

Lecture #2

## ➤ Runtime and Asymptotic Notation

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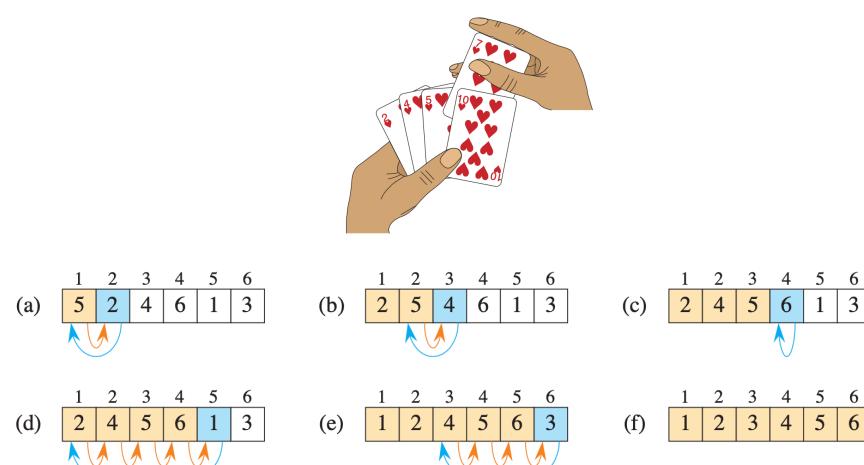
Reading: Chapter 3

## ➤ Aims of this lecture

- To recap and simplify the runtime analysis of InsertionSort.
- To talk about growth of runtime with problem size.
- To introduce asymptotic notation (meet your Greek friends!)
- To show how to apply asymptotic notation

## ➤ Recap: Runtime of InsertionSort

(1)



## ➤ Recap: Runtime of InsertionSort (2)

INSERTIONSORT( $A$ )	Cost	Times
1: <b>for</b> $j = 2$ to $A.length$ <b>do</b>	$c_1$	$n$
2:     key = $A[j]$	$c_2$	$n - 1$
3:     // Insert $A[j]$ into ...	$c_4$	$n - 1$
4: $i = j - 1$	$c_5$	$t_2 + t_3 + \dots = \sum_{j=2}^n t_j$
5: <b>while</b> $i > 0$ and $A[i] > \text{key}$ <b>do</b>	$c_6$	$(t_2 - 1) + (t_3 - 1) + \dots = \sum_{j=2}^n (t_j - 1)$
6: $A[i + 1] = A[i]$	$c_7$	$(t_2 - 1) + (t_3 - 1) + \dots = \sum_{j=2}^n (t_j - 1)$
7: $i = i - 1$		
8: $A[i + 1] = \text{key}$	$c_8$	$n - 1$

Define  $t_j$  as the number of times the while loop is executed for that  $j$ .

## ➤ Recap: Runtime of InsertionSort (3)

- General formula:

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^n t_j + c_6 \sum_{j=2}^n (t_j - 1) + c_7 \sum_{j=2}^n (t_j - 1) + c_8(n-1)$$

- Best case** simplifies to  $T(n) = a n + b$

for constants  $a > 0, b$  composed of  $c_1, c_2$ , etc.

- A **linear** function in  $n$ .

- Worst case** simplifies to  $T(n) = a n^2 + b n + c$

for constants  $a > 0, b, c$  composed of  $c_1, c_2$ , etc.

- A **quadratic** function in  $n$ .

## ➤ On best case and worst case

- The running time of every instance is sandwiched between the best case and the worst case running time.
- ? Best case vs. worst case – which is more important?
- Average case: performance on “average” input.
  - For sorting: assume each permutation is equally likely
  - For other problems it’s not always clear what an average input is
- Why worst case is important:
  - Guarantee that the algorithm will never take longer
  - For some algorithms, the worst case is quite frequent
  - Often (not always) the average case is as bad as the worst case

## ➤ Comparison of two runtimes

- Let’s compare two algorithms:

- Algorithm A has runtime  $2n^2$
- Algorithm B has runtime  $50n \log n$

Which one would you prefer?

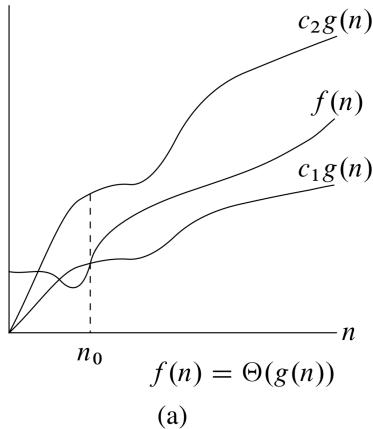
[Using Wolfram Alpha](#)

## ➤ Observations

- The biggest-order term ( $n^2$  vs.  $n \log n$ ) dominates the runtime as  $n$  grows.
- How the runtime scales with  $n$  is more important than constant factors (for large  $n$ ).
- Additive smaller order terms (e.g. “ $+10n$ ” in “ $2n^2 + 10n$ ”) become **irrelevant** for large  $n$ .
- Care about large  $n$ , small problems (small  $n$ ) are easy anyway.
- Recommendations:
  - If your problem is **always very small**, use the simplest algorithm.
  - Otherwise, use most **efficient** algorithm (**best growth** in  $n$ )

## ➤ Asymptotic Notation: $\Theta$

- Idea: capture **asymptotic growth**
- Ignore constant factors
- Ignore small-order terms
- Ignore “blips” for tiny  $n$
- Intuition: “ $\Theta$ ” **captures fastest growing term**  
e.g.  $2n^2 + 3n = \Theta(n^2)$ .
- More details in the book, Section 3.1.



## ➤ Definition of $\Theta(g(n))$

For a given (non-negative) function  $g(n)$  we denote by  $\Theta(g(n))$  the **set of functions**

$\Theta(g(n)) = \{f(n) : \text{there exist constants } 0 < c_1 \leq c_2 \text{ and } n_0 \text{ such that}$   
 $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}$

A function  $f(n)$  belongs to the set  $\Theta(g(n))$  if it can be “sandwiched” between  $c_1 g(n)$  and  $c_2 g(n)$ , for sufficiently large  $n$ .

We could write:  $f(n) \in \Theta(g(n))$ .

However, the common notation is:  $f(n) = \Theta(g(n))$ , the equality being read from left to right!

We say that  $g(n)$  is an **asymptotically tight bound** for  $f(n)$ .

## ➤ Example for $\Theta$ notation

- Example:  $\frac{3}{2}n^2 + \frac{7}{2}n - 4 = \Theta(n^2)$ .

To show this, we need to find constants  $c_1, c_2, n_0$  such that for all

$$n \geq n_0 \quad 0 \leq c_1 n^2 \leq \frac{3}{2}n^2 + \frac{7}{2}n - 4 \leq c_2 n^2$$

- Let's divide by  $n^2$ :

$$0 \leq c_1 \leq \frac{3}{2} + \frac{7}{2n} - \frac{4}{n^2} \leq c_2$$

- This is true, e.g., for  $c_1 = \frac{3}{2}$ ,  $c_2 = 2$ ,  $n_0 = 7$ .  
(Other choices are possible so long as the inequalities hold.)

## ➤ Examples (1)

Task: find constants  $c_1, c_2, n_0 > 0$  from definition of  $\Theta$ .

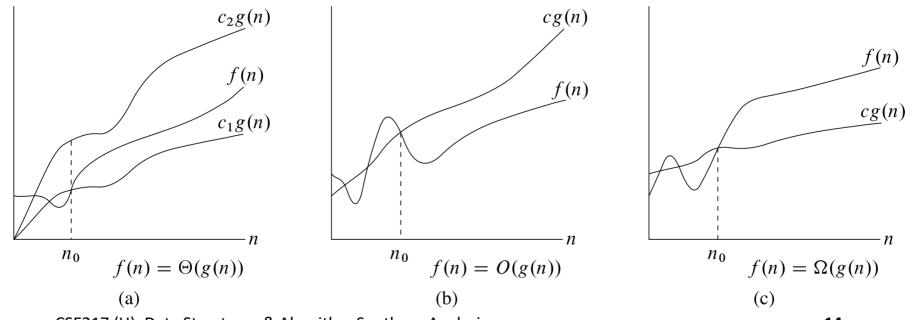
- $2n^2 = \Theta(n^2)$  since for all  $n \geq n_0$   
 $0 \leq c_1 n^2 \leq 2n^2 \leq c_2 n^2$   
when choosing, say,  $c_1 = 1, c_2 = 2, n_0 = 1$
- $2n^2 - 10n = \Theta(n^2)$  since for all  $n \geq n_0$   
 $0 \leq c_1 n^2 \leq 2n^2 - 10n \leq c_2 n^2$   
when choosing, say,  $c_1 = 1, c_2 = 2, n_0 = 10$   
(as after division by  $n^2$  we have  $1 \leq 2 - 10/n \leq 2$  for  $n \geq 10$ )
- $50n \log n = \Theta(n \log n)$  since for all  $n \geq n_0$   
 $0 \leq c_1 n \log n \leq 50n \log n \leq c_2 n \log n$   
when choosing, say,  $c_1 = 50, c_2 = 50, n_0 = 1$

## ➤ Examples (2)

- but:  $2n^2 \neq \Theta(n)$  since there is no constant  $c_2$  such that  $2n^2 \leq c_2 n$  for all  $n \geq n_0$ .
- and:  $2n^2 \neq \Theta(n^3)$  since there is no constant  $c_1$  such that  $2n^2 \geq c_1 n^3$  for all  $n \geq n_0$ .

## ➤ Asymptotic Notation: $\Theta, O, \Omega$

- $\Theta$  expresses tight upper and lower bounds on  $f(n)$ .
- Use  $O$  (“big-Oh”) if we only want to express an upper bound.
- Use  $\Omega$  if we only want to express a lower bound.



## ➤ Definition of $O(g(n)), \Omega(g(n))$

For a given (non-negative) function  $g(n)$  we denote by  $O(g(n))$  and  $\Omega(g(n))$  the following sets of functions:

$O(g(n)) = \{f(n) : \text{there exist constants } 0 < c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$

$\Omega(g(n)) = \{f(n) : \text{there exist constants } 0 < c \text{ and } n_0 \text{ such that } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}$

$O$  and  $\Omega$  are weaker than  $\Theta$ . Together, they give  $\Theta$ :

For any  $f(n)$  and  $g(n)$  we have  $f(n) = \Theta(g(n))$  if and only if  $f(n) = O(g(n))$  and  $f(n) = \Omega(g(n))$ .

## ➤ Faster and slower growth

- Little-Oh “o” and little omega “ω” indicate strictly slower and faster growth, respectively:

$$f(n) = o(g(n)) \text{ if } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) = \omega(g(n)) \text{ if } g(n) = o(f(n))$$

## ➤ Asymptotic Notation: Overview

Notation	Meaning	Analogy
$f(n) = O(g(n))$	$f$ grows at most as fast as $g$	“ $f \leq g$ ”
$f(n) = \Omega(g(n))$	$f$ grows at least as fast as $g$	“ $f \geq g$ ”
$f(n) = \Theta(g(n))$	$f$ grows as fast as $g$	“ $f = g$ ”
$f(n) = o(g(n))$	$f$ grows slower than $g$	“ $f < g$ ”
$f(n) = \omega(g(n))$	$f$ grows faster than $g$	“ $f > g$ ”

- Equalities are to be **read from left to right** – think of  $f(n) = O(g(n))$  as actually meaning  $f(n) \in O(g(n))$
- So  $n = O(n^2)$  is **true** but  $O(n^2) = n$  is **false**!
- We can chain equalities, e. g.  $n = O(n) = O(n^2)$

## ➤ Common runtimes

$\Theta(1)$	constant time
$\Theta(\log n)$	logarithmic time
$\Theta(n)$	linear time
$\Theta(n^2)$	quadratic time
$\Theta(n^3)$	cubic time
$n^k$ for $k = \Theta(1)$	polynomial time
$2^n$	exponential time

- Every polynomial of  $\log n$  grows strictly slower than every polynomial of  $n$ , e. g.  $(\log n)^{100} = o(n^{0.01})$
- Every polynomial of  $n$  grows strictly slower than every exponential function  $2^{n^\varepsilon}$ , e. g.  $n^{100} = o(2^{n^{0.01}})$

## ➤ Examples

Examples of using the various symbols:

- $2n + 1 = O(n)$
- $42 = O(n)$  (but not  $\Theta(n)!$ )
- $n - 9 = \Omega(n)$
- $n^2 + n = \Omega(n)$  (but neither  $O(n)$ , nor  $\Theta(n)!$ )
- $n^3 = o(n^4) = o(2^n)$
- $\sqrt{n} = \omega(\log n)$

## ➤ How to read asymptotic notation

How to read „The runtime of Algorithm XYZ is  $O(n^2)$ “?

“The runtime of Algorithm XYZ is some (anonymous) function that grows at most as fast as  $n^2$ .“

Or, more briefly,

“The runtime of Algorithm XYZ grows at most as fast as  $n^2$ .“

Think of asymptotic notation as a **placeholder** for some anonymous function from the specified class.

- „runtime is  $\Theta(n^2)$ “  $\rightarrow$  „runtime grows as fast as  $n^2$ “
- „runtime is  $\Omega(n^2)$ “  $\rightarrow$  „runtime grows at least as fast as  $n^2$ “
- „runtime is  $o(n^2)$ “  $\rightarrow$  „runtime grows slower than  $n^2$ “
- „runtime is  $\omega(n^2)$ “  $\rightarrow$  „runtime grows faster than  $n^2$ “

## ➤ Asymptotic runtime of InsertionSort

- The runtime of InsertionSort is ...

$$\Omega(n) \quad \text{and} \quad O(n^2)$$

(grows at least as fast as  $n$  and at most as fast as  $n^2$ )

- This is because:

- The best-case runtime is  $\Theta(n)$
- The worst-case runtime is  $\Theta(n^2)$
- So for every input, the runtime is at least  $\Omega(n)$  and at most  $O(n^2)$

## ➤ How to find $c_1, c_2, n_0$

- It is often helpful (though not compulsory) to divide by  $g(n)$ , e.g.

$$c_1 n \leq 10n + 5 \leq c_2 n \Leftrightarrow c_1 \leq 10 + \frac{5}{n} \leq c_2$$

Then try  $c_1, c_2$  **sandwiching the constant term**, e.g.  $c_1 = 10, c_2 = 15$ .

- Remember that  $c_1 > 0$ : to show that  $1 - \frac{3}{n} = \Omega(1)$  we cannot use  $n_0 = 3$  as then there is no suitable  $c_1 > 0$ !

However, say,  $n_0 = 6$  and  $c_1 = \frac{1}{2}$  works as  $1/2 \leq 1 - \frac{3}{n}$  for all  $n \geq 6$ .

- Also remember that inequalities need to hold **for all  $n \geq n_0$** .

For instance, to show  $1 - \frac{3}{n} = O(1)$  we cannot use  $c_2 = \frac{1}{2}$  as

$1 - \frac{3}{n} \leq \frac{1}{2}$  is false for  $n > 6$ ! Need to choose  $c_2 \geq 1$  (e.g.  $c_2 = 1$ ).

- No need to invest time to find the best possible constants.

## ➤ Rules to make runtime analysis simple

- For two non-negative functions  $f(n), g(n)$ :

- Slower functions can be ignored:

$$f(n) + g(n) = \Theta(\max(f(n), g(n)))$$

- Asymptotic times can be multiplied:

$$\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$$

FOO

```

1: foo
2: foo
3: for i = 1 to n do
4:   foo
5:   foo
6:   foo

```

Example of how to use this:

- First two lines take time  $\Theta(1)$
- One iteration of the for loop takes time  $\Theta(1)$
- The for loop is executed  $\Theta(n)$  times
- Total time is:

$$\Theta(1) + \Theta(n) \cdot \Theta(1) = \Theta(n).$$

## ➤ Runtime of InsertionSort

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INSERTIONSORT( $A$ )

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```

1: for j = 2 to A.length do
2:   key = A[j]
3:   // Insert A[j] into ...
4:   i = j - 1
5:   while i > 0 and A[i] > key do
6:     A[i + 1] = A[i]
7:     i = i - 1
8:   A[i + 1] = key

```

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Define  $t_j$  as the number of times the while loop is executed for that  $j$ .

## ➤ Asymptotic Notation: Comparing Sets

- Is  $2n^2 + \Theta(n) = \Theta(n^2)$  true or false?  
(Think of  $\Theta(n)$  as a placeholder for an anonymous function from the set  $\Theta(n)$  of all functions that grow linearly in  $n$ .)
- Such a statement is true if no matter how the anonymous functions are chosen on the left of the equal sign, there is a way to choose the anonymous functions on the right of the equal sign to make the equation valid.
- Example: is  $O(n) = O(n^2)$ ?  
**True, because**  $O(n) \subseteq O(n^2)$
- Example: is  $O(n^2) = O(n)$ ?  
**False**, for example  $n^2$  is in  $O(n^2)$  but not in  $O(n)$ !

## ➤ Summary

- We may consider best-case, average-case, and worst-case runtime. Often the focus is on **worst-case runtime**.
- The most important aspect of efficiency is **scalability**: how the runtime grows with the input size,  $n$ .
  - Asymptotic perspective:  $n \geq n_0$  (smaller problems are easy)
  - Scalability is more important than constant factors
  - Small-order terms become more insignificant as  $n$  grows.
- **Asymptotic notation** ( $O, \Omega, \Theta, o, \omega$ ) hides constant factors and small-order terms, revealing asymptotic runtimes.
- Asymptotic notation refers to **sets of functions**, but for convenience is written with equalities read from left to right.