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Robust portfolios with commodities and stochastic interest rates

JUNHE CHEN †, MATT DAVISON†, M. ESCOBAR-ANEL * and GOLARA ZAFARI‡

†Department of Statistical and Actuarial Sciences, Western University, 1151 Richmond street, London, Ontario, Canada
‡Department of Statistics and Actuarial Sciences, Simon Fraser University, 8888 University Drive, Burnaby, B.C., Canada

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This paper addresses a gap in the literature concerning robust portfolio analysis for commodity markets in the presence of stochastic interest rates. For generality, we study an ambiguity-averse investor with a Cramér-Lundberg surplus to be allocated into a mean-reverting asset representing a commodity and a bond with a Vasicek interest rate model. Our framework allows for closed-form solutions for the optimal strategy, worst case measure, terminal wealth and value functions. We provide necessary conditions for a well-behaved solution. A full estimation is conducted on two commodity representatives: WTI oil prices and gold prices. We find strong evidence that optimal exposures to commodity risk and interest rate risk, as well as the performance of the portfolio, are significantly affected by the level of ambiguity aversion. Our analyses demonstrate that investors who ignore uncertainty incur drastic equivalent welfare losses, in particular ignoring commodity uncertainty is more costly than neglecting interest rate uncertainty. In a comparison between stocks and commodities, ignoring uncertainty on the latter is also more damaging. We also confirm the importance of working in a complete market (investing in Bonds) for commodity investors, otherwise welfare losses could easily reach 45%. In terms of parameter mis-specifications, we find that incorrect large correlation, smaller variance or simply the wrong market price of commodity risk, can lead to drastically large wealth-equivalent losses.

Keywords: Multivariate portfolio choice; Ambiguity aversion; Commodity markets; Welfare-equivalent losses

JEL Classification: G1

1. Introduction

In today's market conditions of persistently low interest rates, high levels of uncertainty about future directions of the economy, and turmoil in key commodity markets due to climate and political circumstances, a thoughtful analysis of robust portfolio allocations involving commodity markets is long overdue. A robust analysis takes into consideration uncertainty in a region of the parameter space. This model uncertainty or parameter mis-specification is known in the literature as ambiguity aversion analysis. Since the seminal work of Maenhout (2004), there have been numerous studies on the topic of robust portfolio analysis, i.e. optimal portfolio allocation for the risk- and ambiguity-averse investor. This author took the setting of Merton (1971) with a single stock and a riskless asset and assumed ambiguity about the expected rate of return on the stock. His adaptation of the robust control framework of Anderson *et al.* (2003) permitted closed-form solutions for the key objects in the analysis, namely allocation, terminal wealth and value function. As every source of risk (probability distribution) conveys its own level of ambiguity aversion, the literature has progressively studied these sources. Liu *et al.* (2005) considers an investor who is ambiguous about the jumps of the process of the state variable and in Liu (2011) a regime-switching expected stock return was treated. Branger and Larsen (2013) model the stock price by a jump-diffusion process with different levels of ambiguity aversion about the diffusion and jump components. Flor and Larsen (2013) consider a stochastic interest rate with an investor allocating between bonds and stocks, they assume different ambiguity aversion levels for the expected returns of short rates and stocks. Munk and Rubtsov (2014) study the impact of ambiguity about expected inflation on the choice of portfolio. Escobar *et al.* (2015) and Bergen *et al.* (2018) treat the cases of ambiguity aversion for stochastic volatility, covariance and stocks for complete and incomplete markets, while (Escobar *et al.* 2018) studies well-posedness of the solutions.

*Corresponding author. Email: marcos.escobar@uwo.ca

All these studies consistently report significant losses and under-performance for investors who acknowledge ambiguity aversion but choose to ignore such uncertainty. As revealed in the literature, some sources of ambiguity, for instance coming from changes in the distribution of interest rates, the distribution of asset returns, or the distribution of volatilities, are more harmful than others. This observation leads to the key question to be addressed in our paper: what is the impact and the most important source of uncertainty for commodity investors?

In this research, we consider a generic investment company with a surplus $X(t)$ that follows a Cramér-Lundberg process and a Vasicek interest rate model correlated with a mean-reverting log-asset process representing a commodity. This is a general setting that addresses banking and insurance companies alike. The company invests the surplus into bonds, commodities and a bank account with the goal of maximizing the expected utility of terminal wealth, but it is concerned with uncertainty in the fixed-income and commodity markets. To the best of our knowledge, we are the first to consider this portfolio optimization problem. A closely related work is (Chiu and Wong 2013), who studied the optimal strategy for a similar type of company with mean-reverting underlyings within expected utility theory; they provided conditions for existence of a solution but no reference to ambiguity-aversion or full estimation. We contribute to the existing literature on several levels:

- By writing the problem at hand in terms of a Hamilton–Jacobi–Bellman-Issac (HJBI) PDE, we obtain closed-form solutions, up to a system of Riccati equations, for all four relevant functions: optimal allocation, worst market conditions, optimal terminal wealth and value function. Explicit conditions for existence and well-behaved solutions are also provided.
- All parameters are estimated using current data from short rates and bond prices as well as two representative commodities (WTI oil and Gold).
- We demonstrate that ambiguity has a significant impact on the optimal trading strategy and terminal wealth. In particular, plausible ambiguity levels result in decreasing kurtosis, standard deviations and excess returns on the optimal portfolio, picturing a more conservative investor.
- We perform a wealth equivalent loss analysis thanks to quasi-closed-form solutions for the relevant suboptimal optimization problems. Our analyses exhibit that investors who ignore model uncertainty incur drastic losses, in particular, ignoring commodity uncertainty is more costly than neglecting interest rate uncertainty. Commodity markets are also more sensitive than stock markets to changes in ambiguity.
- The importance of working in a complete market (by investing in bonds) for commodity investors is confirmed, welfare losses could easily reach 45% when working in incomplete markets.
- We also found that parameter mis-specifications, particularly incorrect large correlations, small variances or simply different market prices of commodity risk, can lead to drastically large wealth-equivalent losses.

The paper is organized as follows. Section 2.1 gives the formulation of surplus $X(t)$ following the Cramér-Lundberg process, Vasicek interest rate model $r(t)$ and the mean-reverting asset with correlation. Section 2.2 formulates the optimization problem using expected terminal surplus. The solution for the value function is found via Riccati ODEs using the exponential affine quadratic ansatz for the derived HJBI PDE. The optimal strategy $\boldsymbol{\pi}^*$ and worst change of measure are also provided explicitly. Section 3 provides the main elements for a full wealth-equivalent loss (suboptimal analysis) $L^{\boldsymbol{\pi}^s}$ in different cases, through solving an HJB PDE as well, but with specially given investment (suboptimal) strategies. The empirical part, Section 4, describes the estimation methodology and reports the numerical findings, in particular, optimal strategies, the probability distribution of terminal surplus and plots of equivalent losses for relevant suboptimal cases. The last section concludes.

2. Formulation of the surplus optimization problem

We consider a company who can invest its surplus on a commodity, a Bond, and a money market account. The model presented here assumes only one commodity in the portfolio but all our results can be easily extended to multiple assets following mean-reverting processes. Details on the evolution of the underlyings and the problem of interest are provided next.

2.1. Assumptions

A classic model for the surplus of a company is the Cramér-Lundberg process which consists of a premium and decreasing jumps, this is:

$$\mathrm{d}X_0(t) = X_0(t^-)\left(c\mathrm{d}t - \int_{\mathbb{R}^+} yN(\mathrm{d}t, \mathrm{d}y)\right), \tag{1}$$

where $N(\mathrm{d}t, \mathrm{d}y)$ is a Poisson random measure with intensity, μ, $\int_{\mathbb{R}^+} \mathbb{E}[N(\mathrm{d}t, \mathrm{d}y)] = \mu \mathrm{d}t$, c is the premium percentage. We assume the initial surplus $X_0(0) = x$ leading to the integral form:

$$X_0(t) = x + c\int_0^t X_0(s)\,\mathrm{d}s - \sum_{i=1}^{N(t)} X_0(T_i^-)Y_i. \tag{2}$$

For $i = 1, 2, \ldots, N(t)$, T_i are the times when a jump occurs, and Y_i are positive random variables with first moment $\mu_1 > 0$ and second moment μ_2. In order to make sure $0 \leq Y_i \leq 1$, we refer to the setting of Chiu and Wong (2013), $Y_i = \mathrm{e}^{-Z_i}$ where Z_i is a non-negative random variable with well-defined MGF. In this process, the claim amounts are $X_0(T_i^-)Y_i$ which is proportional to the amount of surplus.

The company can invest this surplus into three assets: a risk-free bank account, a risky bond and a commodity. We assume the commodity follows a mean-reverting process, i.e. in real-world measure,

$$\mathrm{d}S(t) = (r(t) + \lambda_S\sigma_S - a\ln S(t))S(t)\,\mathrm{d}t + \sigma_S S(t) \times \left(\rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t)\right), \tag{3}$$

where $W_1(t), W_2(t)$ are independent Brownian motions, the drift is $r(t) + \lambda_S\sigma_S - a\ln S(t)$, volatility σ_S and we require $\lambda_S > 0$ to ensure that the return will be larger as the volatility increases. The interest rate $r(t)$ follows a Vasicek model, where the correlation of random part between asset and interest rate is ρ. Mathematically, under the risk-neutral measure, we have:

$$\mathrm{d}r(t) = \kappa(\bar{r} - r(t))\,\mathrm{d}t + \sigma_r \mathrm{d}W_1^Q(t), \tag{4}$$

with infinity yield $y_\infty := \bar{r} - \frac{\sigma_r^2}{2\kappa^2}$. We invest $\pi_S(t)$ into the commodity and $\pi_P(t)$ into a (rollover) bond with a fix time to maturity $T-t$. The price of the bond under the risk-neutral measure is:

$$P(t, r(t); T) := P(t, r(t)) = \exp\left(-I(t;T)r(t) + A(t;T)\right), \tag{5}$$

where

$$I(t;T) = \frac{1-\mathrm{e}^{-\kappa(T-t)}}{\kappa}$$
$$A(t;T) = \left(\bar{r} - \frac{\sigma_r^2}{2\kappa^2}\right)(I(t;T) - (T-t)) - \frac{\sigma_r^2}{4\kappa}I^2(t;T). \tag{6}$$

For simplicity, we transform the dynamic of the rollover bond into

$$\mathrm{d}P(t, r(t)) = (r(t) + \lambda_r I_\tau)P(t, r(t))\,\mathrm{d}t - I_\tau\sigma_r P(t, r(t))\,\mathrm{d}W_1(t), \tag{7}$$

where $I_\tau := I(t; t+\tau) = \frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa}$ is a constant. λ_r is the market price of risk on the Bond return. In the risk-neutral measure, the bond price can be represented via a PDE

$$P_t + P_r(\kappa(\bar{r} - r)) + \frac{\sigma_r^2}{2}P_{rr} = rP. \tag{8}$$

Transforming to real-world measure using Girsanov theorem, the joint dynamics for the bond and the interest rate are

$$\begin{aligned}\mathrm{d}r(t) &= [\kappa(\bar{r} - r(t)) - \lambda_r\sigma_r]\,\mathrm{d}t + \sigma_r \mathrm{d}W_1(t)\\ \mathrm{d}P(t, r(t)) &= (r(t) + \lambda_r I(t;T))P(t, r(t))\,\mathrm{d}t - I(t;T)\sigma_r \times P(t, r(t))\,\mathrm{d}W_1(t).\end{aligned} \tag{9}$$

We assume the jumps of the surplus, hence $N(t)$ and Y_i to be independent of $W_1(t)$ and $W_2(t)$. By letting $Z(t) = \ln S(t)$, we can simplify the representation of the asset price:

$$\mathrm{d}Z(t) = \left(r(t) + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 - aZ(t)\right)\mathrm{d}t + \sigma_S\left(\rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2(t)\right). \tag{10}$$

The model in the previous equations is called the *reference model*. The surplus process becomes

$$\begin{aligned}\mathrm{d}X(t) &= [(c + r(t))X(t) + \pi_S(t)(\lambda_S\sigma_S - aZ(t)) + \pi_P\lambda_r I(t)]\,\mathrm{d}t\\ &\quad + \pi_S(t)\sigma_S\left(\rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2(t)\right)\\ &\quad - \pi_P(t)I(t)\sigma_r\,\mathrm{d}W_1(t) - X(t^-)\int_{\mathbb{R}^+} yN(\mathrm{d}t, \mathrm{d}y).\end{aligned} \tag{11}$$

Note X_t is not self-financing because there are cash inflows, constant premium c and cash outflows, claims Y_i. Moreover if both $c = 0$ and $\mu = 0$, the surplus $X(t)$ will be a constant hence the problem starts simply with an initial budget.

2.2. Optimal portfolio problem

We consider an ambiguous agent with constant relative risk aversion (CRRA) utility:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{12}$$

where the parameter controlling the level of risk aversion is γ. The investor wants to maximize the expected utility from the terminal wealth under worst case conditions parameterized by its level of ambiguity-aversion (ϕ), the objective function in this optimal problem is (see Maenhout 2004):

$$J(t, x, r, z) = \sup_{\pi\in\Pi}\inf_{\boldsymbol{u}} \mathbb{E}^{Q^u}_{x,z,r} \times \left[\int_t^T \frac{1}{2}\left(\frac{u_1^2(s)}{\phi_1(s)} + \frac{u_2^2(s)}{\phi_2(s)}\right)\mathrm{d}s + U(X(T))\middle|\mathcal{F}_t\right], \tag{13}$$

where $\mathbb{E}^Q_{x,z,r}[U(X(T))|\mathcal{F}_t] := \mathbb{E}^Q[U(X(T))|\mathcal{F}_t, X(t) = x, Z(t) = z, r(t) = r]$, Π is the set of all admissible strategies, $(\phi_1(t), \phi_2(t))$ are preference parameters for ambiguity-aversion, and $Q^{\boldsymbol{u}}$ is defined in Equation (14). By setting $\boldsymbol{u} = (u_1, u_2)^T$, $\mathrm{d}\boldsymbol{W}(t) = (\mathrm{d}W_1(t), \mathrm{d}W_2(t))^T$, and letting $\mathcal{Q}$ represent the set of admissible probability measures by Girsanov Theorem, we can mathematically define the Radon-Nikodym derivative as follows:

$$\begin{aligned}\mathcal{Q} = \Bigg\{Q^{\boldsymbol{u}} : \frac{\mathrm{d}Q^u}{\mathrm{d}P}\bigg|_{\mathcal{F}_s} &= \exp\left(-\int_0^s u_1(t)\,\mathrm{d}W_1(t) - \frac{1}{2}\int_0^s u_1^2(t)\,\mathrm{d}t\right.\\ &\quad \left. - \int_0^s u_2(t)\,\mathrm{d}W_2(t) - \frac{1}{2}\int_0^s u_2^2(t)\,\mathrm{d}t\right)\\ &= \exp\left(-\int_0^s \boldsymbol{u}^T(t)\,\mathrm{d}\boldsymbol{W}(t) - \int_0^s \boldsymbol{u}^T(t)\boldsymbol{u}(t)\,\mathrm{d}t\right)\Bigg\}.\end{aligned} \tag{14}$$

This means we consider changes of measure of the form:

$$\begin{cases}\mathrm{d}W_1(t) = \mathrm{d}W_1^Q(t) - u_1(t)\,\mathrm{d}t\\ \mathrm{d}W_2(t) = \mathrm{d}W_2^Q(t) - u_2(t)\,\mathrm{d}t.\end{cases} \tag{15}$$

Hence the dynamics under the new measures, also known as the *alternative models*, are:

$$\begin{cases} \mathrm{d}r(t) = [\kappa(\bar{r}-r(t)) - \sigma_r(\lambda_r + u_1(t))]\,\mathrm{d}t + \sigma_r \mathrm{d}W_1^Q(t) \\ \mathrm{d}Z(t) = \left(r(t) + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 - aZ(t)\right. \\ \left. -\sigma_S\left(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\right)\right)\mathrm{d}t \\ \quad +\sigma_S\left(\rho\,\mathrm{d}W_1^Q(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2^Q(t)\right) \\ \mathrm{d}X(t) = \left[(c + r(t))X(t^-) + \pi_S(t)\right. \\ \quad \left(\lambda_S\sigma_S - aZ(t) - \sigma_S\left(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\right)\right) \\ \quad \left.+(\lambda_r + u_1(t))\pi_P(t)I_\tau\sigma_r\right]\mathrm{d}t + \pi_S(t)\sigma_S \\ \quad \left(\rho\mathrm{d}W_1^Q(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_2^Q(t)\right) \\ \quad -\pi_P(t)I_\tau\sigma_r\,\mathrm{d}W_1^Q(t) - X(t^-)\int_{\mathbb{R}^+} yN(\mathrm{d}t,\mathrm{d}y). \end{cases} \tag{16}$$

Set $\boldsymbol{y}(t) = (r(t), z(t))^T$, $\mathrm{d}\boldsymbol{W}^Q(t) = (\mathrm{d}W_1^Q(t), \mathrm{d}W_2^Q(t))^T$ and $\boldsymbol{\pi} = (\pi_P, \pi_S)^T$. We can represent the dynamics compactly as:

$$\begin{cases} \mathrm{d}\boldsymbol{y}(t) = [(\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y}(t)) - \boldsymbol{\sigma}\boldsymbol{u}]\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}^Q(t) \\ \mathrm{d}X(t) = [(c + r(t))X(t^-) + \boldsymbol{\pi}^T\boldsymbol{B}\boldsymbol{b}(\boldsymbol{y}(t)) - \boldsymbol{\sigma}\boldsymbol{u})]\,\mathrm{d}t \\ \quad +\boldsymbol{\pi}^T\boldsymbol{B}\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}^Q(t) - X(t^-)\int_{\mathbb{R}^+} yN(\mathrm{d}t,\mathrm{d}y). \end{cases} \tag{17}$$

where vectors and matrices are

$$\boldsymbol{A} = \begin{bmatrix} \kappa & 0 \\ -1 & a \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{bmatrix},$$
$$\boldsymbol{B} = \begin{bmatrix} -I_\tau & 0 \\ 0 & 1 \end{bmatrix}, \tag{18}$$
$$\boldsymbol{\theta} = \begin{bmatrix} \kappa\bar{r} - \sigma_r\lambda_r \\ \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 \end{bmatrix}, \quad \boldsymbol{b}(\boldsymbol{y}(t)) = \begin{bmatrix} -\lambda_r\sigma_r \\ \lambda_S\sigma_S - aZ(t) \end{bmatrix}.$$

Then the HJBI equation is:

$$\begin{aligned} &\sup_{\pi\in\Pi}\inf_{\boldsymbol{u}}\Big\{[\kappa\,(\bar{r} - r) - (\lambda_r + u_1)\sigma_r]\,J_r \\ &+ \Big[\Big(r + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2 - az\Big) \\ &-\sigma_S\left(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\right)\Big]J_z \\ &+ \big[(c + r)x + \pi_S\,(\lambda_S\sigma_S - az - \sigma_S \\ &\times\left(\rho u_1(t) + \sqrt{1-\rho^2}u_2(t)\right)\big) + (\lambda_r + u_1)\pi_P I_\tau\sigma_r\big]J_x \\ &+ \frac{1}{2}\sigma_r^2 J_{rr} + \frac{1}{2}\sigma_S^2 J_{zz} + (\rho\pi_S\sigma_S\sigma_r - \pi_P I_\tau\sigma_r^2)J_{rx} + \rho\sigma_S\sigma_r J_{zr} \\ &+ (\pi_S\sigma_S^2 - \pi_P I_\tau\rho\sigma_S\sigma_r)J_{zx} \\ &+ \frac{1}{2}\left((\pi_S\sigma_S\rho - \pi_P I_\tau\sigma_r)^2 + \pi_S^2\sigma_S^2(1-\rho^2)\right)J_{xx} \\ &+ \frac{u_1^2}{2\phi_1} + \frac{u_2^2}{2\phi_2}\Big\} + \mu\mathbb{E}[J(t, x(1-Y), r, z) - J] + J_t = 0, \end{aligned} \tag{19}$$

where Y is a r.v. with the same distribution as Y_i.

For analytical tractability, (Maenhout 2004) provides suitable forms for $\phi_1(t)$ and $\phi_2(t)$ are

$$\begin{aligned} \phi_1(t) &= \frac{\beta_1}{(1-\gamma)J(t,x,r,z)} > 0, \\ \phi_2(t) &= \frac{\beta_2}{(1-\gamma)J(t,x,r,z)} > 0, \end{aligned} \tag{20}$$

where β_1 and β_2 are the ambiguity-aversion parameters. β_1 can be interpreted as ambiguity aversion about the interest rate distribution, while β_2 is ambiguity aversion on the distribution of the commodity return. Let $\boldsymbol{\beta} = \mathrm{diag}(\beta_1, \beta_2)$ and $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$. In matrix form, the HJBI equation becomes:

$$\begin{aligned} &\sup_{\pi\in\Pi}\inf_{\boldsymbol{u}}\Big\{J_t + [(\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y}) - \boldsymbol{\sigma}\boldsymbol{u}]^T J_{\boldsymbol{y}} \\ &+[(c + r)x + \boldsymbol{\pi}^T\boldsymbol{B}(\boldsymbol{b}(\boldsymbol{y}) - \boldsymbol{\sigma}\boldsymbol{u})]J_x + \frac{1}{2}\mathrm{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) \\ &+ \boldsymbol{\pi}^T\boldsymbol{B}\Sigma J_{x\boldsymbol{y}} + \frac{(1-\gamma)J}{2}\boldsymbol{u}^T\boldsymbol{\beta}^{-1}\boldsymbol{u} + \frac{J_{xx}}{2}\boldsymbol{\pi}^T\boldsymbol{B}^T\Sigma\boldsymbol{B}\boldsymbol{\pi}\Big\} \\ &+ \mu\mathbb{E}[J(t, x(1-Y), r, z) - J] = 0. \end{aligned} \tag{21}$$

The proposition next exhibit the implicit solution to the first order conditions.

Proposition 1 *The optimal change of measure is*

$$\boldsymbol{u}^* = \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}. \tag{22}$$

The corresponding optimal investment strategy is

$$\begin{aligned} \boldsymbol{\pi}^* = -\boldsymbol{B}^{-1}&\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right)^{-1} \\ &\times\left(J_x\left(\boldsymbol{b}(\boldsymbol{y}) - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right). \end{aligned} \tag{23}$$

The HJBI PDE becomes

$$\begin{aligned} &J_t + (\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y})^T J_{\boldsymbol{y}} + \frac{1}{2}\mathrm{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) + (c + \boldsymbol{y}^T\boldsymbol{e}_1)xJ_x \\ &- \frac{J_{\boldsymbol{y}}^T\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{2(1-\gamma)J} \\ &- \frac{1}{2}\left(J_x\left(\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y} - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right)^T \\ &\times\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right)^{-1} \\ &\times\left(J_x\left(\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y} - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right) \\ &+ \mu\mathbb{E}[J(t, x(1-Y), r, z) - J] = 0. \end{aligned} \tag{24}$$

where $\boldsymbol{I}$ is 2×2 identity matrix, $\boldsymbol{e}_1 = (1,0)^T$, $\boldsymbol{E}_2 = \mathrm{diag}(0,1)$ and $\boldsymbol{\lambda} = (-\lambda_r\sigma_r, \lambda_S\sigma_S)^T$ so $\boldsymbol{b}(\boldsymbol{y}) = \boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y}$.

Proof The PDE (19) is quadratic w.r.t. $\boldsymbol{u}$. Therefore, it is easy to obtain the optimal change of measure $\boldsymbol{u}^*$ by computing first order derivative of $\boldsymbol{u}$ and set it be $\boldsymbol{0}$. Substituting $\boldsymbol{u}^*$

into the HJB Equation (21), we can also compute the optimal trading strategy for $\boldsymbol{\pi}$ by setting the derivative w.r.t. $\boldsymbol{\pi}$ of RHS to be 0, because Equation (21) is also quadratic w.r.t. $\boldsymbol{\pi}$. We present the straightforward but laborious computational detail in Appendix 1 using a matrix representation. Second order conditions ensure the solutions are indeed a minimum and maximum respectively. ■

Before simplifying and solving the HJBI, we present Lemma 1, which gives a simple way to compute the explicit solution to a Matrix Riccati differential equation (RDE) encountered in the representation of the solution ($\boldsymbol{M}_2$ in Proposition 2.2).

LEMMA 1 *Assume an $n \times n$ Matrix RDE $R(t,T)$ satisfying*

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} &= \boldsymbol{RBR} + \boldsymbol{RA} + \boldsymbol{A}^T\boldsymbol{R} + \boldsymbol{Q} \\ \boldsymbol{R}(T) &= \boldsymbol{S},\end{aligned} \tag{25}$$

where $\boldsymbol{B}$, $\boldsymbol{Q}$ and $\boldsymbol{S}$ are symmetric and non-negative definite. The solution to the RDE will be

$$\boldsymbol{R} = \boldsymbol{K}_2\boldsymbol{K}_1^{-1}, \tag{26}$$

where $n \times n$ matrices $\boldsymbol{K}_1, \boldsymbol{K}_2$ are defined by

$$\boldsymbol{K}(t,T) = \exp\left(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{bmatrix}(T-t)\right)\begin{bmatrix} \boldsymbol{I}_{n\times n} \\ \boldsymbol{S} \end{bmatrix} := \begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \end{bmatrix}. \tag{27}$$

Proof See Abou Kandil and Freiling (2003). ■

Now given the PDE in Proposition 1, we are ready to solve for $J(t,x,z,r)$ and compute explicitly $\boldsymbol{u}^*$ and $\boldsymbol{\pi}^*$. This is provided in the next Proposition.

PROPOSITION 2 *The optimal value function has the representation:*

$$J(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left(M_0(t) + \boldsymbol{y}^T\boldsymbol{M}_1(t) + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2(t)\boldsymbol{y}\right), \tag{28}$$

where $\boldsymbol{M}_1(t)$ is a 2×1 vector and $\boldsymbol{M}_2$ is a 2×2 symmetric matrix with terminal condition $J(T,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}$. Here the matrices $M_0(t)$, $\boldsymbol{M}_1(t)$ and $\boldsymbol{M}_2(t)$ follow Matrix Riccati ODEs:

$$\begin{cases}\boldsymbol{M}_2' + \boldsymbol{D}_0 + \boldsymbol{D}_1\boldsymbol{M}_2 + \boldsymbol{M}_2\boldsymbol{D}_1^T + \boldsymbol{M}_2\boldsymbol{D}_2\boldsymbol{M}_2 = 0 \\ \boldsymbol{M}_2(T) = \boldsymbol{0}_{2\times 2},\end{cases} \tag{29}$$

$$\begin{cases}\boldsymbol{M}_1' + \boldsymbol{C}_0 + \boldsymbol{M}_2(t)\boldsymbol{C}_1 + (\boldsymbol{M}_2(t)\boldsymbol{D}_2 + \boldsymbol{D}_1)\boldsymbol{M}_1 = 0 \\ \boldsymbol{M}_1(T) = \boldsymbol{0}_{2\times 1},\end{cases} \tag{30}$$

$$\begin{cases}M_0' + B_0 + \boldsymbol{M}_1^T(t)\boldsymbol{C}_1 + \frac{1}{2}\boldsymbol{M}_1^T(t)\boldsymbol{D}_2\boldsymbol{M}_1(t) = 0 \\ M_0(T) = 0,\end{cases} \tag{31}$$

where matrices $\boldsymbol{D}_0, \boldsymbol{D}_1, \boldsymbol{D}_2$, vectors $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ and scalar B_0 are given in Appendix 2.

After obtaining the solution $\boldsymbol{M}_2(t)$ in Proposition 2 using Lemma 1, we can simply obtain $\boldsymbol{M}_1(t)$ and $M_0(t)$ sequentially by computing numerical solution to the corresponding ODEs. Now we are ready to compute the optimal strategies and changes of measure.

THEOREM 1 *The optimal trading strategy and change of measure are*

$$\begin{aligned}\boldsymbol{\pi}^* &= x\boldsymbol{B}^{-1}\left(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\right)^{-1} \\ &\quad\times\left[\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T(\boldsymbol{M}_1(t) + \boldsymbol{M}_2(t)\boldsymbol{y})\right] \\ \boldsymbol{u}^* &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T}{1-\gamma}\left((\boldsymbol{M}_1(t) + \boldsymbol{M}_2(t)\boldsymbol{y}) + \frac{1-\gamma}{x}\boldsymbol{B}\boldsymbol{\pi}^*\right).\end{aligned} \tag{32}$$

Proof Plugging the ansatz in Equation (28) into $\boldsymbol{\pi}^*$ and $\boldsymbol{u}^*$ in Proposition 1 can easily derive the equations. The results with plugged $\boldsymbol{\pi}^*$ are shown in Appendix 3. ■

In fact, we can rewrite the trading strategy as follows:

$$\begin{aligned}\frac{\boldsymbol{\pi}^*}{x} =& \boldsymbol{B}^{-1}\left(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\right)^{-1}\left[\boldsymbol{\lambda} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1(t)\right. \\ &\left. + \left(\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2(t) - a\boldsymbol{E}_2\right)\boldsymbol{y}\right].\end{aligned} \tag{33}$$

This demonstrates that the investment fraction $\frac{\boldsymbol{\pi}^*}{x}$, and $\boldsymbol{u}^*$ are linear w.r.t. $\boldsymbol{y}$. Particularly, the part

$$\boldsymbol{B}^{-1}\left(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\right)^{-1}\boldsymbol{\lambda}, \tag{34}$$

represents the standard mean-variance portfolio. Moreover, compatible with (Flor and Larsen 2013), when $a = 0$, i.e. if there is no mean-reverting effect on the asset, the optimal investment strategy will be:

$$\frac{1}{x}\boldsymbol{\pi}^*_{a=0} = \boldsymbol{B}^{-1}\left(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\right)^{-1}\left(\boldsymbol{\lambda} - I_\tau\sigma_r^2[(1-\gamma)-\beta_1]\boldsymbol{e}_1\right), \tag{35}$$

where the second component is from Vasicek stochastic interest rate, and the third component is from ambiguity. Therefore, the value $\frac{1}{x}(\boldsymbol{\pi}^* - \boldsymbol{\pi}^*_{a=0})$ captures the effect of the mean-reverting term.

Proposition 2 provides a candidate for the solution $J(t,x,r,z)$ to the HJBI in equation (21), this solution is expressed in terms of $M_0(t), \boldsymbol{M}_1(t)$ and $\boldsymbol{M}_2(t)$. In order to verify the candidate is well-defined, we must first guarantee that the change of measure determined in the worst-case scenario (the *inf* part) denoted $\boldsymbol{u}^*$ satisfies the Novikov's condition. In a second step we use existing results for a verification theorem on the sup part.

THEOREM 2 *If Frobenius norm $\|\boldsymbol{M}_2(t)\|_F < \infty$ for $0 \le t \le T$, i.e. $\boldsymbol{M}_2$ is a well-defined matrix function, then the Novikov's condition holds for $\boldsymbol{u}^*(t)$. Moreover $\boldsymbol{\pi}^*(t)$ is the optimal strategy in the worst-case scenario $\boldsymbol{u}^*(t)$.*

Proof See Appendix 3. ■

3. Wealth-equivalent losses analysis

In this section we quantify the wealth-equivalent utility loss the investor suffers by following meaningful suboptimal strategies. In particular, we study the effects of ignoring model uncertainty†, the impact of ambiguity on the utility loss incurred from not investing in Bonds (market incompleteness), and the impact of wrongly selecting specific parameters of the model (due to estimation error for instance). Similar to Flor and Larsen (2013), we measure the utility loss in terms of the percentage of wealth lost when using the suboptimal strategies.

First, we need to find a representation for the value function given a suboptimal trading strategy $\boldsymbol{\pi}^s$, this is:

$$J^{\boldsymbol{\pi}^s}(t,x,r,z) = \inf_{Q\in\mathcal{Q}} \mathbb{E}^{Q}_{x,r,z} \times \left[\int_t^T \frac{1}{2}\left(\frac{u_1^2(s)}{\phi_1(s)} + \frac{u_2^2(s)}{\phi_2(s)}\right) \mathrm{d}s + U(X(T);\boldsymbol{\pi}^s)\middle|\mathcal{F}_t\right]. \tag{36}$$

The wealth-equivalent loss of optimal strategy is denoted $L^{\boldsymbol{\pi}^s}$ and it is the solution to the equation:

$$J(t,x(1-L^{\boldsymbol{\pi}^s}),r,z) = J^{\boldsymbol{\pi}^s}(t,x,r,z). \tag{37}$$

Given the strategy $\boldsymbol{\pi}^s$, the HJBI equation is

$$\begin{aligned}\inf_{\boldsymbol{u}} \Big\{ & J_t + [(\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y}) - \boldsymbol{\sigma}\boldsymbol{u}]^T J_{\boldsymbol{y}} \\ & + [(c + \boldsymbol{y}^T\boldsymbol{e}_1)x + (\boldsymbol{\pi}^s)^T\boldsymbol{B}(\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y} - \boldsymbol{\sigma}\boldsymbol{u})]J_x \\ & + \frac{1}{2}\mathrm{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) + (\boldsymbol{\pi}^s)^T\boldsymbol{B}\Sigma J_{x\boldsymbol{y}} + \frac{(1-\gamma)J}{2}\boldsymbol{u}^T\boldsymbol{\beta}^{-1}\boldsymbol{u} \\ & + \frac{J_{xx}}{2}(\boldsymbol{\pi}^s)^T\boldsymbol{B}^T\Sigma\boldsymbol{B}\boldsymbol{\pi}^s \Big\} \\ & + \mu\mathbb{E}[J(t,x(1-Y),r,z) - J] = 0. \end{aligned} \tag{38}$$

The proposition next provides a representation up to Riccati equations of the value function for the suboptimal problem.

Proposition 3 *Assume the suboptimal strategy is of the form $\boldsymbol{\pi}^s = x(\boldsymbol{h}(t) + \boldsymbol{H}(t)\boldsymbol{y})$, i.e. proportional to x and linear w.r.t. $\boldsymbol{y}$, then Equation* (38) *also has an exponential affine form:*

$$J^{\boldsymbol{\pi}^s}(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma} \times \exp\left(M_0^{\boldsymbol{\pi}^s}(t) + \boldsymbol{y}^T\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t) + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{y}\right). \tag{39}$$

The differential equations for $M_0^{\boldsymbol{\pi}^s}(t)$, $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}$ and $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$ are given below:

$$\begin{cases} (\boldsymbol{M}_2^{\boldsymbol{\pi}^s})' - \boldsymbol{H}^T(t)\boldsymbol{D}_0^s\boldsymbol{H}(t) - \boldsymbol{D}_3^s + \boldsymbol{D}_1^s\boldsymbol{M}_2 + \boldsymbol{M}_2(\boldsymbol{D}_1^s)^T \\ \quad + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{D}_2^s\boldsymbol{M}_2^{\boldsymbol{\pi}^s} = 0 \\ \boldsymbol{M}_2^{\boldsymbol{\pi}^s}(T) = \boldsymbol{0}_{2\times 2} \end{cases} \tag{40}$$

$$\begin{cases} (\boldsymbol{M}_1^{\boldsymbol{\pi}^s})' + \boldsymbol{C}_0 + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{C}_1 + (\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{D}_2^s + \boldsymbol{D}_1^s)\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{C}_0^s \\ \quad + [(\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t))^T\boldsymbol{D}_2^s - a\boldsymbol{E}_2]\boldsymbol{B}\boldsymbol{h}(t)(1-\gamma) \\ \quad - (\boldsymbol{H}(t))^T\boldsymbol{D}_0^s\boldsymbol{h}(t) = 0 \\ \boldsymbol{M}_1^{\boldsymbol{\pi}^s}(T) = \boldsymbol{0}_{2\times 1}. \end{cases} \tag{41}$$

$$\begin{cases} (M_0^{\boldsymbol{\pi}^s})' + B_0^s + [(\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t))^T\boldsymbol{D}_2^s + \boldsymbol{\lambda}^T]\boldsymbol{B}\boldsymbol{h}(t)(1-\gamma) \\ \quad + \frac{1}{2}(\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t))^T\boldsymbol{D}_2\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t) \\ \quad - \frac{1}{2}(\boldsymbol{h}(t))^T\boldsymbol{D}_0^s\boldsymbol{h}(t) = 0 \\ M_0^{\boldsymbol{\pi}^s}(T) = 0. \end{cases} \tag{42}$$

where matrices $\boldsymbol{D}_0^s$, $\boldsymbol{D}_1^s$, $\boldsymbol{D}_2^s$, $\boldsymbol{D}_3^s$, vectors $\boldsymbol{C}_0^s$ and scalar B_0^s are given in Appendix 4.

In particular, if $\boldsymbol{\pi}^s = x\boldsymbol{h}(t)$, the ansatz will exclude the quadratic term $\boldsymbol{y}^T\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{y}$ and become

$$J^{\boldsymbol{\pi}^s}(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left(M_0^{\boldsymbol{\pi}^s}(t) + \boldsymbol{y}^T\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)\right). \tag{43}$$

Hence we only need to focus on $M_0^{\boldsymbol{\pi}^s}$ and $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}$ in this case. Moreover Lemma 1 can lead to the solution for $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$.

Proof See Appendix 4. ■

Proposition 3 gives us the wealth-equivalent losses in the form:

$$L^{\boldsymbol{\pi}^s} = 1 - \exp\left(\frac{1}{1-\gamma}\left((M_0^{\boldsymbol{\pi}^s}(t) - M_0(t)) + \boldsymbol{y}^T(\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t) - \boldsymbol{M}_1(t)) + \frac{1}{2}\boldsymbol{y}^T(\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t) - \boldsymbol{M}_2(t))\boldsymbol{y}\right)\right). \tag{44}$$

Plugging a suboptimal parametric set denoted by hat "$(\hat{a}, \hat{\boldsymbol{\sigma}}, \ldots)$" into the ODEs of $M_0(t)$, $\boldsymbol{M}_1(t)$ and $\boldsymbol{M}_2(t)$ in Equation (A8) allow us to compute $M_0^{\boldsymbol{\pi}^s}(t)$, $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)$ and $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$ hence we can obtain the trading strategy $\boldsymbol{\pi}^s(t)$ from Theorem 1.

In other words, with $M_0^{\boldsymbol{\pi}^s}(t)$, $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)$ and $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$, the functions $\boldsymbol{h}(t)$ and $\boldsymbol{H}(t)$ of the suboptimal strategy are

$$\boldsymbol{h}(t) = \hat{\boldsymbol{B}}^{-1}\left(\hat{\boldsymbol{\sigma}}(\hat{\gamma}\boldsymbol{I} + \hat{\boldsymbol{\beta}})\hat{\boldsymbol{\sigma}}^T\right)^{-1} \times \left(\hat{\boldsymbol{\lambda}} + \hat{\boldsymbol{\sigma}}\left(\boldsymbol{I} - \frac{\hat{\boldsymbol{\beta}}}{1-\hat{\gamma}}\right)\hat{\boldsymbol{\sigma}}^T\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)\right) \tag{45}$$

† This is either investors ignoring their own level of uncertainty due to, for example, technical limitations, or ignoring market analyst recommendations about documented levels of ambiguity on the distributions of the assets at hand

$$\boldsymbol{H}(t) = \hat{\boldsymbol{B}}^{-1}\left(\hat{\boldsymbol{\sigma}}(\hat{\gamma}\boldsymbol{I} + \hat{\boldsymbol{\beta}})\hat{\boldsymbol{\sigma}}^{T}\right)^{-1} \times \left(\hat{\boldsymbol{\sigma}}\left(\boldsymbol{I} - \frac{\hat{\boldsymbol{\beta}}}{1-\hat{\gamma}}\right)\boldsymbol{\sigma}^{T}\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t) - \hat{a}\boldsymbol{E}_2\right).$$

where $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t), \boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)$ are solutions presented in Proposition 2 with parametric set denoted generically by $\hat{\theta}$.

We are interested in three families of suboptimal strategies, which will be studied in detail in the upcoming sections:

(i) First, strategies from ambguity averse investors who choose to *ignore ambiguity* on either the distribution of the commodity $S(t)$, the interest rate $r(t)$ or both. This means three cases, $\hat{\beta}_1 = 0$, $\hat{\beta}_2 = 0$, or $\hat{\beta}_1 = \hat{\beta}_2 = 0$. This will allow us to measure the total impact of ignoring ambiguity and which source of ambiguity is more harmful. Given $\boldsymbol{\pi}^s(t)$ from Theorem 1, we can apply Proposition 3 to obtain $M_0^s(t)$, $\boldsymbol{M}_1^s(t)$ and $\boldsymbol{M}_2^s(t)$, then producing $L^{\boldsymbol{\pi}^s}$.

(ii) The second type of suboptimal strategies is derived by considering investor who fail to allocate to Bonds as a way of hedging the interest rate risk. This is a case of incomplete markets. Here we need to find the optimal allocation on commodity given the constraint $\pi_P = 0$. This can be achieved by equivalently substituting $\boldsymbol{E}_2\boldsymbol{\pi}$ for $\boldsymbol{\pi}$ in Equation (21) to force $\pi_P = 0$. Using the notation derives this results leading to:

$$\boldsymbol{u}^s = \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_y + J_x\boldsymbol{E}_2\boldsymbol{\pi})}{(1-\gamma)J}$$
$$\boldsymbol{\pi}^s = -\boldsymbol{E}_2\boldsymbol{P}^{-1}\left(J_x\left(\boldsymbol{b} - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_y}{(1-\gamma)J}\right) + \Sigma J_{xy}\right) \tag{46}$$

where $\boldsymbol{P}$ is the diagnoal matrix with the diagonal elements below,

$$\operatorname{diag}\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right) \tag{47}$$

Plugging $\boldsymbol{u}^s$ and $\boldsymbol{\pi}^s$ produces a new HJBI PDE. The solution $J^{\boldsymbol{\pi}^s}(t,x,r,z)$ and loss $L^{\boldsymbol{\pi}^s}$ can be generated by taking similar steps.

(iii) The third and final group of suboptimal strategies arises from *wrong estimation* of the reference parameters. Some parameters in our model are either very difficult to estimate due to lack of data, like Market Price of rate Risk $\hat{\lambda}_r$ and commodity $\hat{\lambda}_S$, or are not stable due to mis-measurement and inaccuracies in the chosen values, examples of these are volatilities $\hat{\sigma}_r$, $\hat{\sigma}_S$ and the correlation ρ. This wrong values affects the identification of the parameters in the *reference model*, therefore leading to wrongly crafted strategies.This direction is conceptually different to point 1 above as it escapes the worst case analyses of the ambiguity aversion framework. To see this note two aspects, first we work in a setting of equivalent changes of measures to accommodate the worst case preferences of ambiguous investor, this does not capture concerns about covariance mis-specifications (see Fouque *et al.* 2016 for non-equivalent ambiguity framework). Second, by changing the estimates to different values, we effectively work with a different *reference model*, i.e. coming from wrong estimations, hence affecting the level of ambiguity of the *alternative models*.

4. Empirical analysis

This section is divided into three part. First we provide details about the data from commodities and interest rates as well as the estimation approach. Then we report the optimal solutions, i.e. allocation, value and terminal surplus, for the representative commodity markets. The last section studies the impact of various suboptimal strategies.

4.1. Data and estimation methodology.

We target two commodities: WTI oil prices and gold prices, which can be seen as representative of this asset class. In particular, we work with weekly WTI oil prices extracted from https://fred.stlouisfed.org/series/DCOILWTICO, as well as gold prices from https://fred.stlouisfed.org/series/GOLDAMGBD228NLBM. For Bond prices and interest rate calibration we use data from the Federal Reserve Banks of St.louis (FRED) †. We collect weekly US 1-month Treasury Bill rate from https://fred.stlouisfed.org/series/DGS1MO as short rates. For calibration of Market Price of Risk of Bond, we also collected the monthly 10-year govt bond yield from https://fred.stlouisfed.org/series/IRLTLT01USM156N. In all cases we choose the period from Aug 2001 to Sep 2019.

For estimation purposes, we first discretize our model, assuming a time step h with $r_i = r(t_i)$. From dynamic processes in Equation (9) in real-world measure, using explicit solution to OU process, we can obtain:

$$r_{i+1} = r_i e^{-\kappa h} + \bar{r}^{\mathbb{P}}(1 - e^{-\kappa h}) + \sigma_r^{(1)}\varepsilon_{i+1}^{(1)} \tag{48}$$

where for simplicity, we define the real-world interest rate be $r^{\mathbb{P}} := \bar{r} - \frac{\lambda_r\sigma_r}{\kappa}$. Both κ and $r^{\mathbb{P}}$ can be computed via a linear regression analysis. The variance would be:

$$\left(\sigma_r^{(1)}\right)^2 = \operatorname{var}\left(\sigma_r\int_0^h e^{-\kappa(h-s)}\,dW_1(t_i+s)\right) = \frac{1-e^{-2\kappa h}}{2\kappa}\sigma_r^2. \tag{49}$$

Using $r(t_i)$, we can write a discretization of $Z(t)$ as follows:

$$Z_{i+1} = e^{-ah}Z_i + (r_i - \bar{r}^{\mathbb{P}})\frac{e^{-\kappa h} - e^{-ah}}{a-\kappa} + \left(\bar{r}^{\mathbb{P}} + \lambda_S\sigma_S - \frac{1}{2}\sigma_S^2\right)\frac{1-e^{-ah}}{a} + \sigma_Z^{(1)}\varepsilon_{i+1}^{(2)} \tag{50}$$

† see Cooke and Gavin (2015) and literature therein.

Table 1. Regression results.

Regression Result					
	Coef	Value	$p-$values	Sgn.	Overall $p-$values
Interest Rate	Const.	3.442×10^{-5}	0.473		0.000
	$e^{-\kappa h}$	0.9960	0.000	(***)	
	Std.err	0.1116%			
Crude Oil	Const.	0.0259	0.043	(*)	0.000
	e^{-ah}	0.9938	0.000	(***)	
	Std.err	4.1676%			
	Cor.	3.4246%			
Gold	Const.	0.0165	0.037	(*)	0.000
	e^{-ah}	0.9978	0.000	(***)	
	Std.err	1.9818%			
	Cor.	−9.4518%			

where the variance

$$\left(\sigma_Z^{(1)}\right)^2 = \mathrm{var}\Bigg(\int_0^h \sigma_r \frac{e^{-\kappa(h-s)} - e^{-a(h-s)}}{a-\kappa} + \sigma_S \rho e^{-a(h-s)}\,dW_1(t_i+s) + \int_0^h \sigma_S\sqrt{1-\rho^2}e^{-a(h-s)}W_2(t_i+s)\Bigg)$$
$$= \frac{\sigma_r^2}{(a-\kappa)^2}\left(\frac{1-e^{-2\kappa h}}{2\kappa} - \frac{2(1-e^{-(\kappa+a)h})}{\kappa+a} + \frac{1-e^{-2ah}}{2a}\right) + \frac{2\rho\sigma_r\sigma_S}{a-\kappa}\left(\frac{1-e^{-(\kappa+a)h}}{\kappa+a} - \frac{1-e^{-2ah}}{2a}\right) + \sigma_S^2\frac{1-e^{-2ah}}{2a}, \quad (51)$$

and covariance between residuals is

$$\mathrm{cov}(\sigma_r^{(1)}\varepsilon_i^{(1)},\ \sigma_Z^{(1)}\varepsilon_i^{(2)}) = \mathrm{cov}\Bigg(\sigma_r\int_0^h e^{-\kappa(h-s)}\,dW_1(t_i+s), \times \int_0^h \sigma_r\frac{e^{-\kappa(h-s)}-e^{-a(h-s)}}{a-\kappa} + \sigma_S\rho e^{-a(h-s)}\,dW_1(t_i+s) + \int_0^h \sigma_S\sqrt{1-\rho^2}e^{-a(h-s)}W_2(t_i+s)\Bigg)$$
$$= \frac{\sigma_r^2}{a-\kappa}\left(\frac{1-e^{-2\kappa h}}{2\kappa} - \frac{1-e^{-(\kappa+a)h}}{\kappa+a}\right) + \rho\sigma_r\sigma_S\frac{2(1-e^{-(\kappa+a)h})}{\kappa+a}. \quad (52)$$

The two regressions are intended to compute the coefficients $(\kappa, \sigma_r, \bar{r}^{\mathbb{P}}, a, \lambda_S, \sigma_S, \rho)$, 7 parameters in total.

Table 1 gives the regression result for the coefficients. Only $p-$values of const. of interest rate shows no significance, because of long-term quantitative easing between 2009 and 2015.

Along the lines of Appendix B in Flor and Larsen (2013), we compute the infinite yield as $y_\infty = \bar{r} - \frac{\sigma_r^2}{2\kappa^2}$ and the market price of risk of the Bond as:

$$\lambda_r = \frac{\kappa}{\sigma_r}\left(y_\infty + \frac{\sigma_r^2}{2\kappa^2} - \bar{r}^{\mathbb{P}}\right),$$

where $y_\infty = 3.2702\%$ can be represented as the mean value of the long-term yield. From the perspective of $p-$value, the coefficients are all significant.

Table 2 summarizes all parameters from the regressions. For a comparison to existing literature, see Appendix 5.

A visual comparison between the estimates for gold and those for oil (WTI), show important differences between these two commodities. In particular, volatility of WTI price is

Table 2. Parameters of the problem.

Parameters			
Surplus	Premium percentage	c	0.1
	Claim rate	μ	5
	Claim size	e^{-Z}	$Z \sim$ Log-normal $(-4, 1)$
	Initial surplus	$X(0)$	10
Interest Rate	Mean-reverting rate	κ	0.2062
	Risk-neutral measure interest	$\bar{r}$	3.3468%
	Real-world measure interest	$\bar{r}^{\mathbb{P}}$	0.8699%
	Market price of risk	λ_r	0.6331
	Volatility	σ_r	0.8067%
	Initial interest rate	$r(0)$	2%
Crude Oil	Market price of risk	λ_S	4.6109
	Mean-reverting rate	a	0.3229
	Volatility	σ_S	30.1460%
	Correlation	ρ	3.4246%
	Initial price (bear)	$S(0)$	110
	Initial price (bull)	$S(0)$	50
Gold	Market price of risk	λ_S	6.0297
	Mean-reverting rate	a	0.1138
	Volatility	σ_S	14.3068%
	Correlation	ρ	−9.4518%
Objective Function	Time horizon	T	10
	Relative risk aversion	γ	4
	Ambiguity parameter 1	β_1	3
	Ambiguity parameter 2	β_2	3

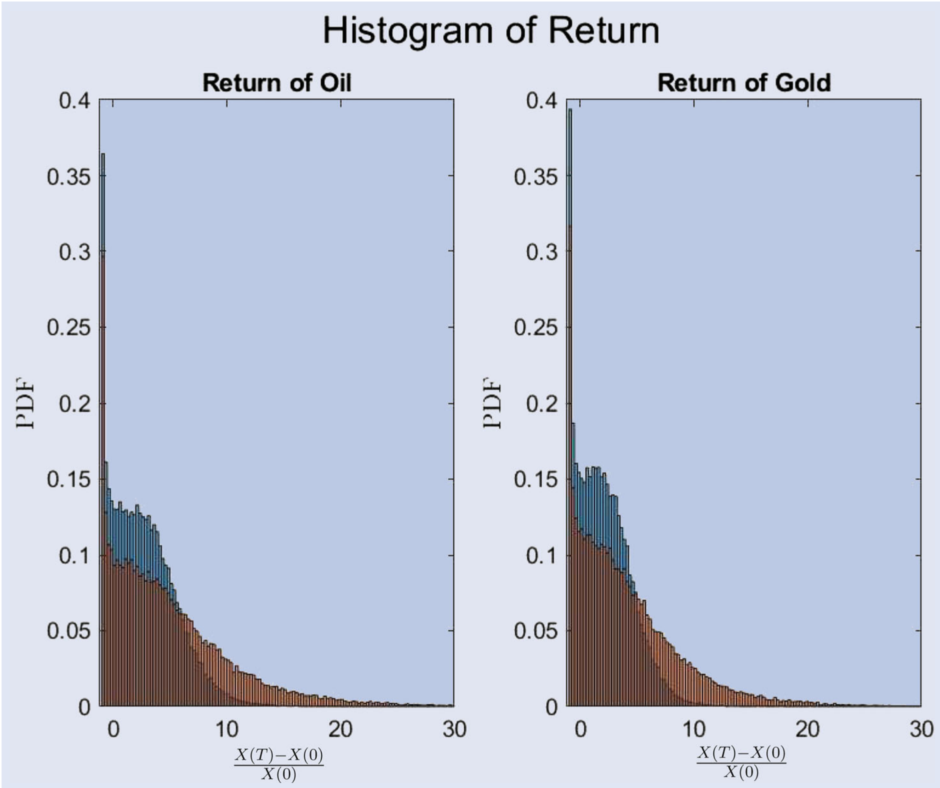

Figure 1. Density of $\frac{X(T)-X(0)}{X(0)}$ for $\beta_1 = \beta_2 = 3$ (blue) and $\beta_1 = \beta_2 = 0$ (orange). Oil portfolio on the left, gold portfolio on the right.

Table 3. Outcome of simulation results.

Commodity	Exp. Return	Std. Deviation	Skewness	Kurtosis	$\pi_S(0)$
Oil (Ambiguity)	13.25%	26.12%	0.7756	3.4581	−14.45%
Oil (Non-ambiguity)	19.86%	56.82%	1.8810	9.2383	−21.99%
Gold (Ambiguity)	11.90%	23.03%	0.7709	3.5443	54.85%
Gold (Non-ambiguity)	17.96%	48.90%	2.0313	11.2388	87.36%

much larger with a higher reversion frequency, on the other hand correlation between gold and interest rate is slightly larger and negative than that of WTI and interest rates. This variety in behaviour helps cement the case for representativeness of our choices of commodity.

4.2. Optimal strategy and terminal surplus

In this section we use the representation of the optimal terminal surplus and allocation in Proposition 2 and Theorem 1 to understand their behaviour. For this we simulate via Monte Carlo the optimal strategy $\boldsymbol{\pi}^*$ and the terminal surplus $X(T)$ given an initial value $\boldsymbol{y}$ and paths for interest rate $r(t)$ and log-price $Z(t)$. For the simulation, we use 50,000 paths and daily time step $\Delta t = 1/252$. We pick up the initial price of $S(0) = 90$ and 1200 for crude oil and gold respectively.

Figure 1 show the density of the terminal return rate $\frac{X(T)-X(0)}{X(0)}$ for the ambiguous (blue) and non-ambiguous (orange) investors, as well as oil-based portfolio (left) and gold-based portfolio on the right. As expected, the absence of ambiguity-aversion ($\beta_1 = \beta_2 = 0$) leads to more extreme behaviour (higher probability on tails). This is also confirmed with the statistics for all four cases (oil, gold; absence and presence of ambiguity) reported in Table 3. Table 3 demonstrates larger moments across the board for non-ambiguous investors (twice in value as those of ambiguity-averse companies). This is also the case for allocations on the commodity, which is 1.5 times as high for non-ambiguous agents. This pictures a less risky and aggressive behaviour for ambiguity-averse investors.

Next we study optimal strategies for two market scenarios: first a bear market, defined as a situation where the initial price is larger than the mean reverting level, therefore the market is very likely to drop. Second a bull market case, initial price lesser than the mean reverting level, hence it is highly likely for prices to go up. Figures 2 and 3 report the optimal investments in Bonds, Commodity and Bank account for a path representing bear market conditions and a path representing a bull market, respectively.

In a bull market, the percentage of wealth invested in the oil portfolio drops significantly from a maximum of long 80% on the commodity to shorting 20%. The situation partially reverse in bear market conditions as per Figure 3. Here the investor allocates more on the commodity at the expense of less investment in Bonds, this is to take advantage of the better returns in the commodity boom. The cash account does not show a clear pattern.

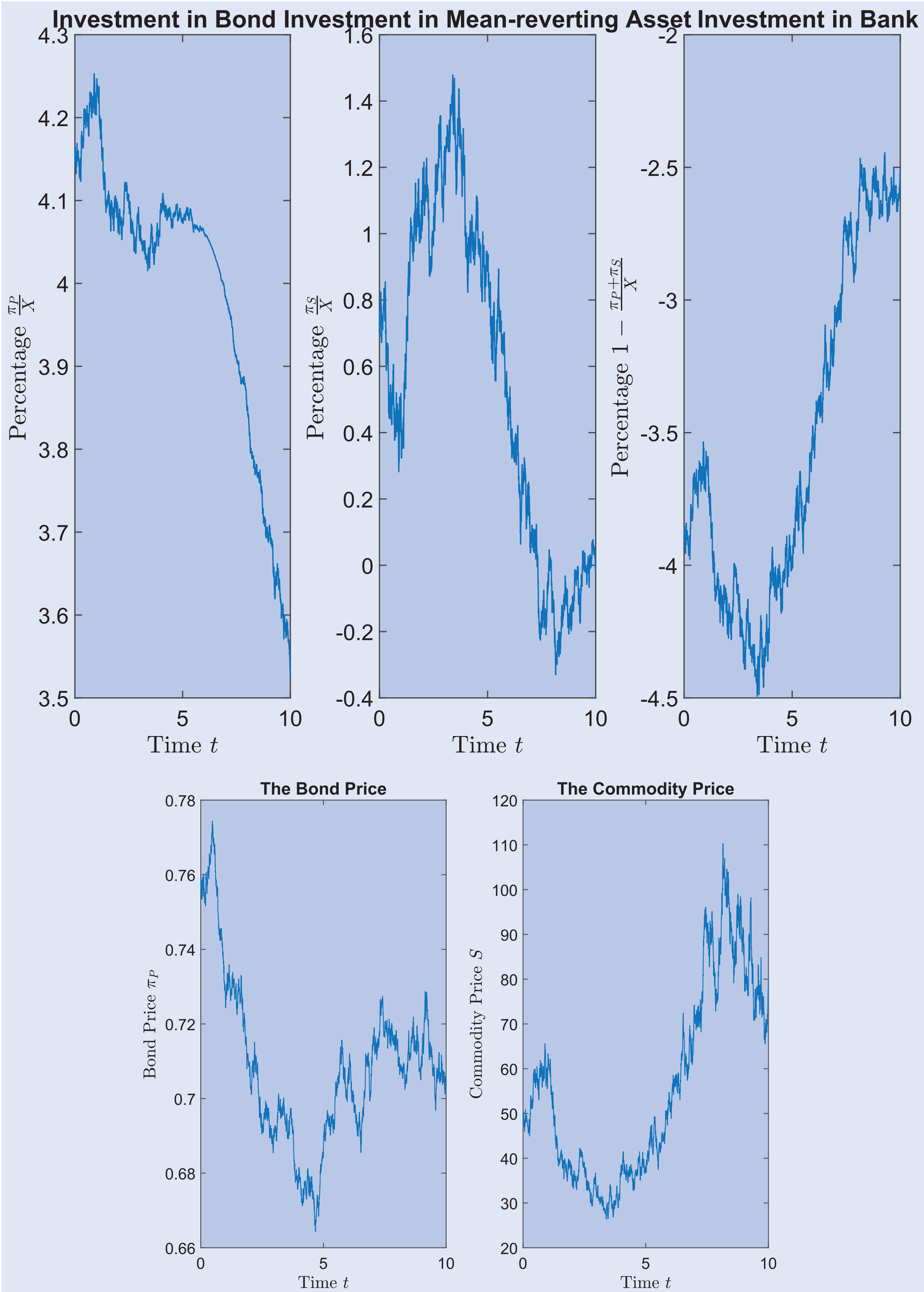

Figure 2. Plot of investment in a bull market, oil portfolio.

4.3. Equivalent losses in suboptimal analysis

Here we study equivalent losses L^{π^s} from the suboptimal strategies described in Section 3.

4.3.1. Group I: ignoring ambiguity, commodities versus stocks. In this section we compare the wealth-equivalent utility loss due to ignoring ambiguity for the two commodities at hand: crude oil and gold. Figure 4 captures the oil portfolio while Figure 5 is about the gold portfolio.

The investor's ambiguity aversion level is depicted in the x-axis while the y-axis displays the wealth-equivalent losses from ignoring such level of ambiguity, e.g using the 'suboptimal' strategy that assumes $\hat{\beta}_1 = \hat{\beta}_2 = 0$. Recall β_1 is about ambiguity-aversion on the interest rate and therefore Bond market, while β_2 captures the ambiguity on the targeted commodity. It is remarkable to see that ignoring ambiguity about

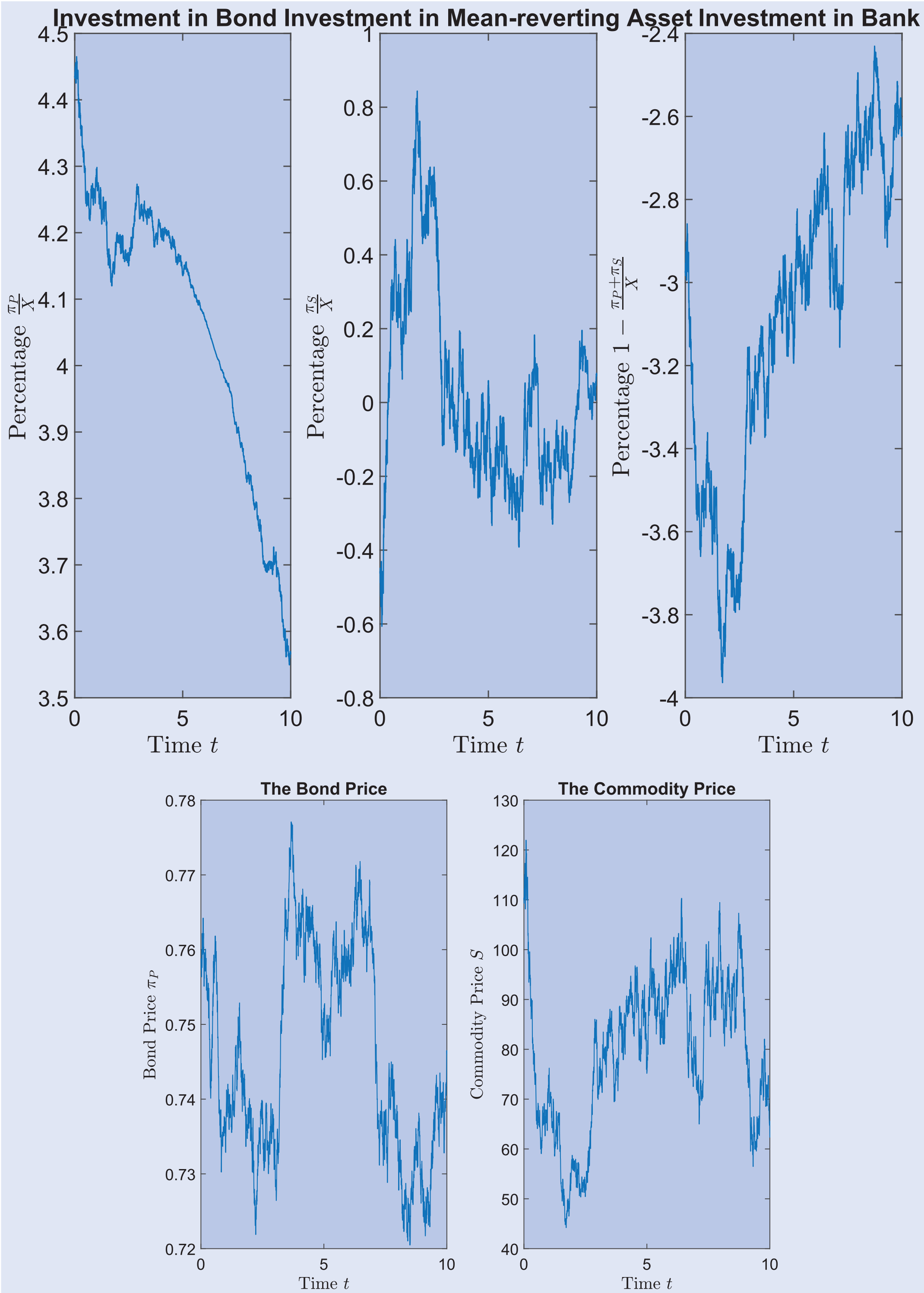

Figure 3. Plot of investment in a bear market, oil portfolio.

the commodity could be quite detrimental for an investor, the right side of Figures 4 and 5 demonstrate quite large losses for even low aversion levels. The damaging effect of ignoring ambiguity depends of the mean-reverting rate a. Heavier mean-reverting effect leads to a more severe wealth-equivalent loss. As for ignoring ambiguity on Bonds, the left hand side of Figures 4 and 5 show a larger impact in the gold portfolio compared to the oil portfolio. This can be attributed to the larger exposure to Bonds in the former portfolio compared to the later.

For the purpose of assessing the impact of ignoring ambiguity in the asset class of commodities compared to the asset class of stocks, we also include the effect of ignoring ambiguity assuming the underlying follows a Geometric Brownian motion (GBM). This is comparable to Flor and Larsen (2013) where the authors study Bonds and Stocks. Such setting can be accommodated here by setting the mean-reverting rate $a = 0$ and lowering the excess return hence treating the dynamics of commodities as a GBM-growth asset. Note, the resulting stocks would have acceptable volatility values (14.3% and

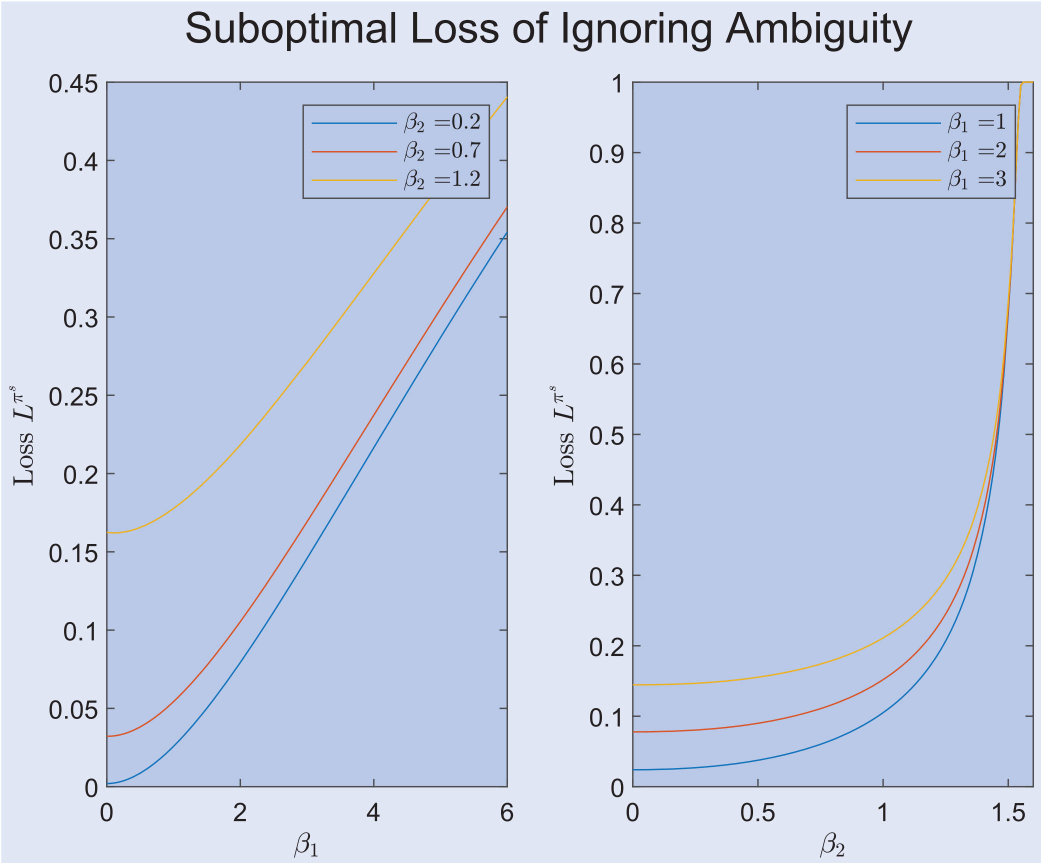

Figure 4. Plot of suboptimal loss given $\boldsymbol{\beta}$ using WTI crude oil price

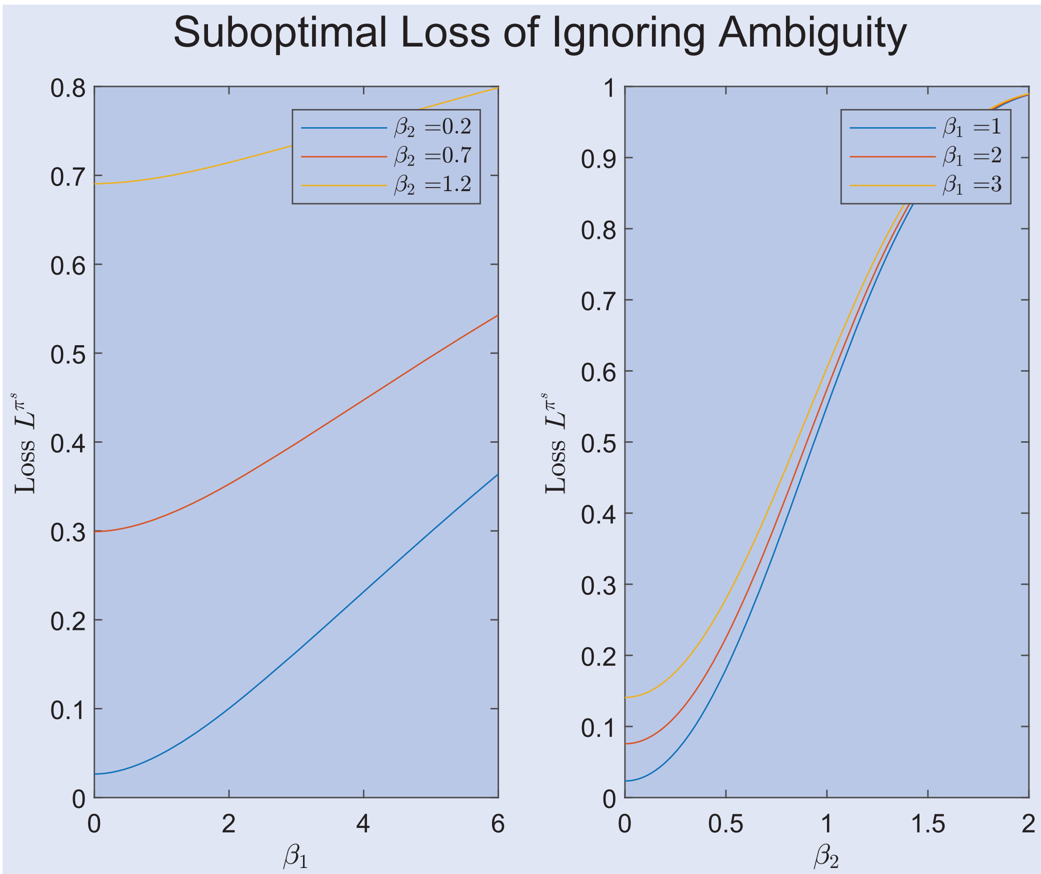

Figure 5. Plot of suboptimal loss given $\boldsymbol{\beta}$ using gold price

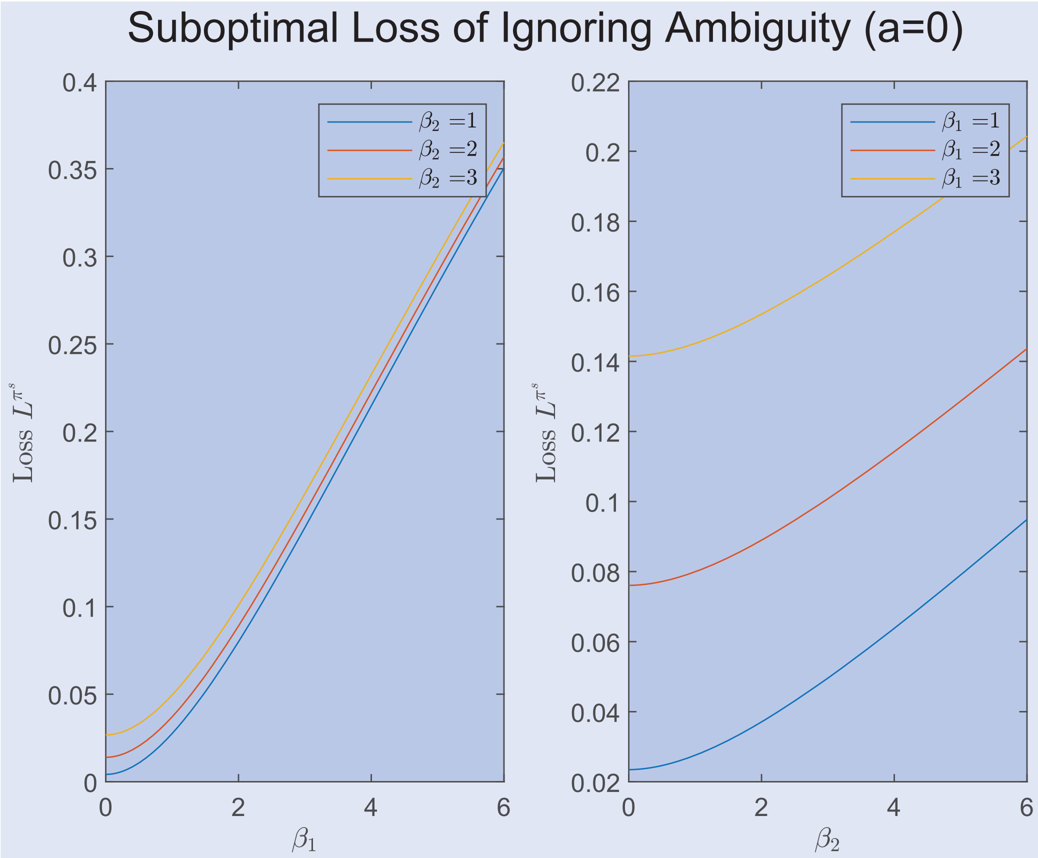

Figure 6. Plot of suboptimal loss given $\boldsymbol{\beta}$ and $a = 0$ using WTI crude oil price

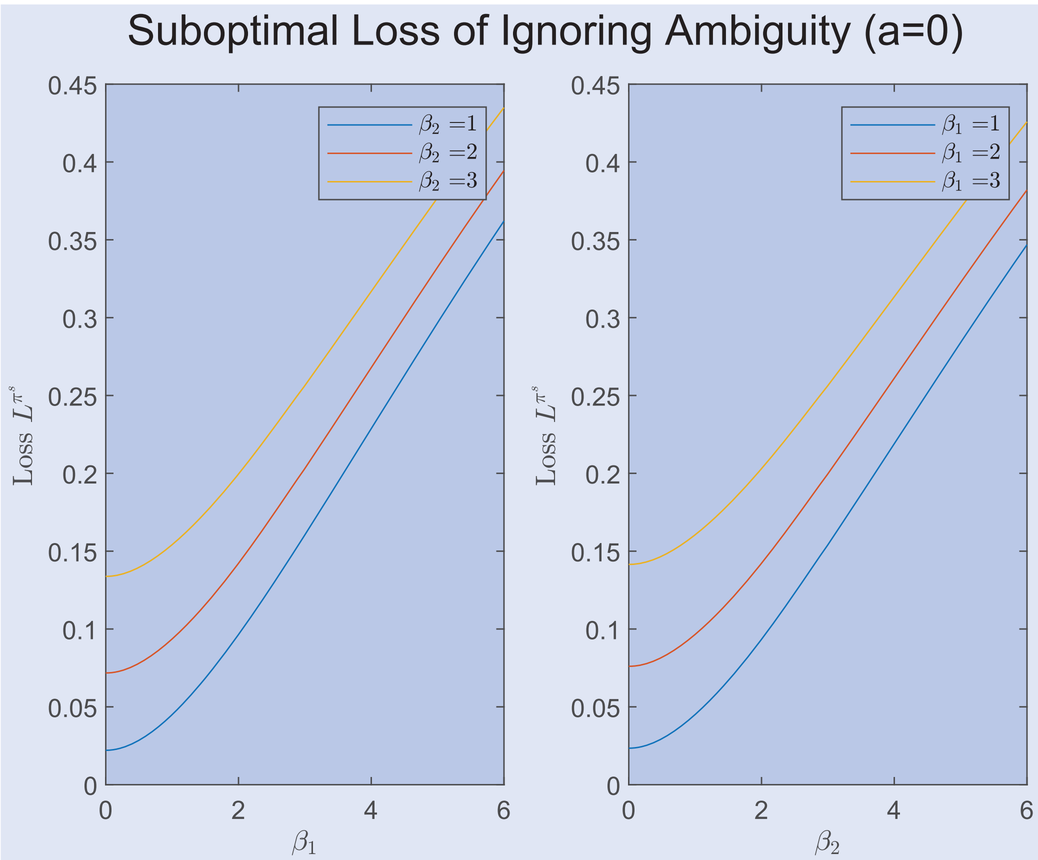

Figure 7. Plot of suboptimal loss given $\boldsymbol{\beta}$ and $a = 0$ using gold price

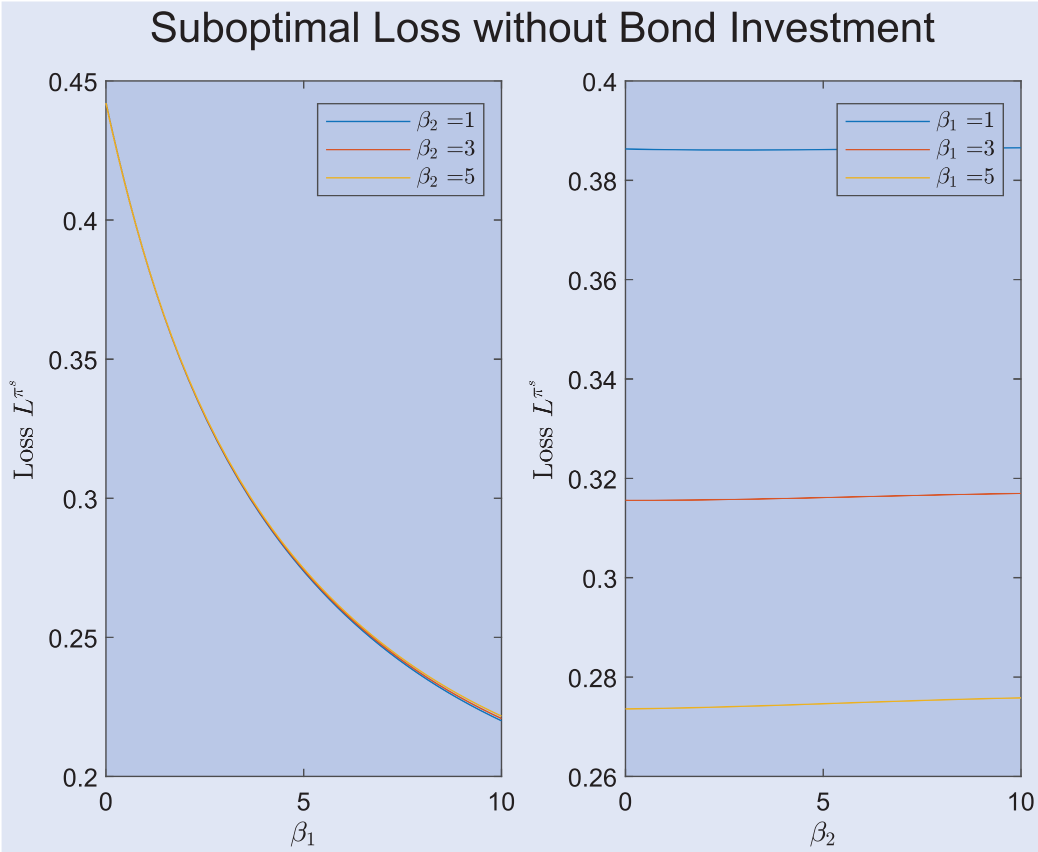

Figure 8. Plot of suboptimal loss without bond using crude oil price investment

30.1%), as well as reasonable excess returns (7.1647% for oil and 9.3748% for gold). The suboptimal losses in this 'stock market scenario' are shown in Figures 6 and 7. Comparing Figures 4 to 6 (oil price) and also Figures 5 to 7 (gold), we can see that for the same level of ambiguity-aversion ($\beta = 1$) the wealth-equivalent utility losses are substantially larger in commodity markets compare to stock markets. The difference can be around the order of four-fold.

4.3.2. Group II: incomplete market: no bond investment. Here we explore the performance of suboptimal strategies obtained due to incomplete markets, i.e. failing to invest in Bonds. Figures 8 and 9 show the wealth-equivalent losses as a function of ambiguity levels: β_1 and β_2, for oil and gold portfolios respectively.

β_1 play a significant role in incomplete markets, with losses of up to 45% (3-% for gold) due to market incompleteness. On the other hand, as β_1 increases the wealth-equivalent losses decrease. This can explained as follows, ambiguity-aversion on $r(t)$ leads to higher expected short rates; in the absence of Bonds, the investor can not take advantage of this improvement in performance hence suffering even worse losses.

Not surprisingly ignoring β_2 (right hand side of Figure 8) plays little role for crude oil, see the right plot in Figure 8. This is because β_2 affects only the expected return of the asset, which has very small correlation (3.42%) with interest rate. However, for larger correlations (−9.45%), the spillover is more important and as Figure 9 shows, one can detect significant impact of up to 25% in losses.

4.3.3. Group III: incorrect parameters. This section focuses on wealth-equivalent losses due to mis-specification of important parameters, in particular market prices of risk, volatilities and correlations. The analysis here displays results only for oil prices, similar observations were produced with the gold portfolio.

Figure 10 shows the losses due to wrongly calibrated market prices of risk. We assume the correct values are those presented in Table 2, then the x-axis represents the chosen values: either $\hat{\lambda}_r$ (left plot) or $\hat{\lambda}_S$ (right plot). We would not be incurring in an error only if $\hat{\lambda}_r = \lambda_r$ or $\hat{\lambda}_S = \lambda_S$, in such cases we would have selected the right parameters. As one can observe, the losses are far more sensitive to a wrong choice of $\hat{\lambda}_S$ than of $\hat{\lambda}_r$. This highlights the importance of a proper estimation exercise.

We also study the losses due to mis-specification of volatilities: $\hat{\sigma}_r$ and $\hat{\sigma}_S$. The patterns in losses are similar to those encountered before, emphasizing the importance of estimating commodity parameters more precisely than Bond's parameters. Interestingly, wrongly assuming lower values of volatilities could be far more consequential than assuming incorrect large values, which tell a story of better overestimating than underestimating the risk.

Lastly, we study the impact of correlation between Bonds and commodities. There is a large body of literature on the absence of such dependence. Hence we assume the correct

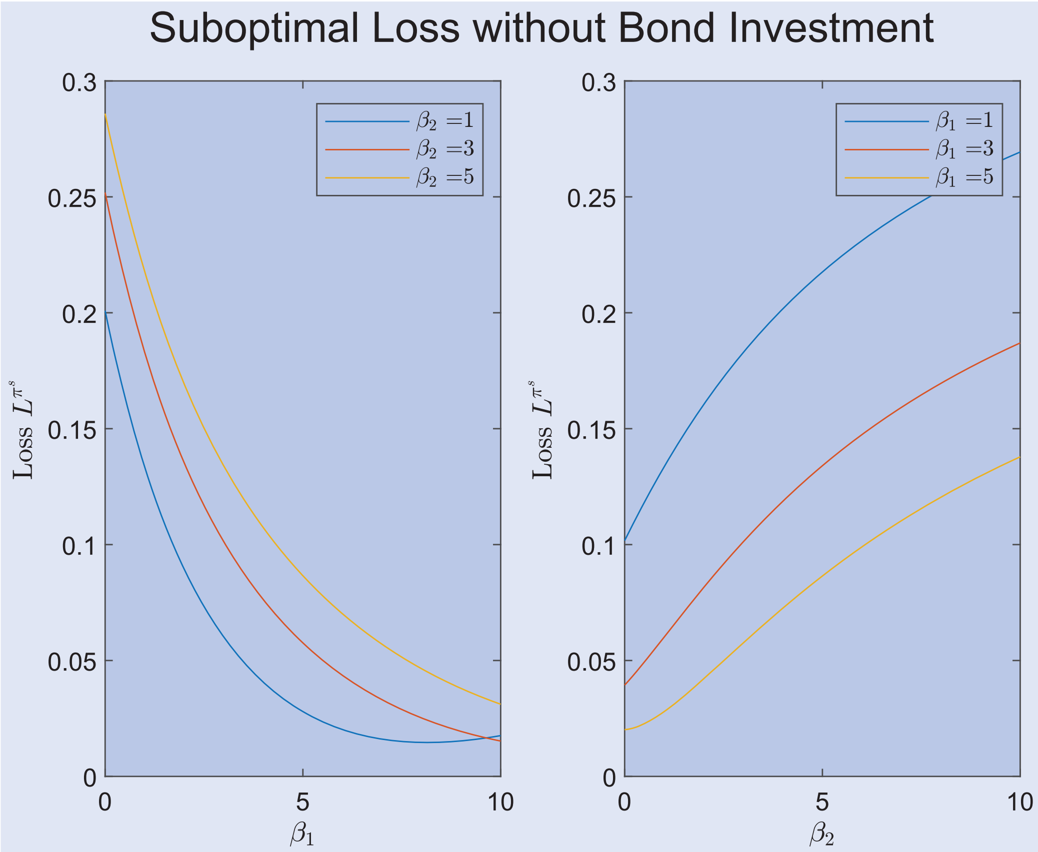

Figure 9. Plot of suboptimal loss without bond using gold investment

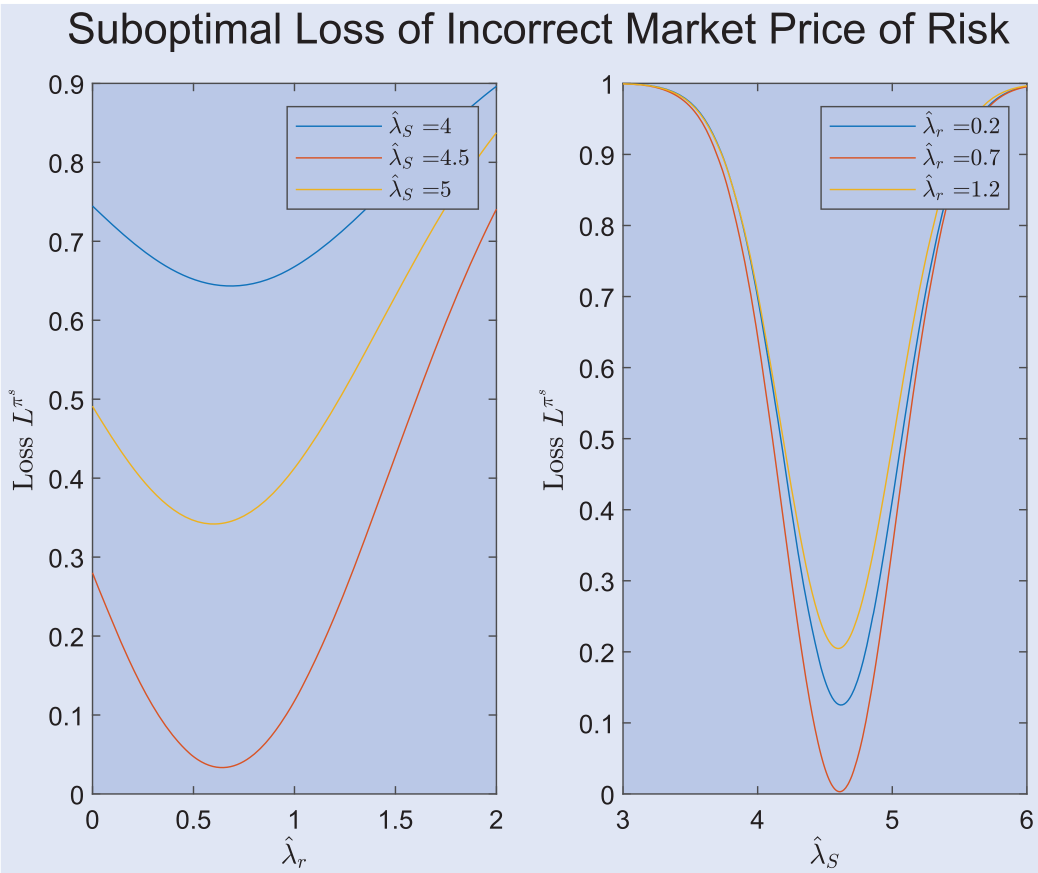

Figure 10. Plot of suboptimal loss given $\hat{\lambda}_r, \hat{\lambda}_S$

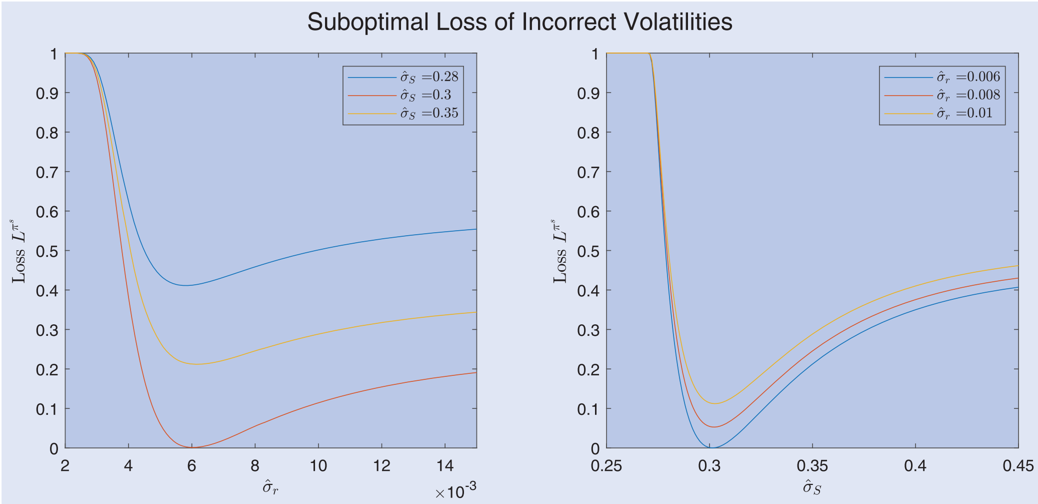

Figure 11. Plot of suboptimal loss given $\hat{\sigma}_r, \hat{\sigma}_S$

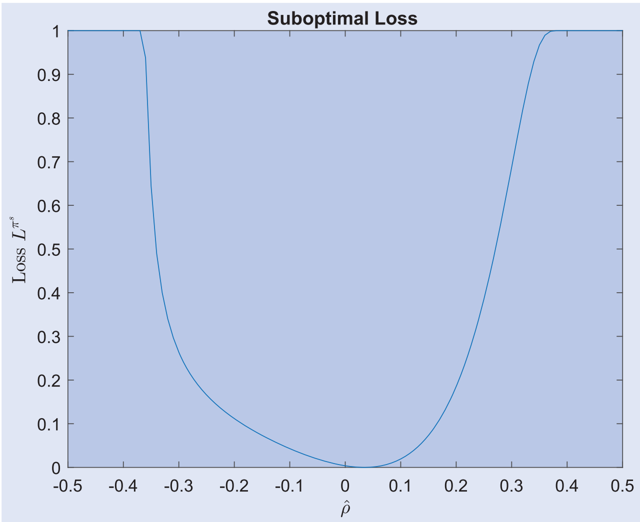

Figure 12. Plot of suboptimal loss given $\hat{\rho}$

value is $\rho = 3.4246\%$ and plot the losses due to incorrectly assuming a value of $\hat{\rho}$. The further away of $\hat{\rho}$ with the true ρ, the higher the utility loss. Incorrect large positive (> 0.4) or negative (< -0.4) correlation creates heavy loss to 100%, while no obvious distinguished pattern is shown between the negative part and the positive part.

5. Conclusion

In this paper, we consider a robust portfolio optimization of a company's surplus consisting of a Bond and a mean-reverting asset representing a commodity. The surplus is assumed to follow a geometric Cramér-Lundberg distribution. We also assume an interest rate to be a Vasicek model and the commodity to be an exponential-OU process. Via a maximization of CRRA utility and applying ambiguity-aversion entropy, we generated analytical, closed-form solutions for the optimal investment, worst-case change of measure, optimal wealth and value function for the insurer (investor).

Thanks to the analytical representations we explore the behaviour of optimal solutions and the impact of various meaningful suboptimal strategies on the investor's portfolio. For these exercises we consider and estimate two assets representatives of commodity market: oil prices and gold prices. Some important findings are: ignoring ambiguity either on Bonds or commodities could lead to drastic wealth-equivalent losses, harsher if ignoring ambiguity-aversion on commodities than on bonds. Even more, ignoring ambiguity-aversion on commodities is more damaging than ignoring such on the asset class of stocks (GBMs). As reported by many authors, working on an incomplete market could be detrimental for a portfolio, we demonstrate losses of up to 45% due to incompleteness in the commodity market. Lastly, mis-specifications in the parameters toward either larger correlation, smaller variances or incorrect market price of commodity risk, could lead to drastically large wealth-equivalent losses, hence an unnecessary under-performances of the investor portfolio.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Junhe Chen http://orcid.org/0000-0001-8258-0456
M. Escobar-Anel http://orcid.org/0000-0001-9691-4322

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Appendices

Appendix 1. Proof of Proposition 1

The first derivative of the HBJI w.r.t. $\boldsymbol{u}$ leads to:

$$\begin{aligned}0 &= (-J_{\boldsymbol{y}}^T\boldsymbol{\sigma} - J_x\boldsymbol{\pi}^T\boldsymbol{B}\boldsymbol{\sigma}) + (1-\gamma)J(\boldsymbol{u}^*)^T\boldsymbol{\beta}^{-1}\\ \boldsymbol{u}^* &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}.\end{aligned} \tag{A1}$$

More explicitly,

$$\begin{aligned}u_1^* &= \frac{\beta_1(\sigma_S\rho J_z + \sigma_r J_r + (\pi_S\sigma_S\rho - \pi_P I_\tau\sigma_r)J_x)}{(1-\gamma)J}\\ u_2^* &= \frac{\beta_2\sqrt{1-\rho^2}(\sigma_S J_z + \pi_S\sigma_S J_x)}{(1-\gamma)J}.\end{aligned} \tag{A2}$$

Plugging $\boldsymbol{u}$ into Equation (21), produces:

$$\begin{aligned}&\sup_{\boldsymbol{\pi}\in\Pi}\Bigg\{J_t + \left[(\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y}) - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}\right]^T J_{\boldsymbol{y}}\\ &\quad + \frac{1}{2}\mathrm{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) + \boldsymbol{\pi}^T\boldsymbol{B}\Sigma J_{x\boldsymbol{y}}\\ &\quad + \left[(c+r)x + \boldsymbol{\pi}^T\boldsymbol{B}\left(\boldsymbol{b}(\boldsymbol{y}) - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{(1-\gamma)J}\right)\right]J_x\\ &\quad + \frac{J_{xx}}{2}\boldsymbol{\pi}^T\boldsymbol{B}^T\Sigma\boldsymbol{B}\boldsymbol{\pi}\\ &\quad + \frac{(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})^T\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_{\boldsymbol{y}} + J_x\boldsymbol{B}\boldsymbol{\pi})}{2(1-\gamma)J}\Bigg\}\\ &\quad + \mu E[J(t, x(1-Y), r, z) - J] = 0.\end{aligned} \tag{A3}$$

This is also quadratic w.r.t. $\boldsymbol{\pi}$. Taking derivative for $\boldsymbol{\pi}$ again, we can obtain

$$\begin{aligned}0 &= \boldsymbol{B}\left(J_x\left(\boldsymbol{b}(\boldsymbol{y}) - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right)\\ &\quad + \boldsymbol{B}\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right)\boldsymbol{B}\boldsymbol{\pi}^*\\ \boldsymbol{\pi}^* &= -\boldsymbol{B}^{-1}\left(J_{xx}\Sigma - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T\right)^{-1}\\ &\quad\times\left(J_x\left(\boldsymbol{b}(\boldsymbol{y}) - \frac{\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) + \Sigma J_{x\boldsymbol{y}}\right).\end{aligned} \tag{A4}$$

By plugging u_1^* and u_2^* into from Equation (A2), the explicit formulation is

$$\boldsymbol{\pi}^* = \begin{bmatrix}\pi_P^*\\ \pi_S^*\end{bmatrix} = \boldsymbol{P}^{-1}\boldsymbol{q} \tag{A5}$$

where the matrix $\boldsymbol{P}$ and vector $\boldsymbol{q}$ are

$$\begin{aligned}\boldsymbol{P} &= \begin{bmatrix} I_\tau^2\sigma_r^2\left(J_{xx} - \frac{\beta_1}{(1-\gamma)J}J_x^2\right) & -\rho\sigma_S\sigma_r I_\tau\left(J_{xx} - \frac{\beta_1}{(1-\gamma)J}J_x^2\right)\\ -\rho\sigma_S\sigma_r I_\tau\left(J_{xx} - \frac{\beta_1}{(1-\gamma)J}J_x^2\right) & \sigma_S^2\left(J_{xx} - \frac{\beta_1\rho^2 + \beta_2(1-\rho^2)}{(1-\gamma)J}J_x^2\right)\end{bmatrix}\\ \boldsymbol{q} &= \begin{bmatrix} I_\tau\left(\rho\sigma_S\sigma_r\left(J_{xz} - \frac{\beta_1}{(1-\gamma)J}J_xJ_z\right) + \sigma_r^2\left(J_{xr} - \frac{\beta_1}{(1-\gamma)J}J_xJ_r\right)\right.\\ \left. -\lambda_r\sigma_r J_x\right)\\ \sigma_S^2\left(\frac{\beta_1\rho^2 + \beta_2(1-\rho^2)}{(1-\gamma)J}J_xJ_z - J_{xz}\right) + \rho\sigma_S\sigma_r\left(\frac{\beta_1}{(1-\gamma)J}J_xJ_r - J_{xr}\right)\\ -(\lambda_S\sigma_S - az)J_x\end{bmatrix}.\end{aligned} \tag{A6}$$

Plugging $\boldsymbol{\pi}^*$ into the HJBI PDE, we can obtain the final PDE in Equation (24).

Appendix 2. Proof of 2

Inserting the ansatz in Equation (28) into the PDE in Equation (24) and grouping conveniently, we obtain a system of ODE

$$\begin{aligned}&M_0' + \boldsymbol{y}^T\boldsymbol{M}_1' + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2'\boldsymbol{y} + \boldsymbol{\theta}^T\boldsymbol{M}_1 + \boldsymbol{y}^T(\boldsymbol{M}_2^T\boldsymbol{\theta} - \boldsymbol{A}^T\boldsymbol{M}_1)\\ &\quad - \frac{1}{2}\boldsymbol{y}^T(\boldsymbol{A}^T\boldsymbol{M}_2 + \boldsymbol{M}_2^T\boldsymbol{A})\boldsymbol{y}\\ &\quad + \frac{1}{2}\mathrm{tr}(\boldsymbol{M}_2\Sigma) + c(1-\gamma) + \boldsymbol{y}^T\boldsymbol{e}_1(1-\gamma)\\ &\quad + \frac{1}{2}\boldsymbol{M}_1^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\\ &\quad + \boldsymbol{y}^T\boldsymbol{M}_2^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1 + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{y}\\ &\quad + \frac{1}{2}\left(\boldsymbol{\lambda} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\right)^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I} + \boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\\ &\quad\times\left(\boldsymbol{\lambda} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\right)\end{aligned}$$

$$+\boldsymbol{y}^T\left(\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2-a\boldsymbol{E}_2\right)^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}$$
$$\times\left(\boldsymbol{\lambda}+\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\right)$$
$$+\frac{1}{2}\boldsymbol{y}^T\left(\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2-a\boldsymbol{E}_2\right)^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}$$
$$\times\left(\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2-a\boldsymbol{E}_2\right)\boldsymbol{y}$$
$$+\mu\mathbb{E}[(1-Y)^{1-\gamma}-1]=0. \tag{A7}$$

Arranging constant parts, first order part $\boldsymbol{y}^T(\,\cdot\,)$ and second order part $\frac{1}{2}\boldsymbol{y}^T(\,\cdot\,)\boldsymbol{y}$ can result in the Riccati ODE:

$$\begin{cases} M_0'+c(1-\gamma)+\mu\mathbb{E}[(1-Y)^{1-\gamma}-1]+\frac{1}{2}\mathrm{tr}(\boldsymbol{M}_2\Sigma) \\ \quad+\frac{1}{2}\boldsymbol{\lambda}^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\boldsymbol{\lambda} \\ \quad+\boldsymbol{M}_1^T\left[\boldsymbol{\theta}+\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\boldsymbol{\sigma}^{-1}\boldsymbol{\lambda}\right] \\ \quad+\frac{1}{2}\boldsymbol{M}_1^T\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\left(\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}+\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\right) \\ \quad\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1=0 \\ M_0(T)=0, \end{cases}$$

$$\begin{cases} \boldsymbol{M}_1'-a\boldsymbol{E}_2\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\boldsymbol{\lambda} \\ \quad-\left(\boldsymbol{A}^T+a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma})\boldsymbol{\sigma}^T\right)\boldsymbol{M}_1 \\ \quad+\boldsymbol{M}_2^T\left(\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\boldsymbol{\sigma}^{-1}\boldsymbol{\lambda}+\boldsymbol{\theta}\right)+(1-\gamma)\boldsymbol{e}_1 \\ \quad+\boldsymbol{M}_2^T\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\left(\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}+\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\right) \\ \left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1=0 \\ \boldsymbol{M}_1(T)=\boldsymbol{0}_{2\times1} \end{cases}$$

$$\begin{cases} \boldsymbol{M}_2'+a^2\boldsymbol{E}_2\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\boldsymbol{E}_2 \\ \quad-\left(\boldsymbol{A}^T+a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)\boldsymbol{M}_2 \\ \quad-\boldsymbol{M}_2^T\left(\boldsymbol{A}^T+a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^T \\ \quad+\boldsymbol{M}_2^T\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\left(\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}+\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\right) \\ \quad\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2=0 \\ \boldsymbol{M}_2(T)=\boldsymbol{0}_{2\times2} \end{cases} \tag{A8}$$

For simplicity, we set the notation

$$\boldsymbol{D}_0:=a^2\boldsymbol{E}_2\left(\boldsymbol{\sigma}(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma})\boldsymbol{\sigma}^T\right)^{-1}\boldsymbol{E}_2$$
$$\boldsymbol{D}_1:=-\left(\boldsymbol{A}^T+a\boldsymbol{E}_2(\boldsymbol{\sigma}^T)^{-1}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)$$
$$\boldsymbol{D}_2:=\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\left(\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}+\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\right)$$
$$\times\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T$$
$$\boldsymbol{C}_0:=(1-\gamma)\boldsymbol{e}_1-a\boldsymbol{E}_2\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\boldsymbol{\lambda}$$
$$\boldsymbol{C}_1:=\left(\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)^{-1}\boldsymbol{\sigma}^{-1}\boldsymbol{\lambda}+\boldsymbol{\theta}\right)$$
$$B_0=c(1-\gamma)+\mu\mathbb{E}[(1-Y)^{1-\gamma}-1]+\frac{1}{2}\mathrm{tr}(\boldsymbol{M}_2(t)\Sigma)$$
$$+\frac{1}{2}\boldsymbol{\lambda}^T\left(\boldsymbol{\sigma}\left(\frac{\gamma\boldsymbol{I}+\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\right)^{-1}\boldsymbol{\lambda}. \tag{A9}$$

to transform the Riccati ODEs in Equations (A8) into the three ODEs in Proposition 2.

Appendix 3. Proof of Lemma 2

We first rewrite the optimal change of measure $\boldsymbol{u}^*(t)$ after inserting explicit $\boldsymbol{\pi}^*(t)$

$$\boldsymbol{u}^*=\boldsymbol{A}_1(t)+\boldsymbol{A}_2(t)\boldsymbol{y}(t) \tag{A10}$$

where

$$\begin{aligned} \boldsymbol{A}_1(t)&=\frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T}{1-\gamma}\Bigg(\boldsymbol{M}_1(t)+(1-\gamma)\left(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\right)^{-1} \\ &\quad\times\left(\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1(t)+\boldsymbol{\lambda}\right) \\ \boldsymbol{A}_2(t)&=\frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T}{1-\gamma}\Bigg[\boldsymbol{M}_2(t)+(1-\gamma)\left(\boldsymbol{\sigma}(\gamma\boldsymbol{I}+\boldsymbol{\beta})\boldsymbol{\sigma}^T\right)^{-1} \\ &\quad\times\left(\boldsymbol{\sigma}\left(\boldsymbol{I}-\frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2(t)-a\boldsymbol{E}_2\right)\Bigg]. \end{aligned} \tag{A11}$$

Now we return to the lemma. In order to prove Novikov's condition, we rewrite

$$\begin{aligned} &\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T\|\boldsymbol{u}^*(t)\|^2\,\mathrm{d}t\right)\right] \\ &\quad=\mathbb{E}^P\Bigg[\exp\Bigg(\frac{1}{2}\int_0^T\boldsymbol{A}_1^T(t)\boldsymbol{A}_1(t)+2\boldsymbol{A}_1^T(t)\boldsymbol{A}_2(t)\boldsymbol{y} \\ &\quad+\boldsymbol{y}^T(t)\boldsymbol{A}_2^T(t)\boldsymbol{A}_2(t)\boldsymbol{y}(t)\,\mathrm{d}t\Bigg)\Bigg]. \end{aligned} \tag{A12}$$

Therefore, we need to prove the boundness of $\boldsymbol{A}_1(t),\boldsymbol{A}_2(t)$ and $\boldsymbol{y}(t)$. By ODE of $\boldsymbol{M}_1$ in Theorem 2, the Lipschitz condition of existence and uniqueness of $\boldsymbol{M}_1$ holds for $0\le t\le T$ as long as the boundness of Frobenius norm $\|\boldsymbol{M}_2\|_F<\infty$ holds. Therefore, $\|\boldsymbol{M}_1(t)\|<\infty$ is a well-defined vector function. This is confirmed also by the form of $\boldsymbol{A}_1(t)$ and $\boldsymbol{A}_2(t)$, i.e. $\|\boldsymbol{A}_1(t)\|<\infty$ and $\|\boldsymbol{A}_2(t)\|_F<\infty$ by Minkovski's inequality and Schwartz's inequality.

Now we plug $\boldsymbol{u}^*(t)$ to obtain the dynamic of $\boldsymbol{y}(t)$ in worst-case scenario

$$\mathrm{d}\boldsymbol{y}(t) = [\boldsymbol{\theta} - \boldsymbol{A}_1(t) - (\boldsymbol{A} + \boldsymbol{A}_2(t))\,\boldsymbol{y}(t)]\ \mathrm{d}t + \boldsymbol{\sigma}\ \mathrm{d}\boldsymbol{W}^Q(t). \tag{A13}$$

The Lipschitz condition for $\boldsymbol{y}(t)$ to be well-defined also holds. Therefore,

$$\mathbb{E}\left[\int_0^T \|\boldsymbol{y}(t)\|^2\ \mathrm{d}t\right] < \infty. \tag{A14}$$

Using Minkovski's inequality and Schwartz's inequality again for Equation (A12), we successfully prove

$$\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \left\|\boldsymbol{u}^*(t)\right\|^2\ \mathrm{d}t\right)\right] < \infty. \tag{A15}$$

With the explicit formula for $\boldsymbol{y}(t)$ in the worst-case scenario, the sup-inf problem degenerates into an HJB problem studied in Chiu and Wong (2013). The proof of the second part follows along the lines of their Section 3.2. The only difference is that in their paper the mean-reverting matrix A is constant, while our mean-reverting matrix is $\boldsymbol{A} + \boldsymbol{A}_2(t)$ in the worst-case scenario.

Appendix 4. Proof of Proposition 3

Taking derivative on the left side of Equation 38, we can obtain

$$\begin{aligned} 0 &= -\boldsymbol{\sigma}^T(J_y + J_x\boldsymbol{B}\boldsymbol{\pi}^s) + (1-\gamma)\boldsymbol{\beta}^{-1}\boldsymbol{u}^* \\ \boldsymbol{u}^* &= \frac{\boldsymbol{\beta}\boldsymbol{\sigma}^T(J_y + J_x\boldsymbol{B}\boldsymbol{\pi}^s)}{(1-\gamma)J}. \end{aligned} \tag{A16}$$

Plugging $\boldsymbol{u}^*$ back into the HJB in Equation (38), we obtain

$$\begin{aligned} &J_t + (\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y})^T J_{\boldsymbol{y}} + (c+r)xJ_x - \frac{J_{\boldsymbol{y}}^T\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{2(1-\gamma)J} + \frac{1}{2}\mathrm{tr}(J_{\boldsymbol{y}\boldsymbol{y}^T}\Sigma) \\ &+ (\boldsymbol{\pi}^s)^T\boldsymbol{B}\left((\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y})J_x + \Sigma J_{x\boldsymbol{y}} - \frac{J_x\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T J_{\boldsymbol{y}}}{(1-\gamma)J}\right) \\ &+ \frac{1}{2}(\boldsymbol{\pi}^s)^T\boldsymbol{B}\boldsymbol{\sigma}\left(J_{xx}\boldsymbol{I} - \frac{J_x^2}{(1-\gamma)J}\boldsymbol{\beta}\right)\boldsymbol{\sigma}^T\boldsymbol{B}\boldsymbol{\pi}^s \\ &+ \mu E[J(t, x(1-Y), r, z) - J] = 0. \end{aligned} \tag{A17}$$

Using the ansatz $J(t,x,r,z) = \frac{x^{1-\gamma}}{1-\gamma}\exp(M_0^{\boldsymbol{\pi}^s}(t) + \boldsymbol{M}_1^{\boldsymbol{\pi}^s}(t)\boldsymbol{y} + \frac{1}{2}\boldsymbol{y}^T\boldsymbol{M}_2^{\boldsymbol{\pi}^s}(t)\boldsymbol{y})$ and the form $\boldsymbol{\pi}^s = x(\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y})$ where $\boldsymbol{h} := \boldsymbol{h}(t)$ and $\boldsymbol{H} := \boldsymbol{H}(t)$, we obtain

$$\begin{aligned} &(M_0^{\boldsymbol{\pi}^s})' + \boldsymbol{y}^T(\boldsymbol{M}_1^{\boldsymbol{\pi}^s})' + \frac{1}{2}\boldsymbol{y}^T(\boldsymbol{M}_2^{\boldsymbol{\pi}^s})'\boldsymbol{y} + (\boldsymbol{\theta} - \boldsymbol{A}\boldsymbol{y})^T(\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{y}) \\ &\quad + (c + \boldsymbol{y}^T\boldsymbol{e}_1)(1-\gamma) \\ &\quad - \frac{(\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{y})^T\boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{y})}{2(1-\gamma)} \\ &\quad + \frac{1}{2}(\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{y})^T\Sigma(\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{y}) \\ &\quad + (\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y})^T\boldsymbol{B}[((\boldsymbol{\lambda} - a\boldsymbol{E}_2\boldsymbol{y}) + \Sigma(\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{y}))(1-\gamma) \\ &\quad - \boldsymbol{\sigma}\boldsymbol{\beta}\boldsymbol{\sigma}^T(\boldsymbol{M}_1^{\boldsymbol{\pi}^s} + \boldsymbol{M}_2^{\boldsymbol{\pi}^s}\boldsymbol{y})] \\ &\quad - \frac{1}{2}(\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y})^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}(\boldsymbol{h} + \boldsymbol{H}\boldsymbol{y}) \\ &\quad + \frac{1}{2}\mathrm{tr}(\boldsymbol{M}_2^{\boldsymbol{\pi}^s}\Sigma) \\ &\quad + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] = 0. \end{aligned} \tag{A18}$$

Then we can arrange Equation (A18) to obtain the differential equation for $M_0^{\boldsymbol{\pi}^s}$, $\boldsymbol{M}_1^{\boldsymbol{\pi}^s}$ and $\boldsymbol{M}_2^{\boldsymbol{\pi}^s}$ in Equation (A19) below. For simplicity, we just represent them as M_0, $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$.

$$\begin{cases} M_0' + c(1-\gamma) + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] + \frac{1}{2}\mathrm{tr}(\boldsymbol{M}_2\Sigma) \\ \quad + \frac{1}{2}\boldsymbol{M}_1^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1 \\ \quad + \boldsymbol{h}^T\boldsymbol{B}\left[\boldsymbol{\lambda} + \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1\right](1-\gamma) + \boldsymbol{\theta}^T\boldsymbol{M}_1 \\ \quad - \frac{1}{2}\boldsymbol{h}^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}\boldsymbol{h} = 0 \\ M_0(T) = 0, \end{cases}$$

$$\begin{cases} \boldsymbol{M}_1' + \left[(1-\gamma)\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T - \boldsymbol{A}^T\right]\boldsymbol{M}_1 \\ \quad + \boldsymbol{y}^T\boldsymbol{M}_2\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_1 \\ \quad + (1-\gamma)\boldsymbol{e}_1 + \boldsymbol{M}_2^T\boldsymbol{\theta} + \left[\boldsymbol{M}_2^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T - a\boldsymbol{E}_2\right] \\ \quad \boldsymbol{B}\boldsymbol{h}(1-\gamma) \\ \quad + \boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\lambda}(1-\gamma) - \boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}\boldsymbol{h} = 0 \\ \boldsymbol{M}_1(T) = \boldsymbol{0}_{2\times 1} \end{cases}$$

$$\begin{cases} \boldsymbol{M}_2' - \boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}[(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})]^{-1}\boldsymbol{\sigma}^T\boldsymbol{B}\boldsymbol{H} \\ \quad - a(1-\gamma)(\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{E}_2 + \boldsymbol{E}_2\boldsymbol{B}\boldsymbol{H}) \\ \quad + \boldsymbol{M}_2^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2 \\ \quad + \left[(1-\gamma)\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T - \boldsymbol{A}^T\right]\boldsymbol{M}_2 \\ \quad + \boldsymbol{M}_2^T\left[(1-\gamma)\boldsymbol{H}^T\boldsymbol{B}\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T - \boldsymbol{A}^T\right]^T = 0 \\ \boldsymbol{M}_2(T) = \boldsymbol{0}_{2\times 2} \end{cases} \tag{A19}$$

For simplicity, if we let

$$\begin{aligned} \boldsymbol{D}_0^s &= \boldsymbol{B}\boldsymbol{\sigma}(1-\gamma)(\gamma\boldsymbol{I} + \boldsymbol{\beta})\boldsymbol{\sigma}^T\boldsymbol{B} \\ \boldsymbol{D}_1^s(t) &= (1-\gamma)\boldsymbol{H}^T(t)\boldsymbol{B}\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T - \boldsymbol{A}^T \\ \boldsymbol{D}_2^s &= \boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T \\ \boldsymbol{D}_3^s(t) &= a(1-\gamma)(\boldsymbol{H}^T(t)\boldsymbol{B}\boldsymbol{E}_2 + \boldsymbol{E}_2\boldsymbol{B}\boldsymbol{H}(t)) \\ \boldsymbol{C}_0^s(t) &= (1-\gamma)\boldsymbol{e}_1 + \boldsymbol{M}_2^T\boldsymbol{\theta} + \boldsymbol{H}^T(t)\boldsymbol{B}\boldsymbol{\lambda}(1-\gamma) \\ \boldsymbol{B}_0^s(t) &= c(1-\gamma) + \mu\mathbb{E}[(1-Y)^{1-\gamma} - 1] + \frac{1}{2}\mathrm{tr}(\boldsymbol{M}_2(t)\Sigma) + \boldsymbol{\theta}^T\boldsymbol{M}_1, \end{aligned} \tag{A20}$$

we can obtain the ODEs in Proposition 3.

Moreover, if we set an strategy with $H(t) = 0$, we can easily derive the differential for $\boldsymbol{M}_2$ in Equation (A19) to obtain:

$$\begin{cases} \boldsymbol{M}_2' + \boldsymbol{M}_2^T\boldsymbol{\sigma}\left(\boldsymbol{I} - \frac{\boldsymbol{\beta}}{1-\gamma}\right)\boldsymbol{\sigma}^T\boldsymbol{M}_2 = 0 \\ \boldsymbol{M}_2(T) = \boldsymbol{0}_{2\times 2} \end{cases} \tag{A21}$$

Which leads to $\boldsymbol{M}_2(t) = \boldsymbol{0}_{2\times 2}$. Then the quadratic term disappears in the exponentially affine form.

Appendix 5. Our Estimates and those in Schwartz (1997)

In order to compare our model with Schwartz's one-factor model of oil, we make a match between the two parameter sets. We extract the stochastic process of oil price using parametric set of 1/2/90 to 2/17/95 in Table IV from Schwartz (1997). Plugging the parameters and assuming a constant interest rate of 6%, we obtain

$$\mathrm{d}X(t) = (0.06 + 6.301 \times 0.326 - 0.694X(t))\,\mathrm{d}t + 0.326\,\mathrm{d}W(t). \tag{A22}$$

Table A1 shows all coefficients of the WTI oil price as well as those from Schwartz's paper in our notation. We observe that the estimated parameters can match parameters from Schwartz's paper quite closely.

Table A1. Parameters, comparison to Schwartz (1997)

Parameters			
Asset	Market price of risk	λ_S	6.5253
	Mean-reverting rate	a	0.6539
	Volatility	σ_S	31.281%
	Correlation	ρ	0.5317%
Schwartz	Market price of risk	λ_S	6.301
	Mean-reverting rate	a	0.694
	Volatility	σ_S	32.6%