Introduction to Data Assimilation
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Abstract

This lecture focuses on Markov jump process and chaotic systems.

1 Chaotic Systems

1.1 Sensitivity to Initial Conditions and Lyapunov Exponents

Chaotic systems are characterized by their extreme sensitivity to initial conditions. Even when the governing equations are entirely deterministic and contain no stochastic term, a small perturbation in the initial condition can lead to vastly different trajectories after a short time. This property makes long-term forecasting of such systems intrinsically difficult.

Consider the linearized model (adapted from equation (1.27)):

$$du = \Big[(-\gamma + i\omega_u)u + f\Big]\,dt + \sigma_u\,dW_t, \tag{1.1}$$

where u is complex, γ represents a (possibly state-dependent) damping coefficient, ω_u is the oscillation frequency, f is a forcing term, and $\sigma_u dW_t$ denotes stochastic noise. The system can intermittently switch between stable and unstable regimes depending on the sign of γ.

Exponential Separation of Trajectories. To illustrate the chaotic separation, let us neglect the stochastic forcing and assume $\gamma = \gamma_- < 0$ (the unstable regime) for some time interval. Let the true solution be $u(t)$ and the perturbed solution (with a small initial deviation $\delta u(0)$) be $u_\delta(t)$. Then

$$u(t) = u(0)\exp\Big[(-\gamma_- + i\omega_u)t\Big], \qquad u_\delta(t) = (u(0) + \delta u(0))\exp\Big[(-\gamma_- + i\omega_u)t\Big].$$

The difference between them satisfies

$$\delta u(t) = u_\delta(t) - u(t) = \delta u(0)\exp\Big[(-\gamma_- + i\omega_u)t\Big]. \tag{1.2}$$

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Since $\gamma_- < 0$, the real part of $-\gamma_-$ is positive, which means

$$|\delta u(t)| = |\delta u(0)| e^{|\gamma_-| t},$$

indicating exponential divergence of nearby trajectories. This exponential rate of separation is quantified by the Lyapunov exponent, a measure of local instability and chaos:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{|\delta u(t)|}{|\delta u(0)|}.$$

If $\lambda > 0$, the system exhibits chaotic behavior.

In the presence of stochastic noise or turbulence, the divergence becomes even faster, as noise effectively enhances local instability and spreads trajectories apart, further reducing the predictability horizon.

1.2 The Lorenz–63 Model

A canonical example of a chaotic system is the Lorenz 63 (L63) model:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= x(\rho - z) - y, \\ \frac{dz}{dt} &= xy - \beta z, \end{aligned} \tag{1.3}$$

where x represents the convection rate, y the horizontal temperature difference, and z the vertical temperature difference. The parameters σ, ρ, and β are proportional to the Prandtl number, Rayleigh number, and certain geometric aspects of the fluid layer.

The model, first proposed by Edward Lorenz in 1963, is a simplified mathematical model for atmospheric convection and serves as the prototype of deterministic chaos. For the classical parameter values

$$\sigma = 10, \quad \rho = 28, \quad \beta = \frac{8}{3},$$

the L63 system displays sensitive dependence on initial conditions, with trajectories diverging rapidly from each other even if their starting points differ only slightly.

For the standard Lorenz–63 parameters $(\sigma, \rho, \beta) = (10, 28, 8/3)$, the Lyapunov spectrum is

$$\lambda_1 \approx 0.9056, \qquad \lambda_2 = 0, \qquad \lambda_3 \approx -14.5723.$$

We compute

$$S_1 = 0.9056 > 0, \qquad S_2 = 0.9056 > 0, \qquad S_3 < 0,$$

so $j = 2$. The Kaplan–Yorke dimension is therefore

$$D_{\mathrm{KY}} = 2 + \frac{S_2}{|\lambda_3|} = 2 + \frac{0.9056}{14.5723} \approx 2.062.$$

This means the Lorenz attractor is a *strange attractor* whose geometry lies between a two-dimensional surface and a three-dimensional volume.

Remark 1.1. The Kaplan–Yorke dimension, also known as the Lyapunov dimension, provides a quantitative connection between the dynamical instability of a system and the fractal geometry of its attractor. It is based on the set of Lyapunov exponents, which measure the average exponential rates of separation of nearby trajectories in phase space.

Suppose an n-dimensional dynamical system has Lyapunov exponents

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Define the partial sums

$$S_j = \sum_{i=1}^{j} \lambda_i.$$

Let j be the largest integer such that $S_j \geq 0$. Then the *Kaplan–Yorke dimension* (or *Lyapunov dimension*) is defined as

$$\boxed{D_{\mathrm{KY}} = j + \frac{S_j}{|\lambda_{j+1}|}.}$$

Intuitively, the first j Lyapunov exponents correspond to the directions in which volumes in phase space expand or remain neutral on average, whereas λ_{j+1} marks the onset of average contraction. The fractional term $\frac{S_j}{|\lambda_{j+1}|}$ interpolates between these directions, yielding a noninteger, fractal dimension for the attractor.

A positive largest Lyapunov exponent, $\lambda_1 > 0$, indicates *sensitive dependence on initial conditions*, a defining property of chaotic systems. The noninteger Kaplan–Yorke dimension captures the *fractal geometry* of the corresponding attractor. Together, they form two complementary signatures of chaos:

- Dynamical randomness — exponential divergence of nearby trajectories;
- Geometric complexity — noninteger, fractal attractor dimension.

Lastly, the Kaplan–Yorke dimension is often regarded as an *upper estimate* or heuristic approximation to the Hausdorff dimension.

1.3 Numerical Illustration and Phase Portrait

In numerical simulations of the L63 model (1.3), two trajectories starting from almost identical initial values, say

$$(x_1(0), y_1(0), z_1(0)) = (5, 0, 25), \qquad (x_2(0), y_2(0), z_2(0)) = (5.01, 0, 25),$$

will initially stay close but diverge exponentially after a few time units.

Panel (a) in Figure 1 shows two simulated trajectories (blue and red curves). The divergence after $t \approx 5$ indicates the system's chaotic unpredictability. Panel (b) shows the phase portrait, where the trajectories spiral around two lobes resembling a butterfly — a visual hallmark of chaos.

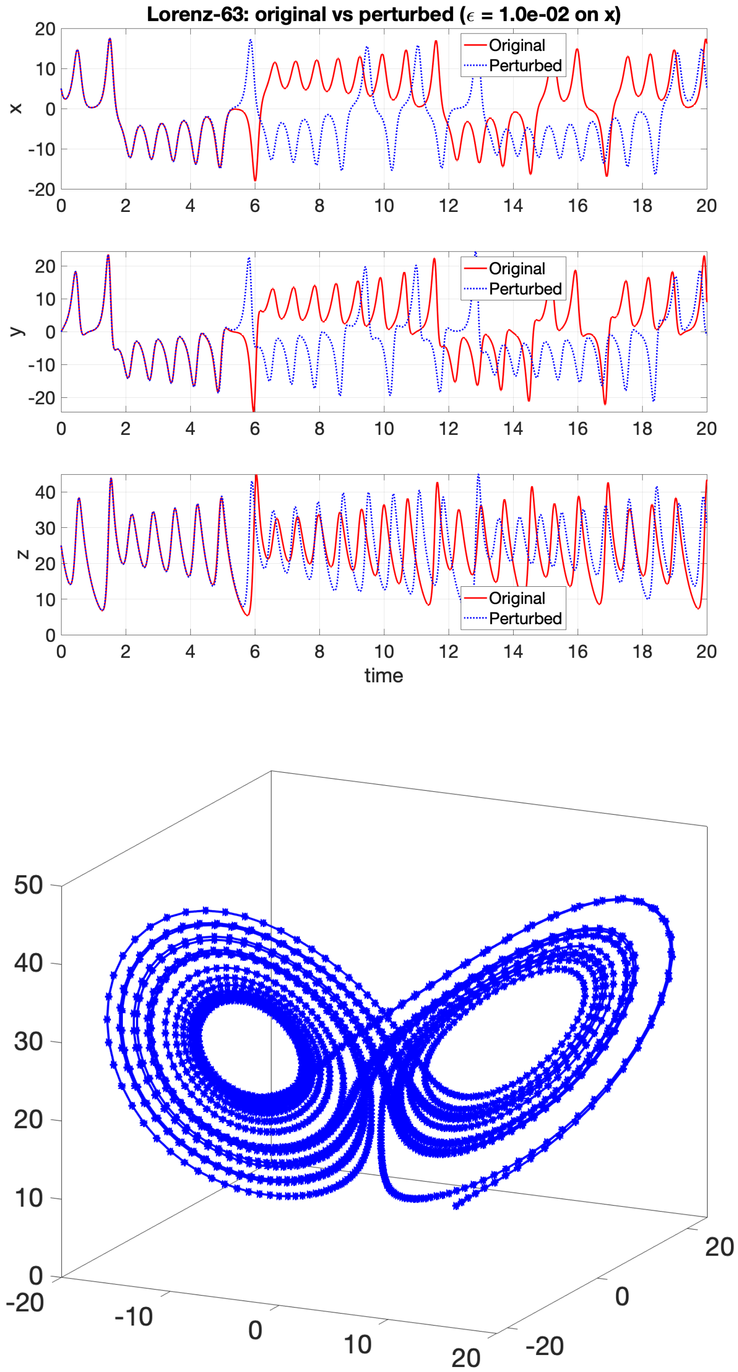

Figure 1: Simulation of the Lorenz 63 model with parameters $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. (a) Two trajectories with slightly different initial conditions diverge quickly. (b) Phase-space projection showing the classical butterfly-shaped Lorenz attractor.

2 Matlab Demo for Kalman Filter for L63 Model

2.1 Interpretation and Forecasting Challenges

Chaotic systems are deterministic in formulation but unpredictable in practice. The underlying equations do not contain randomness, yet small perturbations in the initial state grow exponentially, making long-term prediction impossible. In practical forecasting, this implies:

- Perfect initial conditions are almost never available.
- Forecast trajectories (path-wise forecasts) lose accuracy after a short time.
- Ensemble forecasts, which evolve a collection of trajectories to estimate the probability density function (PDF), provide a more meaningful description of the system's behavior.

Data assimilation techniques (discussed later) are critical in chaotic systems because they combine observational data with model forecasts to produce improved state estimates that delay the divergence of trajectories.

2.2 Remarks

- The Lorenz–63 model exemplifies the butterfly effect: tiny changes in initial conditions can lead to dramatically different outcomes.
- The Lyapunov exponent provides a quantitative measure of predictability — the inverse of λ represents the time horizon over which forecasts remain accurate.
- For chaotic and turbulent systems, the ensemble forecast approach, focusing on the evolution of the PDF rather than single trajectories, is often more appropriate.
- Developing stochastic and reduced-order models that can capture the statistical features of chaos is essential for uncertainty quantification and data assimilation.

References