A Proof on the Asymptotics of Sums

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Abstract

This article provides a formal proof for the interchangeability of the summation operator and the Big-Theta notation under certain conditions. Specifically, we prove that for a set of integers S and a non-negative function $f(k)$, the sum of functions asymptotically bounded by $f(k)$ is, in turn, asymptotically bounded by the sum of $f(k)$.

1 The Proposition

We seek to prove the following proposition regarding asymptotic notation.

Proposition 1. *Let S be a set of integers and $f(k)$ be a function such that $f(k) > 0$ for all $k \in S$. Then,*

$$\sum_{k \in S} \Theta(f(k)) = \Theta\left(\sum_{k \in S} f(k)\right)$$

The notation on the left-hand side, $\sum_{k \in S} \Theta(f(k))$, represents the set of functions formed by summing $g(k)$ over S, where $g(k)$ is any function such that $g(k) \in \Theta(f(k))$.

2 Proof of the Proposition

To prove Proposition 1, we must show that the left-hand side is both an upper bound and a lower bound for the right-hand side. That is, we need to prove:

1. $\sum_{k \in S} \Theta(f(k)) = \mathcal{O}\left(\sum_{k \in S} f(k)\right)$
2. $\sum_{k \in S} \Theta(f(k)) = \Omega\left(\sum_{k \in S} f(k)\right)$

Let $g(k)$ be an arbitrary function such that $g(k) \in \Theta(f(k))$. By the definition of Big-Theta notation, there exist positive constants c_1 and c_2 such that for all $k \in S$:

$$c_1 f(k) \leq g(k) \leq c_2 f(k) \tag{1}$$

2.1 Part 1: The Upper Bound ($\mathcal{O}$)

We use the right-hand side of inequality (1), which is $g(k) \leq c_2 f(k)$.

We begin with the sum of $g(k)$ over all $k \in S$. Since the inequality holds for every term in the summation, we can write:

$$\sum_{k \in S} g(k) \leq \sum_{k \in S} c_2 f(k)$$

Because c_2 is a constant, it can be factored out of the summation:

$$\sum_{k \in S} g(k) \leq c_2 \left(\sum_{k \in S} f(k)\right)$$

This is precisely the definition of Big-O notation. We have shown that $\sum_{k \in S} g(k)$ is bounded above by a constant multiple of $\sum_{k \in S} f(k)$. Therefore,

$$\sum_{k \in S} \Theta(f(k)) = \mathcal{O}\left(\sum_{k \in S} f(k)\right)$$

2.2 Part 2: The Lower Bound (Ω)

Next, we use the left-hand side of inequality (1), which is $c_1 f(k) \leq g(k)$.

Again, we consider the sum over all $k \in S$. The inequality for each term implies:

$$\sum_{k \in S} g(k) \geq \sum_{k \in S} c_1 f(k)$$

Factoring out the constant c_1, we obtain:

$$\sum_{k \in S} g(k) \geq c_1 \left(\sum_{k \in S} f(k) \right)$$

This is the definition of Big-Omega notation. We have shown that $\sum_{k \in S} g(k)$ is bounded below by a constant multiple of $\sum_{k \in S} f(k)$. Therefore,

$$\sum_{k \in S} \Theta(f(k)) = \Omega \left(\sum_{k \in S} f(k) \right)$$

2.3 Conclusion

Since we have shown that the statement is true for both the upper bound ($\mathcal{O}$) and the lower bound (Ω), we can conclude that it is true for Big-Theta (Θ).

Thus, the proposition is proven.

$\square$