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Portfolio choices: comparative statics under both expected return and volatility uncertainty

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This paper studies the comparative statics of an optimal portfolio choice problem for an investor with both expected return and volatility ambiguity about the financial market. The optimal holding of the risky asset depends on risk preference, expected return and volatility ambiguity, yielding a general comparative statistics analysis for all investors with linearly growing absolute risk tolerance.

Keywords: Ambiguity aversion; Risk Aversion; Knightian uncertainty; Comparative statics; Robust portfolio choice; Volatility uncertainty

JEL Classification: D81, G11

1. Introduction

Since the breakthrough work of Arrow (1963) and Pratt (1964), it has been well documented that a more risk-averse investor holds a smaller risky position in the financial market. This property has been significantly extended in several discrete-time economic settings in studies such as Milgron and Shannon (1994), Athey (2002) and Dybvig and Wang (2012). In a continuous-time portfolio choice setting (Merton 1971, Borell 2007, Xia 2011) demonstrate virtually the same property for a large class of investors. While it is economically compelling, however, this monotonic comparative static property no longer holds when the investor has both risk and ambiguity about the financial market (for instance, see Gollier 2011, Tian and Tian 2016). In this paper, we develop a general comparative statics analysis for all investors who have both expected return and volatility ambiguity about the financial market in an important continuous-time Knightian uncertainty model.

When investors know neither the future realization of the risky asset's payoff (risk), nor the probability of its occurring (ambiguity), this distinction between risk and ambiguity is often attributed to Knight (1921). In this paper, investor's preference under Knightian uncertainty is represented essentially by max-min expected utility, given a set of probability measures about the financial market. In other words, investors evaluate the outcome of an investment with respect to a set of models and then choose the model that leads to the lowest expected utility. These preferences exhibit aversion to ambiguity and are axiomatized by Gilboa and Schmeidler (1989), with the dynamic extension developed in Epstein and Schneider (2003). The Gilboa and Schmeidler axioms describe behavior that is consistent with experimental evidence in Ellsberg (1961), and more recent portfolio choice experiments in Bossarts *et al.* (2010). Other models of Knightian uncertainty include rank dependent utility (Quiggin 1982) and the smooth ambiguity model (Klibanoff *et al.* 2005), among others. Knightian uncertainty insists that a single probability measure or belief can not capture the entire uncertainty, but rather requires a set of probability measures to present an agent's uncertainty. There are lots of uncertainty models such as the maxmin utility model (Gilboa and Schmeidler 1989), rank dependent utility (Quiggin 1982), the κ-ignorance model (Chen and Epstein 2002), the smooth ambiguity model (Klibanoff *et al.* 2005), etc.

In this paper, we consider an investor with both expected return ambiguity as in Chen and Epstein (2002) and volatility ambiguity as in Epstein and Ji (2013) in a continuous-time setting. While expected return ambiguity can be interpreted as a set of equivalent probability measures (for instance, see Duffie and Epstein 1992, Chen and Epstein 2002), volatility ambiguity results in a set of *singular* probability measures. Lin and Riedel (2014, 2020) explicitly solve the optimal portfolio choice problem in this general framework with Knightian uncertainty. Here, we develop a robust comparative statics analysis in this continuous-time Knightian uncertainty model.

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Specifically, Lin and Riedel (2014) characterize the optimal portfolio choice rule under Knightian uncertainty regarding both expected return and volatility by establishing a verification theorem. Moreover, they present an analytical solution of the optimal portfolio choice problem for the class of constant relative risk aversion (CRRA) utility functions. This paper provides the comparative statics regarding the optimal holding under Knightian uncertainty for a large class of utility functions. A challenge in this comparative analysis is to verify the smooth condition for a large class of investors in the verification theorem, and there is in fact no analytical expression for these investors in the classical situation in the absence of Knightian uncertainty. We make use of Xia (2011), Tian and Tian (2016) and Riedel (2009) to achieve this task, and we explicitly demonstrate the effect on the optimal holding of various factors such as the risk preference, the risk-free rate, the level of expected return and volatility uncertainty about the financial market.

This paper is closely related to some previous studies on the comparative statics analysis of optimal demand for risky assets. Gollier (2011) obtains monotonicity with respect to wealth in a smooth ambiguity model. He also demonstrates that the monotonic property for a risky holding is invalidated under certain conditions. Cherbonnier and Gollier (2015) consider the optimal demand changes with respect to wealth in the α-maxmin model. Tian and Tian (2016) present a comparative analysis for the same class of investors as is considered in this paper, but these investors are assumed to have only expected return ambiguity. He *et al.* (2017) study the influence of risk aversion and the distortion function for the optimal demand in a rank dependent utility model. Our paper is different from these studies in that both expected return and volatility ambiguity are considered and we derive the comparative static analysis in a setting of Knightian uncertainty about *both* expected return and volatility.

The paper is organized as follows. Section 2 describes motivational examples in a static model. Section 3 presents the continuous-time model under Knightian uncertainty. The corresponding optimal choice problem is solved in Section 4 and an explicit solution for the power utility function is also presented. Section 5 provides the comparative statics analysis. Section 6 concludes the paper and all proofs are presented in the Appendix.

2. Examples in a static model

Consider a static model with two assets. The first asset is risk-free with a zero risk-free interest rate. The second asset is a risky asset with an excess return $\tilde{x}$. The agent has multi-priors on the distribution of $\tilde{x}$, given by a set of probability measures $\mathcal{P}$. The agent's initial wealth is \$ 1. If α is the amount invested in the risky asset, then the agent's final wealth is $1+\alpha\tilde{x}$. The agent's risk-preference is represented by a standard utility function $U(\cdot)$, a strictly increasing, strictly concave and twice continuously differentiable function $U:(0,+\infty)\to\mathbb{R}$, which satisfies the following Inada conditions: $U'(0)=\lim_{x\downarrow 0}U'(x)=\infty$, and $U'(\infty)=\lim_{x\uparrow\infty}U'(x)=0$. Since the investor has multi-priors and ambiguity aversion, the investor solves the optimal portfolio choice problem in the max-min framework of Gilboa and Schmeidler (1989) as follows.

$$\sup_{\alpha}\min_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\left[U(1+\alpha\tilde{x})\right].$$

We present two illustrations in this section to motivate our discussion in a continuous-time framework. In the first illustration, we examine a static binomial model in which the risky asset moves either double or half in the next time period. $U(x)=\frac{x^{1-\gamma}}{1-\gamma}$, where $\gamma>0$ denotes the risk aversion parameter. We consider the following four agents.

- A $\gamma=2$, the subjective probability measure is $(\frac{1}{2},\frac{1}{2})$ for upward and downward, respectively.
- B $\gamma=2.5$, the subjective probability measure is $(\frac{1}{2},\frac{1}{2})$ for upward and downward, respectively.
- C $\gamma=2.5$, the agent has multi-priors with two subjective probability measures $(\frac{1}{2},\frac{1}{2})$ and $(\frac{9}{20},\frac{11}{20})$.
- D $\gamma=3$, the subjective probability measure is $(\frac{1}{2},\frac{1}{2})$ for upward and downward, respectively.

Agents A, B and D have no ambiguity but different risk attitudes. Agent C has ambiguity and his risk attitude is the same as that of Agent B, between the risk attitude of Agent A and Agent D. By a simple numerical calculation, we obtain

$$\alpha_A=0.24,\quad \alpha_B=0.19,\quad \alpha_C=0.14,\quad \alpha_D=0.16.$$

Therefore, the optimal risky positions for these four economic agents satisfy

$$\alpha_A>\alpha_B>\alpha_D>\alpha_C.$$

We make several points below.

(i) $\alpha_A>\alpha_B$ because a more risk-averse Agent B invests less in risky assets than the less risk-averse Agent A, as shown by Arrow (1963) and Pratt (1964) for a general specification of risk preferences. For the same reason, $\alpha_B>\alpha_D$.

(ii) $\alpha_D>\alpha_C$ illustrates the insight of this paper. It demonstrates that although Agent D is more risk averse than agent C from the risk preference perspective, the multi-priors ambiguity of agent C increases her risk aversion *ex post*. Ultimately, Agent C demands less risky asset optimally. Therefore, both the risk-preference and multiple-priors Knightian uncertainty imply that Agent C invests less in the risky assets than Agent D, who is more risk averse but less ambiguous.

(iii) $\alpha_B>\alpha_C$ shows that more ambiguity aversion ensures higher risk premium on the risky asset. Thus, the optimal demand for risky asset is decreased.

Both points (i) and (iii) show that the optimal demand is marginally monotonic with respect to risk aversion and ambiguity aversion, individually. However, when both risk aversion and ambiguity aversion vary, the optimal demand relies on a joint effect. Even for a less risk-averse agent, higher ambiguity might demand a smaller risky position.

In the second illustration, we consider a multi-priors mean-variance setting. The agent's optimal portfolio choice problem is

$$\sup_{\alpha} \min_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[1+\alpha\tilde{x}] - \frac{\gamma}{2} Var^{\mathbb{P}}[\alpha\tilde{x}],$$

where γ is the risk aversion parameter and $1+\alpha\tilde{x}$ is the final wealth for the strategy α. We consider the following two economic agents.

E Agent E has one subjective probability measure $\mathbb{P}$ and $E^{\mathbb{P}}[\tilde{x}] = \mu > 0, Var^{\mathbb{P}}[\tilde{x}] = \sigma^2 > 0$. His risk-aversion parameter is γ_E. It is known that

$$\alpha_E = \frac{1}{\gamma_E} \frac{\mu}{\sigma^2}.$$

F Agent F's multi-prior is represented by a set of probability measures such that $\mathbb{P} \in \mathcal{P}$ if and only if $E^{\mathbb{P}}[\tilde{x}] \in [\underline{\mu}, \overline{\mu}], Var^{\mathbb{P}}[\tilde{x}] \in [\underline{\sigma}^2, \bar{\sigma}^2]$. This set of multi-prior is discussed in Garlappi *et al.* (2007) and Easley and O'Hara (2009). The agent's risk-aversion parameter is γ_F. We assume that $\gamma_F < \gamma_E$ and $\underline{\mu} < \mu < \overline{\mu}$, $\underline{\sigma}^2 < \sigma^2 < \bar{\sigma}^2$.

It is easy to see that the optimal risky demand for Agent F is

$$\alpha_F = \frac{1}{\gamma_F} \frac{\underline{\mu}}{\bar{\sigma}^2}$$

and Agent F chooses the probability measure with the smallest expectation $\underline{\mu}$ and largest variance $\bar{\sigma}^2$ as the worse-case measure in his optimal portfolio choice problem. Consequently, the agent E might have more or less optimal demand than the agent F. Especially, Agent E has more risky investment than Agent F,

$$\alpha_F < \alpha_E, \text{if and only if } \frac{\underline{\mu}}{\gamma_F \bar{\sigma}^2} < \frac{\mu}{\gamma_E \sigma^2}.$$

Clearly, both the risk aversion parameter γ and the multiple-priors $\mathcal{P}$ jointly characterize the comparative statics of the optimal risky demand.

We move to examine a large class of utility functions under Knightian ambiguity in a continuous-time setting.

3. Optimal portfolio choice with ambiguity

3.1. Construction of the set of priors

We start with a continuous-time framework with Knightian uncertainty. Similar to the classical setting in Merton (1971), we consider asset prices with continuous sample paths. Let $C([0,T])$ be the set of all continuous paths with values in $\mathbb{R}$ over the finite time horizon $[0,T]$ endowed with the sup norm. Our state space is

$$\Omega = \Big\{\omega : \omega \in C([0,T]),\ \omega_0 = 1\Big\}.$$

We will consider the canonical process $Y_t(\omega) = \omega_t, t \geq 0$, for all $\omega \in \Omega$. Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by Y.

In the continuous–time diffusion framework, two adapted parameter processes, drift (or expected return) and volatility, describe all uncertainty. We thus model the Knightian uncertainty by a convex and compact subset $\Theta \subset \mathbb{R}^2_+$. The investor is unsure about the distribution of the drift process $\mu = (\mu_t)$ with values in $\mathbb{R}$ and the distribution of the volatility process $\sigma = (\sigma_t)$ with values in $\mathbb{R}$. The restriction is that $(\mu_t, \sigma_t) \in \Theta$. Let us characterize such uncertainty by Γ^{Θ}, which is defined as

$$\Gamma^{\Theta} = \Big\{(\mu,\sigma) \ \Big|\ (\mu,\sigma) \in \Theta, \mu \text{ and } \sigma \text{ are } \mathcal{F}\text{-progressively measurable processes}\Big\}.$$

Let Y be the canonical process $(\Omega, \mathcal{F})$. For $(\mu,\sigma) \in \Gamma^{\Theta}$, let $\mathbb{P}^{\mu,\sigma}$ be a probability measure such that Y is the unique strong solution of the following stochastic differential equation

$$dY_t = \mu_t Y_t dt + \sigma_t Y_t dW_t^{\mu,\sigma},$$

where $W^{\mu,\sigma}$ is a $\mathbb{P}^{\mu,\sigma}$-Brownian motion. Let $\mathcal{P}_0$ be the set of all probability measures $\mathbb{P}^{\mu,\sigma}$ constructed in this way. The set of priors $\mathcal{P}$ is the closure of $\mathcal{P}_0$ under the topology of weak convergence.

3.2. Asset prices

In a financial market, there are two tradable assets: a risk-free asset (bond) and a risky asset (stock). The risky asset denotes a market index in financial market. The price of the risk-free asset is described by

$$dP_t = rP_t dt, \quad P_0 = 1$$

for some constant interest rate r.

Under each probability measure $\mathbb{P}^{\mu,\sigma} \in \mathcal{P}$, $(\mu,\sigma) \in \Gamma^{\Theta}$, the risky stock evolves according to

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t^{\mu,\sigma}) \tag{1}$$

where $W^{\mu,\sigma} = (W_t^{\mu,\sigma})_{t\geq 0}$ is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}^{\mu,\sigma})$.

An investor is initially endowed with some wealth $x_0 > 0$, and allocates her wealth dynamically between the risky stock and the risk-free bond. Let ϕ_t be the amount of her wealth invested in the stock at time t such that ϕ is $\mathcal{F}$-progressively measurable and $\int_0^T \phi_t^2 dt < \infty$, $\mathbb{P}$-a.s., for all $\mathbb{P} \in \mathcal{P}$. The wealth of the investor under a portfolio policy ϕ is given by

$$dX_t = (\mu_t - r)\phi_t dt + rX_t dt + \phi_t \sigma_t dW_t^{\mu,\sigma}, \mathbb{P}^{\mu,\sigma} - a.s., \tag{2}$$

and $X_0 = x_0$, where $W^{\mu,\sigma}$ is a Brownian motion under $\mathbb{P}^{\mu,\sigma} \in \mathcal{P}$ and $(\mu,\sigma) \in \Gamma^{\Theta}$.

A portfolio strategy ϕ is *admissible* if for all $t \in [0,T]$, $X_t \geq 0$, $\mathbb{P}$-a.s., for all $\mathbb{P} \in \mathcal{P}$. Let us denote the set of admissible portfolio strategies by Π.

3.3. Portfolio choice problem

The investor is ambiguity–averse and maximizes the minimal expected utility over her set of priors. The investor is also risk averse and her risk preference is represented by a standard von Neumann-Morgenstern utility function $U(\cdot)$. Under the set of priors, $\mathcal{P}$, the investor's minimal utility across all priors of a terminal wealth X_T is defined by

$$J(x_0;\phi) = \inf_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[U(X_T)],$$

where X is given in the process (2). Therefore, the optimal portfolio choice problem for the investor with both risk aversion and ambiguity aversion is

$$\overline{V}(x_0) = \sup_{\phi\in\mathcal{A}(x_0)} J(x_0;\phi). \tag{3}$$

3.4. A verification theorem

We recall a verification theorem about the optimal portfolio choice problem (3) from Lin and Riedel (2014), which we will need in next section. In brief, the theorem states that a sufficiently smooth solution to the suitably adjusted Hamilton–Jacobi–Bellman-Isaacs (HJBI) equation yields a solution of the optimal portfolio choice problem under Knightian uncertainty. Let $Q = [0,T)\times\mathbb{R}_+$ and $\overline{Q} = Q\cup\partial Q$, where ∂Q is the boundary of Q. $C^{1,2}(Q)$ denotes the set of all functions $f: Q\to\mathbb{R}$ such that all functions $f(t,x)$ are continuously differentiable w.r.t. t and twice differentiable w.r.t. x, for all $(t,x)\in Q$. $C(\overline{Q})$ denotes all the continous functions $f:\overline{Q}\to\mathbb{R}$.

THEOREM 3.1 *Let $V(\cdot,\cdot)\in C^{1,2}(Q)\cap C(\overline{Q})$ be a solution of the following HJBI equation*

$$\sup_{\phi}\Big\{V_t(t,x) + V_x(t,x)(x-\phi)r + \inf_{(\mu,\sigma^2)\in\Theta}\Big\{V_x(t,x)\phi\mu + \frac{1}{2}V_{xx}(t,x)\phi^2\sigma^2\Big\}\Big\} = 0, \tag{4}$$

with boundary condition $V(T,x) = U(x)$.

Let $\hat{\phi}(t,x)\in\mathbb{R}$ satisfy

$$\hat{\phi}(t,x) = \arg\sup_{\phi}\Big\{V_t(t,x) + V_x(t,x)(x-\phi)r + \inf_{(\mu,\sigma^2)\in\Theta}\Big\{V_x(t,x)\phi\mu + \frac{1}{2}V_{xx}(t,x)\phi^2\sigma^2\Big\}\Big\},$$

and $(\hat{\mu}(t,x),\hat{\sigma}^2(t,x))\in\Theta$ satisfy

$$(\hat{\mu}(t,x),\hat{\sigma}^2(t,x)) = \arg\inf_{(\mu,\sigma^2)\in\Theta}\Big\{V_x(t,x)\hat{\phi}\mu + \frac{1}{2}V_{xx}(t,x)(\hat{\phi})^2\sigma^2\Big\}.$$

Let X^* be the unique solution of the stochastic differential equation

$$dX_t^* = (\hat{\mu}(t,X_t^*) - r)\hat{\phi}(t,X_t^*)dt + rX_t^*dt + \hat{\phi}(t,X_t^*)\hat{\sigma}(t,X_t^*)dW_t^{\hat{\mu},\hat{\sigma}},\ X_0 = x_0. \tag{5}$$

Moreover, we define $\phi_t^* = \hat{\phi}(t,X_t^*)$, $\mu_t^* = \hat{\mu}(t,X_t^*)$, and $(\sigma_t^*)^2 = \hat{\sigma}^2(t,X_t^*)$, for all $t\in[0,T]$. If $\phi^*\in\Pi$ and $(\mu^*,(\sigma^*)^2)\in\Gamma^{\Theta}$, then ϕ^* is an optimal portfolio choice.

3.5. The worst case prior in $[\underline{\mu},\bar{\mu}]\times[\underline{\sigma},\bar{\sigma}]$

Beginning in this section, we make use of Theorem 3.1 to further examine the investor's optimal choice problem under Knightian ambiguity when $\Theta = [\underline{\mu},\bar{\mu}]\times[\underline{\sigma},\bar{\sigma}]$, a coupled specification Θ considered in Epstein and Ji (2013) and Epstein and Ji (2014). In this section, we present the relationship between the optimal policy and the value function in Theorem 3.2 for this particular model of Knightian uncertainty from Lin and Riedel (2014). As shown below, Theorem 3.2 plays a crucial role in deriving the analytical expression of the optimal policy for the CRRA utility function and the comparative analysis under Knightian uncertainty for a large class of utility functions.

THEOREM 3.2 *Let $V(\cdot,\cdot)\in C^{1,2}(Q)\cap C(\overline{Q})$ be a solution of the equation (4) with boundary condition $V(T,x) = U(x)$. Assume that $\Theta = [\underline{\mu},\bar{\mu}]\times[\underline{\sigma},\bar{\sigma}]$.*

(i) *If $r<\underline{\mu}$, then the optimal policy is*

$$\hat{\phi}(t;x_0) = -\frac{V_x(t,X_t^1)}{V_{xx}(t,X_t^1)}\frac{\underline{\mu}-r}{\bar{\sigma}^2},$$

where X^1 is the unique solution of the stochastic differential equation

$$dX_t^1 = (\underline{\mu}-r)\hat{\phi}(t,X_t^1)dt + rX_t^1dt + \hat{\phi}(t,X_t^1)\overline{\sigma}dW_t^{\underline{\mu},\overline{\sigma}},\quad X_0^1 = x_0.$$

(ii) *If $\underline{\mu}\le r\le\bar{\mu}$, then the optimal policy is*

$$\hat{\phi} = 0.$$

(iii) *If $r>\bar{\mu}$, then the optimal policy is*

$$\hat{\phi}(t;x_0) = -\frac{V_x(t,X_t^2)}{V_{xx}(t,X_t^2)}\frac{\bar{\mu}-r}{\bar{\sigma}^2},$$

where X^2 is the unique solution of the stochastic differential equation

$$dX_t^2 = (\overline{\mu}-r)\hat{\phi}(t,X_t^2)dt + rX_t^2dt + \hat{\phi}(t,X_t^2)\overline{\sigma}dW_t^{\overline{\mu},\overline{\sigma}},\quad X_0^2 = x_0.$$

Theorem 3.2 states an explicit relationship between $V(t,x)$ in Theorem 3.1 and the optimal policy. There are three situations in Theorem 3.2. If each plausible expected return of the stock is greater than the interest rate, then the optimal policy is the same as a product of the indirect absolute risk tolerance and $\frac{\underline{\mu}-r}{\bar{\sigma}^2}$. On the other hand, if each expected return is smaller than the interest rate, then the optimal policy is the product of the indirect absolute risk tolerance and $\frac{\bar{\mu}-r}{\bar{\sigma}^2}$, a short position on the stock. An interesting situation is when

$r \in [\underline{\mu}, \bar{\mu}]$, the optimal portfolio is zero. The intuition is the following. If the investor has a high level of the ambiguity on the expected return, she cannot decide whether to long or short the risky asset; thus, there is no investment in the risky asset at all because of her strong ambiguity aversion. This no-participation feature in the stock market is similar to Uppal and Wang (2003).

From Theorem 3.2, the robust HJBI equation (4) is equivalent to

$$\begin{cases} V_t(t,x) + V_x(t,x)xr - \frac{V_x^2(t,x)}{V_{xx}(t,x)}\left[\frac{(\underline{\mu}-r)^2}{2\bar{\sigma}^2}I_{r<\underline{\mu}} + \frac{(\bar{\mu}-r)^2}{2\bar{\sigma}^2}I_{r>\bar{\mu}}\right] = 0, \\ V(T,x) = U(x). \end{cases} \tag{6}$$

3.6. Explicit solutions under CRRA utility

For the CRRA utility function, $U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma \neq 1, \gamma > 0$. Lin and Riedel (2014) obtain the explicit solution as follows.

THEOREM 3.3 (i) *If $r < \underline{\mu}$, then the optimal policy is*

$$\hat{\phi}(t;x_0) = \frac{\hat{X}_1(t)}{\gamma}\frac{\underline{\mu}-r}{\bar{\sigma}^2},$$

where $\hat{X}_1(t)$ is the optimal wealth, given explicitly as follows

$$\hat{X}_1(t) = x_0 \exp\left(\left(r + \frac{1}{\gamma}\frac{(\underline{\mu}-r)^2}{\bar{\sigma}^2} - \frac{1}{2}\frac{(\underline{\mu}-r)^4}{\gamma^2\bar{\sigma}^2}\right)t + \frac{1}{\gamma}\frac{(\underline{\mu}-r)^2}{\bar{\sigma}}W_t^{\underline{\mu},\bar{\sigma}}\right).$$

(ii) *If $\underline{\mu} \le r \le \bar{\mu}$, then the optimal policy is*

$$\hat{\phi} = 0.$$

(iii) *If $r > \bar{\mu}$, then the optimal choice is*

$$\hat{\phi}(t;x_0) = \frac{\hat{X}_2(t)}{\gamma}\frac{\bar{\mu}-r}{\bar{\sigma}^2},$$

where $\hat{X}_2(t)$ is the optimal wealth, given explicitly as follows:

$$\hat{X}_2(t) = x_0 \exp\left(\left(r + \frac{1}{\gamma}\frac{(\bar{\mu}-r)^2}{\bar{\sigma}^2} - \frac{1}{2}\frac{(\bar{\mu}-r)^4}{\gamma^2\bar{\sigma}^2}\right)t + \frac{1}{\gamma}\frac{(\bar{\mu}-r)^2}{\bar{\sigma}}W_t^{\bar{\mu},\bar{\sigma}}\right).$$

Theorem 3.3 demonstrates that the verification theorem (Theorems 3.1 and 3.2) can be applied to one particular class of utility function, the CRRA utility, deriving an analytical solution. The idea is simple, and we only explain the situation in which $\underline{\mu} > r$. Under this condition, there is an explicitly expression of the function $V(t,x)$ in Equation (6), given $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ (see Merton 1971), and the function $V(t,x)$ belongs to $C^{1,2}(Q) \cap C(\overline{Q})$. Therefore, Theorems 3.1 and 3.2 can be applied in this situation to derive the solution as presented.

According to Theorem 3.3, the optimal portfolio proportion is

$$\pi(t;x_0) = \frac{\hat{\phi}(t;x_0)}{\hat{X}(t)} = \frac{1}{\gamma}\left(\frac{\underline{\mu}-r}{\bar{\sigma}^2}I_{r<\underline{\mu}} + \frac{\bar{\mu}-r}{\bar{\sigma}^2}I_{r>\bar{\mu}}\right).$$

Hence, the comparative statics for the class of CRRA utility functions follows easily from Theorem 3.3, given by the following corollary.

COROLLARY 3.4 *Assume that $\underline{\mu} > r$. Given two CRRA utility functions U_1 and U_2 with risk aversion γ_1 and γ_2 respectively, and $\gamma_1 > \gamma_2$. Then Agent U_1 always puts a lower proportion of his initial wealth into the risky asset than Agent U_2 in the whole horizon* $[0,T]$.

According to Corollary 3.4, in the class of CRRA utility functions, the more risk averse investor under Knightian uncertainty holds a lower position in the risky asset. We next develop the comparative statics for a much larger class of utility functions in the Knightian uncertainty model.

4. Comparative statics for general utilities

For technical convenience, we assume that the Arrow-Pratt coefficient of absolute risk tolerance grows linearly in the sense that there exists a constant $c > 0$ such that

$$-\frac{U'(x)}{U''(x)} \le c(1+x), \quad \text{for all } x > 0.$$

The class of these utility functions $U(\cdot)$ is denoted by $\mathcal{U}$.

We develop a general comparative statics analysis for any $U(\cdot) \in \mathcal{U}$. For this purpose, we consider two agents in this section. The first agent's risk preference is denoted by a utility function $U_1(\cdot) \in \mathcal{U}$, and his Knightian uncertainty about the financial market is interpreted by $\bar{\Theta}_1 = [\underline{\mu}_1, \bar{\mu}_1] \times [\underline{\sigma}_1, \bar{\sigma}_1] \subset \mathbb{R}_+^2$. Similarly, the second agent's utility function is $U_2(\cdot) \in \mathcal{U}$, and the corresponding Knightian uncertainty is interpreted by $\bar{\Theta}_2 = [\underline{\mu}_2, \bar{\mu}_2] \times [\underline{\sigma}_2, \bar{\sigma}_2] \subset \mathbb{R}_+^2$. For the sake of completeness, we also assume that each agent could face a different risk-free environment, r_1 and r_2, to compare the optimal risky holding in different economies (for instance, domestic and foreign market).

The next result provides the smooth property of the value function that is required in the verification theorem. $C^{1,\infty}(Q)$ denotes the set of all functions $f: Q \to \mathbb{R}$ such that all functions $f(t,x)$ are continuously differentiable w.r.t. t and infinitely many times differentiable w.r.t. x, for all $(t,x) \in Q$.

PROPOSITION 4.1 *For any $U \in \mathcal{U}$ with $\Theta = [\underline{\mu}, \bar{\mu}] \times \underline{\sigma}, \bar{\sigma}] \subset \mathbb{R}_+^2$. Let $V(\cdot,\cdot)$ be the solution of the corresponding robust HJBI equation* (4) *with boundary condition $V(T,\cdot) = U(\cdot)$. Then $V(\cdot,\cdot) \in C^{1,\infty}(Q) \cap C(\overline{Q})$, and $V(t,x)$ is strictly increasing and concave with respect to x, for any t in* $[0,T]$.

We present the comparative statics in several different situations.

4.1. $\underline{\mu}_1 > r_1$ and $\underline{\mu}_2 > r_2$

We first consider the case in which $\underline{\mu}_1 > r_1$ and $\underline{\mu}_2 > r_2$.

THEOREM 4.2 *Let $\underline{\mu}_1 > r_1$ and $\underline{\mu}_2 > r_2$. Suppose*

$$-\frac{U_1'(x)}{U_1''(x)}\frac{\underline{\mu}_1 - r_1}{\bar{\sigma}_1} \leq -\frac{U_2'(x)}{U_2''(x)}\frac{\underline{\mu}_2 - r_2}{\bar{\sigma}_2}, \quad \text{for all } x. \tag{7}$$

Then, under one of the following conditions,

(i) $r_1 = r_2$;
(ii) $r_1 > r_2$, U_1 *or* U_2 *displays increasing relative risk aversion;*
(iii) $r_1 < r_2$, U_1 *or* U_2 *displays decreasing relative risk aversion,*

we have

$$0 < \bar{\sigma}_1\phi_1(0; x_0) \leq \bar{\sigma}_2\phi_2(0; x_0),$$

where $\phi_i(0; x_0), i = 1, 2$, express the amount of money invested in the risky asset at time 0 with initial wealth x_0.

Under the condition that $\underline{\mu}_1 > r_1$ and $\underline{\mu}_2 > r_2$, both Theorem 3.2 and Proposition 4.1 ensure a long position in the risk asset. Theorem 4.2 states that the product of the (Arrow-Pratt) absolute risk tolerance and the Sharpe ratio is positively associated with the product of optimal holding and volatility, *the risk-adjusted optimal holding*.

REMARK 4.1 (i) If there is no ambiguity about either the expected return or volatility, Theorem 4.2 reduces to the main result of Xia (2011).
(ii) If there is only drift ambiguity, then Theorem 4.2 reduces to the comparative static result in Tian and Tian (2016), where the authors consider the κ-ignorance ambiguity. Specifically, let $[\underline{\mu}, \bar{\mu}] = [\mu - \kappa\bar{\sigma}, \mu + \kappa\bar{\sigma}]$, thus

$$\frac{\underline{\mu} - r}{\bar{\sigma}} = \frac{\mu - r}{\bar{\sigma}} - \kappa.$$

(iii) Notice that Theorem 4.2 does not depend on other parameters such as the upper bound of drift and the lower bound of volatility in the continuous-time setting. Thus, the comparative statics in the continuous-time framework are different from the result in the mean-variance model considered in Easley and O'Hara (2009) and Garlappi *et al.* (2007).

Two special situations in Theorem 4.2 are deserved for attention. First, we consider two agents who have the same level of ambiguity about the financial market, but they have different levels of risk preference. In this situation, the comparative statics are given by the next result.

COROLLARY 4.3 *Suppose two agents face the same ambiguity about the market with $\underline{\mu} > r$ and*

$$-\frac{U_1''(x)}{U_1'(x)} \geq -\frac{U_2''(x)}{U_2'(x)}, \quad \text{for all } x.$$

Then we have

$$0 < \phi_1(0; x_0) \leq \phi_2(0; x_0).$$

Corollary 4.3 is remarkable in that the monotonic property of the risky holding is true regardless of the level of expected return and volatility ambiguity, provided that investors are homogeneous about the Knightian uncertainty.

If two agents invest in the same ambiguity market, then we obtain the continuous version of Pratt (1964). The more risk averse an agent is, the less wealth is put in the risky asset. It is investor's risk aversion to lead the comparative statics in the homogeneous environment of the Knightian uncertainty. On the other hand, if the investors are heterogeneous in regard to the Knightian uncertainty, but have the same risk-preference, we obtain the second corollary.

COROLLARY 4.4 *Let $\underline{\mu}_1 > r_1$ and $\underline{\mu}_2 > r_2$. Suppose that two agents have the same risk-preference, interpreted by $U(\cdot) \in \mathcal{U}$ in one financial market. Assume that*

$$\frac{\underline{\mu}_1 - r_1}{\bar{\sigma}_1} \leq \frac{\underline{\mu}_2 - r_2}{\bar{\sigma}_2}.$$

Then we have

$$0 < \bar{\sigma}_1\phi_1(0; x_0) \leq \bar{\sigma}_2\phi_2(0; x_0).$$

The influence of market parameters is described. The agents take the worst case when facing the uncertainty, and only the largest volatility and smallest return rate appear in the results. In contrast to Corollary 4.3 in which the risk aversion plays the role of the risky holding in the monotonic property, Corollary 4.4 shows that the less the robust Sharpe ratio will lead to less risk-adjusted optimal holding.

4.2. $r_1 > \bar{\mu}_1$ and $r_2 > \bar{\mu}_2$

In this section, we consider the situation in which $\bar{\mu}_i < r_i$ for $i = 1, 2$. In this situation, it is plausible that the expected return of the risky asset is smaller than the interest rate. While this assumption might be problematic for the stock market index, it is possible that some irreversible investment offers small expected return under certain circumstance. See Nishimura and Ozaki (2007).

THEOREM 4.5 *Let $r_1 > \bar{\mu}_1$ and $r_2 > \bar{\mu}_2$. Suppose*

$$-\frac{U_1'(x)}{U_1''(x)}\frac{\bar{\mu}_1 - r_1}{\bar{\sigma}_1} \leq -\frac{U_2'(x)}{U_2''(x)}\frac{\bar{\mu}_2 - r_2}{\bar{\sigma}_2}, \quad \text{for all } x. \tag{8}$$

Then under one of the following conditions,

(i) $r_1 = r_2$;
(ii) $r_1 > r_2$, U_1 *or* U_2 *displays increasing relative risk aversion;*

(iii) $r_1 < r_2$, U_1 *or* U_2 *displays decreasing relative risk aversion,*

we have

$$\bar{\sigma}_1\phi_1(0;x_0) \le \bar{\sigma}_2\phi_2(0;x_0) < 0.$$

Assume that $\bar{\mu}_i < r_i$, then each agent holds a short position on the risky asset. Theorem 4.5 is consistent with Theorem 4.2 in that $|\bar{\sigma}_i\phi_i(0;x_0)|$, and the risk-adjusted holding in absolute value, is positive to $-\frac{U_i'(x)}{U_i''(x)}\frac{r_i-\bar{\mu}_i}{\bar{\sigma}_i}$.

4.3. $r_1 > \bar{\mu}_1$ and $r_2 < \underline{\mu}_2$

Lastly, we consider the situation in which $r_1 > \bar{\mu}_1$ and $r_2 < \underline{\mu}_2$.

THEOREM 4.6 *Let* $r_1 > \bar{\mu}_1$ *and* $r_2 < \underline{\mu}_2$. *Suppose*

$$-\frac{U_1'(x)}{U_1''(x)}\frac{r_1-\bar{\mu}_1}{\bar{\sigma}_1} \le -\frac{U_2'(x)}{U_2''(x)}\frac{\underline{\mu}_2-r_2}{\bar{\sigma}_2}, \quad \text{for all } x. \tag{9}$$

Then, under one of the following conditions,

(i) $r_1 = r_2$;
(ii) $r_1 > r_2$, U_1 *or* U_2 *displays increasing relative risk aversion;*
(iii) $r_1 < r_2$, U_1 *or* U_2 *displays decreasing relative risk aversion,*

we have

$$\bar{\sigma}_1|\phi_1(0;x_0)| \le \bar{\sigma}_2\phi_2(0;x_0).$$

Similar to Theorems 4.2 and 4.5, Theorem 4.6 also states that in terms of absolute value, the risk-adjusted optimal holding monotonically depends on the product of the absolute risk aversion and the Sharpe ratio. Its intuition is also similar to that of Theorem 4.2.

REMARK 4.2 When $\underline{\mu}_1 \le r_1 \le \bar{\mu}_1$ (or $\underline{\mu}_2 \le r_2 \le \bar{\mu}_2$), the agent will put all his wealth in the risk-free asset, i.e. the optimal portfolio choice is *zero*. In these cases, it reflects the well-known no-participant feature under uncertainty, which was first demonstrated in Dow and Werlang (1992).

5. Conclusions

In this paper, we study an optimal portfolio choice problem and comparative statics for the general utility function under both expected return and volatility ambiguity. We characterize the optimal choice solution by using the robust HJBI equation. We further demonstrate the comparative statics for an important specification of Knightian uncertainty.

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Appendix: Proofs

Proof of Proposition 4.1

(i) If $\underline{\mu} > r$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(B_t^{\mathbb{P}})$ be a standard Brownian motion with respect to $\mathcal{F}$ under prior $\mathbb{P}$. The risky asset follows as

$$dS_t = S_t(\underline{\mu} dt + \bar{\sigma} dB_t^{\mathbb{P}}), \quad S_0 = 1,$$

and the risk-free bond evolves as

$$dP_t = P_t r dt, \quad P_0 = 1.$$

The investor consumes only at the terminal wealth and her optimal portfolio choice problem is†

$$V(t,x) := \sup_{\phi \in \mathcal{A}_1(x)} J(t,x;\phi) = \sup_{\phi \in \mathcal{A}_1(x)} E_t[U(X(T))]$$

subject to

$$\begin{cases} dX(s) = X(s) r ds + \phi(s)(\underline{\mu} - r) ds + \phi(s) \bar{\sigma} dW_s^{\underline{\mu}, \bar{\sigma}}, \\ X(s) \geq 0, \quad s \in [t,T], \\ X(t) = x, \end{cases}$$

where ϕ_s is the dollar amount invested in the stock at any time $s \in [t,T]$, and $x > 0$ is the wealth at the starting time t. Since the utility satisfies the Inada condition and $U \in \mathcal{U}$, by Lemma 2.3 in Xia (2011), we know that $V(\cdot,\cdot) \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty))$ and $V(t,x)$ is strictly increasing and concave with respect to x, for any t in $[0,T]$. We also know that

$$\hat{\phi}(t;x) = -\frac{V_x(t,x)}{V_{xx}(t,x)} \frac{\underline{\mu} - r}{\bar{\sigma}^2}.$$

And the following equation holds

$$\sup_{\phi} \left\{ V_t(t,x) + V_x(t,x) x r + \frac{1}{2} V_{xx}(t,x) \phi^2 \bar{\sigma}^2 + V_x(t,x) \phi (\underline{\mu} - r) \right\} = 0,$$

with $V(T,x) = U(x)$.

It is easy to see that the function V satisfies the following partial differential equation (PDE)

$$\begin{cases} V_t(t,x) + V_x(t,x) x r - \dfrac{V_x^2(t,x)}{V_{xx}(t,x)} \dfrac{(\underline{\mu} - r)^2}{2\bar{\sigma}_1^2} = 0, \\ V(T,x) = U(x). \end{cases} \tag{A1}$$

Combining Theorems 3.1 and 3.2, we know that V satisfies the HJBI equation.

(ii) If $\underline{\mu} \leq r \leq \bar{\mu}$, we take the following risky asset

$$dS_t = S_t(r dt + \bar{\sigma} dW_t^{r,\bar{\sigma}}), \quad S_0 = 1,$$

and the risk-free bond evolves as

$$dP_t = P_t r dt, \quad P_0 = 1.$$

Then, the optimal policy $\hat{\phi} = 0$, i.e. it means the agent will put all his money in the risk free asset. Therefore,

$$V(t,x) = U(x e^{r(T-t)}).$$

And $V_t(t,x) + V_x(t,x) x r = 0$. From Theorems 3.1 and 3.2, we know that V satisfies the HJBI equation.

(iii) If $\bar{\mu} < r$, the proof is similar to (i).

Before we give the proof of Theorem 4.2, we introduce a comparison theorem for a nonlinear PDE.

LEMMA A.1 *If U_1 and U_2 are in $\mathcal{U}$. Let $f, g \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty))$ be tbe solutions of the following PDEs,*

$$\begin{cases} \frac{1}{2} f^2 f_{xx} + r_1 x f_x + f_t - r_1 f = 0, \quad [0,T) \times (0,\infty), \\ f(T,x) = -\frac{U_1'(x)}{U_1''(x)}, \quad \text{for all } x > 0, \end{cases} \tag{A2}$$

and

$$\begin{cases} \frac{1}{2} g^2 g_{xx} + r_2 x g_x + g_t - r_2 g = 0, \quad [0,T) \times (0,\infty), \\ g(T,x) = -\frac{U_2'(x)}{U_2''(x)}, \quad \text{for all } x > 0, \end{cases} \tag{A3}$$

respectively. If

$$-\frac{U_1'(x)}{U_1''(x)} \leq -\frac{U_2'(x)}{U_2''(x)}, \quad \text{for all } x > 0,$$

and under one of the following conditions,

(i) *$r_1 = r_2$;*
(ii) *$r_1 > r_2$, U_1 or U_2 displays increasing relative risk aversion;*
(iii) *$r_1 < r_2$, U_1 or U_2 displays decreasing relative risk aversion,*

then, for all $x > 0$, $t \in [0,T]$, $f(t,x) \leq g(t,x)$.

Proof (i) If $r_1 = r_2$, then the claim holds based on the results of Theorem 4.2 in Xia (2011).

(ii) $r_1 > r_2$, U_1 or U_2 displays increasing relative risk aversion. For all $x > 0$, $t \in [0,T]$, let us define

$$h(t,x) = e^{(r_1 - r_2)(T-t)} f(t, x e^{-(r_1 - r_2)(T-t)}).$$

It can be shown that h satisfies the following PDE

$$\begin{cases} \frac{1}{2} h^2 h_{xx} + r_2 x h_x + h_t - r_2 h = 0, \quad [0,T) \times (0,\infty), \\ h(T,x) = -\frac{U_1'(x)}{U_1''(x)}, \quad \text{for all } x > 0. \end{cases} \tag{A4}$$

In view of the result of (i), for all $x > 0$, $t \in [0,T]$,

$$h(t,x) \leq g(t,x).$$

If U_1 displays increasing relative risk aversion, then by Theorem 5.4 of Xia (2011), we know that $\frac{f(t,x)}{x}$ is decreasing with respect to x.

† $\mathcal{A}_1(x)$ is the square integrable space, which consists of the $(\mathcal{F}_t)$-measurable processes such that $E[\int_t^T \phi^2(s) ds] < \infty$.

Since $r_1 > r_2$, we know that

$$\frac{h(t,x)}{x} = \frac{f(t, xe^{-(r_1-r_2)(T-t)})}{xe^{-(r_1-r_2)(T-t)}} \geq \frac{f(t,x)}{x}.$$

Therefore,

$$f(t,x) \leq h(t,x).$$

Combining the above two inequalities, we obtain

$$f(t,x) \leq g(t,x).$$

The other cases can be proved in a similar way. ■

Proof of Theorem 4.2

For Agent U_1, the robust HJB equation (6) is reduced as follows:

$$\begin{cases} V_t(t,x) + V_x(t,x)xr_1 - \dfrac{V_x^2(t,x)}{V_{xx}(t,x)} \dfrac{(\underline{\mu}_1 - r_1)^2}{2\bar{\sigma}_1^2} = 0, \\ V(T,x) = U_1(x). \end{cases} \tag{A5}$$

It is the classical HJB equation, thus it can be recognized as the optimal value function of the classical problem. Let $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a probability space and $W^{\underline{\mu},\bar{\sigma}}$ be a standard Brownian motion with respect to $(\mathcal{F}_t)$ under prior $\mathbb{P}$. The risky asset follows as

$$dS_t = S_t(\underline{\mu}_1 dt + \bar{\sigma}_1 dW_t^{\underline{\mu}_1,\bar{\sigma}_1}), \quad S_0 = 1,$$

and the risk-free bond evolves as

$$dP_t = P_t r_1 dt, \quad P_0 = 1.$$

The investor consumes only at the terminal wealth and her optimal portfolio choice problem is†

$$V(t,x) = \sup_{\phi\in\mathcal{A}_2(x)} J(t,x;\phi) = \sup_{\phi\in\mathcal{A}_2(x)} E_t[U_1(X(T))]$$

subject to

$$\begin{cases} dX(s) = X(s)rds + \phi(s)(\underline{\mu}_1 - r_1)ds + \phi(s)\bar{\sigma}_1 dW_s^{\underline{\mu}_1,\bar{\sigma}_1}, \\ X(s) \geq 0, \quad s \in [t,T], \\ X(t) = x, \end{cases} \tag{A6}$$

where ϕ_s expresses the dollar amount invested in the stock at any time $s \in [t,T]$, and $x > 0$ is the wealth at starting time t. From the classical results, the optimal choice for the problem (A6) is

$$\phi(t;x) = -\frac{V_x(t,x)}{V_{xx}(t,x)} \frac{\underline{\mu}_1 - r_1}{\bar{\sigma}_1^2}.$$

For each $(t,x) \in [0,T) \times \mathbb{R}^+$, we define the indirect absolute risk tolerance function

$$f(t,x) := -\frac{V_x(t,x)}{V_{xx}(t,x)} \quad \text{and} \quad f(T,x) := -\frac{U_1'(x)}{U_1''(x)}.$$

Then, from (A5), we can obtain that the indirect absolute risk tolerance function satisfies the following nonlinear PDE

$$\begin{cases} \dfrac{1}{2}\dfrac{(\underline{\mu}_1 - r_1)^2}{\bar{\sigma}_1^2} f^2 f_{xx} + r_1 x f_x + f_t - r_1 f = 0, \quad [0,T) \times (0,\infty), \\ f(T,x) = -\frac{U_1'(x)}{U_1''(x)}, x > 0. \end{cases} \tag{A7}$$

In fact, (A7) can be obtained by the martingale property of $f(s,\hat{X}(s))H(s)$, $s \in [t,T]$, where $H(s) = e^{-rs - \frac{\mu - r}{\bar{\sigma}} W_s^{\underline{\mu},\bar{\sigma}} - \frac{1}{2}\frac{(\mu-r)^2}{\bar{\sigma}^2}s}$. The readers can refer to Proposition 3.4 in Xia (2011).

Changing the problem for Agent U_2 as above, we have that

$$\begin{cases} \dfrac{1}{2}\dfrac{(\underline{\mu}_2 - r_2)^2}{\bar{\sigma}_2^2} g^2 g_{xx} + r_2 x g_x + g_t - r_2 g = 0, \quad [0,T) \times (0,\infty), \\ g(T,x) = -\frac{U_2'(x)}{U_2''(x)}, x > 0, \end{cases} \tag{A8}$$

where $g(t,x)$ is the indirect absolute risk tolerance function for Agent U_2.

For PDEs (A7) and (A8), we define

$$W_1'(x) = U_1'(x)^{\frac{\bar{\sigma}_1}{\underline{\mu}_1 - r_1}}, \quad W_2'(x) = U_2'(x)^{\frac{\bar{\sigma}_2}{\underline{\mu}_2 - r_2}}.$$

Then W_1 and W_2 belong to $\mathcal{U}$, and

$$-\frac{U_1'(x)}{U_1''(x)}\frac{\underline{\mu}_1 - r_1}{\bar{\sigma}_1} = -\frac{W_1'(x)}{W_1''(x)}, \quad -\frac{U_2'(x)}{U_2''(x)}\frac{\underline{\mu}_2 - r_2}{\bar{\sigma}_2} = -\frac{W_2'(x)}{W_2''(x)}.$$

Setting $\bar{f}(t,x) = f(t,x)\frac{\underline{\mu}_1 - r_1}{\bar{\sigma}_1}$ and $\bar{g}(t,x) = g(t,x)\frac{\underline{\mu}_2 - r_2}{\bar{\sigma}_2}$, then $\bar{f}$ and $\bar{g}$ satisfy (A2) and (A3) with terminal conditions $-\frac{W_1'(x)}{W_1''(x)}$ and $-\frac{W_2'(x)}{W_2''(x)}$, respectively. Since $-\frac{W_1'(x)}{W_1''(x)} \leq -\frac{W_2'(x)}{W_2''(x)}$, then, taking $(t,x) = (0,x_0)$, by Lemma A.1 we have

$$\bar{f}(0,x_0) \leq \bar{g}(0,x_0),$$

which means that

$$0 < \bar{\sigma}_1\phi_1(0;x_0) = \bar{f}(0,x_0) \leq \bar{g}(0,x_0) = \bar{\sigma}_2\phi_2(0;x_0).$$

Proof of Theorems 4.5 and 4.6

Proofs of Theorems 4.5 and 4.6 are similar to that of Theorem 4.2, and the details of them are omitted.

† $\mathcal{A}_2(x)$ is the square integrable space, which consists of the $\mathcal{F}_t$-measurable process such that $E[\int_t^T \phi^2(s)ds] < \infty$.