Introduction to Mathematical Logic

For CS Students

CS104

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1 Warm up

Semantics of FOL formulas

1 Warm up

Consider the following formula of FOL. What's its semantics? Can we decide whether it is true or false?

- $y = x + 1$
- $\forall y \exists x (y = x + 1)$
- $c_1 = c_2 + 1$
- $\exists x P(x) \wedge Q(f(a), a)$

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Scope

Definition: Scope (辖域, 量词的作用范围或约束范围)

In a formula $\forall x\ \alpha$ or $\exists x\ \alpha$, x is the *quantified variable* and its scope is the formula α.

The formula α is the scope of the quantifier $\forall x$ ($\exists x$), if $\forall x$ ($\exists x$) is adjacent to α, or, α immediately follows $\forall x$ ($\exists x$).

Identify the scope of each quantifier in the following formulas.

- $\forall x P(x) \land Q(x)$
- $\forall x (P(x) \land Q(x))$
- $\exists y (P(y) \land \forall x Q(x)) \land R(y)$

Free and Bound Variables

Definition

- Free variables (自由变元): an occurrence of a variable x in a formula is *free* iff x is not within the scope of a quantified variable x.
- Bound variables (约束变元): otherwise, the occurrence of this variable is *bound*, i.e., the occurrence of this variable lies in the scope of some quantifier of the same variable.

Note: the variable symbol immediately after $\exists$ or $\forall$ is neither free nor bound.

Examples

In the following formulas, which occurrences are free/bound variables?

- $P(x, y)$
- $\exists y P(x, y)$
- $\forall x \exists y P(x, y)$
- $\forall x P(x) \wedge Q(x)$

We can also use parse trees to understand free and bound variables.

Definition

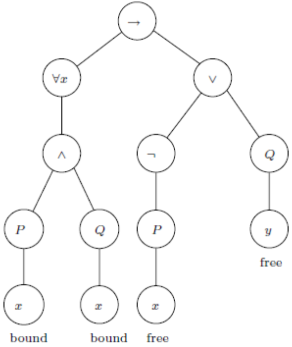
Let φ be a formula in FOL. An occurrence of x in φ is free in φ if it is a leaf node in the parse tree of φ such that there is no path upwards from that node x to a node $\forall x$ or $\exists x$. Otherwise, that occurrence of x is called bound.

In terms of parse trees, the scope of a quantifier is just its subtree, minus any subtree which re-introduce a quantifier for x.

Free and Bound Variables

2 The Scope of Variables, Free and Bound Variables

It is quite possible, and common, that a variable is bound and free in a formula. However, **individual occurrences** of variables are either free or bound, never both at the same time. The parse tree for $\forall x(P(x) \wedge Q(x)) \rightarrow (\neg P(x) \vee Q(y))$.



Sentence

Definition

A formula with no free variables is called a *closed formula* (闭公式), or a *sentence*.

Examples:

- $\exists x \forall y P(x, y) \vee \forall x \exists y Q(x, y)$ is a sentence of FOL.
- $\exists x P(x, y) \vee \forall x Q(x, y)$ is a formula but not a sentence (y is free)

Sentence

The presence of free variables distinguishes formulas from sentences.

It does not make sense to ask whether the formula $y = x + 1$ is true or not. But we can ask whether $\forall y \exists x(y = x + 1)$ or $c_1 = c_2 + 1$ is true or not.

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3 Interpretation

Motivating Example

Consider the following formula of FOL. What's its semantics? Is it true or false?

$$\exists x P(x) \land Q(f(a), a)$$

Consider the following formula of FOL. What's its semantics? Is it true or false?

$$\exists x P(x) \wedge Q(f(a), a)$$

The formula is *true* if we have the following interpretations:

- Domain D: the set of human beings
- a: Meng Zi (孟子)
- $P(x)$: x loves to play
- $Q(x, y)$: x educates y
- $f(x)$: the mother of x

Motivating Example

Consider the following formula of FOL. What's its semantics? Is it true or false?

$$\exists x P(x) \wedge Q(f(a), a)$$

The formula is *false* if we have the following interpretations:

- Domain D: the set of natural numbers
- a: 0
- $P(x)$: $x > 0$
- $Q(x, y)$: $x \leq y$
- $f(x)$: $x + 1$

Meanings of formulas

To assign meanings to formulas of FOL, we need:

- A domain D
- An interpretation of *non-logical symbols*: mapping constants, predicates, and function symbols to specific individuals, properties or relations and functions in the domain D.
- An interpretation of *logical symbols*, which is *fixed*:
  - Logical connectives, punctuations (same as PL)
  - Quantifiers, variable symbols

After interpretation, **terms** in FOL represent **individuals**(个体) in the domain, while **formulas** represent **propositions** with fixed truth values.

Definition

3 Interpretation

An interpretation $\mathcal{I}$ (or structure) consists of:

- A non-empty set D, called the domain (or universe) of $\mathcal{I}$.
- For each constant symbol c, a member $c^{\mathcal{I}}$ of D.
- For each function symbol $f^{(i)}$, an i-ary function $f^{\mathcal{I}}$.
- For each predicate symbol $P^{(i)}$, an i-ary predicate (relation) $P^{\mathcal{I}}$.

The value of a term is always a member of the domain of $\mathcal{I}$.

Consider the term:

$$f(g(a), f(b, c))$$

If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, while symbols a, b, and c are interpreted as $4, 5$, and 6 respectively. The binary function symbol f and unary function symbol g are interpreted as “addition” and “square” respectively. Then the term is interpreted as:

$$4^2 + (5 + 6)$$

which evaluates to 27, a member of $\mathbb{N}$.

Examples

Formulas get values in much the same fashion as terms, except that the values of formulas lie in $\{0, 1\}$.

Consider the formula:

$$f(g(a), g(c)) = g(b)$$

If we have an interpretation $\mathcal{I}$ with domain $\mathbb{N}$, while symbols a, b, and c are interpreted as 4, 5, and 6 respectively. The binary function symbol f and unary function symbol g are interpreted as "addition" and "square" respectively. Then the formula is interpreted as:

$$4^2 + 6^2 = 5^2$$

which is a false proposition.

Let $f^{(1)}$ and $h^{(2)}$ be function symbols, $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, a, b, c be constant symbols.
Define an interpretation $\mathcal{I}$ by

- Domain: $D = \{1, 2, 3\}$
- Constant: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}}(1) = 2, f^{\mathcal{I}}(2) = 3, f^{\mathcal{I}}(3) = 1, h^{\mathcal{I}} : (x, y) \mapsto min\{x, y\}$
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$, $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

(continued from previous slide)

What is the meaning of each of these formulas in this interpretation?

- $f(h(b, f(a)))^{\mathcal{I}}$
- $Q(f(c), a)^{\mathcal{I}}$
- $P(h(f(a), f(c)))^{\mathcal{I}}$

Is there another interpretation $\mathcal{J}$ such that:

- $Q(f(c), a)^{\mathcal{J}}$ is true?
- $P(h(f(a), f(c)))^{\mathcal{J}}$ is false?

Let α_1 be $P(c)$ (where c is a constant), α_2 be $P(x)$ (where x is a variable).

Let $\mathcal{I}$ be the interpretation with domain $\mathbb{N}$, $c^{\mathcal{I}} = 2$, $P^{\mathcal{I}}$ be "is even".
Then $\alpha_1^{\mathcal{I}} = 1$, but $\alpha_2^{\mathcal{I}}$ is undefined.

To give α_2 a value, we must also specify an environment E.
For example, if $E(x) = 2$, then $\alpha_2^{(\mathcal{I},E)} = 1$

Environment

An environment is a function (or, a look-up table) that assigns a value in the domain to every **variable** symbol in the language.

This concept is similar to the *truth valuation* (e.g., $\alpha^v = 1$) in propositional logic.

The combination of an interpretation and an environment supplies a value for every term, atomic formula, and formula.

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The value of terms

Definition: Fix an interpretation $\mathcal{I}$ and an environment E. For each term t, the value of t under $\mathcal{I}$ and E, denoted $t^{(\mathcal{I},E)}$, is defined as follows:

- If t is a constant c, the value $t^{(\mathcal{I},E)}$ is $c^{\mathcal{I}}$.
- If t is a variable x, the value $t^{(\mathcal{I},E)}$ is x^{E}.
- If t is $f(t_1, \ldots, t_n)$, the value $t^{(\mathcal{I},E)}$ is $f^{\mathcal{I}}(t_1^{(\mathcal{I},E)}, \ldots, t_n^{(\mathcal{I},E)})$.

Example

Suppose a language has constant symbol 0, a unary function s and a binary function $+$. An example of interpretations and environments:

- $\mathcal{I}$: $D = \mathbb{N}$, $0^{\mathcal{I}} = 0$, $s^{\mathcal{I}}$ is the successor function and $+^{\mathcal{I}}$ is the addition operation.
- E: $E(x) = 3$

What are the values of the following terms?

- $s((s(0) + s(x)))^{(\mathcal{I},E)} = ?$
- $s((x + s((x + s(0)))))^{(\mathcal{I},E)} = ?$

Once we fix the truth value of terms, the truth value of every atomic formula can be determined.

Definition: Fix an interpretation $\mathcal{I}$ and an environment E. The truth value of an atomic formula in the form $P(t_1, ..., t_n)$ (where P is an n-ary predicate symbol), denoted $P(t_1, ..., t_n)^{(\mathcal{I},E)}$, is $P^{\mathcal{I}}(t_1^{(\mathcal{I},E)}, ..., t_n^{(\mathcal{I},E)})$.

The value of atomic formulas

Example

Let $\mathrm{dom}(\mathcal{I}) = \{a, b\}$, $P^{\mathcal{I}} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$, $E(x) = a$ and $E(y) = b$.

Determine the value/semantics/meanings of the following formulas.

- $P(x, x)^{(\mathcal{I},E)} = 1$, since $\langle E(x), E(x)\rangle = \langle a, a\rangle \in P^{\mathcal{I}}$.
- $P(y, x)^{(\mathcal{I},E)} = 0$, since $\langle E(y), E(x)\rangle = \langle b, a\rangle \notin P^{\mathcal{I}}$.

The value of well-formed formulas

Definition

Fix an interpretation $\mathcal{I}$ and an environment E. The truth value of a well-formed formula φ can be defined recursively as follows:

(i) $$P(t_1, \ldots, t_n)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \mathrm{P}^{\mathcal{I}}(t_1^{(\mathcal{I},E)}, \ldots, t_n^{(\mathcal{I},E)}) = 1 \\ 0 & \text{else} \end{cases}$$

$$(t_1 = t_2)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \mathrm{t}_1^{(\mathcal{I},E)} = t_2^{(\mathcal{I},E)} \\ 0 & \text{else} \end{cases}$$

(ii) $$(\neg\alpha)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \alpha^{(\mathcal{I},E)} = 0 \\ 0 & \text{else} \end{cases}$$

Definition (continued)

(iii) $(\alpha \wedge \beta)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \alpha^{(\mathcal{I},E)} = \beta^{(\mathcal{I},E)} = 1 \\ 0 & \text{else} \end{cases}$

(iv) $(\alpha \vee \beta)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \alpha^{(\mathcal{I},E)} = 1 \text{ or } \beta^{(\mathcal{I},E)} = 1 \\ 0 & \text{else} \end{cases}$

(v) $(\alpha \to \beta)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \alpha^{(\mathcal{I},E)} = 0 \text{ or } \beta^{(\mathcal{I},E)} = 1 \\ 0 & \text{else} \end{cases}$

(vi) $(\alpha \leftrightarrow \beta)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \alpha^{(\mathcal{I},E)} = \beta^{(\mathcal{I},E)} \\ 0 & \text{else} \end{cases}$

The value of well-formed formulas

Definition (continued)

(vii) $(\forall x\ \alpha)^{(\mathcal{I},E)} = \begin{cases} 1 & ?? \\ 0 & \text{else} \end{cases}$

(viii) $(\exists x\ \alpha)^{(\mathcal{I},E)} = \begin{cases} 1 & ?? \\ 0 & \text{else} \end{cases}$

How can we evaluate a formula of the form $(\forall x\ \alpha)$ or $(\exists x\ \alpha)$?

- For $(\forall x\ \alpha)$, we need to verify that α is true for every possible value of x in the domain.
- For $(\exists x\ \alpha)$, we need to verify that α is true for at least one possible value of x in the domain.

We formalize this on the next few slides.

Definition

Definition: For any environment E and domain element d, the new environment "E with x re-assigned to d", denoted $E[x \mapsto d]$, is given by:

$$E[x \mapsto d](y) = \begin{cases} d & \text{if } y \text{ is } x \\ E(y) & \text{if } y \text{ is not } x \end{cases}$$

（新的 E 将 x 赋值给 d，其它变量的赋值不变）

Let $D = \{1, 2, 3\}$ for some interpretation $\mathcal{I}$ and consider E as defined by

$$E(x) = 3 \qquad E(y) = 3 \qquad E(z) = 1$$

Then

$$E[x \mapsto 2](x) = 2 \qquad E[x \mapsto 2](y) = 3 \qquad E[x \mapsto 2](z) = 1$$

What about the following?

$$E[x \mapsto 2][y \mapsto 2](x) \qquad E[x \mapsto 2][y \mapsto 2](y) \qquad E[x \mapsto 2][y \mapsto 2](z)$$

The value of well-formed formulas

Definition (continued)

The values of $(\forall x\ \alpha)$ and $(\exists x\ \alpha)$ are given by:

$$(\forall x\ \alpha)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \alpha^{(\mathcal{I},E[x\mapsto d])} = 1 \text{ for every } d \text{ in the domain of } \mathcal{I} \\ 0 & \text{otherwise} \end{cases}$$

$$(\exists x\ \alpha)^{(\mathcal{I},E)} = \begin{cases} 1 & \text{if } \alpha^{(\mathcal{I},E[x\mapsto d])} = 1 \text{ for some } d \text{ in the domain of } \mathcal{I} \\ 0 & \text{otherwise} \end{cases}$$

Example 1

Let $\mathrm{dom}(\mathcal{I}) = \{a, b\}$, $P^{\mathcal{I}} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$, $E(x) = a$ and $E(y) = b$.

Determine the value/semantics/meanings of the following formulas.

$$(\exists y P(y, x))^{(\mathcal{I}, E)}$$

Because $\langle E[y \mapsto a](y), E[y \mapsto a](x) = a\rangle = \langle a, a\rangle \in P^{\mathcal{I}}$, which means that $P(y, x)^{(\mathcal{I}, E[y \mapsto a])} = 1$. Hence, by definition, the above formula is true.

Example 1

Let dom($\mathcal{I}$) $= \{a, b\}$, $P^{\mathcal{I}} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$, $E(x) = a$ and $E(y) = b$.

Determine the value/semantics/meanings of the following formulas.

$$(\forall x(\forall y P(x, y)))^{(\mathcal{I},E)}$$

Since $\langle b, a\rangle \notin P^{\mathcal{I}}$, so we have $P(x, y)^{(\mathcal{I},E[x\mapsto b][y\mapsto a])} = 0$.
Hence, by definition, the above formula is false.

Example 1

Let $\mathrm{dom}(\mathcal{I}) = \{a, b\}$, $P^{\mathcal{I}} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$, $E(x) = a$ and $E(y) = b$.

Determine the value/semantics/meanings of the following formulas.

$$(\forall x(\exists y P(x, y)))^{(\mathcal{I},E)}$$

Example 2

For a language $\mathscr{L} = \{i, F, R\}$, let i be a constant, F be a unary predicate, while R be a binary predicate. Define an interpretation $\mathcal{I}$:

- Domain $A = \{a, b, c\}$, a set of states of a computer program.
- $i^{\mathcal{I}} = a$: initial states
- $F^{\mathcal{I}} = \{b, c\}$: Final accepting states
- $R^{\mathcal{I}} = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$: State transition relation

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Meaning/value of the formula?

$$\neg F(i)$$

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Meaning/value of the formula?

$$\exists y R(i, y)$$

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Meaning/value of the formula?

$$\forall x \exists y R(x, y)$$

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Meaning/value of the formula?

$$\forall x \forall y \forall z (R(x, y) \wedge R(x, z) \rightarrow y = z)$$

Readings

FOL Semantics

- TextB: Section 2.4.1

Introduction to Mathematical Logic

Thank you for listening!
Any questions?