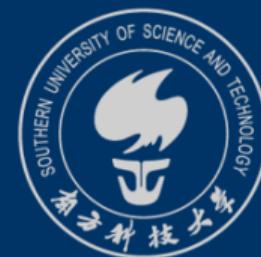# Introduction to Mathematical Logic

For CS Students

CS104

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We have:

$$\neg\forall x A(x) \equiv \exists x \neg A(x)$$
$$\neg\exists x A(x) \equiv \forall x \neg A(x)$$

Hence, $\forall$ is adequate for representing $\exists$, and $\exists$ can be considered as a shorthand for $\neg\forall\neg$.

Logic equivalence in Propositional Logic (PL) still hold after we substitute each occurrence of the same propositional variable by the same FOL formula. For example:

$$\forall x P(x) \equiv \neg\neg\forall x P(x)$$

$$\forall x P(x) \rightarrow \exists y Q(y) \equiv \neg\forall x P(x) \vee \exists y Q(y)$$

Quantifiers of the same type commute:

$$\forall x \forall y A(x, y) \equiv \forall y \forall x A(x, y)$$
$$\exists x \exists y A(x, y) \equiv \exists y \exists x A(x, y)$$

Universal quantifiers distribute over conjunction:

$$\forall x(A(x) \land B(x)) \equiv \forall x A(x) \land \forall x B(x)$$

Existential quantifiers distribute over disjunction:

$$\exists x(A(x) \lor B(x)) \equiv \exists x A(x) \lor \exists x B(x)$$

Example
1 Logical Equivalence

$$\exists x (A(x) \to B(x)) \equiv \exists x (\neg A(x) \lor B(x))$$
$$\equiv \exists x \neg A(x) \lor \exists x B(x)$$
$$\equiv \neg \exists x \neg A(x) \to \exists x B(x)$$
$$\equiv \forall x A(x) \to \exists x B(x).$$

Let $D = \{a_1, a_2, ..., a_n\}$, we may use the following logical equivalence to remove quantifiers:

$$\forall x P(x) \equiv P(a_1) \wedge ... \wedge P(a_n)$$

$$\exists x P(x) \equiv P(a_1) \vee ... \vee P(a_n)$$

Example
1 Logical Equivalence

Let $f^{(1)}$ be function symbols, $P^{(1)}$, $Q^{(2)}$ and $R^{(2)}$ be predicate symbols, a be constant symbols.

Define an interpretation $\mathcal{I}$ by

- Domain: $D = \{2, 3\}$
- Constant: $a^{\mathcal{I}} = 2$
- Functions: $f^{\mathcal{I}}(2) = 3, f^{\mathcal{I}}(3) = 2$
- Predicates: $P^{\mathcal{I}} = \{3\}$, $Q^{\mathcal{I}} = \{\langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle\}$, $R^{\mathcal{I}} = \{\langle 2, 2 \rangle, \langle 3, 3 \rangle\}$

The value of $\forall x(P(x) \land Q(x, a))$ can be given as:

$$(P(2) \land Q(2, 2)) \land (P(3) \land Q(3, 2)) \equiv (0 \land 1) \land (1 \land 1) \equiv 0$$

Satisfiability of a formula

2 Satisfiability

An interpretation $\mathcal{I}$ and environment E satisfy a formula α, denoted $\mathcal{I} \vDash_E \alpha$, iff $\alpha^{(\mathcal{I},E)} = 1$.
They do not satisfy α, denoted $\mathcal{I} \nvDash_E \alpha$, if $\alpha^{(\mathcal{I},E)} = 0$.

Form of α	Condition for $\mathcal{I} \vDash_E \alpha$
$P(t_1, \dots, t_k)$	$\left\langle t_1^{(\mathcal{I},E)}, \dots, t_k^{(\mathcal{I},E)} \right\rangle \in P^{\mathcal{I}}$
$(\neg\beta)$	$\mathcal{I} \nvDash_E \beta$
$(\beta \wedge \gamma)$	both $\mathcal{I} \vDash_E \beta$ and $\mathcal{I} \vDash_E \gamma$
$(\beta \vee \gamma)$	either $\mathcal{I} \vDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\beta \rightarrow \gamma)$	either $\mathcal{I} \nvDash_E \beta$ or $\mathcal{I} \vDash_E \gamma$ (or both)
$(\forall x\ \beta)$	for every $a \in dom(\mathcal{I})$, $\mathcal{I} \vDash_{E[x \mapsto a]} \beta$
$(\exists x\ \beta)$	there is some $a \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x \mapsto a]} \beta$

If $\mathcal{I} \vDash_E \alpha$ for every E, then $\mathcal{I}$ satisfies α, denoted $\mathcal{I} \vDash \alpha$.

Example 1
2 Satisfiability

Let $f^{(1)}$ and $h^{(2)}$ be function symbols, $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, a, b, c be constant symbols.

Define an interpretation $\mathcal{I}$ by

- Domain: $D = \{1, 2, 3\}$
- Constant: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}}(1) = 2, f^{\mathcal{I}}(2) = 3, f^{\mathcal{I}}(3) = 1, h^{\mathcal{I}} : (x, y) \mapsto min\{x, y\}$
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$, $Q^{\mathcal{I}} = \{\langle 1, 2 \rangle, \langle 3, 3 \rangle, \langle 3, 1 \rangle\}$

Define an environment E by

$$E(x) = 3, E(y) = 3, E(z) = 1$$

Example 1
2 Satisfiability

(continued from the previous slide)

We have:

- $\mathcal{I} \vDash_E P(h(f(a), z))$
- $\mathcal{I} \vDash_E Q(y, h(a, b))$

Give a new interpretation $\mathcal{J}$ and environment G such that

- $\mathcal{J} \nvDash_G P(h(f(a), z))$
- $\mathcal{J} \nvDash_G Q(y, h(a, b))$

Example 2
2 Satisfiability

Let L be a language consisting of variables x, y, z, function symbols $f^{(2)}, g^{(1)}$ and predicate symbol $P^{(2)}$.

Define an interpretation $\mathcal{I}$ by

- $dom(\mathcal{I}) : \mathbb{N}$
- $f^{\mathcal{I}}$: sum
- $g^{\mathcal{I}}$: square

Consider the formula $\alpha \stackrel{\text{def}}{=} f(g(x), g(y)) = g(z)$

- Find an environment E to satisfy α.
- Find an environment E such that α is not satisfied.

Example 3
2 Satisfiability

Let's define a language $L = \langle R^{(2)} \rangle$, and the following interpretations:

- $\mathcal{I}_1 = \langle \mathbb{N}, \{(n, m) : n < m\} \rangle$
- $\mathcal{I}_2 = \langle \mathbb{N}, \{(n, m) : n \text{ divides } m\} \rangle$
- $\mathcal{I}_3 = \langle \mathcal{P}(\mathbb{N}), \{(A, B) : A \subseteq B\} \rangle$

Questions:

- Find an environment E such that $\mathcal{I}_1 \vDash_E R(x, y)$ and $\mathcal{I}_2 \nvDash_E R(x, y)$
- Find a sentence α such that $\mathcal{I}_1 \nvDash \alpha$ and $\mathcal{I}_2 \vDash \alpha$
- Find a sentence α such that $\mathcal{I}_2 \vDash \alpha$ and $\mathcal{I}_3 \vDash \alpha$
- Find a sentence α such that $\mathcal{I}_1 \vDash \alpha$ and $\mathcal{I}_2 \vDash \alpha$ and $\mathcal{I}_3 \vDash \alpha$

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Let Σ be a set of FOL formulas.

For interpretation $\mathcal{I}$ and environment E, we write $\mathcal{I} \vDash_E \Sigma$ if and only if for every formula $\varphi \in \Sigma$, we have $\mathcal{I} \vDash_E \varphi$.

Here, $\vDash$ means "satisfy".

Definition

3 Semantic Entailment

Let Σ be a set of FOL formulas and α be a FOL formula. We say $\Sigma \vDash \alpha$ if and only if

For any interpretation $\mathcal{I}$ and environment E, if $\mathcal{I} \vDash_E \Sigma$ then $\mathcal{I} \vDash_E \alpha$ (or $\alpha^{(\mathcal{I}, E)} = 1$).

Interpretation: every pair of interpretation and environment that makes Σ true must also make α true.

Here, $\vDash$ means "semantically entail" or "logically imply".

Example 1
3 Semantic Entailment

Prove that $\forall x(\neg\gamma) \vDash \neg(\exists x\ \gamma)$

Proof: Suppose $\mathcal{I} \vDash_E \forall x(\neg\gamma)$. By definition, we have:

$$\text{For every } a \in dom(\mathcal{I}), \mathcal{I} \vDash_{E[x\mapsto a]} (\neg\gamma)$$

which is equivalent to

$$\text{For every } a \in dom(\mathcal{I}), \mathcal{I} \nvDash_{E[x\mapsto a]} \gamma$$

which means

$$\text{There is no } a \in dom(\mathcal{I}) \text{ such that } \mathcal{I} \vDash_{E[x\mapsto a]} \gamma$$

If $\mathcal{I} \vDash_E (\exists x\ \gamma)$, then there is a $b \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x\mapsto b]} \gamma$, contradiction.
Hence, $\mathcal{I} \vDash_E \neg(\exists x\ \gamma)$ holds as required.

Example 2
3 Semantic Entailment

Prove that:

$$(\forall x P(x)) \rightarrow (\forall x Q(x)) \not\models \forall x(P(x) \rightarrow Q(x))$$

Idea: All we need to do is to find an $\mathcal{I}$ (and E) such that: $\mathcal{I} \models_E (\forall x P(x)) \rightarrow (\forall x Q(x))$ and $\mathcal{I} \not\models_E \forall x(P(x) \rightarrow Q(x))$

A formula α is

- **valid**: if every interpretation and environment satisfy α; that is, if $\mathcal{I} \vDash_E \alpha$ for every $\mathcal{I}$ and E (analogous to "tautology" in PL).

- **satisfiable**: if some interpretation and environment satisfy α; that is, if $\mathcal{I} \vDash_E \alpha$ for some $\mathcal{I}$ and E.

- **unsatisfiable**: if no interpretation and environment satisfy α; that is, if $\mathcal{I} \nvDash_E \alpha$ for every $\mathcal{I}$ and E (analogous to "contradiction" in PL).

$\emptyset \vDash \alpha$ means that α is valid.

Example 1
3 Semantic Entailment

Prove that for any well-formed FOL formulas α and β:

$$\emptyset \vDash (\forall x(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))$$

Proof by contradiction: Suppose there are $\mathcal{I}$ and E such that:

$$\emptyset \nvDash (\forall x(\alpha \to \beta)) \to ((\forall x\ \alpha) \to (\forall x\ \beta))$$

Then we have $\mathcal{I} \vDash_E \forall x(\alpha \to \beta)$, $\mathcal{I} \vDash_E \forall x\ \alpha$, and $\mathcal{I} \nvDash_E \forall x\ \beta$.
By definition, for every $a \in dom(\mathcal{I})$, we have $\mathcal{I} \vDash_{E[x \mapsto a]} (\alpha \to \beta)$, $\mathcal{I} \vDash_{E[x \mapsto a]} \alpha$.
Hence, for every $a \in dom(\mathcal{I})$, we also have $\mathcal{I} \vDash_{E[x \mapsto a]} \beta$, which is $\mathcal{I} \vDash_E \forall x\ \beta$.
Contradiction.

Example 2
3 Semantic Entailment

Whether the following formulas are valid?

- $\exists y \forall x R(x, y) \rightarrow \forall x \exists y R(x, y)$
- $\forall x \exists y R(x, y) \rightarrow \exists y \forall x R(x, y)$
- $\exists x (P(x) \rightarrow \forall x P(x))$

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Given a formula φ in FOL, is φ valid, yes or no?

This problem is not solvable (i.e., we cannot write a correct C or Java program that works for all φ). In other words, first-order logic is not decidable in general.

Propositional logic is decidable, because the truth-table method can be used to determine whether an arbitrary propositional formula is logically valid.

- Text B: chapter 2.4.2
- Text F: chapter 7.3.2

Introduction to Mathematical Logic

Thank you for listening!
Any questions?