

## HW3

September 25, 2025

Recall that the heat kernel is given by

$$G_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^d.$$

**Exercise 1** 1. Let  $\phi \in \mathcal{C}[0, \infty)$  with  $\phi(0) = 0$ . Show that

$$u(t, x) = \int_0^\infty [G_t(x - y) - G_t(x + y)] \phi(y) dy$$

satisfies  $\partial_t u - \partial_{xx} u = 0$  for  $t, x > 0$ ,  $u(t, 0) \equiv 0$  and

$$\lim_{t \rightarrow 0^+} u(t, x) = \phi(x), \quad x > 0.$$

*Hint: extend  $\phi(x)$  to an odd function on  $\mathbb{R}$ .*

2. Find a solution for the half-line heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t u - \Delta u = 0, & t, x > 0, \\ u(0, x) = \phi(x), & x > 0, \\ \partial_x u(t, 0) = 0, & t > 0, \end{cases}$$

where  $\phi \in \mathcal{C}(\mathbb{R})$  and  $\phi'(0) = 0$ .

*Hint: extend  $\phi(x)$  to an even function on  $\mathbb{R}$ .*

**Exercise 2** Let  $g \in \mathcal{C}^\infty[0, \infty)$  satisfy  $g(0) = 0$ . Derive that the solution to the heat equation on half-line

$$\begin{cases} \partial_t u = \partial_{xx} u, & t > 0, \quad x > 0, \\ u(0, x) = 0, & t = 0, \quad x > 0, \\ u(t, 0) = g(t), & x = 0, \quad t \geq 0, \end{cases}$$

is given by

$$u(t, x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds.$$

*Hint: let  $v(t, x) := u(t, x) - g(t)$ ; consider the odd extension of  $v$  and solve the resulting non-homogeneous heat equation.*

**Exercise 3** Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, \pi), \quad t > 0, \\ u(0, x) = \sin x, & x \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, & t \geq 0. \end{cases}$$

**Exercise 4** Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, \ell), \ t > 0, \\ u(0, x) = x^2(\ell - x)^2, & x \in [0, \ell], \\ \partial_x u(t, 0) = \partial_x u(t, \ell) = 0, & t \geq 0. \end{cases}$$

**Exercise 5** Let  $u(t, x)$  be the solution to the following initial-Neumann problem obtained via Separation of Variables:

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, \ell), \ t > 0, \\ u(0, x) = \varphi(x), & x \in [0, \ell], \\ \partial_x u(t, 0) = \partial_x u(t, \ell) = 0, & t \geq 0. \end{cases}$$

Show that  $\lim_{t \rightarrow \infty} u(t, x)$  exists and find the limit.

Can you give a physical interpretation of this limit?

Ex 1. (1) 将  $\phi$  延拓成  $\mathbb{R}$  上的奇函数  $\tilde{\phi} \in C(\mathbb{R})$ , 即  $\tilde{\phi}(x) = \begin{cases} \phi(x) & , x \in [0, \infty) \\ -\phi(-x) & , x \in (-\infty, 0) \end{cases}$

$$\begin{aligned} \text{则 } u(t, x) &= \int_0^\infty [G_t(x-y) - G_t(x+y)] \phi(y) dy \\ &= \int_{-\infty}^\infty G_t(x-y) \tilde{\phi}(y) dy \\ &= G_t * \tilde{\phi}(x) \end{aligned}$$

$$\text{从而, } \textcircled{1} \partial_t u(t, x) - \partial_{xx} u(t, x) = (\partial_t G_t - \partial_{xx} G_t) * \tilde{\phi}(x) = 0$$

$$\textcircled{2} \forall t > 0, u(t, 0) = \int_0^\infty [G_t(-y) - G_t(y)] \phi(y) dy = 0$$

$$\textcircled{3} \forall x > 0, \lim_{t \rightarrow 0^+} u(t, x) = \lim_{t \rightarrow 0^+} G_t * \tilde{\phi}(x) \Big|_{x>0} = \tilde{\phi}(x) \Big|_{x>0} = \phi(x)$$

(2) 将  $\phi$  延拓成  $\mathbb{R}$  上的偶函数  $\tilde{\phi} \in C(\mathbb{R})$ , 即  $\tilde{\phi}(x) = \begin{cases} \phi(x) & , x \in [0, \infty) \\ \phi(-x) & , x \in (-\infty, 0) \end{cases}$

$$\text{注意到, } (\partial_t G_t - \partial_{xx} G_t) * \tilde{\phi}(x) = 0$$

$$\text{故取 } u(t, x) = G_t * \tilde{\phi}(x) = \int_0^\infty [G_t(x-y) + G_t(x+y)] \phi(y) dy$$

$$\text{验证: } \textcircled{1} \forall x > 0, \lim_{t \rightarrow 0^+} u(t, x) = \lim_{t \rightarrow 0^+} G_t * \tilde{\phi}(x) \Big|_{x>0} = \tilde{\phi}(x) \Big|_{x>0} = \phi(x)$$

$$\textcircled{2} \forall t > 0, \partial_x u(t, 0) = \int_0^\infty [\partial_y G_t(-y) + \partial_y G_t(y)] \phi(y) dy = 0$$

$$\text{故 } u(t, x) = \int_0^\infty [G_t(x-y) + G_t(x+y)] \phi(y) dy \text{ 即是所求之解.}$$

Ex 2. 令  $v(t, x) := u(t, x) - g(t)$ ,

将其延拓成  $\mathbb{R}$  上的奇函数  $\tilde{v}(t, x)$ , 即  $\tilde{v}(t, x) = \begin{cases} v(t, x) & , x \in [0, \infty) \\ -v(t, -x) & , x \in (-\infty, 0) \end{cases}$

$$\text{则 } \tilde{v}(t, x) \text{ solves } \begin{cases} \partial_t \tilde{v} = \partial_{xx} \tilde{v} + f & , t > 0, x \in \mathbb{R} \setminus \{0\} \\ \tilde{v}(0, x) = 0 & , x \in \mathbb{R} \\ \tilde{v}(t, 0) = 0 & , t \geq 0 \end{cases} \quad \text{with } f(t, x) := \begin{cases} -g'(t) & , x \geq 0 \\ g'(t) & , x < 0 \end{cases}$$

$$\begin{aligned} \text{从而 } \tilde{v}(t, x) &= \int_0^t ds \int_{-\infty}^\infty \Gamma(t, x; s, y) f(s, y) dy \\ &= \int_0^t ds \int_0^\infty G_{t-s}(x-y) (-g'(s)) dy + \int_0^t ds \int_{-\infty}^0 G_{t-s}(x-y) g'(s) dy \\ &= 2 \int_0^t g'(s) ds \int_{-\infty}^0 G_{t-s}(x-y) dy - \int_0^t ds \int_{-\infty}^\infty G_{t-s}(x-y) g'(s) dy \\ &= 2 \int_0^t g'(s) ds \int_x^\infty G_{t-s}(y) dy - g(t) \\ &= 2 \left[ g(s) \int_x^\infty G_{t-s}(y) dy \right] \Big|_{s=0}^{s=t} - 2 \int_0^t g(s) ds \int_x^\infty \partial_s G_{t-s}(y) dy - g(t) \\ &= 2 \int_0^t g(s) ds \int_x^\infty \partial_y^2 G_{t-s}(y) dy - g(t) \\ &= \frac{x}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds - g(t) \end{aligned}$$

$$\text{故 } u(t, x) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds$$

Ex 3. 令  $u(t, x) = T(t) X(x)$

对应的 S-L 问题为  $\begin{cases} X''(x) + \lambda X(x) = 0, & x \in (0, \pi) \\ X(0) = X(\pi) = 0 \end{cases}$

其通解为  $X(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$

代入边界条件得  $C_2 = 0$ ,  $\lambda_n = n^2$ ,  $X_n(x) = \sin(nx)$ ,  $n = 1, 2, \dots$

从而  $u(t, x) = \sum_{n=1}^{\infty} T_n(t) e^{-n^2 t} \sin(nx)$

代回初始条件:  $u(x, 0) = \sin x = \sum_{n=1}^{\infty} T_n(0) \sin(nx)$  得  $T_n(0) = \begin{cases} 1, & n=1 \\ 0, & n>1 \end{cases}$

故  $u(t, x) = e^{-t} \sin x$

Ex 4. 令  $u(t, x) = T(t) X(x)$

对应的 S-L 问题为  $\begin{cases} X''(x) + \lambda X(x) = 0, & x \in (0, l) \\ X'(0) = X'(l) = 0 \end{cases}$

其通解为  $X(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$

代入边界条件得  $C_1 = 0$ ,  $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ ,  $X_n(x) = \cos\left(\frac{n\pi}{l} x\right)$ ,  $n = 1, 2, \dots$

从而  $u(t, x) = \sum_{n=0}^{\infty} T_n(t) e^{-\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi}{l} x\right)$

代回初始条件:  $u(x, 0) = x^2(l-x)^2 = \sum_{n=0}^{\infty} T_n(0) \cos\left(\frac{n\pi}{l} x\right)$

得  $T_n(0) = \begin{cases} \frac{l^4}{30}, & n=0 \\ \frac{2}{l} \int_0^l x^2(l-x)^2 \cos\left(\frac{n\pi}{l} x\right) dx = -24[1+(-1)^n] \left(\frac{l}{n\pi}\right)^4, & n \geq 1 \end{cases}$

故  $u(t, x) = \frac{l^4}{30} - \sum_{n=1}^{\infty} \frac{3l^4}{(n\pi)^4} e^{-\left(\frac{2n\pi}{l}\right)^2 t} \cos\left(\frac{2n\pi}{l} x\right)$

Ex 5. 由上题可知,  $u(t, x) = \sum_{n=0}^{\infty} T_n(t) e^{-\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi}{l} x\right)$ , 其中  $T_n(t)$  s.t.  $\varphi(x) = \sum_{n=0}^{\infty} T_n(0) \cos\left(\frac{n\pi}{l} x\right)$

易验证  $u(t, x)$  关于  $x$ -级收敛

从而  $\lim_{t \rightarrow \infty} u(t, x) = \sum_{n=0}^{\infty} \lim_{t \rightarrow \infty} T_n(t) e^{-\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi}{l} x\right) = T_0(0) = \frac{1}{l} \int_0^l \varphi(x) dx$

物理意义: 在一个绝热容器内, 热量分布将随时间趋于均匀.

最终内部各点温度会趋于初始温度的空间平均值.