MA337 Real Analysis (H) Notes

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These notes are compatible with the MA337 course (Fall 2025).

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In this chapter, we embark on a journey into the fundamental concepts of measure.

This theory, developed by Henri Lebesgue at the beginning of the 20th century, provides a more robust and general framework for integration than the Riemann integral, allowing us to integrate a wider class of functions and providing powerful convergence theorems essential for modern analysis and probability theory. His theory was published originally in his dissertation *Intégrale, longueur, aire ("Integral, length, area")* at the University of Nancy during 1902.

For a biography of Henri Lebesgue, please see Appendix A.1.

5.1 Systems of Sets: Semi-ring, Ring, Algebra, σ-Algebra, Borel σ-Algebra

Definition 5.1. *(Semi-Ring of Sets)*

A system of sets S is called a ***semi-ring*** *if it satisfies the following two axioms:*

1. *If $A, B \in S$, then $A \cap B \in S$.*
2. *If $A, B \in S$, then there exist disjoint sets $A_1, A_2, \ldots, A_n \in S$ such that $A \setminus B = \bigsqcup_{i=1}^{n} A_i$.*

Example 5.2. *(Semi-open cells in $\mathbb{R}^n$)*

$I_1, I_2, \ldots, I_n$: intervals in $\mathbb{R}$. $C := I_1 \times I_2 \times \ldots \times I_n$ is called a ***cell*** *in $\mathbb{R}^n$.*

semi-open interval*: an interval that is closed at one end and open at the other end, e.g., $[a, b)$ or $(a, b]$.*

Let S be the collection of all semi-open cells in $\mathbb{R}^d$ (not required to be finite!), i.e. $S = \{[a_1, b_1) \times \cdots \times [a_n, b_n) : a_i, b_i \in \mathbb{R}, a_i < b_i\}$. Then S is a semi-ring.

Warning: Be cautious about the directions of semi-open cells! The directions of all cells must coincide.

Remark 5.3. *Question: Can we take all closed/open cells in R^n?*

Answer: NO! For example, $[0, 1] \cap [1, 2] = \{1\}$, $(0, 1) \setminus (1/2, 1) = (0, 1/2]$, both result in some elements not in the original system.

Proposition 5.4. *If S is a semi-ring, then*

1. *$\emptyset \in S$.*
2. *Axiom 2 can be strengthened to: $\forall A \in S$, $\forall A_1, A_2, \ldots, A_n \in S, A_j \in A, \forall j, disjoint$, there exist disjoint sets $A_{m+1}, A_{m+2}, \ldots, A_s \in S$ such that $A = \bigsqcup_{i=1}^{s} A_i$.*

Proof. 1. $\emptyset = A \setminus A. \forall A \in S$.

2. One can prove by induction on m: splitting the whole area A into disjoint parts. It is easier to prove for the semi-ring {all cells in $\mathbb{R}^n$}. □

Remark 5.5. *We now show that with axiom 1 and the strengthened condition above we could say S is a semi-ring.*

Proof. Now axiom 1 is satisfied.

Suppose $A, B \in S$, then $A \setminus B = A \ (A \cap B)$. Let $A_1 = B, n = 1$. By our strengthened condition, one could find disjoint sets $A_2, A_3, \ldots, A_s \in S$, s.t. $A = \bigsqcup_{i=1}^{s} A_i$, i.e. $A \setminus B = \bigsqcup_{i=2}^{s} A_i$. ✓ □

Thus, we have the following equivalent definition for semi-rings.

Definition 5.6. *(Semi-Ring of Sets - Alternative Definition)*

*A system of sets S is called a **semi-ring** if it satisfies the following two axioms:*

1. *If $A, B \in S$, then $A \cap B \in S$.*
2. *$\forall A \in S$, $\forall A_1, A_2, \ldots, A_n \in S, A_j \subset A, \forall j, disjoint$, there exist disjoint sets $A_{m+1}, A_{m+2}, \ldots, A_s \in S$ such that $A = \bigsqcup_{i=1}^{s} A_i$.*

Definition 5.7. *(Semi-ring with Unity)*

*A semi-ring S is called a **semi-ring with unity** if $S \in 2^{\Omega} (\leftrightarrow \forall A \in S, A \in \Omega)$ and $\Omega \in S$ for some set Ω. Ω is called the **unity** of S. Indeed, $\Omega \cap A = A, \forall A \in S$.*

Example 5.8.

1. *(A semi-ring with unity)*

   The semi-ring of all semi-open cells in $\mathbb{R}^n$ (To be more precise, we need to add the element $\mathbb{R}^n$ into it. For convenience, we won't clarify this much in the future. The reader should always keep this unity in mind.) is a semi-ring with unity $\mathbb{R}^n$.

2. *(A semi-ring WITHOUT a unity)*

   The semi-ring of all finite semi-open cells in $\mathbb{R}^n$: NO unity ($\cancel{\mathbb{R}^n}$)!

Definition 5.9. *(Ring of Sets)*

*A system of sets $\mathcal{R}$ is called a **ring** if it satisfies the following two axioms:*

1. *$\forall A, B \in \mathcal{R}$, $A \cap B \in \mathcal{R}$.*
2. *$\forall A, B \in \mathcal{R}$, $A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathcal{R}$.*

Remark 5.10. *In fact, a ring R is closed under set difference and finite unions.*

1. *$\forall A, B \in R$, $A \setminus B = A \triangle (A \cap B) \in R$.*
2. *$\forall A, B \in R$, $A \cup B = (A \triangle B) \triangle (A \cap B) \in R$.*

Conversely, we have

1. *$\forall A, B \in R$, $A \cap B = ((A \cup B) \setminus (A \setminus B)) \setminus (B \setminus A) \in R$.*
2. *$\forall A, B \in R$, $A \triangle B = (A \cup B) \setminus (A \cap B) \in R$.*

Definition 5.11. *(Ring of Sets - Alternative Definition)*

*A system of sets $\mathcal{R}$ is called a **ring** if it satisfies the following two axioms:*

1. *$\forall A, B \in \mathcal{R}$, $A \setminus B \in \mathcal{R}$.*
2. *$\forall A, B \in \mathcal{R}$, $A \cup B \in \mathcal{R}$.*

Remark 5.12. *As a result, we arrive with the same definition of ring requiring closeness under set difference and finite unions.*

Example 5.13. *(A semi-ring but NOT a ring)*

The semi-ring of all cells in $\mathbb{R}^n$: not ensuring the closeness under union!

Definition 5.14. *(Algebra)*

*A ring with unity is called an **algebra of sets**.*

Example 5.15. *(A ring but NOT an algebra)*

Consider $R = \{A \subset \mathbb{N} : |A| < +\infty\}$. R is a ring, but $\mathbb{N} \notin R$, which means it doesn't have a unity.

Proposition 5.16. *1. A ring is a semi-ring.*

*2. $\forall$ system of sets P, $\exists$ a **minimal ring** $\mathcal{R}(P) \supset P$.*

Proof. 1. Let $\mathcal{R}$ be a ring. Then $\forall A, B \in \mathcal{R}$, $A \setminus B = A \setminus B(!) = A \triangle (A \cap B) \in \mathcal{R}$.

2. Start with $\mathcal{R}_0 = 2^\Omega$, where Ω is the union of all sets in P. Let $\{R_\alpha\}$ be the collection of all rings containing P. Then $\mathcal{R}(P) := \bigcap_\alpha R_\alpha$ is the minimal ring containing P (it is clearly again a ring!). $\square$

Proposition 5.17. *Let S be a semi-ring, then*

$$\mathcal{R}(S) = \{\bigcup_{j=1}^{m} A_j, A_j \in S, m \in \mathbb{N} : arbitrary\} \Leftrightarrow \{\bigsqcup_{j=1}^{s} A_j, A_j \in S, s \in \mathbb{N} : arbitrary\}$$

Proof. "$\Leftrightarrow$":

Firstly, the claimed system $\mathcal{R}(S)$ is indeed a ring.

$A = \sqcup_{j=1}^{S} A_j, B = \sqcup_{i=1}^{m} B_i, A \cap B = \sqcup_{i,j}(A_j \cap B_i) \in S \subset \mathcal{R}(S)$.

$\Rightarrow A \triangle B = (A \setminus B) \cap (B \setminus A) = \sqcup_{j=1}^{m} \cap_{i=1}^{m} (A_j \setminus B_i) \in S \subset \mathcal{R}(S)$.

Thus, $\mathcal{R}(S)$ is a ring.

Next, $\forall$ other ring $\tilde{\mathcal{R}}(S)$ containing S, it must contain all elements of $\mathcal{R}(S)$.

i.e. $\tilde{\mathcal{R}}(S) \supset \mathcal{R}(S) \Rightarrow \mathcal{R}(S)$ is the minimal ring containing S. $\square$

Definition 5.18. *(σ-algebra)*

*A system of sets $\mathcal{A}$ is called a σ**-algebra** if*

1. $\mathcal{A} \subset 2^\Omega, \Omega \in \mathcal{A}$;

2. $\mathcal{A}$ is an algebra with unity Ω;

3. $\forall A_1, A_2, \ldots$ (finite or infinite family of sets!) with $\forall j : A_j \in \mathcal{A}$ it holds $\cup_{j=1}^{\infty} A_j \in \mathcal{A}$.

Proposition 5.19. *1. A σ-algebra is closed under taking the implement: $A^c = \Omega \backslash A \in \mathcal{A}$ since a σ-algebra is a ring with unity Ω. It is closed under set difference.*

2. $\emptyset \in \mathcal{A}$ since $\emptyset = \Omega^c$ or $\emptyset = \Omega \setminus \Omega$.

3. *A σ-algebra is closed under finite or countable union thanks to its definition and the fact that $\emptyset \in \mathcal{A}$*

4. *A σ-algebra is closed under finite or countable intersection:*

   $\forall A_1, A_2, \ldots$ (finite or infinite family of sets!) with $\forall j : A_j \in \mathcal{A}$, we have

   $\cap_{j=1}^{\infty} A_j = \Omega \setminus \cup_{j=1}^{\infty}(\Omega \setminus A_j) \in \mathcal{A}$

5. *A σ-algebra is closed under countable symmetric difference.*

Remark 5.20. *Question: What are the minimal conditions we need to define/prove a σ-algebra?*

Answer: I prefer the following three minimal conditions:

1. Unity: $\Omega \in \mathcal{A}$.

2. Closed under set difference: If $A \in \mathcal{A}$, then $A^c = (\Omega \setminus A) \in \mathcal{A}$.

3. σ-additivity: If $A_1, A_2, \ldots \in \mathcal{A}$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

Proposition 5.21. *$\forall S \in 2^{\Omega}, \exists!$ minimum σ-algebra $\mathcal{A}(S) \supset S$.*

Proof. Similar as the proof for $\mathcal{R}(S)$. □

Remark 5.22. *Upshot 1:*

In general:

System of sets$\Rightarrow$ Semi-ring $\Rightarrow$ Ring $\Rightarrow$ Algebra with unity $\Rightarrow$ σ-Algebra.

Now, start with a semi-ring with unity S

$\rightarrow$ could generate a ring $\mathcal{R}(S)$ (still equipped with a unity Ω)

$\rightarrow$ A ring with unity is actually an algebra with unity!

$\rightarrow$ An algebra of sets: $\mathcal{A}(R(S)) = \mathcal{A}(S)$.

Remark 5.23. *Upshot 2:*

A system of sets S

$\rightarrow$ ensuring the two axioms: closeness under intersection and being able to be decomposed into some disjoint subsets

$\rightarrow$ A semi-ring!

$\rightarrow$ could generate a ring $\mathcal{R}(S)$!

$\rightarrow$ A ring which satisfies closeness under: (intersection and symmetric difference) or (union and difference)

$\rightarrow$ equip with a unity

$\rightarrow$ An algebra of sets!

Definition 5.24. *(Borel σ-algebra)*

*The **Borel σ-algebra** on $\mathbb{R}^n$ is defined as the minimum σ-algebra containing all open sets in $\mathbb{R}^n$, denoted as $\mathcal{B}(\mathbb{R}^n)$.*

Remark 5.25. *Note that* $\mathcal{B}(\mathbb{R}^d)$ *also contains all closed sets in* $\mathbb{R}^d$ *since it is closed under difference (open* $\to$ *semi-open* $\to$ *closed).*

Thus, an alternate definition of $\mathcal{B}(\mathbb{R}^d)$ *is the minimum* σ*-algebra containing all closed sets in* $\mathbb{R}^d$*.*

Remark 5.26. *In advanced probability theory, we focus on such Borel* σ*-algebra to study all possible events. One can find more in Foundations of the Theory of Prabability by A.N.Kolomogorov.*

5.2 Measure, Measure Space

5.2.1 Measure

Definition 5.27. *(Measure on a Semi-Ring)*

Let S be a semi-ring. A function $\mu : S \to [0, +\infty)$ is called a ***(finitely additive) measure on*** *S if it satisfies the following two axioms:*

1. *(Non-negativity) $\forall A \in S, \mu(A) \geq 0$.*
2. *(Finite Additivity) If $A, A_1, A_2, \ldots, A_n \in S$ such that $A = \bigsqcup_{j=1}^{n} A_j$, then $\mu(A) = \sum_{j=1}^{n} \mu(A_j)$.*

Proposition 5.28. *1. $\mu(\emptyset) = 0$.*

2. $\forall A, B \in S, A \subset B$, we have $\mu(A) \leq \mu(B)$.

Proof. 1. $\emptyset = \emptyset \cup \emptyset \Rightarrow \mu(\emptyset) = 2\mu(\emptyset)$.

2. Since S is a semi-ring, there exist $A_1, A_2, \ldots, A_m \in S$, s.t. $B \setminus A = \bigsqcup_{l=1}^{p} A_j$

$\Rightarrow B = A \bigsqcup (\bigsqcup_{j=1}^{p} A_j) \Rightarrow \mu(B) = \mu(A) + \Sigma_{j=1}^{p} \mu(A_j) \geq \mu(A)$.

□

Example 5.29. *On the semi-ring {all finite semi-open cells in $\mathbb{R}^n$}, we define a measure as follows:*

A finite semi-open cell $C = I_1 \times I_2 \times \ldots \times I_n$ in $\mathbb{R}^n$, define $\mu(C) := l(I_1) \times l(I_2) \times \ldots \times l(I_n)$, where $l(I)$:=length of I and we are measuring the cell's "volume".

Such μ is called the ***Lebesgue measure on all finite semi-open cells in*** *$\mathbb{R}^n$.*

Proposition 5.30. *$\forall$ measure on a semi-ring S can be extended (with identical proerties) to $R(S)$.*

Proof. For $A = \sqcup_{j=1}^{m} A_j \in \mathcal{R}(S)$ with $A_j \in \mathcal{R}(S)$, define $\mu(A) := \Sigma_{j=1}^{m} \mu(A_j)$. (We need to firstly deal with $A_j \in S$, and then gradually scan the whole $\mathcal{R}(S)$ based on measure-already-defined sets.)

Well-defined (Correctness): Suppose $A = \sqcup_{j=1}^{p} A_j = \sqcup_{i=1}^{s} A'_i$. We have

$\Sigma_{j=1}^{p} \mu(A_j) = \{$using the finite additivity of μ, and $A_j = A_j \cap A = \sqcup_{i=1}^{s}(A_j \cap A'_i)\}$ $= \Sigma_{j=1}^{p}(\Sigma_{i=1}^{s} \mu(A_j \cap A'_i)) = \Sigma_{i=1}^{s}(\Sigma_{j=1}^{p} \mu(A'_i \cap A_j)) = \Sigma_{i=1}^{s} \mu(A'_i)$. ✓

Non-negativity: Clearly, $\mu(A) \geq 0$. ✓

Finite Additivity: Suppose $A, B \in R(S) : A \cap B = \emptyset$. $A = \sqcup_{j=1}^{p} A_j, B = \sqcup_{i=1}^{q} B_j$, with $A_j, B_i \in S$.

$\Rightarrow A \sqcup B = (\sqcup_{j=1}^{p} A_j) \sqcup (\sqcup_{i=1}^{q} B_i)$

$\Rightarrow \mu(A \sqcup B) = \Sigma_{j=1}^{p} \mu(A_j) + \Sigma_{i=1}^{q} \mu(B_i)$

Same for finite union of sets. ✓ □

Proposition 5.31. *(Proerties of a Measure on a ring $\mathcal{R}$)*

1. $\mu(\emptyset) = 0$.
2. *If* $A, B \in R, A \subset B$, *then* $\mu(A) \leq \mu(B)$.
3. ***(Semi-Additivity)*** *If* $A \subset \cup_{j=1}^{n} A_j$, *with* $A, A_j \in R$, *then* $\mu(A) \leq \Sigma_{j=1}^{n} \mu(A_j)$.

   Now, switch from $\bigcup_{j=1}^{n}$ *to* $\bigsqcup_{j=1}^{n}$:

   Set $A_1' := A_1, A_2' := A_2 \setminus A_1, A_3' := A_3 \setminus \cup_{j=1}^{2} A_j, \ldots$

   Now, we have $\bigcup_{j=1}^{n} A_j = \bigsqcup_{j=1}^{n} A_j'$.

   Thus, $A \subset \bigsqcup_{j=1}^{n} A_j'$ *(even more:* $A = (\bigsqcup_{j=1}^{n} A_j') \bigcap A = \bigsqcup_{j=1}^{n} (A_j' \bigcap A)$ *!).*

   Then, $\mu(A) = \bigsqcup_{j=1}^{n} \mu(A_j' \bigcap A) \leq \Sigma_{j=1}^{n} \mu(A_j') \leq \Sigma_{j=1}^{n} \mu(A_j)$.

Remark 5.32. *Question: Could prop. 5.30 (3) maintain for a measure on a semi-ring? Why?*

Answer: NO!!! The key difference between a semi-ring and a ring is that: in a semi-ring S, the diffence between sets may not belong to S, which means though they could be represented as disjoint unions of sets in S, they do NOT have measure defined on them! Then the inequality chain cannot go forward anymore.

Remark 5.33. *Upshot: What we have done so far:*

On a semi-ring S: we can define a finite-additive measure

$\rightarrow$ *extend to the whole ring generated by S: $\mathcal{R}(S)$*

5.2.2 σ-Additivive Measure

Definition 5.34. *(σ-additivity)*

A measure μ on a semi-ring S is called to satisfy ***σ-additivity (countable-additivity)*** *if for any $A \in S$, $\{A_j\}_{j=1}^{\infty} \subset S$ such that $A = \bigsqcup_{j=1}^{\infty} A_j$, we have $\mu(A) = \sum_{j=1}^{\infty} \mu(A_j)$.*

Remark 5.35. ***Warning***: *A σ-algebra is not necessarily σ-additive!*

Also note that σ-additivity always implies ***semi-σ-additivity*** *(sometimes also called* ***subadditivity****):*

$\forall A \subset \bigcup_{j=1}^{\infty} A_j, A, A_j \in S, \mu(A) \leq \Sigma_{j=1}^{\infty} \mu(A_j)$.

And more importantly, finite additivity implies semi-σ-additivity also!

Example 5.36. 1. *Let $\Omega = \mathbb{N}$, $S = 2^{\Omega}$. Define $\mu(A) := \Sigma_{j \in A} p_j$, where p_j is the "weight" assigned to element $j \in \mathbb{N}$ satisfying $\Sigma_{j=1}^{\infty} p_j = 1$ (or any finite number). Then μ is a σ-additive measure on S.*

2. *Let $\Omega = \mathbb{N}$, $S = 2^{\Omega}$. Define $\mu(A) := |A|$ (if A is infinite, $\mu(A) := +\infty$). Then μ is a σ-additive measure on S. (View "weight" being 1 for all elements. This is the case violating the requirement "$\Sigma_{j=1}^{\infty} p_j =$ any finite number" in example 1.)*

3. *(Lebesgue measure on all finite semi-open cells in R^n)*

   Let $S = \{$all finite semi-open cells in $\mathbb{R}^n\}$. We know that S is a semi-ring.

   $\mu(C) := l(I_1) \times l(I_2) \times \ldots \times l(I_n)$, where $l(I)$:=length of I.

   Then μ is a σ-additive measure on S.

   Proof. We already know that μ is a measure on the semi-ring S. μ is finitely additive.

   Suppose $A \in S, \{A_j\}_{j=1}^{\infty} \in S, A = \sqcup_{j=1}^{\infty} A_j$.

   WTS: $\mu(A) = \Sigma_{j=1}^{\infty} A_j$

   Step 1: $\forall n \in \mathbb{N}, A \supset \sqcup_{j=1}^{n} A_j$

   $\Rightarrow \Sigma_{j=1}^{n} \mu(A_j) = \{finit - additivity\} = \mu(\sqcup_{j=1}^{n} A_j) \leq \mu(A)$

   $\Rightarrow$ Take limit $n \to \infty$, we have $\Sigma_{j=1}^{\infty} \mu(A_j) \leq \mu(A)$. ✓

   Step 2: Let $A = [\alpha_1, \beta_1) \times \cdots \times [\alpha_n, \beta_n)$ be a finite semi-open cell in $\mathbb{R}^n$, and suppose $A = \bigsqcup_{j=1}^{\infty} A_j$, where each A_j is also a semi-open cell, and the A_j's are pairwise disjoint.

   Step 2.1: Partition of A into uniform subcells.

   For each integer $m \geq 1$, divide each coordinate interval $[\alpha_i, \beta_i)$ into m equal subintervals: $I_{i,k_i}^{(m)} = \big[\alpha_i + k_i(\beta_i - \alpha_i)/m,\ \alpha_i + (k_i+1)(\beta_i - \alpha_i)/m\big), \qquad k_i = 0, 1, \ldots, m-1.$

   Define the finite family of subcells $\mathcal{Q}_m = \Big\{Q_k^{(m)} = I_{1,k_1}^{(m)} \times \cdots \times I_{n,k_n}^{(m)} : 0 \leq k_i \leq m-1\Big\}$.

   Then the cells in $\mathcal{Q}_m$ are pairwise disjoint and satisfy $A = \bigsqcup_{Q \in \mathcal{Q}_m} Q$.

   In fact, $|\mathcal{Q}_m| = m^n$, which is finite. By finite additivity of μ, $\mu(A) = \sum_{Q \in \mathcal{Q}_m} \mu(Q)$.

   Step 2.2: Classification of subcells.

   For each $Q \in \mathcal{Q}_m$, there are two possibilities:

   1. $Q \subset A_j$ for some j;

   2. Q intersects at least two distinct sets A_{j_1}, A_{j_2}.

   Let $\mathcal{Q}_m^{(1)} = \{Q \in \mathcal{Q}_m : \exists j,\ Q \subset A_j\}, \mathcal{Q}_m^{(2)} = \mathcal{Q}_m \setminus \mathcal{Q}_m^{(1)}$.

   Define $A_m^{(1)} = \bigcup_{Q \in \mathcal{Q}_m^{(1)}} Q, A_m^{(2)} = \bigcup_{Q \in \mathcal{Q}_m^{(2)}} Q$.

   Then $A = A_m^{(1)} \bigsqcup A_m^{(2)}$, and by finite additivity, $\mu(A) = \mu(A_m^{(1)}) + \mu(A_m^{(2)})$.

   Step 2.3: Estimate of $\mu(A_m^{(1)})$.

Since every $Q \in \mathcal{Q}_m^{(1)}$ is contained in some A_j, and all Q's are disjoint, $\mu(A_m^{(1)}) = \sum_{Q \in \mathcal{Q}_m^{(1)}} \mu(Q) \le \sum_{j=1}^{\infty} \mu(A_j)$.

<u>Step 2.4</u>: Estimate of $\mu(A_m^{(2)})$.

Each $Q \in \mathcal{Q}_m^{(2)}$ intersects at least two distinct cells A_{j_1}, A_{j_2}. Thus, every such Q intersects the boundary of some A_j.

Denote $\Gamma = \bigcup_{j=1}^{\infty} \partial A_j$. Each ∂A_j is contained in a finite union of $(n-1)$–dimensional hyperrectangles parallel to the coordinate axes; hence Γ is a countable union of such hyperrectangles. Therefore, $\mu(\Gamma) = 0$.

Let $\delta_m = \max_i \frac{\beta_i - \alpha_i}{m}$ be the mesh size of the partition $\mathcal{Q}_m$. Then $A_m^{(2)}$ is contained in the δ_m–neighborhood of Γ inside A. Because Γ has measure zero, for any $\varepsilon > 0$ there exists $\eta > 0$ such that the η–neighborhood of Γ has μ–measure less than ε. For all sufficiently large m (namely $m > (\max_i(\beta_i - \alpha_i))/\eta)$, we have $\delta_m < \eta$ and hence $\mu(A_m^{(2)}) < \varepsilon$. This shows $\lim_{m\to\infty} \mu(A_m^{(2)}) = 0$.

Combining above, $\mu(A) = \mu(A_m^{(1)}) + \mu(A_m^{(2)}) le \sum_{j=1}^{\infty} \mu(A_j) + \mu(A_m^{(2)})$, and letting $m \to \infty$ gives $\mu(A) \le \sum_{j=1}^{\infty} \mu(A_j)$. ✓ $\square$

4. (Finitely Additivive BUT NOT σ-Additive)

Let $\Omega = (0,1) \cap \mathbb{Q}$. Define the collection $\mathcal{R} = \{A \subset \Omega : A \text{ is finite or co-finite in } \Omega\}$, where "co-finite" means that $\Omega \setminus A$ is finite. Then $\mathcal{R}$ is a ring, since the family of all finite or co-finite subsets of any countable set is closed under finite unions and differences.

Define $\mu : \mathcal{R} \to [0, \infty)$ by $\mu(A) = 0$, if A is finite; 1, if A is co-finite in Ω.

We verify that μ is finitely additive.

If $A, B \in \mathcal{R}$ are disjoint, then:

1. If both A and B are finite, $A \cup B$ is finite, so $\mu(A \cup B) = 0 = \mu(A) + \mu(B)$.

2. If one is finite and the other co-finite, their union is co-finite, so $\mu(A \cup B) = 1 = \mu(A) + \mu(B)$.

3. It is impossible for two disjoint co-finite subsets to exist in Ω, so no contradiction arises.

Hence μ is finitely additive.

Now enumerate $\Omega = \{q_1, q_2, q_3, \dots\}$ and set $A_j = \{q_j\}$.

Then each A_j is finite, hence $\mu(A_j) = 0$. Also note that $\Omega = \bigsqcup_{j=1}^{\infty} A_j$.

If μ were σ-additive, we would have $\mu(\Omega) = \sum_{j=1}^{\infty} \mu(A_j) = 0$. But by definition $\mu(\Omega) = 1$. Therefore μ FAILS to be σ-additive, even though it is finitely additive.

Remark 5.37. *A measure μ with σ-additivity on S could extend to a measure with σ-additivity on $\mathcal{R}(S)$ by defining $\mu\left(\bigsqcup_{j=1}^{m} A_j\right) := \Sigma_{j=1}^{m}\mu(A_j)$, with $A_j \in S$: disjoint.*

While σ-additivity of μ on $\mathcal{R}(S)$ can be derived from σ-additivity on S, note that we still have the weaker condition satisfied: ***semi-σ-additivity****, i.e. $\forall A \subset \cup_{j=1}^{\infty} A_j, A, A_j \in \mathcal{R}(S), \mu(A) \leq \Sigma_{j=1}^{\infty}\mu(A_j)$.*

5.2.3 Outer Measure

Definition 5.38. *(outer Lebesgue measure of a set E)*

Let μ be a σ-additive measure on a semi-ring S with unity Ω (so, $S \subset 2^{\Omega}$). Let $\mathcal{R}(S) = \mathcal{A}(S)$ — the minimum algebra containing S. For any $E \subset \Omega$, define

$$\mu^*(E) := \inf\left\{\Sigma_{j=1}^{\infty}\mu(A_j) : A_j \in S, A \subset \cup_{j=1}^{\infty} A_j\right\}.$$

Then, μ^* is called the ***outer(exterior) Lebesgue measure of a set E*** *induced by μ on Ω.*

Remark 5.39. *The outer measure is to define the measure on sets outside of S beased on the* ***pre-measure*** *on S.*

The outer measure μ^* of a set E ***always exists*** *(may be infinitely many), since*

1. $\left\{\Sigma_{j=1}^{\infty}\mu(A_j) : A_j \in S, A \subset \cup_{j=1}^{\infty} A_j\right\}$ at least contains Ω;

2. Consider the real numbers in $\left\{\Sigma_{j=1}^{\infty}\mu(A_j) : A_j \in S, A \subset \cup_{j=1}^{\infty} A_j\right\}$, they have lower bound 0. By the completeness of $\mathbb{R}$, the infimum exists.

Warning*: In general, one CANNOT claim that $\mathcal{A}(S) \supset \mathcal{A}(\Omega)$. This is also the key problem of out outer measure being not able to capture all the information in the algebra generated by Ω!*

Example 5.40. *(An invisible set under the outer measure)*

Let $S = \{[a, b) : a, b \in \mathbb{Q}, a < b\}$ (S is indeed a semi-ring with unity), and define the pre-measure $\mu([a, b)) = b - a$. The outer measure μ^* on $2^{\mathbb{R}}$ is defined by $\mu^*(E) = \inf\{\sum_{j=1}^{\infty}\mu(A_j) : A_j \in S, E \subseteq \bigcup_{j=1}^{\infty} A_j\}$.

Consider the set $E = \mathbb{Q} \cap [0, 1)$. We will show that $\mu^*(E) = 1$, while $\mu^*(\{q\}) = 0$ for all $q \in E$. Hence, $\mu^*(\bigsqcup_{q\in E}\{q\}) = 1 > 0 = \sum_{q\in E}\mu^*(\{q\})$, which demonstrates that μ^* is not countably additive, even for disjoint sets.

Remark 5.41. *This example shows that μ^* cannot "see" the internal structure of sets outside the algebra $\mathcal{A}(S)$ (But we are still in $\mathcal{A}(\Omega)$!). Although E is a countable, measure-zero set in the intuitive sense, any cover of E by rational half-open intervals must in fact cover the entire interval $[0, 1)$. Hence, the outer measure treats E as if it were as large as $[0, 1)$.*

A simple point of view: We know that there is quite possible to find a set E in $\mathcal{A}(\Omega) \setminus \mathcal{A}(S)$. For such set, we cannot find a quite precise covering of it, so we can only use the whole unity Ω as a part of our approximation.

Remark 5.42. *(The philosophy behind outer measure)*

Why do we call it an "outer measure"?

The name comes from its construction principle: we measure a set from the outside. *Given a subset $E \subseteq \Omega$, we generally cannot measure E directly, because E may be too irregular or may not belong to the algebra $\mathcal{A}(S)$ where the original measure μ is defined. Instead, we approximate E by sets $A_j \in S$ that cover E from the outside and take the smallest possible total measure among all such coverings.*

Formally, $\mu^*(E) = \inf\{\sum_j \mu(A_j) : E \subseteq \bigcup_j A_j,\ A_j \in S\}$, which expresses the idea of an outer approximation. *The measure does not come from the intrinsic structure of E, but from the minimal "outer shell" built using measurable sets in S.*

Philosophically, μ^* represents the best information we can obtain about the size of E given our limited "vocabulary" S. It is an act of estimation under partial visibility: we look at E through a coarse geometric lens and ask, "How small can the total measure of the covering be if I only use shapes I can measure?"

Thus, it is called an outer measure *because it always measures from the* outside, *enclosing E within measurable sets rather than dissecting it from the inside.*

Proposition 5.43. *1. μ^* always $\exists$, and $\mu^*(A) \geq 0, \forall A \subset \Omega$.*

2. We can equivalently say in the definition of μ^* that A_j are disjoint.

3. $\forall A \in \mathcal{A}(S)$, $\mu(A) = \mu^*(A)$

Proof. On one hand, by the semi-σ-additivity, $\mu(A) \leq \Sigma_{j=1}^{\infty}\mu(A_j)$ if $\cup_{j=1}^{\infty} A_j \supset A$.

$\Rightarrow$ Take inf: $\mu(A) \leq \mu^*(A)$;

On the other hand, take the trivial covering: $A_1 = A$,

$\mu(A) = \mu(A_1) = \mu(A_1 \bigsqcup_{j=1}^{\infty} \emptyset) \geq \mu^*(A)$,

$\Rightarrow \mu(A) = \mu^*(A)$. □

4. If $E_1 \subset E_2 \subset \Omega$, then $\mu^*(E_1) \leq \mu^*(E_2)$ (since any covering of E_2 is also a covering of E_1).

*5. (*Semi-σ-additivity of μ^**)*

If $E \subset \cup_{j=1}^{\infty} E_j$, $E, E_j \subset \Omega$, then $\mu^*(E) \leq \Sigma_{j=1}^{\infty}\mu^*(E_j)$. (this CANNOT be improved even if $E = \sqcup_{j=1}^{\infty} E_j$ — check our warning above!)

Proof. $\forall \varepsilon > 0$,

$\forall j$, choose $\{A_{j,k}\}_{k=1}^{\infty} \subset S$ such that $E_j \subset \cup_{k=1}^{\infty} A_{j,k}$ and

$\Sigma_{k=1}^{\infty} \mu(A_{j,k}) \leq \mu^*(E_j) + \frac{\varepsilon}{2^j}$ (thanks to the infimum property).

Thus, $E \subset \cup_{j=1}^{\infty} E_j \subset \cup_{j=1}^{\infty} \cup_{k=1}^{\infty} A_{j,k}$.

Thus, by the definition of μ^* and semi-σ-additivity of μ,

$\mu^*(E) \leq \Sigma_{j=1}^{\infty} \Sigma_{k=1}^{\infty} \mu(A_{j,k}) \leq \Sigma_{j=1}^{\infty} \left(\mu^*(E_j) + \frac{\varepsilon}{2^j}\right) = \Sigma_{j=1}^{\infty} \mu^*(E_j) + \varepsilon$.

Let $\varepsilon \to 0^+$, we get the desired result. $\square$

Example 5.44. *Let's fix a bounded cell Ω in $\mathbb{R}^d$. Let $S = \{\text{all cells } C \subset \Omega\}$.*

Define $\mu(\{p\}) = 0$ for all $p \in \Omega$. Consider $E = \Omega \cap \mathbb{Q}^n, E = \{q_1, q_2, \ldots\}$

$\Rightarrow \mu^*(E) \leq \Sigma_{j=1}^{\infty} \mu^*(\{q_j\}) = \Sigma_{j=1}^{\infty} \mu(\{q_j\}) = 0 \Rightarrow \mu^*(E) = 0.$

$\mu^*(\Omega \setminus E) \leq \mu^*(\Omega) = \mu(\Omega)$

But by semi-σ-additivity, $\mu(\Omega) = \mu^*(\Omega) \leq \mu^*(E) + \mu^*(\Omega \setminus E) = \mu^*(\Omega \setminus E)$.

$\Rightarrow \mu^*(\Omega \setminus E) = \mu(\Omega)$, *which means that the outer measure CANNOT distinguish the counterble but sparce set $\mathbb{Q}^n$.*

5.2.4 Lebesgue Extension of a σ-Additive Measure on $\mathcal{A}(S)$

Definition 5.45. *(Lebesgue Measurable)*

Let S be a semi-ring with unity Ω, and μ be a σ-additive measure on S. $R(S) = \mathcal{A}(S)$ — the minimum algebra containing S, $\mathcal{A}(S) \subset 2^{\Omega}$.

A set $E \subset \Omega$ is called ***(Lebesgue) measurable*** *if and only if $\forall \varepsilon > 0$, $\exists B_\varepsilon \in \mathcal{A}(S)$ such that $\mu^*(E \triangle B_\varepsilon) = \mu^*(E \setminus B_\varepsilon) + \mu^*(B_\varepsilon \setminus E) < \varepsilon$, i.e. the set E can be approximated by a set $B_\varepsilon \in \mathcal{A}(S)$. We call such condition the* ***approximation property****.*

Example 5.46. *In this setting, let $\mu^*(E) = 0$, then E is measurable: Choose $B_\varepsilon = \emptyset$, then $\mu^*(E \triangle B_\varepsilon) = \mu^*(E) < \varepsilon$.*

Definition 5.47. *(Lebesgue Measurable: Altanative Definition)*

Let S be a semi-ring with unity Ω, and μ be a σ-additive measure on S.

A set $E \subset \Omega$ is called ***(Lebesgue) measurable*** *if and only if $\forall A \subset \Omega$, $\mu^*(E) = \mu^*(E \bigcap A) + \mu^*(E \setminus A)$. Such condition is called to be satisfying the* ***Carathéodory***[1] ***criterion****.*

Theorem 5.48. *The two definitions above are equivalent.*

Proof. Let S be a semi-ring with unity Ω, let μ_0 be a σ-additive premeasure on S (we emphasizes that this measure is the pre-measure), and let μ^* be the outer measure obtained from S by the usual covering construction.

[1] For a biography of Constantin Carathéodory, please see Appendix A.2.

$\mathcal{M} := \Big\{E \subset \Omega : \ \forall X \subset \Omega, \ \mu^*(X) = \mu^*(X \cap E) + \mu^*(X \setminus E)\Big\}$ is the Carathéodory σ-algebra.

Auxiliary facts:

(1) For all $X, E, B \subset \Omega$, we have

$\big|\mu^*(X \cap E) - \mu^*(X \cap B)\big| \le \mu^*(E \triangle B), \qquad \big|\mu^*(X \setminus E) - \mu^*(X \setminus B)\big| \le \mu^*(E \triangle B),$

which follows from monotonicity and subadditivity of μ^* (e.g. $X \cap E \subset (X \cap B) \cup (E \triangle B)$).

(2) $A(S) \subset \mathcal{M}$:

First check $S \subset \mathcal{M}$ by the additivity of μ_0 on S and the definition of μ^*; since $\mathcal{M}$ is a σ-algebra, it contains the algebra $A(S)$.

(Approximation $\Rightarrow$ Carathéodory). Assume $E \subset \Omega$ satisfies: for every $\varepsilon > 0$ there is $B_\varepsilon \in A(S)$ with $\mu^*(E \triangle B_\varepsilon) < \varepsilon$.

Fix $X \subset \Omega$. Because $B_\varepsilon \in A(S) \subset \mathcal{M}$, $\mu^*(X) \ge \mu^*(X \cap B_\varepsilon) + \mu^*(X \setminus B_\varepsilon)$.

Applying the first auxiliary fact with $B = B_\varepsilon$ gives $\mu^*(X) \ \ge \ \mu^*(X \cap E) + \mu^*(X \setminus E) - 2\,\mu^*(E \triangle B_\varepsilon)$.

Letting $\varepsilon \downarrow 0$ yields $\mu^*(X) \ge \mu^*(X \cap E) + \mu^*(X \setminus E)$. The reverse inequality is the subadditivity of μ^*, hence equality holds for all X, i.e. $E \in \mathcal{M}$.

(Carathéodory $\Rightarrow$ Approximation). Assume $E \in \mathcal{M}$. Let $\varepsilon > 0$.

By the definition of μ^* choose a cover $E \subset \bigcup_{k\ge1} S_k$ with $S_k \in S$ such that $\sum_{k=1}^{\infty} \mu_0(S_k) \le \mu^*(E) + \varepsilon/3$.

Write $U_N := \bigcup_{k=1}^{N} S_k \in A(S)$ and $U := \bigcup_{k\ge1} S_k$.

Then $\mu^*(U) \le \mu^*(E) + \varepsilon/3$.

Since E is Carathéodory measurable and $E \subset U$, $\mu^*(U) = \mu^*(E) + \mu^*(U \setminus E)$

$\Rightarrow \quad \mu^*(U \setminus E) \le \varepsilon/3$.

By semi-σ-additivity on the tail, choose N so large that $\mu^*(U \setminus U_N) \le \varepsilon/3$.

Hence $\mu^*(U_N \setminus E) \le \mu^*(U \setminus E) + \mu^*(U \setminus U_N) \le \frac{2\varepsilon}{3}, \qquad \mu^*(E \setminus U_N) \le \mu^*(U \setminus U_N) \le \frac{\varepsilon}{3}$, and therefore $\mu^*(E \triangle U_N) \le \varepsilon$.

With $B_\varepsilon := U_N \in A(S)$ we obtain the approximation property.

Combining the two implications proves that the two definitions above are equivalent.

$\square$

Remark 5.49. *Think about it: Does such definition address our problem in the last subsubsection?*

Answer: Yes, the Carathéodory criterion directly and completely addresses this problem!

1. *It provides a filter: The definition provides a precise condition to "sieve" the "measurable" sets from the "non-measurable" ones. A set E is declared measurable if and only if it splits* every *other set A in an additive way with respect to the outer measure: $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$*

2. *It constructs the σ-algebra: The Carathéodory Extension Theorem (which is based on this definition) proves that the collection $\mathcal{M}$ of all sets E that satisfy this criterion forms a σ-algebra.*

3. *It guarantees additivity: The same theorem proves that the outer measure μ^*, when restricted to this σ-algebra $\mathcal{M}$, becomes a* ***countably additive measure****.*

In summary, Definition 5.46 is not just an arbitrary definition; it is the precise tool needed to solve the extension problem. It successfully identifies the exact collection of sets ($\mathcal{M}$, the Lebesgue measurable sets) on which the outer measure μ^* behaves as a true, countably additive measure.

Remark 5.50. *The definition of a (Lebesgue) measurable set captures the idea of* approximability by "nice" sets. *A set $E \subset \Omega$ is called measurable if it can be arbitrarily well approximated by sets B_ε from the algebra $\mathcal{A}(S)$, in the sense that the "disagreement region" between E and B_ε, namely the symmetric difference $E \triangle B_\varepsilon$, has arbitrarily small outer measure: $\mu^*(E \triangle B_\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.*

Intuitively, this means that even if E itself may be irregular or complicated, we can always find a clean, measurable set B_ε that almost coincides with E up to an arbitrarily small "error area." Measurable sets are precisely those whose geometry can be faithfully captured through such approximations.

In the above example, if $\mu^*(E) = 0$, then E is trivially measurable. Indeed, we can take $B_\varepsilon = \emptyset$, so that $\mu^*(E \triangle B_\varepsilon) = \mu^*(E) = 0 < \varepsilon$. This illustrates that every measure-zero set *is measurable: such sets are geometrically "invisible" to the outer measure, since they can be ignored without affecting any measured quantity.*

Setting:
(Ω, S, μ) — Ω - set, S - semi-ring with unity Ω, μ - σ-additive measure on S
$\to$ directly extend to $(\Omega, \mathcal{A}(S), \mu)$, μ: the pre-measure.
$\to$ introduce μ^* on the whole 2^Ω,
$\to$ $(\Omega, \mathcal{M}(\Omega), \mu)$, with $\mathcal{M}(\Omega)$: collection of all measurable sets in Ω.
'measurable': $\forall A \in \mathcal{M}(\Omega)$, $\forall \varepsilon > 0$, $\exists B_\varepsilon \in \mathcal{A}(S)$ such that $\mu^*(A \triangle B_\varepsilon) < \varepsilon$.

Remark 5.51. *To better distinguish μ and μ^*, for those in the original $\mathcal{A}(S)$, we use μ. Otherwise, we use the notation μ^*. Thus, $*$ emphases that the measure on the set is defined by extanding μ.*

Theorem 5.52. *In the above setting, let $\mathcal{M}(S)$ be the collection of all measurable sets and we set $\mu(A) := \mu^*(A), \forall A \in M(S)$. Then,*

1. *$\mathcal{M}(S)$ is a σ-algebra.*

   ($M(S)$ extends the original algebra $\mathcal{A}(S)$.)

2. *μ^* is σ-additive on $\mathcal{M}(S)$.*

   (M extends the original measure μ on $\mathcal{A}(S)$.)

Proof. First of all, we know that $\Omega \in \mathcal{M}(\Omega)$.

Step I: prove if $A \in \mathcal{M}(\Omega)$, then $\Omega \setminus A \in \mathcal{M}(\Omega)$.

Fix $\varepsilon > 0$, $\exists B_\varepsilon \in \mathcal{A}(S)$ such that $\mu^*(A \triangle B_\varepsilon) < \varepsilon$.

Consider $\Omega \setminus B_\varepsilon \in \mathcal{A}$. Then, note $(\Omega \setminus A) \triangle (\Omega \setminus B_\varepsilon) = A \triangle B_\varepsilon$.

Thus, $\mu^*((\Omega \setminus A) \triangle (\Omega \setminus B_\varepsilon)) < \varepsilon \Rightarrow \Omega \setminus A \in \mathcal{M}(\Omega)$.

Step II: prove $\forall A_1, A_2, \ldots, A_n \in \mathcal{M}(\Omega)$, we have $\bigcup_{i=1}^n A_i \in \mathcal{M}(\Omega)$.

Only need to prove for $n = 2$ (others by induction).

$A_1, A_2 \in \mathcal{M}(\Omega)$, $\forall \varepsilon > 0. \exists B_1, B_2 \in \mathcal{A} : \mu^*(A_1 \triangle B_1) < \varepsilon, \mu^*(A_2 \triangle B_2) < \varepsilon$.

$A = A_1 \bigcup A_2$, we will approximate by $B = B_1 \bigcup B_2$.

Since $(A_1 \bigcup A_2) \triangle (B_1 \bigcup B_2) \subset (A_1 \bigcup B_1) \triangle (A_2 \bigcup B_2)$,

$\mu^*(A \triangle B) < \mu^*(A_1 \triangle B_1) + \mu^*(A_2 \triangle B_2) < 2\varepsilon$

$\Rightarrow A_1 \bigcup A_2 \in \mathcal{M}(\Omega)$.

Thus, the first statement is proved.

Corollary 5.53. *$\mathcal{M}(\Omega)$ is an algebra.*

Proof.

- contains Ω.
- closed under taking union: proved above.
- closed under intersection:
- closed under symmetric difference: $A \triangle B =$

□

Step III: prove μ^* is finitely additive on $\mathcal{M}(\Omega)$.

So, $\forall A_1, A_2, \ldots, A_n \in \mathcal{M}(\Omega)$, we need to show $\mu(A_1 \cup A_2 \cup \cdots \cup A_n) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n)$.

Similarly, only need to show for $n = 2$.

Take $A_1, A_2 \in \mathcal{M}(\Omega), A_1 \cap A_2 = \emptyset$.

$\forall \varepsilon > 0, \exists B_1, B_2 \in \mathcal{A}(S) : \mu^*(A_1 \triangle B_1) < \varepsilon, \mu^*(A_2 \triangle B_2) < \varepsilon$.

Since $B_1 \bigcap B_2 \subset (A_1 \triangle B_1) \bigcup (A_2 \triangle B_2)$, we have μ^*

Step IV: prove μ^* is a σ-algebra on $\mathcal{M}(\Omega)$.

...

...

Replace by disjoint union: let $A_1' = A_1, A_2' = A_2 \setminus A_1, A_3' = A_3 \setminus (A_1 \bigcup A_2), \ldots$.

Then, we have $A = \bigsqcup_{i=1}^\infty A_i'$.

We have

Step V: prove μ^* is σ-additive on $\mathcal{M}(\Omega)$.

WTS: $\forall A_1, A_2, \ldots, A_n \in \mathcal{M}(\Omega)$, we have $\mu^*(A_1 \cup A_2 \cup \cdots \cup A_n) = \mu^*(A_1) + \mu^*(A_2) + \cdots + \mu^*(A_n)$.

□

Conclusion: We end up with a triple $(\Omega, \mathcal{M}(\Omega), \mu)$ — (set: Ω, σ-algebra: $\mathcal{M}(\Omega)$, σ-additive measure on $\mathcal{M}(\Omega)$: μ).

5.2.5 Measure Space

Definition 5.54. *(Measure Space)*

Such a triple $(\Omega, \mathcal{A}, \mu)$ ($\mathcal{A}$ is some σ-algebra on the set Ω) is called a ***measure space (spcae with measure)****.*

Definition 5.55. *(Completeness of a Measure Space)*

A ***complete*** *measure (or, more precisely, a complete measure space) is a measure space in which every subset of every null set is measurable (having measure zero).*

More formally, if $(\Omega, \mathcal{A}, \mu)$ is a measure space, then it's called ***complete*** *if and only if $A \subset E \in \mathcal{A}$, $\mu(E) = 0$, $\Rightarrow A \in \mathcal{A}$ (and hence $\mu(A) = 0$).*

Example 5.56. *For $(\Omega, \mathcal{M}(\Omega), \mu)$, we always have completeness:*

$\mu(A) = 0, E \subset A \Rightarrow 0 \leq \mu^*(E) \leq \mu^*(A) = 0 \Rightarrow E \in \mathcal{M}(\Omega)$.

But this FAILS in general. For example, $\exists$ measure 0 non-Borel sets, which is contained in some measure 0 Borel sets, so Lebesgure measure μ on $\mathbb{R}^n$, restricted to Borel σ-algebra is incomplete.

However, any incomplete measure space can extend its measure to attain a complete measure space. One just need to follow the Lebesgue extension of a general measure space $(\Omega, \mathcal{A}, \mu)$.

Theorem 5.57. *For any measure space $(\Omega, \mathcal{A}, \mu)$, the following holds:*

1. *$\forall A_1 \subset A_2 \subset A_3 \subset \ldots$ with $A_i \in \mathcal{A}$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.*
2. *If $A_1 \supset A_2 \supset A_3 \supset \ldots$, $A_i \in \mathcal{A}$, then $\lim_{i\to\infty} \mu(A_i) = \mu(\bigcap_{i=1}^{\infty} A_i)$.*

Both 1 and 2 are called the ***continuity*** *of the measure.*

Proof. □

Question: What about ∞-valued measures?

Consider a space with measure $(\Omega, \mathcal{A}, \mu)$, where μ is a $\bar{\mathbb{R}}$-valued measure. The definition of finite additivity and σ-additivity is repeated word-by-word:

Finite additivity:

1. $\mu(A) \geq 0$;

2. $\mu(\bigcup_{j=1}^{n} A_j) = \Sigma_{j=1}^{n}\mu(A_j)$;

σ-**additivity**:

1. $\mu(A) \geq 0$;

2. $\mu(\bigcup_{j=1}^{\infty} A_j) = \Sigma_{j=1}^{\infty}\mu(A_j)$;

Then we easily deduce several similar properties.

Proposition 5.58. *1.* $\mu(\emptyset) = 0$.

2. If $A \subset B$, *then* $\mu(A) \leq \mu(B)$.

3. If $A \subset \bigcup_{j=1}^{\infty} A_j$, *then* $\mu(\bigcup_{j=1}^{n} A_j) \leq \Sigma_{j=1}^{\infty}\mu(A_j)$.

Definition 5.59. *A measure space with* ∞*-valued measure is called* σ***-finite*** *if*
$\Omega = \bigsqcup_{k=1}^{\infty} \Omega_k, \Omega_k \in \mathcal{A}$, $\mu(\Omega_k) < \infty$.
Then $\forall A \subset \mathcal{A}$, $\mu(A) = \Sigma_{k=1}^{\infty}\mu(A \bigcap \Omega_k) =: \Sigma_{k=1}^{\infty}\mu_k(A)$.

So, essentially, μ is obtained from $\{\mu_k\}_{k=1}^{\infty}$, with each μ_k defined on $\mathcal{A} \cap 2^{\Omega_k}$.

Example 5.60. $\mathbb{R}^n = \bigsqcup_{i_1,i_2,\ldots,i_n}[i_1, i_1+1) \times [i_2, i_2+1) \times \ldots \times [i_n, i_n+1), i_1, \ldots, i_n \in \mathbb{Z}$.

Remark 5.61. *For* $\Omega = \bigsqcup_{k=1}^{\infty} \Omega_k = \bigsqcup_{j=1}^{\infty} \Omega'_j$
Mesures agree:

5.3 Lebesgue Measure in $\mathbb{R}^n$

Goal: Understand $\mathcal{M}(\mathbb{R}^n)$.

Main fact: $\mathcal{B}(\mathbb{R}^n) \subsetneq \mathcal{M}(\mathbb{R}^n) \subsetneq 2^{\mathbb{R}^n}$.

5.3.1 Basic Intuition of Non-Measurable Sets

Proposition 5.62. *1. (Shift-invariance) If $E_\alpha := \{x+\alpha, x \in E, \alpha \in \mathbb{R}^n$: fixed$\}$, then $E_\alpha \in \mathbb{R}^n \Leftrightarrow E \subset \mathcal{M}(R)$; and we also have $\mu(E_\alpha = \mu(E))$.*

It holds since it holds for cells.

2. $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}(\mathbb{R}^n)$

Proposition 5.63. *$\exists$ a non-measurable subset $A \subset [0,1)$.*

Proof. On $[0,1)$, consider the following equivalent relation:

□

Proposition 5.64. *$\forall A \subset \mathbb{R}$ with $\mu(A) > 0$, A contains some $B \subset A$ s.t. $B \notin \mathcal{M}(\mathbb{R})$.*

Proof. □

Remark 5.65. *The same holds in $\mathbb{R}^n$: $\forall A \subset \mathbb{R}^n$ with $\mu(A) > 0$, A contains some $B \subset A$ s.t. $B \notin \mathcal{M}(\mathbb{R}^n)$.*

5.3.2 Cantor Set

We build a sequence of sets:

$E_0 = [0, 1]$

$E_1 = E_0 \setminus I_1$, $I_1 = (\frac{1}{3}, \frac{2}{3})$.

$E_2 = E_1 \setminus I_2$, $I_2 = I_{1,1} \bigcup I_{1,2}$, $I_{1,1} = (\frac{1}{9}, \frac{2}{9})$, $I_{1,2} = (\frac{7}{9}, \frac{8}{9})$.

$E_3 = E_2 \setminus I_3$, $I_3 = I_{2,1} \bigcup I_{2,2} \bigcup I_{2,3} \bigcup I_{2,4}$, $I_{2,1} = (\frac{1}{27}, \frac{2}{27})$, $I_{2,2} = (\frac{4}{27}, \frac{5}{27})$, $I_{2,3} = (\frac{19}{27}, \frac{20}{27})$, $I_{2,4} = (\frac{25}{27}, \frac{26}{27})$.

...

We get a sequence of sets $\{E_k\}$, $\forall E_k$ is closed. $\Rightarrow C_0 := \bigcap_{k=1}^{\infty} E_k$, C_0 is closed and bounded $\Rightarrow C_0$ is compact.

Definition 5.66. *(Cantor Set)*

*Such C_0 is called a **(standard) Cantor set**.*

Proposition 5.67.

1. *C_0 is compact and $C_0 \subset [0, 1]$;*
2. *C_0 is nowhere dense;*
3. *$\mu(C_0) = 0$*

   Proof. □
4. *C_0 is continual.*

   Proof. □

5.3.3 Cantor Staircase Function

Definition 5.68.

Lemma 5.69. *Let $f : \Omega \to \Omega', S' \subset 2^{\Omega'}$, then $\mathcal{A}(f^{-1} * (S')) = f^{-1}(\mathcal{A}(S'))$.*

Proof. □

Corollary 5.70. *(Preimage of Borel set is Borel.)*
If $f : [a, b] \to [c, d]$ is continuous, then $f^{-1}(E')$ is Borel, provided $E' \subset [c, d]$ is Borel.

Proof. Follows from $f^{-1}(G)$ is open if G is open. □

Now, consider $\phi(x) := x + K(x)$, $\phi : [0, 1] \to [0, 2]$, ϕ is strictly increasing.

5.3.4 Construction of a Non-Borel Measurable Set

Appendix A Biographies of Mathematicians

A.1 Henri Lebesgue (1875 - 1941)

Henri Lebesgue's father was a printer. Henri began his studies at the Collège de Beauvais, then he went to Paris where he studied first at the Lycée Saint Louis and then at the Lycée Louis-le-Grand.

Lebesgue entered the École Normale Supérieure in Paris in 1894 and was awarded his teaching diploma in mathematics in 1897. For the next two years he studied in its library where he read Baire's papers on discontinuous functions and realised that much more could be achieved in this area. Later there would be considerable rivalry between Baire and Lebesgue which we refer to below. He was appointed professor at the Lycée Centrale at Nancy where he taught from 1899 to 1902. Building on the work of others, including that of Émile Borel and Camille Jordan, Lebesgue formulated the theory of measure in 1901 and in his famous paper *Sur une généralisation de l'intégrale définie (On a generalization of the definite integral)*, which appeared in the *Comptes Rendus* on 29 April 1901, he gave the definition of the Lebesgue integral that generalises the notion of the Riemann integral by extending the concept of the area below a curve to include many discontinuous functions. This generalisation of the Riemann integral revolutionised the integral calculus. Up to the end of the 19th century, mathematical analysis was limited to continuous functions, based largely on the Riemann method of integration.

His contribution is one of the achievements of modern analysis which greatly expands the scope of Fourier analysis. This outstanding piece of work appears in Lebesgue's doctoral dissertation, *Intégrale, longueur, aire (Integral, length, area)*, presented to the Faculty of Science in Paris in 1902, and the 130 page work was published in Milan in the Annali di Matematica in the same year. Having graduated with his doctorate, Lebesgue obtained his first university appointment when in 1902 he became mâitre de conférences in mathematics at the Faculty of Science in Rennes. This was in keeping with the standard French tradition of a young academic first having appointments in the provinces, then later gaining recognition in being appointed to a more junior post in Paris. On 3 December 1903 he married Louise-Marguerite Vallet and they had two children. However the marriage only lasted until 1916 when they were divorced.

One honour which Lebesgue received at an early stage in his career was an invitation to give the Cours Peccot at the Collège de France. He did so in 1903 and then received an invitation to present the Cours Peccot two years later in 1905. Lebesgue first fell out with Baire in 1904, when Baire gave the Cours Peccot at the Collège de France, over who had the most right to teach such a course. Their rivalry turned into a more serious argument later in their lives. Lebesgue wrote two monographs *Leçons sur l'intégration et la recherche des fonctions primitives (Lectures on integration and research on primitive* *functions)* (1904) and *Leçons sur les séries trigonométriques (Lectures on trigonometric series)* (1906) which arose from these two lecture courses and served to make his important ideas more widely known. However, his work received a hostile reception from classical analysts, especially in France. In 1906 he was appointed to the Faculty of Science in Poitiers and in the following year he was named professor of mechanics there.

Let us attempt to indicate the way that the Lebesgue integral enabled many of the problems associated with integration to be solved. Fourier had assumed that for bounded functions term by term integration of an infinite series representing the function was possible. From this he was able to prove that if a function was representable by a trigonometric series then this series is necessarily its Fourier series. There is a problem here, namely that a function which is not Riemann integrable may be represented as a uniformly bounded series of Riemann integrable functions. This shows that Fourier's assumption for bounded functions does not hold.

In 1905 Lebesgue gave a deep discussion of the various conditions Lipschitz and Jordan had used in order to ensure that a function $f(x)$ is the sum of its Fourier series. What Lebesgue was able to show was that term by term integration of a uniformly bounded series of Lebesgue integrable functions was always valid. This now meant that Fourier's proof that if a function was representable by a trigonometric series then this series is necessarily its Fourier series became valid, since it could now be founded on a correct result regarding term by term integration of series. As Hawkins writes:-

"*In Lebesgue's work ... the generalised definition of the integral was simply the starting point of his contributions to integration theory. What made the new definition important was that Lebesgue was able to recognise in it an analytic tool capable of dealing with - and to a large extent overcoming - the numerous theoretical difficulties that had arisen in connection with Riemann's theory of integration. In fact, the problems posed by these difficulties motivated all of Lebesgue's major results.*"

He was appointed maître de conférences in mathematical analysis at the Sorbonne in 1910. During the first world war he worked for the defence of France, and at this time he fell out with Borel who was doing a similar task. Lebesgue held his post at the Sorbonne until 1918 when he was promoted to Professor of the Application of Geometry to Analysis. In 1921 he was named as Professor of Mathematics at the Collège de France, a position he held until his death in 1941. He also taught at the École Supérieure de Physique et de Chimie Industrielles de la Ville de Paris between 1927 and 1937 and at the École Normale Supérieure in Sèvres.

It is interesting that Lebesgue did not concentrate throughout his career on the field which he had himself started. This was because his work was a striking generalisation, yet Lebesgue himself was fearful of generalisations. He wrote:-

"*Reduced to general theories, mathematics would be a beautiful form without content. It would quickly die.*"

Although future developments showed his fears to be groundless, they do allow us to understand the course his own work followed.

He also made major contributions in other areas of mathematics, including topology, potential theory, the Dirichlet problem, the calculus of variations, set theory, the theory of surface area and dimension theory. By 1922 when he published *Notice sur les travaux scientifique de M Henri Lebesgue* he had written nearly 90 books and papers. This ninety-two page work also provides an analysis of the contents of Lebesgue's papers. After 1922 he remained active, but his contributions were directed towards pedagogical issues, historical work, and elementary geometry.

Lebesgue was honoured with election to many academies. He was elected to the Academy of Sciences on 29 May 1922, to the Royal Society, the Royal Academy of Science and Letters of Belgium (6 June 1931), the Academy of Bologna, the Accademia dei Lincei, the Royal Danish Academy of Sciences, the Romanian Academy of Sciences, and the Kraków Academy of Science and Letters. He was also awarded honorary doctorates from many universities. He also received a number of prizes including the Prix Houllevigue (1912), the Prix Poncelet (1914), the Prix Saintour (1917) and the Prix Petit d'Ormoy (1919).

A.2 Constantin Carathéodory (1873 - 1950)