Introduction to Mathematical Logic

For CS Students

CS104

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Table of Contents

What's a proof?

A proof $\Sigma \vdash A$:

- Starts with a set of premises Σ
- Transforms the premises using a set of inference rules
- Ends with the conclusion

A proof is purely syntactic: one could write a computer program that would verify the proof.

$\Sigma \models A$:

$\Sigma \models A$ if and only if every valuation satisfying $\Sigma^v = 1$ implies that $A^v = 1$.

$\Sigma \vdash A$:

$\Sigma \vdash A$ if and only if there is a proof in the deduction system begining with the premises of Σ and ending with A.

What's the "relations" between (syntactic) proof and (semantic) entailment?
What are desired properties of a proof system?

Table of Contents

Soundness (可靠性)

Soundness means that the conclusion of a proof is always a logical consequence of the premises. That is,

$$\text{If } \Sigma \vdash A, \text{ then } \Sigma \models A$$

If we could syntactically prove a conclusion (e.g., by applying inference rules), then we could also semantically entail the conclusion.

Soundness (可靠性)

Soundness means that the conclusion of a proof is always a logical consequence of the premises. That is,

$$\text{If } \Sigma \vdash A, \text{ then } \Sigma \vDash A$$

Soundness shows that the proof rules are all correct in the sense that valid proof sequents all “preserve truth” computed by our truth-table semantics.

If a proof system is not sound, then it’s pretty much useless.

Completeness (完备性)

Completeness means that all logical consequences in propositional logic are provable in the proof system. That is,

$$\text{If } \Sigma \models A, \text{ then } \Sigma \vdash A$$

If we could semantically entail a conclusion, then we could also prove it syntactically using the proof system.

Table of Contents

Soundness of ND

Let's prove the soundness of ND, that is

$$\text{If } \Sigma \vdash_{ND} \alpha, \text{ then } \Sigma \vDash \alpha$$

Soundness of ND

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$$\text{If } \Sigma \vdash_{ND} \alpha, \text{ then } \Sigma \vDash \alpha$$

The proof is by induction on proof length :The length of a natural deduction proof is the number of lines in it.

Let $P(n)$ be the statement that "For any well-formed formula φ and any set of well-formed formulas Σ, if Σ proves φ in n lines, then $\Sigma \vDash \varphi$." We prove $P(n)$ is true for all positive integers n.

Inductive Proof: Base Case

Base Case: $n = 1$

If $\Sigma \vdash_{ND} \alpha$ with a proof of size 1 (a one-line proof) for any wff α, then $\alpha \in \Sigma$.

1.	α	Premise

Hence, whenever a valuation makes Σ true, since $\alpha \in \Sigma$, it must make α true (by definition of the satisfiability of a set of formulas). Thus, by definition of semantic entailment, $\Sigma \models \alpha$.

Inductive Hypothesis

Assume that $P(i)$ holds for all integers $1 \leq i \leq k$ for some integer k.

Inductive Proof: Inductive Step

Inductive Step

Consider the last line of $\Sigma \vdash_{ND} \alpha$

$$\begin{array}{lll} 1. & & \\ \vdots & \vdots & \\ k & & \\ k+1 & \alpha & \text{Some rule} \end{array}$$

Which rule could give us α?

Inductive Step: Case $\wedge e$

If α is derived by the rule $\wedge e$, then the proof looks like below for some formula β:

$1.$		
$\vdots$	$\vdots$	
j	$\alpha \wedge \beta$	Some rule
$\vdots$	$\vdots$	
k		
$k+1$	α	$\wedge e$: j

The subsequence of this proof from top to $\alpha \wedge \beta$ is a proof of $\alpha \wedge \beta$ of length $\leq k$, i.e., $\Sigma \vdash_{ND} \alpha \wedge \beta$. By inductive hypothesis, $\Sigma \vDash \alpha \wedge \beta$

If $\Sigma \vDash \alpha \wedge \beta$, then $\Sigma \vDash \alpha$ (can be proved by definition).

Inductive Step: Case $\wedge i$

If α is derived by the rule $\wedge i$, then α has the form $\beta_1 \wedge \beta_2$ and the proof looks like:

$$
\begin{array}{lll}
1. & & \\
\vdots & \vdots & \\
c & \beta_1 & \ldots \\
\vdots & \vdots & \\
j & \beta_2 & \ldots \\
\vdots & \vdots & \\
k & & \\
k+1 & \beta_1 \wedge \beta_2 & \wedge i : \text{c,j}
\end{array}
$$

Inductive Step: Case $\wedge i$

$$
\begin{array}{lll}
1. & & \\
\vdots & \vdots & \\
c & \beta_1 & \ldots \\
\vdots & \vdots & \\
j & \beta_2 & \ldots \\
\vdots & \vdots & \\
k & & \\
k+1 & \beta_1 \wedge \beta_2 & \wedge i : \text{c,j}
\end{array}
$$

$\Sigma \vdash_{ND} \beta_1$ is a proof of length $\leq k$. By inductive hypothesis, $\Sigma \vDash \beta_1$.
$\Sigma \vdash_{ND} \beta_2$ is a proof of length $\leq k$. By inductive hypothesis, $\Sigma \vDash \beta_2$.
Hence, we have $\Sigma \vDash \beta_1 \wedge \beta_2$ (can be proved by definition).

Inductive Step: Case $\rightarrow i$

If α is derived by the rule $\rightarrow i$, then α has the form $(\beta \rightarrow \gamma)$ for well-formed formulas β and γ, and the proof looks like:

$$
\begin{array}{r|ll|}
\multicolumn{1}{r}{1.} & \multicolumn{2}{l}{} \\
\multicolumn{1}{r}{\vdots} & \multicolumn{2}{l}{\quad\vdots} \\
\cline{2-3}
j. & \beta & \text{Assumption} \\
\vdots & \quad\vdots & \\
k. & \gamma & \text{Some Rule} \\
\cline{2-3}
\multicolumn{1}{r}{k+1.} & \multicolumn{2}{l}{\alpha \qquad \rightarrow\text{i: } j\text{-}k}
\end{array}
$$

Inductive Step: Case $\to i$

By removing line $k+1$, we no longer have a complete proof. However, if we add β to our list of premises, we can then prove γ in $\leq k$ lines, i.e.,

$$\Sigma \cup \{\beta\} \vdash_{ND} \gamma$$

By the Induction Hypothesis, we have:

$$\Sigma \cup \{\beta\} \models \gamma$$

Notice here that since our induction hypothesis holds for ANY Σ, we could take a new sigma, say $\Sigma_1 = \Sigma \cup \{\beta\}$ and use the IH on this.

Inductive Step: Case $\to i$

Finally, we claim that if $\Sigma \cup \{\beta\} \vDash \gamma$, then $\Sigma \vDash (\beta \to \gamma)$.

Proof by contradiction: assume $\Sigma \nvDash (\beta \to \gamma)$, then get a contradiction.

Inductive Step: Case $\to e$

If α is derived by the rule $\to e$, then the proof looks like ...

Inductive Step: More Cases

There ares a few more Natural Deduction rules we still have to consider, which are left as exercises.

This completes the proof of soundness.

Implications

Once proven, soundness implies that to prove a semantic entailment $\Sigma \vDash \alpha$, we can use:

- True table
- Proof by Definition (Assume we have a valuation that evaluates true for all premises and show that this implies that the conclusion also evaluates to true under the same valuation)
- Proof by Contradiction
- $\Sigma \vdash_{ND} \alpha$ (because of soundness)

Table of Contents

Completeness of ND

Let's prove the completeness of ND, that is

$$\text{If } \Sigma \vDash \alpha, \text{ then } \Sigma \vdash_{ND} \alpha$$

That is, every logical consequence has a proof.

Let $\Sigma = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ for wff α_i (for all $0 \leq i \leq n$). To prove completeness, we want to show for any wff β:

$$\text{If } \Sigma \vDash \beta \text{ holds, then } \Sigma \vdash_{ND} \beta \text{ is valid.}$$

The proof is done if we could prove the 3 lemmas below:

- **Lemma 1**: If $\Sigma \vDash \beta$, then $\emptyset \vDash (\alpha_0 \rightarrow (\alpha_1 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots))$
- **Lemma 2**: For any wff γ, If $\emptyset \vDash \gamma$, then $\emptyset \vdash_{ND} \gamma$ (i.e., tautologies are provable)
- **Lemma 3**: If $\emptyset \vdash_{ND} (\alpha_0 \rightarrow (\alpha_1 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots))$, then $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \vdash_{ND} \beta$, which is exactly $\Sigma \vdash_{ND} \beta$.

Proof of Lemma 1

Lemma 1: If $\Sigma \vDash \beta$, then $\emptyset \vDash (\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta)\ldots))$

Proof: If $\Sigma \vDash \beta$, assume towards a contradiction that

$$\emptyset \nvDash (\alpha_0 \to (\alpha_1 \to (\ldots \to (\alpha_n \to \beta)\ldots))$$

Then, there exists a truth valuation v such that the long implication on the right is *false*. Unwinding implication by implication, this means that $\alpha_i^v = 1$ for all $1 \le i \le n$ and that $\beta^v = 0$. This contradicts with $\Sigma \vDash \beta$.

Hence, we proved lemma 1.

Proof of Lemma 3

Lemma 3: If $\emptyset \vdash_{ND} (\alpha_0 \rightarrow (\alpha_1 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots))$, then $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \vdash_{ND} \beta$.

Proof:

$1.$		
$\vdots$	$\vdots$	
$k.$	$(\alpha_0 \rightarrow (\alpha_1 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)..)))$	Some Rule
$k+1.$	α_0	Premise
$k+2.$	$(\alpha_1 \rightarrow (\alpha_2 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)..)))$	$\rightarrow$ e: $k, k+1$
$k+2.$	α_1	Premise
$k+3.$	$(\alpha_2 \rightarrow (\alpha_3 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)..)))$	$\rightarrow$ e: $k+2, k+3$
$\vdots$	$\vdots$	
$k+(2n+1).$	α_n	Premise
$k+(2n+2).$	β	$\rightarrow$ e: $k+(2n+1), k+(2n+2)$

Lemma 2: For any wff γ, If $\emptyset \vDash \gamma$, then $\emptyset \vdash_{ND} \gamma$ (i.e., tautologies are provable)

Idea of proof: Let's use the tautology $p \land q \rightarrow p$ as an example.
$p \land q \rightarrow p$ has two atoms p and q. First, we want to prove:

$$p, q \vdash p \land q \rightarrow p$$
$$\neg p, q \vdash p \land q \rightarrow p$$
$$p, \neg q \vdash p \land q \rightarrow p$$
$$\neg p, \neg q \vdash p \land q \rightarrow p.$$

Proof of Lemma 2

4 Completeness of the ND Proof System

Then, we could leverage the proof, LEM, and $\vee e$ to prove $p \wedge q \to p$.

1	$p \vee \neg p$							LEM
2	p			ass	$\neg p$			ass
3	$q \vee \neg q$			LEM	$q \vee \neg q$			LEM
4	q	ass	$\neg q$	ass	q	ass	$\neg q$	ass
5	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
6								
7	$p \wedge q \to p$		$p \wedge q \to p$		$p \wedge q \to p$		$p \wedge q \to p$	
8	$p \wedge q \to p$			$\vee$e	$p \wedge q \to p$			$\vee$e
9	$p \wedge q \to p$							$\vee$e

Note that we get line 7 from the result of the previous page.

Proof of Lemma 2

Hence, we only need to prove:

$$
\begin{aligned}
p, q &\vdash p \land q \to p \\
\neg p, q &\vdash p \land q \to p \\
p, \neg q &\vdash p \land q \to p \\
\neg p, \neg q &\vdash p \land q \to p.
\end{aligned}
$$

Here, $p \land q \to p$ could be generalized to any tautology.

Proof of Lemma 2

The observation could be generalize to the following sublemma:

Sublemma: For any formula γ containing atoms $p_1, p_2, ..., p_n$ and any valuation v:

- If $\gamma^v = 1$, then $\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \gamma$
- If $\gamma^v = 0$, then $\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash (\neg\gamma)$

where $\widehat{p}_i$ is defined as follows for all $1 \leq i \leq n$:

$$\widehat{p}_i = \begin{cases} p_i & \text{if } p_i^v = 1 \\ \neg p_i & \text{if } p_i^v = 0 \end{cases}$$

Proof of Lemma 2

Sublemma: For any formula γ containing atoms $p_1, p_2, ..., p_n$ and any valuation v:

- If $\gamma^v = 1$, then $\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \gamma$
- If $\gamma^v = 0$, then $\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash (\neg\gamma)$

Example: $\gamma = (p \to q)$. Consider this truth table:

p	q	$(p \to q)$	Claim in sublemma
T	T	T	$\{p, q\} \vdash_{ND} (p \to q)$
T	F	F	$\{p, (\neg q)\} \vdash_{ND} (\neg(p \to q))$
F	T	T	$\{(\neg p), q\} \vdash_{ND} (p \to q)$
F	F	T	$\{(\neg p), (\neg q)\} \vdash_{ND} (p \to q)$

Proof of Lemma 2

Sublemma: For any formula γ containing atoms $p_1, p_2, ..., p_n$ and any valuation v:

- If $\gamma^v = 1$, then $\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \gamma$
- If $\gamma^v = 0$, then $\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash (\neg\gamma)$

The first case of this sublemma is exactly a generalization of the previous example, in which all the valuations of a tautology is true.

$$p, q \vdash p \wedge q \rightarrow p$$
$$\neg p, q \vdash p \wedge q \rightarrow p$$
$$p, \neg q \vdash p \wedge q \rightarrow p$$
$$\neg p, \neg q \vdash p \wedge q \rightarrow p.$$

So, all we need to do right now is to prove the sublemma.

We prove sublemma by structural induction.

Base case: $\gamma = p_1$ (atom):

- If $\gamma^v = p_1^v = 1$, then $\widehat{p}_1 = p_1$, $p_1 \vdash p_1 = \gamma$
- If $\gamma^v = p_1^v = 0$, then $\widehat{p}_1 = \neg p_1$, $\neg p_1 \vdash \neg p_1 = \neg\gamma$

Induction Hypothesis: Let $P(\gamma)$ be the statement verbatim as above. Assume that $P(\gamma_1)$ and $P(\gamma_2)$ are true for some wff γ_1 and γ_2.

Induction Step:

Case 1: $\gamma = \neg\gamma_1$

If $\gamma^v = 1$, then $(\neg\gamma_1)^v = 1$ so $\gamma_1^v = 0$. Since $P(\gamma_1)$ is true by inductive hypothesis, and γ_1 has the same atoms as γ, we have:

$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \neg\gamma_1 = \gamma$$

If $\gamma^v = 0$, then $(\neg\gamma_1)^v = 0$ so $\gamma_1^v = 1$. Again, by inductive hypothesis, we have:

$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \gamma_1$$

By applying the $\neg\neg i$ rule, we have

$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash (\neg(\neg\gamma_1)) = \neg\gamma$$

which completes the proof of case 1.

Case 2: $\gamma = \gamma_1 \to \gamma_2$.

Case 2a: If $\gamma^v = 0$, then $\gamma_1^v = 1$ and $\gamma_2^v = 0$.
Suppose that γ_1 contains atoms $q_1, ..., q_k$ and γ_2 contains atoms $r_1, ..., r_k$. By induction hypothesis, we have:

$$\{\widehat{q}_1, \widehat{q}_2, ..., \widehat{q}_n\} \vdash \gamma_1$$
$$\{\widehat{r}_1, \widehat{r}_2, ..., \widehat{r}_n\} \vdash \neg\gamma_2$$

Since both $\{\widehat{q}_1, \widehat{q}_2, ..., \widehat{q}_n\}$ and $\{\widehat{r}_1, \widehat{r}_2, ..., \widehat{r}_n\}$ are subsets of $\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\}$, we also have:

$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \gamma_1$$
$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \neg\gamma_2$$

Case 2a (continued): Since we have:

$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \gamma_1$$
$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \neg\gamma_2$$

We could apply the $\wedge i$ rule to get:

$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \gamma_1 \wedge (\neg\gamma_2)$$

We could prove (how?):

$$\gamma_1 \wedge (\neg\gamma_2) \vdash \neg(\gamma_1 \rightarrow \gamma_2)$$

Hence, we have:

$$\{\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_n\} \vdash \neg(\gamma_1 \rightarrow \gamma_2) = \neg\gamma$$

which completes the proof of case 2a.

Case 2: $\gamma = \gamma_1 \to \gamma_2$.
Case 2b: If $\gamma^v = 1$, then one of the following holds.

- $\gamma_1 = 1$ and $\gamma_2 = 1$.
- $\gamma_1 = 0$ and $\gamma_2 = 1$.
- $\gamma_1 = 0$ and $\gamma_2 = 0$.

The proof is similar to case 2a.

In a similar manner, we could prove the claim for other binary connectives, which are left as exercises.
This completes the proof of the sublemma.

Properties of formal proof systems

Each of the 3 formal proof systems we discuss is both sound and complete.

- Hilbert-style system: $\Sigma \vdash_H A$
- Natural Deduction System: $\Sigma \vdash_{ND} A$
- Resolution: $\Sigma \vdash_{Res} A$

Example: A proof system without the properties

Intuitionist's Natural Deduction

- Natural Deduction without the double negation elimination rule
- Intuitionist's Natural Deduction is sound but not complete, since we cannot prove $p \vee \neg p$ without the $\neg\neg e$ rule.

Example: A proof system without the properties

A proof system: Natural Deduction and $p \wedge \neg p$ as an axiom.

- This proof system is complete, but not sound.
- This is not sound since it contains a contradiction and hence we can prove anything.
- It's complete since anything that we can semantically entail will have a proof after we have our contradiction.

Readings

Readings for the proof of soundness and completeness:

- Hilbert-style system: TextF Chapter 3.6
- Natural Deduction System: TextB Chapter 1.4.3, 1.4.4
- Resolution: TextF Chapter 4.4

Reference: CS245 course notes of University of Waterloo.

Introduction to Mathematical Logic

Thank you for listening!
Any questions?