Quantitative Finance, 2021
Vol. 21, No. 6, 881–889, https://doi.org/10.1080/14697688.2020.1837922

© 2021 iStockphoto LP

Geometry of unconditionally efficient portfolios formed with conditioning information: the efficient semicircle

ANDREW F. SIEGEL *

Department of Finance, Department of Information Systems and Operations Management, Department of Statistics, Foster School of Business, University of Washington, Seattle, WA 98195-3200, USA

(*Received 27 March 2019; accepted 9 October 2020; published online 12 February 2021*)

A surprising 'Efficient Semicircle' emerges in risk-return space, restricting all conditional portfolios to this unusual non-hyperbolic frontier

1. Introduction

An unconditionally efficient (UE) portfolio chooses its weights as a function of observed conditioning information in order to achieve the minimum unconditional portfolio variance (the expected conditional variance plus the variance of the conditional mean) among all portfolio weight functions that lead to the same unconditional mean (the expected conditional mean). The UE investor's strategy then consists of a commitment to a particular choice of this weight function that prespecifies, conditionally in every possible case, what the actual portfolio weights would be, in order to solve this variational problem of optimizing an unconditional objective. Thus the UE portfolio concept generalizes the fixed-weight portfolio optimization theory of Markowitz (1959) and Sharpe (1964) to the case where the portfolio is formed using useful conditioning information (whose distribution is correlated with next-period asset returns) but is assessed (as in Dybvig and Ross 1985) through the eyes of an observer who does not see the conditioning information (where this observer might be a client, analyst, or researcher). Because, by definition, the UE portfolio minimizes (unconditional) variance at a specified (unconditional) mean, all of its properties flow most directly from the preferences of an agent with quadratic utility, for which the unconditional-utility-maximizing portfolio has conditional means and conditional variances that also maximize the same utility function.

The UE portfolio concept goes back to Hansen and Richard (1987), who show that a UE portfolio must be conditionally efficient for (almost) all observations of the conditioning information, but that the use of conditionally efficient portfolios will not generally guarantee UE (where a 'conditionally efficient' portfolio is efficient with respect to the conditional distribution of asset returns given the conditioning information). Closed-form UE solutions are provided by Ferson and Siegel (2001) who also show that a UE portfolio must maximize an expected quadratic utility function. UE portfolios have been used in conjunction with asset-pricing tests by Ferson and Siegel (2003); Abhyankar *et al.* (2007); and Ferson and Siegel (2009). Hedging applications were considered by Ferson *et al.* (2006). Peñaranda (2016) explores two alternative characterizations of portfolio efficiency with conditioning information that are distinct from the unconditional

*Email: asiegel@u.washington.edu

efficiency considered here: one maximizes the unconditional Sharpe Ratio of excess returns, the other substitutes the (generally smaller) expected conditional variance in place of the unconditional variance.

The use of conditioning information in portfolio management and performance has been documented by a number of authors. Abhyankar *et al.* (2012) study the empirical performance of unconditionally efficient portfolios, identify factors that significantly improve the risk-return tradeoff, and demonstrate the value of using conditioning information. Avramov and Wermers (2006) assess a variety of mutual fund investment strategies by forming unconditionally efficient portfolio strategies (with constraints) with respect to the subjective investor beliefs (about asset return distributions) implied by each strategy. Brandt and Santa-Clara (2006) use linear approximations to unconditionally efficient portfolios, simplifying the process of conditional portfolio specification by avoiding detailed specification of the conditional distributions of asset returns. Chiang (2015) develops unconditionally efficient tracking strategies portfolio methods for portfolio management. Cotter *et al.* (2017) use portfolios formed with conditioning information (including unconditionally efficient portfolios) to reconsider the ability of commodities and currencies to expand the investment horizon beyond that available with equities and debt. Ferson and Siegel (2015) use conditioning information to develop optimal orthogonal portfolios for active management. Zhou (2008) uses the theory of unconditionally efficient portfolios to identify optimal value-added investment strategies. Agudo *et al.* (2006) demonstrate improvement in Spanish equity investment performance when conditioning information is used. Bhaduri and Saraogi (2010) demonstrate that the yield spread, when used as conditioning information, improves investment results in the India stock market. Fletcher (2011) finds that unconditionally efficient portfolios perform significantly better than a number of alternative methods for incorporating conditioning information, after adjusting for costs, in the UK equity markets.

The main contribution of this article is to define and introduce the Efficient Semicircle, within risk-return space, that surprisingly constrains the conditional moments of a specified UE portfolio. Section 2 demonstrates the strength of the geometric approach, which very directly explains many properties of the conditional portfolios that make up a UE portfolio, and also lends clarity to the known paradoxes (lower investment with a stronger signal, limited conditional mean); while assuming a risk-free asset, we examine the general case of many assets and also the case of one risky asset with constant conditional variance for its helpful intuition. Section 3 summarizes and concludes. Appendix A contains Lemma 1, a technical result to help distinguish mean-variance utility from quadratic utility. Appendix B provides algebraic formulas, with full details and proofs, for these conditional portfolios and the Efficient Semicircle, to accompany and complete the purely geometric results of Section 2. Also included in Appendix B is a curious general result (that does not require UE) that might appear to conflict with Jensen's Inequality: when conditional mean and standard deviation fall on a semicircle with diameter along the expected return axis, then the unconditional moments must also fall on (not inside) this same semicircle (Jensen's Inequality does not apply here because the standard deviation is not an expectation).

2. Geometry and intuition

This section presents geometry and intuition in two cases, illustrating the ability of pure geometrical reasoning to explain conditional portfolio characteristics of a specified unconditionally efficient (UE) portfolio. In the first case of Section 2.1, many assets with a fixed risk-free rate, the Efficient Semicircle (for a specified unconditionally efficient portfolio) is shown to connect the bliss point to the risk-free rate, where the bliss point is identified using expected-quadratic-utility-contour tangency to the unconditionally efficient portfolio on the capital allocation line. In Section 2.2 we focus on the geometry of the simpler case of a single risky asset with constant conditional variance and risk-free rate, providing intuition to visualize results from Ferson and Siegel (2001, Sections I and II) for the percent invested in the risky asset and for the conditional portfolio mean.

2.1. *Geometry with many assets and fixed risk-free rate: the efficient semicircle*

This section presents a purely geometric construction of the Efficient Semicircle of moments (unconditional and conditional, in risk-return space) for a particular UE portfolio P_0, in the case of many assets with a fixed risk-free rate and conditioning information available for portfolio construction, for which the portfolio weights were presented by Ferson and Siegel (2001, first part of Section III). We make use of the fact that a mean-variance efficient portfolio must maximize expected utility of return for a particular quadratic utility, and that expected quadratic utility is characterized by semicircular indifference contours (e.g. Baron 1977; also pages 95–96 of Ingersoll 1987).†

Focusing on a particular UE portfolio P_0, we know that it falls along the unconditional capital allocation line (with slope equal to the unconditional Sharpe Ratio S_U) as shown at left in figure 1 because the UE frontier can be generated from P_0 by combining it with the risk-free asset. The unique bliss point b_0 implied by P_0 is then found along the expected return axis by extending a perpendicular line upward from P_0, as shown. This perpendicularity condition assures tangency with respect to the semicircular indifference contour of expected quadratic utility (with center at b_0) thereby expressing the fact that P_0 maximizes unconditional expected quadratic utility when bliss point b_0 is used.

We next (figure 1, middle, still showing unconditional moments) use the following fact about the geometry of the circle: a point (P_0 here) defines a right angle with respect to two other points (b_0 and r_f here) if and only if the point (P_0) falls on the circle whose diameter is defined by the two other points.‡ We therefore define the 'Efficient Semicircle' so that

† Perhaps the simplest quadratic utility of return parametrization is $-(r_P - b)^2$, with expectation $-(\mu_P - b)^2 - \sigma_P^2$ constant on a circle centered at risk 0 and expected return equal to the bliss point b. It is also clear that maximizing expected utility here must also minimize the risk σ_P when expected return μ_P is held fixed.

‡ This fact about the geometry of circles is known as Thales' Theorem (and its converse) and may be proven, e.g., by augmenting the right triangle formed by the points $(0, b_0)$, $(0, r_f)$, and (σ_0, μ_0) with a copy of itself rotated 180 degrees and joined at the hypotenuse to form a rectangle that may be easily circumscribed by a circle.

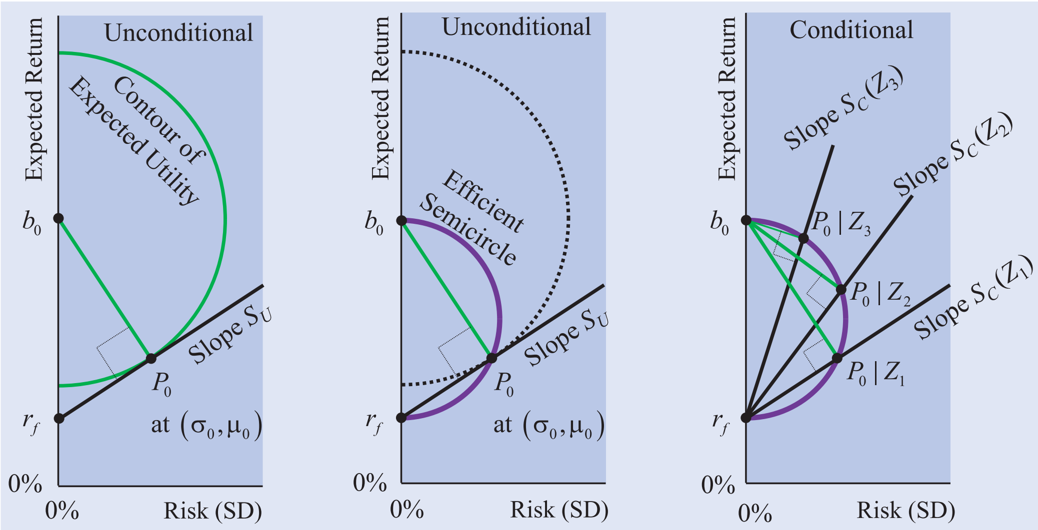

Figure 1. Geometric derivation of the Efficient Semicircle for a particular UE portfolio P_0. Left: The unconditional UE frontier is the ray from r_f through P_0, and we find the bliss point b_0 (for which P_0 maximizes expected quadratic utility) by extending a perpendicular from P_0. Middle: The 'Efficient Semicircle' (with diameter extending from r_f to b_0) must contain the unconditional moments of P_0 due to the geometry of the circle (please see additional details in the text). Right: This same Efficient Semicircle must also contain *all conditional moments* of P_0 because maximizing an unconditional expectation implies maximizing each conditional expectation. Please note, in particular, that the left and middle figures here reference unconditional moments, while the right-hand figure shows conditional moments.

its diameter extends from r_f to b_0 along the expected return axis, as shown in figure 1 (middle). Moreover, it also follows that the closest point to b_0 (maximizing expected utility) along *any* line through r_f (with positive slope) must fall on this Efficient Semicircle because the perpendicularity condition (from circle geometry) implies tangency with a semicircle of constant expected utility. Please note that there are two semicircles involved here: the (fixed) Efficient Semicircle and the tangency indifference contour (which can vary, giving different expected utility if a different line through r_f had been used).

We may use this same bliss point b_0 to characterize conditional moments of P_0 given the conditioning information Z (figure 1, right, now showing conditional moments) because maximization of an unconditional expectation implies maximization of each conditional expectation (almost surely).§ As shown at right in figure 1, the geometry of the circle implies that *the conditional moments of* P_0 *given any information* Z *must also fall on the Efficient Semicircle*. This is because the conditional capital allocation line (with slope $S_C(Z)$ as shown in three cases, where each line is tangent to its conditional efficient frontier of risky assets, which is not shown) has its highest conditional expected utility at the point closest to this same b_0, implying a perpendicularity condition, thereby imposing the Efficient Semicircle as a condition (and because r_f and b_0 remain unchanged, the Efficient Semicircle remains fixed across variations in the conditioning information Z). Note that, while all conditional portfolios for a specified P_0 fall on the Efficient Semicircle, each one of these might fall on a *different* indifference contour, being at varying distances from the bliss point at b_0. An algebraic proof, to supplement this geometric result, is provided in Proposition 4 of Appendix B.

One advantage of this geometric intuition, from figure 1 (right) is that we can immediately see that a conditional Sharpe Ratio $S_C(Z) = 1$ plays a special role because it corresponds to the maximum conditional variance among all conditional portfolios associated with P_0 (this may be seen by creating an inscribed isosceles right triangle within the Efficient Semicircle, whose hypotenuse is the diameter of this semicircle along the expected return axis).

Figures 2–4 show additional detail for three cases of conditioning information for a specified UE portfolio P_0 (please note, in general, that the point of tangency with the conditional frontier does not itself represent a UE portfolio). In each case the conditional moments of the UE portfolio are characterized by the intersection of the conditional capital allocation

§ This makes use of the fact that expected quadratic utility takes the expectation of a function of return. This reasoning cannot be used with mean-variance utility of the form $\mu_P - b\sigma_P^2/2$ because this function cannot be represented as the expectation of a function of return (it requires the square of an expectation) and therefore the conditional portfolios associated with P_0 will not, in general, maximize the same mean-variance utility that P_0 maximizes when conditional mean and conditional variance are used. In particular, the unconditional variance is not the expected conditional variance due to the additional term (the variance of the conditional mean). To see that unconditional optimization of $\mu_P - b\sigma_P^2/2$ cannot be obtained using its conditional optima, note that Lemma 1 in Appendix A shows that the bliss point for a conditional optimum is $r_f + (1 + S_C^2)/b$ and this bliss point will vary according to the conditional Sharpe Ratio S_C, whereas the unconditional optimum requires a fixed bliss point as implied by its unconditional tangency condition. Although a normally-distributed r_P with exponential utility of the form $\exp(-br_p)$ and thus expected utility $E[\exp(-br_p)] = \exp(-b\mu_p + \sigma_p^2 b^2/2)$ has the same set of (noncircular) indifference contours as $\mu_p - b\sigma_p^2/2$, nonlinearity of the exponential function along with Jensen's Inequality interferes with the ability to use conditional expectation inside the exponential function. More generally a non-normally distributed r_p with exponential utility will not even have the same set of indifference contours as those of $\mu_p - b\sigma_p^2/2$ for any b.

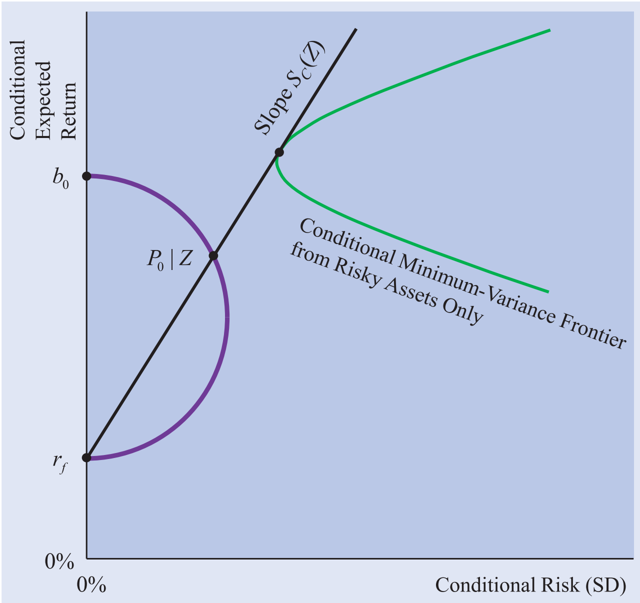

Figure 2. Additional detail for the case of an optimistic signal, with conditioning information creating a high conditional Sharpe Ratio $S_C(Z)$. The conditional capital allocation line starts at r_f and is tangent to the conditional minimum-variance frontier constructed from risky assets only. The conditional portfolio $P_0|Z$ falls where this line intersects the Efficient Semicircle, which is the closest point on the line to the bliss point.

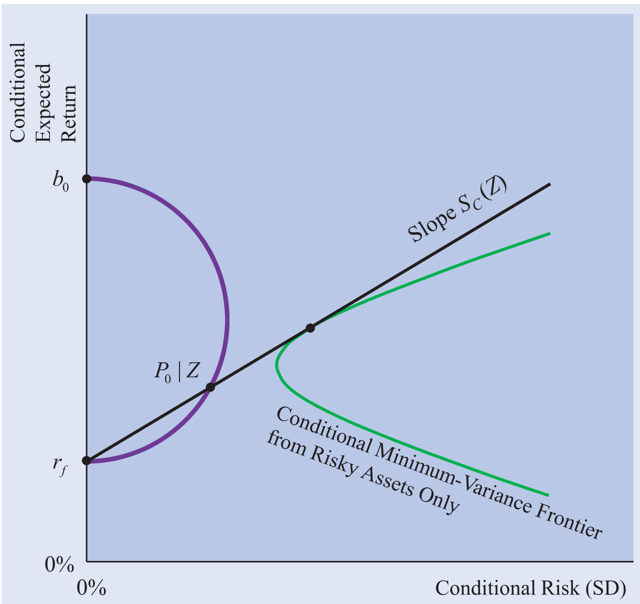

Figure 3. Additional detail for the case of a moderate signal. The conditional portfolio $P_0|Z$ again falls where the conditional capital allocation line intersects the (same) Efficient Semicircle.

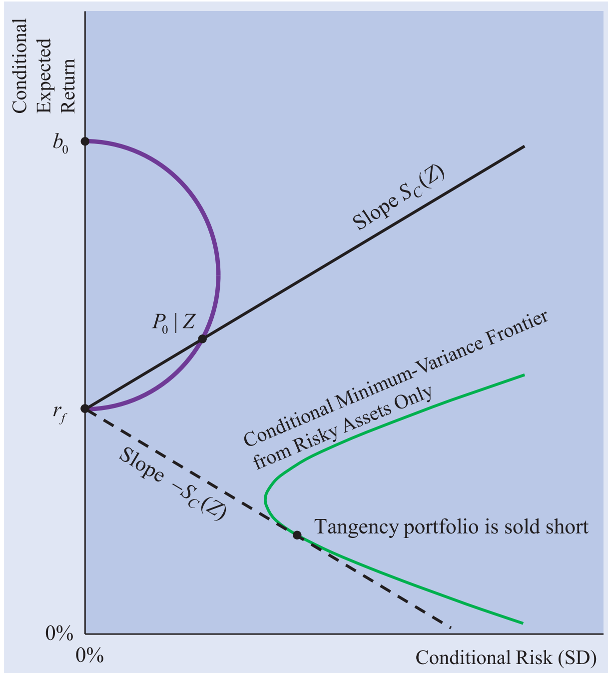

Figure 4. Additional detail for the case of a pessimistic signal, with a negatively sloped tangent line from r_f to the lower portion of the conditional minimum-variance frontier formed from risky assets only. This mediocre tangency portfolio will be sold short, creating an upward-sloping capital allocation line from r_f, whose intersection with the Efficient Semicircle again specifies the location of the conditional portfolio moments.

line with the Efficient Semicircle. With optimistic information (figure 2) the conditional capital allocation line has a high slope; with moderate information (figure 3) the conditional capital allocation line has a lower slope, resulting in a lower conditional expectation. With pessimistic information (figure 4, where the conditional minimum-variance frontier has conditional expected return below r_f) the tangency portfolio is on a negatively sloped line from r_f (and falls on the inefficient lower portion of the conditional minimum-variance frontier) but is sold short to produce an upward-sloping conditional capital allocation line, with conditional moments again falling on the Efficient Semicircle.

2.2. *Geometry with one asset, constant conditional variance, and fixed risk-free rate*

Geometry can de-mystify the paradoxical behavior of the percent invested and of the conditional expectation in the case of a single risky asset having constant conditional variance along with fixed risk-free rate, as was featured in Sections I and II of Ferson and Siegel (2001). Their figure 2 noted an increase followed by a decrease in the percentage invested in the risky asset as the conditional information became more optimistic, along with a continuing increase in the conditional portfolio expectation bounded above by a limiting value (which will be the bliss point b_0).

Figure 5 shows how the conditional portfolio mean of $P_0|Z$ is increasing with the conditional mean $\mu(Z)$ of the risky asset, and is limited above by the bliss point b_0. The percentage invested in the risky asset is zero when the conditional mean of the risky asset is r_f (because the intersection with the Efficient Semicircle will approach the point at r_f with zero risk). This percentage will increase until the conditional Sharpe Ratio reaches 1 (the middle solid line from r_f) at which point it reaches its maximum value and then decreases because at a conditional Sharpe Ratio of 1 the tangency at

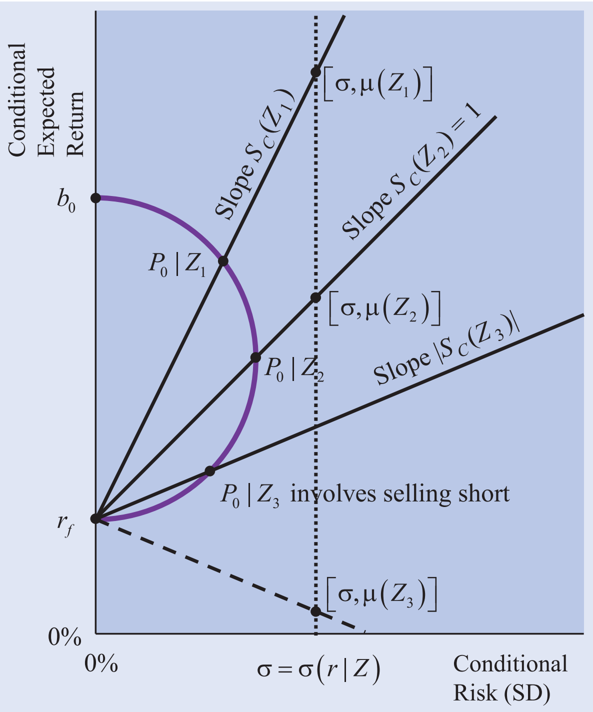

Figure 5. Paradoxical behavior is clarified by geometry for a single risky asset having constant conditional variance σ^2 (the dashed vertical line) along with a risk-free asset. The chosen conditional portfolio $P_0|Z$ is found by projection onto the Efficient Semicircle. The conditional portfolio mean of $P_0|Z$ increases with the conditional mean $\mu(Z)$ of the risky asset, and is limited above by the bliss point b_0. The percentage invested in the risky asset starts at zero when the conditional mean of the risky asset is r_f, and reaches its maximum at a conditional Sharpe Ratio of 1 (at Z_2) due to constant ratios implied by intersections with parallel lines (in this case the circle is locally parallel to the dashed line at a Sharpe Ratio of 1).

the circle is vertical and thus an infinitesimal movement up or down reflects the constant ratios implied by intersections with parallel lines.

3. Summary and conclusion

The conditional risk-return properties for any unconditionally efficient (UE) portfolio, in the presence of a risk-free asset, have been shown to follow directly and intuitively from geometrical principles. In particular, every realization of the conditioning information must lead to a conditional portfolio choice whose conditional risk and conditional expected return fall on the semicircle whose diameter extends on the (conditional) expected return axis from the risk-free rate to the bliss point (which is well-defined because a UE portfolio must maximize expected quadratic utility of return). We refer to this as the ‘Efficient Semicircle’, upon which all conditional portfolio moments (risk and expected return) must fall, with the unconditional moments also on this same Efficient Semicircle.

We note that the Efficient Semicircle, being a semicircular conditional efficient frontier, is unusual because it is not hyperbolic. Moreover, the Efficient Semicircle is *not* a contour of constant expected quadratic utility of return. While each conditional portfolio does satisfy a tangency property between a semicircular conditional utility contour and the conditional capital allocation line, the conditional expected utility will tend to vary (perhaps considerably) according to the conditioning information, leading to multiple semicircular conditional expected utility indifference contours. Nonetheless, a single Efficient Semicircle applies across all realizations of the conditioning information, as is established using geometric principles.

These results also provide intuition to assist with understanding a number of paradoxes that have been observed with a UE portfolio. For example, with a fixed risk-free rate, all conditional portfolio means must naturally be bounded between the fixed risk-free rate and the (also fixed) bliss point, as follows immediately from conditional moments falling on the (fixed) Efficient Semicircle. Another example explains figure 2 of Ferson and Siegel (2001) who studied the case of a single risky asset with fixed conditional variance (and fixed risk-free rate) and found that the percent invested in the risky asset initially increases and then declines as the conditioning information becomes more optimistic; geometrically this corresponds to the steepening of the conditional capital allocation line starting at the risk-free rate, leading to conditional moments that move up along the Efficient Semicircle (hence leading to increasing but bounded conditional expected return) but eventually to decreasing conditional risk (and hence decreasing percent invested in the risky asset) after the conditional Sharpe Ratio reaches 1 (at which point the conditional moments fall on the rightmost point of the Efficient Semicircle and the conditional risk begins to decrease, moving left on the semicircle).

One strength of the geometric approach is highlighted by the inclusion of algebraic proofs in Appendix B for many of the same results. It becomes clear that the geometric approach, very directly and without the need for cumbersome formulae, explains many properties of the conditional portfolios that make up a specified unconditionally efficient portfolio. The geometric approach thus adds intuition and guidance to our understanding of the conditional properties of unconditionally efficient portfolios. One important implication of the Efficient Semicircle is to provide stark evidence to a portfolio manager that in order to receive optimal ratings based on portfolio mean and variance (unconditionally, as seen by a client or a ratings agency), the investment choices are severely constrained by the Efficient Semicircle.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by the Grant I. Butterbaugh Professorship, Foster School of Business, University of Washington.

ORCID

Andrew F. Siegel http://orcid.org/0000-0002-2144-2591

References

Abhyankar, A., Basu, D. and Stremme, A., Portfolio efficiency and discount factor bounds with conditioning information: An empirical study. *J. Bank. Finance*, 2007, **31**(2), 419–437.

Abhyankar, A., Basu, D. and Stremme, A., The optimal use of return predictability: An empirical study. *J. Financ. Quant. Anal.*, 2012, **47**(5), 973–1001.

Agudo, L.F., Magallón, M.V. and Sarto, J.L., Evaluation of performance and conditional information: The case of Spanish mutual funds. *Appl. Financ. Econ.*, 2006, **16**(11), 803–817.

Avramov, D. and Wermers, R., Investing in mutual funds when returns are predictable. *J. Financ. Econ.*, 2006, **81**(2), 339–377.

Baron, D.P., On the utility theoretic foundations of mean-variance analysis. *J. Finance*, 1977, **32**(5), 1683–1697.

Bhaduri, S. and Saraogi, R., The predictive power of the yield spread in timing the stock market. *Emerg. Mark. Rev.*, 2010, **11**(3), 261–272.

Brandt, M.W. and Santa-Clara, P., Dynamic portfolio selection by augmenting the asset space. *J. Finance*, 2006, **61**(5), 2187–2217.

Chiang, I.E., Modern portfolio management with conditioning information. *J. Empir. Finance*, 2015, **33**, 114–134.

Cotter, J., Eyiah-Donkor, E. and Potì, V., Predictability and diversification benefits of investing in commodity and currency futures. *Int. Rev. Financ. Anal.*, 2017, **50**, 52–66.

Dybvig, P.H. and Ross, S.A., Differential information and performance measurement using a security market line. *J. Finance*, 1985, **40**(2), 383–399.

Ferson, W.E. and Siegel, A.F., The efficient use of conditioning information in portfolios. *J. Finance*, 2001, **56**(3), 967–982.

Ferson, W.E. and Siegel, A.F., Stochastic discount factor bounds with conditioning information. *Rev. Financ. Stud.*, 2003, **16**(2), 567–595.

Ferson, W.E. and Siegel, A.F., Testing portfolio efficiency with conditioning information. *Rev. Financ. Stud.*, 2009, **22**(7), 2735–2758.

Ferson, W.E. and Siegel, A.F., Optimal orthogonal portfolios with conditioning information. In *Handbook of Financial Econometrics and Statistics*, edited by C.F. Lee and J.C. Lee, pp. 977–1002, 2015 (Springer: New York).

Ferson, W.E., Siegel, A.F. and Xu, P., Mimicking portfolios with conditioning information. *J. Financ. Quant. Anal.*, 2006, **41**(3), 607–635.

Fletcher, J., An examination of dynamic trading stategies in UK and US stock returns. *J. Bus. Finance Account.*, 2011, **38**(9-10), 1290–1310.

Hansen, L. and Richard, S., The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models. *Econometrica*, 1987, **55**(3), 587–613.

Ingersoll, J.E., *Theory of Financial Decision Making*, 1987 (Rowman & Littlefield Publishers: Lanham, MD).

Markowitz, H.M., *Portfolio Selection: Efficient Diversification of Investments*, 1959 (John Wiley: New York).

Peñaranda, F., Understanding portfolio efficiency with conditioning information. *J. Financ. Quant. Anal.*, 2016, **51**(3), 985–1011.

Sharpe, W.F., Capital asset prices: A theory of market equilibrium under conditions of risk. *J. Finance*, 1964, **19**(3), 425–442.

Zhou, G., On the fundamental law of active portfolio management: How to make conditional investments unconditionally optimal. *J. Portf. Manag.*, 2008, **35**(1), 12–21.

Appendices

Appendix A: Lemma 1

We now state and prove Lemma 1, which was used in Footnote § to show that unconditional optimization of mean-variance utility $\mu_P - b\sigma_P^2/2$ cannot be obtained using its conditional optima, in contrast to the behavior of quadratic utility (which is a function of the random return itself, not a function of an incomplete set of moments).

LEMMA 1 *Given a conditional frontier with conditional Sharpe Ratio S_C along with the risk-free return at fixed rate r_f, the conditional maximum of mean-variance utility $\mu_P - b\sigma_P^2/2$ (which must occur along the tangency frontier from r_f with slope S_C) produces the same portfolio as maximizing quadratic utility with bliss point $r_f + (1 + S_C^2)/b$.*

Proof Contours of constant mean-variance utility have the form $\mu = c + b\sigma^2/2$, and will be conditionally maximized when the slope $b\sigma$ of the contour is equal to the conditional slope S_C of the tangency frontier. Equating these slopes, we obtain the risk of the mean-variance optimal conditional portfolio $\sigma_p = S_C/b$. Using the tangency frontier, we obtain the expected return of the conditionally optimal mean-variance portfolio as $\mu_p = r_f + \sigma_P S_C = r_f + S_C^2/b$. To obtain the bliss point b_0 that quadratic utility would have used to choose this same conditional portfolio, we use orthogonality, of the line connecting $(0, b_0)$ with $(\sigma_p, \mu_p) = (S_C/b, r_f + S_C^2/b)$, with the tangency frontier line having slope S_C. Using negative reciprocal slopes, we obtain the condition $[b_0 - (r_f + S_C^2/b)]/[0 - S_C/b] = -1/S_C$ which simplifies to $b_0 = r_f + (1 + S_C^2)/b$, completing the proof. ■

Appendix B: Algebra and details

In this Appendix B we provide algebraic formulae, with proofs, to accompany and complete the purely geometric results of Section 2. We feature the versatility of multiple expressions, e.g. for conditional moments in terms of conditional Sharpe Ratios instead of conditional asset moments; or using the unconditional Sharpe Ratio instead of using the bliss point b_0. We assume here that there is a fixed risk-free rate r_f as in Sections 2.1 and 2.2, so that the Efficient Semicircle exists and is unique for any UE portfolio P_0. One universal, and perhaps surprising, result is that conditional moments on a semicircle imply unconditional moments on the same semicircle, even when UE portfolios are not involved.

Following Ferson and Siegel (2001) we assume that a vector signal Z (which follows a known distribution) is observed at time $t-1$, and in response a portfolio is constructed (without loss of generality we will omit the subscript on Z_{t-1}) using n risky assets (whose return distribution from time $t-1$ to t depends conditionally upon Z) along with a risk-free asset having known constant rate of return r_f. We do not require that portfolio weights invested in risky assets obey a portfolio constraint, thereby permitting differential percentages placed at risk depending on the information in the signal Z; instead, we impose the portfolio constraint that the total invested (including the risk-free asset) is 1, and allow short sales without cost, so that any portfolio weight vector $x(Z)$ is permitted and implies that we invest $1 - [x(Z)]'\mathbf{1}$ in the risk-free asset, where $\mathbf{1} \equiv (1, 1, \ldots, 1)'$ denotes a column vector of ones. We will observe the n-vector r of risky asset returns at time t (again omitting the subscript, on r_t) and will focus on the unconditional mean and variance of the resulting portfolio return, whose derivation will involve conditional moments. The observed portfolio return is $r_P = r_f + [x(Z)]'(r - r_f\mathbf{1})$. Conditioning on Z, we have the distribution $(r|Z)$ of the asset return vector at time t with conditional mean vector $\mu(Z) = E(r|Z)$ and conditional covariance matrix $V(Z) = Cov(r|Z)$. The conditional portfolio mean is $E(r_P|Z) = r_f + [x(Z)]'[\mu(Z) - r_f\mathbf{1}]$, the conditional portfolio variance is $Var(r_P|Z) = [x(Z)]'V(Z)x(Z)$, while the unconditional portfolio mean is $E(r_P) = E[E(r_P|Z)] = r_f + E\{[x(Z)]'[\mu(Z) - r_f\mathbf{1}]\}$ and the unconditional portfolio variance may be written as

$$\begin{aligned} Var(r_p) &= E[Var(r_P|Z)] + Var[E(r_P|Z)] \\ &= E\{[x(Z)]'V(Z)x(Z)\} + Var\{[x(Z)]'[\mu(Z) - r_f\mathbf{1}]\} \quad \text{(A1)} \end{aligned}$$

Consider a particular unconditionally efficient (UE) portfolio P_0 (assumed to be on the upper UE frontier) constructed using the portfolio weight vector $x_0(Z)$. This UE portfolio P_0 will have realized return $r_0 = r_f + [x_0(Z)]'(r - r_f\mathbf{1})$ whose distribution will depend on both the distribution of Z, which is observed at $t-1$, and the conditional distribution of r which is observed at t. Portfolio P_0 has unconditional expectation denoted $\mu_0 = E(r_0)$, and risk $\sigma_0 = \sqrt{Var(r_0)}$. The UE frontier then may be formed by combining a fixed (i.e. not depending on Z) long or short position in the UE portfolio P_0 (which may itself conditionally involve random amounts of borrowing or lending at r_f, depending on Z) financed by a fixed amount of borrowing or lending at the risk-free rate, leading to the mean-variance frontier lines $\mu_P = r_f \pm \sigma_P \frac{\mu_0 - r_f}{\sigma_0}$ (these are actually rays starting from $\sigma_P = 0$ for which $\mu_P = r_f$, then extending both up and down to the right as σ_P grows, viewing μ_P as a function of the risk σ_P) with slopes $\pm S_U = \pm\frac{\mu_0 - r_f}{\sigma_0}$ where $S_U = S_{0U} = (\mu_0 - r_f)/\sigma_0$ represents the maximized unconditional Sharpe Ratio, which is achieved by all UE portfolios (hence we may omit the subscript '0' from S_U; note that to be efficient, a portfolio must be on the upper line). Theorem 2 of Ferson and Siegel (2001) solves this variational problem by showing that the weights of this portfolio P_0 on the upper UE frontier have the functional form†

$$x_0(Z) = (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right)\Lambda(Z)[\mu(Z) - r_f\mathbf{1}] \tag{A2}$$

where $\Lambda(Z)$ is the inverse of the conditional second-moment matrix:

$$\begin{aligned}\Lambda(Z) &\equiv \{E[(r - r_f\mathbf{1})(r - r_f\mathbf{1})'|Z]\}^{-1} \\ &= \{V(Z) + [\mu(Z) - r_f\mathbf{1}][\mu(Z) - r_f\mathbf{1}']\}^{-1}\end{aligned} \tag{A3}$$

In order to have a more direct formula, the following lemma shows how the weight vector x_0 may be written where the only matrix inverse is of V itself. This representation will be used throughout this Appendix B.

LEMMA 2 *We may write the weight vector $x_0(Z)$ for the UE variational solution from (2), for the UE portfolio P_0 with unconditional mean return μ_0, as*

$$x_0(Z) = (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right) \times \frac{V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{1 + [\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]} \tag{A4}$$

Proof Begin by observing that for any nonsingular $n \times n$ matrix V and any n-vector a, we have $(V + aa')^{-1} = V^{-1} - \frac{V^{-1}aa'V^{-1}}{1+a'V^{-1}a}$ as may be verified by matrix multiplication. We use this with $a = \mu(Z) - r_f\mathbf{1}$, and therefore $\Lambda(Z) = [V(Z) + aa']^{-1}$, along with the fact that $a'V^{-1}(Z)a$ may be factored as a scalar to find

$$\begin{aligned}x_0(Z) &= (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right)\Lambda(Z)[\mu(Z) - r_f\mathbf{1}] \\ &= (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right) \times \left(V^{-1}(Z) - \frac{V^{-1}(Z)aa'V^{-1}(Z)}{1 + a'V^{-1}(Z)a}\right)a \\ &= (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right) \times \left(\frac{1 + a'V^{-1}(Z)a - a'V^{-1}(Z)a}{1 + a'V^{-1}(Z)a}\right)V^{-1}(Z)a \\ &= (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right)\frac{V^{-1}(Z)a}{1 + a'V^{-1}(Z)a} \\ &= (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right) \times \frac{V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{1 + [\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}\end{aligned} \tag{A5}$$

completing the proof. ■

† Notation differences include that we are using $\mathbf{1}$ instead of e to represent the vector of ones, Z instead of S to represent the conditioning information, and μ_0 instead of μ_P for the target mean. To see that the constant term is correct, please note that from (13) on page 976 of Ferson and Siegel (2001), it follows that the unconditional squared Sharpe Ratio is $S_U^2 = (\mu_P - r_f)^2/\sigma_P^2 = 1/(1/\xi - 1)$ from which we find $1/\xi = (1 + S_U^2)/S_U^2$.

The UE portfolio P_0 with return r_0 has conditional properties denoted as follows: The conditional expected return is $\mu_{0C}(Z) = E(r_0|Z)$, the conditional variance is $\sigma_{0C}^2(Z) = Var(r_0|Z)$, the conditional Sharpe Ratio is $S_C(Z) = S_{0C}(Z) = [\mu_{0C}(Z) - r_f]/\sigma_{0C}(Z)$, where we may drop the '0' subscript from the conditional squared Sharpe Ratio because all UE portfolios (including those with unconditional mean other than μ_0) must be conditionally minimum-variance efficient for all Z.

PROPOSITION 1 *Conditional properties associated with the UE portfolio P_0 (conditional moments, conditional squared Sharpe Ratio, and conditional portfolio weight vector) may be written as follows:*

$$\begin{aligned}\mu_{0C}(Z) &= r_f + (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right) \times \frac{[\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{1 + [\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]} \\ &= r_f + (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right)\frac{S_C^2(Z)}{1 + S_C^2(Z)}\end{aligned} \tag{A6}$$

$$\begin{aligned}\sigma_{0C}^2(Z) &= (\mu_0 - r_f)^2\left(\frac{1+S_U^2}{S_U^2}\right)^2 \times \frac{[\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{\{1 + [\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]\}^2} \\ &= (\mu_0 - r_f)^2\left(\frac{1+S_U^2}{S_U^2}\right)^2\frac{S_C^2(Z)}{[1 + S_C^2(Z)]^2} \\ &= \sigma_0^2\frac{(1+S_U^2)^2}{S_U^2}\frac{S_C^2(Z)}{[1 + S_C^2(Z)]^2}\end{aligned} \tag{A7}$$

$$S_C^2(Z) = [\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}] \tag{A8}$$

$$x_0(Z) = (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right)\frac{V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{1 + S_C^2(Z)} \tag{A9}$$

Proof We use the functional form of x_0 from Lemma 2. For the conditional mean we find

$$\begin{aligned}\mu_{0C}(Z) &= E(r_0|Z) = r_f + [x_0(Z)]'[\mu(Z) - r_f\mathbf{1}] \\ &= r_f + (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right) \times \frac{[\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{1 + [\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}\end{aligned} \tag{A10}$$

For the conditional variance we find

$$\begin{aligned}\sigma_C^2(Z) &= [x_0(Z)]'V(Z)x_0(Z)\\ &= (\mu_0 - r_f)^2\left(\frac{1+S_U^2}{S_U^2}\right)^2\\ &\quad\times\frac{[\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)V(Z)V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{\{1+[\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]\}^2}\\ &= (\mu_0 - r_f)^2\left(\frac{1+S_U^2}{S_U^2}\right)^2\\ &\quad\times\frac{[\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{\{1+[\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]\}^2} \qquad \text{(A11)}\end{aligned}$$

For the conditional squared Sharpe Ratio, we use these conditional moments to find

$$\begin{aligned}S_C^2(Z) &= \frac{[\mu_{0C}(Z) - r_f]^2}{\sigma_{0C}^2}\\ &= \frac{(\mu_0 - r_f)^2\left(\frac{1+S_U^2}{S_U^2}\right)^2\frac{\{[\mu(Z)-r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z)-r_f\mathbf{1}]\}^2}{\{1+[\mu(Z)-r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z)-r_f\mathbf{1}]\}^2}}{(\mu_0 - r_f)^2\left(\frac{1+S_U^2}{S_U^2}\right)^2\frac{[\mu(Z)-r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z)-r_f\mathbf{1}]}{\{1+[\mu(Z)-r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z)-r_f\mathbf{1}]\}^2}}\\ &= [\mu(Z) - r_f\mathbf{1}]'V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}] \qquad \text{(A12)}\end{aligned}$$

Finally, using this representation for the conditional squared Sharpe Ratio, we find the required formula for $x_0(Z)$ and the alternative moment formulas, completing the proof. ■

PROPOSITION 2 *The bliss point implied by UE portfolio P_0 may be written variously as*

$$\begin{aligned}b_0 &= \mu_0 + \frac{\sigma_0^2}{\mu_0 - r_f}\\ &= r_f + (\mu_0 - r_f)\frac{1+S_U^2}{S_U^2}\\ &= r_f + \sigma_0\frac{1+S_U^2}{S_U} \qquad \text{(A13)}\end{aligned}$$

and leads to the following simplified expressions for conditional moments and for the conditional portfolio weight, using the bliss point instead of the unconditional Sharpe Ratio as in the previous proposition:

$$\mu_{0C}(Z) = r_f + (b_0 - r_f)\frac{S_C^2(Z)}{1+S_C^2(Z)} \qquad \text{(A14)}$$

$$\sigma_{0C}^2(Z) = (b_0 - r_f)^2\frac{S_C^2(Z)}{[1+S_C^2(Z)]^2} \qquad \text{(A15)}$$

$$x_0(Z) = (b_0 - r_f)\frac{V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{1+S_C^2(Z)} \qquad \text{(A16)}$$

Proof Applying the Pythagorean Theorem to the right triangle formed by r_f, b_0, and P_0 in figure 1 (left), we have

$$[(\mu_0 - r_f)^2 + \sigma_0^2] + [(\mu_0 - b_0)^2 + \sigma_0^2] = (b_0 - r_f)^2 \qquad \text{(A17)}$$

Solving this for the bliss point, we find

$$b_0 = \mu_0 + \frac{\sigma_0^2}{\mu_0 - r_f} \qquad \text{(A18)}$$

The alternative representations for b_0 make use of the fact that the squared unconditional Sharpe Ratio is $S_U^2 = (\mu_0 - r_f)^2/\sigma_0^2$. Using these expressions for b_0 along with the results of Proposition 1 completes the proof. ■

The following two corollaries show that conditional moments are bounded, and follow immediately from the functional forms in the proposition.

COROLLARY 1 *The conditional mean increases monotonically in the squared conditional Sharpe Ratio $S_C^2(Z)$, starting at r_f and approaching b_0 in the limit.*

COROLLARY 2 *The conditional standard deviation increases monotonically (starting from 0) when the squared conditional Sharpe Ratio falls between 0 and 1, whereupon σ_{0C} reaches its maximum value of the radius of the Efficient Semicircle $(b_0 - r_f)/2$, and then decreases monotonically to zero.*

PROPOSITION 3 *The term $(1+S_U^2)/S_U^2$, which appears in the formula for the UE weights, is the harmonic mean of the corresponding conditional quantities $[1+S_C^2(Z)]/S_C^2(Z)$. This follows from $S_U^2/(1+S_U^2) = E\{S_C^2(Z)/[1+S_C^2(Z)]\}$, which takes ordinary expectations of the reciprocals and implies that the squared sine of the angle θ_U formed by the unconditionally efficient frontier line with respect to the horizontal risk axis is equal to the expectation of the squared sines of the corresponding conditional angles $\theta_C(Z)$. Squared sines are natural here because $\cos[2\theta_C(Z)] = 1 - 2\sin^2\theta_C(Z)$, and this cosine term is linear in $\mu_{0C}(Z)$ which in turn has expectation μ_0. Moreover, the harmonic mean representation suggests that large values of the conditional Sharpe Ratio (which are the tangents of these angles) have diminished influence due to the upper bound of 1 inside the expectation.*

Proof Note that $S_U^2 = \tan^2\theta_U$ and that $S_C^2(Z) = \tan^2\theta_C(Z)$, then apply standard trigonometric identities to find $\sin^2[\theta_C(Z)] = S_C^2(Z)/[1+S_C^2(Z)] = \{1 - \cos[2\theta_C(Z)]\}/2$ and similarly for the unconditional case. That the double angle $2\theta_C(Z)$ appears here is quite natural due to the Central Angle Theorem, which tells us that an angle of $2\theta_C(Z)$ is formed at the center of the Efficient Semicircle by segments extending to endpoints at $(0, r_f)$ and at $[\sigma_{0C}(Z), \mu_{0C}(Z)]$. Moreover, the cosine of this double angle is linear in $\mu_{0C}(Z)$ which has expectation μ_0, completing the intuition for the proof. More formally, to prove the identity we take the expectation of (A6) from Proposition 1, while using the fact that $\mu_0 = E[\mu_{0C}(Z)]$, to find

$$\begin{aligned}\mu_0 = E[\mu_{0C}(Z)] &= E\left[r_f + (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right)\frac{S_C^2(Z)}{1+S_C^2(Z)}\right]\\ &= r_f + (\mu_0 - r_f)\left(\frac{1+S_U^2}{S_U^2}\right)E\left(\frac{S_C^2(Z)}{1+S_C^2(Z)}\right) \qquad \text{(A19)}\end{aligned}$$

from which $S_U^2/(1+S_U^2) = E\{S_C^2(Z)/[1+S_C^2(Z)]\}$ follows immediately and completes the proof. ■

PROPOSITION 4 *The Efficient Semicircle: The unconditional moments of the UE portfolio P_0, as well as all of its conditional portfolio moments, fall on a semicircle with center at conditional expected return $(b_0 + r_f)/2$ (with risk 0) and radius $(b_0 - r_f)/2$, where $b_0 = \mu_0 + \sigma_0^2/(\mu_0 - r_f)$ (as was shown in Proposition 2).*

Proof We begin with the unconditional moments. Completing the square with respect to μ_0 after multiplying through by $\mu_0 - r_f$, in $b_0 = \mu_0 + \sigma_0^2/(\mu_0 - r_f)$, leads to

$$\left(\mu_0 - \frac{b_0 + r_f}{2}\right)^2 - \left(\frac{b_0 + r_f}{2}\right)^2 + \sigma_0^2 + b_0 r_f = 0 \qquad \text{(A20)}$$

which simplifies to

$$\left(\mu_0 - \frac{b_0 + r_f}{2}\right)^2 + \sigma_0^2 = \left(\frac{b_0 - r_f}{2}\right)^2 \qquad \text{(A21)}$$

which we recognize as the equation of a circle, indicating that the point (σ_0, μ_0) must fall on the Efficient Semicircle, with center

$[0, (b_0 + r_f)/2]$ and radius $(b_0 - r_f)/2$ (as was shown geometrically in figure 1, middle). Because each conditional realization of Z must imply a conditionally optimal portfolio with respect to this same b_0, this derivation also implies that the conditional moments must (for all Z) fall on this same Efficient Semicircle:

$$\left(\mu_{0C}(Z) - \frac{b_0 + r_f}{2}\right)^2 + \sigma_{0C}^2(Z) = \left(\frac{b_0 - r_f}{2}\right)^2 \tag{A22}$$

as was shown geometrically in figure 1, right, completing the proof. ■

PROPOSITION 5 *In the case of a single risky asset with constant conditional variance, and for any particular UE portfolio P_0, the risky investment $x_0(Z)$ reaches its maximum at a conditional Sharpe ratio of* 1, *i.e.* $S_C(Z) = 1$.

Proof This follows directly from Lemma 2. With a single risky asset and constant conditional variance, the conditional covariance matrix V becomes the fixed scalar σ^2. The portfolio weight from Lemma 2 is then

$$\begin{aligned} x_0(Z) &= (\mu_0 - r_f)\left(\frac{1 + S_U^2}{S_U^2}\right)\frac{V^{-1}(Z)[\mu(Z) - r_f\mathbf{1}]}{1 + S_C^2(Z)} \\ &= (\mu_0 - r_f)\left(\frac{1 + S_U^2}{S_U^2}\right)\frac{[\mu(Z) - r_f]/\sigma^2}{1 + S_C^2(Z)} \\ &= (\mu_0 - r_f)\left(\frac{1 + S_U^2}{S_U^2}\right)\frac{1}{\sigma}\left(\frac{S_C(Z)}{1 + S_C^2(Z)}\right) \end{aligned} \tag{A23}$$

which is a function of the conditional Sharpe Ratio $S_C(Z)$ because σ and P_0 are assumed constant. The conclusion follows from properties of $S_C(Z)/[1 + S_C^2(Z)]$ as a function of $S_C(Z)$. ■

It may seem to be a coincidence that, for any UE portfolio, the conditional risk and conditional expected return always fall on the same semicircle as the unconditional risk and unconditional expected return. One reason this might seem surprising is that when conditional *means* (first moments) fall on a convex curve in two dimensions, the unconditional means fall inside (not on) the curve. However, the Efficient Semicircle involves standard deviations (not just means) and is quite general, as shown in the following proposition (which applies to, but does not require, that the random return be a UE portfolio).

PROPOSITION 6 *For any joint distribution for the vector of random variables (r_P, Z), where r_P might be the random return at time t of portfolio P and Z might be conditioning information at time $t - 1$ used in its formation, if the conditional standard deviation $\sigma(r_P|Z)$ and the conditional expected return $E(r_P|Z)$ fall on the semicircle $[E(r_P|Z) - C_0]^2 + \sigma^2(r_P|Z) = C_1^2$ with center $(0, C_0)$ and radius C_1 in risk-return space for all Z, then the unconditional risk and expected return must fall on this same semicircle: $[E(r_P) - C_0]^2 + \sigma^2(r_P) = C_1^2$.*

Proof Begin by taking expectations in the conditional equation for the semicircle:

$$E[E(r_P|Z) - C_0]^2 + E[\sigma^2(r_P|Z)] = C_1^2 \tag{A24}$$

Now expand the first term, while zeroing out the resulting cross term, as follows, to find

$$\begin{aligned} C_1^2 &= E\{[E(r_P|Z) - E(r_P)] + [E(r_P) - C_0]\}^2 + E[\sigma^2(r_P|Z)] \\ &= E[E(r_P|Z) - E(r_P)]^2 + E[E(r_P) - C_0]^2 + E[\sigma^2(r_P|Z)] \\ &= Var[E(r_P|Z)] + [E(r_P) - C_0]^2 + E[\sigma^2(r_P|Z)] \end{aligned} \tag{A25}$$

Finally, recognizing that the variance is equal to the expected conditional variance plus the variance of the conditional expectation, we find

$$C_1^2 = [E(r_P) - C_0]^2 + \sigma^2(r_P) \tag{A26}$$

completing the proof that the unconditional moments $\sigma(r_P)$ and $E(r_P)$ must also fall on this same semicircle. ■