Week 1, Thursday

△ Root - finding problem:

for function $f(x)$: $\boxed{\text{find } x \text{ s.t. } f(x)=0}$

e.g. $ax+b=0 \quad (a\neq 0) \qquad x=-\frac{b}{a}$

$ax^2+bx+c=0 \quad (a\neq 0) \qquad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

RK: for general polynomials with degree ≥ 5, there is no algebraic formula of the roots. (Abel, Galois)

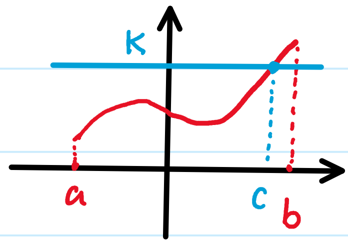
So numerical approach is needed for root-finding problem.

△ Bisection method. (二分法)

$\boxed{\text{Thm}}$ (intermediate value theorem) 中间值定理

$\forall f\in C[a,b]$, $\forall K\in f([a,b])$, $\exists c\in[a,b]$ s.t. $f(c)=K$.

$\boxed{\text{Corollary:}}$



if $f\in C[a,b]$, $f(a)f(b)<0$

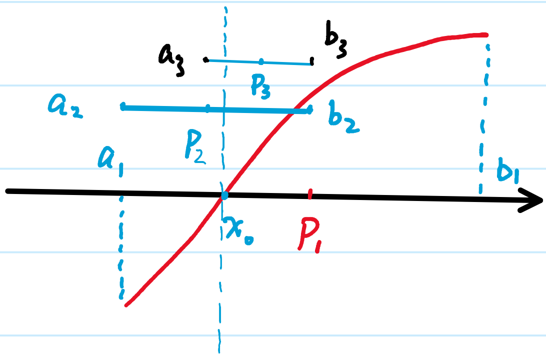
then $\exists x_0\in(a,b)$ s.t. $f(x_0)=0$

proof: $f(a)f(b)<0 \Rightarrow f(a)>0, f(b)<0$ or $f(a)<0, f(b)>0$

$\boxed{\text{idea:}}$ shrinkage the interval that includes the root.

Input: a, b, N_0, ε

Output: approximate root or message of failure



1. find a_1, b_1 s.t. $f(a_1)f(b_1)<0$

2. For $i=1,2,\cdots,N_0$,

   set $p_i=\frac{a_i+b_i}{2}$

3. If $|p_i-p_{i-1}|<\varepsilon$ or $|f(p_i)|<\varepsilon$

   stop, output (p_i)

4. If $f(p_i)f(a_i)>0$, set $a_{i+1}=p_i$, $b_{i+1}=b_i$

elseif $f(p_i)f(a_i)<0$ set $a_{i+1}=a_i$, $b_{i+1}=p_i$

5. set $i=i+1$, go to step 3.

6. Stop, output ("failure").

e.g. $f(x)=x^3+4x^2-10$ find a root in $[1,2]$

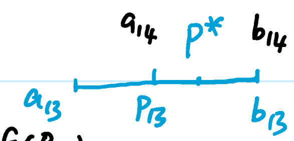
$f(1)=-5$, $f(2)=14$ $\therefore \exists\, x_0\in[1,2]$ s.t. $f(x_0)=0$

① $a_1=1$, $b_1=2$, $p_1=1.5$ $f(p_1)=2.375>0$

② $a_2=1$, $b_2=1.5$, $p_2=1.25$ $f(p_2)\approx-1.8<0$

③ $a_3=1.25$, $b_3=1.5$, $p_3=1.375$ $f(p_3)$ ...

⋮



⑬ $a_{13}=1.364990$, $b_{13}=1.365234$, $p_{13}=1.365112$, $f(p_{13})=-0.002<0$.

(i) $|p^*-p_{13}|\le|b_{14}-a_{14}|=|b_{13}-p_{13}|\approx|1.365234-1.365112|=0.000122$

(ii) Since $0<a_{14}<p^*$

$$\frac{|p_{13}-p^*|}{|p^*|}\le\frac{|p_{13}-p^*|}{|a_{14}|}\le\frac{0.000122}{1.365112}\approx 9\times10^{-5}$$ (4 significant digits)

Correct $p^*=1.365230$

$\boxed{\text{Thm}}$ $|p_n-p^*|\le\dfrac{b-a}{2^n}$

proof: each iteration, the length of the interval becomes half.

$\therefore b_n-a_n=\dfrac{b_1-a_1}{2^{n-1}}$, we also know $p^*\in(a_n,b_n)$

$\therefore |p_n-p^*|\le\frac{1}{2}(b_n-a_n)=\dfrac{b-a}{2^n}$. Done!

RK1: $|p_n-p^*|\le\dfrac{b-a}{2^n}\Rightarrow p_n=p^*+O(2^{-n})$

RK2: this error estimate can be used to determine steps of iterations to achieve certain accuracy.

e.g. in last example, if we require accuracy 10^{-3} then iterations to achieve certain accuracy.

e.g. in last example, if we require accuracy 10^{-3}, then

$\frac{2-1}{2^N} \le 10^{-3} \Rightarrow 2^N \ge 10^3$ i.e. $N \ge 3\log_2 10 \approx 9.96$.

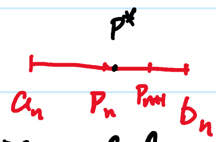
at least it needs 10 steps.

△ advantages:

1. Simple, independent of f, only requires f to be continuous.
2. always converges to a solution (often take it to get a good initial approximation for other method).

△ drawbacks:

1. slow, requires large number of iterations (linearly)



2. Sometimes even p_n is a good approximation, but still discarded.
3. not easy to extend to high dimensional cases. (Chap. 10)

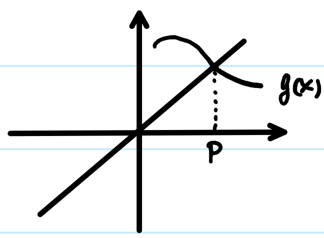
△ fixed-point iteration:

Def p is a fixed point for a function g if $g(p)=p$.

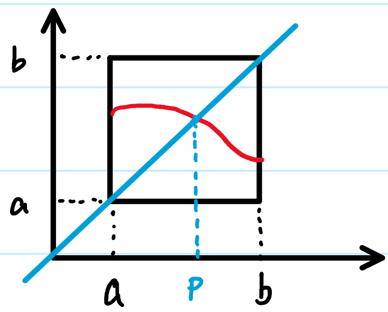
e.g. $g(x)=x^2-2$

$x^2-2=x \Rightarrow x_1=-1, \quad x_2=2$

two fixed points $-1, 2$.



Thm if $g \in C[a,b]$, $g([a,b]) \subset [a,b]$, then g has at least one fixed point in $[a,b]$, i.e. $\exists p$ s.t. $p=g(p)$



Proof: if $g(a)=a$, or $g(b)=b$ done!

otherwise, let $h(x)=g(x)-x$, then $h(x) \in C[a,b]$ since $f \in C[a,b]$

and, $h(a)=g(a)-a>0$, $h(b)=g(b)-b<0$

$\therefore \exists\, p$ s.t. $h(p)=0$, i.e. $g(p)=p$.

e.g. $g(x)=3^{-x}$ on $[0,1]$

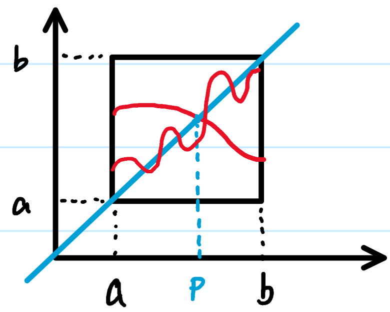
verify: ① $g\in C[0,1]$ ② $g(x)\downarrow$ $g(0)=1$, $g(1)=\frac{1}{3}$ $\therefore g([0,1])\subset[0,1]$

$\therefore \exists\, p\in[0,1]$ s.t. $p=3^{-p}$.

RK1: for any $g\in C[a,b]$, it may have many fixed points

RK2: ensure uniqueness, require $g(x)$ does not vary too rapidly

$\boxed{\text{Thm}}$ (uniqueness)



① $g\in C[a,b]$, $g([a,b])\subset[a,b]$

② $g'(x)$ is defined on (a,b) and $\exists\, 0<K<1$

s.t. $|g'(x)|\le K$, $\forall\, x\in(a,b)$.

Then: g has a unique fixed point in $[a,b]$.

Proof: ① $\Rightarrow \exists\, p$ s.t. $p=g(p)$ existence.

② assume $\exists\, p\ne q$ and $p,q\in[a,b]$ s.t. $p=g(p)$, $q=g(q)$

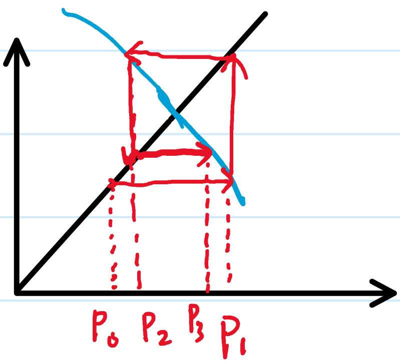
by mean value theorem: $\exists\, \xi\in(p,q)$ s.t. $g'(\xi)=\frac{g(p)-g(q)}{p-q}=1$

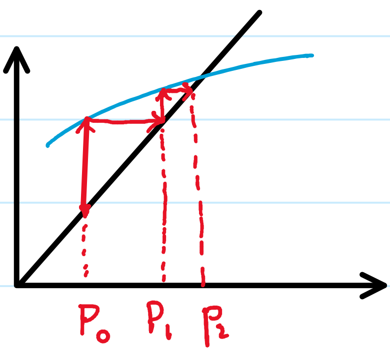
contradicted with $|g'(x)|\le K<1$ $\forall\, x\in(a,b)$

△ fixed-point iteration:

choose p_0, generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n=g(p_{n-1})$

RK: if $p_n\to p$, then $p=\lim_{n\to\infty}p_n=\lim_{n\to\infty}g(p_{n-1})=g(\lim_{n\to\infty}p_{n-1})=g(p)$.





Fixed-point iteration:

Input: P_0, ε, N_0

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output: fixed point P

1. $i=1$ while $i \le N_0$ do 2-5
2. $P = g(P_0)$
3. If $|P-P_0| < \varepsilon$, output (P). STOP
4. $i = i+1$
5. set $P_0 = P$
6. STOP, output ("failure")

e.g. find the positive root of $f(x) = x^2 - x - 1$ $\left(p^* = \frac{1+\sqrt{5}}{2}\right)$

idea: convert $f(x) \Leftrightarrow x = g(x)$ (not unique)

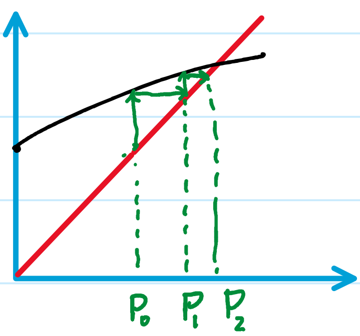
① as $x^2 = x+1$, $x = \pm\sqrt{x+1}$, for positive root, set $g(x) = \sqrt{x+1}$, corresponding fixed-point iteration

$P_n = \sqrt{P_{n-1}+1}$, if we take $P_0 = 1$,

$P_1 = \sqrt{1+1} = \sqrt{2}$, $P_2 = \sqrt{1+\sqrt{2}}$, ...

after 15 iterations $|P_{15} - p^*| \approx 1.5 \times 10^{-8}$

$$|g'(P_{15})| = \frac{1}{2\sqrt{P_{15}+1}} \approx 0.30902$$

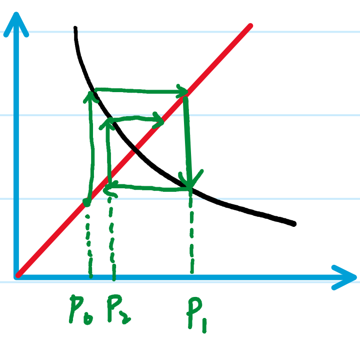


② as $x^2 = x+1$, $x = 1 + \frac{1}{x}$, set $g(x) = 1 + \frac{1}{x}$.

then $P_n = 1 + \frac{1}{P_{n-1}}$, take $P_0 = 1$

$P_1 = 1 + \frac{1}{1} = 2$, $P_2 = 1 + \frac{1}{2} = 1.5$, ...

$|P_{15} - P_*| \approx 4.5 \times 10^{-7}$



Q: how to quantify the convergence speed?

$\boxed{\text{Thm}}$ ① $g \in C[a,b]$, $g([a,b]) \subset [a,b]$

② $g'(x)$ $\exists$ on (a,b) with $|g'(x)| \le k < 1$ $\forall\, x \in (a,b)$

③ $\forall\, p_0 \in [a,b]$, $p_n = g(p_{n-1})$ $\forall\, n \ge 1$.

Conclusions: (i) $p_n \to p$, which is the unique fixed point of $g(x)$

(ii) $|p_n - p| \le k^n \max\{p_0 - a,\ b - p_0\}$ or

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|, \quad \forall\, n \ge 1.$$

Proof: (i) by last theorem, $\exists!\ p \in [a,b]$ s.t. $p = g(p)$

$$\therefore |p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_{n-1})|\,|p_{n-1} - p| \le k|p_{n-1} - p|$$

$$\le \cdots k^n |p_0 - p| \le k^n \max\{p_0 - a,\ b - p_0\}$$

as $n \to \infty$, $k^n \to 0$ $\therefore p_n \to p$.

(ii) easy to show:

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \le k|p_n - p_{n-1}| \le \cdots \le k^n |p_1 - p_0|$$

$$\therefore \forall\, m > n,\ |p_m - p_n| \le \sum_{j=n}^{m-1} |p_{j+1} - p_j| \le \sum_{j=n}^{m-1} k^j |p_1 - p_0|$$

$$\le k^n |p_1 - p_0| \left(1 + k + k^2 + \cdots + k^{m-n-1}\right) \le k^n |p_1 - p_0| \frac{1 - k^{m-n}}{1-k}$$

let $m \to \infty$ left $= |p_0 - p_n| \le$ right $= \frac{k^n |p_1 - p_0|}{1-k}$

e.g. $f(x) = x^3 + 4x^2 - 10 = 0$, $f(1) = -5 < 0$, $f(2) = 14 > 0$, $\exists$ root $\in [1,2]$

① $x = x - f(x)$, $g_1(x) = x - x^3 - 4x^2 + 10$, $p_0 = 1.5$ (not convergent)

$g_1(1) = 6$, $g_1(2) = -12$, $g_1([1,2]) \not\subset [1,2]$

$g_1'(x) = 1 - 3x^2 - 8x$, $g_1'(1) = -10$, $g_1'(2) = -27$, $|g'(x)| > 1$

② $x^2 = \frac{10}{x} - 4x$, $x = \sqrt{\frac{10}{x} - 4x} := g_2(x)$ $p_0 = 1.5$, (not convergent)

$g_2(1) = \sqrt{6}$, $g_2(2) = \sqrt{3}\,i$ $g_2([a,b]) \not\subset [a,b]$

$g'(p) \approx -3.43$

③ $4x^2 = 10 - x^3 \Rightarrow x = \frac{1}{2}\sqrt{10 - x^3} := g_3(x)$ ↘

$g_3(1) = 1.5$, $g_3(2) = \frac{\sqrt{2}}{2}$, $g([1,2]) \not\subset [1,2]$

$g_3'(x) = -\frac{3x^2}{4\sqrt{10-x^3}}$, $g_3'(2) = -\frac{3}{\sqrt{2}}$

but on $[1, 1.7]$, $g_3'([1, 1.7]) < 1$ $P_0 = 1.5$, 30 iterations

$P_{30} = 1.365230013$

④ $x^2(x+4) = 10 \Rightarrow x = \sqrt{\frac{10}{x+4}} := g_4(x)$

$g_4'(x) = -\sqrt{\frac{10}{4(4-x)^3}} \Rightarrow 0.1 < |g_4'(x)| < 0.15 \quad \forall x \in [1,2]$

$|g_4'(x)| < |g_3'(x)|$

$P_0 = 1.5$, $P_{15} = 1.365230013$ faster than $g_3(x)$

⑤ $x(3x^2+8x) = x(3x^2+8x) - (x^3 + 4x^2 - 10)$

$\Rightarrow x = x - \frac{x^3+4x^2-10}{3x^2+8x} := g_5(x)$

$P_0 = 1.5$, $P_4 = 1.365230013$, super fast!

RK: compare with bisection method: $P_{27} \approx 1.365230013$

Q: why g_5 so fast?

$g_5(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g_5'(x) = \frac{f(x) f''(x)}{(f'(x))^2} \Rightarrow g_5'(P) = 0$

HW1-2: 2.1: 1 10 11 13 17, only show how many steps needed

2.2: 2 5 13 a,b 14 21 23 26 — no need to perform the calculations.

2: only two steps