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Testing for structural breaks in factor copula models[☆]

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ABSTRACT

We propose new fluctuation tests for detecting structural breaks in factor copula models and analyse the behaviour under the null hypothesis of no change. In the model, the joint copula is given by the copula of random variables which arise from a factor model. This is particularly useful for analysing data with high dimensions. Parameters are estimated with the simulated method of moments (SMM). The discontinuity of the SMM objective function complicates the derivation of a functional limit theorem for the parameters. We analyse the behaviour of the tests in Monte Carlo simulations and a real data application. It turns out that our test is more powerful than nonparametric tests for copula constancy in high dimensions.

1. Introduction

Dependence models based on copula functions have been an important topic for researchers and practitioner in the last 20 years (see Patton, 2012 and Fan and Patton, 2014 for reviews). These models offer an elegant approach for modelling multivariate distributions that has proven to be useful in many fields such as risk management, asset allocation or option pricing. Multivariate GARCH models (e.g. Engle, 2002 or Bauwens et al., 2006) or multivariate stochastic volatility model (Yu and Meyer, 2006) are the traditional way to model multivariate asset prices, but these models typically come with the drawback that they rely on the multivariate normal distribution, which contrasts stylized facts about the distribution of asset prices, in particular regarding the dependence structure. A number of parametric copula models exist that can capture the tail dependence and asymmetric dependence structure present in financial time series. More recently there have been two key advances in the literature on parametric copula modelling.

First, the need for time-varying dependence has been recognized and a number of modelling approaches have been proposed. Patton (2006) extended Sklar's theorem for conditional distributions and proposed a simple observation driven model for the evolution of the copula parameter over time. Dias and Embrechts (2004) test for structural breaks at unknown dates using a sup LR statistic, whereas Garcia and Tsafack (2011), Stöber and Czado (2014) or Chollete et al. (2009) rely on Markov switching models assuming regime dependent parameters. A model that assumes a smooth evolution over time is proposed by Hafner and Reznikova (2010). A state space approach in which the copula parameter is driven by a latent variable was advocated by Hafner and Manner (2012), whereas Creal et al. (2013) suggest a generalized autoregressive score model for time varying dependence.

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A second innovation in the copula literature has been the availability of parametric models that are applicable in higher dimensional settings. Besides the obvious choice of elliptical copulas, typically Gaussian and Student copulas, three main approaches can be found in the literature. Within the class of Archimedean copulas hierarchical models have been studied by Savu and Trede (2010) and Okhrin et al. (2013). However, in larger dimensions these models are still rather restrictive. A more popular approach is the class of vine copulas studied in Bedford and Cooke (2002), Aas et al. (2009), Stöber and Czado (2011), Stöber et al. (2013) or Brechmann and Czado (2013). A time varying vine copula model has been proposed by Almeida et al. (2016). Finally, Oh and Patton (2017) and Krupskii and Joe (2013) introduced the class of factor copula models. Factor copulas are the copulas implied by a latent factor model, where the difference to traditional factor models is the fact that one is only interested in the copula implied by the factor structure, discarding its marginal information. The advantage of these models is that they can be used in relatively high dimensional applications and nevertheless capture the dependence structure by a low number of parameters. However, the estimation of this model is complicated by the fact that the factors are not observable. Several approaches have been proposed to tackle this problem. Oh and Patton (2013) suggest a simulated method of moments estimator, an approach that we adapt in this paper. Krupskii and Joe (2013) propose maximum likelihood estimation by numerically integrating out the latent factor. This approach has the drawback that it is only applicable when the number of factors is relatively small. Murry et al. (2013) estimate a Gaussian Factor copula model with Bayesian methods. Factor copula models that allow for time-varying parameters have been proposed by Creal and Tsay (2015), who allow for stochastic autoregressive factor loading estimated with a Bayesian approach. An alternative approach can be found in Oh and Patton (2018) where the dynamics of the factor loadings are driven by a generalized autoregressive score model. This model is estimated using maximum likelihood using a multi stage approach.

The aim of this paper is to propose a different approach to allow for time-variation in factor copula models by testing for and dating breakpoints at unknown points in time. Several tests for constant dependencies have recently been developed, see e.g. Bücher and Ruppert (2013) and Giacomini et al. (2009) for the case of copulas, or Dehling et al. (2017) for the case of Kendall's tau. The main motivation for such tests is that dependencies usually increase in times of crises. Therefore, they can be applied to detect and quantify contagion between different financial markets or to construct optimal portfolios in portfolio management.

For the estimation of the model parameters, we rely on the simulated method of moments (SMM), which is different to standard method of moments applications, since the theoretical moment-counterparts are not available analytically and therefore need to be simulated. This complicates the derivation of results regarding the consistency and asymptotic distribution of the estimators. The reason is that the objective function is not continuous and furthermore not differentiable in the parameters and standard asymptotic approaches cannot be used here. We propose a new fluctuation test, where successively parameter estimators are compared to the parameter estimates of the full sample and we then analyse the behaviour of the test under the null hypothesis of no change. In contrast to formerly proposed nonparametric tests for constant copulas by, e.g., Bücher et al. (2014), our test is of parametric nature. The asymptotic distribution of the test statistic is non-trivial. Due to the non-smoothness of the objective function, we cannot make use of a Taylor expansion approach to derive the distribution under the null. To tackle this issue we propose a new construction principle inspired by Newey and McFadden (1994). These new functional limit theorems hold in general for SMM estimation and are therefore of broader interest. As the asymptotic distribution depends on unknown quantities we propose a bootstrap to estimate these.

We propose two possible tests, namely a fluctuation test based on parameter estimates and a test directly based on the moment functions used to estimate the model. We analyse size and power properties of our test in Monte Carlo simulation in various situations and compare our tests with the test proposed by Bücher et al. (2014). While the Bücher et al. (2014) test has better power properties for low dimensions, our test performs better in high dimensions. This reflects the fact that the drawback of having to estimate the model with simulation methods becomes less important as the dimension increases. If the number of dimensions is kept fixed, one simply has more data for estimating the model, while, on the other hand, in a nonparametric copula constancy test, the complexity of the estimated objects increases. Finally, we provide an application to a set of stock returns from the Eurostoxx50.

The rest of the paper is structured as follows. Section 2 presents the test statistic and studies its asymptotic distribution. Results from the Monte Carlo simulations can be found in Section 3. Section 4 presents our empirical application and Section 5 concludes the paper. All proofs are included in the Appendix .

2. Testing for constancy of factor copula models

In this section we describe our theoretical results. Factor copula models and estimation by the simulated method of moments (SMM) are reviewed in Section 2.1. Our null hypothesis and test statistic can be found in Section 2.2, whereas in Section 2.3 the asymptotic behaviour of the test is analysed. Our bootstrap algorithm is presented in Section 2.4. The broader applicability of our results is shortly explained in Section 2.5, whereas Section 2.6 discusses an important assumption that is made.

2.1. Factor copula models and their estimation

We consider the same model setup as in Oh and Patton (2013, 2017) with the difference that we allow underlying dependence parameter to be time-varying. The dynamics of the marginal distributions are determined by a parameter vector ϕ_0 and each variable can have time varying conditional mean $\mu_t(\phi_0)$ and variance $\sigma_t(\phi_0)$. The dependence of the joint distribution of the residuals η_t, captured by the parametric copula $C(., \theta_t)$, depends on the unknown parameters θ_t for $t = 1, \dots, T$. The data-generating process is given by

$$[Y_{1t}, \dots, Y_{Nt}]' =: \boldsymbol{Y}_t = \boldsymbol{\mu}_t(\phi_0) + \boldsymbol{\sigma}_t(\phi_0)\boldsymbol{\eta}_t,$$

with conditional mean $\boldsymbol{\mu}_t(\phi_0) := [\mu_{1t}(\phi_0), \dots, \mu_{Nt}(\phi_0)]'$, conditional variance $\boldsymbol{\sigma}_t(\phi_0) := \mathrm{diag}\{\sigma_{1t}(\phi_0), \dots, \sigma_{Nt}(\phi_0)\}$ and $[\eta_{1t}, \dots, \eta_{Nt}] =: \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} \boldsymbol{F}_\eta = C(F_1(\eta_1), \dots, F_N(\eta_N); \theta_t)$, with marginal distributions F_i, where $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are $\mathcal{F}_{t-1}$-measurable and independent of $\boldsymbol{\eta}_t$. $\mathcal{F}_{t-1}$ is the sigma field containing information from the past $\{\boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \dots\}$. Note that the $r \times 1$ vector ϕ_0 is $\sqrt{T}$ consistently estimable, which is fulfilled by many time series models, e.g. ARMA and GARCH models and the estimator is denoted as $\hat{\phi}$. The marginal distributions of the residuals $F_i(.)$ for $i = 1, \dots, N$ are estimated by the empirical distribution function $\hat{F}_i$.

Using the residual information $\{\hat{\boldsymbol{\eta}}_t := \boldsymbol{\sigma}_t^{-1}(\hat{\boldsymbol{\phi}})[\boldsymbol{Y}_t - \boldsymbol{\mu}_t(\hat{\phi})]\}_{t=1}^T$ from the data, we are interested in estimating the $p \times 1$ vectors $\theta_t \in \Theta$ of the copula $C(., \theta_t)$ for all t. The copula we are interested in is the factor copula that is implied by the following factor structure

$$[X_{1t}, \dots, X_{Nt}]' =: X_t = \boldsymbol{\beta}_t \boldsymbol{Z}_t + \boldsymbol{q}_t, \tag{2.1}$$

with $X_{it} = \sum_{k=1}^K \beta_{ik}^t Z_{kt} + q_{it}$, where $\boldsymbol{q}_t := [q_{1t}, \dots, q_{Nt}]'$, $q_{it} \overset{iid}{\sim} F_q(\alpha_t)$ and $Z_{kt} \overset{iid}{\sim} F_{Z_k}(\gamma_{kt})$ for $i = 1, \dots, N$, $t = 1, \dots, T$ and $k = 1, \dots, K$. Note that Z_{kt} and q_{it} are independent $\forall i, k, t$ and the copula for $\boldsymbol{X}_t$ is given by

$$\boldsymbol{X}_t \sim \boldsymbol{F}_{\boldsymbol{X}_t} = C(G_{1t}(x_{1t}; \theta_t), \dots, G_{Nt}(x_{Nt}; \theta_t); \theta_t),$$

with marginal distributions $G_{it}(., \theta_t)$ and $\theta_t = \left[\{\{\beta_{ik}^t\}_{i=1}^N\}_{k=1}^K, \alpha_t', \gamma_{1t}', \dots, \gamma_{Kt}'\right]'$. Note that the marginal distributions of the factor model $G_{it}(., \theta_t)$ are not of interest and are discarded as one is only interested in the copula implied by this model. We assume that this implied copula governs the dependence of $\boldsymbol{Y}_t$.

In principle, the copula implied by (2.1) offers many possibilities regarding the type and heterogeneity of the dependence. Through the choice of appropriate distributions F_{Z_k} of the common factors and F_q of the idiosyncratic errors one has a lot of flexibility concerning the asymmetry and tail dependence properties of the copula; see Oh and Patton (2017) for details. Furthermore, by imposing the restriction of common factor loadings for specific groups of variables, e.g. those belonging to the same industry, one can reduce the number of parameters in higher dimensional applications.

As the notation suggests, we allow θ_t to be time-varying, having a piecewise constant model in mind. We directly consider the recursive estimation of the model for increasing sample sizes. For this, we denote $s \in (0, 1]$ the fraction of the sample considered and we are interested in the recursively estimated parameter $\theta_{sT,S}$ of $\theta_{\lfloor sT \rfloor} = \theta_t$. Note that the full sample estimator is recovered for $s = 1$. For the estimation we use the simulated method of moments (SMM) estimator defined as

$$\hat{\theta}_{sT,S} := \underset{\theta \in \Theta}{\arg\min}\, Q_{sT,S}(\theta), \tag{2.2}$$

where the objective function is defined as $Q_{sT,S}(\theta) := g_{sT,S}(\theta)' \hat{W}_{sT} g_{sT,S}(\theta)$ with $g_{sT,S}(\theta) := \hat{m}_{sT} - \tilde{m}_S(\theta)$ and $\hat{W}_{sT}$ a $k \times k$ positive definite weight matrix. The $k \times 1$ vectors $\hat{m}_{sT}$ consist of appropriately chosen dependence measures that are potentially averaged from the pairwise measures $\hat{m}_{sT}^{ij}$, computed from the residuals $\{\hat{\eta}_t\}_{t=1}^{\lfloor sT \rfloor}$. As the dependence measures implied by the model are typically not available in closed form they have to be obtained by simulation. Hence, $\tilde{m}_S(\theta)$ is the corresponding vector of dependence measures computed from $\{\tilde{\eta}_l\}_{l=1}^S$, using S simulations from $\boldsymbol{F}_{\boldsymbol{X}_t}$. For the dependence measures of the pair (η_i, η_j) we need to consider copula based dependence measures that do not depend on the marginal distribution of the data. Following Oh and Patton (2013) we consider Spearman's rank correlation ρ^{ij} and quantile dependence λ_q^{ij}. These are defined as

$$\rho^{ij} := 12 \int_0^1 \int_0^1 C_{ij}(u_i, v_j) du_i dv_j - 3$$

$$\lambda_q^{ij} := \begin{cases} P[F_i(\eta_i) \le q | F_j(\eta_j) \le q] = \frac{C_{ij}(q,q)}{q}, & q \in (0, 0.5] \\ P[F_i(\eta_i) > q | F_j(\eta_j) > q] = \frac{1 - 2q + C_{ij}(q,q)}{1-q}, & q \in (0.5, 1). \end{cases}$$

The sample counterparts based on recursive samples are defined as

$$\hat{\rho}^{ij} := \frac{12}{\lfloor sT \rfloor} \sum_{t=1}^{\lfloor sT \rfloor} \hat{F}_i^s(\hat{\eta}_{it}) \hat{F}_j^s(\hat{\eta}_{jt}) - 3$$

$$\hat{\lambda}_q^{ij} := \begin{cases} \frac{\hat{C}_{ij}^s(q,q)}{q}, & q \in (0, 0.5] \\ \frac{1 - 2q + \hat{C}_{ij}^s(q,q)}{1-q}, & q \in (0.5, 1), \end{cases}$$

where $\hat{F}_i^s(y) := \frac{1}{\lfloor sT \rfloor} \sum_{t=1}^{\lfloor sT \rfloor} \mathbb{1}\{\hat{\eta}_{it} \le y\}$ and $\hat{C}_{ij}^s(u, v) := \frac{1}{\lfloor sT \rfloor} \sum_{t=1}^{\lfloor sT \rfloor} \mathbb{1}\{\hat{F}_i^s(\hat{\eta}_{it}) \le u, \hat{F}_j^s(\hat{\eta}_{jt}) \le v\}$. This means that $\hat{F}_j^s$ denotes the marginal empirical distribution function of the jth component calculated from data up to time point $[sT]$. Hence, we are using *sequential ranks*. The sample moments for the simulated data $\{\tilde{\eta}_l\}_{l=1}^S$ are defined analogously and are denoted by $\tilde{\rho}^{ij}$ and $\tilde{\lambda}_q^{ij}$.

Depending on the precise model specification the pairwise dependence measures can be averaged for pairs that are assumed to have the same factor loading as is the case in equidependence or block equidependence models; see Oh and Patton (2017). This reduces the number of moment conditions accordingly.

2.2. Null hypothesis and test statistics

The null hypothesis we are interested in is a constant copula parameter vector against the alternative of a single breakpoint at an unknown point in time,

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_T \qquad H_1 : \theta_t \neq \theta_{t+1} \text{ for some } t = \{1, \ldots, T-1\}.$$

The test statistic we propose is based on the difference between the recursive estimates of the parameter vector and its full sample analogue. Formally, it is defined as

$$\begin{aligned} P := P_{T,S} &:= \sup_{s\in[\varepsilon,1]} P_{sT,S} := \sup_{s\in[\varepsilon,1]} s^2 T(\theta_{sT,S} - \theta_{T,S})'(\theta_{sT,S} - \theta_{T,S}) \\ &\simeq \max_{\lfloor \varepsilon T\rfloor \le t \le T} \left(\frac{t}{T}\right)^2 T(\theta_{t,S} - \theta_{T,S})'(\theta_{t,S} - \theta_{T,S}), \end{aligned} \tag{2.3}$$

where $\theta_{sT,S}$ is the recursive SMM estimator defined above that used the information up to time $t = \lfloor sT \rfloor$, T the sample size of the data, S the number of simulations in the SMM and $\varepsilon > 0$ a trimming parameter. Note that analytically ε has to be chosen strictly greater than zero and thus $s \in [\varepsilon, 1]$ to apply the required limit theorems for our proof of the asymptotic distribution. In the finite sample case ε should be chosen large enough so that the model parameters can be estimated in a reasonable way using $\lfloor \varepsilon T \rfloor$ observations.

Large values of the test statistic (2.3) indicate that the successively estimated parameter vector fluctuates too much over time compared to the full sample estimator, indicating instability.

The test statistic could also be applied to a subset of the parameter vector θ. For example, one may only be interested in testing the stability of the factor loadings assuming constant shape parameters. Another possibility is to consider a block-equidependence model and test for changing factor loadings only for a specific sector such as the financial sector during a financial crisis.

We consider an alternative test statistic that is based on the same principle as (2.3), but is based directly on the moment conditions used to estimate the model.

$$\begin{aligned} M := M_T &:= \sup_{s\in[\varepsilon,1]} M_{sT} := \sup_{s\in[\varepsilon,1]} s^2 T(\hat{m}_{sT} - \hat{m}_T)'(\hat{m}_{sT} - \hat{m}_T) \\ &\simeq \max_{\lfloor \varepsilon T\rfloor \le t \le T} \left(\frac{t}{T}\right)^2 T(\hat{m}_{sT} - \hat{m}_T)'(\hat{m}_{sT} - \hat{m}_T). \end{aligned} \tag{2.4}$$

This statistic is of nonparametric nature and has the advantage that is does not require recursive estimation of the model, which is computationally quite demanding. The disadvantage is that it does not allow testing the constancy of a subset of the parameters, but only can detect breaks in the whole copula. One may, however, consider an appropriate subset of the moment conditions and test for, e.g., breaks in the lower tail quantile dependence. The asymptotic distribution of M comes as a byproduct when deriving the asymptotic distribution of P. The corresponding asymptotic results can be found in the next subsection.

2.3. Asymptotic analysis

For deriving analytical results for the asymptotic distribution of our test statistic we need the following assumptions. The first two ensure that the estimated rank correlation and quantile dependencies converge to their respective population counterparts.

Assumption 1.

(i) The distribution function of the innovations F_η and the joint distribution function of the factors $F_X(\theta)$ are continuous.
(ii) Every bivariate marginal copula $C_{ij}(u_i, u_j; \theta)$ of $\mathbf{C}(u; \theta)$ has continuous partial derivatives with respect to $u_i \in (0, 1)$ and $u_j \in (0, 1)$.

The assumption is similar to Assumption 1 in Oh and Patton (2013), but the assumption on the copula is relaxed in the sense that the restriction of u_i and v_i is relaxed to the open interval $(0, 1)$.

Assumption 2. Define $\gamma_{0t} := \sigma_t^{-1}(\hat{\phi})\dot{\mu}_t(\hat{\phi})$ and $\gamma_{1kt} := \sigma_t^{-1}(\hat{\phi})\dot{\sigma}_{kt}(\hat{\phi})$, where $\dot{\mu}_t(\phi) := \frac{\partial \mu_t(\phi)}{\partial \phi'}$ and $\dot{\sigma}_{kt}(\phi) := \frac{\partial [\sigma_t(\phi)]_{\text{kth column}}}{\partial \phi'}$ for $k = 1, \ldots, N$. Define

$$d_t = \eta_t - \hat{\eta}_t - \left(\gamma_{0t} + \sum_{k=1}^{N} \eta_{kt}\gamma_{1kt}\right)(\hat{\phi} - \phi_0),$$

with η_{kt} is the kth row of η_t and γ_{0t} and γ_{1kt} are $\mathcal{F}_{t-1}$-measurable, where $\mathcal{F}_{t-1}$ contains information from the past as well as possible information from exogenous variables.

(i) $\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor} \gamma_{0t} \xrightarrow{p} s\Gamma_0$ and $\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor} \gamma_{1kt} \xrightarrow{p} s\Gamma_{1k}$, uniformly in $s \in [\varepsilon, 1]$, $\varepsilon > 0$, where Γ_0 and Γ_{1k} are deterministic for $k = 1, \ldots, N$.
(ii) $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{0t}\|)$, $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{0t}\|^2)$, $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{1kt}\|)$ and $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{1kt}\|^2)$ are bounded for $k = 1, \ldots, N$.
(iii) There exists a sequence of positive terms $r_t > 0$ with $\sum_{i=1}^{\infty} r_t < \infty$, such that the sequence $\max_{1\le t\le T} \frac{\|d_t\|}{r_t}$ is tight.
(iv) $\max_{1\le t\le T} \frac{\|\gamma_{0t}\|}{\sqrt{T}} = o_p(1)$ and $\max_{1\le t\le T} \frac{|\eta_{kt}|\|\gamma_{1kt}\|}{\sqrt{T}} = o_p(1)$ for $k = 1, \ldots, N$.
(v) $(\alpha_T(s), \sqrt{T}(\hat{\phi} - \phi_0))$ weakly converges to a continuous Gaussian process in $\mathcal{D}([0, 1]^N) \times \mathbb{R}^r$, where $\mathcal{D}$ is the space of all càdlàg-functions on $[0, 1]^N$, with

$$\alpha_T(s) := \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor sT \rfloor} \left\{ \prod_{k=1}^{N} \mathbb{1}\{U_{kt} \le u_k\} - \mathbf{C}(u; \theta) \right\}.$$

(vi) $\frac{\partial F_\eta}{\partial \eta_k}$ and $\eta_k \frac{\partial F_\eta}{\partial \eta_k}$ are bounded and continuous on $\overline{\mathbb{R}}^N = [-\infty, \infty]^N$ for $k = 1, \ldots, N$.
(vii) For $\mathbf{u} \in [0, 1]^N$ and $\hat{\mathbf{F}}^s(\hat{\eta}_t) = (\hat{F}_1^s(\hat{\eta}_{1t}), \ldots, \hat{F}_N^s(\hat{\eta}_{Nt}))$, the sequential empirical copula process

$$\frac{1}{\sqrt{T}} \left[\sum_{t=1}^{\lfloor sT \rfloor} \mathbb{1}\{\hat{\mathbf{F}}^s(\hat{\eta}_t) \le \mathbf{u}\} - C(\mathbf{u}) \right] \tag{2.5}$$

converges in distribution to some limit process $A^*(s, \mathbf{u})$.

Parts (i) to (vi) of this assumption are similar to Assumption 2 in Oh and Patton (2013), only part (i) is more restrictive. We need this because we consider successively estimated parameters.

Part (vii) ensures that the empirical copula process of the residuals has some well defined limit. Given the literature on this topic, the assumption is plausible, which is discussed in more detail in Section 2.6.

The next assumption is needed for consistency of the successively estimated parameters. It is the same as Assumption 3 in Oh and Patton (2013) with the difference that part (iv) is adapted to our situation.

Assumption 3.

(i) $g_0(\theta) = 0$ only for $\theta = \theta_0$.
(ii) The space Θ of all θ is compact.
(iii) Every bivariate marginal copula $C_{ij}(u_i, u_j; \theta)$ of $\mathbf{C}(u; \theta)$ is Lipschitz-continuous for $(u_i, u_j) \in (0, 1) \times (0, 1)$ on Θ.
(iv) The sequential weighting matrix $\hat{W}_{sT}$ is $O_p(1)$ and $\sup_{s\in[\varepsilon,1]} \|\hat{W}_{sT} - W\| \xrightarrow{p} 0$ for $\varepsilon > 0$, where W is probability limit of W_{sT}.

Finally, we need an assumption for distributional results, which is the same as Assumption 4 in Oh and Patton (2013) with a difference in part (iii).

Assumption 4.

(i) θ_0 is an interior point of Θ.
(ii) $g_0(\theta)$ is differentiable at θ_0 with derivative G such that $G'WG$ is nonsingular.
(iii) $\forall s \in [\varepsilon, 1], \varepsilon > 0$: $g_{sT,S}(\theta_{sT,S})'\hat{W}_{sT}g_{sT,S}(\theta_{sT,S}) \le \inf_{\theta\in\Theta} g_{sT,S}(\theta)'\hat{W}_{sT}g_{sT,S}(\theta) + o_p^*((s^2T)^{-1})$, where $o_p^*((s^2T)^{-1})$ converges on the right hand side to zero and is therefore strictly positive.

With these assumptions, we can formulate our main theorem:

Theorem 1. *Under the null hypothesis $H_0 : \theta_1 = \theta_2 = \cdots = \theta_T$ and if Assumptions 1–4 hold, we obtain for $\varepsilon > 0$*

$$s\sqrt{T}\left(\theta_{sT,S} - \theta_0\right) \xrightarrow{d} A^*(s)$$

as $T, S \to \infty$ in the space of Càdlàg functions on the interval $[\varepsilon, 1]$ and $\frac{S}{T} \to k \in (0, \infty)$ or $\frac{S}{T} \to \infty$. Here, $A^*(s) = \left(G'WG\right)^{-1} G'W(A(s) - \frac{s}{\sqrt{k}}A(1))$, $A(s)$ is a Gaussian process defined in the proof of Lemma 7 in the Appendix and θ_0 the value of all θ_t under the null.

With Theorem 1 we obtain the asymptotic distribution under the null of our parameter test statistic.

Corollary 1. *Under the null hypothesis $H_0 : \theta_1 = \theta_2 = \cdots = \theta_T$ and if Assumptions 1–4 hold, we obtain for our test statistic*

$$P = \sup_{s\in[\varepsilon,1]} s^2 T(\theta_{sT,S} - \theta_{T,S})'(\theta_{sT,S} - \theta_{T,S}) \xrightarrow{d} \sup_{s\in[\varepsilon,1]} (A^*(s) - sA^*(1))'(A^*(s) - sA^*(1))$$

as $T, S \to \infty$ and $\frac{S}{T} \to k \in (0, \infty)$ or $\frac{S}{T} \to \infty$.

The estimation of the change point location is embedded in calculating the test statistic and is given by $\lfloor \tilde{s}T \rfloor$, where $\tilde{s}$ is the maximum point of the quadratic left side of Corollary 1, i.e.

$$\tilde{s} = \operatorname*{argmax}_{s\in[\varepsilon,1]} s^2 T(\theta_{sT,S} - \theta_{T,S})'(\theta_{sT,S} - \theta_{T,S}).$$

For our nonparametric moment test we derive the following asymptotic distribution:

Corollary 2. *Under the null hypothesis $H_0 : \theta_1 = \theta_2 = \cdots = \theta_T$ and if Assumption 1–2 hold, we obtain for our test statistic*

$$M = \sup_{s\in[\varepsilon,1]} s^2 T(\hat{m}_{sT} - \hat{m}_T)'(\hat{m}_{sT} - \hat{m}_T) \xrightarrow{d} \sup_{s\in[\varepsilon,1]} (A(s) - sA(1))'(A(s) - sA(1))$$

as $T, S \to \infty$ and $\frac{S}{T} \to k \in (0, \infty)$ or $\frac{S}{T} \to \infty$.

The location of the changepoint is estimated in the same fashion as for P

Note that the asymptotic distribution of the moment test, as well as the asymptotic distribution of the parameter test, are not known in closed form and depend on the underlying sample. For this reason we cannot compute or simulate the critical values directly and need a bootstrap procedure to overcome this issue.

2.4. Bootstrap distribution

The bootstrap distribution of the test statistics P and M obtained by calculating B versions of the moment process $\frac{t}{T}\sqrt{T}\left(\hat{m}_t^{(b)} - \hat{m}_T^{(b)}\right)$, which can be calculated fast and directly from the data. It is therefore not necessary to solve B minimization problems which would produce a high computational effort.

We estimate the distribution under the null by using an i.i.d. bootstrap with the following steps:

(i) Sample with replacement from the standardized residuals $\{\hat{\eta}_i\}_{i=1}^T$ to obtain a B bootstrap samples $\{\hat{\eta}_i^{(b)}\}_{i=1}^T$, for $b = 1, \ldots, B$.

(ii) Use $\{\hat{\eta}_i^{(b)}\}_{i=1}^t$ to compute $\hat{m}_t^{(b)}$ for $b = 1, \ldots, B$ and $t = \varepsilon T, \ldots, T$ and $\{\hat{\eta}_i\}_{i=1}^T$ to obtain $\hat{m}_T$.

(iii) Calculate the bootstrap analogue of the limiting distribution of Corollary 1.

$$K^{(b)} := \max_{t\in\{\varepsilon T,\ldots,T\}} \left(A_*^{(b)}\left(\frac{t}{T}\right) - \frac{t}{T}A_*^{(b)}(1)\right)' \left(A_*^{(b)}\left(\frac{t}{T}\right) - \frac{t}{T}A_*^{(b)}(1)\right),$$

with $A_*^{(b)}\left(\frac{t}{T}\right) := (\hat{G}'\hat{W}_T\hat{G})^{-1}\hat{G}'\hat{W}_T A^{(b)}(\frac{t}{T})$ and $A^{(b)}(\frac{t}{T}) = \frac{t}{T}\sqrt{T}\left(\hat{m}_t^{(b)} - \hat{m}_T\right)$, where $\hat{G}$ is the two sided numerical derivative estimator of G, evaluated at point $\theta_{T,S}$, computed with the full sample $\{\hat{\eta}_i\}_{i=1}^T$. We can compute the kth column of $\hat{G}$ by

$$\hat{G}^k = \frac{g_{T,S}(\theta_{T,S} + e_k\epsilon_{T,S}) - g_{T,S}(\theta_{T,S} - e_k\epsilon_{T,S})}{2\epsilon_{T,S}}, \quad k \in \{1, \ldots, p\},$$

where e_k is the kth unit vector, whose dimension is $p \times 1$ and $\epsilon_{T,S}$ has to be chosen in a way that it fulfils $\epsilon_{T,S} \to 0$ and $\min\{\sqrt{T}, \sqrt{S}\}\epsilon_{T,S} \to \infty$.

(iv) Compute B versions of $K^{(b)}$ and determine the critical value K such that

$$\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\{K^{(b)} > K\} = 0.05.$$

For our simulation study and the empirical application we use a step size $\epsilon_{T,S} = 0.1$, which worked best following Oh and Patton (2013). Critical values of the moment based test M are obtained similarly by adapting step (iii) of the algorithm.

The intuition for the validity of the bootstrap, beside the fact that we only use the natural estimators for the respective terms, is as follows: Under the null hypothesis, we draw with replacement from the empirical distribution function which is close to the true distribution function. Due to the structure of the limit distribution of the test statistic, we can directly generate realizations from this without having to care about a suitable centring. Under the alternative of one fixed break at time t, the bootstrap quantiles remain bounded because the bootstrap procedure mimics a stationary distribution. By randomly drawing from either the data before or after the break, we effectively draw from stationary distribution which takes the parameters before the break with probability t/T and the ones after the break with probability 1-t/T. Using the above described bootstrap procedure the simulation results indicate that the test results in a reasonable sized and powered testing procedure, cf. Tables 1–3 in Section 3. A formal proof of the bootstrap validity is left for future research.

2.5. Discussion on broader applicability

Although the focus of this paper lies on factor copulas, our tests are not restricted to this case. For example, if the factor copula structure in Eq. (2.1) is replaced by another copula, say an Archimedean copula, the parameter test (2.3) can be performed in a similar way. To obtain a valid test (size control and consistency under fixed alternatives), it is necessary that the SMM procedure yields consistent parameter estimators of the model under the null hypothesis of constant parameters. Oh and Patton (2017) show that many commonly used copulas can be expressed as factor copulas, whereas in Oh and Patton (2013) the estimation of other types of copula models by SMM is considered. On the other hand, we would like to stress that we consider factor copula models as the main application of SMM estimation, at least in financial econometrics. Simpler models can be estimated by ML or GMM, for which the literature already provides change point tests (see e.g. Wied, 2013).

If the model is misspecified (i.e., that the simulated moments do not arise from the correct model), it cannot be expected that the test is valid. We investigate this case in the simulation section and we find that the tests are, in fact, correctly sized in the case of misspecification.

On the other hand, the moment-based test (2.4) can be interpreted as a general constancy test for dependence measures such as Spearman's rho or quantile dependencies (compare e.g. Wied et al., 2013). It is in fact only indirectly linked to the factor copula model, i.e., this test detects changes in the moments which are induced by changes in the parameters. Therefore, the presence of a factor copula model is not necessary for this test.

2.6. Discussion of Assumption 2.(vii)

Bücher et al. (2014) derive a result similar to the one we have in Assumption 2.7. In particular, in their Proposition 3.3, the limit process is given by

$$\mathbb{B}(s,\mathbf{u})-\sum_{j=1}^{N}\partial_j C(\mathbf{u})\mathbb{B}(s,\mathbf{u}^{(j)}).$$

Here, $\mathbf{u}^{(j)}\in[0,1]^N$ is defined by $\mathbf{u}_i^{(j)}=\mathbf{u}_j$, if $i=j$ and 1 otherwise. Moreover $\mathbb{B}(s,\mathbf{u})$ is a tight centred continuous Gaussian process with $\mathbb{B}(0,\mathbf{u})=0$ and

$$\mathrm{Cov}(\mathbb{B}(s,\mathbf{u}),\mathbb{B}(t,\mathbf{v}))=\min(s,t)\mathrm{Cov}(\mathbb{1}(\mathbf{F}(\eta)\le\mathbf{u}),\mathbb{1}(\mathbf{F}(\eta)\le\mathbf{v})).$$

The difference to their setting and ours is that they do not consider residuals, but the original observations. However, we can transfer this result by combining a result by Remillard (2017) and Bücher and Kojadinovic (2016). Remillard (2017) considers the case of residuals. We cannot use his copula results directly, because he only considers the case of $\hat{F}_j^1$, i.e., the case, where the ranks are not calculated sequentially. Nevertheless, Theorem 1 in Remillard (2017) gives a convergence result for the residuals themselves and thus also for the residuals transformed by the (unknown) limit of the empirical distribution function of the residuals. Combined with Theorem 3.4 in Bücher and Kojadinovic (2016), under the additional assumption that the residuals are strictly stationary, we obtain that the process in (2.5) converges to

$$A^*(s,\mathbf{u}):=\mathbb{B}^*(s,\mathbf{u})-\sum_{j=1}^{N}\partial_j C(\mathbf{u})\mathbb{B}^*(s,\mathbf{u}^{(j)}),$$

where $\mathbb{B}^*(s,\mathbf{u})=\mathbb{B}(s,\mathbf{u})+s\mathbb{B}^{**}(\mathbf{u})$ and details about $\mathbb{B}(\mathbf{u})$ can be found in Theorem 1 in Remillard (2017). In particular, it follows that the limit of the sequential empirical copula CUSUM process (where $C(\mathbf{u})$ is replaced by $\frac{1}{T}\sum_{t=1}^{\lfloor T\rfloor}\mathbb{1}\{\hat{\mathbf{F}}^s(\hat{\eta}_t)\le\mathbf{u}\}$) does not depend on whether residuals or the original observations are used.

3. Monte Carlo simulations

In order to study the behaviour of our tests in finite samples and the quality of the bootstrap approximations we perform a small set of Monte Carlo simulations. To this end we consider the one factor copula model

$$[X_{1t},\ldots,X_{Nt}]'=:X_t=\boldsymbol{\beta}_t Z_t+\boldsymbol{q}_t, \tag{3.1}$$

with $\boldsymbol{\beta}_t=(\beta_t,\ldots,\beta_t)'$ a vector of size N, $Z_t\overset{inid}{\sim}\mathrm{Skewt}\left(\nu^{-1},\lambda\right)$[1] and $q_t\overset{iid}{\sim}\mathrm{t}\left(\nu^{-1}\right)$ for $t=1,\ldots,T$. We fix $\nu^{-1}=0.25$ and $\lambda=-0.5$, such that our model is parametrized by the single factor loading $\theta_t=\beta_t$.

For the estimation of the sequential parameters β_t for $t=\varepsilon T,\ldots,T$ in the test statistic we use the SMM approach with $S=25\cdot T$ simulations to match the simulated dependence measures with the dependence measures computed from the data. For this we use five dependence measures, namely Spearman's rank correlation and the 0.05, 0.10, 0.90, 0.95 quantile

[1] As in Oh and Patton (2017) this refers to the skewed t-distribution by Hansen (1994).

Table 1
Size.

	$\theta_0 = 0.5$	$N = 5$	$N = 10$	$N = 20$	$\theta_0 = 1$	$N = 5$	$N = 10$	$N = 20$	$\theta_0 = 2.0$	$N = 5$	$N = 10$	$N = 20$
$T = 500$												
	P	0.102	0.079	0.056	P	0.066	0.056	0.053	P	0.059	0.059	0.056
	M	0.029	0.036	0.046	M	0.030	0.039	0.056	M	0.033	0.043	0.043
	BKRS	0.046	0.036	0.033	BKRS	0.049	0.053	0.049	BKRS	0.056	0.046	0.023
$T = 1000$												
	P	0.089	0.049	0.049	P	0.056	0.046	0.069	P	0.049	0.059	0.049
	M	0.046	0.033	0.056	M	0.049	0.043	0.076	M	0.043	0.056	0.069
	BKRS	0.043	0.046	0.056	BKRS	0.066	0.056	0.076	BKRS	0.069	0.060	0.046
$T = 1500$												
	P	0.073	0.059	0.043	P	0.056	0.069	0.066	P	0.066	0.069	0.066
	M	0.056	0.056	0.049	M	0.049	0.063	0.066	M	0.063	0.069	0.069
	BKRS	0.046	0.056	0.069	BKRS	0.053	0.069	0.066	BKRS	0.046	0.046	0.049

Note: Table 1 reports the rejection rate for $\theta_0 = 0.5, \theta_0 = 1$ and $\theta_0 = 2$ in the model (3.1) for the parameter Test (P) with $\varepsilon = 0.2$, the moment function test (M) and the nonparametric test of Bücher et al. (BKRS).

dependence measures, averaged across all pairs. Note that the burn in period $\lfloor \varepsilon T \rfloor$ has to be chosen sufficiently large in order to obtain reasonable parameter estimates for $\theta_{\lfloor \varepsilon T \rfloor}$ in our test statistic. Unreported simulations suggested that for samples with less than 100 observations highly unreasonable estimates can occur that severely affect the behaviour of our test. We decided to use $\varepsilon = 0.2$. While this is a limitation of our test in the sense that breaks at the beginning of the sample cannot be identified, truncating the sample is common in some tests for structural breaks, see Andrews (1993) or Qu and Perron (2007). Furthermore, breaks at the beginning and the end of the sample are typically hard to detect in any case.

We consider three tests in this simulation exercise, namely the parameter based fluctuation test (P) given in Eq. (2.3), the test based on the moment condition (M) given in (2.4) and the nonparametric test for copula constancy proposed by Bücher et al. (2014) abbreviated as BKRS. The change point detection in the latter test is sensitive to changes in the copula of the multivariate continuous observations and is included as a benchmark. We do note, however, that this test is purely nonparametric in contrast to our test P that is based explicitly on factor copula models. Critical values of our tests are computed using the bootstrap algorithm from Section 2.4 with $B = 1000$ bootstrap replications. The tests are performed at the $\alpha = 0.05$ significance level and we use 301 Monte Carlo replications.[2]

We begin by studying the size of the test for three parameter values $\theta_0 = 0.5$, $\theta_0 = 1$ and $\theta_0 = 2$, sample sizes $T = 500, 1000, 1500$ and cross sectional dimensions $N = 5, 10, 20$. Results are presented in Table 1. All test have acceptable size properties except the parameter based test for small dimensions and sample sizes in the case $\theta_0 = 0.5$. However, as N and T increase the size clearly tends to the nominal level of 5%.[3]

Furthermore, we study the size of the tests when the DGP is not a factor copula. Here we again consider $N = 5, 10, 20$, but only $T = 1000$. For the parameter based test this means that the model is misspecified. The first case is a Clayton copula with parameter $\theta = 1$, a model implying equidependence with Kendall's τ equal to 1/3 and lower tail dependence. The second DGP is a (truncated) Dvine copula model (see Aas et al., 2009). On the first tree all pairs are connected with a Clayton copula with $\theta = 2$, the second tree has Gaussian copulas with $\rho = 0.5$ and the third tree survival Gumbel copulas with parameter $\gamma = 1.25$. All remaining trees have conditional independence, implying the truncation of the model. This model does not imply equidependence, but lower tail dependence is still present for all pairs. The parametric test P is based on the same one-factor copula model (3.1) for both cases. The results in Table 2 show that all tests have good size properties. The parameter based test P is slightly undersized for $N = 5$. From the two examples (Clayton and Dvine copula), it seems that the test is also reliable when the underlying model is misspecified.

To study the power of the test, we generate data with a break point at $\frac{T}{2}$ for all sample sizes, where the data is simulated with $\theta_t = 1$ for $t \in \{\varepsilon T, \ldots, \frac{T}{2}\}$, denoted by θ_0, whereas after the break we increase the parameter to $\theta_t = \{1.2, 1.4, 1.6, 1.8, 2.0\}$ for $t \in \{\frac{T}{2} + 1, \ldots, T\}$, denoted by θ_1. Here we consider $N = 5, 10, 20, 40$, but restrict the sample size to the cases $T = 500{,}1000$. The results can be found in Table 3. Note that the first column of the table reports the size of the tests again. For $N = 40$ the test has good size properties for $T = 500$, but are slightly oversized for $T = 1000$. The BKRS test has the most severe size distortions in this case. We observe that all tests have good power that increases with θ_1 and sample size T. The parameter based test P and the moment test M have increasing power as N increases from 5 to

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2 The computational complexity of the simulations was extremely high due to the fact that for each test $\theta_{sT,S}$ needs to be estimated a large number of times using the computationally heavy SMM estimator and because critical values have to be bootstrapped. This explains why we had to restrict ourselves to a limited number of situations for a fairly simple model. Furthermore, numerical instabilities were present in more complex models when repeatedly estimating the model parameters. Such problems can be dealt with in empirical applications, but further restrict the potential model complexity in simulations. The computations were implemented in Matlab, parallelized and performed using CHEOPS, a scientific High Performance Computer at the Regional Computing Center of the University of Cologne (RRZK) funded by the DFG.

3 Note that a larger burn in period εT leads to a slightly better size properties, in particular for small values of T and N, which can be explained by a lower degree of variation in the numerical minimization procedure.

Table 2
Size under alternative copula DGPs.

		$N = 5$	$N = 10$	$N = 20$	$N = 5$	$N = 10$	$N = 20$
$T = 1000$		Clayton copula			Dvine copula		
	P	0.033	0.037	0.049	0.033	0.056	0.059
	M	0.043	0.047	0.069	0.033	0.063	0.063
	BKRS	0.057	0.049	0.043	0.043	0.053	0.073

Note: Table 2 reports the rejection rate under H_0 for alternative data generating processes for the parameter Test (P) with $\varepsilon = 0.2$ and assuming a one-factor copula, the moment function test (M) and the nonparametric test of Bücher et al. (BKRS), where a Clayton and a Dvine copula are used for the DGP.

Table 3
Power.

	$\theta_0 = 1$	$\theta_1 = 1.2$	$\theta_1 = 1.4$	$\theta_1 = 1.6$	$\theta_1 = 1.8$	$\theta_1 = 2.0$
$N = 5, T = 500$						
P	0.066	0.272	0.551	0.833	0.963	0.993
M	0.030	0.173	0.452	0.771	0.940	0.987
BKRS	0.049	0.272	0.727	0.946	0.996	1.000
$N = 10, T = 500$						
P	0.056	0.266	0.658	0.887	0.993	1.000
M	0.039	0.236	0.558	0.877	0.983	0.997
BKRS	0.053	0.285	0.764	0.973	1.000	1.000
$N = 20, T = 500$						
P	0.053	0.299	0.704	0.907	1.000	1.000
M	0.056	0.259	0.628	0.900	0.993	1.000
BKRS	0.049	0.275	0.750	0.966	1.000	1.000
$N = 40, T = 500$						
P	0.043	0.302	0.691	0.910	0.996	1.000
M	0.059	0.282	0.635	0.920	0.993	1.000
BKRS	0.059	0.225	0.588	0.903	0.996	1.000
$N = 5, T = 1000$						
P	0.056	0.352	0.781	0.980	1.000	1.000
M	0.049	0.285	0.717	0.966	1.000	1.000
BKRS	0.066	0.481	0.946	1.000	1.000	1.000
$N = 10, T = 1000$						
P	0.046	0.415	0.874	0.993	1.000	1.000
M	0.043	0.352	0.801	0.993	1.000	1.000
BKRS	0.056	0.478	0.963	1.000	1.000	1.000
$N = 20, T = 1000$						
P	0.069	0.455	0.887	1.000	1.000	1.000
M	0.076	0.389	0.834	0.993	1.000	1.000
BKRS	0.076	0.465	0.943	0.996	1.000	1.000
$N = 40, T = 1000$						
P	0.076	0.4751	0.927	1.000	1.000	1.000
M	0.073	0.399	0.844	0.993	1.000	1.000
BKRS	0.093	0.398	0.880	0.993	0.996	1.000

Note: Table 3 reports the rejection rate for $\theta_0 = 1.0$ and $\theta_1 = 1.2, 1.4, 1.6, 1.8, 2$ in the model (3.1) for the parameter Test (P) with $\varepsilon = 0.2$, the moment function test (M) and the nonparametric test of Bücher et al. (BKRS).

40, whereas the power of the BKRS test decreases for the higher dimensional case. For $N = 5, 10, 20$ the BKRS test has the highest power followed by the parameter based test. For $N = 40$, however, the P test performs better and even the M test has (mostly) more power than the nonparametric BKRS test. This indicates that the tests based on the factor copula model are preferable for higher dimensional situations. This can be explained by the fact that more available data improves the SMM estimation, while in a nonparametric copula constancy test the complexity of the estimated objects increase.

4. Empirical application

In this section we apply our test to a financial dataset. We use daily stock return data over a time span ranging from July 2005 to May 2009 from the EURO STOXX 50 of the four largest industry sectors Finance, Energy, Telecom and Media and Consumer Retail and we choose the subdivision in Table 4, implying $T = 1000$ and $N = 32$, with group sizes $k_1 = 13, k_2 = 8, k_3 = 5$ and $k_4 = 6$.

Table 4
Included stocks by industry.

Finance	Allianz, Axa, Banco Bilbao, Banco Santander, BNP Paribas, Deutsche Bank, Deutsche Börse, Generali, ING Groep, Intesa, Münchener Rück, Société Générale, Unicredit
Energy	E.ON, ENEL, ENI, SUEZ, Iberdrola, Repsol, RWE, Total
Telecom and media	Deutsche Telekom, France Telecom, Telecom Italia, Telefonica, Vivendi
Consumer retail	Anheuser Busch, Carrefour, Danone, L'Oreal, LVMH, Unilever

To model the conditional mean and variance we estimate an AR(1)-GARCH(1,1) model for each return series and compute the standardized residuals,

$$r_{i,t} = \alpha_i + \beta_i r_{i,t-1} + \sigma_{i,t}\eta_{it},$$
$$\sigma_{it}^2 = \gamma_{i0} + \gamma_{i1}\sigma_{i,t-1}^2 + \gamma_{i2}\eta_{i,t-1}^2,$$

for $t = 1, \ldots, 1000$. The marginal distribution of the residuals is estimated using the empirical CDF. Following Oh and Patton (2017) we specify the following block-equidependence five factor copula model:

$$[X_{1t}, \ldots, X_{Nt}]' =: X_t = \begin{pmatrix} \boldsymbol{\beta}_{1t} \\ \boldsymbol{\beta}_{2t} \\ \boldsymbol{\beta}_{3t} \\ \boldsymbol{\beta}_{4t} \end{pmatrix} Z_{0t} + \begin{pmatrix} \boldsymbol{\beta}_{5t} Z_{1t} \\ \boldsymbol{\beta}_{6t} Z_{2t} \\ \boldsymbol{\beta}_{7t} Z_{3t} \\ \boldsymbol{\beta}_{8t} Z_{4t} \end{pmatrix} + \boldsymbol{q}_t, \tag{4.1}$$

with $\boldsymbol{\beta}_{it} = (\beta_{it}, \ldots, \beta_{it})'$ of size k_i for $i = 1, 2, 3, 4$, where $Z_{0t} \sim \text{Skewt}\left(\nu^{-1}, \lambda\right)$ and $Z_{it} \sim \text{t}\left(\nu^{-1}\right)$ for $i = 1, 2, 3, 4$ and $\boldsymbol{q}_t \overset{iid}{\sim} \text{t}\left(\nu^{-1}\right)$ for $t = 1, \ldots, T$. Thus, we have one common factor with industry specific factor loadings $\boldsymbol{\beta}_{it}$ for $i = 1, \ldots, 4$ and four industry specific factors with corresponding loadings $\boldsymbol{\beta}_{it}$ for $i = 5, \ldots, 8$. We assume identical degrees of freedom for the common factors and the idiosyncratic errors implying a model with tail dependence strictly between zero and one.

For the estimation of the model we use the SMM approach described above with $S = 25 \cdot T$ simulations. The moment conditions are based on five dependence measures, namely Spearman's rank correlation and the 0.05, 0.10, 0.90, 0.95 quantile dependence. In the block equidependence model with four groups and five dependence measures this gives us a total number of $4 \times 5 = 20$ dependence measures.

The full sample estimates can be found in Table 5. For studying the time-variation we fix λ and ν at their full sample estimates to avoid numerical problems as these parameters are difficult to estimate for small samples. As a preliminary analysis, we estimate the model over a rolling window of 200 days. Fig. 4.1 shows that there is some variation over time in the factor loadings with an apparent increase in most parameters towards the end of the sample.

The results of the tests for a structural break in the factor copula parameters can be found in Table 6. The moment based test M finds a significant breakpoint on January 8, 2008. The BKRS test finds a similar break data (Jan. 17), but the statistic is only significant at the 10% level. The parameter test P applied to all factor loadings indicates a break slightly later on March 7, 2008. This is a little earlier than the peak of the financial crisis with Lehman Brothers filing bankruptcy on September 15. Some of the estimated parameters after the break are larger than before the break while other decrease. This makes the direct interpretation of the change in dependence difficult. We return to the implied dependence of the model before and after the break below.

As the dataset contains companies from different sectors we applied the P test to a number of subvectors of the factor loadings. To be precise, we tested for a break in the loading of the market factor alone and of the loadings on the market and group specific factors for each respective sector, while fixing the remaining model parameters at their full sample estimates. For all subsets we find evidence of a structural break. However, the estimated break dates are later than for the full set of loading mainly around the peak of the financial crisis. The break for the loading corresponding to the energy sector is later in December 2008. Comparing the estimated parameters before and after the break, some of the loadings decrease after the estimated breakpoint. Part of the apparent discrepancies between the results for the full loading vector and the analysis on the subsets can be explained by the differences in estimated break dates coupled with the fact that the estimation uncertainty for the relatively small post-break period is quite large, which is due to the fact that factor copulas are difficult to estimate on such small samples.

A direct interpretation of the change in the factor loadings is difficult due to the complex interactive effect the different factors have on the overall dependence structure. Therefore, we computed (by simulation) the rank correlations implied by the different break models. The result can be found in Table 7. As we have a block-equidependence model the implied dependence for assets within each sector is the same, as is the dependence between assets from two sectors. The within sector dependence is given on the main diagonal of the presented matrices, while the between sector dependences are given by the off-diagonal elements.

The results based on the break in all factor loadings indicate increasing (Energy, Telecom) or almost stable (Finance, Consumer) within sector rank correlations and slightly increasing rank correlations between the sectors. The break for the market factor loadings implies a similar change in dependence, but a stronger increase between the Telecom and Energy

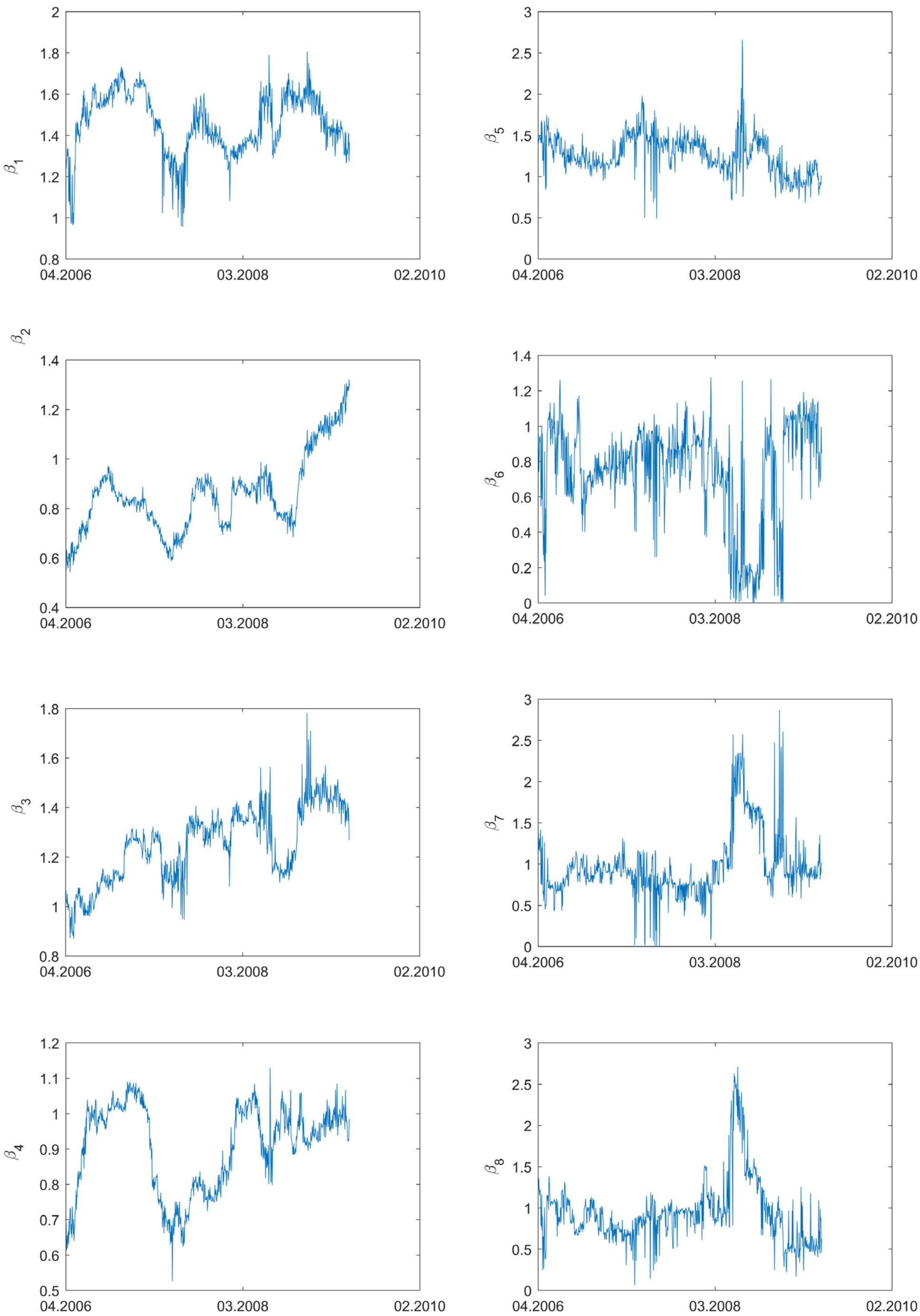

Fig. 4.1. Rolling window parameter estimation for the factor loadings β_i for a window of size 200.

sectors. For the sector specific breaks we note that the results for the finance sector indicate a slight decrease within the finance sector, but increased dependence with the other sectors, which can be interpreted as an indication of contagion from the finance sector to the other sectors. For the energy sector specific case we observe an increase both within the sector and across sectors. The case of the telecommunication sector indicates an increase within the sector, but mostly stable dependence with the other sectors. For the consumption sector-specific breaks we again see stable within sector dependence, but increased dependence across the sectors. Overall, we conclude from this that dependence has indeed increase after the break but the increase in dependence is not as strong and clear cut as one might expect. Estimation uncertainty for the sub-periods may partly explain the mixed results.

Table 5
Full sample Parameter estimates of the model (4.1).

	$\hat{\nu}$	$\hat{\lambda}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$	$\hat{\beta}_7$	$\hat{\beta}_8$
$\theta_{T,S}$	9.263	−0.157	1.491	0.970	1.253	0.889	1.095	0.517	0.678	1.668
std	1.249	0.065	0.132	0.048	0.082	0.073	0.433	0.159	0.169	0.082

Table 6
Breakpoint tests.

		Test stat	p-val	Break date	$\hat{\theta}_{pre}$	$\hat{\theta}_{post}$
θ_{factor}	$\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\\ \beta_6\\ \beta_7\\ \beta_8\end{pmatrix}$	2008.50	0.000	07.03.2008	$\begin{pmatrix}1.36\\ 0.77\\ 1.16\\ 0.86\\ 1.55\\ 0.73\\ 0.76\\ 0.93\end{pmatrix}$	$\begin{pmatrix}1.32\\ 1.16\\ 1.28\\ 1.00\\ 1.22\\ 1.10\\ 1.31\\ 0.56\end{pmatrix}$
θ_{market}	$\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{pmatrix}$	49.71	0.000	15.08.2008	$\begin{pmatrix}1.38\\ 0.74\\ 1.14\\ 0.82\end{pmatrix}$	$\begin{pmatrix}1.24\\ 1.41\\ 1.30\\ 0.93\end{pmatrix}$
$\theta_{finance}$	$\begin{pmatrix}\beta_1\\ \beta_5\end{pmatrix}$	1297.10	0.000	02.09.2008	$\begin{pmatrix}1.24\\ 1.88\end{pmatrix}$	$\begin{pmatrix}1.35\\ 1.16\end{pmatrix}$
θ_{energy}	$\begin{pmatrix}\beta_2\\ \beta_6\end{pmatrix}$	323.98	0.001	05.12.2008	$\begin{pmatrix}0.76\\ 0.65\end{pmatrix}$	$\begin{pmatrix}1.41\\ 1.33\end{pmatrix}$
θ_{tele}	$\begin{pmatrix}\beta_3\\ \beta_7\end{pmatrix}$	907.39	0.000	02.09.2008	$\begin{pmatrix}1.20\\ 1.44\end{pmatrix}$	$\begin{pmatrix}1.52\\ 0.75\end{pmatrix}$
θ_{consum}	$\begin{pmatrix}\beta_4\\ \beta_8\end{pmatrix}$	583.23	0.000	02.07.2008	$\begin{pmatrix}0.65\\ 1.47\end{pmatrix}$	$\begin{pmatrix}0.95\\ 1.20\end{pmatrix}$
M		5.72	0.000	09.01.2008		
$BKRS$		8.25	0.065	17.01.2008		

Note: Table 6 reports tests for a structural break in the factor copula model (4.1). The penultimate row gives the results of the moment based test M. The last row gives the results of the nonparametric test of Bücher et al. (BKRS). The other rows show the results of the parameter based test P for the given subsets of the parameter vector while fixing the remaining parameter values at the full sample estimates. $\hat{\theta}_{pre}$ and $\hat{\theta}_{post}$ denote the parameter estimates before and after the estimated break dates, respectively. We use 1000 bootstrap replications.

In order to get a clearer picture of the evolution of the size and structure of the dependence with respect to the breakpoint we computed the (averaged) dependence measures that were used for estimation before and after the breakpoint indicated by the M test, see Table 8. The results indicate that the overall dependence measured by the rank correlation ρ increases. Similarly, the upper quantile dependence measures $\lambda_{0.9}$ and $\lambda_{0.95}$ increase after the break. Surprisingly, the lower quantile dependence stays approximately constant indicating that the dependence of the (left) tail risk for the data at hand has not increased after the estimated breakpoint while overall the diversification benefits have decreased.

5. Conclusion

We propose new fluctuation tests for detecting structural breaks in factor copula models and analyse the behaviour under the null hypothesis of no change. Due to the discontinuity of the SMM objective function this requires additional effort to derive a functional limit theorem for the model parameters. The presence of nuisance parameters in the asymptotic distribution of the two proposed test statistics requires a bootstrap approximation for parts of the asymptotic distribution. The proposed tests show good size and power properties in finite samples. An empirical application to a set of 32 stock returns indicates the presence of a breakpoint early in 2008, before the Lehman Brothers bankruptcy. Dependence has increased after this break providing evidence of a diversification breakdown and contagion among different stock.

In future research, our work could be extended in several interesting directions. First, one could derive a monitoring procedure for detecting parameter changes in an online-setup. Second, it would be interesting to explore the usefulness of our test in risk management applications like the forecast of value at risk (VaR) and expected shortfall (ES). Finally, it would be worthwhile, but also mathematically demanding to derive appropriate tests in the case of time-varying marginal distributions.

Table 7
Implied rank correlations.

	Pre-break				Post-break			
	Finance	Energy	Telecom	Consumer	Finance	Energy	Telecom	Consumer
Break all factors loadings								
Finance	0.79	0.30	0.38	0.30	0.74	0.37	0.37	0.40
Energy		0.52	0.34	0.27		0.70	0.36	0.38
Telecom			0.63	0.34			0.74	0.39
Consumer				0.59				0.55
Break market factor loadings								
Finance	0.73	0.36	0.44	0.25	0.71	0.47	0.44	0.27
Energy		0.44	0.36	0.21		0.67	0.54	0.32
Telecom			0.61	0.26			0.66	0.30
Consumer				0.75			0.76	
Break financial sector loadings								
Finance	0.82	0.32	0.35	0.20	0.74	0.41	0.45	0.27
Energy		0.53	0.45	0.26		0.53	0.45	0.26
Telecom			0.64	0.29			0.65	0.29
Consumer				0.76			0.76	
Break energy sector loadings								
Finance	0.75	0.36	0.48	0.28	0.75	0.43	0.48	0.28
Energy		0.49	0.37	0.22		0.77	0.44	0.26
Telecom			0.64	0.29			0.65	0.29
Consumer				0.76			0.76	
Break telecommunication sector loadings								
Finance	0.75	0.44	0.38	0.28	0.75	0.44	0.52	0.28
Energy		0.53	0.35	0.26		0.53	0.48	0.26
Telecom			0.75	0.23			0.72	0.31
Consumer				0.76				0.76
Break consumption sector loadings								
Finance	0.75	0.44	0.48	0.23	0.75	0.44	0.48	0.35
Energy		0.53	0.45	0.22		0.53	0.45	0.32
Telecom			0.64	0.24			0.65	0.36
Consumer				0.69				0.68

Note: Table 7 shows the model implied rank correlations before and after the estimated breakpoint corresponding to the subsets of factor loading allowed to break in Table 6 and using the corresponding break date and parameter estimates. The entries on the main diagonal are implied rank correlations between assets within the respective sector, the off-diagonal elements are the implied rank correlations between the sectors.

Appendix. Additional results and proofs

Theorem 1 is proved in different steps. First, we provide a consistency result in Lemma 2. Then, Theorem 4, which is based on Theorem 3, yields a general convergence result for SMM estimators. Lemma 6, which is based on Lemma 5 provides stochastic equicontinuity for the objective function in a general SMM setting. Finally, Lemma 7 yields distribution results for the empirical moments in our specific problem. All these results are then used for proving Theorem 1. Note, that we use the abbreviation *c.s.* to indicate the usage of the Cauchy–Schwarz inequality.

Lemma 2. *If* $\hat{\theta}_{T,S} \xrightarrow{a.s.} \theta_0, \qquad T,S \to \infty$, *then*

$$\sup_{s\in[\varepsilon,1]} \|\hat{\theta}_{sT,S} - \theta_0\| \xrightarrow{p} 0, \qquad \forall \varepsilon > 0, \qquad T,S \to \infty.$$

Proof. Let $\delta > 0$, $\hat{\theta}_{T,S} \xrightarrow{a.s.} \theta_0$ and choose any $\varepsilon > 0 \Rightarrow \forall \gamma > 0$ there exists $T_0^*, S_0^* \in \mathbb{N}_+$, such that for all $T \geq T_0^*, S \geq S_0^*, \qquad \|\hat{\theta}_{T,S} - \theta_0\| < \gamma \Rightarrow$ there exists $T_0, S_0 \in \mathbb{N}_+$ such that for all $T \geq T_0, S \geq S_0, \qquad \|\hat{\theta}_{T,S} - \theta_0\| < \delta$

Choose T, S with $\varepsilon T \geq T_0 \Leftrightarrow T \geq \frac{T_0}{\varepsilon}, S \geq S_0, \quad \forall \varepsilon > 0$ (in all cases $T \geq T_0$)

$\Rightarrow \forall s \in [\varepsilon, 1] : \|\hat{\theta}_{sT,S} - \theta_0\| < \delta$, for all $T \geq \frac{T_0}{\varepsilon}, S \geq S_0, \quad \forall \varepsilon > 0$

$\Rightarrow \sup_{s\in[\varepsilon,1]} \|\hat{\theta}_{sT,S} - \theta_0\| < \delta$, for all $T \geq \frac{T_0}{\varepsilon}, S \geq S_0, \quad \forall \varepsilon > 0$

$\Rightarrow \sup_{s\in[\varepsilon,1]} \|\hat{\theta}_{sT,S} - \theta_0\| \xrightarrow{p} 0, \qquad \forall \varepsilon > 0, \qquad T,S \to \infty. \quad \square$

Theorem 3. *Under the null hypothesis* $H_0 : \theta_1 = \theta_2 = \cdots = \theta_T$, *suppose that* $\forall s \in [\varepsilon, 1], \varepsilon > 0 \quad Q_{sT,S}(\theta_{sT,S}) \geq \sup_{\theta\in\Theta} Q_{sT,S}(\theta) - o_p^*((s^2T)^{-1}), \ \sup_{s\in[\varepsilon,1)} \|\hat{\theta}_{sT,S} - \theta_0\| \xrightarrow{p} 0, T,S \to \infty$ *and:*

Table 8
Average dependence measures.

	Full sample	Pre-break	Post-break
ρ_1	0.46	0.45	0.48
ρ_2	0.39	0.36	0.43
ρ_3	0.45	0.43	0.48
ρ_4	0.39	0.37	0.43
$\lambda^1_{0.05}$	0.31	0.29	0.27
$\lambda^2_{0.05}$	0.27	0.21	0.25
$\lambda^3_{0.05}$	0.28	0.27	0.25
$\lambda^4_{0.05}$	0.25	0.24	0.21
$\lambda^1_{0.1}$	0.40	0.37	0.36
$\lambda^2_{0.1}$	0.33	0.29	0.34
$\lambda^3_{0.1}$	0.38	0.35	0.34
$\lambda^4_{0.1}$	0.35	0.30	0.31
$\lambda^1_{0.9}$	0.34	0.30	0.39
$\lambda^2_{0.9}$	0.28	0.24	0.35
$\lambda^3_{0.9}$	0.31	0.28	0.36
$\lambda^4_{0.9}$	0.28	0.25	0.34
$\lambda^1_{0.95}$	0.27	0.19	0.33
$\lambda^2_{0.95}$	0.22	0.14	0.27
$\lambda^3_{0.95}$	0.23	0.15	0.29
$\lambda^4_{0.95}$	0.22	0.13	0.26

Note: Table 8 contains the (average) empirical moments used for the model estimator for the full sample and the subsamples implied by a structural break on Jan. 9, 2008 that was detected by the moment based structural break test. ρ_i denotes rank correlations, whereas λ^i_q is the quantile q dependence measure for the sectors $i = 1, \ldots, 4$, i.e. finance, energy, telecom and media, consumer retail.

(i) $Q_0(\theta)$ *is maximized on* $\theta_0(= \theta_1 = \cdots = \theta_T)$
(ii) $\theta_0(= \theta_1 = \cdots = \theta_T)$ *are interior points of* Θ
(iii) $Q_0(\theta)$ *is twice differentiable at* θ_0 *with nonsingular second derivative* $H = \nabla_{\theta\theta} Q_0(\theta_0)$
(iv) $s\sqrt{T}\hat{D}_{sT}(\theta_0) \xrightarrow{d} A(s)$
(v) $\forall\delta \to 0 \quad \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \left| \frac{\hat{R}_{sT}(\theta)}{1+s\sqrt{T}\|\theta-\theta_0\|} \right| \xrightarrow{p} 0$ *with* $\hat{R}_{sT} = \frac{s\sqrt{T}[Q_{sT,S}(\theta)-Q_{sT,S}(\theta_0)-\hat{D}_{sT}(\theta-\theta_0)-(Q_0(\theta)-Q_0(\theta_0))]}{\|\theta-\theta_0\|}$

$$\Rightarrow s\sqrt{T}(\theta_{sT,S} - \theta_0) \xrightarrow{d} A^*(s) \qquad \forall s \in [\varepsilon, 1], \varepsilon > 0 \quad \text{and} \quad A^*(s) = H^{-1}A(s),$$

where $A(s)$ *is a continuous Gaussian process.*

Proof. For simplification set $Q := Q_0$ and $\hat{Q} := Q_{sT,S}$. We first show that $s\sqrt{T}\|\theta_{sT,S} - \theta_0\| = O_p(1)$. With a Taylor-expansion of $Q(\theta)$ around θ_0 and knowing $\nabla_\theta Q(\theta_0) = 0$, due to condition (i), we receive $Q(\theta) = Q(\theta_0) + \frac{1}{2}(\theta-\theta_0)'H(\theta-\theta_0) + o(\|\theta-\theta_0\|^3)$. We also know from condition (i) and (iii), that $\exists C > 0 : (\theta-\theta_0)'H(\theta-\theta_0) + o(\|\theta-\theta_0\|^3) \le -C\|\theta-\theta_0\|^2 \Rightarrow Q(\theta_{sT,S}) \le Q(\theta_0) - C\|\theta_{sT,S} - \theta_0\|^2$ and we obtain

$$\begin{aligned}
0 &\le \hat{Q}(\theta_{sT,S}) - \hat{Q}(\theta_0) + o_p^*((s^2T)^{-1}) \\
&= Q(\theta_{sT,S}) - Q(\theta_0) + \hat{D}'_{sT}(\theta_{sT,S} - \theta_0) + \frac{1}{s\sqrt{T}}\|\theta_{sT,S} - \theta_0\|\hat{R}_{sT}(\theta_{sT,S}) + o_p^*((s^2T)^{-1}) \\
&\overset{c.s.}{\le} -C\|\theta_{sT,S} - \theta_0\|^2 + \|\hat{D}'_{sT}\|\|\theta_{sT,S} - \theta_0\| \\
&+ \|\theta_{sT,S} - \theta_0\|(1 + s\sqrt{T}\|\theta_{sT,S} - \theta_0\|)o_p(s^{-1}T^{-\frac{1}{2}}) + o_p^*((s^2T)^{-1}) \\
&= -(C + o_p(1))\|\theta_{sT,S} - \theta_0\|^2 + \|\theta_{sT,S} - \theta_0\|(\|\hat{D}'_{sT}\| + o_p(s^{-1}T^{-\frac{1}{2}})) + o_p^*((s^2T)^{-1}) \\
&\le -(C + o_p(1))\|\theta_{sT,S} - \theta_0\|^2 + \|\theta_{sT,S} - \theta_0\|O_p(s^{-1}T^{-\frac{1}{2}}) + o_p^*((s^2T)^{-1}) \\
&\Rightarrow \|\theta_{sT,S} - \theta_0\|^2 \le \|\theta_{sT,S} - \theta_0\|O_p(s^{-1}T^{-\frac{1}{2}}) + o_p^*((s^2T)^{-1}), \quad \forall s \in [\varepsilon, 1].
\end{aligned} \tag{1}$$

Consider

$$\begin{aligned}
\left(\|\theta_{sT,S} - \theta_0\| + O_p(s^{-1}T^{-\frac{1}{2}})\right)^2 &= \|\theta_{sT,S} - \theta_0\|^2 + \|\theta_{sT,S} - \theta_0\|O_p(s^{-1}T^{-\frac{1}{2}}) + O_p(s^{-2}T^{-1}) \\
&\overset{(1)}{\le} \|\theta_{sT,S} - \theta_0\|O_p(s^{-1}T^{-\frac{1}{2}}) + o_p^*((s^2T)^{-1}) + O_p(s^{-2}T^{-1})
\end{aligned}$$

$$\leq O_p(s^{-2}T^{-1})$$

$$\Rightarrow \left| \|\theta_{sT,S} - \theta_0\| + O_p(s^{-1}T^{-\frac{1}{2}}) \right| \leq O_p(s^{-1}T^{-\frac{1}{2}}), \quad \forall s \in [\varepsilon, 1] \tag{2}$$

and we get

$$\begin{aligned}
\|\theta_{sT,S} - \theta_0\| &= |\|\theta_{sT,S} - \theta_0\| + O_p(s^{-1}T^{-\frac{1}{2}}) - O_p(s^{-1}T^{-\frac{1}{2}})| \\
&\overset{c.s.}{\leq} |\|\theta_{sT,S} - \theta_0\| + O_p(s^{-1}T^{-\frac{1}{2}})| + |-O_p(s^{-1}T^{-\frac{1}{2}})| \\
&\overset{(2)}{\leq} O_p(s^{-1}T^{-\frac{1}{2}})
\end{aligned}$$

$$\Rightarrow s\sqrt{T}\|\theta_{sT,S} - \theta_0\| = O_p(1), \quad \forall s \in [\varepsilon, 1]. \tag{3}$$

Note that for the counter of the remainder Term $\hat{R}_{sT}$, without the factor $s\sqrt{T}$, we get with condition (v) the scale

$$\begin{aligned}
&o_p(1)(1 + s\sqrt{T}\|\theta_{sT,S} - \theta_0\|)\|\theta_{sT,S} - \theta_0\| \frac{1}{s\sqrt{T}} \\
=&o_p\left(\frac{\|\theta_{sT,S} - \theta_0\|}{s\sqrt{T}} + \|\theta_{sT,S} - \theta_0\|^2\right) \\
\overset{(3)}{=}&o_p\left(O_p(s^2T)^{-1} + O_p((s^2T)^{-1})\right) \\
=&o_p((s^2T)^{-1}).
\end{aligned} \tag{4}$$

Now we can show the asymptotic behaviour of $s\sqrt{T}(\hat{\theta}_{sT,S} - \theta_0)$. First let

$$\tilde{\theta}_{sT,S} = \theta_0 - H^{-1}\hat{D}_{sT} \Rightarrow \hat{D}_{sT} = -H(\tilde{\theta}_{sT,S} - \theta_0) \tag{5}$$

be the maximum of the approximation

$$\begin{aligned}
\hat{Q}(\theta) \approx&\hat{Q}(\theta_0) + \hat{D}'_{sT}(\theta - \theta_0) + Q(\theta) - Q(\theta_0) \\
\approx&\hat{Q}(\theta_0) + \hat{D}'_{sT}(\theta - \theta_0)' + \frac{1}{2}(\theta - \theta_0)H(\theta - \theta_0)
\end{aligned} \tag{6}$$

and by construction $s\sqrt{T}$-consistent.

From the previous result (4), we know the convergence ordering of the remainder term of the approximation in (6). So we receive

$$\begin{aligned}
2[\hat{Q}(\theta_{sT,S}) - \hat{Q}(\theta_0)] &= 2\hat{D}'_{sT}(\theta_{sT,S} - \theta_0) + (\theta_{sT,S} - \theta_0)'H(\theta_{sT,S} - \theta_0) + o_p((s^2T)^{-1}) \\
&\overset{(5)}{=} (\theta_{sT,S} - \theta_0)'H(\theta_{sT,S} - \theta_0) - 2(\tilde{\theta}_{sT,S} - \theta_0)'H(\theta_{sT,S} - \theta_0) + o_p((s^2T)^{-1})
\end{aligned}$$

and analogously for $\tilde{\theta}_{sT,S}$

$$\begin{aligned}
2[\hat{Q}(\tilde{\theta}_{sT,S}) - \hat{Q}(\theta_0)] &= 2\hat{D}'_{sT}(\tilde{\theta}_{sT,S} - \theta_0) + (\tilde{\theta}_{sT,S} - \theta_0)'H(\tilde{\theta}_{sT,S} - \theta_0) + o_p((s^2T)^{-1}) \\
&\overset{(5)}{=} -(\tilde{\theta}_{sT,S} - \theta_0)'H(\tilde{\theta}_{sT,S} - \theta_0) + o_p((s^2T)^{-1}).
\end{aligned}$$

Because $\theta_{sT,S}, \tilde{\theta}_{sT,S} \in \Theta$, the convergence ordering of the remainder terms is known and $H = H(\theta_0)$ is negatively definite and nonsingular

$$\begin{aligned}
\Rightarrow o_p((s^2T)^{-1}) \leq&2[\hat{Q}(\theta_{sT,S}) - \hat{Q}(\theta_0)] - 2[\hat{Q}(\tilde{\theta}_{sT,S}) - \hat{Q}(\theta_0)] \\
=&(\theta_{sT,S} - \theta_0)'H(\theta_{sT,S} - \theta_0) - 2(\tilde{\theta}_{sT,S} - \theta_0)'H(\theta_{sT,S} - \theta_0) - (\tilde{\theta}_{sT,S} - \theta_0)'H(\tilde{\theta}_{sT,S} - \theta_0) \\
=&(\theta_{sT,S} - \tilde{\theta}_{sT,S})'H(\theta_{sT,S} - \tilde{\theta}_{sT,S}) \leq -C\|\theta_{sT,S} - \tilde{\theta}_{sT,S}\|^2 \\
\Rightarrow&s\sqrt{T}\|\theta_{sT,S} - \tilde{\theta}_{sT,S}\| = o_p(1).
\end{aligned} \tag{7}$$

So we have $\forall s \in [\varepsilon, 1]$, $\varepsilon > 0$

$$\begin{aligned}
&\|s\sqrt{T}(\theta_{sT,S} - \theta_0) - (-s\sqrt{T}H^{-1}\hat{D}_{sT})\| \\
\overset{(5)}{=}&\|s\sqrt{T}(\theta_{sT,S} - \theta_0) - s\sqrt{T}(\tilde{\theta}_{sT,S} - \theta_0)\| \\
=&\|s\sqrt{T}(\theta_{sT,S} - \tilde{\theta}_{sT,S})\|
\end{aligned}$$

$$=s\sqrt{T}\|(\theta_{sT,S}-\tilde{\theta}_{sT,S})\| \overset{(7)}{=} o_p(1)$$

$$\Rightarrow s\sqrt{T}(\theta_{sT,S}-\theta_0) \xrightarrow{p} -H^{-1}s\sqrt{T}\hat{D}_{sT} \xrightarrow[iv]{d} -H^{-1}A(s) = A^*(s). \quad \square$$

Theorem 4. *Under the null hypothesis* $H_0: \theta_1 = \theta_2 = \cdots = \theta_T$*, suppose that* $\forall s \in [\varepsilon, 1], \varepsilon > 0:\quad g_{sT,S}(\theta_{sT,S})'\hat{W}_{sT}g_{sT,S}(\theta_{sT,S}) \leq \inf_{\theta\in\Theta} g_{sT,S}(\theta)'\hat{W}_{sT}g_{sT,S}(\theta) + o_p^*((s^2T)^{-1})$,

$\sup_{s\in[\varepsilon,1]} \|\hat{\theta}_{sT,S}-\theta_0\| \xrightarrow{p} 0,\ \sup_{s\in[\varepsilon,1]} \|\hat{W}_{sT}-W\| \xrightarrow{p} 0, \quad T, S \to \infty$ *and:*

(i) *There is a* $\theta_0(=\theta_1=\cdots=\theta_T)$ *such that* $g_0(\theta_0)=0$
(ii) $\theta_0(=\theta_1=\cdots=\theta_T)$ *are interior points of* Θ
(iii) $g_0(\theta)$ *is differentiable at* θ_0 *with derivative* G *such that* $G'WG$ *is nonsingular*
(iv) $s\sqrt{T}g_{sT,S}(\theta_0) \xrightarrow{d} A(s)$
(v) $\forall\delta\to 0 \quad \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} s\sqrt{T}\frac{\|g_{sT,S}(\theta)-g_{sT,S}(\theta_0)-g_0(\theta)\|}{1+s\sqrt{T}\|\theta-\theta_0\|} \xrightarrow{p} 0$

$$\Rightarrow s\sqrt{T}(\theta_{sT,S}-\theta_0) \xrightarrow{d} A^*(s) \qquad \forall s\in[\varepsilon,1], \varepsilon>0$$

and $A^*(s) = (G'WG)^{-1}G'WA(s)$,
where $A(s)$ *is a continuous Gaussian process.*

Proof. The Theorem follows by verifying the conditions of Theorem 3. Set $\hat{Q}(\theta) := Q_{sT}(\theta) := -\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta) + \hat{\Delta}_{sT}(\theta)$ with $\hat{g}(\theta) := g_{sT,S}(\theta)$ and $Q(\theta) := Q_0(\theta) := -\frac{1}{2}g(\theta)'Wg(\theta)$ with $g(\theta) := g_0(\theta)$. With a Taylor-expansion of $g(\theta)$ around θ_0

$$g(\theta) = g(\theta_0) + G(\theta-\theta_0) + o(\|\theta-\theta_0\|^2) = G(\theta-\theta_0) + o(\|\theta-\theta_0\|^2), \tag{8}$$

we obtain

$$Q(\theta) = g(\theta)'Wg(\theta) \overset{(8)}{=} [G(\theta-\theta_0)+o(\|\theta-\theta_0\|^2)]'W[G(\theta-\theta_0)+o(\|\theta-\theta_0\|^2)]$$

and comparing this with a Taylor-expansion of $Q(\theta)$ around θ_0

$$Q(\theta) = Q(\theta_0) + \frac{1}{2}(\theta-\theta_0)'H(\theta-\theta_0) + o(\|\theta-\theta_0\|^3),$$

noting that $Q(\theta)$ is maximized at θ_0, it follows $H(\theta_0) = -G'WG$, where H is a nonsingular negative definite matrix. Because H is by construction a nonsingular negative definite matrix, $\exists$ neighbourhood of θ_0, where $Q(\theta)$ has a unique maximum at θ_0 with $Q(\theta_0) = 0$. $\Rightarrow$ Conditions (i), (ii) and (iii) of Theorem 3 are satisfied. By choosing $\hat{D}_{sT} = -G'\hat{W}_{sT}g_{sT,S}(\theta_0)$ it follows, $\forall s\in[\varepsilon,1]$,

$$s\sqrt{T}\hat{D}_{sT} = -s\sqrt{T}G'\hat{W}_{sT}g_{sT,S}(\theta_0) \xrightarrow[(iv)]{d} -G'WA(s),$$

thus condition (iv) of Theorem 3 is fulfilled. Now we define

$$\hat{\varepsilon}(\theta) := \frac{\hat{g}(\theta)-\hat{g}(\theta_0)-g(\theta)}{1+s\sqrt{T}\|\theta-\theta_0\|} \Leftrightarrow \hat{g}(\theta) = [1+s\sqrt{T}\|\theta-\theta_0\|]\hat{\varepsilon}(\theta) + \hat{g}(\theta_0) + g(\theta) \tag{9}$$

and we get

$$\begin{aligned}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta) \overset{(9)}{=} & [1+2s\sqrt{T}\|\theta-\theta_0\| + s^2T\|\theta-\theta_0\|^2]\hat{\varepsilon}(\theta)'\hat{W}_{sT}\hat{\varepsilon}(\theta) \\ & + g(\theta)'\hat{W}_{sT}g(\theta) + \hat{g}(\theta_0)'\hat{W}_{sT}\hat{g}(\theta_0) + 2g(\theta)'\hat{W}_{sT}\hat{g}(\theta_0) \\ & + 2[g(\theta)+\hat{g}(\theta_0)]'\hat{W}_{sT}\hat{\varepsilon}(\theta)[1+s\sqrt{T}\|\theta-\theta_0\|]\end{aligned} \tag{10}$$

Next we define the remainder term of $\hat{Q}(\theta)$

$$\hat{Q}(\theta) = -\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta) + \hat{\Delta}_{sT}(\theta) = -\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta) + \frac{1}{2}\hat{\varepsilon}(\theta)'\hat{W}_{sT}\hat{\varepsilon}(\theta) + \hat{g}(\theta_0)'\hat{W}_{sT}\hat{\varepsilon}(\theta).$$

The remainder term is just chosen in this way, that $\hat{Q}(\theta)$ is consistent with $-\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta)$, which is shown in the next window and that we get the right convergence ordering, when checking condition (v) of Theorem 3. First notice that by condition (v) $\forall\delta>0\ \sup_{s\in[\varepsilon,1]}\sup_{\|\theta-\theta_0\|<\delta}\|\hat{\varepsilon}(\theta)\|=o_p(s^{-1}T^{-\frac{1}{2}})$, furthermore

$$\sup_{s\in[\varepsilon,1]}\sup_{\|\theta-\theta_0\|<\delta}\|\hat{g}(\theta_0)\|=o_p(s^{-1}T^{-\frac{1}{2}}),\quad \sup_{s\in[\varepsilon,1]}\sup_{\|\theta-\theta_0\|<\delta}\|\hat{W}_{sT}\|=O_p(1)\quad\text{and}\quad \frac{\|g(\theta)-g(\theta_0)\|}{\|\theta-\theta_0\|}=O_p(1). \tag{11}$$

$$\begin{aligned}
&\Rightarrow\forall\delta>0\quad \sup_{s\in[\varepsilon,1]}\sup_{\|\theta-\theta_0\|<\delta}\left|\hat{Q}(\theta)-(-\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta))\right|\\
&=\sup_{s\in[\varepsilon,1]}\sup_{\|\theta-\theta_0\|<\delta}\left|\frac{1}{2}\hat{\varepsilon}(\theta)'\hat{W}_{sT}\hat{\varepsilon}(\theta)+\hat{g}(\theta_0)'\hat{W}_{sT}\hat{\varepsilon}(\theta)\right|\\
&\overset{c.s.}{\le}\sup_{s\in[\varepsilon,1]}\sup_{\|\theta-\theta_0\|<\delta}\frac{1}{2}\|\hat{\varepsilon}(\theta)\|\|\hat{W}_{sT}\|\|\hat{\varepsilon}(\theta)\|+\|\hat{g}(\theta_0)\|\|\hat{W}_{sT}\|\|\hat{\varepsilon}(\theta)\|\\
&\overset{(11)}{=}O_p(1)(o_p(s^{-2}T^{-1})+o_p(s^{-2}T^{-1}))=o_p(s^{-2}T^{-1}).
\end{aligned} \tag{12}$$

With the consistency of $\hat{Q}(\theta)$ we can show the initial condition of Theorem 3

$$\begin{aligned}
&\forall s\in[\varepsilon,1],\varepsilon>0\quad \hat{g}(\theta_{sT,S})'\hat{W}_{sT}\hat{g}(\theta_{sT,S})\le\inf_{\theta\in\Theta}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta)+o_p^*((s^2T)^{-1})\\
&\Leftrightarrow\forall s\in[\varepsilon,1],\varepsilon>0\quad -\frac{1}{2}\hat{g}(\theta_{sT,S})'\hat{W}_{sT}\hat{g}(\theta_{sT,S})\ge-\inf_{\theta\in\Theta}\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta)-o_p^*((s^2T)^{-1})\\
&\Leftrightarrow\forall s\in[\varepsilon,1],\varepsilon>0\quad -\frac{1}{2}\hat{g}(\theta_{sT,S})'\hat{W}_{sT}\hat{g}(\theta_{sT,S})\ge-\left(-\inf_{\theta\in\Theta}-\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta)\right)-o_p^*((s^2T)^{-1})\\
&\overset{(12)}{\Leftrightarrow}\forall s\in[\varepsilon,1],\varepsilon>0\quad \hat{Q}(\theta_{sT,S})\ge\sup_{\theta\in\Theta}\hat{Q}(\theta)-o_p^*((s^2T)^{-1}).
\end{aligned}$$

Finally we have to check condition (v) of Theorem 3, for that we calculate

$$\begin{aligned}
&\left|\frac{\hat{R}_{sT}(\theta)}{1+s\sqrt{T}\|\theta-\theta_0\|}\right|\\
&=s\sqrt{T}\left|\frac{\hat{Q}(\theta)-\hat{Q}(\theta_0)-\hat{D}_{sT}(\theta-\theta_0)-(Q(\theta)-Q(\theta_0))}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)}\right|\\
&=s\sqrt{T}\left|\frac{-\frac{1}{2}\hat{g}(\theta)'\hat{W}_{sT}\hat{g}(\theta)+\frac{1}{2}\hat{\varepsilon}(\theta)'\hat{W}_{sT}\hat{\varepsilon}(\theta)+\hat{g}(\theta_0)'\hat{W}_{sT}\hat{\varepsilon}(\theta)+\frac{1}{2}\hat{g}(\theta_0)'\hat{W}_{sT}\hat{g}(\theta_0)}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)}\right.\\
&\quad\left.+\frac{-\hat{D}_{sT}(\theta-\theta_0)-(Q(\theta)-Q(\theta_0))}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)}\right|\qquad(\hat{\varepsilon}(\theta_0)=0),
\end{aligned}$$

inserting (10) and $Q(\theta)=-\frac{1}{2}g(\theta)'Wg(\theta)$, sorting, triangle inequality, choosing $\hat{D}_{sT}=-G'\hat{W}_{sT}\hat{g}(\theta_0)$ and size up the resulting terms, leads to

$$\begin{aligned}
&\le\frac{s\sqrt{T}[2s\sqrt{T}\|\theta-\theta_0\|+s^2T\|\theta-\theta_0\|^2]\left|\hat{\varepsilon}(\theta)'\hat{W}_{sT}\hat{\varepsilon}(\theta)\right|}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)}\qquad(=:r_1(\theta))\\
&\quad+\frac{s\sqrt{T}\left|(-g(\theta)+G(\theta-\theta_0))'\hat{W}_{sT}\hat{g}(\theta_0)\right|}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)}\qquad(=:r_2(\theta))\\
&\quad+\frac{s^2T\left|(g(\theta)+\hat{g}(\theta_0))'\hat{W}_{sT}\hat{\varepsilon}(\theta)\right|}{1+s\sqrt{T}\|\theta-\theta_0\|}\qquad(=:r_3(\theta))\\
&\quad+\frac{s\sqrt{T}\left|g(\theta)'\hat{W}_{sT}\varepsilon(\hat{\theta})\right|}{\|\theta-\theta_0\|}\qquad(=:r_4(\theta))\\
&\quad+\frac{s\sqrt{T}\left|g(\theta)'[W-\hat{W}_{sT}]g(\theta)\right|}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)}.\qquad(=:r_5(\theta))
\end{aligned}$$

Now we have

$$\forall\delta \to 0 \quad \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \left| \frac{\hat{R}_{sT}(\theta)}{1+s\sqrt{T}\|\theta-\theta_0\|} \right| \leq \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \sum_{i=1}^{5} r_i(\theta) = o_p(1)$$

and we just have to check the convergence of the $r_i(\theta)$ terms for $i \in \{1, 2, 3, 4, 5\}$. For r_1, we have

$$\begin{aligned}
\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} r_1(\theta) &= \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s\sqrt{T}[2s\sqrt{T}\|\theta-\theta_0\| + s^2T\|\theta-\theta_0\|^2] \left|\hat{\varepsilon}(\theta)'\hat{W}_{sT}\hat{\varepsilon}(\theta)\right|}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)} \\
&\overset{c.s.}{\leq} \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s\sqrt{T}(s\sqrt{T}\|\theta-\theta_0\|(2+s\sqrt{T}\|\theta-\theta_0\|))\|\hat{\varepsilon}(\theta)\|^2\|\hat{W}_{sT}\|}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)} \\
&\leq \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} cs^2T\|\hat{\varepsilon}(\theta)\|^2\|\hat{W}_{sT}\| \qquad \text{(for a constant } c > 1) \\
&\overset{(11)}{=} o_p(1).
\end{aligned}$$

For r_2, we obtain

$$\begin{aligned}
\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} r_2(\theta) &= \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s\sqrt{T}\left|(-g(\theta)+G(\theta-\theta_0))'\hat{W}_{sT}\hat{g}(\theta_0)\right|}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)} \\
&\overset{c.s.}{\leq} \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s\sqrt{T}o(\|\theta-\theta_0\|^2)\|\hat{W}_{sT}\|\|\hat{g}(\theta_0)\|}{\|\theta-\theta_0\|} \\
&\leq \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} s\sqrt{T}o(\|\theta-\theta_0\|)\|\hat{W}_{sT}\|\|\hat{g}(\theta_0)\| \\
&\overset{(11)}{=} o_p(1).
\end{aligned}$$

Considering r_3 yields

$$\begin{aligned}
\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} r_3(\theta) &= \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s^2T\left|(g(\theta)+\hat{g}(\theta_0))'\hat{W}_{sT}\hat{\varepsilon}(\theta)\right|}{1+s\sqrt{T}\|\theta-\theta_0\|} \\
&\overset{c.s.}{\leq} \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \left(s^2T\|\hat{g}(\theta_0)\| + sT^{\frac{1}{2}}\frac{\|g(\theta)\|}{\|\theta-\theta_0\|}\right)\|\hat{W}_{sT}\|\|\hat{\varepsilon}(\theta)\| \\
&\leq \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \left(s^2To_p(s^{-1}T^{-\frac{1}{2}}) + sT^{\frac{1}{2}}O_p(1)\right)\|\hat{W}_{sT}\|\|\hat{\varepsilon}(\theta)\| \\
&\leq \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} sT^{\frac{1}{2}}O_p(1)\|\hat{W}_{sT}\|\|\hat{\varepsilon}(\theta)\| \\
&\overset{(11)}{=} o_p(1).
\end{aligned}$$

For r_4, it holds

$$\begin{aligned}
\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} r_4(\theta) &= \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s\sqrt{T}\left|g(\theta)'\hat{W}_{sT}\varepsilon(\hat{\theta})\right|}{\|\theta-\theta_0\|} \\
&\overset{c.s.}{\leq} s\sqrt{T}O_p(1)\|\hat{W}_{sT}\|\|\hat{\varepsilon}(\theta)\| \\
&\overset{(11)}{=} o_p(1).
\end{aligned}$$

Finally, for r_5,

$$\begin{aligned}
\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} r_5(\theta) &= \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s\sqrt{T}\left|g(\theta)'[W-\hat{W}_{sT}]g(\theta)\right|}{\|\theta-\theta_0\|(1+s\sqrt{T}\|\theta-\theta_0\|)} \\
&\overset{c.s.}{\leq} \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \frac{s\sqrt{T}\|g(\theta)\|^2\|W-\hat{W}_{sT}\|}{s\sqrt{T}\|\theta-\theta_0\|^2}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \left(\frac{\|g(\theta)\|}{\|\theta-\theta_0\|}\right)^2 o_p(1)\\
&= o_p(1). \quad \square
\end{aligned}
$$

Lemma 5. *Under Assumption 1, 2, 3(ii) and 3(iii)*

(i) *$g_{sT,S}(\theta)$ is stochastically Lipschitz-continuous $\forall s \in [\varepsilon, 1], \varepsilon > 0$, i.e.,*

$$\exists B = O_p(1) \text{ such that } \forall \theta_1, \theta_2 \in \Theta : \qquad \|g_{sT,S}(\theta_1) - g_{sT,S}(\theta_2)\| \le B\|\theta_1 - \theta_2\|$$

(ii) *$\exists \delta > 0$ such that*

$$\lim \sup_{T,S\to\infty} E\left(B^{2+\delta}\right) < \infty.$$

Proof. Without loss of generality suppose $g_{sT,S}(\theta)$ is a one-dimensional function, otherwise show the Lipschitz-continuity for every entry of the vector $g_{sT,S}(\theta)$. (i) We know

$$\tilde{m}_S(\theta) = m_0(\theta) + o_p(1), \tag{13}$$

and from Assumption 3(iii), $m_0(\theta)$ is Lipschitz-continuous, due to combination of Lipschitz-continuous bivariate copulas $C_{ij}(\theta)$,

$$\exists K : |m_0(\theta_1) - m_0(\theta_2)| \le K\|\theta_1 - \theta_2\|. \tag{14}$$

Now consider

$$
\begin{aligned}
|g_{sT,S}(\theta_1) - g_{sT,S}(\theta_2)| &= |\hat{m}_{sT} - \tilde{m}_S(\theta_1) - \hat{m}_{sT} + \tilde{m}_S(\theta_2)|\\
&= |\tilde{m}_S(\theta_2) - \tilde{m}_S(\theta_1)| = |\tilde{m}_S(\theta_1) - \tilde{m}_S(\theta_2)|\\
&\underset{c.s.}{\overset{(13)}{\le}} |m_0(\theta_1) - m_0(\theta_2)| + |o_p(1)|\\
&\overset{(14)}{\le} K\|\theta_1 - \theta_2\| + |o_p(1)|\\
&= \left(K + \frac{|o_p(1)|}{\|\theta_1 - \theta_2\|}\right)\|\theta_1 - \theta_2\|\\
&=: B\|\theta_1 - \theta_2\|.
\end{aligned}
$$

(ii) For some $\delta > 0$

$$\Rightarrow \lim \sup_{T,S\to\infty} E\left(B^{2+\delta}\right) = \lim \sup_{T,S\to\infty} E\left(\left(K + \frac{|o_p(1)|}{\|\theta_1 - \theta_2\|}\right)^{2+\delta}\right) < \infty. \quad \square$$

Lemma 6. *Under Assumption 1, 2, 3(ii) and 3(iii), for $\frac{S}{T} \to \infty$ or $\frac{S}{T} \to k \in (0, \infty)$,*

$$v_{sT,S}(\theta) = \sqrt{sT}[g_{sT,S}(\theta) - g_0(\theta)] \quad \text{is stochastically equicontinuous} \quad \forall s \in [\varepsilon, 1], \varepsilon > 0$$

Proof. By Lemma 5(i) $\{g_{sT,S}(\theta) : \theta \in \Theta\}$ is Lipschitz-continuous $\forall s \in [\varepsilon, 1], \varepsilon > 0$ and so a Type II class of functions in Andrews (1994). By Theorem 2 of Andrews $\{g_{sT,S}(\theta) : \theta \in \Theta\}$ satisfies Pollard's entropy condition with envelope

$$max\{1, \sup_{\theta\in\Theta}\|g_{sT,S}(\theta)\|, \quad B\}, \qquad \forall s \in [\varepsilon, 1], \varepsilon > 0.$$

$\Rightarrow$ Assumption A of Andrews (1994) is satisfied.
Furthermore $g_{sT,S}(\theta)$ is bounded and by Lemma 5(ii) it holds

$$\lim \sup_{T,S\to\infty} E\left(B^{2+\delta}\right) < \infty.$$

$\Rightarrow$ Assumption B of Andrews (1994) is satisfied. Then with Theorem 1 of Andrews (1994) and noting, that Assumption C is fulfilled by construction

$$v_{sT,S}(\theta) = \sqrt{sT}[g_{sT,S}(\theta) - g_0(\theta)] \quad \text{is stochastically equicontinuous} \quad \forall s \in [\varepsilon, 1], \varepsilon > 0. \quad \square$$

Lemma 7. *We consider the dependence measures Spearman's rho and quantile dependence measures, which are functions only depending on bivariate copulas.*

Under the null and Assumptions 1 and 2, for $T \to \infty$,

$$s\sqrt{T}(\hat{m}_{sT} - m_0(\theta_0)) \xrightarrow{d} A(s), \qquad T \to \infty, \quad \forall s \in [\varepsilon, 1], \varepsilon > 0$$

where $A(s)$ is defined in the proof and θ_0 the value of all θ_t under the null.

Proof. By Assumption 2.(vii) the sequential empirical copula of the N-dimensional random vectors fulfils

$$\begin{aligned}\mathbb{C}_{sT} :=& s\sqrt{T}\left[\hat{C}^s(\mathbf{u}) - C(\mathbf{u})\right]\\ =& \frac{1}{\sqrt{T}}\left[\sum_{t=1}^{\lfloor sT \rfloor} \mathbb{1}\{\hat{\mathbf{F}}^s(\hat{\eta}_t) \le \mathbf{u}\} - C(\mathbf{u})\right]\\ \xrightarrow{d}& A^*(s, \mathbf{u}), \qquad T \to \infty, \quad \forall s \in [\varepsilon, 1], \varepsilon > 0,\end{aligned}$$

Note that Spearman's rho between the ith and the jth component is given by

$$12 \int_0^1 \int_0^1 C(1, \dots, 1, u_i, 1, \dots, 1, u_j, 1, \dots, 1) du_i du_j - 3$$

and that the quantile dependencies are projections of the N-dimensional copula onto one specific point divided by some prespecified constant. Define the function $m^{ij}(C)$ as the function which generates a vector of all considered dependence measures (Spearman's rho and/or quantile dependencies for different levels) between the ith and the jth component out of the copula C. Without loss of generality consider the equidependence case (in the same way the argumentation holds for the block equidependence case, only that we average all intra- and inter-group dependence measures), then the function

$$\begin{aligned}&m(C) : D[0, 1]^N \to \mathbb{R}^k\\ &\qquad C \to m(C) = \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} m^{ij*}(C)\end{aligned}$$

is continuous and we directly obtain

$$s\sqrt{T}(\hat{m}_{sT} - m_0(\theta)) = s\sqrt{T}\left[m(C^s) - m(C)\right] \xrightarrow{d} \frac{2}{N(N-1)} \left(\sum_{i,j} m^{ij}(A^*(s, \mathbf{u}))\right) =: A(s)$$

as $T \to \infty$ with $s \in [\varepsilon, 1]$, $\varepsilon > 0$. Here, $m^{ij}(\cdot)$ is the same function as $m^{ij*}(\cdot)$ with the only difference that the formula for Spearman's rho between the ith and the jth component is replaced by

$$12 \int_0^1 \int_0^1 C(1, \dots, 1, u_i, 1, \dots, 1, u_j, 1, \dots, 1) du_i du_j. \quad \square$$

Proof of Theorem 1. The proof follows by checking the conditions of Theorem 4. The initial conditions of Theorem 4 follow by Assumption 4(iii) and Lemma 2.

(i) $g_0(\theta_0) = 0$ follows direct by construction, because $g_0(\theta) = m_0(\theta_0) - m_0(\theta)$.
(ii) $\theta_0 (= \theta_1 = \cdots = \theta_T)$ are interior points of Θ given by Assumption 4(i).
(iii) $g_0(\theta)$ is differentiable at θ_0 with derivative G such that $G'WG$ is nonsingular, given by Assumption 4(ii).
(iv) (1) If $\frac{S}{T} \to \infty$ as $T, S \to \infty$,

$$\begin{aligned}s\sqrt{T} g_{sT,S}(\theta_0) =& s\sqrt{T}(\hat{m}_{sT} - \tilde{m}_S(\theta_0))\\ =& s\sqrt{T}(\hat{m}_{sT} - m_0(\theta_0)) + s\sqrt{T}(m_0(\theta_0) - \tilde{m}_S(\theta_0))\\ =& s\sqrt{T}(\hat{m}_{sT} - m_0(\theta_0)) - \frac{\sqrt{T}}{\sqrt{S}} s\sqrt{S}(\tilde{m}_S(\theta_0) - m_0(\theta_0))\\ &\underset{\text{Lemma 7}}{\xrightarrow{d}} A(s)\end{aligned}$$

(2) If $\frac{S}{T} \to k \in (0, \infty)$ as $T, S \to \infty$,

$$\begin{aligned}s\sqrt{T} g_{sT,S}(\theta_0) =& s\sqrt{T}(\hat{m}_{sT} - \tilde{m}_S(\theta_0))\\ =& s\sqrt{T}(\hat{m}_{sT} - m_0(\theta_0)) + s\sqrt{T}(m_0(\theta_0) - \tilde{m}_S(\theta_0))\\ =& s\sqrt{T}(\hat{m}_{sT} - m_0(\theta_0)) - \frac{\sqrt{T}}{\sqrt{S}} s\sqrt{S}(\tilde{m}_S(\theta_0) - m_0(\theta_0))\end{aligned}$$

$$\underset{\text{Lemma 7}}{\xrightarrow{d}} A(s) - \frac{s}{\sqrt{k}} A(1),$$

combined we get

$$s\sqrt{T} g_{sT,S}(\theta_0) \xrightarrow{d} A(s) - \frac{s}{\sqrt{k}} A(1), \qquad T, S \to \infty, \quad \forall s \in [\varepsilon, 1], \varepsilon > 0.$$

(v) We know by Lemma 6, that for $\frac{S}{T} \to \infty$ or $\frac{S}{T} \to k \in (0, \infty)$
$v_{sT,S}(\theta) = \sqrt{sT}[g_{sT,S}(\theta) - g_0(\theta)]$ is stochastically equicontinuous $\forall s \in [\varepsilon, 1], \varepsilon > 0$.

$$\Rightarrow \forall \varepsilon > 0,\ \eta > 0,\ \exists \delta > 0 : \limsup_{T\to\infty} P\left[\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \|v_{sT,S}(\theta) - v_{sT,S}(\theta_0)\| > \eta\right]$$

$$= \limsup_{T\to\infty} P\left[\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \sqrt{sT}\|g_{sT,S}(\theta) - g_{sT,S}(\theta_0) - g_0(\theta)\| > \eta\right] < \varepsilon. \tag{15}$$

Furthermore the inequality

$$s\sqrt{T}\frac{\|g_{sT,S}(\theta) - g_{sT,S}(\theta_0) - g_0(\theta)\|}{1 + s\sqrt{T}\|\theta - \theta_0\|} \le s\sqrt{T}\|g_{sT,S}(\theta) - g_{sT,S}(\theta_0) - g_0(\theta)\| \tag{16}$$

is valid $\forall s \in [\varepsilon, 1]$.
Finally we obtain

$$\begin{aligned}
&\limsup_{T\to\infty} P\left[\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} s\sqrt{T}\frac{\|g_{sT,S}(\theta) - g_{sT,S}(\theta_0) - g_0(\theta)\|}{1 + s\sqrt{T}\|\theta - \theta_0\|} > \eta\right] \\
&\le \limsup_{T\to\infty} P\left[\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \sqrt{sT}\frac{\|g_{sT,S}(\theta) - g_{sT,S}(\theta_0) - g_0(\theta)\|}{1 + s\sqrt{T}\|\theta - \theta_0\|} > \eta\right] \\
&\overset{(16)}{\le} \limsup_{T\to\infty} P\left[\sup_{s\in[\varepsilon,1]} \sup_{\|\theta-\theta_0\|<\delta} \sqrt{sT}\|g_{sT,S}(\theta) - g_{sT,S}(\theta_0) - g_0(\theta)\| > \eta\right] \overset{(15)}{<} \varepsilon.
\end{aligned}$$

Note that, for the first inequality sign, we use that $0 < s \le \sqrt{s}\ \forall s \in [\varepsilon, 1], \varepsilon > 0$.

This completes the proof. □

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