HW4

October 10, 2025

In what follows, $\Omega \subset \mathbb{R}^d$ will be a bounded domain, $T > 0$, and the parabolic interior Ω_T and boundary $\partial_p \Omega_T$ are given by

$$\Omega_T = (0, T] \times \Omega, \quad \partial_p \Omega_T = \big([0, T] \times \partial\Omega\big) \cup \big(\{0\} \times \Omega\big).$$

Exercise 1 Consider the differential operator

$$(\mathcal{L}u)(t, x) = -\Delta u(t, x) + c(x)u(x),$$

where $c : \Omega \to [-M, +\infty)$ is continuous, $M \geq 0$. The goal is to establish the following weak maximum principle: if $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ and

$$\boxed{(\partial_t + \mathcal{L})u \geq 0 \text{ in } \Omega_T}, \quad \min_{\partial_p \Omega_T} u \geq 0 \implies \min_{\overline{\Omega_T}} u \geq 0. \tag{0.1}$$

1. Prove (0.1) under the condition $\mathcal{L}u > 0$ in Ω_T and $M = 0$.
2. Prove (0.1) under the condition $\mathcal{L}u \geq 0$ in Ω_T and $M = 0$.

   Hint: consider $u_\varepsilon(t, x) = u(t, x) - t\varepsilon$.
3. Prove (0.1) under the condition $\mathcal{L}u \geq 0$ in Ω_T and $M > 0$.

   Hint: consider $v(t, x) = e^{\lambda t}u(t, x)$ *for an appropriate* λ. (0, 1)

Exercise 2 Let $\Omega = (0, \ell)$.

1. Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{1,0}(\partial_p \Omega_T)$ satisfy

$$\begin{cases} u_t - u_{xx} \geq 0, & (t, x) \in \Omega_T, \\ u\big|_{t=0} \geq 0, & x \in \Omega, \\ u(t, 0) \geq 0, & t > 0, \\ u_x(t, \ell) \geq 0, & t > 0. \end{cases}$$

   Show that $u \geq 0$ on $\overline{\Omega_T}$.

   Hint: you may consider $u_\varepsilon(t, x) = u(t, x) + \varepsilon x$.

2. Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{1,0}(\partial_p \Omega_T)$ satisfy

$$\begin{cases} u_t - u_{xx} = f, & (t, x) \in \Omega_T, \\ u\big|_{t=0} = \varphi, & x \in \Omega, \\ u(t, 0) = 0, & t > 0, \\ u_x(t, \ell) = g(t), & t > 0, \end{cases}$$

where f, φ, g are bounded, continuous functions in their domains. Show that

$$\max_{\overline{\Omega_T}} |u| \leq C(|T|+1)(F+G+\Phi)$$

for some constant C depending only on ℓ, where $F = \sup |f|$, $G = \sup |g|$ and $\Phi = \sup |\varphi|$.

Hint: consider $v(t,x) = tF + Gx + \Phi \pm u(t,x)$ and use part 1.

Exercise 3 Let $\Omega = (0,\ell)$. Suppose that $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ solves

$$\begin{cases} u_t - u_{xx} = f(t,x), & (t,x) \in \Omega_T, \\ u(0,x) = 0, & x \in [0,\ell], \\ -u_x + \alpha u = 0, & t > 0,\ x = 0, \\ u_x + \beta u = 0, & t > 0,\ x = \ell, \end{cases}$$

where $\alpha, \beta \geq 0$ are constants. Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t,x)\,dx + \int_0^T \int_0^\ell u_x^2(t,x)\,dxdt \leq C \int_0^T \int_0^\ell f^2(t,x)\,dxdt,$$

for some constant C depending only on T.

Hint: multiply the equation by u on both sides, perform suitable integration by parts in x, then integrate in t; use $|2ab| \leq a^2 + b^2$ and Gronwall at some point.

Exercise 4 Let $\Omega = (0,\ell)$ and $b, c \in \mathcal{C}(\overline{\Omega_T})$. Suppose that $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}^{0,1}(\overline{\Omega_T})$ solves

$$\begin{cases} u_t - u_{xx} + b(t,x)u_x + c(t,x)u = 0, & (t,x) \in \Omega_T, \\ u(0,x) = \varphi(x), & x \in [0,\ell], \\ u(t,0) = u(t,\ell) = 0, & t \in [0,T]. \end{cases}$$

Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t,x)\,dx + \int_0^T \int_0^\ell u_x^2(t,x)\,dxdt \leq C \int_0^\ell \varphi^2(x)\,dx,$$

for some constant C depending only on T, β and γ, where

$$\beta = \sup_{\overline{\Omega_T}} |b(t,x)|, \quad \gamma = \sup_{\overline{\Omega_T}} |c(t,x)|.$$