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ASSESSMENT OF UNCERTAINTY IN HIGH FREQUENCY DATA: THE OBSERVED ASYMPTOTIC VARIANCE

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The availability of high frequency financial data has generated a series of estimators based on intra-day data, improving the quality of large areas of financial econometrics. However, estimating the standard error of these estimators is often challenging. The root of the problem is that traditionally, standard errors rely on estimating a theoretically derived asymptotic variance, and often this asymptotic variance involves substantially more complex quantities than the original parameter to be estimated.

Standard errors are important: they are used to assess the precision of estimators in the form of confidence intervals, to create "feasible statistics" for testing, to build forecasting models based on, say, daily estimates, and also to optimize the tuning parameters.

The contribution of this paper is to provide an alternative and general solution to this problem, which we call *Observed Asymptotic Variance*. It is a general nonparametric method for assessing asymptotic variance (AVAR). It provides consistent estimators of AVAR for a broad class of integrated parameters $\Theta = \int \theta_t \, dt$, where the spot parameter process θ can be a general semimartingale, with continuous and jump components. The observed AVAR is implemented with the help of a two-scales method. Its construction works well in the presence of microstructure noise, and when the observation times are irregular or asynchronous in the multivariate case.

The methodology is valid for a wide variety of estimators, including the standard ones for variance and covariance, and also for more complex estimators, such as, of leverage effects, high frequency betas, and semivariance.

KEYWORDS: Asynchronous times, consistency, discrete observation, edge effect, irregular times, leverage effect, microstructure, observed information, realized volatility, robust estimation, semimartingale, standard error, two-scales estimation, volatility of volatility.

1. INTRODUCTION

1.1. *Two Standard Errors*

AS HIGH FREQUENCY DATA BECOME MORE READILY AVAILABLE, the demand for analyzing such big and noisy data is also increasing. Within the recent decade, we have seen the arrival of novel methodologies for using the high frequency data to estimate volatility, to assess the asymmetric information in financial returns via semivariance and leverage effect, to make inference relating to jumps, and many other objects of interest. As financial markets and global economies evolve, we expect an ongoing need to estimate new parameters of interest from data of the high frequency variety. This process will substantially improve the precision with which we can measure financial and economic quantities.

A typical analysis takes the following form. One seeks to estimate an integrated parameter Θ,

$$\Theta = \int_0^{\mathcal{T}} \theta_t \, dt, \tag{1}$$

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on the basis of n data points, say, $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$, where $\{\theta_t\}$ is a spot parameter process such as volatility, leverage effect, instantaneous regression coefficients, etc. To arrive at feasible inference, one typically needs the following:

THEORETICAL REQUIREMENT 1—Asymptotic Validity of Normal Approximation[2]: As the number of observations n becomes large, one needs

(i) an estimator $\hat{\Theta}_n$ which is consistent;

(ii) a *standard error* $\mathrm{se}(\hat{\Theta}_n)$, that is, a data-based statistic for which

$$\frac{\hat{\Theta}_n - \Theta}{\mathrm{se}(\hat{\Theta}_n)} \xrightarrow{\mathcal{L}} N(0,1) \quad \text{stably.} \tag{2}$$

The conventional way to implement step (ii) is to go through the following additional steps:

THEORETICAL REQUIREMENT 2—Estimated Asymptotic Variance:

(i) Derive a limit theory: $n^{\alpha}(\hat{\Theta}_n - \Theta) \xrightarrow{\mathcal{L}} V^{\frac{1}{2}} N(0,1)$ stably in law, where V is an asymptotic variance.[3]

(ii) Determine the mathematical expression for V.

(iii) Find a consistent estimator $\hat{V}_n$ of V.

(iv) Set $\mathrm{se}(\hat{\Theta}_n) = n^{-\alpha}|\hat{V}_n|^{\frac{1}{2}}$.

Our Alternative. Our purpose in this paper is to circumvent Theoretical Requirement 2, by developing general formulae for $\mathrm{se}(\hat{\Theta}_n)$, which do not depend on knowing the convergence rate α or the asymptotic variance V. We call this *the observed standard error*. We can express $\mathrm{se}(\hat{\Theta}_n) = |\widehat{\mathrm{AVAR}}_n|^{\frac{1}{2}}$, where $\widehat{\mathrm{AVAR}}_n$ is *the Observed Asymptotic Variance*.

Our general formula for observed $\widehat{\mathrm{AVAR}}_n$ is a two-scales construction given in Definition 4 (Section 3.2). The estimator is consistent for the asymptotic variance (it satisfies Theoretical Requirement 2) using Theorem 4 (Section 3.2) and Proposition 1 (Section 3.1). Theoretical Requirement 1 is then satisfied via Proposition 2 (Section 3.1).

Apart from regularity conditions, our only assumption is that the spot parameter process $\{\theta_t\}$ is allowed to be a general semimartingale, hence $\{\theta_t\}$ can have jump or continuous evolution and it can be either an Itô or non-Itô process as in Calvet and Fisher (2008).[4] We shall see in Section 7 that the conditions for our results are satisfied broadly, including on quite exotic quantities such as leverage effect. Additional guidance on theory is provided in Section 6.

Practical guidance to how to use our theory is provided in Section 5. We emphasize that for empirical analysis, one does not need to know the analytical form of V to use the Observed Asymptotic Variance. The technique permits the setting of *prima facie* standard errors by just using our formulae and without any prior theoretical derivation. One can then verify the theoretical conditions afterwards. This is much like the practice in parametric inference (using the observed information) and when bootstrapping.

[2]See Proposition 2 in Section 3.1 for precise conditions. Stable convergence is described in Definition 3 in the same section.

[3]For subsequent decluttering of notation, we set $\mathrm{AVAR}_n = n^{-2\alpha}V$. $\widehat{\mathrm{AVAR}}_n$ is consistent if and only if $\widehat{\mathrm{AVAR}}_n = \mathrm{AVAR}_n(1 + o_p(1))$. Formulae for V and AVAR_n are given explicitly in (16)–(17) in Section 3.1.

[4]See also Rosenbaum, Duvernet, and Robert (2010) for recent interest in this type of evolution.

1.2. *Why Do We Need a Standard Error?*

Currently, the main uses of standard errors are hypothesis testing and self-contained confidence intervals based on (2). In high frequency econometrics, such intervals go back to Barndorff-Nielsen and Shephard (2002a), where $\widehat{\text{AVAR}}_n$ was set as $\frac{2}{3}\times$ the *quarticity* (see Section 3.3). Confidence intervals and tests have been the main spur for pursuing the asymptotics described in Theoretical Requirement 2. Other early contributions to this type of asymptotics are those of Foster and Nelson (1996), Comte and Renault (1998), Jacod and Protter (1998), and Zhang (2001). A substantial amount of work on this problem has followed, as described below and throughout the paper.

There are other applications that require using the standard error. For example,

(i) *Incorporation into forecasting models*: see Andersen, Bollerslev, and Meddahi (2005) and Bollerslev, Patton, and Quaedvlieg (2016).

(ii) *Optimal combination of intra-day high frequency estimators*: see an early draft of Meddahi (2002), as well as Andreou and Ghysels (2002).

(iii) *Model selection in high frequency regression*: see Zhang (2012, Section 4, pp. 268–273). As documented in the cited paper, this problem has applications to the estimation of high frequency betas, as well as to nonparametric options trading.

(iv) *Selection of tuning parameters*: many estimators involve one or more "tuning parameters," such as block or subgrid size. Optimizing the estimator $\hat{\Theta}_n$ as a function of these tuning parameters would naturally involve minimizing the asymptotic variance. We shall see that this optimization can be done on the basis of our proposed $\widehat{\text{AVAR}}_n$. See Section 4 for references and further development.

1.3. *Why Do We Need the Observed Asymptotic Variance?*

The rationale behind this paper is that Theoretical Requirement (TR) 2(ii)–(iii) is a main hindrance to the development and use of inference in high frequency data. Recall that TR 2(ii) entails deriving a central limit theorem for $\hat{\Theta}_n$ in order to obtain the analytical form of the asymptotic variance V, and then (iii) requires finding a consistent estimator for V.

It can already be difficult to construct appropriate estimators $\widehat{\Theta}$, and it is often a substantial work to carry out the steps in TR 2(ii)–(iii). To corroborate this, we draw attention to the large literature that provides estimators $\hat{\Theta}_n$ of Θ, but it lacks feasible (asymptotically pivotal) statistics of the form (2). In particular, it is challenging to build an estimator $\hat{\Theta}_n$ or $\hat{\theta}_n$ that accommodates the presence of microstructure noise and non-synchronousness in observation times. But, the main challenge is to derive the theoretical asymptotic variance AVAR and to find $\widehat{\text{AVAR}}_n$. Examples in the literature include, but are not limited to, semivariance (Barndorff-Nielsen, Kinnebrock, and Shephard (2009)); nearest neighbor truncation (Andersen, Dobrev, and Schaumburg (2012)); estimating the rank of the volatility matrix (Jacod and Podolskij (2013)); principal component analysis (Aït-Sahalia and Xiu (2015)); the volatility of volatility (Vetter (2015); see Remark 4 in Section 2.3 and Example 10 in Section 7); and high frequency regression, and ANOVA (Mykland and Zhang (2006, 2009, 2012); see Example 7 in Section 7). In all these examples, one can obtain a point estimate in the presence of microstructure noise, but one does not have ready access to tests, confidence intervals, and the other methods discussed in Section 1.2. The overall challenge is thus not specific to one estimator, but holds across estimators of various types, which reminds us that we all are in the same boat in searching for how to quantify the uncertainty in the estimators.[5]

It should be emphasized that in many cases, the asymptotic variance is on the form of an integral of a function of volatility. In this case, TR 2(iii) in Section 1.1 can often be met with the theory in Jacod and Protter (2012, Sections 16.4–16.5, pp. 512–554), Jacod and Rosenbaum (2013, 2015), and Mykland and Zhang (2009, Section 4.1, pp. 1421–1426). These papers are important contributions to the AVAR problem. Not all asymptotic variances, however, are on such a form (such as, Examples 5, 9, and 10 in Section 7; and Robert and Rosenbaum (2011, 2012)). Also, even when the AVAR is on this form, it may be difficult to go through step TR 2(ii). In addition, there are cases where the estimation approach may be based on robustness considerations which would make the cited volatility estimators inappropriate (e.g., Andersen, Dobrev, and Schaumburg (2012, 2014)).

1.4. *Connections to the Literature*

The basic principle behind the observed AVAR is to segment the available time line into subperiods, and then compare the estimators in successive subperiods. We show that this difference can be decomposed into two parts. One part reveals the behavior of $\hat{\Theta}$ in the form of its estimation error, and the other part tells us the dynamics of spot parameter process θ alone. We develop estimators to disentangle these two effects and to construct the observed AVAR.

The observed AVAR has a lot in common with the quarticity estimate of AVAR in realized volatility, in the seminal work of Barndorff-Nielsen and Shephard (2002a, 2004a). Also, it resembles observed information in likelihood theory. The difference between the observed AVAR and the estimated AVAR (going through Theoretical Requirement 2 in Section 1.1) corresponds to the difference between the observed and the estimated expected information in parametric inference. We discuss these connections further in Section 3.3.

Our procedure is unlike resampling in that it is not based on the "Russian doll" principle (Hall (1992, Chapter 1.2)), and in particular, it does not involve a second level of nesting. We emphasize that our block parameter K is (typically) unrelated to any block size used to construct the estimator $\widehat{\Theta}_n$. For a precise discussion of this, see Section 5.4.

The comparison of adjacent estimators, however, is also a feature of the subsampling developed for volatility in the work of Kalnina and Linton (2007) and Kalnina (2011), with an important subsequent study by Christensen, Podolskij, Thamrongrat, and Veliyev (2015). Bootstrapping has been developed in the papers by Gonçalves and Meddahi (2009) and Gonçalves, Donovon, and Meddahi (2013), but is further away from the approach of the current paper.

Apart from the overall construction of observed asymptotic variance, there are two other intellectual novelties in the paper. First, the comparison of adjacent values of the integral of θ is given a precise formulation in the *Integral-to-Spot Device* (Theorem 1 and Corollary 1, in Section 2.3) which shows that "realized volatility" of integrals $\int \theta_t \, dt$ converges to the volatility of the spot parameter process θ_t. The only condition is that the spot process be a semimartingale. Second, the estimation of asymptotic variance $\mathrm{AVAR}(\hat{\Theta} - \Theta)$ is reduced to a problem which resembles that of estimating volatility,

[5]To the best of our knowledge, assuming the presence of microstructure noise, the theoretical AVAR and its estimation have been documented only in the case of variance (volatility), covariance, leverage effect (skewness), and, in some instances, of jumps. See Section 7 for references.

with edge effects playing the role of "microstructure noise." We can thus adapt known methods to the current problem of estimating asymptotic variance. It is worth mentioning that edge effects are estimator-specific. As its name suggests, edge effects show up in an estimator whenever the estimator under-uses or over-uses the data at the edge of a sampling interval, relative to the middle portion of the data interval. As we shall see in our examples (Section 7), edge effects are ubiquitous in high frequency inference. The effect is also referred to as burn-in time, and border effect.

We emphasize that our purpose in this paper is to provide a method for getting at observed asymptotic variance, for any estimator of interest. Our proposed approach extends broadly to high frequency inference. The contribution of the current paper is, in particular, to estimators other than volatility. For the latter, much is known both in terms of asymptotic variances and in terms of resampling, as discussed above.

The rest of this paper is organized as follows. Section 2 introduces the intuition behind the observed AVAR and the *Integral-to-Spot Device*. Section 3 develops the general formulae for Observed $\widehat{\text{AVAR}}_n$ using a two-scales construction. It also provides consistent estimators of the quadratic variation of the spot parameter process θ_t. The use of the $\widehat{\text{AVAR}}_n$ to select tuning parameters is discussed in Section 4. Section 5 provides practical guidance to using the theory. The generalization to the multidimensional case is described in Remark 12 in Section 5.2. Section 6 gives advice on how to verify the conditions of the theory. Section 7 provides examples. Finally, Section 8 concludes. Proofs are located in the Appendix in the Supplemental Material (Mykland and Zhang (2017)).

2. FINITE SAMPLE QUADRATIC VARIATIONS OF A PARAMETER PROCESS

2.1. *Setup*

We observe data at high frequency, in a time period from 0 to $\mathcal{T}$. The data will normally take the form of samples from a semimartingale X_t, typically contaminated by microstructure noise. We are interested in estimating integrals of a "parameter" spot process θ_t, which also is assumed to be a semimartingale.

For example, we can take θ_t to be the spot variance of the continuous part X_t^c of the process X_t: $\theta_t = \sigma_t^2$, where $dX_t = \sigma_t\,dW_t + dt$-terms + jump terms, and W is a Brownian motion. In the multivariate case, θ_t can be a function of the instantaneous covariance. The development, however, holds more generally, such as for the leverage effect where $\theta_t = d[X^c, \sigma^2]_t/dt$, the volatility of volatility where $\theta_t = d[\sigma^2, \sigma^2]_t^c/dt$, or other. The case of multivariate θ_t is considered in Remark 12.

DEFINITION 1—Model Structure and Notation: The notation $[X, X]_t$ refers to the continuous-time quadratic variation of semimartingale X from time zero to time t (e.g., Jacod and Shiryaev (2003, pp. 51–52), Protter (2004, p. 66)). The quadratic variation is also known as (ex post) integrated variance (Barndorff-Nielsen and Shephard (2002b)).[6] Semimartingales are defined in, for example, Jacod and Shiryaev (1987, Definition I.4.41, p. 43), as well as Protter (2004, Definitions on p. 52, and Definition and Theorem III.1 on p. 102) and also Dellacherie and Meyer (1982). We assume that all our semimartingales

[6]Similarly, $[X, Z]_t$ refers to the continuous-time quadratic covariation (or integrated covariance) of semimartingales X and Z.

are *càdlàg* (right continuous with left limits). All data generating and latent (such as X_t and θ_t) processes live on a probability space $(\Omega, \mathcal{F}, P)$.[7]

We consider integrated parameters and their estimators[8] over time intervals $(S, T] \subset [0, \mathcal{T}]$:

$$\Theta_{(S,T]} = \int_S^T \theta_t \, dt \quad \text{and} \quad \hat{\Theta}_{(S,T]} = \text{a consistent estimator of } \Theta_{(S,T]}. \tag{3}$$

Even when estimating the spot volatility, one almost invariably estimates such integrals.[9] To get a stab at the asymptotic variance, we shall use the following finite sample quantities.

DEFINITION 2—Rolling Quadratic Variations of Integrated Processes: Divide the time interval $[0, \mathcal{T}]$ into B basic blocks of time periods (days, five minutes, thirty seconds, or other) $(T_{i-1}, T_i]$, with $T_0 = 0$ and $T_B = \mathcal{T}$. The blocks are assumed to be of equal size: Set $\Delta T = \mathcal{T}/B$, and assume that $T_i = i\Delta T$. We shall permit rolling overlapping intervals, and so let K be an integer no greater than $B/2$. We define

(4) the quadratic variation of Θ:

$$\mathrm{QV}_{B,K}(\Theta) = \frac{1}{K} \sum_{i=K}^{B-K} (\Theta_{(T_i, T_{i+K}]} - \Theta_{(T_{i-K}, T_i]})^2, \quad \text{and}$$

the quadratic variation of $\widehat{\Theta}$:

$$\mathrm{QV}_{B,K}(\hat{\Theta}) = \frac{1}{K} \sum_{i=K}^{B-K} (\hat{\Theta}_{(T_i, T_{i+K}]} - \hat{\Theta}_{(T_{i-K}, T_i]})^2.$$

We emphasize that the above discretized quadratic variations are defined on the discrete grid $\{0, \Delta T, 2\Delta T, \dots, \mathcal{T}\}$, as opposed to the continuous-time quadratic variation $[X, X]_t$ discussed above.

Later on, from Section 3 onwards, B, ΔT, and K will depend (explicitly or implicitly) on an index n, which usually denotes the number of observations. We may then write $\Delta T = \Delta T_n$, or omit the index n if the meaning is obvious.

[7]A full specification of the model also involves a filtration $(\mathcal{F}_t)_{0 \le t \le \mathcal{T}}$, $\mathcal{F}_{\mathcal{T}} \subseteq \mathcal{F}$, which we for simplicity shall take to be fixed throughout the paper, until we reach Section 6. Also until then, when we say that X_t is a "semimartingale," we automatically mean a semimartingale relative to $(\mathcal{F}_t)_{0 \le t \le \mathcal{T}}$ and P. The "filtered probability space" $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le \mathcal{T}}, P)$ is also taken to satisfy the "usual conditions" (Jacod and Shiryaev (2003, Definitions I.1.2–I.1.3, p. 2)).

[8]All estimators are implicitly or explicitly indexed by the number of observations n. Consistency, convergence in law, etc., refers to behavior as $n \to \infty$.

[9]The standard spot estimate is $\hat{\theta}_{T_i} = \hat{\Theta}_i/(T_i - T_{i-1})$ for suitable choice of T_{i-1}. See, for example, Foster and Nelson (1996), Comte and Renault (1998), Mykland and Zhang (2008). The theory requires the existence of a "spot" θ_t; cf. Section 5.3. To the extent that the "integral" process has jumps, we assume that such jumps have been suitably removed by the estimation procedure in use, as also discussed at the beginning of Section 7; see also Examples 1 and 9 in the same section. See also Section 5.3. On the other hand, we shall see that the process θ_t can have as many jumps as it wants.

2.2. *The Basic Insight*

The basic insight behind the Observed AVAR is that we can decompose the increment $\hat{\Theta}_{(T_i,T_{i+K}]} - \hat{\Theta}_{(T_{i-K},T_i]}$ into the parts related to estimator behavior and the part solely tied to parameter behavior:

$$\hat{\Theta}_{(T_i,T_{i+K}]} - \hat{\Theta}_{(T_{i-K},T_i]} = \underbrace{(\hat{\Theta}_{(T_i,T_{i+K}]} - \Theta_{(T_i,T_{i+K}]})}_{\text{estimation error}} + \underbrace{(\Theta_{(T_i,T_{i+K}]} - \Theta_{(T_{i-K},T_i]})}_{\text{evolution in parameter}} - \underbrace{(\hat{\Theta}_{(T_{i-K},T_i]} - \Theta_{(T_{i-K},T_i]})}_{\text{estimation error}}. \tag{5}$$

In consequence, we can write the quadratic variation of $\hat{\Theta}$ as

$$\begin{aligned} \mathrm{QV}_{B,K}(\hat{\Theta}) &= \frac{2}{K}\sum_i(\hat{\Theta}_{(T_i,T_{i+K}]} - \Theta_{(T_i,T_{i+K}]})^2 + \frac{1}{K}\sum_i(\Theta_{(T_i,T_{i+K}]} - \Theta_{(T_{i-K},T_i]})^2 \\ &\quad + \text{martingale and negligible terms} \\ &= \big(\underbrace{2\,\mathrm{AVAR}(\hat{\Theta}_{(0,\mathcal{T}]} - \Theta_{(0,\mathcal{T}]})}_{\text{estimation error}} + \underbrace{\mathrm{QV}_{B,K}(\Theta)}_{\text{parameter behavior}}\big)\big(1 + o_p(1)\big) \end{aligned} \tag{6}$$

when ΔT goes to zero.[10]

To turn this from a heuristic to a rigorous theory, we need to

(i) Explain how to go from the first to the second line of (6), and in particular explain how $\frac{1}{K}\sum_i(\hat{\Theta}_{(T_i,T_{i+K}]} - \Theta_{(T_i,T_{i+K}]})^2$ comes to be related to the asymptotic variance of $\hat{\Theta}_{(0,\mathcal{T}]} - \Theta_{(0,\mathcal{T}]}$. We shall do this in Section 3.

(ii) Disentangle $\mathrm{AVAR}(\hat{\Theta}_{(0,\mathcal{T}]} - \Theta_{(0,\mathcal{T}]})$ from $\mathrm{QV}_{B,K}(\Theta)$. We shall do this by finding that the latter is approximately equal to $\frac{2}{3}(K\Delta T)^2[\theta,\theta]_{\mathcal{T}-}$. We shall then be able to write two (or more) distinct linear equations on the form (6), which we can solve for AVAR.

We start with (ii): the approximation of $\mathrm{QV}_{B,K}(\Theta)$.

2.3. *The Integral-to-Spot Device: A General Result for the Quadratic Variation of Integrals of Semimartingales*

A main result is the following, with proof in Appendix B. The Appendix also contains a simplified version of the proof for finite K as $B \to \infty$.

THEOREM 1—The Integral-to-Spot Device, General Case: *Assume that θ_t is a semimartingale on $[0,\mathcal{T}]$. Also suppose that $K\Delta T \to 0$ (K may have subsequences that are $O_p(1)$ or that go to infinity). Set $t_* = \max\{i\Delta T : i\Delta T < t\}$ and $t^* = \min\{i\Delta T : i\Delta T \geq t\}$. Then*

$$\begin{aligned} \frac{1}{(K\Delta T)^2}\mathrm{QV}_{B,K}(\Theta) &= \frac{2}{3}\left(1 - \frac{1}{K^2}\right)[\theta,\theta]_{\mathcal{T}-} \\ &\quad + \frac{1}{K^2}\int_0^{\mathcal{T}}\left(\left(\frac{t^*-t}{\Delta T}\right)^2 + \left(\frac{t-t_*}{\Delta T}\right)^2\right)d[\theta,\theta]_t + o_p(1), \end{aligned} \tag{7}$$

[10]See Footnote 3 in the Introduction for the normalization of AVAR.

where $[\theta, \theta]_{\mathcal{T}-} = \lim_{t \uparrow \mathcal{T}} [\theta, \theta]_t$. *The convergence in probability is uniform in* ΔT, *so long as* $\Delta T > 0$ *and* $K\Delta T \to 0$.[11]

REMARK 1—Consistency for Absolutely Continuous $[\theta, \theta]_t$: If $[\theta, \theta]_t$ is absolutely continuous, the right-hand side of (7) equals $\frac{2}{3}[\theta, \theta]_{\mathcal{T}} + o_p(1)$, also for finite K. The reason is that the limit of the second term in (7) then equals $\frac{2}{3}\frac{1}{K^2}[\theta, \theta]_{\mathcal{T}-}$.

It would seem from Theorem 1 that much nuisance is created when there are jumps in θ. As further analyzed in Section 3.2, however, it is typically meaningful to add the extra restriction that $K \to \infty$. This solves the discontinuity problem, as follows.

COROLLARY 1—The Integral-to-Spot Device, Consistent Case: *In addition to the assumptions of Theorem* 1, *also suppose that* $K \to \infty$. *Then*

$$\text{(8)} \qquad \frac{1}{(K\Delta T)^2} \operatorname{QV}_{B,K}(\Theta) = \frac{2}{3}[\theta, \theta]_{\mathcal{T}-} + o_p(1).$$

REMARK 2—Convergence Fails for Finite K in the Presence of Jumps: To understand the impact of jumps on the above Theorem 1, note that for finite K it will not, in general, produce a limit when θ has jumps.

To see why, suppose for simplicity that θ_t is continuous except for a single jump at (stopping) time $\tau \in (0, \mathcal{T})$. Also assume that $[\theta, \theta]_t^c = [\theta^c, \theta^c]_t$ (the continuous part of $[\theta, \theta]_t$) is absolutely continuous. Recall that $\Delta T = \Delta T_B = \mathcal{T}/B$. For $K = 1$, we get from (7) that

$$\text{(9)} \qquad (\Delta T)^{-2} \sum_i (\Theta_{(T_i.T_{i+1}]} - \Theta_{(T_{i-1}.T_i]})^2 = \frac{2}{3}\big([\theta^c, \theta^c]_{\mathcal{T}}\big) + \frac{1}{2}\big((1-U_B)^2 + U_B^2\big)\Delta\theta_\tau^2 + o_p(1),$$

where $U_B = (\tau - \tau_{B,*})/\Delta T_B$, where $\tau_{B,*} = \max_i\{i\Delta T < \tau\}$. If, for example, the jump happens at a Poisson time independent of the rest of the θ_t process, then one can proceed along the lines of Jacod and Protter (2012, Chapter 4.3) and get that U_B converges in law to a standard uniform random variable. Similar considerations apply more generally to Theorem 1 if θ_t is an Itô-semimartingale in the sense of Jacod and Protter (2012, Chapter 4.4, p. 114).

On the other hand, if τ is a nonrandom time, such as the time of the news release from a (U.S.) Federal Open Market Committee meeting,[12] the right-hand side of (9) simply does not converge, in probability or law.

REMARK 3—Link to Pre-Averaging, and the Factor 2/3: Think of θ_t as X_t. One can relate Theorem 1 to pre-averaging.[13] An integral is much like a sum, and so we are continuously pre-averaging θ_t, and then using the averaged quantity to find the volatility of θ_t. The factor 2/3 originates from the procedure of pre-averaging; cf. the example on page 2255

[11]See Remark 15 in Appendix A. The same holds for Theorem 3 in Section 3.2, and Theorem 8 in Appendix C. In other theorems, the uniformity is valid subject to the needs of other assumptions, such as the balance condition (30) in Section 3.2.

[12]At the time of writing, 2 pm Washington DC time, on the day of the meeting. This time appears to be defined to within single digit milliseconds. See, for example, "Fed probes for leaks ahead of policy news" (*Financial Times*, 24 September 2013).

[13]Jacod, Li, Mykland, Podolskij, and Vetter (2009a), Podolskij and Vetter (2009b).

in Jacod et al. (2009a). A similar factor of 1/2 appears in the estimation of leverage effect; see Mykland and Zhang (2009).[14] This downward bias is typically referred to as "smoothing bias," and is well studied in the literature on nonparametric density estimation (Stoker (1993)). For use of this terminology in the high frequency setting, see Aït-Sahalia, Fan, and Li (2013, Section 4.2, p. 230).

Theorem 1 is concerned with the volatility of a general semimartingale, and this has not been studied in full generality by the pre-averaging literature.[15] It is thus conjectured to have implications for the consistency of pre-averaging estimators of volatility. To see this, consider equidistant discrete observations of θ_{T_i}, and $\overline{\Theta}_{(T_i,T_{i+K}]} = \sum_{j=i+1}^{i+k} \theta_{T_j}\Delta T_n$, and define $\mathrm{QV}_{B,K}(\overline{\Theta})$ in analogy with (4). From the proof of Proposition 5 (in Appendix D.2), it is clear that Theorem 1 yields the following corollary:

$$\begin{aligned}
(10)\qquad \frac{1}{(K\Delta T)^2}\mathrm{QV}_{B,K}(\overline{\Theta}) &= \frac{2}{3}\left(1-\frac{1}{K^2}\right)[\theta,\theta]_{\mathcal{T}-} \\
&\quad + \frac{1}{K^2}\int_0^{\mathcal{T}}\left(\left(\frac{t^*-t}{\Delta T}\right)^2+\left(\frac{t-t_*}{\Delta T}\right)^2\right)d[\theta,\theta]_t + o_p(1).
\end{aligned}$$

With standard calculations, one can get similar results when observing data with microstructure noise, $Y_{T_i} = \theta_{T_i} + \varepsilon_{T_i}$. What is clear from (10), however, is that pre-averaging (followed by the usual two-scales correction) is robust to the most exotic forms of jumps, but with two caveats. One is that (naturally) one cannot capture a jump at the end time $\mathcal{T}$. The other is that if one has reason to scale with a sufficiently small K, one may pick up the $\frac{1}{K^2}$ term in (10) at some point, for example as asymptotic bias. This term would not be a problem with the usual scaling $K = O(B^{1/2})$, but sometimes a smaller K is warranted; see, for example, Jacod and Rosenbaum (2013, 2015), Mykland and Zhang (2016), or in the case of models with shrinking size of noise.

REMARK 4—Link to Volatility of Volatility: We shall see in Section 3 that Theorem 1 is an ingredient in the estimation of $[\theta,\theta]_{\mathcal{T}-}$. A specific procedure is given in Theorem 4 in Section 3.2. In particular, for $\theta_t = \sigma_t^2$, one retrieves an estimator of volatility of volatility. This connects to an earlier estimator of $[\sigma^2,\sigma^2]_{\mathcal{T}}$ by Vetter (2015), which is further discussed in Example 10. An estimator of volatility that is based on different principles can be found in Mykland, Shephard, and Sheppard (2012, Theorem 7 and Corollary 2).

We emphasize that the cited papers in Remarks 3–4 also have central limit theorems (CLT), rather than just consistency. Our main focus is asymptotic variance (AVAR), where only consistency is necessary, and we are interested in the weakest possible conditions for such consistency to hold. Higher order properties of the AVAR would be interesting, cf. the discussion of likelihood methods in Section 3.3, but this seems beyond the scope of this paper.

The particular sharpness of Theorem 1 is due to the following result. Since it may have other applications, we provide the main building block as a separate result. The proof is also in Appendix B. The result is also true for many other processes than semimartingales.

[14]For more on the leverage effect, and further references, see Example 9 in Section 7.

[15]The closest we can find is Chapter 16.2–16.3 of Jacod and Protter (2012), which has several important contributions. We see our statement (10) as a complement to their findings.

THEOREM 2—Rewriting Integral Differences as Semimartingale Increments: *Let θ_t be a semimartingale. We use the following notation. For nonrandom times $S < T$, set*

$$\Theta'_{(S,T]} = \int_S^T (T-t)\,d\theta_t \quad \textit{and} \quad \Theta''_{(S,T]} = \int_S^T (t-S)\,d\theta_t. \tag{11}$$

Then, if $\delta > 0$ is nonrandom,

$$\Theta_{(T,T+\delta]} - \Theta_{(T-\delta,T]} = \Theta'_{(T,T+\delta]} + \Theta''_{(T-\delta,T]}. \tag{12}$$

3. ESTIMATING ASYMPTOTIC VARIANCE IN HIGH FREQUENCY DATA

3.1. *General Principles for the Asymptotic Variance*

Following the notation (3), we have at hand estimators $\hat{\Theta}_{(S,T]} = \hat{\Theta}^{(n)}_{(S,T]}$ of $\Theta_{(S,T]}$.[16]

The typical statistical situation is now as follows: there is a semimartingale $M_{n,t}$ and *edge effects* $e_{n,S}$ and $\tilde{e}_{n,T}$, so that

$$\hat{\Theta}^{(n)}_{(S,T]} - \Theta_{(S,T]} = \underbrace{M_{n,T} - M_{n,S}}_{\text{semimartingale}} + \underbrace{\tilde{e}_{n,T} - e_{n,S}}_{\text{edge effects}} \quad \text{for} \quad S < T \in \mathcal{T}_n, \tag{13}$$

where $\mathcal{T}_n = \{T_{n,i} : i = 0, \dots B_n\}$.[17] The edge effect is essentially anything that messes up the semimartingaleness of the difference $\hat{\Theta}_{(0,\mathcal{T}]} - \Theta_{(0,\mathcal{T}]}$, and it occurs in many shapes, which we shall document in Section 7.[18] The edge effect has a component e_S relating to phasing in the estimator at the beginning of the time interval, and a component $\tilde{e}_T$ for the phasing out at T. For the estimator on the whole interval, we use $\widehat{\Theta}_n = \widehat{\Theta}^{(n)}_{(0,\mathcal{T}]}$ from now on. An important construction leading to (13) relates to half-interval estimators (Section 5.1).

REMARK 5—Edge Effects: To rephrase, the edge effect reflects the difference in behavior of an estimator between the middle and the edges of the interval on which it is defined. For a conceptual illustration, consider the bipower estimator (Barndorff-Nielsen and Shephard (2004b, 2006)) of the integrated volatility of a process X_t, where X_t is observed (without microstructure noise) at equidistant times t_i, $i = 0, \dots, n$, spanning $[0, \mathcal{T}]$. The estimator has the form $\hat{\Theta}_{(S,T]} = \frac{\pi}{2}\sum_{S<t_{i-1}\le t_i\le T} |\Delta X_{t_{i-1}}||\Delta X_{t_i}|$. Each absolute return $|\Delta X_{t_i}|$ appears twice in the summation, except the first and the last such return. This is a case of edge effect. The precise form of this effect is given in Example 2 in Section 7, along with a number of other examples. In fact, the only estimators that we can identify to not have edge effect, are realized volatility and other power variations absent microstructure noise.

Meanwhile, we seek an estimator of the asymptotic variance of $\hat{\Theta}^{(n)}_{(0,\mathcal{T}]}$. For a conceptual path, we turn to the substantial fraction of the high frequency literature which has been concerned with the study of the asymptotic behavior of $\hat{\Theta}^{(n)}_{(S,T]} - \Theta^{(n)}_{(S,T]}$ for all $S < T \in [0, \mathcal{T}]$. This is typically required to achieve *stable convergence.*

[16]See Section 5.1 on how to obtain such estimators from half-interval estimators. The latter are required for stable convergence results; cf. the development in this section and in Section 6.

[17]Until we reach Section 6.

[18]All of $\hat{\Theta}_{(S,T]}$, M_T, e_S, and $\tilde{e}_T$ will depend on the number of observations n. For the most part, n is omitted from our notation to avoid an excessive number of subscripts, but when crucial for understanding we may write $M_{n,T}$, etc. By convention, we use superscript "(n)" when it is too unaesthetic to place n as a subscript.

DEFINITION 3—Stable Convergence: Let $L_n = (L_{n,t})_{0\le t\le \mathcal{T}}$ be a sequence of semimartingales (Definition 1 in Section 2.1). We say that L_n converges stably in law to $L = (L_t)_{0\le t\le \mathcal{T}}$ with respect to a sigma-field $\mathcal{G} \subseteq \mathcal{F}$, and as $n \to \infty$, if (1) L_t is measurable with respect to a sigma-field $\tilde{\mathcal{G}}$ belonging to an extension $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{P})$ of $(\Omega, \mathcal{G}, P)$; and (2) for every $\mathcal{G}$-measurable (real valued) random variable Y, (L_n, Y) converges in law to (L, Y).

For further explanation of stable convergence, and of the upcoming P-UT condition, see Appendix D.1.

As illustrated by a number of examples in Section 7 below, the standard high frequency asymptotic *result* in the literature is now as follows.

CONDITION 1—Standard Convergence Result in the Literature: Assume (13), and that one can show the following. There is an $\alpha > 0$ so that, as $n \to \infty$,

$$n^\alpha M_{n,t} \xrightarrow{\mathcal{L}} L_t \quad \text{stably in law} \tag{14}$$

with respect to a sigma-field $\mathcal{G}$. The quadratic variation $[L, L]_{\mathcal{T}}$ (Section 2.1) is measurable with respect to $\mathcal{G}$, and L_t is a local martingale conditionally on $\mathcal{G}$. Also, $e_{n,T_n} = o_p(n^{-\alpha})$ and $\tilde{e}_{n,S_n} = o_p(n^{-\alpha})$ for any $S_n, T_n \in \mathcal{T}_n$. Finally, the sequence $n^\alpha M_{n,t}$ is Predictably Uniformly Tight (P-UT) (Appendix D.1; Jacod and Shiryaev (2003, Chapter VI.3.b, and Definition VI.6.1, p. 377)).[19]

We recall the basic facts about this situation. First, Condition 1 assures that, with $\hat{\Theta}_n = \hat{\Theta}^{(n)}_{(0,\mathcal{T}]}$ and $\Theta = \Theta_{(0,\mathcal{T}]}$,

$$n^\alpha(\hat{\Theta}_n - \Theta) \xrightarrow{\mathcal{L}} L_{\mathcal{T}} \quad \text{stably in law.} \tag{15}$$

Also, the asymptotic variance of $n^\alpha(\hat{\Theta}_n - \Theta)$ given the underlying data represented by $\mathcal{G}$ is

$$V = \mathrm{AVAR}\big(n^\alpha(\hat{\Theta}_n - \Theta)\big) = \mathrm{Var}(L_{\mathcal{T}}|\mathcal{G}). \tag{16}$$

To declutter the notation, we shall define $\mathrm{AVAR}_n = \mathrm{AVAR}(\hat{\Theta}_n - \Theta)$; formally[20]

$$\mathrm{AVAR}_n = n^{-2\alpha}\,\mathrm{Var}(L_{\mathcal{T}}|\mathcal{G}). \tag{17}$$

Second, we have guidance on how to estimate the asymptotic variance:

PROPOSITION 1—Quadratic Variation and Asymptotic Variance: *Assume Condition 1. Then the conditional variance* $\mathrm{Var}(L_{\mathcal{T}}|\mathcal{G})$ *exists* (*is "well defined"*) *and*

$$[M_n, M_n]_{\mathcal{T}} = \mathrm{AVAR}_n\big(1 + o_p(1)\big). \tag{18}$$

[19]See Section 6 for further explanation of this condition, as well as some standard methods for how to verify it. Examples of verification are also given in Section 7.

[20]As foreshadowed by Footnote 3. In the notation of this earlier footnote, $V = \mathrm{Var}(L_{\mathcal{T}}|\mathcal{G})$.

The proof of Proposition 1 can be found towards the end of Appendix D.1.

In particular, a necessary and sufficient condition for an estimator $\widehat{\mathrm{AVAR}}_n$ of asymptotic variance to be consistent, that is, $\widehat{\mathrm{AVAR}}_n = \mathrm{AVAR}_n(1+o_p(1))$, is that

$$\widehat{\mathrm{AVAR}}_n = [M_n, M_n]_{\mathcal{T}}\big(1+o_p(1)\big). \tag{19}$$

We emphasize that for empirical analysis, one does not need to know the form or value of any of the limiting quantities L_t, $[L,L]_{\mathcal{T}}$, and $\mathcal{G}$ in Condition 1 in order to estimate the asymptotic variance.[21] All one needs is to check the criterion (19). We shall in the sequel use this path to show that our proposed estimator is consistent. The procedure can be used much like observed information or bootstrapping, and recall that practical guidance is provided in Section 5.

Because of its importance, and also to illustrate the simplicity of the approach, we here state the main usage as a corollary to the above development.

PROPOSITION 2—Feasible Estimation: *Assume Condition* 1. *Also assume that L_T is conditionally Gaussian given $\mathcal{G}$. Suppose that* $\widehat{\mathrm{AVAR}}_n = [M_n, M_n]_{\mathcal{T}}(1+o_p(1))$. *Set* $\mathrm{se}(\hat{\Theta}_n) = |\widehat{\mathrm{AVAR}}_n|^{\frac{1}{2}}$. *Then*

$$\frac{\widehat{\Theta}_n - \Theta}{\mathrm{se}(\hat{\Theta}_n)} \xrightarrow{\mathcal{L}} N(0,1) \quad \textit{stably in law}. \tag{20}$$

3.2. *Main Findings: A General Expansion Result for* $\mathrm{QV}_{B,K}(\widehat{\Theta}_n)$, *and the Two-Scales AVAR and* $\widehat{[\theta,\theta]}$

For a given grid, we use the notation

$$\mathrm{ave}\big(e_{T_i}^2\big) \triangleq \frac{1}{B_n}\sum_i e_{T_i}^2 \tag{21}$$

and similarly for $\mathrm{ave}(\tilde{e}_{T_i}^2)$. Observe that $\tilde{e}_0 = e_{\mathcal{T}} = 0$ by convention. We obtain the following:

THEOREM 3—Expansion of $\mathrm{QV}_{B,K}(\hat{\Theta})$: *Assume Condition* 1. *Let $K = K_n$ be positive integers*, *and assume that $K_n\Delta T_n \to 0$. Also assume about the averages of the edge effects that*

$$\mathrm{ave}\big(e_{T_i}^2\big) = o_p\big(K_n \Delta T_n n^{-2\alpha}\big) \quad \textit{and} \quad \mathrm{ave}\big(\tilde{e}_{T_i}^2\big) = o_p\big(K_n \Delta T_n n^{-2\alpha}\big). \tag{22}$$

Then

$$\frac{1}{2K}\sum_{K\le i\le B-K} (\hat{\Theta}_{(T_{i-K},T_{i+K}]} - \Theta_{(T_{i-K},T_{i+K}]})^2 = \mathrm{AVAR}_n\big(1+o_p(1)\big). \tag{23}$$

Also, *if we assume that θ_t is a semimartingale on* $[0,\mathcal{T}]$, *and that*

$$\Delta T_n = o\big(n^{-\alpha}\big), \tag{24}$$

[21] In fact, an automatic minimal $\mathcal{G}$ is provided by Proposition 6 in Appendix D.1.

then

$$\mathrm{QV}_{B,K}(\hat{\Theta}) = 2\mathrm{AVAR}_n + \frac{2}{3}(K_n \Delta T_n)^2 [\theta, \theta]_{\mathcal{T}-} + o_p\big((K_n \Delta T_n)^2\big) + o_p\big(n^{-2\alpha}\big). \tag{25}$$

The convergence in probability is uniform in ΔT, *so long as* $\Delta T > 0$ *and* $K\Delta T \to 0$.

PROOF: See Appendix C, where it is also shown that a related result holds under (occasionally useful) weaker conditions. *Q.E.D.*

On the basis of Theorem 3, we now provide the estimators that we recommend for most situations.

DEFINITION 4—Two-Scales AVAR, and Volatility of Spot θ: Let B, K, and $\mathrm{QV}_{B,K}(\hat{\Theta})$ be as in Definition 2. Let $K_1 < K_2$ be two distinct values of K. The estimators[22]

$$\mathrm{TSAVAR}_n = \frac{1}{2}\left(\frac{1}{K_1^2} - \frac{1}{K_2^2}\right)^{-1}\left(\frac{1}{K_1^2}\mathrm{QV}_{B,K_1}(\hat{\Theta}) - \frac{1}{K_2^2}\mathrm{QV}_{B,K_2}(\hat{\Theta})\right) \quad \text{and} \tag{26}$$

$$\widehat{[\theta, \theta]}_{\mathcal{T}-} = \frac{3}{2}\big(K_2^2 - K_1^2\big)^{-1}(\Delta T)^{-2}\big(\mathrm{QV}_{B,K_2}(\hat{\Theta}) - \mathrm{QV}_{B,K_1}(\hat{\Theta})\big) \tag{27}$$

as well as $\mathrm{se}(\hat{\Theta}_n) = |\mathrm{TSAVAR}_n|^{\frac{1}{2}}$ are referred to as two-scales asymptotic variance, volatility, and standard error. When $K_2 = 2K_1 = 2K$, we shall refer to $(1, 2)$ estimators. Specifically, the (1,2) TSAVAR is

$$\mathrm{TSAVAR}_n = \frac{2}{3}\left(\mathrm{QV}_{B,K}(\hat{\Theta}) - \frac{1}{4}\mathrm{QV}_{B,2K}(\hat{\Theta})\right). \tag{28}$$

The consistency of the two-scales estimators is given by the following result.

THEOREM 4—Consistency of Two-Scales AVAR and Volatility of Spot θ. Feasibility of Inference: *Assume Condition* 1, *and that* θ_t *is a semimartingale on* $[0, \mathcal{T}]$. *Assume that*

$$\mathrm{ave}\big(e_{T_i}^2\big) = o_p\big(n^{-3\alpha}\big) \quad \textit{and} \quad \mathrm{ave}\big(\tilde{e}_{T_i}^2\big) = o_p\big(n^{-3\alpha}\big). \tag{29}$$

Assume that $\Delta T_n = o(n^{-\alpha})$. *Let* $K_{n,1} < K_{n,2}$ *be positive integers, and assume that* $K_{n,i}\Delta T_n \to 0$ *for* $i = 1, 2$. *Assume that both* $K_{n,1}$ *and* $K_{n,2}$ *satisfy the balance condition*

$$K_n \Delta T_n \quad \textit{are of the same order as } n^{-\alpha}, \tag{30}$$

with $\liminf_{n\to\infty}(K_{n,2}/K_{n,1}) > 1$, *which assures that neither of the main terms in* (25) *is ignorable*.

[22]The TSAVAR ((26) and (28)) does not have similar coefficients to the Two-Scales Realized Volatility (TSRV; Zhang, Mykland, and Aït-Sahalia (2005)). For a heuristic explanation of this, consider the left-hand side of (5), and write it as noise + signal − noise from previous interval. This looks like a scene from "inference with microstructure noise," especially if the noise is shrinking (via, say, pre-averaging). The AVAR problem is different, however, in that the "signal" has different properties. In particular, it shrinks at rate $O_p(K_n\Delta T_n)$ by Theorems 1–2. It also has different dependence structure.

Then, TSAVAR_n *and* $\widehat{[\theta,\theta]}_{\mathcal{T}-}$ *are consistent*:

$$\mathrm{TSAVAR}_n = \mathrm{AVAR}_n\big(1+o_p(1)\big) \quad \textit{and} \tag{31}$$

$$\widehat{[\theta,\theta]}_{\mathcal{T}-} \xrightarrow{p} [\theta,\theta]_{\mathcal{T}-}.$$

Finally, if L_T *is conditionally Gaussian given* $\mathcal{G}$, *then*

$$\frac{\widehat{\Theta}_n - \Theta}{\mathrm{se}(\hat{\Theta}_n)} \xrightarrow{\mathcal{L}} N(0,1) \quad \textit{stably in law}. \tag{32}$$

PROOF: Theorem 3 and assumption (30) give rise to (33), for $K = K_1$ and K_2. Ignoring remainder terms gives rise to estimators defined by a system of two equations and two unknowns by letting $K = K_1$ and $= K_2$ in (34). By linear algebra, this system is equivalent to the formulae for the estimators TSAVAR_n and $\widehat{[\theta,\theta]}_{\mathcal{T}-}$ given in (26)–(27) in Definition 4. The estimators are consistent by substituting (34) into (33) and then using that $\liminf_{n\to\infty}(K_{n,2}/K_{n,1}) > 1$. The last part of the result follows from Proposition 2. *Q.E.D.*

REMARK 6—Theoretical and Empirical Decompositions of $\mathrm{QV}_{B,K}$: Under the assumption (30), for $K = K_1$ or K_2, we have the theoretical decomposition:

$$\mathrm{QV}_{B,K}(\hat{\Theta}) = 2\,\mathrm{AVAR}_n + \frac{2}{3}(K\Delta T)^2[\theta,\theta]_{\mathcal{T}-} + o_p\big(n^{-2\alpha}\big). \tag{33}$$

Meanwhile, the two-scales estimators TSAVAR_n and $\widehat{[\theta,\theta]}_{\mathcal{T}-}$ satisfy a corresponding empirical decomposition:

$$\mathrm{QV}_{B,K_i}(\hat{\Theta}) = 2\,\mathrm{TSAVAR}_n + \frac{2}{3}(K_i\Delta T)^2\widehat{[\theta,\theta]}_{\mathcal{T}-}, \quad i = 1, 2. \tag{34}$$

One can think of (34) as an empirical decomposition of $\mathrm{QV}_{B,K_i}(\hat{\Theta})$ into TSAVAR_n and $\widehat{[\theta,\theta]}_{\mathcal{T}-}$; cf. Figure 1.

To get a sense of how the empirical decomposition (34) plays out in real data, we plot the separation of TSAVAR_n and $\widehat{[\theta,\theta]}_{\mathcal{T}-}$ using one month of tick-by-tick data from E-mini S&P 500 futures. As shown in Figure 1, cumulative AVAR is the main component in $\mathrm{QV}_{B,K}(\hat{\Theta})$, and on day 7 and day 21, the dispersion $[\theta,\theta]_{\mathcal{T}}$ of the underlying spot parameter is notable.

REMARK 7—Guidance on ΔT and K, and the Choices That Lead From Theorem 3 to Theorem 4: Both ΔT and K are under the control of the econometrician, and we offer the following main approach to choosing these two tuning parameters.

(i) By linear combination of $\mathrm{QV}_{B,K}$ for two or more K's, one can eliminate either the $[L,L]_{\mathcal{T}}$ or the $[\theta,\theta]_{\mathcal{T}-}$ term in (25). We have seen this in Theorem 4 above. This means that the main question is how to optimize Theorem 3 with respect to ΔT and K.

(ii) On the one hand, ΔT may be arbitrarily small. ΔT is, therefore, limited only by one's computational power. In particular, the assumption (24) is routinely satisfied in practice.

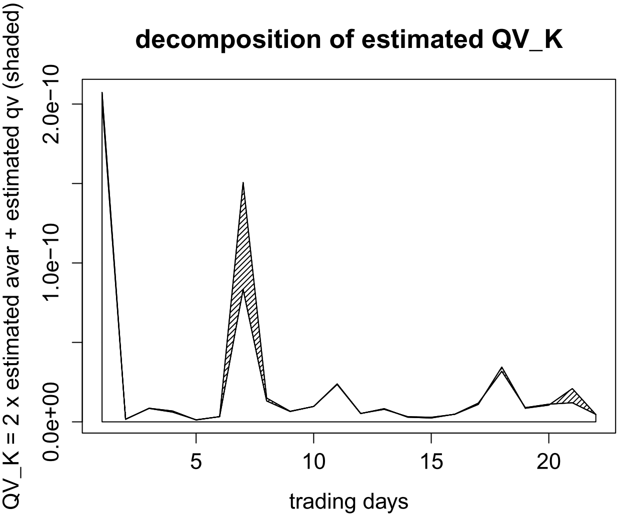

FIGURE 1.—This plot illustrates the empirical decomposition (34) for the S&P E-mini future as traded on the Chicago Mercantile Exchange, across the 22 trading days of May 2007. The top curve is the total volatility $\mathrm{QV}_{B,K}(\hat{\Theta})$ for each day, the lower curve is $2 \times$ TSAVAR for each day, and the shaded part is $\frac{2}{3}(K_1\Delta T)^2\widehat{[\theta,\theta]}_{\mathcal{T}-}$. In the estimation, the underlying parameter is the spot volatility: $\theta_t = \sigma_t^2$. $\widehat{\Theta}_{(S,T]}$ is based on first pre-averaging the data to 15 seconds, and then computing a TSRV on these pre-averages with $j = 20$ and $k = 40$ (see Example 4 in Section 7). The estimator is thus of integrated volatility $\Theta_{(S,T]} = \int_S^T \sigma_t^2\,dt$, and $[\theta,\theta]_{\mathcal{T}} = [\sigma^2,\sigma^2]_{\mathcal{T}}$. For $\mathrm{QV}_{B,K}(\hat{\Theta})$, we take ΔT to be five minutes, and a (1, 2) TSAVAR is computed on this basis for every five minute period, using the forward half-interval method in Section 5.1. The estimation method satisfies the edge condition (29) in Theorem 4 (Example 4).

(iii) In fact, ΔT ought to be small. In particular, by a sufficiency argument, $\mathrm{QV}_{2B,2K}(\hat{\Theta})$ will, under mild conditions, have less variability (given the data) than $\mathrm{QV}_{B,K}(\hat{\Theta})$. This is akin to the desirability of post-averaging after subsampling (Zhang, Mykland, and Aït-Sahalia (2005, Section 3.1, p. 1399)).

(iv) On the other hand, $K\Delta T$ ought not to be very small. As a general rule, we recommend to take $K_n\Delta T_n$ to be of the same order as $n^{-\alpha}$; cf. Condition (30) in Theorem 4. This reflects the need for $K_n\Delta T_n$ to respect both the lower and the upper bounds imposed by Theorem 3.

The reason for this is twofold. First, assumption (22) is a lower bound on $K_n\Delta T_n$. This bound is required to guarantee that the remainder term in (25) is no larger than $o_p((K_n\Delta T_n)^2) + o_p(n^{-2\alpha})$. It can be seen in explicit form from Theorem 8 (in Appendix C) that without assumption (22), one can expect the remainder term in (25) to involve $R_{n,K}$ (eq. (C.33) in the Appendix), which is of the same order as $(K_n\Delta T_n)^{-1}(\mathrm{ave}(e^2_{T_i}) + \mathrm{ave}(\tilde{e}^2_{T_i}))$. Hence assumption (22) is necessary.

Second, choosing $K_n\Delta T_n$ to be of larger order than $n^{-\alpha}$ causes AVAR_n to be dwarfed $[\theta,\theta]_{\mathcal{T}-}$. This is an upper bound on $K_n\Delta T_n$.[23]

Under the assumption (29) in Theorem 4, $K_n\Delta T_n$ is thus chosen to be as large as possible while still satisfying Theorem 3.

[23]A more elaborate development may allow for $K_n\Delta T_n$ to be of order larger than $O(n^{-\alpha})$, as with the cancellation of microstructure noise in two- and multi-scale estimation (Zhang, Mykland, and Aït-Sahalia (2005), Zhang (2006)), but such a development is beyond the scope of this paper.

(v) In summary, one should thus think of ΔT as a computational parameter, while $\delta = K\Delta T$ represents an amount of time over which one can reasonably compute estimators $\hat{\Theta}_{(T,T+\delta]}$.

REMARK 8—Finite Sample Adjustment: Without impacting the asymptotics, one can make finite sample adjustments, and use $\frac{B-2K+1}{B}\mathrm{QV}_{B,K}(\hat{\Theta})$ in lieu of $\mathrm{QV}_{B,K}(\hat{\Theta})$, and $\frac{B-2K-\mathcal{M}'+1}{B}\mathrm{QV}_{B,K,\mathcal{M}'}(\hat{\Theta})$ in lieu of $\mathrm{QV}_{B,K,\mathcal{M}'}(\hat{\Theta})$ from (47). The adjustment will produce "unbiasedness" in Theorem 1 when $[\theta,\theta]_t$ is absolutely continuous with constant derivative.

3.3. *One Scale Standard Error, Quarticity, and the Likelihood Connection*

Tiny Edge Effects. For some estimators, one can choose

$$K_n\Delta T_n = o\big(n^{-\alpha}\big), \tag{35}$$

while (22) remains satisfied. This is most often the case for estimators based on data with no microstructure noise, such as Realized Volatility (RV, Example 1), Bipower Variation (Example 2), and some estimators of integrals of functions of volatility (Example 6). (The examples are further discussed in Section 7, where references to the literature are also given.) We emphasize that the choice (35) may not be possible for estimators based on increasing-size blocks, or on data with microstructure noise.

REMARK 9—A One Scale Standard Error: Assume the conditions of Theorem 3 except condition (24). Assume instead (35). Set $\widehat{\mathrm{AVAR}}_n = \frac{1}{2}\mathrm{QV}_{B,K}(\hat{\Theta})$. Then $\widehat{\mathrm{AVAR}}_n$ is consistent.

Quarticity. The quarticity of Barndorff-Nielsen and Shephard (2002a, 2004a) can be viewed as a one scale estimator in our setup. Instead of (5) in Section 2.2, one writes[24]

$$\hat{\Theta}_{(T_i,T_{i+K}]} = \underbrace{\hat{\Theta}_{(T_i,T_{i+K}]} - \Theta_{(T_i,T_{i+K}]}}_{\text{estimation error}} + \underbrace{\Theta_{(T_i,T_{i+K}]}}_{\text{parameter value}}. \tag{36}$$

This consideration leads to a generalized quarticity, on the form $Q_{B,K} = \frac{1}{K} \times \sum_{i=0}^{B-K}(\hat{\Theta}_{(T_i,T_{i+K}]})^2$.

THEOREM 5—Expansion of Q_{K,B_n}: *Assume Condition* 1, *and that* θ_t *is a continuous semimartingale on* $[0,\mathcal{T}]$. *Suppose that* K *is a finite integer, and that* $\Delta T_n = O(n^{-2\alpha})$. *Also assume* (22) *about the averages of the edge effects*. *Then*

$$Q_{B_n,K} = \mathrm{AVAR}_n + K\Delta T_n\int_0^{\mathcal{T}}\theta_t^2\,dt + o_p\big(n^{-2\alpha}\big). \tag{37}$$

PROOF: By the same method as the proof of Theorem 3, and by the sample-path continuity of θ_t. *Q.E.D.*

[24]This is close to the argument in Barndorff-Nielsen and Shephard (2004a, Appendix B.1.1, pp. 922–923).

In the case where $\hat{\Theta}$ is realized variance based on the observed process X_t, $\theta_t = \sigma_t^2$ (the volatility of X), $B_n = n$, and $\alpha = \frac{1}{2}$. There is no edge effect (cf. Example 1 in Section 7). Thus, $\text{AVAR}_n = 2\Delta T_n \int_0^{\mathcal{T}} \theta_t^2\,dt$. In the case where $K = 1$, one retrieves $Q_{1,B_n} = \frac{3}{2}\text{AVAR}_n(1 + o_p(1))$. We thus retrieve the results of Barndorff-Nielsen and Shephard (2002a, 2004a), also similarly in the case of (synchronous) covariance, correlation, and regression.

A number of estimators have similar behavior in the sense that AVAR is proportional to $\int_0^{\mathcal{T}} \theta_t^2\,dt$. These include Bipower and Multipower Variation (Barndorff-Nielsen and Shephard (2004b, 2006)), and estimation of integrals of $\theta = \sigma^p$ with finite blocks (Mykland and Zhang (2009, Section 4.1, pp. 1421–1426; 2012, Chapter 2.6.2, pp. 170–172)).

In the more general case where $\int_0^{\mathcal{T}} \theta_t^2\,dt$ is not directly related to AVAR_n, one can go to a two scales estimator and obtain that $2Q_{B_n,1} - Q_{B_n,2} = \text{AVAR}_n + o_p(n^{-2\alpha})$. We have not investigated the situation for quarticity where θ_t is discontinuous.

A Likelihood Connection. We think of the observed AVAR as akin to the observed information in likelihood theory. Barndorff-Nielsen and Shephard had a similar view of quarticity.

The observed asymptotic variance is like the observed information in parametric statistical theory, in that there is no need for an intermediate theoretical asymptotic step, involving expectations or similar operations. Just as in likelihood theory, the observed asymptotic variance is easier to use, and it has a more universal form.

In parametric statistics, there has been a lively debate about the relative accuracy properties of observed and estimated expected information. In statistics, *accuracy* refers to the closeness of an approximation to the true distribution of a statistic. For the standard error, accuracy can refer *either* to how close the statistic is to the actual standard deviation of a statistic, *or* to how the $\text{se}(\hat{\Theta}_n)$ best accomplishes the asymptotic approximation of the law of $(\hat{\Theta}_n - \Theta/)\,\text{se}(\hat{\Theta}_n)$ to a normal or other reference distribution. Some of the same considerations may apply to the observed AVAR in this paper, but this question is beyond the scope of this paper.

The subject originally goes back to the debates between Fisher, and Neyman and Pearson. The neo-likelihood wave would seem to have started with Cox (1958, 1980) and Efron and Hinkley (1978), who demonstrated that the observed information in many cases was a more accurate measure of the variance of an estimator. This breakthrough was followed by a large literature, including Barndorff-Nielsen (1986, 1991), Jensen (1992, 1995, 1997), McCullagh (1984, 1987), Skovgaard (1986, 1991), Mykland (1999, 2001).

4. APPLICATION: SELECTION OF TUNING PARAMETERS

Many estimators involve one or more tuning parameters, for example, block or subgrid size. The typical situation is that of a tradeoff between two asymptotic variances. This is unlike the more typical situation in statistics, where the bias-variance tradeoff dominates. Variance-variance tradeoff is explicitly carried out in connection with the estimation of integrated volatility in Zhang, Mykland, and Aït-Sahalia (2005), Zhang (2006), Aït-Sahalia, Mykland, and Zhang (2011), Jacod and Mykland (2015), Podolskij and Vetter (2009a, 2009b), Jacod, Li, Mykland, Podolskij, and Vetter (2009b), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008). The typical question is how many grids to subsample over, or how long a time window to average data over, or how many autocovariances to include. In a twist of this problem, the adaptive method of Jacod and Mykland (2015) does carry out local model selection, but there is still a global tuning parameter which is left to be determined.

Similar tuning involving a variance-variance tradeoff occurs in connection with covariance estimation (Zhang (2011), Bibinger and Mykland (2016), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011)), spot volatility estimation (see Mykland and Zhang (2008)), estimation of the leverage effect (Wang and Mykland (2014), Aït-Sahalia, Fan, Laeven, Wang, and Yang (2016), Kalnina and Xiu (2016)), and estimation of the volatility of volatility (Vetter (2015), Mykland, Shephard, and Sheppard (2012)). These and other inference situations requiring tuning are described in Section 7.

One can think of the tuning problem as involving a parameter c on which the estimators $\hat{\Theta}_{n,c}$ depend.

CONDITION 2: Suppose that there is a tuning parameter c (chosen by the econometrician) upon which $\hat{\Theta}_n = \hat{\Theta}_{n,c}$ and $\mathrm{AVAR}_n = \mathrm{AVAR}_{n,c}$ depend.[25] Assume (as provided by, say, Proposition 1 in Section 3.1, or Theorem 4 in Section 3.2)

$$(38) \qquad \forall c \in \mathcal{C}: \quad \widehat{\mathrm{AVAR}}_{n,c} = \mathrm{AVAR}_c\big(1 + o_p(1)\big) \quad \text{(for fixed } c).$$

We seek $c^* = \arg\min_c \mathrm{AVAR}_{c\in\mathcal{C}}$, which we, for simplicity of discussion, take to be unique. $\mathcal{C}$ is a set of values for the tuning parameters within which one wishes to optimize. For the following *prima facie* discussion, we also take the number of points in $\mathcal{C}$ to be finite.[26]

For given number of observations n, our estimate is accordingly $\hat{c}_n = \arg\min_{c\in\mathcal{C}} \widehat{\mathrm{AVAR}}_{n,c}$, where $\widehat{\mathrm{AVAR}}_{n,c}$ is obtained through our proposals in the preceding sections.

Consistency. Under Condition 2, automatically,

$$(39) \qquad \hat{c}_n \to c^*.$$

Validity. This procedure provides an estimator with asymptotic variance AVAR_{c^*}:

$$(40) \qquad \text{asymptotic variance of} \quad \hat{\Theta}_{n,\hat{c}_n} - \Theta = \mathrm{AVAR}_{n,c^*}.$$

This is the conceptually more complex issue. Since AVAR_c is typically random, so will c^* be random. A priori, the insertion of $\hat{c}_n$ into an estimator might, in principle, create problems for the standard convergence setup discussed in Condition 1. At least in our simple case, however, this difficulty does not arise. We embody this in a formal result.

PROPOSITION 3—Optimization Commutes With Asymptotic Variance: *Assume Conditions 1 and 2. Also suppose that c^* is $\mathcal{G}$-measurable, and that, for each $c \in \mathcal{C}$,*

[25]Observe that Θ does not depend on c, but will normally be (statistically) mutually dependent with c^*. Recall that we assume that $\mathrm{AVAR}_n = n^{-2\alpha}V$ (cf. (16)–(17) in Section 3.1 as well as Footnote 3 in the Introduction).

[26]This case is of practical interest. See the example later in this section. In the more general case, one may imagine that there is a finite partition, say, $\mathcal{P}$ of the space of all c's, and that $\mathcal{C}$ has one representative of each element of $\mathcal{P}$. With a well-chosen $\mathcal{P}$ and $\mathcal{C}$, this construction will normally achieve approximate optimality.

The consistency part below generalizes straightforwardly to more complex $\mathcal{C}$'s, under, say, uniform convergence conditions. The validity part is best left as a separate paper.

$(\hat{\Theta}_{n,c} - \Theta)/\text{AVAR}_c^{1/2}$ *converges stably in law to a* $N(0,1)$ *random variable that is independent of* $\mathcal{G}$.[27] *Then* (40) *holds, and also*

$$(41)\qquad (\hat{\Theta}_{n,\hat{c}_n} - \Theta)/\text{AVAR}_{c^*}^{1/2} \xrightarrow{\mathcal{L}} N(0,1) \quad \textit{and}$$

$$(\hat{\Theta}_{n,\hat{c}_n} - \Theta)/\widehat{\text{AVAR}}_{n,\hat{c}_n}^{1/2} \xrightarrow{\mathcal{L}} N(0,1), \quad \textit{both stably.}$$

PROOF: With probability 1, for n large enough, $n^\alpha(\hat{\Theta}_{n,\hat{c}_n} - \Theta) = \sum_{c\in\mathcal{C}} n^\alpha(\hat{\Theta}_{n,c} - \Theta)I_{\{c=c^*\}}$. We are thus rescued by the stable convergence. *Q.E.D.*

EXAMPLE: A (J,K)-TSRV estimator based on pre-averaged data, with J and K finite, provides an example where the action space $\mathcal{C}$ can indeed be taken to be finite. This estimator is studied in Example 4 in Section 7 below, and it is seen that the two-scales $\widehat{\text{AVAR}}_n$ from Section 3.2 satisfies Theorem 4. The assumptions of Proposition 3 are thus satisfied.

5. GUIDANCE: I. PRACTICE

We here give advice on how to practically carry out the estimation of the asymptotic variance. The situation is that one has a data set and wishes an $\widehat{\text{AVAR}}_n$.

5.1. *Creating Estimators $\hat{\Theta}^{(n)}_{(S,T]}$ in Each Subinterval $(S,T]$*

In practice, a simple way to obtain estimators $\hat{\Theta}^{(n)}_{(S,T]}$ is to start with a given collection of half-interval estimators $\hat{\Theta}^{(n)}_{(0,T]}$, $0 < T \le \mathcal{T}$, and write, for $S < T$,

$$(42)\qquad \hat{\Theta}^{(n)}_{(S,T]} = \hat{\Theta}^{(n)}_{(0,T]} - \hat{\Theta}^{(n)}_{(0,S]}.$$

We call estimators of the form (42) *forward estimators*. If the half-interval estimators have representation $\hat{\Theta}^{(n)}_{(0,T]} - \Theta_{(0,T]} = M_{n,T} + \tilde{e}_{n,T} - \tilde{e}_{n,0}$, then obviously the representation (13) continues to hold for the forward estimators, with $e_T = \tilde{e}_T$.

REMARK 10—Additive Estimators: The forward estimators satisfy

$$(43)\qquad \hat{\Theta}^{(n)}_{(S,T]} - \Theta_{(S,T]} + \hat{\Theta}^{(n)}_{(T,U]} - \Theta_{(T,U]} = \hat{\Theta}^{(n)}_{(S,U]} - \Theta_{(S,U]}, \quad \text{for} \quad S < T < U.$$

Another construction of this type is the *backward estimators*: $\hat{\Theta}^{(n,b)}_{(S,T]} = \hat{\Theta}^{(n)}_{(S,\mathcal{T}]} - \hat{\Theta}^{(n)}_{(T,\mathcal{T}]}$. The development is analogous to that of forward estimators. If estimators are constructed with hindsight, after time $\mathcal{T}$, one can also average the forward and backward estimator, which has slightly better properties by sufficiency considerations. (Similarly to Remark 7(iii).)

5.2. *Irregular Sampling: Validity of the Previous Tick Approach. Several Dimensions*

For simplicity, we discuss this issue for the forward or other additive estimator introduced above. We suppose that data arrive at times $t_{n,i}$, $i = 0, \ldots, B'_n$. We shall take this

[27]In other words, one must check the conditions of Proposition 2 for each $c \in \mathcal{C}$.

to mean that the underlying half-interval estimator $\hat{\Theta}^{(n)}_{(0,T]}$ changes values at times $T = t_{n,i}$. We then set

$$\hat{\Theta}^{(n)}_{(0,T_i]} \triangleq \hat{\Theta}^{(n)}_{(0,T_{n,i,*}]} \quad \text{where} \quad T_{n,i,*} = \max\{t_{n,j} \leq T_{n,i}\}, \tag{44}$$

and proceed as if nothing has happened. This is the previous-tick scheme; see Zhang (2011) and the references therein.

The rationale for this is the following result, which is shown in Appendix D.2.

PROPOSITION 4—Previous Tick Sampling: *Assume that the* $t_{n,i}$, $i = 0, \ldots, B'_n$, *is* (*for each* n) *a nondecreasing sequence of stopping times*. *Suppose that*

$$\sup_i |T_{n,i,*} - T_{n,i}| \xrightarrow{p} 0 \quad as \quad n \to \infty, \tag{45}$$

as well as $T_{n,0,*} = 0$ *and* $T_{n,B'_n,*} = \mathcal{T}$. *In the formal results*[28] *of this paper, the conditions on the microstructure* $\tilde{e}_{n,T_{n,i}}$ *may be replaced by the same conditions on* $\tilde{e}_{n,T_{n,i,*}}$. $\mathcal{F}_{T_{n,i}}$ *may, however, not be replaced by* $\mathcal{F}_{T_{n,i,*}}$.

In practice, this means that the results in Section 3 are unaffected by the previous-tick sampling.

REMARK 11—When there is no Edge Effect: The condition (45) is required even when the microstructure noise $\tilde{e}_{n,T,*}$ is identically zero.

REMARK 12—Several Dimensions: The extension of this theory to several dimensions is straightforward. All our results carry over appropriately for the regular grid $\{T_{n,i}, i = 0, \ldots, B_n\}$, using the identity $xy = \frac{1}{2}((x+y)^2 - x^2 - y^2)$.[29] One can then use Proposition 4 in each dimension, since no time change is involved in our proofs.

5.3. *In Case the Spot Process θ_t Does not Exist*

The theory in this paper requires the existence of a "spot" θ_t, and does not apply, say, to estimating the discontinuous part of the quadratic variation. For example, suppose that $\Theta_{(0,T]} = \int_0^T \theta_t\, dt + \mathfrak{T}_T$, where $\mathfrak{T}_t$ is a process with finitely many jumps in $(0, \mathcal{T}]$ and no other variation. Then, obviously, to first order, $\mathrm{QV}_{B,K}(\Theta) = [\mathfrak{T}, \mathfrak{T}]_T - [\mathfrak{T}, \mathfrak{T}]_0 + o_p(1)$. The same is true for $\mathrm{QV}_{B,K}(\hat{\Theta})$. The situation is not exotic: A simple example would be the estimation of $[X, X]$ when the X process can have jumps. In our setting, the methodology applies to estimating the continuous part $\int \sigma_t^2$ of this quadratic variation.

For this reason, in our examples (Section 7), we consider that the primary estimating procedure removes anything that can cause $\mathfrak{T}_t$ to be nonzero. In the case that the $\mathfrak{T}_t$ process has finitely many jumps, these can alternatively be removed directly with truncation or bi- or multipower methods; cf. the references at the beginning of Section 7. We presently show how one can proceed using truncation.

[28]Theorems, propositions, corollaries, and lemmae. We emphasize that (unless the $t_{n,i}$ are nonrandom, or in certain other circumstances) the $T_{n,i,*}$ may not be stopping times. Hence, for example, the argument in Remark 14 (Section 7) may not be valid. Also, $\mathcal{F}_{T_{n,i,*}}$ will not be defined unless $T_{n,i,*}$ is a stopping time. In case of doubt, please make use of the more specific Proposition 5 in Section 6.

[29]See the definition of multivariate quadratic variation in Jacod and Shiryaev (2003, Eq. (I.4.46), p. 52).

ALGORITHM 1—Jump Removal in $\hat{\Theta}$: If there are ν (finitely many) jumps, truncation creates ν removed intervals[30] $(T_{i_j}, T_{i_j+1}]$, $j = 1, \ldots, \nu$. (These intervals are identified with probability 1 as $n \to \infty$.) One can then proceed as follows. For scale K, omit all $\hat{\Theta}_{(T_i, T_{i+K}]}$ for which $(T_{i_j}, T_{i_j+1}] \subseteq (T_i, T_{i+K}]$ for any of the removed intervals. When $\hat{\Theta}_{(T_i, T_{i+K}]}$ is removed, the relevant squares in $\mathrm{QV}_{B,K}(\hat{\Theta})$ are computed as $(\hat{\Theta}_{(T_{i+K}, T_{i+2K}]} - \hat{\Theta}_{(T_{i-K}, T_i]})^2$. Call this quantity $\mathrm{QV}_{B,K,\text{modified}}(\hat{\Theta})$. Similarly, for the true process θ, denote the modified averaged quadratic variation by $\mathrm{QV}_{B,K,\text{modified}}(\Theta)$.

The critical piece for analyzing the above construction is then the following, which generalizes Theorem 1 in Section 2.3, by the same methods.

THEOREM 6—The Integral-to-Spot Device With Removed Intervals: *Assume that θ_t is a semimartingale on $[0, \mathcal{T}]$. Set $\Delta T = \mathcal{T}/B$, and let $T_i = i\Delta T$. Suppose that $K\Delta T \to 0$, and that $K \to \infty$. Let $\tau_1, \ldots, \tau_v \in (0, T)$ be stopping times. Assume that in Algorithm 1 above, $P(\bigcap_{j=1}^{\nu}\{\tau_j \in (T_{i_j}, T_{i_j+1}]\}) \to 1$ as $B \to \infty$. Then*

$$
\begin{aligned}
(46)\qquad & \frac{1}{(K\Delta T)^2}\,\mathrm{QV}_{B,K,\text{modified}}(\Theta) \\
&= \left(\frac{2}{3}[\theta,\theta]_{\mathcal{T}-} + \frac{2}{3}\sum_{j=1}^{\nu}\bigl([\theta,\theta]_{T_{i_j+1}} - [\theta,\theta]_{T_{i_j}}\bigr)\right)\bigl(1 + o_p(1)\bigr) \\
&\xrightarrow{p} \frac{2}{3}[\theta,\theta]_{\mathcal{T}-} + \frac{2}{3}\sum_{j=1}^{\nu}(\Delta\theta_{\tau_j})^2.
\end{aligned}
$$

Thus, if jump times in $\mathfrak{T}_t$ coincide with those of θ_t, the estimation $[\theta, \theta]_{\mathcal{T}-}$ becomes additionally complicated.

The AVAR estimates, however, are not affected. Under the conditions of Theorem 4, the TSAVAR (26) remains consistent for $\mathrm{AVAR}_n(\hat{\Theta} - \Theta)$. $\mathrm{QV}_{B,K,\text{modified}}(\hat{\Theta})$ will have lost a fraction ν/B_n of its asymptotic variance component; one can consider a small sample multiplicative adjustment of $(1 - \hat{\nu}/B_n)^{-1}$ to the estimated variances, where $\hat{\nu}$ is the number of removed intervals $(T_{i_j}, T_{i_j+1}]$, but this does not impact the asymptotics.

For the case of many small jumps, it is unlikely that all jumps will be detected. The contiguity results of Zhang (2007), however, may mitigate the problem.

5.4. *Block Estimators: The Interface Between Block Sizes $\mathcal{M}_n$ and K_n*

Estimators are often based on rolling blocks of $\mathcal{M}_n$ observations. See, for example, Examples 6, 7, 9, and 10 and Remark 14 in Section 7. We thus have two types of block sizes: (i) $\mathcal{M}_n$ is used to construct the underlying $\hat{\Theta}$, and (ii) K_n (one or more) is used to construct our current $\mathrm{QV}_{B,K}(\hat{\Theta})$, and the resulting AVAR and $[\theta, \theta]_{\mathcal{T}-}$ estimators.

The two fundamental comments on this setup are: (a) it is important to not mix up $\mathcal{M}_n$ and K_n, and (b) there is no need for $\mathcal{M}_n$ and K_n to be related.

[30]The method carrying out the truncation may depend on the estimator.

In the schematic case[31] where observations times are the same as our T_i's, this means that the estimator $\hat{\Theta}_{(0,T_i]}$ is not defined for $i < \mathcal{M}_n$. For $i \geq \mathcal{M}$, however, we can seek relief in forward estimators (Section 5.1), so that no matter what value K_n has, we can define $\hat{\Theta}_{(T_{i-K_n},T_i]} = \hat{\Theta}_{(0,T_i]} - \hat{\Theta}_{(0,T_{i-K_n}]}$ from original forward estimators. These will be defined for $K_n + \mathcal{M}_n \leq i \leq B_n$. With this definition, we can marginally alter $\mathrm{QV}_{B,K}(\hat{\Theta})$ from (4) (Section 2.1) to

$$\mathrm{QV}_{B,K,\mathcal{M}'}(\hat{\Theta}) = \frac{1}{K} \sum_{i=K+\mathcal{M}'}^{B-K} (\hat{\Theta}_{(T_i,T_{i+K}]} - \hat{\Theta}_{(T_{i-K},T_i]})^2, \tag{47}$$

and similarly for $\mathrm{QV}_{B,K}(\Theta)$, where $\mathcal{M}'$ is either $\mathcal{M}_n$ or a slightly larger number (in case an estimator based on a single block is undesirable).

All theorems and other formal results go through unaltered if one replaces $\mathrm{QV}_{B,K}(\hat{\Theta})$ by $\mathrm{QV}_{B,K,\mathcal{M}'}(\hat{\Theta})$, provided $\mathcal{M}'_n \Delta T_n \to 0$ as $n \to \infty$. This is a substantially weaker requirement than the theoretical condition (63) used to analyze edge effects in Remark 14 in Section 7.

6. GUIDANCE: II. THEORY: TOOLS TO VERIFY CONDITION 1

We again emphasize that it is possible to use our methods without first verifying the conditions. This is standard practice in many areas of inference; the observed information, and bootstrapping, are examples where practice is often ahead of theory. We now, however, pass to the question of how to verify conditions. There are three main strategies: discretization, interpolation, and contiguity.

Discretization. For general results, we recommend, in particular, the books by Jacod and Shiryaev (2003), Jacod and Protter (2012), and Aït-Sahalia and Jacod (2014), as well as the many articles cited above, and in these books.

In our context, we assume for greatest generality that data arrive at irregular times, $t_{n,i}$, $i = 0, \ldots, B'_n$. The semimartingale M_n is on the form

$$M_{n,t} = \sum_{j=1}^{i} \chi_j^n, \quad \text{for} \quad t_{n,i} \leq t < t_{n,i+1}. \tag{48}$$

We are now outside the framework of a fixed filtration used in the rest of the paper, but there is a path. Proposition 5 will be proved in Appendix D.2.

CONDITION 3—Alternative Convergence Condition: Let θ_t be a semimartingale on the fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$. Let $t_{i,n}$, $i = 0, \ldots B'_n$, be a nondecreasing sequence of $(\mathcal{F}_t)$-stopping times so that

$$\sup |t_{i+1,n} - t_{i,n}| \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty, \tag{49}$$

as well as $t_{n,0} = 0$ and $t_{n,B'_n} = \mathcal{T}$ for each n. Let $M_{n,t}$ be on the form (48) and assume that $M_{n,t}$ is a semimartingale with respect to filtration $\mathcal{F}_t^n = \mathcal{F}_{t_{n,i}}$ for $t_{n,i} \leq t < t_{n,i+1}$. Assume the rest of the wording of Condition 1 with the proviso that $\{(e_{n,T_{n,i}}, \tilde{e}_{n,T_{n,i}}) : T_{n,i} \in \mathcal{T}_n\}$ be replaced with the set of random variables $\{(e_{n,T_{n,i,*}}, \tilde{e}_{n,T_{n,i,*}}) : T_{n,i,*} = \max\{t_{n,j} \leq T_{n,i}\}\}$.

[31] Otherwise see Section 5.2.

PROPOSITION 5—Satisfying Conditions With a Discrete-Time Martingale: *In the formal results*[32] *of this paper, Condition 1 may be replaced by Condition 3. At the same time, the conditions on the microstructure* $(e_{n,T_{n,i}}, \tilde{e}_{n,T_{n,i}})$ *should be replaced by the same conditions on* $(e_{n,T_{n,i,*}}, \tilde{e}_{n,T_{n,i,*}})$, *while* $\mathcal{F}_{T_{n,i}}$ *may not be replaced by* $\mathcal{F}_{T_{n,i,*}}$. *With these modifications, all formal results remain valid.*

We can now avail ourselves of the standard Jacod structure. For example, to satisfy Condition 1, we can check the assumptions of Theorem IX.7.19 (pp. 589–590), or of Theorem IX.7.28 (pp. 590–591), of Jacod and Shiryaev (2003, Chapter IX.7b), with $B \equiv Z \equiv G \equiv 0$. To additionally satisfy P-UT, we additionally need, respectively,

$$\sum_{i=1}^{n} \left| E\big(h(\chi_i^n) | \mathcal{F}_{(i-1)/n}\big) \right| = O_p(1) \quad \text{or} \quad \sum_{i=1}^{n} \left| E\big(\chi_i^n | \mathcal{F}_{(i-1)/n}\big) \right| = O_p(1). \tag{50}$$

Furthermore, if $L_{n,t} = n^{\alpha} M_{n,t}$ can be written as $L_{n,t} = L_{n,t}^{(1)} + L_{n,t}^{(2)}$, Condition 1 is satisfied for $L_{n,t}$ provided it is satisfied for $L_{n,t}^{(1)}$, and provided $L_{n,t}^{(2)} \to 0$ uniformly in probability (ucp), with (for P-UT)

$$\sum_{i=1}^{n} \left| E\big(L_{n,t_{n,i}}^{(2)} - L_{n,t_{n,i-1}}^{(2)} | \mathcal{F}_{n,t_{n,i-1}}\big) \right| = O_p(1), \tag{51}$$

again by Jacod and Shiryaev (2003, Theorem VI.6.21, p. 382). Both ucp and P-UT are additive (ibid., Remark 6.4, p. 377).

Incidentally, Theorem IX.7.19, or Theorem 7.28, of Jacod and Shiryaev (2003) also guarantees the conditions of Proposition 2 (feasible estimation).

The methodology is illustrated by Example 6, where the paper by Jacod and Rosenbaum (2013) verifies the stable convergence with the help of Jacod and Shiryaev (2003, Theorem IX.7.19, p. 590) and where ignorable terms are ucp, and where it remains to show P-UT-ness. The example illustrates that the P-UT property often follows from the same arguments that give rise to stable convergence.

Interpolation. This has to a great extent been the approach of the current authors. Even if the data are discrete, one can create a continuous martingale by interpolation. One can verify Condition 1 by checking the assumptions of Zhang (2001, Theorem B.4, pp. 65–67) or Mykland and Zhang (2012, Theorem 2.28, pp. 152–153). The P-UT property is here automatic, by Jacod and Shiryaev (2003, Corollary VI.6.30, p. 385). We have included a procedure of this type in Example 1 in Section 7.

The idea of interpolation goes back to Heath (1977), and is related to embedding; cf. the references in Mykland (1995). In our current case, however, one has to be particularly precise, since the process θ_t already lives on the relevant filtration.

Contiguity. The contiguity approach (Mykland and Zhang (2009, 2011, 2012, 2016)) may, when applicable, reduce high frequency martingales to ones that are locally Gaussian. We refer to the cited papers for further discussion.

7. EXAMPLES: CORROBORATION OF CONCEPT

The purpose of this section is to document that the assumptions in this paper are widely satisfied in the existing literature. The relevant papers will typically have expressions

[32]See Footnote 28 in Section 5.2 for caveats.

for AVAR_n and an estimator thereof. In most cases, however, the alternative Observed $\widehat{\text{AVAR}}_n$ is much easier to implement when constructing a feasible statistic of the form (2). We also in many cases describe carefully the separation into martingale and edge effect, thereby hopefully assisting the understanding of the concept.

Unless the opposite is indicated, we suppose that X_t is an Itô-semimartingale, either with no jumps ($dX_t = \mu_t\,dt + \sigma_t\,dW_t$), or with jumps that are removed by bi- and multipower methods (Barndorff-Nielsen and Shephard (2004b, 2006), Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2006a, 2006b)), or by truncation[33] (Mancini (2001), Aït-Sahalia and Jacod (2007, 2008, 2009, 2012), Jacod and Todorov (2010), Lee and Mykland (2008, 2012), Jing, Kong, Liu, and Mykland (2012)), as appropriate. See also Zhang (2007), Christensen, Oomen, and Podolskij (2014), and Bajgrowicz, Scaillet, and Treccani (2016). We emphasize that θ can be a general semimartingale,[34] so that, for example, the Lévy driven volatility model in Barndorff-Nielsen and Shephard (2001) is covered by the examples. We either observe X_{t_i} at times t_i, $i = 0, \ldots, n$ spanning $[0, \mathcal{T}]$, or we observe Y_{t_i}, which is a version of X_{t_i} that is contaminated by microstructure noise.

In implementation, we assume that $\hat{\Theta}_{(S,T]}$ is the forward estimator from Section 5.1. For examples with irregular observations, we assume the previous-tick scheme from Section 5.2, and in particular that (45) is satisfied. We shall omit the subscript n on t: t_i means $t_{n,i}$.

REMARK 13—Two Types of Conditions: To see how our examples fit into the theory, we need to check two classes of conditions. One is on the martingale $M_{n,t}$, and they are all in Condition 1 or in the alternative Condition 3. We recall that they are

$$
\begin{aligned}
&n^{\alpha} M_n \xrightarrow{\mathcal{L}} L \quad \text{stably}, \quad n^{\alpha} M_n \quad \text{is P-UT}, \quad [L, L]_{\mathcal{T}} \in \mathcal{G}, \quad \text{and} \\
&L \text{ is a martingale conditionally on } [L, L]_{\mathcal{T}}.
\end{aligned} \tag{52}
$$

The edge effects have various conditions attached to them depending on their order of magnitude. They all need to satisfy that $\tilde{e}_{T_i} = o_p(n^{-\alpha})$. The edge conditions are in Sections 3.2–3.3. The easiest condition to satisfy is (29) in Theorem 4, which makes the two-scales AVAR and $\widehat{[\theta, \theta]}_{\mathcal{T}-}$ consistent. This condition also implies (22) in Theorem 3 for the choice of K_n that satisfies the balance condition (30).

EXAMPLE 1—Realized Volatility, No Microstructure Noise: The parameter is $\theta_t = \sigma_t^2$. The convergence rate is $\alpha = 1/2$. In the straightforward X-is-continuous case, a popular estimator for the $\int_0^t \theta\,ds$ is the standard realized volatility (RV), $\sum_{t_{i+1} \le t} (X_{t_{i+1}} - X_{t_i})^2$ (Andersen, Bollerslev, Diebold, and Ebens (2001), Andersen, Bollerslev, Diebold, and Labys (2001), Barndorff-Nielsen and Shephard (2002a)). There is no edge effect, that is, $\tilde{e}_{T_{n,i,*}} \equiv 0$. By Remark 13, we need to check (52). The stable convergence has been shown by Jacod and Protter (1998) using discretization. We here use interpolation because it gives the P-UT property directly. The interpolated semimartingale has the form $M_{n,t} = \sum_{t_{n,j+1} \le t} (X_{t_{n,j+1}} - X_{t_{n,j}})^2 + (X_t - X_{t_{n,*}})^2 - \int_0^t \sigma_s^2\,ds$, where $t_{n,*} = \max_j \{t_{n,j} \le t\}$. See Zhang (2001), Mykland and Zhang (2006, 2012). The requirements $[L, L] \in \mathcal{G}$, and that L be a martingale conditional on $\mathcal{G}$, also follow from the construction in the cited papers, and all theorems in the current paper can be used.

[33]For the case of removal by truncation, please consult Section 5.3.

[34]In all our examples, the spot values of θ_t exists. See Section 5.3 for further discussion of this.

EXAMPLE 2—Bipower Variation, No Microstructure Noise: The bipower variation $\hat{\Theta}_{(0,T]} = \frac{\pi}{2}\sum_{0<t_i\leq T}|\Delta X_{t_{i-1}}||\Delta X_{t_i}|$ (and more generally, multipower variation; Barndorff-Nielsen and Shephard (2004b, 2006)) estimates the integrated volatility in a way that is robust to jumps. Since jumps are of the essence in this model, we specify that $dX_t = \mu_t\,dt + \sigma_t\,dW_t + dJ_t$, where J_t is a semimartingale for which $[J, J]_t$ is purely discontinuous.

The parameter is $\theta_t = \sigma_t^2$. The convergence rate is $\alpha = 1/2$. We here study the case of equidistant sampling, $t_{n,i} - t_{n,i-1} = \Delta t_n = \mathcal{T}/n$, and for convenience we take $\Delta T_n = \Delta t_n$. Our semimartingale is

$$M_{n,T_i} = \frac{\pi}{2}\sum_{j=2}^{i}|\Delta X_{t_{j-1}}||\Delta X_{t_j}| - \int_0^{T_{i-1}} \sigma_t^2\,dt. \tag{53}$$

The papers by Barndorff-Nielsen and Shephard (2004b, 2006), Barndorff-Nielsen et al. (2006a, 2006b), and Barndorff-Nielsen, Shephard, and Winkel (2006) have shown stable convergence and the other conditions of (52), with the exception of the P-UT property. For details about the P-UT property and its proof, see Appendix D.3.

Edge Effect. There is some variability between proofs of whether the integral in (53) has upper limit T_{i-1} or T_i. In the latter case, there is no edge effect. In the former case, by Remark 14 below, $\text{ave}(\tilde{e}^2_{T_i}) = O_p((\Delta T_n)^3)$, by (64).

In conclusion, all theorems and estimators in the current paper can be used to estimate the AVAR of bipower variation.

EXAMPLE 3—Classical Two-Scales Realized Volatility: The parameter remains $\theta_t = \sigma_t^2$. There is now microstructure noise, and observations are of the form

$$Y_{t_i} = X_{t_i} + \varepsilon_{t_i}, \tag{54}$$

which we here for simplicity take to be i.i.d., or to be stationary with fast mixing dependence. X_t is assumed to be a continuous Itô-semimartingale.

The classical Two-Scales Realized Volatility (TSRV; Zhang, Mykland, and Aït-Sahalia (2005), Aït-Sahalia, Mykland, and Zhang (2011)) has a convergence rate of $\alpha = 1/6$. It is easy to see that Conditon 1 is satisfied. Edge effects, whether alone or by averages, are of order $O_p(n^{-2\alpha})$, cf. Zhang, Mykland, and Aït-Sahalia (2005, eq. (A.21), p. 1409), whence Theorem 4 applies. The two-scales AVAR and $\widehat{[\theta, \theta]}_{T-}$ are thus consistent. In fact, Theorem 3 is valid so long as $n^{2\alpha}K_n\Delta T_n \to \infty$.

EXAMPLE 4—Pre-Averaging Followed by TSRV: The parameter remains $\theta_t = \sigma_t^2$. The observations are as in (54). The convergence rate is $\alpha = 1/4$. The estimator is constructed as follows. One pre-averages observations across blocks of size $O(n^{1/2})$ observations, and then calculates a (j, k) TSRV on the basis of the pre-averaged observations, where $1 \leq J < K$ are finite. One can show that this estimator of integrated volatility converges stably at rate $\alpha = 1/4$, the semimartingale is P-UT, and the edge effects are benign, of exact order $O_p(n^{-1/2})$. The edge effects are thus small enough to satisfy the edge conditions (22) and (29) in Theorems 3–4 (Section 3.2). We have used this method in Figure 1. Note that in the terms of Section 5.4, and Remark 14 below, $\mathcal{M} = k$.

It is conjectured that the same type of situation pertains to classical pre-averaging (Jacod et al. (2009a), Podolskij and Vetter (2009b)), but we have not investigated this.

EXAMPLE 5—Multi-Scale and Kernel Realized Volatility: The parameter remains $\theta_t = \sigma_t^2$. The observations are as in (54). The convergence rate is $\alpha = 1/4$. We here show that the Multi-Scale Realized Volatility (MSRV, Zhang (2006)) is covered by our current development. Following Bibinger and Mykland (2016), the result also covers Realized Kernel estimators (RK; Barndorff-Nielsen et al. (2008)).

We shall go through this case in some detail since it illustrates many of the issues. From eq. (15), p. 1024, and eq. (51), p. 1039, in Zhang (2006),

$$M_{n,t} = M_{n,t}^{(1)} + M_{n,t}^{(2)} + M_{n,t}^{(3)}, \tag{55}$$

where[35]

$$M_{n,t}^{(1)} = -2\sum_{i=1}^{\mathcal{M}_n} a_{n,i}\frac{1}{i}\sum_{t_{i+1}\le t} \varepsilon_{t_{n,j}}\varepsilon_{t_{n,j-i}}, \tag{56}$$

$$M_{n,t}^{(2)} = \sum_{i=1}^{\mathcal{M}_n} a_{n,i}[X,X]_t^{(n,i)} - \int_0^t \sigma_s^2\,ds, \quad \text{and}$$

$$M_{n,t}^{(3)} = 2\sum_{i=1}^{\mathcal{M}_n} a_{n,i}[X,\varepsilon]_t^{(i)}.$$

The edge effects, e and $\tilde{e}$, are given by (ibid., eq. (51), p. 1039, and rewritten form (53), p. 1040)

$$e_{n,0} = \sum_{j=0}^{\mathcal{M}_n-1} \varpi_{n,j}\varepsilon_{t_j}^2 - E\varepsilon^2 \quad \text{and} \tag{57}$$

$$\tilde{e}_{n,t_k} = \sum_{j=0}^{\mathcal{M}_n-1} \varpi_{n,j}\varepsilon_{t_{k-j}}^2 - E\varepsilon^2, \quad \text{where} \quad \varpi_{n,j} = \sum_{i=j+1}^{\mathcal{M}_n}\frac{a_{n,i}}{i}.$$

With these definitions, and with $\mathcal{M}_n = O(n^{1/2})$, eq. (13) in the current paper is satisfied up to $O_p(n^{-1/2})$ (ibid., Proposition 1, p. 1023).

The terms in (57) are of order $O_p(n^{-1/4})$, and so Condition 1 is violated. Since this magnitude of edge effects is in any case undesirable, we propose to amend the MSRV by estimating the edge effects:

$$\text{adjusted MSRV}_{n,t_k} = \text{original MSRV}_{n,t_k} - \hat{e}_{n,t_k} + \hat{e}_{n,0}, \quad \text{where} \tag{58}$$

$$\hat{e}_{n,t_k} = \sum_{j=0}^{\mathcal{M}_n-1} \varpi_{n,j}(Y_{t_{k-j}} - \bar{Y}_{t_k})^2 - \frac{1}{2}[X,X]_{\mathcal{T}}^{(n,1)},$$

[35]Except that we use $\mathcal{M}_n$ to denote the number of scales (called M_n in Zhang (2006)). The square brackets in (56) are discrete sums. The $a_{n,i}$ are given by ibid., eqs. (21)–(22) p. 1026.

and similarly for $\hat{e}_{n,0}$, where $\bar{Y}_{t_k}$ is the mean of $Y_{t_{k-\mathcal{M}_n+1}}, \dots, Y_{t_k}$. From Zhang (2006, Condition 1, p. 1023, and eq. (54), p. 1040),

$$\sum_{j=0}^{\mathcal{M}_n-1} \varpi_{n,j} = 1 \quad \text{and} \quad \sum_{j=0}^{\mathcal{M}_n-1} \varpi_{n,j} = O_p\big(\mathcal{M}_n^{-1}\big), \tag{59}$$

we obtain $\hat{e}_{n,t_k} = \tilde{e}_{n,t_k} + O_p(n^{-1/2})$. Hence,

$$\text{adjusted MSRV}_{n,t_k} = M_{n,t_k} + O_p\big(n^{-1/2}\big), \tag{60}$$

and so the new edge effect is of size $O_p(n^{-1/2})$. Under the conditions of ibid., Theorem 4 (p. 1031), including $\mathcal{M}_n/n^{1/2} \to c$, it is easy to see that Condition 1 is satisfied.[36] The edge effects are thus small enough to satisfy the edge conditions (22) and (29) in Theorems 3–4 (Section 3.2).

Similar arguments would extend to the dependent but mixing noise in Aït-Sahalia, Mykland, and Zhang (2011).

REMARK 14—Edge Effects in Block Based Estimation: Estimators are often based on rolling blocks of $\mathcal{M}_n$ observations.[37] This is the case in the following Examples 6, 7, 9, and 10.[38] See also Section 5.4 to the effect that our K_n is unrelated to $\mathcal{M}_n$.

Rolling block estimators frequently have the common feature that the edge effect is (exactly or approximately) on the form $\tilde{e}_{T_i} = -\Theta_{(T_{i-\mathcal{M}_n+1}, T_i]}$. We here present a general strategy for dealing with edge effects on this form, and we shall comment on specifics in connection with individual examples. For simplicity, we assume that observations are an equidistant sample every $\Delta t_n = \mathcal{T}/n$ units of time, and we also set $\Delta T_n = \mathcal{T}/n$. (This is the case for all the papers we cite on block estimation.) Assume that the conditions (52) on the martingale $M_{n,t}$ are satisfied.

First of all, use (B.9) in Appendix B to write $\tilde{e}_{T_i} = \Theta''_{(T_{i-\mathcal{M}_n+1}, T_i]} - \theta_{T_i}(\mathcal{M}_n - 1)\Delta T_n$, where $\Theta''_{(T_{i-\mathcal{M}_n+1}, T_i]}$ is as defined in (11) in Section 2.3. Then absorb $-\theta_{T_i}(\mathcal{M}_n - 1)\Delta T_n$ in the semimartingale M_n, so that

$$M_{n,T_i}^{\text{adjusted}} = M_{n,T_i}^{\text{original}} - \theta_{T_i}(\mathcal{M}_n - 1)\Delta T_n, \tag{61}$$

and redefine the edge effect as

$$\tilde{e}_{T_i} = \Theta''_{(T_{i-\mathcal{M}_n+1}, T_i]}. \tag{62}$$

So long as[39]

$$\mathcal{M}_n \Delta T_n = o\big(n^{-\alpha}\big), \tag{63}$$

[36]The second term in (58) is only available at time $\mathcal{T}$. This means that it can be used to estimate the MSRV at time $\mathcal{T}$. For the intermediate calculations at times $T_{n,i}$ or $T_{n,i,*}$, this is not a concern, however, since the term is constant in i and thus will cancel when computing $\hat{\Theta}_{(T_i, T_{i+K}]} - \hat{\Theta}_{(T_{i-K}, T_i]}$. For purposes of verifying the conditions of our results, we therefore proceed as if $\frac{1}{2}[X, X]_{\mathcal{T}}^{(n,1)}$ is replaced by $E\varepsilon^2$.

[37]Many papers use k_n or M_n to denote what we here call $\mathcal{M}_n$. We use the latter symbol to avoid overlap with our own notation.

[38]The block structure is also present in most of our other examples, even if we have not used the structure explicitly. To some extent, this is a question of technique of proof.

[39]See Jacod and Rosenbaum (2015) and Theorem 3.1 in Jacod and Rosenbaum (2013) for an important contribution on what can happen otherwise.

the limiting martingale and the mode of convergence are unchanged (Jacod and Shiryaev (2003, Lemma VI.3.31, p. 532)). The P-UT property is also not affected (ibid., Remark VI.6.4, p. 377). Also, by the same methods as in the Proof of Theorem 1 (see Appendix B), $\tilde{e}_{T_i} = O_p(\mathcal{M}_n \Delta T_n) = o_p(n^{-\alpha})$. Hence, Condition 1, or alternative Condition 3, is satisfied.

As an application of Theorem 7 in Appendix A (the proof is similar to that of Theorem 2 (Appendix B)), we obtain that

$$\operatorname{ave}\left(\tilde{e}_{T_i}^2\right) = \begin{cases} \dfrac{1}{3\mathcal{T}}\left((\mathcal{M}_n - 1)\Delta T_n\right)^3 [\theta, \theta]_{\mathcal{T}-}\left(1 + o_p(1)\right) \\ \text{when } \mathcal{M}_n \to \infty \text{ as } n \to \infty, \text{ and} \\ O_p\left((\Delta T_n)^3\right) \text{ when } \mathcal{M}_n \text{ remains finite as } n \to \infty, \end{cases} \tag{64}$$

whence assumption (29) in Theorem 4 is satisfied. The two-scales AVAR and $\widehat{[\theta, \theta]}_{\mathcal{T}-}$ are thus consistent. Depending on the size of $\mathcal{M}_n$, further small edge conditions are satisfied.

EXAMPLE 6—Block Estimation of Higher Powers of Volatility: The parameter is $\theta_t = g(\sigma_t^2)$, with g not being the identity function. In the absence of microstructure noise, the convergence rate is $\alpha = 1/2$. If microstructure noise is present, the convergence rate is $\alpha = 1/4$. We are here concerned with the former case.[40] The estimation of integrals of σ_t^p goes back to Barndorff-Nielsen and Shephard (2002a), who showed that the case $g(x) = x^2$ is related to the asymptotic variance of the realized volatility. See also Barndorff-Nielsen et al. (2006a), Mykland and Zhang (2012, Proposition 2.17, p. 138), and Renault, Sarisoy, and Werker (2013) for related developments.

Block estimation (Mykland and Zhang (2009, Section 4.1, pp. 1421–1426)) has the ability to make these estimators approximately or fully efficient. One path is to keep the block size $\mathcal{M}_n$ finite. This avoids bias. When using overlapping (rolling) blocks (or moving windows), however, the asymptotic variance is hard to compute (Mykland and Zhang (2012, Chapter 2.6.2, pp. 170–172)). This is an instance where the observed AVAR would seem to be particularly appealing. Conditions (52) are clearly satisfied, by the derivation in the cited papers. Also, by Remark 14, we can use all of the results: Theorem 3–5, and Remark 9.

Another path is to let the block size increase with n; cf. Mykland and Zhang (2011, Section 5, pp. 224–229), and Jacod and Rosenbaum (2013, 2015). As seen in the cited papers, for increasing block size, there is a bias that can be corrected for. In Jacod and Rosenbaum (2013), the corrected estimator is (in their notation) $V'(g)^n$ (eq. (3.7), p. 1469), which satisfies assumptions (52). We now discuss how to verify these assumptions. The stable convergence is stated in ibid., Theorem 3.2 (pp. 1469–1470). The P-UT condition is satisfied by noting that in the proof of their Lemma 4.4 (pp. 1478–1480), each of the four components obviously also satisfies our equation (51), by being bounded term-wise. In their Lemma 4.5 (pp. 1478, 1480–1481), they proceeded by verifying the conditions of Jacod and Shiryaev (2003, Theorem IX.7.28, p. 591), and it is easy to see that the second part of (our) eq. (50) is satisfied, guaranteeing P-UT also for this term in view of Section 6.

The edge effect is part of $V_t^{n,2}$ in Jacod and Rosenbaum (2013, p. 1478). Ibid., assumption (3.6) (p. 1469) yields that condition (63) in Remark 14 is satisfied, whence at least the two-scales AVAR and $\widehat{[\theta, \theta]}_{T-}$ are consistent.

[40]Inference in the presence of noise was considered in Jacod and Protter (2012, Sections 16.4–16.5, pp. 512–554).

As a final comment, n is typically given for fixed data. When this is the case, it is entirely in the mind of the econometrician whether the block size is finite or not as $n \to \infty$. This raises the question of which asymptotics to use. This conundrum may also be a reason for using the observed asymptotic variance, and other small sample methods.

EXAMPLE 7—High Frequency Regression, and ANOVA: We are here concerned with systems on the form $dV_t = \beta_t \, dX_t + dZ_t$, where V_t and X_t can be observed at high frequency, either with or without microstructure noise. The coefficient process β_t can either be the "beta" from portfolio optimization, with Z_t in the role of idiosyncratic noise, or β_t can be the hedging "delta" for an option, with Z_t as tracking error. Nonparametric estimates can be used directly, or for forecasting, or for model checking. X_t can be multidimensional. The regression problem seeks to estimate or make tests about $\int_0^{\mathcal{T}} \beta_t \, dt$ (Mykland and Zhang (2009, Section 4.2, pp. 1424–1426), Zhang (2012, Section 4, pp. 268–273), Reiss, Todorov, and Tauchen (2015)). The ANOVA problem seeks to estimate $[Z, Z]_{\mathcal{T}}$ (Zhang (2001) and Mykland and Zhang (2006)). Convergence rates are as for realized or other powers of volatility, with $\alpha = 1/2$ when there is no microstructure noise, and $\alpha = 1/4$ otherwise. When there is no microstructure noise, Condition 1 is satisfied by a slight extension of the derivations in the cited papers. Both regression and ANOVA have edge effects due to blocking, as in Example 6. Since $\mathcal{M}_n$ is finite, and according to Remark 14, we can use all results: Theorems 3–5, and Remark 9.

EXAMPLE 8—Estimation of Covolatility (ex post Covariance) From Asynchronous Observations: A popular estimator is due to Hayashi and Yoshida (2005); see also Podolskij and Vetter (2009a), Christensen, Podolskij, and Vetter (2013), and Bibinger and Vetter (2015) for microstructure, jumps, and asymptotic distributions. Alternatives include the previous-tick estimator (Zhang (2011), Bibinger and Mykland (2016)), and quasi-likelihood (Shephard and Xiu (2014)). The estimator in Mykland and Zhang (2012, Chapter 2.6.3, pp. 172–175) is a hybrid of Hayashi–Yoshida and quasi-likelihood. The asymptotic distributions, however, are often quite complex, and the estimation of AVAR is daunting. In comparison, the approach of observed AVAR offers a pleasing alternative to assessing the asymptotic variance of covolatility. In all these cases, it is quite clear that the stable convergence holds, and that the current paper's Condition 1 is satisfied, including the P-UT property. In terms of edge effects, the previous-tick Two-Scales Covariance (TSCV; Zhang (2011)) has exactly the same properties as the classical TSRV (Example 3). This is because of the strong representation property of one in terms of the other (Zhang (2011, eq. (39), p. 41; see also eq. (8), p. 35)). The two-scales AVAR and $\widehat{[\theta, \theta]}_{\mathcal{T}-}$ based on the previous-tick TSCV are thus consistent. Due to the large number of covariance estimators, however, we have not investigated edge effects for the full spectrum of these.

EXAMPLE 9—Continuous Leverage Effect, With or Without Microstructure Noise: The parameter is $\theta_t = d[\sigma^2, X^c]_t/dt$. If there is no microstructure noise, the convergence rate is $\alpha = 1/4$. If microstructure noise is present, the convergence rate is $\alpha = 1/8$. The estimation of leverage effect was discussed in Mykland and Zhang (2009, Section 4.3, pp. 1426–1428) and Wang and Mykland (2014) for the case where X_t is continuous, and in Aït-Sahalia et al. (2016) and Kalnina and Xiu (2016) for the case where the process X_t can also have jumps.[41] Wang and Mykland (2014) and Aït-Sahalia et al. (2016) studied

[41] Aït-Sahalia, Fan, and Li (2013) discussed leverage effect in the parametric framework.

both the case where there is microstructure noise, and the case where there is none. All estimators are based on blocks.

We here study the procedure of Aït-Sahalia et al. (2016). Jumps are removed as in Jacod and Todorov (2010). The relevant central limit theorems are Theorem 3 (no microstructure noise) and Theorem 7 (with microstructure noise). The conditions (52) are satisfied by a slight extension of the proofs of these results. The optimal rates ($\alpha = 1/4$ and $\alpha = 1/8$) are attained in both cases (with choice of parameter $b = 1/2$). The edge effects are essentially on the form described in Remark 14; cf. $D(2)_t^n$ (in (their) Appendix B.1 for the no-microstructure case, and in (their) Appendix B.4 for the case with microstructure noise). In both cases, $\mathcal{M}_n$ (called k_n in this paper) is of order $O(n^{2\alpha})$. Thus condition (63) in Remark 14 is satisfied. The two-scales AVAR and $\widehat{[\theta, \theta]}_{T-}$ are thus consistent.

EXAMPLE 10—Volatility of Volatility, No Microstructure Noise: The process X is assumed to be a continuous Itô-semimartingale, with volatility $\sigma_t^2 = d[X, X]_t/dt$ which is itself assumed to be a continuous Itô-semimartingale. The parameter is $\theta_t = d[\sigma^2, \sigma^2]_t/dt$. The convergence rate is $\alpha = 1/4$. The results in the literature on this inference problem are Vetter (2015, Theorems 2.5 and 2.6) and Mykland, Shephard, and Sheppard (2012, Theorem 7 and Corollary 2).

We here focus on the estimator of Vetter (2015). It is on the form (25) in Section 3.2 above, with AVAR_n replaced by the quarticity estimator of Barndorff-Nielsen and Shephard (2002a, 2004a). The estimator is thus a special case of Theorem 3.

Turning to the question of whether the estimator satisfies the conditions of this paper, observe that this is also a rolling block estimator. The conditions (52) are satisfied by a slight extension of the proof of Vetter (2015, Theorem 2.5). $\mathcal{M}_n$ is of order $O(n^{1/2})$, and hence condition (63) in Remark 14 is satisfied. The two-scales AVAR and $\widehat{[\theta, \theta]}_{T-}$ (the estimator of the volatility of the volatility of the volatility) are thus consistent, as are the multi-scale estimators.

It should be noted that by computing the two-scales estimate $\widehat{[\theta, \theta]}_{T-}$ for any of the estimators in Examples 3–5, one obtains an estimator of $[\sigma^2, \sigma^2]_{T-}$ that is consistent in the presence of microstructure.

8. CONCLUSION

The paper introduces a nonparametric estimator of estimation error which we call the observed asymptotic variance. In analogy with the "observed information" of parametric inference, our statistic estimates the asymptotic variance without needing a formula for the theoretical quantity. As we have seen in our examples, the estimator is consistent in all of them.

We emphasize that the method has a strong applied motivation, and that it meets a need. Assessing the standard error of a high-frequency-based estimator is challenging to implement. We hope our proposed methodology will be a useful tool at the disposal of everyone who works with high frequency data.

On the mathematical side, the basic insight is Equation (5) in Section 2.2. To operationalize this insight, the two main tools are the Integral-to-Spot Device (Section 2.3), and the mathematical similarity between edge effects and microstructure noise (Section 3.1). The estimation of asymptotic variance (AVAR) is implemented with the help of a two-scales method in Section 3.2, and examples are given in Section 7. Practical and theoretical guidance to how to use the procedure is given in Sections 5–6.

The observed AVAR can also be used for the selection of tuning parameters, also in the non-obvious case of stable convergence and random variance (Section 4). As part of the theoretical development, we show how to feasibly disentangle the impact of estimation error $\hat{\Theta}_{(0,\mathcal{T}]} - \Theta_{(0,\mathcal{T}]}$ and the variation $[\theta, \theta]_{\mathcal{T}-}$ in the parameter process alone. For the latter, we also obtain a new estimator of quadratic variation of target parameters. The methods generalize readily to several dimensions.

A number of issues have been left for later. Consistency is only the first-order requirement on estimators of AVAR. Further optimization may involve the convergence rate, and the AVAR of $\widehat{\text{AVAR}}$. A main question remains of whether there is added benefit in going to a multi-scale procedure. There is also room for a more complete theory of tuning parameter selection, and of multivariate inference. Additional insight may be gained by letting $\Delta T \to 0$ for fixed $\delta = K\Delta T$. It would also be interesting to extend Observed AVAR to the case where the spot process θ_t is not a semimartingale, and to the case where it does not exist (see Section 5.3).

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