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Factor High-Frequency-Based Volatility (HEAVY) Models*

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Abstract

We propose a new class of multivariate volatility models utilizing realized measures of asset variance and covariance extracted from high-frequency data. Dimension reduction for estimation of large covariance matrices is achieved by imposing a factor structure with time-varying conditional factor loadings. Statistical properties of the model, including conditions that ensure covariance stationarity of returns, are established. The performance of the model is assessed using a panel of large U.S. financial institutions during the financial crisis, where empirical results show that the new model has both superior in- and out-of-sample properties. We show that the superior performance applies to a wide range of quantities of interest, including volatilities, covariances, betas, and scenario-based risk measures. The model's performance is particularly strong at short forecast horizons.

Key words: conditional beta, conditional covariance, forecasting, realized covariance, realized Kernel, systematic risk

JEL classification: C32, C53, C58, G17, G21

Conditional covariances are key inputs in risk management and portfolio optimization. Multivariate GARCH models using daily data have been traditionally used to measure and forecast second-moment dynamics of asset returns [see Bauwens, Laurent, and Rombouts (2006) for a survey]. While most multivariate volatility models are feasible when the

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number of assets is small—five or fewer—only a small subset remain feasible when applied to large, empirically realistic portfolios. There are a number of difficulties in high-dimension covariance modeling, including the computational effort required to invert large conditional covariance matrices when evaluating the likelihood and the high-dimensionality of the parameter space. Recent contributions to the literature have attempted to side-step these issues by using alternative estimators or carefully designed models. Pakel et al. (2017) construct the composite likelihood by summing up the log-likelihoods of pairs of assets in order to avoid the inversion of high-dimensional matrices. The Dynamic Equicorrelation model proposed in Engle and Kelly (2012) leads to a simple analytic form of the inverse of the conditional covariance by assuming that the time-varying correlations are identical across all pairs of assets. A third, and older, approach to achieve dimension reduction is to assume a strong factor structure for asset returns when modeling conditional covariance. Early examples include Engle, Ng, and Rothschild (1990), who introduced the factor-ARCH model to measure the covariance of Treasury bills.

Recently, intra-daily estimators of volatility—collectively known as realized measures—have been used to improve volatility models. Realized measures incorporate information from asset price paths to improve the measurement of volatility over a fixed horizon, typically 1 day. The simplest and most common realized measure is realized variance, which estimates the quadratic variation of the intra-daily log-price process using the sum of the squared high-frequency returns. When prices follow a diffusion process with stochastic volatility and can be directly observed without error, realized variance converges to the daily integrated variance of the underlying volatility process (Andersen et al., 2001a; Barndorff-Nielsen and Shephard, 2002). However, microstructure noise such as bid–ask bounce is ubiquitous in high-frequency data which limits the sampling frequency which can be used with simple estimators. Several alternative solutions exist in the literature to control microstructure effect, including two- and multi-scale realized volatility (Aït-Sahalia, Mykland, and Zhang, 2005; Zhang, 2006), realized kernel estimators (Barndorff-Nielsen et al., 2008), and the pre-averaging approach (Jacod et al., 2009).

The multivariate extension of realized variance, known as realized covariance, was first introduced to econometrics in Andersen et al. (2001b) and the asymptotic theory was first studied in Barndorff-Nielsen and Shephard (2004). In the absence of market microstructure noise, and when prices are synchronously observed, realized covariance estimates the quadratic covariation of prices. Estimators that are robust to microstructure noise and non-synchronous trading include multivariate realized kernels (Barndorff-Nielsen et al., 2011) and pre-averaging estimators (Christensen, Kinnebrock, and Podolskij, 2010) and realized QMLE (Shephard and Xiu, 2017). Like their low-frequency counterparts, multivariate realized measures can be transformed to estimate other quantities, such as realized correlation or realized factor loadings (also known as realized beta). Realized beta, in particular, is of particular economic interest and has been widely studied. Bollerslev and Zhang (2003) employ realized factor loadings constructed in the Fama–French three-factor model to improve asset pricing predictions. Barndorff-Nielsen and Shephard (2004) derive the asymptotic distribution of realized betas. Bandi and Russell (2005) study the finite-sample properties of realized betas in the presence of market microstructure noise. Andersen et al. (2006) find that realized betas are less persistent than realized variances and covariances and suggest modeling them as short-memory processes, and Patton and Verardo (2012) study the effect of earnings announcement on realized betas.

In this paper, we introduce Factor HEAVY models, a new class of multivariate volatility models that exploit high-frequency data and utilize a factor approach to facilitate estimation in empirically relevant scenarios. We exploit a factor decomposition of asset prices to build a model which is feasible in high dimensions and estimable with an imbalanced panel. Our model resembles β-GARCH, which models the factor variance, conditional β, and idiosyncratic variance each with a GARCH-type evolution (Braun, Nelson, and Sunier, 1995). The covariance dynamics in the β-GARCH model are given by

$$\begin{aligned}\sigma^2_{f,t} &= \theta_0 + \theta_1 r^2_{f,t-1} + \theta_2\sigma^2_{f,t-1}\\ \beta_{i,t} &= \delta_{i,0} + \delta_{i,1}\frac{r_{f,t-1}r_{i,t-1}}{\sigma^2_{f,t-1}} + \delta_{i,2}\beta_{i,t-1}\\ \sigma^2_{i,t} &= \alpha_{i,0} + \alpha_{i,1}(r_{i,t-1} - \beta_{i,t-1}r_{f,t-1})^2 + \alpha_{i,2}\sigma^2_{i,t-1},\end{aligned} \tag{1}$$

where returns on individual assets as related through a common factor, $r_{i,t} = \beta_{i,t}r_{f,t} + \epsilon_{i,t}$ and $\sigma^2_{f,t}$ is the variance of the factor, or in a conditional CAPM, the market return. This specification uses natural proxies of the left-hand-side variables to act as shocks—squared returns of the factor for the factor variance, standardized cross-products for the β dynamics, and squares of the innovation for the idiosyncratic variance.

We contribute to a growing literature that combines multivariate GARCH-type dynamics and realized measures, and explicitly model the close-to-close return, not just the intra-daily return.[1] In multivariate HEAVY models (Noureldin, Shephard, and Sheppard, 2012), the conditional covariance is modeled as a smoothed function of recent lags of the realized covariance matrix. Jin and Maheu (2013) model daily returns using a Wishart-autoregressive-like structure and propose a Bayesian estimation method. Hansen Lunde, and Voev (2014) propose the realized β GARCH model in which variance and covariance are separated modeled. In our models, daily returns on individual assets are driven by common factors with time-varying factor loadings.[2] The dynamics of factor volatility, conditional factor loadings, and idiosyncratic volatility all follow the HEAVY structure where realized measures drive the dynamics of daily covariance (Shephard and Sheppard, 2010). Factor HEAVY models have two advantages over multivariate HEAVY models. First, multivariate HEAVY models are directly parameterized on variances and covariance and so specify common dynamics for all second moments of asset returns. Second, multivariate HEAVY models suffer from the curse of dimensionality, not only in terms of the number of parameters in the model but also in the dimension of the realized measure required to drive

1 Initial models that included realized measures focused only on modeling intraday realized measures (Halbleib-Chiriac and Voev, 2016; Golosnoy, Gribisch, and Liesenfeld, 2012). While this is an interesting topic, models that omit the overnight return dependence are not appropriate for most applications in portfolio allocation or risk management.

2 The time-variation in conditional βs has been debated in the literature for the last two decades. Braun, Nelson, and Sunier (1995) use bivariate EGARCH models and find weak evidence of time-varying conditional βs. Ferson and Harvey (1993), Bali and Engle (2010), and Hansen Lunde, and Voev (2014) find significant time-series variation in the conditional βs. Bali, Engle, and Tang (2017) document substantial time-varying conditional βs in the cross-section of daily stock returns.

the dynamics. For example, when modeling the conditional covariance of 50 assets, a 50-dimensional realized covariance is required each day. Our model only requires estimating low-dimensional realized measures irrespective of the number of assets in the model, and so can easily scale to empirically relevant dimensions. In the empirical analysis, we show that Factor HEAVY dominates other competing models in terms of in-sample performance. We also compare the out-of-sample ability of Factor HEAVY and the cDCC GARCH models (Aielli, 2013) when forecasting variance and covariance, βs, and marginal expected shortfall (MES). The results show that Factor HEAVY outperforms cDCC models and that the gains are particularly substantial in short-term forecasting. The Factor HEAVY has a more pronounced advantage over the long memory RiskMetrics model (Zumbach, 2007) than over cDCC models when forecasting variance and covariance. This superior performance at short horizons is particularly useful from a regulatory point-of-view since accurate and timely detection of changes in the covariance structure of returns is required when considering interventions.

The remainder of the paper is structured as follows. Section 1 introduces the Factor HEAVY models and discusses their properties. We initially focus the exposition on the one-factor version of the model before describing the full multi-factor extension. Section 2 discusses estimation and asymptotic properties. Section 3 describes the data used in the paper and presents our empirical results. Section 4 concludes the paper.

1 Factor HEAVY Models

1.1 Notation and Model Setup

To facilitate the exposition of the model, the initial focus is on a one-factor specification. The full K-factor specification is presented in Section 1.7. Let $r_t = (r_{f,t}, r_{1,t}, r_{2,t}, ..., r_{N,t})'$ denote a $N+1$ by 1 vector of low-frequency, typically daily, returns. The first return is a pervasive factor and the remaining N are returns on individual assets assumed to be related through the factor.[3] We denote the information set formed by the history of low-frequency returns with $\mathcal{F}_t^{\mathrm{LF}}$, the natural filtration containing all past low-frequency returns. In the standard multivariate ARCH literature, returns are typically assumed to conditionally follow some unspecified distribution F with mean zero and covariance matrix Ω_t. That is,

$$r_t|\mathcal{F}_{t-1}^{\mathrm{LF}} \sim F(0, \Omega_t).$$

Our interest is in modeling the conditional covariance of low-frequency returns using high-frequency realized measures, and so we augment the information set with a realized measure that estimates the quadratic covariation of the factor and individual assets. We denote the time t value of this $N+1$ by $N+1$ matrix-valued random variable RM_t. The realized measure could be a realized covariance or a more sophisticated noise-robust measure such as a realized kernel (Barndorff-Nielsen et al., 2011), pre-averaged realized variance (Christensen, Kinnebrock, and Podolskij, 2010), or realized QMLE covariance (Shephard

3 While mixing intra-daily and daily information is the natural application of the Factor HEAVY models, this class of models is well suited useful for building models that mix data sampled at two other frequencies, for example, daily and weekly or monthly.

and Xiu, 2017).[4] We use $\mathcal{F}_t^{\mathrm{HF}}$ to denote the filtration that contains the current (t) and all lags ($s < t$) of both the realized measures and low-frequency returns, so that $\mathcal{F}_t^{\mathrm{LF}} \subset \mathcal{F}_t^{\mathrm{HF}}$.

We assume that returns, conditional on the high-frequency information set, follow the distribution F with mean zero and covariance matrix Σ_t:

$$r_t|\mathcal{F}_{t-1}^{\mathrm{HF}} \sim F(0, \Sigma_t).$$

However, since the model now contains both high- and low-frequency data, it is necessary to specify a model for the process generating the realized measures. We assume that the realized measure is conditionally distributed as a Wishart with ν degrees of freedom, so that

$$\mathrm{RM}_t|\mathcal{F}_{t-1}^{\mathrm{HF}} \sim W_{N+1}\left(\nu, \frac{1}{\nu}M_t\right),$$

where M_t is a positive definite matrix so that $\mathrm{RM}_t = M_t^{1/2}\Xi_t(M_t^{1/2})'$, where $M_t^{1/2}$ is the Cholesky decomposition of M_t and $\Xi_t|\mathcal{F}_{t-1}^{\mathrm{HF}} \sim W_{N+1}(\nu, \frac{1}{\nu}I_{N+1})$.[5] We use a partitioning of the realized measures so that

$$\mathrm{RM}_t = \begin{bmatrix} \mathrm{RM}_{ff,t} & \mathrm{RM}_{f1,t} & \dots & \mathrm{RM}_{fN,t} \\ \mathrm{RM}_{1f,t} & \mathrm{RM}_{11,t} & \dots & \mathrm{RM}_{1N,t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{RM}_{Nf,t} & \mathrm{RM}_{1N,t} & \dots & \mathrm{RM}_{NN,t} \end{bmatrix}, \tag{2}$$

where $\mathrm{RM}_{ff,t}$ is a scalar measuring the quadratic variation of the factor, $\mathrm{RM}_{ii,t}$ is a scalar realized measure estimating the quadratic variation of the ith individual asset and $\mathrm{RM}_{if,t}$ is a scalar measuring the quadratic covariation between the ith asset and the factor.

We propose to model the conditional covariance of returns, Σ_t, as well as the conditional expectation of the realized measure, M_t, using a factor structure. The conditional covariance of returns and the conditional expectation of the realized measures can be written as

$$\Sigma_t = \begin{bmatrix} \sigma_{f,t}^2 & \beta_{1,t}\sigma_{f,t}^2 & \cdots & \beta_{N,t}\sigma_{f,t}^2 \\ \beta_{1,t}\sigma_{f,t}^2 & \beta_{1,t}^2\sigma_{f,t}^2 + \sigma_{1,t}^2 & \cdots & \beta_{1,t}\beta_{N,t}\sigma_{f,t}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N,t}\sigma_{f,t}^2 & \beta_{N,t}\beta_{1,t}\sigma_{f,t}^2 & \cdots & \beta_{N,t}^2\sigma_{f,t}^2 + \sigma_{N,t}^2 \end{bmatrix}, \tag{3}$$

4 While the choice of realized measure does not affect the description of the model, the steps required to estimate the model may differ depending on the estimator used. These issues are discussed in Section 3.

5 The Wishart distribution is in the linear exponential family and is the natural candidate for positive definite matrix-valued shocks. This choice is made as a matter of convenience, and aside from using the Wishart as a useful quasi-likelihood which has the property that the conditional expectation is the solution to the first-order condition, we do not make use of any specific features of this distribution. ν is fixed and not estimated.

and

$$M_t = \begin{bmatrix} \mu_{f,t} & \lambda_{1,t}\mu_{f,t} & \cdots & \lambda_{N,t}\mu_{f,t} \\ \lambda_{1,t}\mu_{f,t} & \lambda_{1,t}^2\mu_{f,t} + \mu_{1,t} & \cdots & \lambda_{1,t}\lambda_{N,t}\mu_{f,t} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N,t}\mu_{f,t} & \lambda_{N,t}\lambda_{1,t}\mu_{f,t} & \cdots & \lambda_{N,t}^2\mu_{f,t} + \mu_{N,t} \end{bmatrix}. \quad (4)$$

The model then requires specifications for the dynamics of the factor variance, $\sigma_{f,t}^2$ (and the corresponding realized measure $\mu_{f,t}$), the factor loadings $\beta_{i,t}$ ($\lambda_{i,t}$), and the idiosyncratic variances $\sigma_{i,t}^2$ ($\mu_{i,t}$). Throughout the paper, we use the notation $\mathrm{E}_{t-1}(.) = \mathrm{E}(.|\mathcal{F}_{t-1}^{\mathrm{HF}})$ to denote the expectation conditional on the high-frequency information set. Moreover, the transformations required when building factor-based models are known in closed form and lead to simple expressions for conditional expectations of the quantities of interest.

Proposition 1. *Define the realized β and realized idiosyncratic variance as* $R\beta_{i,t} = \frac{\mathrm{RM}_{if,t}}{\mathrm{RM}_{ff,t}}$ *and* $\mathrm{RIV}_{i,t} = \mathrm{RM}_{ii,t} - (R\beta_{i,t})^2\mathrm{RM}_{ff,t}$, *respectively. Under the assumption* $\mathrm{RM}_t|\mathcal{F}_{t-1}^{\mathrm{HF}} \sim W_{N+1}(\nu, \frac{1}{\nu}M_t)$, *we have* $\mathrm{E}_{t-1}[R\beta_{i,t}] = \lambda_{i,t}$ *and* $\mathrm{E}_{t-1}[\mathrm{RIV}_{i,t}] = \frac{\nu-1}{\nu}\mu_{i,t}$.

1.2 Factor Variance Dynamics

The factor dynamics are assumed to follow a HEAVY structure. Suppose there is a single factor and that the factor return can be decomposed as

$$r_{f,t} = \sigma_{f,t}\xi_{f,t},$$

so that the conditional mean of the factor return is zero, the conditional variance of the factor return is measurable with respect to $\mathcal{F}_{t-1}^{\mathrm{HF}}$ and $\mathrm{E}_{t-1}[\xi_{f,t}^2] = 1$. The conditional variance of the factor is driven by the realized measure corresponding to the quadratic variation of the factor, and so

$$\sigma_{f,t}^2 = \theta_0 + \theta_1\mathrm{RM}_{ff,t-1} + \theta_2\sigma_{f,t-1}^2. \quad (5)$$

This is not a complete model and only allows for one-step forecasts, and so we complete the model by specifying a model for the realized measure. The dynamics of the conditional mean of the realized measure follow a similar process to the factor conditional variance,

$$\mu_{f,t} = \theta_0^M + \theta_1^M\mathrm{RM}_{ff,t-1} + \theta_2^M\mu_{f,t-1}. \quad (6)$$

These two equations correspond to the HEAVY model (Shephard and Sheppard, 2010) for the volatility of the factor.

1.3 Factor Loading Dynamics

Returns on individual assets are related through exposure to a common factor through a time-varying loading. The factor loading of asset *i* is

$$\beta_{i,t} = \frac{\sigma_{if,t}}{\sigma_{f,t}^2} = \frac{\mathrm{Cov}[r_{f,t}, r_{i,t}]}{\mathrm{V}[r_{f,t}]}.$$

Factor loadings are also driven by realized measures, namely “realized betas” are used. Without microstructure noise, Barndorff-Nielsen and Shephard (2004) show that when the realized measure is realized covariance, then the realized β is a consistent estimator of the ratio between integrated equity covariance with the factor and integrated variance of the factor return. The dynamic factor loadings evolve as

$$\beta_{i,t} = \delta_{i,0} + \delta_{i,1} R\beta_{i,t-1} + \delta_{i,2}\beta_{i,t-1}, \tag{7}$$

and we complete the model by specifying dynamics for the realized β:

$$\lambda_{i,t} = \delta^M_{i,0} + \delta^M_{i,1} R\beta_{i,t-1} + \delta^M_{i,2}\lambda_{i,t-1}. \tag{8}$$

There is no observation equation for the realized β and its measurement occurs jointly with the idiosyncratic variances, which are described next.

1.4 Idiosyncratic Dynamics

The final component of the model is the idiosyncratic variance, defined as $\mathrm{E}_{t-1}[(r_{i,t} - \beta_{i,t} r_{f,t})^2]$. Using a standard decomposition into a common factor and an orthogonal component, we can define the idiosyncratic shock as

$$\epsilon_{i,t} = r_{i,t} - \beta_{i,t} r_{f,t},$$

where $\epsilon_{i,t} = \sigma_{i,t}\xi_{i,t}$ and $\sigma^2_{i,t}$ are the conditional variance of $r_{i,t}$ given the factor return. By construction, the N by 1 vector $\epsilon_t = (\epsilon_{1,t}, \ldots, \epsilon_{N,t})'$ is contemporaneously uncorrelated, and we further assume that there are no volatility spillovers between the idiosyncratic shocks. We use the realized idiosyncratic variance to drive the dynamics of the idiosyncratic variance. We assume a univariate HEAVY-like structure for the idiosyncratic variance, so that

$$\sigma^2_{i,t} = \alpha_{i,0} + \alpha_{i,1}\mathrm{RIV}_{i,t-1} + \alpha_{i,2}\sigma^2_{i,t-1}. \tag{9}$$

The model is completed by specifying the evolution of the realized idiosyncratic volatility as

$$\mu_{i,t} = \alpha^M_{i,0} + \alpha^M_{i,1}\mathrm{RIV}_{i,t-1} + \alpha^M_{i,2}\mu_{i,t-1}. \tag{10}$$

1.5 Full Specifications

The model specifications contain two distinct components. The first component, which determines the dynamics of the conditional covariance of the low-frequency data, is collectively referred to as the HEAVY-P equations,

$$\begin{aligned}
\sigma^2_{f,t} &= \theta_0 + \theta_1 \mathrm{RM}_{ff,t-1} + \theta_2 \sigma^2_{f,t-1} \\
\beta_{i,t} &= \delta_{i,0} + \delta_{i,1} R\beta_{i,t-1} + \delta_{i,2}\beta_{i,t-1} \quad \text{for } i = 1, 2, \ldots, N \\
\sigma^2_{i,t} &= \alpha_{i,0} + \alpha_{i,1}\mathrm{RIV}_{i,t} + \alpha_{i,2}\sigma^2_{i,t-1} \quad \text{for } i = 1, 2, \ldots, N.
\end{aligned} \tag{11}$$

We refer to the second component, which governs the dynamics of the realized measure, as the HEAVY-M equations,

$$\begin{aligned}
\mu_{f,t} &= \theta^M_0 + \theta^M_1 \mathrm{RM}_{ff,t-1} + \theta^M_2 \mu_{f,t-1} \\
\lambda_{i,t} &= \delta^M_{i,0} + \delta^M_{i,1} R\beta_{i,t-1} + \delta^M_{i,2}\lambda_{i,t-1} \quad \text{for } i = 1, 2, \ldots, N \\
\mu_{i,t} &= \alpha^M_{i,0} + \alpha^M_{i,1}\mathrm{RIV}_{i,t-1} + \alpha^M_{i,2}\mu_{i,t-1} \quad \text{for } i = 1, 2, \ldots, N.
\end{aligned} \tag{12}$$

1.6 Forecasting

We focus on forecasting conditional factor variances, βs, and idiosyncratic variances. One-step forecasts are directly given in the HEAVY-P equations. To compute multi-step forecasts, we utilize the recursive structure of the HEAVY-M equations.

Proposition 2. *The s-step forecasts in the Factor HEAVY model is given by*

$$\sigma^2_{f,t+s|t} = \mathrm{E}_t(\sigma^2_{f,t+s}) = \theta_2^{s-1}\sigma^2_{f,t+1} + \theta_0 \frac{1-\theta_2^{s-1}}{1-\theta_2}$$
$$+\theta_1 \sum_{i=1}^{s-1} \theta_2^{i-1}\left(\theta_0^M \frac{1-(\theta_1^M+\theta_2^M)^{s-1-i}}{1-(\theta_1^M+\theta_2^M)} + (\theta_1^M+\theta_2^M)^{s-1-i}\mu_{f,t+1}\right)$$

$$\beta_{i,t+s|t} = \mathrm{E}_t(\beta_{i,t+s}) = \delta_{i,2}^{s-1}\beta_{i,t+1} + \delta_{i,0}\frac{1-\delta_{i,2}^{s-1}}{1-\delta_{i,2}}$$
$$+\delta_{i,1}\sum_{i=1}^{s-1}\delta_{i,2}^{i-1}\left(\delta_{i,0}^M \frac{1-(\delta_{i,1}^M+\delta_{i,2}^M)^{s-1-i}}{1-(\delta_{i,1}^M+\delta_{i,2}^M)} + (\delta_{i,1}^M+\delta_{i,2}^M)^{s-1-i}\lambda_{i,t+1}\right)$$

$$\sigma^2_{i,t+s|t} = \mathrm{E}_t(\sigma^2_{i,t+s}) = \alpha_{i,2}^{s-1}\sigma^2_{i,t+1} + \alpha_{i,0}\frac{1-\alpha_{i,2}^{s-1}}{1-\alpha_{i,2}}$$
$$+\alpha_{i,1}\sum_{i=1}^{s-1}\alpha_{i,2}^{i-1}\left(\frac{\nu-1}{\nu}\alpha_{i,0}^M \frac{1-\left(\frac{\nu-1}{\nu}\alpha_{i,1}^M+\alpha_{i,2}^M\right)^{s-1-i}}{1-\left(\frac{\nu-1}{\nu}\alpha_{i,1}^M+\alpha_{i,2}^M\right)} + \left(\frac{\nu-1}{\nu}\alpha_{i,1}^M+\alpha_{i,2}^M\right)^{s-1-i}\mu_{i,t+1}\right). \tag{13}$$

The proof is presented in the Online Appendix. While these are unbiased forecasts for the components of the conditional covariance, they do not directly lead to unbiased forecasts of the conditional covariance. It arises because the covariance is a nonlinear function of beta, factor variance, and idiosyncratic variance, and forecasts of these components separately do not guarantee an unbiased forecast of such a function. An unbiased estimate could be computed using simulation, although we find that this error is sufficiently small so that simulation is not required.

1.7 Multiple Factors

The one-factor model can be directly extended to include multiple factors. In the multi-factor HEAVY model, returns on individual assets are determined by K factors and an idiosyncratic shock,

$$r_{i,t} = \sum_{k=1}^{K}\beta_{i,k,t}r_{fk,t} + \epsilon_{i,t} = \sum_{k=1}^{K}\beta_{i,k,t}\sigma_{fk,t}e_{fk,t} + \sigma_{i,t}e_{i,t},$$

in which $r_{fk,t}$ and $\sigma_{fk,t}$ denote the daily return on the kth factor and its volatility, respectively; $\beta_{i,k,t}$ represents the conditional loading of $r_{i,t}$ with respect to $r_{fk,t}$; $\{e_{fk,t}\}$ is an i.i.d. innovation sequence with zero mean and unit variance. $\epsilon_{i,t}$ is uncorrelated with each factor return given $\mathcal{F}^{\mathrm{HF}}_{t-1}$.

We now assume that the return vector r_t contains K factors and N individual assets where the K factors are in the first K positions, and the realized measure is $K + N$ by $K + N$ where the upper K by K block measures the quadratic covariation of the factors. We use block-partitions of the conditional covariance and the realized measure

$$\Sigma_t = \begin{bmatrix} \Sigma_{f,t} & \Sigma_{f1,t} & \dots & \Sigma_{fN,t} \\ \Sigma_{1f,t} & \Sigma_{11,t} & \dots & \Sigma_{1N,t} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{Nf,t} & \Sigma_{N1,t} & \dots & \Sigma_{NN,t} \end{bmatrix}, \mathrm{RM}_t = \begin{bmatrix} \mathrm{RM}_{ff,t} & \mathrm{RM}_{f1,t} & \dots & \mathrm{RM}_{fN,t} \\ \mathrm{RM}_{1f,t} & \mathrm{RM}_{11,t} & \dots & \mathrm{RM}_{1N,t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{RM}_{Nf,t} & \mathrm{RM}_{1N,t} & \dots & \mathrm{RM}_{NN,t} \end{bmatrix},$$

where the upper left block is K by K, the "*if*" blocks are 1 by K, and the "*ii*" blocks are scalars. We allow for a general dependence structure of factors, and so the conditional loadings for asset i are

$$\beta_{i,t} = \Sigma_{f,t}^{-1}\Sigma_{f1,t}.$$

Similarly, define the multi-factor realized βs for asset i as

$$R\beta_{i,t} = \mathrm{RM}_{ff,t}^{-1}\mathrm{RM}_{fi,t},$$

where $\mathrm{RM}_{ff,t}$ is K by K and the $\mathrm{RM}_{fi,t}$ is K by 1, and the realized idiosyncratic variance as

$$\mathrm{RIV}_{i,t} = \mathrm{RM}_{ii,t} - R\beta_{i,t}'\mathrm{RM}_{ff,t}R\beta_{i,t}.$$

In the K-factor HEAVY model, the multiple factors follow a multivariate HEAVY model (Noureldin, Shephard, and Sheppard, 2012), while each of the factor loadings and idiosyncratic volatilities follow the same dynamics as in the one-factor model. The HEAVY-P equations are then

$$\begin{aligned} \Sigma_{f,t} &= CC' + A\mathrm{RM}_{ff,t-1}A' + B\Sigma_{f,t-1}B' \\ \beta_{i,t} &= \delta_{i,0} + \delta_{i,1}R\beta_{i,t-1} + \delta_{i,2}\beta_{i,t-1} \\ \sigma_{i,t}^2 &= \alpha_{i,0} + \alpha_{i,1}\mathrm{RIV}_{i,t-1} + \alpha_{i,2}\sigma_{i,t-1}^2, \end{aligned} \tag{14}$$

where $\beta_{i,t} = (\beta_{i,1,t}, \dots, \beta_{i,K,t})$, $\delta_{i,0}$ is a K by 1 vector, $\delta_{i,1}$ and $\delta_{i,2}$ are diagonal matrices, and A, B, and C are parameter matrices which satisfy the assumptions given in Noureldin, Shephard, and Sheppard (2012). The HEAVY-M equations are similarly modified, and are

$$\begin{aligned} M_{f,t} &= C^M(C^M)' + A^M\mathrm{RM}_{ff,t-1}(A^M)' + B^M M_{f,t-1}(B^M)' \\ \lambda_{i,t} &= \delta_{i,0}^M + \delta_{i,1}^M R\beta_{i,t-1} + \delta_{i,2}^M\lambda_{i,t-1} \\ \mu_{i,t} &= \alpha_{i,0}^M + \alpha_{i,1}^M\mathrm{RIV}_{i,t-1} + \alpha_{i,2}^M\mu_{i,t-1}, \end{aligned} \tag{15}$$

where $M_{f,t} = \mathrm{E}_{t-1}(\mathrm{RM}_{ff,t})$. As before, the final two equations in Equations (14) and (15) are repeated for each asset. Equations (14) and (15) constitute the K-factor HEAVY model.

2 Estimation and Inference

2.1 Estimation

This section initially describes estimation in a one-factor model before turning to the full K-factor model. The conditional likelihood of the returns and the realized measures are the natural method to estimate the parameters. The parameters in the HEAVY-P equations are estimated by maximizing conditional likelihood

$$L = \sum_{t=1}^{T} l_t(\psi; r_t), \tag{16}$$

where $\psi = (\theta', \phi'_1, \ldots, \phi'_N)'$, $\theta = (\theta_0, \theta_1, \theta_2)'$, $\phi_i = (\delta_0, \delta_1, \delta_2, \alpha_0, \alpha_1, \alpha_2)'$,

$$l_t(\psi; r_t) = -\frac{1}{2}\left(\ln|\Sigma_t| + r'_t \Sigma_t^{-1} r_t\right) + c, \tag{17}$$

and c is a term which does not depend on the model parameters.[6] The model structure can be directly exploited to simplify estimation by expressing the log-likelihood in two components—one which measures the likelihood of the common factor and one which measures the likelihood of the idiosyncratic errors, $r_{i,t} - \beta_{i,t} r_{f,t}$.

Proposition 3. *The joint quasi log-likelihood of the daily returns can be equivalently expressed*

$$\begin{aligned} l_t = &\underbrace{-\frac{1}{2}\left(\ln(\sigma^2_{f,t}) + \frac{r^2_{f,t}}{\sigma^2_{f,t}}\right)}_{l_{f,t}:\text{ factor}} + \\ &\sum_{i=1}^{N} \underbrace{-\frac{1}{2}\left(\ln(\sigma^2_{i,t}) + \frac{(r_{i,t} - \beta_{i,t} r_{f,t})^2}{\sigma^2_{i,t}}\right)}_{l_{i,t}:\text{ idiosyncratic } i} + c. \end{aligned} \tag{18}$$

This decomposition leads to a natural two-step estimator, where the parameters of the $l_{f,t}$ are first maximized, and then the parameters governing the conditional factor loadings and idiosyncratic volatilities are estimated.[7]

The parameters of the HEAVY-M equations are estimated by maximizing the standardized Wishart log-likelihood,

$$L^M = \sum_{t=1}^{T} l_t^M\left(\psi^M; \mathrm{RM}_t\right),$$

where $\psi^M = [(\theta^M)', (\phi_1^M)', \ldots, (\phi_N^M)']'$, $\theta^M = (\theta_0^M, \theta_1^M, \theta_2^M)'$, $\phi_i^M = (\delta_0^M, \delta_1^M, \delta_2^M, \alpha_0^M, \alpha_1^M, \alpha_2^M)'$,

$$l_t^M = -\frac{\nu}{2}\left(\ln|M_t| + \mathrm{tr}(M_t^{-1}(\mathrm{RM}_t))\right) + c_\nu^M, \tag{19}$$

and c_ν^M is a constant conditional on the shape parameter of the standardized Wishart, ν. Our interest is in the parameters of the dynamics, and so we do not estimate this parameter.

6 The parameter dynamics are all recursive and so depend on values for initial observations. We use a backward exponentially weighted moving average based on the first $\tau = \lceil T^{0.25} \rceil$ observations of the realized measure to initialize the process so that $M_0 = \sum_{t=1}^{\tau} w_t M_t$, where $w_t = 0.06(0.94)^{t-1}/(1 - 0.94^{\tau})$.

7 The second step of the estimation process involves N optimizations, although we refer to this is a single step since the ordering of the assets does not matter.

After ignoring the shape parameter terms, Wishart log-likelihood is equivalent to the multivariate normal log-likelihood (up to location and scale constants). That is, it can be regarded as the quasi log-likelihood. Using the structure of M_t, this log-likelihood can be similarly decomposed into two components,

$$l_t^M = \underbrace{-\frac{\nu}{2}\left(\ln(\mu_{f,t}) + \frac{\mathrm{RM}_{ff,t}}{\mu_{f,t}}\right)}_{l_{f,t}^M:\ \text{factor}} + \sum_{i=1}^{N} \underbrace{-\frac{\nu}{2}\left(\ln(\mu_{i,t}) + \frac{\left(\lambda_{i,t}^2 \mathrm{RM}_{ff,t} - 2\lambda_{i,t}\mathrm{RM}_{fi,t} + \mathrm{RM}_{ii,t}\right)}{\mu_{i,t}}\right)}_{l_{i,t}^M:\ \text{idiosyncratic } i} + c_\nu^M. \tag{20}$$

This decomposition also leads to a natural two-step estimation strategy, where the parameters governing the factor dynamics are first estimated and then the parameters of the idiosyncratic volatility are estimated. This decomposition comes directly from the model and does not require correct specification or other restrictions on the realized measure. This is particularly useful since estimation of the model parameters only requires storing the realized measure for the factor, the assets, and between the factor and assets, and not the complete $N + 1$ by $N + 1$ realized measure. The likelihood structure is particularly useful when using noise-robust realized measures which are known to suffer from data attrition due to refresh-time sampling when the number of assets, N, is large.

The joint likelihoods of the K-factor model can be similarly decomposed so that

$$l_t = \underbrace{-\frac{1}{2}\left(\ln|\Sigma_{f,t}| + r_{f,t}'\Sigma_{f,t}^{-1}r_{f,t}\right)}_{l_{f,t}:\,\text{factor}} + \sum_{i=1}^{N} \underbrace{-\frac{1}{2}\left(\ln\left(\sigma_{i,t}^2\right) + \frac{\left(r_{i,t} - \sum_{k=1}^{K}\beta_{i,k,t}r_{fk,t}\right)^2}{\sigma_{i,t}^2}\right)}_{l_{i,t}:\ \text{idiosyncratic } i} + c$$

and

$$l_t^M = \underbrace{-\frac{\nu}{2}\left(\ln|M_{f,t}| - \mathrm{tr}(M_{f,t}^{-1}(\mathrm{RM}_{ff,t}))\right)}_{l_{f,t}^M:\,\text{factor}}$$
$$+ \sum_{i=1}^{N} \underbrace{-\frac{\nu}{2}\left(\ln(\mu_{i,t}) - \frac{\lambda_{i,t}'\mathrm{RM}_{ff,t}\lambda_{i,t} - 2\lambda_{i,t}'\mathrm{RM}_{fi,t} + \mathrm{RM}_{ii,t}}{\mu_{i,t}}\right)}_{l_{i,t}^M:\ \text{idiosyncratic } i} + c_\nu^M,$$

where $\lambda_{i,t} = (\lambda_{i,1,t}, \ldots, \lambda_{i,K,t})'$. The Factor-HEAVY structure preserves the variation-free nature of the HEAVY framework in the sense that the lagged terms from one set of equations do not appear in the other.

2.2 Quasi-likelihood-Based Asymptotic Inference

Similar to the HEAVY models, the parameter estimators in Factor HEAVY equations have the usual asymptotic properties of quasi-maximum-likelihood estimators. Since there are no parameters common to both the factor and equity equations, and between HEAVY-P and HEAVY-M equations, we can consider their asymptotic properties separately.

Here, we focus on the one-factor HEAVY model for simplicity. The score equations for the corresponding likelihood, evaluated at the estimated parameters, are

$$\sum_{t=1}^{T} S_{f,t}\left(\widehat{\theta}\right) = 0 \quad \sum_{t=1}^{T} S_{f,t}^{M}\left(\widehat{\theta^{M}}\right) = 0$$

$$\sum_{t=1}^{T} S_{i,t}\left(\widehat{\phi}_i\right) = 0 \quad \sum_{t=1}^{T} S_{i,t}^{M}\left(\widehat{\phi_i^{M}}\right) = 0,$$

where $S_{f,t}(\theta) = \partial l_{f,t}/\partial\theta$, $S_{i,t}(\phi_i) = \partial l_{i,t}/\partial\phi_i$, $S_{f,t}^{M}(\theta^{M}) = \partial l_{f,t}^{M}/\partial(\theta^{M})$, $S_{i,t}^{M}(\phi_i^{M}) = \partial l_{i,t}^{M}/\partial(\phi_i^{M})$. Let θ_o, $\phi_{i,o}$, θ_o^{M}, and $\phi_{i,o}^{M}$ indicate the true parameter values. The scores evaluated at these values are martingale difference sequences with respect to $\mathcal{F}_{t-1}^{\mathrm{HF}}$. Under regularity conditions [see Bollerslev and Wooldridge (1992), Newey and McFadden (1994), White (1996), *inter alia*.], we have

$$\sqrt{T}(\widehat{\theta} - \theta_o) \rightarrow N(0, \mathcal{I}_{\theta}^{-1}\mathcal{J}_{\theta}\mathcal{I}_{\theta}^{-1})$$

$$\sqrt{T}(\widehat{\theta^{M}} - \theta_o^{M}) \rightarrow N(0, (\mathcal{I}_{\theta^{M}}^{M})^{-1}\mathcal{J}_{\theta^{M}}^{M}(\mathcal{I}_{\theta^{M}}^{M})^{-1})$$

$$\sqrt{T}(\widehat{\phi}_i - \phi_{i,o}) \rightarrow N(0, \mathcal{I}_{\phi_i}^{-1}\mathcal{J}_{\phi_i}\mathcal{I}_{\phi_i}^{-1})$$

$$\sqrt{T}(\widehat{\phi_i^{M}} - \phi_{i,o}^{M}) \rightarrow N(0, (\mathcal{I}_{\phi_i^{M}}^{M})^{-1}\mathcal{J}_{\phi_i^{M}}^{M}(\mathcal{I}_{\phi_i^{M}}^{M})^{-1}),$$

where

$$\mathcal{I}_{\theta} = -\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left(\frac{\partial S_{f,t}(\theta)}{\partial\theta'}\Big|_{\theta=\theta_o}\right),\ \mathcal{J}_{\theta} = \mathrm{avar}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} S_{f,t}(\theta_o)\right),$$

$$\mathcal{I}_{\phi_i} = -\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left(\frac{\partial S_{i,t}(\phi_i)}{\partial\phi_i'}\Big|_{\phi_i=\phi_{i,o}}\right),\ \mathcal{J}_{\phi_i} = \mathrm{avar}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} S_{i,t}(\phi_{i,o})\right),$$

$$\mathcal{I}_{\theta^{M}}^{M} = -\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left(\frac{\partial S_{f,t}^{M}(\theta^{M})}{\partial(\theta^{M})'}\Bigg|_{\theta^{M}=\theta_o^{M}}\right),\ \mathcal{J}_{\theta}^{\mathcal{M}} = \mathrm{avar}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} S_{f,t}^{M}\left(\theta_o^{M}\right)\right),$$

$$\mathcal{I}_{\phi_i^{M}}^{M} = -\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left(\frac{\partial S_{i,t}^{M}\left(\phi_i^{M}\right)}{\partial\left(\phi_i^{M}\right)'}\Bigg|_{\phi_i^{M}=\phi_{i,o}^{M}}\right),\ \mathcal{J}_{\phi_i^{M}}^{M} = \mathrm{avar}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} S_{i,t}^{M}\left(\phi_{i,o}^{M}\right)\right).$$

We have omitted calculation of the cross-terms in the variance–covariance between components of the model. While these are generally not of interest, calculation of these is straightforward. Inference in the Factor HEAVY model is simplified when compared with typical multivariate volatility models since the estimating equations have a natural block structure due to the factor structure.

3 Empirical Analysis and Model Evaluation

3.1 Data and Descriptive Statistics

We apply the model to a sample containing 40 large U.S. financial firms from July 1, 2000, to June 30, 2010. A maximum of 2511 daily observations was available for any firm. The panel is slightly heterogeneous since some of the included firms stop trading during the financial crisis. High-frequency quote and trade data are extracted from the Trade and Quote (TAQ) database, and daily price data are extracted from Center for Research in Security Prices stock database. High-frequency data typically contain mis-recordings and other erroneous data, and so all price data were cleaned using a slightly modified set of rules proposed in Barndorff-Nielsen et al. (2009).[8] Our initial focus is on a one-factor model similar to a conditional CAPM (Jagannathan and Wang, 1996; Lewellen and Nagel, 2006). We choose the S&P 500 as the market proxy and use the SPDR S&P 500 (SPY), a highly liquid ETF that tracks the S&P 500 index.

We use realized covariance as the realized measure as our baseline measure and explore more sophisticated alternatives in a subsequent section. We sample all prices using last-price interpolation and use a sparse sampling scheme. Our preferred estimator is based on 10-min sampling with subsampling every minute. Suppose $p_{j,t}$ is the jth log price vector on day t containing the log prices of the factor and the 40 firms. Then the sub-sampled realized covariance is defined as

$$RC_t^{SS} = \frac{m-1}{s(m-s)} \sum_{i=1}^{m-s} \bar{r}_{i,t}\bar{r}'_{i,t},$$

where $\bar{r}_{i,t} = \sum_{j=i}^{s} r_{i+j,t}$, s is the length of the block (e.g., 10), $r_{j,t} = p_{j,t} - p_{j-1,t}$ are the high-frequency returns, and m is the number of price samples (e.g., 390 when using 1-min returns in U.S. equity data). In Section 3.4.2, we consider alternative specifications where we vary $s \in \{5, 10, 15, 30\}$, as well as a realized kernel where we use the non-flat Parzen kernel and the bandwidth selection procedure outlined in Barndorff-Nielsen et al. (2011).

Figure 1 shows annualized realized volatility of the factor, average annualized realized volatility of all firm equity returns, average realized correlations between SPY and equity returns, and average realized βs. At the beginning of the financial crisis in the summer of 2007, the volatility of daily returns increases substantially. The returns on the factor are less volatile than those on equity returns. During the crisis, the average realized correlations increased relative to their pre-crisis values, although the changes in dependence are far more striking in terms of the average realized βs.[9]

One of the primary difficulties encountered when incorporating realized measures into a model of the conditional covariance of low-frequency data is the absence of overnight data. In models of the conditional variance, it is common to assume that the full-day variance can be simply scaled from the within trading-day variance. When studying multivariate quantities, there are more possibilities, and in particular, the covariance and the variances may not scale with a common factor. We examine the choice of modeling space in Figure 2, where we compare the intra-daily and overnight βs and as well as the intra-daily and overnight correlations between the factor and equity returns. We compute these using only the open-to-close and close-to-open returns (not the other intra-daily data). The figure shows β that behaves very differently from correlation—most βs lie near the 45-degree line, indicating that βs are very stable throughout the entire day. The relationship between intra-daily and overnight correlations appears to be more complicated, and many of the firms have

8 The sole modification is to use the market each day that has the highest trading volume rather than always selecting NYSE.

9 Since we only consider 40 firms, there is no reason to expect the cross-sectional average beta to be one.

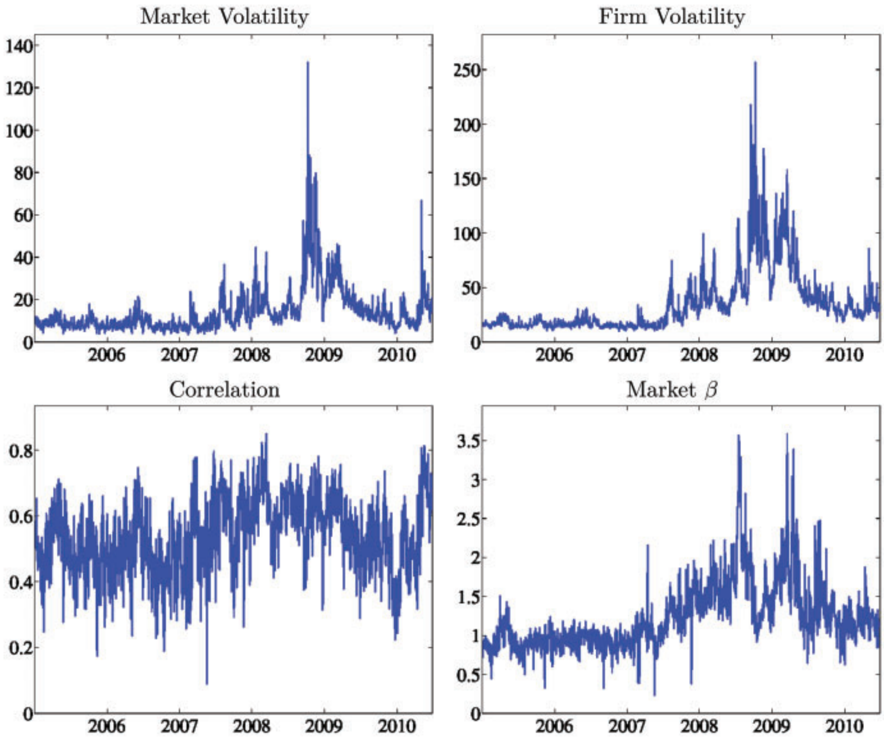

Figure 1 Top left: Annualized realized volatility of SPY. Top right: Average annualized realized volatility of all firm equity returns. Bottom left: Average realized correlations between SPY and firm equity returns. Bottom right: Average realized βs.

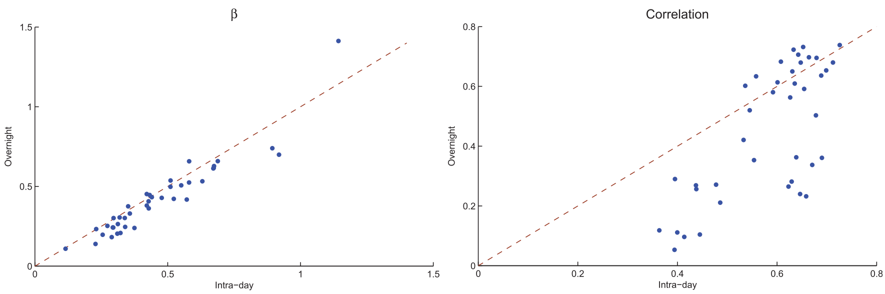

Figure 2 The left panel shows intra-daily and overnight market βs for the 40 firms in the application. The right panel shows the intra-daily and overnight correlations between the firm returns and the market for the same firms.

stronger correlations with the factor during the market opening hours than during the overnight period. This is consistent with systematic news arriving during market hours and idiosyncratic news arriving after the market closes. This difference indicates that models incorporating high-frequency data that parameterize the conditional β are better suited than models which use other transformations of the conditional covariance.

We are primarily interested in the covariance of returns over the entire day, including both the period where markets are active and the overnight return. Realized measures can only be computed for the period where the market is actively traded, so we use a transformation to ensure that the average value of the realized measure is the same as the average value of the outer product of returns (see Section B.1 in the Online Appendix for details).

Finally, a particular difficulty arises when using noise-robust estimators constructed using only the K factors and the individual assets. This occurs since these estimators typically use some form of refresh time sampling and so will not produce identical estimates of the quadratic covariation of the factors. We propose to first transform these so that they have the same estimate of the quadratic covariation of the factors, and then standardize these modified realized measures (see Section B.2 in the Online Appendix).

3.2 In-Sample Performance

The Factor HEAVY model is estimated using the complete sample available for each firm. Table 1 presents the minimum, the 25th percentile, the mean, the median, the 75th percentile, and the maximum of the parameter estimates across all firms. The estimates of the dynamics of the conditional β lie in a narrow range, with most sensitivity parameters estimated to be close to 0.10. The corresponding estimates of the parameters in the HEAVY-M equations fall into a similarly narrow band. The estimated dynamics of the conditional idiosyncratic volatilities are more dispersed, although most series are highly responsive to idiosyncratic news. The estimates from the idiosyncratic components of the HEAVY-M are also more similar to each other than those from the HEAVY-P, with both high sensitivity to news and large persistence.

In traditional ARCH-type models, both volatilities and βs are driven by daily shocks. We consider an augmented model of the HEAVY-P equations which allow for the natural daily shock to enter the model. The modified dynamics are then

$$
\begin{aligned}
\sigma_{f,t}^2 &= \theta_0 + \theta_1 \mathrm{RM}_{ff,t-1} + \theta_2 \sigma_{f,t-1}^2 + \theta_3 r_{f,t-1}^2 \\
\beta_{i,t} &= \delta_{i,0} + \delta_{i,1} \mathrm{R}\beta_{i,t-1} + \delta_{i,2} \beta_{i,t-1} + \delta_{i,3} r_{f,t-1} r_{i,t-1} / \sigma_{f,t-1}^2 \\
\sigma_{i,t}^2 &= \alpha_{i,0} + \alpha_{i,1} \mathrm{RIV}_{i,t-1} + \alpha_{i,2} \sigma_{i,t-1}^2 + \alpha_{i,3} \epsilon_{i,t-1}^2,
\end{aligned}
$$

where θ_3 allows the daily factor return to influence the factor volatility, $\delta_{i,3}$ allows a daily β shock to enter the β dynamics, and $\alpha_{i,3}$ allows for an idiosyncratic shock where $\epsilon_{i,t} = r_{i,t} - \beta_{i,t} r_{f,t}$. These are tested one-at-a-time, and all are conducted at the 5% level. Estimation is conducted using the two-step estimator described in Section 2.1, and inference is conducted using robust standard errors constructed as described in Section 2.2. The daily shock in the factor volatility equation is not significant. In the β equations, $\delta_{i,3}$ is significant for eight (out of 40) firms but has a much smaller coefficient than $\delta_{i,1}$ and so the changes in the fit value are typically small. In the idiosyncratic equations, $\alpha_{i,3}$ is significant for seven firms and the coefficient was smaller in magnitude than the coefficient on the realized measure.

We conduct a number of other experiments to assess whether all components of the models are necessary. We first compare the in-sample fit with that of the pairwise fit of the cDCC model, which has recently been used to fit dynamic β models (Engle and Kelly, 2012; Bali, Engle, and Tang, 2017). The cDCC model specifies dynamics for the conditional

Table 1 Cross-sectional statistics of full-sample parameter estimates for the 40 financial firms

	β				Idiosyncratic			
	$\delta_{i,1}$	$\delta_{i,2}$	$\boldsymbol{\delta}_{i,1}^M$	$\boldsymbol{\delta}_{i,1}^M+\boldsymbol{\delta}_{i,2}^M$	$\alpha_{i,1}$	$\boldsymbol{\alpha}_{i,2}$	$\boldsymbol{\alpha}_{i,1}^M$	$\boldsymbol{\alpha}_{i,1}^M+\boldsymbol{\alpha}_{i,2}^M$
Min	0.01	0.00	0.07	0.96	0.07	0.06	0.08	0.90
$Q_{0.25}$	0.07	0.83	0.12	0.98	0.42	0.29	0.39	1.00
Mean	0.12	0.82	0.15	0.98	0.51	0.43	0.44	0.99
Median	0.10	0.89	0.15	0.99	0.49	0.42	0.45	1.00
$Q_{0.75}$	0.13	0.92	0.17	0.99	0.64	0.54	0.50	1.00
Max	0.53	0.99	0.26	1.00	0.92	0.93	0.63	1.00

Note: The left panel contains estimates from the β component of the HEAVY-P and HEAVY-M equations, and the right panel contains estimates from the idiosyncratic volatility component.

variances of the market and the individual asset as standard GARCH models, and the conditional correlation is modeled using the standardized residuals.

We also compare the Factor HEAVY with the daily β-GARCH model [Equation (1)], as well as the multivariate HEAVY estimated on the market and each individual asset. Finally, we compare the Factor HEAVY to two nested specifications. The first assumes that the conditional factor loading is constant, which corresponds to restrictions that $\delta_{i,1}=\delta_{i,2}=0$, and the second which assumes that the idiosyncratic volatility is constant, corresponding to the restrictions $\alpha_{i,1}=\alpha_{i,2}=0$.

Table 2 contains the value of the difference in the quasi log-likelihood of these models,

$$\sum_{t=1}^{T}\left(l_{t,\text{Factor HEAVY}}-l_{t,\text{Alternative}}\right),$$

evaluated using only daily data in the case of HEAVY models. Positive values indicate that the Factor HEAVY produced a superior in-sample fit.[10] The Factor HEAVY produces much larger log-likelihoods than either of the daily-only models, and the closest daily model differs by over 80 log-likelihood points. Moreover, the typical difference is more than 150 points, indicating that there is substantial information in the high-frequency measures. The models nested in the Factor HEAVY—either with a constant factor loading or a constant idiosyncratic volatility—are also considerably worse, although the assumption of constant factor loading does not always lead to large changes in the log-likelihood. On the other hand, idiosyncratic volatilities appear to be time-varying with typical log-likelihood differences of 1000 points. Finally, the results comparing the Factor HEAVY with bivariate multivariate HEAVY models indicate some preference for the Factor HEAVY, with better performance in more than 75% of the series examined.

We also examine the fit of the model using residual-based tests. We are primarily concerned with misspecification of the conditional covariance between the returns on a pair of

10 Standard likelihood ratio tests are not directly applicable since the null hypothesis restricts parameters to be on the boundary of the parameter space and, under the null, there is an unidentified nuisance parameter.

Table 2 In-sample log-likelihood comparison between factor HEAVY and competing models for all equities

	Daily		High-Frequency based		
	cDCC	β GARCH	Constant factor loading	Constant idiosyncratic variance	Multivariate HEAVY
Min	85.5	82.48	4.277	211.6	–23.340
25%	137.9	135.8	31.77	483.9	1.220
Median	157.4	164.6	65.39	956.6	6.000
Mean	169.6	169.8	67.33	1004	6.356
75%	201.0	190.7	101.7	1206	9.074
Max	307.0	309.7	149.0	2441	42.05

Note: All values are the total log-likelihood difference between the factor HEAVY and a competing model comparing only the daily component of the model and were computed using the bivariate covariance of the market and one of the financial firms.

equities since the model does not explicitly attempt to fit these. We are interested in testing the null that

$$H_0 : \mathrm{E}_{t-1}(r_{i,t}r_{j,t}) = \beta_{i,t}\beta_{j,t}\sigma^2_{f,t},$$

where we have suppressed the dependence of the factor loadings and market variance on the model parameters,

$$\zeta = \left(\theta'_o, \delta'_{i,o}, \delta'_{j,o}\right)',$$

which includes the parameters of the conditional variance of the factor as well as the conditional factor loadings of assets i and j. Since the parameter estimates are variation-free across models, it is not necessary to include the parameters for the other components in ζ.

We use the robust regression-based specification tests developed in Wooldridge (1990). We use $z_{i,j,t} = \left(1, r_{i,t-1}r_{j,t-1}, r_{i,t-2}r_{j,t-2}\right)$ as the vector of misspecification indicator variables. These indicators will have power against static misspecification (through the constant term) as well as persistence in the cross-products. The test is implemented in two steps. First, we regress each element of $z_{i,j,t}$ on the gradient vector $\nabla_\zeta \eta_t$ of $\eta_t = r_{i,t}r_{j,t} - \beta_{i,t}\beta_{j,t}\sigma^2_{f,t}$ with respect to ζ evaluated at the estimate $\widehat{\zeta}$. We define $\widehat{z}_{i,j,t}$ as the residual vector obtained from this regression. Second, we regress unit on the vector $\widehat{\eta}_t\widehat{z}_{i,j,t}$ where $\widehat{\eta}_t$ is the value of η_t evaluated at $\widehat{\zeta}$. The test statistic is then computed as $T \times R^2$ where the R^2 comes from the second regression. The test statistics is asymptotically distributed as χ^2_3. Figure 3 contains a histogram of the test statistics. We fail to reject the null of correct conditional specification in 93% of the pairs tested, and so the rejection rate is close to size.

3.3 Out-of-Sample Forecasting Results

We next turn attention to assessing the out-of-sample performance of the Factor HEAVY, and focus the out-of-sample comparisons on the cDCC model, as a leading example of low-frequency models. All models are fitted using a recursive scheme where parameters are

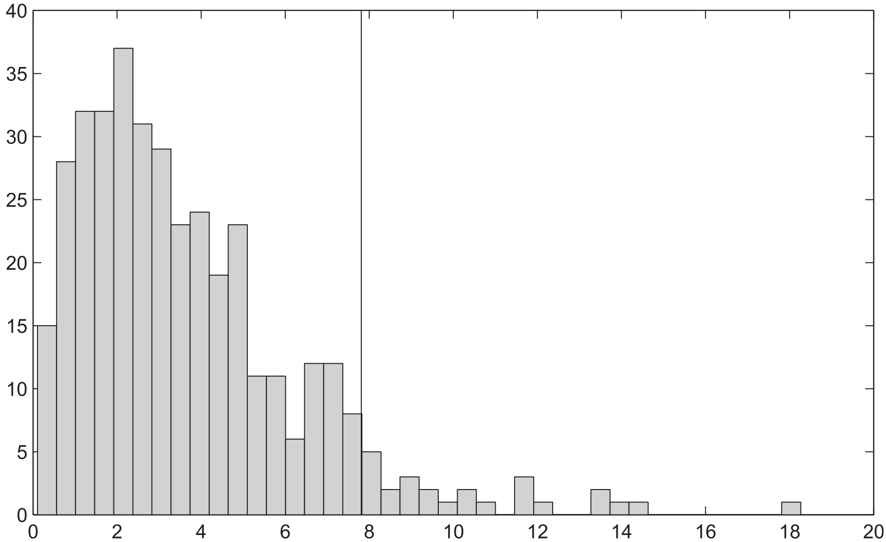

Figure 3 Histogram of misspecification test statistics. The vertical line is at the 95% critical value (7.81).

updated once a week (on Fridays) starting from July 2005, which allows for a minimum of 1250 days in the smallest models. The cDCC models are always estimated pairwise with only the market and one of the financial firms. We also present a volatility forecasting comparison with the long memory RiskMetrics model (Zumbach, 2007). We evaluate the performance of the model using 1-day, 1-week, and 2-week forecast horizons, which are all important for risk management.[11]

Figure 4 shows the one-step forecasts of cDCC and Factor HEAVY models for Capital One. The conditional β forecasts from the Factor HEAVY are more persistent than those from the cDCC. This higher persistence originates from the direct modeling of conditional β in Equation (7) where the coefficients indicate considerable persistence and a slow response to news. Both series of conditional βs show large increases during the financial crisis and the continuing turmoil of early 2009, although they differ markedly in the pre-crisis period of 2007 until mid-2008, where the HEAVY model indicates substantial increases in conditional β while the cDCC does not.

3.3.1 Statistical accuracy

We compare the statistical accuracy of all models using out-of-sample comparisons. The QLIK loss function has recently emerged as a sensible method to evaluate variance and covariance forecasts in the presence of noisy proxies (see Patton and Sheppard, 2009; Patton, 2011; Laurent, Rombouts, and Violante, 2013). The QLIK loss function uses the kernel of the Gaussian log-likelihood to evaluate forecasts,

$$l_t^s\left(\widehat{\Sigma}_{t+s|t}, C_{t+s}\right) = \ln|\widehat{\Sigma}_{t+s|t}| + \mathrm{tr}(\widehat{\Sigma}_{t+s|t}^{-1} C_{t+s}), \tag{21}$$

11 Results for longer horizons, up to 1 month, indicate there is little difference between the models.

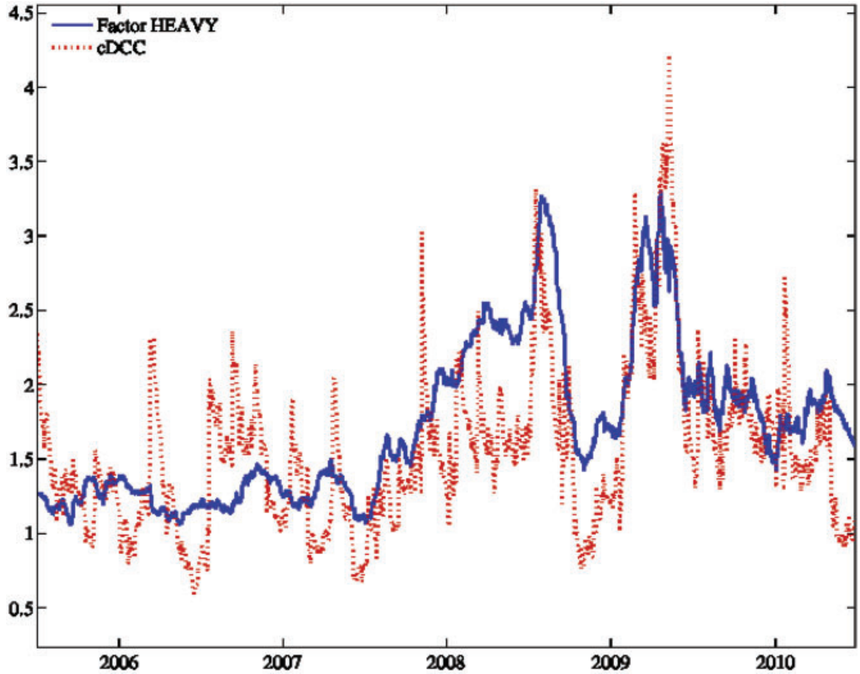

Figure 4 One-step β forecasts of cDCC and factor HEAVY models for Capital One.
Note: Solid lines denote the factor HEAVY forecasts and dashed lines represent the cDCC forecasts.

where s is the number of steps ahead, $\widehat{\Sigma}_{t+s|t}$ is a model-based time-t conditional forecast of the covariance of r_{t+s}, and C_{t+s} is the proxy for the unobserved covariance at time $t + s$. We are interested in the covariance over the entire day including the overnight period and so the proxy is $r_t r_t'$. The proxy must satisfy the condition that $\mathrm{E}_t(C_{t+s}) = \Sigma_{t+s}$, so that it is unbiased. When forecasting the covariance over the entire day, realized measures based on intra-daily data will fail to meet this criterion.

Suppressing the explicit dependence on $\widehat{\Sigma}_{t+s|t}$ and C_{t+s}, the QLIK loss function can be decomposed to isolate various component losses,

$$\begin{aligned} l_t^s &= \underbrace{\left(\ln\left(\sigma^2_{f,t+s|t}\right) - \frac{V_{f,t+s}}{\sigma^2_{f,t+s|t}}\right)}_{\text{factor}} + \sum_{i=1}^{N} \underbrace{\left(\ln\left(\sigma^2_{i,t+s|t}\right) - \frac{\beta^2_{i,t+s|t}V_{f,t+s} - 2\beta_{i,t+s|t}C_{fi,t+s} + V_{i,t+s}}{\sigma^2_{i,t+s|t}}\right)}_{\text{idiosyncratic } i} \\ &= \underbrace{\left(\ln\left(\sigma^2_{f,t+s|t}\right) - \frac{V_{f,t+s}}{\sigma^2_{f,t+s|t}}\right)}_{\text{factor}} + \sum_{i=1}^{N} \underbrace{\left(\ln\left(\beta^2_{i,t+s|t}\sigma^2_{f,t+s|t} + \sigma^2_{i,t+s|t}\right) - \frac{V_{i,t+s}}{\beta^2_{i,t+s|t}\sigma^2_{f,t+s|t} + \sigma^2_{i,t+s|t}}\right)}_{\text{individual equity } i} \\ &\qquad + \text{term corresponding to the dependence structure (copula)} \\ &= l^s_{f,t} + \sum_{i=1}^{N} l^s_{i,t} + l^s_{C,t}, \end{aligned} \tag{22}$$

where $V_{f,t+s}$, $V_{i,t+s}$, and $C_{fi,t+s}$ are the proxies for the factor variance, equity variance, and their covariance on day $t + s$. While this decomposition is generally only applicable to factor models, it also holds for any bivariate model which contains the factor, such as the daily cDCC models.

All comparisons are implemented using Diebold and Mariano (1995) (DM) tests of equal predictive accuracy where the covariance forecasts come from the Factor HEAVY and bivariate cDCC models. The null in DM test statistics is of equal expected loss,

$$H_0 : \mathrm{E}[l_t^s\left(\widehat{\Sigma}_{t+s|t}^{\mathrm{HEAVY}}, C_{t+s}\right)] = \mathrm{E}[l_t^s\left(\widehat{\Sigma}_{t+s|t}^{\mathrm{cDCC}}, C_{t+s}\right)],$$

and the alternative is two-sided. Define the loss differential as $\delta_t = l_t^s(\widehat{\Sigma}_{t+s|t}^{\mathrm{cDCC}}, C_{t+s}) - l_t^s(\widehat{\Sigma}_{t+s|t}^{\mathrm{HEAVY}}, C_{t+s})$, then positive values correspond to the superior performance of the Factor HEAVY while negative values correspond to the superior performance of the cDCC. The test statistic is implemented as a simple t-statistic

$$\mathrm{DM} = \sqrt{T - \tau + 1}\frac{\overline{\delta}}{\sqrt{\mathrm{avar}(\delta_t)}},$$

where $\overline{\delta} = \frac{1}{T-\tau+1}\sum_{t=\tau}^{T}\delta_t$ and $\mathrm{avar}(\delta_t)$ is a consistent estimator of the long-run variance of δ_t. This is estimated using a Newey–West covariance estimator with 10 lags.

Table 3 contains the value of the DM test statistics for three distinct comparisons. The first compares the bivariate loss [Equation (21)] applied to the market and the individual firm, the firm only loss ($l_{i,t}^s$) and the copula ($l_{C,t}^s$). All comparisons are conducted for 1-day, 5-day, and 2-week horizons, and are implemented on point-wise forecasts and not cumulative—that is, predicting $\Sigma_{t+s|t}$ not $\sum_{j=1}^{s}\Sigma_{t+j|t}$. The out-performance of these models is so strong at short horizons that cumulative comparisons are not useful. At the 1-day horizon, the Factor HEAVY model is dominant in both the joint and firm tests, with strong rejections for almost all pairs. The copula-based comparisons are more mixed, although the null is rejected in favor of the Factor HEAVY for 12 of the 40 series, and only once in favor of cDCC. The results using 1-week forecasts are unsurprisingly weaker, although the Factor HEAVY is still preferred. Finally, at the 2-week horizon, a similar pattern is found, with many statistically significant rejections both in the joint and firm volatility. The DM tests for the factor variance also uniformly prefer the HEAVY model, and the test statistic at the 1-day horizon is 3.72. The Factor HEAVY has a larger advantage over the long memory RiskMetrics model than over cDCC. In particular, at the 1-day horizon, Factor HEAVY forecasts are significantly more accurate than RiskMetrics forecasts for 21 firms in the copula-based comparison at the 10% significance level. Furthermore, there are still 23 firms for which the Factor HEAVY is preferred significantly in the joint test at the 1-week horizon.[12]

The ability of models to forecast conditional β is also examined using the estimated idiosyncratic variance as the loss function. The conditional β represents the optimal hedge ratio and so an accurate forecast should lead to a small tracking error. This leads to the loss function

$$l_{i,t}^s = \left(r_{i,t+s} - \beta_{i,t+s|t} r_{f,t+s}\right)^2.$$

12 The detailed result is omitted to save space, but available on request.

Table 3 Diebold–Mariano statistics for the null hypothesis that factor HEAVY and cDCC forecasts are equally accurate when forecasting 1-day, 1-week, and 2-week ahead volatility of firm equity returns, the dependence structure, and the covariance matrix

	1 Day			1 Week			2 Weeks		
	Joint	Firm	Copula	Joint	Firm	Copula	Joint	Firm	Copula
AET	**4.42**	**3.25**	1.44	**2.40**	**2.64**	−0.39	**2.82**	**2.39**	−0.53
AFL	**3.14**	**1.79**	**3.30**	1.37	1.35	0.36	1.62	1.47	−0.05
AIG	−0.09	−1.00	1.20	−0.81	−1.27	1.60	−1.17	−1.42	−0.07
ALL	**3.21**	**2.25**	0.98	**1.99**	1.61	1.45	**2.39**	**1.90**	1.09
AXP	**3.88**	**2.18**	1.48	1.55	0.99	1.42	1.04	0.42	0.69
BAC	1.37	0.81	−0.24	−0.27	−0.67	0.41	−0.70	−1.07	0.34
BBT	**4.22**	**2.61**	1.06	1.05	0.36	0.99	**2.45**	**2.33**	0.84
BEN	**3.70**	**2.10**	0.72	**1.86**	1.24	1.14	**2.11**	1.43	1.08
BK	**5.14**	**2.65**	**2.77**	**2.15**	**1.91**	0.58	**2.28**	**2.05**	0.72
C	**2.45**	1.02	1.41	0.28	−0.73	1.37	−1.11	*−1.77*	−0.19
CB	**5.56**	**2.96**	**4.69**	**2.73**	1.63	**1.89**	**3.40**	**1.99**	0.47
CMA	**4.00**	**2.06**	**1.86**	**2.36**	1.64	**1.68**	**3.16**	**2.91**	1.45
COF	**5.90**	**3.70**	**2.35**	**2.93**	**2.86**	0.95	**1.83**	**2.24**	0.14
ETFC	**1.88**	**2.08**	0.01	1.36	**1.66**	0.76	**1.81**	**1.65**	1.22
EV	**2.87**	0.83	0.34	0.79	−0.70	1.61	**1.69**	0.97	0.47
FITB	**4.07**	**2.69**	**2.01**	0.98	0.29	1.58	**1.86**	1.28	1.03
FNM	1.37	1.12	0.44	1.38	1.11	1.31	**1.70**	1.46	0.55
FRE	0.88	0.99	0.00	−0.38	−0.59	1.36	−0.49	−0.73	1.50
GS	1.44	1.51	−1.30	0.65	**1.76**	−1.09	0.15	0.12	−0.48
HIG	0.85	**1.99**	−1.21	−0.41	−0.15	−0.73	1.13	1.30	0.59
HRB	**2.73**	**2.34**	−0.08	0.70	0.65	−0.04	−0.27	−0.20	−0.91
JNS	**4.02**	**2.27**	**2.64**	1.32	1.12	0.73	1.02	0.91	0.81
JPM	**4.74**	**2.81**	1.39	0.77	0.48	0.76	0.15	−0.36	0.66
KEY	**2.24**	−0.05	**1.88**	−0.29	−1.23	0.85	0.02	−1.05	1.03
LEH	**2.05**	**1.94**	0.17	1.20	**1.87**	0.29	−0.93	−0.79	*−1.72*
LM	3.08	1.28	0.92	**2.04**	**2.28**	−0.10	**3.16**	**2.54**	0.29
MER	**4.34**	**3.99**	**1.75**	1.08	0.83	0.76	−1.21	−1.16	−1.40
MET	**5.00**	**1.72**	**3.10**	**2.15**	1.10	**1.99**	0.92	0.66	0.30
MMC	**5.34**	**3.63**	0.25	0.61	**3.93**	−1.16	0.95	**3.05**	−1.12
MS	**3.01**	**2.60**	−0.61	0.47	0.94	−0.62	−0.98	−0.87	−1.30
PGR	**2.17**	**2.43**	*−1.66*	0.35	**1.75**	−1.61	0.52	1.49	*−2.14*
PNC	**4.74**	**2.82**	**2.31**	**1.82**	0.47	1.46	0.36	−0.13	**1.85**
PRU	**3.13**	**2.71**	−1.07	−0.46	0.57	*−1.91*	0.49	0.32	−0.19
SCHW	**3.22**	**2.55**	−0.54	1.36	1.52	0.11	0.89	0.64	0.00
SLM	0.01	−0.51	0.53	−0.16	−0.24	−0.02	0.91	1.42	−1.28
STI	**3.56**	**2.64**	0.69	0.78	0.48	0.49	0.11	0.00	−0.23
STT	**3.78**	**2.12**	**3.14**	0.22	0.70	−0.27	0.70	0.93	0.00
TROW	**2.53**	0.44	0.62	−0.09	−0.80	0.45	0.98	−0.08	0.97
UNH	**2.76**	1.23	0.88	1.24	0.77	1.14	1.41	0.86	0.49
USB	**2.01**	0.69	0.10	−0.27	−0.70	0.22	−0.59	−0.71	−0.39

Notes: Numbers in bold font indicate that factor HEAVY forecasts are significantly more accurate than cDCC forecasts at the 10% significance level. Italic numbers indicate the converse.

Table 4 Diebold–Mariano statistics for the null hypothesis that factor HEAVY and cDCC forecasts are equally accurate when forecasting 1-day, 1-week, and 2-week ahead βs

	1 Day	1 Week	2 Weeks		1 Day	1 Week	2 Weeks
AET	1.56	**1.71**	**1.83**	HRB	**1.75**	**1.79**	**1.73**
AFL	**2.10**	**2.33**	**2.70**	JNS	**1.85**	**1.76**	0.89
AIG	1.07	1.24	1.14	JPM	−0.27	1.09	1.17
ALL	−0.07	0.73	0.76	KEY	0.64	1.04	1.16
AXP	−0.01	0.37	0.96	LEH	−1.08	−0.91	−0.96
BAC	−0.12	0.30	0.57	LM	**1.92**	**1.92**	**2.23**
BBT	1.12	1.20	1.36	MER	**1.66**	**1.70**	1.23
BEN	1.35	**2.04**	**2.18**	MET	0.60	0.66	1.24
BK	0.77	2.81	**2.68**	MMC	1.07	0.92	1.13
C	0.79	0.80	0.83	MS	1.35	1.13	0.61
CB	0.01	1.71	**2.27**	PGR	1.00	1.10	1.22
CMA	1.06	1.65	**1.99**	PNC	−1.27	−0.97	0.40
COF	**1.70**	**2.31**	**2.60**	PRU	−0.91	−0.62	−0.65
ETFC	−0.73	0.17	1.61	SCHW	−0.65	−0.61	−0.78
EV	−0.18	0.62	1.28	SLM	**1.98**	**2.54**	**2.25**
FITB	−0.35	0.72	1.48	STI	0.87	1.55	**1.83**
FNM	−0.76	−0.72	−0.29	STT	1.09	1.33	**2.15**
FRE	−1.12	−0.14	1.06	TROW	−0.91	−0.59	0.91
GS	0.82	0.06	0.10	UNH	1.11	0.88	1.33
HIG	1.30	0.85	1.17	USB	−0.99	−0.10	0.99

Notes: Numbers in bold font indicate that factor HEAVY forecasts are significantly more accurate than cDCC forecasts at the 10% significance level. Italic numbers indicate the converse.

Table 4 contains the result of DM tests of the null of equal idiosyncratic variance of the residual, comparing the Factor HEAVY to the cDCC. In most cases, the null cannot be rejected. However, when it is, the rejections indicate superior performance of the Factor HEAVY. These results are weaker than in the QLIK tests, which may be due to heteroskedasticity of the idiosyncratic error. The statistical tests all indicate that the out-of-sample performance of the Factor HEAVY is superior to the leading low-frequency model. Moreover, the gains are particularly striking in terms of predicting variance. The gains in terms of predicting dependence, whether from the copula term which arises from the QLIK loss function, or from measuring the idiosyncratic variance, are smaller. We hypothesize that the gains to forecasting volatility may be larger since the range of conditional variances is much larger than that of conditional correlations or βs.

3.3.2 MES forecasting

Risk forecasting, and in particular systemic risk forecasting, is an important application of multivariate volatility models. MES, introduced by Acharya et al. (2016), measures the expected loss conditional on a factor being in a state of stress and is defined as

$$\mathrm{MES}_{i,t} = -\mathrm{E}_t\left(r_{i,t+1} | r_{f,t+1} < c\right),$$

where c is a threshold which determine whether the factor is showing signs of stress. The longer term s-step ahead MES is defined similarly but based on cumulative returns:

$$\mathrm{MES}^{s}_{i,t} = -\mathrm{E}_t\left(R_{i,t+1:s}|R_{f,t+1:s} < c_s\right),$$

where $R_{i,t+1:s} = \exp(\sum_{\tau=1}^{s} r_{i,t+\tau}) - 1$ is the cumulative return on firm equity from $t+1$ to $t+s$, $R_{f,t+1:s}$ is similarly defined and c_s allows for explicit dependence between the horizon and the threshold for a stress event. We follow a similar procedure to that introduced in Brownlees and Engle (2017) to estimate MES (see the Online Appendix for details). MES prediction accuracy is evaluated using Relative Mean Square Error (RMSE), which is used in place of MSE to control for strong heteroskedasticity of time-varying losses. The RMSE for s-step ahead MES is defined as

$$\mathrm{RMSE}^{s}_{i,t_c} = \left(\frac{-R_{i,t_c+1:s} - \mathrm{MES}^{s}_{i,t_c}}{\mathrm{MES}^{s}_{i,t_c}}\right)^2,$$

where the period from $t_c + 1$ to $t_c + s$ represents the event period of length s during which the market experiences a loss larger than a predetermined threshold. Since MES can only be evaluated when the market suffers a relatively large loss, few data points are available to evaluate the models.

Relative predictive accuracy is assessed using Diebold–Mariano tests where the loss differential is $\delta_{i,t_c} = \mathrm{RMSE}^{\mathrm{cDCC}}_{i,t_c} - \mathrm{RMSE}^{\mathrm{HEAVY}}_{i,t_c}$. The test statistic is

$$\sqrt{T_c}\frac{\overline{\delta_i}}{\sqrt{\mathrm{avar}(\delta_t)}},$$

where T_c denotes the number of event days.

Figure 5 shows the actual event days and event periods and the corresponding market index losses. The corresponding threshold market index losses are 2%, 4%, and 6.5%, respectively.[13] Under these settings, there are 84 event days, 90 1-week event periods, and 51 2-week event periods in the sample. Figure 5 compares MES forecasts with the losses for Capital One on event days.[14] While it is not clear from the figure which model performs better on average, Factor HEAVY forecasts are significantly more accurate at the 10% significance level. The figure shows that each model can track actual losses well on some event days. At the crisis peak around autumn 2008, however, Factor HEAVY forecasts outperform cDCC forecasts where cDCC forecasts tend to seriously underestimate the actual losses. Forecasts from Factor HEAVY models respond quickly to the shock while forecasts from cDCC models adjust slowly. This fact may provide insight into the source of the forecasting gains of Factor HEAVY models.

Table 5 presents the Diebold–Mariano statistics associated with the 1-day, 1-week, and 2-week MES forecasts.[15] One-day Factor HEAVY forecasts are significantly more accurate

13 These values are chosen since larger values of the threshold index require extremely large bootstrap samples, especially in the period when volatility is low.

14 The bandwidth in kernel estimation is set equal to 0.05. This value is chosen to give stable results.

15 When doing bootstrapping for s-step forecasting, we fix the number of remaining simulation paths to 1000 which satisfy $R_{f,t+1:s} < c_s$. Under this setting, the MES simulation result was found to be stable.

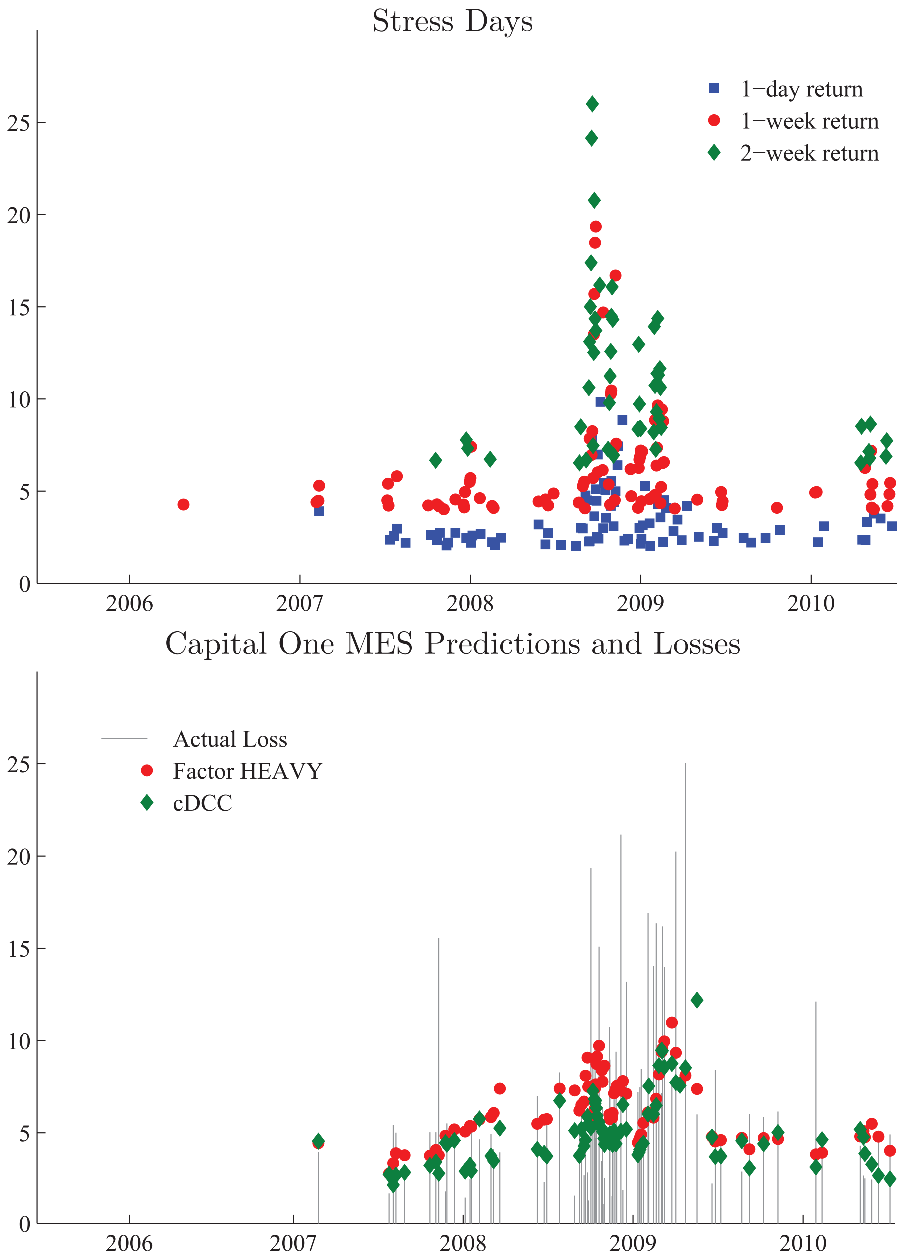

Figure 5 Event days and the beginning days of event periods from July 2005 and the corresponding market index losses.

Notes: Event days are the days on which the daily market index return declines by at least 2%. Event periods are those during which the market index experiences a loss larger than a specific horizon-dependent value. Actual equity losses of Capital One and 1-day MES forecasts on event days from July 2005. Solid lines denote the actual losses; circle and star markers represent factor HEAVY and cDCC forecasts, respectively.

Table 5 Diebold–Mariano statistics associated with the 1-day, 1-week, and 2-week MES forecasts

	1 Day	1 Week	2 Weeks		1 Day	1 Week	2 Weeks
AET	**3.80**	**2.34**	**2.92**	HRB	0.04	**1.98**	**2.74**
AFL	**2.83**	0.81	1.41	JNS	**2.58**	**2.02**	**2.00**
AIG	*–3.17*	–1.48	0.31	JPM	0.89	0.29	–1.01
ALL	0.20	–0.73	*–1.84*	KEY	*–2.22*	–1.58	*–2.38*
AXP	0.56	–0.99	*–1.83*	LEH	–0.96	–1.21	–1.39
BAC	*–2.82*	*–1.72*	*–1.75*	LM	**2.71**	**2.08**	1.19
BBT	1.31	–0.51	0.78	MER	–1.51	–1.19	–0.44
BEN	0.79	–1.14	–0.70	MET	–1.28	–1.19	*–2.05*
BK	**2.72**	0.73	0.04	MMC	**2.95**	**2.70**	**2.82**
C	–1.60	*–1.87*	*–2.44*	MS	0.65	–1.07	*–2.32*
CB	1.52	1.33	0.18	PGR	0.89	0.76	1.10
CMA	**1.95**	0.94	1.54	PNC	0.13	–1.16	–1.01
COF	**3.16**	**2.19**	**2.26**	PRU	–1.34	–1.64	*–2.17*
ETFC	–0.43	–0.21	–1.61	SCHW	**3.16**	**1.95**	**2.27**
EV	–0.67	–1.09	–1.62	SLM	1.00	–1.03	**3.36**
FITB	–1.61	*–1.72*	*–1.94*	STI	–1.57	*–1.98*	*–2.36*
FNM	–1.36	–0.92	1.22	STT	1.16	0.24	–1.05
FRE	*–2.98*	0.36	–1.12	TROW	–0.35	–1.00	–0.66
GS	0.98	0.71	0.46	UNH	**1.66**	**2.81**	**2.47**
HIG	–1.18	–1.10	0.99	USB	–0.14	*–2.05*	–1.48

Notes: Numbers in bold font indicate that factor HEAVY forecasts are significantly more accurate than cDCC forecasts at the 10% significance level for this firm. Italic numbers indicate the superior performance of cDCC forecasts.

than the cDCC forecasts for 23 firms and 10 of these are significant at the 10% level. On the other hand, cDCC forecasts are significantly more accurate for four firms. When looking at the 1-week MES, Factor HEAVY still enjoys an advantage over cDCC in terms of the number of significant forecasts, but the advantage is somewhat smaller than for 1-day-ahead forecasts. The gains from using Factor HEAVY models, however, are further reduced at the 2-week horizon, although for some firms like Capital One, Factor HEAVY forecasts are significantly more accurate than cDCC forecasts at all forecast horizons of interests.

It is noted that Capital One has consistently good performance when forecasting volatility, covariance, β, and MES. Since MES is the expected equity loss of the firm conditional on a threshold market index loss, we suspect that these all play a role in the 1-day MES gains, although the performance in MES forecasting is stronger in models where the Factor HEAVY performed better in terms of the copula. Factor HEAVY models are significantly more accurate when forecasting both the MES and copula for the firms such as Aetna, Aflac, Comerica Inc., Capital One, Janus Capital, Legg Mason, Marsh & McLennan, and UnitedHealth Group. This relation between MES and copula forecasting remains but weakens at longer horizons.

3.4 Extensions

Three extensions to the one-factor HEAVY model are explored. The first examines the gains from including multiple factors. The second examines the effect of changing the realized measure. The third examines the evidence for including asymmetries in the model.

3.4.1 Multiple factors

We consider two specifications for the multiple-factor models. In the first specification, we have prior knowledge of the number of factors K, and one of the factors is an observable market proxy. The one-factor specification is therefore augmented to a K-factor structure where the additional $(K-1)$ factors are constructed portfolios. The portfolio weights are computed using principle component analysis on CAPM residuals on the daily data. Define $\widehat{\epsilon}_{i,t} = r_{i,t} - \widehat{\beta}_i r_{f,t}$ to be CAPM residuals where $\widehat{\beta}_i$ is the usual OLS estimator. Portfolios are then formed using the $N \times (K-1)$ weight matrix W which corresponds to eigenvectors associated with the $K-1$ largest eigenvalues in the outer-product of $\widehat{\Omega} = T^{-1}\sum_{t=1}^{T} \widehat{\epsilon}_t \widehat{\epsilon}_t'$. The new factors have daily returns $r_{\text{PCA},t} = \sum W_i \widehat{\epsilon}_{i,t}$, where W_i is the ith row of W. High-frequency returns are similarly constructed using the intra-daily data. Finally, realized covariances are computed using the high-frequency returns of the constructed portfolios, along with the market and each individual firm.

Alternatively, we can assume that all factors are unobserved. In this specification, the observable market proxy is not one of the factors and all factors need to be extracted by principle component analysis. In addition, the number of factors is unknown. Various methods in the literature of approximate factor models can be employed to estimate the number of factors, such as Bai and Ng (2002), Alessi, Barigozzi, and Capasso (2010), Onatski (2010), and Ahn and Horenstein (2013), among others [see Choi and Jeong (2018) for a review]. In the empirical discussion below, we will apply Onatski (2010)'s estimator based on the differenced eigenvalues, which is found to be more accurate than Bai and Ng (2002)'s information criteria when the idiosyncratic errors are substantially correlated. The estimator of the number of factors is specifically $\widehat{K} = \max\{K \le K_{\max}, \varpi_k - \varpi_{k+1} \ge c\}$ where ϖ_k is the kth largest eigenvalue of $T^{-1}\sum_{t=1}^{T} r_t r_t'$, c is a threshold value estimated from the empirical distribution of eigenvalues [see Onatski (2010) for the calibration algorithm], and $K_{\max}$ is a given upper bound which is set equal to 10 in our empirical discussion. The factors for daily returns are portfolios formed using principal component analysis on the daily returns, given the previously estimated number of factors. Specifically, the $N \times K$ weight matrix $\tilde{W}$ corresponds to eigenvectors associated with the K largest eigenvalues of $T^{-1}\sum_{t=1}^{T} r_t r_t'$. The daily returns of the factors are $\tilde{r}_{\text{PCA},t} = \sum \tilde{W}_i r_{i,t}$. High-frequency returns of the factors are constructed using the intra-daily returns of the firms in a similar way.

Empirically, we present the results of a multiple-factor model with the second specification for the 40 firms.[16] The estimated number of factors is 2 by Onatski (2010)'s method. Table 6 shows that the dispersion of the estimates in the β and idiosyncratic volatility equations is similar to that in the one-factor model in Table 1. The fit is also examined based on the marginal log-likelihood of the firm. We find that the model with two factors improves

16 If we use S&P 500 and SPY as the market proxy in the first model specification, the results of the first model specification are close to those of the second model specification, as the returns of S&P 500 and SPY are highly correlated with the corresponding returns of the extracted factor associated with the largest eigenvalue in the second model specification.

Table 6 The top panel lists the cross-sectional statistics of full-sample parameter estimates of the two-factor HEAVY model with the second specification for the 40 financial firms

Panel A: Cross-sectional statistics of parameter estimates

	Factor							
	A_{11}	A_{22}	B_{11}	B_{22}	A_{11}^M	A_{22}^M	B_{11}^M	B_{22}^M
	0.42	0.35	0.90	0.94	0.62	0.36	0.79	0.93

	β							
	$\delta_{i,1,1}$	$\delta_{i,2,1}$	$\delta_{i,1,2}$	$\delta_{i,2,2}$	$\delta_{i,1,1}^M$	$\delta_{i,1,1}^M + \delta_{i,2,1}^M$	$\delta_{i,1,2}^M$	$\delta_{i,1,2}^M + \delta_{i,2,2}^M$
Min	0.01	0.14	0.00	0.00	0.02	0.97	0.00	0.72
$Q_{0.25}$	0.07	0.82	0.01	0.91	0.10	0.99	0.01	0.92
Median	0.10	0.90	0.02	0.97	0.12	0.99	0.02	0.94
Mean	0.12	0.84	0.05	0.92	0.11	0.99	0.02	0.97
$Q_{0.75}$	0.16	0.92	0.08	0.99	0.13	0.99	0.03	0.98
Max	0.39	0.98	0.19	1.00	0.19	1.00	0.09	1.00

	Idiosyncratic			
	$\alpha_{i,1}$	$\alpha_{i,2}$	$\alpha_{i,1}^M$	$\alpha_{i,1}^M + \alpha_{i,2}^M$
Min	0.03	0.00	0.23	0.92
$Q_{0.25}$	0.47	0.22	0.39	1.00
Median	0.60	0.37	0.47	1.00
Mean	0.58	0.38	0.46	1.00
$Q_{0.75}$	0.70	0.47	0.52	1.00
Max	1.00	0.97	0.66	1.00

Panel B: Diebold–Mariano statistics

	DM		DM		DM		DM		DM
AET	–0.69	BK	**2.07**	FNM	–0.86	LEH	–0.08	PRU	**2.91**
AFL	0.74	C	–0.63	FRE	0.42	LM	**2.55**	SCHW	0.75
AIG	–0.60	CB	0.26	GS	0.61	MER	**1.90**	SLM	0.33
ALL	1.03	CMA	1.32	HIG	*–2.04*	MET	0.92	STI	**2.65**
AXP	**2.43**	COF	0.27	HRB	–0.10	MMC	1.30	STT	1.50
BAC	0.75	ETFC	0.60	JNS	1.30	MS	**2.52**	TROW	**2.82**
BBT	**2.12**	EV	**3.01**	JPM	**1.97**	PGR	0.70	UNH	–0.86
BEN	0.41	FITB	–0.04	KEY	**3.21**	PNC	**1.69**	USB	**1.81**

Notes: The bottom panel shows the Diebold–Mariano statistics for the null hypothesis that the two-factor HEAVY model with the second specification and the one-factor HEAVY model are equally accurate when forecasting 1-day ahead volatility of firm equity returns. Numbers in bold font indicate that the two-factor HEAVY forecasts are significantly more accurate than one-factor HEAVY forecasts at the 10% significance level for this firm. Italic numbers indicate the superior performance of one-factor HEAVY forecasts.

in-sample fit for 26 out of the 40 firms.[17] When forecasting 1-day ahead volatility of firm equity returns, two-factor model forecasts are significantly more accurate than one-factor model forecasts for 31 firms, and 14 of these are significant at the 10% level, while one-factor model forecasts are significant for only 1 firm. These in-sample and out-of-sample results indicate that multi-factor models may indeed yield gains.

3.4.2 Alternative realized measures

The analysis made so far is based on 10-min sampling with subsampling every minute. As a robustness check, the model is fit using alternative realized measures. The top panel of Table 7 reports parameter estimates and the (joint) log-likelihood across five choices of realized measure: realized covariance sampled between 5 and 30 min, all using one-minute subsampling, and multivariate realized kernels. The second panel reports DM test statistics from 1-day forecasts for Capital One against the cDCC model using the same measures. All results are stable across alternative realized measures, and realized measures which sample more often perform somewhat better both in- and out-of sample, with realized kernel producing the best in-sample fit. All key parameters which measure the sensitivity to the realized measure in the HEAVY-P equations, θ_1 (market), $\delta_{i,1}$ (factor loading), and $\alpha_{i,1}$ (idiosyncratic variance), decline as the realized measure deteriorates, which is consistent with additional measurement noise when sampling less-frequently. All estimators consistently out-perform cDCC forecasts, and a similar pattern appears in terms of using more precise estimators.

Table 8 shows the relative distribution of parameters using the 25th quantile, median, and 75th quantile of the estimates across all 40 assets, as well as the distribution of likelihood differences relative to the likelihood of the 10-min realized covariance-based model (negative indicates better performance of the 10-min model). These are broadly consistent with the pattern for Capital One, where less frequent sampling leads to smaller sensitivity to news as well as smaller likelihoods. The performance of the 5-min and 10-min realized covariance is extremely similar.

3.4.3 Conditional asymmetries in volatility

The third extension is to examine whether asymmetries are needed in the model. GJR-GARCH and TARCH models have been broadly found to fit conditional variances better than symmetric GARCH models (Glosten, Jagannathan, and Runkle, 1993; Zakoian, 1994). A GJR-like specification can be constructed in the HEAVY-P using an indicator variable based on the daily return,

$$\sigma^2_{f,t} = \underset{\substack{0.00 \\ (0.00)}}{\theta_0} + \underset{\substack{0.12 \\ (0.04)}}{\theta_1} \mathrm{RM}_{ff,t-1} + \underset{\substack{0.76 \\ (0.04)}}{\theta_2} \sigma^2_{f,t-1} + \underset{\substack{0.19 \\ (0.05)}}{\theta_3} \mathrm{RM}_{ff,t-1} I\left(r_{f,t-1} < 0\right). \tag{23}$$

We test the null that $\theta_3 = 0$ which is strongly rejected. Without the asymmetry (imposing $\theta_3 = 0$), $\widehat{\theta}_1 = 0.26$. When the asymmetry is introduced, $\widehat{\theta}_1 = 0.12$ and $\widehat{\theta}_3 = 0.19$.

17 Larger model does not uniformly improve the fit over the smaller model since the one-factor model and the two-factor models are not nested and we are comparing the marginal log-likelihood which is not the objective of the in-sample fit.

Table 7 Parameter estimates, in-sample likelihood, and Diebold–Mariano statistics of 1-day forecasts for Capital One using different realized measures, that is, 5-, 10-, 15-, or 30-min realized covariance matrices with 1-min subsampling or realized kernels

Panel A: In-sample results for Capital One

	5-Min	10-Min	15-Min	30-Min	Kernel
θ_1	0.28	0.26	0.26	0.24	0.28
$\delta_{i,1}$	0.12	0.10	0.10	0.09	0.13
$\alpha_{i,1}$	0.64	0.62	0.60	0.52	0.63
Likelihood	–9125	–9138	–9163	–9229	–9115

Panel B: Out-of-sample Diebold–Mariano tests for Capital One

	5-Min	10-Min	15-Min	30-Min	Kernel
Joint	5.93	5.90	6.07	6.56	6.03
Factor	3.57	3.72	3.74	3.99	3.74
Firm	3.67	3.70	3.75	3.89	3.77
Copula	2.36	2.35	2.54	3.09	2.42
β	1.81	1.70	1.54	1.19	1.78
MES	2.89	3.16	3.12	2.96	2.71

Notes: The null hypothesis of the Diebold–Mariano statistics is that factor HEAVY and cDCC forecasts are equally accurate.

Table 8 The distribution of parameter estimates as well as the distribution of the differences between the log-likelihood of the alternative model and that of the 10-min realized covariance-based model across all 40 firms

	5-Min			15-Min			30-Min			Kernel		
	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.25}$	Median	$Q_{0.75}$
$\delta_{i,1}$	0.07	0.11	0.17	0.06	0.08	0.11	0.05	0.06	0.08	0.08	0.13	0.20
$\delta_{i,2}$	0.74	0.86	0.91	0.85	0.90	0.92	0.90	0.92	0.94	0.72	0.85	0.90
$\delta_{i,1}^M$	0.15	0.18	0.21	0.10	0.12	0.14	0.06	0.08	0.10	0.14	0.17	0.21
$\alpha_{i,1}$	0.41	0.54	0.69	0.35	0.45	0.59	0.24	0.34	0.53	0.34	0.52	0.64
$\alpha_{i,2}$	0.25	0.38	0.48	0.33	0.49	0.59	0.46	0.61	0.71	0.30	0.40	0.57
$\alpha_{i,1}^M$	0.41	0.48	0.52	0.35	0.41	0.47	0.29	0.34	0.40	0.41	0.47	0.53
ΔL	–2.94	–1.00	3.78	–4.27	–1.99	1.46	–19.33	–14.79	–5.97	17.35	30.98	43.63

These coefficient changes are large, and the full-sample marginal log-likelihood increases from –3674 to –3611.

Following Braun, Nelson, and Sunier (1995), asymmetries are added to the conditional factor loading through three terms,

$$\begin{aligned}\beta_{i,t} = \delta_{i,0} + \delta_{i,1} R\beta_{i,t-1} + \delta_{i,2}\beta_{i,t-1} + \delta_{i,3} R\beta_{i,t-1} I\left(r_{f,t-1} < 0\right) \\ + \delta_{i,4} R\beta_{i,t-1} I(\epsilon_{i,t-1} < 0) + \delta_{i,5} R\beta_{i,t-1} I\left(r_{f,t-1}\epsilon_{i,t-1} < 0\right),\end{aligned}$$

where $\delta_{i,3}$ allows for asymmetries due to the market return, $\delta_{i,4}$ allows asymmetries through the idiosyncratic shock, and $\delta_{i,5}$ allows asymmetries through the interaction of the market return and the idiosyncratic shock. This specification is fit to all 40 firms and the null $\delta_{i,3} = \delta_{i,4} = \delta_{i,5} = 0$ is tested using a Wald test with a sandwich variance–covariance for the parameters. The null is rejected for 5 out of the 40, indicating little evidence of conditional asymmetries in β. We also test whether each of asymmetry parameters $(\delta_{i,3}, \delta_{i,4}, \delta_{i,5})$ is individually significant. The numbers of times each of $\delta_{i,3} = 0$, $\delta_{i,4} = 0$, and $\delta_{i,5} = 0$ is rejected are, respectively, 2, 11, and 2 out of the 40. Finally, conditional asymmetries are added to the idiosyncratic variance

$$\sigma^2_{i,t} = \alpha_{i,0} + \alpha_{i,1}\mathrm{RIV}_{i,t-1} + \alpha_{i,2}\sigma^2_{i,t-1} + \alpha_{i,3}\mathrm{RIV}_{i,t-1}I\left(r_{f,t-1} < 0\right) + \alpha_{i,4}\mathrm{RIV}_{i,t-1}I(\epsilon_{i,t-1} < 0),$$

where $\alpha_{i,3}$ allows for asymmetries due to the factor return and $\alpha_{i,4}$ allows for asymmetries due to the sign of the idiosyncratic shock. When tested with a Wald test, only 3 of the 40 series reject the null $\alpha_{i,3} = \alpha_{i,4} = 0$, indicating that models for the idiosyncratic variance do not require conditional asymmetries.[18] We also test whether $\alpha_{i,3}$ and $\alpha_{i,4}$ are individually significant. The numbers of times each of $\alpha_{i,3} = 0$ and $\alpha_{i,4} = 0$ is rejected are both 5 out of the 40. In sum, the evidence suggests that the asymmetry is strong in the dynamics of the factor variance, but is rather weak in the dynamics of factor loadings and idiosyncratic variances.

4 Conclusion

The paper introduces a new class of multivariate high-frequency-based volatility models utilizing a factor structure for both the high- and low-frequency conditional covariances. This structure allows for parsimonious models to be fit while simplifying the task of incorporating high-frequency realized measures. The factor volatility, β, and idiosyncratic volatility are modeled as separate HEAVY-type processes and the daily shocks in low-frequency models are replaced with the corresponding realized measures. This modification is supported by the empirical evidence that realized measures dominate daily measures for most equities. The Factor HEAVY is also validated as a covariance model using the specification tests. We show that this model improves on existing multivariate HEAVY models.

In an empirical analysis of 40 large U.S. financial firms, we find that the dynamics of the conditional covariance are accurately captured by a one-factor model. The Factor HEAVY model performs better both in- and out-of-sample when compared with a leading low-frequency model. We find that Factor HEAVY has a large advantage over cDCC in covariance matrix forecasting and that the advantage appears in a variety of dimensions including the forecast copula, β, and MES. The out-of-sample forecast ability is more pronounced at shorter forecast horizons which are the most relevant from a risk-management point-of-view. We consider extensions to multiple-factor models, which is shown to be empirically

18 These results contrast sharply with what is typically found in standard low-frequency models, such as the conditional variance models in the cDCC. This difference between the low-frequency models and the high-frequency models is due to the decomposition which separates the market, which has a strong asymmetry from the idiosyncratic volatility, which does not. This additional parsimony is not applicable in standard low- (or high-) frequency cDCC-type models since the conditional variance estimated includes both components.

more successful than the one-factor model. Furthermore, we check the robustness of the performance to some selected realized measures. We also examine a number of extensions to the base model including the role of asymmetries in the conditional variance, β, and idiosyncratic variance, where we find that asymmetries are only needed for the conditional variance of the market.

Supplementary Data

Supplementary data are available at *Journal of Financial Econometrics* online.

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