Solution for HM

yang_du@foxmail.com , M5001

Remark: If you find any mistakes, typos, or have any other feedback, please let me know.

1 Conditional Expectation

Grade Distribution (Total=5+5+10+10=30).

1. $\forall \epsilon > 0$, take $A_\epsilon = \{\mathbb{E}(Y|\mathcal{A}) - \mathbb{E}(X|\mathcal{A}) \geq \epsilon\}$; it is clear that $A_\epsilon \in \mathcal{A}$. We can see

$$\mathbb{E}\left[(\mathbb{E}[Y|\mathcal{A}] - \mathbb{E}[X|\mathcal{A}])1_{A_\epsilon}\right] \geq \epsilon\mathbb{E}(1_{A_\epsilon}) = \epsilon\mathbb{P}(A_\epsilon) \geq 0. \tag{1.1}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left[(\mathbb{E}[Y|\mathcal{A}] - \mathbb{E}[X|\mathcal{A}])1_{A_\epsilon}\right] &= \mathbb{E}\left(\mathbb{E}[Y - X|\mathcal{A}]1_{A_\epsilon}\right) \\ &= \mathbb{E}\left(\mathbb{E}[(Y - X)1_{A_\epsilon}|\mathcal{A}]\right) = \mathbb{E}[(Y - X)1_{A_\epsilon}] \leq 0,\end{aligned} \tag{1.2}$$

where we use the fact that $A_\epsilon \in \mathcal{A}$ and $X \geq Y a.s.$.
Combine (1.1) and (1.2), we get $\mathbb{P}(A_\epsilon) = \mathbb{P}(\{\mathbb{E}(Y|\mathcal{A}) - \mathbb{E}(X|\mathcal{A}) \geq \epsilon\}) = 0$ which implies

$$\mathbb{P}(\mathbb{E}(Y|\mathcal{A}) - \mathbb{E}(X|\mathcal{A}) > 0) = \mathbb{P}(\bigcup_{n=1}^{\infty}\{\mathbb{E}(Y|\mathcal{A}) - \mathbb{E}(X|\mathcal{A}) \geq \frac{1}{n}\}) \leq \sum_{n=1}^{\infty}\mathbb{P}(A_{\frac{1}{n}}) = 0 .$$

Hence, we finally get $\mathbb{E}[X|\mathcal{A}] \geq \mathbb{E}[Y|\mathcal{A}] \quad a.s.$.

2. (i) $\alpha X + \beta Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

   (ii) $\alpha\mathbb{E}[X|\mathcal{A}] + \beta\mathbb{E}[Y|\mathcal{A}] \in \mathcal{A}$. For $\forall A \in \mathcal{A}$, we have

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}[\alpha X + \beta Y \mid \mathcal{A}]1_A\right] = \mathbb{E}\left(\mathbb{E}\left[(\alpha X + \beta Y)1_A \mid \mathcal{A}\right]\right) \\ &= \mathbb{E}\left[(\alpha X + \beta Y)1_A\right] \\ &= \alpha\mathbb{E}\left[X1_A\right] + \beta\mathbb{E}\left[Y1_A\right] \quad \text{(by linearity of expectation)} \\ &= \alpha\mathbb{E}\left(\mathbb{E}[X \mid \mathcal{A}]1_A\right) + \beta\mathbb{E}\left(\mathbb{E}[Y \mid \mathcal{A}]1_A\right) \\ &= \mathbb{E}\left(\alpha\mathbb{E}[X \mid \mathcal{A}]1_A\right) + \mathbb{E}\left(\beta\mathbb{E}[Y \mid \mathcal{A}]1_A\right) \quad \text{(by linearity of expectation)} \\ &= \mathbb{E}\left(\{\alpha\mathbb{E}[X \mid \mathcal{A}] + \beta\mathbb{E}[Y \mid \mathcal{A}]\}\, 1_A\right) \quad \text{(by linearity of expectation)} \\ &= \mathbb{E}\left[\mathbb{E}\left[\alpha\mathbb{E}(X|\mathcal{A}) + \beta\mathbb{E}(Y|\mathcal{A}) \mid \mathcal{A}\right] 1_A\right].\end{aligned}$$

For parts without special instructions, we just use the definition of conditional expectation.

3.**Remark:** Here, we refer to the proof of Theorem 8.3 in Liying Li's lecture notes on Advanced Probability Theory.

Recall that in proving the unconditional version , we took expectation of the inequality $\varphi(x) \geq ax + b$, where we chose a and b so that $ax + b$ is a tangent line at $x = \mathbb{E}(X)$. For the conditional expectation, such a strategy has a measurability problem. To take the conditional expectation of the inequality $\varphi(x) \geq ax+b$, the numbers a and b will vary with the choice of $\mathbb{E}[X \mid \mathcal{G}]$, which is not a fixed number and can take values in an uncountable set such as $\mathbb{R}$. The resulting inequality $\mathbb{E}[\varphi(X) \mid \mathcal{G}] \geq a\mathbb{E}[X \mid \mathcal{G}] + b$ only holds outside a zero measure set depending on a and b, and since the uncountable union of negligible sets can fail to be negligible (or even measurable), we cannot argue that

$$\varphi\left(\mathbb{E}\left[X\middle|\mathcal{A}\right]\right) \leq \mathbb{E}\left[\varphi(X)\middle|\mathcal{A}\right]$$

holds a.s.

The idea of the actual proof is to deal with this issue.

Proof: All the straight lines below a convex function φ fully characterize it. We can do better by using a countable number of them, that is,

$$\varphi(x) = \sup\{ax + b : a, b \in \mathbb{Q}, \varphi(t) \geq at + b, \forall t\}. \tag{1.3}$$

For every (a, b) in (8.4), by Question 1.1 above, there is a $N_{a,b}$ with $\mathbb{P}(N_{a,b}) = 0$ such that

$$\mathbb{E}[\varphi(X) \mid \mathcal{G}](\omega) \geq \mathbb{E}[aX + b \mid \mathcal{G}] = a\mathbb{E}[X \mid \mathcal{G}](\omega) + b, \quad \forall\omega \in N^c_{a,b}. \tag{1.4}$$

Hence, when $\omega \notin N = \bigcup_{a,b} N_{a,b}$, by (1.3) and (1.4) we have

$$\mathbb{E}[\varphi(X) \mid \mathcal{G}](\omega) \geq \varphi(\mathbb{E}[X \mid \mathcal{G}](\omega)). \tag{1.5}$$

On the other hand, by σ-subadditivity, $\mathbb{P}(N) \leq \sum_{a,b} \mathbb{P}(N_{a,b}) = 0$, and hence (1.5) holds a.s..

4. **Method 1.**The function $|\cdot|$ is convex , so from Question 1.3, we have $\left|\mathbb{E}\left[XY\middle|\mathcal{A}\right]\right| \leq \mathbb{E}\left[\,|XY|\,\middle|\mathcal{A}\right]$. Hence we just need to check that

$$\mathbb{E}\left[\,|XY|\,\middle|\mathcal{A}\right] \leq \left(\mathbb{E}\left[|X|^p\middle|\mathcal{A}\right]\right)^{1/p}\left(\mathbb{E}\left[|Y|^q\middle|\mathcal{A}\right]\right)^{1/q}.$$

We split $\Omega = U_1 \cup U_2 \cup U_3$ where

$$U_1 = \{\omega, \left(\mathbb{E}\left[|X|^p\middle|\mathcal{A}\right]\right)^{1/p}(\omega) = 0\},$$

$$U_2 = \{\omega, \left(\mathbb{E}\left[|Y|^p\middle|\mathcal{A}\right]\right)^{1/p}(\omega) = 0\},$$

$$U_3 = \{\omega, \left(\mathbb{E}\left[|X|^p \middle| \mathcal{A}\right]\right)^{1/p}(\omega) \left(\mathbb{E}\left[|Y|^p \middle| \mathcal{A}\right]\right)^{1/p}(\omega) \neq 0\}.$$

Similarly as Question 1.1 , we can prove the following lemma,
Lemma 1. If $X_1 = X_2$ a.s. on $B \in \mathcal{A}$, then $\mathbb{E}(X_1|\mathcal{A}) = \mathbb{E}(X_2|\mathcal{A})$ a.s. on B.

Case 1. $\int_{U_1} |X|^p d\mathbb{P} = \int_{U_1} \mathbb{E}(|X|^p\,|\mathcal{A})\,d\mathbb{P} = 0$ which implies $X(\omega) = 0$ a.s. on U_1. Hence by Lemma 1 , we have $\mathbb{E}(|XY|\big|\mathcal{A}) = \mathbb{E}(0|\mathcal{A}) = 0$ a.s. on U_1. Therefore, both sides of the inequality are zero, so the inequality a.s. holds naturally on U_1.

Case 2. Similarly as above, the inequality a.s. holds on U_2.

Case 3. Consider $\omega \in U_3$. To simplify the notation, we set $x = \left(\mathbb{E}\left[|X|^p \middle| \mathcal{A}\right]\right)^{1/p}$ and $y = \left(\mathbb{E}\left[|Y|^p \middle| \mathcal{A}\right]\right)^{1/p}$. The Young's inequality tells us that for $a, b > 0$, we have $ab \le \frac{a^p}{p} + \frac{b^q}{q}$. Hence

$$\frac{|X||Y|}{xy} \le \frac{1}{p}\frac{|X|^p}{x^p} + \frac{1}{q}\frac{|Y|^q}{y^q}. \tag{1.6}$$

Take conditional expection on both sides of (1.6) on U_3, then

$$\mathbb{E}(\frac{|X||Y|}{xy}|\mathcal{A}) \le \mathbb{E}(\frac{1}{p}\frac{|X|^p}{x^p} + \frac{1}{q}\frac{|Y|^q}{y^q}\Big|\mathcal{A}) = \frac{1}{p\cdot x^p}\mathbb{E}(|X|^p|\mathcal{A}) + \frac{1}{q\cdot y^q}\mathbb{E}(|Y|^q|\mathcal{A}) = \frac{1}{p} + \frac{1}{q} = 1.$$

So $\mathbb{E}(|XY|\big|\mathcal{A}) \le xy = \left(\mathbb{E}\left[|X|^p \middle| \mathcal{A}\right]\right)^{1/p} \left(\mathbb{E}\left[|Y|^p \middle| \mathcal{A}\right]\right)^{1/p}$ a.s. on U_3 .

From above , we have $\mathbb{E}(|XY|\big|\mathcal{A}) \le \left(\mathbb{E}\left[|X|^p \middle| \mathcal{A}\right]\right)^{1/p} \left(\mathbb{E}\left[|Y|^q \middle| \mathcal{A}\right]\right)^{1/q}$ a.s. on Ω .

Method 2. In fact, we can consider the regular conditional probability to get the inequality from the classical unconditional version directly . We recommend that interested readers refer to Section 4.1.3 of the 5th edition of Durrett's book.

2 Discrete-time stochastic processes

Grade Distribution (Total=12+8+12+8+8+12+5=65).

1. **Remark:** We will provide a detailed answer to (a), and the trade-off is that, since the derivation processes of (b) and (c) are entirely similar to that of (a), both are omitted here. But it does not mean the part can be omitted in your assignment.

(a)($\Rightarrow$) Assume $X = \{X_n\}$ is a $\{\mathcal{F}_n\}$-supermartingale. For any fixed $k \in \mathbb{N}$, we have:

$$\begin{aligned}
E[X_{n+k}|\mathcal{F}_n] &= E\Big[E[X_{n+k}|\mathcal{F}_{n+k-1}]\Big|\mathcal{F}_n\Big] && \text{(by } \mathcal{F}_n \subset \mathcal{F}_{n+k-1}\text{)} \\
&\le E[X_{n+k-1}|\mathcal{F}_n] && \text{(by supermartingale def. and monotonicity)} \\
&\le E[X_{n+k-2}|\mathcal{F}_n] && \text{(repeating the argument)} \\
&\vdots \\
&\le E[X_{n+1}|\mathcal{F}_n] \le X_n && \text{(by supermartingale def.)}
\end{aligned}$$

Thus, the condition holds.

($\Leftarrow$) Conversely, assume that for any fixed $k \in \mathbb{N}$, we have $X_n \ge E[X_{n+k}|\mathcal{F}_n]$. The process is given as adapted and integrable. To show it is a supermartingale, we only need to verify the defining inequality. The assumption holds for any $k \in \mathbb{N}$, so we may choose $k = 1$. This directly yields:

$$X_n \ge E[X_{n+1}|\mathcal{F}_n]$$

This is the definition of a supermartingale.

(b) Omitted.

(c) Omitted.

(d)Similarly, we just consider the case of supermartingale.

$\Longrightarrow$ If X is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-backward supermartingale ,then $X_{n+1} \ge \mathbb{E}[X_n \mid \mathcal{F}_{n+1}]$. Then we have that:

$$\begin{aligned}
X_{n+k} &\ge \mathbb{E}[X_{n+k-1} \mid \mathcal{F}_{n+k}] \\
&\ge \mathbb{E}[\mathbb{E}[X_{n+k-2} \mid \mathcal{F}_{n+k-1}] \mid \mathcal{F}_{n+k}] = \mathbb{E}[X_{n+k-2} \mid \mathcal{F}_{n+k}] \\
&\ge \cdots \ge \mathbb{E}[X_n \mid \mathcal{F}_{n+k}]
\end{aligned}$$

$\Longleftarrow$ Just take $k = 1$.

2.(1) Since X_n is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-martingale and $X_{n+1} \in \mathcal{F}_n$, we get

$$X_n = \mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_{n+1}.$$

Since n is arbitrary, we can get $X_n = \cdots = X_1$.

(2) In class, we have proved the Existence of Doob's decomposition. Suppose we have two decompositions $X_n = Y_n + Z_n = Y_n' + Z_n'$ where Y_n, Y_n' are $\{\mathcal{F}_n : n \in \mathbb{N}\}$-martingales and Z_n, Z_n' are predictable . So $Y_n - Y_n' = Z_n' - Z_n \in \mathcal{F}_{n-1}$ and is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-martingale. Hence from (1), we get

$$Y_n - Y_n' = Z_n - Z_n' = Z_1 - Z_1' = 0,$$

where we use $Z_1 = Z_1' = 0$ in the last equality by the definition of increasing process in class.

From above, $\forall n \in \mathbb{N}, Y_n = Y'_n$ and $Z_n = Z'_n$. We complete the proof of Uniqueness.

3.Recall that $\mathcal{F}_\alpha = \{A \in \mathcal{F}_\infty, A \cap \{\alpha \leq n\} \in \mathcal{F}_n, \forall n \in \mathbb{N} \cup \{\infty\}\}$ and for optional r.v. α, we have (or we can prove)

$$\{\omega, \alpha(\omega) = n\} \in \mathcal{F}_n, \forall n \in \mathbb{N} \cup \{\infty\} \iff \{\omega, \alpha(\omega) \leq n\} \in \mathcal{F}_n, \forall n \in \mathbb{N} \cup \{\infty\}. \tag{2.1}$$

(i)$\Omega \in \mathcal{F}_\alpha$, since $\Omega \cap \{\alpha \leq n\} = \{\alpha \leq n\} \in \mathcal{F}_n$, $\forall n \in \mathbb{N} \cup \{\infty\}$ by (2.1).

(ii)$\forall A \in \mathcal{F}_\alpha$, $\forall n \in \mathbb{N} \cup \{\infty\}$, $A^c \cap \{\alpha \leq n\} = (\Omega \cap \{\alpha \leq n\})/(A \cap \{\alpha \leq n\}) \in \mathcal{F}_n$, because $\mathcal{F}_n$ is closed under complement. Hence $A^c \in \mathcal{F}_\alpha$.

(iii) If $A_1, A_2, \cdots \in \mathcal{F}_\alpha$, then

$$\left(\bigcup_{k=1}^{\infty} A_k\right) \cap \{\alpha \leq n\} = \bigcup_{k=1}^{\infty} (A_k \cap \{\alpha \leq n\}) \in \mathcal{F}_n$$

since $\mathcal{F}_n$ is closed under countable unions and by (2.1). Hence $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_\alpha$.

At last , we want to prove that $\alpha \in \mathcal{F}_\alpha$. $\forall k \in \mathbb{N} \cup \{\infty\}$, then $\{\alpha \leq k\} \cap \{\alpha \leq n\} = \{\alpha \leq k \wedge n\} \in \mathcal{F}_n, \forall n \in \mathbb{N} \cup \{\infty\}$ by (2.1). Hence we complete the proof.

4. (1)It's easy to check that $X_n \in L^1(\Omega, \mathcal{F}, P)$ and X_n is $\mathcal{F}_n$-measurable. Then we just need to show the following point.

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(Y|\mathcal{F}_{n+1})|\mathcal{F}_n) = \mathbb{E}(Y|\mathcal{F}_n) = X_n.$$

(2)We are asked to prove that if $\alpha \leq \beta$, then $\{X_\alpha, X_\beta\}$ is a martingale relative to $\{\mathcal{F}_\alpha, \mathcal{F}_\beta\}$.

To show that $\{X_\alpha, X_\beta\}$ is an $\{\mathcal{F}_\alpha, \mathcal{F}_\beta\}$-martingale, we need to verify two conditions:

(i) X_α is $\mathcal{F}_\alpha$-measurable.

(ii) $\mathbb{E}[X_\beta|\mathcal{F}_\alpha] = X_\alpha$.

Note that the integrability of X_α and X_β is guaranteed because the martingale $X_n = \mathbb{E}[Y|\mathcal{F}_n]$ is uniformly integrable.

(i)X_α is $\mathcal{F}_\alpha$-measurable. We must show that for any Borel set B, $\{X_\alpha \in B\} \cap \{\alpha = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. We have

$$\{X_\alpha \in B\} \cap \{\alpha = n\} = \{X_n \in B\} \cap \{\alpha = n\}.$$

Since X_n is $\mathcal{F}_n$-measurable, $\{X_n \in B\} \in \mathcal{F}_n$. As α is a stopping time, $\{\alpha = n\} \in \mathcal{F}_n$. The intersection of two sets in $\mathcal{F}_n$ is also in $\mathcal{F}_n$. Thus, X_α is $\mathcal{F}_\alpha$-measurable. An identical argument shows X_β is $\mathcal{F}_\beta$-measurable.

(ii) $\mathbb{E}[X_\beta|\mathcal{F}_\alpha] = X_\alpha$. To prove this equality, we show that for any $A \in \mathcal{F}_\alpha$, we have $\mathbb{E}[X_\beta 1_A] = \mathbb{E}[X_\alpha 1_A]$.

First, let's evaluate $\mathbb{E}[X_\alpha 1_A]$. By definition of X_α,

$$\mathbb{E}[X_\alpha 1_A] = \mathbb{E}\left[\sum_{n=0}^{\infty} X_n 1_{\{\alpha=n\}} 1_A\right] = \sum_{n=0}^{\infty} \mathbb{E}[X_n 1_{A\cap\{\alpha=n\}}].$$

Since $A \in \mathcal{F}_\alpha$, the set $A_n := A \cap \{\alpha = n\}$ is in $\mathcal{F}_n$. Using the definition $X_n = \mathbb{E}[Y|\mathcal{F}_n]$ and the property of conditional expectation,

$$\mathbb{E}[X_n 1_{A_n}] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_n] 1_{A_n}] = \mathbb{E}[Y 1_{A_n}].$$

Summing over all n, we get

$$\mathbb{E}[X_\alpha 1_A] = \sum_{n=0}^{\infty} \mathbb{E}[Y 1_{A\cap\{\alpha=n\}}] = \mathbb{E}[Y 1_A].$$

Next, we evaluate $\mathbb{E}[X_\beta 1_A]$ in a similar manner.

$$\mathbb{E}[X_\beta 1_A] = \sum_{n=0}^{\infty} \mathbb{E}[X_\beta 1_{A\cap\{\alpha=n\}}].$$

Let's analyze the term $\mathbb{E}[X_\beta 1_{A_n}]$ where $A_n = A \cap \{\alpha = n\} \in \mathcal{F}_n$. Since $\alpha \le \beta$, on the set A_n, we have $\beta \ge n$.

$$\begin{aligned}\mathbb{E}[X_\beta 1_{A_n}] &= \mathbb{E}\left[\sum_{m=n}^{\infty} X_m 1_{\{\beta=m\}} 1_{A_n}\right] \\ &= \sum_{m=n}^{\infty} \mathbb{E}[X_m 1_{A_n\cap\{\beta=m\}}].\end{aligned}$$

For each $m \ge n$, $A_n \in \mathcal{F}_n \subseteq \mathcal{F}_m$. Since β is a stopping time, $\{\beta = m\} \in \mathcal{F}_m$. Therefore, the set $A_n \cap \{\beta = m\}$ is $\mathcal{F}_m$-measurable.

$$\mathbb{E}[X_m 1_{A_n\cap\{\beta=m\}}] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_m] 1_{A_n\cap\{\beta=m\}}] = \mathbb{E}[Y 1_{A_n\cap\{\beta=m\}}].$$

Summing over $m \ge n$,

$$\mathbb{E}[X_\beta 1_{A_n}] = \sum_{m=n}^{\infty} \mathbb{E}[Y 1_{A_n\cap\{\beta=m\}}] = \mathbb{E}[Y 1_{A_n\cap\{\beta\ge n\}}] = \mathbb{E}[Y 1_{A_n}],$$

where the last equality holds because on A_n, we have $\alpha = n$, which implies $\beta \ge n$.

Finally, summing over all n:

$$\mathbb{E}[X_\beta 1_A] = \sum_{n=0}^{\infty} \mathbb{E}[Y 1_{A\cap\{\alpha=n\}}] = \mathbb{E}[Y 1_A].$$

We have shown that for any $A \in \mathcal{F}_\alpha$, $\mathbb{E}[X_\alpha 1_A] = \mathbb{E}[Y 1_A]$ and $\mathbb{E}[X_\beta 1_A] = \mathbb{E}[Y 1_A]$. Therefore, $\mathbb{E}[X_\alpha 1_A] = \mathbb{E}[X_\beta 1_A]$. Since X_α is $\mathcal{F}_\alpha$-measurable, this implies $\mathbb{E}[X_\beta|\mathcal{F}_\alpha] = X_\alpha$.

$$\mathbb{E}(X_\beta|\mathcal{F}_\alpha) = \mathbb{E}(\mathbb{E}(Y|\mathcal{F}_\beta)|\mathcal{F}_\alpha) = \mathbb{E}(Y|\mathcal{F}_\alpha) = X_\alpha$$

since $\alpha \leq \beta$ and then $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$.

5. X is a martingale transformation of Y through Z means

$$X_n = \sum_{k=1}^{n} Z_k(Y_k - Y_{k-1}).$$

And we know that

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\sum_{k=1}^{n+1} Z_k(Y_k - Y_{k-1}) \mid \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{n} Z_k(Y_k - Y_{k-1}) + Z_{n+1}(Y_{n+1} - Y_n) \mid \mathcal{F}_n\right] \\
&= X_n + \mathbb{E}\left[Z_{n+1}(Y_{n+1} - Y_n) \mid \mathcal{F}_n\right] \\
&= X_n + Z_{n+1}\left(\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] - Y_n\right) \\
&\geq (\text{or } \leq) X_n.
\end{aligned}$$

where we use $\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] \geq (\text{or } \leq) Y_n$ and $Z_{n+1} \in \mathcal{F}_n$ since Y is a submartingale (or supermartingale) and Z is a predictable non-negative process .

6. For any $n \geq m$, $X_m = \mathbb{E}[X_n|\mathcal{F}_m]$. By choosing $n = 1$, we get

$$X_n = \mathbb{E}[X_1|\mathcal{F}_n].$$

Since X_1 is integrable, $Z \in L^1$. By the conditional Jensen's inequality:

$$|X_n| = |\mathbb{E}[X_1|\mathcal{F}_n]| \leq \mathbb{E}[|X_1||\mathcal{F}_n] = \mathbb{E}[Z|\mathcal{F}_n]$$

To prove u.i., we must show that $\lim_{K\to\infty} \sup_n \mathbb{E}[|X_n| 1_{\{|X_n|>K\}}] = 0$. Let $A_{n,K} = \{|X_n| > K\}$. Note that $A_{n,K} \in \mathcal{F}_n$.

$$\begin{aligned}
\mathbb{E}[|X_n| 1_{A_{n,K}}] &\leq \mathbb{E}[\mathbb{E}[Z|\mathcal{F}_n] 1_{A_{n,K}}] \\
&= \mathbb{E}[Z 1_{A_{n,K}}] = \mathbb{E}[|X_1| 1_{\{|X_n|>K\}}]
\end{aligned} \tag{2.2}$$

Now, we show that the probability of the set $A_{n,K}$ vanishes uniformly in n as $K \to \infty$. By Markov's inequality:

$$\mathbb{P}(A_{n,K}) = \mathbb{P}(|X_n| > K) \leq \frac{\mathbb{E}[|X_n|]}{K}$$

Using Jensen's inequality again, $\mathbb{E}[|X_n|] \leq \mathbb{E}[\mathbb{E}[|X_1||\mathcal{F}_n]] = \mathbb{E}[|X_1|]$. So, for any n, $\mathbb{P}(A_{n,K}) \leq \frac{\mathbb{E}[|X_1|]}{K}$. This shows that $\sup_n \mathbb{P}(A_{n,K}) \to 0$ as $K \to \infty$.

Since $X_1 \in L^1$, the integral of $|X_1|$ is absolutely continuous with respect to the measure $\mathbb{P}$. This means for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mathbb{P}(A) < \delta$, then $\mathbb{E}[|X_1|1_A] < \epsilon$.

Let K large enough such that $\sup_n \mathbb{P}(A_{n,K}) < \delta$. For such a K, by (2.2) it follows that:

$$\sup_n \mathbb{E}[|X_n|1_{\{|X_n|>K\}}] \leq \sup_n \mathbb{E}[|X_1|1_{A_{n,K}}] \leq \epsilon$$

Since ϵ was arbitrary, we conclude that $\{X_n\}$ is u.i. by

$$\lim_{K\to\infty} \sup_n \mathbb{E}[|X_n|1_{\{|X_n|>K\}}] = 0.$$

7.This only requires some basic calculations.

$$\begin{aligned}
\mathbb{E}\left[\frac{X_1 + \cdots + X_{n+1}}{n+1} \mid \mathcal{F}_n\right] &= \mathbb{E}\left[\frac{X_1 + \cdots + X_n}{n+1} + \frac{X_{n+1}}{n+1} \mid \mathcal{F}_n\right] \\
&= \frac{X_1 + \cdots + X_n}{n+1} + \frac{1}{n+1}\mathbb{E}(X_{n+1}|\mathcal{F}_n) \\
&= \frac{X_1 + \cdots + X_n}{n+1} + \frac{X_1 + \cdots + X_n}{n(n+1)} \\
&= (X_1 + \cdots + X_n)(\frac{1}{n+1} + \frac{1}{n(n+1)}). \\
&= \frac{X_1 + \cdots + X_n}{n}.
\end{aligned}$$