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PDE (H) HW1.

Ex1. $\begin{cases} \partial_t u + \partial_x u = -u + e^{x+2t} \\ u(0,x) = 0. \end{cases}$

Let $w(t) = u(t, \eta(t))$

$w'(t) = \partial_t u + \eta'(t)\, \partial_x u(t, \underset{x}{\eta(t)})$.

Let $\eta'(t) = 1 \Rightarrow \eta(t) = t + C_1$. $\quad \eta(0) \triangleq x_0 \Rightarrow C_1 = x_0$.

$w'(t) \underset{\substack{\uparrow \\ \text{condition}}}{=} -w(t) + e^{x+2t} = -w(t) + e^{3t+x_0}$.

$\Rightarrow w(t) = C e^{-t} + \frac{1}{4} e^{x_0+3t}$.

$u(0,x) = 0 \Rightarrow w(0) = C + \frac{1}{4} e^{x_0} = 0. \Rightarrow C = -\frac{1}{4} e^{x_0}$.

$\Rightarrow w(t) = \frac{1}{4}(e^{x_0+3t} - e^{x_0-t})$.

Since on the char. curve: $x = x_0 + t$,

$u(t,x) = \frac{1}{4}(e^{x+2t} - e^{x-2t})$.

(If there is a sol., then it must be this)

Verification: if $u(t,x)$ is like above, then

$\partial_t u + \partial_x u + u = \frac{1}{2} e^{x+2t} + \cancel{\frac{1}{2} e^{x-2t}} + \frac{1}{4} e^{x+2t} \cancel{- \frac{1}{4} e^{x-2t}}$
$+ \frac{1}{4} e^{x+2t} - \cancel{\frac{1}{4} e^{x-2t}}$
$= e^{x+2t}$ ✓

$u(0,x) = \frac{1}{4}(e^x - e^x) = 0$. ✓

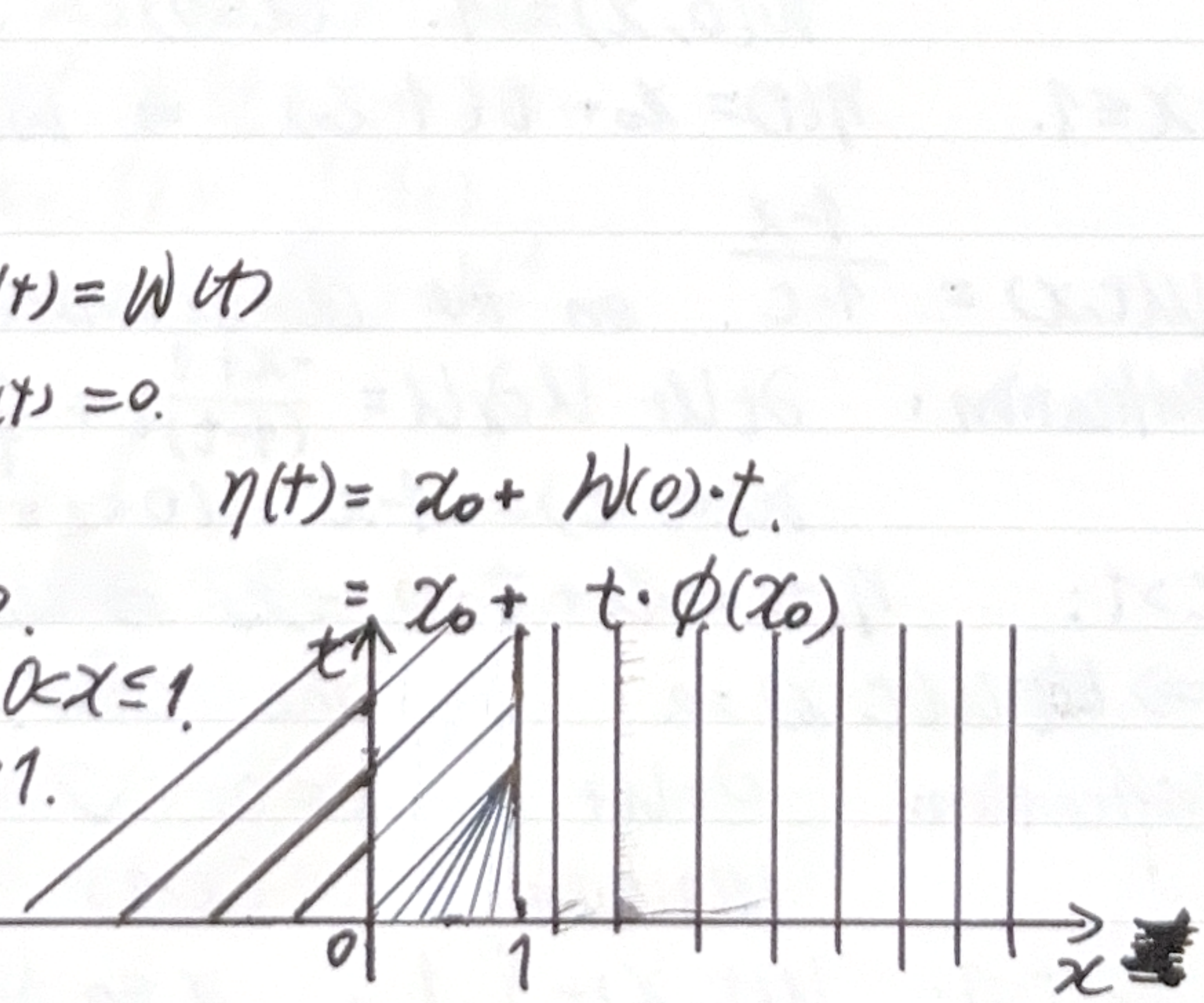
Thus, the sol. is $u(t,x) = \frac{1}{4}(e^{x+2t} - e^{x-2t})$.

Ex2.

1. char ODE: $\begin{cases} \eta'(t) = w(t) \\ w'(t) = 0. \end{cases}$

$\Rightarrow w(t) = \text{const}. \qquad \eta(t) = x_0 + w(0) \cdot t.$
$\qquad\qquad\qquad\qquad\qquad = x_0 + t \cdot \phi(x_0)$

$\phi(x) = \begin{cases} 1, & x \le 0. \\ 1 - x, & 0 < x \le 1. \\ 0, & x > 1. \end{cases}$



To more precisely consider the largest time, we derive the following:

~~Claim:~~

~~Proof of claim:~~ Consider the map $F_t : x_0 \mapsto \eta(t) = x_0 + t \cdot \phi(x_0)$.
To make sure no two characteristics intersect, we need to make sure there doesn't exist a time t, $\exists\ x_1 \neq x_2$ s.t. $F_t(x_1) = F_t(x_2)$.

$\Rightarrow F_t$ must be monotone in x_0.

Now, check the case $x_0 > 1$: $\phi(x_0) = 0$.

$\Rightarrow F_t : x_0 \longmapsto \eta(t) = x_0$.

F_t is monotone increasing in x_0.

$\Rightarrow$ For $x_0 \in \mathbb{R}$, find where F_t ~~starting~~ stop monotone increasing.

$$\frac{\partial}{\partial x_0} F_t = 1 + t\,\phi'(x_0) > 0 \Rightarrow \begin{cases} 1 + t > 0 \ \checkmark \\ 1 - t < 0 \end{cases} \Rightarrow t < 1.$$

($\phi'(x_0)$: $+1$, -1.)

$\Rightarrow t_s = 1$.

2. for $t < 1$:

① $x \leq 0$. $\eta(t) = x_0 + t \Rightarrow w(t) = \dot\eta(t) = 1$

$\Rightarrow u(t,x) = 1$ on the characteristics.

Verification: $\partial_t u + u\partial_x u = 0 + 1 \cdot 0 = 0$. ✓

$u(0,x) = 1$. $(x \leq 0)$ ✓.

② $0 < x \leq 1$. $\eta(t) = x_0 + t(1 - x_0) \Rightarrow w(t) = \dot\eta(t) = 1 - x_0$
$= \dfrac{1-x}{1-t}$.

$\Rightarrow u(t,x) = \dfrac{1-x}{1-t}$. on the characteristics.

Verification: $\partial_t u + u\partial_x u = \dfrac{-x+1}{(1-t)^2} + \dfrac{1-x}{1-t}\,\dfrac{-1}{1-t} = 0$. ✓

$u(0,x) = 1 - x$ $(0 < x \leq 1)$ ✓

③ $x > 1$: $\eta(t) = x_0 + t \cdot 0 = x_0 \Rightarrow w(t) = \dot\eta(t) = 0$.

$\Rightarrow$ ~~the~~ $u(t,x) = 0$. on the char.

Verification: $\partial_t u + u\partial_x u = 0$. ✓

$u(0,x) = 0$. $(x > 1)$ ✓.

Thus, for $t < 1$: $u(t,x) = \begin{cases} 1, & x \leq 0. \\ \frac{1-x}{1-t}, & 0 < x \leq 1. \\ 0, & x > 1. \end{cases}$

Ex3.

1. $\begin{cases} x_1 \partial_{x_1} u + x_2 \partial_{x_2} u = 2u \\ u(x_1, 1) = g(x_1) \end{cases} \Leftrightarrow \partial_{x_1} u + \frac{x_2}{x_1} \partial_{x_2} u = \frac{2u}{x_1}$

$\Leftrightarrow \frac{x_1}{x_2} \partial_{x_1} u + \partial_{x_2} u = \frac{2u}{x_2}.$

(Conditions: $\underset{\substack{\uparrow \\ x}}{u(x_1}, \underset{\substack{\uparrow \\ t}}{x_2)}: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$).

Let $w(x_2) = u(\eta(x_2), x_2)$.

$\dot{w}(x_2) = \eta'(x_2) \partial_{x_1} u + \partial_{x_2} u$

Let $\eta'(x_2) = \frac{\eta(x_2)}{x_2} \Rightarrow \eta(x_2) = x_2 \cdot C_1$. $\quad x_2 \cdot x_{21}$.

$\eta(1) = x_{21} \Rightarrow C_1 = x_{21} \Rightarrow \eta(x_2) =$ ... x_{21}

$\dot{w}(x_2) = \frac{2}{x_2} w(x_2). \Rightarrow w(x_2) = e^{C_2} \cdot x_2^2.$

$w(1) = u(x_{21}, 1) = g(x_{21}) = e^{C_2}.$

$\Rightarrow w(x_2) = g(x_{21}) \cdot x_2^2.$

$\Rightarrow u(x_1, x_2) = g(\frac{x_1}{x_2}) \cdot x_2^2$. (on the char.)

Verification: $x_1 \partial_{x_1} u + x_2 \partial_{x_2} u$

$= x_1 \cdot g'(\frac{x_1}{x_2}) \cdot \frac{1}{x_2} \cdot x_2^2 + x_2 \cdot g'(\frac{x_1}{x_2}) (-\frac{x_1}{x_2^2}) x_2^2$

$+ x_2 \cdot g(\frac{x_1}{x_2}) \cdot 2x_2$

$= 2x_2^2 \cdot g(\frac{x_1}{x_2}) = 2u.$ ✓.

$u(x_1, 1) = g(x_1).$ ✓

Thus, $u(x_1, x_2) = g(\frac{x_1}{x_2}) \cdot x_2^2.$

2. $\begin{cases} \partial_{x_2} u + u \partial_{x_1} u = 1 \\ u(x_1, x_1) = \frac{1}{2} x_1. \end{cases}$

($u(x_2, x_1): \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$).

Let $u(x_2, \eta(x_2)) = w(x_2)$.

$\dot{w}(x_2) = \partial_{x_2} u + \eta'(x_2) \partial_{x_1} u.$

Let $\eta'(x_2) = w(x_2)$.

$\dot{w}(x_2) = 1. \Rightarrow w(x_2) = x_2 + C_1 = \eta'(x_2) \Rightarrow \eta(x_2) = \frac{1}{2} x_2^2 + C_1 x_2 + C_2.$

$= u(x_2, \eta(x_2)).$

$\cancel{\Rightarrow u(x_2, \tfrac{1}{2}x_2^2 + C_1x_2 + C_2) = x_2 + C_1}$

Consider a characteristic which intersects the line $x_1 = x_2$ at a point (defined as $(x_0, x_{20} = x_0)$).

$\eta(x_{20}) = \eta(x_0) = x_0.$

$w(x_{20}) = w(x_0) = u(x_0, x_0) = \frac{1}{2}x_0.$

$\Rightarrow x_0 + C_1 = \frac{1}{2}x_0 \qquad C_1 = -\frac{1}{2}x_0.$

$\eta(x_0) = x_0 = \frac{1}{2}x_0^2 - \frac{1}{2}x_0^2 + C_2 \Rightarrow C_2 = x_0.$

$\Rightarrow \eta(x_2) = \frac{1}{2}x_2^2 - \frac{1}{2}x_0x_2 + x_0.$

$w(x_2) = x_2 - \frac{1}{2}x_0.$

$\Rightarrow \cancel{\eta(x_2, x_1) = x_2 - \tfrac{1}{2}(x_1 - \tfrac{1}{2}x_2^2}$

$u(x_2, \eta(x_2)) = w(x_2) = x_2 - \frac{1}{2}x_0.$

$x_1 = \eta(x_2) = \frac{1}{2}x_2^2 - \frac{1}{2}x_0x_2 + x_0 = \frac{1}{2}x_2^2 + (1 - \frac{1}{2}x_2)x_0.$

$$u(x_2, x_1) = x_2 - \frac{1}{2}\left(\frac{x_1 - \frac{1}{2}x_2^2}{1 - \frac{1}{2}x_2}\right) = x_2 - \frac{x_1 - \frac{1}{2}x_2^2}{2 - x_2}.$$

$$= \frac{2x_2 - x_2^2 - x_1 + \frac{1}{2}x_2^2}{2 - x_2} = \frac{-x_2^2 + 4x_2 - 2x_1}{4 - 2x_2}.$$

$$= \frac{2x_1 + x_2^2 - 4x_2}{2x_2 - 4}.$$

Verification: $\partial_{x_2}u + u\,\partial_{x_1}u$

$$= \frac{(2x_2-4)^2 - 2(\cancel{2x_1} + \cancel{x_2^2} - \cancel{4x_2})}{(2x_2-4)^2} + \frac{\cancel{4x_1} + \cancel{2x_2^2} - \cancel{8x_2}}{(2x_2-4)^2}$$

$= 1.$ ✓

$$u(x_1, x_1) = \frac{2x_1 + x_1^2 - 4x_1}{2x_1 - 4} = \frac{x_1(x_1-2)}{2(x_1-2)} = \frac{1}{2}x_1 \; ✓.$$

Thus, $$u(x_2, x_1) = \frac{2x_1 + x_2^2 - 4x_2}{2x_2 - 4}.$$

Ex 4.

1. $u_\lambda(t,x) := u(\lambda^2 t, \lambda x)$, with $u(t,x)$ being a solution.

$\cancel{\partial_t u_\lambda - \Delta u_\lambda = \lambda^2 \partial_t u(\lambda^2 t, \lambda x) - \lambda^2 \Delta u(\lambda^2 t, \lambda x)}$

$\cancel{= \lambda^2(\partial_t u(\lambda^2 t, \lambda x) - \Delta u(\lambda^2 t, \lambda x)) = \lambda^2 \cdot 0 = 0}$

$$\frac{\partial u_\lambda}{\partial t} = \partial_t u(\lambda^2 t, \lambda x)\cdot \lambda^2.$$

$$\Delta_x u_\lambda = \nabla\cdot\nabla u_\lambda = \nabla\cdot\left(\frac{\partial u_\lambda}{\partial x_1}, \dots, \frac{\partial u_\lambda}{\partial x_d}\right)$$

$$= \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)\cdot\left(\lambda \partial_{\xi_1} u(\lambda^2 t, \lambda x), \dots, \lambda\, \partial_{\xi_d} u(\lambda^2 t, \lambda x)\right)$$

$$= \sum_{i=1}^{d} \lambda^2 \frac{\partial^2}{\partial \xi_i^2} u(\lambda^2 t, \lambda x) = \lambda^2 \sum_{i=1}^{d} \frac{\partial^2}{\partial \xi_i^2} u(\lambda^2 t, \lambda x).$$

$$= \lambda^2 \Delta_\xi u(\lambda^2 t, \lambda x)$$

$$\Rightarrow \partial_t u_\lambda - \Delta_x u_\lambda = \lambda^2\left(\partial_t u(\lambda^2 t, \lambda x) - \Delta_\xi u(\lambda^2 t, \lambda x)\right).$$

Since u satisfies $\partial_t u - \Delta_\xi u(\tau, \xi) = 0$ for any arguments (τ, ξ), we know

$$\partial_t u_\lambda - \Delta_x u_\lambda = \lambda^2\left(\partial_\tau u(\lambda^2 t, \lambda x) - \Delta_\xi u(\lambda^2 t, \lambda x)\right)$$

$$= \lambda^2 \cdot 0 = 0. \qquad \square.$$

2. Take derivative in λ:

$$\frac{\partial}{\partial \lambda} u_\lambda(t,x) = 2\lambda t \cdot \frac{\partial}{\partial \tau} u(\lambda^2 t, \lambda x) + x \frac{\partial}{\partial \xi} u(\lambda^2 t, \lambda x).$$

$$= x\cdot \nabla_\xi u(\lambda^2 t, \lambda x) + 2t\cdot\lambda\cdot \partial_\tau u(\lambda^2 t, \lambda x).$$

Define the heat eq. operator: $H(\cdot) = \partial_t - \Delta_x$.

then, we have $H(u_\lambda(t,x)) = H(u(\lambda^2 t, \lambda x)) = 0.\ \forall \lambda.$

$$\Rightarrow \partial_\lambda\left(H(u_\lambda(t,x))\right) = \frac{\partial}{\partial \lambda}(0) = 0$$

$$\|$$

Since u is smooth $\quad H\left(\partial_\lambda(u_\lambda(t,x))\right) \quad \Rightarrow \partial_\lambda(u_\lambda(t,x))$ solves the heat equation, $\forall \lambda \in \mathbb{R}$.

Take $\lambda = 1$. $\Rightarrow x\cdot\nabla_\xi u(t, x) + 2t\cdot \partial_\tau u(t, x)$ solves the eq.

i.e. $x\cdot\nabla_x u(t,x) + 2t\cdot\partial_t u(t,x)$ solves the eq.

i.e. $v(t,x)$ also solves the equation $\square$.