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Estimation of the discontinuous leverage effect: Evidence from the NASDAQ order book

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ABSTRACT

An extensive empirical literature documents a generally negative relation, named the "leverage effect," between asset returns and changes of volatility. It is more challenging to establish such a return–volatility relationship for jumps in high-frequency data. We propose new nonparametric methods to assess and test for a discontinuous leverage effect — i.e. a covariation between contemporaneous jumps in prices and volatility. The methods are robust to market microstructure noise and build on a newly developed price-jump localization and estimation procedure. Our empirical investigation of six years of transaction data from 320 NASDAQ firms displays no unconditional negative covariation between price and volatility cojumps. We show, however, that there is a strong and significant discontinuous leverage effect if one conditions on the sign of price jumps and whether the price jumps are market-wide or idiosyncratic.

1. Introduction

Understanding the relation between asset returns and volatility is among the most enduring and highly active research topics in finance. From an economic point of view, there seems to be a consensus that stock market returns and changes in volatility should be negatively related.[2] The linear, inverse return–volatility relationship is usually attributed to both changes in financial leverage and a time-varying risk premium; see Black (1976), French et al. (1987), Duffee (1995), Bekaert and Wu (2000) and Bollerslev et al. (2006). The financial leverage explanation motivates labeling statistical dependence between stock returns and volatility as the "leverage effect". Following Wang and Mykland (2014) and Aït-Sahalia et al. (2017), this paper measures the leverage effect with a covariation statistic.[3]

Estimation of the leverage effect is challenging. Aït-Sahalia et al. (2013) document that the leverage effect fades out when using data sampled at increasing observation frequencies. Specifically, in the framework of the Heston model, they show

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[1] The views expressed are those of the individual authors and do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors.

[2] Some papers define the leverage effect as the relation between returns and the *level* of volatility. Duffee (1995) discusses the relation between the two definitions.

[3] We also provide methodological and empirical evidence for a correlation statistic in Sections 3.4 and 5.2, respectively.

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that discretization errors, volatility estimates and market microstructure noise bias the naïve return–volatility correlation estimator toward zero. Recent research has tried hard to establish the leverage effect for intraday data.

If the asset price and volatility processes have both Brownian and jump components, then the relation between returns and volatility splits into continuous and discontinuous parts. Continuous leverage refers to the covariation between the Brownian components of the price and volatility processes. Vetter (2012), Wang and Mykland (2014), Aït-Sahalia et al. (2017) and Kalnina and Xiu (2017) study measures of continuous leverage. These papers document a negative and usually time-varying continuous leverage effect. The discontinuous leverage effect (DLE) measures the covariation between sizes of contemporaneous price and volatility jumps. Bandi and Renò (2016) highlight the importance of both leverage components for asset pricing and risk management. They show how the intensity of price–volatility cojumps, as well as their signs and magnitudes, affects the return and variance risk premia.

The existence of the DLE appears controversial, however. Several previous studies reached different conclusions regarding a DLE. Jacod et al. (2017) use truncated returns and increments of local spot-volatility estimates to construct correlation statistics for one-minute S&P 500 Exchange-Traded Funds (ETF) data from 2005 to 2011. These correlation statistics indicate little evidence of a DLE. In contrast, Bandi and Renò (2016) focus on a relatively small set of very large price jumps and a spot variance estimator based on infinitesimal cross-moments for high-frequency S&P 500 futures from 1982 to 2009. Their parametric estimates suggest a significant DLE with correlations from −0.6 to −1. Aït-Sahalia et al. (2017) find that the DLE for five-second Dow Jones index data from 2003 to 2013 is usually different from zero. Their empirical analysis does not recover the sign and magnitude of the discontinuous leverage, however. Finally, Todorov and Tauchen (2011) use five-minute, option-implied volatility index (VIX) data to evaluate volatility jumps in the S&P 500 index from 2003 to 2008. The authors find that squared jumps in the S&P 500 index are strongly positively correlated with jumps in the VIX. All these papers focus on stock market indexes, not individual stocks, and only use methods that are not robust to market microstructure noise.

Our paper makes both methodological and empirical contributions. We introduce novel methods to estimate the covariation of contemporaneous price and volatility jumps—denoted by Aït-Sahalia et al. (2017) as the DLE. A direct extension of our covariation estimator consistently estimates the corresponding correlation. Aït-Sahalia et al. (2017) derive a limit theorem for the DLE estimator that only applies to a setting without market microstructure noise. Christensen et al. (2014) point out, however, that it is important to use noise–robust methods and thereby to avoid downsampling the data to lower observation frequencies. Downsampling may result in spurious jump detection and affect the accuracy of discontinuous leverage estimates. Using noise–robust estimators for jumps in log prices and volatility, we establish a stable central limit theorem under market microstructure noise for the DLE for finite activity price jumps or large jumps of an infinite activity jump component. We provide a consistent, asymptotic test for the presence of the DLE.

We estimate the covariation using only the physical measure, i.e., observed stock prices. DLE estimation requires three steps: price-jump localization, price-jump estimation and estimation of changes in the spot-volatility process at price-jump times. In the presence of microstructure noise, none of the three steps is standard. We use spectral methods for the estimation. Reiß (2011) introduces spectral estimation of the quadratic variation from noisy observations. Bibinger et al. (2014) and Altmeyer and Bibinger (2015) establish the asymptotic efficiency of spectral estimators of the integrated volatility matrix in the multivariate case with noisy and non-synchronous observations. Bibinger et al. (2017) propose a related spot-volatility estimator. Although spectral and the popular pre-average estimators have some similarities, they belong to different classes of estimators; see Remark 1 of Bibinger and Winkelmann (2018). Our theoretical contribution is to provide methods to detect and estimate price jumps and to combine the three steps to infer the DLE.

To detect price-jump times, we refine the adaptive thresholding approach of Bibinger and Winkelmann (2015). We construct an argmax-estimator, as is often used in change-point analysis. This refinement of the jump localization is motivated by the fact that estimation of price jumps becomes more difficult in cases where the jump times are not precisely determined, see Vetter (2014) for a related problem. To estimate the price-jump size at a detected jump time, we first review the pre-average method of Lee and Mykland (2012), which extends the Lee and Mykland (2008) approach to a model with market microstructure noise. While Lee and Mykland (2012) mainly focus on a global test for jumps, we focus on local jump estimates. We generalize their stable central limit theorem from a jump–diffusion to more general semimartingale models. Estimating the entire quadratic variation with jumps or testing for jumps over a whole day are related yet different problems. Jacod et al. (2010) and Koike (2017) have developed rate-optimal consistent pre-average estimators for the quadratic variation and Bibinger and Winkelmann (2015) provide spectral estimators for this purpose. These methods neither locate nor estimate the size of individual price jumps. The Lee and Mykland (2012) method utilizes natural *local* average statistics to address inference on price jumps under noise. Their pre-average method attains the optimal rate of convergence for local price-jump estimation. As one ingredient of the price-jump localization, we exploit the simple structure of these pre-average statistics that permits an asymptotic theory based on Gaussian approximations. Using spectral local statistics for price-jump estimation, we derive a superior estimator with a smaller variance than the pre-average estimator. The asymptotic variance of the spectral estimator attains the lower bound. Thus, we provide the first feasible, asymptotically efficient estimator of price jumps from noisy observations. To estimate changes in the spot volatility at a price-jump time, we employ the jump-robust techniques of Bibinger and Winkelmann (2018). Finally, we plug the price-jump and volatility-jump estimates into the DLE statistic of Aït-Sahalia et al. (2017).

Our methods provide new empirical evidence about the DLE for 320 individual stocks, which were actively traded at the NASDAQ stock exchange from 2010 to 2015. We find no prevalent evidence of an unconditional DLE in individual stock data, but we identify two forces that prevent significant unconditional discontinuous leverage estimates: First, while downward price jumps usually covary negatively with contemporaneous volatility jumps, upward price jumps covary positively with contemporaneous volatility jumps. Second, market jumps, i.e. price jumps that coincide with jumps of a market portfolio, display a larger DLE. In contrast, idiosyncratic price jumps, which occur without a contemporaneous jump of the market portfolio, are associated with a smaller and less significant DLE. We establish an economically and statistically significant covariation between cojumps in stock prices and volatility by conditioning on the sign of price jumps and whether those jumps are systematic or idiosyncratic.

Our failure to find an unconditionally negative DLE is consistent with the asset pricing models of Pástor and Veronesi (2012, 2013) in which specific events trigger jumps. That is, the continuous leverage effect and the DLE are fundamentally different in that model. Their learning model implies that changes in monetary or government policy trigger market-wide price and volatility cojumps, where the uncertainty about the impact of a new policy regime on the profitability of private firms always raises volatility, regardless of the effect on prices. News that causes asset prices to jump up while causing volatility to jump down is incompatible with their model. Our results are also consistent with Pelger (2017), who studies systematic and non-systematic risk factors in S&P500 high-frequency firm data. That is, we confirm that the DLE appears predominantly for systematic risk, while being weaker and more often non-significant for idiosyncratic risk.

The rest of the paper is organized as follows. Section 2 introduces the model and assumptions. Section 3 presents the price-jump estimators, spot-volatility estimation and the DLE estimator. We compare the spectral approach for price jumps with the (Lee and Mykland, 2012) pre-average estimator. Section 4 provides Monte Carlo evidence and Section 5 the empirical findings. Section 6 concludes. The Appendix contains the proofs.

2. Statistical model and assumptions

We work with a very general class of continuous-time processes, namely Itô semimartingales. Its implicit no-arbitrage properties make it the most popular model for log-price processes in financial econometrics. The model is formulated for a log price, X_t, and its volatility, σ_t, over a fixed time period $t \in [0, 1]$, on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$:

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s + \int_0^t \int_{\mathbb{R}} \delta(s,z)\mathbb{1}_{\{|\delta(s,z)|\le 1\}}(\mu-\nu)(ds,dz)$$
$$+ \int_0^t \int_{\mathbb{R}} \delta(s,z)\mathbb{1}_{\{|\delta(s,z)|>1\}}\mu(ds,dz)\,, \tag{1}$$

with a standard Brownian motion W_s, the jump size function δ, defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$, and the Poisson random measure μ, which is compensated by $\nu(ds,dz) = \lambda(dz)\otimes ds$ with a σ-finite measure λ. We write $\Delta X_t = X_t - X_{t-}$ with $X_{t-} = \lim_{s<t,s\to t} X_s$ for the process of jumps in X_t and $\Delta\sigma_t^2 = \sigma_t^2 - \sigma_{t-}^2$ for jumps of the squared volatility. Our notation follows that of Jacod and Protter (2012). We impose mild regularity assumptions on the characteristics of X_t.

Assumption 1. The drift $(b_t)_{t\ge 0}$ is a locally bounded process. The volatility never vanishes, $\inf_{t\in[0,1]}\sigma_t > 0$ almost surely. For all $0 \le t+s \le 1$, $t \ge 0$, some constants $C_n, \tilde{C}_n > 0$, some $\alpha > 1/2$ and for a sequence of stopping times T_n, increasing to ∞, we have that

$$\left|\mathbb{E}\Big[\sigma_{(t+s)\wedge T_n} - \sigma_{t\wedge T_n}\,|\mathcal{F}_t\Big]\right| \le C_n\, s^{\alpha}\,, \tag{2}$$

$$\mathbb{E}\Big[\sup_{t\in[0,s]} |\sigma_{(t+t)\wedge T_n} - \sigma_{t\wedge T_n}|^2\Big] \le \tilde{C}_n\, s\,. \tag{3}$$

Assumption 1 requires some smoothness of the volatility process. It does not exclude volatility jumps, only fixed times of discontinuity are excluded. We impose the following regularity condition on the jumps.

Assumption 2. Assume for the predictable function δ in (1) that $\sup_{\omega,x}|\delta(t,x)|/\gamma(x)$ is locally bounded with a non-negative, deterministic function γ that satisfies

$$\int_{\mathbb{R}} (\gamma^r(x)\wedge 1)\lambda(dx) < \infty\,. \tag{4}$$

The index r, $0 \le r \le 2$, in (4) measures the jump activity. Smaller values of r make Assumption 2 more restrictive. In particular, $r = 0$ results in finite-activity jumps and $r = 1$ implies that jumps are summable.

Remark 1. Assumption 1 is satisfied in a very general model, where the volatility process σ_t is an Itô semimartingale

$$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s\,ds + \int_0^t \tilde{\sigma}_s\,d\tilde{W}_s + \int_0^t \int_{\mathbb{R}} \tilde{\delta}(s,z)\mathbb{1}_{\{|\tilde{\delta}(s,z)|\le 1\}}(\mu-\nu)(ds,dz)$$
$$+ \int_0^t \int_{\mathbb{R}} \tilde{\delta}(s,z)\mathbb{1}_{\{|\tilde{\delta}(s,z)|>1\}}\mu(ds,dz)\,, \tag{5}$$

with a standard Brownian motion $\tilde{W}_s$, when the characteristics in (5) are locally bounded and when an analogous condition as (4) holds for $\tilde{\delta}$ in (5) with $r = 2$. We may use the same μ in (1) and (5), such that μ is a jump measure on $\mathbb{R}_+ \times \mathbb{R}$ governing the jumps in the log price and its volatility. The predictable functions, δ and $\tilde{\delta}$, defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$, then determine common jumps of σ_t and X_t. Whenever $\delta\tilde{\delta} \equiv 0$, there is no price–volatility cojump. Our asymptotic theory and Assumption 1 allow for generalizations of (5), such as inclusion of long-memory fractional volatility components. Thus, our theoretical setup nests almost any popular stochastic volatility model that allows for both continuous and discontinuous leverage effects.

In practice, one cannot observe the efficient price (1) directly and one must account for market microstructure noise in analyzing price and volatility jumps. To efficiently exploit high-frequency prices, we posit a latent discrete observation model with noise:

$$\text{Observe} \quad Y_{t_i^n}, i = 0, \ldots, n, \quad \text{with} \quad Y_t = X_t + \epsilon_t \,, \tag{6}$$

where ϵ_t captures the market microstructure noise. We use the typical notation, $\Delta_i^n Y = Y_{t_i^n} - Y_{t_{i-1}^n}$, $i = 1, \ldots, n$, for noisy returns and analogous notation for the processes X_t and ϵ_t. In our baseline setup, market microstructure noise is a white noise process $(\epsilon_t)_{t\geq 0}$, independent of X_t, with $\mathbb{E}[\epsilon_t] = 0$ and $\mathbb{E}[\epsilon_t^2] = \eta^2$, as well as $\mathbb{E}[\epsilon_t^{4+\delta}] < \infty$ for some $\delta > 0$, for all $t \in [0, 1]$. The process Y_t is accommodated on the product space $(\bar{\Omega}, \mathcal{G}, (\mathcal{G}_t), \bar{\mathbb{P}})$, where $\mathcal{G}_t = \mathcal{F}_t \otimes \sigma(\epsilon_s, s \leq t)$ contains information about the signal and noise. Below we extend the model to more general setups with serially correlated, heteroscedastic noise. Because we apply our methods to locally infer price and volatility jumps of individual stock prices, non-synchronicity of the multivariate data is of less importance here.

3. Inference on the discontinuous leverage effect

The DLE is defined as the covariation of contemporaneous price and volatility jumps. We estimate it in three steps. We first address noise–robust estimation of price jumps in Section 3.1, then we turn to noise–robust estimation of spot-volatility changes in Section 3.2. In Section 3.3, we show how to detect a priori unknown price-jump times in noisy data and how to refine price-jump estimation in this case. The estimated DLE is the covariation of the price-jump and spot-volatility estimates at detected jump times. This estimator is presented in Section 3.4.

3.1. Price-jump estimation

3.1.1. Local jump estimator and test using pre-averaged log prices

Consider the statistic

$$T^{LM}(\tau; \Delta_1^n Y, \ldots, \Delta_n^n Y) = \hat{P}(t_l^n) - \hat{P}(t_{l-M_n}^n)\,, \; l = \lfloor \tau n \rfloor + 1\,, \tag{7}$$

at a (stopping) time $\tau \in (0, 1)$ and with pre-processed price estimates

$$\hat{P}(t_j^n) = M_n^{-1} \sum_{i=j}^{(j+M_n-1)\vee n} Y_{t_i^n}\,. \tag{8}$$

Lee and Mykland (2012) propose a test for price jumps at time τ based on (7). The window length for the pre-averaging is $M_n = c\sqrt{n}$, with a proportionality constant c. The following proposition generalizes Lemma 1 in Lee and Mykland (2012), where the authors assume that they observe discrete, noisy observations from a jump–diffusion model.

Proposition 3.1. *Under Assumptions 1 and 2 with $r < 4/3$ for equidistant observations, $t_i^n = i/n$, the Lee–Mykland statistic (7) obeys the stable*[4] *central limit theorem,*

$$\sqrt{M_n}\left(T^{LM}(\tau; \Delta_1^n Y, \ldots, \Delta_n^n Y) - \Delta X_\tau\right) \xrightarrow{(st)} MN\Big(0, \frac{1}{3}(\sigma_\tau^2 + \sigma_{\tau-}^2)c^2 + 2\eta^2\Big)\,, \tag{9}$$

as $n \to \infty$, where MN stands for mixed normal.

Thus, in case of a price jump at τ, (7) consistently estimates the price-jump size. The central limit theorem accounts for a contemporaneous volatility jump. If there is no volatility jump, then $\sigma_\tau^2 = \sigma_{\tau-}^2$ in (9). With the null hypothesis, $\Delta X_\tau = 0$, and alternative, $|\Delta X_\tau| > 0$, Proposition 3.1 facilitates a consistent test for a jump in the stock price at time point $\tau \in (0, 1)$.

4 Stable means stable convergence in law with respect to $\mathcal{F}$.

3.1.2. Local jump estimator and test using spectral statistics

To estimate price jumps using spectral statistics, we consider an orthogonal system of sine functions that are localized on a window around τ:

$$\Phi_{j,\tau}(t)=\sqrt{\frac{2}{h_n}}\sin\left(j\pi h_n^{-1}(t-(\tau-h_n/2))\right)\mathbb{1}_{[\tau-h_n/2,\tau+h_n/2]}(t)\,,\;j\geq 1\,. \tag{10}$$

Asymptotically efficient volatility estimation from noisy observations (6) motivates consideration of local averages of noisy log prices in the frequency domain; see Reiß (2011) and Bibinger et al. (2014). Intuitively, *spectral statistics*,

$$S_j(\tau)=\sum_{i=1}^{n}\Delta_i^n Y\,\Phi_{j,\tau}((t_{i-1}^n+t_i^n)/2)\,,\;j\geq 1\,, \tag{11}$$

maximize the local information load about the signal process and thereby allow for local estimates of the efficient prices: X_τ and $X_{\tau-}$. The scaling factor in front of the sine in (10) ensures that $\int_{\tau-h_n/2}^{\tau+h_n/2}\Phi_{j,\tau}^2(t)\,dt=1$. We propose the following statistic:

$$\mathcal{T}(\tau;\Delta_1^nY,\ldots,\Delta_n^nY)=\sum_{j=1}^{J_n}(-1)^{j+1}a_{2j-1}S_{2j-1}(\tau)\sqrt{h_n/2}\,, \tag{12}$$

with weights $(a_{2j-1})_{j\geq 1}$, to infer price jumps. (12) is a rescaled weighted sum of spectral statistics over odd spectral frequencies up to some spectral cut-off frequency $2J_n-1$. Excluding even frequencies and alternating the signs of addends facilitate a consistent estimation of price jumps, ΔX_τ, as in (9).

The window length is set to be $h_n=\kappa\log(n)/\sqrt{n}$, for some constant κ. Despite the logarithmic factor, the window length resembles the one in (8). We derive optimal oracle weights by minimizing the variance, which vary with the volatility σ_t. Yet, under Assumption 1, the error of approximating σ_t^2 constant on $[\tau-h_n/2,\tau)$ and $[\tau,\tau+h_n/2]$ is asymptotically negligible. Then, as in the weighted least squares approach, this leads to optimal weights

$a_j\propto 1/\mathbb{V}\mathrm{ar}(S_j(\tau))$.[5]

In order to consistently estimate the jump $(X_\tau-X_{\tau-})$, we set $\sum_{j=1}^{J_n}a_{2j-1}=1$ such that

$$\begin{aligned}a_{2j-1}&=\frac{\left(\mathbb{V}\mathrm{ar}\left(S_{2j-1}(\tau)\right)\right)^{-1}}{\left(\sum_{u=1}^{J_n}\left(\mathbb{V}\mathrm{ar}\left(S_{2u-1}(\tau)\right)\right)^{-1}\right)}\\&=\frac{\left(\frac{1}{2}(\sigma_\tau^2+\sigma_{\tau-}^2)+\pi^2(2j-1)^2h_n^{-2}n^{-1}\eta^2\right)^{-1}}{\left(\sum_{u=1}^{J_n}\left(\frac{1}{2}(\sigma_\tau^2+\sigma_{\tau-}^2)+\pi^2(2u-1)^2h_n^{-2}n^{-1}\eta^2\right)^{-1}\right)}\,.\end{aligned} \tag{13}$$

For an adaptive method, we estimate these oracle optimal weights by plugging in the estimated noise variance,

$$\hat{\eta}^2=-n^{-1}\sum_{i=1}^{n-1}\Delta_i^nY\Delta_{i-1}^nY=\eta^2+\mathcal{O}_{\mathbb{P}}\left(n^{-1/2}\right), \tag{14a}$$

and the pre-estimated spot squared volatility,

$$\begin{aligned}\hat{\sigma}_{\tau-,pil}^2&=\frac{k_n}{J_p}\sum_{k=1}^{k_n^{-1}}\sum_{j=1}^{J_p}\left(S_j^2(\tau-kh_n)-\pi^2j^2h_n^{-2}n^{-1}\hat{\eta}^2\right)\\&\qquad\times\mathbb{1}\left(\left|(J_p)^{-1}\sum_{j=1}^{J_p}\left(S_j^2(\tau-kh_n)-\pi^2j^2h_n^{-2}n^{-1}\hat{\eta}^2\right)\right|\leq u_n\right)\\&=\sigma_{\tau-}^2+\mathcal{O}_{\mathbb{P}}\left(n^{-1/8}\right),\end{aligned} \tag{14b}$$

with $k_n^{-1}=\mathcal{R}n^{1/4}$ for a constant $\mathcal{R}$, a threshold sequence, $u_n=h_n^{\varpi}$, $0<\varpi<1$, and maximal spectral frequency, J_p, leading to the above rate-optimal estimators under Assumptions 1 and 2 with $r<3/2$. The notation $S_j^2(\tau-kh_n)$ refers to squared spectral statistics computed from k_n^{-1} bins with sine functions centered around times, $\tau-kh_n$, before τ. $\hat{\sigma}_{\tau,pil}^2$ is the analog of (14b), replacing $\tau-kh_n$ by $\tau+kh_n$. Bibinger and Winkelmann (2018) detail the construction and prove the asymptotic properties of pre-estimators (14a) and (14b) and also suggest how to choose $\mathcal{R}$ and J_p.

Next, we state asymptotic results for $\mathcal{T}(\tau;\Delta_1^nY,\ldots,\Delta_n^nY)$, which refers to statistic (12) with estimated optimal weights.

5 We write $A\propto B$ for proportionality, i.e. $A=c\,B$ for some non-zero constant c.

Proposition 3.2. *Under Assumptions 1 and 2 with $r < 4/3$ for equidistant observations, $t_i^n = i/n$, our statistic (12) obeys the stable central limit theorem as $n \to \infty$ and $J_n \to \infty$:*

$$n^{1/4}\left(\mathcal{T}(\tau; \Delta_1^n Y, \ldots, \Delta_n^n Y) - \Delta X_\tau\right) \xrightarrow{(st)} MN\Big(0, 2\Big(\frac{\sigma_\tau^2 + \sigma_{\tau-}^2}{2}\Big)^{1/2}\eta\Big). \tag{15}$$

In the case of no volatility jump at τ, $\sigma_\tau = \sigma_{\tau-}$ and the asymptotic variance is $2\sigma_\tau\eta$. Finally, we extend Proposition 3.2 to a more realistic model that incorporates serially correlated, heteroscedastic noise and irregular sampling.

Assumption 3. Assume the existence of a differentiable, cumulative distribution function, F, that determines the observation times via a quantile transformation, $t_i^n = F^{-1}(i/n)$, $i = 0, \ldots, n$. Assume $(F^{-1})'$ is α-Hölder continuous for some $\alpha > 1/2$, i.e., $|(F^{-1})'(t) - (F^{-1})'(s)| \le |t-s|^\alpha$ for all s, t.

The noise process ϵ_t is independent of X_t. For all t, we have $\mathbb{E}[\epsilon_t] = 0$ and $\mathbb{E}\big[\epsilon_t^{4+\delta}\big] < \infty$, for some $\delta > 0$. Further, assume $(\epsilon_{t_i^n})$ is an R-dependent process, such that $\mathbb{C}\mathrm{ov}(\epsilon_{t_i^n}, \epsilon_{t_{i+u}^n}) = 0$ for all i and all $u > R$ for some $R < \infty$, then the long-run variance process converges as follows:

$$\sum_{l=-\lfloor tn\rfloor}^{n-\lfloor tn\rfloor} \mathbb{C}\mathrm{ov}\big(\epsilon_{\lfloor tn\rfloor}, \epsilon_{\lfloor tn\rfloor + l}\big) \to \eta_t^2\,, \tag{16}$$

for $t \in [0, 1]$, uniformly in probability. The process $(\eta_t^2)_{t\in[0,1]}$ is locally bounded and satisfies, for all $t, (t+s) \in [0, 1]$, the mild smoothness condition:

$$|\eta_{t+s}^2 - \eta_t^2| \le K s^\alpha\,, \tag{17}$$

for some constant K. The noise does not vanish: $\eta_t^2 > 0$ for all $t \in [0, 1]$.

Proposition 3.3. *Under Assumptions 1, 2 with $r < 4/3$ and 3, the statistic (12) obeys the stable central limit theorem as $n \to \infty$ and $J_n \to \infty$*

$$n^{1/4}\left(\mathcal{T}(\tau; \Delta_1^n Y, \ldots, \Delta_n^n Y) - \Delta X_\tau\right) \xrightarrow{(st)} MN\Big(0, 2\Big(\frac{\sigma_\tau^2 + \sigma_{\tau-}^2}{2}\Big)^{1/2}\eta_\tau\big((F^{-1})'(\tau)\big)^{1/2}\Big). \tag{18}$$

In analogy to Proposition 3.1, Propositions 3.2 and 3.3 show the consistency of the spectral jump estimator and give a consistent test for a price jump at time τ. One can construct standardized, feasible versions of (15) and (18) by inserting spot squared volatility and long-run noise variance estimators. See (14a) and Bibinger et al. (2017) for such estimators. In fact, the pre-estimation of optimal weights also estimates the variances of (12). The asymptotic variance of the Lee–Mykland statistic in (9) generalizes to $(1/3)(\sigma_\tau^2 + \sigma_{\tau-}^2)(F^{-1})'(\tau)c^2 + 2\eta_\tau^2$ under the conditions from Proposition 3.3. Lee and Mykland (2012) provide a generalization to R-dependent noise using sub-sampling, and this directly applies to our general setup with Assumption 3. The spectral price-jump estimator (12) and the pre-average jump estimator (7) have the same optimal convergence rate and similar asymptotic properties.

Remark 2. Writing (9) with rate $n^{1/4}$ instead of $M_n^{1/2}$, the variance of the Lee–Mykland estimator (7) with $\sigma_\tau = \sigma_{\tau-}$ becomes $\frac{2}{3}\sigma_\tau^2 c + 2\eta^2 c^{-1}$. The variance is minimized by the constant $c = \sqrt{3}\eta\sigma_\tau^{-1}$, which yields $4\sigma_\tau\eta/\sqrt{3}$ in (9). Since $4/\sqrt{3} \approx 2.31$, this optimized variance of (an infeasible) Lee–Mykland estimator is about 16% larger than the variance $2\sigma_\tau\eta$ of the spectral estimator in (15). Moreover, according to the LAN result of Koike (2017), the latter is optimal. That is, the variance of the spectral estimator coincides with a lower bound for the asymptotic variance, which is given by the inverse of the Fisher information from Proposition 5.2 of Koike (2017). Our estimator is hence the first feasible, asymptotically efficient estimator for price jumps in the semimartingale model with market microstructure noise.

We caution, however, that estimates via spectral statistics (12) and pre-averages (7) are biased when a jump is not located close to time τ but instead close to one of the edges of the local window. Fig. 1 illustrates this. The bias for the Lee–Mykland estimator is linear. This effect directly relates to the so-called "pulverisation" of jumps by pre-averages described in Mykland and Zhang (2016). For our statistic, the bias hinges on the weights and the spectral cut-off. The lower panel of Fig. 1 reveals that the bias is similar for both methods. The bias becomes important when studying price jumps at a priori unknown times, such as when one is estimating the DLE. Section 3.3 discusses our solution. Related problems by not knowing the exact timing of jumps arise and have been addressed in different ways in Vetter (2014) and Bibinger and Winkelmann (2015).

3.2. Spot-volatility estimation

We estimate the contemporaneous volatility adjustment to a price jump at time $\tau \in (0, 1)$. We employ the spectral spot squared volatility estimators of Bibinger and Winkelmann (2018), smoothed over local windows before τ and after τ, to consistently estimate the volatilities σ_τ^2 and $\sigma_{\tau-}^2$. Based on estimated oracle optimal weights for volatility estimation

$$w_{jk} = I_k^{-1} I_{jk} = \frac{\Big(\sigma_{(k-1)h_n}^2 + \pi^2 j^2 h_n^{-2}\frac{\eta_{(k-1)h_n}^2}{n}\Big)^{-2}}{\sum_{m=1}^{J_n}\Big(\sigma_{(k-1)h_n}^2 + \pi^2 m^2 h_n^{-2}\frac{\eta_{(k-1)h_n}^2}{n}\Big)^{-2}}\,, \tag{19}$$

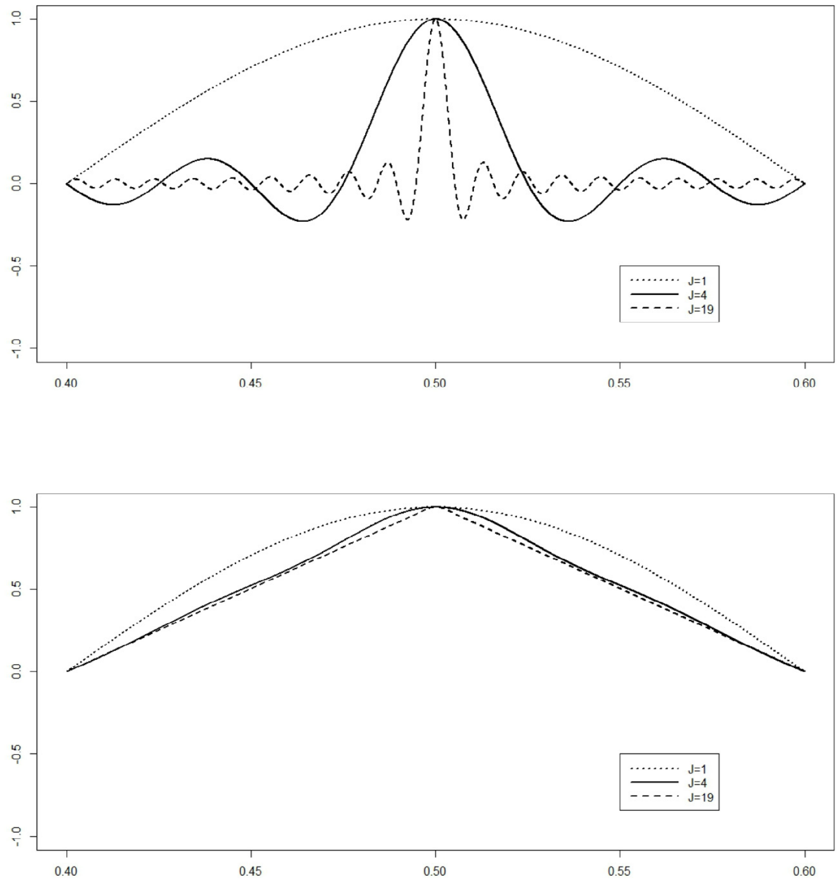

Fig. 1. Example for $\sum_{j=1}^{J} J^{-1}(-1)^{j+1}\Phi_{2j-1,\tau}(t)\sqrt{h_n/2}$ (top) and, with oracle optimal weights a_{2j-1} from the simulation setup in Section 4, $\sum_{j=1}^{J} a_{2j-1}(-1)^{j+1}\Phi_{2j-1,\tau}(t)\sqrt{h_n/2}$ (bottom), as functions of time t for three values of J on window $[\tau - h_n/2, \tau + h_n/2]$, $\tau = 0.5$, $h_n = 0.2$. The plots illustrate that, if a jump occurs on the interval and is not sufficiently close to τ, the estimation of ΔX_τ becomes biased for the actual jump.

inserting spot squared volatility and noise variance estimators, with

$$\zeta_k^{ad}(Y) = \sum_{j=1}^{J_n} \hat{w}_{jk}\Big(S_{jk}^2 - \pi^2 j^2 h_n^{-2}\frac{\hat{\eta}^2_{(k-1)h_n}}{n}\Big), \tag{20}$$

the spectral estimator of the spot squared volatility at time $\tau-$ is

$$\hat{\sigma}^2_{\tau-} = k_n \sum_{k=\lfloor \tau h_n^{-1}\rfloor - k_n^{-1}}^{\lfloor \tau h_n^{-1}\rfloor - 1} \zeta_k^{ad}(Y)\mathbb{1}_{\{h_n|\zeta_k^{ad}(Y)|\le u_n\}}. \tag{21}$$

To estimate the noise variance η^2 and pre-estimate the spot squared volatility in (19), we use (14a) and (14b), respectively. To obtain (20), we adapt $S_{jk} = S_j((k-1/2)h_n)$ from (11). Analogously to $\hat{\sigma}^2_{\tau-}$, $\hat{\sigma}^2_{\tau}$ is defined by summing over $k \in \{\lfloor \tau h_n^{-1}\rfloor + 1, \ldots, \lfloor \tau h_n^{-1}\rfloor + k_n^{-1}\}$. The theory by Bibinger and Winkelmann (2018) renders the following result:

Corollary 3.4. *Under Assumptions 1 and 2 with $r < 3/2$ for equidistant observations, $t_i^n = i/n$, and under Assumption 3 for the noise, the statistics (21) with $k_n \propto n^{-\beta}\log(n)$ satisfy*

$$n^{\beta/2}\left(\left(\hat{\sigma}^2_\tau - \hat{\sigma}^2_{\tau-}\right) - \Delta\sigma^2_\tau\right) \xrightarrow{(st)} MN\left(0, 8(\sigma^3_\tau + \sigma^3_{\tau-})\eta_\tau\right) \tag{22}$$

for all

$$0 < \beta < \left(1/4 \wedge \varpi\left(1 - \frac{r}{2}\right)\right), \tag{23}$$

with ϖ from the truncation sequence u_n, such that we come arbitrarily close to the optimal rate $n^{1/8}$ in (22).

Theorem 10.30 of Aït-Sahalia and Jacod (2014) provides a related result for volatility-jump estimation without microstructure noise, where the first line in their equation (10.81) for one fixed point in time corresponds to our result under condition (23). The corollary provides an asymptotic test of the hypothesis of no volatility jump, $\Delta\sigma_\tau^2 = 0$, against the alternative that $\Delta\sigma_\tau^2 \neq 0$. The statistic (27) in Bibinger and Winkelmann (2018) gives an efficient test. For non-equidistant observations, the noise level in (22) includes $((F^{-1})'(\tau))^{1/2}$, analogous to (18) for price jumps.

3.3. Price-jump estimation at unknown price-jump times

The time of a price jump is usually unknown. Hence the jump times need to be estimated from the observed high-frequency data. Estimation of price jumps at a priori unknown times poses an intricate problem due to the fact that price-jump times, τ_k, can only be located on (asymptotically small) time intervals and not determined exactly. Given the bias problem emphasized at the end of Section 3.1, it is crucial to detect jump times very precisely. Since price jumps are one component of the DLE, biased price-jump estimates produce a bias in DLE estimates. To estimate price-jump times, we propose the following procedure.

We begin our jump-detection procedure with local quadratic variation estimates, as in Bibinger and Winkelmann (2015) and related works on spectral volatility estimation. The local quadratic variation estimates enable volatility estimation (21) and a thresholding procedure to locate bins $((k-1)h_n, kh_n)$ that contain a (large) price jump. We apply a bin-wise threshold, $u_n(kh_n) = 2\log(h_n^{-1})h_n\hat{\sigma}^2_{(k-1)h_n,pil}$, with pre-estimated squared volatility, as defined in (14b). The moving threshold accounts for intraday volatility patterns. Besides the threshold u_n, DLE estimation with infinite active jumps requires the detection of bins with an increase in quadratic variation larger than a finite constant a^2. For the price jump detection, this requires unbiased estimation of the changes in the quadratic variation, $\Delta_k[X,X] = [X,X]_{kh_n} - [X,X]_{(k-1)h_n}$, on bins with price jumps. To detect bins with $\Delta_k[X,X] > a^2 \vee u_n$, we adapt the statistics from Section 3.1.3 of Bibinger and Winkelmann (2015) and define

$$\begin{aligned}\widetilde{\zeta}^{ad}_{k,l} &= \sum_{j\in\mathcal{J}_n} \hat{w}_{jk}\Big(\tfrac{1}{2}S^2_{jk} + \tfrac{1}{2}\tilde{S}^2_{jl} - \pi^2 j^2 h_n^{-2}\frac{\hat{\eta}^2_{(k-1)h_n}}{n}\Big)\,,\\ \widetilde{\zeta}^{ad}_{k} &= \max\big(\widetilde{\zeta}^{ad}_{k,k}, \widetilde{\zeta}^{ad}_{k,k+1}\big)\,,\end{aligned} \tag{24}$$

by summing over the set $\mathcal{J}_n$ of odd numbers up to the cut-off J_n and with spectral statistics $\tilde{S}_{jk} = S_j((k-1)h_n)$ shifted by $h_n/2$ in comparison to $S_{jk} = S_j((k-1/2)h_n)$. This adjustment of (20) allows for unbiased estimation of the increase in quadratic variation on bins with jumps. Due to the overlapping nature of shifted bins and the maximum operator in (24), a jump on a bin also affects a neighboring bin. The weighting of a jump on a neighboring bin is always smaller, however, than the weighting on the bin containing the jump. Thus, the increment in jump variation on a bin containing a jump is estimated by

$$\Delta_k\widehat{[X,X]} = h_n\,\widetilde{\zeta}^{ad}_k\,\mathbb{1}_{\{\widetilde{\zeta}^{ad}_k > \max(\widetilde{\zeta}^{ad}_{k-1},\widetilde{\zeta}^{ad}_{k+1})\}}\,. \tag{25}$$

The thresholding procedure detects asymptotically small bins with jumps. The exact position of the unknown jump time within the detected bin remains unknown, however.

To account for the bias problem in price-jump estimation, emphasized in Section 3.1, one can adjust the spectral estimator (12) by cutting out detected jump bins. For the following discussion, we focus on a single unknown price-jump time τ. As illustrated in the upper part of Fig. 2, the price before and after the jump then could be estimated based on observations on two neighboring time windows to the left and right of the jump bin. A similar construction is used in Section 10.5 of Aït-Sahalia and Jacod (2014) to locate volatility-jump times. However, the bin-widths determined by thresholding decay with order h_n and are of the same size as the bins on which the price-jump estimation is conducted. For efficient price-jump estimation, we therefore refine the localization of price-jump times to smaller intervals. The lower part of Fig. 2 illustrates this.

To more precisely locate the jump time $\hat{\tau} \in ((k-1)h_n, kh_n)$, on a bin with $\Delta_k\widehat{[X,X]} > a^2 \vee u_n$, we partition this jump bin into R_n sub-intervals of lengths $(r_n+l_n)/n$ with (r_n+l_n) an even integer. The jump window $(t^n_{l-l_n}, t^n_{l+r_n})$, with $l = \lfloor \tau n\rfloor + 1$, includes the price jump. r_n determines the number of observations to the right of the price jump up to the end of the jump window, l_n or l_n+1 is the number of observations to the left of the price jump down to the beginning of the jump window. For the price-jump estimation, we then cut out this jump window that contains τ. We identify the jump window by comparing R_n increments of pre-averages

$$i = \operatorname{argmax}_{i=1,\ldots,R_n}\left|T^{LM}\Big((k-1)h_n + (i-1/2)\frac{r_n+l_n}{n};\,\Delta^n_1 Y,\ldots,\Delta^n_n Y\Big)\right|\,, \tag{26}$$

with the statistics from (7), averaging over $(r_n+l_n)/2 \ll \sqrt{n}$ instead of M_n observations. We exploit the simple asymptotics of the argmax of pre-averages to detect the sub-interval with the jump.[6] The price-jump size is estimated with spectral statistics, computed from observations on a window of length h_n around the cut out sub-interval with the jump. That is,

6 Note that the asymptotics of maximum statistics of spectral estimators are much more intricate. We therefore mix spectral and pre-average detection techniques for the refined detection of jumps.

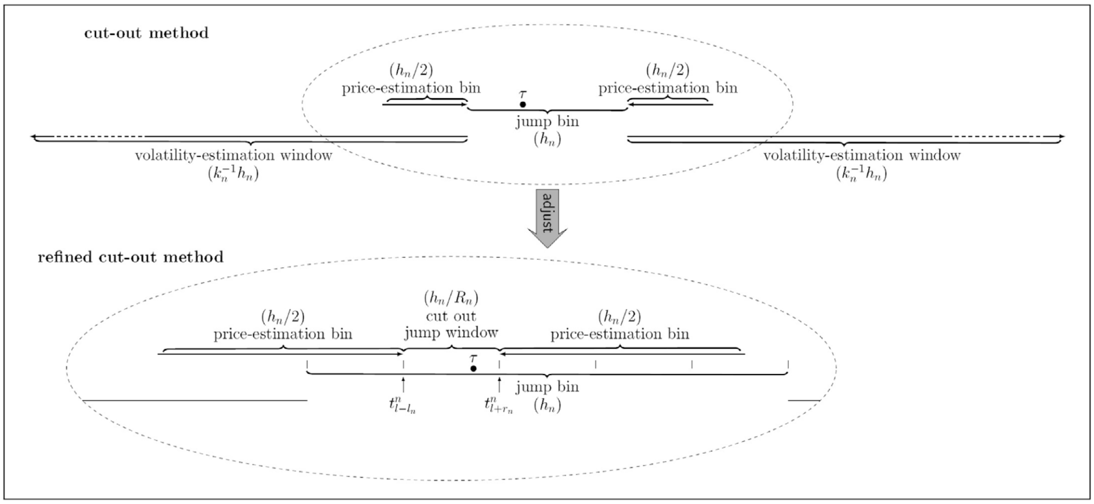

Fig. 2. Illustration of the cut-out method and the three tuning parameters. On the top the bin with a located (large) price jump is cut out (Example 1). The proposed refined cut-out procedure with $R_n = 5$ is illustrated below. Interval lengths are given in parentheses.

given $t^n_{l+r_n}$ and $t^n_{l-l_n}$, we use (12) with the basis (10) centered around $Y_{t^n_{l+r_n}} - Y_{t^n_{l-l_n}}$ and with returns $\Delta^n_i Y$ in an interval $[t^n_{l-l_n} - h_n/2, t^n_{l-l_n}]$ to the left of the jump window and $[t^n_{l+r_n}, t^n_{l+r_n} + h_n/2]$ to the right of the jump window. This is the same as deleting observations $Y_{t^n_i}$ on $(t^n_{l-l_n}, t^n_{l+r_n})$ and shifting observations $Y_{t^n_i}$ from the left and right toward the center. This construction of the refined cut-out procedure is illustrated in Fig. 2.

Proposition 3.5. *When $R_n \to \infty$, with $R_n = \mathcal{O}(\sqrt{n})$, such that $(r_n + l_n) \propto n^\delta \to \infty$, for some $0 < \delta < 1/2$, the adjusted price-jump estimation with the cut-out window determined by*

$$t^n_{l-l_n} = (k-1)h_n + (i-1)\frac{r_n + l_n}{n}, \quad t^n_{l+r_n} = (k-1)h_n + i\,\frac{r_n + l_n}{n}\,, \tag{27}$$

with i defined in (26), satisfies under Assumptions 1, 2 and 3, for a jump time τ when $r = 0$ or with $|\Delta X_\tau| > a$ for some $a > 0$ in that the Lévy measure μ does not have an atom, the central limit theorem in Proposition 3.3.

While $(r_n + l_n)$ is set by the econometrician, the two summands l_n and r_n are unknown and depend on the true value of τ. The refinement of the cut-out method uses one additional tuning parameter, R_n, or equivalently $(r_n + l_n)$, which fixes the lengths of the sub-intervals. Proposition 3.5 establishes asymptotically efficient (large) price-jump estimation under noise, even if the jump time τ is unknown.

Analogously, the Lee–Mykland statistic (7) can be adjusted for the unknown time point, τ, in the jump window $(t^n_{l-l_n}, t^n_{l+r_n})$. We estimate the price to the left of the jump window with $\hat{P}(t^n_{l-l_n-M_n})$ and to the right of the jump window with $\hat{P}(t^n_{l+r_n})$. This adjustment is also robust in the sense that Proposition 3.1 remains valid.

Consider the two illustrative extreme examples in determining jump windows:

Example 1. $R_n = 1$ implies cutting out the whole bin with $t^n_{l-l_n} = (k-1)h_n$ and $t^n_{l+r_n} = kh_n$. This is the cut-out method without refinement as sketched in the upper part of Fig. 2. We can show that the price-jump estimator (12) is consistent and preserves (almost) the optimal convergence rate in this case. However, the constant in the variance in (15) increases when the jump window is of order $n^{-1/2}$.

Example 2. $R_n = nh_n - 1$, when $\hat{\tau}_k = \text{argmax}_i \{t^n_i \in [(k-1)h_n, kh_n) | \,|\Delta^n_i Y|\}$, implies centering (12) around the largest absolute return on a bin. Since the noise is centered and its variance, $\eta^2_{\tau_k}$, typically is rather small (see Hansen and Lunde (2006)), the time of the largest absolute return might be considered a good candidate for the jump arrival and the method would require one fewer tuning parameter. In particular, if one restricts attention to jumps much larger than η_{τ_k}, the method could also perform well in practice. Theoretically, however, centering the jump window around the largest absolute return is only suitable if one assumes that $\eta_{\tau_k} \to 0$ when $n \to \infty$.

3.4. Discontinuous leverage effect

This section establishes inference on a covariation measure for contemporaneous price and volatility jumps. Aït-Sahalia et al. (2017) introduced it as the *tail discontinuous leverage effect* in their equation (2.7):

$$[X,\sigma^2]^d_T(a)=\sum_{s\le T}\Delta X_s\big(\sigma_s^2-\sigma_{s-}^2\big)\,\mathbb{1}_{\{|\Delta X_s|>a\}}\,. \tag{28}$$

Combining the above jump-localization procedure with the spectral jump and volatility estimators and setting $T=1$, we consider the DLE estimator

$$\widehat{[X,\sigma^2]}^d_1(a)=\sum_{k=2}^{h_n^{-1}-1}\widehat{\Delta X}_{\hat\tau_k}\big(\hat\sigma^2_{\hat\tau_k}-\hat\sigma^2_{\hat\tau_k-}\big)\,\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}\,, \tag{29}$$

where $\hat\tau_k$ are the estimated price-jump times, $\widehat{\Delta X}_{\hat\tau_k}$ are the estimated log-price jumps and $\hat\sigma^2_{\hat\tau_k-}$ and $\hat\sigma^2_{\hat\tau_k}$ are the spot-volatility estimates. Only finitely many addends with (large) price jumps in (29) are non-zero. The corresponding bins are determined by thresholding. If the time of one (large) price jump τ_k is ascribed to a bin, we can directly estimate the associated volatility jump based on $\hat\sigma^2_{\tau_k-}$ from (21) and $\hat\sigma^2_{\tau_k}$. In contrast, the price-jump estimation relies on (12) with the refined cut-out method and Proposition 3.5.

Aït-Sahalia et al. (2017) point out that a central limit theorem for the DLE in the presence of market microstructure noise cannot generally be obtained with pre-averaging or related approaches. However, by focusing either on the *tail* DLE, with some $a>0$ or assuming $r=0$ in Assumption 2, we derive the following asymptotic result:

Proposition 3.6. *Under Assumptions 1, 2 and 3, for any $a>0$ in that the Lévy measure μ does not have an atom, the estimator for the DLE (29) satisfies the feasible (self-scaling) central limit theorem*

$$n^{\beta/2}\frac{\big(\widehat{[X,\sigma^2]}^d_1(a)-[X,\sigma^2]^d_1(a)\big)}{\Big(\sum_{k=2}^{h_n^{-1}-1}\big(\widehat{\Delta X}_{\hat\tau_k}\big)^2\,8\hat\eta_{\hat\tau_k}\big(\hat\sigma^3_{\hat\tau_k}+\hat\sigma^3_{\hat\tau_k-}\big)\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}\Big)^{1/2}}\xrightarrow{(d)}N(0,1)\,, \tag{30}$$

with β as in (23). If no price jump is detected, we set the estimate equal to zero. In particular, the limit theorem facilitates, for some $\alpha\in(0,1)$, an asymptotic level α test with asymptotic power 1 for testing the hypothesis $\tilde H_0:[X,\sigma^2]^d_1(a)=0$, against the alternative $\tilde H_1:[X,\sigma^2]^d_1(a)\neq0$:

$$\varphi=\mathbb{1}_{\Big\{|n^{\beta/2}\widehat{[X,\sigma^2]}^d_1(a)|>q_{1-\alpha/2}\sqrt{\sum_{k=2}^{h_n^{-1}-1}(\widehat{\Delta X}_{\hat\tau_k})^2\,8\hat\eta_{\hat\tau_k}(\hat\sigma^3_{\hat\tau_k}+\hat\sigma^3_{\hat\tau_k-})\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}}\Big\}}\,, \tag{31}$$

where $q_{1-\alpha/2}$ denotes the $(1-\alpha/2)$ quantile of the standard normal law.

One loses no generality by imposing the scaling $T=1$; any fixed $T\in\mathbb{R}_+$ can be considered. The condition that the Lévy measure μ does not have an atom in a is analogous to (10.76) in Aït-Sahalia and Jacod (2014). There are only atoms in at most countably many values. According to Aït-Sahalia and Jacod (2014), the condition holds for any $a>0$ if μ has a density. This applies to all models used in finance with infinite jump activity.

Proposition 3.7. *Under Assumptions 1, 2 and 3 and under the specific case of finite jump activity, $r=0$ in Assumption 2, the estimator for the DLE,*

$$\widehat{[X,\sigma^2]}^d_1=\sum_{k=2}^{h_n^{-1}-1}\widehat{\Delta X}_{\hat\tau_k}\big(\hat\sigma^2_{\hat\tau_k}-\hat\sigma^2_{\hat\tau_k-}\big)\,\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>u_n\}},$$

together with β as in (23) and $\varpi<\frac{1+\delta/2-1/4}{2+\delta/2}$, satisfies the feasible central limit theorem,

$$n^{\beta/2}\frac{\big(\widehat{[X,\sigma^2]}^d_1-[X,\sigma^2]^d_1\big)}{\Big(\sum_{k=2}^{h_n^{-1}-1}\big(\widehat{\Delta X}_{\hat\tau_k}\big)^2\,8\hat\eta_{\hat\tau_k}\big(\hat\sigma^3_{\hat\tau_k}+\hat\sigma^3_{\hat\tau_k-}\big)\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>u_n\}}\Big)^{1/2}}\xrightarrow{(d)}N(0,1)\,. \tag{32}$$

The upper bound on ϖ relates to Assumption 3 and the existence of higher moments of ϵ_t. If all moments of the noise exist, the bound imposes no condition on the truncation. For $\delta\to0$ in Assumption 3, $\varpi<3/8$ leads to more conservative thresholds. Since $r=0$ in (23), we also derive the optimal rate in this case. Although we conjecture that this upper bound on ϖ is not needed, it simplifies the proof considerably.

Proposition 3.6 follows from combining our results on jump localization, the estimation of price jumps at detected jump times and Corollary 3.4 about volatility-jump estimation. However, the proof cannot be extended in a similar way to the case $r\neq0$ and $a=0$ when considering infinitely many small price jumps. It is unknown if an asymptotic distribution theory is possible in this general case. Propositions 3.6 and 3.7 give us exactly the statistics we require for our data study, however.

Remark 3. Propositions 3.6 and 3.7 indicate that, in the asymptotic results of the estimated DLE, the estimation error for the volatility jumps dominates the error for the price jumps. Consequently, the length of the jump window in Proposition 3.5 for price-jump estimation has asymptotically no effect on DLE estimation. Nevertheless, choosing $R_n > 1$ is of interest from an applied point of view. Removing jump windows has a locally similar effect as downsampling the data to a lower observation frequency. Given the discussion by Christensen et al. (2014) about spurious jump detection via downsampling, one would like to avoid deleting large jump windows in the empirical application. The refined method is superior to cutting out larger windows in that it poses less risk of estimating spuriously large jumps.

The leverage effect is also often defined as a correlation, rather than as the covariation (28). To gain further insights across individual firms in the empirical Section 5, we follow Jacod et al. (2017) and consider a scaled measure of the DLE:

$$\frac{[X,\sigma^2]_T^d(a)}{\sqrt{[X,X]_T^d(a)[\sigma^2,\sigma^2]_T^d(a)}} = \frac{\sum_{s\le T}\Delta X_s\Delta\sigma_s^2\mathbb{1}_{\{|\Delta X_s|>a\}}}{\sqrt{\sum_{s\le T}(\Delta X_s)^2\mathbb{1}_{\{|\Delta X_s|>a\}}}\sqrt{\sum_{s\le T}(\Delta\sigma_s^2)^2\mathbb{1}_{\{|\Delta X_s|>a\}}}}\,, \tag{33}$$

that is, the correlation between contemporaneous price and volatility jumps. We may use $a = 0$ in case of finite activity jumps, $r = 0$ in Assumption 2. Note that (33) is a path-wise defined, integrated measure. (33) is a scalar parameter only under the restriction to time-homogeneous jump measures. Using Propositions 3.6 and 3.7, and setting $T = 1$, we obtain the following result:

Corollary 3.8. *Under all conditions from Proposition 3.6 and with*

$$\widehat{[\sigma^2,\sigma^2]}_1^d(a) = \sum_{k=2}^{h_n^{-1}-1}\left(\hat\sigma^2_{\hat\tau_k}-\hat\sigma^2_{\hat\tau_k-}\right)^2\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}\,, \tag{34a}$$

$$\widehat{[X,X]}_1^d(a) = \sum_{k=2}^{h_n^{-1}-1}\left(\widehat{\Delta X}_{\hat\tau_k}\right)^2\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}\,, \tag{34b}$$

we derive a consistent estimator of (33) *with*

$$\frac{\widehat{[X,\sigma^2]}_1^d(a)}{\sqrt{\widehat{[X,X]}_1^d(a)\widehat{[\sigma^2,\sigma^2]}_1^d(a)}} - \frac{[X,\sigma^2]_1^d(a)}{\sqrt{[X,X]_1^d(a)[\sigma^2,\sigma^2]_1^d(a)}} = \mathcal{O}_{\tilde{\mathbb{P}}}(n^{-\beta/2}),$$

with β as in (23)*. Analogously, in the setup of Proposition 3.7, we obtain the same result for $a = 0$.*

Remark 4. In the general model (1), the covariation (28) and its normalized correlation (33) appear to be the natural quantities to measure discontinuous leverage. This is in line with previous works. For a continuous leverage effect, Kalnina and Xiu (2017) point in their equation (4) at another different (integrated) spot correlation measure. As explained by the authors, this quantity is only well-defined for purely continuous semimartingales and hence not feasible in our setup. Kalnina and Xiu (2017) propose a second correlation measure in their equation (5), which corresponds to (33). Aït-Sahalia et al. (2017) discuss difficulties in estimating a correlation like (33) for the continuous leverage effect. A source of the problem, identified in their Section 7.5, is a bias in the estimation of the volatility of volatility. In contrast to a correlation based on the continuous leverage effect, the corresponding correlation for the DLE (33) does not suffer from such problems. In particular, the consistency of the variation of the finitely many volatility jumps in (34a) follows from Corollary 3.4.

4. Simulations

This section reports the results of simulation studies of the finite-sample properties of the price-jump estimators, the corresponding price-jump tests and the discontinuous leverage statistics. The simulation study in Bibinger and Winkelmann (2018) evaluates the finite-sample inference on volatility jumps.

This simulation study emulates that of Lee and Mykland (2012). Although their theory only applies to the jump–diffusion setup, they simulate a more complex and realistic model, including stochastic volatility and time-varying noise. The efficient price follows

$$X_t = 1 + \int_0^t \sigma_s\,dW_s\,,\, t\in[0,1], \tag{35}$$

with Heston-type stochastic volatility,

$$d\sigma_s^2 = 0.0162\left(0.8465-\sigma_s^2\right)ds + 0.117\,\sigma_s\,dB_s\,, \tag{36}$$

where B and W are two independent standard Brownian motions. We adopt the parameter values of Lee and Mykland (2012) in (36) and assume 252 trading days per year and 6.5 trading hours a day. The model for the market microstructure noise is

$$\epsilon_{t_i^n} = 0.0861\Delta_i^n X + 0.06\left(\Delta_i^n X + \Delta_{i-1}^n X\right)U_i\,,\ i = 0,\ldots,n, \tag{37}$$

Table 1
Comparison of size and power of the two tests.

Frequency (n)	$\Delta X_\tau = 0$		$\Delta X_\tau = q$		$\Delta X_\tau = 2q$		$\Delta X_\tau = 3q$	
Test	LM	BNW	LM	BNW	LM	BNW	LM	BNW
Moderate noise case, $q = 0.0005$								
3 sec (1200)	0.049 (0.034)*	0.045	0.199 (0.059)*	0.274	0.473 (0.320)*	0.677	0.777 (0.786)*	0.924
2 sec (1800)	0.050 (0.030)*	0.053	0.280 (0.071)*	0.382	0.695 (0.483)*	0.828	0.937 (0.920)*	0.988
1 sec (3600)	0.049 (0.046)*	0.056	0.281 (0.091)*	0.594	0.697 (0.709)*	0.982	0.950 (0.988)*	1
Large noise case, $q = 0.005$								
3 sec (1200)	0.052 (0.046)*	0.049	0.296 (0.275)*	0.996	0.803 (0.889)*	1	0.997 (0.997)*	1
2 sec (1800)	0.053 (0.046)*	0.052	0.465 (0.593)*	0.999	0.937 (0.998)*	1	0.988 (1)*	1
1 sec (3600)	0.050 (0.041)*	0.049	0.829 (0.918)*	1	0.994 (1)*	1	0.997 (1)*	1

The table lists the simulated values of standardized test statistics (7) and (12), from 6000 iterations for each configuration, exceeding the 0.05-quantile of the standard normal. "LM" marks the Lee–Mykland test and "BNW" our proposed spectral test. We simulated from the model given by (35), (36) and (37). In parentheses ( )*, we report the values from Table 4 in Lee and Mykland (2012) of their analogous simulation study. Following Table 5 in Lee and Mykland (2012), we used constants $c = 1/19$ for $q = 0.0005$ and $c = 1/9$ for $q = 0.005$ to determine M_n in (7) (for $\Delta X_\tau = 2q, 3q$ and $q = 0.005$, we doubled M_n, which increased the power). For (12), we used $h_n = \kappa \log(n)/\sqrt{n}$ with $\kappa \approx 5/12$ for $q = .0005$ and $\kappa \approx 2/3$ for $q = .005$.

with $(U_i)_{0\le i\le n}$ being a sequence of normally distributed random variables with mean 0 and variance q^2. We consider two parameterizations of q, which governs the noise level (market quality parameter). The cross-correlation between X_t and noise violates one of our theoretical assumptions, but we expect no degradation in the performance of our approach. We estimate q in the presence of serial correlation with the noise estimator suggested in Proposition 1 of Lee and Mykland (2012).

We implement the self-scaling adaptive version of (12) with pre-estimated optimal weights. The notes to Table 1 give values of h_n. The pre-averaging for the Lee–Mykland statistics (7) refers to a block-size, $M_n = c\sqrt{n/k}$, where k denotes the order of serial correlation in the simulated noise. The constant c is chosen according to Table 5 of Lee and Mykland (2012).

Evaluation of the pre-average and spectral tests to infer price jumps

Lee and Mykland (2012) compare the performance of the noise–robust local jump tests in Lee and Mykland (2012) to those in Lee and Mykland (2008), which are not designed to be robust to noise. We replicate this simulation study and compare the finite-sample performances of the statistics defined in (7) and (12). Considering the power of the tests associated with Proposition 3.1 (Lee–Mykland) and Proposition 3.2 (our spectral method) allows us to compare our results to those in Table 4 of Lee and Mykland (2012). We generate realizations of $Y_i = X_{t_i^n} + \epsilon_{t_i^n}$ for one trading hour using time resolutions of 1, 2 and 3 s, respectively ($n = 3600, 1800, 1200$). The jump size in τ is related to the noise level q, i.e., $\Delta X_\tau = 0$ under the hypothesis and $\Delta X_\tau = q, 2q, 3q$ under the alternative.

Table 1 shows the simulation results, along with the values reported by Lee and Mykland (2012) in parentheses. Most of our results for the Lee–Mykland test closely track those reported by Lee and Mykland (2012). Our results for the power under moderate noise and smaller jumps are a bit better than expected from Lee and Mykland (2012), while some results in the large noise case are smaller. In the large noise case, we report values where M_n is doubled compared to the constant adopted from Lee and Mykland (2012), which led to higher power. The windows used for the spectral method are much larger than the values M_n/n for the Lee–Mykland statistics.

At first glance it might seem surprising that the power in Table 1 increases for larger noise. This is not, however, because of large noise that makes precise testing and estimation more difficult, but because the jump sizes increase along with q. Large jumps naturally produce better testing results. The size of both tests on the hypothesis appears to be accurate. The new spectral test (10) attains considerably better power in all cases.

Evaluation of the pre-average and spectral estimators for price-jump sizes

In the same setup, we compare the performance of the jump-size estimators. Table 2 confirms that our spectral estimator attains a smaller root mean square error (RMSE) than the Lee–Mykland estimator, in all configurations, with the same optimal parameter choice as above. Efficiency gains are most relevant for the configuration with moderate noise and the smallest jump size. In this setup, our new estimator has a RMSE that is almost 50% smaller for $n = 3600$. For large noise and jump size q, our new estimator reduces the RMSE by 20%. These significant improvements of estimation accuracy are particularly relevant because the moderate noise setting is realistic for current high-frequency data.

Table 2
Comparison of RMSEs for the two price-jump size estimators.

Frequency (n)	$\Delta X_\tau = q$		$\Delta X_\tau = 2q$		$\Delta X_\tau = 3q$	
Estimator	LM	BNW	LM	BNW	LM	BNW
Moderate noise case, $q = 0.0005$						
3 sec (1200)	11.0	9.9	11.1	10.2	11.9	10.8
2 sec (1800)	6.8	5.3	6.9	6.0	7.9	6.8
1 sec (3600)	4.7	2.6	4.8	3.6	6.3	4.7
Large noise case, $q = 0.005$						
3 sec (1200)	14.8	14.4	15.0	14.5	15.2	14.5
2 sec (1800)	10.0	9.4	10.2	9.5	10.6	9.5
1 sec (3600)	5.6	4.5	5.9	4.6	6.4	4.6

The table lists the root mean square errors, multiplied by 10^4, of the estimators (7) and (12), from 6000 iterations for each configuration under the alternative when price jumps are present. "LM" marks the Lee–Mykland estimator and "BNW" our proposed estimator. We simulated from the model given by (35), (36) and (37). Tuning parameters are reported in Table 1.

Fig. 3 demonstrates the finite-sample accuracy of the normal limit laws in (9) and (15). The empirical distributions closely approximate their normal asymptotic limit.

Evaluation of the discontinuous leverage estimator

We modify the simulation setup by adding one jump at a random time to the volatility in (36). The volatility-jump size is set to the median value from the empirical sample described in Table 3. To create discontinuous leverage, we implement a contemporaneous downward price jump of 0.2%, which is comparable to the sizes in Fig. 5. Using a rather large price jump and average volatility-jump size allows us to study the finite-sample accuracy of the result (30). We can analyze the DLE estimator (29) because thresholding reliably detects jumps of this size. We simulate one trading day with observation frequencies of 1, 2 and 3 s, frequencies that generate 23400, 11700 and 7800 observations, respectively, over the day.

We analyze the performance of the DLE in a model with moderate microstructure noise. We first estimate spectral statistics over a partition of the whole day, identifying price jumps by thresholding. Next, we estimate the squared volatility before and after the jump by local averages of the bin-wise, parametric estimates over 8 bins. Then, we estimate the local jump size using (12) and implement the refinement from Section 3.3 for unknown jump times. We partition the bin with the detected jump in $R = 6$ equidistant sub-intervals and apply the adjusted jump size estimation using (26). The window sizes for the first step and the price-jump estimation are equal: we use $h^{-1} = 100$ for the 1-second frequency and $h^{-1} = 50$ for the two smaller frequencies. The spectral cut-off frequency is set to $J = 30$ in all cases. Estimates are reasonably robust to different values of h and J.

For the fixed, true value -2.324 of the DLE (28),[7] we obtain the following results:

Frequency	1 s	2 s	3 s
Bias	-0.04	-0.02	-0.03
Variance	0.16	0.19	0.21

The inherently slow convergence rate of the estimation leads to pronounced finite sample variances. Fig. 4 shows QQ-normal plots for the test statistic obeying the central limit theorem (30). The normal distribution fits reasonably well for all frequencies. Our test for the DLE attains very high power (approximately 99%) in the case of one observation per second and only slightly smaller power for the lower observation frequencies. Overall, simulations indicate that the DLE estimation performs well in this complex environment.

5. The discontinuous leverage effect in stock prices

This section presents results of applying the spectral methods of Section 3 to stock price data. We first introduce the dataset and discuss how to estimate price and volatility jumps on these data. Finally, we investigate the DLE, i.e., a covariation measure, and the associated correlations of price and volatility cojumps.

5.1. Price and volatility cojumps

We use NASDAQ order book data from the LOBSTER database. Initially, we pick the 30 stocks with the largest market capitalizations from each of the 12 NASDAQ industries for a total of $12 * 30 = 360$ stocks.[8] The sample spans January 1, 2010 to December 31, 2015, 1509 days with trading from 9:30 to 16:00 EST. The tick-by-tick data shows evidence of market

7 We rescale all DLE values by multiplying by 10^7.

8 The industries can be found on www.nasdaq.com/screening/industries.aspx. The year 2013 serves as the baseline year.

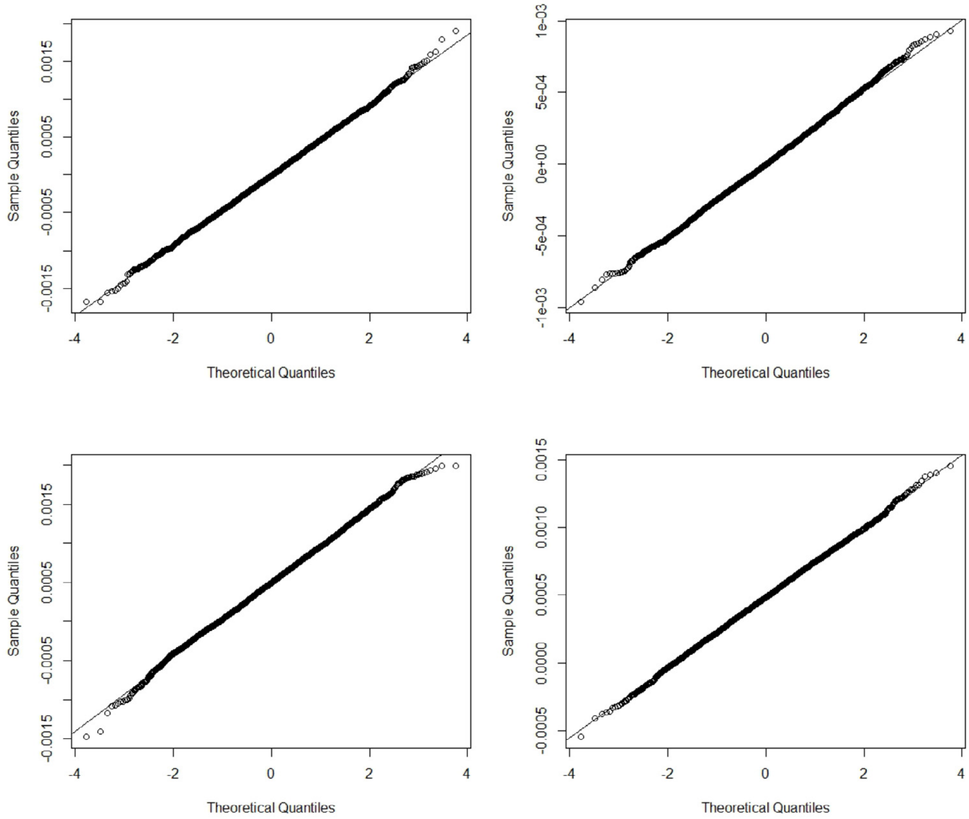

Fig. 3. QQ-normal plots for the Lee–Mykland statistic (left) and the spectral, jump statistic (12) (right). The top panels depict the 6000 iterations when $\Delta X_\tau = 0$. The bottom panels show results for the iterations when $\Delta X_\tau = q = 0.0005$.

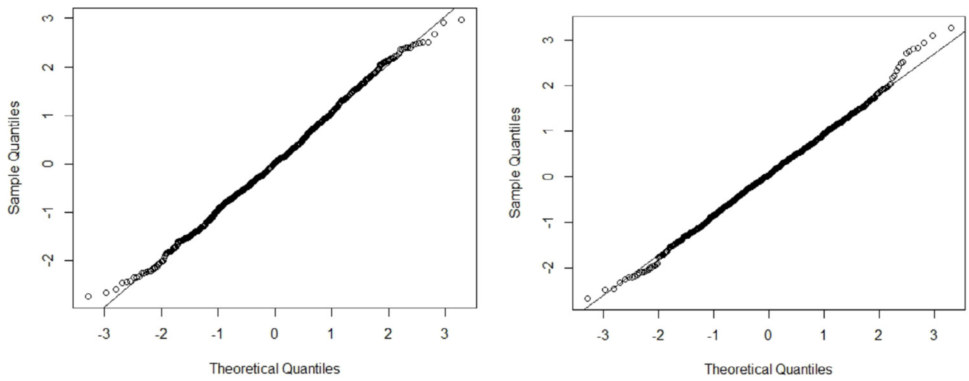

Fig. 4. QQ-normal plots for the DLE (30), with 1-second (left) and 3-second (right) observation frequencies.

microstructure noise, such as significant negative first-order autocorrelation. The test of Aït-Sahalia and Xiu (2017), equation (40), displays significant noise for 50% of all stocks, across all trading days.[9] As shown in the simulations of Winkelmann et al. (2016), spectral estimators perform particularly well with liquid stocks, i.e., those having at least about one trade every 15 s. To restrict the analysis to very liquid stocks, we exclude trading days with fewer than 1500 trades for a given stock.[10] This

9 Aït-Sahalia and Xiu (2017) report similar percentages for the S&P100 in their Table 4. To control the overall significance level of tests across firms and trading days, we use the (Benjamini and Hochberg, 1995) step-up procedure at level $\alpha = 0.1$. In case of no market microstructure noise our methods remain valid.

10 Results are robust to higher (2000) and lower (1000) thresholds.

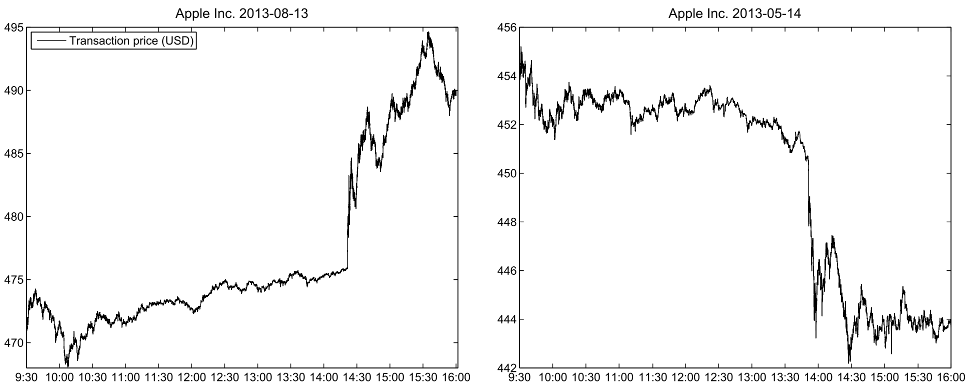

Fig. 5. Price process at the NASDAQ stock exchange of Apple Inc. on two different days with price–volatility cojumps. Number of trades: 87,445 (left), 40,707 (right).

Table 3
Price and volatility cojumps: NASDAQ order book, 2010–2015.

Conditioning criteria	# of cojumps	Price-jump size			Volatility-jump size		
		$Q_{0.25}$	$Q_{0.5}$	$Q_{0.75}$	$Q_{0.25}$	$Q_{0.5}$	$Q_{0.75}$
Panel A: Apple Inc. stock							
All jumps	209	−0.095	−0.033	0.048	43.8	88.6	193.4
Positive price jumps							
· All	83	0.029	0.070	0.152	−28.4	84.8	180.3
· Market	20	0.073	0.152	0.266	−28.2	124.7	266.0
· Idiosyncratic	63	0.028	0.061	0.119	−28.4	80.6	153.3
Negative price jumps							
· All	126	−0.137	−0.084	−0.046	51.4	89.6	216.9
· Market	19	−0.172	−0.110	−0.050	86.2	316.5	504.9
· Idiosyncratic	107	−0.130	−0.084	−0.046	49.4	82.5	174.4
Panel B: Mean across all stocks							
All jumps	73.8	−0.115	0.014	0.152	39.2	137.3	299.1
Positive price jumps							
· All	38.5	0.108	0.175	0.286	−23.3	114.4	290.7
· Market	9.9	0.139	0.206	0.327	154.2	269.5	361.9
· Idiosyncratic	28.6	0.101	0.164	0.266	−21.8	108.9	230.1
Negative price jumps							
· All	35.4	−0.254	−0.160	−0.101	76.7	145.6	315.0
· Market	7.6	−0.283	−0.194	−0.141	168.8	276.8	444.6
· Idiosyncratic	27.9	−0.242	−0.154	−0.098	74.7	119.5	304.2

Quantiles (Q) of the jump distributions are in percent. Market jumps refer to days with jumps in the NASDAQ composite index. Idiosyncratic jumps refer to days without jumps in the NASDAQ composite index.

selection procedure reduces the number of firms to 320. We focus on transactions with non-zero returns but do not adjust the data further; that is, we do not clean or synchronize trades. The number of observed trades varies substantially across stocks and days. There is a maximum of 227,139 intradaily observations for the Apple Inc. stock on September 9, 2014; the median number of daily transactions across stocks is much smaller, only 5977.

The local jump detection and estimation take the time-varying trading activity into account. We partition each trading day d into $h^{-1,(d,s)}$ bins for every stock s. As suggested by our theoretical results, the number of bins $k = 1, \ldots, h^{-1,(d,s)}$ grows with the number of trades $n^{(d,s)}$ with $h^{-1,(d,s)} = \lfloor 3\sqrt{n^{(d,s)}}\log(n^{(d,s)})^{-1}\rfloor$. We detect price jumps by applying the adaptive threshold, $u_k^{(d,s)} = 2\log(h^{-1,(d,s)})/h^{-1,(d,s)}\hat{\sigma}^{2,(d,s)}_{k,pil}$, to bin-wise quadratic variation estimates (25). It is well-known that the number of detected price jumps depends on the thresholding procedure in that a lower threshold usually increases the number of detected small price jumps.[11] We find that relatively small volatility changes at price-jump time points strongly influence the DLE estimates. For that reason, we apply the test for volatility jumps, as proposed by Bibinger and Winkelmann (2018), to focus on price jumps with significant contemporaneous volatility jumps.[12] The tests for volatility jumps reduce the influence of the price-jump-detection threshold on the DLE estimates. For price-jump estimation, we partition jump bins

[11] The main results about the DLE are robust against different threshold choices. As a robustness check, we substitute the $\log(h^{-1,(d,s)})$ term of the threshold with $\log(\log(h^{-1,(d,s)}))$, which increases the number of price-jump days per stock from around 14% to 29%.

[12] Note that in (29), summands without volatility jumps "automatically" cancel out because $\sigma_\tau^2 = \sigma_{\tau-}^2$. To control the overall significance level of tests across firms and price jumps at level $\alpha = 0.1$, we use the (Benjamini and Hochberg, 1995) step-up procedure (the false discovery rate).

Table 4
The discontinuous leverage across NASDAQ firms.

Row	Conditioning criteria	Rejection rate	DLE quantiles			Correlation quantiles		
			$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$	$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$
1	All jumps	0.10	−0.37	0.17	1.01	−0.12	0.01	0.15
	Positive price jumps							
2	· All	0.65	1.14	2.08	3.87	0.11	0.28	0.44
3	· Market	0.89	1.64	2.88	4.48	0.12	0.32	0.55
4	· Idiosyncratic	0.66	0.84	1.63	3.15	0.09	0.26	0.47
	Negative price jumps							
5	· All	0.63	−3.72	−1.75	−0.88	−0.47	−0.29	−0.09
6	· Market	0.85	−3.75	−2.02	−1.03	−0.62	−0.31	−0.15
7	· Idiosyncratic	0.67	−3.53	−1.69	−0.92	−0.52	−0.32	−0.14

The rejection rate indicates the percentage of firms having a significant DLE. We control the overall significance at level $\alpha = 0.1$ with the step-up procedure of Benjamini and Hochberg (1995). DLE quantiles refer to a firm's average DLE, rescaled by $\times 10^7$. The empirical quantiles contain all DLE and correlation estimates across firms.

into $R = 6$ sub-intervals and center (12) around the cut-out return obtained via (26).[13] The number of frequencies studied on each bin is $J^{(d,s)} = 5\log(n^{(d,s)})$. We average the truncated spectral statistics over $\lceil 3\sqrt[4]{n^{(d,s)}}/\log(n^{(d,s)})\rceil$ bins to estimate spot volatility to the right and left of the detected price jump.

Fig. 5 shows two examples of price–volatility cojumps of the Apple Inc. stock, an upward price jump in the left panel and a downward price jump in the right panel. The estimates of the price jumps are 0.27% and -0.24%, respectively. Note that if one would approximate the price-jump sizes just by looking at Fig. 5 and assuming a small noise level, one may expect much larger price-jump estimates. Christensen et al. (2014) and Barndorff-Nielsen et al. (2009) explain that seemingly large returns often consist of smaller, unidirectional returns on a short time interval.[14] This explains how downsampling to lower observation frequencies can affect both jump detection and the estimation of price-jump sizes. Fig. 5 also suggests that volatility jumped contemporaneously with the price jump. That is, the variability of the stock price appears in both cases much smaller before the price jump than afterwards. This apparent jump in volatility is not directly determined by the price jump mechanically feeding through to higher volatility. Indeed estimated changes in volatility only use log-price information from bins that neighbor the price-jump bin. The increase in spot volatility evaluated approximately 30 min before and after the price jumps is 184% (left panel of Fig. 5) and 163% (right panel of Fig. 5). Note that the strong upward jumps in both the price and volatility processes, in the left panel of Fig. 5, is not consistent with the negative price–volatility cojump correlation in high-frequency data that Bandi and Renò (2016) report for S&P 500 futures.

To get deeper insights about price and volatility cojumps, Table 3 shows summary statistics for detected cojumps and quantiles of the respective jump distributions. Panel A of Table 3 shows summary statistics for the Apple Inc. stock; Panel B displays averages across the 320 stocks. We condition results on the sign of price jumps and whether they are market jumps or idiosyncratic. Following Li et al. (2017), we use a market index to measure market jumps. The market proxy is the NASDAQ Composite Index, which is the market capitalization-weighted index of about 3000 equities listed on the NASDAQ stock exchange. The price-jump detection method proposed in Section 3.3 finds these market jumps on 8% of the days in the sample. We define idiosyncratic jumps as discontinuities where the index displays no contemporaneous market jump.

The top row of Panel A of Table 3 shows that the Apple Inc. stock price displays 209 contemporaneous price–volatility cojumps, with more downward price jumps (126 or 60%) than upward price jumps (83 or 40%). Panel B of Table 3 shows that the average number of price–volatility cojumps in the six-year sample across all individual stocks is 73.8. Columns three to five and six to eight of Table 3 show the quantiles of the price-jump and volatility-jump distributions. They indicate that idiosyncratic jumps tend to be smaller than market jumps. The magnitude of price jumps is in line with the sizes of −0.15 to 0.18% reported by Lee and Mykland (2012) for the IBM stock in 2007.

The magnitude of volatility jumps is striking. The 0.75 empirical quantile of the volatility-jump distribution of the Apple Inc. stock for negative market price jumps is about 505%. That is, volatility frequently jumps to more than five times its pre-jump size when prices jump down. The analogous 0.75 quantile for volatility jumps, conditional on a negative market price jump, averaged across all firms is 445%. Scheduled news announcements are known to reduce trading and volatility right before the announcement but portend a strong response afterwards, which is manifested in large volatility jumps. The rows labeled "market", in Panel B of Table 3, show that volatility jumps are usually positive for both positive and negative price jumps. Overall, the volatility-jump distribution is right-skewed, indicating the important role of upward jumps in volatility.

5.2. The discontinuous leverage effect

This subsection characterizes the DLE of contemporaneous price and volatility jumps. Fig. 6 illustrates a typical relation between price jumps and contemporaneous volatility jumps using data from Apple Inc. from 2010 to 2015. Following

[13] Note that centering the jump estimator around the largest absolute return on a detected bin, as described in Example 2, does not change the main conclusions about the DLE. However, individual estimates of price-jump sizes can differ quite substantially.

[14] While (Christensen et al., 2014) attribute a local drift to such phenomena, Barndorff-Nielsen et al. (2009) explain this characteristic by the microstructure of the orderbook and call it "gradual jumps".

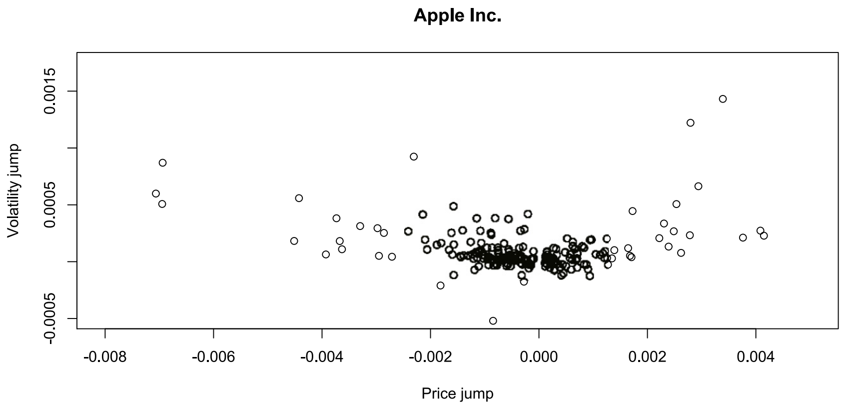

Fig. 6. Scatter plot of price–volatility cojumps. Sample period 2010–2015.

Duffie et al. (2000) and Bandi and Renò (2016), one would expect an unconditional, negative linear relation between the price and volatility-jump sizes. However, the figure does not depict such a uniformly negative relation. Row 1 of Table 4 documents the absence of an unconditional price–volatility cojump relation across firms. That is, the test (30) rejects the null hypothesis of no DLE for only 10% of the 320 firms. The DLE estimates and correlation (33) are usually close to zero, with inconsistent signs across firms. The median DLE across all firms is 0.17; the corresponding correlation is 0.01. In other words, there is no prevalent unconditional leverage effect using either the covariation or correlation measure of leverage.

This result confirms previous negative findings of parametric asset pricing models by Chernov et al. (2003), Eraker et al. (2003) and Eraker (2004), who use U.S. stock index and option data. Jacod et al. (2017) also find no significant correlation of price–volatility cojumps in one-minute S&P 500 ETF data. Row 1 of Table 4 thus extends the literature's negative results on discontinuous leverage to the cross-section of individual stock price processes.

Given that it is difficult to reject the hypothesis of no DLE, the question arises if we should expect the discontinuous relation to be similar to that of the continuous leverage. As discovered by Lahaye et al. (2011), specific events cause large jumps and those jumps are relatively rare. Volatility jumps are very large on impact, but the level of volatility often subsequently decays quickly toward a pre-event level. The impact of news potentially drives common price and volatility jumps, as described by Pástor and Veronesi (2012, 2013). We conjecture that news effects usually trigger upward jumps in volatility, regardless of the effect on prices, and thus produce a positive (negative) correlation of volatility jumps with contemporaneous upward (downward) price jumps. To investigate this response pattern, we condition the DLE estimates on the signs of the price jumps.

Rows 2 and 5 of Table 4 show the outcomes of the DLE test (30) conditional on upward and downward price jumps, respectively. We focus on stocks with more than 10 price-volatility cojumps and exclude jumps larger than six standard deviations, which leaves us with 307 firms. Quantiles of the DLE estimates and correlations indicate that the DLE is negative for downward price jumps and positive for upward price jumps. That is, the leverage statistic quantiles are uniformly positive (negative) for positive (negative) price jumps. Row 2 of Table 4 shows that 65% of the firms display a significant DLE if prices jump up. Similarly, row 5 of Table 4 shows that 63% of the firms have a statistically significant DLE for negative price jumps. The positive (negative) relation between positive (negative) price jumps and contemporaneous volatility jumps is also visible in the scatter plot in Fig. 6.

In addition to conditioning on the sign of the price jump, we consider the fact that standard asset pricing models price different sources of risk differently. Systematic jumps are often related to macroeconomic news announcements and trigger cojumps across a large fraction of all firms while firm-specific jumps likely reflect idiosyncratic risk.

Conditioning on whether price jumps are market-wide or idiosyncratic reveals a strong conditional relation between discontinuities in prices and volatility (see rows 3 and 6 of Table 4). We focus on firms having more than 10 market price–volatility cojumps and omit jumps larger than six times its standard deviation. This shrinks the number of firms to 230. For this sample, positive, market-wide, price jumps and contemporaneous volatility jumps (see row 3 of Table 4) display a significant DLE for 89% of the firms. The median DLE estimate across all firms for a single, positive price–volatility cojump is 2.88. The median correlation between positive market jumps and volatility jumps is 0.32. Downward market jumps (see row 6 of Table 4) exhibit a significant downward sloping relation for 85% of the firms. The median DLE estimate for a single price–volatility cojump is -2.02 with a corresponding correlation of -0.31. A comparison of rows 3 to 4 and 6 to 7 of Table 4 shows that market jumps usually covary more strongly with contemporaneous volatility jumps than do idiosyncratic price jumps. Market jumps show a stronger conditional DLE than do idiosyncratic jumps because market events coincide with large price and volatility cojumps. This allows us to conclude that the tail DLE is particularly strong.

In contrast to market jumps, idiosyncratic jumps are smaller, coming more from the center of the jump distributions, and display a weaker DLE. Rows 4 and 7 of Table 4 indicate that about 66% of the stocks have a significant DLE for idiosyncratic jumps.

In summary, two forces prevent an unconditionally negative DLE: First, the sign of the price–volatility cojump depends on the sign of the price jump. That is, positive (negative) price jumps are positively (negatively) correlated with contemporaneous volatility jumps. Second, the DLE is stronger for market price jumps than for idiosyncratic price jumps.

The positive (negative) covariation between upward (downward) price jumps and contemporaneous volatility jumps might explain why Jacod et al. (2017) find no significant, unconditional correlation between price and volatility jumps, while Todorov and Tauchen (2011) report a strong positive relation between squared price jumps and jumps in volatility. Our results indicate that one would expect a positive unconditional DLE between squared price jumps and volatility.

The weaker relation between idiosyncratic price jumps and volatility jumps relates to Yu (2012), who models a time-varying leverage effect in a (semi)parametric stochastic volatility model where the time-variation is associated with the size of returns. By conditioning on positive and negative price jumps, we focus on Yu's positive and negative extreme states. Our analysis indicates that it is important to distinguish market jumps and idiosyncratic jumps, which roughly implies distinguishing the tail from the rest of the price-jump distribution.

6. Conclusion

This paper makes both methodological and empirical contributions to the literature on contemporaneous price and volatility jumps. We propose a nonparametric estimator of the discontinuous leverage effect (DLE) in high-frequency data that is robust to the presence of market microstructure noise. The new estimator allows us to study transactions data from the order book without down-sampling to a lower, regular observation frequency. For DLE estimation, we develop an efficient jump estimator for unknown jump times. We document that the new estimator has superior asymptotic and finite sample qualities compared to a method utilizing pre-average jump-size estimation.

Previous research has found it difficult to empirically document a DLE. Studying contemporaneous price and volatility jumps of 320 individual NASDAQ stocks from 2010 to 2015, we also find mixed and mostly insignificant, unconditional DLEs when considering all detected price and volatility cojumps. We show that the event-specific nature and distinct sources of jumps obscure the true relation between price and contemporaneous volatility jumps. We establish that a strong and significant DLE exists by conditioning on the sign of price jumps and on whether the price jumps are market or idiosyncratic jumps.

The DLE is different than its continuous counterpart, which was studied by Kalnina and Xiu (2017) and Aït-Sahalia et al. (2017), for example. In line with the model of Pástor and Veronesi (2012, 2013) the sign of the DLE depends on the sign of the price jump: a negative DLE across stocks exists for market downward price jumps but DLE estimates are consistently positive for market upward price jumps.

Our findings have implications for the parametric modeling of asset prices. Our empirical results cast doubt on the unconditional bivariate normality assumption of Bandi and Renò (2016), which implies tail independence and a generally linear relation around the center of the price–volatility cojump distribution. On the contrary, our results indicate that price–volatility cojumps around the center of the joint jump distribution—i.e. smaller jumps—are usually only weakly related, while jumps of the upper and the lower quantiles exhibit a strong and significant DLE. The linear dependence, which was introduced by Duffie et al. (2000), allows for tail dependence but appears to conflict with the data because it imposes one linear relation for both upward and downward price jumps. A specification that combines the uncorrelatedness assumption of Broadie et al. (2007) and a price jump sign dependence, as modeled by Maneesoonthorn et al. (2017), may adequately capture jump sizes of contemporaneous price and volatility cojumps. Working out the pricing implications of such a parametric model might be a path for future research.

Finally, one would like to explore the cross-sectional and time series dimension of the estimated DLE in detail. It is natural to ask if an asset pricing framework, such as that in Cremers et al. (2015), prices discontinuous leverage.

Appendix. Proofs

Standard localization techniques allow us to assume that there exists a constant Λ, such that

$$\max\{|b_s(\omega)|, |\sigma_s(\omega)|, |X_s(\omega)|, |\delta_\omega(s,x)|/\gamma(x)\} \le \Lambda,$$

for all $(\omega, s, x) \in (\Omega, \mathbb{R}_+, \mathbb{R})$, i.e., characteristics are uniformly bounded. We refer to Jacod and Protter (2012), Section 4.4.1, for a proof.

A.1. Proof of Proposition 3.1

We decompose the observations $Y_{t_i^n}$ into signal $X_{t_i^n}$ and noise $\epsilon_{t_i^n}$. In order to analyze the discretization variance from the signal terms, an illustration of the pre-processed price estimates (8) as a function in the efficient log-returns $\Delta_i^n X$ is helpful.

Reordering addends, similar as in the proofs of Zhang (2006), we obtain the identity

$$M_n^{-1}\Big(\sum_{i=l}^{l+M_n-1} Y_{t_i^n}-\sum_{i=l-M_n}^{l-1} Y_{t_i^n}\Big)=M_n^{-1}\sum_{i=l}^{l+M_n-1}\big(Y_{t_i^n}-Y_{t_{i-M_n}^n}\big)$$
$$=\sum_{k=1}^{M_n-1}\Delta_{l+k}^n Y\,\frac{M_n-k}{M_n}+\sum_{k=0}^{M_n-1}\Delta_{l-k}^n Y\,\frac{M_n-k}{M_n}\,. \tag{A.1}$$

The expectation and variance of noise terms are readily derived using the left-hand side of (A.1) and the fact that $\epsilon_{t_i^n}$ is i.i.d. with mean zero and variance η^2. For the signal part, we exploit the above identity and consider the right-hand side of (A.1). Considering the drift part $B_t=\int_0^t b_s\,ds$ in the pre-processed price estimates (8), we can bound the right-hand side above by

$$\Big|\sum_{k=1}^{M_n-1}\Delta_{l+k}^n B\,\frac{M_n-k}{M_n}+\sum_{k=0}^{M_n-1}\Delta_{l-k}^n B\,\frac{M_n-k}{M_n}\Big|\le K\,M_n\,n^{-1}=\scriptstyle{\mathcal{O}}(n^{-1/4}),$$

$\mathbb{P}$-almost surely, with a constant K, using the fact that

$$\sum_{k=1}^{M_n-1}(1-k/M_n)+\sum_{k=0}^{M_n-1}(1-k/M_n)=M_n\,.$$

We decompose the signal process, $X_t=B_t+C_t+J_t$, into its jump component, $(J_t)_{t\ge 0}$, and the continuous Itô semimartingale, $(C_t)_{t\ge 0}$. Under Assumption 2 and for $r\ge 1$, we can, with some constant K_p depending on p, use the estimate

$$\forall s,t\ge 0:\ \mathbb{E}\left[|J_t-J_s|^p\big|\mathcal{F}_s\right]\le K_p\,\mathbb{E}\Big[\Big(\int_s^t\int_{\mathbb{R}}(\gamma^r(x)\wedge 1)\lambda(dx)ds\Big)^{1/r}\Big]$$
$$\le K_p|t-s|^{(1/r)} \tag{A.2}$$

to find that the jump terms in the right-hand side of (A.1) satisfy

$$\mathbb{E}\Big[\Big|\sum_{k=1}^{M_n-1}\Delta_{l+k}^n J\,\frac{M_n-k}{M_n}+\sum_{k=1}^{M_n-1}\Delta_{l-k}^n J\,\frac{M_n-k}{M_n}\Big|\Big]=\mathcal{O}\big(M_n n^{-1/r}\big)\,,$$

with some $r<4/3$, where we omit $\Delta_l^n J$ for $l=\lfloor\tau n\rfloor+1$. Thus, the terms multiplied with $n^{1/4}$ tend to zero in probability by Markov's inequality. Because the expectations of all increments $\Delta_i^n C$ vanish and $\mathbb{E}[\Delta_l^n(C+J+\epsilon)|\Delta X_\tau]=\Delta X_\tau$ for $l=\lfloor\tau n\rfloor+1$, we conclude that

$$\mathbb{E}\big[T^{LM}\big(\tau;\Delta_1^n Y,\ldots,\Delta_n^n Y\big)|\Delta X_\tau\big]=\Delta X_\tau+\scriptstyle{\mathcal{O}}_{\mathbb{P}}\big(n^{-1/4}\big)\,.$$

In the case that $t_i^n=i/n$, Itô isometry and the smoothness of the volatility granted by (2) and (3) imply that for $l=\lfloor\tau n\rfloor+1$,

$$\mathbb{E}[(\Delta_{l+k}^n C)^2|\mathcal{F}_\tau]=\mathbb{E}\Big[\int_{(l+k-1)/n}^{(l+k)/n}\sigma_s^2\,ds\big|\mathcal{F}_\tau\Big]+\mathcal{O}_{\mathbb{P}}(n^{-2})$$
$$=\frac{\sigma_\tau^2}{n}+\mathcal{O}_{\mathbb{P}}\big(n^{-1}\sqrt{M_n/n}\big)\,,$$

for all $k=1,\ldots,M_n-1$. Analogously, we obtain that

$$\mathbb{E}[(\Delta_{l-k}^n C)^2|\mathcal{F}_{\tau-M_n/n}]=\frac{\sigma_{\tau-}^2}{n}+\mathcal{O}_{\mathbb{P}}\big(n^{-1}\sqrt{M_n/n}\big)\,,$$

for all $k=1,\ldots,M_n-1$. Use of the identities

$$\sum_{k=0}^{M_n-1}(1-k/M_n)^2=\frac{1}{3}M_n+\frac{1}{2}+\frac{1}{6}M_n^{-1}\,,$$
$$\sum_{k=1}^{M_n-1}(1-k/M_n)^2=\frac{1}{3}M_n-\frac{1}{2}+\frac{1}{6}M_n^{-1}\,,$$

and the independence of the noise and signal terms yield the asymptotic variance,

$$\mathbb{V}\mathrm{ar}\big(\sqrt{M_n}\,T^{LM}(\tau;\Delta_1^n Y,\ldots,\Delta_n^n Y)\big)\to\frac{1}{3}(\sigma_\tau^2+\sigma_{\tau-}^2)c^2+2\eta^2$$

of the rescaled statistic. The form of the variance in (9) follows from the above. Using that

$$\mathbb{E}[(\Delta_{l+k}^n C)^4|\mathcal{F}_\tau] = \frac{3\,\sigma_\tau^4}{n^2} + \mathcal{O}_{\mathbb{P}}\big(n^{-2}\big)\,,\;\; \mathbb{E}[(\Delta_{l-k}^n C)^4|\mathcal{F}_{\tau-M_n/n}] = \frac{3\,\sigma_{\tau-}^4}{n^2} + \mathcal{O}_{\mathbb{P}}\big(n^{-2}\big)$$

and with the assumed existence of $\mathbb{E}[\epsilon_t^4]$, the Lyapunov criterion with fourth moments obtained from (A.1) yields, together with the above considerations, the central limit theorem (9).

Next, we prove that the convergence is stable in law. The latter is equivalent to the joint weak convergence of $\alpha_n = \sqrt{M_n}\big(T^{LM}(\tau;\Delta_1^n Y,\ldots,\Delta_n^n Y) - \Delta X_\tau\big)$ with any $\mathcal{G}$-measurable bounded random variable Z:

$$\mathbb{E}\left[Zg(\alpha_n)\right] \to \mathbb{E}\left[Zg(\alpha)\right] = \mathbb{E}[Z]\mathbb{E}\left[g(\alpha)\right] \tag{A.3}$$

for any continuous bounded function, g, and

$$\alpha = \big((\sigma_\tau^2 + \sigma_{\tau-}^2)c^{2/3} + 2\eta^2\big)^{1/2} U\,, \tag{A.4}$$

with U, a standard normally distributed, random variable that is independent of $\mathcal{G}$. In order to verify (A.3), consider the sequence $A_n = [(\tau - M_n/n)\vee 0, (\tau + M_n/n)\wedge 1]$. Each α_n is measurable with respect to the σ-field $\mathcal{G}_1$. The sequence of decompositions

$$\tilde{C}(n)_t = \int_0^t \mathbb{1}_{A_n}(s)\sigma_s\,dW_s\,,\;\; \bar{C}(n)_t = C_t - \tilde{C}(n)_t\,,$$

$$\tilde{\epsilon}(n)_t = \mathbb{1}_{A_n}(t)\epsilon_t\,,\;\; \bar{\epsilon}(n)_t = \epsilon_t - \tilde{\epsilon}(n)_t\,,$$

of $(C_t)_{t\ge 0}$ and $(\epsilon_t)_{t\ge 0}$ are well-defined. If $\mathcal{H}_n$ denotes the σ-field generated by $\bar{C}(n)_t$, $\bar{\epsilon}(n)_t$ and $\mathcal{F}_0$, then $\big(\mathcal{H}_n\big)_n$ is an isotonic sequence with $\bigvee_n \mathcal{H}_n = \mathcal{G}_1$. Since $\mathbb{E}[Z|\mathcal{H}_n] \to Z$ in $L^1(\mathbb{P})$, it thus suffices that

$$\mathbb{E}[Zg(\alpha_n)] \to \mathbb{E}[Z\,g(\alpha)] = \mathbb{E}[Z]\mathbb{E}[g(\alpha)]\,, \tag{A.5}$$

for Z being $\mathcal{H}_q$ measurable for some q. Note that we can approximate the volatility to be constant over local intervals $[\tau - M_n/n,\tau)$ and $[\tau,\tau + M_n/n]$. Then, for all $n \ge q$, conditional on $\mathcal{H}_q$, α_n has a law independent of $\bar{C}(n)_t$ and $\bar{\epsilon}(n)_t$, such that the ordinary central limit theorem implies the claimed convergence.

A.2. Proof of Proposition 3.2

A neat decomposition of the spectral statistics into observation errors and returns of the efficient price is obtained with *summation by parts*

$$S_j(\tau) = \Big(\sum_{i=1}^n \Delta_i^n X\,\Phi_{j,\tau}((t_{i-1}^n + t_i^n)/2) - \sum_{i=1}^{n-1}\epsilon_{t_i^n}\Phi_{j,\tau}'(t_i^n)\,\frac{t_{i+1}^n - t_{i-1}^n}{2}\Big)(1 + \mathcal{O}_{\mathbb{P}}(1))\,, \tag{A.6}$$

where the asymptotically negligible remainder comes from approximating $\Phi_{j,\tau}((t_{i+1}^n + t_i^n)/2) - \Phi_{j,\tau}((t_{i-1}^n + t_i^n)/2)$ with the derivative and end-effects. The system of derivatives $(\Phi_{j,\tau}')_{j\ge 1}$ is again orthogonal such that covariances between different spectral frequencies vanish.

First, we prove that the drift is asymptotically negligible under Assumption 1. Because $\int_0^1 \Phi_{j,\tau}(t)\,dt = 2\sqrt{2h_n}/(\pi j)$ and $\int_0^1 \big|\Phi_{j,\tau}(t)\big|\,dt = 2\sqrt{2h_n}/\pi$, we get with generic constant K that $\mathbb{P}$-almost surely

$$\Big|\sum_{i=1}^n \Delta_i^n B\,\Phi_{j,\tau}((t_{i-1}^n + t_i^n)/2)\Big| \le K\sum_{i=1}^n (t_i^n - t_{i-1}^n)\big|\Phi_{j,\tau}((t_{i-1}^n + t_i^n)/2)\big| \le K\,\frac{\sqrt{h_n}}{\pi}\,,$$

and thus

$$\begin{aligned}\Big|\sum_{j=1}^{J_n}(-1)^{j+1}a_{2j-1}\sum_{i=1}^n \Delta_i^n B\,\Phi_{j,\tau}((t_{i-1}^n + t_i^n)/2)\Big| &\le K\sum_{j=1}^{J_n}(1 + j^2h_n^{-2}/n)^{-1}\sqrt{h_n}\\ &= K\sum_{j=1}^{J_n}\Big(1 + \frac{j^2}{\kappa^2\log^2(n)}\Big)^{-1}\sqrt{h_n}\\ &\le K\Big(\sum_{j=1}^{\log(n)}\sqrt{h_n} + \sum_{j=1}^{J_n} j^{-2}\sqrt{h_n}\log^2(n)\Big)\\ &\le K\log^2(n)\sqrt{h_n}\,.\end{aligned}$$

This yields that $\mathbb{P}$-almost surely

$$n^{1/4}\sqrt{\frac{h_n}{2}}\Big|\sum_{j=1}^{J_n}(-1)^{j+1}a_{2j-1}\sum_{i=1}^{n}\Delta_i^n B\,\Phi_{j,\tau}((t_{i-1}^n+t_i^n)/2)\Big|\to 0\,,$$

which ensures that we can neglect the drift in the asymptotic analysis of (12).

Next, we analyze the variance of (12) with oracle optimal weights (13). A locally constant approximation of $\sigma_s, s\in[\tau-h_n/2,\tau)$ and $\sigma_s, s\in[\tau,\tau+h_n/2]$ is asymptotically negligible under Assumption 1. Based on (A.6), using the fact that

$$\int_{\tau-h_n/2}^{\tau}\Phi_{j,\tau}^2(t)\,dt=\int_{\tau}^{\tau+h_n/2}\Phi_{j,\tau}^2(t)\,dt=1/2\,,$$

yields the following variances of spectral statistics:

$$\mathbb{V}\mathrm{ar}\big(S_j(\tau)\big)=\frac{1}{2}(\sigma_\tau^2+\sigma_{\tau-}^2)+\frac{\pi^2j^2}{h_n^2}\frac{\eta^2}{n}\,.$$

We thus obtain the conditional variance,

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}\Big(n^{1/4}\,\mathcal{T}(\tau;\Delta_1^nY,\ldots,\Delta_n^nY)\big|\mathcal{F}_\tau\Big)\\
&=n^{1/2}\Big(\sum_{j=1}^{J_n}\Big(\frac{1}{2}(\sigma_\tau^2+\sigma_{\tau-}^2)+\pi^2(2j-1)^2h_n^{-2}n^{-1}\eta^2\Big)^{-1}\Big)^{-1}h_n/2+\mathcal{O}_{\tilde{\mathbb{P}}}(1)\\
&=\frac{1}{2}\Bigg(\frac{\sum_{j=1}^{J_n}(\frac{1}{2}(\sigma_\tau^2+\sigma_{\tau-}^2)+\pi^2(2j-1)^2h_n^{-2}n^{-1}\eta^2)^{-1}}{\log(n)}\Bigg)^{-1}+\mathcal{O}_{\tilde{\mathbb{P}}}(1)\\
&=\frac{1}{2}\Big(\int_0^\infty\frac{1}{\frac{1}{2}(\sigma_\tau^2+\sigma_{\tau-}^2)+\pi^2(2z)^2\eta^2}\,dz\Big)^{-1}(1+\mathcal{O}(1))+\mathcal{O}_{\tilde{\mathbb{P}}}(1)\\
&=2\Big(\frac{\sigma_\tau^2+\sigma_{\tau-}^2}{2}\Big)^{1/2}\eta+\mathcal{O}_{\tilde{\mathbb{P}}}(1)\,.
\end{aligned}$$

With $\delta_n\le n^{-1}$, $l=\lfloor\tau n\rfloor+1$, we have that

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{T}(\tau;\Delta_1^nY,\ldots,\Delta_n^nY)|\Delta X_\tau\big]\\
&\quad=\sqrt{\frac{h_n}{2}}\sum_{j=1}^{J_n}a_{2j-1}(-1)^{j+1}\Phi_{2j-1,\tau}(\tau+\delta_n)\mathbb{E}[\Delta_l^nY|\Delta X_\tau]+\mathcal{O}_{\tilde{\mathbb{P}}}(n^{-1/4})\\
&\quad=\sqrt{\frac{h_n}{2}}\sum_{j=1}^{J_n}a_{2j-1}(-1)^{j+1}\Phi_{2j-1,\tau}(\tau+\delta_n)\Delta X_\tau+\mathcal{O}_{\tilde{\mathbb{P}}}(n^{-1/4})\\
&\quad=(1+\mathcal{O}(\delta_n))\Delta X_\tau+\mathcal{O}_{\tilde{\mathbb{P}}}(n^{-1/4})\,.
\end{aligned}$$

Considering further jumps on the estimation window, (A.2) yields for some constant K

$$\mathbb{E}\Big[\Big|\sum_{j=1}^{J_n}(-1)^{j+1}a_{2j-1}\sum_{i\neq l}\Delta_i^nJ\,\Phi_{j,\tau}((t_{i-1}^n+t_i^n)/2)\Big|\Big]\le K\log^2(n)\sqrt{h_n}\sup_i|t_i^n-t_{i-1}^n|^{1/r-1}$$

by the triangle inequality, decomposing $|t_i^n-t_{i-1}^n|^{1/r}=(t_i^n-t_{i-1}^n)|t_i^n-t_{i-1}^n|^{1/r-1}$ and using the same Riemann sum approximation as for the drift terms above. As for the Lee–Mykland statistic, $r<4/3$ ensures asymptotic negligibility of further jumps on $[\tau-h_n/2,\tau+h_n/2]$.

Since we assume $\mathbb{E}[\epsilon_t^4]<\infty$, we can establish a Lyapunov condition with fourth moments. Integral approximations with $\int_0^1\Phi_{j,\tau}^4(t)\,dt$ and $\int_0^1(\Phi_{j,\tau}')^4(t)\,dt$ yield, with generic constant C, for all j,

$$n\frac{h_n^2}{4}\sum_{i=1}^{n}\mathbb{E}[(\Delta_i^nX)^4]\Phi_{j,\tau}^4((t_{i-1}^n+t_i^n)/2)\le Cnh_n^2n^{-1}\frac{3}{2}h_n^{-1}=\mathcal{O}(h_n)$$

as well as,

$$\begin{aligned}
n\frac{h_n^2}{4}\sum_{i=1}^{n}\mathbb{E}[(\epsilon_{t_i^n})^4](\Phi_{j,\tau}')^4(t_i^n)(t_{i+1}^n-t_{i-1}^n)^4/16&\le Cnh_n^2h_n^{-5}n^{-3}\log^5(n)\\
&\le Cn^{-1/2}\log^2(n)\,.
\end{aligned}$$

Considering signal and noise terms separately, we derive for the signal terms with Jensen's inequality for weighted sums,

$$
\begin{aligned}
& n\frac{h_n^2}{4}\sum_{i=1}^{n}\mathbb{E}\Big[\Big(\Delta_i^n X\sum_{j=1}^{J_n}(-1)^{j+1}a_{2j-1}\Phi_{2j-1,\tau}((t_{i-1}^n+t_i^n)/2)\Big)^4\Big]\\
&\leq n\frac{h_n^2}{4}\sum_{i=1}^{n}\mathbb{E}[(\Delta_i^n X)^4]\sum_{j=1}^{J_n}a_{2j-1}\Phi^4_{2j-1,\tau}((t_{i-1}^n+t_i^n)/2)=\mathcal{O}\Big(\sum_{j=1}^{J_n}a_{2j-1}\,h_n\Big)=\mathcal{O}(h_n)\,.
\end{aligned}
$$

An analogous bound by Jensen's inequality for the noise terms implies the Lyapunov condition.

Stability of weak convergence is proved along the same lines as for Proposition 3.1 and we omit the proof. It remains to show that

$$
\mathbb{E}\Big[\Big|\sum_{j=1}^{J_n}(-1)^{j+1}\big(\hat{a}_{2j-1}-a_{2j-1}\big)S_{2j-1}(\tau)\sqrt{h_n/2}\Big|\Big]=\scriptstyle{\mathcal{O}}_{\tilde{\mathbb{P}}}\big(n^{-1/4}\big)\,, \tag{A.7}
$$

where $\hat{a}_{2j-1}$ denotes the estimated oracle weights, to prove the claimed result. Using the triangle and Hölder's inequalities, we can bound the right-hand side of (A.7) by

$$
\sum_{j=1}^{J_n}\mathbb{E}\Big[\big|\hat{a}_{2j-1}-a_{2j-1}\big|\,\big|S_{2j-1}(\tau)\big|\Big]\sqrt{\frac{h_n}{2}}\leq\sum_{j=1}^{J_n}\Big(\mathbb{E}\Big[\big|\hat{a}_{2j-1}-a_{2j-1}\big|^2\Big]\mathbb{E}\Big[\big|S_{2j-1}(\tau)\big|^2\Big]\Big)^{1/2}\sqrt{\frac{h_n}{2}}\,.
$$

In order to analyze the magnitude of the error of pre-estimating the weights, $\big|\hat{a}_{2j-1}-a_{2j-1}\big|^2$, we interpret (13) as a function of the variables $\sigma_\tau^2,\sigma_{\tau-}^2$ and η^2. Differential calculus and the delta method yield the upper bound,

$$
\begin{aligned}
&\sum_{j=1}^{J_n}K\Big(a_{2j-1}^2\big(\delta_n(\sigma_\tau^2)+\delta_n(\eta^2)\big)^2\,\mathbb{V}\mathrm{ar}(S_{2j-1}(\tau))\Big)^{1/2}\sqrt{h_n/2}\\
&\leq\sum_{j=1}^{J_n}K\,\delta_n(\sigma_\tau^2)\frac{\big(\mathbb{V}\mathrm{ar}(S_{2j-1}(\tau))\big)^{-1/2}}{\sum_{u=1}^{J_n}\big(\mathbb{V}\mathrm{ar}(S_{2u-1}(\tau))\big)^{-1}}\sqrt{h_n/2}\\
&\leq\sum_{j=1}^{J_n}K\Big(1+\frac{j^2}{\log^2(n)}\Big)^{-1/2}\delta_n(\sigma_\tau^2)\sqrt{h_n/2}=\mathcal{O}\Big(\log^3(n)\delta_n(\sigma_\tau^2)\sqrt{h_n}\Big)=\scriptstyle{\mathcal{O}}\big(n^{-1/4}\big),
\end{aligned}
$$

for the right-hand side of (A.7) with generic constant K and bounds $\delta_n(\sigma_\tau^2)\leq Kn^{-1/8}$ and $\delta_n(\eta^2)\leq Kn^{-1/2}$ for the errors of pre-estimating $\sigma_\tau^2,\sigma_{\tau-}^2$ and η^2 with (14a) and (14b), respectively. This ensures (A.7) and completes the proof of Proposition 3.2.

A.3. Proof of Proposition 3.3

The proof reduces to generalizing the analysis of the asymptotic variance and fourth moments for a Lyapunov condition. Consider the noise term on the right-hand side of (A.6) under R-dependent noise and for $t_i^n=F^{-1}(i/n)$. The expectation still vanishes and the variance becomes the following:

$$
\begin{aligned}
&\mathbb{E}\Bigg[\Big(\sum_{i=1}^{n-1}\epsilon_{t_i^n}\Phi'_{j,\tau}(t_i^n)\tfrac{t_{i+1}^n-t_{i-1}^n}{2}\Big)^2\Bigg]\\
&=\mathbb{E}\Bigg[\sum_{i=1}^{n-1}\epsilon_{t_i^n}^2(\Phi'_{j,\tau}(t_i^n))^2\big(\tfrac{t_{i+1}^n-t_{i-1}^n}{2}\big)^2+2\sum_{i=1}^{n-1}\sum_{u=1}^{R\wedge(n-i)}\epsilon_{t_i^n}\epsilon_{t_{i+u}^n}\Phi'_{j,\tau}(t_i^n)\Phi'_{j,\tau}(t_{i+u}^n)\tfrac{t_{i+1}^n-t_{i-1}^n}{2}\,\tfrac{t_{i+u+1}^n-t_{i+u-1}^n}{2}\Bigg]\\
&=\mathbb{E}\Bigg[\sum_{i=1}^{n-1}(\Phi'_{j,\tau}(t_i^n))^2\tfrac{t_{i+1}^n-t_{i-1}^n}{2}(F^{-1})'(\tau)n^{-1}\Big(\epsilon_{t_i^n}^2+\sum_{u=1}^{R\wedge(n-i)}\epsilon_{t_i^n}\epsilon_{t_{i+1}^n}\Big)\Bigg](1+\scriptstyle{\mathcal{O}}(1))\\
&=\eta_\tau^2(F^{-1})'(\tau)n^{-1}\int_0^1\Phi'_{j,\tau}(t)\,dt(1+\scriptstyle{\mathcal{O}}(1))=\eta_\tau^2(F^{-1})'(\tau)n^{-1}\pi^2j^2h_n^{-2}(1+\scriptstyle{\mathcal{O}}(1))\,.
\end{aligned}
$$

We used the smoothness of $(F^{-1})'$ and $\Phi'_{j,\tau}$ for approximations. The same Riemann sum approximation as in the equidistant observations case applies for the signal term. Using a (double-Riemann sum) integral approximation as $J_n\to\infty$, analogously as in the proof of Proposition 3.2, yields the asymptotic variance in (18). Introducing the shortcut, $\delta_{i,v}^R=\mathbb{1}_{\{|i-v|\leq R\}}$, we obtain the following estimates for the fourth moments:

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{i=1}^{n-1}\epsilon_{t_i^n}\Phi'_{j,\tau}(t_i^n)\tfrac{t_{i+1}^n-t_{i-1}^n}{2}\Big)^4\Big]\\
&=\mathbb{E}\Big[\sum_{i,v,u,r=1}^{n-1}\epsilon_{t_i^n}\epsilon_{t_v^n}\epsilon_{t_u^n}\epsilon_{t_r^n}\Phi'_{j,\tau}(t_i^n)\Phi'_{j,\tau}(t_v^n)\Phi'_{j,\tau}(t_u^n)\Phi'_{j,\tau}(t_r^n)\tfrac{t_{i+1}^n-t_{i-1}^n}{2}\,\tfrac{t_{v+1}^n-t_{v-1}^n}{2}\,\tfrac{t_{u+1}^n-t_{u-1}^n}{2}\,\tfrac{t_{r+1}^n-t_{r-1}^n}{2}\Big]\\
&=\sum_{i,v,u,r=1}^{n-1}\mathbb{E}\big[\epsilon_{t_i^n}\epsilon_{t_v^n}\epsilon_{t_u^n}\epsilon_{t_r^n}\big]\big(\delta_{i,v}^R\delta_{u,r}^R+\delta_{i,u}^R\delta_{v,r}^R+\delta_{i,r}^R\delta_{v,u}^R\big)\Phi'_{j,\tau}(t_i^n)\Phi'_{j,\tau}(t_v^n)\\
&\qquad\qquad\times\Phi'_{j,\tau}(t_u^n)\Phi'_{j,\tau}(t_r^n)\tfrac{t_{i+1}^n-t_{i-1}^n}{2}\,\tfrac{t_{v+1}^n-t_{v-1}^n}{2}\,\tfrac{t_{u+1}^n-t_{u-1}^n}{2}\,\tfrac{t_{r+1}^n-t_{r-1}^n}{2}\\
&=\big((F^{-1}(\tau))'\big)^2 3\,\eta_\tau^4\,n^{-2}-\mathcal{R}_n\,,
\end{aligned}
$$

with a remainder, $\mathcal{R}_n$, that satisfies for some constant C that

$$
\begin{aligned}
\mathcal{R}_n&\le\sum_{i,v,u,r=1}^{np}C\Big(\delta_{i,v}^R\delta_{u,r}^R\big(\delta_{i,u}^R+\delta_{v,r}^R+\delta_{i,r}^R+\delta_{v,u}^R\big)+\delta_{i,u}^R\delta_{v,r}^R\big(\delta_{i,v}^R+\delta_{u,r}^R+\delta_{i,r}^R+\delta_{v,u}^R\big)\\
&\qquad\qquad+\delta_{i,r}^R\delta_{v,u}^R\big(\delta_{i,v}^R+\delta_{u,r}^R+\delta_{i,u}^R+\delta_{v,r}^R\big)\Big)n^{-4}\\
&=\mathcal{O}\big(nR^3n^{-4}\big)=\mathcal{O}\big(n^{-3}\big)=\scriptstyle{\mathcal{O}}\big(n^{-2}\big)\,,
\end{aligned}
$$

such that $\mathcal{R}_n$ is asymptotically negligible. Inserting the estimate, the Lyapunov condition is ensured in the generalized setting. Under R-dependence, the convergence of the generalized variance and the generalized Lyapunov criterion imply the central limit theorem (18) and stability is proved analogously as above.

A.4. Proof of Proposition 3.5

Suppose that $\tau\in((k-1)h_n,kh_n)$ and we run the procedure from (26) to find a sub-interval that contains the jump. The variances of the statistics $T^{LM}\big((k-1)h_n+(i-1/2)\frac{r_n+l_n}{n};\Delta_1^nY,\ldots,\Delta_n^nY\big)$, $i=1,\ldots,R_n$, defined as in (7) with M_n replaced by $(r_n+l_n)/2$, are readily obtained from (A.1) and given by

$$
\begin{aligned}
&\mathbb{V}\mathrm{ar}\Big(T^{LM}\Big((k-1)h_n+(i-1/2)\frac{r_n+l_n}{n};\Delta_1^nY,\ldots,\Delta_n^nY\Big)\Big)\\
&\qquad\qquad=\frac{4\eta^2_{(k-1)h_n}}{r_n+l_n}+\frac{l_n}{n}\frac{\sigma^2_{\tau-}}{3}+\frac{r_n}{n}\frac{\sigma^2_{\tau}}{3}+\scriptstyle{\mathcal{O}}\Big(\frac{r_n+l_n}{n}\Big)\,.
\end{aligned}
$$

In particular, for $r_n+l_n=\scriptstyle{\mathcal{O}}(\sqrt{n})$, remainders in the proof of Proposition 3.1 become even smaller and the noise term prevails in the variance, such that, for all i,

$$
\sqrt{r_n+l_n}\,T^{LM}\big((k-1)h_n+(i-1/2)\tfrac{r_n+l_n}{n};\Delta_1^n(C+\epsilon),\ldots,\Delta_n^n(C+\epsilon)\big)\xrightarrow{(st)}MN\big(0,4\eta^2_{(k-1)h_n}\big),
$$

with C_t the continuous semimartingale part of X_t. Since, under Assumption 3, covariances between

$$
\Big(\frac{2}{r_n+l_n}\sum_{j=T_i}^{T_i+(r_n+l_n)/2-1}\big(\epsilon_{t_j^n}-\epsilon_{t^n_{j-(r_n+l_n)/2}}\big)\Big)_{i=1,\ldots,R_n}\,,\quad T_i=\lfloor(k-1)h_n\,n\rfloor+(i-1/2)(r_n+l_n)+1,
$$

are negligible, we deduce joint weak convergence to i.i.d. Gaussian limit random variables. Similarly as in Lee and Mykland (2012), using basic extreme value theory, we derive that

$$
B_n^{-1}\Big(\max_{i=1,\ldots,R_n}\sqrt{r_n+l_n}\,T^{LM}\big((k-1)h_n+(i-1/2)\tfrac{r_n+l_n}{n};\Delta_1^n(C+\epsilon),\ldots,\Delta_n^n(C+\epsilon)\big)-A_n\Big)\xrightarrow{(st)}\xi\,,
$$

with ξ a standard Gumbel random variable and

$$
A_n=2\eta_{(k-1)h_n}\sqrt{2\log(R_n)}-\eta_{(k-1)h_n}\frac{\log(4\pi\,\log(R_n))}{\sqrt{2\log(R_n)}}\,,\quad B_n^{-1}=\frac{\sqrt{\log(R_n)}}{\sqrt{2}\eta_{(k-1)h_n}}\,.
$$

For $(r_n+l_n)\propto n^\delta$, $\delta>0$, and if the jump is not located very close to the edges between the sub-intervals, the statistic on the sub-interval with the jump tends to infinity. That is, for

$$
M_n=\max_{i=1,\ldots,R_n}T^{LM}\big((k-1)h_n+(i-1/2)\tfrac{r_n+l_n}{n};\Delta_1^nY,\ldots,\Delta_n^nY\big)\,,
$$

where now $(\Delta_j^n Y)$ are inserted, not $(\Delta_j^n(C+\epsilon))$, we have that

$$\sqrt{r_n+l_n}\,M_n - A_n \to \infty \;.$$

As long as $M_n > (r_n+l_n)^{-1/2+\epsilon}$ for some $\epsilon > 0$, this holds true. We need to carefully consider the potential bias issue discussed at the end of Section 3.1. The probability that $M_n \le (r_n+l_n)^{-1/2+\epsilon}$ translates to the probability that a jump is located in some small (in n decreasing) vicinity of the block edges. Using that jump times are locally uniformly distributed, we obtain that

$$\begin{aligned}\mathbb{P}\big(M_n \le (r_n+l_n)^{-\frac{1}{2}+\epsilon}\big) &= \mathbb{P}\Big(\min_{i=1,\ldots,R_n}\Big|\tau-(k-1)h_n-(i-1)\tfrac{r_n+l_n}{n}\Big| \le \frac{(r_n+l_n)^{\frac{1}{2}+\epsilon}}{n}\Big)\\ &= \mathbb{P}\big(U\in(0,2(r_n+l_n)^{-1/2+\epsilon})\big) = \mathcal{O}\big((r_n+l_n)^{-1/2+\epsilon}\big)\,,\end{aligned}$$

with U a random variable uniformly distributed on $[0,1]$ and using the symmetry. Apparently, the probability converges to zero for ϵ sufficiently small. This implies that, for any such choice of R_n and r_n+l_n, the procedure asymptotically almost surely detects the sub-interval which contains the jump.

Assigning jump times to a bin by thresholding induces a negligible error. This is proved analogously as in the proof of Proposition 3.6 in the next paragraph.

Cutting out noisy prices in the window $(t^n_{l-l_n}, t^n_{l+r_n})$ around τ, the adjusted statistics (12) are asymptotically unbiased estimators of ΔX_τ. Considering their asymptotic properties, we can exploit most parts of the proof of Proposition 3.2. The only relevant difference is due to the increment over the cut-out window in the spectral statistics

$$\sqrt{\frac{h_n}{2}}\sum_{j=1}^{J_n}(-1)^{j+1}a_{2j-1}(Y_{t^n_{l+r_n}}-Y_{t^n_{l-l_n}})\Phi_{2j-1,\tau}(\tau)\,.$$

The increments $Y_{t^n_{l+r_n}}-Y_{t^n_{l-l_n}}$ take the role of $\Delta_l^n Y$, $l=\lfloor \tau n\rfloor+1$, where the window of statistics (12) is centered. Using Jensen's inequality, we obtain that

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sqrt{\frac{h_n}{2}}\sum_{j=1}^{J_n}a_{2j-1}(-1)^{j+1}\Phi_{2j-1,\tau}(\tau)(X_{t^n_{l+r_n}}-X_{t^n_{l-l_n}})\Big)^2\Big]\\ &\le \frac{h_n}{2}\sum_{j=1}^{J_n}a_{2j-1}\Phi^2_{2j-1,\tau}(\tau)\mathbb{E}\big[(X_{t^n_{l+r_n}}-X_{t^n_{l-l_n}})^2\big]\\ &\le \max(\sigma_\tau^2,\sigma_{\tau-}^2)(t^n_{l+r_n}-t^n_{l-l_n}) = \scriptstyle{\mathcal{O}}(n^{-1/2})\,.\end{aligned}$$

Since $\Phi'_{2j-1,\tau}(\tau)=0$, the summation by parts transformation (A.6) shows that the variance due to noise is not affected by the adjustment. Overall, we conclude that for the adjusted estimator,

$$|\widehat{\Delta X}_\tau - \mathcal{T}(\tau;\Delta_1^n Y,\ldots,\Delta_n^n Y)| = \scriptstyle{\mathcal{O}}_{\tilde{\mathbb{P}}}(n^{-1/4})\,.$$

We conclude the result with Proposition 3.3.

A.5. Proof of Propositions 3.6 and 3.7

Denoting the finitely many stopping times with $|\Delta X_{\tau_k}| > a$, as $\tau_1,\ldots,\tau_N$, (28) can be written

$$[X,\sigma^2]_T^d(a) = \sum_{k=1}^N \Delta X_{\tau_k}\big(\sigma^2_{\tau_k}-\sigma^2_{\tau_k-}\big)\,.$$

The estimator (29) then becomes

$$\widehat{[X,\sigma^2]}_1^d(a) = \sum_{k=1}^{\hat N}\widehat{\Delta X}_{\hat\tau_k}\big(\hat\sigma^2_{\hat\tau_k}-\hat\sigma^2_{\hat\tau_k-}\big)\,.$$

The case without price jumps on the considered interval, $N=0$, is trivial. Consider the set

$$\begin{aligned}\tilde\Omega_n &= \big\{\omega\in\Omega\,|\,\tau_1 > k_n^{-1}h_n,\ \tau_N < 1-k_n^{-1}h_n,\ \tau_i-\tau_{i-1} > 2k_n^{-1}h_n\,,\ i=1,\ldots,N-1\big\}\\ &\cup\ \big\{\omega\in\Omega\,|\,\tau_i = k\cdot h_n\,,\ i=1,\ldots,N-1,\ k=0,\ldots,h_n^{-1}-1\big\}^{\complement}\,.\end{aligned}$$

We can restrict to the subset $\tilde\Omega_n$, since $\mathbb{P}(\tilde\Omega_n)\to 1$ as $n\to\infty$. We infer the jump times $\{\tau_i, i=1,\ldots,N\}$, or the respective bins on which jumps occur by thresholding. To show that this identification of jump times only induces an asymptotically negligible error, we prove that

$$\Big|\sum_{k=2}^{h_n^{-1}-1}\widehat{\Delta X}_{\hat\tau_k}\big(\hat\sigma^2_{\hat\tau_k}-\hat\sigma^2_{\hat\tau_k-}\big)\,\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}-\sum_{k=1}^{N}\widehat{\Delta X}_{\tau_k}\big(\hat\sigma^2_{\tau_k}-\hat\sigma^2_{\tau_k-}\big)\Big|=\mathcal{o}_{\bar{\mathbb{P}}}\big(n^{-\beta/2}\big)\,.$$

This is ensured by Corollary 3.4 and by Proposition 3.5 if

$$\sum_{k=2}^{h_n^{-1}-1}\big|\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}-\mathbb{1}_{\{\tau_i\in((k-1)h_n,kh_n)\}}\big|=\mathcal{o}_{\bar{\mathbb{P}}}\big(n^{-1/8}\big)\,.$$

Denote $\mathcal{K}=\{1\le k\le h_n^{-1}|\tau_i\in((k-1)h_n,kh_n)\}$ and $\mathcal{K}^{\complement}=\{2,\ldots,h_n^{-1}-1\}\setminus\mathcal{K}$. The last relation can be rewritten

$$\sum_{k\in\mathcal{K}}\mathbb{1}_{\{\Delta_k\widehat{[X,X]}\le a^2\vee u_n\}}+\sum_{k\in\mathcal{K}^{\complement}}\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}=\mathcal{o}_{\bar{\mathbb{P}}}\big(n^{-1/8}\big)\,. \tag{A.8}$$

For each k in the finite set $\mathcal{K}$, we prove that

$$\mathbb{1}_{\{\Delta_k\widehat{[X,X]}\le a^2\vee u_n\}}=\mathcal{o}_{\bar{\mathbb{P}}}\big(n^{-1/8}\big)\,.$$

The restriction to $\tilde\Omega_n$ ensures that the considered jumps cannot occur on neighboring bins. Corollary 3.3 and its proof in Bibinger and Winkelmann (2015) establishes that $\Delta_k\widehat{[X,X]}=(\Delta X_{\tau_i})^2+\chi_i$ with $\mathbb{V}\mathrm{ar}(\chi_i)=\mathcal{O}(n^{-1/2})$. More precisely, as outlined in Section 3.1.3 of Bibinger and Winkelmann (2015), for $\tau_i\in((k-1)h_n,kh_n)$, we have that

$$\mathbb{E}[h_n\,S^2_{jk}]=2\sin^2(\pi jh_n^{-1}(\tau_i-(k-1)h_n))(\Delta X_{\tau_i})^2+\mathcal{O}(h_n)\,,$$
$$\mathbb{E}[h_n\,\max(\tilde S^2_{jk},\tilde S^2_{j(k+1)})]=2\cos^2(\pi jh_n^{-1}(\tau_i-(k-1)h_n))(\Delta X_{\tau_i})^2+\mathcal{O}(h_n)\,.$$

The contribution with the cosine term is by $\tilde S^2_{jk}$ when $\tau_i\in((k-1)h_n,(k-1/2)h_n)$ and by $\tilde S^2_{j(k+1)}$ when $\tau_i\in((k-1/2)h_n,kh_n)$. When $\tau_i=(k-1/2)h_n$, the cosine vanishes. Since the Lévy measure does not have an atom in a, it thus holds that, for some fixed $\epsilon>0$,

$$\Delta_k\widehat{[X,X]}=a^2+\epsilon+\chi_i\,.$$

Using Chebyshev's inequality, we derive that

$$\bar{\mathbb{P}}\big(\Delta_k\widehat{[X,X]}\le a^2\vee u_n\big)\le\bar{\mathbb{P}}\big(|\chi_i|>\epsilon-u_n\big)=\mathcal{O}\big(n^{-1/2}\big)\,.$$

Considering indicator functions $\mathbb{1}_{A_n}$ with $p_n=\bar{\mathbb{P}}(A_n)\to 0$, using that $\mathbb{E}[\mathbb{1}_{A_n}]=p_n$ and $\mathbb{V}\mathrm{ar}(\mathbb{1}_{A_n})\le p_n$, we obtain that

$$\sum_{k\in\mathcal{K}}\mathbb{1}_{\{\Delta_k\widehat{[X,X]}\le a^2\vee u_n\}}=\mathcal{O}_{\bar{\mathbb{P}}}\big(n^{-1/4}\big)=\mathcal{o}_{\bar{\mathbb{P}}}\big(n^{-1/8}\big)\,. \tag{A.9}$$

Due to the maximum operator in (24), the term with the square cosine factor above feeds in two successive statistics. The cosine giving some factor bounded from above by one, we have for $\tau_i\in((k-1)h_n,kh_n)$ that

$$\widetilde\zeta^{ad}_k>\max(\widetilde\zeta^{ad}_{k-1},\widetilde\zeta^{ad}_{k+1})\,,$$

asymptotically almost surely. We conclude that the first sum in (A.8) is asymptotically negligible.

For $k\in\mathcal{K}^{\complement}$, neighboring a bin with $k\pm1\in\mathcal{K}$, it holds asymptotically almost surely that $\widetilde\zeta^{ad}_k<\max(\widetilde\zeta^{ad}_{k-1},\widetilde\zeta^{ad}_{k+1})$, such that the indicator function sets it to zero. For all other $k\in\mathcal{K}^{\complement}$ we have that $\Delta_k\widehat{[X,X]}=h_n\tilde\zeta_k$ with $\tilde\zeta_k$ the local estimate for $\sigma^2_{(k-1)h_n}$ satisfying by Lemma 2 of Bibinger and Winkelmann (2018) the upper moment bound

$$\mathbb{E}\big[|\tilde\zeta_k|^{4+\delta}\big]=\mathcal{O}\big(\log(n)\big)$$

for δ from Assumption 3. Markov's inequality yields for $k\in\mathcal{K}^{\complement}$

$$\bar{\mathbb{P}}\big(\Delta_k\widehat{[X,X]}>a^2\vee u_n\big)=\mathcal{O}\Big(h_n^{4+\delta}\log(n)\big(a^2\vee u_n\big)^{(4+\delta)}\Big)\,.$$

Thereby we obtain that

$$\sum_{k\in\mathcal{K}^{\complement}}\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}=\mathcal{O}_{\bar{\mathbb{P}}}\Big(h_n^{-1}h_n^{2+\delta/2}\sqrt{\log(n)}\big(a^2\vee u_n\big)^{(2+\delta/2)}\Big)\,. \tag{A.10}$$

If $a>0$, then this sum decays very fast as $n\to\infty$ and we clearly have $\mathcal{o}_{\bar{\mathbb{P}}}(n^{-1/8})$. If $a=0$, then the resulting order is $h_n^{1+\delta/2}\sqrt{\log(n)}h_n^{-\varpi(2+\delta/2)}$ and the term is $\mathcal{o}_{\bar{\mathbb{P}}}(n^{-1/8})$ when

$$\varpi<\frac{1+\delta/2-1/4}{2+\delta/2}\,.$$

A slightly stronger condition even implies the summability,

$$\sum_{n\in\mathbb{N}}\bar{\mathbb{P}}\Big(\sum_{k\in\mathcal{K}^{\complement}}\mathbb{1}_{\{\Delta_k\widehat{[X,X]}>a^2\vee u_n\}}>0\Big)<\infty\,,$$

and thus the almost sure convergence by Borel–Cantelli. With (A.10), we deduce (A.8) and are left to consider price and volatility-jump estimates at times $\hat{\tau}_i$, $i=1,\ldots,\hat{N}$. In both cases, $a>0$ or $a=0$ when $r=0$ in Assumption 2, $N<\infty$ holds almost surely. On $\tilde{\Omega}_n$, all involved local estimates for different price-jump times τ_i, $i=1,\ldots,N$, are computed from disjoint datasets. The latter are not necessarily independent, but all covariations converge to zero asymptotically. For the single price-jump estimates, we have by Proposition 3.5 that

$$\widehat{\Delta X}_{\hat{\tau}_i}=\Delta X_{\hat{\tau}_i}+\mathcal{O}_{\bar{\mathbb{P}}}\big(n^{-1/4}\big)\,.$$

Based on Corollary 3.4, we obtain that

$$\Big(\big(\hat{\sigma}^2_{\hat{\tau}_i}-\hat{\sigma}^2_{\hat{\tau}_i-}\big)-\Delta\sigma^2_{\hat{\tau}_i}\Big)\xrightarrow{(st)}\sqrt{8(\sigma^3_{\hat{\tau}_i}+\sigma^3_{\hat{\tau}_i-})\eta_{\hat{\tau}_i}}Z_i\,,$$

for all $i=1,\ldots,\hat{N}$, with β from (23) and (Z_i) i.i.d. standard normals. By the asymptotic negligibility of covariations, the vector

$$n^{\beta/2}\Big(\hat{\sigma}^2_{\hat{\tau}_1}-\hat{\sigma}^2_{\hat{\tau}_1-},\ldots,\hat{\sigma}^2_{\hat{\tau}_{\hat{N}}}-\hat{\sigma}^2_{\hat{\tau}_{\hat{N}}-}\Big)\xrightarrow{(st)}\big(U_1,\ldots,U_{\hat{N}}\big)\,,\tag{A.12}$$

converges stably in law, where $U_1,\ldots,U_{\hat{N}}$ are independent and $U_i=\sqrt{8(\sigma^3_{\hat{\tau}_i}+\sigma^3_{\hat{\tau}_i-})\eta_{\hat{\tau}_i}}Z_i$. Altogether, the asymptotic orders of different error terms and standard relations for weak and stochastic convergences imply (30).

A.6. Proof of Corollary 3.8

According to the proof of Propositions 3.6 and 3.7, the identification of bins with (large) jumps only induces an asymptotically negligible error. We are thus left to consider

$$\frac{\sum_{k=1}^{\hat{N}}\widehat{\Delta X}_{\hat{\tau}_k}\big(\hat{\sigma}^2_{\hat{\tau}_k}-\hat{\sigma}^2_{\hat{\tau}_k-}\big)}{\Big(\sum_{k=1}^{\hat{N}}\big(\widehat{\Delta X}_{\hat{\tau}_k}\big)^2\sum_{k=1}^{\hat{N}}\big(\hat{\sigma}^2_{\hat{\tau}_k}-\hat{\sigma}^2_{\hat{\tau}_k-}\big)^2\Big)^{1/2}}-\frac{\sum_{k=1}^{\hat{N}}\Delta X_{\hat{\tau}_k}\Delta\sigma^2_{\hat{\tau}_k}}{\Big(\sum_{k=1}^{\hat{N}}\big(\Delta X_{\hat{\tau}_k}\big)^2\sum_{k=1}^{\hat{N}}\big(\Delta\sigma^2_{\hat{\tau}_k}\big)^2\Big)^{1/2}}\,.$$

From (A.12), we adopt that

$$n^{\beta/2}\Big(\sum_{k=1}^{\hat{N}}\Big(\widehat{\Delta X}_{\hat{\tau}_k}\big(\hat{\sigma}^2_{\hat{\tau}_k}-\hat{\sigma}^2_{\hat{\tau}_k-}\big)-\Delta X_{\hat{\tau}_k}\Delta\sigma^2_{\hat{\tau}_k}\Big)\Big)\xrightarrow{(st)}\sum_{k=1}^{\hat{N}}\Delta X_{\hat{\tau}_k}U_k\,.$$

In the estimation of the discontinuous leverage, the estimation error of $(\Delta\sigma^2_{\hat{\tau}_k})$ dominates the smaller error of estimating $(\Delta X_{\hat{\tau}_k})$. Analogously, for estimating (33)

$$\sum_{k=1}^{\hat{N}}\Big(\big(\widehat{\Delta X}_{\hat{\tau}_k}\big)^2-\big(\Delta X_{\hat{\tau}_k}\big)^2\Big)=\mathcal{O}_{\bar{\mathbb{P}}}\big(n^{-1/4}\big)\,,$$

readily obtained from Proposition 3.3 and the delta method, induces an error that is negligible at first asymptotic order. For the second variation, we deduce that

$$n^{\beta/2}\sum_{k=1}^{\hat{N}}\Big(\big(\hat{\sigma}^2_{\hat{\tau}_k}-\hat{\sigma}^2_{\hat{\tau}_k-}\big)^2-\big(\Delta\sigma^2_{\hat{\tau}_k}\big)^2\Big)\xrightarrow{(st)}\sum_{k=1}^{\hat{N}}2\Delta\sigma^2_{\hat{\tau}_k}U_k\,,$$

from (A.12) and applying the binomial formula or the delta method for the square function. Another application of the delta method yields

$$\begin{aligned}&n^{\beta/2}\left(\frac{\widehat{[X,\sigma^2]}_1^d(a)}{\sqrt{\widehat{[X,X]}_1^d(a)\widehat{[\sigma^2,\sigma^2]}_1^d(a)}}-\frac{[X,\sigma^2]_1^d(a)}{\sqrt{[X,X]_1^d(a)[\sigma^2,\sigma^2]_1^d(a)}}\right)\\&=\sum_{k=1}^{\hat{N}_k}U_k\left(\frac{\Delta X_{\hat{\tau}_k}}{\big([X,X]_1^d(a)[\sigma^2,\sigma^2]_1^d(a)\big)^{1/2}}-\frac{\Delta\sigma^2_{\hat{\tau}_k}[X,\sigma^2]_1^d(a)}{\big([X,X]_1^d(a)\big)^{1/2}\big([\sigma^2,\sigma^2]_1^d(a)\big)^{3/2}}\right)+\mathcal{O}_{\bar{\mathbb{P}}}(1)\,,\end{aligned}$$

such that we obtain a stable central limit theorem with rate $n^{\beta/2}$ and asymptotic variance

$$\frac{8(\sigma^3_{\hat{\tau}_k}+\sigma^3_{\hat{\tau}_k-})\eta_{\hat{\tau}_k}}{[X,X]^d_1(a)[\sigma^2,\sigma^2]^d_1(a)}\Big(\Delta X_{\hat{\tau}_k}-\frac{\Delta\sigma^2_{\hat{\tau}_k}[X,\sigma^2]^d_1(a)}{[\sigma^2,\sigma^2]^d_1(a)}\Big)^2 .$$

This implies Corollary 3.8.

References

Aït-Sahalia, Y., Fan, J., Laeven, R.J.A., Wang, C.D., Yang, X., 2017. Estimation of the continuous and discontinuous leverage effect. J. Amer. Statist. Assoc. 112, 1744–1758.
Aït-Sahalia, Y., Fan, J., Li, Y., 2013. The leverage effect puzzle: Disentangling sources of bias at high frequency. J. Financ. Econ. 109 (1), 224–249.
Aït-Sahalia, Y., Jacod, J., 2014. High-Frequency Financial Econometrics. Princeton University Press, Princeton, NJ.
Aït-Sahalia, Y., Xiu, D., 2017. A Hausman test for the presence of market microstructure noise in high frequency data. J. Econometrics (forthcoming).
Altmeyer, R., Bibinger, M., 2015. Functional stable limit theorems for quasi-efficient spectral covolatility estimators. Stochastic Process. Appl. 125 (12), 4556–4600.
Bandi, F.M., Renò, R., 2016. Price and volatility co-jumps. J. Financ. Econ. 119 (1), 107–146.
Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A., Shephard, N., 2009. Realized kernels in practice: Trades and quotes. Econom. J. 12 (3), C1–C32.
Bekaert, G., Wu, G., 2000. Asymmetric volatility and risk in equity markets. Rev. Financ. Stud. 13 (1), 1–42.
Benjamini, Y., Hochberg, Y., 1995. Controlling the false discovery rate: A practical and powerful approach to multiple testing. J. R. Stat. Soc. Ser. B Stat. Methodol. 57 (1), 289–300.
Bibinger, M., Hautsch, N., Malec, P., Reiß, M., 2014. Estimating the quadratic covariation matrix from noisy observations: Local method of moments and efficiency. Ann. Statist. 42 (4), 80–114.
Bibinger, M., Hautsch, N., Malec, P., Reiß, M., 2017. Estimating the spot covariation of asset prices – statistical theory and empirical evidence. J. Bus. Econom. Statist. (forthcoming).
Bibinger, M., Winkelmann, L., 2015. Econometrics of cojumps in high-frequency data with noise. J. Econometrics 184 (2), 361–378.
Bibinger, M., Winkelmann, L., 2018. Common price and volatility jumps in noisy high-frequency data. Electron. J. Stat. 12 (1), 2018–2073.
Black, F., 1976. Studies of stock price volatility changes. In: Proceedings of the 1976 Meeting of the American Statistical Association. Bus. Econ. Stat. 6, 177–181.
Bollerslev, T., Litvinova, J., Tauchen, G., 2006. Leverage and volatility feedback effects in high-frequency data. J. Financ. Econ. 4 (3), 353–384.
Broadie, M., Chernov, M., Johannes, M., 2007. Model specification and risk premia: Evidence from futures options. J. Finance 62 (3), 1453–1490.
Chernov, M., Gallant, A.G., Ghysels, E., Tauchen, G., 2003. Alternative models for stock price dynamics. J. Econometrics 116 (1–2), 225–257.
Christensen, K., Oomen, R., Podolskij, M., 2014. Fact or friction: Jumps at ultra high frequency. J. Financ. Econ. 114 (3), 576–599.
Cremers, M., Halling, M., Weinbaum, D., 2015. Aggregate jump and volatility risk in the cross-section of stock returns. J. Finance 70 (2), 577–614.
Duffee, G.R., 1995. Stock returns and volatility – A firm-level analysis. J. Financ. Econ. 37 (3), 399–420.
Duffie, D., Pan, J., Singleton, K., 2000. Transform analysis and asset pricing for affine-diffusions. Econometrica 68 (6), 1343–1376.
Eraker, B., 2004. Do stock prices and volatility jump? Reconciling evidence from spot and option prices. J. Finance 59 (3), 1367–1403.
Eraker, B., Johannes, M., Polson, N., 2003. The impact of jumps in volatility and returns. J. Finance 58 (3), 1269–1300.
French, K.R., Schwert, W., Stambaugh, R., 1987. Expected stock returns and volatility. J. Financ. Econ. 19 (1), 3–29.
Hansen, P.R., Lunde, A., 2006. Realized variance and market microstructure noise. J. Bus. Econom. Statist. 24 (2), 127–161.
Jacod, J., Klüppelberg, C., Müller, G., 2017. Testing for non-correlation between price and volatility jumps. J. Econometrics 197 (2), 284–297.
Jacod, J., Podolskij, M., Vetter, M., 2010. Limit theorems for moving averages of discretized processes plus noise. Ann. Statist. 38 (3), 1478–1545.
Jacod, J., Protter, P., 2012. Discretization of Processes. Springer.
Kalnina, I., Xiu, D., 2017. Nonparametric estimation of the leverage effect: A trade-off between robustness and efficiency. J. Amer. Statist. Assoc. 112 (517), 384–396.
Koike, Y., 2017. Time endogeneity and an optimal weight function in pre-averaging covariance estimation. Stat. Inference Stoch. Process. 20 (1), 15–56.
Lahaye, J., Laurent, S., Neely, C.J., 2011. Jumps, cojumps and macro announcements. J. Appl. Econometrics 26 (6), 893–921.
Lee, S., Mykland, P.A., 2008. Jumps in financial markets: A new nonparametric test and jump dynamics. Rev. Financ. Stud. 21 (6), 2535–2563.
Lee, S., Mykland, P.A., 2012. Jumps in equilibrium prices and market microstructure noise. J. Econometrics 168 (2), 396–406.
Li, J., Todorov, V., Tauchen, G., 2017. Jump regressions. Econometrica 85 (1), 173–195.
Maneesoonthorn, W., Forbes, C.S., Martin, G.M., 2017. Inference on self-exciting jumps in prices and volatility using high frequency measures. J. Appl. Econometrics 32 (3), 504–532.
Mykland, P.A., Zhang, L., 2016. Between data cleaning and inference: Pre-averaging and robust estimators of the efficient price. J. Econometrics 194 (2), 242–262.
Pástor, L., Veronesi, P., 2012. Uncertainty about government policy and stock prices. J. Finance 67 (4), 1219–1264.
Pástor, L., Veronesi, P., 2013. Political uncertainty and risk premia. J. Financ. Econ. 110 (3), 520–545.
Pelger, M., 2017. Understanding systematic risk: A high-frequency approach (working paper). Department of Management Science & Engineering, Stanford University, Stanford, CA 94305.
Reiß, M., 2011. Asymptotic equivalence for inference on the volatility from noisy observations. Ann. Statist. 39 (2), 772–802.
Todorov, V., Tauchen, G., 2011. Volatility jumps. J. Bus. Econom. Statist. 29 (3), 356–371.
Vetter, M., 2012. Estimation of correlation for continuous semimartingales. Scand. J. Stat. 39 (4), 757–771.
Vetter, M., 2014. Inference on the Lévy measure in case of noisy observations. Statist. Probab. Lett. 87, 125–133.
Wang, C.D., Mykland, P.A., 2014. The estimation of the leverage effect with high-frequency data. J. Amer. Statist. Assoc. 109 (505), 197–215.
Winkelmann, L., Bibinger, M., Linzert, T., 2016. ECB monetary policy surprises: Identification through cojumps in interest rates. J. Appl. Econometrics 31 (4), 613–629.
Yu, J., 2012. A semiparametric stochastic volatility model. J. Econometrics 167 (2), 473–482.
Zhang, L., 2006. Efficient estimation of stochastic volatility using noisy observations: A multi-scale approach. Bernoulli 12 (6), 1019–1043.