# Introduction to Mathematical Logic

For CS Students

CS104

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2025 年 5 月 19 日

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# Natural Deduction for FOL



Natural Deduction for FOL extends Natural Deduction for propositional logic by including rules for introduction and elimination of quantifiers.

Other proof techniques and tricks remain the same as Natural Deduction for propositional logic. In fact, all the rules from Natural Deduction extend to our FOL setting, however we need new rules to handle quantifiers.

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When writing natural deduction proofs in predicate logic, it is often useful to replace a variable in a formula with a term.

Suppose that the following sentences are true:

$$\forall x(Man(x) \to Mortal(x))$$
$$Man(Socrates)$$

To conclude that “Socrates is mortal”, we need to replace every occurrence of the variable $x$ in the implication by the term *Socrates*.

$$Man(Socrates) \to Mortal(Socrates)$$

By the $\to$e rule, we conclude that $Mortal(Socrates)$.

Formally, we use substitution to refer to this process of replace $x$ by $Socrates$ in the formula.

For a variable $x$, a term $t$, and a formula $\alpha$, $\alpha[t/x]$ denotes the resulting formula by replacing each free occurrence of $x$ in $\alpha$ with $t$.
In other words, substitution does NOT affect bound occurrences of the variable.

Intuitively, $\alpha[t/x]$ answers the question:

"What happens to $\alpha$ if $x$ has the value specified by term $t$?"

# Examples



Let $\alpha$ be $P(f(x))$:

- $\alpha[(y+y)/x]$ is $P(f((y+y)))$
- $\alpha[f(x)/x]$ is $P(f(f(x)))$.

Let $\beta$ be $P(x) \wedge (\exists x Q(x))$:

- $\beta[y/x]$ is $P(y) \wedge (\exists x Q(x))$ (only the free $x$ gets substituted).

Let $\gamma$ be $\forall x(E(f(x)) \wedge S(x,y))$:

- $\gamma[g(x,y)/x]$ is still $\gamma$, since $\gamma$ has no free occurrence of $x$.

# Avoid Capture



If $\alpha$ is $\forall x(\exists y((x + y) = z))$, what is $\alpha[(y - 1)/z]$?

There is a problem if we have:

$$\forall x(\exists y((x + y) = (y - 1)))$$

Because the free variable $y$ in the term $(y - 1)$ got "captured" by the quantifier $\exists y$.

We want to avoid this capture to prevent accidentally changing the semantics.

# Avoid Capture



If $\alpha$ is $\forall x(\exists y((x+y)=z))$, what is $\alpha[(y-1)/z]$?

We can prevent capture by renaming the quantified variable to something harmless, that is, a variable that occurs in neither $\alpha$ nor $t$.
For example, $\alpha$ can be rewritten by renaming $y$ by a new variable $w$, without changing its meaning.

$$\forall x(\exists w((x+w)=z))$$

Now, $\alpha[(y-1)/z]$ is:

$$\forall x(\exists w((x+w)=(y-1)))$$

Or, we could simply substitute $z$ with a fresh variable (e.g., $u$, $v$) that doesn't occur in $\alpha$.

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# $\forall$-elimination

| Name | $\vdash$-notation | inference notation |
|---|---|---|
| $\forall$-elimination ($\forall$e) | If $\Sigma \vdash_{ND} (\forall x\ \alpha)$ then $\Sigma \vdash_{ND} \alpha[t/x]$ | $\dfrac{(\forall x\ \alpha)}{\alpha[t/x]}$ |

Intuition: Given that a formula is true for every value of $x$, we can conclude it is true for any particular value, such as that of $t$.

| | | |
|---|---|---|
| ∃-introduction ($\exists$i) | If $\Sigma \vdash_{ND} \alpha[t/x]$, then $\Sigma \vdash_{ND} (\exists x\ \alpha)$ | $\dfrac{\alpha[t/x]}{(\exists x\ \alpha)}$ |

Intuition: Given that a formula is true for a particular value $(t)$, we can conclude it is true for some value.

Prove that

$$\forall x P(x) \vdash_{ND} \exists x P(x)$$

Proof:

| | | |
|---|---|---|
| 1. | $\forall x P(x)$ | Premise |
| 2. | $P(u)$ | $\forall e : 1$ |
| 3. | $\exists x P(x)$ | $\exists i : 2$ |

Note: $u$ represents an individual object in the domain.

Prove that:

$$\{P(t), \forall x(P(x) \to \neg Q(x))\} \vdash_{ND} \neg Q(t)$$

Proof:

| | | |
|---|---|---|
| 1. | $P(t)$ | Premise |
| 2. | $\forall x(P(x) \to \neg Q(x))$ | Premise |
| 3. | $P(t) \to \neg Q(t)$ | $\forall e : 2$ |
| 4. | $\neg Q(t)$ | $\to e : 1, 3$ |

Prove that:

$$\neg P(y) \vdash_{ND} \exists x(P(x) \rightarrow Q(y))$$

Proof:

| | | |
|---|---|---|
| 1. | $\neg P(y)$ | Premise |
| 2. | $P(y)$ | Assumption |
| 3. | $\perp$ | $\neg e : 1, 2$ |
| 4. | $Q(y)$ | $\perp e : 3$ |
| 5. | $P(y) \rightarrow Q(y)$ | $\rightarrow i : 2-4$ |
| 6. | $\exists x(P(x) \rightarrow Q(y))$ | $\exists i : 5$ |

Note: here we took $P(x) \rightarrow Q(y)$ for $\alpha$ in the $\exists i$ rule, i.e., $\alpha[y/x]$ is $P(y) \rightarrow Q(y)$.

# $\forall$-introduction



| Name | $\vdash$-notation | inference notation |
|---|---|---|
| $\forall$-introduction ($\forall$i) | If $\Sigma \vdash_{ND} \alpha[y/x]$ and $y$ not free in $\Sigma$ or $\alpha$, then $\Sigma \vdash_{ND} (\forall x\ \alpha)$ | $\dfrac{\boxed{\begin{array}{c} y \text{ fresh} \\ \vdots \\ \alpha[y/x] \end{array}}}{(\forall x\ \alpha)}$ |

To prove $\forall x\ \alpha$, prove $\alpha[y/x]$ for arbitrary $y$. This rule follows ordinary mathematical usage:

- To prove a property holds for all integers, one often starts with "Let $x$ be an integer", then one proves that $x$ has the property.
- Since we know nothing about the value $x$, except that it is an integer, this justifies that every integer has the property.

# $\forall$-introduction



| Name | $\vdash$-notation | inference notation |
|---|---|---|
| $\forall$-introduction ($\forall$i) | If $\Sigma \vdash_{ND} \alpha[y/x]$ and $y$ not free in $\Sigma$ or $\alpha$, then $\Sigma \vdash_{ND} (\forall x\ \alpha)$ | $\dfrac{\boxed{\begin{array}{c} y \text{ fresh} \\ \vdots \\ \alpha[y/x] \end{array}}}{(\forall x\ \alpha)}$ |

A variable is fresh in a subproof if it occurs nowhere outside the box of the subproof.

- It's safest to always use variables that aren't in any formula in $\Sigma$ and not in $\alpha$.
- Your fresh variable must be used only in the subproof. They cannot escape boxes.
- Use different fresh variables in different subproofs to avoid confusion.

# Example: $\forall$i



Prove that:

$$\forall x P(x) \vdash \forall y P(y)$$

Proof:

| | | |
|---|---|---|
| 1. | $\forall x P(x)$ | Premise |
| 2. | $u$ fresh | |
| 3. | $P(u)$ | $\forall$ e:1 |
| 4. | $\forall y P(y)$ | $\forall$ i: 2-3 |

# Example: $\forall$i

Prove that:

$$\emptyset \vdash \forall x(P(x) \rightarrow P(x))$$

Proof:

| | | |
|---|---|---|
| 1. | $u$ fresh | |
| 2. | $P(u)$ | Assumption |
| 3. | $P(u)$ | Reflexive: 2 |
| 4. | $P(u) \rightarrow P(u)$ | $\rightarrow$i:2-3 |
| 5. | $\forall x(P(x) \rightarrow P(x))$ | $\forall$i: 1-4 |

# Example: $\forall$i

Prove that:

$$\{\forall x(P(x) \to Q(x)), \forall x P(x)\} \vdash \forall x Q(x)$$

# $\exists$-elimination



| Name | $\vdash$-notation | inference notation |
|---|---|---|
| $\exists$-elimination ($\exists$e) | If $\Sigma, \alpha[u/x] \vdash_{ND} \beta$, with $u$ fresh, then $\Sigma, (\exists x\ \alpha) \vdash_{ND} \beta$ | $\dfrac{(\exists x\ \alpha) \qquad \boxed{\begin{array}{c}\alpha[u/x],\ u \text{ fresh} \\ \vdots \\ \beta\end{array}}}{\beta}$ |

Intuition: You know something exists for $\alpha$, but you don't know which one. So you:

- Pretend one such object exists by giving it a fresh name like $u$.
- Use $\alpha[u/x]$ to derive something, like $\beta$.
- As long as $\beta$ doesn't depend on which object you picked (i.e., $\beta$ doesn't contain $u$), you're allowed to conclude $\beta$.

# $\exists$-elimination



| Name | $\vdash$-notation | inference notation |
| --- | --- | --- |
| $\exists$-elimination ($\exists$e) | If $\Sigma, \alpha[u/x] \vdash_{ND} \beta$, with $u$ fresh, then $\Sigma, (\exists x\ \alpha) \vdash_{ND} \beta$ | $\dfrac{(\exists x\ \alpha) \qquad \boxed{\begin{array}{c}\alpha[u/x],\ u \text{ fresh} \\ \vdots \\ \beta\end{array}}}{\beta}$ |

Restriction: In $\exists e$, the fresh variable $u$ should not occur in $\Sigma$, $\alpha$, or $\beta$.

# Example: $\exists$e



Prove that:

$$\exists x P(x) \vdash \exists y P(y)$$

Proof:

| | | |
|---|---|---|
| 1. | $\exists x P(x)$ | Premise |
| 2. | $P(u)$ $u$ fresh | Assumption |
| 3. | $\exists y P(y)$ | $\exists$ i: 2 |
| 4. | $\exists y P(y)$ | $\exists$ e:1, 2-3 |

# Example: $\exists$e

Prove that:

$$\exists y(\forall x P(x,y)) \vdash \forall x(\exists y P(x,y))$$

Proof:

| | | |
|---|---|---|
| 1. | $\exists y(\forall x P(x,y))$ | Premise |
| 2. | $\forall x P(x,w)$ $w$ fresh | Assumption |
| 3. | $u$ fresh | |
| 4. | $P(u,w)$ | $\forall$e:2 |
| 5. | $\exists y P(u,y)$ | $\exists$i:4 |
| 6. | $\forall x(\exists y P(x,y))$ | $\forall$i:3-5 |
| 7. | $\forall x(\exists y P(x,y))$ | $\exists$e:1,2-6 |

# Example: $\exists$e

Prove that:

$$\{\exists x P(x), \forall x(P(x) \to Q(x))\} \vdash \exists x Q(x)$$

Proof:

| | | |
|---|---|---|
| 1. | $\exists x P(x)$ | Premise |
| 2. | $\forall x(P(x) \to Q(x))$ | Premise |
| 3. | $P(u)$, $u$ fresh | Assumption |
| 4. | $P(u) \to Q(u)$ | $\forall$e:2 |
| 5. | $Q(u)$ | $\to$e:3,4 |
| 6. | $\exists x Q(x)$ | $\exists$i: 5 |
| 7. | $\exists x Q(x)$ | $\exists$e:1,3-6 |

# Example: $\exists$e

Prove that:

$$\exists x(P(x) \vee Q(x)) \vdash \exists x P(x) \vee \exists x Q(x)$$

Proved in class.

# Exercises

Prove the following:

- $\exists x(\neg P(x)) \vdash \neg(\forall x P(x))$
- $\neg(\forall x P(x)) \vdash \exists x(\neg P(x))$

# Coursework

- Assignment 6

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# Soundness and Completeness



The ND system of FOL is sound and complete.

# Normal Form



It is often more convenient to deal with formulas in which all quantifiers have been moved to the front of the expression. These types of formulas are said to be in prenex normal form (前束范式):

$$Q_1 x_1 Q_2 x_2 \ldots Q_k x_k B$$

where $Q_i (1 \leq i \leq k)$ is $\forall$ or $\exists$, and $B$ is quantifier free.

Every formula in FOL is logically equivalent to a formula in prenex normal form.

# Higher-Order Logic



FOL is very powerful, but it cannot express statements about properties, relations, or functions themselves; FOL also has trouble with set theory concepts such as membership or cardinality, especially for infinite sets or higher-level set constructions

- For every property $P$, there is an element $x$ such that $P(x)$ holds.
- There is a function $f$ such that for all $x$, $f(x)$ is greater than $10$.
- The power set of a set exists.

Higher-Order Logic (HOL) allows for quantification over predicates, functions, and sets, enabling more complex and expressive statements.

# Temporal Logic

FOL does not account for time explicitly.

- Event A will eventually happen.
- Event A always holds at some point in the future.

Temporal logic is logic for reasoning about how truths change over time.

| Symbol | Meaning |
|---|---|
| $\bigcirc\varphi$ | **Next:** $\varphi$ holds in the next state |
| $\Diamond\varphi$ | **Eventually:** $\varphi$ holds sometime in the future |
| $\Box\varphi$ | **Globally:** $\varphi$ holds in all future states |
| $\varphi\,\mathcal{U}\,\psi$ | **Until:** $\varphi$ holds until $\psi$ holds |
| $\varphi\,\mathcal{R}\,\psi$ | **Release:** $\psi$ must hold until (and if) $\varphi$ holds |

FOL is limited in expressing modalities such as necessity and possibility.

Modal logic extends classical logic with operators for necessity ($\Box$) and possibility ($\Diamond$).

$$\text{“It is necessary that every square is a rectangle”} \quad \Rightarrow \quad \Box(\forall x\ (S(x) \to R(x)))$$
$$\text{“It is possible that it will rain tomorrow”} \quad \Rightarrow \quad \Diamond R$$

Hoare Logic can be used for proving the correctness of computer programs.

# Readings

- TextB: chapter 2.2.4, 2.3.1
- Reference: CS245 course notes at University of Waterloo

# Introduction to Mathematical Logic

*Thank you for listening!*
*Any questions?*