HW3

September 25, 2025

Recall that the heat kernel is given by

$$G_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0,\ x \in \mathbb{R}^d.$$

Exercise 1 1. Let $\phi \in \mathcal{C}[0,\infty)$ with $\phi(0) = 0$. Show that

$$u(t,x) = \int_0^\infty \big[G_t(x-y) - G_t(x+y)\big]\phi(y)\,dy$$

satisfies $\partial_t u - \partial_{xx} u = 0$ for $t, x > 0$, $u(t,0) \equiv 0$ and

$$\lim_{t\to 0+} u(t,x) = \phi(x), \quad x > 0.$$

Hint: extend $\phi(x)$ to an odd function on $\mathbb{R}$.

2. Find a solution for the half-line heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t u - \Delta u = 0, & t, x > 0, \\ u(0,x) = \phi(x), & x > 0, \\ \partial_x u(t,0) = 0, & t > 0, \end{cases}$$

where $\phi \in \mathcal{C}(\mathbb{R})$ and $\phi'(0) = 0$.

Hint: extend $\phi(x)$ to an even function on $\mathbb{R}$.

Exercise 2 Let $g \in \mathcal{C}^\infty[0,\infty)$ satisfy $g(0) = 0$. Derive that the solution to the heat equation on half-line

$$\begin{cases} \partial_t u = \partial_{xx} u, & t > 0,\ x > 0, \\ u(0,x) = 0, & t = 0,\ x > 0, \\ u(t,0) = g(t), & x = 0,\ t \ge 0, \end{cases}$$

is given by

$$u(t,x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\,ds.$$

Hint: let $v(t,x) := u(t,x) - g(t)$; consider the odd extension of v and solve the resulting non-homogeneous heat equation.

Exercise 3 Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0,\pi),\ t > 0, \\ u(0,x) = \sin x, & x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \ge 0. \end{cases}$$

Exercise 4 Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, \ell),\ t > 0, \\ u(0, x) = x^2(\ell - x)^2, & x \in [0, \ell], \\ \partial_x u(t, 0) = \partial_x u(t, \ell) = 0, & t \geq 0. \end{cases}$$

Exercise 5 Let $u(t, x)$ be the solution to the following initial-Neumann problem obtained via Separation of Variables:

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0, \ell),\ t > 0, \\ u(0, x) = \varphi(x), & x \in [0, \ell], \\ \partial_x u(t, 0) = \partial_x u(t, \ell) = 0, & t \geq 0. \end{cases}$$

Show that $\lim_{t \to \infty} u(t, x)$ exists and find the limit.

Can you give a physical interpretation of this limit?

HW3. P.1.

Ex 1. (1) 将ϕ延拓成$\mathbb{R}$上的奇函数$\tilde{\phi}\in C(\mathbb{R})$，即 $\tilde{\phi}(x)=\begin{cases}\phi(x) & ,x\in[0,\infty)\\ -\phi(-x) & ,x\in(-\infty,0)\end{cases}$

则 $u(t,x)=\int_0^\infty [G_t(x-y)-G_t(x+y)]\phi(y)\,dy$

$=\int_{-\infty}^\infty G_t(x-y)\tilde{\phi}(y)\,dy$

$=G_t*\tilde{\phi}(x)$

从而，① $\partial_t u(t,x)-\partial_{xx}u(t,x)=(\partial_t G_t-\partial_{xx}G_t)*\tilde{\phi}(x)=0$

② $\forall t>0$, $u(t,0)=\int_0^\infty [G_t(-y)-G_t(y)]\phi(y)\,dy=0$

③ $\forall x>0$, $\lim\limits_{t\to 0+}u(t,x)=\lim\limits_{t\to 0+}G_t*\tilde{\phi}(x)\big|_{x>0}=\tilde{\phi}(x)\big|_{x>0}=\phi(x)$

(2) 将ϕ延拓成$\mathbb{R}$上的偶函数$\tilde{\phi}\in C(\mathbb{R})$，即 $\tilde{\phi}(x)=\begin{cases}\phi(x) & ,x\in[0,\infty)\\ \phi(-x) & ,x\in(-\infty,0)\end{cases}$

注意到，$(\partial_t G_t-\partial_{xx}G_t)*\tilde{\phi}(x)=0$

故取 $u(t,x)=G_t*\tilde{\phi}(x)=\int_0^\infty [G_t(x-y)+G_t(x+y)]\phi(y)\,dy$

验证：① $\forall x>0$, $\lim\limits_{t\to 0+}u(t,x)=\lim\limits_{t\to 0+}G_t*\tilde{\phi}(x)\big|_{x>0}=\tilde{\phi}(x)\big|_{x>0}=\phi(x)$

② $\forall t>0$, $\partial_x u(t,0)=\int_0^\infty [\partial_y G_t(-y)+\partial_y G_t(y)]\phi(y)\,dy=0$

故 $u(t,x)=\int_0^\infty [G_t(x-y)+G_t(x+y)]\phi(y)\,dy$ 即是所求之解.

Ex 2. 令 $v(t,x):=u(t,x)-g(t)$,

将其延拓成$\mathbb{R}$上的奇函数$\tilde{v}(t,x)$，即 $\tilde{v}(t,x)=\begin{cases}v(t,x) & ,x\in[0,\infty)\\ -v(t,-x) & ,x\in(-\infty,0)\end{cases}$

则 $\tilde{v}(t,x)$ solves $\begin{cases}\partial_t\tilde{v}=\partial_{xx}\tilde{v}+f & ,t>0,x\in\mathbb{R}\setminus\{0\}\\ \tilde{v}(0,x)=0 & ,x\in\mathbb{R}\\ \tilde{v}(t,0)=0 & ,t\geq 0\end{cases}$ with $f(t,x):=\begin{cases}-g'(t) & ,x\geq 0\\ g'(t) & ,x<0\end{cases}$

从而 $\tilde{v}(t,x)=\int_0^t ds\int_{-\infty}^\infty \Gamma(t,x;s,y)f(s,y)\,dy$

$=\int_0^t ds\int_0^\infty G_{t-s}(x-y)(-g'(s))\,dy+\int_0^t ds\int_{-\infty}^0 G_{t-s}(x-y)g'(s)\,dy$

$=2\int_0^t g'(s)\,ds\int_{-\infty}^0 G_{t-s}(x-y)\,dy-\int_0^t ds\int_{-\infty}^\infty G_{t-s}(x-y)g'(s)\,dy$

$=2\int_0^t g'(s)\,ds\int_x^\infty G_{t-s}(y)\,dy-g(t)$

$=2\left[g(s)\int_x^\infty G_{t-s}(y)\,dy\right]\Big|_{s=0}^{s=t}-2\int_0^t g(s)\,ds\int_x^\infty \partial_s G_{t-s}(y)\,dy-g(t)$

$=2\int_0^t g(s)\,ds\int_x^\infty \partial_y^2 G_{t-s}(y)\,dy-g(t)$

$=\frac{x}{2\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}g(s)\,ds-g(t)$

故 $u(t,x)=\frac{x}{2\sqrt{\pi}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}g(s)\,ds$

Ex 3. 令 $u(t,x)=T(t)X(x)$

对应的S-L问题为 $\begin{cases} X''(x)+\lambda X(x)=0, & x\in(0,\pi) \\ X(0)=X(\pi)=0 \end{cases}$

其通解为 $X(x)=C_1\sin(\sqrt{\lambda}x)+C_2\cos(\sqrt{\lambda}x)$

代入边界条件得 $C_2=0$，$\lambda_n=n^2$，$X_n(x)=\sin(nx)$，$n=1,2,\cdots$

从而 $u(t,x)=\sum_{n=1}^{\infty}T_n(0)e^{-n^2t}\sin(nx)$

代回初始条件：$u(x,0)=\sin x=\sum_{n=1}^{\infty}T_n(0)\sin(nx)$ 得 $T_n(0)=\begin{cases}1, & n=1\\ 0, & n>1\end{cases}$

故 $u(t,x)=e^{-t}\sin x$

Ex 4. 令 $u(t,x)=T(x)X(x)$

对应的S-L问题为 $\begin{cases} X''(x)+\lambda X(x)=0, & x\in(0,l) \\ X'(0)=X'(l)=0 \end{cases}$

其通解为 $X(x)=C_1\sin(\sqrt{\lambda}x)+C_2\cos(\sqrt{\lambda}x)$

代入边界条件得 $C_1=0$，$\lambda_n=\left(\frac{n\pi}{l}\right)^2$，$X_n(x)=\cos\left(\frac{n\pi}{l}x\right)$，$n=1,2,\cdots$

从而 $u(t,x)=\sum_{n=0}^{\infty}T_n(0)e^{-\left(\frac{n\pi}{l}\right)^2t}\cos\left(\frac{n\pi}{l}x\right)$

代回初始条件：$u(x,0)=x^2(l-x)^2=\sum_{n=0}^{\infty}T_n(0)\cos\left(\frac{n\pi}{l}x\right)$

得 $T_n(0)=\begin{cases}\frac{l^4}{30}, & n=0\\ \frac{2}{l}\int_0^l x^2(l-x)^2\cos\left(\frac{n\pi}{l}x\right)dx=-24\left[1+(-1)^n\right]\left(\frac{l}{n\pi}\right)^4, & n\geq 1\end{cases}$

故 $u(t,x)=\frac{l^4}{30}-\sum_{n=1}^{\infty}\frac{3l^4}{(n\pi)^4}e^{-\left(\frac{2n\pi}{l}\right)^2t}\cos\left(\frac{2n\pi}{l}x\right)$

Ex 5. 由上题可知，$u(t,x)=\sum_{n=0}^{\infty}T_n(0)e^{-\left(\frac{n\pi}{l}\right)^2t}\cos\left(\frac{n\pi}{l}x\right)$，其中 $T_n(0)$ s.t. $\varphi(x)=\sum_{n=0}^{\infty}T_n(0)\cos\left(\frac{n\pi}{l}x\right)$

易验证 $u(t,x)$ 关于 x 一致收敛

从而 $\lim_{t\to\infty}u(t,x)=\sum_{n=0}^{\infty}\lim_{t\to\infty}T_n(0)e^{-\left(\frac{n\pi}{l}\right)^2t}\cos\left(\frac{n\pi}{l}x\right)=T_0(0)=\frac{1}{l}\int_0^l\varphi(x)dx$

物理意义：在一个绝热容器内，热量分布将随时间趋于均匀，

最终内部各点温度会趋于初始温度的空间平均值。