Introduction to Mathematical Logic

For CS Students

CS104

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2025 年 5 月 29 日

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Decision Problems

Definition

A **decision problem** is a problem which has a yes or no answer on a given input.

Example: The problem "Is an integer x positive?" is a decision problem.

Definition

A decision problem is **decidable** (可判定的) if there exists an algorithm that always halts with a yes/no answer for every input.

Otherwise, we say that the decision problem is **undecidable** (不可判定的).

The Collatz Conjecture

For even numbers, divide by 2; for odd numbers, multiply by 3 and add 1. With enough repetition, do all positive integers converge to 1?

```
while (n > 1){
    if ( (n % 2) == 0){
        n = n/2;
    } else {
        n = 3 * n + 1;
    }
}
```

The Collatz Conjecture is one of the most famous unsolved problems in math, and is algorithmically undecidable.

The Halting Problem

1 Decidability

The Halting Problem (停机问题) is a decision problem in computability theory.

Given a computer program P and an input I, determine if P halts on input I.

Theorem (Turing, 1936): The Halting Problem is undecidable.

Theorem proved in class.

A nice animation for the proof: `https://www.youtube.com/watch?v=92WHN-pAFCs`

The Partial Correctness Problem

1 Decidability

Given a Hoare Triple $(\!| P |\!)\ C\ (\!| Q |\!)$, does C satisfy the triple under partial correctness?

Theorem: The Partial Correctness Problem is undecidable.

Proof Sketch: Assume towards a contradiction that there is an algorithm B that returns "yes" when $(\!| P |\!)\ C\ (\!| Q |\!)$ is satisfied under partial correctness and "no" otherwise. Then, we can construct an algorithm A that decides the Halting Problem. Contradiction.

Introduction to Mathematical Logic

Thank you for listening!
Any questions?