HW6

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Exercise 1 For $\Omega = B_r(0) \subset \mathbb{R}^2$, let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases}$$

where f, g are continuous. Show that u satisfies

$$u(0) = \frac{1}{2\pi r} \int_{\partial\Omega} g(x)\, dS(x) + \frac{1}{2\pi} \int_{\Omega} (\ln r - \ln|x|) f(x)\, dx.$$

Hint: consider

$$\varphi(t) = \frac{1}{2\pi t} \int_{\partial B_t(0)} g(x)\, dS(x) + \frac{1}{2\pi} \int_{B_t(0)} (\ln t - \ln|x|) f(x)\, dx,$$

and show that $\varphi'(t) = 0$, $\lim_{t \to 0+} \varphi(t) = u(0)$.

Exercise 2 Let $v \in \mathcal{C}^2(\Omega)$. We say that Ω is subharmonic if $-\Delta v \leq 0$ in Ω.

1. Show that if v is subharmonic, then for any $B_r(x) \subset \Omega$,

   $$v(x) \leq \fint_{B_r(x)} v(y)\, dy.$$

   Hint: let $\varphi(r) = \fint_{B_r(x)} v(y)\, dy$ *and consider* $\varphi'(r)$.

2. Show that if $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and v is subharmonic, then

   $$\max_{\bar{\Omega}} v(x) = \max_{\partial\Omega} v(x).$$

3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth convex function. Show that if u is harmonic, then $v = \phi(u)$ is subharmonic.

4. Show that $v = |\nabla u|^2$ is subharmonic if u is harmonic.

Exercise 3 Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $u(x) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u = 1, & \Omega, \\ u = 0, & \partial\Omega. \end{cases}$$

Show that for any $x_0 \in \Omega$,

$$\frac{1}{2d} \min_{x \in \partial\Omega} |x - x_0|^2 \leq u(x_0) \leq \frac{1}{2n} \max_{x \in \partial\Omega} |x - x_0|^2.$$

Hint: consider $v(x) = u(x) - \dfrac{1}{2d}|x - x_0|^2$.

Exercise 4 Let $\Omega_0 \subset \mathbb{R}^d$ be a bounded domain, and $\Omega := \mathbb{R}^d \setminus \overline{\Omega_0}$. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$ satisfy

$$\begin{cases} -\Delta u + c(x)u = 0, & \Omega, \\ \qquad\qquad u = g(x), & \partial\Omega, \\ \quad \lim\limits_{|x|\to\infty} u(x) = \ell \in \mathbb{R}, & \end{cases}$$

where $c(x) \geq 0$ is bounded on any bounded subset of Ω. Show that

$$\sup_{\Omega} |u(x)| \leq \max\big\{|\ell|, \max_{\partial\Omega} |g(x)|\big\}.$$

Hint: obtain an L^∞-estimate on $B_R \setminus \Omega_0$ for any $R > 0$, and then take $R \to \infty$.

Exercise 5 Let $\Omega \subset \mathbb{R}^d$ be bounded. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u + u^3 - u = 0, & \Omega, \\ u = g, & \partial\Omega. \end{cases}$$

Show that if $\max\limits_{\partial\Omega} |g(x)| \leq 1$, then $\max\limits_{\bar{\Omega}} |u(x)| \leq 1$.

Hint: let $x_0 = \operatorname*{argmax}\limits_{x\in\bar{\Omega}} u(x)$; use the fact that

$$u > 1 \implies u^3 - u > 0,$$

to get a contradiction.

Exercise 6 Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ solve

$$\begin{cases} -\Delta u + c(x)u = f(x), & \Omega, \\ \dfrac{\partial u}{\partial n} + \alpha(x)u = 0, & \partial\Omega, \end{cases}$$

where $\alpha(x) \geq 0$ and $c(x) \geq c_0 > 0$. Show that there exists a constant $M = M(c_0)$,

$$\int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{c_0}{2} \int_{\Omega} |u(x)|^2 \, dx + \int_{\partial\Omega} \alpha(x) u^2(x) \, dS(x) \leq M \int_{\Omega} |f(x)|^2 \, dx.$$

Ex 1. 考虑 $\varphi(t) := \underbrace{\frac{1}{2\pi t}\int_{\partial B_t(0)} u(x)\,dS(x)}_{=\frac{1}{2\pi}\int_{\partial B_1(0)} u(tx)\,dS(x)} + \frac{1}{2\pi}\int_{B_t(0)} (\ln t - \ln|x|) f(x)\,dx$

注意到 $\varphi'(t) = \frac{1}{2\pi}\int_{\partial B_1(0)} \nabla u(tx)\cdot x\,dS(x) + \frac{1}{2\pi}\int_{B_t(0)} \frac{1}{t} f(x)\,dx$

$= \frac{1}{2\pi t}\int_{\partial B_t(0)} \frac{\partial}{\partial n} u(x)\,dS(x) + \frac{1}{2\pi t}\int_{B_t(0)} f(x)\,dx$

$= \frac{1}{2\pi t}\int_{B_t(0)} \Delta u(x)\,dx + \frac{1}{2\pi t}\int_{B_t(0)} f(x)\,dx = 0$, $\forall t \in (0,r]$

故 $\varphi(t)$ 在 $(0,r]$ 上恒为常数

又 $\frac{1}{2\pi t}\int_{\partial B_t(0)} u(x)\,dS(x) = \frac{1}{2\pi}\int_{\partial B_1(0)} u(tx)\,dS(x) \to u(0)$, as $t \to 0^+$

$\left|\frac{1}{2\pi}\int_{B_t(0)} (\ln t - \ln|x|) f(x)\,dx\right| \le \frac{\|f\|_\infty}{2\pi}\int_{B_t(0)} \ln\left(\frac{t}{|x|}\right) dx = \frac{\|f\|_\infty}{4} t^2 \to 0$, as $t \to 0^+$

故 $\lim_{t\to 0^+} \varphi(t) = u(0)$，取 $t = r$ 即得结论.

Ex 2. (1) 考虑 $\varphi(r) = ⨍_{B_r(x)} v(y)\,dy = ⨍_{B_1(0)} v(x+ry)\,dy$

因 $\varphi'(r) = ⨍_{B_1(0)} y\cdot \nabla v(x+ry)\,dy = \frac{1}{|B_1|}\int_0^1 t\,dt \int_{\partial B_t} \vec{n}\cdot \nabla v(x+ry)\,dS(y)$

$= \frac{1}{|B_1|}\int_0^1 t\,dt \int_{B_t(0)} \Delta v(x+ry)\,dy \ge 0$

又 $\lim_{r\to 0^+} \varphi(r) = \lim_{r\to 0^+} ⨍_{B_r(x)} v(y)\,dy = v(x)$

故 $v(x) \le \varphi(r) = ⨍_{B_r(x)} v(y)\,dy$

(2) 假设 v 最大值在 $x_0 \in \Omega$ 处取到，且 $v(x_0) > \max_{\partial\Omega} v(x)$

若 $v(x) = v(x_0)$，则对 $\forall B_r(x) \subset \Omega$ 有 $v(x) \le ⨍_{B_r(x)} v(y)\,dy \le v(x_0) = v(x)$

从而 $v(y) = v(x_0)$，$\forall y \in B_r(x)$

故 $A := \{x \in \Omega : v(x) = \max_{\Omega} v\}$ 是开集

又由 $x_0 \in A$ 知 A 是非空闭集，从而 $A = \Omega$，即 v 在 Ω 上为常数

由连续性知，此时 v 在 $\bar{\Omega}$ 上恒为常数，与 $v(x_0) > \max_{\partial\Omega} v(x)$ 矛盾.

故 $\max_{\Omega} v(x) = \max_{\partial\Omega} v(x)$

(3) 因 $\frac{\partial v}{\partial x_i} = \phi'(u)\cdot\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 v}{\partial x_i^2} = \phi''(u)\cdot\left(\frac{\partial u}{\partial x_i}\right)^2 + \phi'(u)\cdot\frac{\partial^2 u}{\partial x_i^2}$

故 $-\Delta v = -\phi''(u)\cdot|\nabla u|^2 - \phi'(u)\cdot\Delta u \le 0$ ($\because \phi$ convex)，即 v 是 subharmonic.

(4) 因 $\frac{\partial v}{\partial x_i} = \sum_i \left(\frac{\partial u}{\partial x_i}\right)^2$, $\frac{\partial^2 v}{\partial x_i^2} = 2\sum_{i,j}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + 2\sum_{i,j}\frac{\partial u}{\partial x_i}\cdot\frac{\partial^3 u}{\partial x_i \partial x_j^2}$

故 $-\Delta v = -2\sum_{j=1}^n\sum_{i=1}^n\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 - 2\sum_{i=1}^n \frac{\partial u}{\partial x_i}\cdot\frac{\partial}{\partial x_i}(\Delta u)$

$= -2\sum_{j=1}^n\sum_{i=1}^n\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 \le 0$ ($\because u$ harmonic)

即 v 是 subharmonic

Ex3. 考虑 $v(x) = u(x) + \frac{1}{2d}|x-x_0|^2$

则 $\Delta v(x) = \Delta u(x) + \frac{1}{2d}\cdot\Delta|x-x_0|^2 = \Delta u + 1 = 0$

故由 maximum principle 知，$u(x_0) = v(x_0) \le \max\limits_{\partial\Omega} v(x) = \frac{1}{2d}\max\limits_{\partial\Omega}|x-x_0|^2$

类似可得 $u(x_0) \ge \frac{1}{2d}\min\limits_{\partial\Omega}|x-x_0|^2$，结论得证.

Ex4. 考虑充分大的 R，使 $\bar{\Omega}_0 \subset B_R$，记 $\Omega_R := B_R / \bar{\Omega}_0 \subset \mathbb{R}^d / \bar{\Omega}_0$

记 $M_R := \max\left\{\max\limits_{\partial B_R}|u(x)|, \max\limits_{\partial\Omega}|g(x)|\right\}$

则 $-\Delta(u-M_R) + c(x)(u(x)-M_R) \le 0$ in Ω_R

故由 Weak maximum principle 知，$\max\limits_{\Omega_R} u(x) \le M_R$

类似可得 $\max\limits_{\Omega_R} -u(x) \le M_R$，从而 $\max\limits_{\Omega_R}|u(x)| \le M_R$

两边令 $R\to\infty$ 即得结论.

Ex5. 假设 $u(x)$ 在 x_0 处取到最大值 $u(x_0) > 1$.

则由 $\max\limits_{\partial\Omega}|g(x)| \le 1$ 知 $x_0 \in \Omega$

又 $u(x)$ 在 $x=x_0$ 处 Hessian 矩阵半负定，故 $\Delta u(x_0) \le 0$

从而 $-\Delta u(x_0) + u^3(x_0) - u(x_0) > 0$，矛盾.

故 $\max\limits_{\Omega}|u(x)| \le 1$.

Ex6. 注意到 $\int_\Omega(-\Delta u)u\,dx + \int_\Omega c(x)u^2\,dx = \int_\Omega fu\,dx$

分部积分得 $LHS = \int_\Omega|\nabla u|^2dx - \int_{\partial\Omega} u\cdot\frac{\partial u}{\partial n}dS + \int_\Omega c(x)u^2dx$

$$= \int_\Omega|\nabla u|^2dx + \int_{\partial\Omega}\alpha(x)\cdot u^2dS + \int_\Omega c(x)u^2dx$$

$$\ge \int_\Omega|\nabla u|^2dx + \int_{\partial\Omega}\alpha(x)\cdot u^2dS + c_0\int_\Omega u^2dx$$

又 $RHS \le \frac{1}{2c_0}\int_\Omega f^2dx + \frac{c_0}{2}\int_\Omega u^2dx$

从而取 $M = \frac{1}{2c_0}$ 即得结论.