

Lecture #4

## ➤ HeapSort

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## ➤ Aims of this lecture

- To introduce the **HeapSort** algorithm.
- To show how a **clever data structure**, a **heap**, can lead to a **fast** and **in place** sorting algorithm
  - **In place:  $O(1)$  additional space.**
- To **practice the design and analysis of algorithms.**

Reading: Chapter 6

## ➤ Idea behind HeapSort

- Idea:
  - Find the largest element.
  - Move it to the end of the array (put another one in its place).
  - Repeat with remaining elements.
- Like SelectionSort but ...
  - SelectionSort compares lots of elements to find the largest.
  - **Can we store knowledge gained from these comparisons for the future?**
  - Use this knowledge to make future iterations faster!

## ➤ Use your imagination...

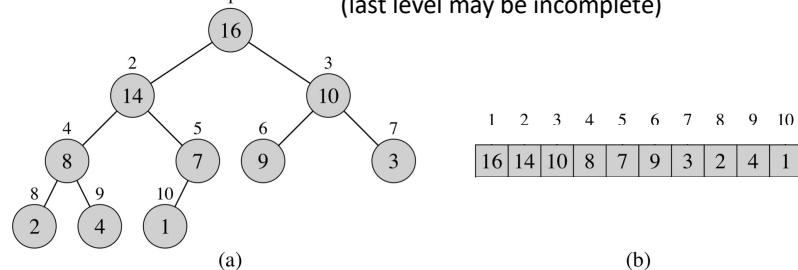


Photo : Thomas Bresson

## ➤ A Heap

- Essentially an array **imagined** as being a **binary tree**!
  - Elements are arranged row by row from left to right.  


(last level may be incomplete)



- Navigate through the array/imaginary tree using these operations:

- $\text{Parent}(i) = \left\lfloor \frac{i}{2} \right\rfloor$  (“floor of  $i/2$ ”),  $\text{Left}(i) = 2i$ ,  $\text{Right}(i) = 2i + 1$

数组的索引顺序就是节点的“排队”顺序。一个节点的子节点，只有在

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第*i*个节点的子节点：前面有*i*-1个节点作为父节点，生成 $2i-2$ 个子节点，加上最前面的第一个根节点，一共 $2i-1$ 个节点，后面就是第 $2i$ 和

## ► Procedures (what do we need)

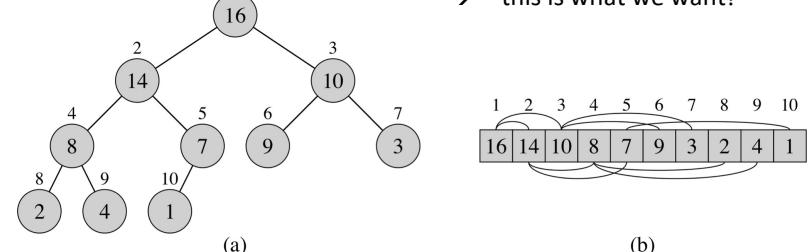
1. **Build-Max-Heap**: produces a Max-Heap from an unordered array
  2. **Max-Heapify**: maintains the max-heap property once the maximum has been removed
  3. **HeapSort**: sorts an array in place

- New variable  $A.\text{heap-size}$  indicates how many elements of  $A$  are stored in a heap:  $0 \leq A.\text{heap-size} \leq A.\text{length}$ .
    - Decreasing  $A.\text{heap-size}$  by 1 effectively removes the last element from the heap (we imagine a heap without it)
  - There are analogous operations for min-heaps:  
Min-Heapify and Build-Min-Heap.



## ➤ Heap Properties

- **Max-heap property:** for every node other than the root, the parent is no smaller than the node,  $A[\text{Parent}(i)] \geq A[i]$ .
  - In a max-heap, the **root** always stores a **largest** element.  
→ this is what we want!



- **Min-heap property:** for every node other than the root, the parent is no larger than the node,  $A[\text{Parent}(i)] \leq A[i]$ .

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## ➤ Procedures (what do we need)

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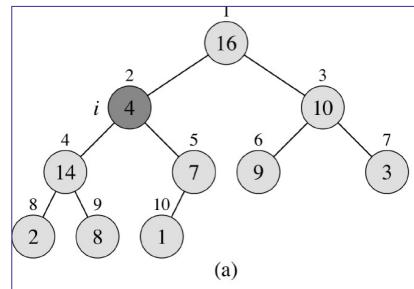
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注意这边，Max-Heapify是一个用来递归构造一个Max-Heap的方法，也就是from bottom to up建立。上面的1、2、3步并不是步骤，2 Max-Heapify也被包含在1中的build过程中！

## ➤ Max-Heapify( $A, i$ )

- Assumes subtrees Left( $i$ ) and Right( $i$ ) are max-heaps**, but max-heap property might be violated in root of subtree at  $i$ .
  - “Subtree x”: the part of the tree including x and everything below.
- Lets the value at  $A[i]$  “float down” if necessary, to restore max-heap property at  $i$
- At the end of Max-Heapify the subtree at  $i$  is a max-heap.

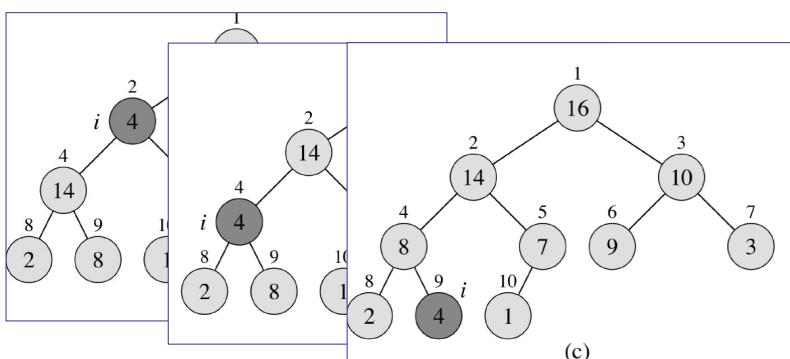


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## ➤ Max-Heapify: Example

- Compare  $A[i]$  with all existing children
- If **largest child** is larger than  $A[i]$ , swap and recurse on child



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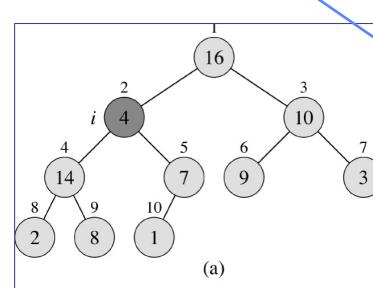
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注意： $h$ 实际上是与 $n$ 有关的数，所以并不能  $O(h)=O(1)$ 。

- $i$ 是一个叶子节点： $i=2i$ 会超过堆的边界，也就是没有子节点。
- 在堆排序过程中，堆的大小动态缩小时，比如 $i = \text{LEFT}(5) = 10 \leq 9$ ，会显示 False，这时候子节点是上一回合中最大的那个数，已不在当前回合的范围。

## ➤ Max-Heapify: informal and in pseudocode

- Compare  $A[i]$  with **all** existing children
- If **largest child** is larger than  $A[i]$ , swap and recurse on child



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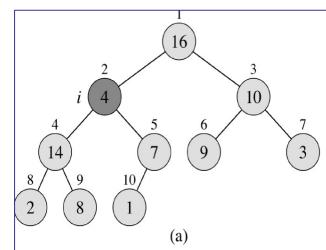
```
MAX-HEAPIFY( $A, i$ )
1:  $l = \text{Left}(i)$ 
2:  $r = \text{Right}(i)$ 
3: if  $l \leq A.\text{heap-size}$  and  $A[l] > A[i]$  then
4:   largest =  $l$ 
5: else
6:   largest =  $i$ 
7: if  $r \leq A.\text{heap-size}$  and  $A[r] > A[\text{largest}]$  then
8:   largest =  $r$ 
9: if largest  $\neq i$  then
10:  exchange  $A[i]$  with  $A[\text{largest}]$ 
11:  MAX-HEAPIFY( $A, \text{largest}$ )
```

现在的largest这个位置还是前面的那个位置，现在这个位置上的数字变成了之前 $i$ （左边或右边的一个子节点）上的数字。我们对这个子树再重复验证它的这个子节点所连带的subtree是不是还有上面的violation。

## ➤ Runtime of Max-Heapify

- Define the **height** of a node as the longest number of simple downward edges from the node to a leaf.
- Leaf**: a node without children.
- Max-Heapify takes constant time,  $\Theta(1)$ , on each level.
- Running time of Max-Heapify on a node of height  $h$  is  $O(h)$ .
- It's not  $\Omega(h)$  as Max-Heapify may stop early, e.g. if heap-property holds at  $i$ .
- For leaves  $h = 0$  and the time is  $O(1)$ .

```
MAX-HEAPIFY( $A, i$ )
1:  $l = \text{Left}(i)$ 
2:  $r = \text{Right}(i)$ 
3: if  $l \leq A.\text{heap-size}$  and  $A[l] > A[i]$  then
4:   largest =  $l$ 
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## ➤ Bounding the height of a heap

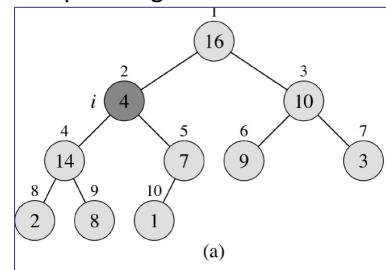
- **Claim:** the height of a heap = height of the root is at most  $\log n$ .

- **Proof:** the number  $n$  of elements in a heap of height  $h$  is

- Doubling on each level
  - At least 1 node on the last level
  - Hence in total at least

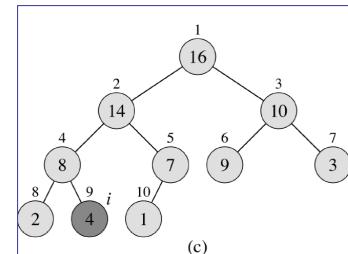
$$1 + 2 + 4 + \cdots + 2^{h-1} + 1 = 2^h$$

(we used  $\sum_{i=0}^{k-1} 2^i = 2^k - 1$ )



- So size and height are related as  $n \geq 2^h \Leftrightarrow \log n \geq h$
  - “the height of the root is at most  $\log n$ ”
  - So the runtime of Max-Heapify is  $O(\log n)$

## ► Max-Heapify: Correctness



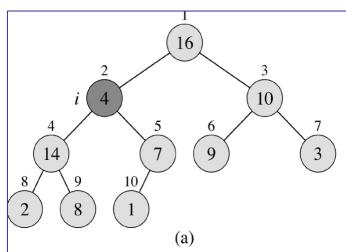
```

MAX-HEAPIFY( $A, i$ )
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```

- By induction (on the height):
  - Base case: height = 0 ( $i$  is a leaf)
  - Then left( $i$ ) and right( $i$ ) are larger than A.heap-size and the algorithm returns a heap!

## ➤ Max-Heapify: Correctness



- By induction (on the height):  
10: exchange  $A[i]$  with  $A[\text{largest}]$   
11: MAX-HEAPIFY( $A$ , largest)
  - **Inductive case:** assume it works for height  $h = i - 1$  and show it works for  $h = i$
  - Then the algorithm swaps  $A[i]$  with the larger between  $\text{Left}(i)$  and  $\text{Right}(i)$  (if any) and one subtree was already a heap and the other will be by inductive hypothesis.

## ► Procedures (what do we need)

1. **Build-Max-Heap**: produces a Max-Heap from an unordered array

- 2. Max-Heapify:** maintains the max-heap property once the maximum has been removed ✓

- 3. HeapSort:** sorts an array in place

## ➤ Building a Heap

- Idea: use Max-Heapify repeatedly to create a heap.
- Which order of nodes: top-down or bottom-up?
- Answer: **bottom-up** – Max-Heapify assumes Left( $i$ ) and Right( $i$ ) are heaps. Top-down wouldn't work, bottom-up does.
- Note: nodes in  $A \left[ \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right), \dots, n \right]$  are all leaves. Leaves are max-heaps, so no work required.

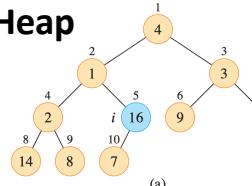
BUILD-MAX-HEAP( $A, n$ )

```

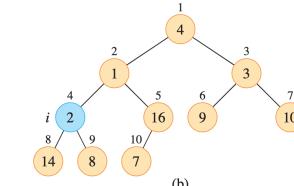
1  A.heap-size = n
2  for i = ⌊n/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

$A [4 | 1 | 3 | 2 | 16 | 9 | 10 | 14 | 8 | 7]$

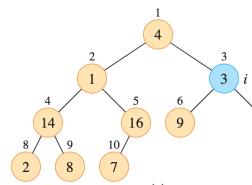
## ➤ Build-Max-Heap



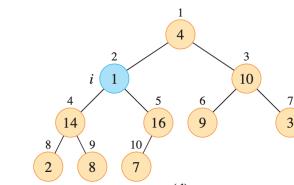
(a)



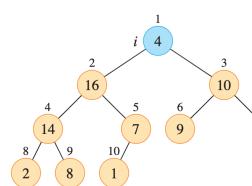
(b)



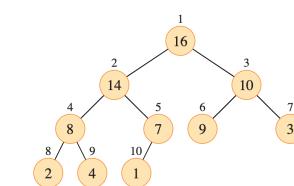
(c)



(d)



(e)



(f)

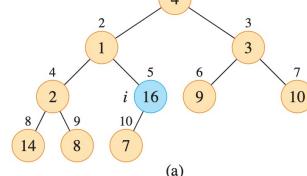
## ➤ Correctness of Build-Max-Heap

BUILD-MAX-HEAP( $A, n$ )

```

1  A.heap-size = n
2  for i = ⌊n/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

- Loop invariant:** At the start of each iteration  $i$  of the for loop, each node  $i + 1, i + 2, \dots, n$  is the root of a max-heap.
- Initialisation:** true for leaves  $\left\lfloor \frac{n}{2} \right\rfloor + 1, \dots, n$ .



(a)

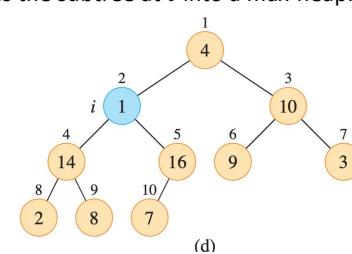
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- Loop invariant:** At the start of each iteration  $i$  of the for loop, each node  $i + 1, i + 2, \dots, n$  is the root of a max-heap.
- Maintenance:** by loop invariant, all children of  $i$  are roots of max-heaps (as their numbers are larger than  $i$ ).  
Then Max-Heapify( $A, i$ ) turns the subtree at  $i$  into a max-heap.



(d)

## ➤ Correctness of Build-Max-Heap

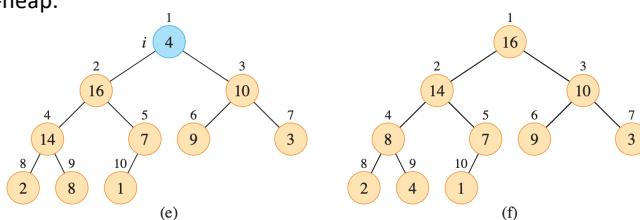
BUILD-MAX-HEAP( $A, n$ )

```

1    $A.heap\_size = n$ 
2   for  $i = \lfloor n/2 \rfloor$  downto 1
3       MAX-HEAPIFY( $A, i$ )

```

- Loop invariant:** At the start of each iteration  $i$  of the for loop, each node  $i + 1, i + 2, \dots, n$  is the root of a max-heap.
- Termination:** the loop terminates at  $i = 0$ , hence node 1 is the root of a max-heap.



## ➤ Runtime of Build-Max-Heap

- The **height of a heap** = height of the root is at most  $\log n$ .
- So all nodes have height at most  $\log n$ .
- Every call to Max-Heapify takes time  $O(\log n)$ .
- Build-Max-Heap calls Max-Heapify  $O(n)$  times.
- Total time is at most  $O(n) \cdot O(\log n) = O(n \log n)$ .
  - The time can be improved to  $O(n)$  since most nodes have small height.
  - $O(n \log n)$  is sufficient for us, though.

## ➤ Refined Analysis of Build-Max-Heap

- Observation: most nodes have small height**

One can show: there are at most  $\left\lceil \frac{n}{2^{h+1}} \right\rceil$  nodes of height  $h$ .

$O(\log n)$  time bound is correct, but crude for most nodes.

A better bound:

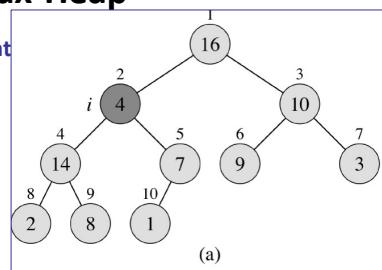
$$\sum_{h=1}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=1}^{\lfloor \log n \rfloor} \frac{h}{2^h}\right) = O\left(n \sum_{h=1}^{\infty} \frac{h}{2^h}\right) = O(n)$$

as the infinite series of  $\frac{h}{2^h}$  is 2.

- 1<sup>st</sup> equality, we used that:  $[x] \leq 2x$  for  $x \geq 1/2$

$\Rightarrow$  for  $h \leq \log n$ ,  $\frac{n}{2^{h+1}} \geq 1/2$  because  $n \geq 2^h$  (see slide 13)

- 2nd equality, we used that  $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$  for  $|x| < 1$



## ➤ Procedures (what do we need)

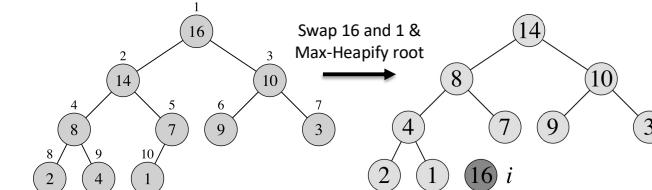
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**2. Max-Heapify:** maintains the max-heap property once the maximum has been removed ✓

**3. HeapSort:** sorts an array in place

## ➤ HeapSort

- Ideas:



- Build a max-heap, such that the root contains largest element.
- Swap the root with the last element of the heap/array.
- Discard the last element from the heap by reducing heap.size.  
(We simply imagine a smaller heap.)
- Call Max-Heapify( $A, 1$ ) to restore heap property at the root.

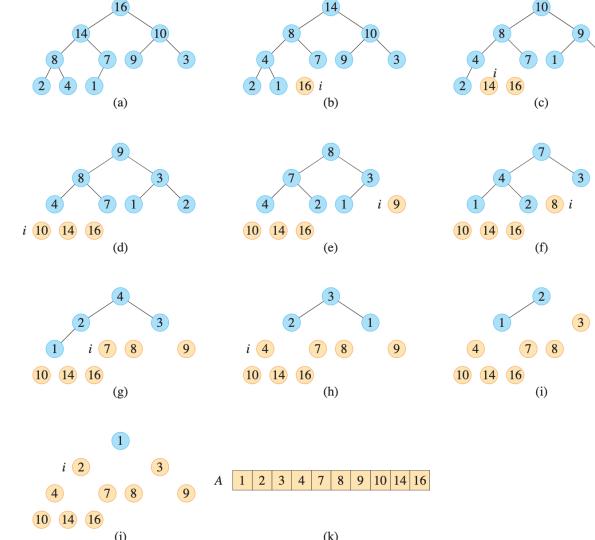
HEAPSORT( $A$ )

```

1: BUILD-MAX-HEAP( $A$ )
2: for  $i = A.length$  downto 2 do
3:   exchange  $A[1]$  with  $A[i]$ 
4:    $A.heap-size = A.heap-size - 1$ 
5:   MAX-HEAPIFY( $A, 1$ )

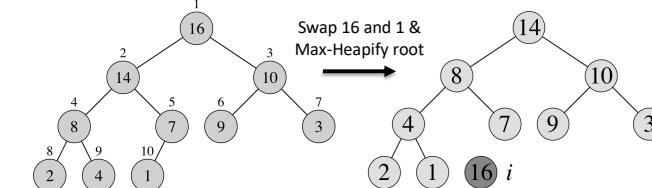
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## ➤ HeapSort: Example



## ➤ HeapSort

- Ideas:



- Build a max-heap, such that the root contains largest element.
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```

**Runtime:**

$$\begin{aligned}
 & O(n \log n) \\
 & + (n - 1) \cdot O(\log n) \\
 & = O(n \log n)
 \end{aligned}$$

## ➤ Correctness of HeapSort

**Loop Invariant:** “At the start of each iteration of the for loop of lines 2-5, the subarray  $A[1..i]$  is a max-heap containing the  $i$  smallest elements of  $A[1..n]$ , and the subarray  $A[i+1..n]$  contains the  $n-i$  largest elements of  $A[1..n]$ , sorted.”

- Initialization:** The subarray  $A[i+1..n]$  is empty, thus the invariant holds.

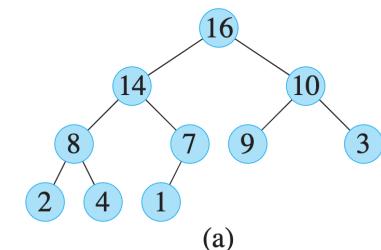
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HEAPSORT( $A$ )

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## ➤ Correctness of HeapSort

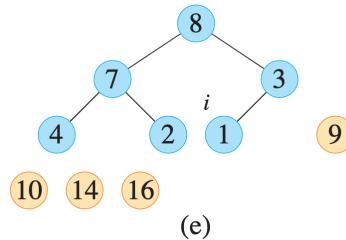
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**Maintenance:**  $A[1]$  is the largest element in  $A[1..i]$  and it is smaller than the elements in  $A[i+1..n]$ . When we put it in the  $i$ th position, then  $A[i..n]$  contains the largest elements, sorted. Decreasing the heap size and calling Max-Heapify turns  $A[1..i-1]$  into a max-heap. Decrementing  $i$  sets up the invariant for the next iteration.

---

```
HEAPSORT( $A$ )
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2: for  $i = A.length$  downto 2 do
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5:   MAX-HEAPIFY( $A, 1$ )
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## ➤ Correctness of HeapSort

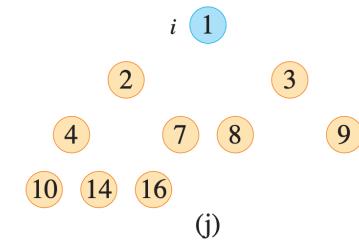
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- **Termination:** After the loop  $i=1$ . This means that  $A[2..n]$  is sorted and  $A[1]$  is the smallest element in the array, which makes the array sorted.

---

```
HEAPSORT( $A$ )
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## ➤ Summary

- Heapsort sorts in place in time  $O(n \log n)$ .
  - Building a Heap in time  $O(n)$ .
  - Extracting the largest element and restoring the heap-property in total time  $O(n \log n)$ .
- The use of appropriate **data structures** can speed up computation (in contrast to SelectionSort).
  - The heap “memorises” information about comparisons of elements.
  - The heap is imaginary, no objects/pointers required!
- Heaps also play a role in **Priority Queues**.

CSE217: Data Structures & Algorithm Analysis

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关键区别：重建 (Rebuild) vs. 修复 (Repair)

让我们来对比一下这两种思路：

### 1. 浪费时间的思路 (重建整个堆)

我们有一个  $n$  个元素的最大堆。取出最大值  $A[1]$ 。

现在我们剩下  $n-1$  个元素，它们是  $A[2], A[3], \dots, A[n]$ 。

(浪费时间的步骤) 把这  $n-1$  个元素当成一个全新的、无序的数组，然后调用 Build-Max-Heap 算法，花费  $O(n)$  的时间重新把它们组织成一个堆。

重复这个过程...

如果按照这个思路，每次取出一个元素都要花费  $O(n)$  的时间来重建堆，总时间复杂度会是  $O(n^2)$ ，这和低效的选择排序、冒泡排序就没什么区别了。

### 2. 高效的思路 (Heapsort的实际做法：仅修复)

Heapsort的实际做法是：在取出最大值后，堆的结构几乎是完好的，只有一个地方出了问题。我们只需要修复那个小问题就行了。

发生了什么？交换：我们将堆顶  $A[1]$ （最大值）和堆的最后一个元素  $A[\text{heap.size}]$  交换。缩小： $\text{heap.size}--$ 。

现在的局面是什么？根节点 (Root):  $A[1]$  现在存放的是原来那个很小的叶子节点的值。这导致最大堆的性质在根部被破坏了。 $A[1]$  很可能比它的子节点  $A[2]$  和  $A[3]$  要小。其他部分 (The Rest of the Tree): 根节点的两个子树（以  $A[2]$  和  $A[3]$  为根的子树）本身仍然是完美的最大堆！因为我们根本没有动它们内部的结构。

我们要做的工作：我们不需要重建一切。我们只需要把那个被安插在CEO位置上的“实习生”（原来那个小的值）一路“下沉”或“筛选”到它合适的位置就行了。这个操作就是 Max-Heapify。

Max-Heapify( $A, 1$ ) 的工作效率：

它从根节点  $i=1$  开始。

它将  $A[1]$  的值和它的子节点比较，然后和较大的那个子节点交换。

现在，那个小的值下沉了一层。它可能又破坏了下一层的堆性质。

所以 Max-Heapify 会递归地沿着一条路径继续往下走，直到这个元素不再小于它的子节点，或者它自己变成了叶子节点。

这个过程有多快？

这个“下沉”的路径最长能有多长？就是这个堆的高度。

一个包含  $k$  个元素的完全二叉树，它的高度大约是  $\log_2(k)$ 。

所以，每一次“修复”操作的时间复杂度仅仅是  $O(\log n)$ 。