Week 5, Thursday

Δ Review: interpolation (插值), given data $\{x_j, f(x_j)\}_{j=0}^{n}$

find $p(x) \in P_n = \{$all polynomials of degree $\le n\}$ s.t. $p(x_j) = f(x_j)$

- Lagrange interpolating polynomials:

$$p(x) = \sum_{j=0}^{n} f(x_j) L_j(x), \quad L_j(x) = \prod_{i=0, i\ne j}^{n} \frac{x - x_i}{x_j - x_i}.$$

- Newton's divided difference

$$p(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \cdots, x_k] w_k(x), \quad w_k(x) = (x - x_0)\cdots(x - x_{k-1})$$

- divided difference.

0th d-d: $f[x_i] = f(x_i)$

first d-d: $f[x_i, x_{i+1}] = \dfrac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$

$\vdots$

kth d-d: $f[x_i, \cdots, x_{i+k}] = \dfrac{f[x_{i+1}, \cdots, x_{i+k}] - f[x_i, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}$

advantage: if add one more point, only need add one more term.

add x_{n+1}, then add $f[x_0, x_1, \cdots, x_{n+1}](x - x_0)\cdots(x - x_n)$

- reminder $R(x) = f(x) - p(x)$

Lagrange: $R(x) = \dfrac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$

d-d : $R(x) = f[x, x_0, x_1, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_n)$

$\boxed{\text{Thm}}$ if $f \in C^n[a,b]$, $\exists\, \xi \in (a,b)$ s.t. $f[x_0, x_1, \cdots, x_n] = \dfrac{f^{(n)}(\xi)}{n!}$

equal spacing: $x_i = x_0 + ih$, $i = 1, 2, \cdots, n$.

forward difference formula: $\Delta f(x_k) = f(x_{k+1}) - f(x_k)$

backward difference formula: $\nabla f(x_k) = f(x_k) - f(x_{k-1})$

$x = x_0 + sh$ $\quad p(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$

$x = x_n + sh \qquad P(x) = f(x_n) + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n).$

Hermite interpolation: given $\{x_j, f(x_j), f'(x_j)\}_{j=0}^{n}$

find a polynomial $H(x)$ agree with f and f' at $x_0, x_1, \cdots, x_n$.

$\boxed{\text{Thm}}$ $(\exists !)$ the least degree of the polynomial $H(x)$ is degree of at most $2n+1$ given by:

$$H(x) = \sum_{j=0}^{n} f(x_j) H_j(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_j(x), \text{ where}$$

$$H_j(x) = [1 - 2(x - x_j) L_j'(x_j)] L_j^2(x), \quad \hat{H}_j(x) = (x - x_j) L_j^2(x)$$

RK: $f(x) = H(x) + \dfrac{w_n^2(x)}{(2n+2)!} f^{(2n+2)}(\xi(x))$

Q: Do we expect we can approximate as accurate as possible with polynomials of very high degree by enough data $\{x_j, f(x_j)\}_{j=0}^{n}$?

Problem: high-degree polynomials can oscillate dramatically.

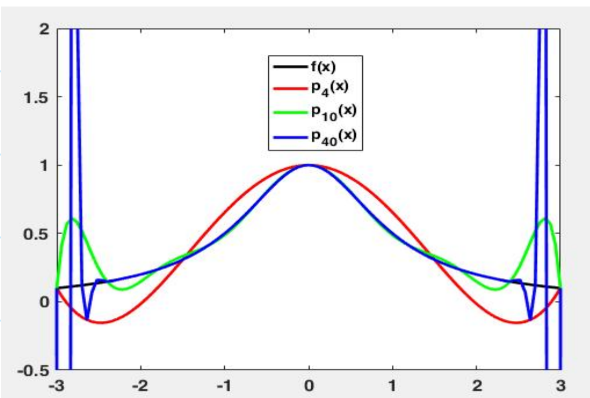
e.g. Runge phenomenon, take $f(x) = \dfrac{1}{1+x^2}$, in $[-5, 5]$,

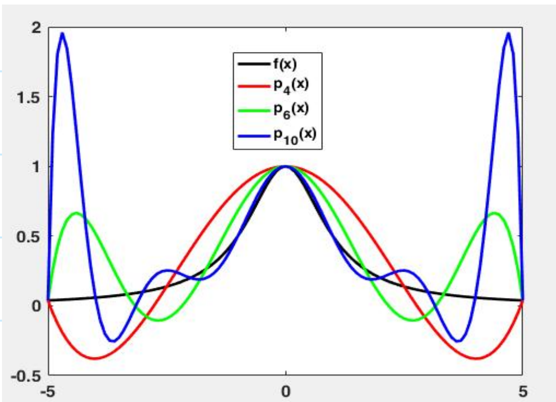
$x_j = -5 + \frac{10}{n} j$, $j = 0, 1, \cdots, n$. then $P_n(x) = \sum_{j=0}^{n} \dfrac{1}{1+x_j^2} L_j(x)$

RK: Clearly $f(x) \in C^\infty[-5,5]$, but not analytically at $x = \pm i$

Runge theoretically proved that, $\exists\, C \approx 3.63$, such that

$\lim_{n\to\infty} L_n(x) = f(x)$, $|x| \le C$, but if $|x| > C$, $L_n(x)$ diverges.





$(|x| \le 3,\ n \nearrow,\ P_n(x) \to f(x))$ $\qquad$ $(|x| > C$, close to 5, $n \nearrow$, $P_n(x)$ diverges$)$

RK: if we choose Chebyshev points as interpolating nodal points

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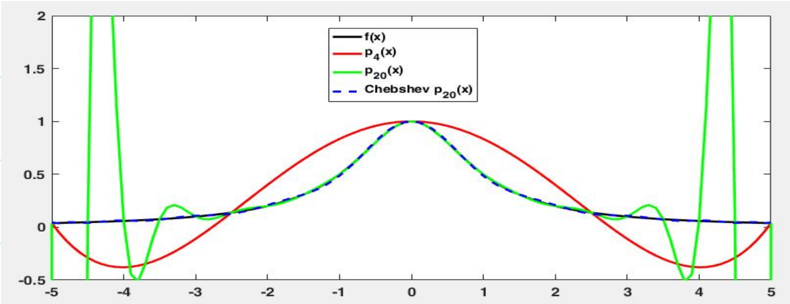
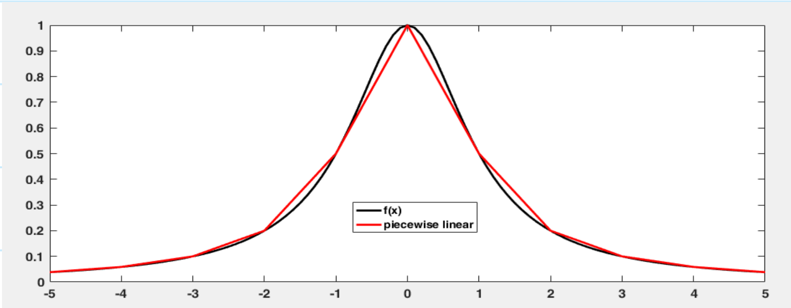
$x_k = 5\cos\left(\frac{k\pi - \frac{1}{2}\pi}{n}\right), k=1,2,\cdots,n$. above oscillation disappear.



△ another Solution: piecewise-polynomial approximation



① $I_h(x) \in C[x_0, x_n]$ ② $I_h(x_j) = f(x_j)$ ③ $I_h(x)\big|_{[x_j, x_{j+1}]}$ is linear

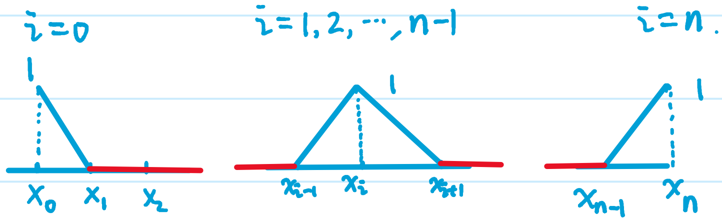
$$I_h(x) = f(x_j)\frac{x - x_{j+1}}{x_j - x_{j+1}} + f(x_{j+1})\frac{x - x_j}{x_{j+1} - x_j}, \quad x \in [x_j, x_{j+1}]$$

Q: how to find a basis for $X_n = \{\text{piecewise linear on } [x_{j-1}, x_j]\}$

$$\begin{cases} \varphi_i(x_j) = \delta_{ij}, & 0 \le i, j \le N. \\ \varphi_i\big|_{[x_j, x_{j+1}]} \text{ is linear} \end{cases}$$

"Hat" function:

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases}$$



Define: interpolation: $I_h: C[0,1] \to X_n$ s.t.

$$I_h u(x_i) = u(x_i), \quad i = 0, 1, \cdots, n.$$

RK: $I_h f(x) = \sum_{i=0}^{n} f(x_i)\varphi_i(x)$.

$x \in [x_j, x_{j+1}]$, $f(x) = I_h(x) + \frac{1}{2}f''(\xi(x))(x - x_j)(x - x_{j+1})$, $\xi(x) \in (x_j, x_{j+1})$.

$x \in [x_j, x_{j+1}]$, $f(x) = I_h(x) + \frac{1}{2} f''(\xi(x))(x-x_j)(x-x_{j+1})$, $\xi(x) \in (x_j, x_{j+1})$.

error bound:

$\therefore \max_{x \in [x_0, x_n]} |f(x) - I_h f(x)| \le \frac{1}{8} h^2 \max_{x \in [x_0, x_n]} |f''(x)|$, $h = \max_{0 \le j \le n} |x_{j+1} - x_j|$

RK: applications of this piecewise linear approximation

e.g. two-point boundary value problem $\begin{cases} -u''(x) = f(x), & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$

Weak formulation: find u in $H_0^1(0,1)$ s.t.

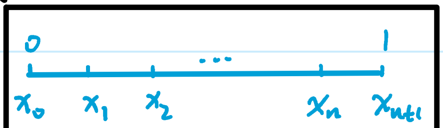
$$(u', v') = (f, v), \quad \forall v \in H_0^1(0,1)$$

where $H_0^1(0,1) = \{ v(x) \mid \int_0^1 |v|^2 dx, \int_0^1 |v'|^2 dx < \infty, v(0) = v(1) = 0 \}$.

piecewise linear finite element method: (有限元)

find u_h in X_n, i.e. $u_h = \sum_{j=1}^{n} c_j \varphi_j(x)$ s.t.

$(u_h', v_h') = (f, v_h)$, $\forall v_h \in X_n$



$c_1 = c_{n+1} = 0$, i.e.

$u_h(0) = u_h(1) = 0$

$\Leftrightarrow$ $\forall i = 1, 2, \cdots, n$, take $v_h = \varphi_i$

$\sum_{j=1}^{n} (\varphi_j', \varphi_i') c_j = (f, \varphi_i)$,

denote $m_{ij} = (\varphi_j', \varphi_i')$, $(f, \varphi_i) = f_i$

$$M \vec{c} = \vec{f}$$

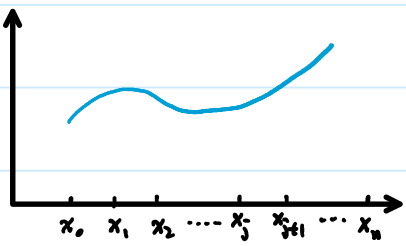
RK: it is easy to prove M is symmetric and positive definite, so it is invertible. (try it)

$\boxed{\text{Thm}}$ For linear FEM: $\|u - u_h\|_{L^2} + ch \|u' - u_h'\|_{L^2} \le 2(ch)^2 \|f\|_{L^2}$.

(more about FEM, refer to Feng Kang 冯康)

△ Cubic splines: piecewise cubic

Def: $S(x) \in C^2[x_0, x_n]$ such that $S(x)$ is an interpolation of $f(x)$, on each interval $S(x)$ is a cubic polynomial



Cubic spline interpolant S for f satisfies:

$S_j(x) = S(x)\big|_{[x_j, x_{j+1}]}$ is a cubic polynomial, $j=0,1,\cdots,n-1$

$S_j(x) = a_j + b_j x + c_j x^2 + d_j x^3$ $4n$ coefficients to be determined.

(a) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$, $j=0,1,\cdots,n-1$

also imply: $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ $(S \in C[x_0, x_n])$ $2n$

(b) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$, $j=0,1,\cdots,n-2$ $(S \in C^1(x_0, x_n))$ $n-1$

(c) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$, $j=0,1,\cdots,n-2$ $(S \in C^2(x_0, x_n))$ $n-1$

(d) one of following boundary condition is satisfied 2

$4n$ conditions

(i) $S''(x_0) = S''(x_n) = 0$ (natural (free) boundary)

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (clamped boundary)

△ Construction:

$$S_j(x) = S(x)\big|_{[x_j, x_{j+1}]} = a_j + b_j(x-x_j) + c_j(x-x_j) + d_j(x-x_j)^3 \quad j=0,1,\cdots,n-1$$

define: $h_j = x_{j+1} - x_j$, $j=0,1,\cdots,n-1$

as $S''_j(x)$ is a linear function

$$S_j''(x) = M_j \frac{x_{j+1}-x}{h_j} + M_{j+1}\frac{x-x_j}{h_j} \qquad (c)$$

$$\therefore S_j(x) = M_j \frac{(x_{j+1}-x)^3}{6h_j} + M_{j+1}\frac{(x-x_j)^3}{6h_j} + P_j x + Q_j$$

as $S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1})$ (a)

$$S_j(x) = M_j \frac{(x_{j+1}-x)^3}{6h_j} + M_{j+1}\frac{(x-x_j)^3}{6h_j} + \left(f(x_j) - \frac{M_j h_j^2}{6}\right)\frac{x_{j+1}-x}{h_j}$$

$$+ \left(f(x_{j+1}) - \frac{M_{j+1}h_j^2}{6}\right)\frac{x-x_j}{h_j} \qquad j=0,1,\cdots,n-1.$$

$$S_j'(x) = -M_j \frac{(x_{j+1}-x)^2}{2h_j} + M_{j+1}\frac{(x-x_j)^2}{2h_j} + \frac{f(x_{j+1}) - f(x_j)}{h_j} - \frac{M_{j+1}-M_j}{6} h_j$$

(b) $S_j'(x_{j+1}) = S_{j+1}'(x_{j+1}) \qquad j = 0, 1, 2, \cdots, n-2$

$$\Rightarrow \quad \frac{1}{2}h_j M_{j+1} + \frac{f(x_{j+1}) - f(x_j)}{h_j} - \frac{M_{j+1} - M_j}{6} h_j$$

$$= -\frac{1}{2}h_{j+1} M_{j+1} + \frac{f(x_{j+2}) - f(x_{j+1})}{h_{j+1}} - \frac{M_{j+2} - M_{j+1}}{6} h_{j+1}$$

i.e. $\frac{1}{6}h_j M_j + \frac{1}{3}(h_j + h_{j+1}) M_{j+1} + \frac{1}{6}h_{j+1} M_{j+2} = f[x_{j+1}, x_{j+2}] - f[x_j, x_{j+1}]$

$$\Leftrightarrow \quad \frac{1}{6}h_j M_j + \frac{1}{3}(h_j + h_{j+1}) M_{j+1} + \frac{1}{6}h_{j+1} M_{j+2} = \underbrace{(h_j + h_{j+1})}_{= x_{j+2} - x_j} f[x_j, x_{j+1}, x_{j+2}]$$

$j = 0, 1, 2, \cdots, n-2.$

(d) natural: $M_0 = f''(x_0) = 0$, $M_n = f''(x_n) = 0$ (known)

$$\begin{bmatrix} \frac{1}{3}(h_0+h_1) & \frac{1}{6}h_1 & & & \\ \frac{1}{6}h_1 & \frac{1}{3}(h_1+h_2) & \frac{1}{6}h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{6}h_{n-3} & \frac{1}{3}(h_{n-3}+h_{n-2}) & \frac{1}{6}h_{n-2} \\ & & & \frac{1}{6}h_{n-2} & \frac{1}{3}(h_{n-2}+h_{n-1}) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} (h_0+h_1) f[x_0, x_1, x_2] - \cancelto{0}{\frac{1}{6}h_0 M_0} \\ (h_1+h_2) f[x_1, x_2, x_3] \\ \vdots \\ (h_{n-3}+h_{n-2}) f[x_{n-3}, x_{n-2}, x_{n-1}] \\ (h_{n-2}+h_{n-1}) f[x_{n-2}, x_{n-1}, x_n] - \cancelto{0}{\frac{1}{6}h_{n-1} M_n} \end{bmatrix}$$

$$A\vec{M} = \vec{F}$$

Thm $a = x_0 < x_1 < \cdots < x_n = b$, the f has a unique natural cubic spline interpolant S on $x_0, x_1, \cdots, x_n$.

proof: as A given above is strictly diagonal dominant, then A is invertible. $\therefore$ $A\vec{M} = \vec{F}$ has a unique solution.

RK: if $f \in C^4[a,b]$, $|f^{(4)}(x)|_{max\ x\in[a,b]} = M$.

$$|f(x) - S(x)| \le \frac{5M}{384} \max_{0 \le j \le n-1} h_j^4$$

HW5-1: Sec 3.4 <u>11</u> Sec 3.5 4 <u>a</u> 6 <u>a</u> 13 29

Consider the following function

$$f(x) = \frac{1}{1+25x^2}, \quad x \in [-1, 1].$$

a. Set $t_k = -1 + 0.01 * k$, $k = 0, 1, \cdots, 200$.

b. Set $x_j = -1 + 0.4 * j$, $j = 0, 1, \cdots, 5$. Use these nodes to get the interpolation $L_5(x)$ and compute the error $e_k^{[1]} = |f(t_k) - L_5(t_k)|$, $k = 0, 1, \cdots, 200$.

c. Set $x_j = -1 + 0.2 * j$, $j = 0, 1, \cdots, 10$. Use these nodes to get the interpolation $L_{10}(x)$ and compute the error $e_k^{[2]} = |f(t_k) - L_{10}(t_k)|$, $k = 0, 1, \cdots, 200$.

d. Set $x_j = \cos\frac{(2j-1)\pi}{22}$, $j = 1, \cdots, 11$. Use these nodes to get the interpolation $C_{10}(x)$ and compute the error $e_k^{[3]} = |f(t_k) - C_{10}(t_k)|$, $k = 0, 1, \cdots, 200$.

e. Set $x_j = -1 + 0.2 * j$, $j = 0, 1, \cdots, 10$. Use these nodes to get a cubic spline with natural boundary condition $S_{10}(x)$ and compute the error $e_k^{[4]} = |f(t_k) - S_{10}(t_k)|$, $k = 0, 1, \cdots, 200$.

f. Plot the errors $e_k^{[\cdot]}$s, (learn "subplot" to present your results in an elegant way).

print out the figure in ℓf's, no need to submit the matlab code.

Due: 2025. 10. 21