Introduction to Mathematical Logic

For CS Students

CS104

Yida TAO (陶伊达)

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Types of Proof Systems

We will introduce 3 formal proof systems.

- Hilbert-style system ($\Sigma \vdash_H A$): many axioms and only one rule. The deduction is linear.
- **Natural Deduction System ($\Sigma \vdash_{ND} A$): Few axioms (even none) and many rules. The deductions are tree-like.**[1]
- Resolution ($\Sigma \vdash_{Res} A$): used to prove contradictions.

[1] Part of this slide is based on the course notes of UWaterloo CS245.

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Alphabet of ND

$\Sigma = \{(,), \neg, \wedge, \vee, \rightarrow, \leftrightarrow, p, q, r, \ldots\}$

Formulas of ND

1. Atoms $p, q, r, \ldots$ are formulas.
2. If A, B are formulas, then $(\neg A), (A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ are also formulas.
3. Only expressions of Σ that are generated by 1 and 2 are formulas.

Reflexivity (Premise)

$\Sigma \cup \{\alpha\} \vdash \alpha$ (or $\Sigma, \alpha \vdash \alpha$)

If you want to write down a previous formula in the proof again, you can do it by reflexivity (自反).

An example of using reflexivity

A proof of $\{p, q\} \vdash_{ND} p$

1.	p	Premise
2.	q	Premise
3.	p	Reflexivity: 1

Alternatively, we could simply write:

1.	p	Premise

Inference Rules

For each logical symbol, the rules come in pairs.

- An “introduction rule” adds the symbol to the formula.
- An “elimination rule” removes the symbol from the formula.

Inference Rules for Conjunction

Name	$\vdash$-notation	inference notation
$\wedge$-introduction ($\wedge$i)	If $\Sigma \vdash_{ND} \alpha$ and $\Sigma \vdash_{ND} \beta$, then $\Sigma \vdash_{ND} (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Name	$\vdash$-notation	inference notation
$\wedge$-elimination ($\wedge$e)	If $\Sigma \vdash_{ND} (\alpha \wedge \beta)$, then $\Sigma \vdash_{ND} \alpha$ and $\Sigma \vdash_{ND} \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha} \quad \dfrac{(\alpha \wedge \beta)}{\beta}$

Intuition from tautology: $\alpha \to (\beta \to \alpha \wedge \beta), \alpha \wedge \beta \to \alpha, \alpha \wedge \beta \to \beta$

Inference Rules for Conjunction

Example: Show that $\{(p \wedge q)\} \vdash_{ND} (q \wedge p)$

Proof:

1.	$(p \wedge q)$	Premise
2.	q	$\wedge$e: 1
3.	p	$\wedge$e: 1
4.	$(q \wedge p)$	$\wedge$i: 2,3

Name	$\vdash$-notation	inference notation
$\to$-elimination ($\to$e) (*modus ponens*)	If $\Sigma \vdash_{ND} (\alpha \to \beta)$ and $\Sigma \vdash_{ND} \alpha$, then $\Sigma \vdash_{ND} \beta$	$\dfrac{(\alpha \to \beta) \quad \alpha}{\beta}$

Intuition: If you assume α is true and α implies β, then you may conclude β.

Inference Rules for Implication

Name	$\vdash$-notation	inference notation
$\rightarrow$-introduction ($\rightarrow$i)	If $\Sigma, \alpha \vdash_{ND} \beta$, then $\Sigma \vdash_{ND} (\alpha \rightarrow \beta)$	$\dfrac{\boxed{\begin{array}{c}\alpha \\ \vdots \\ \beta\end{array}}}{(\alpha \rightarrow \beta)}$

Intuition: If by assuming α is true we can get β, then α implies β.

Important: The "box" denotes a sub-proof. Nothing inside the sub-proof may come out. Outside of the sub-proof, we could only use the sub-proof as a whole.

Inference Rules for Implication

Example: Give a proof of $\{(p \to q), (q \to r)\} \vdash_{ND} (p \to r)$

Proof:

1.	$(p \to q)$	Premise
2.	$(q \to r)$	Premise
3.	p	Assumption
4.	q	$\to$e: 1, 3
5.	r	$\to$e: 2, 4
6.	$(p \to r)$	$\to$i: 3–5

Inference Rules for Implication

Prove: $p \to (q \to r) \vdash q \to (p \to r)$

Inference Rules for Disjunction

Name	**$\vdash$-notation**	**inference notation**
$\vee$-introduction ($\vee$i)	If $\Sigma \vdash_{ND} \alpha$, then $\Sigma \vdash_{ND} (\alpha \vee \beta)$ and $\Sigma \vdash_{ND} (\beta \vee \alpha)$	$\dfrac{\alpha}{(\alpha \vee \beta)} \qquad \dfrac{\alpha}{(\beta \vee \alpha)}$
$\vee$-elimination ($\vee$e)	If $\Sigma, \alpha_1 \vdash_{ND} \beta$ and $\Sigma, \alpha_2 \vdash_{ND} \beta$, then $\Sigma, (\alpha_1 \vee \alpha_2) \vdash_{ND} \beta$	$\dfrac{(\alpha_1 \vee \alpha_2) \quad \boxed{\begin{matrix}\alpha_1 \\ \vdots \\ \beta\end{matrix}} \quad \boxed{\begin{matrix}\alpha_2 \\ \vdots \\ \beta\end{matrix}}}{\beta}$

$\vee$e is also known as "proof by cases".

Intuition: from the tautology $(\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \to \beta) \wedge (\alpha_2 \to \beta) \to \beta$

Inference Rules for Implication

Example: Give a proof of $\{(p \vee q)\} \vdash_{ND} ((p \rightarrow q) \vee (q \rightarrow p))$

1.	$(p \vee q)$	Premise
2.	p	Assumption
3.	q	Assumption
4.	p	Reflexivity: 2
5.	$(q \rightarrow p)$	$\rightarrow$i: 3–4
6.	$((p \rightarrow q) \vee (q \rightarrow p))$	$\vee$i: 5
7.	q	Assumption
8.	p	Assumption
9.	q	Reflexivity: 7
10.	$(p \rightarrow q)$	$\rightarrow$i: 8–9
11.	$((p \rightarrow q) \vee (q \rightarrow p))$	$\vee$i: 10
12.	$((p \rightarrow q) \vee (q \rightarrow p))$	$\vee$e: 1, 2–6, 7–11

Examples

Prove that $\{p \to q\} \vdash (r \lor p) \to (r \lor q)$

Inference Rules for Negation

If an assumption α leads to a contradiction, then we have $(\neg\alpha)$.

Name	$\vdash$-notation	inference notation
$\neg$-introduction ($\neg$i)	If $\Sigma, \alpha \vdash_{ND} \bot$, then $\Sigma \vdash_{ND} (\neg\alpha)$	$\dfrac{\boxed{\begin{array}{c}\alpha \\ \vdots \\ \bot\end{array}}}{(\neg\alpha)}$

We shall use the notation $\bot$ to represent any contradiction.
It may appear in proofs as if it were a formula.

If we have both α and $(\neg\alpha)$, then we have a contradiction, also known as $\neg$e ($\neg$-elimination).

Name	$\vdash$-notation	inference notation
$\perp$-introduction	$\Sigma, \alpha, (\neg\alpha) \vdash_{ND} \perp$	$\dfrac{\alpha \quad (\neg\alpha)}{\perp}$

Inference Rules for Negation

Example. Show that $\{\alpha \to (\neg\alpha)\} \vdash_{ND} (\neg\alpha)$

1.	$(\alpha \to (\neg\alpha))$	Premise
2.	α	Assumption
3.	$(\neg\alpha)$	$\to$e: 1, 2
4.	$\perp$	$\perp$i: 2, 3
5.	$(\neg\alpha)$	$\neg$i: 2–4

Inference Rules for Negation

The elimination rule for *double negations*:

Name	$\vdash$-notation	inference notation
$\neg\neg$-elimination ($\neg\neg$e)	If $\Sigma \vdash_{ND} (\neg(\neg\alpha))$, then $\Sigma \vdash_{ND} \alpha$	$\dfrac{(\neg(\neg\alpha))}{\alpha}$

Contradiction elimination:

Name	$\vdash$-notation	inference notation
$\perp$-elimination ($\perp$e)	If $\Sigma \vdash_{ND} \perp$, then $\Sigma \vdash_{ND} \alpha$	$\dfrac{\perp}{\alpha}$

Prove: $\{A \vee B, \neg A\} \vdash B$

Derived Rules

Any proof that uses $\perp$e

27.	$\perp$	$\langle$*some rule*$\rangle$
28.	α	$\perp$e: 27.

...can be derived by existing inference rules.

27.	$\perp$	$\langle$*some rule*$\rangle$
28.	$(\neg\alpha)$	Assumption
29.	$\perp$	Reflexivity: 27
30.	$(\neg(\neg\alpha))$	$\neg$i: 28–29
31.	α	$\neg\neg$e: 30.

Whenever we have a proof of the form $\Sigma \vdash_{ND} \alpha$, we can consider it as a derived rule:

$$\frac{\Sigma}{\alpha}$$

If we use this in a proof, it can be replaced by the original proof of $\Sigma \vdash_{ND} \alpha$. The result is a proof using only the basic rules.

Using derived rules **does not expand the things that can be proved**. But they can make it easier to find a proof.

Modus tollens (MT, 否定后件): $\{(p \to q), (\neg q)\} \vdash_{ND} (\neg p)$

1.	$(p \to q)$	Premise
2.	$(\neg q)$	Premise
3.		
4.		
5.		
6.	$(\neg p)$	

Please finish the proof.

Derived Rules

Modus tollens can be used as a derived rule:

$$\frac{\alpha \to \beta \quad \neg\beta}{\neg\alpha}\ \text{MT}$$

Double-negation introduction:

$$\frac{\alpha}{(\neg(\neg\alpha))} \quad \neg\neg i$$

1.	α	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\perp$	$\perp$i: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: 2–3

Derived Rules

Proof by contradiction (reductio ad absurdum):

$$\dfrac{\boxed{\begin{array}{c}(\neg\alpha)\\ \vdots \\ \bot\end{array}}}{\alpha}\ \text{PBC}$$

1.	$((\neg\alpha) \to \bot)$	Premise
2.	$(\neg\alpha)$	Assumption
3.	$\bot$	$\to$e: 1, 2
4.	$(\neg(\neg\alpha))$	$\neg$i: 2–3
5.	α	$\neg\neg$e: 4

Law of Excluded Middle (tertiam non datur, 排中律):

$$\frac{}{(\alpha \vee (\neg\alpha))} \quad \text{LEM}$$

Proved in class.

Assigments

- Assignment 4 released.

Readings

Finished in 2 weeks

- TextB: Section 1.2
- Text1: 第二章 2.6

Introduction to Mathematical Logic

Thank you for listening!
Any questions?