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A second-order discretization with Malliavin weight and Quasi-Monte Carlo method for option pricing

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This paper shows a second-order discretization scheme for expectations of stochastic differential equations. We introduce a smart Malliavin weight which is given by a sum of simple polynomials of Brownian motions as an improvement of the scheme of Yamada [*J. Comput. Appl. Math.*, 2017, **321**, 427–447]. A new quasi-Monte Carlo simulation is proposed to obtain an efficient option pricing scheme. Numerical examples for the SABR model are shown to illustrate the validity of the scheme.

Keywords: Option pricing; European option; Digital option; Quasi-Monte Carlo method; SABR model; Weak approximation; Stochastic differential equations; Malliavin calculus

1. Introduction

Improving weak approximation is an urgent task for the financial industry since many important quantities in trading and risk management are given by the expectation of solutions to stochastic differential equations (SDEs). The existing and popular method is the Euler–Maruyama (EM) scheme. In industry, the scheme is often used with Monte Carlo (MC) simulation due to its ease of the implementation. The EM scheme is known as the order one weak approximation, that is, the accuracy is $O(n^{-1})$ if we discretize the SDE n-times based on a simple stochastic process. Bally and Talay (1996) showed that the convergence rate $O(n^{-1})$ is still valid even when the test functions are non-smooth. This result is important in finance because the test functions can be irregular in pricing and risk management. While the EM scheme is an order one weak scheme, Kusuoka (2001, 2004) showed a mathematical result for a higher order method under a general condition that can be applied to finance. Independently, Lyons and Victoir (2004) constructed the cubature method on Wiener space which has a similarity to Kusuoka's method. The scheme proposed in Kusuoka (2001, 2004), Lyons and Victoir (2004) is called the KLV method. Since then, practical implementation or extension of the higher order method has been studied by many authors (see Ninomiya and Victoir 2008, Tanaka and Kohatsu-Higa 2009, Gyurkó and Lyons 2011, Bayer *et al*. 2013a, 2013b, Crisan *et al*. 2013, Tanaka 2014, Morimoto and Sasada 2017, for example).

In Yamada (2017), an another type of higher order weak approximation scheme was shown, by extending the method in Takahashi and Yamada (2016). The author obtained a scheme for multidimensional SDEs using Malliavin weights given explicitly by polynomials of Brownian motions which come from the use of integration by parts on Wiener space. Then, discretization of the expectation of SDEs comes down to the computation of Malliavin weights. The author introduced an implementation method using semi-closed form approximation with numerical integration and showed numerical examples for an option pricing problem.

This paper describes an improvement of the scheme of Yamada (2017). The main purpose of the paper is to show a weak approximation with a new Malliavin weight scheme and a Quasi-Monte Carlo (QMC) method. We introduce a *smart Malliavin weight* constructed with a sum of simple polynomials of Brownian motions to discretize the expectation of SDEs efficiently. The weak approximation is justified using an improved error estimate over that in Yamada (2017). The QMC method for expectation with Malliavin weights will be a suitable simulation scheme for the weak approximation since the convergence rate of statistical error is better than the usual MC method and the expectation is efficiently computed in a high dimension setting. Non-smooth (Lipschitz continuous or bounded measurable) test functions are considered in the weak approximation to deal with numerical issues in finance. Our methodology can be applied to pricing European options under a stochastic volatility model and also to pricing digital options in multidimensional models. We perform numerical studies for European and digital call option pricing under the SABR

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stochastic volatility model to check the validity of the proposed weak approximation scheme.

After we prepare notation and the basics on stochastic calculus, section 3 shows the main result on the weak approximation. In section 4, the numerical scheme with the QMC method is introduced. Numerical experiments for option pricing are shown to confirm the effectiveness and validity of the method. Section 5 concludes and discusses future works. The technical proofs of the mathematical results in section 3 are given in appendix A.1.

2. Preliminaries

In this section, we summarize notations and basic results, which are necessary for the subsequent analysis. See Ikeda and Watanabe (1989), Nualart (2006), Yamada (2017) for more details.

Let $\mathscr{B}_b(\mathbb{R}^N)$ be the space of $\mathbb{R}$-valued bounded Borel measurable functions on $\mathbb{R}^N$ and $C_{Lip}(\mathbb{R}^N)$ be the space of $\mathbb{R}$-valued Lipschitz continuous functions on $\mathbb{R}^N$. For $f \in C_{Lip}(\mathbb{R}^N)$, denote by $C_{Lip}[f]$ the Lipschitz constant of f. Let $C_b^\infty(\mathbb{R}^N)$ be the set of all infinitely continuously differentiable functions $f : \mathbb{R}^N \to \mathbb{R}$ such that f and all of its partial derivatives are bounded. For $f \in C_b^\infty(\mathbb{R}^N)$, we write $\|\nabla^i f\|_\infty = \sup_{j_1,\dots,j_i \in \{1,\dots,N\}} \left\| \frac{\partial^i f}{\partial x_{j_1} \dots \partial x_{j_i}} \right\|_\infty$, $i \in \mathbb{N}$.

Let $(\mathscr{W}, H, P)$ be the d-dimensional Wiener space, that is, $\mathscr{W}$ is the space of continuous functions $w : [0, T] \to \mathbb{R}^d$ such that $w(0) = 0$, H is the Cameron–Martin space of all absolutely continuous functions $h : [0, T] \to \mathbb{R}^d$ with a square integrable derivative and P is the Wiener measure. H is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$. Let $W : [0, T] \times \mathscr{W} \to \mathbb{R}^d$ be the coordinate process defined by $W(t, w) = W_t(w) = w(t)$, $t \in [0, T]$, $w \in \mathscr{W}$, i.e. $(W_t)_t$ is a d-dimensional Brownian motion. We define $W(h)$ the Wiener integral by $W(h) = \sum_{i=1}^d \int_0^T \dot{h}^i(t) \mathrm{d}W_t^i$, $h \in H$. Let $\mathcal{S}$ denote the class of smooth random variables of the form $F = f(W(h_1), \dots, W(h_n))$ where $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in H$, $n \geq 1$. For $F \in \mathcal{S}$, we define the derivative DF as the H-valued random variable $DF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) h_i$, where $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$. For $F \in \mathcal{S}$, we set $D^j F = D \cdots DF$, $j \in \mathbb{N}$, as the $H^{\otimes j}$-valued random variable obtained iterating j-times the operator D. For $k \in \mathbb{N}$, $p \in [1, \infty)$, we define $\|F\|_{k,p}^p = E[|F|^p] + \sum_{j=1}^k E[\|D^j F\|_{H^{\otimes j}}^p]$, $F \in \mathcal{S}$. Then, the space $\mathbb{D}^{k,p}$ is defined as the completion of $\mathcal{S}$ respect to the seminorm $\|\cdot\|_{k,p}$. Further, let $\mathbb{D}^\infty$ be the space of smooth Wiener functionals in the sense of Malliavin $\mathbb{D}^\infty = \cap_{p \geq 1} \cap_{k \in \mathbb{N}} \mathbb{D}^{k,p}$. For $\mathbb{D}^{1,2}$, the stochastic process $D_t F = (D_{1,t} F, \dots, D_{d,t} F)$, $t \in [0, T]$ denotes the Malliavin derivative of F satisfying $\langle DF, h \rangle_H = \sum_{k=1}^d \int_0^T D_{k,t} F \dot{h}^k(t) \mathrm{d}t$, $h \in H$. Let δ be an unbounded operator from $L^2(\mathscr{W}; H)$ into $L^2(\mathscr{W})$ such that the domain of δ, denoted by $\mathrm{Dom}(\delta)$, is the set of H-valued square integrable random variables u such that $|E[\langle DF, u \rangle_H]| \leq C \|F\|_2$, for all $F \in \mathbb{D}^{1,2}$ where C is some constant depending on u, and if $u \in \mathrm{Dom}(\delta)$, $\delta(u) \in L^2(\mathscr{W})$ is characterized by

$$E[F\delta(u)] = E[\langle DF, u \rangle_H], \quad \text{for all} \quad F \in \mathbb{D}^{1,2}. \tag{2.1}$$

$\delta(u)$ is called Skorohod integral of the process u. The operator δ is continuous from $\mathbb{D}^\infty(H) \subset \mathrm{Dom}(\delta)$ into $\mathbb{D}^\infty$. For $F = (F_1, \dots, F_N) \in (\mathbb{D}^\infty)^N$, define the Malliavin covariance matrix $\sigma^F = (\sigma_{i,j}^F)_{1 \leq i,j \leq N}$ as $\sigma_{i,j}^F = \langle DF_i, DF_j \rangle_H = \sum_{k=1}^d \int_0^T D_{k,s} F_i D_{k,s} F_j \mathrm{d}s$, $1 \leq i, j \leq N$. We say that $F \in (\mathbb{D}^\infty)^N$ is nondegenerate if the matrix σ^F is invertible *a.s.* and $\|(\det \sigma^F)^{-1}\|_p < \infty$, $1 \leq p < \infty$. We summarize the integration by parts on Wiener space (see Nualart 2006, Proposition 2.14).

PROPOSITION 1 *Let $F \in (\mathbb{D}^\infty)^N$ be a nondegenerate and $G \in \mathbb{D}^\infty$. Then, for any f belonging to the space $C_b^\infty(\mathbb{R}^N)$ and multi-index $(\kappa_1, \dots, \kappa_l) \in \{1, \dots, N\}^l$, $l \geq 1$, there exists $H_{(\kappa_1,\dots,\kappa_l)}(F, G)$ such that*

$$E[\partial_{\kappa_1} \cdots \partial_{\kappa_l} f(F) G] = E[f(F) H_{(\kappa_1,\dots,\kappa_l)}(F, G)] \tag{2.2}$$

where $\partial_{\kappa_1} \cdots \partial_{\kappa_l} = \frac{\partial^l f}{\partial x_{\kappa_1} \cdots \partial x_{\kappa_l}}$. Moreover, $H_{(\kappa_1,\dots,\kappa_l)}(F, G)$ is recursively given by

$$H_{(i)}(F, G) = \delta\left(\sum_{j=1}^N G \gamma_{ij}^F DF_j\right), \quad i = 1, \dots, N, \tag{2.3}$$

$$H_{(\kappa_1,\dots,\kappa_l)}(F, G) = H_{(\kappa_l)}(F, H_{(\kappa_1,\dots,\kappa_{l-1})}(F, G)), \tag{2.4}$$

where γ^F is the inverse matrix of the Malliavin covariance of F.

The random variable $H_{(\kappa_1,\dots,\kappa_l)}(F, G)$ in (2.2) is called Malliavin weight.

3. Weak approximation

This section shows a new weak approximation scheme of the order $O(1/n^2)$, where non-smooth test functions are considered so that we can apply it to numerical problems in finance.

3.1. Stochastic differential equations

On d-dimensional Wiener space, let $W = (W_t)_t = ((W_t^1, \dots, W_t^d))_t$ be a d-dimensional Brownian motion and $X = (X_t(x))_t$ be the solution to following SDE

$$\mathrm{d}X_t(x) = V_0(X_t(x))\mathrm{d}t + \sum_{i=1}^d V_i(X_t(x))\mathrm{d}W_t^i,$$
$$X_0(x) = x \in \mathbb{R}^N, \tag{3.1}$$

where $V_i^j \in C_b^\infty(\mathbb{R}^N)$, $i = 0, 1, \dots, d$, $j = 1, \dots, N$. We assume the following elliptic condition on the vector fields: there exists $\epsilon > 0$ such that $\sum_{i=1}^d V_i(x) \otimes V_i(x) \geq \epsilon I$ for any $x \in \mathbb{R}^N$ where I denotes the identity matrix. Let $(P_t)_t$ be a semigroup of linear operators given by

$$(P_t \varphi)(x) = E[\varphi(X_t(x))], \quad (t, x) \in [0, T] \times \mathbb{R}^N,$$

where $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a test function appropriately chosen.

Let L^i, $i = 0, 1, \dots, d$ be the differential operators defined as follows: for $\varphi \in C_b^\infty(\mathbb{R}^N)$,

$$L^i \varphi(x) = \sum_{k=1}^N V_i^k(x) \frac{\partial \varphi}{\partial x_k}(x), \quad i = 1, \dots, d,$$

$$L^0\varphi(x) = \sum_{k=1}^{N} V_0^k(x)\frac{\partial \varphi}{\partial x_k}(x) + \frac{1}{2}\sum_{k,l=1}^{N}\sum_{j=1}^{d} V_j^k(x)V_j^l(x)\frac{\partial^2\varphi}{\partial x_k \partial x_l}(x).$$

We define a norm of multi-index $\alpha = (\alpha_1, \dots, \alpha_r) \in \{0, 1, \dots, d\}^r$, $r \in \mathbb{N}$ as $\|\alpha\| = 2\#\{i; \alpha_i = 0\} + \#\{i; \alpha_i \neq 0\}$. Let $(\overline{X}_t^{\mathrm{EM}}(x))_t$ the (one-shot) Euler–Maruyama scheme given by

$$\overline{X}_t^{\mathrm{EM}}(x) = x + V_0(x)t + \sum_{i=1}^{d} V_i(x)W_t^i, \quad t \in [0, T], \quad x \in \mathbb{R}^N.$$

For $G \in \mathbb{D}^\infty$, we define the Malliavin weight with respect to $\overline{X}_t^{\mathrm{EM}}(x)$ given by

$$H_{(i)}(\overline{X}_t^{\mathrm{EM}}(x), G) = \sum_{k=1}^{d}\sum_{j=1}^{N}[A^{-1}]_{ij}V_k^j(x)[GW_t^k - \int_0^t D_{k,s}G\mathrm{d}s], \quad 1 \le i \le N. \tag{3.2}$$

where $A = (A_{ij})_{1\le i,j\le N}$ is the matrix defined by $A_{ij} = t\sum_{k=1}^{d} V_k^i(x)V_k^j(x)$, $1 \le i, j \le N$, and $D_{k,s}G$, $k = 1, \dots, d$, $s \in [0, t]$ is the Malliavin derivative of Wiener functional $G \in \mathbb{D}^\infty$ with respect to k-th Brownian motion. Here after, we shall write $W_t^0 = t$ and use the notation for iterated Itô integral

$$I_{(\alpha_1,\dots,\alpha_r)}(t) = \int_{0<t_1<\cdots<t_r<t} \mathrm{d}W_{t_1}^{\alpha_1}\cdots \mathrm{d}W_{t_r}^{\alpha_r}$$

for multi-index $\alpha = (\alpha_1, \dots, \alpha_r) \in \{0, 1, \dots, d\}^r$, $r \in \mathbb{N}$.

This paper shows an efficient weak scheme with a Malliavin weight for the target expectation given by

$$P_T f(x) = E[f(X_T(x))], \quad T > 0, \ x \in \mathbb{R}^N, \tag{3.3}$$

where f could be non-smooth. The aim is to give an improvement of the scheme in Yamada (2017). We shall construct a Malliavin weight which have potential to be second-order discretization and will be simpler than one in Theorem 1 of Yamada (2017).

Before we show main results, we briefly review the approach and the result of Yamada (2017). In Yamada (2017), the following Taylor formula **[T]** is shown to obtain a second-order discretization:

[T] $E[\varphi(X_t(x))]$

$$= E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))] + \sum_{\beta_1=2}^{5}\sum_{\kappa_1\in\{1,\dots,N\}}\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_r)\in\{0,1,\dots,d\}^r, r\ge 2,\\ \|\alpha\|=\beta_l}} L^{\alpha_1}\cdots L^{\alpha_{r-1}}V_{\alpha_r}^{\kappa_1}(x) \times E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1)}(\overline{X}_t^{\mathrm{EM}}(x), I_{(\alpha_1,\dots,\alpha_r)}(t))] \tag{3.4}$$

$$+ \sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}\sum_{\alpha=(\alpha_1,\dots,\alpha_4)\in\{1,\dots,d\}^4} \frac{1}{2}L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)L^{\alpha_3}V_{\alpha_4}^{\kappa_2}(x)E[\varphi(\overline{X}_t^{\mathrm{EM}}(x)) H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x), I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4)}(t))] \tag{3.5}$$

$$+ \sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_4)\in\{0,1,\dots,d\}^4,\\ \alpha_i=0,\ 1\le\alpha_j,\alpha_3,\alpha_4\le d,\ (i,j)=(1,2),(2,1)}} L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)L^{\alpha_3}V_{\alpha_4}^{\kappa_2}(x) \times E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x), I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4)}(t))] \tag{3.6}$$

$$+ \sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}\sum_{\alpha=(\alpha_1,\dots,\alpha_5)\in\{1,\dots,d\}^5} L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)L^{\alpha_3}L^{\alpha_4}V_{\alpha_5}^{\kappa_2}(x) \times E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x), I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4,\alpha_5)}(t))] + R_\varphi(t, x), \tag{3.7}$$

where $R_\varphi(t, x)$ satisfies

$$\sup_{x\in\mathbb{R}^N} |R_\varphi(t, x)| \le \begin{cases} C\sum_{i=1}^{3}\|\nabla^i\varphi\|_\infty t^3, & \text{if } \varphi \in C_b^\infty(\mathbb{R}^N), \\ CC_{Lip}[\varphi]t^2, & \text{if } \varphi \in C_{Lip}(\mathbb{R}^N), \\ C\|\varphi\|_\infty t^{\frac{3}{2}}, & \text{if } \varphi \in \mathscr{B}_b(\mathbb{R}^N), \end{cases} \tag{3.8}$$

for some constant $C > 0$. The residual analysis in the above is due to the following estimates for expectations: for nondegenerate, $F \in (\mathbb{D}^\infty)^N$ and $G \in \mathbb{D}^\infty$ and multi-index $(\kappa_1, \dots, \kappa_k) \in \{1, \dots, N\}^k$,

$$|E[\varphi(F)H_{(\kappa_1,\dots,\kappa_k)}(F, G)]| \le C\|\nabla^k\varphi\|_\infty\|G\|_p, \quad \varphi \in C_b^\infty(\mathbb{R}^N), \tag{3.9}$$

$$|E[\varphi(F)H_{(\kappa_1,\dots,\kappa_k)}(F, G)]| \le CC_{Lip}[\varphi]\|H_{(\kappa_1,\dots,\kappa_{k-1})}(F, G)\|_p, \quad \varphi \in C_{Lip}(\mathbb{R}^N), \tag{3.10}$$

$$|E[\varphi(F)H_{(\kappa_1,\dots,\kappa_k)}(F, G)]| \le C\|\varphi\|_\infty\|H_{(\kappa_1,\dots,\kappa_k)}(F, G)\|_p, \quad \varphi \in \mathscr{B}_b(\mathbb{R}^N), \tag{3.11}$$

for $p \ge 2$. Then, by the principle that local approximation of order $O(t^3)$ for smooth test function $\varphi \in C_b^\infty(\mathbb{R}^N)$ gives discretization of order $O(n^{-2})$, a weak approximation can be obtained based on **[T]** (see Theorem 1 of Yamada (2017)). Although the weak approximation in Yamada (2017) is explicitly computed, the Malliavin weights in the local approximation are bit complicated. This is why we want to introduce an improvement of local approximation in this paper.

3.2. Main result

We use a new sharp local approximation of the order $O(t^3)$ in order to obtain an efficient weak approximation of order $O(n^{-2})$. This will be constructed by a smaller number of Malliavin weights by reducing some approximation terms in **[T]** in (A10). Throughout this paper, we call the sum of weights in the new local approximation *smart Malliavin weight*. To construct smart Malliavin weight, we firstly prepare the following key estimate of the order $O(t^3)$.

LEMMA 1 *Assume $\varphi \in C_b^\infty(\mathbb{R}^N)$. Let $\alpha = (\alpha_1, \dots, \alpha_r) \in \{0, 1, \dots, d\}^r$, $3 \le r \le 5$, $3 \le \|\alpha\| \le 5$ and $\kappa = (\kappa_1, \dots, \kappa_k) \in \{1, \dots, N\}^k$, $k \in \mathbb{N}$. There exists $C = C_V > 0$ depending only on the family of vector fields $V = (V_1, \dots, V_d)$*

such that

$$\sup_{x\in\mathbb{R}^N}\left|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\dots,\kappa_k)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\dots,\alpha_r)}(t))]\right| \le C\|\nabla^{k+n(\alpha)}\varphi\|_\infty t^3,\quad t\in(0,1], \tag{3.12}$$

where $n(\alpha)=6-\|\alpha\|$.

Proof See appendix A.1. □

Remark 1 In (3.12), we attain $O(t^3)$-bound although the standard estimates (3.9) and (3.11) (as in Kusuoka and Stroock (1984), Nualart (2006)) give the order less than $O(t^3)=O(t^{6/2})$ as

$$\sup_{x\in\mathbb{R}^N}\left|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\dots,\kappa_k)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\dots,\alpha_r)}(t))]\right| \le C\|\varphi\|_\infty t^{(\|\alpha\|-k)/2}, \tag{3.13}$$

$$\sup_{x\in\mathbb{R}^N}\left|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\dots,\kappa_k)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\dots,\alpha_r)}(t))]\right| \le C\|\nabla^k\varphi\|_\infty t^{\|\alpha\|/2}, \tag{3.14}$$

for $\|\alpha\|=3,4,5$. The estimate (3.12) is shaper than (3.13) and (3.14) in small $t\in(0,1]$. Lemma 1 suggests that the term such as $E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\dots,\kappa_k)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\dots,\alpha_r)}(t))]$, $3\le r\le 5$, $3\le\|\alpha\|\le 5$ can be dealt with the remainder term of the local approximation since the order of the expectation is $O(t^3)$. Then, we can reduce some Malliavin weights in **[T]**. The estimate (3.12) of Lemma 1 will be a key result in the construction of smart weak approximation.

Using Lemma 1, we have the following local approximation of the order $O(t^3)$ for the case of the test function $\varphi\in C_b^\infty(\mathbb{R}^N)$, which is much simpler than [**T**].

PROPOSITION 2 *We have*

[**T**′]

$$\begin{aligned}E[\varphi(X_t(x))] &= E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))] + \sum_{k\in\{1,\dots,N\}}\sum_{(\alpha_1,\alpha_2)\in\{0,1,\dots,d\}^2} L^{\alpha_1}V_{\alpha_2}^k(x)E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(k)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2)}(t))] \qquad (3.15)\\ &\quad+\sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}\sum_{(\alpha_1,\alpha_2)\in\{1,\dots,d\}^2}\frac{1}{2}L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)\times L^{\alpha_1}V_{\alpha_2}^{\kappa_2}(x)E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(0,0)}(t))]+\hat{R}_\varphi(t,x), \qquad (3.16)\end{aligned}$$

where $\hat{R}_\varphi(t,x)$ *satisfies*

$$\sup_{x\in\mathbb{R}^N}|\hat{R}_\varphi(t,x)|\le\begin{cases}C\sum_{i=1}^4\|\nabla^i\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^{3/2}, & \text{if } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N),\end{cases} \tag{3.17}$$

for some constant $C>0$.

Proof See appendix A.2. □

Remark 2 The correction terms in the above approximation [**T**′] become simpler than those in [**T**], by controlling $O(t^3)$-bound when $\varphi\in C_b^\infty(\mathbb{R}^N)$. Then, we can attain $O(t^3)$ -approximation for $\varphi\in C_b^\infty(\mathbb{R}^N)$ with a smaller number of Malliavin weights. We also note that the error bounds for $\varphi\in C_{Lip}(\mathbb{R}^N)$ and $\varphi\in\mathscr{B}_b(\mathbb{R}^N)$ change.

We now show a new weak approximation. Let $\gamma>0$, $n\in\mathbb{N}$ and define $t_i=T(1-(1-i/n)^\gamma)$, $i=0,1,\dots,n$ and $s_i=t_i-t_{i-1}$, $i=1,\dots,n$. The partition $(t_0,t_1,\dots,t_n)$ was introduced in Kusuoka (2001, 2004), Lyons and Victoir (2004). By choosing the parameter γ appropriately, we can attain second-order discretization using smart Malliavin weight $\pi_t(x,W_t)$ defined in the below.

The following is our main result on weak approximation.

THEOREM 1 *We have*

$$\left\|P_Tf-Q_{(s_1)}Q_{(s_2)}\cdots Q_{(s_n)}f\right\|_\infty=O\left(\frac{1}{n^2}\right), \tag{3.18}$$

when $f\in C_{Lip}(\mathbb{R}^N)$ *and* $\gamma>4/3$ *or when* $f\in\mathscr{B}_b(\mathbb{R}^N)$ *and* $\gamma>2$, *where* $(Q_{(t)})_t$ *is operators given by* $Q_{(t)}\varphi(x)=E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))\pi_t(x,W_t)]$, $t\in(0,T]$, $x\in\mathbb{R}^N$ *with Malliavin weight* $\pi_t(x,W_t)$

$$\begin{aligned}\pi_t(x,W_t) &= 1+\sum_{\kappa_1\in\{1,\dots,N\}}c_{\kappa_1}(t,x)H_{(\kappa_1)}(\overline{X}_t^{\mathrm{EM}}(x),1) \qquad (3.19)\\ &\quad+\sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}c_{\kappa_1,\kappa_2}(t,x)H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),1) \qquad (3.20)\\ &\quad+\sum_{(\kappa_1,\kappa_2,\kappa_3)\in\{1,\dots,N\}^3}c_{\kappa_1,\kappa_2,\kappa_3}(t,x)H_{(\kappa_1,\kappa_2,\kappa_3)}(\overline{X}_t^{\mathrm{EM}}(x),1), \qquad (3.21)\end{aligned}$$

with coefficients

$$\begin{aligned}c_{\kappa_1}(t,x)&=L^0V_0^{\kappa_1}(x)\frac{1}{2}t^2,\\ c_{\kappa_1,\kappa_2}(t,x)&=\sum_{(\alpha_1,\alpha_2)\in\{1,\dots,d\}^2}\frac{1}{2}t^2\left\{\{L^0V_{\alpha_1}^{\kappa_1}(x)+L^{\alpha_1}V_0^{\kappa_1}(x)\}\times V_{\alpha_1}^{\kappa_2}(x)+\frac{1}{2}L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)L^{\alpha_1}V_{\alpha_2}^{\kappa_2}(x)\right\},\\ c_{\kappa_1,\kappa_2,\kappa_3}(t,x)&=\sum_{(\alpha_1,\alpha_2)\in\{1,\dots,d\}^2}L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)V_{\alpha_2}^{\kappa_2}(x)V_{\alpha_1}^{\kappa_3}(x)\frac{1}{2}t^2.\end{aligned}$$

Proof See appendix A.3. □

Remark 3 The result of the second-order discretization in Theorem 1 is remarkable since when $f\in\mathscr{B}_b(\mathbb{R}^N)$ usual small time estimate $\|P_tf-Q_{(t)}f\|_\infty\le Ct\|f\|_\infty$ (see the bottom of (A10) in appendix A.2) tells us $\|P_Tf-Q_{(T/n)}^nf\|_\infty\le\sum_{i=1}^nC(1/n)\|f\|_\infty=C\|f\|_\infty=O(1)$, that is, $Q_{(T/n)}^nf(x)$ cannot be a discretization. However, Theorem 1 claims that $\|P_Tf-Q_{(s_1)}Q_{(s_2)}\cdots Q_{(s_n)}f\|_\infty=O(n^{-2})$, then the order of discretization substantially rises. This is a consequence of use of Lemma 1, Proposition 2 and Kusuoka's partition. Also, the weak approximation is an improvement of Yamada (2017) in the sense that the Malliavin weight is much simplified. The result should be compared with Theorem 1 of Yamada (2017). Moreover, we note that the term (3.21) is typically appeared in the lowest order expansion of expectation of SDEs via Malliavin calculus (e.g. see Takahashi and Yamada 2012) and is very easy to compute. Indeed, $H_{(\kappa_1,\dots,\kappa_r)}(\overline{X}_t^{\mathrm{EM}}(x),1)$, $(\kappa_1,\dots,\kappa_r)\in\{1,\dots,N\}^r$, $r\le 3$, is obtained by polynomials of Brownian motions up to third order by explicit computation using (3.2). The approximation (3.18) suggests that we could attain higher order weak approximation using the lowest order expansion of expectation of SDEs with some additional correction terms (3.19), (3.20). Then, the implementation of the weak approximation is quite easy.

For financial applications, we state the following result as a corollary of the above theorem.

COROLLARY 1 *For $x \in \mathbb{R}^N$, we have*

$$\sup_{K\in\mathbb{R}} \left| P_T f_K(x) - Q_{(s_1)} Q_{(s_2)} \cdots Q_{(s_n)} f_K(x) \right| = O\left(\frac{1}{n^2}\right), \tag{3.22}$$

where f_K is the pay-off function given by $f_K(x) = (x_1 - K)^+ := \max\{x_1 - K, 0\}$ or $f_K(x) = \mathbf{1}_{\{x_1 \geq K\}}$, and γ is chosen as $\gamma > 4/3$ when $f_K(x) = (x_1 - K)^+$ or $\gamma > 2$ when $f_K(x) = \mathbf{1}_{\{x_1 \geq K\}}$. Here, $(Q_{(t)})_t$ is same operators in Theorem 1.

Proof Since $C_{Lip}[f_K] = 1$ when $f_K(x) = (x_1 - K)^+$ and $\| f_K \|_\infty = 1$ when $f_K(x) = \mathbf{1}_{\{x_1 \geq K\}}$ for all $K \in \mathbb{R}$, the rate of convergence of the weak approximation in Theorem 1 is uniform in K, by the proof in appendix A.3. □

4. Numerical scheme with Quasi-Monte Carlo simulation

4.1. Algorithm

We introduce a QMC method for the weak approximation in Theorem 1 (and Corollary 1): $Q_{(s_1)} Q_{(s_2)} \cdots Q_{(s_n)} f(x)$, $x \in \mathbb{R}^N$, $s_i = t_i - t_{i-1}$, $i = 1, \ldots, n$ where $t_i = T(1 - (1 - i/n)^\gamma)$, $i = 0, 1, \ldots, n$. The quantity $Q_{(s_1)} Q_{(s_2)} \cdots Q_{(s_n)} f(x)$, the approximation for $P_T f(x)$, can be represented by

$$\begin{aligned} P_T f(x) &\simeq Q_{(s_1)} Q_{(s_2)} \cdots Q_{(s_n)} f(x) \\ &= E\left[f\left(\overline{X}_T^{\mathrm{EM},(n)}\right) \prod_{i=1}^n \pi_{s_i}\left(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n)}, \Delta W_{s_i}\right) \right] \end{aligned} \tag{4.1}$$

where $\left(\overline{X}_{t_i}^{\mathrm{EM},(n)}\right)_{i=0,1,\ldots,n}$ is the EM scheme

$$\begin{aligned} \overline{X}_0^{\mathrm{EM},(n)} &= x, \\ \overline{X}_{t_i}^{\mathrm{EM},(n)} &= \overline{X}_{t_{i-1}}^{\mathrm{EM},(n)} + V_0(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n)}) s_i \\ &\quad + \sum_{k=1}^d V_k(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n)}) \Delta W_{s_i}^k, \quad i = 1, \ldots, n, \end{aligned}$$

with $\Delta W_{s_i} = (\Delta W_{s_i}^1, \ldots, \Delta W_{s_i}^d)$, $\Delta W_{s_i}^k = W_{t_i}^k - W_{t_{i-1}}^k$, $k = 1, \ldots, d$. Note that the EM scheme is a functional of ΔW_{s_i}, $i = 1, \ldots, n$ and denote it by $\overline{X}_{t_i}^{\mathrm{EM},(n)} = \overline{X}_{t_i}^{\mathrm{EM},(n)}(\Delta W_{s_1}, \ldots, \Delta W_{s_i})$. Then, (4.1) is approximated by the QMC method as

$$\begin{aligned} &E\left[f\left(\overline{X}_T^{\mathrm{EM},(n)}\right) \prod_{i=1}^n \pi_{s_i}\left(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n)}, \Delta W_{s_i}\right) \right] \\ &\simeq \frac{1}{M} \sum_{j=1}^M f\left(\overline{X}_T^{\mathrm{EM},(n),[j]}\right) \prod_{i=1}^n \pi_{s_i}\left(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]}, \sqrt{s_i}\xi_i^j\right), \end{aligned} \tag{4.2}$$

with d-dimensional vector ξ_i^j, $i = 1, \ldots, n$ of which component is given by $\xi_{i,l}^j = \Phi^{-1}(y_{(i-1)d+l}^j)$ $(1 \leq l \leq d)$ through the inverse of the normal distribution function Φ^{-1} and $(n \times d)$-dimensional vector $y^j = (y_1^j, \ldots, y_{n\times d}^j)$ from a low discrepancy sequence, in our case the *Sobol' sequence*. Here, $\overline{X}_{t_i}^{\mathrm{EM},(n),[j]}$, $1 \leq i \leq n$, $1 \leq j \leq M$ represents $\overline{X}_{t_i}^{\mathrm{EM},(n),[j]} = \overline{X}_{t_i}^{\mathrm{EM},(n)}(\sqrt{s_1}\xi_1^j, \ldots, \sqrt{s_i}\xi_i^j)$. We note that ξ_i^j is directly obtained from *rnorm.sobol* function in fOptions library in R, for example. To be more clear in the numerical method, we show its precise algorithm below. Also, to see how the proposed scheme works as an extension of the EM scheme, we describe the algorithms of the EM scheme (**Algorithm 1**) and the Malliavin weight scheme (**Algorithm 2**) for comparison. Here fter, $X \sim N(0, 1)$ represents a Gaussian random variable with mean 1 and variance 0.

Algorithm 1 The Euler–Maruyama scheme

for $j = 1$ **to** M **do**
  $\overline{X}_{t_0}^{\mathrm{EM},(n),[j]} = x.$
  for $i = 1$ **to** n **do**
    Simulate i.i.d. $Z = (Z^1, \ldots, Z^d)$, $Z^k \sim N(0, 1)$
    Update $\overline{X}_{t_i}^{\mathrm{EM},(n),[j]} = \overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]} + V_0(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]})(t_i - t_{i-1}) + \sum_{k=1}^d V_k(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]})\sqrt{t_i - t_{i-1}} Z^k$
  end for
end for
Return $\frac{1}{M}\sum_{j=1}^M f(\overline{X}_T^{\mathrm{EM},(n),[j]})$.

Algorithm 2 The Malliavin weight scheme

Generate $\xi_i^j = (\xi_{i,1}^j, \ldots, \xi_{i,d}^j)$, $\xi_{i,k}^j = \Phi^{-1}(y_{(i-1)d+k}^j)$, $1 \leq i \leq n$, $1 \leq k \leq d$ from $(n \times d)$-dimensional vector $y^j = (y_1^j, \ldots, y_{n\times d}^j)$ from a low discrepancy sequence, $j = 1, \ldots, M$
for $j = 1$ **to** M **do**
  $\overline{X}_{t_0}^{\mathrm{EM},(n),[j]} = x$, $\mathcal{W}_{t_0}^{[j]} = 1.$
  for $i = 1$ **to** n **do**
    Update Malliavin weight $\mathcal{W}_{t_i}^{[j]} = \mathcal{W}_{t_{i-1}}^{[j]} \times \pi_{t_i - t_{i-1}}(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]}, \sqrt{t_i - t_{i-1}}\xi_i^j)$
    Update $\overline{X}_{t_i}^{\mathrm{EM},(n),[j]} = \overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]} + V_0(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]})(t_i - t_{i-1}) + \sum_{k=1}^d V_k(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]})\sqrt{t_i - t_{i-1}}\xi_{i,k}^j$
  end for
end for
Return $\frac{1}{M}\sum_{j=1}^M f(\overline{X}_T^{\mathrm{EM},(n),[j]})\mathcal{W}_T^{[j]}$.

By **Algorithm 2**, the Malliavin weight scheme is regarded as a natural extension of the EM scheme. Indeed, the proposed scheme is easily implemented by generating the paths of the EM scheme and the Malliavin weight. The Malliavin weight scheme enables us to give a smart computation since

- we can attain very accurate result with a few time steps,
- we only need the standard EM scheme and the product of Malliavin weights,
- QMC provides a faster convergence than the standard MC, in practice,
- the methodology is durable in a multidimensional integration problem.

More precisely, we have the following as a corollary of Theorem 1.

COROLLARY 2 *Let $f \in C_b^\infty(\mathbb{R}^N)$. Then, there exists $C > 0$ which is independent to n, and $c = c(n \times d) > 0$ depending on $n \times d$ such that*

$$\left| P_T f(x) - \frac{1}{M} \sum_{j=1}^{M} f\left(\overline{X}_T^{\mathrm{EM},(n),[j]}\right) \times \prod_{i=1}^{n} \pi_{s_i}\left(\overline{X}_{t_{i-1}}^{\mathrm{EM},(n),[j]}, \sqrt{s_i}\xi_i^j\right) \right| \leq C\frac{1}{n^2} + c\frac{(\log M)^{n\times d}}{M},$$

where γ and the Malliavin weight $\pi_t(x, W_t)$ are same as in Theorem 1.

Proof The weak approximation error has been already obtained in Theorem 1. The integration error on the approximation (4.2) is given as $c\frac{(\log M)^{n\times d}}{M}$ where $c = c(n \times d) > 0$ depends on the dimension $n \times d$ of the target integral. □

Remark 4 To theoretically ensure that the variation of integrand in the sense of Hardy and Krause is finite, we put sufficient smoothness condition on f in the above (not for weak approximation). See Glasserman (2004) for more details on QMC. Although the condition may be restrictive, QMC usually works well even if the integrands are less regular. The result of Corollary 2 would hold for non-smooth pay-offs when smoothing procedure can be appropriately applied. See section 4.3.2 for numerical experiment on QMC of our scheme.

Under appropriate conditions, it is known the rate of convergence of QMC method behave nearly $O(1/M)$ or $O(1/M^{1-\epsilon})$ where $\epsilon > 0$ depends on the dimension of the integral, while standard MC method converges as $O(1/\sqrt{M})$.

4.2. Features compared with existing second-order scheme

We now turn to compare our method with the *Ninomiya-Victoir* (NV, in short) scheme, the well-known second-order weak method introduced by Ninomiya and Victoir (2008), that can be applied to numerical problems in finance. Although the proposed Malliavin weight scheme has same weak order to the NV scheme and also both are implemented by QMC, the algorithms and their features are quite different. The NV scheme is based on the following one-shot process

$$\overline{X}_t^{\mathrm{NV}}(x) = \begin{cases} \exp(\frac{t}{2}V_0) \circ \exp(\sqrt{t}Z^1 V_1) \circ \cdots \circ \exp(\sqrt{t}Z^d V_d) \\ \quad \circ \exp(\frac{t}{2}V_0)(x), \quad \text{if } N = 1, \\ \exp(\frac{t}{2}V_0) \circ \exp(\sqrt{t}Z^d V_d) \circ \cdots \circ \exp(\sqrt{t}Z^1 V_1) \\ \quad \circ \exp(\frac{t}{2}V_0)(x), \quad \text{if } N = -1, \end{cases} \tag{4.3}$$

where N is a Bernoulli random variable with the distribution $P(N = 1) = P(N = -1) = 1/2$ and $(Z^i)_{i=1,\ldots,d}$ is a family of independent random variables with $Z^i \sim N(0, 1)$ such that n and $(Z^i)_{i=1,\ldots,d}$ are independent. Then, the NV scheme is given by an approximation $(Q_{(T/n)}^{\mathrm{NV}})^n f(x)$ toward $E[f(X_T(x))]$, where $(Q_{(t)}^{\mathrm{NV}})_t$ is a local operator defined by $(Q_{(t)}^{\mathrm{NV}}\varphi)(x) = E[\varphi(\overline{X}_t^{\mathrm{NV}}(x))]$ through $(d + 1)$-dimensional random variable $(N, Z^1, \ldots, Z^d)$. The algorithm of the NV scheme differs from those of our method and both methodologies have the following features:

- The NV scheme needs to solve $(n\times(d+1))$-dimensional integral by QMC whose dimension is higher than ours since the Malliavin weight scheme targets $(n \times d)$-dimensional integral. In theoretical view point, the integration error of the NV scheme might be larger than that of the Malliavin weight scheme.
- The NV scheme holds under hypoelliptic (or uniform finite generation) condition while we put ellipticity on the vector fields for the Malliavin weight scheme. Then, the NV scheme theoretically works on a weaker condition. This is an advantage of the NV scheme.
  However, in the practical view point, the numerical algorithm of the NV scheme depends on the solvability of ordinary differential equation (ODE) $\frac{dy_t}{dt} = V_i(y_t)$ with the vector field $V_i, i = 0, 1, \ldots, d$. This is because the NV scheme needs to compose the solution flows of vector field $\exp(tV_0)(x)$ or of random vector field $\exp(tZV_i)(x)$, $i = 1, \ldots, d$, $Z \sim N(0, 1)$ as in (4.3). When the vector fields are too simple or of specific type, ODE flows could be analytically solved (this case is called 'lucky case' in Bayer *et al.* (2013a), Morimoto and Sasada (2017)) and then the NV scheme can be computed faster in such case. The problem occurs when the ODE flows $\exp(tV_i)(x)$, $i = 1, \ldots, d$ do not have analytical solution. This case typically appears in finance models, for example, the ODE flows of the SABR stochastic volatility model are not analytically tractable. The case of ODE flows has no analytical solution, the NV scheme has to rely on some numerical method of ODE, which may cause time-consuming computation. We mention that this problem was partially relaxed by some ideas in Bayer *et al.* (2013a), Morimoto and Sasada (2017). Consequently, in the NV scheme, the form of the vector fields and corresponding ODEs has to be checked and the numerical method should be chosen according to SDEs or finance models, i.e. the NV scheme is model dependent.
  On the other hand, the Malliavin weight scheme of this paper works whether ODE flows can be solved analytically or not because only we need to compute the EM scheme and the Malliavin weight $\pi_t(x, W_t)$. Although we impose the elliptic condition on the vector fields, the proposed scheme can be applied to a wide class of SDEs or models in a unified way.
- Both second-order weak approximation schemes work on non-smooth test functions. The applicability to non-smooth test functions of Malliavin weight scheme was just shown using Kusuoka's partition in the previous section. The NV scheme is implemented by uniform partition, which was completely justified by the recent work of Kusuoka (2013).

We summarize the features on the NV scheme and the Malliavin weight scheme in the table 1.

We could see these differences on the schemes through a numerical experiment in Section 4.3.3 (see table 2).

Table 1. The Ninomiya–Victoir scheme and the Malliavin weight scheme.

Method	Ninomiya–Victoir	Malliavin weight
Discretization error	$O\left(\frac{1}{n^2}\right)$	$O\left(\frac{1}{n^2}\right)$
Integration error	$\mathrm{const}(n \times (d+1))\ \frac{(\log M)^{n\times(d+1)}}{M}$	$\mathrm{const}(n \times d)\ \frac{(\log M)^{n\times d}}{M}$
Condition on vector fields	Hypoellipticity	Ellipticity
ODE flows $e^{sV_i}(x)$	Need analytical or numerical solution	Do Not need analytical tractability
Test function	Smooth or non-smooth	Smooth or non-smooth
Time partition	Uniform partition	Kusuoka's partition

4.3. *Numerical experiments for SABR model*

We apply the weak approximation with QMC method to the SABR stochastic volatility model. Let $(W_t^1, W_t^2)_t$ be a two-dimensional standard Brownian motion and $(X_t^1(x), X_t^2(x))_t = (S_t, \sigma_t)_t$ be the following two-dimensional process

$$\mathrm{d}S_t = \sigma_t C(S_t)\mathrm{d}W_t^1,$$
$$\mathrm{d}\sigma_t = \nu\sigma_t(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2),$$

where $\nu > 0$, $\rho \in [-1, 1]$ and C is the local volatility function, that is, we consider the case $d = N = 2$ in section 2. The target expectation is the option price $E[f_K(S_T)]$ with the European call pay-off $f_K(x) = \max\{x - K, 0\}$ (Lipschitz continuous function) or digital call option pay-off $f_K(x) = \mathbf{1}_{\{x\geq K\}}$ (non-smooth bounded Borel function of discontinuous type). The parameters are specified as $T = 2.0$, $S_0 = 100$, $C(x) = x^{\beta}$, $\sigma_0 = 0.3 \times S_0^{1-\beta}$ (lognormal scaling), $\beta = 0.5$, $\nu = 0.1$, $\rho = -0.5$. Although this example may not satisfy the mathematical condition imposed in section 2 rigorously, the method can be still applied to the model by borrowing the modification method of Takahashi (2015) (see Section 4.3 in Takahashi (2015)). The partition parameter γ is chosen as $\gamma = 1.34 > 4/3$ for the European call option and $\gamma = 2.01 > 2$ for the digital call option to give the examples of Theorem 1 and Corollary 1. The numerical experiments of the scheme are performed by MacBook Pro with Intel Core i7 CPU with 3.3 GHz and 16 GB RAM.

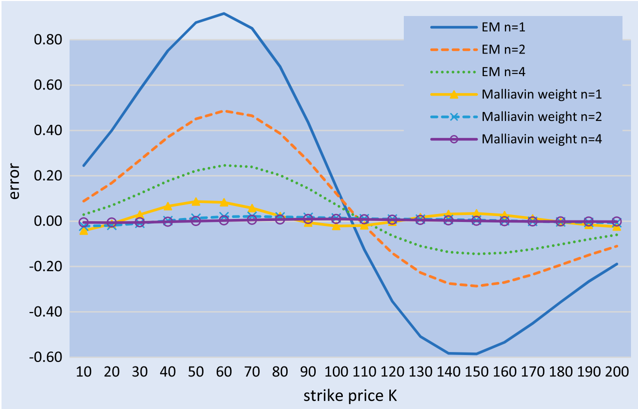

Figure 1. Numerical error for European call option under SABR model.

4.3.1. Weak approximation error. First, we analyse weak convergence rate of the Malliavin weight scheme. We perform numerical experiments for the Malliavin weight scheme by varying the strike price $K = 10, 20, \ldots, 190, 200$ to see the effectiveness of Corollary 1 numerically and to check the accuracy for deep In-The-Money (ITM) and Out-of-The-Money (OTM) cases. The Malliavin weight scheme is computed using QMC simulation scheme (4.2) with number of time steps $n = 2^0, 2^1, \ldots, 2^3$ and number of paths $M = 10^7$. The benchmark value or exact value (Exact) is computed by the Euler–Maruyama scheme (EM) and Monte Carlo simulation (MC) with number of time steps $n = 2^{11} = 2{,}048$ and number of paths $M = 10^7$. In order to compare the methodology, we compute same option price using EM scheme and MC simulation with $n = 2^0, 2^1, \ldots, 2^7$ and $M = 10^7$. We perform these computations for each strike price $K = 10, 20, \ldots, 190, 200$. We shall write Malliavin weight=Malliavin weight(n, K), EM=EM(n, K), Exact=Exact(n, K) to emphasis the condition on n and K.

To check the validity of Theorem 1 or Corollary 1, we plot numerical errors (Malliavin Weight(n, K)) − (Exact(n, K)) and (EM(n, K) − (Exact(n, K)) for n and K, and uniform errors $\max_{K=10,\ldots,200} |\text{Malliavin Weight}(n, K)-\text{Exact}(n, K)|$ and $\max_{K=10,\ldots,200} |\text{EM}(n, K) - \text{Exact}(n, K)|$ for n. The numerical errors are in figures 1 and 2 and the uniform errors are in figures 3 and 4, for the European call option and the digital call option, respectively.

By figures 1 and 2, we see that the proposed scheme (Malliavin weight) substantially improves accuracy for all strike prices compared with the EM scheme (EM). Moreover, by figures 3 and 4, we can observe that the Malliavin weight scheme attains smaller numerical error with a few time steps. Indeed, to control the errors within 5.0×10^{-3} for the Lipschitz continuous pay-off case and within 3.0×10^{-4} for the bounded Borel pay-off case, the Malliavin weight scheme requires only a time steps $n = 8$ for the both cases while the EM scheme needs $n = 128$ or more than it. This suggests that the Malliavin weight scheme can attain enough accuracy through lower dimensional integration, i.e. $(n \times d)$-dimensional integration with a smaller n, and the fact reinforces the advantage of the proposed QMC method. Further, by figures 3 and 4, it can be confirmed that the rate of convergence with respect to discretization number n is almost close to the order two, i.e. $O(n^{-2})$.

4.3.2. Integration error. Next, we perform a convergence analysis for QMC simulation of the Malliavin weight scheme to check the performance of Corollary 2. We compute the ATM European call option price ($K = 100$) for each set of paths $M = i \times 2.0 \times 10^4, i = 1, 2, \ldots, 500$ by the Malliavin weight

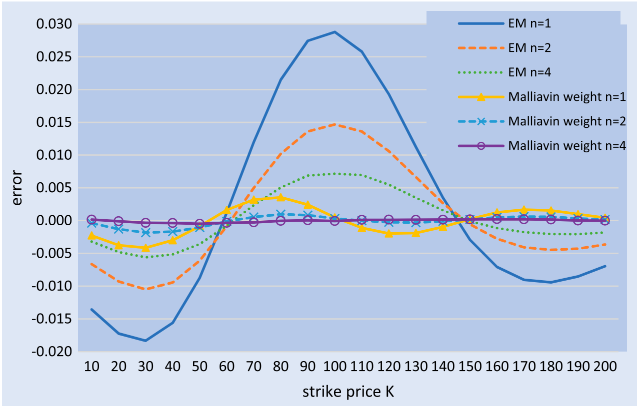

Figure 2. Numerical error for digital call option under SABR model.

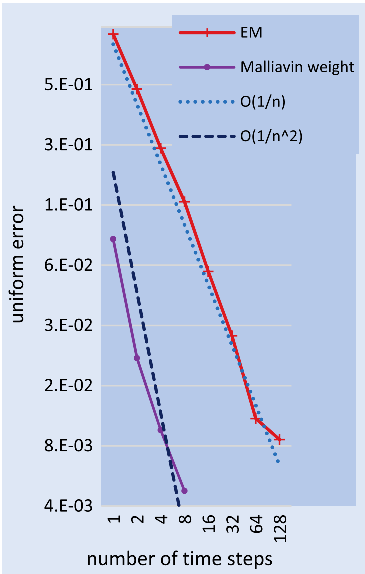

Figure 3. Uniform error for European call option under SABR model.

scheme with time steps $n = 8$. For comparison, we perform the same computation for the EM scheme with MC simulation where the number of time steps is taken large enough, $n = 2^{11} = 2,048$, to ensure the accuracy of discretization. Here, the exact price is computed by the EM scheme and MC simulation with large numbers of paths $M = 10^7$ and time steps $n = 2^{11} = 2,048$. The result is shown in figure 5. By the figure, we can see that the accuracy of QMC simulation of the Malliavin weight scheme is quite good since the scheme attains fast convergence with a smaller simulation number as we expected, while the EM scheme needs a larger number of paths. Moreover, it is worth remarking that the difference between (Malliavin weight $n = 8$) and (Exact price) in figure 5 still involves weak approximation error because the Malliavin weight scheme is performed with a small number of time steps $n = 8$, but the difference is almost zero when the number of paths increases. Then, we can again check that the Malliavin weight scheme works well as weak approximation, by this experiment. These results suggest the effectiveness and the validity of the Malliavin weight scheme with QMC simulation.

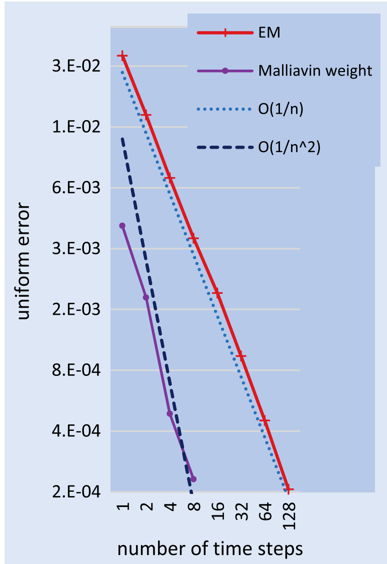

Figure 4. Uniform error for digital call option under SABR model.

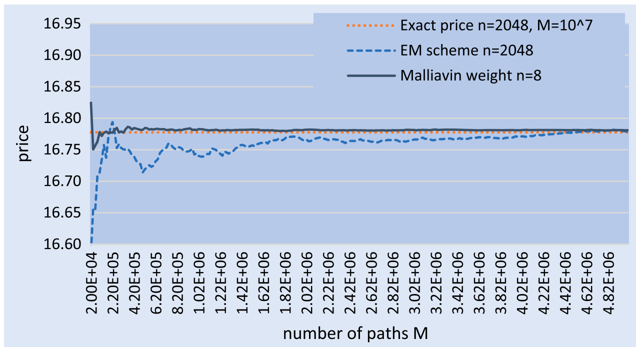

Figure 5. The rate of convergence of the Malliavin weight scheme with respect to number of paths.

4.3.3. Comparison with the Ninomiya–Victoir scheme. We finally perform a comparison analysis for the second-order weak approximations, the Ninomiya–Victoir (NV) scheme and the Malliavin weight scheme. For the comparison, we will use the numerical results of the NV scheme and its variant, the 'NV with drift' scheme, reported in Bayer *et al.* (2013a). See Bayer *et al.* (2013a) for the details of the 'NV with drift' scheme. Let us take the parameters for the SABR model as $T = 1.0, K = 1.05, S_0 = 1.0, C(x) = x^{\beta}, \sigma_0 = 0.3, \beta = 0.9$, $\nu = 0.4$, $\rho = -0.7$ as in Bayer *et al.* (2013a). The exact price of European call option is 0.09400046. In Bayer *et al.* (2013a), the authors performed the EM scheme, the NV scheme and the 'NV with drift' scheme and compared these numerical values with the exact price. We shall perform the same analysis for the Malliavin weight scheme.

The results are summarized in table 2 below. In the table, we list the relative errors for the exact price and the computation times for the numerical schemes (EM, NV, NV with drift, Malliavin weight) with numbers of time steps n and paths M. By comparing the relative errors, we can check the accuracy of the Malliavin weight scheme outperform that of the NV scheme and the 'NV with drift' scheme. Also, our results are

Table 2. The accuracy and the computation time of the NV scheme and the Malliavin weight scheme.

Method	Number of time steps n	Number of paths M	Relative Error	Time (s)
Euler–Maruyama	32	512000	0.00150	5.87
Ninomiya–Victoir	2	512000	0.00134	2.44
Ninomiya–Victoir with drift	2	128000	0.00140	0.28
Malliavin weight	2	512000	0.00004	0.38
Malliavin weight	2	128000	0.00010	0.10

stable in terms of integration error because the numerical value of the Malliavin weight scheme with a number of paths $M = 1.28 \times 10^3$ is very close to the value with $M = 5.12 \times 10^3$. Further, it is obvious to see the Malliavin weight scheme can be computed in a short time. Indeed, the Malliavin weight scheme takes 0.38 s when $M = 5.12 \times 10^3$ and also takes 0.10 s when $M = 1.28 \times 10^3$ while the NV scheme and the 'NV with drift' scheme take 2.44 s (the case $M = 5.12 \times 10^3$) and 0.28 s (the case $M = 1.28 \times 10^3$), respectively. These results must be a consequence of the differences of the methods as well as their algorithms, as we discussed in section 4.2.

5. Concluding remarks

This paper showed a new weak approximation scheme as an improvement of Yamada (2017). We introduced a smart Malliavin weight and a QMC method to obtain an efficient numerical methodology. Our method will be suitable for computation of option prices with irregular pay-offs since the scheme is constructed and justified for non-smooth test functions. The validity and the effectiveness of the proposed scheme are illustrated through numerical examples for European and digital call options. Further, we confirmed that numerical results are compatible with the theoretical part of the paper.

The applications and the extensions of the proposed scheme (e.g. for multilevel Monte Carlo method (Tanaka and Yamada 2014, Debrabant and Rößler 2015) will be interesting topics. Also, it is important to consider whether mathematical conditions imposed in the paper can be weakened so that we can extend the application range or deal with more wide class of models such as the hypoelliptic case. These should be discussed in future work.

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Appendix 1.

In the appendix, we denote by $C > 0$ a generic constant whose value might change from line to line.

A.1. Proof of Lemma 1

In this proof, we will mainly use the relation between the operators, the Skorohod integral δ and the Malliavin derivative D, in the integration by parts (2.1), especially,

$$E[g(\overline{X}_t^{\mathrm{EM}}(x))\int_0^t u_s^k \mathrm{d}W_s^k] = E[\int_0^t D_{k,s}g(\overline{X}_t^{\mathrm{EM}}(x))u_s^k ds],\quad k=1,\ldots,d, \tag{A1}$$

where g is a function that belongs to $C_b^\infty(\mathbb{R}^N)$ and $u \in L^2([0,t]\times\mathscr{W})$ is a $\mathbb{R}^d$-valued adapted process. Here, we note that the sum of the Itô integrals $\int_0^t u_s^k \mathrm{d}W_s^k$, $k=1,\ldots,d$ in the left-hand side of (A1) corresponds to the Skorohod integral, in the case that the integrand is the adapted process. We apply (A1) to the expectation

$$E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))I_{(\alpha_1,\ldots,\alpha_r)}(t)] = E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\int_{0<t_1<\cdots<t_r<t}\mathrm{d}W_{t_1}^{\alpha_1}\cdots\mathrm{d}W_{t_r}^{\alpha_r}],$$

where $g=\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi$ and $\alpha=(\alpha_1,\ldots,\alpha_r)$ with $3\le r\le 5$ and $3\le\|\alpha\|\le 5$. If $\alpha_r\neq 0$, we can see

$$\begin{aligned}
&E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\int_{0<t_1<\cdots<t_r<t}\mathrm{d}W_{t_1}^{\alpha_1}\cdots\mathrm{d}W_{t_r}^{\alpha_r}]\\
&= E[\int_0^t D_{\alpha_r,t_r}\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\\
&\quad \int_{0<t_1<\cdots<t_{r-1}<t_r}\mathrm{d}W_{t_1}^{\alpha_1}\cdots\mathrm{d}W_{t_{r-1}}^{\alpha_{r-1}}dt_r]\\
&= E[\int_0^t \sum_{1\le\kappa_{k+1}\le N}\partial_{\kappa_1}\cdots\partial_{\kappa_k}\partial_{\kappa_{k+1}}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\\
&\quad D_{\alpha_r,t_r}\overline{X}_t^{\mathrm{EM},\kappa_{k+1}}(x)\\
&\quad \int_{0<t_1<\cdots<t_{r-1}<t_r}\mathrm{d}W_{t_1}^{\alpha_1}\cdots\mathrm{d}W_{t_{r-1}}^{\alpha_{r-1}}dt_r]\\
&= \sum_{1\le\kappa_{k+1}\le N}V_{\alpha_r}^{\kappa_{k+1}}(x)\int_0^t E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\partial_{\kappa_{k+1}}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\\
&\quad \int_{0<t_1<\cdots<t_{r-1}<t_r}\mathrm{d}W_{t_1}^{\alpha_1}\cdots\mathrm{d}W_{t_{r-1}}^{\alpha_{r-1}}]dt_r.
\end{aligned}$$

Also, if $\alpha_r=0$, we have

$$\begin{aligned}
&E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\int_{0<t_1<\cdots<t_r<t}\mathrm{d}W_{t_1}^{\alpha_1}\cdots\mathrm{d}W_{t_r}^{\alpha_r}]\\
&= \int_0^t E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\int_{0<t_1<\cdots<t_{r-1}<t_r}\\
&\quad \mathrm{d}W_{t_1}^{\alpha_1}\cdots\mathrm{d}W_{t_{r-1}}^{\alpha_{r-1}}]dt_r.
\end{aligned}$$

Define $\alpha_{-1} := (\alpha_1\ldots,\alpha_{r-1})$ and apply the above computations for $\alpha_{r-1}=0$ or $\neq 0$. Define $\alpha_{-k} := (\alpha_1\ldots,\alpha_{r-k})$, $k=1,\ldots,r$ and iterate this procedure k times until $k=n(\alpha)=6-\|\alpha\|$. Then, there exists a multi-index $J=(j_1,\ldots,j_r)\in\{0,1,\ldots,d\}^r$ with $\|J\|=6$ such that

$$\begin{aligned}
&E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))I_{(\alpha_1,\ldots,\alpha_r)}(t)]\\
&= \sum_{1\le\kappa_{k+1},\ldots,\kappa_{k+n(\alpha)}\le N}E\Big[\partial_{\kappa_1}\cdots\partial_{\kappa_{k+n(\alpha)}}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\\
&\quad \times\int_{0<t_1<\cdots<t_r<t}\prod_{i=1}^{n(\alpha)}D_{\alpha_{l_i},t_{l_i}}\overline{X}_t^{\mathrm{EM},\kappa_{k+i}}(x)\mathrm{d}W_{t_1}^{j_1}\cdots\mathrm{d}W_{t_r}^{j_r}\Big],
\end{aligned}$$

where $\alpha_{l_i}\neq 0$ and $j_{l_i}=0$ for $l_i\in\{r-n(\alpha),\ldots,r\}$ satisfying $l_i\neq l_k$, $1\le i,k\le n(\alpha)$. Since we can see

$$\begin{aligned}
&\int_{0<t_1<\cdots<t_r<t}\prod_{i=1}^{n(\alpha)}D_{\alpha_{l_i},t_{l_i}}\overline{X}_t^{\mathrm{EM},\kappa_{k+i}}(x)\mathrm{d}W_{t_1}^{j_1}\cdots\mathrm{d}W_{t_r}^{j_r}\\
&= \prod_{i=1}^{n(\alpha)}V_{\alpha_{l_i}}^{\kappa_{k+i}}(x)\int_{0<t_1<\cdots<t_r<t}\mathrm{d}W_{t_1}^{j_1}\cdots\mathrm{d}W_{t_r}^{j_r},
\end{aligned}$$

it holds

$$\begin{aligned}
&E[\partial_{\kappa_1}\cdots\partial_{\kappa_k}\varphi(\overline{X}_t^{\mathrm{EM}}(x))I_{(\alpha_1,\ldots,\alpha_r)}(t)]\\
&= \sum_{1\le\kappa_{k+1},\ldots,\kappa_{k+n(\alpha)}\le N}E\Big[\partial_{\kappa_1}\cdots\partial_{\kappa_{k+n(\alpha)}}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\\
&\quad \times\prod_{i=1}^{n(\alpha)}V_{\alpha_{l_i}}^{\kappa_{k+i}}(x)\int_{0<t_1<\cdots<t_r<t}\mathrm{d}W_{t_1}^{j_1}\cdots\mathrm{d}W_{t_r}^{j_r}\Big].
\end{aligned}$$

With the standard estimate for $\int_{0<t_1<\cdots<t_r<t}\mathrm{d}W_{t_1}^{j_1}\cdots\mathrm{d}W_{t_r}^{j_r}$, $(j_1,\ldots,j_r)\in\{0,1,\ldots,d\}^r$, i.e. for $p\ge 2$, $\left\|\int_{0<t_1<\cdots<t_r<t}\mathrm{d}W_{t_1}^{j_1}\cdots\mathrm{d}W_{t_r}^{j_r}\right\|_p\le Ct^{\|J\|/2}$, we therefore have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^N}\left|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\ldots,\kappa_k)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\ldots,\alpha_r)}(t))]\right|\\
&\le\|\nabla^{k+n(\alpha)}\varphi\|_\infty Ct^{\|J\|/2}=\|\nabla^{k+n(\alpha)}\varphi\|_\infty Ct^3,
\end{aligned}$$

where $C=C_V>0$ depending only on the family of vector fields $V=(V_1,\ldots,V_d)$. □

A.2. Proof of Proposition 2

Using the estimate of Lemma 1, we will reduce some approximation terms in Taylor formula **[T]** but still keep the rate of convergence of approximation error at the order $O(t^3)$. We directly apply Lemma 1 to some terms in (3.4) in **[T]** as follows: for $\kappa_1=1,\ldots,N$ and $\alpha=(\alpha_1,\ldots,\alpha_r)\in\{0,1,\ldots,d\}^r$, $3\le r\le 5$, $3\le\|\alpha\|\le 5$,

$$\begin{aligned}
&\left|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\ldots,\alpha_r)}(t))]\right|\\
&\le C\|\nabla^{1+n(\alpha)}\varphi\|_\infty t^3,\quad n(\alpha)=6-\|\alpha\|.
\end{aligned}$$

We can also apply Lemma 1 to the terms (3.5), (3.6) and (3.7) in **[T]**. By Yamada (2017), the term $I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4)}(t)$ in (3.5) and (3.6) in **[T]** can be decomposed as

$$\begin{aligned}
&I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4)}(t)\\
&= I_{(\alpha_3,\alpha_4,\alpha_1,\alpha_2)}(t)+I_{(\alpha_3,\alpha_1,\alpha_4,\alpha_2)}(t)\\
&\quad +I_{(\alpha_1,\alpha_3,\alpha_4,\alpha_2)}(t)+I_{(\alpha_3,0,\alpha_2)}(t)\mathbf{1}_{\alpha_1=\alpha_4\neq 0}\\
&\quad +I_{(0,\alpha_4,\alpha_2)}(t)\mathbf{1}_{\alpha_1=\alpha_3\neq 0}+I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(t)\\
&\quad +I_{(\alpha_1,\alpha_3,\alpha_2,\alpha_4)}(t)+I_{(\alpha_3,\alpha_1,\alpha_2,\alpha_4)}(t)\\
&\quad +I_{(\alpha_1,0,\alpha_4)}(t)\mathbf{1}_{\alpha_2=\alpha_3\neq 0}+I_{(0,\alpha_2,\alpha_4)}(t)\mathbf{1}_{\alpha_1=\alpha_3\neq 0}\\
&\quad +I_{(\alpha_3,\alpha_1,0)}(t)\mathbf{1}_{\alpha_2=\alpha_4\neq 0}+I_{(\alpha_1,\alpha_3,0)}(t)\mathbf{1}_{\alpha_2=\alpha_4\neq 0}\\
&\quad +I_{(0,0)}(t)\mathbf{1}_{\alpha_1=\alpha_3\neq 0,\alpha_2=\alpha_4\neq 0}.
\end{aligned}$$

Also, $I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4,\alpha_5)}(t)$ in (3.7) in Taylor formula [**T**] is represented as

$$I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4,\alpha_5)}(t)=\int_0^t I_{(\alpha_1)}(s)I_{(\alpha_3,\alpha_4,\alpha_5)}(s)\mathrm{d}W_s^{\alpha_2}+\int_0^t I_{(\alpha_1,\alpha_2)}(s)I_{(\alpha_3,\alpha_4)}(s)\mathrm{d}W_s^{\alpha_5}+\int_0^t I_{(\alpha_1)}(s)I_{(\alpha_3,\alpha_4)}(s)ds\mathbf{1}_{\alpha_2=\alpha_5}. \tag{A2}$$

The first term of right-hand side of (A2) is given by

$$\begin{aligned}\int_0^t & I_{(\alpha_1)}(s)I_{(\alpha_3,\alpha_4,\alpha_5)}(s)\mathrm{d}W_s^{\alpha_2}\\ &= I_{(\alpha_3,\alpha_4,\alpha_5,\alpha_1,\alpha_2)}(t)+I_{(\alpha_3,\alpha_4,\alpha_1,\alpha_5,\alpha_2)}(t)\\ &\quad+I_{(\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_2)}(t)+I_{(\alpha_3,\alpha_1,\alpha_4,\alpha_5,\alpha_2)}(t)\\ &\quad+I_{(0,\alpha_4,\alpha_5,\alpha_2)}(t)\mathbf{1}_{\alpha_1=\alpha_3}+I_{(\alpha_3,0,\alpha_5,\alpha_2)}(t)\mathbf{1}_{\alpha_1=\alpha_4}\\ &\quad+I_{(\alpha_3,\alpha_4,0,\alpha_2)}(t)\mathbf{1}_{\alpha_1=\alpha_5}.\end{aligned}$$

The second term of right-hand side of (A2) is given by

$$\begin{aligned}\int_0^t & I_{(\alpha_1,\alpha_2)}(s)I_{(\alpha_3,\alpha_4)}(s)\mathrm{d}W_s^{\alpha_5}\\ &= I_{(\alpha_3,\alpha_4,\alpha_1,\alpha_2,\alpha_5)}(t)+I_{(\alpha_3,\alpha_1,\alpha_4,\alpha_2,\alpha_5)}(t)\\ &\quad+I_{(\alpha_1,\alpha_3,\alpha_4,\alpha_2,\alpha_5)}(t)+I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)}(t)\\ &\quad+I_{(\alpha_1,\alpha_3,\alpha_2,\alpha_4,\alpha_5)}(t)+I_{(\alpha_3,\alpha_1,\alpha_2,\alpha_4,\alpha_5)}(t)\\ &\quad+I_{(0,\alpha_2,\alpha_4,\alpha_5)}(t)\mathbf{1}_{\alpha_1=\alpha_3}+I_{(\alpha_1,0,\alpha_4,\alpha_5)}(t)\mathbf{1}_{\alpha_2=\alpha_5}\\ &\quad+I_{(0,\alpha_4,\alpha_2,\alpha_5)}(t)\mathbf{1}_{\alpha_1=\alpha_3}+I_{(\alpha_3,0,\alpha_2,\alpha_5)}(t)\mathbf{1}_{\alpha_1=\alpha_4}\\ &\quad+I_{(\alpha_3,\alpha_1,0,\alpha_5)}(t)\mathbf{1}_{\alpha_2=\alpha_4}+I_{(\alpha_1,\alpha_3,0,\alpha_5)}(t)\mathbf{1}_{\alpha_2=\alpha_4}\\ &\quad+I_{(0,0,\alpha_5)}(t)\mathbf{1}_{\alpha_1=\alpha_3,\alpha_2=\alpha_4}.\end{aligned}$$

The third term of right-hand side of (A2) is given by

$$\begin{aligned}\int_0^t & I_{(\alpha_1)}(s)I_{(\alpha_3,\alpha_4)}(s)ds\mathbf{1}_{\alpha_2=\alpha_5}\\ &= I_{(\alpha_3,\alpha_4,\alpha_1,0)}(t)\mathbf{1}_{\alpha_2=\alpha_5}+I_{(\alpha_3,\alpha_1,\alpha_4,0)}(t)\mathbf{1}_{\alpha_2=\alpha_5}\\ &\quad+I_{(\alpha_1,\alpha_3,\alpha_4,0)}(t)\mathbf{1}_{\alpha_2=\alpha_5}+I_{(\alpha_3,0,0)}(t)\mathbf{1}_{\alpha_1=\alpha_4}\mathbf{1}_{\alpha_2=\alpha_5}\\ &\quad+I_{(0,\alpha_4,0)}(t)\mathbf{1}_{\alpha_1=\alpha_3}\mathbf{1}_{\alpha_2=\alpha_5}.\end{aligned}$$

We estimate $E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4)}(t)]$ and $E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))\allowbreak I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4,\alpha_5)}(t)]$, using the above decomposition and Lemma 1.

- Estimate for $E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2)}(t))]$.
  - When $\|(\alpha_1,\alpha_2)\|=4$, i.e. $(\alpha_1,\alpha_2)=(0,0)$, we obviously see the followings:

$$\left|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(0,0)}(t)]\right| \le \begin{cases}\|\nabla^2\varphi\|_\infty\frac{1}{2}t^2, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^{3/2}, & \text{if } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases}$$

- Estimate for $E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))I_{(\alpha_1,\alpha_2,\alpha_3)}(t)]$.
  - When $\#\{1\le i\le 3;\alpha_i\neq 0\}=2$ and $\#\{1\le i\le 3;\alpha_i=0\}=1$,

$$\left|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2,\alpha_3)}(t))]\right| \le \begin{cases}C\|\nabla^4\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^{3/2}, & \text{if } C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases} \tag{A3}$$

  - When $\#\{1\le i\le 3;\alpha_i\neq 0\}=3$,

$$|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2,\alpha_3)}(t))]| \le \begin{cases}C\|\nabla^3\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^2, & \text{if } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t^{3/2}, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases} \tag{A4}$$

- Estimate for $E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(t)]$.
  - When $\alpha=(\alpha_1,\dots,\alpha_4)\in\{0,1,\dots,d\}^4$ and $\#\{i;\alpha_i=0\}\ge 2$, i.e. $\|\alpha\|\ge 6$,

$$|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(t))]| \le \begin{cases}C\|\nabla^2\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^{5/2}, & \text{if } C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t^2, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases} \tag{A5}$$

  - When $\alpha=(\alpha_1,\dots,\alpha_4)\in\{0,1,\dots,d\}^4$, $\#\{i;\alpha_i=0\}=1$ and $\#\{i;\alpha_i\neq 0\}=3$,

$$|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(t))]| \le \begin{cases}C\|\nabla^3\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^2, & \text{if } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t^{3/2}, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases} \tag{A6}$$

  - When $\alpha=(\alpha_1,\dots,\alpha_4)\in\{1,\dots,d\}^4$,

$$|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(t)]| \le \begin{cases}C\|\nabla^4\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^{3/2}, & \text{if } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases} \tag{A7}$$

- Estimate for $E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)}(t)]$.
  - When $\alpha=(\alpha_1,\dots,\alpha_5)\in\{0,1,\dots,d\}^5$ and $\#\{i;\alpha_i=0\}\ge 1$, i.e. $\|\alpha\|\ge 6$,

$$|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)}(t)]| \le \begin{cases}C\|\nabla^2\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^2, & \text{if } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t^{3/2}, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases} \tag{A8}$$

  - When $\alpha=(\alpha_1,\dots,\alpha_5)\in\{1,\dots,d\}^5$, we have

$$|E[\varphi(\overline{X}_t^{\mathrm{EM}}(x))H_{(\kappa_1,\kappa_2)}(\overline{X}_t^{\mathrm{EM}}(x),I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)}(t)]| \le \begin{cases}C\|\nabla^3\varphi\|_\infty t^3, & \text{if } \varphi\in C_b^\infty(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^{3/2}, & \text{if } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t, & \text{if } \varphi\in\mathscr{B}_b(\mathbb{R}^N).\end{cases} \tag{A9}$$

From these observations, we will decompose the expansion as

$$\begin{aligned}E[\varphi(X_t(x))] &= E[\varphi(\overline{X}_t^{\mathrm{EM}}(x)]\\ &\quad+\sum_{1\le\kappa_1\le N}E\Bigg[\partial_{\kappa_1}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\sum_{(\alpha_1,\alpha_2)\in\{0,1,\dots,d\}^2}L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)I_{(\alpha_1,\alpha_2)}(t)\Bigg]\\ &\quad+\frac{1}{2}\sum_{1\le\kappa_1,\kappa_2\le N}E\Bigg[\partial_{\kappa_1}\partial_{\kappa_2}\varphi(\overline{X}_t^{\mathrm{EM}}(x))\sum_{(\alpha_1,\alpha_2)\in\{1,\dots,d\}^2}L^{\alpha_1}V_{\alpha_2}^{\kappa_1}(x)L^{\alpha_1}V_{\alpha_2}^{\kappa_2}(x)I_{(0,0)}(t)\Bigg]\\ &\quad+\hat{R}_\varphi(t,x),\end{aligned}$$

where

$$\hat{R}_\varphi(t,x) = \sum_{\beta_1=3,4,5}\sum_{\kappa_1\in\{1,\dots,N\}}\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_r)\in\{0,1,\dots,d\}^r,\ r\geq 3,\\ \|\alpha\|=\beta_l}} L^{\alpha_1}\cdots L^{\alpha_{r-1}}V^{\kappa_1}_{\alpha_r}(x) E[\varphi(\overline{X}^{\mathrm{EM}}_t(x))H_{(\kappa_1)}(\overline{X}^{\mathrm{EM}}_t(x), I_{(\alpha_1,\dots,\alpha_r)}(t))]$$
$$+ \sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}\sum_{\alpha=(\alpha_1,\dots,\alpha_4)\in\{1,\dots,d\}^4}\frac{1}{2}L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x) L^{\alpha_3}V^{\kappa_2}_{\alpha_4}(x)$$
$$E\Big[\varphi(\overline{X}^{\mathrm{EM}}_t(x))H_{(\kappa_1,\kappa_2)}\Big(\overline{X}^{\mathrm{EM}}_t(x), I_{(\alpha_3,\alpha_4,\alpha_1,\alpha_2)}(t) + I_{(\alpha_3,\alpha_1,\alpha_4,\alpha_2)}(t) + I_{(\alpha_1,\alpha_3,\alpha_4,\alpha_2)}(t) + I_{(\alpha_3,0,\alpha_2)}(t)\mathbf{1}_{\alpha_1=\alpha_4} + I_{(0,\alpha_4,\alpha_2)}(t)\mathbf{1}_{\alpha_1=\alpha_3} + I_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(t) + I_{(\alpha_1,\alpha_3,\alpha_2,\alpha_4)}(t) + I_{(\alpha_3,\alpha_1,\alpha_2,\alpha_4)}(t) + I_{(\alpha_1,0,\alpha_4)}(t)\mathbf{1}_{\alpha_2=\alpha_3} + I_{(0,\alpha_2,\alpha_4)}(t)\mathbf{1}_{\alpha_1=\alpha_3} + I_{(\alpha_3,\alpha_1,0)}(t)\mathbf{1}_{\alpha_2=\alpha_4} + I_{(\alpha_1,\alpha_3,0)}(t)\mathbf{1}_{\alpha_2=\alpha_4}\Big)\Big]$$
$$+ \sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_4)\in\{0,1,\dots,d\}^4,\\ \alpha_i=0,\ 1\leq\alpha_j,\alpha_3,\alpha_4\leq d,\ (i,j)=(1,2),(2,1)}} L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x)L^{\alpha_3}V^{\kappa_2}_{\alpha_4}(x)E[\varphi(\overline{X}^{\mathrm{EM}}_t(x)) H_{(\kappa_1,\kappa_2)}(\overline{X}^{\mathrm{EM}}_t(x), I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4)}(t))]$$
$$+ \sum_{(\kappa_1,\kappa_2)\in\{1,\dots,N\}^2}\sum_{\alpha=(\alpha_1,\dots,\alpha_5)\in\{1,\dots,d\}^5} L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x) L^{\alpha_3}L^{\alpha_4}V^{\kappa_2}_{\alpha_5}(x)E[\varphi(\overline{X}^{\mathrm{EM}}_t(x)) H_{(\kappa_1,\kappa_2)}(\overline{X}^{\mathrm{EM}}_t(x), I_{(\alpha_1,\alpha_2)}(t)I_{(\alpha_3,\alpha_4,\alpha_5)}(t))] + R_\varphi(t,x).$$

By (A3)–(A9) with (3.8), we obtain

$$\sup_{x\in\mathbb{R}^N}|\hat{R}_\varphi(t,x)| \leq \begin{cases} C\sum_{i=1}^4\|\nabla^i\varphi\|_\infty t^3, & \text{when } \varphi\in C^\infty_b(\mathbb{R}^N),\\ CC_{Lip}[\varphi]t^{3/2}, & \text{when } \varphi\in C_{Lip}(\mathbb{R}^N),\\ C\|\varphi\|_\infty t, & \text{when } \varphi\in\mathscr{B}_b(\mathbb{R}^N),\end{cases} \tag{A10}$$

for some constant $C > 0$. □

A.3. Proof of Theorem 1

The approximation terms are computed as follows.

Proof

- Computation of $E\Big[\partial_{\kappa_1}\varphi(\overline{X}^{\mathrm{EM}}_t(x))\sum_{(\alpha_1,\alpha_2)\in\{0,1,\dots,d\}^2} L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x)I_{(\alpha_1,\alpha_2)}(t)\Big]$, $1\leq\kappa_1\leq N$.
  - When $1\leq\alpha_1,\alpha_2\leq d$,
$$E\Big[\partial_{\kappa_1}\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x)I_{(\alpha_1,\alpha_2)}(t)\Big] = \sum_{1\leq\kappa_2,\kappa_3\leq N}E\Big[\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^{\alpha_1} V^{\kappa_1}_{\alpha_2}(x)V^{\kappa_2}_{\alpha_1}(x)V^{\kappa_3}_{\alpha_2}(x)H_{(\kappa_1,\kappa_2,\kappa_3)}(\overline{X}^{\mathrm{EM}}_t,1)\Big].$$
  - When $\alpha_1=0$, $1\leq\alpha_2\leq d$,
$$E\Big[\partial_{\kappa_1}\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^0V^{\kappa_1}_{\alpha_2}(x)I_{(0,\alpha_2)}(t)\Big] = \sum_{1\leq\kappa_2\leq N}E\Big[\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^0V^{\kappa_1}_{\alpha_2}(x)V^{\kappa_2}_{\alpha_2}(x) H_{(\kappa_1,\kappa_2)}(\overline{X}^{\mathrm{EM}}_t,1)\Big].$$
  - When $\alpha_2=0$, $1\leq\alpha_1\leq d$,
$$E\Big[\partial_{\kappa_1}\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^{\alpha_1}V^{\kappa_1}_0(x)I_{(\alpha_1,0)}(t)\Big] = \sum_{1\leq\kappa_2\leq N}E\Big[\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^{\alpha_1}V^{\kappa_1}_0(x)V^{\kappa_2}_{\alpha_1}(x) H_{(\kappa_1,\kappa_2)}(\overline{X}^{\mathrm{EM}}_t,1)\Big].$$
  - When $\alpha_1,\alpha_2=0$,
$$E\Big[\partial_{\kappa_1}\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^0V^{\kappa_1}_{\alpha_2}(x)I_{(0,0)}(t)\Big] = E\Big[\varphi(\overline{X}^{\mathrm{EM}}_t(x))L^0V^{\kappa_1}_0(x)H_{(\kappa_1)}(\overline{X}^{\mathrm{EM}}_t,1)\Big].$$
- Computation of $E\Big[\partial_{\kappa_1}\partial_{\kappa_2}\varphi(\overline{X}^{\mathrm{EM}}_t(x))\sum_{(\alpha_1,\alpha_2)\in\{1,\dots,d\}^2} L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x)L^{\alpha_1}V^{\kappa_2}_{\alpha_2}(x)I_{(0,0)}(t)\Big]$, $1\leq\kappa_1,\kappa_2\leq N$.
$$E\Big[\partial_{\kappa_1}\partial_{\kappa_2}\varphi(\overline{X}^{\mathrm{EM}}_t(x)) \sum_{(\alpha_1,\alpha_2)\in\{1,\dots,d\}^2}L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x)L^{\alpha_1}V^{\kappa_2}_{\alpha_2}(x)I_{(0,0)}(t)\Big] = E\Big[\varphi(\overline{X}^{\mathrm{EM}}_t(x))H_{(\kappa_1,\kappa_2)}(\overline{X}^{\mathrm{EM}}_t,1)\Big]\frac{1}{2}t^2 \sum_{(\alpha_1,\alpha_2)\in\{1,\dots,d\}^2}L^{\alpha_1}V^{\kappa_1}_{\alpha_2}(x)L^{\alpha_1}V^{\kappa_2}_{\alpha_2}(x).$$

Then, we obtain the representation $Q_{(t)}$ in Theorem 1 and the estimate

$$\|P_t\varphi - Q_{(t)}\varphi\|_\infty \leq Ct^3\sum_{i=1}^4\|\nabla^i\varphi\|_\infty,\quad t\in(0,1],\ \varphi\in C^\infty_b(\mathbb{R}^N), \tag{A11}$$
$$\|P_t\varphi - Q_{(t)}\varphi\|_\infty \leq Ct^{3/2}C_{Lip}[\varphi],\quad t\in(0,1],\ \varphi\in C_{Lip}(\mathbb{R}^N), \tag{A12}$$
$$\|P_t\varphi - Q_{(t)}\varphi\|_\infty \leq Ct\|\varphi\|_\infty,\quad t\in(0,1],\ \varphi\in\mathscr{B}_b(\mathbb{R}^N), \tag{A13}$$

by combining (A10) of Proposition 1. We also have the upper estimate of $Q_{(t)}\varphi$

$$\|Q_{(t)}\varphi\|_\infty \leq \|\varphi\|_\infty\{1 + C\sum_{i=1}^3 t^{i/2}\}. \tag{A14}$$

using the estimates: for $p\geq 2$,

$$\sup_{x\in\mathbb{R}^N}\|H_{(\kappa_1)}(\overline{X}^{\mathrm{EM}}_t(x), I_{(\alpha_1,\alpha_2)}(t))\|_p, \leq Ct^{(\|\alpha\|-1)/2},\ 2\leq\|\alpha\|\leq 4,$$
$$\sup_{x\in\mathbb{R}^N}\|H_{(\kappa_1,\kappa_2)}(\overline{X}^{\mathrm{EM}}_t(x), I_{(0,0)}(t))\|_p \leq Ct^{(4-2)/2} = Ct.$$

We note that the following decomposition holds

$$\begin{aligned} &P_Tf(x) - Q_{(s_1)}Q_{(s_2)}\cdots Q_{(s_n)}f(x)\\ &= P_{s_1}P_{T-s_1}f(x) - Q_{(s_1)}P_{T-s_1}f(x)\\ &\quad + Q_{(s_1)}P_{s_2}P_{T-s_1-s_2}f(x) - Q_{(s_1)}Q_{(s_2)}P_{T-s_1-s_2}f(x)\\ &\quad + \cdots + Q_{(s_1)}Q_{(s_2)}\cdots P_{s_n}f(x) - Q_{(s_1)}Q_{(s_2)}\cdots Q_{(s_n)}f(x)\end{aligned}$$

and thereby the following estimate holds

$$
\begin{aligned}
&\|P_T f - Q_{(s_1)}Q_{(s_2)}\cdots Q_{(s_n)}f\|_\infty \\
&\quad \le \|P_{s_1}P_{T-t_1}f - Q_{(s_1)}P_{T-t_1}f\|_\infty \\
&\quad + \sum_{k=2}^{n-1}\|P_{s_k}P_{T-t_k}f - Q_{(s_k)}P_{T-t_k}f\|_\infty\left(1+O(\max_{i<k}s_i^{1/2})\right) \\
&\quad + \|P_{s_n}f - Q_{(s_n)}f\|_\infty\left(1+O(\max_{i<n}s_i^{1/2})\right)
\end{aligned}
\tag{A15}
$$

using (A14). Here, by (A11) and Proposition 3 in Yamada (2017), we can see

$$
\|P_s P_t f - Q_{(s)}P_t f\|_\infty \le \begin{cases} C C_{Lip}[f]\sum_{i=1}^4 \frac{s^3}{t^{(i-1)/2}}, & f\in C_{Lip}(\mathbb{R}^N), \\ C\|f\|_\infty \sum_{i=1}^4 \frac{s^3}{t^{i/2}}, & f\in \mathscr{B}_b(\mathbb{R}^N). \end{cases}
$$

Then, the global error is estimated as

$$
\begin{aligned}
&\|P_T f - Q_{(s_n)}Q_{(s_{n-1})}\cdots Q_{(s_1)}f\|_\infty \\
&\le \begin{cases} \Big\{C_{Lip}[f]C\sum_{k=2}^n\sum_{i=1}^4\frac{s_k^3}{(T-t_k)^{(i-1)/2}} \\ \quad +\|P_{s_n}f - Q_{(s_n)}f\|_\infty\Big\}\Big(1+O(\max_{i<n}s_i^{1/2})\Big), \\ \quad f\in C_{Lip}(\mathbb{R}^N), \\ \Big\{\|f\|_\infty C\sum_{k=2}^n\sum_{i=1}^4\frac{s_k^3}{(T-t_k)^{i/2}} \\ \quad +\|P_{s_n}f - Q_{(s_n)}f\|_\infty\Big\}\Big(1+O(\max_{i<n}s_i^{1/2})\Big), \\ \quad f\in \mathscr{B}_b(\mathbb{R}^N), \end{cases} \qquad \text{(A16)} \\
&= O\left(\frac{1}{n^2}\right). \qquad \text{(A17)}
\end{aligned}
$$

Here we used similar argument in Yamada (2017):

$$
\sum_{k=2}^n\sum_{i=1}^4\frac{s_k^3}{(T-t_k)^{(i-1)/2}} = O\left(\frac{1}{n^2}\right), \quad \text{if } f\in C_{Lip}(\mathbb{R}^N),\ \gamma>4/3, \tag{A18}
$$

$$
\sum_{k=2}^n\sum_{i=1}^4\frac{s_k^3}{(T-t_k)^{i/2}} = O\left(\frac{1}{n^2}\right), \quad \text{if } f\in \mathscr{B}_b(\mathbb{R}^N),\ \gamma>2, \tag{A19}
$$

and

$$
\begin{aligned}
\|P_{s_n}f - Q_{(s_n)}f\|_\infty &\le C_{Lip}[f]Cs_n^{3/2} = O\left(\frac{1}{n^{\frac{3\gamma}{2}}}\right) \\
&= O\left(\frac{1}{n^2}\right),\ f\in C_{Lip}(\mathbb{R}^N), \gamma>4/3,
\end{aligned}
\tag{A20}
$$

$$
\begin{aligned}
\|P_{s_n}f - Q_{(s_n)}f\|_\infty &\le \|f\|_\infty C s_n = O\left(\frac{1}{n^{\frac{2\gamma}{2}}}\right) \\
&= O\left(\frac{1}{n^2}\right),\ f\in \mathscr{B}_b(\mathbb{R}^N), \gamma>2,
\end{aligned}
\tag{A21}
$$

with $O\left(\max_{i<n}s_i^{1/2}\right) = O\left(\frac{1}{n^{1/2}}\right)$. Therefore, we obtained assertion. □