Week 2, Thursday

Review: root-finding problem: find p s.t $f(p)=0$

① bisection method: shrinkage the interval including the root

1. find a_1, b_1 s.t. $f(a_1)f(b_1)<0$, $P_1=\frac{a_1+b_1}{2}$

2. For $i=1, \cdots, N_0$

if $f(a_i)f(P_i)<0$, set $a_{i+1}=a_i$, $b_{i+1}=P_i$

else if $f(a_i)f(P_i)>0$, set $a_{i+1}=P_i$, $b_{i+1}=b_i$

$\boxed{\text{Thm}}$ if $f\in C[a,b]$, $f(a)f(b)<0$, then the sequence $\{P_n\}_{n=1}^{\infty}$ generated by the bisection method converges to p, which is the root of $f(x)$, i.e. $f(p)=0$ and $|P_n-P|\leq\frac{b-a}{2^n}$.

② fixed-point iteration: convert root-finding to fixed-point-finding

$f(x)=0 \Leftrightarrow x=g(x)$ (or at least $f(p)=0 \Leftrightarrow p=g(p)$)

$\boxed{\text{Thm}}$ (existence) $g(x)\in C[a,b]$, $g([a,b])\subset[a,b]$

(uniqueness) $g'(x)\ \exists$ on (a,b), and $|g'(x)|\leq K<1$, $\forall x\in(a,b)$.

(fixed-point iteration)

The sequence: take $P_0\in[a,b]$, $P_n=g(P_{n-1})$, $n=1,2,\cdots$.

$\boxed{\text{Thm}}$ under the same condition in above theorem, the sequence $\{P_n\}_{n=0}^{\infty}$ converges to the unique fixed point, with speed:

(i) $|p^n-p|\leq K^n\max\{p_0-a, b-p_0\}$

(ii) $|p^n-p|\leq\frac{K^n}{1-K}|P_1-P_0|$

e.g. $f(x)=x^3+4x^2-10=0$, $f(1)=-5<0$, $f(2)=14>0$, $\exists$ root $\in[1,2]$

$$x(3x^2+8x) = x(3x^2+8x)-(x^3+4x^2-10)$$

$$\Rightarrow\ x = x - \frac{x^3+4x^2-10}{3x^2+8x} := g_5(x)$$

$P_0=1.5$, $P_4=1.36523001 3$, super fast!

compare with bisection method: $P_{27}\approx 1.365230013$

RK: essentially, the convergence speed depends on $|g'(p)|$.

$$g_5(x)=x-\frac{f(x)}{f'(x)} \Rightarrow g_5'(x)=\frac{f(x)f''(x)}{(f'(x))^2} \Rightarrow g_5'(p)=0$$

$\triangle$ Q: how to set $|g'(p)|$ as small as possible?

A general formula to construct fixed-point function:

$x = x-\phi(x)f(x) := g(x)$, easy to check:

$p=g(p) \Leftrightarrow f(p)=0$ if $\phi(p)\neq 0$. To this end:

$g'(x) = 1-(\phi'(x)f(x)+\phi(x)f'(x))$ if we want to have:

$g'(p)=0 \Leftrightarrow 1-(\phi'(p)f(p)+\phi(p)f'(p))=0 \Rightarrow \phi(p)=\frac{1}{f'(p)}$

so it is reasonable to take $\phi(x)=\frac{1}{f'(x)}$, i.e.

$$g(x) = x-\frac{f(x)}{f'(x)}, \quad \boxed{P_n = P_{n-1}-\frac{f(P_{n-1})}{f'(P_{n-1})}} \rightarrow (\text{Newton's iteration})$$

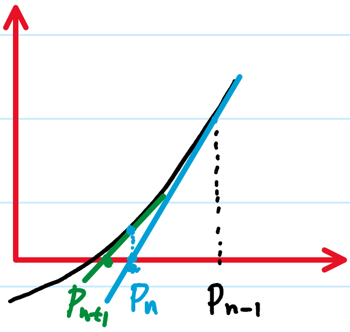
$\triangle$ another way to derive Newton's iteration: for $P_0\approx P$

$$f(p) = f(P_0)+(P-P_0)f'(P_0)+\frac{1}{2}(P-P_0)^2f''(\xi_0)$$

when $|P_0-P|\ll 1$, $|P-P_0|^2\ll|P-P_0| \Rightarrow (P-P_0)f'(P_0)\approx -f(P_0)$

$$P\approx P_0-\frac{f(P_0)}{f'(P_0)} \quad \boxed{P_n = P_{n-1}-\frac{f(P_{n-1})}{f'(P_{n-1})}} \rightarrow (\text{Newton's iteration})$$

△ Geometrically: $f'(P_{n-1}) = \dfrac{f(P_{n-1}) - 0}{P_{n-1} - P_n}$



$y - f(P_{n-1}) = f'(P_{n-1})(x - P_{n-1})$: the line tangent $f(x)$ at $(P_{n-1}, f(P_{n-1}))$

iteract x-axis at P_n

Input: P_0, ε, N_0

Output: approximation of the root.

1. set $i=0$, while $i \le N_0$, do 2-5
2. If $f'(P_0)=0$. STOP, output ("failure")
3. set $p = P_0 - \dfrac{f(P_0)}{f'(P_0)}$
4. If $|P-P_0|<\varepsilon$ / or $|f(p)|<\varepsilon$. STOP. output (p).
5. set $i=i+1$, $P_0 = p$
6. STOP, output ("failure")

e.g. $f(x) = \cos(x) - x$

(a) fixed point with: $g(x) = \cos x$

$P_0 = \frac{\pi}{4}$; $P_n = \cos(P_{n-1})$ $n = 1, 2, \cdots$

(b) Newton's iteration with: $g(x) = x - \dfrac{\cos x - x}{-\sin x - 1}$

$P_0 = \frac{\pi}{4}$; $P_n = P_{n-1} - \dfrac{P_{n-1} - \cos(P_{n-1})}{1 + \sin(P_{n-1})}$

	(a)	(b)
n=0	0.7853	0.7853981635 ←
n=1	0.7071	0.7395361337
n=2	0.7602	0.7390851781
	0.7247	0.7390851332
	0.7487	0.7390851332
	0.7325	

0.7481

0.7325 much faster!

0.7435

0.7361

Q: How to quantify the speed of convergence?

Def: $\{p_n\} \to p$ and $p_n \neq p$, $\forall n$, $\exists\ \lambda, \alpha$ s.t.

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda \text{, then we say}$$

$\{p_n\}$ converges to p of order α with asymptotic error constant λ.

(i) $\alpha = 1$, $\lambda < 1$, the sequence is linearly convergent

(ii) $\alpha = 2$, quadratically convergent

e.g. $\{\alpha_n\}$, $\{\beta_n\} \to 0$, and

$$\lim_{n\to\infty} \frac{|\alpha_{n+1}|}{|\alpha_n|} = 0.5, \qquad \lim_{n\to\infty} \frac{|\beta_{n+1}|}{|\beta_n|^2} = 0.5$$

$$|\alpha_n| \approx 0.5|\alpha_{n-1}| \approx \cdots \approx 0.5^n |\alpha_0|$$

$$|\beta_n| \approx 0.5|\beta_{n-1}|^2 \approx 0.5\left(0.5|\beta_{n-2}|^2\right)^2 \approx \cdots \approx 0.5^{2^0+2^1+2^2+\cdots+2^{n-1}} |\beta_0|^{2^n}$$
$$= (0.5)^{2^n-1} (\beta_0)^{2^n}$$

if $|\alpha_0| = |\beta_0| = 1$

	α_n	β_n
1	0.5×10^0	0.5×10^0
2	0.25×10^0	0.125×10^{-1}
3	0.125×10^0	0.78125×10^{-2}
4	0.625×10^{-1}	3.0515×10^{-5}
5	0.3125×10^{-1}	10^{-10}
6	0.15625×10^{-1}	10^{-19}
7	0.78125×10^{-2}	10^{-39}

※ $\boxed{\text{Thm}}$ Let $f \in C^2[a,b]$ $p \in (a,b)$ s.t. $f(p) = 0$ and $f'(p) \neq 0$

$\boxed{\text{Thm}}$ ※ let $f \in C^2[a,b]$, $p \in (a,b)$ s.t. $f(p)=0$ and $f'(p) \neq 0$,

then there exist a δ s.t. Newton's method generates a sequence converges to p for any $p_0 \in [p-\delta, p+\delta]$.

RK: basic idea is same as the fixed-point iteration, i.e. find $(p-\delta, p+\delta)$ s.t. $|g'(x)| \leq K < 1$, $\forall x \in (p-\delta, p+\delta)$

Proof: as $g(x) = x - \dfrac{f(x)}{f'(x)}$, $\quad g'(x) = \dfrac{f(x) f''(x)}{(f'(x))^2}$

Since $f \in C^2[a,b]$ and $f'(p) \neq 0$,

$\exists \delta_1 > 0$, s.t. $f'(x) \neq 0$, $\forall x \in [p-\delta_1, p+\delta_1] \subset [a,b]$

on the other hand, $g'(p) = 0$ and $g \in C^1[p-\delta_1, p+\delta_1]$

$\therefore \forall\ 0 < K < 1$, $\exists\ 0 < \delta \leq \delta_1$ s.t. $|g'(x)| \leq K$, $\forall x \in [p-\delta, p+\delta]$

Next show $g([p-\delta, p+\delta]) \subset [p-\delta, p+\delta]$: $\forall x \in [p-\delta, p+\delta]$

$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)|\,|x-p| \leq K|x-p| < \delta$

$\therefore g(x) \in [p-\delta, p+\delta]$, $\forall x \in [p-\delta, p+\delta]$

By the theorem for fixed-point iteration, we have $p_n \to p$ as $n \to \infty$.

$\boxed{\text{Thm}}$ ① $g \in C[a,b]$, $g([a,b]) \subset [a,b]$

② $g'(x)$ is $\boxed{\text{continuous}}$ on (a,b) and $|g'(x)| \leq K < 1$, $\forall x \in (a,b)$

③ $g'(p) \neq 0$, p is the fixed point

then $\{p_n\}$ generated by the fixed-point iteration converges $\boxed{\text{only linearly}}$ to the fixed point.

key idea: $\lim\limits_{n\to\infty} \dfrac{|p_{n+1} - p|}{|p_n - p|} = \lim\limits_{n\to\infty} \dfrac{|g(p_n) - g(p)|}{|p_n - p|} = \boxed{\lim\limits_{n\to\infty} |g'(\xi_n)|} = |g'(p)| < 1$

g' is continuous

key idea: $\lim_{n\to\infty} \frac{\text{[illegible]}}{|p_n - p|} = \lim_{n\to\infty} \frac{\text{[illegible]}}{|p_n - p|} = \boxed{\lim_{n\to\infty} |g'(\xi_n)|} = |g'(p)| < 1$

mean value theorem.

$\boxed{\text{Thm}}$ ① $p = g(p)$, $g'(p) = 0$

② g'' is continuous and $|g''(x)| < M$ on an open interval I, $p \in I$,

then $\exists\ \delta > 0$ s.t. $\forall\ p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$ $\forall n \geq 1$, converges to p $\boxed{\text{at least quadratically}}$

key idea: Since g'' is continuous on I and $g'(p) = 0$,

$\forall\ 0 < K < 1$, $\exists\ \delta > 0$, s.t. $[p-\delta, p+\delta] \subset I$ and $|g'(x)| \leq K$, $\forall x \in [p-\delta, p+\delta]$

and $\forall p_0 \in [p-\delta, p+\delta]$, $p_n = g(p_{n-1}) \to p$ as $n \to \infty$ (by thm ※)

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(\xi)}{2}(x-p)^2$$

$$= p + \frac{1}{2} g''(\xi)(x-p)^2$$

$$\Rightarrow p_{n+1} = p + \frac{1}{2} g''(\xi_n)(p_n - p)^2$$

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{1}{2}|g''(p)| \quad \text{at least quadratically}$$

$$\leq \frac{1}{2} M$$

Summary:

① bisection $\begin{cases} \text{simple, always convergent, a good initial data} \\ \text{producer for more faster methods, slow} \end{cases}$

② fixed-point iteration: $\begin{cases} \text{faster than bisection with suitable} \\ \text{choice of } g\text{. usually linearly convergent.} \\ \text{not easy to construct } g. \end{cases}$

③ Newton's method (can be viewed as a special fixed-point iteration with $g'(p) = 0$).

$\boxed{\text{drawbacks}}$ 1) require g to be differentiable, and $g'(p) \neq 0$

2) require g'' to be continuous to get quadratic convergence rate

3) Sensitive to initial guess

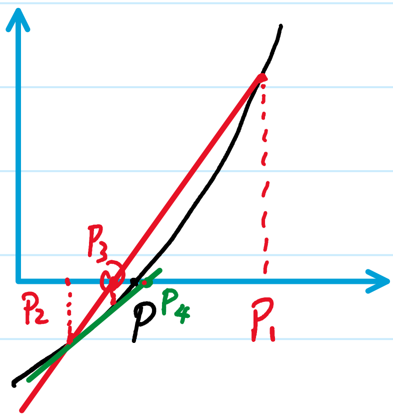
$\boxed{\text{advantage}}$ quadratically convergent, a uniform formula.

RK: ① what if f is not differentiable or difficult to compute $f'(x)$

(割线法)

Scant method: replace: $f'(P_{n-1}) \approx \dfrac{f(P_{n-1})-f(P_{n-2})}{P_{n-1}-P_{n-2}}$, i.e.

$$\boxed{P_n = P_{n-1} - \frac{P_{n-1}-P_{n-2}}{f(P_{n-1})-f(P_{n-2})} f(P_{n-1})}$$



red line: go through $(P_1, f(P_1)), (P_2, f(P_2))$ i.e. scant line

P_3 is the scant line intercross with x-axis.

② sensitive to initial guess (close enough to the root)

use bisection method to get a good initial.

③ Q: what if $f'(p)=0$? (HW 2.4 10)

HW 2 : 2.3 1, 2, 3a, 4a, 31, 32

2.4 8, 9, 10, 12

Using Newton's method, see what will happen:

① $f(x) = 1-x^2$, $P_0=0$ ② $f(x) = x^3-2x+2$, $P_0=0$

③ $f(x)=\sqrt[3]{x}$, $P_0=1$ ④ $f(x)=\sqrt{|x|}$, $P_0=1$