DSAA (H). Lab 06.

Q6.1

1. The algorithm is correct.

Proof: Loop Invariant: At the end of each iteration of the first for loop. (the i-th iteration). the subarray $A[1, \cdots, i]$ is sorted in increasing order.

Initialization: At the end of the first iteration of the first for loop. the largest element in the whole array is swapped to the first place: $A[1]$ is trivially sorted.

Maintenance: Suppose the loop invariant is true for the i-th iteration. Consider the $(i+1)$-th iteration. Now, $A[1, \cdots, i]$ is sorted in increasing order. Then, $A[i]$'s value is inserted into $A[1, \cdots, i+1]$ s.t. the elements to its left are ~~___~~ smaller than it and the elements of $A[1, \cdots, i+1]$ to its right are greater than it. And element to its ~~right~~ left don't move a bit. After this. insertion, the remaining right part of $A[1, \cdots, i+1]$ continues this process, until every element ever been at the $(i+1)$-th place has been ~~___~~ inserted to the right place. $\Rightarrow$ the subarray $A[1, \cdots, i+1]$ is sorted in increasing order.

Termination: At the end of the n-th iteration, the subarray $A[1, \cdots, n]$ is sorted in increasing order. $\Rightarrow$ the whole array is sorted! $\square$

2. $n \times n \times 1 \leq T(n) \leq n \times n \times 2$

$\Rightarrow T(n) = O(n^2)$ and $T(n) = \Omega(n^2)$

$\Rightarrow T(n) = \Theta(n^2)$

NO.

DATE

$$\begin{array}{ccccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 \\ 2 < & 4 < & 5 < & 6 < & 8 < & 9 < & 10 < & 12 < & 25. \end{array}$$

Q 6.2.

1. $\mathbb{P}(10 \text{ and } 4 \text{ are compared})$
   $= \mathbb{P}(z_2 \text{ and } z_7 \text{ are compared}) = \frac{2}{7-2+1} = \frac{1}{3}.$
2. $\mathbb{P}(12 \text{ and } 25 \text{ are compared})$
   $= \mathbb{P}(z_8 \text{ and } z_9 \text{ are compared}) = \frac{2}{9-8+1} = 1.$
3. $\mathbb{P}(2 \text{ and } 25 \text{ are compared})$
   $= \mathbb{P}(z_1 \text{ and } z_9 \text{ are compared}) = \frac{2}{9-1+1} = \frac{2}{9}.$
4. $\mathbb{P}(10 \text{ and } 5 \text{ are compared})$
   $= \mathbb{P}(z_3 \text{ and } z_7 \text{ are compared}) = \frac{2}{7-3+1} = \frac{2}{5}.$

Q 6.3.

Proof: In class, we have shown $\sum_{k=1}^{n} \frac{1}{k} = O(\log n)$. → 2-based

We now prove: $\sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$.

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots\cdots}_{(\lfloor \log n \rfloor - 1) \text{ times.}} \geq \frac{1}{2} \log n. \Rightarrow \sum_{k=1}^{n} \frac{1}{k} = \Omega(\log n)$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n).$$

$$\Rightarrow \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \geq 2 \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=1}^{n-i} \frac{1}{k+1}$$

$$\geq 2 \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1} = 2 \lfloor \tfrac{n}{2} \rfloor \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1}$$

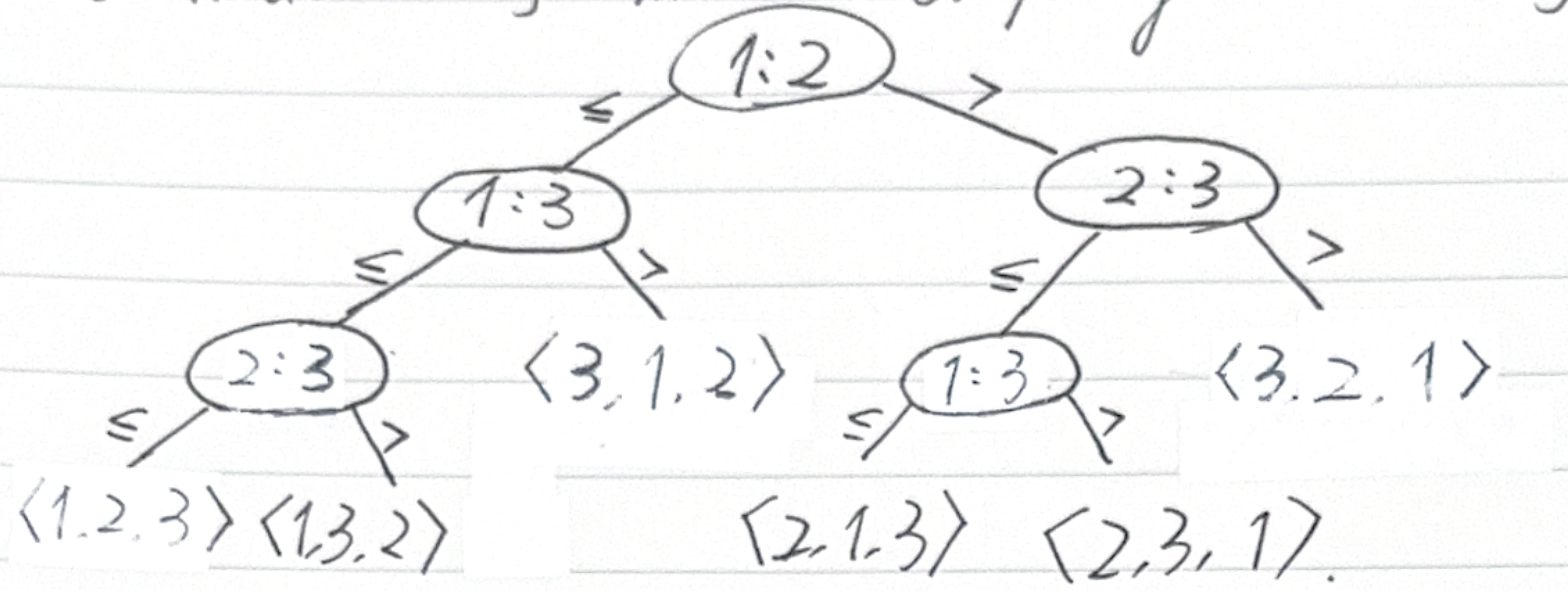
$$= 2 \lfloor \tfrac{n}{2} \rfloor \Theta\left(\log \lfloor \tfrac{n}{2} \rfloor\right) = 2 \lfloor \tfrac{n}{2} \rfloor \Theta(\log n) = n\,\Theta(\log n)$$
$$= \Theta(n \log n).$$

$$\Rightarrow \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \Omega(n \log n).$$

□.

Q6.4: $A=[a_1, \cdots, a_n]$. $a_i \neq a_j$, $\forall i \neq j$. $n=3$.

inner node: $i:j$ means comparing a_i and a_j.



Q6.5:

The smallest possible depth of a leaf in a decision tree

$\Leftrightarrow$ The smallest possible number of comparisons made to sort the whole array.

The answer is $n-1$, with n denoting the length of the array.

Proof: (Proof by induction).

① base case: $n=1$, no comparison. $\Rightarrow \# = 0 = n-1$. ✓

② Suppose the statement holds for n.

$\Rightarrow$ Consider array of length $(n+1)$.

Its subarray of length n needs at least $n-1$ comparisons

The $(n+1)$-th element needs at least 1 comparison.

[Specifically, it's compared with the smallest element (must be less or equal than it) or the largest element (must be greater than it) of the privious subarray.]

$\Rightarrow$ In total, it needs at least n comparisons. ✓. $\square$.