



Introduction to Mathematical Logic

For CS Students

CS104

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1 Warm Up

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Recall that logic is the science of reasoning.

One important goal of logic is to infer that a **conclusion** is true based on a set of **premises**.

A logical argument:

Premise 1

Premise 2

.....

Premise 3

Conclusion

A common problem is to prove that an argument is valid, that is the set of premises **semantically entails** (可推导) the conclusion.



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2 Semantic Entailment

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Satisfiability of a set of formulas

2 Semantic Entailment

Let $\Sigma \subseteq \text{Form}(\mathcal{L}^p)$ (a set of well-formed formulas). v is a truth valuation. Define:

$$\Sigma^v = \begin{cases} 1 & \text{if for each } B \in \Sigma, B^v = 1 \\ 0 & \text{else} \end{cases}$$

Σ is **satisfiable** iff there is some valuation v such that $\Sigma^v = 1$; we say v **satisfies** Σ .

Example: The set $\{(p \rightarrow q) \vee r, p \vee (q \vee s)\}$ is satisfiable.



Satisfiability of a set of formulas

2 Semantic Entailment

Find all truth valuations that satisfy Σ .

- $\Sigma_1 = \{p_1, p_2, \dots, p_n\}$
- $\Sigma_2 = \{(p_i \rightarrow p_j) \mid i, j \in \mathbb{N}\}$
- $\Sigma_3 = \{(p_i \rightarrow p_j) \mid i \leq j \in \mathbb{N}\}$



Definition

2 Semantic Entailment

Let Σ be a set of formulas ($\Sigma \subseteq \text{Form}(\mathcal{L}^p)$), A be a formula ($A \in \text{Form}(\mathcal{L}^p)$). We say:

- A is a **logical consequence** (逻辑推论) of Σ , or
- Σ **(semantically) entails** (逻辑蕴含) A , or
- $\Sigma \models A$

if and only if

For all truth valuation v , if $\Sigma^v = 1$ then $A^v = 1$.



Definition

2 Semantic Entailment

We use $\Sigma \not\models A$ to denote “not $\Sigma \models A$ ”, which is:

There exists a truth valuation v such that $\Sigma^v = 1$ and $A^v = 0$.



Proving entailment

2 Semantic Entailment

How do we prove $\Sigma \models A$?

- **Direct proof:** For every truth valuation under which all of the premises are true, show that the conclusion is also true under this valuation.
- **Using a truth table:** Consider all rows of the truth table in which all of the formulas in Σ are true. Verify that A is true in all of these rows.

Example: Prove $\neg p \models p \wedge \neg q \rightarrow p \wedge q$



Proving entailment

2 Semantic Entailment

How do we prove $\Sigma \models A$?

- **Proof by contradiction:** Assume that the entailment does not hold, which means that there is a truth valuation under which all of the premises are true and the conclusion is false. Derive a contradiction.

Example: Prove $A \rightarrow B, B \rightarrow C \models A \rightarrow C$



Disproving entailment

2 Semantic Entailment

How do we prove $\Sigma \not\models A$?

- Find one truth valuation v under which all of the premises in Σ are true and the conclusion A is false.

Example: $\neg(A \leftrightarrow B) \vee C, B \wedge \neg C \not\models \neg A \wedge (B \rightarrow C)$



Example

2 Semantic Entailment

Socrate says:

“If I’m guilty, I must be punished; I’m guilty. Thus I must be punished.”

“If I’m guilty, I must be punished; I’m not guilty. Thus I must not be punished.”

Which argument(s) is logically correct?



Properties

2 Semantic Entailment

Logical Equivalence

$A \equiv B$ if and only if both $A \models B$ and $B \models A$.

Empty set

$\emptyset \models A$ means that A is tautology.

Semantic Entailment

$A \models B$ if and only if $A \rightarrow B$ is tautology.

Theorem (extended)

$A_1, \dots, A_n \models A$ if and only if $\emptyset \models A_1 \wedge \dots \wedge A_n \rightarrow A$.



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N-ary Logical Connectives

3 Logical Connectives

So far, we have discussed:

- One unary connective symbol: \neg
- Four binary connective symbols: $\wedge, \vee, \rightarrow, \leftrightarrow$

In fact, there are more unary and binary connective symbols, as well as n -ary connectives for $n > 2$.



N-ary Boolean Functions

3 Logical Connectives

To generalize, n -ary logical connectives are n -ary Boolean Functions (n 元布尔函数), which takes n boolean inputs and returns a single boolean output.

$$f: \{T, F\}^n \rightarrow \{T, F\}$$

For any $n \geq 1$, there are 2^{2^n} different n -ary boolean functions.



Unary Connectives

3 Logical Connectives

We temporarily use f, g (with subscripts) to represent any connective. $fA_1 \dots A_n$ denotes a formula formed by connecting formulas A_1, \dots, A_n using an n -ary connective f .

When $n = 1$, there are $2^{2^1} = 4$ different unary connectives, f_1, f_2, f_3, f_4 .

A	f_1A	f_2A	f_3A	f_4A
1	1	1	0	0
0	1	0	1	0



Binary Connectives

3 Logical Connectives

When $n = 2$, there are $2^{2^2} = 16$ binary connectives, g_1, g_2, \dots, g_{16} .

A	B	g_1AB	g_2AB	g_3AB	g_4AB	g_5AB	g_6AB	g_7AB	g_8AB
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

A	B	g_9AB	$g_{10}AB$	$g_{11}AB$	$g_{12}AB$	$g_{13}AB$	$g_{14}AB$	$g_{15}AB$	$g_{16}AB$
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1



Binary Connective

3 Logical Connectives

- g_9 : Joint denial (或非词, neither...nor...), denoted by \downarrow , $p \downarrow q \equiv \neg(p \vee q)$
- g_{15} : Alternative denial (与非词, not both), denoted by \uparrow , $p \uparrow q \equiv \neg(p \wedge q)$
- g_7 : Exclusive disjunction (异或词, either...or...), denoted by \otimes , $p \otimes q \equiv \neg(p \leftrightarrow q)$



Redundancy

3 Logical Connectives

Formulas $(p \rightarrow q)$ and $((\neg p) \vee q)$ are logically equivalent.

Hence, \rightarrow is said to be **definable** in terms of \neg and \vee .

We don't need to use \rightarrow ; we can always write an equivalent formula without it.



Adequate Set

3 Logical Connectives

A set of connectives is said to be **adequate** (完备的) iff every well-formed formula is logically equivalent to a well-formed formula using only connectives from the set.



Adequate Set

3 Logical Connectives

Theorem 5.1

$\{\neg, \wedge, \vee\}$ is an adequate set of connectives.

Proof: By structural induction. Let $P(\gamma)$ be the statement that γ is logically equivalent to a well-formed formula that uses only connectives from the set $S = \{\neg, \wedge, \vee\}$. We prove this by structural induction.

Base case: If γ is a propositional formula, the statement holds since there are no connectives.

Inductive Hypothesis: Assume $P(\alpha)$ and $P(\beta)$ holds for some α and β .



Adequate Set

3 Logical Connectives

Theorem 5.1

$\{\neg, \wedge, \vee\}$ is an adequate set of connectives.

Proof (Continued):

Induction step:

If $\gamma = (\neg\alpha)$, then since α is logically equivalent to a formula using only connectives from S , we have that γ must also be since it is the negation of this formula.

If $\gamma = (\alpha \star \beta)$:

- If $\star \in S$, we are done by the logic above.
- If $\star = \rightarrow$, then $\gamma \equiv ((\neg\alpha) \vee \beta)$ by the implication rule, and we're done.
- If $\star = \leftrightarrow$, then $\gamma \equiv ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$ and using the implication rule, we're done.



Adequate Set

3 Logical Connectives

Lemma 5.2

Each of the sets $\{\neg, \wedge\}$, $\{\neg, \vee\}$ and $\{\neg, \rightarrow\}$ is adequate.

Proof Sketch: The first two follow from DeMorgan's Laws. The third can follow from the first two. They can all be proven by structural induction or reduced to another known adequate set of connectives.



Adequate Set

3 Logical Connectives

Prove that the set $\{\wedge, \vee, \rightarrow\}$ is not adequate.



Adequate Set

3 Logical Connectives

Every n -ary boolean function is **definable** in terms of only the connectives from the adequate set (discussed later).



Assignment

Coursework

- Assignment 3 on PL semantics.



Readings

Optional

- Text1: 1.3
- Text1: 2.5, 2.8



Introduction to Mathematical Logic

Thank you for listening!
Any questions?