

# Introduction to Data Assimilation Lecture Notes

## Lecture 3 on Oct. 23, 2025

Quanling Deng\*

### Abstract

This file prepares and documents the lecture 3 notes for the 2025 Fall YMSC public course: Introduction to Data Assimilation.

**Keywords** Data assimilation, Kalman filter in 1D, numerical algorithm

## 1 Kalman Filter: 1D Example [1]

In the last lecture, we gave a derivation of the Kalman filter. In this lecture, we present the second derivation.

### 1.1 Forecast and Observation Models

**Forecast model.** The complex linear dynamics with complex Gaussian noise is

$$u_{m+1} = F u_m + \sigma_{m+1}, \quad F \in \mathbb{C}, \quad \sigma_{m+1} \sim \mathcal{N}(0, r), \quad r = \mathbb{E}[\sigma_{m+1} \sigma_{m+1}^*] > 0. \quad (1.1)$$

**Observation model.** A linear observation operator with complex Gaussian observation noise:

$$v_{m+1} = g u_{m+1} + \sigma_{m+1}^o, \quad \sigma_{m+1}^o \sim \mathcal{N}(0, r^o), \quad r^o = \mathbb{E}[\sigma_{m+1}^o (\sigma_{m+1}^o)^*] > 0, \quad (1.2)$$

independent of  $\sigma_{m+1}$  and of past states/noises.

**Forecast (Prediction) Step** Forecast formulas

$$\bar{u}_{m+1|m} = F \bar{u}_{m|m}, \quad r_{m+1|m} = F r_{m|m} F^* + r. \quad (1.3)$$

---

\*Corresponding author. Yau Mathematical Science Center, Tsinghua University, Beijing, China 100048; School of Computing, Australian National University, Canberra, ACT 2601, Australia. E-mail addresses: qldeng@tsinghua.edu.cn; Quanling.Deng@anu.edu.au

## 1.2 Analysis/Filtering Step: Two Derivations of the Kalman Gain

**Form of the affine update.** Given  $v_{m+1}$ , we update the mean with an affine rule

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1}(v_{m+1} - g\bar{u}_{m+1|m}), \quad K_{m+1} \in \mathbb{R}, \quad (1.4)$$

and define the posterior covariance

$$r_{m+1|m+1} = \mathbb{E}[(u_{m+1} - \bar{u}_{m+1|m+1})(u_{m+1} - \bar{u}_{m+1|m+1})^*]. \quad (1.5)$$

Herein,  $K_{m+1}$  is called the Kalman gain, and  $v_{m+1} - g\bar{u}_{m+1|m}$  is called the innovation.

### 1.2.1 (A) Bayesian/Conditional-Density Derivation via Minimizing $J(u)$

**Bayes' rule (density form).** Let  $u$  be the (latent) state and  $v$  the observation. Assume joint density  $p(u, v)$  with marginals  $p(u)$  and  $p(v)$  and conditional densities  $p(u | v) = \frac{p(u, v)}{p(v)}$  and  $p(v | u) = \frac{p(u, v)}{p(u)}$ . Then

$$p(u | v) = \frac{p(u, v)}{p(v)} = \frac{p(v | u)p(u)}{p(v)} \propto p(v | u)p(u).$$

**Kalmain gain derivation.** The likelihood and prior at time  $m+1$  are Gaussian:

$$p(v_{m+1} | u_{m+1}) \propto \exp\left(-\frac{|v_{m+1} - gu_{m+1}|^2}{r^o}\right), \quad p(u_{m+1}) \propto \exp\left(-\frac{|u_{m+1} - \bar{u}_{m+1|m}|^2}{r_{m+1|m}}\right).$$

Hence the posterior density is proportional to  $\exp(-\frac{1}{2}J(u))$  with

$$J(u) = \frac{|u - \bar{u}_{m+1|m}|^2}{r_{m+1|m}} + \frac{|v_{m+1} - gu|^2}{r^o}.$$

Since  $J$  is real-valued and strictly convex in  $u \in \mathbb{C}$ , the minimizer satisfies the first-order optimality condition (using, e.g., Wirtinger calculus  $\partial J / \partial u^* = 0$ ):

$$\frac{\partial J}{\partial u^*} = \frac{u - \bar{u}_{m+1|m}}{r_{m+1|m}} - \frac{g(v_{m+1} - gu)}{r^o} = 0.$$

Rearranging,

$$\left(\frac{1}{r_{m+1|m}} + \frac{g^2}{r^o}\right)u = \frac{\bar{u}_{m+1|m}}{r_{m+1|m}} + \frac{gv_{m+1}}{r^o}.$$

Solving for  $u$  and comparing with (1.4) yields the Kalman gain

$$K_{m+1} = \frac{g r_{m+1|m}}{r^o + g^2 r_{m+1|m}} \in [0, 1/g] \quad (1.6)$$

and the posterior mean update (1.4).

### 1.2.2 (B) Least-Squares (Minimum-Error-Variance) Derivation

We keep the notation: the prior (forecast) error at time  $m+1$  is

$$e_{m+1|m} := u_{m+1} - \bar{u}_{m+1|m}, \quad r_{m+1|m} := \mathbb{E}[e_{m+1|m} e_{m+1|m}^*] \in \mathbb{R}_{\geq 0},$$

and the observation model is  $v_{m+1} = g u_{m+1} + \sigma_{m+1}^o$  with  $g \in \mathbb{C}$ , independent, zero-mean complex Gaussian noise  $\sigma_{m+1}^o$ , and variance  $r^o := \mathbb{E}[\sigma_{m+1}^o (\sigma_{m+1}^o)^*] > 0$ . Define the innovation

$$d_{m+1} := v_{m+1} - g \bar{u}_{m+1|m} = g e_{m+1|m} + \sigma_{m+1}^o,$$

and consider the estimator

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1} d_{m+1}, \quad K_{m+1} \in \mathbb{C}.$$

The posterior error is

$$e_{m+1|m+1} := u_{m+1} - \bar{u}_{m+1|m+1} = (1 - K_{m+1}g) e_{m+1|m} - K_{m+1} \sigma_{m+1}^o.$$

We minimize the mean-square error

$$\phi(K) := \mathbb{E}[|e_{m+1|m+1}|^2] = \mathbb{E}[(1 - Kg)(1 - Kg)^*].$$

Independence and zero means imply  $\mathbb{E}[e(\sigma^o)^*] = 0$ , so the cross terms vanish and

$$\phi(K) = |1 - Kg|^2 r_{m+1|m} + |K|^2 r^o.$$

Treat  $K$  and  $K^*$  as independent (Wirtinger calculus) and set the stationary condition

$$\frac{\partial \phi}{\partial K^*} = 0.$$

Compute

$$\frac{\partial}{\partial K^*}(|1 - Kg|^2) = \frac{\partial}{\partial K^*}((1 - Kg)(1 - Kg)^*) = (1 - Kg) \frac{\partial}{\partial K^*}(1 - Kg)^* = -(1 - Kg)g^*,$$

and  $\frac{\partial}{\partial K^*}(|K|^2) = \frac{\partial}{\partial K^*}(KK^*) = K$ . Hence

$$\frac{\partial \phi}{\partial K^*} = r_{m+1|m} (-g^*(1 - Kg)) + r^o K = -r_{m+1|m} g^* + r_{m+1|m} |g|^2 K + r^o K.$$

Setting this to zero yields

$$(|g|^2 r_{m+1|m} + r^o) K = r_{m+1|m} g^*,$$

and therefore the complex scalar Kalman gain

$$K_{m+1} = \frac{r_{m+1|m} g^*}{r^o + |g|^2 r_{m+1|m}} \in \mathbb{C}.$$

If  $g \in \mathbb{R}$ , this reduces to  $K_{m+1} = \frac{g r_{m+1|m}}{r^o + g^2 r_{m+1|m}}$ .

## Posterior Covariance Update (complex scalar) and Joseph Form

Let the prior (forecast) error and covariance at time  $m+1$  be

$$e_{m+1|m} := u_{m+1} - \bar{u}_{m+1|m}, \quad r_{m+1|m} := \mathbb{E}[e_{m+1|m} e_{m+1|m}^*] \in \mathbb{R}_{\geq 0}.$$

With the innovation  $d_{m+1} = v_{m+1} - g \bar{u}_{m+1|m} = g e_{m+1|m} + \sigma_{m+1}^o$  and the affine analysis

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1} d_{m+1}, \quad K_{m+1} \in \mathbb{C},$$

the posterior (analysis) error is

$$e_{m+1|m+1} := u_{m+1} - \bar{u}_{m+1|m+1} = (1 - K_{m+1}g) e_{m+1|m} - K_{m+1} \sigma_{m+1}^o.$$

Independence and zero means give  $\mathbb{E}[e_{m+1|m} (\sigma_{m+1}^o)^*] = 0$ . Thus

$$\begin{aligned} r_{m+1|m+1} &:= \mathbb{E}[e_{m+1|m+1} e_{m+1|m+1}^*] \\ &= \mathbb{E}\left[\left((1 - Kg)e - K\sigma^o\right)\left((1 - Kg)e - K\sigma^o\right)^*\right] \\ &= (1 - K_{m+1}g) r_{m+1|m} (1 - K_{m+1}g)^* + K_{m+1} r^o K_{m+1}^* \\ &= (1 - Kg)(1 - Kg)^* r_{m+1|m} + |K|^2 r^o \quad (\text{assuming real}) \\ &= (1 - K_{m+1}g) r_{m+1|m} \quad (\text{after substituting } K_{m+1} \text{ from (1.6)}) \end{aligned} \tag{1.7}$$

Equation (1.7) is the *Joseph form* in the complex scalar case; it holds for any (possibly suboptimal)  $K$  and guarantees  $r_{m+1|m+1} \geq 0$ .

*Posterior covariance check (Joseph form).* With the optimal  $K_{m+1}$ ,

$$r_{m+1|m+1} = \mathbb{E}[|e_{m+1|m+1}|^2] = (1 - K_{m+1}g)(1 - K_{m+1}g)^* r_{m+1|m} + |K_{m+1}|^2 r^o = \frac{r_{m+1|m} r^o}{r^o + |g|^2 r_{m+1|m}}.$$

## Equivalence of the Joseph Form and the Simplified Posterior-Covariance Update (1D KF, Complex Scalar)

**Setup and notation.** Let  $u \in \mathbb{C}$  be the state,  $v \in \mathbb{C}$  the observation,  $g \in \mathbb{C}$  the (scalar) observation operator, and let  $r := r_{m+1|m} \in \mathbb{R}_{\geq 0}$  be the prior (forecast) variance at time  $m+1$ . The observation-noise variance is  $r^o > 0$ . For a complex scalar gain  $K \in \mathbb{C}$ , the *Joseph form* of the posterior variance is

$$r_{m+1|m+1} = (1 - Kg)(1 - Kg)^* r + |K|^2 r^o. \tag{1.8}$$

The *simplified form* is

$$r_{m+1|m+1} = (1 - Kg) r, \tag{1.9}$$

which is valid *at the optimal Kalman gain*. The complex scalar Kalman gain is

$$K^* = \frac{rg^*}{r^o + |g|^2 r}, \quad D := r^o + |g|^2 r > 0. \tag{1.10}$$

**Theorem 1.1** (Equivalence at the optimal gain). *For the one-dimensional complex scalar Kalman filter, if  $K = K^*$  in (1.10), then the Joseph form (1.8) and the simplified form (1.9) coincide:*

$$(1 - K^* g)(1 - K^* g)^* r + |K^*|^2 r^o = (1 - K^* g) r = \frac{r r^o}{r^o + |g|^2 r}.$$

*Proof.* Insert  $K^*$  from (1.10). Then

$$K^* g = \frac{r g^*}{D} g = \frac{|g|^2 r}{D} \in \mathbb{R}_{\geq 0}, \quad |K^*|^2 = \frac{r^2 |g|^2}{D^2}.$$

Hence

$$(1 - K^* g)(1 - K^* g)^* = (1 - K^* g)^2 = \left(1 - \frac{|g|^2 r}{D}\right)^2 = 1 - 2 \frac{|g|^2 r}{D} + \frac{|g|^4 r^2}{D^2}.$$

Using these identities in the Joseph form (1.8) gives

$$\begin{aligned} r_{m+1|m+1} &= \left(1 - 2 \frac{|g|^2 r}{D} + \frac{|g|^4 r^2}{D^2}\right) r + \frac{r^2 |g|^2}{D^2} r^o \\ &= r - 2 \frac{|g|^2 r^2}{D} + \frac{|g|^4 r^3}{D^2} + \frac{|g|^2 r^2 r^o}{D^2}. \end{aligned}$$

Factor the last two terms:

$$\frac{|g|^4 r^3}{D^2} + \frac{|g|^2 r^2 r^o}{D^2} = \frac{|g|^2 r^2}{D^2} (|g|^2 r + r^o) = \frac{|g|^2 r^2}{D^2} D = \frac{|g|^2 r^2}{D}.$$

Therefore

$$r_{m+1|m+1} = r - 2 \frac{|g|^2 r^2}{D} + \frac{|g|^2 r^2}{D} = r - \frac{|g|^2 r^2}{D} = r \left(1 - \frac{|g|^2 r}{D}\right) = r \frac{r^o}{D} = \frac{r r^o}{r^o + |g|^2 r}.$$

But with  $K^* g = \frac{|g|^2 r}{D}$ , the simplified form (1.9) gives

$$(1 - K^* g) r = \left(1 - \frac{|g|^2 r}{D}\right) r = \frac{r r^o}{D}.$$

Thus the two expressions are equal at  $K = K^*$ .  $\square$

**Corollary 1.2** (Equivalent characterizations). *The following are equivalent for the 1D complex scalar KF:*

1.  $K = K^*$  in (1.10) (Kalman gain solves the normal equation  $(|g|^2 r + r^o)K = g^* r$ ).
2. The posterior covariance equals the simplified form (1.9).
3. The Joseph form (1.8) reduces to the simplified value  $r r^o / (r^o + |g|^2 r)$ .

Aspect	Joseph form	Simplified form
Exact equality	Equals simplified only at $K = K^*$	Valid only at $K^*$
Inputs needed	Uses $r, r^o, g, K$	Uses $r, g, K$
PSD guarantee	Always $r_+ \geq 0$ for any $K$	Not guaranteed for arbitrary $K$
Numerical stability	More robust to roundoff/approx. $K$	More compact; may lose PSD
Closed form at $K^*$	$r_+ = \frac{r r^o}{r^o +  g ^2 r}$	Same value at $K^*$
Cost (1D)	Slightly higher (extra multiplies)	Minimal
Matrix generalization	$(I - KH)P(I - KH)^* + KRK^*$	$(I - KH)P$ (valid at $K^*$ )

Table 1: Posterior-variance updates in 1D complex-valued Kalman filtering. Here  $r_+ := r_{m+1|m+1}$ .

### Analysis formulas.

$$K_{m+1} = \frac{g r_{m+1|m}}{r^o + g^2 r_{m+1|m}},$$

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1}(v_{m+1} - g \bar{u}_{m+1|m}),$$

$$r_{m+1|m+1} = (1 - K_{m+1}g) r_{m+1|m}.$$

### 1.3 Algorithmic Summary (1D Complex Kalman Filter)

Combining two steps, the 1D Kalman filter algorithm is: Given  $(\bar{u}_{m|m}, r_{m|m})$ , the next step is estimated as:

$$\text{Forecast: } \bar{u}_{m+1|m} = F \bar{u}_{m|m}, \quad r_{m+1|m} = F r_{m|m} F^* + r.$$

$$\text{Analysis: } K_{m+1} = \frac{g r_{m+1|m}}{r^o + g^2 r_{m+1|m}},$$

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1}(v_{m+1} - g \bar{u}_{m+1|m}),$$

$$r_{m+1|m+1} = (1 - K_{m+1}g) r_{m+1|m}.$$

### 1.4 Remarks on Complex Setting

All updates are scalar and maintain the covariance real and nonnegative. Complex arithmetic enters through  $F \in \mathbb{C}$  and the noises; the observation operator  $g \in \mathbb{R}$  and the covariances  $r, r^o, r_{(.)}$  are real and nonnegative. The derivations above are unchanged from the real case after replacing transposes by conjugate-transposes and using the scalar identity  $|z|^2 = zz^*$  for  $z \in \mathbb{C}$ .

## 2 Complex Ornstein–Uhlenbeck (OU) process

We consider a complex Ornstein–Uhlenbeck (OU) process

$$du(t) = (-\gamma + i\omega)u(t) dt + \sigma dW(t), \quad \gamma, \sigma > 0, \omega \in \mathbb{R}, \quad (2.1)$$

with complex white noise

$$dW(t) = \frac{dW_1(t) + i dW_2(t)}{\sqrt{2}},$$

where  $W_j$  are independent Wiener processes; formally  $\dot{W}_j(t)$  are zero-mean white noises with

$$\langle \dot{W}_j(t) \rangle = 0, \quad \langle \dot{W}_j(t) \dot{W}_j(s) \rangle = \delta(t-s), \quad \langle \dot{W}_i(t) \dot{W}_j(s) \rangle = 0 \quad (i \neq j).$$

Time-discretizing this SDE (e.g. Euler–Maruyama) yields the standard 1D complex Kalman filter testbed.

### Derivation of the exact discrete parameters for the OU step

The discrete-time linear state model sampled on a uniform grid with observation interval  $\Delta t = t_{m+1} - t_m$  and its exact (in-time) parameters over one step are

$$u_{m+1} = F u_m + \sigma_{m+1}, \quad F = e^{(-\gamma+i\omega)\Delta t}, \quad \sigma_{m+1} \sim \mathcal{N}(0, r), \quad (2.2)$$

with complex Gaussian system noise variance

$$r = \frac{\sigma^2}{2\gamma} (1 - e^{-2\gamma\Delta t}). \quad (2.3)$$

The observation model is

$$v_{m+1} = g u_{m+1} + \sigma_{m+1}^o, \quad \sigma_{m+1}^o \sim \mathcal{N}(0, r^o), \quad (2.4)$$

with  $g \in \mathbb{R}$  here.

Variation-of-constants (integrating factor) solution Let  $t \geq s$  and define the integrating factor  $e^{-at}$ . By Itô's product rule,

$$d(e^{-at} u(t)) = e^{-at} du(t) - a e^{-at} u(t) dt = e^{-at} (au(t) dt + \sigma dW(t)) - a e^{-at} u(t) dt = \sigma e^{-at} dW(t).$$

Integrate from  $s$  to  $t$ :

$$e^{-at} u(t) - e^{-as} u(s) = \sigma \int_s^t e^{-a\tau} dW(\tau).$$

Hence the mild solution is

$$u(t) = e^{a(t-s)} u(s) + \sigma \int_s^t e^{a(t-\tau)} dW(\tau). \quad (2.5)$$

Thus, the mild solution on  $[t_m, t_{m+1}]$ :

$$u(t_{m+1}) = e^{(-\gamma+i\omega)\Delta t} u(t_m) + \sigma \int_{t_m}^{t_{m+1}} e^{(-\gamma+i\omega)(t_{m+1}-s)} dW(s). \quad (2.6)$$

Set  $F = e^{(-\gamma+i\omega)\Delta t}$  and define  $\sigma_{m+1}$  as the stochastic convolution term. By Itô isometry for complex noise,

$$\begin{aligned}\mathbb{E}[\sigma_{m+1} \sigma_{m+1}^*] &= \sigma^2 \int_{t_m}^{t_{m+1}} (e^{(-\gamma+i\omega)(t_{m+1}-s)})^2 ds = \sigma^2 \int_0^{\Delta t} e^{-2\gamma\tau} d\tau \\ &= \frac{\sigma^2}{2\gamma} (1 - e^{-2\gamma\Delta t}) =: r,\end{aligned}$$

which proves (2.2)–(2.3).

### Summary (discrete-time parameters over one step)

Sampling at interval  $\Delta t$  gives the exact one-step state-space model

$$u_{m+1} = F u_m + \sigma_{m+1}, \quad F := e^{a\Delta t} = e^{(-\gamma+i\omega)\Delta t}, \quad \sigma_{m+1} \sim \mathcal{N}(0, r),$$

with process-noise variance

$$r = \mathbb{E}[\sigma_{m+1} \sigma_{m+1}^*] = \sigma^2 \int_0^{\Delta t} e^{-2\gamma\tau} d\tau = \frac{\sigma^2}{2\gamma} (1 - e^{-2\gamma\Delta t}).$$

(2.7)

These formulas are the continuous-to-discrete exact map for the OU process and underlie the Kalman filter design in discrete time.

## 3 Numerical example and 1D Kalman filter algorithm

### Kalman filter (complex scalar) — forecast/analysis

Let  $(\bar{u}_{m|m}, r_{m|m})$  denote the analysis mean and variance at time  $t_m$ .

**Forecast:**  $\bar{u}_{m+1|m} = F \bar{u}_{m|m}, \quad r_{m+1|m} = |F|^2 r_{m|m} + r.$  (3.1)

**Gain:**  $K_{m+1} = \frac{r_{m+1|m} g^*}{r^o + |g|^2 r_{m+1|m}}.$  (3.2)

**Analysis:**  $\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1} (v_{m+1} - g \bar{u}_{m+1|m}).$  (3.3)

**Variance:**  $r_{m+1|m+1} = (1 - K_{m+1}g)(1 - K_{m+1}g)^* r_{m+1|m} + |K_{m+1}|^2 r^o = (1 - K_{m+1}g) r_{m+1|m}.$  (3.4)

### MATLAB prototype for the example

```
% 1D complex OU + complex Kalman filter
clear; rng(1);

% Parameters
gamma = 0.5; omega = 10; sigma = 1;
```

```

dt = 2; M = 400;
g = 1; ro = 0.25;

% Exact one-step model
F = exp((-gamma + 1i*omega)*dt);
r = (sigma^2/(2*gamma))*(1 - exp(-2*gamma*dt));

% Truth and observations
u = zeros(1,M+1); u(1) = 0;      % mean-zero start
v = zeros(1,M);
for m=1:M
    w = sqrt(r/2)*(randn + 1i*randn);    % CN(0,r)
    u(m+1) = F*u(m) + w;
    vo = sqrt(ro/2)*(randn + 1i*randn); % CN(0,ro)
    v(m) = g*u(m+1) + vo;
end

% KF initialization
ubar = 0; rbar = r;                  % prior guesses
uhat = zeros(1,M+1); uhat(1) = ubar;
RMS = zeros(1,M);

for m=1:M
    % Forecast
    ufor = F*ubar;
    rfor = abs(F)^2 * rbar + r;

    % Gain
    K = (rfor*conj(g)) / (ro + abs(g)^2 * rfor);

    % Analysis
    innov = v(m) - g*ufor;
    ubar = ufor + K*innov;
    rbar = (1 - K*g) * rfor;    % 1D simplified covariance update

    % Store
    uhat(m+1) = ubar;
    RMS(m) = abs(ubar - u(m+1));
end

fprintf('Temporal RMS error ~ %.3f\n', mean(RMS));

```

## References

- [1] Andrew J Majda and John Harlim, *Filtering complex turbulent systems*, Cambridge University Press, 2012.