HW5

October 14, 2025

Exercise 1 Show that $\Delta u = 0$ is invariant under rotation, that is, if $\Delta u = 0$ and $O \in O(d)$ is a $d \times d$ orthogonal matrix, then

$$v(x) := u(Ox), \quad x \in \mathbb{R}^d$$

also solves $\Delta v = 0$.

Exercise 2 Show that if there exists a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ to the Neumann problem

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = g, & x \in \partial\Omega, \end{cases}$$

then

$$\int_\Omega f \, dx = -\int_{\partial\Omega} g \, dS.$$

Hint: use integration by parts on $\displaystyle\int_\Omega (\Delta u)v \, dx$ with $v \equiv 1$.

Exercise 3 Let $f \in \mathcal{C}^2(\mathbb{R}^3)$ be supported on $B_1(0)$.

1. Use the fundamental solution in $\mathbb{R}^3$ to find a solution to the equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \mathbb{R}^3, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

(You can leave the answer as an integral.)

2. Show that $u(x) \sim \dfrac{c}{|x|}$ for $|x|$ large, and determine the constant c.

3. (Optional) Give a physical interpretation of the result.

Exercise 4 Let $\Omega = \{(x_1, x_2) : x_1, x_2 > 0\}$ be the first quadrant. Use reflection symmetry to find the Green's function in Ω, i.e., for each $y \in \Omega$, solve

$$\begin{cases} -\Delta G(x) = \delta(x - y), & x \in \Omega, \\ G(x) = 0, & x \in \partial\Omega. \end{cases}$$

Ex1. 令 $O = (O_{ij})_{d \times d}$. 注意到 $OO^T = \left(\sum\limits_{k=1}^{d} O_{ik} O_{jk}\right)_{d \times d} = I$

则 $\Delta v(x) = \sum\limits_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} v(x)$

$= \sum\limits_{k=1}^{d} \sum\limits_{i=1}^{d} \sum\limits_{j=1}^{d} O_{ik} O_{jk} \frac{\partial^2}{\partial y_i \partial y_j} u(y) = \sum\limits_{i=1}^{d} \frac{\partial^2}{\partial y_i^2} u(y) = \Delta u(x) = 0$

Ex2. 由分部积分得

$\int_{\Omega} f \, dx = -\int_{\Omega} \Delta u \, dx = -\int_{\partial \Omega} \frac{\partial u}{\partial n} \, dS = -\int_{\partial \Omega} g \, dS$

Ex3. (1) 注意到, 基本解 $\Phi(x) = \frac{1}{4\pi |x|}$, $x \in \mathbb{R}^3$

考虑 $u(x) = (\Phi * f)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy = \frac{1}{4\pi} \int_{B_1(0)} \frac{f(y)}{|x-y|} dy$

一方面, $-\Delta u(x) = -\int_{\mathbb{R}^3} \Delta \Phi(y) f(x-y) dy = \int_{\mathbb{R}^3} \delta(y) f(x-y) dy = f(x)$

另一方面, $|u(x)| \leq \|f\|_{L^\infty} \int_{B_1(0)} \frac{1}{4\pi (|x|-|y|)} dy \leq \frac{1}{3} \|f\|_{L^\infty} \cdot \frac{1}{|x|-1}$, for $|x| > 1$

故 $u(x) \to 0$, as $|x| \to \infty$.

从而 $u(x)$ 即是所求之解.

(2) 对于足够大的 x, $\exists \tilde{c} < \infty$ s.t. $\left| \frac{|x|}{|x-y|} f(y) \right| \leq \tilde{c} |f(y)|$

故由控制收敛定理知, $|x| u(x) = \int_{B_1(0)} \frac{|x|}{4\pi |x-y|} f(y) dy \to \frac{1}{4\pi} \int_{B_1(0)} f(y) dy$, as $|x| \to \infty$.

从而 $u(x) \sim \frac{c}{|x|}$ as $|x| \to \infty$. 其中 $c = \frac{1}{4\pi} \int_{B_1(0)} f(y) dy$

(3) 一个带电体在很远的地方产生的电势, 只取决于其总电荷量, 与其内部电荷分布无关.

从远处看, 一个带电的球 近似于一个点电荷.

Ex4. 考虑 $G(x,y) = \Phi(x,y) - \Phi(x, y^{(1)}) - \Phi(x, y^{(2)}) + \Phi(x, y^{(3)})$

其中 $y = (y_1, y_2) \in \Omega$, $y^{(1)} = (-y_1, y_2)$, $y^{(2)} = (y_1, -y_2)$, $y^{(3)} = (-y_1, -y_2)$

一方面, $-\Delta G(x,y) = -\Delta_x \Phi(x,y) - \Delta_x \Phi(x, y^{(1)}) - \Delta_x \Phi(x, y^{(2)}) + \Delta_x \Phi(x, y^{(3)})$

$= \delta(x-y)$, $x \in \Omega$

(事实上, 只需注意 $y^{(1)}, y^{(2)}, y^{(3)} \notin \Omega$)

另一方面, 对 $\forall x = (x_1, x_2) \in \partial \Omega$

若 $x_1 = 0$ 而 $x_2 > 0$, 则 $\Phi(x,y) = \Phi(|x-y|) = \Phi(|x-y^{(1)}|) = \Phi(x, y^{(1)})$

$\Phi(x, y^{(2)}) = \Phi(|x-y^{(2)}|) = \Phi(|x-y^{(3)}|) = \Phi(x, y^{(3)})$

从而 $G(x,y) = 0$

类似可证 $x_1 > 0$ 而 $x_2 = 0$ 时 $G(x,y) = 0$

从而 $G(x,y) = 0$, $x \in \partial \Omega$

从而 $G(x,y)$ 即是所求之 Green 函数.