Introduction to Mathematical Logic

For CS Students

CS104

Yida TAO (陶伊达)

2025 年 3 月 24 日

Table of Contents

What's reasoning?

- Everyday reasoning: Natural Language and intuition.
- Semantic entailment: $\Sigma \models A$, use semantic truth table (tableaux) to determine if A follows from Σ (i.e., to reason semantically).

Problems with a purely semantical approach?

Problems with a purely semantical approach:

- Entries in the semantic truth table could be large or even infinite (think about common sense in real life and axioms in math).
- Semantic approach directly produces a “yes/no” answer, so it is difficult to recognize intermediate results.

The *deductive* approach overcomes the above problems, by reasoning syntactically or symbolically (don’t care about the semantics).

What's a proof?

A formal proof is a "logical chain" from assumptions to conclusions.

- First, the "chain" must be finite.
- Second, each "link" in the "chain" may be:
  - Axioms (common sense)
  - Assumptions (premises)
  - Intermediate conclusions derived by using inference rules.

A formal proof is syntactic, rather than semantic.

Remember Leibniz's vision: Let's calculate!

Table of Contents

2 Overview of Proof Systems

Definition

A formal **proof system** (deductive system, 形式推演系统) consists of the language part and the inference part.

- Language part
  - Symbols, alphabet
  - Set of formulas
- Inference part
  - Set of axioms
  - Inference rules

Proof: Starting from a set of formulas and a set of axioms, obtaining new formulas through finitely many times of mechanically applying inference rules.

Notation

We notate "there is a proof with premises Σ and conclusion A" by

$$\Sigma \vdash A$$

Types of Proof Systems

We will introduce 3 formal proof systems.

- Hilbert-style system ($\Sigma \vdash_H A$): many axioms and only one rule. The deduction is linear.
- Natural Deduction System ($\Sigma \vdash_{ND} A$): Few axioms (even none) and many rules. The deductions are tree-like.
- Resolution ($\Sigma \vdash_{Res} A$): used to prove contradictions.

Language

Alphabet of $\mathscr{H}$

$\Sigma = \{(,), \neg, \rightarrow, p, q, r, \ldots\}$

Formulas of $\mathscr{H}$

1. Atoms $p, q, r, \ldots$ are formulas.
2. If A, B are formulas, then $(\neg A), (A \rightarrow B)$ are also formulas.
3. Only expressions of Σ that are generated by 1 and 2 are formulas.

Axioms & Inference Rules

Axioms

A, B, C are arbitrary well-formed formulas.

- $A_1 : A \to (B \to A)$
- $A_2 : (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- $A_3 : (\neg A \to \neg B) \to (B \to A)$

Inference rule

Modus ponens (MP, 分离规则): A and $A \to B$ imply B.

$$r_{mp} : \frac{A, A \to B}{B}$$

Definition 6.1

A **proof** of formula A in $\mathscr{H}$ is a finite sequence of formulas

$$A_1, A_2, \ldots, A_n$$

such that for any $i \leq n$, A_i

- is either an axiom in $\mathscr{H}$
- or is A_j $(j < i)$
- or is derived from A_j, A_k $(j, k < i)$ by the MP rule (e.g., $A_k = A_j \to A_i$)

Definition 6.1 (cont.)

A **proof** of formula A in $\mathscr{H}$ is a finite sequence of formulas

$$A_1, A_2, \ldots, A_n$$

For $A_n = A$, the last formula in the sequence, we say that:

- A_n is a theorem in $\mathscr{H}$
- The sequence $A_1, A_2, \ldots, A_n$ is a proof of A_n
- A_n is *provable*, denoted by $\vdash A_n$.
- If $\vdash A$, then A may be used like an axiom in a subsequent proof

Theorem H1

$\vdash A \rightarrow A$

Proof:

1.	$\vdash (A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$	Axiom 2
2.	$\vdash A \to ((A \to A) \to A)$	Axiom 1
3.	$\vdash (A \to (A \to A)) \to (A \to A)$	MP 1, 2
4.	$\vdash A \to (A \to A)$	Axiom 1
5.	$\vdash A \to A$	MP 3, 4

■

The proof here is purely syntactic (mechanic), no semantic involved.

Link between syntax and semantics

Theorem 6.1 (Soundness)

If $\vdash \alpha$, then $\models \alpha$.

Proved by induction.

Intuition: The proof system can only prove "correct" statement (tautology).

Proof from Assumptions

Definition 6.2

If a formula A has a proof with a set of assumptions (premises) Σ, we say "A can be proved from Σ", denoted as $\Sigma \vdash_H A$, or simply $\Sigma \vdash A$, which is a finite sequence of formulas:

$$A_1, A_2, \ldots, A_n$$

such that for any $i \leq n$, A_i

- is either an axiom in $\mathscr{H}$ or a formula in Σ
- or is A_j $(j < i)$
- or is derived from A_j, A_k $(j, k < i)$ by the MP rule (e.g., $A_k = A_j \to A_i$)

If $\Sigma = \varnothing$, then $\Sigma \vdash A$ is simply $\vdash A$.

Prove that $\{\alpha, \neg\alpha\} \vdash \beta$

Proof from Assumptions

Theorem 6.2 (Monotonicity)

A proof system is monotonic if for any set of formulas Σ and formulas A, B:

$$\Sigma \vdash A \text{ implies } \Sigma \cup \{B\} \vdash A$$

In other words, if $\Sigma \vdash A$ and $\Sigma \subseteq \Sigma'$, then $\Sigma' \vdash A$.

Intuition: If we can prove A using Σ, adding more assumptions to Σ should not prevent A from still being provable.

Derived rules in $\mathscr{H}$

The proof of $\vdash A \to A$ is rather complicated for such a trivial formula.

To make our life easier, we introduce new rules of inference called derived rules.

Deduction rule

For any set of formulas Σ and formulas A, B in $\mathscr{H}$:

$$\Sigma \cup \{A\} \vdash B \text{ if and only if } \Sigma \vdash A \to B.$$

Specifically, $\{A\} \vdash B$ if and only if $\vdash A \to B$.

Intuition: If assuming A leads to B, then A implies B.

Instead of proving $A \to B$ directly from axioms (which can be cumbersome), we can temporarily assume A, derive B, and then conclude $A \to B$ by the deduction rule.

Derived rules in $\mathscr{H}$

Theorem H2

$$\vdash (A \to B) \to ((B \to C) \to (A \to C))$$

Proof

1.	$\{A \to B, B \to C, A\} \vdash A$	Assumption
2.	$\{A \to B, B \to C, A\} \vdash A \to B$	Assumption
3.	$\{A \to B, B \to C, A\} \vdash B$	MP 1, 2
4.	$\{A \to B, B \to C, A\} \vdash B \to C$	Assumption
5.	$\{A \to B, B \to C, A\} \vdash C$	MP 3, 4
6.	$\{A \to B, B \to C\} \vdash A \to C$	Deduction 5
7.	$\{A \to B\} \vdash [(B \to C) \to (A \to C)]$	Deduction 6
8.	$\vdash (A \to B) \to [(B \to C) \to (A \to C)]$	Deduction 7

Derived rules in $\mathscr{H}$

Contrapositive rule

In $\vdash$ notation:

$$\text{if } \Sigma \vdash \neg B \to \neg A, \text{ then } \Sigma \vdash A \to B.$$

In inference notation:

$$\frac{\Sigma \vdash \neg B \to \neg A}{\Sigma \vdash A \to B}, \quad \frac{\Sigma \vdash A \to B}{\Sigma \vdash \neg B \to \neg A}$$

This rule immediately follows from Axiom 3 and MP.

Derived rules in $\mathscr{H}$

Theorem H3

$\vdash \neg\neg A \to A$

Proved in class.

Derived rules in $\mathscr{H}$

Transitivity rule

$$\frac{\Sigma \vdash A \to B \qquad \Sigma \vdash B \to C}{\Sigma \vdash A \to C}$$

Can be proved using A as an additional assumption (monotonicity) and MP.

Derived rules in $\mathscr{H}$

Exchange of antecedent rule

$$\frac{\Sigma \vdash A \to (B \to C)}{\Sigma \vdash B \to (A \to C)}$$

Exchanging the antecedent simply means that it doesn't matter in which order we use the formulas necessary in a proof.

Theorem H4

$\vdash \neg A \to (A \to B)$

$\vdash A \to (\neg A \to B)$

Proof

1.	$\{\neg A\} \vdash \neg A \to (\neg B \to \neg A)$	Axiom 1
2.	$\{\neg A\} \vdash \neg A$	Assumption
3.	$\{\neg A\} \vdash \neg B \to \neg A$	MP 1, 2
4.	$\{\neg A\} \vdash (\neg B \to \neg A) \to (A \to B)$	Axiom 3
5.	$\{\neg A\} \vdash A \to B$	MP 3, 4
6.	$\vdash \neg A \to (A \to B)$	Deduction 5

If you can prove some formula A and its negation $\neg A$, then you can prove any formula B.

Double negation rule

$$\frac{\Sigma \vdash \neg\neg A}{\Sigma \vdash A}, \frac{\Sigma \vdash A}{\Sigma \vdash \neg\neg A}$$

Double negation is a very intuitive rule. We expect that “it is raining” and “it is not true that it is not raining” will have the same truth value.

Derived rules in $\mathscr{H}$

Theorem H5

$\vdash$ true

$\vdash \neg$false

Note: *true* and *false* are not part of the syntax of the Hilbert system; they are introduced as derived concepts for convenience in proofs.

Derived rules in $\mathscr{H}$

Reductio ad absurdum

$$\frac{\Sigma \vdash \neg A \to \text{false}}{\Sigma \vdash A}$$

Reductio ad absurdum is a very useful rule in mathematics: Assume the negation of what you wish to prove and show that it leads to a contradiction.

Derived rules in $\mathscr{H}$

Theorem H6

$\vdash (A \to \neg A) \to \neg A$

Proved in class.

Readings

Optional

- TextF: Section 3.1, 3.3, 3.4
- Text3: 第二章 2.6

Introduction to Mathematical Logic

Thank you for listening!
Any questions?