COMPLEX ANALYSIS (H) MIDTERM. APRIL 19, 2025.

] **1. (5 points)** Does there exist an *invertible* holomorphic mapping f of the unit square $\{|\text{Re } z| < 1, |\text{Im } z| < 1\}$ onto the entire complex plane $\mathbb{C}$?

2. (10 points) Let $f(z)$ be a continuous function in a domain D which is holomorphic in $D \setminus \gamma$, where $\gamma \subset \mathbb{C}$ is a broken line. Prove that f is in fact holomorphic in D.

3. (10 points) Can the function $f(z) = \bar{z}$ be uniformly approximated by polynomials on the circle $\{|z| = 1\}$?

4. (7 points) We say that a sequence of meromorphic functions in a domain D *normally converges to* f *in* D, if for any compact $K \subset D$, the sequence $(f_j - f)$ doesn't have singularities on K for all $j \geq N = N(K)$, and $f_j - f$ converges to 0 uniformly on K. Prove that the normal limit f defined in this way is also meromorphic in D.

5. (7 points) Construct an example of a holomorphic mapping f of the unit disc $B_1(0)$ with (i) $f(B_1(0)) = \mathbb{C}$; (**extra 3 points**) (ii) $f(B_1(0)) = \overline{\mathbb{C}}$.

6. (10 points) Let $f(z)$ be holomorphic in the punctured unit disc $B_1^*(0) = \{0 < |z| < 1\}$, and the function $|z|^{1/2} \cdot |f(z)|$ be bounded. Prove that $f(z)$ extends holomorphically to the unit disc $B_1(0)$.

7. (7 points) Let $f \in \mathcal{O}(\mathbb{C})$. Assume $|f(z)| = O(|z|^a)$ for some $a > 0$ holds. Prove that f is a polynomial.

8. (7 points) Is there a holomorphic function $f(z)$ in $\{|z| < 1\}$ such that $f(1/n) = (-1)^n/n$ for $n \in \mathbb{N}$?

9. (7 points) Prove that all C^1 functions $f(z)$ in $\mathbb{C}$ such that $\frac{\partial f}{\partial \bar{z}} = z$ are actually of class C^∞ .

10. (7 points) For which values of $R > 0$ is the set $\{|z^2 - 1| < R\}$ connected?

11. (10 points) Let $f_n \in \mathcal{O}(\overline{B_R(a)})$ be a uniformly bounded sequence of functions, i.e. $|f_n(z)| \leq C$ for all $z \in \overline{B_R(a)}$ and a constant C. Prove the sequence f_n' is also uniformly bounded in some $\overline{B_r(a)}$, $r \leq R$.

12. (7 points) Can the function $f(x) = x\ln(1+x)$ be extended holomorphically from the positive ray $\mathbb{R}^+$ to a domain in complex plane? To the entire complex plane?

13. (10 points) Does there exist a function $f(z)$ holomorphic in $\{|z| > 0\}$ such that for all z we have $|f(z)| > \exp(1/|z|)$?

14. (10 points) Prove that the function $f(z) := \sum\limits_{n\geq 1} z^{n!}$ is holomorphic in the disc $\{|z| < 1\}$ but cannot be extended holomorphically to a neighborhood of any point a in the boundary of the disc.

15. (7 points) Prove that there exists a holomorphic function $f(z)$ in $\{|z| > 1\}$ such that for every z, $f(z)$ equals to one of the values of $\sqrt{1+z^2}$.

16. (7 points) A function $f(z)$ is holomorphic in $\{|z| > 1\}$ and is bounded there from below: $|f(z)| \geq M > 0$ for all z. Prove that there exists a (finite or infinite) $\lim_{z\to\infty} f(z)$.

17. (7 points) Let $f \in C(\overline{D}) \cup \mathcal{O}(D)$, where $D = \{0 < |z| < 1\}$. Assume $f|_{\partial D} = 1$. Prove that f is constant.