_1, we have:
a. $p_{11} = 2.69065$ **b.** $p_7 = -2.87939$ **c.** $p_6 = 0.73909$ **d.** $p_5 = 0.96433$

9. Using the endpoints of the intervals as p_0 and p_1, we have:
a. $p_{16} = 2.69060$ **b.** $p_6 = -2.87938$ **c.** $p_7 = 0.73908$ **d.** $p_6 = 0.96433$

11. **a.** Newton's method with $p_0 = 1.5$ gives $p_3 = 1.51213455$.
The Secant method with $p_0 = 1$ and $p_1 = 2$ gives $p_{10} = 1.51213455$.
The Method of False Position with $p_0 = 1$ and $p_1 = 2$ gives $p_{17} = 1.51212954$.
b. Newton's method with $p_0 = 0.5$ gives $p_5 = 0.976773017$.
The Secant method with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 10.976773017$.
The Method of False Position with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 0.976772976$.

13. **a.** For $p_0 = -1$ and $p_1 = 0$, we have $p_{17} = -0.04065850$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_9 = 0.9623984$.
b. For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$.
c. For $p_0 = -0.5$, we have $p_5 = -0.04065929$, and for $p_0 = 0.5$, we have $p_{21} = 0.9623989$.

15. **a.** $p_0 = -10$, $p_{11} = -4.30624527$ **b.** $p_0 = -5$, $p_5 = -4.30624527$
c. $p_0 = -3$, $p_5 = 0.824498585$ **d.** $p_0 = -1$, $p_4 = -0.824498585$
e. $p_0 = 0$, p_1 cannot be computed since $f'(0) = 0$ **f.** $p_0 = 1$, $p_4 = 0.824498585$
g. $p_0 = 3$, $p_5 = -0.824498585$ **h.** $p_0 = 5$, $p_5 = 4.30624527$
i. $p_0 = 10$, $p_{11} = 4.30624527$

17. For $f(x) = \ln(x^2 + 1) - e^{0.4x}\cos\pi x$, we have the following roots.
a. For $p_0 = -0.5$, $p_3 = -0.4341431$.
b. For $p_0 = 0.5$, $p_3 = 0.4506567$.
For $p_0 = 1.5$, $p_3 = 1.7447381$.
For $p_0 = 2.5$, $p_5 = 2.2383198$.
For $p_0 = 3.5$, $p_4 = 3.7090412$.
c. The initial approximation $n - 0.5$ is quite reasonable.
d. For $p_0 = 24.5$, $p_2 = 24.4998870$.

19. For $p_0 = 1$, $p_5 = 0.589755$. The point has the coordinates $(0.589755, 0.347811)$.

21. The two numbers are approximately 6.512849 and 13.487151.

23. The borrower can afford to pay at most 8.10%.

25. We have $P_L = 265816$, $c = -0.75658125$, and $k = 0.045017502$. The 1980 population is $P(30) = 222{,}248{,}320$, and the 2010 population is $P(60) = 252{,}967{,}030$.

27. Using $p_0 = 0.5$ and $p_1 = 0.9$, the Secant method gives $p_5 = 0.842$.

29. **a.** We have, approximately,

$$A = 17.74, \quad B = 87.21, \quad C = 9.66, \quad \text{and} \quad E = 47.47$$

With these values, we have

$$A \sin\alpha \cos\alpha + B \sin^2\alpha - C \cos\alpha - E \sin\alpha = 0.02.$$

b. Newton's method gives $\alpha \approx 33.2^\circ$.

31. The equation of the tangent line is

$$y - f(p_{n-1}) = f'(p_{n-1})(x - p_{n-1}).$$

To complete this problem, set $y = 0$ and solve for $x = p_n$.

Exercise Set 2.4 (Page 84)

1. **a.** For $p_0 = 0.5$, we have $p_{13} = 0.567135$. **b.** For $p_0 = -1.5$, we have $p_{23} = -1.414325$.
c. For $p_0 = 0.5$, we have $p_{22} = 0.641166$. **d.** For $p_0 = -0.5$, we have $p_{23} = -0.183274$.

3. Modified Newton's method in Equation (2.11) gives the following:
a. For $p_0 = 0.5$, we have $p_3 = 0.567143$. **b.** For $p_0 = -1.5$, we have $p_2 = -1.414158$.
c. For $p_0 = 0.5$, we have $p_3 = 0.641274$. **d.** For $p_0 = -0.5$, we have $p_5 = -0.183319$.

5. Newton's method with $p_0 = -0.5$ gives $p_{13} = -0.169607$. Modified Newton's method in Eq. (2.11) with $p_0 = -0.5$ gives $p_{11} = -0.169607$.

7. **a.** For $k > 0$,

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = 1,$$

so the convergence is linear.

b. We need to have $N > 10^{m/k}$.

9. Typical examples are
a. $p_n = 10^{-3^n}$ **b.** $p_n = 10^{-\alpha^n}$

11. This follows from the fact that $\lim_{n\to\infty} \frac{\left|\frac{b-a}{2^{n+1}}\right|}{\left|\frac{b-a}{2^n}\right|} = \frac{1}{2}$.

13. If $\frac{|p_{n+1}-p|}{|p_n-p|^3} = 0.75$ and $|p_0 - p| = 0.5$, then $|p_n - p| = (0.75)^{(3^n-1)/2}|p_0 - p|^{3^n}$.
To have $|p_n - p| \le 10^{-8}$ requires that $n \ge 3$.

Exercise Set 2.5 (Page 89)

1. The results are listed in the following table.

	a	b	c	d
$\hat{p}_0$	0.258684	0.907859	0.548101	0.731385
$\hat{p}_1$	0.257613	0.909568	0.547915	0.736087
$\hat{p}_2$	0.257536	0.909917	0.547847	0.737653
$\hat{p}_3$	0.257531	0.909989	0.547823	0.738469
$\hat{p}_4$	0.257530	0.910004	0.547814	0.738798
$\hat{p}_5$	0.257530	0.910007	0.547810	0.738958

3. $p_0^{(1)} = 0.826427$

5. $p_1^{(0)} = 1.5$

7. For $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0^{(0)} = 1$, we have $p_0^{(3)} = 1.32472$.

9. For $g(x) = 0.5(x + \frac{3}{x})$ and $p_0^{(0)} = 0.5$, we have $p_0^{(4)} = 1.73205$.

11. **a.** For $g(x) = \left(2 - e^x + x^2\right)/3$ and $p_0^{(0)} = 0$, we have $p_0^{(3)} = 0.257530$.

b. For $g(x) = 0.5(\sin x + \cos x)$ and $p_0^{(0)} = 0$, we have $p_0^{(4)} = 0.704812$.

c. With $p_0^{(0)} = 0.25$, $p_0^{(4)} = 0.910007572$.

d. With $p_0^{(0)} = 0.3$, $p_0^{(4)} = 0.469621923$.

13. Aitken's Δ^2 method gives:

a. $\hat{p}_{10} = 0.0\overline{45}$

b. $\hat{p}_2 = 0.0363$

15. We have

$$\frac{|p_{n+1} - p_n|}{|p_n - p|} = \frac{|p_{n+1} - p + p - p_n|}{|p_n - p|} = \left|\frac{p_{n+1} - p}{p_n - p} - 1\right|,$$

so

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = \lim_{n\to\infty} \left|\frac{p_{n+1} - p}{p_n - p} - 1\right| = 1.$$

17. **a.** *Hint*: First show that $p_n - p = -\frac{1}{(n+1)!}e^{\xi}x^{n+1}$, where ξ is between 0 and 1.

b.

n	p_n	$\hat{p}_n$
0	1	3
1	2	2.75
2	2.5	$2.7\overline{2}$
3	$2.\overline{6}$	2.71875
4	$2.708\overline{3}$	$2.718\overline{3}$
5	$2.71\overline{6}$	2.7182870
6	$2.7180\overline{5}$	2.7182823
7	2.7182539	2.7182818
8	2.7182787	2.7182818
9	2.7182815	
10	2.7182818	

Exercise Set 2.6 (Page 99)

1. **a.** For $p_0 = 1$, we have $p_{22} = 2.69065$.

b. For $p_0 = 1$, we have $p_5 = 0.53209$; for $p_0 = -1$, we have $p_3 = -0.65270$; and for $p_0 = -3$, we have $p_3 = -2.87939$.

c. For $p_0 = 1$, we have $p_5 = 1.32472$.

d. For $p_0 = 1$, we have $p_4 = 1.12412$; and for $p_0 = 0$, we have $p_8 = -0.87605$.

e. For $p_0 = 0$, we have $p_6 = -0.47006$; for $p_0 = -1$, we have $p_4 = -0.88533$; and for $p_0 = -3$, we have $p_4 = -2.64561$.

f. For $p_0 = 0$, we have $p_{10} = 1.49819$.

3. The following table lists the initial approximation and the roots.

	p_0	p_1	p_2	Approximate Roots	Complex Conjugate Roots
a	-1	0	1	$p_7 = -0.34532 - 1.31873i$	$-0.34532 + 1.31873i$
	0	1	2	$p_6 = 2.69065$	
b	0	1	2	$p_6 = 0.53209$	
	1	2	3	$p_9 = -0.65270$	
	-2	-3	-2.5	$p_4 = -2.87939$	
c	0	1	2	$p_5 = 1.32472$	
	-2	-1	0	$p_7 = -0.66236 - 0.56228i$	$-0.66236 + 0.56228i$
d	0	1	2	$p_5 = 1.12412$	
	2	3	4	$p_{12} = -0.12403 + 1.74096i$	$-0.12403 - 1.74096i$
	-2	0	-1	$p_5 = -0.87605$	
e	0	1	2	$p_{10} = -0.88533$	
	1	0	-0.5	$p_5 = -0.47006$	
	-1	-2	-3	$p_5 = -2.64561$	
f	0	1	2	$p_6 = 1.49819$	
	-1	-2	-3	$p_{10} = -0.51363 - 1.09156i$	$-0.51363 + 1.09156i$
	1	0	-1	$p_8 = 0.26454 - 1.32837i$	$0.26454 + 1.32837i$

5. a. The roots are 1.244, 8.847, and -1.091, and the critical points are 0 and 6.

b. The roots are 0.5798, 1.521, 2.332, and -2.432, and the critical points are 1, 2.001, and -1.5.

7. The methods all find the solution 0.23235.

9. The minimal material is approximately 573.64895 cm^2.

Exercise Set 3.1 (Page 112)

1. a. $P_1(x) = -0.148878x + 1$; $P_2(x) = -0.452592x^2 - 0.0131009x + 1$; $P_1(0.45) = 0.933005$; $|f(0.45) - P_1(0.45)| = 0.032558$; $P_2(0.45) = 0.902455$; $|f(0.45) - P_2(0.45)| = 0.002008$

b. $P_1(x) = 0.467251x + 1$; $P_2(x) = -0.0780026x^2 + 0.490652x + 1$; $P_1(0.45) = 1.210263$; $|f(0.45) - P_1(0.45)| = 0.006104$; $P_2(0.45) = 1.204998$; $|f(0.45) - P_2(0.45)| = 0.000839$

c. $P_1(x) = 0.874548x$; $P_2(x) = -0.268961x^2 + 0.955236x$; $P_1(0.45) = 0.393546$; $|f(0.45) - P_1(0.45)| = 0.0212983$; $P_2(0.45) = 0.375392$; $|f(0.45) - P_2(0.45)| = 0.003828$

d. $P_1(x) = 1.031121x$; $P_2(x) = 0.615092x^2 + 0.846593x$; $P_1(0.45) = 0.464004$; $|f(0.45) - P_1(0.45)| = 0.019051$; $P_2(0.45) = 0.505523$; $|f(0.45) - P_2(0.45)| = 0.022468$

3. a. $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.135$; $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.00397$

b. $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.03375$; $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.001898$

c. $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.135$; $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.010125$

d. $\left|\frac{f''(\xi)}{2}(0.45-0)(0.45-0.6)\right| \le 0.06779$; $\left|\frac{f'''(\xi)}{6}(0.45-0)(0.45-0.6)(0.45-0.9)\right| \le 0.151$

5. a.

n	$x_0, x_1, \ldots, x_n$	$P_n(8.4)$
1	8.3, 8.6	17.87833
2	8.3, 8.6, 8.7	17.87716
3	8.3, 8.6, 8.7, 8.1	17.87714

b.

n	$x_0, x_1, \ldots, x_n$	$P_n(-1/3)$
1	$-0.5, -0.25$	0.21504167
2	$-0.5, -0.25, 0.0$	0.16988889
3	$-0.5, -0.25, 0.0, -0.75$	0.17451852

c.

n	$x_0, x_1, \ldots, x_n$	$P_n(0.25)$
1	0.2, 0.3	-0.13869287
2	0.2, 0.3, 0.4	-0.13259734
3	0.2, 0.3, 0.4, 0.1	-0.13277477

d.

n	$x_0, x_1, \ldots, x_n$	$P_n(0.9)$
1	0.8, 1.0	0.44086280
2	0.8, 1.0, 0.7	0.43841352
3	0.8, 1.0, 0.7, 0.6	0.44198500

7. a.

n	Actual Error	Error Bound
1	1.180×10^{-3}	1.200×10^{-3}
2	1.367×10^{-5}	1.452×10^{-5}

b.

n	Actual Error	Error Bound
1	4.052×10^{-2}	4.515×10^{-2}
2	4.630×10^{-3}	4.630×10^{-3}

c.

n	Actual Error	Error Bound
1	5.921×10^{-3}	6.097×10^{-3}
2	1.746×10^{-4}	1.813×10^{-4}

d.

n	Actual Error	Error Bound
1	2.730×10^{-3}	1.408×10^{-2}
2	5.179×10^{-3}	9.222×10^{-3}

9. $y = 4.25$

11. We have $f(1.09) \approx 0.2826$. The actual error is 4.3×10^{-5}, and an error bound is 7.4×10^{-6}. The discrepancy is due to the fact that the data are given to only four decimal places, and only four-digit arithmetic is used.

13. a. $P_2(x) = -11.22388889x^2 + 3.810500000x + 1$, and an error bound is 0.11371294.

b. $P_2(x) = -0.1306344167x^2 + 0.8969979335x - 0.63249693$, and an error bound is 9.45762×10^{-4}.

c. $P_3(x) = 0.1970056667x^3 - 1.06259055x^2 + 2.532453189x - 1.666868305$, and an error bound is 10^{-4}.

d. $P_3(x) = -0.07932x^3 - 0.545506x^2 + 1.0065992x + 1$, and an error bound is 1.591376×10^{-3}.

15. a. 1.32436

b. 2.18350

c. 1.15277, 2.01191

d. Parts (a) and (b) are better due to the spacing of the nodes.

17. The largest possible step size is 0.004291932, so 0.004 would be a reasonable choice.

19. a. The interpolating polynomial is $P_5(x) = -0.00252225x^5 + 0.286629x^4 - 10.7938x^3 + 157.312x^2 + 1642.75x + 179323$. The year 1960 corresponds to $x = 0$, so the results are:

YEAR	1950	1975	2014	2020
x	-10	15	54	60
$P_5(x)$	192, 539	215, 526	306, 211	266, 161
U.S. CENSUS	150, 697	215, 973(*EST.*)	317, 298(*EST.*)	341, 000(*EST.*)

b. Based on the value of 1950, we would not put much faith in the values for 1975, 2014, and 2020. However, the 1975 value is close to the estimated population, but the 2014 value is not quite as good. The 2020 value is unrealistic.

21. Since $g'\left(\left(j + \frac{1}{2}\right)h\right) = 0$,

$$\max|g(x)| = \max\left\{|g(jh)|, \left|g\left(\left(j + \frac{1}{2}\right)h\right)\right|, |g((j+1)h)|\right\} = \max\left(0, \frac{h^2}{4}\right),$$

so $|g(x)| \leq h^2/4$.

23. a. (i) $B_3(x) = x$ (ii) $B_3(x) = 1$

d. $n \geq 250{,}000$

Exercise Set 3.2 (Page 120)

1. The approximations are the same as in Exercise 5 of Section 3.1.

3. a. We have $\sqrt{3} \approx P_4(1/2) = 1.708\overline{3}$.

b. We have $\sqrt{3} \approx P_4(3) = 1.690607$.

c. Absolute error in part (a) is approximately 0.0237, and the absolute error in part (b) is 0.0414, so part (a) is more accurate.

5. $P_2 = f(0.5) = 4$

7. $P_{0,1,2,3}(2.5) = 2.875$

9. The incorrect approximation is $-f(2)/6 + 2f(1)/3 + 2/3 + 2f(-1)/3 - f(-2)/6$ and the correct approximation is $-f(2)/6 + 2f(1)/3 + 2f(-1)/3 - f(-2)/6$, so the incorrect approximation is 2/3 too large.

11. The first 10 terms of the sequence are 0.038462, 0.333671, 0.116605, -0.371760, -0.0548919, 0.605935, 0.190249, -0.513353, -0.0668173, and 0.448335.
Since $f(1 + \sqrt{10}) = 0.0545716$, the sequence does not appear to converge.

13. Change Algorithm 3.1 as follows:

INPUT numbers $y_0, y_1, \ldots, y_n$; values $x_0, x_1, \ldots, x_n$ as the first column $Q_{0,0}, Q_{1,0}, \ldots, Q_{n,0}$ of Q.

OUTPUT the table Q with $Q_{n,n}$ approximating $f^{-1}(0)$.

STEP 1 For $i = 1, 2, \ldots, n$
for $j = 1, 2, \ldots, i$
set

$$Q_{i,j} = \frac{y_i Q_{i-1,j-1} - y_{i-j} Q_{i,j-1}}{y_i - y_{i-j}}.$$

Exercise Set 3.3 (Page 130)

1. a. $P_1(x) = 16.9441 + 3.1041(x - 8.1)$; $P_1(8.4) = 17.87533$; $P_2(x) = P_1(x) + 0.06(x - 8.1)(x - 8.3)$; $P_2(8.4) = 17.87713$; $P_3(x) = P_2(x) - 0.00208333(x - 8.1)(x - 8.3)(x - 8.6)$; $P_3(8.4) = 17.87714$

b. $P_1(x) = -0.1769446 + 1.9069687(x - 0.6)$; $P_1(0.9) = 0.395146$;
$P_2(x) = P_1(x) + 0.959224(x - 0.6)(x - 0.7)$; $P_2(0.9) = 0.4526995$;
$P_3(x) = P_2(x) - 1.785741(x - 0.6)(x - 0.7)(x - 0.8)$; $P_3(0.9) = 0.4419850$

3. In the following equations, we have $s = \frac{1}{h}(x - x_0)$.

a. $P_1(s) = -0.718125 - 0.0470625s$; $P_1\left(-\frac{1}{3}\right) = -0.006625$
$P_2(s) = P_1(s) + 0.312625s(s - 1)/2$; $P_2\left(-\frac{1}{3}\right) = 0.1803056$
$P_3(s) = P_2(s) + 0.09375s(s - 1)(s - 2)/6$; $P_3\left(-\frac{1}{3}\right) = 0.1745185$

b. $P_1(s) = -0.62049958 + 0.3365129s$; $P_1(0.25) = -0.1157302$
$P_2(s) = P_1(s) - 0.04592527s(s - 1)/2$; $P_2(0.25) = -0.1329522$
$P_3(s) = P_2(s) - 0.00283891s(s - 1)(s - 2)/6$; $P_3(0.25) = -0.1327748$

5. In the following equations, we have $s = \frac{1}{h}(x - x_n)$.

a. $P_1(s) = 1.101 + 0.7660625s$; $f(-\frac{1}{3}) \approx P_1(-\frac{4}{3}) = 0.07958333$;
$P_2(s) = P_1(s) + 0.406375s(s + 1)/2$; $f(-\frac{1}{3}) \approx P_2(-\frac{4}{3}) = 0.1698889$;
$P_3(s) = P_2(s) + 0.09375s(s + 1)(s + 2)/6$; $f(-\frac{1}{3}) \approx P_3(-\frac{4}{3}) = 0.1745185$

b. $P_1(s) = 0.2484244 + 0.2418235s$; $f(0.25) \approx P_1(-1.5) = -0.1143108$
$P_2(s) = P_1(s) - 0.04876419s(s + 1)/2$; $f(0.25) \approx P_2(-1.5) = -0.1325973$
$P_3(s) = P_2(s) - 0.00283891s(s + 1)(s + 2)/6$; $f(0.25) \approx P_3(-1.5) = -0.1327748$

7. a. $P_3(x) = 5.3 - 33(x + 0.1) + 129.8\overline{3}(x + 0.1)x - 556.\overline{6}(x + 0.1)x(x - 0.2)$

b. $P_4(x) = P_3(x) + 2730.243387(x + 0.1)x(x - 0.2)(x - 0.3)$

9. a. $f(0.05) \approx 1.05126$ **b.** $f(0.65) \approx 1.91555$ **c.** $f(0.43) \approx 1.53725$

11. The coefficient of x^2 is 3.5.

13. The approximation to $f(0.3)$ should be increased by 5.9375.

15. $\Delta^2 P(10) = 1140$

17. a. The interpolating polynomial is $P_5(x) = 179323 + 2397.4x - 3.695x(x - 10) + 0.098\overline{3}x(x - 10)(x - 20) + 0.0344042x(x - 10)(x - 20)(x - 30) - 0.00252225x(x - 10)(x - 20)(x - 30)(x - 40)$, where $x = 0$ corresponds to 1960.

$$\begin{aligned}
P_5(-10) &= 192,539 \text{ approximates the population in 1950.}\\
P_5(15) &= 215,526 \text{ approximates the population in 1975.}\\
P_5(54) &= 306,215 \text{ approximates the population in 2014.}\\
P_5(60) &= 266,165 \text{ approximates the population in 2020.}
\end{aligned}$$

b. Based on the value of 1950, we would not place much credence on the 1975, 2014, and 2020 approximations. Although 1975 and 2014 are not bad, 2020 seems unrealistic.

19. $\Delta^3 f(x_0) = -6$, $\Delta^4 f(x_0) = \Delta^5 f(x_0) = 0$, so the interpolating polynomial has degree 3.

21. Since $f[x_2] = f[x_0] + f[x_0, x_1](x_1 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$,

$$a_2 = \frac{f[x_2] - f[x_0]}{(x_2 - x_0)(x_2 - x_1)} - \frac{f[x_0, x_1]}{(x_2 - x_1)}.$$

This simplifies to $f[x_0, x_1, x_2]$.

23. Let $\tilde{P}(x) = f[x_{i_0}] + \sum_{k=1}^{n} f[x_{i_0}, \ldots, x_{i_k}](x - x_{i_0}) \cdots (x - x_{i_k})$ and $\hat{P}(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k](x - x_0) \cdots (x - x_k)$. The polynomial $\tilde{P}(x)$ interpolates $f(x)$ at the nodes $x_{i_0}, \ldots, x_{i_n}$, and the polynomial $\hat{P}(x)$ interpolates $f(x)$ at the nodes $x_0, \ldots, x_n$. Since both sets of nodes are the same and the interpolating polynomial is unique, we have $\tilde{P}(x) = \hat{P}(x)$. The coefficient of x^n in $\tilde{P}(x)$ is $f[x_{i_0}, \ldots, x_{i_n}]$, and the coefficient of x^n in $\hat{P}(x)$ is $f[x_0, \ldots, x_n]$. Thus, $f[x_{i_0}, \ldots, x_{i_n}] = f[x_0, \ldots, x_n]$.

Exercise Set 3.4 (Page 139)

1. The coefficients for the polynomials in divided-difference form are given in the following tables. For example, the polynomial in part (a) is

$$H_3(x) = 17.56492 + 3.116256(x - 8.3) + 0.05948(x - 8.3)^2 - 0.00202222(x - 8.3)^2(x - 8.6).$$

a	b	c	d
17.56492	0.22363362	−0.02475	−0.62049958
3.116256	2.1691753	0.751	3.5850208
0.05948	0.01558225	2.751	−2.1989182
−0.00202222	−3.2177925	1	−0.490447
		0	0.037205
		0	0.040475
			−0.0025277777
			0.0029629628

3. The following table shows the approximations.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
a	8.4	17.877144	17.877146	2.33×10^{-6}
b	0.9	0.44392477	0.44359244	3.3323×10^{-4}
c	$-\frac{1}{3}$	0.1745185	0.17451852	1.85×10^{-8}
d	0.25	−0.1327719	−0.13277189	5.42×10^{-9}

5. a. We have $\sin 0.34 \approx H_5(0.34) = 0.33349$.

b. The formula gives an error bound of 3.05×10^{-14}, but the actual error is 2.91×10^{-6}. The discrepancy is due to the fact that the data are given to only five decimal places.

c. We have $\sin 0.34 \approx H_7(0.34) = 0.33350$. Although the error bound is now 5.4×10^{-20}, the inaccuracy of the given data dominates the calculations. This result is actually less accurate than the approximation in part (b), since $\sin 0.34 = 0.333487$.

7. $H_3(1.25) = 1.169080403$ with an error bound of 4.81×10^{-5}, and $H_5(1.25) = 1.169016064$ with an error bound of 4.43×10^{-4}.

9. $H_3(1.25) = 1.169080403$ with an error bound of 4.81×10^{-5}, and $H_5(1.25) = 1.169016064$ with an error bound of 4.43×10^{-4}.

11. a. Suppose $P(x)$ is another polynomial with $P(x_k) = f(x_k)$ and $P'(x_k) = f'(x_k)$, for $k = 0, \ldots, n$, and the degree of $P(x)$ is at most $2n + 1$. Let

$$D(x) = H_{2n+1}(x) - P(x).$$

Then $D(x)$ is a polynomial of degree at most $2n + 1$ with $D(x_k) = 0$, and $D'(x_k) = 0$, for each $k = 0, 1, \ldots, n$. Thus, D has zeros of multiplicity 2 at each x_k and

$$D(x) = (x - x_0)^2 \ldots (x - x_n)^2\, Q(x).$$

Hence, $D(x)$ must be of degree $2n$ or more, which would be a contradiction, or $Q(x) \equiv 0$ which implies that $D(x) \equiv 0$. Thus, $P(x) \equiv H_{2n+1}(x)$.

b. First note that the error formula holds if $x = x_k$ for any choice of ξ. Let $x \neq x_k$, for $k = 0, \ldots, n$, and define

$$g(t) = f(t) - H_{2n+1}(t) - \frac{(t-x_0)^2 \ldots (t-x_n)^2}{(x-x_0)^2 \ldots (x-x_n)^2}\left[f(x) - H_{2n+1}(x)\right].$$

Note that $g(x_k) = 0$, for $k = 0, \ldots, n$, and $g(x) = 0$. Thus, g has $n+2$ distinct zeros in $[a, b]$. By Rolle's Theorem, g' has $n+1$ distinct zeros $\xi_0, \ldots, \xi_n$, which are between the numbers $x_0, \ldots, x_n, x$. In addition, $g'(x_k) = 0$, for $k = 0, \ldots, n$, so g' has $2n+2$ distinct zeros $\xi_0, \ldots, \xi_n, x_0, \ldots, x_n$. Since g' is $2n+1$ times differentiable, the Generalized Rolle's Theorem implies that a number ξ in$[a, b]$ exists with $g^{(2n+2)}(\xi) = 0$. But,

$$g^{(2n+2)}(t) = f^{(2n+2)}(t) - \frac{d^{2n+2}}{dt^{2n+2}}H_{2n+1}(t) - \frac{[f(x) - H_{2n+1}(x)] \cdot (2n+2)!}{(x-x_0)^2 \cdots (x-x_n)^2}$$

and

$$0 = g^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - \frac{(2n+2)![f(x) - H_{2n+1}(x)]}{(x-x_0)^2 \cdots (x-x_n)^2}.$$

The error formula follows.

Exercise Set 3.5 (Page 158)

1. $S(x) = x$ on $[0, 2]$.
3. The equations of the respective free cubic splines are

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following tables.

a.

i	a_i	b_i	c_i	d_i
0	17.564920	3.13410000	0.00000000	0.00000000

b.

i	a_i	b_i	c_i	d_i
0	0.22363362	2.17229175	0.00000000	0.00000000

c.

i	a_i	b_i	c_i	d_i
0	−0.02475000	1.03237500	0.00000000	6.50200000
1	0.33493750	2.25150000	4.87650000	−6.50200000

d.

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.45508693	0.00000000	−8.9957933
1	−0.28398668	3.18521313	−2.69873800	−0.94630333
2	0.00660095	2.61707643	−2.98262900	9.9420966

5. The following tables show the approximations.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
a	8.4	17.87833	17.877146	1.1840×10^{-3}
b	0.9	0.4408628	0.44359244	2.7296×10^{-3}
c	$-\frac{1}{3}$	0.1774144	0.17451852	2.8959×10^{-3}
d	0.25	−0.1315912	−0.13277189	1.1807×10^{-3}

	x	Approximation to $f'(x)$	Actual $f'(x)$	Error
a	8.4	3.134100	3.128232	5.86829×10^{-3}
b	0.9	2.172292	2.204367	0.0320747
c	$-\frac{1}{3}$	1.574208	1.668000	0.093792
d	0.25	2.908242	2.907061	1.18057×10^{-3}

7. The equations of the respective clamped cubic splines are

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following tables.

a.

i	a_i	b_i	c_i	d_i
0	17.564920	3.116256	0.0608667	−0.00202222

b.

i	a_i	b_i	c_i	d_i
0	0.22363362	2.1691753	0.65914075	−3.2177925

c.

i	a_i	b_i	c_i	d_i
0	−0.02475000	0.75100000	2.5010000	1.0000000
1	0.33493750	2.18900000	3.2510000	1.0000000

d.

i	a_i	b_i	c_i	d_i
0	−0.62049958	3.5850208	−2.1498407	−0.49077413
1	−0.28398668	3.1403294	−2.2970730	−0.47458360
2	0.006600950	2.6666773	−2.4394481	−0.44980146

9.

	x	Approximation to $f(x)$	Actual $f(x)$	Error
a	8.4	17.877152	17.877146	5.910×10^{-6}
b	0.9	0.4439248	0.44359244	3.323×10^{-4}
c	$-\frac{1}{3}$	0.17451852	0.17451852	0
d	0.25	−0.13277221	−0.13277189	3.19×10^{-7}

	x	Approximation to $f'(x)$	Actual $f'(x)$	Error
a	8.4	3.128369	3.128232	1.373×10^{-4}
b	0.9	2.204470	2.204367	1.0296×10^{-4}
c	$-\frac{1}{3}$	1.668000	1.668000	0
d	0.25	2.908242	2.907061	1.18057×10^{-3}

11. $b = -1,\ c = -3,\ d = 1$

13. $a = 4,\ b = 4,\ c = -1,\ d = \frac{1}{3}$

15. The piecewise linear approximation to f is given by

$$F(x) = \begin{cases} 20(e^{0.1} - 1)x + 1, & \text{for } x \text{ in } [0, 0.05] \\ 20(e^{0.2} - e^{0.1})x + 2e^{0.1} - e^{0.2}, & \text{for } x \text{ in } (0.05, 1]. \end{cases}$$

We have

$$\int_0^{0.1} F(x)\,dx = 0.1107936 \quad \text{and} \quad \int_0^{0.1} f(x)\,dx = 0.1107014.$$

17. The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.0	−0.7573593	0.0	−6.627417
0.25	0.7071068	−2.0	−4.970563	6.627417
0.5	0.0	−3.242641	0.0	6.627417
0.75	−0.7071068	−2.0	4.970563	−6.627417

$\int_0^1 S(x)dx = 0.000000$, $S'(0.5) = -3.24264$, and $S''(0.5) = 0.0$

19. The equation of the spline is

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	1.0	0.0	−5.193321	2.028118
0.25	0.7071068	−2.216388	−3.672233	4.896310
0.5	0.0	−3.134447	0.0	4.896310
0.75	−0.7071068	−2.216388	3.672233	2.028118

$\int_0^1 s(x)\,dx = 0.000000$, $s'(0.5) = -3.13445$, and $s''(0.5) = 0.0$

21. **a.** On $[0, 0.05]$, we have $s(x) = 1.000000 + 1.999999x + 1.998302x^2 + 1.401310x^3$, and on $(0.05, 0.1]$, we have $s(x) = 1.105170 + 2.210340(x - 0.05) + 2.208498(x - 0.05)^2 + 1.548758(x - 0.05)^3$.

b. $\int_0^{0.1} s(x)\,dx = 0.110701$

c. 1.6×10^{-7}

d. On $[0, 0.05]$, we have $S(x) = 1 + 2.04811x + 22.12184x^3$, and on $(0.05, 0.1]$, we have $S(x) = 1.105171 + 2.214028(x - 0.05) + 3.318277(x - 0.05)^2 - 22.12184(x - 0.05)^3$. $S(0.02) = 1.041139$ and $S(0.02) = 1.040811$.

23. The spline has the equation

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

x_i	a_i	b_i	c_i	d_i
0	0	75	−0.659292	0.219764
3	225	76.9779	1.31858	−0.153761
5	383	80.4071	0.396018	−0.177237
8	623	77.9978	−1.19912	0.0799115

The spline predicts a position of $s(10) = 774.84$ ft and a speed of $s'(10) = 74.16$ ft/s. To maximize the speed, we find the single critical point of $s'(x)$, and compare the values of $s(x)$ at this point and the endpoints. We find that max $s'(x) = s'(5.7448) = 80.7$ ft/s $= 55.02$ mi/h. The speed 55 mi/h was first exceeded at approximately 5.5 s.

25. The equation of the spline is

$$S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the coefficients are given in the following table.

	Sample 1				Sample 2			
x_i	a_i	b_i	c_i	d_i	a_i	b_i	c_i	d_i
0	6.67	−0.44687	0	0.06176	6.67	1.6629	0	−0.00249
6	17.33	6.2237	1.1118	−0.27099	16.11	1.3943	−0.04477	−0.03251
10	42.67	2.1104	−2.1401	0.28109	18.89	−0.52442	−0.43490	0.05916
13	37.33	−3.1406	0.38974	−0.01411	15.00	−1.5365	0.09756	0.00226
17	30.10	−0.70021	0.22036	−0.02491	10.56	−0.64732	0.12473	−0.01113
20	29.31	−0.05069	−0.00386	0.00016	9.44	−0.19955	0.02453	−0.00102

27. The three clamped splines have equations of the form

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for x in $[x_i, x_{i+1}]$, where the values of the coefficients are given in the following tables.

Spline 1

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	1	3.0	1.0	−0.347	−0.049	1.0
1	2	3.7	0.447	−0.206	0.027	
2	5	3.9	−0.074	0.033	0.342	
3	6	4.2	1.016	1.058	−0.575	
4	7	5.7	1.409	−0.665	0.156	
5	8	6.6	0.547	−0.196	0.024	
6	10	7.1	0.048	−0.053	−0.003	
7	13	6.7	−0.339	−0.076	0.006	
8	17	4.5				−0.67

Spline 2

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	17	4.5	3.0	−1.101	−0.126	3.0
1	20	7.0	−0.198	0.035	−0.023	
2	23	6.1	−0.609	−0.172	0.280	
3	24	5.6	−0.111	0.669	−0.357	
4	25	5.8	0.154	−0.403	0.088	
5	27	5.2	−0.401	0.126	−2.568	
6	27.7	4.1				−4.0

Spline 3

i	x_i	$a_i = f(x_i)$	b_i	c_i	d_i	$f'(x_i)$
0	27.7	4.1	0.330	2.262	−3.800	0.33
1	28	4.3	0.661	−1.157	0.296	
2	29	4.1	−0.765	−0.269	−0.065	
3	30	3.0				−1.5

29. Let $f(x) = a + bx + cx^2 + dx^3$. Clearly, f satisfies properties (a), (c), (d), and (e) of Definition 3.10, and f interpolates itself for any choice of $x_0, \ldots, x_n$. Since (ii) of property (f) in Definition 3.10 holds, f must be its own clamped cubic spline. However, $f''(x) = 2c + 6dx$ can be zero only at $x = -c/3d$. Thus, part (i) of property (f) in Definition 3.10 cannot hold at two values x_0 and x_n. Thus, f cannot be a natural cubic spline.

31. Insert the following before Step 7 in Algorithm 3.4 and Step 8 in Algorithm 3.5:

For $j = 0, 1, \ldots, n-1$ set

$l_1 = b_j$; (*Note that* $l_1 = s'(x_j)$.)
$l_2 = 2c_j$; (*Note that* $l_2 = s''(x_j)$.)
OUTPUT (l_1, l_2)

Set

$l_1 = b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}^2$;(*Note that* $l_1 = s'(x_n)$.)
$l_2 = 2c_{n-1} + 6d_{n-1}h_{n-1}$;(*Note that* $l_2 = s''(x_n)$.)
OUTPUT (l_1, l_2).

33. We have

$$|f(x) - F(x)| \leq \frac{M}{8} \max_{0\leq j\leq n-1} |x_{j+1} - x_j|^2,$$

where $M = \max_{a\leq x\leq b} |f''(x)|$.
Error bounds for Exercise 15 are on $[0, 0.1]$, $|f(x) - F(x)| \leq 1.53 \times 10^{-3}$, and

$$\left| \int_0^{0.1} F(x)\, dx - \int_0^{0.1} e^{2x}\, dx \right| \leq 1.53 \times 10^{-4}.$$

35. $S(x) = \begin{cases} 2x - x^2, & 0 \leq x \leq 1 \\ 1 + (x-1)^2, & 1 \leq x \leq 2 \end{cases}$

Exercise Set 3.6 (Page 167)

1. a. $x(t) = -10t^3 + 14t^2 + t, \quad y(t) = -2t^3 + 3t^2 + t$
b. $x(t) = -10t^3 + 14.5t^2 + 0.5t, \quad y(t) = -3t^3 + 4.5t^2 + 0.5t$
c. $x(t) = -10t^3 + 14t^2 + t, \quad y(t) = -4t^3 + 5t^2 + t$
d. $x(t) = -10t^3 + 13t^2 + 2t, \quad y(t) = 2t$

3. a. $x(t) = -11.5t^3 + 15t^2 + 1.5t + 1, \quad y(t) = -4.25t^3 + 4.5t^2 + 0.75t + 1$

b. $x(t) = -6.25t^3 + 10.5t^2 + 0.75t + 1, \quad y(t) = -3.5t^3 + 3t^2 + 1.5t + 1$

c. For t between $(0, 0)$ and $(4, 6)$, we have

$$x(t) = -5t^3 + 7.5t^2 + 1.5t, \quad y(t) = -13.5t^3 + 18t^2 + 1.5t,$$

and for t between $(4, 6)$ and $(6, 1)$, we have

$$x(t) = -5.5t^3 + 6t^2 + 1.5t + 4, \quad y(t) = 4t^3 - 6t^2 - 3t + 6.$$

d. For t between $(0, 0)$ and $(2, 1)$, we have

$$x(t) = -5.5t^3 + 6t^2 + 1.5t, \quad y(t) = -0.5t^3 + 1.5t,$$

for t between $(2, 1)$ and $(4, 0)$, we have

$$x(t) = -4t^3 + 3t^2 + 3t + 2, \quad y(t) = -t^3 + 1,$$

and for t between $(4, 0)$ and $(6, -1)$, we have

$$x(t) = -8.5t^3 + 13.5t^2 - 3t + 4, \quad y(t) = -3.25t^3 + 5.25t^2 - 3t.$$

5. a. Using the forward divided difference gives the following table.

0	u_0			
0	u_0	$3(u_1 - u_0)$		
1	u_3	$u_3 - u_0$	$u_3 - 3u_1 + 2u_0$	
1	u_3	$3(u_3 - u_2)$	$2u_3 - 3u_2 + u_0$	$u_3 - 3u_2 + 3u_1 - u_0$

Therefore,

$$\begin{aligned} u(t) &= u_0 + 3(u_1 - u_0)t + (u_3 - 3u_1 + 2u_0)t^2 + (u_3 - 3u_2 + 3u_1 - u_0)t^2(t-1) \\ &= u_0 + 3(u_1 - u_0)t + (-6u_1 + 3u_0 + 3u_2)t^2 + (u_3 - 3u_2 + 3u_1 - u_0)t^3. \end{aligned}$$

Similarly, $v(t) = v_0 + 3(v_1 - v_0)t + (3v_2 - 6v_1 + 3v_0)t^2 + (v_3 - 3v_2 + 3v_1 - v_0)t^3$.

b. Using the formula for Bernstein polynomials gives

$$\begin{aligned} u(t) &= u_0(1-t)^3 + 3u_1t(1-t)^2 + 3u_2t^2(1-t) + u_3t^3 \\ &= u_0 + 3(u_1 - u_0)t + (3u_2 - 6u_1 + 3u_0)t^2 + (u_3 - 3u_2 + 3u_1 - u_0)t^3. \end{aligned}$$

Similarly,

$$\begin{aligned} v(t) &= v_0(1-t)^3 + 3v_1 + (1-t)^2 + 3v_2t^2(1-t) + v_0t^3 \\ &= v_0 + 3(v_1 - v_0)t + (3v_2 - 6v_1 + 3v_0)t^2 + (v_3 - 3v_2 + 3v_1 - v_0)t^3. \end{aligned}$$

Exercise Set 4.1 (Page 180)

1. From the forward-backward difference formula (4.1), we have the following approximations:

a. $f'(0.5) \approx 0.8520$, $f'(0.6) \approx 0.8520$, $f'(0.7) \approx 0.7960$

b. $f'(0.0) \approx 3.7070$, $f'(0.2) \approx 3.1520$, $f'(0.4) \approx 3.1520$

3. a.

x	Actual Error	Error Bound
0.5	0.0255	0.0282
0.6	0.0267	0.0282
0.7	0.0312	0.0322

b.

x	Actual Error	Error Bound
0.0	0.2930	0.3000
0.2	0.2694	0.2779
0.4	0.2602	0.2779

5. For the endpoints of the tables, we use Formula (4.4). The other approximations come from Formula (4.5).

a. $f'(1.1) \approx 17.769705$, $f'(1.2) \approx 22.193635$, $f'(1.3) \approx 27.107350$, $f'(1.4) \approx 32.150850$

b. $f'(8.1) \approx 3.092050$, $f'(8.3) \approx 3.116150$, $f'(8.5) \approx 3.139975$, $f'(8.7) \approx 3.163525$

c. $f'(2.9) \approx 5.101375$, $f'(3.0) \approx 6.654785$, $f'(3.1) \approx 8.216330$, $f'(3.2) \approx 9.786010$

d. $f'(2.0) \approx 0.13533150$, $f'(2.1) \approx -0.09989550$, $f'(2.2) \approx -0.3298960$, $f'(2.3) \approx -0.5546700$

7. a.

x	Actual Error	Error Bound
1.1	0.280322	0.359033
1.2	0.147282	0.179517
1.3	0.179874	0.219262
1.4	0.378444	0.438524

c.

x	Actual Error	Error Bound
2.9	0.011956	0.0180988
3.0	0.0049251	0.00904938
3.1	0.0004765	0.00493920
3.2	0.0013745	0.00987840

b.

x	Actual Error	Error Bound
8.1	0.00018594	0.000020322
8.3	0.00010551	0.000010161
8.5	9.116×10^{-5}	0.000009677
8.7	0.00020197	0.000019355

d.

x	Actual Error	Error Bound
2.0	0.00252235	0.00410304
2.1	0.00142882	0.00205152
2.2	0.00204851	0.00260034
2.3	0.00437954	0.00520068

9. The approximations and the formulas used are:

a. $f'(2.1) \approx 3.899344$ from (4.7), $f'(2.2) \approx 2.876876$ from (4.7), $f'(2.3) \approx 2.249704$ from (4.6), $f'(2.4) \approx 1.837756$ from (4.6), $f'(2.5) \approx 1.544210$ from (4.7), $f'(2.6) \approx 1.355496$ from (4.7)

b. $f'(-3.0) \approx -5.877358$ from (4.7), $f'(-2.8) \approx -5.468933$ from (4.7), $f'(-2.6) \approx -5.059884$ from (4.6), $f'(-2.4) \approx -4.650223$ from (4.6), $f'(-2.2) \approx -4.239911$ from (4.7), $f'(-2.0) \approx -3.828853$ from (4.7)

11. a.

x	Actual Error	Error Bound
2.1	0.0242312	0.109271
2.2	0.0105138	0.0386885
2.3	0.0029352	0.0182120
2.4	0.0013262	0.00644808
2.5	0.0138323	0.109271
2.6	0.0064225	0.0386885

b.

x	Actual Error	Error Bound
-3.0	1.55×10^{-5}	6.33×10^{-7}
-2.8	1.32×10^{-5}	6.76×10^{-7}
-2.6	7.95×10^{-7}	1.05×10^{-7}
-2.4	6.79×10^{-7}	1.13×10^{-7}
-2.2	1.28×10^{-5}	6.76×10^{-7}
-2.0	7.96×10^{-6}	6.76×10^{-7}

13. $f'(3) \approx \frac{1}{12}[f(1) - 8f(2) + 8f(4) - f(5)] = 0.21062$, with an error bound given by

$$\max_{1 \le x \le 5} \frac{|f^{(5)}(x)|h^4}{30} \le \frac{23}{30} = 0.7\overline{6}.$$

15. From the forward-backward difference formula (4.1), we have the following approximations:

a. $f'(0.5) \approx 0.852$, $f'(0.6) \approx 0.852$, $f'(0.7) \approx 0.7960$

b. $f'(0.0) \approx 3.707$, $f'(0.2) \approx 3.153$, $f'(0.4) \approx 3.153$

17. For the endpoints of the tables, we use Formula (4.7). The other approximations come from Formula (4.6).

a. $f'(2.1) \approx 3.884$, $f'(2.2) \approx 2.896$, $f'(2.3) \approx 2.249$, $f'(2.4) \approx 1.836$, $f'(2.5) \approx 1.550$, $f'(2.6) \approx 1.348$

b. $f'(-3.0) \approx -5.883$, $f'(-2.8) \approx -5.467$, $f'(-2.6) \approx -5.059$, $f'(-2.4) \approx -4.650$, $f'(-2.2) \approx -4.208$, $f'(-2.0) \approx -3.875$

19. The approximation is -4.8×10^{-9}. $f''(0.5) = 0$. The error bound is 0.35874. The method is very accurate since the function is symmetric about $x = 0.5$.

21. a. $f'(0.2) \approx -0.1951027$ **b.** $f'(1.0) \approx -1.541415$ **c.** $f'(0.6) \approx -0.6824175$

23. The three-point formulas give the results in the following table.

Time	0	3	5	8	10	13
Speed	79	82.4	74.2	76.8	69.4	71.2

25. $f'(0.4) \approx -0.4249840$ and $f'(0.8) \approx -1.032772$.

27. The approximations eventually become zero because the numerator becomes zero.

29. Since $e'(h) = -\varepsilon/h^2 + hM/3$, we have $e'(h) = 0$ if and only if $h = \sqrt[3]{3\varepsilon/M}$. Also, $e'(h) < 0$ if $h < \sqrt[3]{3\varepsilon/M}$ and $e'(h) > 0$ if $h > \sqrt[3]{3\varepsilon/M}$, so an absolute minimum for $e(h)$ occurs at $h = \sqrt[3]{3\varepsilon/M}$.

Exercise Set 4.2 (Page 189)

1. a. $f'(1) \approx 1.0000109$ **b.** $f'(0) \approx 2.0000000$ **c.** $f'(1.05) \approx 2.2751459$ **d.** $f'(2.3) \approx -19.646799$

3. a. $f'(1) \approx 1.001$ **b.** $f'(0) \approx 1.999$ **c.** $f'(1.05) \approx 2.283$ **d.** $f'(2.3) \approx -19.61$

5. $\int_0^\pi \sin x\, dx \approx 1.999999$

7. With $h = 0.1$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{1.2}\left[1.8e^{1.8} - 8\left(1.9e^{1.9}\right) + 8(2.1)e^{2.1} - 2.2e^{2.2}\right] = 22.166995.$$

With $h = 0.05$, Formula (4.6) becomes

$$f'(2) \approx \frac{1}{0.6}\left[1.9e^{1.9} - 8\left(1.95e^{1.95}\right) + 8(2.05)e^{2.05} - 2.1e^{2.1}\right] = 22.167157.$$

9. Let $N_2(h) = N\left(\frac{h}{3}\right) + \left(\frac{N\left(\frac{h}{3}\right)-N(h)}{2}\right)$ and $N_3(h) = N_2\left(\frac{h}{3}\right) + \left(\frac{N_2\left(\frac{h}{3}\right)-N_2(h)}{8}\right)$. Then $N_3(h)$ is an $O(h^3)$ approximation to M.

11. Let $N(h) = (1+h)^{1/h}$, $N_2(h) = 2N\left(\frac{h}{2}\right) - N(h)$, $N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{1}{3}\left(N_2\left(\frac{h}{2}\right) - N_2(h)\right)$.

a. $N(0.04) = 2.665836331$, $N(0.02) = 2.691588029$, $N(0.01) = 2.704813829$

b. $N_2(0.04) = 2.717339727$, $N_2(0.02) = 2.718039629$. The $O(h^3)$ approximation is $N_3(0.04) = 2.718272931$.

c. Yes, since the errors seem proportional to h for $N(h)$, to h^2 for $N_2(h)$, and to h^3 for $N_3(h)$.

13. a. We have

$$P_{0,1}(x) = \frac{\left(x-h^2\right)N_1\left(\frac{h}{2}\right)}{\frac{h^2}{4}-h^2} + \frac{\left(x-\frac{h^2}{4}\right)N_1(h)}{h^2-\frac{h^2}{4}}, \quad \text{so} \quad P_{0,1}(0) = \frac{4N_1\left(\frac{h}{2}\right)-N_1(h)}{3}.$$

Similarly,

$$P_{1,2}(0) = \frac{4N_1\left(\frac{h}{4}\right)-N_1\left(\frac{h}{2}\right)}{3}.$$

b. We have

$$P_{0,2}(x) = \frac{\left(x-h^4\right)N_2\left(\frac{h}{2}\right)}{\frac{h^4}{16}-h^4} + \frac{\left(x-\frac{h^4}{16}\right)N_2(h)}{h^4-\frac{h^4}{16}}, \quad \text{so} \quad P_{0,2}(0) = \frac{16N_2\left(\frac{h}{2}\right)-N_2(h)}{15}.$$

15. c.

k	4	8	16	32	64	128	256	512
p_k	$2\sqrt{2}$	3.0614675	3.1214452	3.1365485	3.1403312	3.1412723	3.1415138	3.1415729
P_k	4	3.3137085	3.1825979	3.1517249	3.144184	3.1422236	3.1417504	3.1416321

d. Values of p_k and P_k are given in the following tables, together with the extrapolation results:

For p_k, we have:

2.8284271				
3.0614675	3.1391476			
3.1214452	3.1414377	3.1415904		
3.1365485	3.1415829	3.1415926	3.1415927	
3.1403312	3.1415921	3.1415927	3.1415927	3.1415927

For P_k, we have:

4				
3.3137085	3.0849447			
3.1825979	3.1388943	3.1424910		
3.1517249	3.1414339	3.1416032	3.1415891	
3.1441184	3.1415829	3.1415928	3.1415926	3.1415927

Exercise Set 4.3 (Page 200)

1. The Trapezoidal rule gives the following approximations.

a. 0.265625 **b.** −0.2678571 **c.** 0.228074 **d.** 0.1839397
e. −0.8666667 **f.** −0.1777643 **g.** 0.2180895 **h.** 4.1432597

3. The errors are shown in the table.

	Actual Error	Error Bound
a	0.071875	0.125
b	7.943×10^{-4}	9.718×10^{-4}
c	0.0358147	0.0396972
d	0.0233369	0.1666667
e	0.1326975	0.5617284
f	9.443×10^{-4}	1.0707×10^{-3}
g	0.0663431	0.0807455
h	1.554631	2.298827

5. Simpson's rule gives the following approximations.

a. 0.1940104 **b.** −0.2670635 **c.** 0.1922453 **d.** 0.16240168
e. −0.7391053 **f.** −0.1768216 **g.** 0.1513826 **h.** 2.5836964

7. The errors are shown in the table.

	Actual Error	Error Bound
a	2.604×10^{-4}	2.6042×10^{-4}
b	7.14×10^{-7}	9.92×10^{-7}
c	1.406×10^{-5}	2.170×10^{-5}
d	1.7989×10^{-3}	4.1667×10^{-4}
e	5.1361×10^{-3}	0.063280
f	1.549×10^{-6}	2.095×10^{-6}
g	3.6381×10^{-4}	4.1507×10^{-4}
h	4.9322×10^{-3}	0.1302826

9. The Midpoint rule gives the following approximations.

a. 0.1582031 **b.** −0.2666667 **c.** 0.1743309 **d.** 0.1516327
e. −0.6753247 **f.** −0.1768200 **g.** 0.1180292 **h.** 1.8039148

11. The errors are shown in the table.

	Actual Error	Error Bound
a	0.0355469	0.0625
b	3.961×10^{-4}	4.859×10^{-4}
c	0.0179285	0.0198486
d	8.9701×10^{-3}	0.0833333
e	0.0564448	0.2808642
f	4.698×10^{-4}	5.353×10^{-4}
g	0.0337172	0.0403728
h	0.7847138	1.1494136

13. $f(1) = \frac{1}{2}$

15. The following approximations are obtained from Formula (4.23) through Formula (4.30), respectively.

a. 0.1024404, 0.1024598, 0.1024598, 0.1024598, 0.1024695, 0.1024663, 0.1024598, and 0.1024598

b. 0.7853982, 0.7853982, 0.7853982, 0.7853982, 0.7853982, 0.7853982, 0.7853982, and 0.7853982

c. 1.497171, 1.477536, 1.477529, 1.477523, 1.467719, 1.470981, 1.477512, and 1.477515

d. 4.950000, 2.740909, 2.563393, 2.385700, 1.636364, 1.767857, 2.074893, and 2.116379

17.

i	t_i	w_i	$y(t_i)$	
(4.23)	(4.24)	(4.26)	(4.27)	(4.29)
5.43476	5.03420	5.03292	4.83393	5.03180

19. The degree of precision is three.

21. $c_0 = \frac{1}{3}$, $c_1 = \frac{4}{3}$, $c_2 = \frac{1}{3}$

23. $c_0 = \frac{1}{4}$, $c_1 = \frac{3}{4}$, $x_1 = \frac{2}{3}$ gives degree of precision 2.

25. If $E(x^k) = 0$, for all $k = 0, 1, \ldots, n$ and $E(x^{n+1}) \neq 0$, then with $p_{n+1}(x) = x^{n+1}$, we have a polynomial of degree $n+1$ for which $E(p_{n+1}(x)) \neq 0$. Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be any polynomial of degree less than or equal to n. Then $E(p(x)) = a_n E(x^n) + \cdots + a_1 E(x) + a_0 E(1) = 0$. Conversely, if $E(p(x)) = 0$ for all polynomials of degree less than or equal to n, it follows that $E(x^k) = 0$, for all $k = 0, 1, \ldots, n$. Let $p_{n+1}(x) = a_{n+1}x^{n+1} + \cdots + a_0$ be a polynomial of degree $n+1$ for which $E(p_{n+1}(x)) \neq 0$. Since $a_{n+1} \neq 0$, we have

$$x^{n+1} = \frac{1}{a_{n+1}} p_{n+1}(x) - \frac{a_n}{a_{n+1}} x^n - \cdots - \frac{a_0}{a_{n+1}}.$$

Then

$$E(x^{n+1}) = \frac{1}{a_{n+1}} E(p_{n+1}(x)) - \frac{a_n}{a_{n+1}} E(x^n) - \cdots - \frac{a_0}{a_{n+1}} E(1) = \frac{1}{a_{n+1}} E(p_{n+1}(x)) \neq 0.$$

Thus, the quadrature formula has degree of precision n.

27. With $x_{-1} = a$, $x_2 = b$, and $h = \frac{b-a}{3}$ the formula for odd n in Theorem 4.3 gives

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \sum_{i=0}^{1} a_i f(x_i) + \frac{h^3 f''(\xi)}{2!} \int_{-1}^{2} t(t-1)\,dt.$$

So,

$$a_0 = \int_{x_{-1}}^{x_2} L_0(x)\,dx = \int_{x_{-1}}^{x_2} \frac{(x - x_1)}{(x_0 - x_1)}\,dx = \left.\frac{(x - x_1)^2}{2(x_0 - x_1)}\right|_{x_{-1}}^{x_2} = \frac{3}{2}h,$$

$$a_1 = \int_{x_{-1}}^{x_2} L_1(x)\,dx = \int_{x_{-1}}^{x_2} \frac{(x - x_0)}{(x_1 - x_0)}\,dx = \left.\frac{(x - x_0)^2}{2(x_1 - x_0)}\right|_{x_{-1}}^{x_2} = \frac{3}{2}h,$$

and

$$\frac{h^3 f''(\xi)}{2}\int_{-1}^{2}(t^2-t)\,dt = \frac{h^3 f''(\xi)}{2}\left[\frac{1}{3}t^3-\frac{1}{2}t^2\right]_{-1}^{2} = \frac{3h^3}{4}f''(\xi).$$

The formula becomes

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}[f(x_0)+f(x_1)] + \frac{3h^3}{4}f''(\xi).$$

Exercise Set 4.4 (Page 208)

1. The Composite Trapezoidal rule approximations are:

a. 0.639900 **b.** 31.3653 **c.** 0.784241 **d.** −6.42872

e. −13.5760 **f.** 0.476977 **g.** 0.605498 **h.** 0.970926

3. The Composite Trapezoidal rule approximations are:

a. 0.6363098 **b.** 22.477713 **c.** 0.7853980 **d.** −6.274868

e. −14.18334 **f.** 0.4777547 **g.** 0.6043941 **h.** 0.9610554

5. The Composite Midpoint rule approximations are:

a. 0.633096 **b.** 11.1568 **c.** 0.786700 **d.** −6.11274

e. −14.9985 **f.** 0.478751 **g.** 0.602961 **h.** 0.947868

7. a. 3.15947567 **b.** 3.10933713 **c.** 3.00906003

9. $\alpha = 1.5$

11. a. The Composite Trapezoidal rule requires $h < 0.000922295$ and $n \geq 2168$.

b. The Composite Simpson's rule requires $h < 0.037658$ and $n \geq 54$.

c. The Composite Midpoint rule requires $h < 0.00065216$ and $n \geq 3066$.

13. a. The Composite Trapezoidal rule requires $h < 0.04382$ and $n \geq 46$. The approximation is 0.405471.

b. The Composite Simpson's rule requires $h < 0.44267$ and $n \geq 6$. The approximation is 0.405466.

c. The Composite Midpoint rule requires $h < 0.03098$ and $n \geq 64$. The approximation is 0.405460.

15. a. Because the right and left limits at 0.1 and 0.2 for f, f', and f'' are the same, the functions are continuous on $[0, 0.3]$. However,

$$f'''(x) = \begin{cases} 6, & 0 \leq x \leq 0.1 \\ 12, & 0.1 < x \leq 0.2 \\ 12, & 0.2 < x \leq 0.3 \end{cases}$$

is discontinuous at $x = 0.1$.

b. We have 0.302506 with an error bound of 1.9×10^{-4}.

c. We have 0.302425, and the value of the actual integral is the same.

17. The length is approximately 15.8655.

19. Composite Simpson's rule with $h = 0.25$ gives 2.61972 s.

21. The length is approximately 58.47082, using $n = 100$ in the Composite Simpson's rule.

23. To show that the sum

$$\sum_{j=1}^{n/2} f^{(4)}(\xi_j)2h$$

is a Riemann Sum, let $y_i = x_{2i}$, for $i = 0, 1, \ldots \frac{n}{2}$. Then $\Delta y_i = y_{i+1} - y_i = 2h$ and $y_{i-1} \leq \xi_i \leq y_i$. Thus,

$$\sum_{j=1}^{n/2} f^{(4)}(\xi_j)\Delta y_j = \sum_{j=1}^{n/2} f^{(4)}(\xi_j)2h$$

is a Riemann Sum for $\int_a^b f^{(4)}(x)dx$. Hence,

$$E(f) = -\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j) = -\frac{h^4}{180}\left[\sum_{j=1}^{n/2} f^{(4)}(\xi_j)2h\right] \approx -\frac{h^4}{180}\int_a^b f^{(4)}(x)\,dx = -\frac{h^4}{180}\left[f'''(b) - f'''(a)\right].$$

25. **a.** Composite Trapezoidal Rule: With $h = 0.0069669$, the error estimate is 2.541×10^{-5}.

b. Composite Simpson's Rule: With $h = 0.132749$, the error estimate is 3.252×10^{-5}.

c. Composite Midpoint Rule: With $h = 0.0049263$, the error estimate is 2.541×10^{-5}.

Exercise Set 4.5 (Page 217)

1. Romberg integration gives $R_{3,3}$ as follows:

a. 0.1922593 **b.** 0.1606105 **c.** −0.1768200 **d.** 0.08875677

e. 2.5879685 **f.** −0.7341567 **g.** 0.6362135 **h.** 0.6426970

3. Romberg integration gives $R_{4,4}$ as follows:

a. 0.1922594 **b.** 0.1606028 **c.** −0.1768200 **d.** 0.08875528

e. 2.5886272 **f.** −0.7339728 **g.** 0.6362134 **h.** 0.6426991

5. Romberg integration gives:

a. 0.19225936 with $n = 4$ **b.** 0.16060279 with $n = 5$ **c.** −0.17682002 with $n = 4$ **d.** 0.088755284 with $n = 5$

e. 2.5886286 with $n = 6$ **f.** −0.73396918 with $n = 6$ **g.** 0.63621335 with $n = 4$ **h.** 0.64269908 with $n = 5$

7. $R_{33} = 11.5246$

9. $f(2.5) \approx 0.43459$

11. $R_{31} = 5$

13. Romberg integration gives:

a. 62.4373714, 57.2885616, 56.4437507, 56.2630547, and 56.2187727 yields a prediction of 56.2.

b. 55.5722917, 56.2014707, 56.2055989, and 56.2040624 yields a prediction of 56.20.

c. 58.3626837, 59.0773207, 59.2688746, 59.3175220, 59.3297316, and 59.3327870 yields a prediction of 59.330.

d. 58.4220930, 58.4707174, 58.4704791, and 58.4704691 yields a prediction of 58.47047.

e. Consider the graph of the function.

15. $R_{10,10} = 58.47046901$

17. We have

$$\begin{aligned}
R_{k,2} &= \frac{4R_{k,1} - R_{k-1,1}}{3} \\
&= \frac{1}{3}\left[R_{k-1,1} + 2h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (i - 1/2))h_{k-1})\right], \quad \text{from (4.35),} \\
&= \frac{1}{3}\left[\frac{h_{k-1}}{2}(f(a) + f(b)) + h_{k-1}\sum_{i=1}^{2^{k-2}-1} f(a + ih_{k-1})\right. \\
&\quad \left. + 2h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (i - 1/2)h_{k-1})\right], \quad \text{from (4.34) with } k-1 \text{ instead of } k, \\
&= \frac{1}{3}\left[h_k(f(a) + f(b)) + 2h_k\sum_{i=1}^{2^{k-2}-1} f(a + 2ih_k) + 4h_k\sum_{i=1}^{2^{k-2}} f(a + (2i - 1)h)\right] \\
&= \frac{h}{3}\left[f(a) + f(b) + 2\sum_{i=1}^{M-1} f(a + 2ih) + 4\sum_{i=1}^{M} f(a + (2i - 1)h)\right],
\end{aligned}$$

where $h = h_k$ and $M = 2^{k-2}$.

19. Equation (4.35) follows from

$$\begin{aligned}
R_{k,1} &= \frac{h_k}{2}\left[f(a)+f(b)+2\sum_{i=1}^{2^{k-1}-1} f(a+ih_k)\right] \\
&= \frac{h_k}{2}\left[f(a)+f(b)+2\sum_{i=1}^{2^{k-1}-1} f(a+\frac{i}{2}h_{k-1})\right] \\
&= \frac{h_k}{2}\left[f(a)+f(b)+2\sum_{i=1}^{2^{k-1}-1} f(a+ih_{k-1})+2\sum_{i=1}^{2^{k-2}} f(a+(i-1/2)h_{k-1})\right] \\
&= \frac{1}{2}\left\{\frac{h_{k-1}}{2}\left[f(a)+f(b)+2\sum_{i=1}^{2^{k-2}-1} f(a+ih_{k-1})\right]+h_{k-1}\sum_{i=1}^{2^{k-2}} f(a+(i-1/2)h_{k-1})\right\} \\
&= \frac{1}{2}\left[R_{k-1,1}+h_{k-1}\sum_{i=1}^{2^{k-2}} f(a+(i-1/2)h_{k-1})\right].
\end{aligned}$$

Exercise Set 4.6 (Page 226)

1. Simpson's rule gives:

a. $S(1, 1.5) = 0.19224530$, $S(1, 1.25) = 0.039372434$, $S(1.25, 1.5) = 0.15288602$, and the actual value is 0.19225935.

b. $S(0, 1) = 0.16240168$, $S(0, 0.5) = 0.028861071$, $S(0.5, 1) = 0.13186140$, and the actual value is 0.16060279.

c. $S(0, 0.35) = -0.17682156$, $S(0, 0.175) = -0.087724382$, $S(0.175, 0.35) = -0.089095736$, and the actual value is -0.17682002.

d. $S(0, \frac{\pi}{4}) = 0.087995669$, $S(0, \frac{\pi}{8}) = 0.0058315797$, $S(\frac{\pi}{8}, \frac{\pi}{4}) = 0.082877624$, and the actual value is 0.088755285.

3. Adaptive quadrature gives:

a. 0.19226 **b.** 0.16072 **c.** -0.17682 **d.** 0.088709

5. Adaptive quadrature gives:

a. 108.555281 **b.** -1724.966983 **c.** -15.306308 **d.** -18.945949

7.

	Simpson's Rule	Number Evaluation	Error	Adaptive Quadrature	Number Evaluation	Error
a	-0.21515695	57	6.3×10^{-6}	-0.21515062	229	1.0×10^{-8}
b	0.95135226	83	9.6×10^{-6}	0.95134257	217	1.1×10^{-7}

9. Adaptive quadrature gives

$$\int_{0.1}^{2} \sin\frac{1}{x}\,dx \approx 1.1454 \quad\text{and}\quad \int_{0.1}^{2} \cos\frac{1}{x}\,dx \approx 0.67378.$$

11. $\int_0^{2\pi} u(t)\,dt \approx 0.00001$

13. We have, for $h = b - a$,

$$\left|T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right| \approx \frac{h^3}{16}\left|f''(\mu)\right|$$

and

$$\left|\int_a^b f(x)\,dx - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right| \approx \frac{h^3}{48}\left|f''(\mu)\right|.$$

So,

$$\left|\int_a^b f(x)\,dx - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right| \approx \frac{1}{3}\left|T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right|.$$

Exercise Set 4.7 (Page 234)

1. Gaussian quadrature gives:
a. 0.1922687 **b.** 0.1594104 **c.** −0.1768190 **d.** 0.08926302

3. Gaussian quadrature with $n = 3$ gives:
a. 0.1922594 **b.** 0.1605954 **c.** −0.1768200 **d.** 0.08875385

5. Gaussian quadrature gives:
a. 0.1922594 **b.** 0.1606028 **c.** −0.1768200 **d.** 0.08875529

7. Gaussian quadrature with $n = 5$ gives:
a. 0.1922594 **b.** 0.1606028 **c.** −0.1768200 **d.** 0.08875528

9. The approximation is 3.743713701 with absolute error 0.2226462.

11. $a = 1$, $b = 1$, $c = \frac{1}{3}$, $d = -\frac{1}{3}$

13. The Legendre polynomials $P_2(x)$ and $P_3(x)$ are given by

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) \quad \text{and} \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right),$$

so their roots are easily verified.
For $n = 2$,

$$c_1 = \int_{-1}^{1} \frac{x + 0.5773502692}{1.1547005}\,dx = 1$$

and

$$c_2 = \int_{-1}^{1} \frac{x - 0.5773502692}{-1.1547005}\,dx = 1.$$

For $n = 3$,

$$c_1 = \int_{-1}^{1} \frac{x(x + 0.7745966692)}{1.2}\,dx = \frac{5}{9},$$

$$c_2 = \int_{-1}^{1} \frac{(x + 0.7745966692)(x - 0.7745966692)}{-0.6}\,dx = \frac{8}{9},$$

and

$$c_3 = \int_{-1}^{1} \frac{x(x - 0.7745966692)}{1.2}\,dx = \frac{5}{9}.$$

Exercise Set 4.8 (Page 248)

1. Algorithm 4.4 with $n = m = 4$ gives:
a. 0.3115733 **b.** 0.2552526 **c.** 16.50864 **d.** 1.476684

3. Algorithm 4.5 with $n = m = 2$ gives:
a. 0.3115733 **b.** 0.2552446 **c.** 16.50863 **d.** 1.488875

5. Algorithm 4.4 with $n = 4$ and $m = 8$, $n = 8$ and $m = 4$, and $n = m = 6$ gives:
a. 0.5119875, 0.5118533, 0.5118722 **b.** 1.718857, 1.718220, 1.718385
c. 1.001953, 1.000122, 1.000386 **d.** 0.7838542, 0.7833659, 0.7834362
e. −1.985611, −1.999182, −1.997353 **f.** 2.004596, 2.000879, 2.000980
g. 0.3084277, 0.3084562, 0.3084323 **h.** −22.61612, −19.85408, −20.14117

7. Algorithm 4.5 with $n = m = 3$, $n = 3$ and $m = 4$, $n = 4$ and $m = 3$, and $n = m = 4$ gives:
a. 0.5118655, 0.5118445, 0.5118655, 0.5118445, 2.1×10^{-5}, 1.3×10^{-7}, 2.1×10^{-5}, 1.3×10^{-7}
b. 1.718163, 1.718302, 1.718139, 1.718277, 1.2×10^{-4}, 2.0×10^{-5}, 1.4×10^{-4}, 4.8×10^{-6}
c. 1.000000, 1.000000, 1.0000000, 1.000000, 0, 0, 0, 0
d. 0.7833333, 0.7833333, 0.7833333, 0.7833333, 0, 0, 0, 0
e. −1.991878, −2.000124, −1.991878, −2.000124, 8.1×10^{-3}, 1.2×10^{-4}, 8.1×10^{-3}, 1.2×10^{-4}
f. 2.001494, 2.000080, 2.001388, 1.999984, 1.5×10^{-3}, 8×10^{-5}, 1.4×10^{-3}, 1.6×10^{-5}
g. 0.3084151, 0.3084145, 0.3084246, 0.3084245, 10^{-5}, 5.5×10^{-7}, 1.1×10^{-5}, 6.4×10^{-7}
h. −12.74790, −21.21539, −11.83624, −20.30373, 7.0, 1.5, 7.9, 0.564

9. Algorithm 4.4 with $n = m = 14$ gives 0.1479103, and Algorithm 4.5 with $n = m = 4$ gives 0.1506823.

11. Algorithm 4.6 with $n = m = p = 2$ gives the first listed value.
a. 5.204036, $e(e^{0.5} - 1)(e - 1)^2$ **b.** 0.08429784, $\frac{1}{12}$ **c.** 0.08641975, $\frac{1}{14}$
d. 0.09722222, $\frac{1}{12}$ **e.** 7.103932, $2 + \frac{1}{2}\pi^2$ **f.** 1.428074, $\frac{1}{2}(e^2 + 1) - e$

13. Algorithm 4.6 with $n = m = p = 4$ gives the first listed value. The second is from Algorithm 4.6 with $n = m = p = 5$.
a. 5.206447 **b.** 0.08333333 **c.** 0.07142857

15. The approximation 20.41887 requires 125 functional evaluations.

17. The approximation to the center of mass is $(\overline{x}, \overline{y})$, where $\overline{x} = 0.3806333$ and $\overline{y} = 0.3822558$.

19. The area is approximately 1.0402528.

Exercise Set 4.9 (Page 255)

1. The Composite Simpson's rule gives:
a. 0.5284163 **b.** 4.266654 **c.** 0.4329748 **d.** 0.8802210

3. The Composite Simpson's rule gives:
a. 0.4112649 **b.** 0.2440679 **c.** 0.05501681 **d.** 0.2903746

5. The escape velocity is approximately 6.9450 mi/s.

7. a. $\int_0^\infty e^{-x} f(x)\,dx \approx 0.8535534\ f(0.5857864) + 0.1464466\ f(3.4142136)$

b. $\int_0^\infty e^{-x} f(x)\,dx \approx 0.7110930\ f(0.4157746) + 0.2785177\ f(2.2942804) + 0.0103893\ f(6.2899451)$

9. $n = 2$: 2.9865139
$n = 3$: 2.9958198

Exercise Set 5.1 (Page 264)

1. a. Since $f(t, y) = y \cos t$, we have $\frac{\partial f}{\partial y}(t, y) = \cos t$, and f satisfies a Lipschitz condition in y with $L = 1$ on

$$D = \{(t, y) | 0 \le t \le 1, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is $y(t) = e^{\sin t}$.

b. Since $f(t, y) = \frac{2}{t}y + t^2 e^t$, we have $\frac{\partial f}{\partial y} = \frac{2}{t}$, and f satisfies a Lipschitz condition in y with $L = 2$ on

$$D = \{(t, y) | 1 \le t \le 2, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is $y(t) = t^2(e^t - e)$.

c. Since $f(t, y) = -\frac{2}{t}y + t^2 e^t$, we have $\frac{\partial f}{\partial y} = -\frac{2}{t}$, and f satisfies a Lipschitz condition in y with $L = 2$ on

$$D = \{(t, y) | 1 \le t \le 2, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is

$$y(t) = (t^4 e^t - 4t^3 e^t + 12t^2 e^t - 24te^t + 24e^t + (\sqrt{2} - 9)e)/t^2.$$

d. Since $f(t, y) = \frac{4t^3 y}{1+t^4}$, we have $\frac{\partial f}{\partial y} = \frac{4t^3}{1+t^4}$, and f satisfies a Lipschitz condition in y with $L = 2$ on

$$D = \{(t, y) | 0 \le t \le 1, -\infty < y < \infty\}.$$

Also, f is continuous on D, so there exists a unique solution, which is $y(t) = 1 + t^4$.

3. a. Lipschitz constant $L = 1$; it is a well-posed problem.

b. Lipschitz constant $L = 1$; it is a well-posed problem.

c. Lipschitz constant $L = 1$; it is a well-posed problem.

d. The function f does not satisfy a Lipschitz condition, so Theorem 5.6 cannot be used.

5. a. Differentiating $y^3 t + yt = 2$ gives $3y^2 y' t + y^3 + y' t + y = 0$. Solving for y' gives the original differential equation, and setting $t = 1$ and $y = 1$ verifies the initial condition. To approximate $y(2)$, use Newton's method to solve the equation $y^3 + y - 1 = 0$. This gives $y(2) \approx 0.6823278$.

b. Differentiating $y \sin t + t^2 e^y + 2y - 1 = 0$ gives $y' \sin t + y \cos t + 2te^y + t^2 e^y y' + 2y' = 0$. Solving for y' gives the original differential equation, and setting $t = 1$ and $y = 0$ verifies the initial condition. To approximate $y(2)$, use Newton's method to solve the equation $(2 + \sin 2)y + 4e^y - 1 = 0$. This gives $y(2) \approx -0.4946599$.

7. Let the point (t, y) be on the line. Then $\frac{(y-y_1)}{(t-t_1)} = \frac{(y_2-y_1)}{(t_2-t_1)}$ so $\frac{(y-y_1)}{(y_2-y_1)} = \frac{(t-t_1)}{(t_2-t_1)}$. If $\lambda = \frac{(t-t_1)}{(t_2-t_1)}$, then $t = (1-\lambda)t_1 + \lambda t_2$. Similarly, if $\lambda = \frac{(y-y_1)}{(y_2-y_1)}$, then $y = (1-\lambda)y_1 + \lambda y_2$. So the choice $\lambda = \frac{(t-t_1)}{(t_2-t_1)} = \frac{(y-y_1)}{(y_2-y_1)}$ is the value of λ needed to place $(t, y) = ((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ on the line.

9. Let (t_1, y_1) and (t_2, y_2) be in D, with $a \le t_1 \le b$, $a \le t_2 \le b$, $-\infty < y_1 < \infty$, and $-\infty < y_2 < \infty$. For $0 \le \lambda \le 1$, we have $(1-\lambda)a \le (1-\lambda)t_1 \le (1-\lambda)b$ and $\lambda a \le \lambda t_2 \le \lambda b$. Hence, $a = (1-\lambda)a + \lambda a \le (1-\lambda)t_1 + \lambda t_2 \le (1-\lambda)b + \lambda b = b$. Also, $-\infty < (1-\lambda)y_1 + \lambda y_2 < \infty$, so D is convex.

Exercise Set 5.2 (Page 272)

1. Euler's method gives the approximations in the following table.

a.

i	t_i	w_i	$y(t_i)$
1	0.500	0.0000000	0.2836165
2	1.000	1.1204223	3.2190993

b.

i	t_i	w_i	$y(t_i)$
1	2.500	2.0000000	1.8333333
2	3.000	2.6250000	2.5000000

c.

i	t_i	w_i	$y(t_i)$
1	1.250	2.7500000	2.7789294
2	1.500	3.5500000	3.6081977
3	1.750	4.3916667	4.4793276
4	2.000	5.2690476	5.3862944

d.

i	t_i	w_i	$y(t_i)$
1	0.250	1.2500000	1.3291498
2	0.500	1.6398053	1.7304898
3	0.750	2.0242547	2.0414720
4	1.000	2.2364573	2.1179795

3. a.

t	Actual Error	Error Bound
0.5	0.2836165	11.3938
1.0	2.0986771	42.3654

b.

t	Actual Error	Error Bound
2.5	0.166667	0.429570
3.0	0.125000	1.59726

c.

t	Actual Error	Error Bound
1.25	0.0289294	0.0355032
1.50	0.0581977	0.0810902
1.75	0.0876610	0.139625
2.00	0.117247	0.214785

d.

t	Actual Error
0.25	0.0791498
0.50	0.0906844
0.75	0.0172174
1.00	0.118478

For Part (d), error bound formula (5.10) cannot be applied since $L = 0$.

5. Euler's method gives the approximations in the following tables.

a.

i	t_i	w_i	$y(t_i)$
2	1.200	1.0082645	1.0149523
4	1.400	1.0385147	1.0475339
6	1.600	1.0784611	1.0884327
8	1.800	1.1232621	1.1336536
10	2.000	1.1706516	1.1812322

b.

i	t_i	w_i	$y(t_i)$
2	1.400	0.4388889	0.4896817
4	1.800	1.0520380	1.1994386
6	2.200	1.8842608	2.2135018
8	2.600	3.0028372	3.6784753
10	3.000	4.5142774	5.8741000

c.

i	t_i	w_i	$y(t_i)$
2	0.400	−1.6080000	−1.6200510
4	0.800	−1.3017370	−1.3359632
6	1.200	−1.1274909	−1.1663454
8	1.600	−1.0491191	−1.0783314
10	2.000	−1.0181518	−1.0359724

d.

i	t_i	w_i	$y(t_i)$
2	0.2	0.1083333	0.1626265
4	0.4	0.1620833	0.2051118
6	0.6	0.3455208	0.3765957
8	0.8	0.6213802	0.6461052
10	1.0	0.9803451	1.0022460

7. The actual errors for the approximations in Exercise 3 are in the following tables.

a.

t	Actual Error
1.2	0.0066879
1.5	0.0095942
1.7	0.0102229
2.0	0.0105806

b.

t	Actual Error
1.4	0.0507928
2.0	0.2240306
2.4	0.4742818
3.0	1.3598226

c.

t	Actual Error
0.4	0.0120510
1.0	0.0391546
1.4	0.0349030
2.0	0.0178206

d.

t	Actual Error
0.2	0.0542931
0.5	0.0363200
0.7	0.0273054
1.0	0.0219009

9. Euler's method gives the approximations in the following table.

a.

i	t_i	w_i	$y(t_i)$
1	1.1	0.271828	0.345920
5	1.5	3.18744	3.96767
6	1.6	4.62080	5.70296
9	1.9	11.7480	14.3231
10	2.0	15.3982	18.6831

b. Linear interpolation gives the approximations in the following table.

t	Approximation	$y(t)$	Error
1.04	0.108731	0.119986	0.01126
1.55	3.90412	4.78864	0.8845
1.97	14.3031	17.2793	2.976

c. $h < 0.00064$

11. a. Euler's method produces the following approximation to $y(5) = 5.00674$.

	$h = 0.2$	$h = 0.1$	$h = 0.05$
w_N	5.00377	5.00515	5.00592

b. $h = \sqrt{2 \times 10^{-6}} \approx 0.0014142$.

13. a. $1.021957 = y(1.25) \approx 1.014978$, $1.164390 = y(1.93) \approx 1.153902$

b. $1.924962 = y(2.1) \approx 1.660756$, $4.394170 = y(2.75) \approx 3.526160$

c. $-1.138277 = y(1.3) \approx -1.103618$, $-1.041267 = y(1.93) \approx -1.022283$

d. $0.3140018 = y(0.54) \approx 0.2828333$, $0.8866318 = y(0.94) \approx 0.8665521$

15. a. $h = 10^{-n/2}$

b. The minimal error is $10^{-n/2}(e-1) + 5e10^{-n-1}$.

c.

t	$w(h = 0.1)$	$w(h = 0.01)$	$y(t)$	Error ($n = 8$)
0.5	0.40951	0.39499	0.39347	1.5×10^{-4}
1.0	0.65132	0.63397	0.63212	3.1×10^{-4}

17. b. $w_{50} = 0.10430 \approx p(50)$

c. Since $p(t) = 1 - 0.99e^{-0.002t}$, $p(50) = 0.10421$.

Exercise Set 5.3 (Page 280)

1. a.

t_i	w_i	$y(t_i)$
0.50	0.12500000	0.28361652
1.00	2.02323897	3.21909932

b.

t_i	w_i	$y(t_i)$
2.50	1.75000000	1.83333333
3.00	2.42578125	2.50000000

c.

t_i	w_i	$y(t_i)$
1.25	2.78125000	2.77892944
1.50	3.61250000	3.60819766
1.75	4.48541667	4.47932763
2.00	5.39404762	5.38629436

d.

t_i	w_i	$y(t_i)$
0.25	1.34375000	1.32914981
0.50	1.77218707	1.73048976
0.75	2.11067606	2.04147203
1.00	2.20164395	2.11797955

3. a.

t_i	w_i	$y(t_i)$
0.50	0.25781250	0.28361652
1.00	3.05529474	3.21909932

b.

t_i	w_i	$y(t_i)$
2.50	1.81250000	1.83333333
3.00	2.48591644	2.50000000

c.

t_i	w_i	$y(t_i)$
1.25	2.77897135	2.77892944
1.50	3.60826562	3.60819766
1.75	4.47941561	4.47932763
2.00	5.38639966	5.38629436

d.

t_i	w_i	$y(t_i)$
0.25	1.32893880	1.32914981
0.50	1.72966730	1.73048976
0.75	2.03993417	2.04147203
1.00	2.11598847	2.11797955

5. a.

i	t_i	Order 2 w_i	$y(t_i)$
1	1.1	1.214999	1.215886
2	1.2	1.465250	1.467570

b.

i	t_i	Order 2 w_i	$y(t_i)$
1	0.5	0.5000000	0.5158868
2	1.0	1.076858	1.091818

c.

i	t_i	Order 2 w_i	$y(t_i)$
1	1.5	−2.000000	−1.500000
2	2.0	−1.777776	−1.333333
3	2.5	−1.585732	−1.250000
4	3.0	−1.458882	−1.200000

d.

i	t_i	Order 2 w_i	$y(t_i)$
1	0.25	1.093750	1.087088
2	0.50	1.312319	1.289805
3	0.75	1.538468	1.513490
4	1.0	1.720480	1.701870

7. a.

i	t_i	Order 4 w_i	$y(t_i)$
1	1.1	1.215883	1.215886
2	1.2	1.467561	1.467570

b.

i	t_i	Order 4 w_i	$y(t_i)$
1	0.5	0.5156250	0.5158868
2	1.0	1.091267	1.091818

c.

i	t_i	Order 4 w_i	$y(t_i)$
1	1.5	−2.000000	−1.500000
2	2.0	−1.679012	−1.333333
3	2.5	−1.484493	−1.250000
4	3.0	−1.374440	−1.200000

d.

i	t_i	Order 4 w_i	$y(t_i)$
1	0.25	1.086426	1.087088
2	0.50	1.288245	1.289805
3	0.75	1.512576	1.513490
4	1.0	1.701494	1.701870

9. a. Taylor's method of order two gives the results in the following table.

i	t_i	w_i	$y(t_i)$
1	1.1	0.3397852	0.3459199
5	1.5	3.910985	3.967666
6	1.6	5.643081	5.720962
9	1.9	14.15268	14.32308
10	2.0	18.46999	18.68310

b. Linear interpolation gives $y(1.04) \approx 0.1359139$, $y(1.55) \approx 4.777033$, and $y(1.97) \approx 17.17480$. Actual values are $y(1.04) = 0.1199875$, $y(1.55) = 4.788635$, and $y(1.97) = 17.27930$.

c. Taylor's method of order four gives the results in the following table.

i	t_i	w_i
1	1.1	0.3459127
5	1.5	3.967603
6	1.6	5.720875
9	1.9	14.32290
10	2.0	18.68287

d. Cubic Hermite interpolation gives $y(1.04) \approx 0.1199704$, $y(1.55) \approx 4.788527$, and $y(1.97) \approx 17.27904$.

11. Taylor's method of order two gives the following:

t_i	w_i	$y(t_i)$
5	0.5	0.5146389
10	1.0	1.249305
15	1.5	2.152599
20	2.0	2.095185

13. a. Rate in rate out = 2 gal/min. An increase of 10 gallons requires 5 minutes.

b. 49.75556 pounds of salt

Exercise Set 5.4 (Page 291)

1. a.

t	Modified Euler	$y(t)$
0.5	0.5602111	0.2836165
1.0	5.3014898	3.2190993

b.

t	Modified Euler	$y(t)$
2.5	1.8125000	1.8333333
3.0	2.4815531	2.5000000

c.

t	Modified Euler	$y(t)$
1.25	2.7750000	2.7789294
1.50	3.6008333	3.6081977
1.75	4.4688294	4.4793276
2.00	5.3728586	5.3862944

d.

t	Modified Euler	$y(t)$
0.25	1.3199027	1.3291498
0.50	1.7070300	1.7304898
0.75	2.0053560	2.0414720
1.00	2.0770789	2.1179795

3. a.

t_i	Modified Euler w_i	$y(t_i)$
1.2	1.0147137	1.0149523
1.5	1.0669093	1.0672624
1.7	1.1102751	1.1106551
2.0	1.1808345	1.1812322

b.

t_i	Modified Euler w_i	$y(t_i)$
1.4	0.4850495	0.4896817
2.0	1.6384229	1.6612818
2.4	2.8250651	2.8765514
3.0	5.7075699	5.8741000

c.

t_i	Modified Euler w_i	$y(t_i)$
0.4	−1.6229206	−1.6200510
1.0	−1.2442903	−1.2384058
1.4	−1.1200763	−1.1146484
2.0	−1.0391938	−1.0359724

d.

t_i	Modified Euler w_i	$y(t_i)$
0.2	0.1742708	0.1626265
0.5	0.2878200	0.2773617
0.7	0.5088359	0.5000658
1.0	1.0096377	1.0022460

5. a.

t	Midpoint	$y(t)$
0.5	0.2646250	0.2836165
1.0	3.1300023	3.2190993

b.

t	Midpoint	$y(t)$
2.5	1.7812500	1.8333333
3.0	2.4550638	2.5000000

c.

t	Midpoint	$y(t)$
1.25	2.7777778	2.7789294
1.50	3.6060606	3.6081977
1.75	4.4763015	4.4793276
2.00	5.3824398	5.3862944

d.

t	Midpoint	$y(t)$
0.25	1.3337962	1.3291498
0.50	1.7422854	1.7304898
0.75	2.0596374	2.0414720
1.00	2.1385560	2.1179795

7. a.

t_i	Midpoint w_i	$y(t_i)$
1.2	1.0153257	1.0149523
1.5	1.0677427	1.0672624
1.7	1.1111478	1.1106551
2.0	1.1817275	1.1812322

b.

t_i	Midpoint w_i	$y(t_i)$
1.4	0.4861770	0.4896817
2.0	1.6438889	1.6612818
2.4	2.8364357	2.8765514
3.0	5.7386475	5.8741000

c.

t_i	Midpoint w_i	$y(t_i)$
0.4	−1.6192966	−1.6200510
1.0	−1.2402470	−1.2384058
1.4	−1.1175165	−1.1146484
2.0	−1.0382227	−1.0359724

d.

t_i	Midpoint w_i	$y(t_i)$
0.2	0.1722396	0.1626265
0.5	0.2848046	0.2773617
0.7	0.5056268	0.5000658
1.0	1.0063347	1.0022460

9. a.

t_i	Heun w_i	$y(t_i)$
0.50	0.2710885	0.2836165
1.00	3.1327255	3.2190993

b.

t_i	Heun w_i	$y(t_i)$
2.50	1.8464828	1.8333333
3.00	2.5094123	2.5000000

c.

t_i	Heun w_i	$y(t_i)$
1.25	2.7788462	2.7789294
1.50	3.6080529	3.6081977
1.75	4.4791319	4.4793276
2.00	5.3860533	5.3862944

d.

t_i	Heun w_i	$y(t_i)$
0.25	1.3295717	1.3291498
0.50	1.7310350	1.7304898
0.75	2.0417476	2.0414720
1.00	2.1176975	2.1179795

11. a.

t_i	Heun w_i	$y(t_i)$
1.2	1.0149305	1.0149523
1.5	1.0672363	1.0672624
1.7	1.1106289	1.1106551
2.0	1.1812064	1.1812322

b.

t_i	Heun w_i	$y(t_i)$
1.4	0.4895074	0.4896817
2.0	1.6602954	1.6612818
2.4	2.8741491	2.8765514
3.0	5.8652189	5.8741000

c.

t_i	Heun w_i	$y(t_i)$
0.4	−1.6201023	−1.6200510
1.0	−1.2383500	−1.2384058
1.4	−1.1144745	−1.1146484
2.0	−1.0357989	−1.0359724

d.

t_i	Heun w_i	$y(t_i)$
0.2	0.1614497	0.1626265
0.5	0.2765100	0.2773617
0.7	0.4994538	0.5000658
1.0	1.0018114	1.0022460

13. a.

t_i	Runge-Kutta w_i	$y(t_i)$
0.5	0.2969975	0.2836165
1.0	3.3143118	3.2190993

b.

t_i	Runge-Kutta w_i	$y(t_i)$
2.5	1.8333234	1.8333333
3.0	2.4999712	2.5000000

c.

t_i	Runge-Kutta w_i	$y(t_i)$
1.25	2.7789095	2.7789294
1.50	3.6081647	3.6081977
1.75	4.4792846	4.4793276
2.00	5.3862426	5.3862944

d.

t_i	Runge-Kutta w_i	$y(t_i)$
0.25	1.3291650	1.3291498
0.50	1.7305336	1.7304898
0.75	2.0415436	2.0414720
1.00	2.1180636	2.1179795

15. a.

t_i	Runge-Kutta w_i	$y(t_i)$
1.2	1.0149520	1.0149523
1.5	1.0672620	1.0672624
1.7	1.1106547	1.1106551
2.0	1.1812319	1.1812322

b.

t_i	Runge-Kutta w_i	$y(t_i)$
1.4	0.4896842	0.4896817
2.0	1.6612651	1.6612818
2.4	2.8764941	2.8765514
3.0	5.8738386	5.8741000

c.

t_i	Runge-Kutta w_i	$y(t_i)$
0.4	−1.6200576	−1.6200510
1.0	−1.2384307	−1.2384058
1.4	−1.1146769	−1.1146484
2.0	−1.0359922	−1.0359724

d.

t_i	Runge-Kutta w_i	$y(t_i)$
0.2	0.1627655	0.1626265
0.5	0.2774767	0.2773617
0.7	0.5001579	0.5000658
1.0	1.0023207	1.0022460

17. a. $1.0221167 \approx y(1.25) = 1.0219569, \quad 1.1640347 \approx y(1.93) = 1.1643901$

b. $1.9086500 \approx y(2.1) = 1.9249616, \quad 4.3105913 \approx y(2.75) = 4.3941697$

c. $-1.1461434 \approx y(1.3) = -1.1382768, \quad -1.0454854 \approx y(1.93) = -1.0412665$

d. $0.3271470 \approx y(0.54) = 0.3140018, \quad 0.8967073 \approx y(0.94) = 0.8866318$

19. **a.** $1.0227863 \approx y(1.25) = 1.0219569$, $1.1649247 \approx y(1.93) = 1.1643901$

b. $1.91513749 \approx y(2.1) = 1.9249616$, $4.3312939 \approx y(2.75) = 4.3941697$

c. $-1.1432070 \approx y(1.3) = -1.1382768$, $-1.0443743 \approx y(1.93) = -1.0412665$

d. $0.3240839 \approx y(0.54) = 0.3140018$, $0.8934152 \approx y(0.94) = 0.8866318$

21. **a.** $1.02235985 \approx y(1.25) = 1.0219569$, $1.16440371 \approx y(1.93) = 1.1643901$

b. $1.88084805 \approx y(2.1) = 1.9249616$, $4.40842612 \approx y(2.75) = 4.3941697$

c. $-1.14034696 \approx y(1.3) = -1.1382768$, $-1.04182026 \approx y(1.93) = -1.0412665$

d. $0.31625699 \approx y(0.54) = 0.3140018$, $0.88866134 \approx y(0.94) = 0.8866318$

23. **a.** $1.0223826 \approx y(1.25) = 1.0219569$, $1.1644292 \approx y(1.93) = 1.1643901$

b. $1.9373672 \approx y(2.1) = 1.9249616$, $4.4134745 \approx y(2.75) = 4.3941697$

c. $-1.1405252 \approx y(1.3) = -1.1382768$, $-1.0420211 \approx y(1.93) = -1.0412665$

d. $0.31716526 \approx y(0.54) = 0.3140018$, $0.88919730 \approx y(0.94) = 0.8866318$

25. **a.** $1.0219569 = y(1.25) \approx 1.0219550$, $1.1643902 = y(1.93) \approx 1.1643898$

b. $1.9249617 = y(2.10) \approx 1.9249217$, $4.3941697 = y(2.75) \approx 4.3939943$

c. $-1.138268 = y(1.3) \approx -1.1383036$, $-1.0412666 = y(1.93) \approx -1.0412862$

d. $0.31400184 = y(0.54) \approx 0.31410579$, $0.88663176 = y(0.94) \approx 0.88670653$

27. In 0.2 s, we have approximately 2099 units of KOH.

29. With $f(t, y) = -y + t + 1$, we have

$$w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right) = w_i\left(1 - h + \frac{h^2}{2}\right) + t_i\left(h - \frac{h^2}{2}\right) + h$$

and

$$w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))]$$
$$= w_i\left(1 - h + \frac{h^2}{2}\right) + t_i\left(h - \frac{h^2}{2}\right) + h.$$

31. The appropriate constants are

$$\alpha_1 = \delta_1 = \alpha_2 = \delta_2 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \frac{1}{2} \quad \text{and} \quad \alpha_3 = \delta_3 = 1.$$

Exercise Set 5.5 (Page 300)

1. The Runge-Kutta-Fehlberg Algorithm gives the results in the following tables.

a.

i	t_i	w_i	h_i	y_i
1	0.2093900	0.0298184	0.2093900	0.0298337
3	0.5610469	0.4016438	0.1777496	0.4016860
5	0.8387744	1.5894061	0.1280905	1.5894600
7	1.0000000	3.2190497	0.0486737	3.2190993

b.

i	t_i	w_i	h_i	y_i
1	2.2500000	1.4499988	0.2500000	1.4500000
2	2.5000000	1.8333332	0.2500000	1.8333333
3	2.7500000	2.1785718	0.2500000	2.1785714
4	3.0000000	2.5000005	0.2500000	2.5000000

c.

i	t_i	w_i	h_i	y_i
1	1.2500000	2.7789299	0.2500000	2.7789294
2	1.5000000	3.6081985	0.2500000	3.6081977
3	1.7500000	4.4793288	0.2500000	4.4793276
4	2.0000000	5.3862958	0.2500000	5.3862944

d.

i	t_i	w_i	h_i	y_i
1	0.2500000	1.3291478	0.2500000	1.3291498
2	0.5000000	1.7304857	0.2500000	1.7304898
3	0.7500000	2.0414669	0.2500000	2.0414720
4	1.0000000	2.1179750	0.2500000	2.1179795

3. The Runge-Kutta-Fehlberg Algorithm gives the results in the following tables.

a.

i	t_i	w_i	h_i	y_i
1	1.1101946	1.0051237	0.1101946	1.0051237
5	1.7470584	1.1213948	0.2180472	1.1213947
7	2.3994350	1.2795396	0.3707934	1.2795395
11	4.0000000	1.6762393	0.1014853	1.6762391

b.

i	t_i	w_i	h_i	y_i
4	1.5482238	0.7234123	0.1256486	0.7234119
7	1.8847226	1.3851234	0.1073571	1.3851226
10	2.1846024	2.1673514	0.0965027	2.1673499
16	2.6972462	4.1297939	0.0778628	4.1297904
21	3.0000000	5.8741059	0.0195070	5.8741000

c.

i	t_i	w_i	h_i	y_i
1	0.1633541	−1.8380836	0.1633541	−1.8380836
5	0.7585763	−1.3597623	0.1266248	−1.3597624
9	1.1930325	−1.1684827	0.1048224	−1.1684830
13	1.6229351	−1.0749509	0.1107510	−1.0749511
17	2.1074733	−1.0291158	0.1288897	−1.0291161
23	3.0000000	−1.0049450	0.1264618	−1.0049452

d.

i	t_i	w_i	h_i	y_i
1	0.3986051	0.3108201	0.3986051	0.3108199
3	0.9703970	0.2221189	0.2866710	0.2221186
5	1.5672905	0.1133085	0.3042087	0.1133082
8	2.0000000	0.0543454	0.0902302	0.0543455

5. a. The number of infectives is $y(30) \approx 80295.7$.

b. The limiting value for the number of infectives for this model is $\lim_{t\to\infty} y(t) = 100{,}000$.

7. Steps 3 and 6 must use the new equations. Step 4 must now use

$$R = \frac{1}{h}\left| -\frac{1}{160}K_1 - \frac{125}{17952}K_3 + \frac{1}{144}K_4 - \frac{12}{1955}K_5 - \frac{3}{44}K_6 + \frac{125}{11592}K_7 + \frac{43}{616}K_8 \right|,$$

and in Step 8 we must change to $\delta = 0.871(TOL/R)^{1/5}$. Repeating Exercise 3 using the Runge-Kutta-Verner method gives the results in the following tables.

a.

i	t_i	w_i	h_i	y_i
1	1.42087564	1.05149775	0.42087564	1.05150868
3	2.28874724	1.25203709	0.50000000	1.25204675
5	3.28874724	1.50135401	0.50000000	1.50136369
7	4.00000000	1.67622922	0.21125276	1.67623914

b.

i	t_i	w_i	h_i	y_i
1	1.27377960	0.31440170	0.27377960	0.31440111
4	1.93610139	1.50471956	0.20716801	1.50471717
7	2.48318866	3.19129592	0.17192536	3.19129017
11	3.00000000	5.87411325	0.05925262	5.87409998

c.

i	t_i	w_i	h_i	y_i
1	0.50000000	−1.53788271	0.50000000	−1.53788284
5	1.26573379	−1.14736319	0.17746598	−1.14736283
9	1.99742532	−1.03615509	0.19229794	−1.03615478
14	3.00000000	−1.00494544	0.10525374	−1.00494525

d.

i	t_i	w_i	h_i	y_i
1	0.50000000	0.29875168	0.50000000	0.29875178
2	1.00000000	0.21662609	0.50000000	0.21662642
4	1.74337091	0.08624885	0.27203938	0.08624932
6	2.00000000	0.05434531	0.03454832	0.05434551

Exercise Set 5.6 (Page 314)

1. The Adams-Bashforth methods give the results in the following tables.

a.

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	0.0268128	0.0268128	0.0268128	0.0268128	0.0268128
0.4	0.1200522	0.1507778	0.1507778	0.1507778	0.1507778
0.6	0.4153551	0.4613866	0.4960196	0.4960196	0.4960196
0.8	1.1462844	1.2512447	1.2961260	1.3308570	1.3308570
1.0	2.8241683	3.0360680	3.1461400	3.1854002	3.2190993

b.

t	2-step	3-step	4-step	5-step	$y(t)$
2.2	1.3666667	1.3666667	1.3666667	1.3666667	1.3666667
2.4	1.6750000	1.6857143	1.6857143	1.6857143	1.6857143
2.6	1.9632431	1.9794407	1.9750000	1.9750000	1.9750000
2.8	2.2323184	2.2488759	2.2423065	2.2444444	2.2444444
3.0	2.4884512	2.5051340	2.4980306	2.5011406	2.5000000

c.

t	2-step	3-step	4-step	5-step	$y(t)$
1.2	2.6187859	2.6187859	2.6187859	2.6187859	2.6187859
1.4	3.2734823	3.2710611	3.2710611	3.2710611	3.2710611
1.6	3.9567107	3.9514231	3.9520058	3.9520058	3.9520058
1.8	4.6647738	4.6569191	4.6582078	4.6580160	4.6580160
2.0	5.3949416	5.3848058	5.3866452	5.3862177	5.3862944

d.

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	1.2529306	1.2529306	1.2529306	1.2529306	1.2529306
0.4	1.5986417	1.5712255	1.5712255	1.5712255	1.5712255
0.6	1.9386951	1.8827238	1.8750869	1.8750869	1.8750869
0.8	2.1766821	2.0844122	2.0698063	2.0789180	2.0789180
1.0	2.2369407	2.1115540	2.0998117	2.1180642	2.1179795

3. The Adams-Bashforth methods give the results in the following tables.

a.

t	2-step	3-step	4-step	5-step	$y(t)$
1.2	1.0161982	1.0149520	1.0149520	1.0149520	1.0149523
1.4	1.0497665	1.0468730	1.0477278	1.0475336	1.0475339
1.6	1.0910204	1.0875837	1.0887567	1.0883045	1.0884327
1.8	1.1363845	1.1327465	1.1340093	1.1334967	1.1336536
2.0	1.1840272	1.1803057	1.1815967	1.1810689	1.1812322

b.

t	2-step	3-step	4-step	5-step	$y(t)$
1.4	0.4867550	0.4896842	0.4896842	0.4896842	0.4896817
1.8	1.1856931	1.1982110	1.1990422	1.1994320	1.1994386
2.2	2.1753785	2.2079987	2.2117448	2.2134792	2.2135018
2.6	3.5849181	3.6617484	3.6733266	3.6777236	3.6784753
3.0	5.6491203	5.8268008	5.8589944	5.8706101	5.8741000

c.

t	2-step	3-step	4-step	5-step	$y(t)$
0.5	−1.5357010	−1.5381988	−1.5379372	−1.5378676	−1.5378828
1.0	−1.2374093	−1.2389605	−1.2383734	−1.2383693	−1.2384058
1.5	−1.0952910	−1.0950952	−1.0947925	−1.0948481	−1.0948517
2.0	−1.0366643	−1.0359996	−1.0359497	−1.0359760	−1.0359724

d.

t	2-step	3-step	4-step	5-step	$y(t)$
0.2	0.1739041	0.1627655	0.1627655	0.1627655	0.1626265
0.4	0.2144877	0.2026399	0.2066057	0.2052405	0.2051118
0.6	0.3822803	0.3747011	0.3787680	0.3765206	0.3765957
0.8	0.6491272	0.6452640	0.6487176	0.6471458	0.6461052
1.0	1.0037415	1.0020894	1.0064121	1.0073348	1.0022460

5. The Adams-Moulton methods give the results in the following tables.

a.

t_i	2-step	3-step	4-step	$y(t_i)$
0.2	0.0268128	0.0268128	0.0268128	0.0268128
0.4	0.1533627	0.1507778	0.1507778	0.1507778
0.6	0.5030068	0.4979042	0.4960196	0.4960196
0.8	1.3463142	1.3357923	1.3322919	1.3308570
1.0	3.2512866	3.2298092	3.2227484	3.2190993

c.

t_i	2-step	3-step	4-step	$y(t_i)$
1.2	2.6187859	2.6187859	2.6187859	2.6187859
1.4	3.2711394	3.2710611	3.2710611	3.2710611
1.6	3.9521454	3.9519886	3.9520058	3.9520058
1.8	4.6582064	4.6579866	4.6580211	4.6580160
2.0	5.3865293	5.3862558	5.3863027	5.3862944

d.

t_i	2-step	3-step	4-step	$y(t_i)$
0.2	1.2529306	1.2529306	1.2529306	1.2529306
0.4	1.5700866	1.5712255	1.5712255	1.5712255
0.6	1.8738414	1.8757546	1.8750869	1.8750869
0.8	2.0787117	2.0803067	2.0789471	2.0789180
1.0	2.1196912	2.1199024	2.1178679	2.1179795

7. a.

t_i	w_i	$y(t_i)$
0.2	0.0269059	0.0268128
0.4	0.1510468	0.1507778
0.6	0.4966479	0.4960196
0.8	1.3408657	1.3308570
1.0	3.2450881	3.2190993

b.

t_i	w_i	$y(t_i)$
2.2	1.3666610	1.3666667
2.4	1.6857079	1.6857143
2.6	1.9749941	1.9750000
2.8	2.2446995	2.2444444
3.0	2.5003083	2.5000000

c.

t_i	w_i	$y(t_i)$
1.2	2.6187787	2.6187859
1.4	3.2710491	3.2710611
1.6	3.9519900	3.9520058
1.8	4.6579968	4.6580160
2.0	5.3862715	5.3862944

d.

t_i	w_i	$y(t_i)$
0.2	1.2529350	1.2529306
0.4	1.5712383	1.5712255
0.6	1.8751097	1.8750869
0.8	2.0796618	2.0789180
1.0	2.1192575	2.1179795

9. The Adams Fourth-order Predictor-Corrector Algorithm gives the results in the following tables.

a.

t	w	$y(t)$
1.2	1.0149520	1.0149523
1.4	1.0475227	1.0475339
1.6	1.0884141	1.0884327
1.8	1.1336331	1.1336536
2.0	1.1812112	1.1812322

b.

t	w	$y(t)$
1.4	0.4896842	0.4896817
1.8	1.1994245	1.1994386
2.2	2.2134701	2.2135018
2.6	3.6784144	3.6784753
3.0	5.8739518	5.8741000

c.

t	w	$y(t)$
0.5	−1.5378788	−1.5378828
1.0	−1.2384134	−1.2384058
1.5	−1.0948609	−1.0948517
2.0	−1.0359757	−1.0359724

d.

t	w	$y(t)$
0.2	0.1627655	0.1626265
0.4	0.2048557	0.2051118
0.6	0.3762804	0.3765957
0.8	0.6458949	0.6461052
1.0	1.0021372	1.0022460

11. Milne-Simpson's Predictor-Corrector method gives the results in the following tables.

a.

i	t_i	w_i	$y(t_i)$
2	1.2	1.01495200	1.01495231
5	1.5	1.06725997	1.06726235
7	1.7	1.11065221	1.11065505
10	2.0	1.18122584	1.18123222

b.

i	t_i	w_i	$y(t_i)$
2	1.4	0.48968417	0.48968166
5	2.0	1.66126150	1.66128176
7	2.4	2.87648763	2.87655142
10	3.0	5.87375555	5.87409998

c.

i	t_i	w_i	$y(t_i)$
5	0.5	−1.53788255	−1.53788284
10	1.0	−1.23840789	−1.23840584
15	1.5	−1.09485532	−1.09485175
20	2.0	−1.03597247	−1.03597242

d.

i	t_i	w_i	$y(t_i)$
2	0.2	0.16276546	0.16262648
5	0.5	0.27741080	0.27736167
7	0.7	0.50008713	0.50006579
10	1.0	1.00215439	1.00224598

13. a. With $h = 0.01$, the three-step Adams-Moulton method gives the values in the following table.

i	t_i	w_i
10	0.1	1.317218
20	0.2	1.784511

b. Newton's method will reduce the number of iterations per step from three to two, using the stopping criterion

$$|w_i^{(k)} - w_i^{(k-1)}| \leq 10^{-6}.$$

15. The new algorithm gives the results in the following tables.

a.

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
1.2	1.0149520	1.0149520	1.0149520	1.0149523
1.5	1.0672499	1.0672499	1.0672499	1.0672624
1.7	1.1106394	1.1106394	1.1106394	1.1106551
2.0	1.1812154	1.1812154	1.1812154	1.1812322

b.

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
1.4	0.4896842	0.4896842	0.4896842	0.4896817
2.0	1.6613427	1.6613509	1.6613517	1.6612818
2.4	2.8767835	2.8768112	2.8768140	2.8765514
3.0	5.8754422	5.8756045	5.8756224	5.8741000

c.

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
0.4	−1.6200494	−1.6200494	−1.6200494	−1.6200510
1.0	−1.2384104	−1.2384105	−1.2384105	−1.2384058
1.4	−1.1146533	−1.1146536	−1.1146536	−1.1146484
2.0	−1.0359139	−1.0359740	−1.0359740	−1.0359724

d.

t_i	$w_i(p=2)$	$w_i(p=3)$	$w_i(p=4)$	$y(t_i)$
0.2	0.1627655	0.1627655	0.1627655	0.1626265
0.5	0.2774037	0.2773333	0.2773468	0.2773617
0.7	0.5000772	0.5000259	0.5000356	0.5000658
1.0	1.0022473	1.0022273	1.0022311	1.0022460

17. Using the notation $y = y(t_i)$, $f = f(t_i, y(t_i))$, $f_t = f_t(t_i, y(t_i))$, etc., we have

$$\begin{aligned}
y + hf + \frac{h^2}{2}(f_t + ff_y) + \frac{h^3}{6}\left(f_{tt} + f_t f_y + 2ff_{yt} + ff_y^2 + f^2 f_{yy}\right) \\
= y + ahf + bh\left[f - h(f_t + ff_y) + \frac{h^2}{2}\left(f_{tt} + f_t f_y + 2ff_{yt} + ff_y^2 + f^2 f_{yy}\right)\right] \\
+ ch\left[f - 2h(f_t + ff_y) + 2h^2\left(f_{tt} + f_t f_y + 2ff_{yt} + ff_y^2 + f^2 f_{yy}\right)\right].
\end{aligned}$$

Thus,

$$a + b + c = 1, \quad -b - 2c = \frac{1}{2}, \quad \text{and} \quad \frac{1}{2}b + 2c = \frac{1}{6}.$$

This system has the solution

$$a = \frac{23}{12}, \quad b = -\frac{16}{12}, \quad \text{and} \quad c = \frac{5}{12}.$$

19. We have

$$y(t_{i+1}) - y(t_{i-1}) = \int_{t_{i-1}}^{t_{i+1}} f(t, y(t))dt$$

$$= \frac{h}{3}[f(t_{i-1}, y(t_{i-1})) + 4f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))] - \frac{h^5}{90} f^{(4)}(\xi, y(\xi)).$$

This leads to the difference equation

$$w_{i+1} = w_{i-1} + \frac{h[f(t_{i-1}, w_{i-1}) + 4f(t_i, w_i) + f(t_{i+1}, w_{i+1})]}{3},$$

with local truncation error

$$\tau_{i+1}(h) = \frac{-h^4 y^{(5)}(\xi)}{90}.$$

21. The entries are generated by evaluating the following integrals:

$$k = 0 : (-1)^k \int_0^1 \binom{-s}{k} ds = \int_0^1 ds = 1,$$

$$k = 1 : (-1)^k \int_0^1 \binom{-s}{k} ds = -\int_0^1 -s\, ds = \frac{1}{2},$$

$$k = 2 : (-1)^k \int_0^1 \binom{-s}{k} ds = \int_0^1 \frac{s(s+1)}{2} ds = \frac{5}{12},$$

$$k = 3 : (-1)^k \int_0^1 \binom{-s}{k} ds = -\int_0^1 \frac{-s(s+1)(s+2)}{6} ds = \frac{3}{8},$$

$$k = 4 : (-1)^k \int_0^1 \binom{-s}{k} ds = \int_0^1 \frac{s(s+1)(s+2)(s+3)}{24} ds = \frac{251}{720}, \quad \text{and}$$

$$k = 5 : (-1)^k \int_0^1 \binom{-s}{k} ds = -\int_0^1 -\frac{s(s+1)(s+2)(s+3)(s+4)}{120} ds = \frac{95}{288}.$$

Exercise Set 5.7 (Page 321)

1. The Adams Variable Step-Size Predictor-Corrector Algorithm gives the results in the following tables.

a.

i	t_i	w_i	h_i	y_i
1	0.04275596	0.00096891	0.04275596	0.00096887
5	0.22491460	0.03529441	0.05389076	0.03529359
12	0.60214994	0.50174348	0.05389076	0.50171761
17	0.81943926	1.45544317	0.04345786	1.45541453
22	0.99830392	3.19605697	0.03577293	3.19602842
26	1.00000000	3.21912776	0.00042395	3.21909932

b.

i	t_i	w_i	h_i	y_i
1	2.06250000	1.12132350	0.06250000	1.12132353
5	2.31250000	1.55059834	0.06250000	1.55059524
9	2.62471924	2.00923157	0.09360962	2.00922829
13	2.99915773	2.49895243	0.09360962	2.49894707
17	3.00000000	2.50000535	0.00021057	2.50000000

c.

i	t_i	w_i	h_i	y_i
1	1.06250000	2.18941363	0.06250000	2.18941366
4	1.25000000	2.77892931	0.06250000	2.77892944
8	1.85102559	4.84179835	0.15025640	4.84180141
12	2.00000000	5.38629105	0.03724360	5.38629436

d.

i	t_i	w_i	h_i	y_i
1	0.06250000	1.06817960	0.06250000	1.06817960
5	0.31250000	1.42861668	0.06250000	1.42861361
10	0.62500000	1.90768386	0.06250000	1.90767015
13	0.81250000	2.08668486	0.06250000	2.08666541
16	1.00000000	2.11800208	0.06250000	2.11797955

3. The following tables list representative results from the Adams Variable Step-Size Predictor-Corrector Algorithm.

a.

i	t_i	w_i	h_i	y_i
5	1.10431651	1.00463041	0.02086330	1.00463045
15	1.31294952	1.03196889	0.02086330	1.03196898
25	1.59408142	1.08714711	0.03122028	1.08714722
35	2.00846205	1.18327922	0.04824992	1.18327937
45	2.66272188	1.34525123	0.07278716	1.34525143
52	3.40193112	1.52940900	0.11107035	1.52940924
57	4.00000000	1.67623887	0.12174963	1.67623914

b.

i	t_i	w_i	h_i	y_i
5	1.18519603	0.20333499	0.03703921	0.20333497
15	1.55558810	0.73586642	0.03703921	0.73586631
25	1.92598016	1.48072467	0.03703921	1.48072442
35	2.29637222	2.51764797	0.03703921	2.51764743
45	2.65452689	3.92602442	0.03092051	3.92602332
55	2.94341188	5.50206466	0.02584049	5.50206279
61	3.00000000	5.87410206	0.00122679	5.87409998

c.

i	t_i	w_i	h_i	y_i
5	0.16854008	−1.83303780	0.03370802	−1.83303783
17	0.64833341	−1.42945306	0.05253230	−1.42945304
27	1.06742915	−1.21150951	0.04190957	−1.21150932
41	1.75380240	−1.05819340	0.06681937	−1.05819325
51	2.50124702	−1.01335240	0.07474446	−1.01335258
61	3.00000000	−1.00494507	0.01257155	−1.00494525

d.

i	t_i	w_i	h_i	y_i
5	0.28548652	0.32153668	0.05709730	0.32153674
15	0.85645955	0.24281066	0.05709730	0.24281095
20	1.35101725	0.15096743	0.09891154	0.15096772
25	1.66282314	0.09815109	0.06236118	0.09815137
29	1.91226786	0.06418555	0.06236118	0.06418579
33	2.00000000	0.05434530	0.02193303	0.05434551

5. The current after 2 seconds is approximately $i(2) = 8.693$ amperes.

7. The population after 5 years is 56,751.

Exercise Set 5.8 (Page 329)

1. The Extrapolation Algorithm gives the results in the following tables.

a.

i	t_i	w_i	h	k	y_i
1	0.25	0.04543132	0.25	3	0.04543123
2	0.50	0.28361684	0.25	3	0.28361652
3	0.75	1.05257634	0.25	4	1.05257615
4	1.00	3.21909944	0.25	4	3.21909932

b.

i	t_i	w_i	h	k	y_i
1	2.25	1.44999987	0.25	3	1.45000000
2	2.50	1.83333321	0.25	3	1.83333333
3	2.75	2.17857133	0.25	3	2.17857143
4	3.00	2.49999993	0.25	3	2.50000000

c.

i	t_i	w_i	h	k	y_i
1	1.25	2.77892942	0.25	3	2.77892944
2	1.50	3.60819763	0.25	3	3.60819766
3	1.75	4.47932759	0.25	3	4.47932763
4	2.00	5.38629431	0.25	3	5.38629436

d.

i	t_i	w_i	h	k	y_i
1	0.25	1.32914981	0.25	3	1.32914981
2	0.50	1.73048976	0.25	3	1.73048976
3	0.75	2.04147203	0.25	3	2.04147203
4	1.00	2.11797954	0.25	3	2.11797955

3. The Extrapolation Algorithm gives the results in the following tables.

a.

i	t_i	w_i	h	k	y_i
1	1.50	1.06726237	0.50	4	1.06726235
2	2.00	1.18123223	0.50	3	1.18123222
3	2.50	1.30460372	0.50	3	1.30460371
4	3.00	1.42951608	0.50	3	1.42951607
5	3.50	1.55364771	0.50	3	1.55364770
6	4.00	1.67623915	0.50	3	1.67623914

b.

i	t_i	w_i	h	k	y_i
1	1.50	0.64387537	0.50	4	0.64387533
2	2.00	1.66128182	0.50	5	1.66128176
3	2.50	3.25801550	0.50	5	3.25801536
4	3.00	5.87410027	0.50	5	5.87409998

c.

i	t_i	w_i	h	k	y_i
1	0.50	−1.53788284	0.50	4	−1.53788284
2	1.00	−1.23840584	0.50	5	−1.23840584
3	1.50	−1.09485175	0.50	5	−1.09485175
4	2.00	−1.03597242	0.50	5	−1.03597242
5	2.50	−1.01338570	0.50	5	−1.01338570
6	3.00	−1.00494526	0.50	4	−1.00494525

d.

i	t_i	w_i	h	k	y_i
1	0.50	0.29875177	0.50	4	0.29875178
2	1.00	0.21662642	0.50	4	0.21662642
3	1.50	0.12458565	0.50	4	0.12458565
4	2.00	0.05434552	0.50	4	0.05434551

5. Extrapolation predicts the coordinates of capture to be (100,145.59). The actual coordinates are (100, 145.59). All coordinates are in feet.

Exercise Set 5.9 (Page 337)

1. The Runge-Kutta for Systems Algorithm gives the results in the following tables.

a.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.200	2.12036583	2.12500839	1.50699185	1.51158743
0.400	4.44122776	4.46511961	3.24224021	3.26598528
0.600	9.73913329	9.83235869	8.16341700	8.25629549
0.800	22.67655977	23.00263945	21.34352778	21.66887674
1.000	55.66118088	56.73748265	56.03050296	57.10536209

b.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.500	0.95671390	0.95672798	−1.08381950	−1.08383310
1.000	1.30654440	1.30655930	−0.83295364	−0.83296776
1.500	1.34416716	1.34418117	−0.56980329	−0.56981634
2.000	1.14332436	1.14333672	−0.36936318	−0.36937457

c.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}	w_{3i}	u_{3i}
0.5	0.70787076	0.70828683	−1.24988663	−1.25056425	0.39884862	0.39815702
1.0	−0.33691753	−0.33650854	−3.01764179	−3.01945051	−0.29932294	−0.30116868
1.5	−2.41332734	−2.41345688	−5.40523279	−5.40844686	−0.92346873	−0.92675778
2.0	−5.89479008	−5.89590551	−8.70970537	−8.71450036	−1.32051165	−1.32544426

d.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}	w_{3i}	u_{3i}
0.2	1.38165297	1.38165325	1.00800000	1.00800000	−0.61833075	−0.61833075
0.5	1.90753116	1.90753184	1.12500000	1.12500000	−0.09090565	−0.09090566
0.7	2.25503524	2.25503620	1.34300000	1.34000000	0.26343971	0.26343970
1.0	2.83211921	2.83212056	2.00000000	2.00000000	0.88212058	0.88212056

3. The Runge-Kutta for Systems Algorithm gives the results in the following tables.

a.

t_i	w_{1i}	y_i
0.200	0.00015352	0.00015350
0.500	0.00742968	0.00743027
0.700	0.03299617	0.03299805
1.000	0.17132224	0.17132880

b.

t_i	w_{1i}	y_i
1.200	0.96152437	0.96152583
1.500	0.77796897	0.77797237
1.700	0.59373369	0.59373830
2.000	0.27258237	0.27258872

c.

t_i	w_{1i}	y_i
1.000	3.73162695	3.73170445
2.000	11.31424573	11.31452924
3.000	34.04395688	34.04517155

d.

t_i	w_{1i}	w_{2i}
1.200	0.27273759	0.27273791
1.500	1.08849079	1.08849259
1.700	2.04353207	2.04353642
2.000	4.36156675	4.36157780

5. The predicted number of prey, x_{1i}, and predators, x_{2i}, are given in the following table.

i	t_i	x_{1i}	x_{2i}
10	1.0	4393	1512
20	2.0	288	3175
30	3.0	32	2042
40	4.0	25	1258

7. The approximations for the swinging pendulum problems are given in the following table.

a.

t_i	θ
1.0	−0.365903
2.0	−0.0150563

b.

t_i	θ
1.0	−0.338253
2.0	−0.0862680

9. The Adams fourth-order predictor-corrector method for systems applied to the problems in Exercise 1 gives the results in the following tables.

a.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.200	2.12036583	2.12500839	1.50699185	1.51158743
0.400	4.44122776	4.46511961	3.24224021	3.26598528
0.600	9.73913329	9.83235869	8.16341700	8.25629549
0.800	22.52673210	23.00263945	21.20273983	21.66887674
1.000	54.81242211	56.73748265	55.20490157	57.10536209

b.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.500	0.95675505	0.95672798	−1.08385916	−1.08383310
1.000	1.30659995	1.30655930	−0.83300571	−0.83296776
1.500	1.34420613	1.34418117	−0.56983853	−0.56981634
2.000	1.14334795	1.14333672	−0.36938396	−0.36937457

c.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}	w_{3i}	u_{3i}
0.5	0.70787076	0.70828683	−1.24988663	−1.25056425	0.39884862	0.39815702
1.0	−0.33691753	−0.33650854	−3.01764179	−3.01945051	−0.29932294	−0.30116868
1.5	−2.41332734	−2.41345688	−5.40523279	−5.40844686	−0.92346873	−0.92675778
2.0	−5.88968402	−5.89590551	−8.72213325	−8.71450036	−1.32972524	−1.32544426

d.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}	w_{3i}	u_{3i}
0.2	1.38165297	1.38165325	1.00800000	1.00800000	−0.61833075	−0.61833075
0.5	1.90752882	1.90753184	1.12500000	1.12500000	−0.09090527	−0.09090566
0.7	2.25503040	2.25503620	1.34300000	1.34300000	0.26344040	0.26343970
1.0	2.83211032	2.83212056	2.00000000	2.00000000	0.88212163	0.88212056

Exercise Set 5.10 (Page 348)

1. Let L be the Lipschitz constant for ϕ. Then

$$u_{i+1} - v_{i+1} = u_i - v_i + h[\phi(t_i, u_i, h) - \phi(t_i, v_i, h)],$$

so

$$|u_{i+1} - v_{i+1}| \leq (1 + hL)|u_i - v_i| \leq (1 + hL)^{i+1}|u_0 - v_0|.$$

3. By Exercise 32 in Section 5.4, we have

$$\begin{aligned}\phi(t,w,h) &= \frac{1}{6}f(t,w) + \frac{1}{3}f\left(t+\frac{1}{2}h, w+\frac{1}{2}hf(t,w)\right)\\ &+ \frac{1}{3}f\left(t+\frac{1}{2}h, w+\frac{1}{2}hf\left(t+\frac{1}{2}h, w+\frac{1}{2}hf(t,w)\right)\right)\\ &+ \frac{1}{6}f\left(t+h, w+hf\left(t+\frac{1}{2}h, w+\frac{1}{2}hf\left(t+\frac{1}{2}h, w+\frac{1}{2}hf(t,w)\right)\right)\right),\end{aligned}$$

so

$$\phi(t,w,0) = \frac{1}{6}f(t,w) + \frac{1}{3}f(t,w) + \frac{1}{3}f(t,w) + \frac{1}{6}f(t,w) = f(t,w).$$

5. a. The local truncation error is $\tau_{i+1} = \frac{1}{4}h^3y^{(4)}(\xi_i)$, for some ξ, where $t_{i-2} < \xi_i < t_{i+1}$.

b. The method is consistent but unstable and not convergent.

7. The method is unstable.

Exercise Set 5.11 (Page 355)

1. Euler's method gives the results in the following tables.

a.

t_i	w_i	y_i
0.200	0.027182818	0.449328964
0.500	0.000027183	0.030197383
0.700	0.000000272	0.004991594
1.000	0.000000000	0.000335463

b.

t_i	w_i	y_i
0.200	0.373333333	0.046105213
0.500	−0.093333333	0.250015133
0.700	0.146666667	0.490000277
1.000	1.333333333	1.000000001

c.

t_i	w_i	y_i
0.500	16.47925	0.479470939
1.000	256.7930	0.841470987
1.500	4096.142	0.997494987
2.000	65523.12	0.909297427

d.

t_i	w_i	y_i
0.200	6.128259	1.000000001
0.500	−378.2574	1.000000000
0.700	−6052.063	1.000000000
1.000	387332.0	1.000000000

3. The Runge-Kutta fourth-order method gives the results in the following tables.

a.

t_i	w_i	y_i
0.200	0.45881186	0.44932896
0.500	0.03181595	0.03019738
0.700	0.00537013	0.00499159
1.000	0.00037239	0.00033546

b.

t_i	w_i	y_i
0.200	0.07925926	0.04610521
0.500	0.25386145	0.25001513
0.700	0.49265127	0.49000028
1.000	1.00250560	1.00000000

c.

t_i	w_i	y_i
0.500	188.3082	0.47947094
1.000	35296.68	0.84147099
1.500	6632737	0.99749499
2.000	1246413200	0.90929743

d.

t_i	w_i	y_i
0.200	−215.7459	1.00000000
0.500	−555750.0	1.00000000
0.700	−104435653	1.00000000
1.000	−269031268010	1.00000000

5. The Adams Fourth-Order Predictor-Corrector Algorithm gives the results in the following tables.

a.

t_i	w_i	y_i
0.200	0.4588119	0.4493290
0.500	−0.0112813	0.0301974
0.700	0.0013734	0.0049916
1.000	0.0023604	0.0003355

b.

t_i	w_i	y_i
0.200	0.0792593	0.0461052
0.500	0.1554027	0.2500151
0.700	0.5507445	0.4900003
1.000	0.7278557	1.0000000

c.

t_i	w_i	y_i
.500	188.3082	0.4794709
1.000	38932.03	0.8414710
1.500	9073607	0.9974950
2.000	2115741299	0.9092974

d.

t_i	w_i	y_i
0.200	−215.7459	1.000000001
0.500	−682637.0	1.000000000
0.700	−159172736	1.000000000
1.000	−566751172258	1.000000000

7. The Trapezoidal Algorithm gives the results in the following tables.

a.

t_i	w_i	k	y_i
0.200	0.39109643	2	0.44932896
0.500	0.02134361	2	0.03019738
0.700	0.00307084	2	0.00499159
1.000	0.00016759	2	0.00033546

b.

t_i	w_i	k	y_i
0.200	0.04000000	2	0.04610521
0.500	0.25000000	2	0.25001513
0.700	0.49000000	2	0.49000028
1.000	1.00000000	2	1.00000000

c.

t_i	w_i	k	y_i
0.500	0.66291133	2	0.47947094
1.000	0.87506346	2	0.84147099
1.500	1.00366141	2	0.99749499
2.000	0.91053267	2	0.90929743

d.

t_i	w_i	k	y_i
0.200	−1.07568307	4	1.00000000
0.500	−0.97868360	4	1.00000000
0.700	−0.99046408	3	1.00000000
1.000	−1.00284456	3	1.00000000

9. a.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.100	−96.33011	0.66987648	193.6651	−0.33491554
0.200	−28226.32	0.67915383	56453.66	−0.33957692
0.300	−8214056	0.69387881	16428113	−0.34693941
0.400	−2390290586	0.71354670	4780581173	−0.35677335
0.500	−695574560790	0.73768711	1391149121600	−0.36884355

b.

t_i	w_{1i}	u_{1i}	w_{2i}	u_{2i}
0.100	0.61095960	0.66987648	−0.21708179	−0.33491554
0.200	0.66873489	0.67915383	−0.31873903	−0.33957692
0.300	0.69203679	0.69387881	−0.34325535	−0.34693941
0.400	0.71322103	0.71354670	−0.35612202	−0.35677335
0.500	0.73762953	0.73768711	−0.36872840	−0.36884355

11. The Backward Euler method applied to $y' = \lambda y$ gives

$$w_{i+1} = \frac{w_i}{1 - h\lambda}, \quad \text{so} \quad Q(h\lambda) = \frac{1}{1 - h\lambda}.$$

13. The following tables list the results of the Backward Euler method applied to the problems in Exercise 2.

a.

i	t_i	w_i	k	y_i
2	0.2	1.67216224	2	1.58928220
4	0.4	1.69987544	2	1.62715998
6	0.6	1.92400672	2	1.87190587
8	0.8	2.28233119	2	2.24385657
10	1.0	2.75757631	2	2.72501978

b.

i	t_i	w_i	k	y_i
2	0.2	0.87957046	2	0.56787944
4	0.4	0.56989261	2	0.44978707
6	0.6	0.64247315	2	0.60673795
8	0.8	0.81061829	2	0.80091188
10	1.0	1.00265457	2	1.00012341

c.

i	t_i	w_i	k	y_i
1	1.25	0.55006309	2	0.51199999
3	1.75	0.19753128	2	0.18658892
5	2.25	0.09060118	2	0.08779150
7	2.75	0.04900207	2	0.04808415

d.

i	t_i	w_i	k	y_i
1	0.25	0.79711852	2	0.96217447
3	0.75	0.72203841	2	0.73168856
5	1.25	0.31248267	2	0.31532236
7	1.75	−0.17796016	2	−0.17824606

Exercise Set 6.1 (Page 371)

1. a. Intersecting lines with solution $x_1 = x_2 = 1$.

b. One line, so there is an infinite number of solutions with $x_2 = \frac{3}{2} - \frac{1}{2}x_1$.

c. One line, so there is an infinite number of solutions with $x_2 = -\frac{1}{2}x_1$.

d. Intersecting lines with solution $x_1 = \frac{2}{7}$ and $x_2 = -\frac{11}{7}$.

3. a. $x_1 = 1.0, x_2 = -0.98, x_3 = 2.9$

b. $x_1 = 1.1, x_2 = -1.1, x_3 = 2.9$

5. Gaussian elimination gives the following solutions.

a. $x_1 = 1.1875, x_2 = 1.8125, x_3 = 0.875$ with one row interchange required

b. $x_1 = -1, x_2 = 0, x_3 = 1$ with no interchange required

c. $x_1 = 1.5, x_2 = 2, x_3 = -1.2, x_4 = 3$ with no interchange required

d. No unique solution

7. Gaussian elimination with single precision arithmetic gives the following solutions:

a. $x_1 = -227.0769, x_2 = 476.9231, x_3 = -177.6923$;

b. $x_1 = 1.001291, x_2 = 1, x_3 = 1.00155$;

c. $x_1 = -0.03174600, x_2 = 0.5952377, x_3 = -2.380951, x_4 = 2.777777$;

d. $x_1 = 1.918129, x_2 = 1.964912, x_3 = -0.9883041, x_4 = -3.192982, x_5 = -1.134503$.

9. a. When $\alpha = -1/3$, there is no solution.

b. When $\alpha = 1/3$, there is an infinite number of solutions with $x_1 = x_2 + 1.5$, and x_2 is arbitrary.

c. If $\alpha \neq \pm 1/3$, then the unique solution is

$$x_1 = \frac{3}{2(1+3\alpha)} \quad \text{and} \quad x_2 = \frac{-3}{2(1+3\alpha)}.$$

11. a. There is sufficient food to satisfy the average daily consumption.

b. We could add 200 of species 1, or 150 of species 2, or 100 of species 3, or 100 of species 4.

c. Assuming none of the increases indicated in part (b) was selected, species 2 could be increased by 650, or species 3 could be increased by 150, or species 4 could be increased by 150.

d. Assuming none of the increases indicated in parts (b) or (c) were selected, species 3 could be increased by 150, or species 4 could be increased by 150.

13. Suppose $x'_1, \ldots, x'_n$ is a solution to the linear system (6.1).

a. The new system becomes

$$E_1 : a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$E_i : \lambda a_{i1}x_1 + \lambda a_{i2}x_2 + \cdots + \lambda a_{in}x_n = \lambda b_i$$
$$\vdots$$
$$E_n : a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.$$

Clearly, $x'_1, \ldots, x'_n$ satisfies this system. Conversely, if $x^*_1, \ldots, x^*_n$ satisfies the new system, dividing E_i by λ shows $x^*_1, \ldots, x^*_n$ also satisfies (6.1).

b. The new system becomes

$$E_1 : a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$E_i : (a_{i1} + \lambda a_{j1})x_1 + (a_{i2} + \lambda a_{j2})x_2 + \cdots + (a_{in} + \lambda a_{jn})x_n = b_i + \lambda b_j$$
$$\vdots$$
$$E_n : a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.$$

Clearly, $x'_1, \ldots, x'_n$ satisfies all but possibly the ith equation. Multiplying E_j by λ gives

$$\lambda a_{j1}x'_1 + \lambda a_{j2}x'_2 + \cdots + \lambda a_{jn}x'_n = \lambda b_j,$$

which can be subtracted from E_i in the new system results in the system (6.1). Thus, $x'_1, \ldots, x'_n$ satisfies the new system. Conversely, if $x^*_1, \ldots, x^*_n$ is a solution to the new system, then all but possibly E_i of (6.1) are satisfied by $x^*_1, \ldots, x^*_n$. Multiplying E_j of the new system by $-\lambda$ gives

$$-\lambda a_{j1}x^*_1 - \lambda a_{j2}x^*_2 - \cdots - \lambda a_{jn}x^*_n = -\lambda b_j.$$

Adding this to E_i in the new system produces E_i of (6.1). Thus, $x^*_1, \ldots, x^*_n$ is a solution of (6.1).

c. The new system and the old system have the same set of equations to satisfy. Thus, they have the same solution set.

15. The Gauss-Jordan method gives the following results.

a. $x_1 = 0.98, x_2 = -0.98, x_3 = 2.9$ **b.** $x_1 = 1.1, x_2 = -1.0, x_3 = 2.9$

17. b. The results for this exercise are listed in the following table. (The abbreviations M/D and A/S are used for multiplications/divisions and additions/subtractions, respectively.)

	Gaussian Elimination		Gauss-Jordan	
n	M/D	A/S	M/D	A/S
3	17	11	21	12
10	430	375	595	495
50	44150	42875	64975	62475
100	343300	338250	509950	499950

19. The Gaussian-Elimination–Gauss-Jordan hybrid method gives the following results.

a. $x_1 = 1.0, x_2 = -0.98, x_3 = 2.9$ **b.** $x_1 = 1.0, x_2 = -1.0, x_3 = 2.9$

Exercise Set 6.2 (Page 383)

1. a. none **b.** Interchange rows 2 and 3. **c.** none **d.** Interchange rows 1 and 2.

3. a. Interchange rows 1 and 2. **b.** Interchange rows 1 and 3.
c. Interchange rows 1 and 2, then interchange rows 2 and 3. **d.** Interchange rows 1 and 2.

5. a. Interchange rows 1 and 3, then interchange rows 2 and 3. **b.** Interchange rows 2 and 3.
c. Interchange rows 2 and 3. **d.** Interchange rows 1 and 3, then interchange rows 2 and 3.

7. a. Interchange rows 1 and 2, and columns 1 and 3, then interchange rows 2 and 3, and columns 2 and 3.
b. Interchange rows 1 and 2, and columns 1 and 3, then interchange rows 2 and 3.
c. Interchange rows 1 and 2, and columns 1 and 3, then interchange rows 2 and 3.
d. Interchange rows 1 and 2, and columns 1 and 2, then interchange rows 2 and 3; and columns 2 and 3.

9. Gaussian elimination with three-digit chopping arithmetic gives the following results.
a. $x_1 = 30.0, x_2 = 0.990$ **b.** $x_1 = 0.00, x_2 = 10.0, x_3 = 0.142$
c. $x_1 = 0.206, x_2 = 0.0154, x_3 = -0.0156, x_4 = -0.716$ **d.** $x_1 = 0.828, x_2 = -3.32, x_3 = 0.153, x_4 = 4.91$

11. Gaussian elimination with three-digit rounding arithmetic gives the following results.
a. $x_1 = -10.0, x_2 = 1.01$ **b.** $x_1 = 0.00, x_2 = 10.0, x_3 = 0.143$
c. $x_1 = 0.185, x_2 = 0.0103, x_3 = -0.0200, x_4 = -1.12$ **d.** $x_1 = 0.799, x_2 = -3.12, x_3 = 0.151, x_4 = 4.56$

13. Gaussian elimination with partial pivoting and three-digit chopping arithmetic gives the following results.
a. $x_1 = 10.0, x_2 = 1.00$ **b.** $x_1 = -0.163, x_2 = 9.98, x_3 = 0.142$
c. $x_1 = 0.177, x_2 = -0.0072, x_3 = -0.0208, x_4 = -1.18$ **d.** $x_1 = 0.777, x_2 = -3.10, x_3 = 0.161, x_4 = 4.50$

15. Gaussian elimination with partial pivoting and three-digit rounding arithmetic gives the following results.
a. $x_1 = 10.0, x_2 = 1.00$ **b.** $x_1 = 0.00, x_2 = 10.0, x_3 = 0.143$
c. $x_1 = 0.178, x_2 = 0.0127, x_3 = -0.0204, x_4 = -1.16$ **d.** $x_1 = 0.845, x_2 = -3.37, x_3 = 0.182, x_4 = 5.07$

17. Gaussian elimination with scaled partial pivoting and three-digit chopping arithmetic gives the following results.
a. $x_1 = 10.0, x_2 = 1.00$ **b.** $x_1 = -0.163, x_2 = 9.98, x_3 = 0.142$
c. $x_1 = 0.171, x_2 = 0.0102, x_3 = -0.0217, x_4 = -1.27$ **d.** $x_1 = 0.687, x_2 = -2.66, x_3 = 0.117, x_4 = 3.59$

19. Gaussian elimination with scaled partial pivoting and three-digit rounding arithmetic gives the following results.
a. $x_1 = 10.0, x_2 = 1.00$ **b.** $x_1 = 0.00, x_2 = 10.0, x_3 = 0.143$
c. $x_1 = 0.180, x_2 = 0.0128, x_3 = -0.0200, x_4 = -1.13$ **d.** $x_1 = 0.783, x_2 = -3.12, x_3 = 0.147, x_4 = 4.53$

21. a. $x_1 = 9.98, x_2 = 1.00$ **b.** $x_1 = 0.0724, x_2 = 10.0, x_3 = 0.0952$
c. $x_1 = 0.161, x_2 = 0.0125, x_3 = -0.0232, x_4 = -1.42$ **d.** $x_1 = 0.719, x_2 = -2.86, x_3 = 0.146, x_4 = 4.00$

23. a. $x_1 = 10.0, x_2 = 1.00$ **b.** $x_1 = 0.00, x_2 = 10.0, x_3 = 0.143$
c. $x_1 = 0.179, x_2 = 0.0127, x_3 = -0.0203, x_4 = -1.15$ **d.** $x_1 = 0.874, x_2 = -3.49, x_3 = 0.192, x_4 = 5.33$

25. b. $i_1 = 2.43478$ amps, $i_2 = 4.53846$ amps, $i_3 = -0.23077$ amps
c. $i_1 = 23.0$ amps, $i_2 = 6.54$ amps, $i_3 = 2.97$ amps
d. Actual (c) $i_1 = 9.53$ amps, $i_2 = 6.56$ amps, $i_3 = 2.97$ amps. With pivoting $i_1 = 9.52$ amps, $i_2 = 6.55$ amps, $i_3 = 2.97$ amps.

Exercise Set 6.3 (Page 394)

1. a. $\begin{bmatrix} 4 \\ -18 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 7 & -16 \end{bmatrix}$

3. a. $\begin{bmatrix} -4 & 10 \\ 1 & 15 \end{bmatrix}$ **b.** $\begin{bmatrix} 11 & 4 & -8 \\ 6 & 13 & -12 \end{bmatrix}$ **c.** $\begin{bmatrix} -1 & 5 & -3 \\ 3 & 4 & -11 \\ -6 & -7 & -4 \end{bmatrix}$ **d.** $\begin{bmatrix} -2 & 1 \\ -14 & 7 \\ 6 & 1 \end{bmatrix}$

5. a. The matrix is singular. **b.** $\begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{bmatrix}$ **c.** The matrix is singular. **d.** $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ -\frac{3}{14} & \frac{1}{7} & 0 & 0 \\ \frac{3}{28} & -\frac{11}{7} & 1 & 0 \\ -\frac{1}{2} & 1 & -1 & 1 \end{bmatrix}$

7. The solutions to the linear systems obtained in parts (a) and (b) are, from left to right,

$$x_1 = 3, x_2 = -6, x_3 = -2, x_4 = -1 \quad \text{and} \quad x_1 = x_2 = x_3 = x_4 = 1$$

9. No, since the products $A_{ij}B_{jk}$, for $1 \le i, j, k \le 2$, cannot be formed.
The following are necessary and sufficient conditions:
a. The number of columns of A is the same as the number of rows of B.
b. The number of vertical lines of A equals the number of horizontal lines of B.
c. The placement of the vertical lines of A is identical to placement of the horizontal lines of B.

11. a. $A^2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ \frac{1}{6} & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A^4 = A$, $A^5 = A^2$, $A^6 = I, \ldots$

b.

	Year 1	Year 2	Year 3	Year 4
Age 1	6000	36000	12000	6000
Age 2	6000	3000	18000	6000
Age 3	6000	2000	1000	6000

c.

$$A^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ \frac{1}{6} & 0 & 0 \end{bmatrix}.$$

The i, j-entry is the number of beetles of age i necessary to produce one beetle of age j.

13. a. Suppose $\tilde{A}$ and $\hat{A}$ are both inverses of A. Then $A\tilde{A} = \tilde{A}A = I$ and $A\hat{A} = \hat{A}A = I$. Thus,

$$\tilde{A} = \tilde{A}I = \tilde{A}(A\hat{A}) = (\tilde{A}A)\hat{A} = I\hat{A} = \hat{A}.$$

b. $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$ and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$, so $(AB)^{-1} = B^{-1}A^{-1}$ since there is only one inverse.

c. Since $A^{-1}A = AA^{-1} = I$, it follows that A^{-1} is nonsingular. Since the inverse is unique, we have $(A^{-1})^{-1} = A$.

15. a. We have

$$\begin{bmatrix} 7 & 4 & 4 & 0 \\ -6 & -3 & -6 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2(x_0 - x_1) + \alpha_0 + \alpha_1 \\ 3(x_1 - x_0) - \alpha_1 - 2\alpha_0 \\ \alpha_0 \\ x_0 \end{bmatrix} = \begin{bmatrix} 2(x_0 - x_1) + 3\alpha_0 + 3\alpha_1 \\ 3(x_1 - x_0) - 3\alpha_1 - 6\alpha_0 \\ 3\alpha_0 \\ x_0 \end{bmatrix}$$

b. $B = A^{-1} = \begin{bmatrix} -1 & -\frac{4}{3} & -\frac{4}{3} & 0 \\ 2 & \frac{7}{3} & 2 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

17. The answers are the same as those in Exercise 5.

Exercise Set 6.4 (Page 403)

1. The determinants of the matrices are:
a. −8 **b.** 14 **c.** 0 **d.** 3

3. The answers are the same as in Exercise 1.

5. $\alpha = -\frac{3}{2}$ and $\alpha = 2$

7. $\alpha = -5$

9. a. $\bar{x} = x_1 + ix_2 = re^{i\alpha}$, where $r = \sqrt{x_1^2 + x_2^2}$, $\alpha = \tan^{-1}\frac{x_2}{x_1}$. So,

$R_\theta \bar{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$. However,

$\bar{y} = re^{i(\alpha+\theta)} = r(\cos(\alpha+\theta) + i\sin(\alpha+\theta)) = (x_1\cos\theta - x_2\sin\theta) + i(x_2\cos\theta - x_1\sin\theta) = y_1 + iy_2$. So, $\bar{y} = R_0\bar{x}$

b. $R_\theta^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R_{-\theta}$

c. $R_{\frac{\pi}{6}}\bar{x} = \begin{bmatrix} \frac{1}{2}\sqrt{3} - 1 \\ \sqrt{3} + \frac{1}{2} \end{bmatrix}$ and $R_{\frac{-\pi}{6}}\bar{x} = \begin{bmatrix} \frac{1}{2}\sqrt{3} + 1 \\ \sqrt{3} - \frac{1}{2} \end{bmatrix}$

d. $\det R_\theta = \det R_\theta^{-1} = 1$

11. a. $\det A = 0$

b. If $\det A \neq 0$, the system would have the unique solution $(0, 0, 0, 0)^t$ which would not make sense in the context

c. $x_1 = \frac{1}{2}x_4$, $x_2 = x_4$, $x_3 = \frac{1}{2}x_4$, x_4 is any positive, even integer.

13. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \tilde{A} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Expanding along the third rows gives

$$\begin{aligned}\det A =& a_{31}\det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} - a_{32}\det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} + a_{33}\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ =& a_{31}(a_{12}a_{23} - a_{13}a_{22}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})\end{aligned}$$

and

$$\begin{aligned}\det \tilde{A} =& a_{31}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{bmatrix} - a_{32}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{11} & a_{13} \end{bmatrix} + a_{33}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \\ =& a_{31}(a_{13}a_{22} - a_{12}a_{23}) - a_{32}(a_{13}a_{21} - a_{11}a_{23}) + a_{33}(a_{12}a_{21} - a_{11}a_{22}) = -\det A.\end{aligned}$$

The other two cases are similar.

15. a. The solution is $x_1 = 0$, $x_2 = 10$, and $x_3 = 26$.

b. We have $D_1 = -1$, $D_2 = 3$, $D_3 = 7$, and $D = 0$, and there are no solutions.

c. We have $D_1 = D_2 = D_3 = D = 0$, and there are infinitely many solutions.

d. Cramer's rule requires 39 multiplications/divisions and 20 additions/subtractions.

Exercise Set 6.5 (Page 413)

1. a. $x_1 = -3$, $x_2 = 3$, $x_3 = 1$

b. $x_1 = \frac{1}{2}$, $x_2 = -\frac{9}{2}$, $x_3 = \frac{7}{2}$

3. a. $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d. $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

5. a. $L = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix}$

b. $L = \begin{bmatrix} 1 & 0 & 0 \\ -2.106719 & 1 & 0 \\ 3.067193 & 1.197756 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1.012 & -2.132 & 3.104 \\ 0 & -0.3955257 & -0.4737443 \\ 0 & 0 & -8.939141 \end{bmatrix}$

c. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & -1.33333 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.849190 & 1 & 0 & 0 \\ -0.4596433 & -0.2501219 & 1 & 0 \\ 2.768661 & -0.3079435 & -5.352283 & 1 \end{bmatrix}$

and

$$U = \begin{bmatrix} 2.175600 & 4.023099 & -2.173199 & 5.196700 \\ 0 & 13.43947 & -4.018660 & 10.80698 \\ 0 & 0 & -0.8929510 & 5.091692 \\ 0 & 0 & 0 & 12.03614 \end{bmatrix}$$

7. a. $x_1 = 1$, $x_2 = 2$, $x_3 = -1$

b. $x_1 = 1$, $x_2 = 1$, $x_3 = 1$

c. $x_1 = 1.5$, $x_2 = 2$, $x_3 = -1.199998$, $x_4 = 3$

d. $x_1 = 2.939851$, $x_2 = 0.07067770$, $x_3 = 5.677735$, $x_4 = 4.379812$

9. a. $P^tLU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$ **b.** $P^tLU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 4 \end{bmatrix}$

11. a. $A = PLU = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix}$ The initial population must be $(200, 200, 200, 200)^t$

b. The initial population must be $(200, 400, 800, -300)^t$. The negative entry shows that the population after 1 year can never be 100 females of each age.

13. a. To compute P^tLU requires $\frac{1}{3}n^3 - \frac{1}{3}n$ multiplications/divisions and $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$ additions/subtractions.

b. If $\tilde{P}$ is obtained from P by a simple row interchange, then $\det \tilde{P} = -\det P$. Thus, if $\tilde{P}$ is obtained from P by k interchanges, we have $\det \tilde{P} = (-1)^k \det P$.

c. Only $n - 1$ multiplications are needed in addition to the operations in part (a).

d. We have $\det A = -741$. Factoring and computing $\det A$ requires 75 multiplications/divisions and 55 additions/subtractions.

Exercise Set 6.6 (Page 429)

1. a. The only symmetric matrix is (a).

b. All are nonsingular.

c. Matrices (a) and (b) are strictly diagonally dominant.

d. The only positive definite matrix is (a).

3. a.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

b.

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 1.0 & 0.0 & 0.0 \\ 0.25 & -0.45454545 & 1.0 & 0.0 \\ 0.25 & 0.27272727 & 0.076923077 & 1.0 \end{bmatrix},$$

$$D = \begin{bmatrix} 4.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 2.75 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.1818182 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.5384615 \end{bmatrix}$$

c.

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.25 & 1.0 & 0.0 & 0.0 \\ -0.25 & -0.27272727 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.44 & 1.0 \end{bmatrix},$$

$$D = \begin{bmatrix} 4.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 2.75 & 0.0 & 0.0 \\ 0.0 & 0.0 & 4.5454545 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.12 \end{bmatrix}$$

d.

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.33333333 & 1.0 & 0.0 & 0.0 \\ 0.16666667 & 0.2 & 1.0 & 0.0 \\ -0.16666667 & 0.1 & -0.24324324 & 1.0 \end{bmatrix},$$

$$D = \begin{bmatrix} 6.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3.3333333 & 0.0 & 0.0 \\ 0.0 & 0.0 & 3.7 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.5810811 \end{bmatrix}$$

5. Choleski's Algorithm gives the following results.

a. $L = \begin{bmatrix} 1.414213 & 0 & 0 \\ -0.7071069 & 1.224743 & 0 \\ 0 & -0.8164972 & 1.154699 \end{bmatrix}$

b. $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0.5 & 1.658311 & 0 & 0 \\ 0.5 & -0.7537785 & 1.087113 & 0 \\ 0.5 & 0.4522671 & 0.08362442 & 1.240346 \end{bmatrix}$

c. $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0.5 & 1.658311 & 0 & 0 \\ -0.5 & -0.4522671 & 2.132006 & 0 \\ 0 & 0 & 0.9380833 & 1.766351 \end{bmatrix}$

d. $L = \begin{bmatrix} 2.449489 & 0 & 0 & 0 \\ 0.8164966 & 1.825741 & 0 & 0 \\ 0.4082483 & 0.3651483 & 1.923538 & 0 \\ -0.4082483 & 0.1825741 & -0.4678876 & 1.606574 \end{bmatrix}$

7. The modified factorization algorithm gives the following results.

a. $x_1 = 1, x_2 = -1, x_3 = 0$

b. $x_1 = 0.2, x_2 = -0.2, x_3 = -0.2, x_4 = 0.25$

c. $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = 2$

d. $x_1 = -0.8586387, x_2 = 2.418848, x_3 = -0.9581152, x_4 = -1.272251$

9. The modified Choleski's algorithm gives the following results.

a. $x_1 = 1, x_2 = -1, x_3 = 0$

b. $x_1 = 0.2, x_2 = -0.2, x_3 = -0.2, x_4 = 0.25$

c. $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = 2$

d. $x_1 = -0.85863874, x_2 = 2.4188482, x_3 = -0.95811518, x_4 = -1.2722513$

11. The Crout Factorization Algorithm gives the following results.

a. $x_1 = 0.5$, $x_2 = 0.5$, $x_3 = 1$

b. $x_1 = -0.9999995$, $x_2 = 1.999999$, $x_3 = 1$

c. $x_1 = 1$, $x_2 = -1$, $x_3 = 0$

d. $x_1 = -0.09357798$, $x_2 = 1.587156$, $x_3 = -1.167431$, $x_4 = 0.5412844$

13. We have $x_i = 1$, for each $i = 1, \ldots, 10$.

15. Only the matrix in (d) is positive definite.

17. $-2 < \alpha < \frac{3}{2}$

19. $0 < \beta < 1$ and $3 < \alpha < 5 - \beta$

21. **a.** Since $\det A = 3\alpha - 2\beta$, A is singular if and only if $\alpha = 2\beta/3$.

b. $|\alpha| > 1$, $|\beta| < 1$

c. $\beta = 1$

d. $\alpha > \frac{2}{3}$, $\beta = 1$

23. $i_1 = 0.6785047$, $i_2 = 0.4214953$, $i_3 = 0.2570093$, $i_4 = 0.1542056$, $i_5 = 0.1028037$

25. **a.** No, for example, consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

b. Yes, since $A = A^t$.

c. Yes, since $\mathbf{x}^t(A + B)\mathbf{x} = \mathbf{x}^t A\mathbf{x} + \mathbf{x}^t B\mathbf{x}$.

d. Yes, since $\mathbf{x}^t A^2 \mathbf{x} = \mathbf{x}^t A^t A\mathbf{x} = (A\mathbf{x})^t(A\mathbf{x}) \geq 0$, and because A is nonsingular, equality holds only if $\mathbf{x} = \mathbf{0}$.

e. No, for example, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$.

27. One example is $A = \begin{bmatrix} 1.0 & 0.2 \\ 0.1 & 1.0 \end{bmatrix}$.

29. The Crout Factorization Algorithm can be rewritten as follows:

Step 1 Set $l_1 = a_1$; $u_1 = c_1/l_1$.
Step 2 For $i = 2, \ldots, n-1$ set $l_i = a_i - b_i u_{i-1}$; $u_i = c_i/l_i$.
Step 3 Set $l_n = a_n - b_n u_{n-1}$.
Step 4 Set $z_1 = d_1/l_1$.
Step 5 For $i = 2, \ldots, n$ set $z_i = (d_i - b_i z_{i-1})/l_i$.
Step 6 Set $x_n = z_n$.
Step 7 For $i = n-1, \ldots, 1$ set $x_i = z_i - u_i x_{i+1}$.
Step 8 OUTPUT $(x_1, \ldots, x_n)$;
STOP.

31. The Crout Factorization Algorithm requires $5n - 4$ multiplications/divisions and $3n - 3$ additions/subtractions.

Exercise Set 7.1 (Page 447)

1. **a.** We have $||\mathbf{x}||_\infty = 4$ and $||\mathbf{x}||_2 = 5.220153$.

b. We have $||\mathbf{x}||_\infty = 4$ and $||\mathbf{x}||_2 = 5.477226$.

c. We have $||\mathbf{x}||_\infty = 2^k$ and $||\mathbf{x}||_2 = (1 + 4^k)^{1/2}$.

d. We have $||\mathbf{x}||_\infty = 4/(k+1)$ and $||\mathbf{x}||_2 = (16/(k+1)^2 + 4/k^4 + k^4 e^{-2k})^{1/2}$.

3. **a.** We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0, 0, 0)^t$.

b. We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0, 1, 3)^t$.

c. We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (0, 0, \frac{1}{2})^t$.

d. We have $\lim_{k\to\infty} \mathbf{x}^{(k)} = (1, -1, 1)^t$.

5. The l_∞ norms are as follows:

a. 25 **b.** 16 **c.** 4 **d.** 12

7. **a.** We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 8.57 \times 10^{-4}$ and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 2.06 \times 10^{-4}$.

b. We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 0.90$ and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 0.27$.

c. We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 0.5$ and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 0.3$.

d. We have $||\mathbf{x} - \hat{\mathbf{x}}||_\infty = 6.55 \times 10^{-2}$, and $||A\hat{\mathbf{x}} - \mathbf{b}||_\infty = 0.32$.

9. a. Since $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \geq 0$ with equality only if $x_i = 0$ for all i, properties (i) and (ii) in Definition 7.1 hold. Also,

$$\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^{n} |\alpha x_i| = \sum_{i=1}^{n} |\alpha||x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha|\|\mathbf{x}\|_1,$$

so property (iii) holds.
Finally,

$$\|\mathbf{x}+\mathbf{y}\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n}(|x_i| + |y_i|) = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1,$$

so property (iv) also holds.

b. (1a) 8.5 (1b) 10 (1c) $|\sin k| + |\cos k| + e^k$ (1d) $4/(k+1) + 2/k^2 + k^2 e^{-k}$

c. We have

$$\|\mathbf{x}\|_1^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2 = (|x_1| + |x_2| + \cdots + |x_n|)^2$$
$$\geq |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = \sum_{i=1}^{n} |x_i|^2 = \|\mathbf{x}\|_2^2.$$

Thus, $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2$.

11. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $\|AB\|_{\otimes} = 2$, but $\|A\|_{\otimes} \cdot \|B\|_{\otimes} = 1$.

13. b. We have

5a. $\|A\|_F = \sqrt{326}$
5b. $\|A\|_F = \sqrt{326}$
5c. $\|A\|_F = 4$
5d. $\|A\|_F = \sqrt{148}$.

15. That $\|\mathbf{x}\| \geq 0$ follows easily. That $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$ follows from the definition of positive definite. In addition,

$$\|\alpha\mathbf{x}\| = \left[(\alpha\mathbf{x}^t)\, S(\alpha\mathbf{x})\right]^{\frac{1}{2}} = \left[\alpha^2\mathbf{x}^t S\mathbf{x}\right]^{\frac{1}{2}} = |\alpha| \left(\mathbf{x}^t S\mathbf{x}\right)^{\frac{1}{2}} = |\alpha|\|\mathbf{x}\|.$$

From Cholesky's factorization, let $S = LL^t$. Then

$$\begin{aligned}
\mathbf{x}^t S\mathbf{y} &= \mathbf{x}^t LL^t\mathbf{y} = \left(L^t\mathbf{x}\right)^t \left(L^t\mathbf{y}\right) \\
&\leq \left[\left(L^t\mathbf{x}\right)^t \left(L^t\mathbf{x}\right)\right]^{1/2} \left[\left(L^t\mathbf{y}\right)^t \left(L^t\mathbf{y}\right)\right]^{1/2} \\
&= \left(\mathbf{x}^t LL^t\mathbf{x}\right)^{1/2} \left(\mathbf{y}^t LL^t\mathbf{y}\right)^{1/2} = \left(\mathbf{x}^t S\mathbf{x}\right)^{1/2} \left(\mathbf{y}^t S\mathbf{y}\right)^{1/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= \left[(\mathbf{x}+\mathbf{y})^t\, S\,(\mathbf{x}+\mathbf{y})\right] = \left[\mathbf{x}^t S\mathbf{x} + \mathbf{y}^t S\mathbf{x} + \mathbf{x}^t S\mathbf{y} + \mathbf{y}^t S\mathbf{y}\right] \\
&\leq \mathbf{x}^t S\mathbf{x} + 2\left(\mathbf{x}^t S\mathbf{x}\right)^{1/2} \left(\mathbf{y}^t S\mathbf{y}\right)^{1/2} + \left(\mathbf{y}^t S\mathbf{y}\right)^{1/2} \\
&= \mathbf{x}^t S\mathbf{x} + 2\|\mathbf{x}\|\|\mathbf{y}\| + \mathbf{y}^t S\mathbf{y} = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.
\end{aligned}$$

This demonstrates properties (*i*) – (*iv*) of Definition 7.1.

17. It is not difficult to show that (*i*) holds. If $\|A\| = 0$, then $\|A\mathbf{x}\| = 0$ for all vectors $\mathbf{x}$ with $\|\mathbf{x}\| = 1$. Using $\mathbf{x} = (1, 0, \ldots, 0)^t$, $\mathbf{x} = (0, 1, 0, \ldots, 0)^t, \ldots,$ and $\mathbf{x} = (0, \ldots, 0, 1)^t$ successively implies that each column of A is zero. Thus, $\|A\| = 0$ if and only if $A = 0$. Moreover,

$$\|\alpha A\| = \max_{\|\mathbf{x}\|=1} \|(\alpha A\mathbf{x})\| = |\alpha| \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = |\alpha| \cdot \|A\|,$$

$$\|A+B\| = \max_{\|\mathbf{x}\|=1} \|(A+B)\mathbf{x}\| \leq \max_{\|\mathbf{x}\|=1}(\|A\mathbf{x}\| + \|B\mathbf{x}\|),$$

so

$$||A+B|| \le \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| + \max_{\|\mathbf{x}\|=1} \|B\mathbf{x}\| = \|A\| + \|B\|$$

and

$$\|AB\| = \max_{\|\mathbf{x}\|=1} \|(AB)\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|A(B\mathbf{x})\|.$$

Thus,

$$||AB|| \le \max_{\|\mathbf{x}\|=1} \|A\| \; \|B\mathbf{x}\| = \|A\| \max_{\|\mathbf{x}\|=1} \|B\mathbf{x}\| = \|A\| \; \|B\|.$$

19. First note that the right-hand side of the inequality is unchanged if $\mathbf{x}$ is replaced by any vector $\hat{\mathbf{x}}$ with $|x_i| = |\hat{x}_i|$ for each $i = 1, 2, \ldots n$. Then choose the new vector $\hat{\mathbf{x}}$ so that $\hat{x}_i y_i \ge 0$ for each i, and apply the inequality to $\hat{\mathbf{x}}$ and $\mathbf{y}$.

Exercise Set 7.2 (Page 454)

1. a. The eigenvalue $\lambda_1 = 3$ has the eigenvector $\mathbf{x}_1 = (1, -1)^t$, and the eigenvalue $\lambda_2 = 1$ has the eigenvector $\mathbf{x}_2 = (1, 1)^t$.

b. The eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$ has the eigenvector $\mathbf{x} = \left(1, \frac{1+\sqrt{5}}{2}\right)^t$, and the eigenvalue $\lambda_2 = \frac{1-\sqrt{5}}{2}$ has the eigenvector $\mathbf{x} = \left(1, \frac{1-\sqrt{5}}{2}\right)^t$.

c. The eigenvalue $\lambda_1 = \frac{1}{2}$ has the eigenvector $\mathbf{x}_1 = (1, 1)^t$, and the eigenvalue $\lambda_2 = -\frac{1}{2}$ has the eigenvector $\mathbf{x}_2 = (1, -1)^t$.

d. The eigenvalue $\lambda_1 = \lambda_2 = 3$ has the eigenvectors $\mathbf{x}_1 = (0, 0, 1)^t$ and $\mathbf{x}_2 = (1, 1, 0)^t$, and the eigenvalue $\lambda_3 = 1$ has the eigenvector $\mathbf{x}_3 = (-1, 1, 0)^t$.

e. The eigenvalue $\lambda_1 = 7$ has the eigenvector $\mathbf{x}_1 = (1, 4, 4)^t$, the eigenvalue $\lambda_2 = 3$ has the eigenvector $\mathbf{x}_2 = (1, 2, 0)^t$, and the eigenvalue $\lambda_3 = -1$ has the eigenvector $\mathbf{x}_3 = (1, 0, 0)^t$.

f. The eigenvalue $\lambda_1 = 5$ has the eigenvector $\mathbf{x}_1 = (1, 2, 1)^t$, and the eigenvalue $\lambda_2 = \lambda_3 = 1$ has the eigenvectors $\mathbf{x}_2 = (-1, 0, 1)^t$ and $\mathbf{x}_3 = (-1, 1, 0)^t$.

3. a. The eigenvalues $\lambda_1 = 2 + \sqrt{2}i$ and $\lambda_2 = 2 - \sqrt{2}i$ have eigenvectors $\mathbf{x}_1 = (-\sqrt{2}i, 1)^t$ and $\mathbf{x}_2 = (\sqrt{2}i, 1)^t$.

b. The eigenvalues $\lambda_1 = (3 + \sqrt{7}i)/2$ and $\lambda_2 = (3 - \sqrt{7}i)/2$ have eigenvectors $\mathbf{x}_1 = ((1 - \sqrt{7}i)/2, 1)^t$ and $\mathbf{x}_2 = ((1 + \sqrt{7}i)/2, 1)$.

5. a. 3 **b.** $\frac{1+\sqrt{5}}{2}$ **c.** $\frac{1}{2}$ **d.** 3 **e.** 7 **f.** 5

7. Only the matrix in 1(c) is convergent.

9. a. 3 **b.** 1.618034 **c.** 0.5 **d.** 3 **e.** 8.224257 **f.** 5.203527

11. Since

$$A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{2^k-1}{2^{k+1}} & 2^{-k} \end{bmatrix}, \text{ we have } \lim_{k\to\infty} A_1^k = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Also,

$$A_2^k = \begin{bmatrix} 2^{-k} & 0 \\ \frac{16k}{2^{k-1}} & 2^{-k} \end{bmatrix}, \text{ so } \lim_{k\to\infty} A_2^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

13. a. We have the real eigenvalue $\lambda = 1$ with the eigenvector $\mathbf{x} = (6, 3, 1)^t$.

b. Choose any multiple of the vector $(6, 3, 1)^t$.

15. Let A be an $n \times n$ matrix. Expanding across the first row gives the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)M_{11} + \sum_{j=2}^{n} (-1)^{j+1} a_{1j} M_{1j}.$$

The determinants M_{1j} are of the form

$$M_{1j} = \det \begin{bmatrix} a_{21} & a_{22}-\lambda & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,j-1}-\lambda & a_{j-1,j+1} & \cdots & a_{j-1,n} \\ a_{j,1} & a_{j,2} & \cdots & a_{j,j-1} & a_{j,j+1} & \cdots & a_{j,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,j-1} & a_{j+1,j+1}-\lambda & \cdots & a_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn}-\lambda \end{bmatrix},$$

for $j = 2, \ldots, n$. Note that each M_{1j} has $n-2$ entries of the form $a_{ii} - \lambda$. Thus,

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)M_{11} + \{\text{terms of degree } n-2 \text{ or less}\}.$$

Since

$$M_{11} = \det \begin{bmatrix} a_{22}-\lambda & a_{23} & \cdots & \cdots & a_{2n} \\ a_{32} & a_{33}-\lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{n-1,n} \\ a_{n2} & \cdots & \cdots & a_{n,n-1} & a_{nn}-\lambda \end{bmatrix}$$

is of the same form as $\det(A - \lambda I)$, the same argument can be repeatedly applied to determine

$$p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) + \{\text{terms of degree } n-2 \text{ or less in } \lambda\}.$$

Thus, $p(\lambda)$ is a polynomial of degree n.

17. a. $\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I)$

b. If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$, and, by induction, $A^k\mathbf{x} = \lambda^k\mathbf{x}$.

c. If $A\mathbf{x} = \lambda\mathbf{x}$ and A^{-1} exists, then $\mathbf{x} = \lambda A^{-1}\mathbf{x}$. By Exercise 16 (b), $\lambda \neq 0$, so $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$.

d. Since $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$, we have $(A^{-1})^2\mathbf{x} = \frac{1}{\lambda}A^{-1}\mathbf{x} = \frac{1}{\lambda^2}\mathbf{x}$. Mathematical induction gives

$$(A^{-1})^k\mathbf{x} = \frac{1}{\lambda^k}\mathbf{x}.$$

e. If $A\mathbf{x} = \lambda\mathbf{x}$, then

$$q(A)\mathbf{x} = q_0\mathbf{x} + q_1A\mathbf{x} + \cdots + q_kA^k\mathbf{x} = q_0\mathbf{x} + q_1\lambda\mathbf{x} + \cdots + q_k\lambda^k\mathbf{x} = q(\lambda)\mathbf{x}.$$

f. Let $A - \alpha I$ be nonsingular. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$(A - \alpha I)\mathbf{x} = A\mathbf{x} - \alpha I\mathbf{x} = \lambda\mathbf{x} - \alpha\mathbf{x} = (\lambda - \alpha)\mathbf{x}.$$

Thus,

$$\frac{1}{\lambda - \alpha}\mathbf{x} = (A - \alpha I)^{-1}\mathbf{x}.$$

19. For

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

we have $\rho(A) = \rho(B) = 1$ and $\rho(A + B) = 3$.

Exercise Set 7.3 (Page 465)

1. Two iterations of Jacobi's method gives the following results.

a. $(0.1428571, -0.3571429, 0.4285714)^t$ **b.** $(0.97, 0.91, 0.74)^t$

c. $(-0.65, 1.65, -0.4, -2.475)^t$ **d.** $(1.325, -1.6, 1.6, 1.675, 2.425)^t$

3. Two iterations of the Gauss-Seidel method give the following results.

a. $\mathbf{x}^{(2)} = (0.1111111, -0.2222222, 0.6190476)^t$ **b.** $\mathbf{x}^{(2)} = (0.979, 0.9495, 0.7899)^t$

c. $\mathbf{x}^{(2)} = (-0.5, 2.64, -0.336875, -2.267375)^t$

d. $\mathbf{x}^{(2)} = (1.189063, -1.521354, 1.862396, 1.882526, 2.255645)^t$

5. Jacobi's Algorithm gives the following results.

a. $\mathbf{x}^{(8)} = (0.0351008, -0.2366338, 0.6581273)^t$ **b.** $\mathbf{x}^{(6)} = (0.9957250, 0.9577750, 0.7914500)^t$

c. $\mathbf{x}^{(21)} = (-0.7971058, 2.7951707, -0.2593958, -2.2517930)^t$

d. $\mathbf{x}^{(12)} = (0.7870883, -1.003036, 1.866048, 1.912449, 1.985707)^t$

7. The Gauss-Seidel Algorithm gives the following results.

a. $\mathbf{x}^{(6)} = (0.03535107, -0.2367886, 0.6577590)^t$ **b.** $\mathbf{x}^{(4)} = (0.9957475, 0.9578738, 0.7915748)^t$

c. $\mathbf{x}^{(10)} = (-0.7973091, 2.794982, -0.2589884, -2.251798)^t$

d. $\mathbf{x}^{(7)} = (0.7866825, -1.002719, 1.866283, 1.912562, 1.989790)^t$

9. a.

$$T_j = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \text{ and } \det(\lambda I - T_j) = \lambda^3 + \frac{5}{4}x.$$

Thus, the eigenvalues of T_j are 0 and $\pm\frac{\sqrt{5}}{2}i$, so $\rho(T_j) = \frac{\sqrt{5}}{2} > 1$.

b. $\mathbf{x}^{(25)} = (-20.827873, 2.0000000, -22.827873)^t$

c.

$$T_g = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \text{ and } \det(\lambda I - T_g) = \lambda\left(\lambda + \frac{1}{2}\right)^2.$$

Thus, the eigenvalues of T_g are 0, $-1/2$, and $-1/2$; and $\rho(T_g) = 1/2$.

d. $\mathbf{x}^{(23)} = (1.0000023, 1.9999975, -1.0000001)^t$ is within 10^{-5} in the l_∞ norm.

11. a. A is not strictly diagonally dominant.

b.

$$T_g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0.75 \\ 0 & 0 & -0.625 \end{bmatrix} \quad \text{and} \quad \rho(T_g) = 0.625.$$

c. With $\mathbf{x}^{(0)} = (0, 0, 0)^t$, $\mathbf{x}^{(13)} = (0.89751310, -0.80186518, 0.7015543)^t$

d. $\rho(T_g) = 1.375$. Since T_g is not convergent, the Gauss-Seidel method will not converge.

13. The results for this exercise are listed on page 847 in Exercise 9, where additional results are given for a method presented in Section 7.4.

15. a. The equations were reordered so all $a_{ii} \neq 0$ for $i = 1, 2, \ldots, 8$.

b.(i). $F_1 \approx -0.00265$
$F_2 \approx -6339.745$
$F_3 \approx -3660.255$
$f_1 \approx -8965.753$
$f_2 \approx 6339.748$
$f_3 \approx 10000$
$f_4 \approx -7320.507$
$f_5 \approx 6339.748$
Jacobi Iterative method required 57 iterations.

(ii). $F_1 \approx -0.003621$
$F_2 \approx -6339.745$
$F_3 \approx -3660.253$
$f_1 \approx -8965.756$
$f_2 \approx 6339.745$
$f_3 \approx 10000$
$f_4 \approx -7320.509$
$f_5 \approx 6339.747$
Gauss-Seidel method required 30 iterations.

17. The matrix $T_j = (t_{ik})$ has entries given by

$$t_{ik} = \begin{cases} 0, & i = k \text{ for } 1 \le i \le n \text{ and } 1 \le k \le n \\ -\dfrac{a_{ik}}{a_{ii}}, & i \ne k \text{ for } 1 \le i \le n \text{ and } 1 \le k \le n. \end{cases}$$

Since A is strictly diagonally dominant,

$$\|T_j\|_\infty = \max_{1\le i\le n} \sum_{\substack{k=1\\k\ne i}}^{n} \left| \frac{a_{ik}}{a_{ii}} \right| < 1.$$

19. a. Since A is a positive definite, $a_{ii} > 0$ for $1 \le i \le n$, and A is symmetric. Thus, A can be written as $A = D - L - L^t$, where D is diagonal with $d_{ii} > 0$ and L is lower triangular. The diagonal of the lower triangular matrix $D - L$ has the positive entries $d_{11} = a_{11}$, $d_{22} = a_{22}, \ldots, d_{nn} = a_{nn}$, so $(D-L)^{-1}$ exists.

b. Since A is symmetric,

$$P^t = \left(A - T_g^t A T_g\right)^t = A^t - T_g^t A^t T_g = A - T_g^t A T_g = P.$$

Thus, P is symmetric.

c. $T_g = (D-L)^{-1}L^t$, so

$$(D-L)T_g = L^t = D - L - D + L + L^t = (D-L) - (D - L - L^t) = (D-L) - A.$$

Since $(D-L)^{-1}$ exists, we have $T_g = I - (D-L)^{-1}A$.

d. Since $Q = (D-L)^{-1}A$, we have $T_g = I - Q$. Note that Q^{-1} exists. By the definition of P we have

$$\begin{aligned} P =& A - T_g^t A T_g = A - \left[I - (D-L)^{-1}A\right]^t A \left[I - (D-L)^{-1}A\right] \\ =& A - [I-Q]^t A [I-Q] = A - \left(I - Q^t\right) A (I-Q) \\ =& A - \left(A - Q^t A\right)(I-Q) = A - \left(A - Q^t A - AQ + Q^t AQ\right) \\ =& Q^t A + AQ - Q^t AQ = Q^t \left[A + \left(Q^t\right)^{-1} AQ - AQ\right] \\ =& Q^t \left[AQ^{-1} + \left(Q^t\right)^{-1} A - A\right] Q. \end{aligned}$$

e. Since

$$AQ^{-1} = A\left[A^{-1}(D-L)\right] = D - L \quad \text{and} \quad \left(Q^t\right)^{-1} A = D - L^t,$$

we have

$$AQ^{-1} + \left(Q^t\right)^{-1} A - A = D - L + D - L^t - \left(D - L - L^t\right) = D.$$

Thus,

$$P = Q^t \left[AQ^{-1} + \left(Q^t\right)^{-1} A - A\right] Q = Q^t D Q.$$

So, for $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x}^t P\mathbf{x} = \mathbf{x}^t Q^t DQ\mathbf{x} = (Q\mathbf{x})^t D(Q\mathbf{x})$.
Since D is a positive diagonal matrix, $(Q\mathbf{x})^t D(Q\mathbf{x}) \geq 0$ unless $Q\mathbf{x} = \mathbf{0}$. However, Q is nonsingular, so $Q\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Thus, P is positive definite.

f. Let λ be an eigenvalue of T_g with the eigenvector $\mathbf{x} \neq \mathbf{0}$. Since $\mathbf{x}^t P\mathbf{x} > 0$,

$$\mathbf{x}^t \left[A - T_g^t A T_g\right] \mathbf{x} > 0$$

and

$$\mathbf{x}^t A\mathbf{x} - \mathbf{x}^t T_g^t A T_g \mathbf{x} > 0.$$

Since $T_g\mathbf{x} = \lambda\mathbf{x}$, we have $\mathbf{x}^t T_g^t = \lambda\mathbf{x}^t$, so

$$\left(1 - \lambda^2\right) \mathbf{x}^t Ax = \mathbf{x}^t A\mathbf{x} - \lambda^2 \mathbf{x}^t Ax > 0.$$

Since A is positive definite, $1 - \lambda^2 > 0$, and $\lambda^2 < 1$. Thus, $|\lambda| < 1$.

g. For any eigenvalue λ of T_g, we have $|\lambda| < 1$. This implies $\rho(T_g) < 1$ and T_g is convergent.

Exercise Set 7.4 (Page 473)

1. Two iterations of the SOR method give the following results.

a. $(-0.0173714, -0.1829986, 0.6680503)^t$ **b.** $(0.9876790, 0.9784935, 0.7899328)^t$

c. $(-0.71885, 2.818822, -0.2809726, -2.235422)^t$

d. $(1.079675, -1.260654, 2.042489, 1.995373, 2.049536)^t$

3. Two iterations of the SOR method with $\omega = 1.3$ give the following results.

a. $\mathbf{x}^{(2)} = (-0.1040103, -0.1331814, 0.6774997)^t$ **b.** $\mathbf{x}^{(2)} = (0.957073, 0.9903875, 0.7206569)^t$

c. $\mathbf{x}^{(2)} = (-1.23695, 3.228752, -0.1523888, -2.041266)^t$

d. $\mathbf{x}^{(2)} = (0.7064258, -0.4103876, 2.417063, 2.251955, 1.061507)^t$

5. The SOR Algorithm gives the following results.

a. $\mathbf{x}^{(11)} = (0.03544356, -0.23718333, 0.65788317)^t$ **b.** $\mathbf{x}^{(7)} = (0.9958341, 0.9579041, 0.7915756)^t$

c. $\mathbf{x}^{(8)} = (-0.7976009, 2.795288, -0.2588293, -2.251768)^t$

d. $\mathbf{x}^{(10)} = (0.7866310, -1.002807, 1.866530, 1.912645, 1.989792)^t$

7. The tridiagonal matrices are in parts (b) and (c).
(9b): For $\omega = 1.012823$ we have $\mathbf{x}^{(4)} = (0.9957846, 0.9578935, 0.7915788)^t$.
(9c): For $\omega = 1.153499$ we have $\mathbf{x}^{(7)} = (-0.7977651, 2.795343, -0.2588021, -2.251760)^t$.

9.

	Jacobi 33 Iterations	Gauss-Seidel 8 Iterations	SOR ($\omega = 1.2$) 13 Iterations
x_1	1.53873501	1.53873270	1.53873549
x_2	0.73142167	0.73141966	0.73142226
x_3	0.10797136	0.10796931	0.10797063
x_4	0.17328530	0.17328340	0.17328480
x_5	0.04055865	0.04055595	0.04055737
x_6	0.08525019	0.08524787	0.08524925
x_7	0.16645040	0.16644711	0.16644868
x_8	0.12198156	0.12197878	0.12198026
x_9	0.10125265	0.10124911	0.10125043
x_{10}	0.09045966	0.09045662	0.09045793
x_{11}	0.07203172	0.07202785	0.07202912
x_{12}	0.07026597	0.07026266	0.07026392
x_{13}	0.06875835	0.06875421	0.06875546
x_{14}	0.06324659	0.06324307	0.06324429
x_{15}	0.05971510	0.05971083	0.05971200
x_{16}	0.05571199	0.05570834	0.05570949
x_{17}	0.05187851	0.05187416	0.05187529
x_{18}	0.04924911	0.04924537	0.04924648
x_{19}	0.04678213	0.04677776	0.04677885
x_{20}	0.04448679	0.04448303	0.04448409
x_{21}	0.04246924	0.04246493	0.04246597
x_{22}	0.04053818	0.04053444	0.04053546
x_{23}	0.03877273	0.03876852	0.03876952
x_{24}	0.03718190	0.03717822	0.03717920
x_{25}	0.03570858	0.03570451	0.03570548
x_{26}	0.03435107	0.03434748	0.03434844
x_{27}	0.03309542	0.03309152	0.03309246
x_{28}	0.03192212	0.03191866	0.03191958
x_{29}	0.03083007	0.03082637	0.03082727
x_{30}	0.02980997	0.02980666	0.02980755
x_{31}	0.02885510	0.02885160	0.02885248
x_{32}	0.02795937	0.02795621	0.02795707
x_{33}	0.02711787	0.02711458	0.02711543
x_{34}	0.02632478	0.02632179	0.02632262
x_{35}	0.02557705	0.02557397	0.02557479
x_{36}	0.02487017	0.02486733	0.02486814
x_{37}	0.02420147	0.02419858	0.02419938
x_{38}	0.02356750	0.02356482	0.02356560
x_{39}	0.02296603	0.02296333	0.02296410
x_{40}	0.02239424	0.02239171	0.02239247
x_{41}	0.02185033	0.02184781	0.02184855
x_{42}	0.02133203	0.02132965	0.02133038
x_{43}	0.02083782	0.02083545	0.02083615
x_{44}	0.02036585	0.02036360	0.02036429
x_{45}	0.01991483	0.01991261	0.01991324
x_{46}	0.01948325	0.01948113	0.01948175
x_{47}	0.01907002	0.01906793	0.01906846
x_{48}	0.01867387	0.01867187	0.01867239
x_{49}	0.01829386	0.01829190	0.01829233
x_{50}	0.71792896	0.01792707	0.01792749
x_{51}	0.01757833	0.01757648	0.01757683

	Jacobi 33 iterations	Gauss-Seidel 8 iterations	SOR ($\omega = 1.2$) 13 iterations
x_{52}	0.01724113	0.01723933	0.01723968
x_{53}	0.01691660	0.01691487	0.01691517
x_{54}	0.01660406	0.01660237	0.01660267
x_{55}	0.01630279	0.01630127	0.01630146
x_{56}	0.01601230	0.01601082	0.01601101
x_{57}	0.01573198	0.01573087	0.01573077
x_{58}	0.01546129	0.01546020	0.01546010
x_{59}	0.01519990	0.01519909	0.01519878
x_{60}	0.01494704	0.01494626	0.01494595
x_{61}	0.01470181	0.01470085	0.01470077
x_{62}	0.01446510	0.01446417	0.01446409
x_{63}	0.01423556	0.01423437	0.01423461
x_{64}	0.01401350	0.01401233	0.01401256
x_{65}	0.01380328	0.01380234	0.01380242
x_{66}	0.01359448	0.01359356	0.01359363
x_{67}	0.01338495	0.01338434	0.01338418
x_{68}	0.01318840	0.01318780	0.01318765
x_{69}	0.01297174	0.01297109	0.01297107
x_{70}	0.01278663	0.01278598	0.01278597
x_{71}	0.01270328	0.01270263	0.01270271
x_{72}	0.01252719	0.01252656	0.01252663
x_{73}	0.01237700	0.01237656	0.01237654
x_{74}	0.01221009	0.01220965	0.01220963
x_{75}	0.01129043	0.01129009	0.01129008
x_{76}	0.01114138	0.01114104	0.01114102
x_{77}	0.01217337	0.01217312	0.01217313
x_{78}	0.01201771	0.01201746	0.01201746
x_{79}	0.01542910	0.01542896	0.01542896
x_{80}	0.01523810	0.01523796	0.01523796

11. a. We have $P_0 = 1$, so the equation $P_1 = \frac{1}{2}P_0 + \frac{1}{2}P_2$ gives $P_1 - \frac{1}{2}P_2 = \frac{1}{2}$. Since $P_i = \frac{1}{2}P_{i-1} + \frac{1}{2}P_{i+1}$, we have $-\frac{1}{2}P_{i-1} + P_i + \frac{1}{2}P_{i+1} = 0$, for $i = 1, \ldots, n-2$. Finally, since $P_n = 0$ and $P_{n-1} = \frac{1}{2}P_{i-2} + \frac{1}{2}P_n$, we have $-\frac{1}{2}P_{n-2} + P_{i-1} = 0$. This gives the linear system.

b. The solution vector is

$$(0.90906840, 0.81814162, 0.72722042, 0.63630504, 0.54539520, 0.45449021, 0.36358911, 0.18179385, 0.27269073, 0.90897290)^t$$

using 62 iterations with $w = 1.25$ and a tolerance of 10^{-5} in the l_∞-norm For $n = 10$.

c. The equations are $P_i = \alpha P_{i-1} + (1-\alpha)P_{i+1}$ for $i = 1, \ldots, n-1$ and the linear system becomes

$$\begin{bmatrix} 1 & -(1-\alpha) & 0 & \cdots & 0 & \cdots & 0 \\ -\alpha & 1 & -(1-\alpha) & \ddots & & & \vdots \\ 0 & -\alpha & 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & -\alpha & 1 & & -(1-\alpha) \\ 0 & \cdots & \cdots & \cdots & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ \vdots \\ \vdots \\ P_{n-1} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

d. The solution vector is (0.49973968, 0.24961354, 0.1245773, 0.62031557, 0.30770075, 0.15140201, 0.73256883, 0.14651284, 0.34186112, 0.48838809)^t using 21 iterations with $w = 1.25$ and a tolerance of 10^{-5} in the l_∞-norm for $n = 10$.

13. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T_ω. Then

$$\begin{aligned}\prod_{i=1}^{n} \lambda_i &= \det T_\omega = \det\left((D-\omega L)^{-1}[(1-\omega)D + \omega U]\right)\\ &= \det(D-\omega L)^{-1}\det((1-\omega)D+\omega U) = \det\left(D^{-1}\right)\det((1-\omega)D)\\ &= \left(\frac{1}{(a_{11}a_{22}\ldots a_{nn})}\right)\left((1-\omega)^n a_{11}a_{22}\ldots a_{nn}\right) = (1-\omega)^n.\end{aligned}$$

Thus,

$$\rho(T_\omega) = \max_{1\le i\le n} |\lambda_i| \ge |\omega - 1|,$$

and $|\omega - 1| < 1$ if and only if $0 < \omega < 2$.

Exercise Set 7.5 (Page 484)

1. The $\|\cdot\|_\infty$ condition numbers are:

a. 50 **b.** 241.37 **c.** 600,002 **d.** 339,866

3.

	$\|\mathbf{x}-\hat{\mathbf{x}}\|_\infty$	$K_\infty(A)\|\mathbf{b}-A\hat{\mathbf{x}}\|_\infty/\|A\|_\infty$
a	8.571429×10^{-4}	1.238095×10^{-2}
b	0.1	3.832060
c	0.04	0.8
d	20	1.152440×10^{5}

5. Gaussian elimination and iterative refinement give the following results.

a. (i) $(-10.0, 1.01)^t$, (ii) $(10.0, 1.00)^t$

b. (i) $(12.0, 0.499, -1.98)^t$, (ii) $(1.00, 0.500, -1.00)^t$

c. (i) $(0.185, 0.0103, -0.0200, -1.12)^t$, (ii) $(0.177, 0.0127, -0.0207, -1.18)^t$

d. (i) $(0.799, -3.12, 0.151, 4.56)^t$, (ii) $(0.758, -3.00, 0.159, 4.30)^t$

7. The matrix is ill conditioned since $K_\infty = 60002$. We have $\tilde{\mathbf{x}} = (-1.0000, 2.0000)^t$.

9. a. $K_\infty\left(H^{(4)}\right) = 28,375$

b. $K_\infty\left(H^{(5)}\right) = 943,656$

c. Actual solution $\mathbf{x} = (-124, 1560, -3960, 2660)^t$;
Approximate solution $\tilde{\mathbf{x}} = (-124.2, 1563.8, -3971.8, 2668.8)^t$; $\|\mathbf{x}-\tilde{\mathbf{x}}\|_\infty = 11.8$; $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_\infty}{\|\mathbf{x}\|_\infty} = 0.02980$;

$$\frac{K_\infty(A)}{1-K_\infty(A)\left(\frac{\|\delta A\|_\infty}{\|A\|_\infty}\right)}\left[\frac{\|\delta b\|_\infty}{\|b\|_\infty} + \frac{\|\delta A\|_\infty}{\|A\|_\infty}\right] = \frac{28375}{1-28375\left(\frac{6.\overline{6}\times 10^{-6}}{2.08\overline{3}}\right)}\left[0 + \frac{6.\overline{6}\times 10^{-6}}{2.08\overline{3}}\right]$$
$$= 0.09987.$$

11. For any vector $\mathbf{x}$, we have

$$\|\mathbf{x}\| = \left\|A^{-1}A\mathbf{x}\right\| \le \left\|A^{-1}\right\| \ \|A\mathbf{x}\|, \quad \text{so} \quad \|A\mathbf{x}\| \ge \frac{\|\mathbf{x}\|}{\left\|A^{-1}\right\|}.$$

Let $\mathbf{x} \ne \mathbf{0}$ be such that $\|\mathbf{x}\| = 1$ and $B\mathbf{x} = \mathbf{0}$. Then

$$\|(A - B)\mathbf{x}\| = \|A\mathbf{x}\| \ge \frac{\|\mathbf{x}\|}{\left\|A^{-1}\right\|}$$

and

$$\frac{\|(A - B)\mathbf{x}\|}{\|A\|} \ge \frac{1}{\left\|A^{-1}\right\| \ \|A\|} = \frac{1}{K(A)}.$$

Since $\|\mathbf{x}\| = 1$,

$$\|(A - B)\mathbf{x}\| \le \|A - B\| \ \|\mathbf{x}\| = \|A - B\| \quad \text{and} \quad \frac{\|A - B\|}{\|A\|} \ge \frac{1}{K(A)}.$$

Exercise Set 7.6 (Page 499)

1. a. $(0.18, 0.13)^t$

b. $(0.19, 0.10)^t$

c. Gaussian elimination gives the best answer since $\mathbf{v}^{(2)} = (0, 0)^t$ in the conjugate gradient method.

d. $(0.13, 0.21)^t$. There is no improvement, although $\mathbf{v}^{(2)} \ne \mathbf{0}$.

3. a. $(1.00, -1.00, 1.00)^t$

b. $(0.827, 0.0453, -0.0357)^t$

c. Partial pivoting and scaled partial pivoting also give $(1.00, -1.00, 1.00)^t$.

d. $(0.776, 0.238, -0.185)^t$;
The residual from (3b) is $(-0.0004, -0.0038, 0.0037)^t$, and the residual from part (3d) is $(0.0022, -0.0038, 0.0024)^t$. There does not appear to be much improvement, if any. Rounding error is more prevalent because of the increase in the number of matrix multiplications.

5. a. $\mathbf{x}^{(2)} = (0.1535933456, -0.1697932117, 0.5901172091)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0.221$.

b. $\mathbf{x}^{(2)} = (0.9993129510, 0.9642734456, 0.7784266575)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0.144$.

c. $\mathbf{x}^{(2)} = (-0.7290954114, 2.515782452, -0.6788904058, -2.331943982)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 2.2$.

d. $\mathbf{x}^{(2)} = (-0.7071108901, -0.0954748881, -0.3441074093, 0.5256091497)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0.39$.

e. $\mathbf{x}^{(2)} = (0.5335968381, 0.9367588935, 1.339920949, 1.743083004, 1.743083004)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 1.3$.

f. $\mathbf{x}^{(2)} = (0.35714286, 1.42857143, 0.35714286, 1.57142857, 0.28571429, 1.57142857)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0$.

7. a. $\mathbf{x}^{(3)} = (0.06185567013, -0.1958762887, 0.6185567010)^t$, $\|\mathbf{r}^{(3)}\|_\infty = 0.4 \times 10^{-9}$.

b. $\mathbf{x}^{(3)} = (0.9957894738, 0.9578947369, 0.7915789474)^t$, $\|\mathbf{r}^{(3)}\|_\infty = 0.1 \times 10^{-9}$.

c. $\mathbf{x}^{(4)} = (-0.7976470579, 2.795294120, -0.2588235305, -2.251764706)^t$, $\|\mathbf{r}^{(4)}\|_\infty = 0.39 \times 10^{-7}$.

d. $\mathbf{x}^{(4)} = (-0.7534246575, 0.04109589039, -0.2808219179, 0.6917808219)^t$, $\|\mathbf{r}^{(4)}\|_\infty = 0.11 \times 10^{-9}$.

e. $\mathbf{x}^{(5)} = (0.4516129032, 0.7096774197, 1.677419355, 1.741935483, 1.806451613)^t$, $\|\mathbf{r}^{(5)}\|_\infty = 0.2 \times 10^{-9}$.

f. $\mathbf{x}^{(2)} = (0.35714286, 1.42857143, 0.35714286, 1.57142857, 0.28571429, 1.57142857)^t$, $\|\mathbf{r}^{(2)}\|_\infty = 0$.

9.

a.	Jacobi 49 Iterations	Gauss-Seidel 28 Iterations	SOR ($\omega = 1.3$) 13 Iterations	Conjugate Gradient 9 Iterations
x_1	0.93406183	0.93406917	0.93407584	0.93407713
x_2	0.97473885	0.97475285	0.97476180	0.97476363
x_3	1.10688692	1.10690302	1.10691093	1.10691243
x_4	1.42346150	1.42347226	1.42347591	1.42347699
x_5	0.85931331	0.85932730	0.85933633	0.85933790
x_6	0.80688119	0.80690725	0.80691961	0.80692197
x_7	0.85367746	0.85370564	0.85371536	0.85372011
x_8	1.10688692	1.10690579	1.10691075	1.10691250
x_9	0.87672774	0.87674384	0.87675177	0.87675250
x_{10}	0.80424512	0.80427330	0.80428301	0.80428524
x_{11}	0.80688119	0.80691173	0.80691989	0.80692252
x_{12}	0.97473885	0.97475850	0.97476265	0.97476392
x_{13}	0.93003466	0.93004542	0.93004899	0.93004987
x_{14}	0.87672774	0.87674661	0.87675155	0.87675298
x_{15}	0.85931331	0.85933296	0.85933709	0.85933979
x_{16}	0.93406183	0.93407462	0.93407672	0.93407768

b.	Jacobi 60 Iterations	Gauss-Seidel 35 Iterations	SOR ($\omega = 1.2$) 23 Iterations	Conjugate Gradient 11 Iterations
x_1	0.39668038	0.39668651	0.39668915	0.39669775
x_2	0.07175540	0.07176830	0.07177348	0.07178516
x_3	−0.23080396	−0.23078609	−0.23077981	−0.23076923
x_4	0.24549277	0.24550989	0.24551535	0.24552253
x_5	0.83405412	0.83406516	0.83406823	0.83407148
x_6	0.51497606	0.51498897	0.51499414	0.51500583
x_7	0.12116003	0.12118683	0.12119625	0.12121212
x_8	−0.24044414	−0.24040991	−0.24039898	−0.24038462
x_9	0.37873579	0.37876891	0.37877812	0.37878788
x_{10}	1.09073364	1.09075392	1.09075899	1.09076341
x_{11}	0.54207872	0.54209658	0.54210286	0.54211344
x_{12}	0.13838259	0.13841682	0.13842774	0.13844211
x_{13}	−0.23083868	−0.23079452	−0.23078224	−0.23076923
x_{14}	0.41919067	0.41923122	0.41924136	0.41925019
x_{15}	1.15015953	1.15018477	1.15019025	1.15019425
x_{16}	0.51497606	0.51499318	0.51499864	0.51500583
x_{17}	0.12116003	0.12119315	0.12120236	0.12121212
x_{18}	−0.24044414	−0.24040359	−0.24039345	−0.24038462
x_{19}	0.37873579	0.37877365	0.37878188	0.37878788
x_{20}	1.09073364	1.09075629	1.09076069	1.09076341
x_{21}	0.39668038	0.39669142	0.39669449	0.39669775
x_{22}	0.07175540	0.07177567	0.07178074	0.07178516
x_{23}	−0.23080396	−0.23077872	−0.23077323	−0.23076923
x_{24}	0.24549277	0.24551542	0.24551982	0.24552253
x_{25}	0.83405412	0.83406793	0.83407025	0.83407148

c.	Jacobi 15 Iterations	Gauss-Seidel 9 Iterations	SOR ($\omega = 1.1$) 8 Iterations	Conjugate Gradient 8 Iterations
x_1	−3.07611424	−3.07611739	−3.07611796	−3.07611794
x_2	−1.65223176	−1.65223563	−1.65223579	−1.65223582
x_3	−0.53282391	−0.53282528	−0.53282531	−0.53282528
x_4	−0.04471548	−0.04471608	−0.04471609	−0.04471604
x_5	0.17509673	0.17509661	0.17509661	0.17509661
x_6	0.29568226	0.29568223	0.29568223	0.29568218
x_7	0.37309012	0.37309011	0.37309011	0.37309011
x_8	0.42757934	0.42757934	0.42757934	0.42757927
x_9	0.46817927	0.46817927	0.46817927	0.46817927
x_{10}	0.49964748	0.49964748	0.49964748	0.49964748
x_{11}	0.52477026	0.52477026	0.52477026	0.52477027
x_{12}	0.54529835	0.54529835	0.54529835	0.54529836
x_{13}	0.56239007	0.56239007	0.56239007	0.56239009
x_{14}	0.57684345	0.57684345	0.57684345	0.57684347
x_{15}	0.58922662	0.58922662	0.58922662	0.58922664
x_{16}	0.59995522	0.59995522	0.59995522	0.59995523
x_{17}	0.60934045	0.60934045	0.60934045	0.60934045
x_{18}	0.61761997	0.61761997	0.61761997	0.61761998
x_{19}	0.62497846	0.62497846	0.62497846	0.62497847
x_{20}	0.63156161	0.63156161	0.63156161	0.63156161
x_{21}	0.63748588	0.63748588	0.63748588	0.63748588
x_{22}	0.64284553	0.64284553	0.64284553	0.64284553
x_{23}	0.64771764	0.64771764	0.64771764	0.64771764
x_{24}	0.65216585	0.65216585	0.65216585	0.65216585
x_{25}	0.65624320	0.65624320	0.65624320	0.65624320
x_{26}	0.65999423	0.65999423	0.65999423	0.65999422
x_{27}	0.66345660	0.66345660	0.66345660	0.66345660
x_{28}	0.66666242	0.66666242	0.66666242	0.66666242
x_{29}	0.66963919	0.66963919	0.66963919	0.66963919
x_{30}	0.67241061	0.67241061	0.67241061	0.67241060
x_{31}	0.67499722	0.67499722	0.67499722	0.67499721
x_{32}	0.67741692	0.67741692	0.67741691	0.67741691
x_{33}	0.67968535	0.67968535	0.67968535	0.67968535
x_{34}	0.68181628	0.68181628	0.68181628	0.68181628
x_{35}	0.68382184	0.68382184	0.68382184	0.68382184
x_{36}	0.68571278	0.68571278	0.68571278	0.68571278
x_{37}	0.68749864	0.68749864	0.68749864	0.68749864
x_{38}	0.68918652	0.68918652	0.68918652	0.68918652
x_{39}	0.69067718	0.69067718	0.69067718	0.69067717
x_{40}	0.68363346	0.68363346	0.68363346	0.68363349

11. a.

Solution	Residual
2.55613420	0.00668246
4.09171393	−0.00533953
4.60840390	−0.01739814
3.64309950	−0.03171624
5.13950533	0.01308093
7.19697808	−0.02081095
7.68140405	−0.04593118
5.93227784	0.01692180
5.81798997	0.04414047
5.85447806	0.03319707
5.94202521	−0.00099947
4.42152959	−0.00072826
3.32211695	0.02363822
4.49411604	0.00982052
4.80968966	0.00846967
3.81108707	−0.01312902

This converges in 6 iterations with tolerance 5.00×10^{-2} in the l_∞ norm and $\|\mathbf{r}^{(6)}\|_\infty = 0.046$.

b.

Solution	Residual
2.55613420	0.00668246
4.09171393	−0.00533953
4.60840390	−0.01739814
3.64309950	−0.03171624
5.13950533	0.01308093
7.19697808	−0.02081095
7.68140405	−0.04593118
5.93227784	0.01692180
5.81798996	0.04414047
5.85447805	0.03319706
5.94202521	−0.00099947
4.42152959	−0.00072826
3.32211694	0.02363822
4.49411603	0.00982052
4.80968966	0.00846967
3.81108707	−0.01312902

This converges in 6 iterations with tolerance 5.00×10^{-2} in the l_∞ norm and $\|\mathbf{r}^{(6)}\|_\infty = 0.046$.

c. All tolerances lead to the same convergence specifications.

13. a. We have $P_0 = 1$, so the equation $P_1 = \frac{1}{2}P_0 + \frac{1}{2}P_2$ gives $P_1 - \frac{1}{2}P_2 = \frac{1}{2}$. Since $P_i = \frac{1}{2}P_{i-1} + \frac{1}{2}P_{i+1}$, we have $-\frac{1}{2}P_{i-1} + P_i + \frac{1}{2}P_{i+1} = 0$, for $i = 1, \ldots, n-2$. Finally, since $P_n = 0$ and $P_{n-1} = \frac{1}{2}P_{i-2} + \frac{1}{2}P_n$, we have $-\frac{1}{2}P_{n-2} + P_{i-1} = 0$. This gives the linear system which contains a positive definite matrix A.

b. For $n = 10$, the solution vector is $(0.909009091, 0.81818182, 0.72727273, 0, 63636364, 0.54545455, 0.45454545, 0.36363636, 0.27272727, 0.18181818, 0.09090909)^t$ using 10 iterations with $C^{-1} = I$ and a tolerance of 10^{-5} in the l_∞-norm.

c. The resulting matrix is not positive definite and the method fails.

d. The method fails.

15. a. Let $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ be a set of nonzero A-orthogonal vectors for the symmetric positive definite matrix A. Then $\langle \mathbf{v}^{(i)}, A\mathbf{v}^{(j)} \rangle = 0$, if $i \neq j$. Suppose

$$c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)} + \cdots + c_n\mathbf{v}^{(n)} = \mathbf{0},$$

where not all c_i are zero. Suppose k is the smallest integer for which $c_k \neq 0$. Then

$$c_k \mathbf{v}^{(k)} + c_{k+1}\mathbf{v}^{(k+1)} + \cdots + c_n \mathbf{v}^{(n)} = \mathbf{0}.$$

We solve for $\mathbf{v}^{(k)}$ to obtain

$$\mathbf{v}^{(k)} = -\frac{c_{k+1}}{c_k}\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}\mathbf{v}^{(n)}.$$

Multiplying by A gives

$$A\mathbf{v}^{(k)} = -\frac{c_{k+1}}{c_k}A\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}A\mathbf{v}^{(n)},$$

so

$$\begin{aligned}(\mathbf{v}^{(k)})^t A\mathbf{v}^{(k)} &= -\frac{c_{k+1}}{c_k}(\mathbf{v}^{(k)})^t A\mathbf{v}^{(k+1)} - \cdots - \frac{c_n}{c_k}(\mathbf{v}^{(k)t})A\mathbf{v}^{(n)} \\ &= -\frac{c_{k+1}}{c_k}\langle \mathbf{v}^{(k)}, A\mathbf{v}^{(k+1)}\rangle - \cdots - \frac{c_n}{c_k}\langle \mathbf{v}^{(k)}, A\mathbf{v}^{(n)}\rangle \\ &= -\frac{c_{k+1}}{c_k}\cdot 0 - \cdots - \frac{c_n}{c_k}\cdot 0.\end{aligned}$$

Since A is positive definite, $\mathbf{v}^{(k)} = \mathbf{0}$, which is a contradiction. Thus, all c_i must be zero, and $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ is linearly independent.

b. Let $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ be a set of nonzero A-orthogonal vectors for the symmetric positive definite matrix A and let $\mathbf{z}$ be orthogonal to $\mathbf{v}^{(i)}$, for each $i = 1, \ldots, n$. From part (a), the set $\{\mathbf{v}^{(1)}, \ldots \mathbf{v}^{(n)}\}$ is linearly independent, so there is a collection of constants $\beta_1, \ldots, \beta_n$ with

$$\mathbf{z} = \sum_{i=1}^{n} \beta_i \mathbf{v}^{(i)}.$$

Hence,

$$\langle \mathbf{z}, \mathbf{z}\rangle = \mathbf{z}^t\mathbf{z} = \sum_{i=1}^{n} \beta_i \mathbf{z}^t \mathbf{v}^{(i)} = \sum_{i=1}^{n} \beta_i \cdot 0 = 0,$$

and Theorem 7.30, part (v), implies that $\mathbf{z} = \mathbf{0}$.

17. If A is a positive definite matrix whose eigenvalues are $0 < \lambda_1 \leq \cdots \leq \lambda_n$, then $||A||_2 = \lambda_n$ and $||A^{-1}||_2 = \frac{1}{\lambda_1}$, so $K_2(A) = \lambda_n/\lambda_1$.
For the matrix A in Example 3, we have

$$K_2(A) = \frac{\lambda_5}{\lambda_1} = \frac{700.031}{0.0570737} = 12265.2,$$

and the matrix AH has

$$K_2(AH) = \frac{\lambda_5}{\lambda_1} = \frac{1.88052}{0.156370} = 12.0261.$$

Exercise Set 8.1 (Page 514)

1. The linear least squares polynomial is $1.70784x + 0.89968$.

3. The least squares polynomials with their errors are, respectively, $0.6208950 + 1.219621x$, with $E = 2.719 \times 10^{-5}$; $0.5965807 + 1.253293x - 0.01085343x^2$, with $E = 1.801 \times 10^{-5}$; and $0.6290193 + 1.185010x + 0.03533252x^2 - 0.01004723x^3$, with $E = 1.741 \times 10^{-5}$.

5. a. The linear least squares polynomial is $72.0845x - 194.138$, with error 329.

b. The least squares polynomial of degree two is $6.61821x^2 - 1.14352x + 1.23556$, with error 1.44×10^{-3}.

c. The least squares polynomial of degree three is $-0.0136742x^3 + 6.84557x^2 - 2.37919x + 3.42904$, with error 5.27×10^{-4}.

d. The least squares approximation of the form be^{ax} is $24.2588e^{0.372382x}$, with error 418.

e. The least squares approximation of the form bx^a is $6.23903x^{2.01954}$, with error 0.00703.

7. a. $k = 0.8996, \quad E(k) = 0.295$

b. $k = 0.9052, E(k) = 0.128$. Part (b) fits the total experimental data best.

9. The least squares line for the point average is 0.101 (ACT score) + 0.487.

11. The linear least squares polynomial gives $y \approx 0.17952x + 8.2084$.

13. a. $\ln R = \ln 1.304 + 0.5756 \ln W$ **b.** $E = 25.25$

c. $\ln R = \ln 1.051 + 0.7006 \ln W + 0.06695(\ln W)^2$ **d.** $E = \sum_{i=1}^{37}\left(R_i - bW_i^a e^{c(\ln W_i)^2}\right)^2 = 20.30$

Exercise Set 8.2 (Page 524)

1. The linear least squares approximations are:

a. $P_1(x) = 1.833333 + 4x$ **b.** $P_1(x) = -1.600003 + 3.600003x$ **c.** $P_1(x) = 1.140981 - 0.2958375x$

d. $P_1(x) = 0.1945267 + 3.000001x$ **e.** $P_1(x) = 0.6109245 + 0.09167105x$ **f.** $P_1(x) = -1.861455 + 1.666667x$

3. The least squares approximations of degree two are:

a. $P_2(x) = 2.000002 + 2.999991x + 1.000009x^2$ **b.** $P_2(x) = 0.4000163 - 2.400054x + 3.000028x^2$

c. $P_2(x) = 1.723551 - 0.9313682x + 0.1588827x^2$ **d.** $P_2(x) = 1.167179 + 0.08204442x + 1.458979x^2$

e. $P_2(x) = 0.4880058 + 0.8291830x - 0.7375119x^2$ **f.** $P_2(x) = -0.9089523 + 0.6275723x + 0.2597736x^2$

5. a. 0.3427×10^{-9} **b.** 0.0457142 **c.** 0.000358354 **d.** 0.0106445 **e.** 0.0000134621 **f.** 0.0000967795

7. The Gram-Schmidt process produces the following collections of polynomials:

a. $\phi_0(x) = 1, \phi_1(x) = x - 0.5, \quad \phi_2(x) = x^2 - x + \frac{1}{6}$, and $\phi_3(x) = x^3 - 1.5x^2 + 0.6x - 0.05$

b. $\phi_0(x) = 1, \phi_1(x) = x - 1, \quad \phi_2(x) = x^2 - 2x + \frac{2}{3}$, and $\phi_3(x) = x^3 - 3x^2 + \frac{12}{5}x - \frac{2}{5}$

c. $\phi_0(x) = 1, \phi_1(x) = x - 2, \quad \phi_2(x) = x^2 - 4x + \frac{11}{3}$, and $\phi_3(x) = x^3 - 6x^2 + 11.4x - 6.8$

9. The least squares polynomials of degree two are:

a. $P_2(x) = 3.833333\phi_0(x) + 4\phi_1(x) + 0.9999998\phi_2(x)$

b. $P_2(x) = 2\phi_0(x) + 3.6\phi_1(x) + 3\phi_2(x) + \phi_3(x)$

c. $P_2(x) = 0.5493061\phi_0(x) - 0.2958369\phi_1(x) + 0.1588785\phi_2(x) + 0.013771507\phi_3(x)$

d. $P_2(x) = 3.194528\phi_0(x) + 3\phi_1(x) + 1.458960\phi_2(x) + 0.4787957\phi_3(x)$

e. $P_2(x) = 0.6567600\phi_0(x) + 0.09167105\phi_1(x) - 0.73751218\phi_2(x) - 0.18769253\phi_3(x)$

f. $P_2(x) = 1.471878\phi_0(x) + 1.666667\phi_1(x) + 0.2597705\phi_2(x) + 0.059387393\phi_3(x)$

11. The Laguerre polynomials are $L_1(x) = x - 1$, $L_2(x) = x^2 - 4x + 2$ and $L_3(x) = x^3 - 9x^2 + 18x - 6$.

13. Let $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ be a linearly independent set of polynomials in $\prod_n$. For each $i = 0, 1, \ldots, n$, let $\phi_i(x) = \sum_{k=0}^n b_{ki}x^k$. Let $Q(x) = \sum_{k=0}^n a_k x^k \in \prod_n$. We want to find constants $c_0, \ldots, c_n$ so that

$$Q(x) = \sum_{i=0}^n c_i\phi_i(x).$$

This equation becomes

$$\sum_{k=0}^n a_k x^k = \sum_{i=0}^n c_i\left(\sum_{k=0}^n b_{ki}x^k\right),$$

so we have both

$$\sum_{k=0}^n a_k x^k = \sum_{k=0}^n\left(\sum_{i=0}^n c_i b_{ki}\right)x^k, \quad \text{and} \quad \sum_{k=0}^n a_k x^k = \sum_{k=0}^n\left(\sum_{i=0}^n b_{ki}c_i\right)x^k.$$

But $\{1, x, \ldots, x^n\}$ is linearly independent, so, for each $k = 0, \ldots, n$, we have

$$\sum_{i=0}^n b_{ki}c_i = a_k,$$

which expands to the linear system

$$\begin{bmatrix} b_{01} & b_{02} & \cdots & b_{0n} \\ b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

This linear system must have a unique solution $\{c_0, c_1, \ldots, c_n\}$, or else there is a nontrivial set of constants $\left\{c_0', c_1', \ldots, c_n'\right\}$, for which

$$\begin{bmatrix} b_{01} & \cdots & b_{0n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} c_0' \\ \vdots \\ c_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus,

$$c_0'\phi_0(x) + c_1'\phi_1(x) + \cdots + c_n'\phi_n(x) = \sum_{k=0}^{n} 0x^k = 0,$$

which contradicts the linear independence of the set $\{\phi_0, \ldots, \phi_n\}$. Thus, there is a unique set of constants $\{c_0, \ldots, c_n\}$, for which

$$Q(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x).$$

15. The normal equations are

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx, \quad \text{for each } j = 0, 1, \ldots, n.$$

Let

$$b_{jk} = \int_a^b x^{j+k} dx, \quad \text{for each } j = 0, \ldots, n, \quad \text{and} \quad k = 0, \ldots, n,$$

and let $B = (b_{jk})$. Further, let

$$\mathbf{a} = (a_0, \ldots, a_n)^t \quad \text{and} \quad \mathbf{g} = \left(\int_a^b f(x)dx, \ldots, \int_a^b x^n f(x)\, dx \right)^t.$$

Then the normal equations produce the linear system $B\mathbf{a} = \mathbf{g}$.

To show that the normal equations have a unique solution, it suffices to show that if $f \equiv 0$ then $\mathbf{a} = \mathbf{0}$. If $f \equiv 0$, then

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = 0, \quad \text{for } j = 0, \ldots, n, \quad \text{and} \quad \sum_{k=0}^{n} a_j a_k \int_a^b x^{j+k} dx = 0, \quad \text{for } j = 0, \ldots, n,$$

and summing over j gives

$$\sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \int_a^b x^{j+k} dx = 0.$$

Thus,

$$\int_a^b \sum_{j=0}^{n} \sum_{k=0}^{n} a_j x^j a_k x^k dx = 0 \quad \text{and} \quad \int_a^b \left(\sum_{j=0}^{n} a_j x^j \right)^2 dx = 0.$$

Define $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then $\int_a^b [P(x)]^2\, dx = 0$ and $P(x) \equiv 0$. This implies that $a_0 = a_1 = \cdots = a_n = 0$, so $\mathbf{a} = \mathbf{0}$. Hence, the matrix B is nonsingular, and the normal equations have a unique solution.

Exercise Set 8.3 (Page 534)

1. The interpolating polynomials of degree two are:

a. $P_2(x) = 2.377443 + 1.590534(x - 0.8660254) + 0.5320418(x - 0.8660254)x$

b. $P_2(x) = 0.7617600 + 0.8796047(x - 0.8660254)$

c. $P_2(x) = 1.052926 + 0.4154370(x - 0.8660254) - 0.1384262x(x - 0.8660254)$

d. $P_2(x) = 0.5625 + 0.649519(x - 0.8660254) + 0.75x(x - 0.8660254)$

3. Bounds for the maximum errors of polynomials in Exercise 1 are:

a. 0.1132617 **b.** 0.04166667 **c.** 0.08333333 **d.** 1.000000

5. The zeros of $\tilde{T}_3$ produce the following interpolating polynomials of degree two.

a. $P_2(x) = 0.3489153 - 0.1744576(x - 2.866025) + 0.1538462(x - 2.866025)(x - 2)$

b. $P_2(x) = 0.1547375 - 0.2461152(x - 1.866025) + 0.1957273(x - 1.866025)(x - 1)$

c. $P_2(x) = 0.6166200 - 0.2370869(x - 0.9330127) - 0.7427732(x - 0.9330127)(x - 0.5)$

d. $P_2(x) = 3.0177125 + 1.883800(x - 2.866025) + 0.2584625(x - 2.866025)(x - 2)$

7. The cubic polynomial $\frac{383}{384}x - \frac{5}{32}x^3$ approximates $\sin x$ with error at most 7.19×10^{-4}.

9. **a.** $n = 1 : \det T_1 = x$

b. $n = 2 : \det T_2 = \det \begin{pmatrix} x & 1 \\ 1 & 2x \end{pmatrix} = 2x^2 - 1$

c. $n = 3: \det T_3 = \det \begin{pmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{pmatrix} = x \det \begin{pmatrix} 2x & 1 \\ 1 & 2x \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ 0 & 2x \end{pmatrix} = x(4x^2 - 1) - 2x = 4x^3 - 3x$

11. The change of variable $x = \cos\theta$ produces

$$\int_{-1}^{1} \frac{T_n^2(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{[\cos(n \arccos x)]^2}{\sqrt{1-x^2}}\,dx = \int_{0}^{\pi} (\cos(n\theta))^2\,dx = \frac{\pi}{2}.$$

13. It was shown in text (see Eq. (8.13)) that the zeros of $T_n'(x)$ occur at $x_k' = \cos(k\pi/n)$ for $k = 1, \ldots, n-1$. Because $x_0' = \cos(0) = 1$, $x_n' = \cos(\pi) = -1$, and all values of the cosine lie in the interval $[-1, 1]$ it remains only to show that the zeros are distinct. This follows from the fact that for each $k = 1, \ldots, n-1$, we have x_k' in the interval $(0, \pi)$ and on this interval $D_x \cos(x) = -\sin x < 0$. As a consequence, $T_n'(x)$ is one to one on $(0, \pi)$, and these $n-1$ zeros of $T_n'(x)$ are distinct.

Exercise Set 8.4 (Page 544)

1. The Padé approximations of degree two for $f(x) = e^{2x}$ are:

$$n = 2, m = 0 : r_{2,0}(x) = 1 + 2x + 2x^2$$

$$n = 1, m = 1 : r_{1,1}(x) = (1 + x)/(1 - x)$$

$$n = 0, m = 2 : r_{0,2}(x) = (1 - 2x + 2x^2)^{-1}$$

i	x_i	$f(x_i)$	$r_{2,0}(x_i)$	$r_{1,1}(x_i)$	$r_{0,2}(x_i)$
1	0.2	1.4918	1.4800	1.5000	1.4706
2	0.4	2.2255	2.1200	2.3333	1.9231
3	0.6	3.3201	2.9200	4.0000	1.9231
4	0.8	4.9530	3.8800	9.0000	1.4706
5	1.0	7.3891	5.0000	undefined	1.0000

3. $r_{2,3}(x) = (1 + \frac{2}{5}x + \frac{1}{20}x^2)/(1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3)$

i	x_i	$f(x_i)$	$r_{2,3}(x_i)$
1	0.2	1.22140276	1.22140277
2	0.4	1.49182470	1.49182561
3	0.6	1.82211880	1.82213210
4	0.8	2.22554093	2.22563652
5	1.0	2.71828183	2.71875000

5. $r_{3,3}(x) = (x - \frac{7}{60}x^3)/(1 + \frac{1}{20}x^2)$

i	x_i	$f(x_i)$	MacLaurin Polynomial of Degree 6	$r_{3,3}(x_i)$
0	0.0	0.00000000	0.00000000	0.00000000
1	0.1	0.09983342	0.09966675	0.09938640
2	0.2	0.19866933	0.19733600	0.19709571
3	0.3	0.29552021	0.29102025	0.29246305
4	0.4	0.38941834	0.37875200	0.38483660
5	0.5	0.47942554	0.45859375	0.47357724

7. The Padé approximations of degree five are:

a. $r_{0,5}(x) = (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5)^{-1}$

b. $r_{1,4}(x) = (1 - \frac{1}{5}x)/(1 + \frac{4}{5}x + \frac{3}{10}x^2 + \frac{1}{15}x^3 + \frac{1}{120}x^4)$

c. $r_{3,2}(x) = (1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3)/(1 + \frac{2}{5}x + \frac{1}{20}x^2)$

d. $r_{4,1}(x) = (1 - \frac{4}{5}x + \frac{3}{10}x^2 - \frac{1}{15}x^3 + \frac{1}{120}x^4)/(1 + \frac{1}{5}x)$

i	x_i	$f(x_i)$	$r_{0,5}(x_i)$	$r_{1,4}(x_i)$	$r_{2,3}(x_i)$	$r_{4,1}(x_i)$
1	0.2	0.81873075	0.81873081	0.81873074	0.81873075	0.81873077
2	0.4	0.67032005	0.67032276	0.67031942	0.67031963	0.67032099
3	0.6	0.54881164	0.54883296	0.54880635	0.54880763	0.54882143
4	0.8	0.44932896	0.44941181	0.44930678	0.44930966	0.44937931
5	1.0	0.36787944	0.36809816	0.36781609	0.36781609	0.36805556

9. $r_{T_{2,0}}(x) = (1.266066T_0(x) - 1.130318T_1(x) + 0.271495 3T_2(x))/T_0(x)$

$r_{T_{1,1}}(x) = (0.994570 5T_0(x) - 0.456904 6T_1(x))/(T_0(x) + 0.480387 45T_1(x))$

$r_{T_{0,2}}(x) = 0.794022 0T_0(x)/(T_0(x) + 0.877857 5T_1(x) + 0.177426 6T_2(x))$

i	x_i	$f(x_i)$	$r_{T_{2,0}}(x_i)$	$r_{T_{1,1}}(x_i)$	$r_{T_{0,2}}(x_i)$
1	0.25	0.77880078	0.74592811	0.78595377	0.74610974
2	0.50	0.60653066	0.56515935	0.61774075	0.58807059
3	1.00	0.36787944	0.40724330	0.36319269	0.38633199

11. $r_{T_{2,2}}(x) = \dfrac{0.91747T_1(x)}{T_0(x) + 0.088914T_2(x)}$

i	x_i	$f(x_i)$	$r_{T_{2,2}}(x_i)$
0	0.00	0.00000000	0.00000000
1	0.10	0.09983342	0.09093843
2	0.20	0.19866933	0.18028797
3	0.30	0.29552021	0.26808992
4	0.40	0.38941834	0.35438412

13. a. $e^x = e^{M\ln\sqrt{10}+s} = e^{M\ln\sqrt{10}}e^s = e^{\ln 10^{\frac{M}{2}}}e^s = 10^{\frac{M}{2}}e^s$

b. $e^s \approx \left(1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3\right) / \left(1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3\right)$, with $|\text{error}| \le 3.75\times 10^{-7}$.

c. Set $M = \text{round}(0.8685889638x)$, $s = x - M/(0.8685889638)$, and $\hat{f} = \left(1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3\right) / \left(1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3\right)$. Then $f = (3.16227766)^M\hat{f}$.

Exercise Set 8.5 (Page 553)

1. $S_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x$

3. $S_3(x) = 3.676078 - 3.676078\cos x + 1.470431\cos 2x - 0.7352156\cos 3x + 3.676078\sin x - 2.940862\sin 2x$

5. $S_n(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{n-1}\frac{1-(-1)^k}{k}\sin kx$

7. The trigonometric least squares polynomials are:

a. $S_2(x) = \cos 2x$

b. $S_2(x) = 0$

c. $S_3(x) = 1.566453 + 0.5886815\cos x - 0.2700642\cos 2x + 0.2175679\cos 3x + 0.8341640\sin x - 0.3097866\sin 2x$

d. $S_3(x) = -2.046326 + 3.883872\cos x - 2.320482\cos 2x + 0.7310818\cos 3x$

9. The trigonometric least squares polynomial is $S_3(x) = -0.4968929 + 0.2391965\cos x + 1.515393\cos 2x + 0.2391965\cos 3x - 1.150649\sin x$, with error $E(S_3) = 7.271197$.

11. The trigonometric least squares polynomials and their errors are

a. $S_3(x) = -0.08676065 - 1.446416\cos\pi(x-3) - 1.617554\cos 2\pi(x-3) + 3.980729\cos 3\pi(x-3) - 2.154320\sin\pi(x-3) + 3.907451\sin 2\pi(x-3)$ with $E(S_3) = 210.90453$

b. $S_3(x) = -0.0867607 - 1.446416\cos\pi(x-3) - 1.617554\cos 2\pi(x-3) + 3.980729\cos 3\pi(x-3) - 2.354088\cos 4\pi(x-3) - 2.154320\sin\pi(x-3) + 3.907451\sin 2\pi(x-3) - 1.166181\sin 3\pi(x-3)$ with $E(S_4) = 169.4943$

13. a. $T_4(x) = 15543.19 + 141.1964\cos(\frac{2}{15}\pi t - \pi) - 203.4015\cos(\frac{4}{15}\pi t - 4\pi) + 274.6943\cos(\frac{2}{5}\pi t - 6\pi) - 210.75\cos(\frac{8}{15}\pi t - 4\pi) + 716.5316\sin(\frac{2}{15}\pi t - \pi) - 286.7289\sin(\frac{4}{15}\pi t - 2\pi) + 453.1107\sin(\frac{2}{5}\pi t - 3\pi)$

b. April 8, 2013, corresponds to $t = 1.27$ with $P_4(1.27) = 14374$, and April 8, 2014, corresponds to $t = 13.27$ with $P_4(13.27) = 16906$

c. $|14374 - 14613| = 239$ and $|16906 - 16256| = 650$. It does not seem to approximate well with a relative error of about 3%

d. June 17, 2014, corresponds to $t = 15.57$ with $P_4(15.57) = 14298$. Since the actual closing was 16808, the approximation was way off.

15. Let $f(-x) = -f(x)$. The integral $\int_{-a}^{0} f(x)\,dx$ under the change of variable $t = -x$ transforms to

$$-\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = -\int_0^a f(t)\,dt = -\int_0^a f(x)\,dx.$$

Thus,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$$

17. The following integrations establish the orthogonality.

$$\int_{-\pi}^{\pi} [\phi_0(x)]^2\,dx = \frac{1}{2}\int_{-\pi}^{\pi} dx = \pi,$$

$$\int_{-\pi}^{\pi} [\phi_k(x)]^2\,dx = \int_{-\pi}^{\pi} (\cos kx)^2\,dx = \int_{-\pi}^{\pi} \left[\frac{1}{2} + \frac{1}{2}\cos 2kx\right]dx = \pi + \left[\frac{1}{4k}\sin 2kx\right]_{-\pi}^{\pi} = \pi,$$

$$\int_{-\pi}^{\pi} [\phi_{n+k}(x)]^2\,dx = \int_{-\pi}^{\pi} (\sin kx)^2\,dx = \int_{-\pi}^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos 2kx\right]dx = \pi - \left[\frac{1}{4k}\sin 2kx\right]_{-\pi}^{\pi} = \pi,$$

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_0(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos kx\,dx = \left[\frac{1}{2k}\sin kx\right]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_0(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin kx\,dx = \left[\frac{-1}{2k}\cos kx\right]_{-\pi}^{\pi} = \frac{-1}{2k}[\cos k\pi - \cos(-k\pi)] = 0,$$

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_j(x)\,dx = \int_{-\pi}^{\pi} \cos kx \cos jx dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(k+j)x + \cos(k-j)x]\,dx = 0,$$

$$\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_{n+j}(x)\,dx = \int_{-\pi}^{\pi} \sin kx \sin jx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(k-j)x - \cos(k+j)x]\,dx = 0,$$

and

$$\int_{-\pi}^{\pi} \phi_k(x)\phi_{n+j}(x)\,dx = \int_{-\pi}^{\pi} \cos kx \sin jx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(k+j)x - \sin(k-j)x]\,dx = 0.$$

19. The steps are nearly identical to those for determining the constants b_k except for the additional constant term a_0 in the cosine series. In this case,

$$0 = \frac{\partial E}{\partial a_0} = 2\sum_{j=0}^{2m-1}[y_j - S_n(x_j)](-1/2) = \sum_{j=0}^{2m-1} y_j - \sum_{j=0}^{2m-1}\left(\frac{a_0}{2} + a_n\cos nx_j + \sum_{k=1}^{n-1}(a_k\cos kx_j + b_k\sin kx_j)\right),$$

The orthogonality implies that only the constant term remains in the second sum, and we have

$$0 = \sum_{j=0}^{2m-1} y_j - \frac{a_0}{2}(2m), \quad \text{which implies that} \quad a_0 = \frac{1}{m}\sum_{j=0}^{2m-1} y_j.$$

Exercise Set 8.6 (Page 565)

1. The trigonometric interpolating polynomials are:

a. $S_2(x) = -12.33701 + 4.934802\cos x - 2.467401\cos 2x + 4.934802\sin x$

b. $S_2(x) = -6.168503 + 9.869604\cos x - 3.701102\cos 2x + 4.934802\sin x$

c. $S_2(x) = 1.570796 - 1.570796\cos x$

d. $S_2(x) = -0.5 - 0.5\cos 2x + \sin x$

3. The Fast Fourier Transform Algorithm gives the following trigonometric interpolating polynomials.

a. $S_4(x) = -11.10331 + 2.467401\cos x - 2.467401\cos 2x + 2.467401\cos 3x - 1.233701\cos 4x + 5.956833\sin x - 2.467401\sin 2x + 1.022030\sin 3x$

b. $S_4(x) = 1.570796 - 1.340759\cos x - 0.2300378\cos 3x$

c. $S_4(x) = -0.1264264 + 0.2602724\cos x - 0.3011140\cos 2x + 1.121372\cos 3x + 0.04589648\cos 4x - 0.1022190\sin x + 0.2754062\sin 2x - 2.052955\sin 3x$

d. $S_4(x) = -0.1526819 + 0.04754278\cos x + 0.6862114\cos 2x - 1.216913\cos 3x + 1.176143\cos 4x - 0.8179387\sin x + 0.1802450\sin 2x + 0.2753402\sin 3x$

5.

	Approximation	Actual
a.	−69.76415	−62.01255
b.	9.869602	9.869604
c.	−0.7943605	−0.2739383
d.	−0.9593287	−0.9557781

7. The b_j terms are all zero. The a_j terms are as follows:

$$\begin{array}{llll}
a_0 = -4.0008033 & a_1 = 3.7906715 & a_2 = -2.2230259 & a_3 = 0.6258042 \\
a_4 = -0.3030271 & a_5 = 0.1813613 & a_6 = -0.1216231 & a_7 = 0.0876136 \\
a_8 = -0.0663172 & a_9 = 0.0520612 & a_{10} = -0.0420333 & a_{11} = 0.0347040 \\
a_{12} = -0.0291807 & a_{13} = 0.0249129 & a_{14} = -0.0215458 & a_{15} = 0.0188421 \\
a_{16} = -0.0166380 & a_{17} = 0.0148174 & a_{18} = -0.0132962 & a_{19} = 0.0120123 \\
a_{20} = -0.0109189 & a_{21} = 0.0099801 & a_{22} = -0.0091683 & a_{23} = 0.0084617 \\
a_{24} = -0.0078430 & a_{25} = 0.0072984 & a_{26} = -0.0068167 & a_{27} = 0.0063887 \\
a_{28} = -0.0060069 & a_{29} = 0.0056650 & a_{30} = -0.0053578 & a_{31} = 0.0050810 \\
a_{32} = -0.0048308 & a_{33} = 0.0046040 & a_{34} = -0.0043981 & a_{35} = 0.0042107 \\
a_{36} = -0.0040398 & a_{37} = 0.0038837 & a_{38} = -0.0037409 & a_{39} = 0.0036102 \\
a_{40} = -0.0034903 & a_{41} = 0.0033803 & a_{42} = -0.0032793 & a_{43} = 0.0031866 \\
a_{44} = -0.0031015 & a_{45} = 0.0030233 & a_{46} = -0.0029516 & a_{47} = 0.0028858 \\
a_{48} = -0.0028256 & a_{49} = 0.0027705 & a_{50} = -0.0027203 & a_{51} = 0.0026747 \\
a_{52} = -0.0026333 & a_{53} = 0.0025960 & a_{54} = -0.0025626 & a_{55} = 0.0025328 \\
a_{56} = -0.0025066 & a_{57} = 0.0024837 & a_{58} = -0.0024642 & a_{59} = 0.0024478 \\
a_{60} = -0.0024345 & a_{61} = 0.0024242 & a_{62} = -0.0024169 & a_{63} = 0.0024125
\end{array}$$

9. a. The trigonometric interpolating polynomial is $S(x) = \frac{31086.25}{2} - \frac{240.25}{2}\cos(\pi x - 8\pi) + 141.0809\cos(\frac{\pi}{8}x - \pi) - 203.4989\cos(\frac{\pi}{4}x - 2\pi) + 274.6464\cos(\frac{3\pi}{8}x - 3\pi) - 210.75\cos(\frac{\pi}{2}x - 4\pi) + 104.2019\cos(\frac{5\pi}{8}x - 5\pi) - 155.7601\cos(\frac{3\pi}{4}x - 6\pi) + 243.0707\cos(\frac{7\pi}{8}x - 7\pi) + 716.5795\sin(\frac{\pi}{8}x - \pi) - 286.6405\sin(\frac{\pi}{4}x - 2\pi) + 453.2262\sin(\frac{3\pi}{8}x - 3\pi) + 22.5\sin(\frac{\pi}{2}x - 4\pi) + 138.9449\sin(\frac{5\pi}{8}x - 5\pi) - 223.8905\sin(\frac{3\pi}{4}x - 6\pi) - 194.2018\sin(\frac{7\pi}{8}x - 7\pi)$

b. April 8, 2013, corresponds to $x = 1.27$ with $S(1.27) = 14721$, and April 8, 2014, corresponds to $x = 13.27$ with $S(13.27) = 16323$

c. $|14613 - 14721| = 108$ with relative error 0.00734 and $|16256 - 16323| = 67$ with relative error 0.00412. The approximations are not that bad.

d. June 17, 2014 corresponds to $x = 15.57$ with $S(15.57) = 15073$. The actual closing was 16808 so the approximation was not good.

11. From Eq. (8.28),

$$c_k = \sum_{j=0}^{2m-1} y_j e^{\frac{\pi i j k}{m}} = \sum_{j=0}^{2m-1} y_j(\zeta)^{jk} = \sum_{j=0}^{2m-1} y_j \left(\zeta^k\right)^j.$$

Thus,

$$c_k = \left(1, \zeta^k, \zeta^{2k}, \ldots, \zeta^{(2m-1)k}\right)^t \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{2m-1} \end{bmatrix},$$

and the result follows.

Exercise Set 9.1 (Page 576)

1. a. The eigenvalues and associated eigenvectors are $\lambda_1 = 2$, $\mathbf{v}^{(1)} = (1, 0, 0)^t$; $\lambda_2 = 1$, $\mathbf{v}^{(2)} = (0, 2, 1)^t$; and $\lambda_3 = -1$, $\mathbf{v}^{(3)} = (-1, 1, 1)^t$. The set is linearly independent.

b. The eigenvalues and associated eigenvectors are $\lambda_1 = 2$, $\mathbf{v}^{(1)} = (0, 1, 0)^t$; $\lambda_2 = 3$, $\mathbf{v}^{(2)} = (1, 0, 1)^t$; and $\lambda_3 = 1$, $\mathbf{v}^{(3)} = (1, 0, -1)^t$. The set is linearly independent.

c. The eigenvalues and associated eigenvectors are $\lambda_1 = 1$, $\mathbf{v}^{(1)} = (0, -1, 1)^t$; $\lambda_2 = 1 + \sqrt{2}$, $\mathbf{v}^{(2)} = (\sqrt{2}, 1, 1)^t$; and $\lambda_3 = 1 - \sqrt{2}$, $\mathbf{v}^{(3)} = (-\sqrt{2}, 1, 1)^t$; The set is linearly independent.

d. The eigenvalues and associated eigenvectors are $\lambda_1 = \lambda_2 = 2$, $\mathbf{v}^{(1)} = \mathbf{v}^{(2)} = (1, 0, 0)^t$; $\lambda_3 = 3$ with $\mathbf{v}^{(3)} = (0, 1, 1)^t$. There are only 2 linearly independent eigenvectors.

3. a. The three eigenvalues are within $\{\lambda \mid |\lambda| \le 2\} \cup \{\lambda \mid |\lambda - 2| \le 2\}$ so $\rho(A) \le 4$.

b. The three eigenvalues are within $\{\lambda \mid |\lambda - 4| \le 2\}$ so $\rho(A) \le 6$.

c. The three real eigenvalues satisfy $0 \le \lambda \le 6$ so $\rho(A) \le 6$.

d. The three real eigenvalues satisfy $1.25 \le \lambda \le 8.25$ so $1.25 \le \rho(A) \le 8.25$.

5. The vectors are linearly dependent since $-2\mathbf{v}_1 + 7\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$.

7. a. (i) $\mathbf{0} = c_1(1, 1)^t + c_2(-2, 1)^t$ implies that $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. But $\det\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = 3 \neq 0$, so by Theorem 6.7, we have $c_1 = c_2 = 0$.
(ii) $\{(1, 1)^t, (-3/2, 3/2)^t\}$.
(iii) $\{(\sqrt{2}/2, \sqrt{2}/2)^t, (-\sqrt{2}/2, \sqrt{2}/2)^t\}$.

b. (i) The determinant of this matrix is $-2 \neq 0$, so $\{(1, 1, 0)^t, (1, 0, 1)^t, (0, 1, 1)^t\}$ is a linearly independent set.
(ii) $\{(1, 1, 0)^t, (1/2, -1/2, 1)^t, (-2/3, 2/3, 2/3)^t\}$
(iii) $\{(\sqrt{2}/2, \sqrt{2}/2, 0)^t, (\sqrt{6}/6, -\sqrt{6}/6, \sqrt{6}/3)^t, (-\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)^t\}$

c. (i) If $\mathbf{0} = c_1(1, 1, 1, 1)^t + c_2(0, 2, 2, 2)^t + c_3(1, 0, 0, 1)^t$, then we have

$$(E_1): c_1 + c_3 = 0, \quad (E_2): c_1 + 2c_2 = 0, \quad (E_3): c_1 + 2c_2 = 0, \quad (E_4): c_1 + 2c_2 + c_3 = 0.$$

Subtracting (E_3) from (E_4) implies that $c_3 = 0$. Hence, from (E_1), we have $c_1 = 0$, and from (E_2), we have $c_2 = 0$. The vectors are linearly independent.
(ii) $\{(1, 1, 1, 1)^t, (-3/2, 1/2, 1/2, 1/2)^t, (0, -1/3, -1/3, 2/3)^t\}$
(iii) $\{(1/2, 1/2, 1/2, 1/2)^t, (-\sqrt{3}/2, \sqrt{3}/6, \sqrt{3}/6, \sqrt{3}/6)^t, (0, -\sqrt{6}/6, -\sqrt{6}/6, \sqrt{6}/3)^t\}$

d. (i) If A is the matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$, then $\det A = 60 \neq 0$, so the vectors are linearly independent.
(ii) $\{(2, 2, 3, 2, 3)^t, (2, -1, 0, -1, 0)^t, (0, 0, 1, 0, -1)^t, (1, 2, -1, 0, -1)^t, (-2/7, 3/7, 2/7, -1, 2/7)^t\}$
(iii) $\{(\sqrt{30}/15, \sqrt{30}/15, \sqrt{30}/10, \sqrt{30}/15, \sqrt{30}/10)^t, (\sqrt{6}/3, -\sqrt{6}/6, 0, -\sqrt{6}/6, 0)^t,$
$(0, 0, \sqrt{2}/2, 0, -\sqrt{2}/2)^t, (\sqrt{7}/7, 2\sqrt{7}/7, -\sqrt{7}/7, 0, -\sqrt{7}/7)^t, (-\sqrt{70}/35, 3\sqrt{70}/70, \sqrt{70}/35, -\sqrt{70}/10, \sqrt{70}/35)^t\}$

9. a. Let μ be an eigenvalue of A. Since A is symmetric, μ is real, and Theorem 9.13 gives $0 \le \mu \le 4$. The eigenvalues of $A - 4I$ are of the form $\mu - 4$. Thus,

$$\rho(A - 4I) = \max|\mu - 4| = \max(4 - \mu) = 4 - \min\mu = 4 - \lambda = |\lambda - 4|.$$

b. The eigenvalues of $A - 4I$ are -3.618034, -2.618034, -1.381966, and -0.381966, so $\rho(A - 4I) = 3.618034$ and $\lambda = 0.381966$. An eigenvector is $(0.618034, 1, 1, 0.618034)^t$.

c. As in part (a), $0 \le \mu \le 6$, so $|\lambda - 6| = \rho(B - 6I)$.

d. The eigenvalues of $B - 6I$ are -5.2360673, -4, -2, and -0.76393202, so $\rho(B - 6I) = 5.2360673$ and $\lambda = 0.7639327$. An eigenvector is $(0.61803395, 1, 1, 0.6180395)^t$.

11. If $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, then for any j, with $1 \le j \le k$, we have $c_1\mathbf{v}_j^t\mathbf{v}_1 + \cdots + c_k\mathbf{v}_j^t\mathbf{v}_k = \mathbf{0}$. But orthogonality gives $c_i\mathbf{v}_j^t\mathbf{v}_i = 0$, for $i \neq j$, so $c_j\mathbf{v}_j^t\mathbf{v}_j = 0$, and since $\mathbf{v}_j^t\mathbf{v}_j \neq 0$, we have $c_j = 0$.

13. Since $\{\mathbf{v}_i\}_{i=1}^n$ is linearly independent in $\mathbb{R}^n$, there exist numbers $c_1, \ldots, c_n$ with

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

Hence, for any k, with $1 \le k \le n$, $\mathbf{v}_k^t\mathbf{x} = c_1\mathbf{v}_k^t\mathbf{v}_1 + \cdots + c_n\mathbf{v}_k^t\mathbf{v}_n = c_k\mathbf{v}_k^t\mathbf{v}_k = c_k$.

15. A strictly diagonally dominant matrix has all its diagonal elements larger in magnitude than the sum of the magnitudes of all the other elements in its row. As a consequence, the magnitude of the center of each Geršgorin circle exceeds in magnitude the radius of the circle. No circle can therefore include the origin. Hence, 0 cannot be an eigenvalue of the matrix, and the matrix is nonsingular.

Exercise Set 9.2 (Page 582)

1. In each instance, we will compare the characteristic polynomial of A, denoted $p(A)$ to that of B, denoted $p(B)$. They must agree if the matrices are to be similar.

a. $p(A) = x^2 - 4x + 3 \neq x^2 - 2x - 3 = p(B)$.

b. $p(A) = x^2 - 5x + 6 \neq x^2 - 6x + 6 = p(B)$.

c. $p(A) = x^3 - 4x^2 + 5x - 2 \neq x^3 - 4x^2 + 5x - 6 = p(B)$.

d. $p(A) = x^3 - 5x^2 + 12x - 11 \neq x^3 - 4x^2 + 4x + 11 = p(B)$.

3. In each case, we have $A^3 = (PDP^{(-1)})(PDP^{(-1)})(PDP^{(-1)}) = PD^3P^{(-1)}$.

a. $\begin{bmatrix} \frac{26}{5} & -\frac{14}{5} \\ -\frac{21}{5} & \frac{19}{5} \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 9 \\ 0 & -8 \end{bmatrix}$ **c.** $\begin{bmatrix} \frac{9}{5} & -\frac{8}{5} & \frac{7}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} & -\frac{6}{5} \end{bmatrix}$ **d.** $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

5. They are all diagonalizable with P and D as follows.

a. $P = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$ **b.** $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

c. $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d. $P = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1+\sqrt{2} & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. All the matrices except (d) have 3 linearly independent eigenvectors. The matrix in part (d) has only 2 linearly independent eigenvectors. One choice for P is each case is

a. $\begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

9. Only the matrices in parts (a) and (c) are positive definite.

a. $Q = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ **c.** $Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11. In each case, the matrix fails to have 3 linearly independent eigenvectors.

a. $\det(A) = 12$, so A is nonsingular. **b.** $\det(A) = -1$, so A is nonsingular.

c. $\det(A) = 12$, so A is nonsingular. **d.** $\det(A) = 1$, so A is nonsingular.

13. The matrix A has an eigenvalue of multiplicity 1 at $\lambda_1 = 3$ with eigenvector $\mathbf{s}_1 = (0, 1, 1)^t$, and an eigenvalue of multiplicity 2 at $\lambda_2 = 2$ with linearly independent eigenvectors $\mathbf{s}_2 = (1, 1, 0)^t$ and $\mathbf{s}_3 = (-2, 0, 1)^t$. Let $S_1 = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$, $S_2 = \{\mathbf{s}_2, \mathbf{s}_1, \mathbf{s}_3\}$, and $S_3 = \{\mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_1\}$. Then $A = S_1^{-1}D_1S_1 = S_2^{-1}D_2S_2 = S_3^{-1}D_3S_3$, so A is similar to D_1, D_2, and D_3.

15. a. The eigenvalues and associated eigenvectors are
$\lambda_1 = 5.307857563,\ (0.59020967, 0.51643129, 0.62044441)^t$;
$\lambda_2 = -0.4213112993,\ (0.77264234, -0.13876278, -0.61949069)^t$;
$\lambda_3 = -0.1365462647,\ (0.23382978, -0.84501102, 0.48091581)^t$.

b. A is not positive definite because $\lambda_2 < 0$ and $\lambda_3 < 0$.

17. Because A is similar to B and B is similar to C, there exist invertible matrices S and T with $A = S^{-1}BS$ and $B = T^{-1}CT$. Hence, A is similar to C because

$$A = S^{-1}BS = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS).$$

19. a. Let the columns of Q be denoted by the vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, which are also the rows of Q^t. Because Q is orthogonal, $(\mathbf{q}_i)^t \cdot \mathbf{q}_j$ is zero when $i \neq j$ and 1 when $i = j$. But the ij- entry of Q^tQ is $(\mathbf{q}_i)^t \cdot \mathbf{q}_j$ for each i and j, so $Q^tQ = I$. Hence, $Q^t = Q^{-1}$.

b. From part (i), we have $Q^tQ = I$, so

$$(Q\mathbf{x})^t(Q\mathbf{y}) = (\mathbf{x}^tQ^t)(Q\mathbf{y}) = \mathbf{x}^t(Q^tQ)\mathbf{y} = \mathbf{x}^t(I)\mathbf{y} = \mathbf{x}^t\mathbf{y}.$$

c. This follows from part (ii) with $\mathbf{x}$ replacing $\mathbf{y}$ since then

$$||Q\mathbf{x}||_2^2 = (Q\mathbf{x})^t(Q\mathbf{x}) = \mathbf{x}^t\mathbf{x} = ||\mathbf{x}||_2^2.$$

Exercise Set 9.3 (Page 599)

1. The approximate eigenvalues and approximate eigenvectors are:

a. $\mu^{(3)} = 3.666667$, $\mathbf{x}^{(3)} = (0.9772727, 0.9318182, 1)^t$

b. $\mu^{(3)} = 2.000000$, $\mathbf{x}^{(3)} = (1, 1, 0.5)^t$

c. $\mu^{(3)} = 5.000000$, $\mathbf{x}^{(3)} = (-0.2578947, 1, -0.2842105)^t$

d. $\mu^{(3)} = 5.038462$, $\mathbf{x}^{(3)} = (1, 0.2213741, 0.3893130, 0.4045802)^t$

3. The approximate eigenvalues and approximate eigenvectors are:

a. $\mu^{(3)} = 1.027730$, $\mathbf{x}^{(3)} = (-0.1889082, 1, -0.7833622)^t$

b. $\mu^{(3)} = -0.4166667$, $\mathbf{x}^{(3)} = (1, -0.75, -0.6666667)^t$

c. $\mu^{(3)} = 17.64493$, $\mathbf{x}^{(3)} = (-0.3805794, -0.09079132, 1)^t$

d. $\mu^{(3)} = 1.378684$, $\mathbf{x}^{(3)} = (-0.3690277, -0.2522880, 0.2077438, 1)^t$

5. The approximate eigenvalues and approximate eigenvectors are:

a. $\mu^{(3)} = 3.959538$, $\mathbf{x}^{(3)} = (0.5816124, 0.5545606, 0.5951383)^t$

b. $\mu^{(3)} = 2.0000000$, $\mathbf{x}^{(3)} = (-0.6666667, -0.6666667, -0.3333333)^t$

c. $\mu^{(3)} = 7.189567$, $\mathbf{x}^{(3)} = (0.5995308, 0.7367472, 0.3126762)^t$

d. $\mu^{(3)} = 6.037037$, $\mathbf{x}^{(3)} = (0.5073714, 0.4878571, -0.6634857, -0.2536857)^t$

7. The approximate eigenvalues and approximate eigenvectors are:

a. $\lambda_1 \approx \mu^{(9)} = 3.999908$, $\mathbf{x}^{(9)} = (0.9999943, 0.9999828, 1)^t$

b. $\lambda_1 \approx \mu^{(13)} = 2.414214$, $\mathbf{x}^{(13)} = (1, 0.7071429, 0.7070707)^t$

c. $\lambda_1 \approx \mu^{(9)} = 5.124749$, $\mathbf{x}^{(9)} = (-0.2424476, 1, -0.3199733)^t$

d. $\lambda_1 \approx \mu^{(24)} = 5.235861$, $\mathbf{x}^{(24)} = (1, 0.6178361, 0.1181667, 0.4999220)^t$

9. a. $\mu^{(9)} = 1.00001523$ with $\mathbf{x}^{(9)} = (-0.19999391, 1, -0.79999087)^t$

b. $\mu^{(12)} = -0.41421356$ with $\mathbf{x}^{(12)} = (1, -0.70709184, -0.707121720)^t$

c. The method did not converge in 25 iterations. However, convergence occurred with $\mu^{(42)} = 1.63663642$ with $\mathbf{x}^{(42)} = (-0.57068151, 0.3633658, 1)^t$

d. $\mu^{(9)} = 1.38195929$ with $\mathbf{x}^{(9)} = (-0.38194003, -0.23610068, 0.23601909, 1)^t$

11. The approximate eigenvalues and approximate eigenvectors are:

a. $\mu^{(8)} = 4.0000000$, $\mathbf{x}^{(8)} = (0.5773547, 0.5773282, 0.5773679)^t$

b. $\mu^{(13)} = 2.414214$, $\mathbf{x}^{(13)} = (-0.7071068, -0.5000255, -0.4999745)^t$

c. $\mu^{(16)} = 7.223663$, $\mathbf{x}^{(16)} = (0.6247845, 0.7204271, 0.3010466)^t$

d. $\mu^{(20)} = 7.086130$, $\mathbf{x}^{(20)} = (0.3325999, 0.2671862, -0.7590108, -0.4918246)^t$

13. The approximate eigenvalues and approximate eigenvectors are:

a. $\lambda_2 \approx \mu^{(1)} = 1.000000$, $\mathbf{x}^{(1)} = (-2.999908, 2.999908, 0)^t$

b. $\lambda_2 \approx \mu^{(1)} = 1.000000$, $\mathbf{x}^{(1)} = (0, -1.414214, 1.414214)^t$

c. $\lambda_2 \approx \mu^{(6)} = 1.636734$, $\mathbf{x}^{(6)} = (1.783218, -1.135350, -3.124733)^t$

d. $\lambda_2 \approx \mu^{(10)} = 3.618177$, $\mathbf{x}^{(10)} = (0.7236390, -1.170573, 1.170675, -0.2763374)^t$

15. The approximate eigenvalues and approximate eigenvectors are:

a. $\mu^{(8)} = 4.000001$, $\mathbf{x}^{(8)} = (0.9999773, 0.99993134, 1)^t$

b. The method fails because of division by zero.

c. $\mu^{(7)} = 5.124890$, $\mathbf{x}^{(7)} = (-0.2425938, 1, -0.3196351)^t$

d. $\mu^{(15)} = 5.236112$, $\mathbf{x}^{(15)} = (1, 0.6125369, 0.1217216, 0.4978318)^t$

17. a. We have $|\lambda| \leq 6$ for all eigenvalues λ.

b. The approximate eigenvalue and approximate eigenvector are $\mu^{(133)} = 0.69766854$, $\mathbf{x}^{(133)} = (1, 0.7166727, 0.2568099, 0.04601217)^t$.

c. The characteristic polynomial is $P(\lambda) = \lambda^4 - \frac{1}{4}\lambda - \frac{1}{16}$, and the eigenvalues are $\lambda_1 = 0.6976684972$, $\lambda_2 = -0.2301775942 + 0.56965884i$, $\lambda_3 = -0.2301775942 - 0.56965884i$, and $\lambda_4 = -0.237313308$.

d. The beetle population should approach zero since A is convergent.

19. Using the Inverse Power method with $\mathbf{x}^{(0)} = (1, 0, 0, 1, 0, 0, 1, 0, 0, 1)^t$ and $q = 0$ gives the following results:

a. $\mu^{(49)} = 1.0201926$, so $\rho(A^{-1}) \approx 1/\mu^{(49)} = 0.9802071$;

b. $\mu^{(30)} = 1.0404568$, so $\rho(A^{-1}) \approx 1/\mu^{(30)} = 0.9611163$;

c. $\mu^{(22)} = 1.0606974$, so $\rho(A^{-1}) \approx 1/\mu^{(22)} = 0.9427760$.
The method appears to be stable for all α in $[\frac{1}{4}, \frac{3}{4}]$.

21. Forming $A^{-1}B$ and using the Power method with $\mathbf{x}^{(0)} = (1, 0, 0, 1, 0, 0, 1, 0, 0, 1)^t$ gives the following results:

a. The spectral radius is approximately $\mu^{(46)} = 0.9800021$.

b. The spectral radius is approximately $\mu^{(25)} = 0.9603543$.

c. The spectral radius is approximately $\mu^{(18)} = 0.9410754$.

23. The approximate eigenvalues and approximate eigenvectors are:

a. $\mu^{(2)} = 1.000000$, $\mathbf{x}^{(2)} = (0.1542373, -0.7715828, 0.6171474)^t$

b. $\mu^{(13)} = 1.000000$, $\mathbf{x}^{(13)} = (0.00007432, -0.7070723, 0.7071413)^t$

c. $\mu^{(14)} = 4.961699$, $\mathbf{x}^{(14)} = (-0.4814472, 0.05180473, 0.8749428)^t$

d. $\mu^{(17)} = 4.428007$, $\mathbf{x}^{(17)} = (0.7194230, 0.4231908, 0.1153589, 0.5385466)^t$

25. Since

$$\mathbf{x}^t = \frac{1}{\lambda_1 v_i^{(1)}}(a_{i1}, a_{i2}, \ldots, a_{in}),$$

the ith row of B is

$$(a_{i1}, a_{i2}, \ldots, a_{in}) - \frac{\lambda_1}{\lambda_1 v_i^{(1)}}\left(v_i^{(1)}a_{i1}, v_i^{(1)}a_{i2}, \ldots, v_i^{(1)}a_{in}\right) = \mathbf{0}.$$

Exercise Set 9.4 (Page 609)

1. Householder's method produces the following tridiagonal matrices.

a. $\begin{bmatrix} 12.00000 & -10.77033 & 0.0 \\ -10.77033 & 3.862069 & 5.344828 \\ 0.0 & 5.344828 & 7.137931 \end{bmatrix}$ **b.** $\begin{bmatrix} 2.0000000 & 1.414214 & 0.0 \\ 1.414214 & 1.000000 & 0.0 \\ 0.0 & 0.0 & 3.0 \end{bmatrix}$

c. $\begin{bmatrix} 1.0000000 & -1.414214 & 0.0 \\ -1.414214 & 1.000000 & 0.0 \\ 0.0 & 0.0 & 1.000000 \end{bmatrix}$ **d.** $\begin{bmatrix} 4.750000 & -2.263846 & 0.0 \\ -2.263846 & 4.475610 & -1.219512 \\ 0.0 & -1.219512 & 5.024390 \end{bmatrix}$

3. Householder's method produces the following upper Hessenberg matrices.

a. $\begin{bmatrix} 2.0000000 & 2.8284271 & 1.4142136 \\ -2.8284271 & 1.0000000 & 2.0000000 \\ 0.0000000 & 2.0000000 & 3.0000000 \end{bmatrix}$ **b.** $\begin{bmatrix} -1.0000000 & -3.0655513 & 0.0000000 \\ -3.6055513 & -0.23076923 & 3.1538462 \\ 0.0000000 & 0.15384615 & 2.2307692 \end{bmatrix}$

c. $\begin{bmatrix} 5.0000000 & 4.9497475 & -1.4320780 & -1.5649769 \\ -1.4142136 & -2.0000000 & -2.4855515 & 1.8226448 \\ 0.0000000 & -5.4313902 & -1.4237288 & -2.6486542 \\ 0.0000000 & 0.0000000 & 1.5939865 & 5.4237288 \end{bmatrix}$

d. $\begin{bmatrix} 4.0000000 & 1.7320508 & 0.0000000 & 0.0000000 \\ 1.7320508 & 2.3333333 & 0.23570226 & 0.40824829 \\ 0.0000000 & -0.47140452 & 4.6666667 & -0.57735027 \\ 0.0000000 & 0.0000000 & 0.0000000 & 5.0000000 \end{bmatrix}$

Exercise Set 9.5 (Page 621)

1. Two iterations of the QR method without shifting produce the following matrices.

a. $A^{(3)} = \begin{bmatrix} 3.142857 & -0.559397 & 0.0 \\ -0.559397 & 2.248447 & -0.187848 \\ 0.0 & -0.187848 & 0.608696 \end{bmatrix}$ **b.** $A^{(3)} = \begin{bmatrix} 4.549020 & 1.206958 & 0.0 \\ 1.206958 & 3.519688 & 0.000725 \\ 0.0 & 0.000725 & -0.068708 \end{bmatrix}$

c. $A^{(3)} = \begin{bmatrix} 4.592920 & -0.472934 & 0.0 \\ -0.472934 & 3.108760 & -0.232083 \\ 0.0 & -0.232083 & 1.298319 \end{bmatrix}$

d. $A^{(3)} = \begin{bmatrix} 3.071429 & 0.855352 & 0.0 & 0.0 \\ 0.855352 & 3.314192 & -1.161046 & 0.0 \\ 0.0 & -1.161046 & 3.331770 & 0.268898 \\ 0.0 & 0.0 & 0.268898 & 0.282609 \end{bmatrix}$

e. $A^{(3)} = \begin{bmatrix} -3.607843 & 0.612882 & 0.0 & 0.0 \\ 0.612882. & -1.395227 & -1.111027 & 0.0 \\ 0 & -1.111027 & 3.133919 & 0.346353 \\ 0.0 & 0.0 & 0.346353 & 0.869151 \end{bmatrix}$

f. $A^{(3)} = \begin{bmatrix} 1.013260 & 0.279065 & 0.0 & 0.0 \\ 0.279065 & 0.696255 & 0.107448 & 0.0 \\ 0.0 & 0.107448 & 0.843061 & 0.310832 \\ 0.0 & 0.0 & 0.310832 & 0.317424 \end{bmatrix}$

3. The matrices in Exercise 1 have the following eigenvalues, accurate to within 10^{-5}.

a. $3.414214, 2.000000, 0.58578644$

b. $-0.06870782, 5.346462, 2.722246$

c. $1.267949, 4.732051, 3.000000$

d. $4.745281, 3.177283, 1.822717, 0.2547188$

e. $3.438803, 0.8275517, -1.488068, -3.778287$

f. $0.9948440, 1.189091, 0.5238224, 0.1922421$

5. The matrices in Exercise 1 have the following eigenvectors, accurate to within 10^{-5}.

a. $(-0.7071067, 1, -0.7071067)^t$, $(1, 0, -1)^t$, $(0.7071068, 1, 0.7071068)^t$

b. $(0.1741299, -0.5343539, 1)^t$, $(0.4261735, 1, 0.4601443)^t$, $(1, -0.2777544, -0.3225491)^t$

c. $(0.2679492, 0.7320508, 1)^t$, $(1, -0.7320508, 0.2679492)^t$, $(1, 1, -1)^t$

d. $(-0.08029447, -0.3007254, 0.7452812, 1)^t$, $(0.4592880, 1, -0.7179949, 0.8727118)^t$, $(0.8727118, 0.7179949, 1, -0.4592880)^t$ $(1, -0.7452812, -0.3007254, 0.08029447)^t$

e. $(-0.01289861, -0.07015299, 0.4388026, 1)^t$, $(-0.1018060, -0.2878618, 1, -0.4603102)^t$, $(1, 0.5119322, 0.2259932, -0.05035423)^t$ $(-0.5623391, 1, 0.2159474, -0.03185871)^t$

f. $(-0.1520150, -0.3008950, -0.05155956, 1)^t$, $(0.3627966, 1, 0.7459807, 0.3945081)^t$, $(1, 0.09528962, -0.6907921, 0.1450703)^t$, $(0.8029403, -0.9884448, 1, -0.1237995)^t$

7. a. To within 10^{-5}, the eigenvalues are 2.618034, 3.618034, 1.381966, and 0.3819660.

b. In terms of p and ρ the eigenvalues are $-65.45085p/\rho$, $-90.45085p/\rho$, $-34.54915p/\rho$, and $-9.549150p/\rho$.

9. The actual eigenvalues are as follows:

a. When $\alpha = 1/4$, we have 0.97974649, 0.92062677, 0.82743037, 0.70770751, 0.57115742, 0.42884258, 0.29229249, 0.17256963, 0.07937323, and 0.02025351.

b. When $\alpha = 1/2$, we have 0.95949297, 0.84125353, 0.65486073, 0.41541501, 0.14231484, −0.14231484, −0.41541501, −0.65486073, −0.84125353, and −0.95949297.

c. When $\alpha = 3/4$, we have 0.93923946, 0.76188030, 0.48229110, 0.12312252, −0.28652774, −0.71347226, −1.12312252, −1.48229110, −1.76188030, and −1.93923946. The method appears to be stable for $\alpha \le \frac{1}{2}$.

11. a. Let

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and $\mathbf{y} = P\mathbf{x}$. Show that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2$. Use the relationship $x_1 + ix_2 = re^{i\alpha}$, where $r = \|\mathbf{x}\|_2$ and $\alpha = \tan^{-1}(x_2/x_1)$ and $y_1 + iy_2 = re^{i(\alpha+\theta)}$.

b. Let $\mathbf{x} = (1, 0)^t$ and $\theta = \pi/4$.

13. Let $C = RQ$, where R is upper triangular and Q is upper Hessenberg. Then $c_{ij} = \sum_{k=1}^{n} r_{ik}q_{kj}$. Since R is an upper-triangular matrix, $r_{ik} = 0$ if $k < i$. Thus, $c_{ij} = \sum_{k=i}^{n} r_{ik}q_{kj}$. Since Q is an upper-Hessenberg matrix, $q_{kj} = 0$ if $k > j + 1$. Thus, $c_{ij} = \sum_{k=i}^{j+1} r_{ik}q_{kj}$. The sum will be zero if $i > j + 1$. Hence, $c_{ij} = 0$ if $i \ge j + 2$. This means that C is an upper-Hessenberg matrix.

15. INPUT: dimension n, matrix $A = (a_{ij})$, tolerance TOL, maximum number of iterations N.

OUTPUT: eigenvalues $\lambda_1, \ldots, \lambda_n$ of A or a message that the number of iterations was exceeded.

Step 1 Set $FLAG = 1$; $k1 = 1$.

Step 2 While ($FLAG = 1$) do Steps 3–10

Step 3 For $i = 2, \ldots, n$ do Steps 4–8.

Step 4 For $j = 1, \ldots, i-1$ do Steps 5–8.

Step 5 If $a_{ii} = a_{jj}$ then set

$\text{CO} = 0.5\sqrt{2}$;
$\text{SI} = \text{CO}$

else set

$b = |a_{ii} - a_{jj}|$;
$c = 2a_{ij}\ \text{sign}(a_{ii} - a_{jj})$;
$\text{CO} = 0.5\left(1 + b/\left(c^2 + b^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$;
$\text{SI} = 0.5c/\left(CO\left(c^2 + b^2\right)^{\frac{1}{2}}\right)$.

Step 6 For $k = 1, \ldots, n$
if $(k \neq i)$ and $(k \neq j)$ then
set $x = a_{k,j}$;
$y = a_{k,i}$;
$a_{k,j} = \text{CO}\cdot x + \text{SI}\cdot y$;
$a_{k,i} = \text{CO}\cdot y + \text{SI}\cdot x$;
$x = a_{j,k}$;
$y = a_{i,k}$;
$a_{j,k} = \text{CO}\cdot x + \text{SI}\cdot y$;
$a_{i,k} = \text{CO}\cdot y - \text{SI}\cdot x$.

Step 7 Set $x = a_{j,j}$;
$y = a_{i,i}$;
$a_{j,j} = \text{CO}\cdot\text{CO}\cdot x + 2\cdot\text{SI}\cdot CO\cdot a_{j,i} + \text{SI}\cdot\text{SI}\cdot y$;
$a_{i,i} = \text{SI}\cdot\text{SI}\cdot x - 2\cdot\text{SI}\cdot\text{CO}\cdot a_{i,j} + \text{CO}\cdot\text{CO}\cdot y$.

Step 8 Set $a_{i,j} = 0$; $a_{j,i} = 0$.

Step 9 Set

$$s = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|.$$

Step 10 If $s < TOL$ then for $i = 1, \ldots, n$ set
$\lambda_i = a_{ii}$;
OUTPUT $(\lambda_1, \ldots, \lambda_n)$;
set $FLAG = 0$.

else set $k1 = k1 + 1$;
if $k1 > N$ then set $FLAG = 0$.

Step 11 If $k1 > N$ then
OUTPUT ('Maximum number of iterations exceeded');
STOP.

Exercise Set 9.6 (Page 636)

1. a. $s_1 = 1 + \sqrt{2}$, $s_2 = -1 + \sqrt{2}$ **b.** $s_1 = 2.676243$, $s_2 = 0.9152717$

c. $s_1 = 3.162278$, $s_2 = 2$ **d.** $s_1 = 2.645751$, $s_2 = 1$, $s_3 = 1$

3. a.

$$U = \begin{bmatrix} -0.923880 & -0.382683 \\ -0.3826831 & 0.923880 \end{bmatrix}, \quad S = \begin{bmatrix} 2.414214 & 0 \\ 0 & 0.414214 \end{bmatrix},$$

$$V^t = \begin{bmatrix} -0.923880 & -0.382683 \\ -0.382683 & 0.923880 \end{bmatrix}$$

b.

$$U = \begin{bmatrix} 0.8247362 & -0.3913356 & 0.4082483 \\ 0.5216090 & 0.2475023 & -0.8164966 \\ 0.2184817 & 0.8863403 & 0.4082483 \end{bmatrix}, \quad S = \begin{bmatrix} 2.676243 & 0 \\ 0 & 0.9152717 \\ 0 & 0 \end{bmatrix},$$

$$V^t = \begin{bmatrix} 0.8112422 & 0.5847103 \\ -0.5847103 & 0.8112422 \end{bmatrix}$$

c.

$$U = \begin{bmatrix} -0.632456 & -0.500000 & -0.5 & 0.3162278 \\ 0.316228 & -0.500000 & 0.5 & 0.6324555 \\ -0.316228 & -0.500000 & 0.5 & -0.6324555 \\ -0.632456 & 0.500000 & 0.5 & 0.3162278 \end{bmatrix}, \quad S = \begin{bmatrix} 3.162278 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$V^t = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

d.

$$U = \begin{bmatrix} -0.436436 & 0.707107 & 0.408248 & -0.377964 \\ 0.436436 & 0.707107 & -0.408248 & 0.377964 \\ -0.436436 & 0 & -0.816497 & -0.377964 \\ -0.654654 & 0 & 0 & 0.755929 \end{bmatrix}, \quad S = \begin{bmatrix} 2.645751 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V^t = \begin{bmatrix} -0.577350 & -0.577350 & 0.577350 \\ 0 & 0.707107 & 0.707107 \\ 0.816497 & -0.408248 & 0.408248 \end{bmatrix}$$

5. For the matrix A in Example 2, we have

$$A^tA = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

So $A^tA(1,2,1)^t = (5,10,5)^t = 5(1,2,1)^t$, $A^tA(1,-1,1)^t = (2,-2,2)^t = 2(1,-1,1)^t$, and $A^tA(-1,0,1)^t = (-1,0,1)^t$.

7. a. Use the tabulated values to construct

$$\mathbf{b} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix} = \begin{bmatrix} 1 & 1.0 & 1.0 \\ 1 & 1.1 & 1.21 \\ 1 & 1.3 & 1.69 \\ 1 & 1.5 & 2.25 \\ 1 & 1.9 & 3.61 \\ 1 & 2.1 & 4.41 \end{bmatrix}.$$

The matrix A has the singular value decomposition $A = U\,S\,V^t$, where

$$U = \begin{bmatrix} -0.203339 & -0.550828 & 0.554024 & 0.055615 & -0.177253 & -0.560167 \\ -0.231651 & -0.498430 & 0.185618 & 0.165198 & 0.510822 & 0.612553 \\ -0.294632 & -0.369258 & -0.337742 & -0.711511 & -0.353683 & 0.177288 \\ -0.366088 & -0.20758 & -0.576499 & 0.642950 & -0.264204 & -0.085730 \\ -0.534426 & 0.213281 & -0.200202 & -0.214678 & 0.628127 & -0.433808 \\ -0.631309 & 0.472467 & 0.414851 & 0.062426 & -0.343809 & 0.289864 \end{bmatrix},$$

$$S = \begin{bmatrix} 7.844127 & 0 & 0 \\ 0 & 1.223790 & 0 \\ 0 & 0 & 0.070094 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad V^t = \begin{bmatrix} -0.288298 & -0.475702 & -0.831018 \\ -0.768392 & -0.402924 & 0.497218 \\ 0.571365 & -0.781895 & 0.249363 \end{bmatrix}.$$

So,

$$\mathbf{c} = U^t\mathbf{b} = \begin{bmatrix} -5.955009 \\ -1.185591 \\ -0.044985 \\ -0.003732 \\ -0.000493 \\ -0.001963 \end{bmatrix},$$

and the components of $\mathbf{z}$ are

$$z_1 = \frac{c_1}{s_1} = \frac{-5.955009}{7.844127} = -0.759168, \quad z_2 = \frac{c_2}{s_2} = \frac{-1.185591}{1.223790} = -0.968786,$$

and

$$z_3 = \frac{c_3}{s_3} = \frac{-0.044985}{0.070094} = -0.641784.$$

This gives the least squares coefficients in $P_2(x) = a_0 + a_1x + a_2x^2$ as

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \mathbf{x} = V\,\mathbf{z} = \begin{bmatrix} 0.596581 \\ 1.253293 \\ -0.010853 \end{bmatrix}.$$

The least squares error using these values uses the last three components of $\mathbf{c}$, and is

$$||A\mathbf{x} - \mathbf{b}||_2 = \sqrt{c_4^2 + c_5^2 + c_6^2} = \sqrt{(-0.003732)^2 + (-0.000493)^2 + (-0.001963)^2} = 0.004244.$$

b. Use the tabulated values to construct

$$\mathbf{b} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \\ 1 & x_5 & x_5^2 & x_5^3 \end{bmatrix} = \begin{bmatrix} 1 & 1.0 & 1.0 & 1.0 \\ 1 & 1.1 & 1.21 & 1.331 \\ 1 & 1.3 & 1.69 & 2.197 \\ 1 & 1.5 & 2.25 & 3.375 \\ 1 & 1.9 & 3.61 & 6.859 \\ 1 & 2.1 & 4.41 & 9.261 \end{bmatrix}.$$

The matrix A has the singular value decomposition $A = U\,S\,V^t$, where

$$U = \begin{bmatrix} -0.116086 & -0.514623 & 0.569113 & -0.437866 & -0.381082 & 0.246672 \\ -0.143614 & -0.503586 & 0.266325 & 0.184510 & 0.535306 & 0.578144 \\ -0.212441 & -0.448121 & -0.238475 & 0.48499 & 0.180600 & -0.655247 \\ -0.301963 & -0.339923 & -0.549619 & 0.038581 & -0.573591 & 0.400867 \\ -0.554303 & 0.074101 & -0.306350 & -0.636776 & 0.417792 & -0.115640 \\ -0.722727 & 0.399642 & 0.390359 & 0.363368 & -0.179026 & 0.038548 \end{bmatrix},$$

$$S = \begin{bmatrix} 14.506808 & 0 & 0 & 0 \\ 0 & 2.084909 & 0 & 0 \\ 0 & 0 & 0.198760 & 0 \\ 0 & 0 & 0 & 0.868328 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$V^t = \begin{bmatrix} -0.141391 & -0.246373 & -0.449207 & -0.847067 \\ -0.639122 & -0.566437 & -0.295547 & 0.428163 \\ 0.660862 & -0.174510 & -0.667840 & 0.294610 \\ -0.367142 & 0.766807 & -0.514640 & 0.111173 \end{bmatrix}.$$

So,

$$\mathbf{c} = U^t\mathbf{b} = \begin{bmatrix} -5.632309 \\ -2.268376 \\ 0.036241 \\ 0.005717 \\ -0.000845 \\ -0.004086 \end{bmatrix},$$

and the components of $\mathbf{z}$ are

$$z_1 = \frac{c_1}{s_1} = \frac{-5.632309}{14.506808} = -0.388253, \quad z_2 = \frac{c_2}{s_2} = \frac{-2.268376}{2.084909} = -1.087998,$$

$$z_3 = \frac{c_3}{s_3} = \frac{0.036241}{0.198760} = 0.182336, \quad \text{and} \quad z_4 = \frac{c_4}{s_4} = \frac{0.005717}{0.868328} = 0.65843.$$

This gives the least squares coefficients in $P_2(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ as

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{x} = V\,\mathbf{z} = \begin{bmatrix} 0.629019 \\ 1.185010 \\ 0.035333 \\ -0.010047 \end{bmatrix}.$$

The least squares error using these values uses the last two components of $\mathbf{c}$, and is

$$||A\mathbf{x} - \mathbf{b}||_2 = \sqrt{c_5^2 + c_6^2} = \sqrt{(-0.000845)^2 + (-0.004086)^2} = 0.004172.$$

9. $P_2(x) = 19.691025 - 0.0065112585x + 6.3494753 \times 10^{-7}x^2$. The least squares error is 0.42690171.

11. Let A be an $m \times n$ matrix. Theorem 9.25 implies that $\text{Rank}(A) = \text{Rank}(A^t)$, so $\text{Nullity}(A) = n - \text{Rank}(A)$ and $\text{Nullity}(A^t) = m - \text{Rank}(A^t) = m - \text{Rank}(A)$. Hence, $\text{Nullity}(A) = \text{Nullity}(A^t)$ if and only if $n = m$.

13. $\text{Rank}(S)$ is the number of nonzero entries on the diagonal of S. This corresponds to the number of nonzero eigenvalues (counting multiplicities) of A^tA. So $\text{Rank}(S) = \text{Rank}(A^tA)$, and by part (ii) of Theorem 9.26, this is the same as $\text{Rank}(A)$.

15. Because $U^{-1} = U^t$ and $V^{-1} = V^t$ both exist, $A = USV^t$ implies that $A^{-1} = (USV^t)^{-1} = VS^{-1}U^t$ if and only if S^{-1} exists.

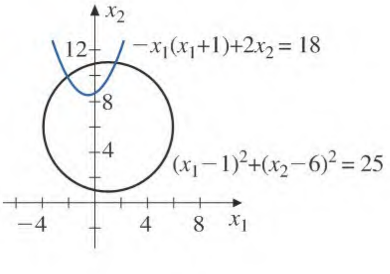
17. Yes. By Theorem 9.25 we have $\text{Rank}(A^tA) = \text{Rank}((A^tA)^t) = \text{Rank}(AA^t)$. Applying part (iii) of Theorem 9.26 gives $\text{Rank}(AA^t) = \text{Rank}(A^tA) = \text{Rank}(A)$.

19. If the $n \times n$ matrix A has the singular values $s_1 \geq s_2 \geq \cdots \geq s_n > 0$, then $||A||_2 = \sqrt{\rho(A^tA)} = s_1$. In addition, the singular values of A^{-1} are $\frac{1}{s_n} \geq \cdots \geq \frac{1}{s_2} \geq \frac{1}{s_1} > 0$, so $||A^{-1}||_2 = \sqrt{\frac{1}{s_n^2}} = \frac{1}{s_n}$. Hence, $K_2(A) = ||A||_2 \cdot ||A^{-1}||_2 = s_1/s_n$.

Exercise Set 10.1 (Page 648)

1. The solutions are near $(-1.5, 10.5)$ and $(2, 11)$.

a. The graphs are shown in the figure below.



b. Use

$$\mathbf{G}_1(\mathbf{x}) = \left(-0.5 + \sqrt{2x_2 - 17.75}, 6 + \sqrt{25 - (x_1 - 1)^2}\right)^t$$

and

$$\mathbf{G}_2(\mathbf{x}) = \left(-0.5 - \sqrt{2x_2 - 17.75}, 6 + \sqrt{25 - (x_1 - 1)^2}\right)^t.$$

For $\mathbf{G}_1(\mathbf{x})$ with $\mathbf{x}^{(0)} = (2, 11)^t$, we have $\mathbf{x}^{(9)} = (1.5469466, 10.969994)^t$, and for $\mathbf{G}_2(\mathbf{x})$ with $\mathbf{x}^{(0)} = (-1.5, 10.5)$, we have $\mathbf{x}^{(34)} = (-2.000003, 9.999996)^t$.

3. b. With $\mathbf{x}^{(0)} = (0, 0)^t$ and tolerance 10^{-5}, we have $\mathbf{x}^{(13)} = (0.9999973, 0.9999973)^t$.

c. With $\mathbf{x}^{(0)} = (0, 0)^t$ and tolerance 10^{-5}, we have $\mathbf{x}^{(11)} = (0.9999984, 0.9999991)^t$.

5. a. With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, we have $\mathbf{x}^{(5)} = (5.0000000, 0.0000000, -0.5235988)^t$.

b. With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, we have $\mathbf{x}^{(9)} = (1.0364011, 1.0857072, 0.93119113)^t$.

c. With $\mathbf{x}^{(0)} = (0, 0, 0.5)^t$, we have $\mathbf{x}^{(5)} = (0.00000000, 0.09999999, 1.0000000)^t$.

d. With $\mathbf{x}^{(0)} = (0, 0, 0)^t$, we have $\mathbf{x}^{(5)} = (0.49814471, -0.19960600, -0.52882595)^t$.

7. a. With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, we have $\mathbf{x}^{(3)} = (0.5000000, 0, -0.5235988)^t$.

b. With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, we have $\mathbf{x}^{(4)} = (1.036400, 1.085707, 0.9311914)^t$.

c. With $\mathbf{x}^{(0)} = (0, 0, 0)^t$, we have $\mathbf{x}^{(3)} = (0, 0.1000000, 1.0000000)^t$.

d. With $\mathbf{x}^{(0)} = (0, 0, 0)^t$, we have $\mathbf{x}^{(4)} = (0.4981447, -0.1996059, -0.5288260)^t$.

9. A stable solution occurs when $x_1 = 8000$ and $x_2 = 4000$.

11. Use Theorem 10.5.

13. Use Theorem 10.5 for each of the partial derivatives.

15. In this situation we have, for any matrix norm,

$$||\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)|| = ||A\mathbf{x} - A\mathbf{x}_0|| = ||A(\mathbf{x} - \mathbf{x}_0)|| \le ||A|| \cdot ||\mathbf{x} - \mathbf{x}_0||.$$

The result follows by selecting $\delta = \varepsilon/||A||$, provided that $||A|| \neq 0$. When $||A|| = 0$, δ can be arbitrarily chosen because A is the zero matrix.

Exercise Set 10.2 (Page 655)

1. a. $\mathbf{x}^{(2)} = (0.4958936, 1.983423)^t$

b. $\mathbf{x}^{(2)} = (-0.5131616, -0.01837622)^t$

c. $\mathbf{x}^{(2)} = (-23.942626, 7.6086797)^t$

d. $\mathbf{x}^{(1)}$ cannot be computed since $J(0)$ is singular.

3. a. $(0.5, 0.2)^t$ and $(1.1, 6.1)^t$

b. $(-0.35, 0.05)^t$, $(0.2, -0.45)^t$, $(0.4, -0.5)^t$ and $(1, -0.3)^t$

c. $(-1, 3.5)^t$, $(2.5, 4)^t$

d. $(0.11, 0.27)^t$

5. a. With $\mathbf{x}^{(0)} = (0.5, 2)^t$, $\mathbf{x}^{(3)} = (0.5, 2)^t$. With $\mathbf{x}^{(0)} = (1.1, 6.1)$, $\mathbf{x}^{(3)} = (1.0967197, 6.0409329)^t$.

b. With $\mathbf{x}^{(0)} = (-0.35, 0.05)^t$, $\mathbf{x}^{(3)} = (-0.37369822, 0.056266490^t$.
With $\mathbf{x}^{(0)} = (0.2, -0.45)^t$, $\mathbf{x}^{(4)} = (0.14783924, -0.43617762)^t$.
With $\mathbf{x}^{(0)} = (0.4, -0.5)^t$, $\mathbf{x}^{(3)} = (0.40809566, -0.49262939)^t$.
With $\mathbf{x}^{(0)} = (1, -0.3)^t$, $\mathbf{x}^{(4)} = (1.0330715, -0.27996184)^t$.

c. With $\mathbf{x}^{(0)} = (-1, 3.5)^t$, $\mathbf{x}^{(1)} = (-1, 3.5)^t$ and $\mathbf{x}^{(0)} = (2.5, 4)^t$, $\mathbf{x}^{(3)} = (2.546947, 3.984998)^t$.

d. With $\mathbf{x}^{(0)} = (0.11, 0.27)^t$, $\mathbf{x}^{(6)} = (0.1212419, 0.2711051)^t$.

7. a. $\mathbf{x}^{(5)} = (0.5000000, 0.8660254)^t$

b. $\mathbf{x}^{(6)} = (1.772454, 1.772454)^t$

c. $\mathbf{x}^{(5)} = (-1.456043, -1.664230, 0.4224934)^t$

d. $\mathbf{x}^{(4)} = (0.4981447, -0.1996059, -0.5288260)^t$

9. With $\mathbf{x}^{(0)} = (1, 1 - 1)^t$ and $TOL = 10^{-6}$, we have $\mathbf{x}^{(20)} = (0.5, 9.5 \times 10^{-7}, -0.5235988)^t$.

11. With $\theta_i^{(0)} = 1$, for each $i = 1, 2, \ldots, 20$, the following results are obtained.

i	1	2	3	4	5	6
$\theta_i^{(5)}$	0.14062	0.19954	0.24522	0.28413	0.31878	0.35045

i	7	8	9	10	11	12	13
$\theta_i^{(5)}$	0.37990	0.40763	0.43398	0.45920	0.48348	0.50697	0.52980

i	14	15	16	17	18	19	20
$\theta_i^{(5)}$	0.55205	0.57382	0.59516	0.61615	0.63683	0.65726	0.67746

13. When the dimension n is 1, $\mathbf{F}(\mathbf{x})$ is a one-component function $f(\mathbf{x}) = f_1(\mathbf{x})$, and the vector $\mathbf{x}$ has only one component $x_1 = x$. In this case, the Jacobian matrix $J(\mathbf{x})$ reduces to the 1×1 matrix $\left[\frac{\partial f_1}{\partial x_1}(\mathbf{x})\right] = f'(\mathbf{x}) = f'(x)$. Thus, the vector equation

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - J(\mathbf{x}^{(k-1)})^{-1}\mathbf{F}(\mathbf{x}^{(k-1)})$$

becomes the scalar equation

$$x_k = x_{k-1} - f(x_{k-1})^{-1} f(x_{k-1}) = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}.$$

Exercise Set 10.3 (Page 664)

1. **a.** $\mathbf{x}^{(2)} = (0.4777920, 1.927557)^t$ **b.** $\mathbf{x}^{(2)} = (-0.3250070, -0.1386967)^t$
c. $\mathbf{x}^{(2)} = (0.52293721, 0.82434906)^t$ **d.** $\mathbf{x}^{(2)} = (1.77949990, 1.74339606)^t$

3. **a.** $\mathbf{x}^{(8)} = (0.5, 2)^t$ **b.** $\mathbf{x}^{(9)} = (-0.3736982, 0.05626649)^t$
c. $\mathbf{x}^{(9)} = (0.5, 0.8660254)^t$ **d.** $\mathbf{x}^{(8)} = (1.772454, 1.772454)^t$

5. **a.** With $\mathbf{x}^{(0)} = (2.5, 4)^t$, we have $\mathbf{x}^{(3)} = (2.546947, 3.984998)^t$.
b. With $\mathbf{x}^{(0)} = (0.11, 0.27)^t$, we have $\mathbf{x}^{(4)} = (0.1212419, 0.2711052)^t$.
c. With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, we have $\mathbf{x}^{(3)} = (1.036401, 1.085707, 0.9311914)^t$.
d. With $\mathbf{x}^{(0)} = (1, -1, 1)^t$, we have $\mathbf{x}^{(8)} = (0.9, -1, 0.5)^t$; and with $\mathbf{x}^{(0)} = (1, 1, -1)^t$, we have $\mathbf{x}^{(8)} = (0.5, 1, -0.5)^t$.

7. With $\mathbf{x}^{(0)} = (1, 1 - 1)^t$, we have $\mathbf{x}^{(56)} = (0.5000591, 0.01057235, -0.5224818)^t$.

9. With $\mathbf{x}^{(0)} = (0.75, 1.25)^t$, we have $\mathbf{x}^{(4)} = (0.7501948, 1.184712)^t$. Thus, $a = 0.7501948, b = 1.184712$, and the error is 19.796.

11. Let λ be an eigenvalue of $M = \left(I + \mathbf{u}\mathbf{v}^t\right)$ with eigenvector $\mathbf{x} \neq \mathbf{0}$. Then $\lambda\mathbf{x} = M\mathbf{x} = \left(I + \mathbf{u}\mathbf{v}^t\right)\mathbf{x} = \mathbf{x} + \left(\mathbf{v}^t\mathbf{x}\right)\mathbf{u}$. Thus, $(\lambda - 1)\mathbf{x} = \left(\mathbf{v}^t\mathbf{x}\right)\mathbf{u}$. If $\lambda = 1$, then $\mathbf{v}^t\mathbf{x} = 0$. So, $\lambda = 1$ is an eigenvalue of M with multiplicity $n - 1$ and eigenvectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n-1)}$, where $\mathbf{v}^t\mathbf{x}^{(j)} = 0$, for $j = 1, \ldots, n - 1$. Assume $\lambda \neq 1$ implies that $\mathbf{x}$ and $\mathbf{u}$ are parallel. Suppose $\mathbf{x} = \alpha\mathbf{u}$. Then $(\lambda - 1)\alpha\mathbf{u} = \left(\mathbf{v}^t(\alpha\mathbf{u})\right)\mathbf{u}$. Thus, $\alpha(\lambda - 1)\mathbf{u} = \alpha\left(\mathbf{v}^t\mathbf{u}\right)\mathbf{u}$, which implies that $\lambda - 1 = \mathbf{v}^t\mathbf{u}$ or $\lambda = 1 + \mathbf{v}^t\mathbf{u}$. Hence, M has eigenvalues λ_i, $1 \le i \le n$, where $\lambda_i = 1$, for $i = 1, \ldots, n - 1$, and $\lambda_n = 1 + \mathbf{v}^t\mathbf{u}$. Since $\det M = \prod_{i=1}^n \lambda_i$, we have $\det M = 1 + \mathbf{v}^t\mathbf{u}$.

Exercise Set 10.4 (Page 672)

1. **a.** With $\mathbf{x}^{(0)} = (0, 0)^t$, we have $\mathbf{x}^{(11)} = (0.4943541, 1.948040)^t$.
b. With $\mathbf{x}^{(0)} = (1, 1)^t$, we have $\mathbf{x}^{(2)} = (0.4970073, 0.8644143)^t$.
c. With $\mathbf{x}^{(0)} = (2, 2)^t$, we have $\mathbf{x}^{(1)} = (1.736083, 1.804428)^t$.
d. With $\mathbf{x}^{(0)} = (0, 0)^t$, we have $\mathbf{x}^{(2)} = (-0.3610092, 0.05788368)^t$.

3. **a.** $\mathbf{x}^{(3)} = (0.5, 2)^t$ **b.** $\mathbf{x}^{(3)} = (0.5, 0.8660254)^t$
c. $\mathbf{x}^{(4)} = (1.772454, 1.772454)^t$ **d.** $\mathbf{x}^{(3)} = (-0.3736982, 0.05626649)^t$

5. **a.** With $\mathbf{x}(0) = (0, 0)^t$, $g(3.3231994, 0.11633359) = -0.14331228$ in two iterations
b. With $\mathbf{x}(0) = (0, 0)^t$, $g(0.43030383, 0.18006958) = 0.32714638$ in 38 iterations
c. With $\mathbf{x}(0) = (0, 0, 0)^t$, $g(-0.66340113, 0.31453697, 0.50007629) = 0.69215167$ in five iterations
d. With $\mathbf{x}(0) = (0.5, 0.5, 0.5)^t$, $g(-0.03338762, 0.00401587, -0.00093451) = 1.01000124$ in three iterations

7. **a.** $b = 1.5120985$, $a = 0.87739838$ **b.** $b = 21.014867$, $a = -3.7673246$
c. Part (b) does. **d.** Part (a) predicts 86%; part (b) predicts 39%.

Exercise Set 10.5 (Page 680)

1. a. $(3, -2.25)^t$ **b.** $(0.42105263, 2.6184211)^t$ **c.** $(2.173110, -1.3627731)^t$

3. Using $\mathbf{x}(0) = \mathbf{0}$ in all parts gives:

a. $(0.44006047, 1.8279835)^t$ **b.** $(-0.41342613, 0.096669468)^t$

c. $(0.49858909, 0.24999091, -0.52067978)^t$ **d.** $(6.1935484, 18.532258, -21.725806)^t$

5. a. With $x^{(}0) = (-1, 3.5)^t$ the result is $(-1, 3.5)^t$.
With $\mathbf{x}(0) = (2.5, 4)^t$ the result is $(-1, 3.5)^t$.

b. With $\mathbf{x}(0) = (0.11, 0.27)^t$ the result is $(0.12124195, 0.27110516)^t$.

c. With $\mathbf{x}(0) = (1, 1, 1)^t$ the result is $(1.03640047, 1.08570655, 0.93119144)^t$.

d. With $\mathbf{x}(0) = (1, -1, 1)^t$ the result is $(0.90016074, -1.00238008, 0.496610937)^t$.
With $\mathbf{x}(0) = (1, 1, -1)^t$ the result is $(0.50104035, 1.00238008, -0.49661093)^t$.

7. a. With $\mathbf{x}(0) = (-1, 3.5)^t$ the result is $(-1, 3.5)^t$.
With $\mathbf{x}(0) = (2.5, 4)^t$ the result is $(2.5469465, 3.9849975)^t$.

b. With $\mathbf{x}(0) = (0.11, 0.27)^t$ the result is $(0.12124191, 0.27110516)^t$.

c. With $\mathbf{x}(0) = (1, 1, 1)^t$ the result is $(1.03640047, 1.08570655, 0.93119144)^t$.

d. With $\mathbf{x}(0) = (1, -1, 1)^t$ the result is $(0.90015964, -1.00021826, 0.49968944)^t$.
With $\mathbf{x}(0) = (1, 1, -1)^t$ the result is $(0.5009653, 1.00021826, -0.49968944)^t$.

9. $(0.50024553, 0.078230039, -0.52156996)^t$

11. With $\mathbf{x}^{(0)} = (0.75, 0.5, 0.75)^t$, $\mathbf{x}^{(2)} = (0.52629469, 0.52635099, 0.52621592)^t$

13. For each λ, we have

$$0 = G(\lambda, \mathbf{x}(\lambda)) = F(\mathbf{x}(\lambda)) - e^{-\lambda} F(\mathbf{x}(0)),$$

so

$$0 = \frac{\partial F(\mathbf{x}(\lambda))}{\partial \mathbf{x}} \frac{d\mathbf{x}}{d\lambda} + e^{-\lambda} F(\mathbf{x}(0)) = J(\mathbf{x}(\lambda))\mathbf{x}'(\lambda) + e^{-\lambda} F(\mathbf{x}(0))$$

and

$$J(\mathbf{x}(\lambda))\mathbf{x}'(\lambda) = -e^{-\lambda} F(\mathbf{x}(0)) = -F(\mathbf{x}(0)).$$

Thus,

$$\mathbf{x}'(\lambda) = -J(\mathbf{x}(\lambda))^{-1} F(\mathbf{x}(0)).$$

With $N = 1$, we have $h = 1$ so that

$$\mathbf{x}(1) = \mathbf{x}(0) - J(\mathbf{x}(0))^{-1} F(\mathbf{x}(0)).$$

However, Newton's method gives

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - J(\mathbf{x}^{(0)})^{-1} F(\mathbf{x}^{(0)}).$$

Since $\mathbf{x}(0) = \mathbf{x}^{(0)}$, we have $\mathbf{x}(1) = \mathbf{x}^{(1)}$.

Exercise Set 11.1 (Page 692)

1. The Linear Shooting Algorithm gives the results in the following tables.

a.

i	x_i	w_{1i}	$y(x_i)$
1	0.5	0.82432432	0.82402714

b.

i	x_i	w_{1i}	$y(x_i)$
1	0.25	0.3937095	0.3936767
2	0.50	0.8240948	0.8240271
3	0.75	1.337160	1.337086

3. The Linear Shooting Algorithm gives the results in the following tables.

a.

i	x_i	w_{1i}	$y(x_i)$
3	0.3	0.7833204	0.7831923
6	0.6	0.6023521	0.6022801
9	0.9	0.8568906	0.8568760

b.

i	x_i	w_{1i}	$y(x_i)$
5	1.25	0.1676179	0.1676243
10	1.50	0.4581901	0.4581935
15	1.75	0.6077718	0.6077740

c.

i	x_i	w_{1i}	$y(x_i)$
3	0.3	−0.5185754	−0.5185728
6	0.6	−0.2195271	−0.2195247
9	0.9	−0.0406577	−0.0406570

d.

i	x_i	w_{1i}	$y(x_i)$
3	1.3	0.0655336	0.06553420
6	1.6	0.0774590	0.07745947
9	1.9	0.0305619	0.03056208

5. The Linear Shooting Algorithm with $h = 0.05$ gives the following results.

i	x_i	w_{1i}
6	0.3	0.04990547
10	0.5	0.00673795
16	0.8	0.00033755

The Linear Shooting Algorithm with $h = 0.1$ gives the following results.

i	x_i	w_{1i}
3	0.3	0.05273437
5	0.5	0.00741571
8	0.8	0.00038976

7. For Eq. (11.3), let $u_1(x) = y$ and $u_2(x) = y'$. Then

$$u_1'(x) = u_2(x), \quad a \le x \le b, \quad u_1(a) = \alpha$$

and

$$u_2'(x) = p(x)u_2(x) + q(x)u_1(x) + r(x), \quad a \le x \le b, \quad u_2(a) = 0.$$

For Eq. (11.4), let $v_1(x) = y$ and $v_2(x) = y'$. Then

$$v_1'(x) = v_2(x), \quad a \le x \le b, \quad v_1(a) = 0$$

and

$$v_2'(x) = p(x)v_2(x) + q(x)v_1(x), \quad a \le x \le b, \quad v_2(a) = 1.$$

Using the notation $u_{1,i} = u_1(x_i)$, $u_{2,i} = u_2(x_i)$, $v_{1,i} = v_1(x_i)$, and $v_{2,i} = v_2(x_i)$ leads to the equations in Step 4 of Algorithm 11.1.

9. **a.** There are no solutions if b is an integer multiple of π and $B \neq 0$.

b. A unique solution exists whenever b is not an integer multiple of π.

c. There is an infinite number of solutions if b is an multiple integer of π and $B = 0$.

Exercise Set 11.2 (Page 699)

1. The Nonlinear Shooting Algorithm gives $w_1 = 0.405505 \approx \ln 1.5 = 0.405465$.

3. The Nonlinear Shooting Algorithm gives the results in the following tables.

a.

i	x_i	w_{1i}	$y(x_i)$	w_{2i}
2	1.20000000	0.18232094	0.18232156	0.83333370
4	1.40000000	0.33647129	0.33647224	0.71428547
6	1.60000000	0.47000243	0.47000363	0.62499939
8	1.80000000	0.58778522	0.58778666	0.55555468

Convergence in 4 iterations $t = 1.0000017$.

b.

i	x_i	w_{1i}	$y(x_i)$	w_{2i}
2	0.31415927	1.36209813	1.36208552	1.29545926
4	0.62831853	1.80002060	1.79999746	1.45626846
6	0.94247780	2.24572329	2.24569937	1.32001776
8	1.25663706	2.58845757	2.58844295	0.79988757

Convergence in 4 iterations $t = 1.0000301$.

c.

i	x_i	w_{1i}	$y(x_i)$	w_{2i}
1	0.83775804	0.86205941	0.86205848	0.38811718
2	0.89011792	0.88156057	0.88155882	0.35695076
3	0.94247780	0.89945618	0.89945372	0.32675844
4	0.99483767	0.91579268	0.91578959	0.29737141

Convergence in 3 iterations $t = 0.42046725$.

d.

i	x_i	w_{1i}	$y(x_i)$	w_{2i}
4	0.62831853	2.58784539	2.58778525	0.80908243
8	1.25663706	2.95114591	2.95105652	0.30904693
12	1.88495559	2.95115520	2.95105652	−0.30901625
16	2.51327412	2.58787536	2.58778525	−0.80904433

Convergence in 6 iterations $t = 1.0001253$.

5.

i	x_i	w_{1i}	w_{2i}
3	0.6	0.71682963	0.92122169
5	1.0	1.00884285	0.53467944
8	1.6	1.13844628	−0.11915193

Exercise Set 11.3 (Page 704)

1. The Linear Finite-Difference Algorithm gives the following results.

a.

i	x_i	w_{1i}	$y(x_i)$
1	0.5	0.83333333	0.82402714

b.

i	x_i	w_{1i}	$y(x_i)$
1	0.25	0.39512472	0.39367669
2	0.5	0.82653061	0.82402714
3	0.75	1.33956916	1.33708613

c. $\dfrac{4(0.82653061) - 0.83333333}{3} = 0.82426304$

3. The Linear Finite-Difference Algorithm gives the results in the following tables.

a.

i	x_i	w_i	$y(x_i)$
2	0.2	1.018096	1.0221404
5	0.5	0.5942743	0.59713617
7	0.7	0.6514520	0.65290384

b.

i	x_i	w_i	$y(x_i)$
5	1.25	0.16797186	0.16762427
10	1.50	0.45842388	0.45819349
15	1.75	0.60787334	0.60777401

c.

i	x_i	w_{1i}	$y(x_i)$
3	0.3	−0.5183084	−0.5185728
6	0.6	−0.2192657	−0.2195247
9	0.9	−0.0405748	−0.04065697

d.

i	x_i	w_{1i}	$y(x_i)$
3	1.3	0.0654387	0.0655342
6	1.6	0.0773936	0.0774595
9	1.9	0.0305465	0.0305621

5. The Linear Finite-Difference Algorithm gives the results in the following tables.

i	x_i	$w_i(h = 0.1)$
3	0.3	0.05572807
6	0.6	0.00310518
9	0.9	0.00016516

i	x_i	$w_i(h = 0.05)$
6	0.3	0.05132396
12	0.6	0.00263406
18	0.9	0.00013340

7. a. The approximate deflections are shown in the following table.

i	x_i	w_{1i}
5	30	0.0102808
10	60	0.0144277
15	90	0.0102808

b. Yes.

c. Yes. Maximum deflection occurs at $x = 60$. The exact solution is within tolerance, but the approximation is not.

9. First, we have

$$\left|\frac{h}{2}p(x_i)\right| \le \frac{hL}{2} < 1,$$

so

$$\left|-1-\frac{h}{2}p(x_i)\right| = 1+\frac{h}{2}p(x_i) \quad \text{and} \quad \left|-1+\frac{h}{2}p(x_i)\right| = 1-\frac{h}{2}p(x_i).$$

Therefore,

$$\left|-1-\frac{h}{2}p(x_i)\right| + \left|-1+\frac{h}{2}p(x_i)\right| = 2 \le 2+h^2q(x_i),$$

for $2 \le i \le N-1$.
Since

$$\left|-1+\frac{h}{2}p(x_1)\right| < 2 \le 2+h^2q(x_1) \quad \text{and} \quad \left|-1-\frac{h}{2}p(x_N)\right| < 2 \le 2+h^2q(x_N),$$

Theorem 6.31 implies that the linear system (11.19) has a unique solution.

Exercise Set 11.4 (Page 711)

1. The Nonlinear Finite-Difference Algorithm gives the following results.

i	x_i	w_i	$y(x_i)$
1	1.5	0.4067967	0.4054651

3. The Nonlinear Finite-Difference Algorithm gives the results in the following tables.

a.

i	x_i	w_i	$y(x_i)$
2	1.20000000	0.18220299	0.18232156
4	1.40000000	0.33632929	0.33647224
6	1.60000000	0.46988413	0.47000363
8	1.80000000	0.58771808	0.58778666

Convergence in 3 iterations

b.

i	x_i	w_i	$y(x_i)$
2	0.31415927	1.36244080	1.36208552
4	0.62831853	1.80138559	1.79999746
6	0.94247780	2.24819259	2.24569937
8	1.25663706	2.59083695	2.58844295

Convergence in 3 iterations

c.

i	x_i	w_i	$y(x_i)$
1	0.83775804	0.86205907	0.86205848
2	0.89011792	0.88155964	0.88155882
3	0.94247780	0.89945447	0.89945372
4	0.99483767	0.91579005	0.91578959

Convergence in 2 iterations

d.

i	x_i	w_i	$y(x_i)$
4	0.62831853	2.58932301	2.58778525
8	1.25663706	2.95378037	2.95105652
12	1.88495559	2.95378037	2.95105652
16	2.51327412	2.58932301	2.58778525

Convergence in 4 iterations

5. b. For (4a)

x_i	$w_i(h = 0.2)$	$w_i(h = 0.1)$	$w_i(h = 0.05)$	$EXT_{1,i}$	$EXT_{2,i}$	$EXT_{3,i}$
1.2	0.45458862	0.45455753	0.45454935	0.45454717	0.45454662	0.45454659
1.4	0.41672067	0.41668202	0.41667179	0.41666914	0.41666838	0.41666833
1.6	0.38466137	0.38462855	0.38461984	0.38461761	0.38461694	0.38461689
1.8	0.35716943	0.35715045	0.35714542	0.35714412	0.35714374	0.35714372

For (4c)

x_i	$w_i(h = 0.2)$	$w_i(h = 0.1)$	$w_i(h = 0.05)$	$EXT_{1,i}$	$EXT_{2,i}$	$EXT_{3,i}$
1.2	2.0340273	2.0335158	2.0333796	2.0333453	2.0333342	2.0333334
1.4	2.1148732	2.1144386	2.1143243	2.1142937	2.1142863	2.1142858
1.6	2.2253630	2.2250937	2.2250236	2.2250039	2.2250003	2.2250000
1.8	2.3557284	2.3556001	2.3555668	2.3555573	2.3555556	2.3355556

7. The Jacobian matrix $J = (a_{i,j})$ is tridiagonal with entries given in (11.21). So,

$$\begin{aligned}
a_{1,1} &= 2 + h^2 f_y\left(x_1, w_1, \frac{1}{2h}(w_2 - \alpha)\right),\\
a_{1,2} &= -1 + \frac{h}{2} f_{y'}\left(x_1, w_1, \frac{1}{2h}(w_2 - \alpha)\right),\\
a_{i,i-1} &= -1 - \frac{h}{2} f_{y'}\left(x_i, w_i, \frac{1}{2h}(w_{i+1} - w_{i-1})\right), \quad \text{for } 2 \le i \le N-1\\
a_{i,i} &= 2 + h^2 f_y\left(x_i, w_i, \frac{1}{2h}(w_{i+1} - w_{i-1})\right), \quad \text{for } 2 \le i \le N-1\\
a_{i,i+1} &= -1 + \frac{h}{2} f_{y'}\left(x_i, w_i, \frac{1}{2h}(w_{i+1} - w_{i-1})\right), \quad \text{for } 2 \le i \le N-1\\
a_{N,N-1} &= -1 - \frac{h}{2} f_{y'}\left(x_N, w_N, \frac{1}{2h}(\beta - w_{N-1})\right),\\
a_{N,N} &= 2 + h^2 f_y\left(x_N, w_N, \frac{1}{2h}(\beta - w_{N-1})\right).
\end{aligned}$$

Thus, $|a_{i,i}| \ge 2 + h^2\delta$, for $i = 1, \ldots, N$. Since $|f_{y'}(x, y, y')| \le L$ and $h < 2/L$,

$$\left|\frac{h}{2} f_{y'}(x, y, y')\right| \le \frac{hL}{2} < 1.$$

So,

$$|a_{1,2}| = \left| -1 + \frac{h}{2} f_{y'}\left(x_1, w_1, \frac{1}{2h}(w_2 - \alpha)\right)\right| < 2 < |a_{1,1}|,$$

$$
\begin{aligned}
|a_{i,i-1}| + |a_{i,i+1}| &= - a_{i,i-1} - a_{i,i+1} \\
&= 1 + \frac{h}{2} f_{y'}\left(x_i, w_i, \frac{1}{2h}(w_{i+1} - w_{i-1})\right) + 1 - \frac{h}{2} f_{y'}\left(x_i, w_i, \frac{1}{2h}(w_{i+1} - w_{i-1})\right) \\
&= 2 \le |a_{i,i}|,
\end{aligned}
$$

and

$$
|a_{N,N-1}| = -a_{N,N-1} = 1 + \frac{h}{2} f_{y'}\left(x_N, w_N, \frac{1}{2h}(\beta - w_{N-1})\right) < 2 < |a_{N,N}|.
$$

By Theorem 6.31, the matrix J is nonsingular.

Exercise Set 11.5 (Page 726)

1. The Piecewise Linear Algorithm gives $\phi(x) = -0.07713274\phi_1(x) - 0.07442678\phi_2(x)$. The actual values are $y(x_1) = -0.07988545$ and $y(x_2) = -0.07712903$.

3. The Piecewise Linear Algorithm gives the results in the following tables.

a.

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	−0.212333	−0.21
6	0.6	−0.241333	−0.24
9	0.9	−0.090333	−0.09

b.

i	x_i	$\phi(x_i)$	$y(x_i)$
3	0.3	0.1815138	0.1814273
6	0.6	0.1805502	0.1804753
9	0.9	0.05936468	0.05934303

c.

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.3585989	−0.3585641
10	0.50	−0.5348383	−0.5347803
15	0.75	−0.4510165	−0.4509614

d.

i	x_i	$\phi(x_i)$	$y(x_i)$
5	0.25	−0.1846134	−0.1845204
10	0.50	−0.2737099	−0.2735857
15	0.75	−0.2285169	−0.2284204

5. The Cubic Spline Algorithm gives the results in the following tables.

i	x_i	$\phi(x_i)$	y_i
1	0.25	−0.1875	−0.1875
2	0.5	−0.25	−0.25
3	0.75	−0.1875	−0.1875

7. The Piecewise Linear Algorithm gives the results in the following table.

i	x_i	$\phi(x_i)$	$w(x_i)$
4	24	0.00071265	0.0007
8	48	0.0011427	0.0011
10	60	0.00119991	0.0012
16	96	0.00071265	0.0007

9. With $z(x) = y(x) - \beta x - \alpha(1 - x)$, we have

$$
z(0) = y(0) - \alpha = \alpha - \alpha = 0 \quad \text{and} \quad z(1) = y(1) - \beta = \beta - \beta = 0.
$$

Further, $z'(x) = y'(x) - \beta + \alpha$. Thus,

$$
y(x) = z(x) + \beta x + \alpha(1 - x) \quad \text{and} \quad y'(x) = z'(x) + \beta - \alpha.
$$

Substituting for y and y' in the differential equation gives

$$
-\frac{d}{dx}(p(x)z' + p(x)(\beta - \alpha)) + q(x)(z + \beta x + \alpha(1 - x)) = f(x).
$$

Simplifying gives the differential equation

$$-\frac{d}{dx}(p(x)z') + q(x)z = f(x) + (\beta - \alpha)p'(x) - [\beta x + \alpha(1 - x)]q(x).$$

11. The Cubic Spline Algorithm gives the results in the following table.

x_i	$\phi_i(x)$	$y(x_i)$
0.3	1.0408183	1.0408182
0.5	1.1065307	1.1065301
0.9	1.3065697	1.3065697

13. If $\sum_{i=1}^{n} c_i\phi_i(x) = 0$, for $0 \le x \le 1$, then for any j, we have $\sum_{i=1}^{n} c_i\phi_i(x_j) = 0$.
But

$$\phi_i(x_j) = \begin{cases} 0 & i \ne j, \\ 1 & i = j, \end{cases}$$

so $c_j\phi_j(x_j) = c_j = 0$. Hence, the functions are linearly independent.

15. Let $\mathbf{c} = (c_1, \ldots, c_n)^t$ be any vector and let $\phi(x) = \sum_{j=1}^{n} c_j\phi_j(x)$. Then

$$\begin{aligned}
\mathbf{c}^t A\mathbf{c} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}c_ic_j = \sum_{i=1}^{n}\sum_{j=i-1}^{i+1} a_{ij}c_ic_j \\
&= \sum_{i=1}^{n}\left[\int_0^1 \{p(x)c_i\phi_i'(x)c_{i-1}\phi_{i-1}'(x) + q(x)c_i\phi_i(x)c_{i-1}\phi_{i-1}(x)\}\,dx\right. \\
&\quad + \int_0^1 \left\{p(x)c_i^2[\phi_i'(x)]^2 + q(x)c_i^2[\phi_i'(x)]^2\right\}\,dx \\
&\quad \left. + \int_0^1 \{p(x)c_i\phi_i'(x)c_{i+1}\phi_{i+1}'(x) + q(x)c_i\phi_i(x)c_{i+1}\phi_{i+1}(x)\}\,dx\right] \\
&= \int_0^1 \left\{p(x)[\phi'(x)]^2 + q(x)[\phi(x)]^2\right\}dx.
\end{aligned}$$

So, $\mathbf{c}^t A\mathbf{c} \ge 0$ with equality only if $\mathbf{c} = \mathbf{0}$. Since A is also symmetric, A is positive definite.

Exercise Set 12.1 (Page 741)

1. The Poisson Equation Finite-Difference Algorithm gives the following results.

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
1	1	0.5	0.5	0.0	0
1	2	0.5	1.0	0.25	0.25
1	3	0.5	1.5	1.0	1

3. The Poisson Equation Finite-Difference Algorithm gives the following results.

a. 30 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
2	2	0.4	0.4	0.1599988	0.16
2	4	0.4	0.8	0.3199988	0.32
4	2	0.8	0.4	0.3199995	0.32
4	4	0.8	0.8	0.6399996	0.64

b. 29 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
2	1	1.256637	0.3141593	0.2951855	0.2938926
2	3	1.256637	0.9424778	0.1830822	0.1816356
4	1	2.513274	0.3141593	−0.7721948	−0.7694209
4	3	2.513274	0.9424778	−0.4785169	−0.4755283

c. 126 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
4	3	0.8	0.3	1.2714468	1.2712492
4	7	0.8	0.7	1.7509414	1.7506725
8	3	1.6	0.3	1.6167917	1.6160744
8	7	1.6	0.7	3.0659184	3.0648542

d. 127 iterations required:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
2	2	1.2	1.2	0.5251533	0.5250861
4	4	1.4	1.4	1.3190830	1.3189712
6	6	1.6	1.6	2.4065150	2.4064186
8	8	1.8	1.8	3.8088995	3.8088576

5. The approximate potential at some typical points gives the following results.

i	j	x_i	y_j	$w_{i,j}$
1	4	0.1	0.4	88
2	1	0.2	0.1	66
4	2	0.4	0.2	66

7. To incorporate the SOR method, make the following changes to Algorithm 12.1:

Step 1 Set $h = (b - a)/n$;

$$k = (d - c)/m;$$

$$\omega = 4/\left(2 + \sqrt{4 - (\cos \pi/m)^2 - (\cos \pi/n)^2}\right);$$

$$\omega_0 = 1 - w;$$

In each of Steps 7, 8, 9, 11, 12, 13, 14, 15, and 16 after

set . . .

insert

set $E = w_{\alpha,\beta} - z$;

if ($|E| >$ NORM) then set NORM $= |E|$;

set $w_{\alpha,\beta} = \omega_0 E + z$.

where α and β depend on which step is being changed.

Exercise Set 12.2 (Page 754)

1. The Heat Equation Backward-Difference Algorithm gives the following results.

a.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
1	1	0.5	0.05	0.632952	0.652037
2	1	1.0	0.05	0.895129	0.883937
3	1	1.5	0.05	0.632952	0.625037
1	2	0.5	0.1	0.566574	0.552493
2	2	1.0	0.1	0.801256	0.781344
3	2	1.5	0.1	0.566574	0.552493

3. The Crank-Nicolson Algorithm gives the following results.

a.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
1	1	0.5	0.05	0.628848	0.652037
2	1	1.0	0.05	0.889326	0.883937
3	1	1.5	0.05	0.628848	0.625037
1	2	0.5	0.1	0.559251	0.552493
2	2	1.0	0.1	0.790901	0.781344
3	2	1.5	0.1	0.559252	0.552493

5. The Forward-Difference Algorithm gives the following results.

a. For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	5	0.8	0.5	3.035630	0
3	5	1.2	0.5	−3.035630	0
4	5	1.6	0.5	1.876122	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	10	0.8	0.5	0	0
3	10	1.2	0.5	0	0
4	10	1.6	0.5	0	0

b. For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
3	10	0.94247780	0.5	0.4926589	0.4906936
6	10	1.88495559	0.5	0.5791553	0.5768449
9	10	2.82743339	0.5	0.1881790	0.1874283

7. a. For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
2	5	0.8	0.5	−0.00258	0
3	5	1.2	0.5	0.00258	0
4	5	1.6	0.5	−0.00159	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
2	10	0.8	0.5	-4.93×10^{-4}	0
3	10	1.2	0.5	4.93×10^{-4}	0
4	10	1.6	0.5	-3.05×10^{-4}	0

b. For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	$w_{i,j}$	$u(x_i, t_j)$
3	10	0.94247780	0.5	0.4986092	0.4906936
6	10	1.88495559	0.5	0.5861503	0.5768449
9	10	2.82743339	0.5	0.1904518	0.1874283

9. The Crank-Nicolson Algorithm gives the following results.

a. For $h = 0.4$ and $k = 0.1$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	5	0.8	0.5	8.2×10^{-7}	0
3	5	1.2	0.5	-8.2×10^{-7}	0
4	5	1.6	0.5	5.1×10^{-7}	0

For $h = 0.4$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	10	0.8	0.5	-2.6×10^{-6}	0
3	10	1.2	0.5	2.6×10^{-6}	0
4	10	1.6	0.5	-1.6×10^{-6}	0

b. For $h = \frac{\pi}{10}$ and $k = 0.05$:

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
3	10	0.94247780	0.5	0.4926589	0.4906936
6	10	1.88495559	0.5	0.5791553	0.5768449
9	10	2.82743339	0.5	0.1881790	0.1874283

11. a. Using $h = 0.4$ and $k = 0.1$ leads to meaningless results. Using $h = 0.4$ and $k = 0.05$ again gives meaningless answers. Letting $h = 0.4$ and $k = 0.005$ produces the following:

i	j	x_i	t_j	w_{ij}
1	100	0.4	0.5	-165.405
2	100	0.8	0.5	267.613
3	100	1.2	0.5	-267.613
4	100	1.6	0.5	165.405

b.

i	j	x_i	t_j	$w(x_{ij})$
3	10	0.94247780	0.5	0.46783396
6	10	1.8849556	0.5	0.54995267
9	10	2.8274334	0.5	0.17871220

13. a. The approximate temperature at some typical points is given in the table.

i	j	r_i	t_j	$w_{i,j}$
1	20	0.6	10	137.6753
2	20	0.7	10	245.9678
3	20	0.8	10	340.2862
4	20	0.9	10	424.1537

b. The strain is approximately $I = 1242.537$.

15. We have

$$a_{11}v_1^{(i)} + a_{12}v_2^{(i)} = (1-2\lambda)\sin\frac{i\pi}{m} + \lambda\sin\frac{2\pi i}{m}$$

and

$$\begin{aligned}\mu_i v_1^{(i)} &= \left[1 - 4\lambda\left(\sin\frac{i\pi}{2m}\right)^2\right]\sin\frac{i\pi}{m} = \left[1 - 4\lambda\left(\sin\frac{i\pi}{2m}\right)^2\right]\left(2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m}\right)\\ &= 2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m} - 8\lambda\left(\sin\frac{i\pi}{2m}\right)^3\cos\frac{i\pi}{2m}.\end{aligned}$$

However,

$$\begin{aligned}(1-2\lambda)\sin\frac{i\pi}{m} + \lambda\sin\frac{2\pi i}{m} =&2(1-2\lambda)\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m} + 2\lambda\sin\frac{i\pi}{m}\cos\frac{i\pi}{m}\\ =&2(1-2\lambda)\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m}\\ &+ 2\lambda\left[2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m}\right]\left[1 - 2\left(\sin\frac{i\pi}{2m}\right)^2\right]\\ =&2\sin\frac{i\pi}{2m}\cos\frac{i\pi}{2m} - 8\lambda\cos\frac{i\pi}{2m}\left[\sin\frac{i\pi}{2m}\right]^3.\end{aligned}$$

Thus,

$$a_{11}v_1^{(i)} + a_{12}v_2^{(i)} = \mu_i v_1^{(i)}.$$

Further,

$$\begin{aligned}a_{j,j-1}v_{j-1}^{(i)} + a_{j,j}v_j^{(i)} + a_{j,j+1}v_{j+1}^{(i)} =&\lambda\sin\frac{i(j-1)\pi}{m} + (1-2\lambda)\sin\frac{ij\pi}{m} + \lambda\sin\frac{i(j+1)\pi}{m}\\ =&\lambda\left(\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m} - \sin\frac{i\pi}{m}\cos\frac{ij\pi}{m}\right) + (1-2\lambda)\sin\frac{ij\pi}{m}\\ &+ \lambda\left(\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m} + \sin\frac{i\pi}{m}\cos\frac{ij\pi}{m}\right)\\ =&\sin\frac{ij\pi}{m} - 2\lambda\sin\frac{ij\pi}{m} + 2\lambda\sin\frac{ij\pi}{m}\cos\frac{i\pi}{m}\\ =&\sin\frac{ij\pi}{m} + 2\lambda\sin\frac{ij\pi}{m}\left(\cos\frac{i\pi}{m} - 1\right)\end{aligned}$$

and

$$\begin{aligned}\mu_i v_j^{(i)} &= \left[1 - 4\lambda\left(\sin\frac{i\pi}{2m}\right)^2\right]\sin\frac{ij\pi}{m} = \left[1 - 4\lambda\left(\frac{1}{2} - \frac{1}{2}\cos\frac{i\pi}{m}\right)\right]\sin\frac{ij\pi}{m}\\ &= \left[1 + 2\lambda\left(\cos\frac{i\pi}{m} - 1\right)\right]\sin\frac{ij\pi}{m},\end{aligned}$$

so

$$a_{j,j-1}v_{j-1}^{(i)} + a_{j,j}v_j^{(i)} + a_{j,j+1}v_j^{(i)} = \mu_i v_j^{(i)}.$$

Similarly,

$$a_{m-2,m-1}v_{m-2}^{(i)} + a_{m-1,m-1}v_{m-1}^{(i)} = \mu_i v_{m-1}^{(i)},$$

so $A\mathbf{v}^{(i)} = \mu_i \mathbf{v}^{(i)}$.

17. To modify Algorithm 12.2, change the following:

Step 7 Set

$$t = jk;$$
$$z_1 = (w_1 + kF(h))/l_1.$$

Step 8 For $i = 2, \dots, m-1$ set

$$z_i = (w_i + kF(ih) + \lambda z_{i-1})/l_i.$$

To modify Algorithm 12.3, change the following:

Step 7 Set

$$t = jk;$$
$$z_1 = \left[(1-\lambda)w_1 + \frac{\lambda}{2}w_2 + kF(h)\right] \Big/ l_1.$$

Step 8 For $i = 2, \dots, m-1$ set

$$z_i = \left[(1-\lambda)w_i + \frac{\lambda}{2}(w_{i+1} + w_{i-1} + z_{i-1}) + kF(ih)\right] \Big/ l_i.$$

19. To modify Algorithm 12.2, change the following:

Step 7 Set

$t = jk;$
$w_0 = \phi(t);$
$z_1 = (w_1 + \lambda w_0)/l_1.$
$w_m = \psi(t).$

Step 8 For $i = 2, \dots, m-2$ set

$$z_i = (w_i + \lambda z_{i-1})/l_i;$$

Set

$z_{m-1} = (w_{m-1} + \lambda w_m + \lambda z_{m-2})/l_{m-1}.$

Step 11 OUTPUT (t);
For $i = 0, \dots, m$ set $x = ih$;
OUTPUT (x, w_i).

To modify Algorithm 12.3, change the following:

Step 1 Set

$h = l/m;$
$k = T/N;$
$\lambda = \alpha^2 k/h^2;$
$w_m = \psi(0);$
$w_0 = \phi(0).$

Step 7 Set

$t = jk;$
$z_1 = \left[(1-\lambda)w_1 + \frac{\lambda}{2}w_2 + \frac{\lambda}{2}{}_0 + \frac{\lambda}{2}\phi(t)\right]/l_1;$
$w_0 = \phi(t).$

Step 8 For $i = 2, \dots, m-2$ set

$$z_i = \left[(1-\lambda)w_i + \tfrac{\lambda}{2}(w_{i+1} + w_{i-1} + z_{i-1})\right]/l_i;$$

Set

$z_{m-1} = \left[(1-\lambda)w_{m-1} + \frac{\lambda}{2}(w_m + w_{m-2} + z_{m-2} + \psi(t))\right]/l_{m-1};$
$w_m = \psi(t).$

Step 11 OUTPUT (t);
For $i = 0, \dots, m$ set $x = ih$;
OUTPUT (x, w_i).

Exercise Set 12.3 (Page 763)

1. The Wave Equation Finite-Difference Algorithm gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	4	0.25	1.0	−0.7071068	−0.7071068
3	4	0.50	1.0	−1.0000000	−1.0000000
4	4	0.75	1.0	−0.7071068	−0.7071068

3. The Wave Equation Finite-Difference Algorithm with $h = \frac{\pi}{10}$ and $k = 0.05$ gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	10	$\frac{\pi}{5}$	0.5	0.5163933	0.5158301
5	10	$\frac{\pi}{2}$	0.5	0.8785407	0.8775826
8	10	$\frac{4\pi}{5}$	0.5	0.5163933	0.5158301

The Wave Equation Finite-Difference Algorithm with $h = \frac{\pi}{20}$ and $k = 0.1$ gives the following results.

i	j	x_i	t_j	w_{ij}
4	5	$\frac{\pi}{5}$	0.5	0.5159163
10	5	$\frac{\pi}{2}$	0.5	0.8777292
16	5	$\frac{4\pi}{5}$	0.5	0.5159163

The Wave Equation Finite-Difference Algorithm with $h = \frac{\pi}{20}$ and $k = 0.05$ gives the following results.

i	j	x_i	t_j	w_{ij}
4	10	$\frac{\pi}{5}$	0.5	0.5159602
10	10	$\frac{\pi}{2}$	0.5	0.8778039
16	10	$\frac{4\pi}{5}$	0.5	0.5159602

5. The Wave Equation Finite-Difference Algorithm gives the following results.

i	j	x_i	t_j	w_{ij}	$u(x_i, t_j)$
2	3	0.2	0.3	0.6729902	0.61061587
5	3	0.5	0.3	0	0
8	3	0.8	0.3	−0.6729902	−0.61061587

7. a. The air pressure for the open pipe is $p(0.5, 0.5) \approx 0.9$ and $p(0.5, 1.0) \approx 2.7$.

b. The air pressure for the closed pipe is $p(0.5, 0.5) \approx 0.9$ and $p(0.5, 1.0) \approx 0.9187927$.

Exercise Set 12.4 (Page 777)

1. With $E_1 = (0.25, 0.75)$, $E_2 = (0, 1)$, $E_3 = (0.5, 0.5)$, and $E_4 = (0, 0.5)$, the basis functions are

$$\phi_1(x, y) = \begin{cases} 4x & \text{on } T_1 \\ -2 + 4y & \text{on } T_2, \end{cases}$$

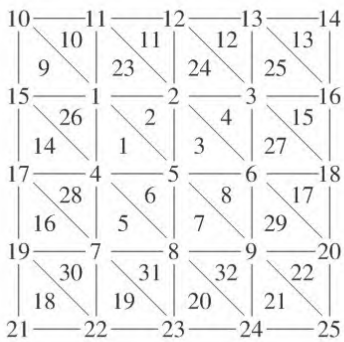
$$\phi_2(x, y) = \begin{cases} -1 - 2x + 2y & \text{on } T_1 \\ 0 & \text{on } T_2, \end{cases}$$

$$\phi_3(x, y) = \begin{cases} 0 & \text{on } T_1 \\ 1 + 2x - 2y & \text{on } T_2, \end{cases}$$

$$\phi_4(x, y) = \begin{cases} 2 - 2x - 2y & \text{on } T_1 \\ 2 - 2x - 2y & \text{on } T_2, \end{cases}$$

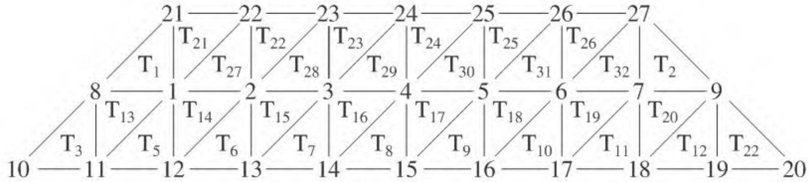
and $\gamma_1 = 0.323825$, $\gamma_2 = 0$, $\gamma_3 = 1.0000$, and $\gamma_4 = 0$.

3. The Finite-Element Algorithm with $K = 8$, $N = 8$, $M = 32$, $n = 9$, $m = 25$, and $NL = 0$ gives the following results, where the labeling is as shown in the diagram.



$$\gamma_1 = 0.511023$$
$$\gamma_2 = 0.720476$$
$$\gamma_3 = 0.507899$$
$$\gamma_4 = 0.720476$$
$$\gamma_5 = 1.01885$$
$$\gamma_6 = 0.720476$$
$$\gamma_7 = 0.507896$$
$$\gamma_8 = 0.720476$$
$$\gamma_9 = 0.511023$$
$$\gamma_i = 0 \quad 10 \le i \le 25$$
$$u(0.125, 0.125) \approx 0.614187$$
$$u(0.125, 0.25) \approx 0.690343$$
$$u(0.25, 0.125) \approx 0.690343$$
$$u(0.25, 0.25) \approx 0.720476$$

5. The Finite-Element Algorithm with $K = 0$, $N = 12$, $M = 32$, $n = 20$, $m = 27$, and $NL = 14$ gives the following results, where the labeling is as shown in the diagram.



$\gamma_1 = 21.40335$	$\gamma_8 = 24.19855$	$\gamma_{15} = 20.23334$	$\gamma_{22} = 15$
$\gamma_2 = 19.87372$	$\gamma_9 = 24.16799$	$\gamma_{16} = 20.50056$	$\gamma_{23} = 15$
$\gamma_3 = 19.10019$	$\gamma_{10} = 27.55237$	$\gamma_{17} = 21.35070$	$\gamma_{24} = 15$
$\gamma_4 = 18.85895$	$\gamma_{11} = 25.11508$	$\gamma_{18} = 22.84663$	$\gamma_{25} = 15$
$\gamma_5 = 19.08533$	$\gamma_{12} = 22.92824$	$\gamma_{19} = 24.98178$	$\gamma_{26} = 15$
$\gamma_6 = 19.84115$	$\gamma_{13} = 21.39741$	$\gamma_{20} = 27.41907$	$\gamma_{27} = 15$
$\gamma_7 = 21.34694$	$\gamma_{14} = 20.52179$	$\gamma_{21} = 15$	

$$u(1, 0) \approx 22.92824$$

$$u(4, 0) \approx 22.84663$$

$$u\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right) \approx 18.85895$$

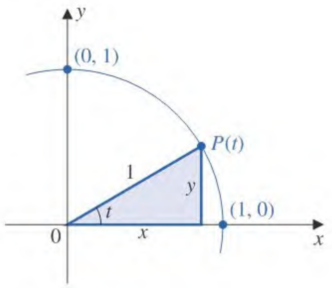
Index

Index of Algorithms

Glossary of Notation

$C(X)$	Set of all functions continuous on X	*3*
$\mathbb{R}$	Set of real numbers	*3*
$C^n(X)$	Set of all functions having n continuous derivatives on X	*4*
$C^\infty(X)$	Set of all functions having derivatives of all orders on X	*4*
$0.\bar{3}$	A decimal in which the numeral 3 repeats indefinitely	*10*
$fl(y)$	Floating-point form of the real number y	*17*
$O(\cdot)$	Order of convergence	*34*
$\lfloor\ \ \rfloor$	Floor function, $\lfloor x \rfloor$, the greatest integer less than or equal to x	*50*
$\lceil\ \ \rceil$	Ceiling function, $\lceil x \rceil$, the smallest integer greater than or equal to x	*42*
$\operatorname{sgn}(x)$	Sign of the number x: 1 if $x > 0$, -1 if $x < 0$	*52*
Δ	Forward difference	*87*
$\bar{z}$	Complex conjugate of the complex number z	*95*
$\binom{n}{k}$	The kth binomial coefficient of order n	*115*
$f[\cdot]$	Divided difference of the function f	*123*
∇	Backward difference	*127*
$\mathbb{R}^n$	Set of ordered n-tuples of real numbers	*261*
τ_i	Local truncation error at the ith step	*276*
$\rightarrow$	Equation replacement	*362*
$\leftrightarrow$	Equation interchange	*362*
(a_{ij})	Matrix with a_{ij} as the entry in the ith row and jth column	*363*
$\mathbf{x}$	Column vector or element of $\mathbb{R}^n$	*364*
$[A, \mathbf{b}]$	Augmented matrix	*364*
O	A matrix with all zero entries	*386*
δ_{ij}	Kronecker delta: 1 if $i = j$, 0 if $i \neq j$	*390*
I_n	$n \times n$ identity matrix	*390*
A^{-1}	Inverse matrix of the matrix A	*391*
A^t	Transpose matrix of the matrix A	*394*
M_{ij}	Minor of a matrix	*400*
$\det A$	Determinant of the matrix A	*400*
$\mathbf{0}$	Vector with all zero entries	*386*
$\lVert\mathbf{x}\rVert$	Arbitrary norm of the vector $\mathbf{x}$	*438*
$\lVert\mathbf{x}\rVert_2$	The l_2 norm of the vector $\mathbf{x}$	*438*
$\lVert\mathbf{x}\rVert_\infty$	The l_∞ norm of the vector $\mathbf{x}$	*438*
$\lVert A\rVert$	Arbitrary norm of the matrix A	*444*
$\lVert A\rVert_2$	The l_2 norm of the matrix A	*445*
$\lVert A\rVert_\infty$	The l_∞ norm of the matrix A	*445*
$\rho(A)$	The spectral radius of the matrix A	*452*
$K(A)$	The condition number of the matrix A	*478*
$\langle\mathbf{x}, \mathbf{y}\rangle$	Inner product of the n-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$	*487*
Π_n	Set of all polynomials of degree n or less	*521*
$\widetilde{\Pi}_n$	Set of all monic polynomials of degree n	*528*
$\mathcal{T}_n$	Set of all trigonometric polynomials of degree n or less	*546*
$\mathcal{C}$	Set of complex numbers	*570*
$\mathbf{F}$	Function mapping $\mathbb{R}^n$ into $\mathbb{R}^n$	*642*
$A(\mathbf{x})$	Matrix whose entries are functions form $\mathbb{R}^n$ into $\mathbb{R}$	*651*
$J(\mathbf{x})$	Jacobian matrix	*652*
∇g	Gradient of the function g	*667*

Trigonometry



$$\sin t = y \qquad \cos t = x$$

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

$$\sec t = \frac{1}{\cos t} \qquad \csc t = \frac{1}{\sin t}$$

$$(\sin t)^2 + (\cos t)^2 = 1$$

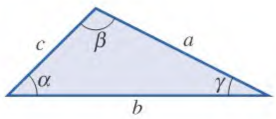
$$\sin(t_1 \pm t_2) = \sin t_1 \cos t_2 \pm \cos t_1 \sin t_2$$

$$\cos(t_1 \pm t_2) = \cos t_1 \cos t_2 \mp \sin t_1 \sin t_2$$

$$\sin t_1 \sin t_2 = \frac{1}{2}[\cos(t_1 - t_2) - \cos(t_1 + t_2)]$$

$$\cos t_1 \cos t_2 = \frac{1}{2}[\cos(t_1 - t_2) + \cos(t_1 + t_2)]$$

$$\sin t_1 \cos t_2 = \frac{1}{2}[\sin(t_1 - t_2) + \sin(t_1 + t_2)]$$



Law of Sines: $\dfrac{\sin \alpha}{\alpha} = \dfrac{\sin \beta}{\beta} = \dfrac{\sin \gamma}{\gamma}$

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Common Series

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

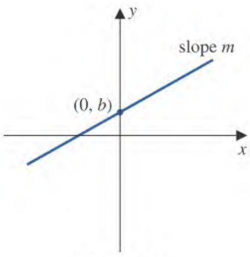
$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + \cdots, \qquad |t| < 1$$

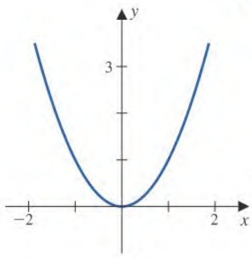
The Greek Alphabet

Alpha	A	α	Eta	H	η	Nu	N	ν	Tau	T	τ
Beta	B	β	Theta	Θ	θ	Xi	Ξ	ξ	Upsilon	Υ	υ
Gamma	Γ	γ	Iota	I	ι	Omicron	O	o	Phi	Φ	ϕ
Delta	Δ	δ	Kappa	K	κ	Pi	Π	π	Chi	X	χ
Epsilon	E	ϵ	Lambda	Λ	λ	Rho	P	ρ	Psi	Ψ	ψ
Zeta	Z	ζ	Mu	M	μ	Sigma	Σ	σ	Omega	Ω	ω

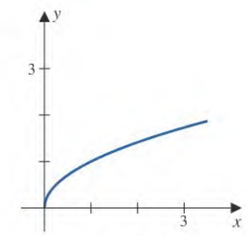
Common Graphs



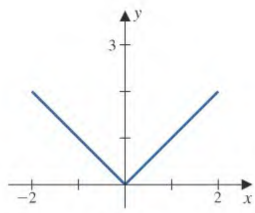
$y = mx + b$



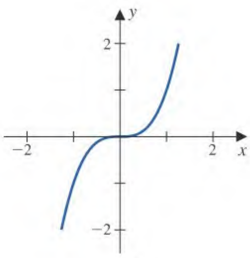
$y = x^2$



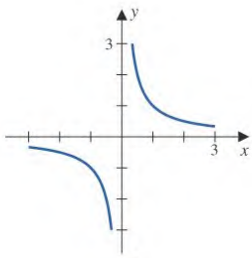
$y = \sqrt{x}$



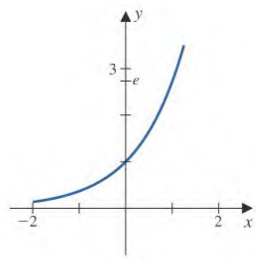
$y = |x|$



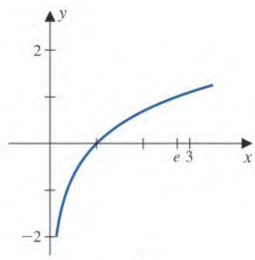
$y = x^3$



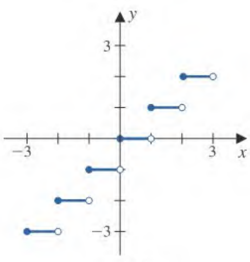
$y = \dfrac{1}{x}$



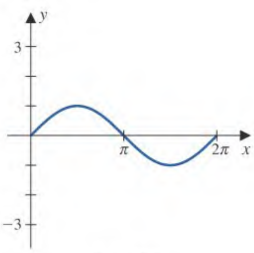
$y = e^x$



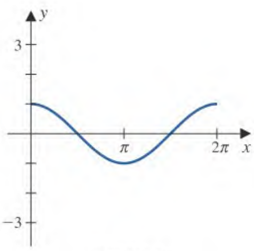
$y = \ln x$



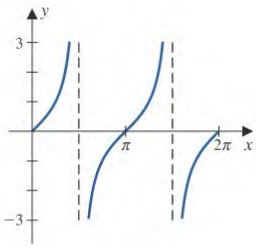
$y = \lfloor x \rfloor$



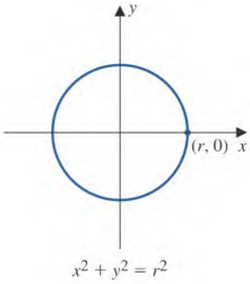
$y = \sin x$



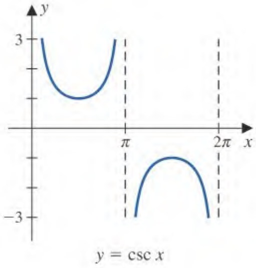
$y = \cos x$



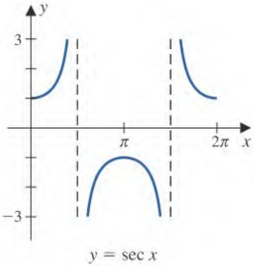
$y = \tan x$



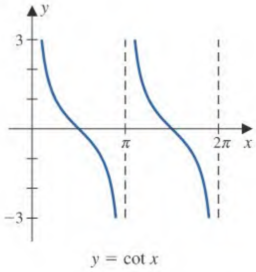
$x^2 + y^2 = r^2$



$y = \csc x$



$y = \sec x$



$y = \cot x$