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Portfolio selection with tail nonlinearly transformed risk measures—a comparison with mean-CVaR analysis

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We compare portfolio selection under the tail nonlinearly transformed risk measure (TNT) with mean-CVaR analysis for normally distributed returns. TNT arises as a natural extension to CVaR that additionally distorts the portfolio returns' outcomes by means of a concave transformation function. We first address the portfolio setting where no risk-free asset exists. Here, the global minimum CVaR portfolio, also under fairly realistic asset returns, may not exist. TNT is able to resolve this shortcoming: its additional transformation of the portfolio outcomes ensures that the global minimum TNT portfolio and, accordingly, the (μ, TNT)-efficient frontiers, do always exist. Second, we analyze the setting where the risk-free asset is available. In this case, Tobin's theorem holds both for CVaR and TNT, under the condition that the respective investors are sufficiently risk averse. Under TNT, this condition is less restrictive, Tobin's theorem holds more often. We third address the choice of optimal portfolios under CVaR and TNT. Under CVaR, optimal portfolios exhibit plunging: either the investor invests entirely in the risky tangency portfolio or she invests entirely in the risk-free asset; diversification is never optimal. TNT, in contrast, yields more realistic portfolio structures. Below a certain minimum risk premium, we do not find any risky investment. Once this minimum risk premium is passed, the risky investment is continuous and strictly monotonously increasing in the risk premium. This pattern is in line with empirical evidence on the stock market participation puzzle and the equity premium puzzle.

Keywords: Portfolio selection; Conditional value-at-risk; Axiomatic risk measures; Risk management

1. Introduction

Portfolio selection is a major field of application in modern risk management. One central research question in this field involves the definition of adequate measures of risk and the corresponding determination of optimal portfolios. The seminal work of Markowitz (1952) employed the variance as the relevant measure of risk. Since then, many scholars have developed this approach further by, e.g. providing a rigorous analytical derivation of mean–variance efficient frontiers (Merton 1972), addressing the issue of equilibrium capital asset pricing (Lintner 1969, Mossin 1968, Sharpe 1964) or by including a background risk (e.g. Jiang *et al.* 2010).

At the same time, the variance as the relevant measure of risk has been challenged, as it implies that investors weight under performance and over performance of their portfolios equally, whereas it is actually only downside risk which should be relevant. In the last two decades, therefore, downside risk measures have become a popular alternative to the variance. Typical examples include Value-at-Risk (VaR) (e.g. Baumol 1963, Jorion 1997 and Conditional Value-at-Risk (CVaR) (e.g. Acerbi and Tasche 2002, Rockafellar and Uryasev 2002). In particular, VaR since the 1990s has emerged as an industry standard by choice or by regulation (e.g. Holton 2003). VaR indicates the portfolio loss that is only exceeded with a (small) pre-specified probability. While VaR exhibits various advantages such as the possibility to apply it to any kind of financial risk, expressing the risk in monetary units, and easy estimability from statistical data, it has also been criticized. Among others, VaR does not consider losses beyond the pre-specified probability, and it may violate the property of subadditivity, that is, VaR fails to give

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credit for diversification. In the sequel, CVaR has been proposed as a natural extension. CVaR indicates the expected portfolio loss incurred with a pre-specified probability. Therefore, unlike VaR, CVaR takes into account extreme losses, and it is subadditive. Various scholars have applied CVaR to portfolio selection and have also compared the results to traditional mean–variance analysis (e.g. Alexander and Baptista 2002, 2004, Brandtner 2013, De Giorgi 2002, Deng *et al.* 2009).

Meanwhile, CVaR has been called into question in the literature as well. While initially it has been considered a major advantage that CVaR is a so-called coherent risk measure satisfying the properties of monotonicity (M), translation invariance (TI), positive homogeneity (PH), and subadditivity (S) (Artzner *et al.* 1999), the properties of (TI) and (PH) have been criticized by various scholars in recent times. For example, Brandtner (2013) shows that it is exactly the interplay of (TI) and (PH) that induces counter-intuitive all-or-nothing decisions instead of diversification in portfolio selection. Föllmer and Schied (2002) criticize (PH) for being too restrictive when it comes to considering liquidity risks. El Karoui and Ravanelli (2009) question (TI), as it requires the existence of a risk-free asset, which is virtually non-existent. Additionally, CVaR has also been criticized from a decision theoretic perspective. This is due to the fact that CVaR, as being a simple expected portfolio loss, is linear in a financial position's outcomes. Gains (or small losses) and high losses can, therefore, linearly compensate each other. Experimental evidence, in contrast, indicates that investors exhibit loss aversion, i.e. they turn out to be more risk averse with respect to high losses (e.g. Bosch-Doménech and Silvestre 2006). Therefore, an appropriate risk measure not only should transform the probabilities of a financial position as is done by CVaR, but should also consider a non-linear transformation of the outcomes itself as is done under traditional expected utility theory and prospect theory (Morgenstern and Von Neumann 1949, Kahneman and Tversky 1979). Interestingly, CVaR's restrictive linearity in the outcomes can be attributed to the underlying properties of (TI) and (PH) again.

This recent criticism has lead scholars to advance extensions to CVaR that come without (TI) and (PH). Chen and Yang (2011) have suggested Weighted Expected Shortfall (WES) which, following the concept of expected utility theory, additionally penalizes tail outcomes by means of a non-linear transformation function. Chen *et al.* (2012) follow this approach but propose another (exponential) transformation function, which yields the so-called Tail Nonlinearly Transformed Risk Measure (TNT). While Chen *et al.* (2012) have empirically analyzed optimal portfolio selection under TNT, the literature lacks a rigorous theoretical analysis and a comparison with the variance and CVaR as of yet. In order to fill this research gap, in this paper we study portfolio selection under the new TNT and its relation to CVaR for normally distributions returns. We further include the standard mean–variance results for two reasons: first and foremost, they serve as an analytical benchmark and for comparison. In this sense, our work is closely related to Alexander and Baptista (2002) and De Giorgi (2002), who have conducted this type of comparison for VaR and CVaR. Second, as is well known, for normally distributed returns and an exponential utility function (as employed by TNT), efficient and optimal portfolios are identical under expected utility theory and the linear mean–variance utility function (CARA-normal case). As such, our work can also be understood as comparing efficient and optimal portfolios under expected utility theory, CVaR and TNT, which itself extends both expected utility theory and CVaR.

Our contribution and findings are as follows: first, we address the portfolio setting where no risk-free asset is available and derive the (μ, TNT)-boundary, the global minimum TNT portfolio, and the corresponding (μ, TNT)-efficient frontier. We find that the (μ, TNT)-boundary coincides with the $(\mu, CVaR)$-boundary and the (μ, σ)-boundary. We further provide a closed-form solution for the portfolio vector of the global minimum TNT portfolio and show that this portfolio always exists. This constitutes a major improvement over CVaR, where the global minimum risk portfolio may not exist, even for realistic asset returns. In formal terms, the existence of the global minimum TNT portfolio can be attributed to TNT's additional transformation of the portfolio return's outcomes. We further show that the (μ, TNT)-efficient frontier is a subset of the (μ, σ)-efficient frontier and a superset of the $(\mu, CVaR)$-efficient frontier.

Second, we analyze the portfolio setting where a risk-free asset is available. We find that Tobin (1958) theorem holds both for CVaR and TNT, under the condition that the investors' degree of risk aversion is sufficiently strong. Otherwise, the $(\mu, CVaR)$-efficient frontier and the (μ, TNT)-efficient frontier are empty. We show that, in the former case, the tangency portfolio under the variance, CVaR and TNT coincide. TNT, however, again proves to be advantageous when compared to CVaR, due to its additional transformation of the portfolio outcomes: for a given pre-specified probability (or confidence level), Tobin's theorem may already hold for TNT, while the $(\mu, CVaR)$-efficient frontier is still empty.

Third, we compare the choice of specific optimal portfolios that maximizes an investor's individual utility function under the variance, CVaR and TNT. Under the variance, it is well known that diversification between the risky tangency portfolio and the risk-free asset is optimal as soon as the risk premium is positive. CVaR, by contrast, yields more restrictive and counter-intuitive results: investors who maximize a $(\mu, CVaR)$-utility function will never diversify, but either invest completely in the tangency portfolio or in the risk-free asset only. This restrictive pattern again originates in CVaR's linearity properties of (TI) and (PH). TNT, by transforming the portfolio returns' outcomes non-linearly, yields more plausible results. Below a certain minimum risk premium, investors only hold the risk-free asset. Once this minimum risk premium is reached, the risky investment is continuous and strictly monotonously increasing in the risk premium. This pattern is in line with empirical evidence on the stock market participation puzzle (e.g. Mankiw and Zeldes 1991, Haliassos and Bertaut 1995) and the equity premium puzzle (e.g. Mehra and Prescott 1985).

The remainder of the paper is organized as follows: Section 2 introduces the risk measures under consideration. Section 3 addresses portfolio selection when no risk-free asset exists and compares the (μ, σ)-, $(\mu, CVaR)$-, and (μ, TNT)-boundaries and -efficient frontiers. Section 4 considers the case when a risk-free asset is available. Section 5 analyzes the choice of optimal portfolios under the three relevant risk measures. Section 6 illustrates the economic implications that arise under the three risk measures by means of an empirical study. Section 7 concludes.

2. CVaR vs. TNT: representation and properties

We consider a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the set of all events, $\mathcal{F}$ is the σ-algebra of subsets of $\mathcal{F}$, and P is a probability measure on $\mathcal{F}$. Let $\mathcal{X}$ be the set of all measurable, real valued random variables X; in the remainder of the paper, we will identify the random variables with (uncertain) financial positions, and portfolio returns in particular. The cumulative distribution function and the quantile function of a random variable X are given by $F_X(x) = F(x) = P(X \leq x)$ and $F_X^{-1}(p) = F^{-1}(p) = \sup\{x \in \mathbb{R} | F(x) < p\}$, $p \in (0, 1]$, and $F^{-1}(0) = \text{ess}\inf\{X\}$. A risk measure ρ is a mapping from $\mathcal{X}$ to the real line $\mathbb{R}$. As such, a risk measure assigns a single numerical value to a random variable. For the risk measures under consideration in this paper to be finite, it is assumed that the means and the variances of the random variables are finite.

We start with a list of well-known properties (or axioms), which commonly serve as a basis for defining modern risk measures (see, e.g. Artzner *et al.* 1999 or Acerbi (2002) for a detailed discussion):

- (M) Monotonicity: $X \leq Y \Rightarrow \rho(X) \geq \rho(Y), \forall X, Y \in \mathcal{X}$
- (TI) Translation invariance: $\rho(X + m) = \rho(X) - m, m \in \mathbb{R}, \forall X \in \mathcal{X}$
- (PH) Positive homogeneity: $\rho(\delta \cdot X) = \delta \cdot \rho(X)$, $\delta \geq 0$, $\forall X \in \mathcal{X}$
- (S) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y), \forall X, Y \in \mathcal{X}$
- (CA) Comonotonic additivity: For non-decreasing functions f, g and random variables $Z, X = f(Z), Y = g(Z) \in \mathcal{X}$, $\rho(X + Y) = \rho(X) + \rho(Y)$
- (LI) Law invariance: $F_X(x) = F_Y(x)$, $\forall x \in \mathbb{R} \Rightarrow \rho(X) = \rho(Y), \forall X, Y \in \mathcal{X}$
- (C) Convexity: $\rho(\delta \cdot X + (1 - \delta) \cdot Y) \leq \delta \cdot \rho(X) + (1 - \delta) \cdot \rho(Y), \delta \in [0, 1], \forall X, Y \in \mathcal{X}$.

Note that the first four properties define the class of coherent risk measures (Artzner *et al.* 1999), while the first six properties characterize the subclass of spectral risk measures (Acerbi 2002). Both classes additionally satisfy (C), which is immediately implied by (PH) and (S).

The variance,

$$\sigma^2(X) = \int_{-\infty}^{\infty} (x - E(X))^2 \, \mathrm{d}F(x), \tag{1}$$

has been the financial standard risk measure since Markowitz (1952) first applied it to portfolio selection. Despite its advantages, the variance has also been criticized for not adequately accounting for investors' attitudes towards risk: first, the variance is a symmetric risk measure and, accordingly, does not explicitly capture the downside risk, which is particularly relevant to investors. Second, the variance violates the above axioms of (M), (TI), (PH), and (S), and, therefore, is not a coherent risk measure in the sense of Artzner *et al.* (1999) at all. Third, in terms of decision theoretic foundation, the variance is consistent with expected utility theory only if investors' preferences exhibit constant absolute risk aversion (CARA) or if returns follow a normal distribution. Note, however, that these conditions of consistency will be satisfied in our paper. We assume normally distributed returns, and, following the concept of TNT, employ an exponential utility function. Under these assumptions, maximizing expected utility is equivalent to maximizing the mean–variance utility function $\mu(X) - \lambda/2 \cdot \sigma^2(X)$, where $\lambda > 0$ is the exponential utility function's risk parameter. Therefore, in this paper's setting, the mean–variance framework is consistent with the traditional expected utility framework.

In order to address these shortcomings, and, above all, to overcome the lack of coherence, scholars have proposed to replace traditional variance by coherent and spectral risk measures. In particular, Conditional Value-at-Risk (CVaR) at the confidence level $\alpha \in (0, 1]$ has emerged as an industry standard in the last two decades, both in academia (e.g. Acerbi and Tasche 2002, Rockafellar and Uryasev 2002) and practice (e.g. Basel III, Swiss Solvency Test). CVaR indicates the expected loss of a financial position X incurred in the $\alpha \cdot 100\%$ worst cases, that is

$$CVaR_\alpha(X) = -\frac{1}{\alpha} \cdot \int_0^\alpha F_X^{-1}(p) \, \mathrm{d}p. \tag{2}$$

As such, it explicitly accounts for downside risks, and it also satisfies any of the above properties. Note that CVaR distorts the physical probabilities by assigning a constant weight of $\frac{1}{\alpha}$ to the $\alpha \cdot 100\%$ worst cases and zero weight else. At the same time, CVaR is linear in (i.e. does not transform) the financial position's outcomes.

For normally distributed positions, $X \sim N(\mu_X, \sigma_X)$, it is well known that CVaR by means of the properties of (TI) and (PH) can be written as a linear combination of the position's mean and standard deviation,

$$CVaR_\alpha(X) = CVaR_\alpha(\mu_X + \sigma_X \cdot X_0) = -\mu_X + \sigma_X \cdot k(\alpha), \tag{3}$$

where X_0 is a standard normal random variable and $k(\alpha) := CVaR(X_0) = \frac{n(N^{-1}(\alpha))}{\alpha}$ (e.g. Alexander and Baptista 2004, Section 2). Several studies have addressed the use of CVaR for portfolio selection under normal and elliptical return distributions (see, e.g. Alexander and Baptista (2002, 2004), De Giorgi (2002), Deng *et al.* (2009)).

Despite its advantages over the variance, CVaR has also been called into question in the recent literature, mainly due to the linearity properties of (TI) and (PH) (the detailed arguments and references have been given in the Introduction). This recent criticism has lead scholars to propose extensions to CVaR that come without (TI) and (PH). Chen and Yang (2011) have introduced Weighted Expected Shortfall (WES) which additionally penalizes tail-outcomes by means of a non-linear transformation function. Chen *et al.* (2012) also follow this approach but propose another transformation function, which yields the so-called Tail Nonlinearly Transformed Risk Measure (TNT). Formally, TNT is defined as

$$TNT_\alpha(X) = -\frac{1}{\alpha} \cdot \int_0^\alpha u(F_X^{-1}(p)) \, \mathrm{d}p, \tag{4}$$

where u is a negative, monotonically increasing, concave and continuous function that transforms a financial position's outcomes. The concavity of the transformation function u ensures that larger losses are penalized more severely, which allows capturing investors' loss aversion. In the remainder of the paper, we follow Chen *et al.* (2012) and will employ the exponential transformation function $u(x) = -\exp(-\lambda \cdot x)$, where $\lambda > 0$ is some risk parameter. This continuous transformation function offers via λ an additional degree of freedom for the individual loss of the investor. The higher λ, the more concave u and, thus, the more strongly higher losses are penalized. TNT satisfies a number of properties that will turn out to be relevant for the derivation of our results on efficient and optimal portfolios.

PROPOSITION 2.1 *TNT as given in* (4) *satisfies*

- (M) *Monotonicity:* $X \leq Y \quad \Rightarrow TNT_\alpha(X) \geq TNT_\alpha(Y), \forall X, Y \in \mathcal{X}$
- (C) *Convexity:* $TNT_\alpha(\delta \cdot X + (1-\delta) \cdot Y) \leq \delta \cdot TNT_\alpha(X) + (1-\delta) \cdot TNT_\alpha(Y), \delta \in [0,1], \forall X, Y \in \mathcal{X}$
- (SSD) *Consistency with second-order stochastic dominance:* $\int_{-\infty}^{t} F_X(x)dx \leq \int_{-\infty}^{t} F_Y(x)\,\mathrm{d}x$ *for all* $t \Rightarrow TNT_\alpha(X) \leq TNT_\alpha(Y)$.

When initiating the class of generalized convex risk measures, with TNT among them, Chen and Yang (2011), Chen *et al.* (2012) proposed two axioms as reasonable properties a risk measure should satisfy: Monotonicity (M) and Convexity (C). (M) has a straightforward economic interpretation: it captures the idea that if the outcomes of a financial position Y is always at least as high as that of X, Y cannot be riskier than X. (C) encourages diversification: a convex combination of two financial positions is no more risky than the weighted sum of the single risks. TNT further is consistent with (SSD) (for the proof, see Bäuerle and Müller 2006 Theorem 4.4, Brandtner *et al.* 2020). In their seminal paper, Rothschild and Stiglitz (1970) have shown that X second-order stochastically dominates Y if and only if Y arises from adding uncorrelated noise to X, and that this is equivalent to the fact that any risk averse expected utility decision maker prefers X over Y. The same holds true for TNT, which is an important feature of risk measures in numerous finance and insurance applications (see, e.g. Gollier and Schlesinger 1996).

For X normally distributed, $X \sim N(\mu_X, \sigma_X^2)$, it holds that $\exp(-\lambda X) \sim LN(-\lambda\mu_X, \lambda^2\sigma_X^2)$, and TNT (for the exponential transformation function) becomes the mean of a truncated log-normally distributed random variables with parameters $(-\lambda\mu_X, \lambda^2\sigma_X^2)$:

$$
\begin{aligned}
&TNT_{\alpha,\lambda}(X) \\
&= E\Big(\exp(-\lambda X) | -\exp(-\lambda X) \leq F^{-1}_{-\exp(-\lambda X)}(\alpha)\Big) \\
&= E\Big(\exp(-\lambda X) | \exp(-\lambda X) \geq F^{-1}_{-\exp(-\lambda X)}(1-\alpha)\Big) \\
&= \exp\left(-\lambda\mu_X + \frac{1}{2}\lambda^2\sigma_X^2\right) \cdot \frac{N\Big(\lambda\sigma_X - N^{-1}(1-\alpha)\Big)}{\alpha} \qquad (5)
\end{aligned}
$$

(e.g. Lee and Krutchkoff 1980, Arismendi 2013). It is worth emphasizing that in comparison to CVaR, the representation of TNT under normally distributed returns is non-linear in the mean and standard deviation, which will prove to be advantageous in the remainder of the paper.

3. $(\mu, risk)$-boundaries and $(\mu, risk)$-efficient frontiers under variance, CVaR, and TNT: the case of n risky assets

3.1. The portfolio problem

We start with analyzing the standard portfolio problem with $n \geq 2$ risky assets and, for the moment, assume that no risk-free asset exists. Assume that the rates of return follow a multivariate normal distribution where $\mu \in \mathbb{R}^n$ is the vector of the means of the rates of returns and Σ is the $n \times n$ positive definite variance–covariance matrix of the rates of returns. $\mathbb{1} \in \mathbb{R}^n$ denotes the n-dimensional unit vector $(1, \ldots, 1)$. $\Theta_n = \{\theta \in \mathbb{R}^n | \sum_{j=1}^n \theta_j = 1\}$ denotes the set of possible portfolios, with θ_j as the fraction of wealth invested in asset j and $\theta = (\theta_1, \ldots, \theta_n)^T$ as the vector of these shares. For any $\theta \in \Theta_n$, $\mu(\theta)$ and $risk(\theta)$ denote the mean and the risk of the rate of return of portfolio θ, respectively. In the following, we compare the characteristics of the efficient and optimal portfolios under the risk measures variance, CVaR, and TNT. Recall that, for normally distributed returns and an exponential utility function, the variance represents the relevant risk measure in the expected utility framework.

In this paper, we assume normally distributed returns for the following reasons: first, under normally distributed returns, risk measures such as CVaR and TNT are completely determined by the vector of the means and the variance–covariance matrix; as such, they are based on the exact same data as the traditional (μ, σ)-approach, which allows an intuitive comparison (e.g. Alexander and Baptista 2002, De Giorgi 2002, Deng *et al.* 2009). Likewise, the optimal portfolios under TNT and normally distributed returns may serve as a reliable benchmark for empirical analyses under real-world historical data. Second, the normal distribution assumption allows to derive explicit solutions for the optimal portfolio vector. These explicit solutions are relevant both from a theoretical and a practitioner's point of view. Theoretically, they serve as a basis for comparative static analysis and the derivation of corresponding normative policy implications (in just the same way as this has been accomplish by the papers of Alexander and Baptista (2004) and Alexander and Baptista (2006). From a practitioner's point of view, explicit solutions considerably simplify the implementation of modern risk measures in internal and regulatory risk management tools; among others, normally distributed returns serve as a building block in the Basel III IRB risk weight functions.

3.2. The $(\mu, risk)$-boundaries

This section provides the definitions of the risk–return boundaries for the three relevant risk measures variance, CVaR, and TNT. First, we briefly recall the traditional (μ, σ)-framework (Markowitz 1952) as a benchmark.

DEFINITION 3.1 *A portfolio* $\theta \in \Theta_n$ *belongs to the* (μ, σ)-*boundary if and only if, for some fixed return* $\mu(\theta) = \bar{\mu}$, θ *solves* $\min_{\theta \in \Theta_n} \sigma(\theta)$.

Merton (1972) has shown that a portfolio θ belongs to the (μ, σ)-boundary if and only if

$$\frac{\sigma^2(\theta)}{1/C} - \frac{(\mu(\theta) - A/C)^2}{D/C^2} = 1, \tag{6}$$

where $A \equiv \mathbb{1}^T \Sigma^{-1} \mu$, $B \equiv \mu^T \Sigma^{-1} \mu$, $C \equiv \mathbb{1}^T \Sigma^{-1} \mathbb{1}$ and $D \equiv BC - A^2$ are constants, with B, C and D being positive.

We now consider the $(\mu, CVaR)$-framework (e.g. De Giorgi 2002, Alexander and Baptista 2004). The $(\mu, CVaR)$-boundary is defined as follows.

Definition 3.2 *A portfolio $\theta \in \Theta_n$ belongs to the $(\mu, CVaR)$-boundary if and only if, for some fixed return $\mu(\theta) = \bar{\mu}$, θ solves $\min_{\theta \in \Theta_n} CVaR_\alpha(\theta)$.*

For normally distributed portfolio returns, CVaR is given by a linear combination of the portfolio return's mean and its standard deviation, and, thus, θ solves

$$\min_{\theta \in \Theta_n} CVaR_\alpha(\theta) = -\theta^T \mu + \sqrt{\theta^T \Sigma \theta} \cdot k(\alpha) \tag{7}$$

$$\text{s.t.} \quad \theta^T \mu = \bar{\mu}, \; \theta^T \mathbb{1} = 1. \tag{8}$$

As the portfolio return's mean is fixed when determining the $(\mu, CVaR)$-boundary, minimizing CVaR is equivalent to minimizing the standard deviation for this fixed return. This immediately yields

Proposition 3.1 *A portfolio belongs to the $(\mu, CVaR)$-boundary if and only if it belongs to the (μ, σ)-boundary. That is, any portfolio on the $(\mu, CVaR)$-boundary satisfies Equation (6).*

Likewise, for TNT, we have

Definition 3.3 *A portfolio $\theta \in \Theta_n$ belongs to the (μ, TNT)-boundary if and only if, for some fixed return $\mu(\theta) = \bar{\mu}$, θ solves $\min_{\theta \in \Theta_n} TNT_{\alpha,\lambda}(\theta)$.*

For normally distributed portfolio returns, the (μ, TNT)-boundary is obtained via

$$\min_{\theta \in \Theta_n} TNT_{\alpha,\lambda}(\theta) = \exp\left(-\lambda \cdot \theta^T \mu + \frac{\lambda^2}{2} \cdot \theta^T \Sigma \theta\right) \cdot \frac{N(\lambda \cdot \sqrt{\theta^T \Sigma \theta} + N^{-1}(\alpha))}{\alpha} \tag{9}$$

$$\text{s.t.} \quad \theta^T \mu = \bar{\mu}, \; \theta^T \mathbb{1} = 1. \tag{10}$$

Assuming normally distributed portfolio returns, TNT is a positive function of the portfolio return's mean and is negatively related to its standard deviation. Determining the (μ, TNT)-boundary via minimizing TNT thus, again, is equivalent to minimizing the standard deviation for a fixed return $\bar{\mu}$. This immediately yields

Proposition 3.2 *A portfolio belongs to the (μ, TNT)-boundary if and only if it belongs to the (μ, σ)-boundary. That is, any portfolio on the (μ, TNT)-boundary satisfies Equation (6).*

3.3. *The $(\mu, risk)$-efficient frontiers*

This section derives the characteristics of the global minimum risk portfolios and the $(\mu, risk)$-efficient frontiers for the three different risk measures variance, CVaR and TNT.

First, we again recall the standard results from Markowitz' portfolio theory.

Definition 3.4 *A portfolio $\theta \in \Theta_n$ belongs to the (μ, σ)-efficient frontier if and only if no portfolio $\upsilon \in \Theta_n$ exists such that $\mu(\upsilon) \geq \mu(\theta)$ and $\sigma(\upsilon) \leq \sigma(\theta)$, where at least one of the inequalities is strict.*

As (μ, σ)-efficient portfolios exhibit the maximum expected return for a given standard deviation, they contain any portfolio of the (μ, σ)-boundary with a mean greater than the one of the global minimum variance portfolio. This global minimum variance portfolio, θ_σ, and the (μ, σ)-efficient frontier are characterized as follows:

Proposition 3.3 *The global minimum variance portfolio, $\theta_\sigma := \arg\min_{\theta \in \Theta_n} \sigma(\theta)$, is given by $\theta_\sigma = \frac{1}{C} \Sigma^{-1} \mathbb{1}$. It exhibits a standard deviation of $\sigma(\theta_\sigma) = 1/\sqrt{C}$, and a mean of $\mu(\theta_\sigma) = A/C$. The (μ, σ)-efficient frontier contains any portfolio on the (μ, σ)-boundary with $\mu \geq \mu(\theta_\sigma)$.*

For the proof, see, for example, De Giorgi (2002), proposition 4.1.

Alexander and Baptista (2004) have provided the corresponding results for CVaR as the relevant measure of risk, which we recall briefly. Already here, the linearity properties (TI) and (PH) underlying CVaR will turn out to be problematic, as the minimum CVaR portfolio may not exist.

Definition 3.5 *A portfolio $\theta \in \Theta_n$ belongs to the $(\mu, CVaR)$-efficient frontier if and only if no portfolio $\upsilon \in \Theta$ exists such that $\mu(\upsilon) \geq \mu(\theta)$ and $CVaR(\upsilon) \leq CVaR(\theta)$, where at least one of the inequalities is strict.*

In order to characterize the $(\mu, CVaR)$-efficient frontier, the global minimum CVaR portfolio, θ_{CVaR_α}, needs to be determined first.

Proposition 3.4 *Let the global minimum CVaR portfolio at the $\alpha \in (0, 1]$ confidence level be defined as $\theta_{CVaR_\alpha} := \arg\min_{\theta \in \Theta_n} CVaR_\alpha(\theta)$. Let further g and h be n-dimensional vectors defined by $g = (1/D)[B(\Sigma^{-1}\mathbb{1}) - A(\Sigma^{-1}\mu)]$ and $h = (1/D)[C(\Sigma^{-1}\mu) - A(\Sigma^{-1}\mathbb{1})]$, and $k(\alpha) := n(N^{-1}(\alpha))/\alpha$. Then it holds that*

(i) The global minimum CVaR portfolio at the $\alpha \in (0, 1]$ confidence level, θ_{CVaR_α}, does not exist if and only if $k(\alpha) < \sqrt{D/C}$.

(ii) The global minimum CVaR portfolio exists if and only if $k(\alpha) \geq \sqrt{D/C}$. In this case,

$$\theta_{CVaR_\alpha} = g + h\left(\mu(\theta_\sigma) + \sqrt{\frac{D^2/C^2}{k^2(\alpha) - D/C}} \cdot \sigma(\theta_\sigma)\right) \tag{11}$$

and

$$CVaR_\alpha(\theta_{CVaR_\alpha}) = \sqrt{k^2(\alpha) - D/C} \cdot \sigma(\theta_\sigma) - \mu(\theta_\sigma), \tag{12}$$

and the mean and the standard deviation are given by $\mu(\theta_{CVaR_\alpha}) = \mu(\theta_\sigma) + \sqrt{(D^2/C^2)/(k^2(\alpha) - D/C)} \cdot \sigma(\theta_\sigma)$ *and* $\sigma(\theta_{CVaR_\alpha}) = \sqrt{k^2(\alpha)/(C \cdot k^2(\alpha) - D)}$, *respectively.*

For the proof, see Alexander and Baptista (2004), proposition 1. The existence of the global minimum CVaR portfolio depends on the choice of the confidence level α. If $k(\alpha) < \sqrt{D/C}$, the square root in (11) becomes negative and, accordingly, the $(\mu, CVaR)$-boundary is negatively sloped. As a result, the set of $(\mu, CVaR)$-efficient portfolios is empty. This possible non-existence of the global minimum CVaR portfolio originates from the properties of (TI) and (PH). Due to these properties, CVaR can be written as a linear combination of the portfolio return's mean and standard deviation, $CVaR_\alpha(\theta) = -\mu_\theta + k(\alpha) \cdot \sigma(\theta)$. Now moving upward the (μ, σ)-efficient frontier starting from the global minimum variance portfolio yields two effects. First, the portfolio return's mean increases which, c.p., yields a decrease in the portfolio return's CVaR (mean effect). Second, the portfolio return's standard deviation increases which, c.p., yields an increase in the portfolio return's CVaR (standard deviation effect). If the investor's risk aversion is weak, the mean effect outweighs the standard deviation effect and, therefore, the portfolio risk can be reduced ad infinitum, so the global minimum CVaR portfolio does not exist.

If the investor's risk aversion is strong, $k(\alpha) \geq \sqrt{D/C}$, the global minimum CVaR portfolio exists, and comparing the (μ, σ)- and the $(\mu, CVaR)$-efficient frontiers yields

Corollary 3.1 *If* $k(\alpha) \geq \sqrt{D/C}$, *the global minimum CVaR portfolio is* (μ, σ)*-efficient. Moreover, each* $(\mu, CVaR)$*-efficient portfolio is, at the same time,* (μ, σ)*-efficient, while the converse is not true. In particular, the global minimum variance portfolio is never* $(\mu, CVaR)$*-efficient.*

If the global minimum CVaR portfolio exists, it satisfies $\mu(\theta_{CVaR_\alpha}) > \mu(\theta_\sigma)$ (see proposition 3.4). Therefore, it lies strictly above the global minimum variance portfolio in the (σ, μ)-plane, and the $(\mu, CVaR)$-efficient frontier is a subset of the (μ, σ)-efficient frontier.

We now turn to TNT, which explicitly considers investors' loss aversion by non-linearly transforming the portfolio returns' outcomes. In line with this objective, TNT does not satisfy the critical properties of (TI) and (PH). Hence, for investors using TNT as their risk measure, the problem of the non-existence of the global minimum risk portfolio will no longer prevail.

Definition 3.6 *A portfolio* $\theta \in \Theta_n$ *belongs to the* (μ, TNT)*-efficient frontier if and only if no portfolio* $\upsilon \in \Theta$ *exists such that* $\mu(\upsilon) \geq \mu(\theta)$ *and* $TNT(\upsilon) \leq TNT(\theta)$, *where at least one of the inequalities is strict.*

We now characterize the global minimum TNT portfolio. Interestingly, it closely resembles the global minimum CVaR portfolio if one replaces the constant $k(\alpha)$ by $s_{\alpha,\lambda}(\theta) := \lambda \cdot \sigma(\theta) + n(\lambda \cdot \sigma(\theta) + N^{-1}(\alpha))/N(\lambda \cdot \sigma(\theta) + N^{-1}(\alpha))$ (for the technical details, we refer the reader to appendix 1).

Proposition 3.5 *Let the global minimum TNT portfolio at the* $\alpha \in (0, 1]$ *confidence level and for risk parameter* $\lambda > 0$ *be defined as* $\theta_{TNT_{\alpha,\lambda}} := \arg\min_{\theta \in \Theta_n} TNT_{\alpha,\lambda}(\theta)$. *Then it holds that*

$$\theta_{TNT_{\alpha,\lambda}} = g + h\left(\mu(\theta_\sigma) + \sqrt{\frac{D^2/C^2}{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C}} \cdot \sigma(\theta_\sigma)\right) \tag{13}$$

and

$$\begin{aligned} TNT_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) &= \exp\left(-\lambda \cdot \mu(\theta_\sigma) - \lambda \cdot \sqrt{\frac{D^2/C^2}{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C}} \cdot \sigma(\theta_\sigma)\right. \\ &\quad \left. + \frac{\lambda^2}{2} \cdot \frac{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})}{\cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C} \cdot \sigma^2(\theta_\sigma)\right) \\ &\quad \cdot \frac{N(\lambda \cdot \sqrt{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})/(C \cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D)} - N^{-1}(1-\alpha))}{\alpha}, \end{aligned} \tag{14}$$

and the mean and the standard deviation are given by $\mu(\theta_{TNT_{\alpha,\lambda}}) = \mu(\theta_\sigma) + \sqrt{(D^2/C^2)/(s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C)} \cdot \sigma(\theta_\sigma)$ *and* $\sigma(\theta_{TNT_{\alpha,\lambda}}) = \sqrt{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})/(C \cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D)}$, *respectively.*

For the proof, see appendix 2. Unlike it was the case with CVaR, the global minimum TNT portfolio now always exists. The reason is that TNT's non-linear transformation of the portfolio return's outcomes by means of the exponential function ensures that the mean effect from proposition 3.4 is always outweighed by the standard deviation effect. Note that, while equation (13) provides a closed form solution for the TNT-minimizing portfolio vector, the portfolio vector is only implicitly defined (as $\theta_{TNT_{\alpha,\lambda}}$ is also contained on the right hand side of the equation).

We are now prepared to compare the efficient frontiers under variance, CVaR and TNT:

Corollary 3.2 *The global minimum TNT portfolio (13) at the* $\alpha \in (0, 1]$ *confidence level and risk parameter* $\lambda > 0$ *is always* (μ, σ)*-efficient. Moreover, each* (μ, TNT)*-efficient portfolio is, at the same time,* (μ, σ)*-efficient, while the converse is not true. In particular, the global minimum variance portfolio is never* (μ, TNT)*-efficient.*

Corollary 3.2 immediately follows as $\mu(\theta_{TNT_{\alpha,\lambda}}) > \mu(\theta_\sigma)$, see proposition 3.5. Like the global minimum CVaR portfolio, the global minimum TNT portfolio lies above the global minimum variance portfolio in the (σ, μ)-plane; it holds.

Corollary 3.3 *The global minimum TNT portfolio lies (strictly) below the global minimum CVaR portfolio, but (strictly) above the global minimum variance portfolio in the* (σ, μ)*-plane:* $\mu(\theta_{CVaR_\alpha}) > \mu(\theta_{TNT_{\alpha,\lambda}}) > \mu(\theta_\sigma)$. *Accordingly, the* ($\mu$, *CVaR)-efficient frontier is a subset of the* (μ, *TNT)-efficient frontier, which, in turn, is a subset of the* (μ, σ)*-efficient frontier.*

Figure 1 provides an illustration.

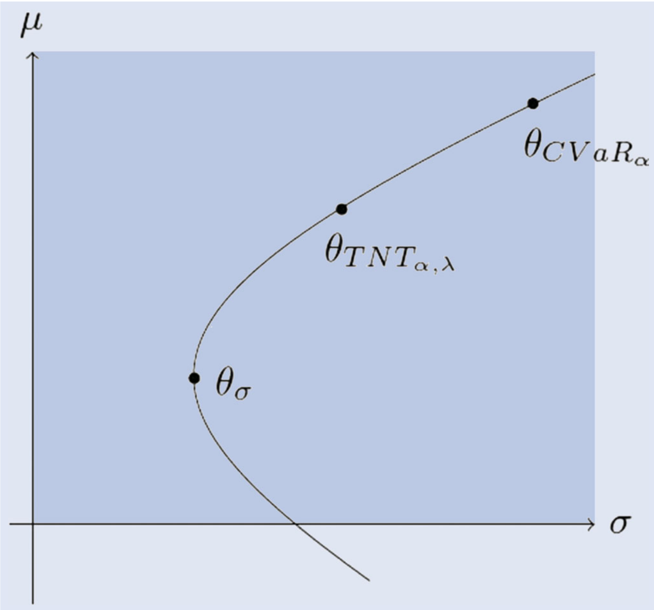

Figure 1. Location of the global minimum variance portfolio, the global minimum TNT portfolio and the global minimum CVaR portfolio in the (σ, μ)-plane.

4. $(\mu, risk)_R$-efficient frontiers and Tobin's theorem: the case of n risky and a risk-free asset

4.1. The portfolio problem

In this section, we extend the portfolio problem by adding a risk-free asset with return μ_0. The new set of possible portfolios is denoted by $\Theta_{n+1} = \{\theta \in \mathbb{R}^{n+1} | \sum_{j=1}^{n+1} \theta_j = 1\}$, where θ_j denotes the share invested in asset j, and θ_{n+1} denotes the share invested in the risk-free asset. As before, we start our analysis with the (μ, σ)-framework and then proceed towards CVaR and TNT, respectively. Below, we will add the subscript R to the $(\mu, risk)$-boundaries and $(\mu, risk)$-efficient frontiers in order to indicate that the risk-free asset is available now. Note that while we address the standard static portfolio selection setting, recent literature has analyzed dynamic portfolio selection settings. For recent contribution regarding the dynamic setting, we refer to Battauz *et al.* (2015), Battauz *et al.* (2017a), Ling *et al.* (2020) and Ma *et al.* (2019).

4.2. The $(\mu, risk)_R$-efficient frontiers

We start our analysis by recalling the well-known (Markowitz 1952) results. In line with the relevant literature, we assume that the return of the risk-free asset is smaller than the return of the global minimum variance portfolio, $\mu_0 < A/C$.

For the variance as the measure of risk, any (μ, σ)-investor splits her wealth between the risk-free asset and the risky tangency portfolio (Tobin 1958 theorem). Proposition 4.1 summarizes the results:

PROPOSITION 4.1 *The portfolio $\theta^*(\bar{\mu})$ that solves $\min_{\theta \in \Theta_{n+1}} \sigma(\theta)$, s.t. $\mu(\theta) = \bar{\mu}$, is given by*

$$\theta^*(\bar{\mu}) = (1 - \lambda(\bar{\mu})) \cdot \begin{pmatrix} \not{\mathbb{1}} \\ 1 \end{pmatrix} + \lambda(\bar{\mu}) \cdot \begin{pmatrix} \theta_T \\ 0 \end{pmatrix}, \tag{15}$$

where $\not{\mathbb{1}} = (0, \ldots, 0)^T$, $\lambda(\bar{\mu}) = (A - \mu_0 \cdot C)\frac{\bar{\mu} - \mu_0}{D(\mu_0)}$, $D(\mu_0) = B - 2 \cdot A \cdot \mu_0 + C \cdot \mu_0^2$, and θ_T denotes the so-called tangency portfolio which is characterized by

$$\theta_T = \frac{1}{A - \mu_0 \cdot C} \Sigma^{-1}(\mu - \mu_0 \mathbb{1}), \tag{16}$$

with

$$\mu(\theta_T) = \frac{B - \mu_0 \cdot A}{A - \mu_0 \cdot C} \quad \text{and} \quad \sigma(\theta_T) = \frac{\frac{B - \mu_0 \cdot A}{A - \mu_0 \cdot C} - \mu_0}{\sqrt{D(\mu_0)}} \tag{17}$$

as the mean and the standard deviation of the tangency portfolio, respectively. Correspondingly, the $(\mu, \sigma)_R$-boundary and the $(\mu, \sigma)_R$-efficient frontier are given by

$$\mu(\sigma) = \sqrt{D(\mu_0)} \cdot \sigma \pm \mu_0 \tag{18}$$

and

$$\mu(\sigma) = \sqrt{D(\mu_0)} \cdot \sigma + \mu_0, \tag{19}$$

respectively.

For a proof, see, e.g. De Giorgi (2002), section 5.1. Note that Rockafellar *et al.* (2006), Theorem 2 has generalized this one fund theorem (Tobin separation) to so-called generalized deviation measures. These risk measures still satisfy the critical properties of positive homogeneity and translation equivariance, while TNT comes without these properties, and therefore is more general. However, in order to prove our Tobin-separation next, we need to rely on the assumption of normally distributed returns.

The next step is to replace the variance by the modern axiomatic risk measures of CVaR and TNT, respectively. First, we recall the results for CVaR.

PROPOSITION 4.2 *The portfolio $\theta^*(\bar{\mu})$ that solves $\min_{\theta \in \Theta_{n+1}}$ CVaR(θ), s.t. $\mu(\theta) = \bar{\mu}$, is given by (15)–(17).*

Further, it holds that:

(i) The $(\mu, CVaR)_R$-boundary, in the (σ, μ)-plane, is given by

$$\mu(\sigma) = \sqrt{D(\mu_0)} \cdot \sigma \pm \mu_0. \tag{20}$$

(ii) If $k(\alpha) < \sqrt{D(\mu_0)}$, then no portfolio exists which is $(\mu, CVaR)_R$-efficient.

(iii) If $k(\alpha) \geq \sqrt{D(\mu_0)}$, the $(\mu, CVaR)_R$-efficient frontier, in the (σ, μ)-plane, is given by

$$\mu(\sigma) = \sqrt{D(\mu_0)} \cdot \sigma + \mu_0. \tag{21}$$

For the proof, see, e.g. De Giorgi (2002, section 5.2). Proposition 4.2 states that the $(\mu, CVaR)_R$-boundary and the $(\mu, \sigma)_R$-boundary coincide. This finding comes as no surprise as CVaR, under the assumption of normally distributed returns, is a function of the portfolio return's mean and standard deviation only. There is, however, a significant deviation between the two risk measures with respect to the efficient frontiers: under CVaR, the efficient frontier is empty if the investor's risk aversion, $k(\alpha)$, is weak, $k(\alpha) < \sqrt{D(\mu_0)}$. The tangency portfolio θ_T in (16) then lies below the (risky) global minimum CVaR portfolio θ_{CVaR_α} as given in (11) on the $(\mu, CVaR)$-boundary. The tangency portfolio thus is not $(\mu, CVaR)$-efficient and, accordingly, the $(\mu, CVaR)_R$-efficient frontier is empty. Investors, in this case, can counter-intuitively increase their portfolio return *and* decrease their portfolio risk by borrowing money and investing the funds in the tangency portfolio. In contrast, if the investor's risk aversion is strong, $k(\alpha) \geq \sqrt{D(\mu_0)}$, the $(\mu, CVaR)_R$-efficient frontier coincides with the $(\mu, \sigma)_R$-efficient frontier. Both are given by a straight line between the risk-free asset and the tangency portfolio in the (σ, μ)-plane. Tobin's theorem remains valid in this latter case only. In Section 6, we will show that both cases may be economically relevant.

We finally address the new risk measure of TNT. As already mentioned, TNT, by means of a non-linear transformation of the outcomes, will help to overcome the problem of non-existence of the efficient frontier that prevails under CVaR.

Proposition 4.3 *The portfolio $\theta^*(\bar{\mu})$ that solves $\min_{\theta \in \Theta_{n+1}}$ $TNT(\theta)$, s.t. $\mu(\theta) = \bar{\mu}$, is given by (15)–(17).*

Further, it holds that

(i) The $(\mu, TNT)_R$-boundary, in the (σ, μ)-plane, is given by

$$\mu(\sigma) = \sqrt{D(\mu_0)} \cdot \sigma \pm \mu_0. \tag{22}$$

(ii) If $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) < \sqrt{D(\mu_0)}$, then no portfolio exists which is $(\mu, TNT)_R$-efficient.

(iii) If $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) \geq \sqrt{D(\mu_0)}$, the $(\mu, TNT)_R$-efficient frontier, in the (σ, μ)-plane, is given by

$$\mu(\sigma) = \sqrt{D(\mu_0)} \cdot \sigma + \mu_0. \tag{23}$$

The proof is given in Appendix 2. Again, the $(\mu, TNT)_R$-boundary coincides with the $(\mu, \sigma)_R$-boundary, and, again, the problem of the existence of the $(\mu, TNT)_R$-efficient frontier arises. Now, the $(\mu, TNT)_R$-efficient frontier is non-empty if $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) \geq \sqrt{D(\mu_0)}$. Interestingly, the new TNT proves to be superior to CVaR in that TNT mitigates the problem of non-existing efficient frontiers. This is because the relevant threshold under TNT, $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})$, is always greater than its CVaR-counterpart, $k(\alpha)$:

Proposition 4.4 *For any given $\alpha \in (0, 1]$ and $\lambda > 0$, it holds that $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) > k(\alpha)$.*

For the proof, see appendix 3. The $(\mu, TNT)_R$-efficient frontier (for a given α confidence level) exists more often than the corresponding $(\mu, CVaR)_R$-efficient frontier. Additionally transforming the portfolio return's outcomes as is done by TNT, thus, partially helps to overcome the shortcomings of CVaR. If the $(\mu, TNT)_R$-efficient frontier exists, it is identical to the $(\mu, \sigma)_R$-efficient frontier; Tobin's theorem still applies, and the tangency portfolio remains unchanged compared to traditional variance as measure of risk.

5. Optimal portfolio choice and (non-)diversification

5.1. *Optimal portfolios*

We are now interested in investors' optimal portfolio choice: among the $(\mu, risk)$-efficient portfolios, we search for the specific portfolio that maximizes an investor's individual $(\mu, risk)$-utility function. As in the previous section, we assume the existence of n risky assets and a risk-free asset. As Tobin's theorem holds for all three risk measures variance, CVaR, and TNT, any $(\mu, risk)$-efficient portfolio comprises of the tangency portfolio and the risk-free asset.† The investment opportunity set thus is given by $\{(\beta, 1-\beta)^T | \beta \in \mathbb{R}\}$, where β denotes the share invested in the risky tangency portfolio, θ_T, and $(1-\beta)$ is the share invested in the risk-free asset. The portfolio returns' mean and standard deviation come to $\mu(\beta) = \beta \cdot \mu(\theta_T) + (1-\beta) \cdot \mu_0$ and $\sigma(\beta) = \beta \cdot \sigma(\theta_T)$. In the remainder of this section, we us the notion of 'diversification' if investors invest a positive fraction $(1-\beta) > 0$ in the risk-free asset and the positive fraction $\beta > 0$ in the tangency portfolio θ_T. Note that diversification of this kind refers to the basic liquidity preference problem in the sense of Tobin (1958) and may not be confused with diversification between risky assets in the sense of Markowitz (1952), where there is a smaller portfolio risk the more risky assets are involved.

5.2. *(Non-)diversification and minimum risk premium*

We start with the optimal portfolio choice of a (μ, σ)-investor and assume the utility function given by

$$U_1(\mu(\beta), \sigma(\beta)) = \mu(\beta) - \frac{\gamma}{2} \cdot \sigma^2(\beta), \gamma > 0. \tag{24}$$

This utility function is well established in the portfolio selection literature: first, it has a solid decision theoretic foundation as being the certainty equivalent of EU-investors with exponential utility function for normally distributed returns (CARA-normal case, Freund 1956, Bamberg 1986). Second, from an optimization point of view, a one-to-one correspondence holds between the constrained program of minimizing the portfolio variance for a given expected portfolio return (as employed in sections 3 and 4) and maximizing the (unconstrained) utility function $U_1(\mu, \sigma)$ (e.g. Steinbach 2001, theorem 1.9, Brandtner 2013, section 4.1); as such, our approach of finding optimal portfolios by means of the utility function U_1 is consistent with the previous determination of the (μ, σ)-efficient frontiers. Third, this utility function has striking analytical advantages also for related asset pricing purposes (see, e.g.Lintner 1969 inverse additivity rule, or Bamberg 1986).

For the (μ, σ)-utility function U_1 in (24), it is easily verified that the optimal investment in the tangency portfolio, $\beta_{\mu,\sigma} := \arg\max_\beta U_1(\mu(\beta), \sigma(\beta))$, is given by

$$\beta_{\mu,\sigma} = \frac{1}{\gamma} \cdot \frac{\mu(\theta_T) - \mu_0}{\sigma^2(\theta_T)}. \tag{25}$$

The (μ, σ)-investor will invest a positive share in the risky tangency portfolio if and only if the risk premium is positive, $\mu(\theta_T) - \mu_0 > 0$. The risky investment share is continuous and strictly monotonously increasing in the risk premium, and continuous and strictly monotonously decreasing in the tangency portfolio's variance, $\sigma^2(\theta_T)$, and the investor's risk aversion, γ.

† In order to ensure that the $(\mu, risk)$-efficient frontiers are non-empty for CVaR and TNT, we assume $k(\alpha) > \sqrt{D(\mu_0)}$ and $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) > \sqrt{D(\mu_0)}$, respectively.

We next come to CVaR, where we apply the utility function

$$U_2(\mu(\beta), CVaR(\beta)) = \lambda \cdot \mu(\beta) - (1-\lambda) \cdot CVaR_\alpha(\beta), \quad \lambda \in [0,1]. \tag{26}$$

This utility function has first been proposed by Acerbi and Simonetti (2002), and it has several advantages: First, like CVaR itself, the convex combination of the mean and (negative) CVaR still is a (negative) coherent and spectral risk measure (i.e. it satisfies any of the properties (M), (TI), (PH), (S), (LI), and (C)). Therefore, the determination of $(\mu, CVaR)$-efficient frontiers and the corresponding optimal portfolios is based on the same axiomatic framework. Second, minimizing CVaR for a given expected return (as has been done in sections 3 and 4) and maximizing the utility function U_2 in (26) yields the same $(\mu, CVaR)$-efficient frontiers (e.g. Krokhmal *et al.* 2002, theorem 3, Acerbi and Simonetti 2002, proposition 4.2). Third, consistent with the (μ, σ)-case, the relationship between risk and return is again linear. For a more detailed discussion of the utility function (26), we refer the reader to Brandtner (2013, section 4.1).

Under CVaR as the measure of risk, the optimal investment in the tangency portfolio, $\beta_{\mu,CVaR} := \arg\max_\beta U_2(\mu(\beta), CVaR(\beta))$, can be derived as follows: The first derivative of the portfolio return's utility with respect to β is given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\beta} U_2(\mu(\beta), CVaR(\beta)) \\
&= \frac{\mathrm{d}}{\mathrm{d}\beta}(\mu(\beta) - (1-\lambda) \cdot k(\alpha) \cdot \sigma(\theta_T)) \\
&= \frac{\mathrm{d}}{\mathrm{d}\beta}(\beta \cdot \mu(\theta_T) + (1-\beta) \cdot \mu_0 \\
&\quad - (1-\lambda) \cdot k(\alpha) \cdot \beta \cdot \sigma(\theta_T)) \\
&= \mu(\theta_T) - \mu_0 - (1-\lambda) \cdot k(\alpha) \cdot \sigma(\theta_T).
\end{aligned} \tag{27}$$

The derivative does not contain the decision variable β, and, therefore, we obtain

$$\beta_{\mu,CVaR} = \begin{cases} +\infty & \text{if } \dfrac{\mu(\theta_T) - \mu_0}{\sigma(\theta_T)} \geq (1-\lambda) \cdot k(\alpha) \\ -\infty & \text{if } \dfrac{\mu(\theta_T) - \mu_0}{\sigma(\theta_T)} < (1-\lambda) \cdot k(\alpha) \end{cases} \tag{28}$$

(for a more general analysis, see Brandtner 2013). $(\mu, CVaR)$-investors do not diversify. If the tangency portfolio's Sharpe-ratio is greater than the investor's risk aversion, infinite investment in the tangency portfolio, financed by borrowing at the risk-free rate, is optimal. If, in contrast, the tangency portfolio's Sharpe-ratio is smaller than the investor's risk aversion, we find infinite short-sales in the tangency portfolio and investment in the risk-free asset. If shortsales and borrowing are ruled out, i.e. $\beta \in [0,1]$, $(\mu, CVaR)$-investors will either invest their entire wealth in the tangency portfolio or in the risk-free asset. Formally, the all-or-nothing investment decision can be attributed to the critical linearity properties of (TI) and (PH); due to these properties, the portfolio return is linear in the share β, and accordingly, the first-order condition does not contain β.

The all-or-nothing decision under CVaR appears problematic for several reasons: first, it does not reflect the empirical evidence which clearly shows that investors hold diversified portfolios (e.g. Polkovnichenko 2005, section 1, Calvet *et al.* 2009, tables 1 and 2). Second, the comparative statics with respect to the risk premium and the investor's risk aversion are implausible, as they exhibit constant intervals and jump discontinuities: up to a certain degree of risk premium and risk aversion, we find no investment in the risky asset. Once a certain threshold is passed, the optimal investment entirely switches towards the risky tangency portfolio. Third, unlike it was the case under the variance, the set of $(\mu, CVaR)$-efficient portfolios does no longer coincide with the set of optimal portfolios; except of the two corner positions, no $(\mu, CVaR)$-efficient portfolio can be optimal under the utility function U_2 given in equation (27). Interestingly, Battauz *et al.* (2017a) find a similar counter-intuitive all-or nothing pattern in a dynamic investment context. There, if the investment horizon is long enough (investment horizon effect), aggressive investors take extreme long positions in a default-prone asset with excess expected return, which the authors refer to as nirvana optimal portfolio solution.

Finally, we analyze optimal portfolios under new TNT. TNT's additional non-linear transformation of the portfolio returns avoids the restrictive properties of (TI) and (PH), and, thus, will help to overcome CVaR's counterintuitive non-diversification. For the (μ, TNT)-utility function, we adopt the line of argument taken with the variance and CVaR, and employ the utility function

$$U_3(\mu(\beta), TNT_{\alpha,\lambda}(\beta)) = \delta \cdot \mu(\beta) - (1-\delta) \cdot TNT_{\alpha,\lambda}(\beta), \quad \delta \in [0,1]. \tag{29}$$

The (negative) utility function U_3 preserves the properties of (M), (C), and (LI) that are underlying TNT (Chen *et al.* 2012, proposition 5). At the same time, (29) can be derived from the constrained optimization in sections 3 and 4, and it again represents a linear risk-reward tradeoff. Regarding the optimal portfolio, we obtain

PROPOSITION 5.1 *The optimal investment in the tangency portfolio,*

$$\beta_{\mu,TNT} := \arg\max_\beta U_3(\mu(\beta), TNT(\beta)),$$

is implicitly given by

$$\frac{(\mu(\theta_T) - \mu_0)}{\sigma(\theta_T)} = \frac{(1-\delta) \cdot \left[(\lambda \cdot \beta_{\mu,TNT} \cdot \sigma(\theta_T)) + \frac{n(\lambda \cdot \beta_{\mu,TNT} \cdot \sigma(\theta_T) + N^{-1}(\alpha))}{N(\lambda \cdot \beta_{\mu,TNT} \cdot \sigma(\theta_T) + N^{-1}(\alpha))}\right]}{\frac{\alpha \cdot \delta}{\lambda \cdot \exp(-\lambda \cdot (\beta_{\mu,TNT} \cdot \mu(\theta_T) + (1-\beta_{\mu,TNT}) \cdot \mu_0) + \frac{\lambda^2}{2} \cdot \beta_{\mu,TNT}^2 \cdot \sigma^2(\theta_T)) \cdot N(\lambda \cdot \beta_{\mu,TNT} \cdot \sigma(\theta_T) + N^{-1}(\alpha))} + (1-\delta)}. \tag{30}$$

The share of risky investment is positive if and only if

$$\frac{(\mu(\theta_T) - \mu_0)}{\sigma(\theta_T)} > \frac{(1-\delta) \cdot \left[\frac{n(N^{-1}(\alpha))}{\alpha}\right]}{\left(\frac{\delta}{\lambda \cdot e^{-\lambda \cdot \mu_0}} + (1-\delta)\right)}, \tag{31}$$

and, in this case, is continuous and strictly monotonously increasing in the risk parameters α and δ, and the risk premium $(\mu(\theta_T) - \mu_0)$, and is continuous and strictly monotonously decreasing in the risk parameter λ and the tangency portfolio's standard deviation, $\sigma(\theta_T)$.

For the proof, see appendix 4. Recall the optimal portfolio decisions of (μ, σ)-investors and (μ, TNT)-investors: The former start investing in the risky tangency portfolio as soon as the risk premium is positive and then continuously increase the risky investment with increasing risk premium. The latter require a minimum risk premium, and, once it is reached, jump towards investing entirely risky in the tangency portfolio. The optimal portfolio decision of (μ, TNT)-investors now stands between these two cases: unlike under the variance and like CVaR, a positive risk premium is not sufficient for positive risky investment in the tangency portfolio. Instead, a minimum risk premium as defined in (31) is required. Once this minimum risk premium is reached, however, diversification prevails: the risky investment, in line with economic intuition, is continuous and strictly monotonously increasing in the risk premium and decreasing in the investor's risk aversion.

Note that TNT's portfolio pattern of diversification after a minimum risk premium is reached, finds strong empirical support. The well-known stock market participation puzzle describes the fact that even though stocks offer high expected returns (i.e. a positive risk premium), many investors do not allocate money to them (Mankiw and Zeldes 1991, Haliassos and Bertaut 1995). Among others, stock market participation is positively affected by age, wealth, and education. Now, if investors' preferences follow the (μ, TNT)-approach, the absence from the stock market can be explained by a sufficiently strong degree of risk aversion; for α, λ, and δ such that

$$\frac{(1-\delta)\cdot\left[\frac{n(N^{-1}(\alpha))}{\alpha}\right]}{\left(\frac{\delta}{\lambda\cdot e^{-\lambda\cdot\mu_0}}+(1-\delta)\right)} > \frac{(\mu(\theta_T)-\mu_0)}{\sigma(\theta_T)}, \tag{32}$$

it is rational for (μ, TNT)-investors, to invest in the risk-free asset only. If investor's risk aversion is reduced by means of increasing wealth (decreasing absolute risk aversion, which is well known to have strong empirical support) or some kind of higher education, this may serve as an explanation for the participation or absence of (μ, TNT)-investors in stock markets. Likewise, the portfolio structures chosen by (μ, TNT)-investors may serve as an explanation for the closely related equity premium puzzle (Mehra and Prescott 1985), according to which the difference between (risky) stock returns and (less risky or risk-free) bonds is considerably higher than proposed by standard asset pricing models. (μ, TNT)-investors, in particular when being strongly risk averse, ask for a significant risk premium before they are willing to invest in the risky (tangency) portfolio, which matches the investment pattern observed in practice.

6. Case study

In order to illustrate the theoretical results, we conduct an empirical portfolio selection study next. To this end, an investor can invest her wealth in three major stock indices, the US S&P500, the German DAX, and UK's FTSE100. Our analysis restricts to these three indices in order to highlight fundamental differences in measuring downside risk between the variance, CVaR, and new TNT.

Table 1. Summary statistics of S&P500, DAX, and FTSE 100. The means and standard deviations are reported in percentage terms.

	Expected rate of return	Standard deviation	Correlation coefficient		
	(%)	(%)	(1)	(2)	(3)
(1) S&P500	9.40	12.12	1.0	0.56	0.57
(2) DAX	4.90	18.62		1.00	0.79
(3) FTSE100	0.60	14.00			1.00

Table 1 reports the annualized means, annualized standard deviations, and correlation coefficients, which were computed using daily returns during the period 2014–2018. In this sample period, the S&P500 has performed superior in having both the highest expected return and the lowest standard deviation. DAX and FTSE100, in contrast, show the common risk–return pattern: DAX exhibits a higher expected return and, at the same time, a higher standard deviation when compared to FTSE100. Therefore, it can be expected that investors will show a strong tendency towards investing in the S&P500 under all three risk measures, while the investment decisions should differ regarding DAX and FTSE100.

We first assume that a risk-free asset is not available and determines the global minimum risk portfolios under the variance, CVaR, and TNT. First recall that under CVaR, this portfolio may not exist. For our sample period, the relevant condition $k(\alpha) > \sqrt{D/C}$ (see proposition 3.4) becomes $k(\alpha) > \sqrt{D/C} = 77.98\%$, and, therefore, $\alpha \geq 51.13\%$. As the confidence level chosen in actual risk management is rather low (the Basel III framework and the Swiss Solvency Test have implemented confidence levels of 2.5% and 1%, respectively), the question of existence of the global minimum CVaR portfolio can be neglected in our study. Nonetheless, TNT can still be seen superior to CVaR, as the question of existence does not arise at all under TNT.

Table 2 reports the composition of the three global minimum risk portfolios. For CVaR, we have chosen the standard confidence levels of $\alpha = 1\%$ and $\alpha = 5\%$, and for TNT we have set the additional risk parameter to $\lambda = 10$ and $\lambda = 30$. Figure 2 depicts TNT's exponential transformation function on the relevant domain of returns between -5% and 10%. For risk parameter $\lambda = 10$, the exponential transformation function's curvature is still low, while it becomes significantly more concave for $\lambda = 30$. Accordingly, the global minimum risk portfolios can be expected to not deviate between CVaR and TNT for $\lambda = 10$, while there may be a significant deviation for $\lambda = 30$.

The main findings are as follows: first, the global minimum CVaR portfolio at both confidence levels exhibits a significantly higher return and risk than the global minimum variance portfolio; the expected return increases by 2.86% ($\alpha = 0.01$) and 3.57% ($\alpha = 0.05$) and the standard deviation increases by 0.51% and 0.90%, respectively. Second, in terms of return and risk, the global minimum TNT portfolio lies between the global minimum variance portfolio and the global minimum CVaR portfolio. Third the global minimum TNT portfolio for the low risk parameter $\lambda = 10$ is almost

Table 2. Composition of the global minimum risk portfolios. The shares of the respective indices are reported in percentage terms.

		CVaR		TNT			
	Variance	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 1\%$ $\lambda = 10$	$\alpha = 1\%$ $\lambda = 30$	$\alpha = 5\%$ $\lambda = 10$	$\alpha = 5\%$ $\lambda = 30$
S&P500	68.66	93.48	101.78	91.98	86.19	97.90	87.00
DAX	− 18.23	− 6.71	− 2.86	− 7.40	− 10.09	− 4.66	− 9.71
FTSE100	49.57	13.23	1.07	15.42	23.91	6.76	22.71
μ	5.86	8.54	9.43	8.38	7.75	9.02	7.84
σ	11.23	11.74	12.13	11.69	11.49	11.94	11.51

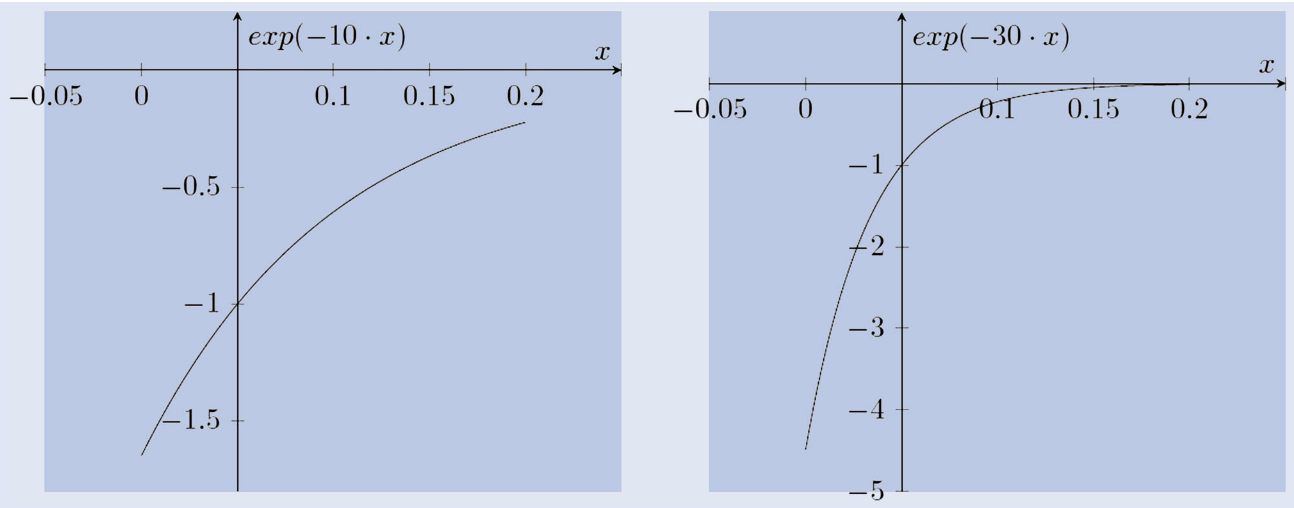

Figure 2. The exponential transformation function of TNT on the relevant domain of returns between −5% and 10% assuming $\lambda = 10$ (left) and $\lambda = 30$ (right).

identical to the one under CVaR, while they significantly differ for $\lambda = 30$. In the latter case, the global minimum TNT portfolio's expected return and standard deviation decrease by 0.79% and 0.25%, respectively. These findings confirm the significant impact of the additional transformation of the outcomes on the risk–return characteristics of the global minimum TNT portfolio. Fourth, investors for any of the risk measures invest a large share of their wealth in the S&P500, and the effect is particularly pronounced under CVaR and TNT; the reason is that these two risk measures implicitly contain components both of risk (standard deviation) and expected return, and that the S&P500 in the sample period simultaneously exhibits the smallest standard deviation and the highest expected return. Fifth, when increasing TNT's risk parameter λ from 10 to 30, we find a significant increase in the investment in FTSE100 of 8.49% and 15.95%, respectively, which is due to FTSE100's lower standard deviation, and which becomes more important when increasing the investor's risk aversion.

Now suppose that the risk-free asset is available. Following Battauz *et al.* (2017b), we use the EU Bund for different maturities on 14th December 2016 as a surrogate and fix the return μ_0 at an intermediate level of −0.80%. As such, the case study captures current market conditions in the eurozone.† For all three risk measures, the tangency portfolio is identical and has the following portfolio weights (in parentheses): S&P500 (175.36%), DAX (31.29%), and FTSE100 (−106.65%). Accordingly, the tangency portfolio has an expected rate of return of $\mu(\theta_T) = 17.37\%$, a standard deviation of $\sigma(\theta_T) = 18.55\%$, and a Sharpe-ratio of $\frac{\mu(\theta_T)-\mu_0}{\sigma(\theta_T)} = 0.98$; the latter will become relevant for investors' optimal portfolio choice below.

With respect to the risk measures of CVaR and TNT, we are also interested in the conditions under which the tangency portfolio exists. For CVaR, the relevant condition reads $k(\alpha) > \sqrt{D(\mu_0)} = 97.95\%$, which yields that the tangency portfolio is $(\mu, CVaR)$-efficient for $\alpha < 39.24\%$. For TNT, we obtain the condition $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) > \sqrt{D(\mu_0)} = 97.95\%$, assuming $\lambda = 10$ or $\lambda = 30$, the tangency portfolio always exists. Decreasing the risk parameter λ, the chosen confidence level becomes relevant, for example, if $\lambda = 1$ the tangency portfolio is (μ, TNT)-efficient for $\alpha < 41.33\%$. The results show that the condition imposed under CVaR is more strict than are the corresponding ones under TNT, i.e. the $(\mu, TNT)_R$-efficient frontier exists even if the $(\mu, CVaR)_R$-efficient frontier may not.

Finally, we examine the optimal portfolio choice of an investor when using the variance, CVaR and TNT. In case of the (μ, σ)-utility function U_1, the investor's optimal investment share in the tangency portfolio is given by

$$\beta_{\mu,\sigma} = \frac{1}{\gamma} \cdot \frac{\mu(\theta_T) - \mu_0}{\sigma^2(\theta_T)} = \frac{5.279}{\gamma}; \tag{33}$$

it is continuously and strictly decreasing in the investor's risk aversion γ. Additionally, the share of risky investment

† For a more comprehensive analysis of the impact of (negative) interest rates with practical applications, we refer the reader to Battauz *et al.* (2012) and Battauz and Rotondi (2019).

is positive if and only if the risk premium is positive, which is satisfied with our data, as $\mu(\theta_T) - \mu_0 = 0.1737 - (-0.08) = 0.1818 > 0$.

In contrast, under CVaR, the investor with the $(\mu, CVaR)$-utility function U_2 either invests entirely risky or entirely risk-free. Based on the data of our study, and the tangency portfolio's Sharpe-ratio in particular, we obtain

$$\beta_{\mu,CVaR} = \begin{cases} +\infty & \text{if } 0.979 \geq (1-\lambda) \cdot k(\alpha) \\ -\infty & \text{if } 0.979 < (1-\lambda) \cdot k(\alpha) \end{cases}. \tag{34}$$

For a 'pure' CVaR-minimizing investor (i.e. $\lambda = 0$), the threshold confidence level at which the investor switches from one corner position to the other is given by $k(\alpha) = 0.979$, i.e. $\alpha = 0.392$. For a 'mixed' mean-CVaR investor with $\lambda = 0.5$, the relevant confidence level satisfies $k(\alpha) = 1.958$, i.e. it decreases to $\alpha = 0.063$.

As pointed out in the previous section, (μ, TNT)-investors who maximize the utility function U_3 require a positive minimum risk premium before they start investing in the risky tangency portfolio. This minimum risk premium is given by

$$\frac{(\mu(\theta_T) - \mu_0)}{\sigma(\theta_T)} = \frac{(1-\delta) \cdot \left[\frac{n(N^{-1}(\alpha))}{\alpha}\right]}{\left(\frac{\delta}{\lambda \cdot e^{-\lambda \cdot \mu_0}} + (1-\delta)\right)}. \tag{35}$$

A 'pure' TNT-minimizing investor (i.e. $\delta = 0$), therefore, requires a minimum risk premium of

$$\mu(\theta_T) - \mu_0 = \frac{n(N^{-1}(\alpha))}{\alpha} \cdot \sigma(\theta_T), \tag{36}$$

which, for our tangency portfolio's standard deviation of $\sigma(\theta_T) = 18.55\%$ and confidence level $\alpha = 0.05$, comes to 38.27%. The 'mixed' mean-TNT investor with $\delta = 0.75$, $\alpha = 0.05$, and $\lambda = 10$ requires a smaller risk premium of 29.97%. The high value of the minimum risk premium is due to the high expected return and standard deviation of the tangency portfolio in the sample period.

7. Conclusion

In this paper, we have analyzed the Tail Nonlinearly Transformed Risk Measure (TNT) with respect to important properties and the application to portfolio selection. In particular, by assuming normally distributed portfolio returns, we have compared to optimal portfolios under TNT with those that arise under well-established Conditional Value-at-Risk. Our main conclusions are TNT is superior to CVaR with respect to (1) the portfolio selection between n risky assets, as the global minimum TNT portfolio always exists, which is not the case for CVaR, (2) the portfolio selection between n risky assets and a risk-free asset, as Tobin's theorem holds under less restrictive conditions on the investor's risk aversion under TNT when compared to CVaR, and (3) the choice of optimal portfolios, as CVaR yields counter-intuitive all-or-nothing decisions and non-diversification, while TNT investors diversify between a risky and a risk-free asset once a required minimum risk premium is reached.

Our analyses and findings may serve as a starting point for future research: (1) from a practitioner's point of view, additional evidence is required on the characteristics of optimal portfolios concerning the required minimum premium. Especially, empirical evidence is needed to determine the parameter setting in case of the (μ, TNT)-utility function in order to incorporate the equity premium puzzle or the market participation puzzle by using, for example, different datasets or simulation. (2) Interesting would be an extension of our single-period results to a multi-period setting. (3) We assume normal distribution as a first step to analyze TNT, so the extension of our normality-based results to the case of non-normality would be another point to investigate.

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No potential conflict of interest was reported by the author(s).

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Appendices

Appendix 1. Proof of proposition 2.3

First, we show that the global minimum-TNT portfolio always exists. For the proof, we use the result that the (μ, σ)-boundary and the (μ, TNT)-boundary coincide. Equation (6) implies that for any (μ, σ)-efficient portfolio the mean is given by

$$\mu(\theta) = \frac{A}{C} + \sqrt{\frac{D}{C} \cdot \left(\sigma^2(\theta) - \frac{1}{C}\right)} \tag{A1}$$

Further, the minimum-TNT portfolio is (μ, σ)-efficient since given a fixed mean we minimize the standard deviation and vice versa. Hence, using (9) and (A1), yields

$$\begin{aligned} &\min_{\sigma(\theta)\in[1/\sqrt{C},\infty)} \frac{1}{\alpha} \\ &\quad \cdot \exp\left(-\lambda(A/C + \sqrt{(D/C)(\sigma^2(\theta) - 1/C)} + \frac{\lambda}{2}\sigma^2(\theta)\right) \\ &\quad \cdot N(\lambda \cdot \sigma(\theta) + N^{-1}(\alpha)). \end{aligned} \tag{A2}$$

Note that

$$\begin{aligned} &\frac{\partial TNT(\sigma(\theta))}{\partial\sigma(\theta)} \\ &\quad = \left(\frac{-\lambda\sigma(\theta)\sqrt{D/C}}{\sqrt{\sigma^2(\theta) - 1/C}} + \lambda^2\sigma(\theta)\right) \\ &\qquad \cdot \exp\left(-\lambda(A/C + \sqrt{(D/C)(\sigma^2(\theta) - 1/C)} + \frac{\lambda}{2}\sigma^2(\theta)\right) \quad (A3) \\ &\qquad \cdot N(\lambda\sigma(\theta) + N^{-1}(\alpha)) \\ &\qquad + \exp\left(-\lambda(A/C + \sqrt{(D/C)(\sigma^2(\theta) - 1/C)} + \frac{\lambda}{2}\sigma^2(\theta)\right) \quad (A4) \\ &\qquad \cdot n(\lambda\sigma(\theta) + N^{-1}(\alpha)) \cdot \lambda \overset{!}{=} 0, \quad (A5) \end{aligned}$$

simplifying and replacing $s_{\alpha,\lambda}(\theta) := \lambda \cdot \sigma(\theta) + n(\lambda \cdot \sigma(\theta) + N^{-1}(\alpha))/N(\lambda \cdot \sigma(\theta) + N^{-1}(\alpha))$, we arrive at

$$\sigma(\theta)^2 - 1/C = \frac{\sigma^2(\theta)D/C}{s^2_{\alpha,\lambda}(\theta)}. \tag{A6}$$

Finally, the standard deviation of the minimum TNT portfolio is obtained by

$$\sigma(\theta_{TNT_{\alpha,\lambda}}) = \sqrt{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})/(C \cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D)}. \tag{A7}$$

It follows that the global minimum-TNT portfolio exists if $C \cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D > 0$ or, equivalently, $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) > \sqrt{\frac{D}{C}}$. We now show that the inequality always holds. The desired result follows from

$$s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) > \sqrt{\frac{D}{C}}. \tag{A8}$$

Replacing $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})$ by $\frac{\sigma\sqrt{D/C}}{\sqrt{\sigma^2-1/C}}$ yields

$$\frac{\sigma\sqrt{D/C}}{\sqrt{\sigma^2-1/C}} > \sqrt{\frac{D}{C}} \tag{A9}$$

$$\frac{\sigma}{\sqrt{\sigma^2-1/C}} > 1 \tag{A10}$$

$$\sigma^2 > \sigma^2 - 1/C. \tag{A11}$$

Since $C > 0$, the condition is always fulfilled and, as such, the global minimum-TNT portfolio always exists.

For $\lambda = 0$, the condition of existence of the minimum CVaR portfolio comes to

$$\frac{n(N^{-1}(\alpha))}{\alpha} > \sqrt{\frac{D}{C}}. \tag{A12}$$

This completes the first part of our proof.

Second, we show that equation (13) holds. It follows from equation (A1) and equation (A7) that

$$\mu(\theta_{TNT_{\alpha,\lambda}}) = \frac{A}{C} + \sqrt{\frac{D}{C}\cdot\left(\frac{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})}{Cs^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D} - \frac{1}{C}\right)} \tag{A13}$$

$$= \mu(\theta_\sigma) + \sqrt{\frac{D^2/C^2}{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C}}\cdot\sigma(\theta_\sigma) \tag{A14}$$

Equation (13) results from equation (A14) and the fact that for any mean $\mu \in \mathbb{R}$ there exists a unique portfolio on the (μ, σ)-boundary $\theta = g + h \cdot \mu$ (see Huang and Litzenberger 1988). This completes the second part of our proof.

Third, we show that equation (14) holds. Equation (5), equation (A7) and equation (A14) imply that

$$\begin{aligned} &TNT(\theta_{TNT_{\alpha,\lambda}}) \\ &= \exp\left(-\lambda\cdot\left(\mu(\theta_\sigma) + \sqrt{\frac{D^2/C^2}{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C}}\cdot\sigma(\theta_\sigma)\right)\right. \\ &\quad \left. + \frac{1}{2}\lambda^2\cdot\left(\frac{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})}{C\cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D}\right)\right) \\ &\quad \cdot \frac{N(\lambda\cdot\sqrt{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})/(C\cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D)}) - N^{-1}(1-\alpha))}{\alpha} \end{aligned} \tag{A15}$$

$$\begin{aligned} &= \exp\left(-\lambda\cdot\mu(\theta_\sigma) - \lambda\cdot\sqrt{\frac{D^2/C^2}{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C}}\cdot\sigma(\theta_\sigma)\right. \\ &\quad \left. + \frac{\lambda^2}{2}\cdot\frac{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})}{\cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D/C}\cdot\sigma^2(\theta_\sigma)\right) \\ &\quad \cdot \frac{N(\lambda\cdot\sqrt{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})/(C\cdot s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D)}) - N^{-1}(1-\alpha))}{\alpha}. \end{aligned} \tag{A16}$$

This completes the third part of our proof.

Appendix 2. Proof of proposition 4.3

We assume that $\mu_0 < A/C$. The set of $(\mu, TNT_{\alpha,\lambda})$-efficient portfolios is the subset of (μ, σ)-efficient portfolios with a mean greater than or equal to $\mu(\theta_{TNT_{\alpha,\lambda}})$. In order to be $(\mu, TNT_{\alpha,\lambda})$-efficient, the tangency portfolio has to fulfill the condition

$$\mu(\theta_T) \geq \mu(\theta_{TNT_{\alpha,\lambda}}) = \frac{A}{C} + \sqrt{\frac{D}{C}\cdot\left(\frac{s^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}})}{Cs^2_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) - D} - \frac{1}{C}\right)}. \tag{A17}$$

The mean of the tangency portfolio corresponds to $\frac{B-\mu_0\cdot A}{A-\mu_0\cdot C}$ and, as such, it can be easily shown that the mean of the tangency portfolio is greater than the mean of the global minimum TNT portfolio if $s_{\alpha,\lambda}(\theta_{TNT_{\alpha,\lambda}}) > \sqrt{D(\mu_0)}$.

Appendix 3. Proof of proposition 4.4

We show that for any given confidence level $\alpha \in (0, 1]$, $s_{\alpha,\lambda}(\theta) > k(\alpha)$ always holds. Note that $s_{\alpha,\lambda}(\theta) = \lambda\cdot\sigma(\theta) + \frac{n(\lambda\cdot\sigma(\theta)-N^{-1}(\alpha))}{N(\lambda\cdot\sigma(\theta)-N^{-1}(\alpha))}$. Comparing the latter term with $k(\alpha) = \frac{n(N^{-1}(\alpha))}{\alpha}$ yields

$$\frac{n(\lambda\cdot\sigma(\theta) - N^{-1}(\alpha))}{N(\lambda\cdot\sigma(\theta) - N^{-1}(\alpha))} > \frac{n(N^{-1}(\alpha))}{\alpha} \tag{A18}$$

$$\frac{n(\lambda\cdot\sigma(\theta) - N^{-1}(\alpha))}{N(\lambda\cdot\sigma(\theta) - N^{-1}(\alpha))} > \frac{n(-N^{-1}(1-\alpha))}{N(-N^{-1}(1-\alpha))} \tag{A19}$$

$$h(\lambda\cdot\sigma(\theta) - N^{-1}(\alpha)) > h(-N^{-1}(1-\alpha)), \tag{A20}$$

$$\text{with } h(\cdot) \text{ being the reversed hazard rate, we get} \tag{A21}$$

$$\lambda\cdot\sigma(\theta) - N^{-1}(\alpha) > -N^{-1}(1-\alpha) \tag{A22}$$

$$\lambda\cdot\sigma(\theta) > 0, \tag{A23}$$

which always holds. Since the first term of $s_{\alpha,\lambda}(\theta)$ is always greater than zero ($\lambda\cdot\sigma(\theta) > 0$), it immediately follows that $s_{\alpha,\lambda}(\theta) > k(\alpha)$. This completes our proof.

Appendix 4. Proof of equation proposition 5.1

We show that equation (31) holds. Differentiating equation (29) with respect to β yields

$$\begin{aligned} &\frac{\partial U_3(\mu(\beta), TNT(\beta))}{\partial\beta} \\ &\quad = \delta(\mu(\theta_T) - \mu_0) - (1-\delta)\Big[(-\lambda\cdot\mu(\theta_T) + \lambda\cdot\mu_0 \\ &\quad + \lambda^2\cdot\beta\cdot\sigma^2(\theta_T)) \end{aligned} \tag{A24}$$

$$\cdot\exp(-\lambda\cdot\beta\cdot\mu(\theta_T) - \lambda\cdot(1-\beta)\cdot\mu_0 + \frac{\lambda^2}{2}\cdot\beta^2\cdot\sigma^2(\theta_T)) \tag{A25}$$

$$\cdot\frac{N(\lambda\cdot\beta\cdot\sigma(\theta_T) - N^{-1}(1-\alpha))}{\alpha} \tag{A26}$$

$$+\exp(-\lambda\cdot\theta\cdot\mu_X - \lambda\cdot(1-\theta)\cdot\mu_0 + \frac{\lambda^2}{2}\cdot\theta^2\cdot\sigma^2_X) \tag{A27}$$

$$\cdot\frac{n(\lambda\cdot\beta\cdot\sigma(\theta_T) - N^{-1}(1-\alpha))}{\alpha}\cdot\lambda\cdot\sigma(\theta_T) \stackrel{!}{=} 0. \tag{A28}$$

Simplifying results in the following implicit relationship between the risk premium and the risky investment share

$$\frac{(\mu(\theta_T) - \mu_0)}{\sigma(\theta)}$$

$$= \frac{(1-\delta)\cdot\left[(\lambda\cdot\theta\cdot\sigma_X)+\frac{n(\lambda\cdot\beta\cdot\sigma(\theta_T)-N^{-1}(1-\alpha))}{N(\lambda\cdot\beta\cdot\sigma(\theta_T)-N^{-1}(1-\alpha))}\right]}{\left(\frac{\alpha\cdot\gamma}{\lambda\cdot\exp(-\lambda\cdot\beta\cdot\mu(\theta_T)-\lambda\cdot(1-\beta)\cdot\mu_0+\frac{\lambda^2}{2}\cdot\beta^2\cdot\sigma^2(\theta_T))\cdot N(\lambda\cdot\beta\cdot\sigma(\theta_T)-N^{-1}(1-\alpha))}+(1-\delta)\right)}. \tag{A29}$$

Setting $\beta = 0$ gives the minimum risk premium

$$\frac{(\mu(\theta_T)-\mu_0)}{\sigma(\theta_T)} = \frac{(1-\delta)\cdot\left[\frac{n(N^{-1}(\alpha))}{\alpha}\right]}{\left(\frac{\delta}{\lambda\cdot\exp(-\lambda\cdot\mu_0)}+(1-\delta)\right)}, \tag{A30}$$

which completes our proof.