Introduction to Data Assimilation Lecture Notes
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Abstract

This file prepares and documents the lecture 2 notes for the 2025 Fall YMSC public course: Introduction to Data Assimilation.

Keywords Data assimilation, Kalman filter

1 Kalman Filter: 1D Example [1]

Real-time prediction with inaccurate or intermittent measurements requires combining prior (model) forecasts with observations. In this chapter we derive the closed-form one-dimensional Kalman filter as the Bayesian analysis step for a simplified yet instructive setting and outline its use for a complex Ornstein–Uhlenbeck (OU) process. We also state stability considerations and the role of model/observation error covariances.

1.1 Overview and Notation

We work with complex-valued states and noises while the observation operator is real (as in the reference). Let $u_m \in \mathbb{C}$ denote the true state at time index $m \in \mathbb{Z}_{\geq 0}$ and let $v_m \in \mathbb{C}$ be the observation. We adopt the forecast/analysis notation

$$\bar{u}_{m|m} \in \mathbb{C} \quad \text{(posterior/analysis mean at time } m), \qquad \bar{u}_{m+1|m} \in \mathbb{C} \quad \text{(prior/forecast mean at } m+1),$$

with corresponding scalar error covariances

$$r_{m|m} = \mathbb{E}\Big[(u_m - \bar{u}_{m|m})(u_m - \bar{u}_{m|m})^*\Big] \in \mathbb{R}_{\geq 0}, \qquad r_{m+1|m} = \mathbb{E}\Big[(u_{m+1} - \bar{u}_{m+1|m})(u_{m+1} - \bar{u}_{m+1|m})^*\Big].$$

1.2 Stochastic Model

Forecast (state) model. The complex linear dynamics with complex Gaussian system noise is

$$u_{m+1} = F\, u_m + \sigma_{m+1}, \qquad F \in \mathbb{C}, \quad \sigma_{m+1} \sim \mathcal{N}(0, r), \;\; r = \mathbb{E}[\sigma_{m+1}\sigma_{m+1}^*] > 0. \tag{1.1}$$

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Herein, the system noise is understood as

$$\sigma_{m+1} \;=\; \frac{\sigma_{1,m+1} + i\,\sigma_{2,m+1}}{\sqrt{2}},$$

where $\sigma_{j,m+1} \in \mathbb{R}$ are independent, unbiased Gaussians with variances $\langle \sigma_{j,m+1}\sigma^*_{j,m+1}\rangle$ $(j = 1, 2)$, so that

$$r \;:= \mathbb{E}[\sigma_{m+1}\sigma^*_{m+1}] := \;\langle \sigma_{m+1}\sigma^*_{m+1}\rangle \;=\; \frac{1}{2}\sum_{j=1}^{2}\langle \sigma_{j,m+1}\sigma^*_{j,m+1}\rangle \;>\; 0. \tag{1.2}$$

Observation model. A linear real observation operator $g \in \mathbb{R}$ with complex Gaussian observation noise:

$$v_{m+1} = g\,u_{m+1} + \sigma^o_{m+1}, \qquad \sigma^o_{m+1} \sim \mathcal{N}(0, r^o),\;\; r^o = \mathbb{E}[\sigma^o_{m+1}(\sigma^o_{m+1})^*] > 0, \tag{1.3}$$

independent of σ_{m+1} and of past states/noises.

The likelihood at t_{m+1} is Gaussian:

$$p(v_{m+1} \mid u_{m+1}) \sim \mathcal{N}(g\,u_{m+1},\, r^o).$$

1.3 Complex Gaussian Basics (Scalar)

A scalar complex normal random variable $z \in \mathbb{C}$ with mean μ and (Hermitian) variance $s > 0$ is written as $z \sim \mathcal{N}(\mu, s)$ with density

$$p(z) = \frac{1}{\pi s}\exp\!\left(-\frac{|z-\mu|^2}{s}\right).$$

Independence of complex Gaussian variables is characterized by vanishing cross-covariances, e.g., $\mathbb{E}[\sigma_{m+1}(\sigma^o_{m+1})^*] = 0$.

1.4 Forecast (Prediction) Step

Let $e_{m|m} := u_m - \bar{u}_{m|m}$ and $e_{m+1|m} := u_{m+1} - \bar{u}_{m+1|m}$ and denote the prior/forecast mean and covariance by

$$\bar{u}_{m+1|m} := \mathbb{E}[u_{m+1} \mid v_{1:m}], \qquad r_{m+1|m} := \mathbb{E}\big[(u_{m+1} - \bar{u}_{m+1|m})(u_{m+1} - \bar{u}_{m+1|m})^*\big],$$

and the posterior/analysis mean and covariance by $\bar{u}_{m|m}$ and $r_{m|m}$ respectively. Propagating (1.1) gives the forecast formulas

$$\bar{u}_{m+1|m} \;=\; F\,\bar{u}_{m|m}, \qquad r_{m+1|m} \;=\; F\,r_{m|m}\,F^* + r. \tag{1.4}$$

Forecast (Prediction) covariance derivation. From the state model

$$u_{m+1} = F\,u_m + \sigma_{m+1}, \qquad \mathbb{E}[\sigma_{m+1}] = 0, \quad r := \mathbb{E}[\sigma_{m+1}\sigma_{m+1}^*] > 0,$$

and the forecast mean $\bar{u}_{m+1|m} = F\,\bar{u}_{m|m}$, define the prior error

$$e_{m+1|m} := u_{m+1} - \bar{u}_{m+1|m}.$$

Using $\bar{u}_{m+1|m} = F\,\bar{u}_{m|m}$ and $e_{m|m} := u_m - \bar{u}_{m|m}$, we have

$$e_{m+1|m} = (F\,u_m + \sigma_{m+1}) - F\,\bar{u}_{m|m} = F\,(u_m - \bar{u}_{m|m}) + \sigma_{m+1} = F\,e_{m|m} + \sigma_{m+1}.$$

Hence the prior covariance is

$$\begin{aligned} r_{m+1|m} &:= \mathbb{E}\Big[\, e_{m+1|m}\, e_{m+1|m}^* \,\Big] = \mathbb{E}\Big[\, \big(Fe_{m|m} + \sigma_{m+1}\big)\big(Fe_{m|m} + \sigma_{m+1}\big)^* \,\Big] \\ &= \mathbb{E}\Big[\, Fe_{m|m}e_{m|m}^*F^* \,\Big] + \mathbb{E}\Big[\, Fe_{m|m}\sigma_{m+1}^* \,\Big] + \mathbb{E}\Big[\, \sigma_{m+1}e_{m|m}^*F^* \,\Big] + \mathbb{E}\Big[\, \sigma_{m+1}\sigma_{m+1}^* \,\Big]. \end{aligned}$$

By independence of σ_{m+1} from $e_{m|m}$ and $\mathbb{E}[\sigma_{m+1}] = 0$, the cross terms vanish:

$$\mathbb{E}\big[Fe_{m|m}\sigma_{m+1}^*\big] = F\,\mathbb{E}[e_{m|m}]\,\mathbb{E}[\sigma_{m+1}^*] = 0, \qquad \mathbb{E}\big[\sigma_{m+1}e_{m|m}^*F^*\big] = \mathbb{E}[\sigma_{m+1}]\,\mathbb{E}[e_{m|m}^*]\,F^* = 0.$$

Therefore,

$$r_{m+1|m} = F\,\mathbb{E}\big[e_{m|m}e_{m|m}^*\big]\,F^* + \mathbb{E}\big[\sigma_{m+1}\sigma_{m+1}^*\big] = F\,r_{m|m}\,F^* + r,$$

which is the desired forecast (prior) covariance update.

1.5 Analysis (Filtering) Step: Two Derivations of the Kalman Gain

Form of the affine update. Given v_{m+1}, we update the mean with an affine rule

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1}\big(v_{m+1} - g\,\bar{u}_{m+1|m}\big), \qquad K_{m+1} \in \mathbb{R}, \tag{1.5}$$

and define the posterior covariance

$$r_{m+1|m+1} = \mathbb{E}\big[(u_{m+1} - \bar{u}_{m+1|m+1})(u_{m+1} - \bar{u}_{m+1|m+1})^*\big]. \tag{1.6}$$

Herein, K_{m+1} is called the Kalman gain, and $v_{m+1} - g\,\bar{u}_{m+1|m}$ is called the innovation.

1.5.1 (A) Bayesian/Conditional-Density Derivation via Minimizing $J(u)$

Bayes' rule (density form). Let u be the (latent) state and v the observation. Assume joint density $p(u,v)$ with marginals $p(u)$ and $p(v)$ and conditional densities $p(u \mid v) = \dfrac{p(u,v)}{p(v)}$ and $p(v \mid u) = \dfrac{p(u,v)}{p(u)}$. Then

$$p(u \mid v) = \frac{p(u,v)}{p(v)} = \frac{p(v \mid u)\,p(u)}{p(v)} \;\propto\; p(v \mid u)\,p(u),$$

where the proportionality constant

$$p(v) = \int p(v \mid u)\, p(u)\, du$$

depends only on v (the data) and not on u (the state). Thus the posterior density is proportional to the likelihood times the prior:

$$\boxed{p(u \mid v) = \frac{1}{Z(v)}\, p(v \mid u)\, p(u), \qquad Z(v) = \int p(v \mid u)\, p(u)\, du}.$$

Kalmain gain derivation. The likelihood and prior at time $m{+}1$ are Gaussian:

$$p(v_{m+1} \mid u_{m+1}) \propto \exp\!\left(-\frac{|v_{m+1} - g u_{m+1}|^2}{r^o}\right), \qquad p(u_{m+1}) \propto \exp\!\left(-\frac{|u_{m+1} - \bar{u}_{m+1|m}|^2}{r_{m+1|m}}\right).$$

Hence the posterior density is proportional to $\exp(-\frac{1}{2} J(u))$ with

$$J(u) = \frac{|u - \bar{u}_{m+1|m}|^2}{r_{m+1|m}} + \frac{|v_{m+1} - gu|^2}{r^o}.$$

Since J is real-valued and strictly convex in $u \in \mathbb{C}$, the minimizer satisfies the first-order optimality condition (using, e.g., Wirtinger calculus $\partial J / \partial u^* = 0$):

$$\frac{\partial J}{\partial u^*} = \frac{u - \bar{u}_{m+1|m}}{r_{m+1|m}} - \frac{g\,(v_{m+1} - gu)}{r^o} = 0.$$

Rearranging,

$$\Big(\frac{1}{r_{m+1|m}} + \frac{g^2}{r^o}\Big) u = \frac{\bar{u}_{m+1|m}}{r_{m+1|m}} + \frac{g\, v_{m+1}}{r^o}.$$

Solving for u and comparing with (1.5) yields the Kalman gain

$$\boxed{K_{m+1} = \frac{g\, r_{m+1|m}}{r^o + g^2 r_{m+1|m}} \in [0, 1/g]} \tag{1.7}$$

and the posterior mean update (1.5).

1.5.2 (B) Least-Squares (Minimum-Error-Variance) Derivation

We keep the notation: the prior (forecast) error at time $m{+}1$ is

$$e_{m+1|m} := u_{m+1} - \bar{u}_{m+1|m}, \qquad \mathbb{E}[e_{m+1|m}] = 0, \qquad r_{m+1|m} := \mathbb{E}\big[e_{m+1|m} e^*_{m+1|m}\big] \in \mathbb{R}_{\geq 0},$$

and the observation model is $v_{m+1} = g\, u_{m+1} + \sigma^o_{m+1}$ with $g \in \mathbb{C}$ (real g is a special case), independent, zero-mean complex Gaussian noise σ^o_{m+1}, and variance $r^o := \mathbb{E}\big[\sigma^o_{m+1} (\sigma^o_{m+1})^*\big] > 0$. Define the innovation

$$d_{m+1} := v_{m+1} - g\, \bar{u}_{m+1|m} = g\, e_{m+1|m} + \sigma^o_{m+1},$$

and consider the affine estimator

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1}\, d_{m+1}, \qquad K_{m+1} \in \mathbb{C}.$$

The posterior error is

$$e_{m+1|m+1} := u_{m+1} - \bar{u}_{m+1|m+1} = (1 - K_{m+1} g)\, e_{m+1|m} - K_{m+1}\, \sigma^o_{m+1}.$$

We minimize the mean-square error

$$\phi(K) := \mathbb{E}\big[\, |e_{m+1|m+1}|^2 \,\big] = \mathbb{E}\Big[\big((1-Kg)e - K\sigma^o\big)\big((1-Kg)e - K\sigma^o\big)^*\Big].$$

Independence and zero means imply $\mathbb{E}[e\,(\sigma^o)^*] = 0$, so the cross terms vanish and

$$\phi(K) = |1 - Kg|^2\, r_{m+1|m} + |K|^2\, r^o.$$

Treat K and K^* as independent (Wirtinger calculus) and set the stationary condition

$$\frac{\partial \phi}{\partial K^*} = 0.$$

Compute

$$\frac{\partial}{\partial K^*}\big(|1-Kg|^2\big) = \frac{\partial}{\partial K^*}\big((1-Kg)(1-Kg)^*\big) = (1-Kg)\,\frac{\partial}{\partial K^*}(1-Kg)^* = -(1-Kg)\, g^*,$$

and $\frac{\partial}{\partial K^*}(|K|^2) = \frac{\partial}{\partial K^*}(KK^*) = K$. Hence

$$\frac{\partial \phi}{\partial K^*} = r_{m+1|m}\,\big(-g^*(1-Kg)\big) + r^o\, K = -\,r_{m+1|m}\, g^* + r_{m+1|m}\, |g|^2 K + r^o K.$$

Setting this to zero yields

$$\big(|g|^2 r_{m+1|m} + r^o\big)\, K \;=\; r_{m+1|m}\, g^*,$$

and therefore the complex scalar Kalman gain

$$\boxed{K_{m+1} = \frac{r_{m+1|m}\, g^*}{r^o + |g|^2\, r_{m+1|m}} \in \mathbb{C}.}$$

If $g \in \mathbb{R}$, this reduces to $K_{m+1} = \dfrac{g\, r_{m+1|m}}{r^o + g^2 r_{m+1|m}}$.

Posterior covariance check (Joseph form). With the optimal K_{m+1},

$$r_{m+1|m+1} = \mathbb{E}\big[|e_{m+1|m+1}|^2\big] = (1-K_{m+1}g)(1-K_{m+1}g)^*\, r_{m+1|m} + |K_{m+1}|^2 r^o = \frac{r_{m+1|m}\, r^o}{r^o + |g|^2 r_{m+1|m}}.$$

Posterior Covariance Update (complex scalar) and Joseph Form

Let the prior (forecast) error and covariance at time $m{+}1$ be

$$e_{m+1|m} := u_{m+1} - \bar{u}_{m+1|m}, \qquad r_{m+1|m} := \mathbb{E}\big[e_{m+1|m}\, e^*_{m+1|m}\big] \in \mathbb{R}_{\geq 0}.$$

With the innovation $d_{m+1} = v_{m+1} - g\,\bar{u}_{m+1|m} = g\,e_{m+1|m} + \sigma^o_{m+1}$ and the affine analysis

$$\bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1}\, d_{m+1}, \qquad K_{m+1} \in \mathbb{C},$$

the posterior (analysis) error is

$$e_{m+1|m+1} := u_{m+1} - \bar{u}_{m+1|m+1} = (1 - K_{m+1} g)\, e_{m+1|m} - K_{m+1}\, \sigma^o_{m+1}.$$

Independence and zero means give $\mathbb{E}[\, e_{m+1|m}(\sigma^o_{m+1})^* \,] = 0$. Thus

$$\begin{aligned}
r_{m+1|m+1} &:= \mathbb{E}\big[e_{m+1|m+1}\, e^*_{m+1|m+1}\big] \\
&= \mathbb{E}\Big[\big((1-Kg)e - K\sigma^o\big)\big((1-Kg)e - K\sigma^o\big)^*\Big] \\
&= (1 - K_{m+1} g)\, r_{m+1|m}\, (1 - K_{m+1} g)^* + K_{m+1} r^o K^*_{m+1} \qquad (1.8) \\
&= (1 - Kg)(1 - Kg)^*\, r_{m+1|m} \; + \; |K|^2\, r^o \qquad \text{(assuming real)} \\
&= (1 - K_{m+1} g)\, r_{m+1|m} \qquad \text{(after substituting } K_{m+1} \text{ from (1.7))}
\end{aligned}$$

Equation (1.8) is the *Joseph form* in the complex scalar case; it holds for any (possibly suboptimal) K and guarantees $r_{m+1|m+1} \geq 0$.

Equivalence of the Joseph Form and the Simplified Posterior-Covariance Update (1D KF, Complex Scalar)

Setup and notation. Let $u \in \mathbb{C}$ be the state, $v \in \mathbb{C}$ the observation, $g \in \mathbb{C}$ the (scalar) observation operator, and let $r := r_{m+1|m} \in \mathbb{R}_{\geq 0}$ be the prior (forecast) variance at time $m{+}1$. The observation-noise variance is $r^o > 0$. For a complex scalar gain $K \in \mathbb{C}$, the *Joseph form* of the posterior variance is

$$r_{m+1|m+1} = (1 - Kg)(1 - Kg)^*\, r + |K|^2\, r^o. \qquad (1.9)$$

The *simplified form* is

$$r_{m+1|m+1} = (1 - Kg)\, r, \qquad (1.10)$$

which is valid *at the optimal Kalman gain.* The complex scalar Kalman gain is

$$K^\star = \frac{r\, g^*}{r^o + |g|^2 r}, \qquad D := r^o + |g|^2 r > 0. \qquad (1.11)$$

Theorem 1.1 (Equivalence at the optimal gain)**.** *For the one–dimensional complex scalar Kalman filter, if $K = K^\star$ in* (1.11)*, then the Joseph form* (1.9) *and the simplified form* (1.10) *coincide:*

$$(1 - K^\star g)(1 - K^\star g)^*\, r + |K^\star|^2 r^o \;=\; (1 - K^\star g)\, r \;=\; \frac{r\, r^o}{r^o + |g|^2 r}.$$

Proof. Insert $K^\star$ from (1.11). Then

$$K^\star g = \frac{r\,g^*}{D}\,g = \frac{|g|^2 r}{D} \in \mathbb{R}_{\geq 0}, \qquad |K^\star|^2 = \frac{r^2|g|^2}{D^2}.$$

Hence

$$(1 - K^\star g)(1 - K^\star g)^* = (1 - K^\star g)^2 = \Big(1 - \frac{|g|^2 r}{D}\Big)^2 = 1 - 2\frac{|g|^2 r}{D} + \frac{|g|^4 r^2}{D^2}.$$

Using these identities in the Joseph form (1.9) gives

$$\begin{aligned} r_{m+1|m+1} &= \Big(1 - 2\frac{|g|^2 r}{D} + \frac{|g|^4 r^2}{D^2}\Big) r \;+\; \frac{r^2|g|^2}{D^2}\, r^o \\ &= r - 2\,\frac{|g|^2 r^2}{D} + \frac{|g|^4 r^3}{D^2} + \frac{|g|^2 r^2 r^o}{D^2}. \end{aligned}$$

Factor the last two terms:

$$\frac{|g|^4 r^3}{D^2} + \frac{|g|^2 r^2 r^o}{D^2} = \frac{|g|^2 r^2}{D^2}\,(|g|^2 r + r^o) = \frac{|g|^2 r^2}{D^2}\,D = \frac{|g|^2 r^2}{D}.$$

Therefore

$$r_{m+1|m+1} = r - 2\,\frac{|g|^2 r^2}{D} + \frac{|g|^2 r^2}{D} = r - \frac{|g|^2 r^2}{D} = r\Big(1 - \frac{|g|^2 r}{D}\Big) = r\,\frac{r^o}{D} = \frac{r\,r^o}{r^o + |g|^2 r}.$$

But with $K^\star g = \frac{|g|^2 r}{D}$, the simplified form (1.10) gives

$$(1 - K^\star g)\,r = \Big(1 - \frac{|g|^2 r}{D}\Big) r = \frac{r\,r^o}{D}.$$

Thus the two expressions are equal at $K = K^\star$. □

Corollary 1.2 (Equivalent characterizations)**.** *The following are equivalent for the 1D complex scalar KF:*

1. $K = K^\star$ *in* (1.11) *(Kalman gain solves the normal equation* $(|g|^2 r + r^o)K = g^* r$*).*
2. *The posterior covariance equals the simplified form* (1.10)*.*
3. *The Joseph form* (1.9) *reduces to the simplified value* $r\,r^o/(r^o + |g|^2 r)$*.*

Aspect	Joseph form	Simplified form
Exact equality	Equals simplified only at $K = K^\star$	Valid only at $K^\star$
Inputs needed	Uses r, r^o, g, K	Uses r, g, K
PSD guarantee	Always $r_+ \geq 0$ for any K	Not guaranteed for arbitrary K
Numerical stability	More robust to roundoff/approx. K	More compact; may lose PSD
Closed form at $K^\star$	$r_+ = \dfrac{r\, r^o}{r^o + \lvert g\rvert^2 r}$	Same value at $K^\star$
Cost (1D)	Slightly higher (extra multiplies)	Minimal
Matrix generalization	$(I - KH)P(I - KH)^* + KRK^*$	$(I - KH)P$ (valid at $K^\star$)

Table 1: Posterior-variance updates in 1D complex-valued Kalman filtering. Here $r_+ := r_{m+1|m+1}$.

Closed form with the optimal Kalman gain. For the complex scalar setting, the optimal gain is

$$K_{m+1} = \frac{r_{m+1|m}\, g^*}{r^o + |g|^2 r_{m+1|m}}.$$

Let $D := r^o + |g|^2 r_{m+1|m}$. Then

$$Kg = \frac{r_{m+1|m}\,|g|^2}{D} \in \mathbb{R}_{\geq 0}, \qquad |K|^2 = \frac{r_{m+1|m}^2\,|g|^2}{D^2}.$$

Insert these into (1.8):

$$\begin{aligned} r_{m+1|m+1} &= \Big(1 - \tfrac{r_{m+1|m}|g|^2}{D}\Big)^2 r_{m+1|m} + \tfrac{r_{m+1|m}^2|g|^2}{D^2}\, r^o \\ &= r_{m+1|m} - 2\,\tfrac{r_{m+1|m}^2|g|^2}{D} + \tfrac{r_{m+1|m}^3|g|^4}{D^2} + \tfrac{r_{m+1|m}^2|g|^2 r^o}{D^2} \\ &= r_{m+1|m} - \tfrac{r_{m+1|m}^2|g|^2}{D} \quad = \quad \frac{r_{m+1|m}\, r^o}{r^o + |g|^2 r_{m+1|m}}. \end{aligned}$$

Equivalently, since $1 - Kg = \dfrac{r^o}{D}$,

$$\boxed{r_{m+1|m+1} = (1 - K_{m+1} g)\, r_{m+1|m} = \frac{r_{m+1|m}\, r^o}{r^o + |g|^2 r_{m+1|m}}\,.}$$

The first equality highlights the contraction of the prior covariance by the factor $(1-Kg)$; the second gives the explicit posterior variance in terms of $r_{m+1|m}$, r^o, and g.

1.6 Algorithmic Summary (1D Complex Kalman Filter)

Given $(\bar{u}_{m|m}, r_{m|m})$:

$$
\begin{aligned}
\text{Forecast:}\quad & \bar{u}_{m+1|m} = F\,\bar{u}_{m|m}, \qquad r_{m+1|m} = F r_{m|m} F^* + r. \\
\text{Analysis:}\quad & K_{m+1} = \frac{g\,r_{m+1|m}}{r^o + g^2 r_{m+1|m}}, \\
& \bar{u}_{m+1|m+1} = \bar{u}_{m+1|m} + K_{m+1}\big(v_{m+1} - g\,\bar{u}_{m+1|m}\big), \\
& r_{m+1|m+1} = (1 - K_{m+1} g)\, r_{m+1|m}.
\end{aligned}
$$

1.7 Remarks on Complex Setting

All updates are scalar and maintain the covariance real and nonnegative. Complex arithmetic enters through $F \in \mathbb{C}$ and the noises; the observation operator $g \in \mathbb{R}$ and the covariances $r, r^o, r_{(\cdot)}$ are real and nonnegative. The derivations above are unchanged from the real case after replacing transposes by conjugate-transposes and using the scalar identity $|z|^2 = zz^*$ for $z \in \mathbb{C}$.

2 Complex OU example (for simulation)

Consider the complex OU SDE

$$
\mathrm{d}u(t) = (-\gamma + i\omega)u(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \qquad \gamma, \sigma > 0,\ \omega \in \mathbb{R}, \tag{2.1}
$$

with complex white noise

$$
\mathrm{d}W(t) = \frac{\mathrm{d}W_1(t) + i\,\mathrm{d}W_2(t)}{\sqrt{2}},
$$

where W_j are independent Wiener processes; formally $\dot{W}_j(t)$ are zero-mean white noises with

$$
\langle \dot{W}_j(t)\rangle = 0, \qquad \langle \dot{W}_j(t)\dot{W}_j(s)\rangle = \delta(t-s), \qquad \langle \dot{W}_i(t)\dot{W}_j(s)\rangle = 0\ \ (i \neq j).
$$

Time-discretizing this SDE (e.g. Euler–Maruyama) and using (1.4)–(**??**) yields the standard 1D complex Kalman filter testbed.

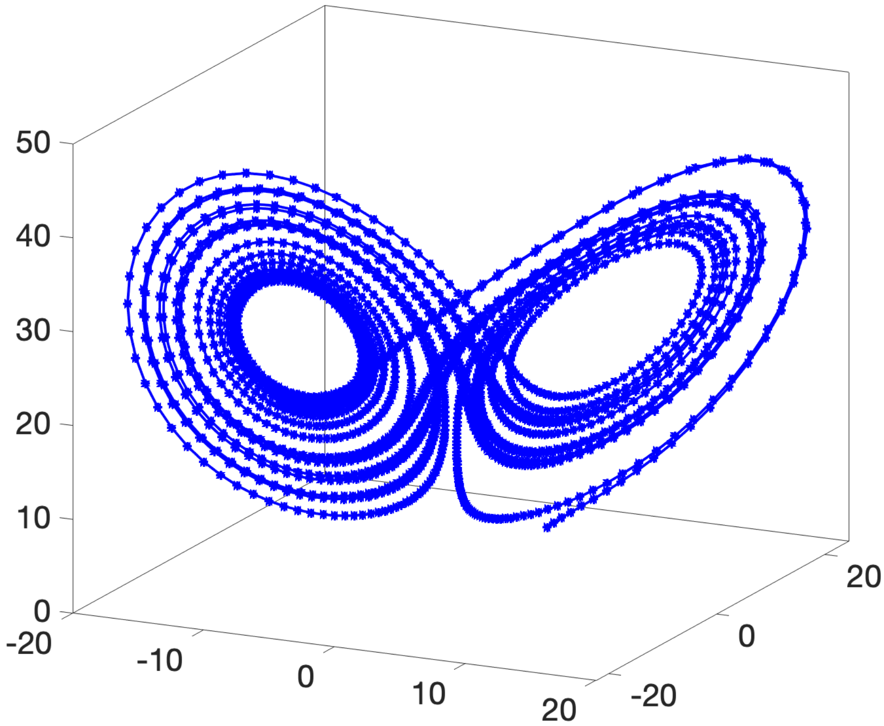
Remarks.

- In the scalar complex setting, all covariances $(r, r^o, r_\cdot)$ are real and nonnegative; conjugation appears only in inner-product forms $|z|^2 = zz^*$.
- The analysis step is optimal in the least-squares sense under linear/Gaussian assumptions and coincides with the Bayesian posterior mean.

The Lorenz-63 system is a classic chaotic model defined by:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Typical parameters:
$\sigma = 10,\ \rho = 28,\ \beta = 8/3$



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