MAT7092 - Stochastic Processes

2025 Autumn

1 Conditional Expectation

Let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a r.v. and $\mathcal{A} \subseteq \mathcal{F}$ be a σ-algebra.

1. Prove that if $X \geq Y$ a.s., then $\mathbb{E}[X|\mathcal{A}] \geq \mathbb{E}[Y|\mathcal{A}]$.
2. For any $\alpha, \beta \in \mathbb{R}$, prove that $\alpha\mathbb{E}[X|\mathcal{A}] + \beta\mathbb{E}[Y|\mathcal{A}] = \mathbb{E}[\alpha X + \beta Y|\mathcal{A}]$.
3. For convex integrable function φ, prove that

$$\varphi\left(\mathbb{E}\left[X\middle|\mathcal{A}\right]\right) \leq \mathbb{E}\left[\varphi(X)\middle|\mathcal{A}\right].$$

4. For $1 < p, q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, prove that

$$\left|\mathbb{E}\left[XY\middle|\mathcal{A}\right]\right| \leq \left(\mathbb{E}\left[|X|^p\middle|\mathcal{A}\right]\right)^{1/p} \left(\mathbb{E}\left[|Y|^q\middle|\mathcal{A}\right]\right)^{1/q}$$

2 Discrete-time stochastic processes

1. Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a filtration, $X = \{X_n : n \in \mathbb{N}\}$ be an adapted process relative to $\mathcal{F}$ such that $X_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

   (a) X is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-supermartingale if and only if $X_n \geq \mathbb{E}[X_{n+k} \big| \mathcal{F}_n]$ for any fixed $k \in \mathbb{N}$.

   (b) X is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-submartingale if and only if $X_n \leq \mathbb{E}[X_{n+k} \big| \mathcal{F}_n]$ for any fixed $k \in \mathbb{N}$.

   (c) X is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-martingale if and only if $X_n = \mathbb{E}[X_{n+k} \big| \mathcal{F}_n]$ for any fixed $k \in \mathbb{N}$.

   (d) What is the case of backward supermartingale/submartingale/martingale if $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a family of decreasing σ-algebra? State the results and prove them.

2. If X_n is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-martingale and $X_{n+1} \in \mathcal{F}_n$ for $\forall n \in \mathbb{N}$, prove that $X_n = X_1$ for all $n \in \mathbb{N}$ a.s.. Use this result to show that the uniqueness of Doob's decomposition $X = Y + Z$ for martingale Y and increasing predictable process Z.

3. For a filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ and an optional r.v. α, prove that $\mathcal{F}_\alpha$ is a σ-algebra and $\alpha \in \mathcal{F}_\alpha$.

4. Let $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a r.v. and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a filtration. Set $X_n = \mathbb{E}[Y \big| \mathcal{F}_n]$ for $n \in \mathbb{N} \cup \{\infty\}$.

   (a) Prove that $\{X_n : n \in \mathbb{N} \cup \{\infty\}\}$ is a $\{\mathcal{F}_n : n \in \mathbb{N} \cup \{\infty\}\}$-martingale.

   (b) If $\alpha \leq \beta$ are optional r.v.s relative to $\{\mathcal{F}_n : n \in \mathbb{N}\}$, then $\{X_\alpha, X_\beta\}$ is a martingale relative to $\{\mathcal{F}_\alpha, \mathcal{F}_\beta\}$.

5. Let Y be a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-submartingale/$\mathcal{F}$-supermartingale, Z be a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-predictable non-negative process, X be a martingale transformation of Y through Z. If X is integrable, then prove that X is a $\{\mathcal{F}_n : n \in \mathbb{N}\}$-submartingale/$\{\mathcal{F}_n : n \in \mathbb{N}\}$-supermartingale.

6. Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of decreasing σ-algebra, $X = \{X_n : n \in \mathbb{N}\}$ be a backward martingale relative to $\{\mathcal{F}_n : n \in \mathbb{N}\}$. Prove that X is uniformly integrable.

7. Let X_n be an integrable r.v. for all $n \in \mathbb{N}$ satisfying

$$\mathbb{E}\left[X_{n+1} \middle| X_1, \ldots, X_n\right] = \frac{X_1 + \ldots + X_n}{n}.$$

Prove that $\{\frac{X_1+\ldots+X_n}{n} : n \in \mathbb{N}\}$ is a martingale relative to $\{\mathcal{F}_n^X : n \in \mathbb{N}\}$.