

Mechanics of good trade execution in the framework of linear temporary market impact

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We define the concept of good trade execution and we construct explicit adapted good trade execution strategies in the framework of linear temporary market impact. Good trade execution strategies are dynamic, in the sense that they react to the actual realisation of the traded asset price path over the trading period; this is paramount in volatile regimes, where price trajectories can considerably deviate from their expected value. Remarkably, however, the implementation of our strategies does not require the full specification of an SDE evolution for the traded asset price, making them robust across different models. Moreover, rather than minimising the *expected* trading cost, good trade execution strategies minimise trading costs in a *pathwise* sense, a point of view not yet considered in the literature. The mathematical apparatus for such a pathwise minimisation hinges on certain random Young differential equations that correspond to the Euler–Lagrange equations of the classical Calculus of Variations. These Young differential equations characterise our good trade execution strategies in terms of an initial value problem that allows for easy implementations.

Keywords: Quantitative trading strategies; Market microstructure; Quantitative finance; Quantitative finance techniques

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1. Introduction

Executions of large trades can affect the price of the traded asset, a phenomenon known as *market impact*. The price is affected in the direction unfavourable to the trade: while selling, the market impact decreases the price; while buying, the market impact increases the price. Therefore, a trader who wishes to minimise her trading costs has to split her order into a sequence of smaller suborders which are executed over a finite time horizon. How to optimally split a large order is a question that naturally arises.

Academically, the literature discussing such an optimal split was initiated by the seminal papers by Almgren and Chriss (2001) and by Bertsimas and Lo (1998). Both papers deal with the trading process of one large market participant who would like to buy or sell a large amount of shares or contracts during a specified duration. The optimisation problem is formulated as a trade-off between two pressures. On the one hand, market impact demands to trade slowly in order to minimise the unfavourable impact that the execution itself has on the price. On the other hand, traders have an incentive

to trade rapidly, because they do not want to carry the risk of adverse price movements away from their decision price. Such a trade-off between market impact and market risk is usually translated into a stochastic control problem where the trader’s strategy (i.e. the control) is the trading speed. The class of admissible strategies defines the set over which the risk-cost functional is optimised.

In the design of mathematical models for optimal trade execution we identify two phases. The first phase is the description of trading costs. This refers to the choice of a function F that depends on time, asset price, quantity to execute and trading speed, and models the instantaneous cost of trading. The overall cost during the time window $[0, T]$ is then expressed as the time integral

$$J(q) = \int_0^T F(t, x_t, q_t, \dot{q}_t) dt,$$

where the path $t \mapsto x_t$ is the evolution of the asset price during the trading period. The letter q stands for quantity of the asset and the trajectory $t \mapsto q(t)$, $[0, T] \rightarrow \mathbb{R}$, is referred to as inventory trajectory. Its time derivative \dot{q} is the rate of execution and it represents the control variable that a trader modulates while executing the trade.

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The minimisation of the trading cost J faces the challenge that the price path $t \mapsto x_t$ is not known at the beginning of the trading period. Hence, in order to gain some predictive power, a stochastic model for the evolution of the asset price is introduced. This is the second phase in the design of mathematical models for trade execution. Concretely, it means that a stochastic process $\{S_t : 0 \leq t \leq T\}$ is introduced and the actual price trajectory (x_t) is thought of as a realisation of this stochastic process. Then, the mathematical optimisation focuses on the expected trading cost

$$\mathbb{E} \left[\int_0^T F(t, S_t, q_t, \dot{q}_t) dt \right]. \quad (1)$$

Notice that this entails a considerable degree of model dependency, in that the optimisation is based on the distributional assumptions on the price process. Relevant references for the discussion to follow are Denny and Suchanek (1986), Xu and Zhou (2013), Cartea *et al.* (2017), Kentia and Kühn (2018), Roux and Zastawniak (2018), Huang *et al.* (2019), Casgrain and Jaimungal (2019a, 2019b), Cordoni and Persio (2020), and Mitchell and Chen (2020).

Two alternatives exist for the minimisations of the expected trading cost in equation (1). These alternatives are static minimisation (giving rise to static trading strategies) and dynamic minimisation (giving rise to dynamic trading strategies).

Static strategies are completely decided at the beginning of the trading period; they are based only on the information available at the initial time of the trade. Mathematically, this is formulated by considering q as a deterministic path. In this case, it is often observed that, by interchanging expectation and time integral in equation (1), the actual realisation of the price process disappears from the formulae, replaced by its expected trajectory. When the expected price path is the only feature of the price process that enters the formulae (as in Almgren and Chriss 2001), the static strategy does not take into account the volatility of asset prices, whose role however is paramount in financial markets. A visual representation of the relevance of volatility in the context of trade execution is provided in figure 1.

In figure 1, Almgren and Chriss's framework is adopted. The price process is a standard one-dimensional Brownian motion and two price paths are considered, one with low volatility and the other with high volatility. Notwithstanding the remarkable difference between the two, they have the same expected path (dashed blue line in the first quadrant) and, as a consequence, the static liquidation strategy is the same for both price paths (dashed blue line in the second quadrant). The simplicity of the model is such that it compromises on the possibility to distinguish rather different market regimes. This is made clear by comparing the static optimal solution with the a-posteriori one.

The a-posteriori solution is the minimiser q of the cost functional J given the actual price trajectory x . This is not implementable in real trading because it is anticipative, in that it assumes that the entire price trajectory is known at the beginning of the trading period. However, since it is independent of the choice of the price process, the a-posteriori solution constitutes a useful term of comparison for the stochastic model. In the example of figure 1, we observe how different the

two a-posteriori solutions corresponding to the two market regimes are. In the case of low volatility, the a-posteriori solution is close to the static one, because the price path does not depart significantly from its expected trajectory. Instead, in the case of high volatility, the a-posteriori solution deviates from the static one: the inventory trajectory is considerably steeper where the price is above its expected value, and it is almost flat when the price is below its expected value.

In order to take into account more features of the price process (such as its volatility), the literature on optimal trade execution has utilised the mathematical techniques of stochastic optimal control. This has produced the second alternative the minimisation of the expected cost in equation (1), and dynamic trading strategies proliferated since Bertsimas and Lo (1998) (in discrete time) and Gatheral and Schied (2011) (in continuous time). An excellent presentation of the techniques of stochastic optimal control applied to trade execution is contained in the textbook by Cartea *et al.* (2015).

Dynamic trading strategies take fully into account the distributional features of the price process because they are obtained via the Hamilton–Jacobi–Bellman equation, in which the generator of the diffusion that models the price enters.[†] Furthermore, dynamic strategies are random when seen from the initial time, in that they depend on the information that is revealed to the trader during the trading period. Mathematically, this means that dynamic strategies are stochastic processes adapted to the relevant market information filtration. Since deterministic strategies are in particular adapted stochastic processes, the class of static strategies is a subset of the class of dynamic strategies. Therefore, the minimisation over the class of dynamic strategies is expected to improve the result obtained when minimising over the smaller class of static strategies.

This, however, is not always confirmed in the models. Indeed, despite the mathematical sophistication, cases exist in which optimal trading strategies, although sought among dynamic ones, are in fact static. One of such cases is for example the ‘Liquidation without penalties only temporary impact’ in Cartea *et al.* (2015, section 6.3), another is the ‘Optimal acquisition with terminal penalty and temporary impact’ in Cartea *et al.* (2015, section 6.4). This reduction to static optimal solutions clashes with the intuition for which trading strategies should take into account actual realisations of price paths, as the a-posteriori solutions in figure 1 suggest.

A second drawback of applying the technique of HJB equation to the problem of optimal trade execution is the heavy model dependence. Optimality of the trading strategies holds under the assumption that the price follows some specified dynamics, and this invests of considerable importance the second phase in the design of mathematical models.

In this paper, we propose a new alternative for the minimisation of trading costs. This new alternative considers the pathwise optimisation of the cost functional J without taking

[†] In the case of linear temporary market impact and quadratic inventory cost, a recent work by Belak *et al.* (2018) actually discusses techniques that can be more generally applied to the case of general semimartingales. In this case, there is no HJB equation; instead the authors rely on forward–backward stochastic differential equations. In section 2, we will review this general solution.

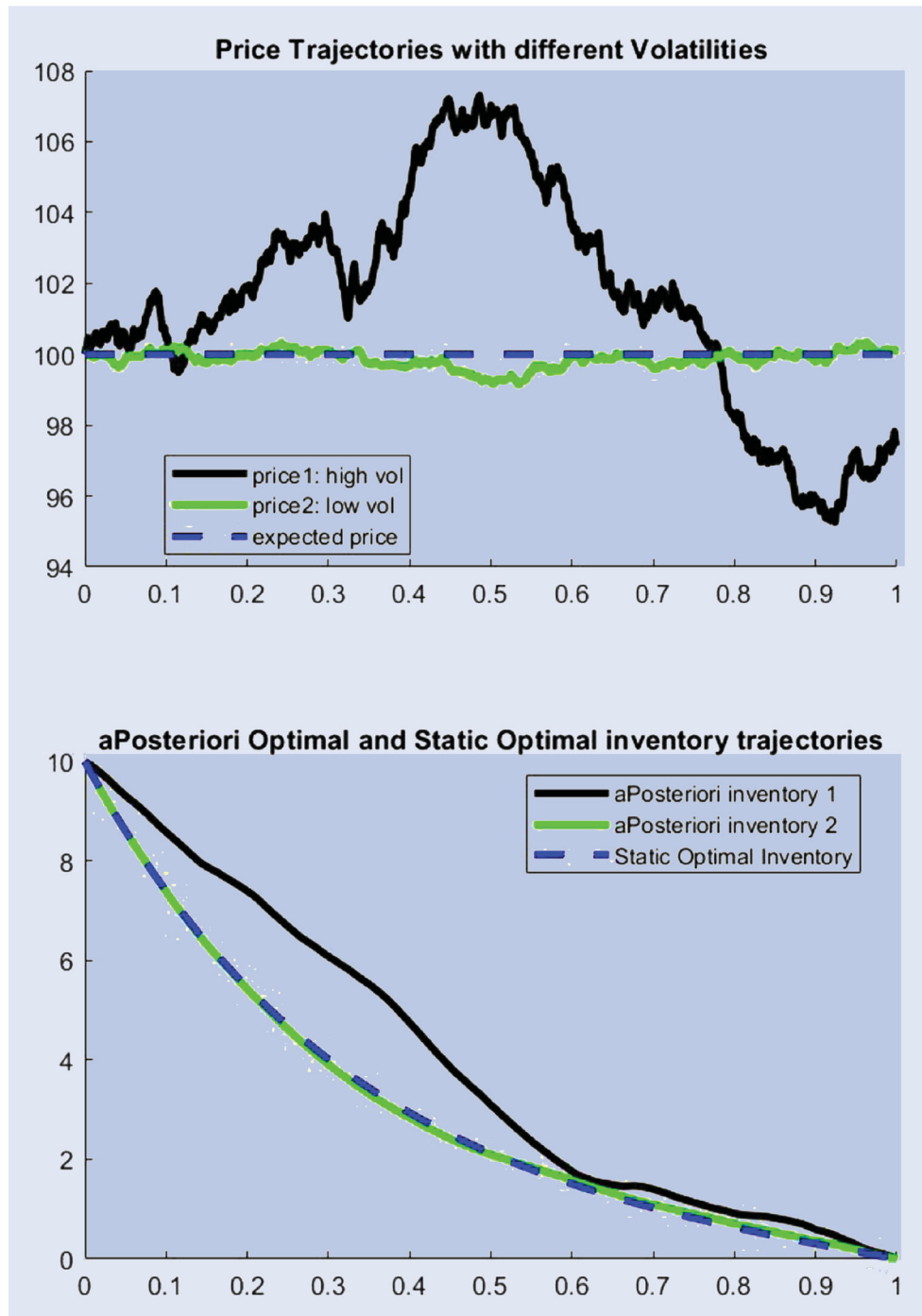


Figure 1. A-posteriori optimal and static optimal inventories in two different volatility regimes.

expectation. We observe that the reason for the anticipativeness of a-posteriori solutions is the imposition of the constraint that the liquidation terminates exactly at the (arbitrarily fixed) trading horizon. Relaxing this constraint enables to produce adapted pathwise solutions that display two remarkable features. On the one hand, they avoid the degeneracy to static trajectories even in the cases where the techniques of HJB equation do not produce genuinely dynamic strategies; on the other hand, their model dependence is moderate and confined to the expected trajectory of the price path, as was the case for static strategies, rather than to the full law of the price process.

Our trading strategies give rise to inventory trajectories that are obtained in closed-form formulae. Moreover, we can

characterise these trajectories as solutions to certain random Young differential equations, inspired by the second-order Euler–Lagrange equations in the classical Calculus of Variations. Such a characterisation allows one to implement the inventory trajectories via an easily-simulated initial value problem.

The rest of the paper is organised as follows. Section 2 describes in detail the mathematical framework in which the problem of optimal trade execution is formulated. Our descriptions examine in particular two aspects of the mathematical models. The first aspect (section 2.1) is the reduction to static optimal inventories that happens in the context of stochastic optimal control of the expected quantity in

equation (1). Proposition 2.2 examines such a reduction, listing its causes. This is novel in the literature and answers the questions raised in Brigo and Di Graziano (2014), Brigo and Piat (2018) and Bellani *et al.* (2018) about the comparison between static and dynamic solutions to the problem of optimal trade execution. The second aspect is the unbiasedness of liquidation errors (section 2.2).

Section 3 presents the concept of good trade execution. Section 3.1 specialises good trade executions in the case of linear temporary market impact and quadratic inventory cost. In particular, section 3.1.1 derives a closed-form formula for good trade executions, and section 3.1.2 characterises it in terms of a Cauchy problem with random Young differential equations. Uniqueness of the good trade execution follows from this characterisation.

Section 3.2 presents good trade executions with risk criteria other than the quadratic inventory cost. More precisely, section 3.2.1 considers a time-dependent variant of the quadratic inventory cost, whereas section 3.2.2 presents good trade executions when the risk criterion is inspired by the value-at-risk adopted in Gatheral and Schied (2011).

Finally, two applications are given in section 4, and section 5 concludes the paper. Appendix presents the mathematical apparatus on which the characterisation of good trade executions is based.

2. Framework

We adopt the perspective of liquidation; the case of acquisition is *mutatis mutandis* the same. Let x_0 denote initial inventory, and let $x_T = 0$ be the liquidation target. The letter q stands for the quantity of the asset and the trajectory $t \mapsto q(t)$, $[0, T] \rightarrow \mathbb{R}$, shall be referred to as inventory trajectory. Its time derivative \dot{q} is the rate of execution and it represents the control variable that a trader modulates while executing the trade. Without yet referring to any probabilistic structure, let us introduce the space of such inventory trajectories:

$$\mathcal{Q}_{pw}^{0,x_0} := \left\{ q : [0, T] \rightarrow \mathbb{R}, \quad q \text{ absolutely continuous,} \right. \\ \left. q(0) = x_0, \quad q(T) = x_T \right\}.$$

The subscript ‘pw’ stands for ‘pathwise’ and emphasises the non-probabilistic perspective.

DEFINITION 2.1 Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, and let $\{S_t : 0 \leq t \leq T\}$ be a stochastic process defined on it. We say that S is a price process if: 1. for all $0 \leq t \leq T$ the second moment of S_t is finite; 2. the maps $t \mapsto \mathbb{E}[S_t]$ and $t \mapsto \mathbb{E}[S_t^2]$ are in $L^1[0, T]$; 3. there exists some $p \geq 1$ such that all the paths of S are of finite p -variation, i.e. for all ω in Ω ,

$$\|S(\omega)\|_{p\text{-var}, [0, T]} < \infty.$$

Notice that the paths of the price process are not necessarily assumed to be continuous.

Given a price process $\{S_t : 0 \leq t \leq T\}$, we let $\{\mathfrak{F}_t : 0 \leq t \leq T\}$ be the minimal \mathbb{P} -completed right-continuous filtration generated by S . It is always assumed that \mathfrak{F}_0 is trivial.

If the price process is a semimartingale, we additionally introduce the following terminology. We say that the semimartingale S is a totally square integrable special semimartingale if the following two conditions hold:

- (1) the semimartingale S is a special semimartingale, i.e. it admits a canonical decomposition

$$S_t = S_0 + A_t + M_t, \quad 0 \leq t \leq T,$$

where A is a predictable bounded variation process, M is a local martingale, and $A_0 = M_0 = 0$;

- (2) the following integrability holds:

$$\mathbb{E}[\langle M \rangle_T] + \mathbb{E}[\|A\|_{2, [0, T]}^2] < \infty,$$

where $\langle M \rangle$ denotes the quadratic variation of the local martingale M , and $\|A\|_{2, [0, T]}$ denotes the 2-variation of the path A on the time interval $[0, T]$.[†]

Execution rates are progressively measurable square-integrable processes; more precisely, we define the space of execution rates as

$$\mathcal{R} := \left\{ r \in L^2([0, T] \times \Omega, \mathcal{B}[0, T] \otimes \mathfrak{F}_T, dt \otimes \mathbb{P}) : \right. \\ \left. r \text{ is } (\mathfrak{F}_t)_t\text{-progressively measurable} \right\}. \quad (2)$$

Notice that the measurability depends on the filtration of the price process.

Admissible inventory trajectories are first integrals of execution rates with initial value x_0 . More precisely, we define the space \mathcal{Q}^{0,x_0} of admissible inventory trajectories as

$$\mathcal{Q}^{0,x_0} := \left\{ (q_t)_{t \in [0, T]} : \exists r \in \mathcal{R}, q_t = x_0 + \int_0^t r_u du \quad \forall 0 \leq t \leq T \right\}. \quad (3)$$

Among admissible inventory trajectories we distinguish those that are fuel-constrained, namely such that their terminal value is $x_T = 0$. Thus a fuel-constrained admissible inventory trajectory is an $(\mathfrak{F}_t)_t$ -adapted process with absolutely continuous paths, with deterministic initial value x_0 , terminal value $x_T = 0$, and such that its derivative is in \mathcal{R} . More precisely, we define the space $\mathcal{Q}_{\text{fuel}}^{0,x_0}$ of fuel-constrained admissible

[†] Recall that the 2-variation $\|x\|_{2, [0, T]}$ of a path $x : t \mapsto x_t \in \mathbb{R}^d$ is defined as

$$\|x\|_{2, [0, T]} := \sup \left\{ \left(\sum_{t_i} |x_{t_{i+1}} - x_{t_i}|^2 \right)^{1/2} : \right. \\ \left. 0 = t_0 < t_1 < \dots < t_n = T \right\},$$

where the supremum is taken over all the partitions of the interval $[0, T]$.

inventory trajectories as

$$\mathcal{Q}_{\text{fuel}}^{0, x_0} := \{(q_t)_{t \in [0, T]} \in \mathcal{Q}^{0, x_0} : q_T = x_T \text{ } \mathbb{P}\text{-a.s.}\}.$$

Notice that every realisation of a generic q in $\mathcal{Q}_{\text{fuel}}^{0, x_0}$ is a path in $\mathcal{Q}_{\text{pw}}^{0, x_0}$, namely for all q in $\mathcal{Q}_{\text{fuel}}^{0, x_0}$ and all ω in Ω it holds

$$(q_t(\omega))_{0 \leq t \leq T} \in \mathcal{Q}_{\text{pw}}^{0, x_0}.$$

In the space of fuel-constrained inventory trajectories we isolate the subspace of static trajectories, given by

$$\mathcal{Q}_{\text{static}}^{0, x_0} = \left\{ q \in \mathcal{Q}_{\text{fuel}}^{0, x_0} : q_t \text{ is } \mathfrak{F}_0\text{-measurable for all } t \geq 0 \right\}.$$

These are the execution strategies whose entire trajectories are \mathfrak{F}_0 -measurable, namely deterministic. We say that the admissible inventory trajectories not in $\mathcal{Q}_{\text{static}}^{0, x_0}$ are non-static (or dynamic): therefore, the admissible inventory trajectory q is non-static if q is in $\mathcal{Q}^{0, x_0} \setminus \mathcal{Q}_{\text{static}}^{0, x_0}$.

It is convenient to extend the definitions of the spaces of inventory trajectories to the case where the initial time is not zero. The symbols $\mathcal{Q}_{\text{pw}}^{t, x_t}$, $\mathcal{Q}_{\text{fuel}}^{t, x_t}$, $\mathcal{Q}_{\text{fuel}}^{t, x_t}$ and $\mathcal{Q}_{\text{static}}^{t, x_t}$ will denote the straightforward generalisations of the definitions above to the case where the initial time is t in $[0, T)$ and the trajectories are pinned to the value x_t at time t .

With the notation introduced so far, we now formulate the classical stochastic optimisation problem associated with optimal trade execution.

Let $X = (S, q)$ denote the state variable, which keeps track of the fundamental price S and of the inventory q . The dynamics of X is controlled by an execution rate \dot{q} in \mathcal{R} . In order to emphasise this dependence, we can write $X = X^r$, where r is the control in the space \mathcal{R} of execution rates. With this notation, we express the objective function $H = H^{\dot{q}}$ of the classical stochastic optimisation problem as

$$H^{\dot{q}}(t, x_1, x_2) := \mathbb{E}_{t, S_t = x_1, q_t = x_2} \left[\int_t^T F(s, X_s^{\dot{q}}, \dot{q}_s) ds \right], \quad (4)$$

where q is in $\mathcal{Q}_{\text{fuel}}^{0, x_0}$, and where $F = F(t, X, r) = F(t, S, q, r)$ is a Lagrangian that describes risk-adjusted execution-impacted costs from trade. The stochastic optimisation problem for fuel-constrained inventory trajectories is therefore written as

$$\inf \left\{ H^{\dot{q}}(0, S_0, x_0) : q \in \mathcal{Q}_{\text{fuel}}^{0, x_0} \right\}. \quad (5)$$

An important aspect in the definition of the Lagrangian F in equation (4) is the description of how the trade execution impacts the price, i.e. the market impact. In this work, we focus on the so-called temporary market impact.

Let S_t denote the price process at time t . We say that the liquidator exerts a temporary market impact on S_t if for some function g in $C(\mathbb{R}^2)$ the execution price of her order at time t is

$$g(S_t, \dot{q}_t),$$

where $t \mapsto q_t$ is the liquidator's inventory trajectory, and \dot{q}_t denotes its time derivative at time t . A well-known example of temporary market impact is given by $g(S, r) = S + c_1^2 r$,

for some coefficient $c_1 > 0$ of market impact. In this case, the execution cost is a linear function of the rate of execution \dot{q} ; since in a liquidation q is decreasing, the steepest the inventory trajectory is at time t , the smaller the execution price is at time t . The classical formulation in Almgren and Chriss (2001) utilises this linear temporary market impact.

In the following sections 2.1 and 2.2, we introduce the concepts of reduction to static optimal strategies and the concept of liquidation error. We show that, in the context of linear temporary market impact with quadratic inventory cost, fuel-constrained optimal liquidation strategies are bound to be static, and non-fuel constrained optimal liquidation strategies commit biased errors of liquidation. This motivates the search for a formulation of the problem of optimal execution that is alternative to the classical one of equation (5). A possible alternative will then be presented in section 3; under this alternative, optimal liquidation strategies will be non-static, and—despite being non-fuel constrained—they will have unbiased liquidation errors.

2.1. Reduction to static optimal strategies

In a temporary market impact model, trade revenues gained in the infinitesimal time dt are $-g(S_t, \dot{q}_t)\dot{q}_t dt$. When the temporary market impact is linear, this becomes

$$(-S_t \dot{q}_t - c_1^2 \dot{q}_t^2) dt,$$

where revenues decompose in a first summand $S_t \dot{q}_t$ where the price process appears, and a second summand $c_1^2 \dot{q}_t^2$ that does not comprise the price process. Clearly, such a decomposition holds in more general situations than the one of linear market impact. If this decomposition holds for the whole Lagrangian F and if the bounded variation component A of the price process S is deterministic, then we observe the reduction of optimal dynamic solutions to optimal static ones. This happens in some cases studied in the literature (see Cartea *et al.* 2015, sections 6.3 and 6.4), where the optimal inventory trajectory, although sought dynamic, is eventually found to be static. It means that the optimiser of (5) is in the space $\mathcal{Q}_{\text{static}}^{0, x_0}$ of static inventory trajectories. The following proposition explains this phenomenon, pointing out those aspects of the model that cause the reduction to static trade executions.

PROPOSITION 2.2 ('Reduction to static optimal trade executions') *Assume that*

$$F(t, X, r) = rS + L(t, q, r), \quad (6)$$

for some Caratheodory function[†] L that does not depend on S . Assume that there exist an integrable function α on $[0, T]$

[†] See definition A.2 in appendix for the definition of Caratheodory function. The function L in the statement of proposition 2.2 is assumed to be a Caratheodory function with the choices: 1. the open interval $(0, T)$ as the subset U of \mathbb{R}^n in definition A.2; 2. the two-dimensional variable (q, r) as the variable ξ in definition A.2.

and a constant $\beta \geq 0$ such that

$$|L(t, q, r)| \leq \alpha(t) + \beta(q^2 + r^2).$$

Let the price process S be a totally square integrable continuous canonical semimartingale with canonical decomposition

$$S_t = S_0 + A_t + M_t, \quad 0 \leq t \leq T. \quad (7)$$

Assume that A is \mathcal{F}_0 -measurable, namely that the drift of the price process is deterministic. Then, for all $0 \leq t \leq T$ it holds

$$\inf \left\{ H^{\dot{q}}(t, S_t, \mathbf{x}_t) : q \in \mathcal{Q}_{\text{static}}^{t, \mathbf{x}_t} \right\} = \inf \left\{ H^{\dot{q}}(t, S_t, \mathbf{x}_t) : q \in \mathcal{Q}_{\text{fuel}}^{t, \mathbf{x}_t} \right\}.$$

Proof We give first the proof in the case where M in the canonical decomposition of S is a martingale.

Let q be in $\mathcal{Q}_{\text{fuel}}^{t, \mathbf{x}_t}$. Let X be the state variable $X = (S, q)$ and let Y be the two-dimensional path $Y = (q, A)$. Let φ in $C^\infty(\mathbb{R}^2)$ be the function $\varphi(x, y) = xy$. Notice that

$$\varphi(X_r^{\dot{q}}) - \varphi(X_t^{\dot{q}}) - \int_t^r X_s^{\dot{q}} dY_s, \quad t \leq r,$$

is a centred martingale. Hence,

$$\begin{aligned} H^{\dot{q}}(t, S_t, \mathbf{x}_t) &= \mathbb{E}_t \left[\int_t^T F(s, X_s^{\dot{q}}, \dot{q}_s) ds \right. \\ &\quad \left. + \varphi(X_T^{\dot{q}}) - \varphi(X_t^{\dot{q}}) - \int_t^T X_r^{\dot{q}} dY_r \right] \\ &= -\mathbf{x}_t S_t + \mathbb{E}_t \left[\int_t^T (-q_r dA_s + L(r, q_r, \dot{q}_r) dr) \right]. \end{aligned}$$

It holds

$$\begin{aligned} &\inf_{q \in \mathcal{Q}_{\text{fuel}}^{t, \mathbf{x}_t}} H^{\dot{q}}(t, S_t, \mathbf{x}_t) \\ &\geq -\mathbf{x}_t S_t + \mathbb{E}_t \left[\inf_{q \in \mathcal{Q}_{\text{pw}}^{t, \mathbf{x}_t}} \int_t^T (-q_r dA_s + L(s, q_s, \dot{q}_s) ds) \right], \end{aligned}$$

where the infimum on the right-hand side is taken in a path-wise sense for each realisation of the price S . In fact, the integrand does not depend on such a realisation (i.e. it does not depend on ω in Ω) because A is non-random. Therefore, any minimising sequence for the infimum inside the expectation is actually independent of ω and we have

$$\begin{aligned} &\inf_{q \in \mathcal{Q}_{\text{fuel}}^{t, \mathbf{x}_t}} H^{\dot{q}}(t, S_t, \mathbf{x}_t) \\ &\geq -\mathbf{x}_t S_t + \inf_{q \in \mathcal{Q}_{\text{static}}^{t, \mathbf{x}_t}} \mathbb{E}_t \left[\int_t^T (-q_r dA_s + L(s, q_s, \dot{q}_s) ds) \right] \\ &= \inf_{q \in \mathcal{Q}_{\text{static}}^{t, \mathbf{x}_t}} H^{\dot{q}}(t, S_t, \mathbf{x}_t). \end{aligned}$$

This yields the stated equality in the case where M is a martingale. If instead M is only a local martingale, a standard localisation argument concludes the proof. ■

REMARK 2.3 The classical optimal trade execution proposed by Almgren and Chriss (2001) was originally formulated with optimality claimed over the set of static inventory trajectories and under the assumption that the price process is an arithmetic Brownian motion. However, it is easy to show that the same solution of the static optimisation is obtained if the Brownian motion is replaced by any square-integrable martingale. In this sense, the static optimal solution of Almgren and Chriss is robust. In view of proposition 2.2, this robustness actually extends to the case where the liquidation strategy is regarded as the optimiser over the class of fuel-constrained inventory trajectories.

REMARK 2.4 A simple case where the optimal trading strategy is non-static is discussed by Gatheral and Schied (2011). This means that the optimal inventory trajectory obtained from the stochastic control problem is in the space $\mathcal{Q}_{\text{fuel}}^{0, \mathbf{x}_0} \setminus \mathcal{Q}_{\text{static}}^{0, \mathbf{x}_0}$. In view of proposition 2.2, we understand the dynamism of their solution by noticing the following. The risk measure adopted by those authors (see Gatheral and Schied 2011, section 2.1) is the value-at-risk of the position $q_t S_t$, and this has the consequence of disrupting the assumption that the Lagrangian F can be decomposed as in equation (6). Indeed, Gatheral and Schied consider the optimisation

$$\inf \left\{ \mathbb{E} \left[\int_0^T (\dot{q}_t^2 + \lambda q_t S_t) dt \right] : q \in \mathcal{Q}_{\text{fuel}}^{0, \mathbf{x}_0} \right\}, \quad (8)$$

where the price process $S_t = \exp(\sigma W_t - \sigma^2 t/2)$ is the exponential martingale of σW , where W denotes the standard one-dimensional Brownian motion, and where λ and σ are positive coefficients. Equation (8) is Gatheral and Schied (2011, equation (2.7)). Alternatively, it can be noticed that the same minimisation as in equation (8) is produced by choosing $F(t, X, r) = rS + r^2$ and $dS = \lambda S dt + \sigma S dW$. Indeed, the expected cost

$$\mathbb{E} \left[\int_0^T (\dot{q}_t^2 + \dot{q}_t S_t) dt \right], \quad \text{with } dS_t = \lambda S_t dt + \sigma S_t dW_t, \quad (9)$$

differs from the expected cost in equation (8) (where the price process is the exponential martingale) only by a constant. With the modelling choices in equation (9), the Lagrangian F does not incorporate any risk criterion and thus it satisfies the assumptions of proposition 2.2, but the price process S has a position dependent drift coefficient, violating the assumption that A in equation (7) is deterministic.

REMARK 2.5 In view of proposition 2.2, we understand why incorporating signals (i.e. short-term price predictors) in the framework of optimal trade execution leads to dynamic optimal strategies (see Cartea and Jaimungal 2016 and Lehalle and Neuman 2019). Indeed, signals are incorporated by modelling the price evolution as

$$dS_t = I_t dt + \sigma(t, S_t) dW_t,$$

where I_t is a Markov process that represents the signal. The stochasticity of I disrupts the assumption on the drift A in proposition 2.2.

COROLLARY 2.6 Assume the setting of proposition 2.2. Assume that the price process is modelled as the diffusion

$$dS_t = \mu(t) dt + \sigma(t, S_t) dW_t, \quad (10)$$

for some measurable Lipschitz coefficients μ and σ . Assume that the drift coefficient μ is a deterministic function of time only. Assume that

- (1) for all t the map $(q, r) \mapsto -\mu(t)q + L(t, q, r)$ is strictly convex;
- (2) there exist exponents $p > m \geq 1$ and coefficients $\alpha_1 > 0, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$-\mu(t)q + L(t, q, r) \geq \alpha_1 |r|^p + \alpha_2 |q|^m + \alpha_3,$$

for all t, q and r .

Then, the infimum in equation (5) is attained for some optimal deterministic q in $\mathcal{Q}_{\text{static}}^{0, x_0} \cap W^{1,p}(0, T)$, where $W^{1,p}(0, T)$ denotes the Sobolev space of absolutely continuous function such that their p th power and the p th power of their derivative are integrable on the time interval $(0, T)$.

REMARK 2.7 The assumptions on F in corollary 2.6 are satisfied in particular by the classical choice

$$F(t, S, q, r) = rS + c_1^2 r^2 + c_2^2 q^2, \quad (11)$$

where $c_1 > 0$ is a coefficient of temporary market impact and $c_2 \geq 0$ is a coefficient of risk aversion (or of inventory cost). Therefore, corollary 2.6 explains why in Cartea *et al.* (2015, section 6.3) the optimal solution is sought dynamic and eventually found to be static. This also says that, although in Almgren and Chriss (2001) the optimal trade execution was sought only over the class $\mathcal{Q}_{\text{static}}$ for tractability, this was in fact without loss of generality (remark 2.3).

Proof of Corollary 2.6 The fact that the infimum over $\mathcal{Q}_{\text{fuel}}^{0, x_0}$ is actually the same as the infimum over $\mathcal{Q}_{\text{static}}^{0, x_0}$ follows from proposition 2.2. Existence, uniqueness and p -integrability of the minimiser follow from the two assumptions on the function $(t, q, r) \mapsto -\mu(t)q + L(t, q, r)$; see Dacorogna (2008, theorem 4.1). ■

2.2. Errors of liquidation

The space $\mathcal{Q}_{\text{fuel}}^{0, x_0}$ of fuel-constrained admissible inventory trajectories has been isolated from the space \mathcal{Q}^{0, x_0} of first integrals of execution rates. An inventory trajectory q in $\mathcal{Q}^{0, x_0} \setminus \mathcal{Q}_{\text{fuel}}^{0, x_0}$ is said to commit a liquidation error, because with positive probability $q_T \neq x_T$. Liquidation errors are common among dynamic solutions to optimal trade execution problems. This is because the mathematical techniques used for dynamic solutions are not well-suited to simultaneously impose the two constraints $q_0 = x_0$ and $q_T = x_T$. Clearly, the constraint $q_0 = x_0$ has the priority and hence the constraint $q_T = x_T$ is relaxed. The usual relaxation entails to introduce a terminal penalisation for the outstanding inventory

at final time. Hence, if F is the Lagrangian describing risk-adjusted cost of trade, it is custom to relax the minimisation in equation (5) and consider instead the problem

$$\inf \left\{ \mathbb{E} \left[\int_0^T F(t, X_t^q, \dot{q}_t) dt + c_5^2 (q_T - x_T)^2 \right] : q \in \mathcal{Q}^{0, x_0} \right\}, \quad (12)$$

where $c_5 \geq 0$ is a coefficient of penalisation for outstanding terminal inventory. Notice that the minimisation is performed over the broad class \mathcal{Q}^{0, x_0} of first integrals of execution rates. Notice also that the objective function in equation (12) can be expressed in the general form discussed so far because

$$\begin{aligned} & \mathbb{E} \left[\int_0^T F(t, X_t^q, \dot{q}_t) dt + c_5^2 (q_T - x_T)^2 \right] \\ &= \mathbb{E} \left[\int_0^T G(t, X_t^q, \dot{q}_t) dt \right], \end{aligned}$$

where $G(t, X_t^q, \dot{q}_t) = F(t, X_t^q, \dot{q}_t) + 2c_5^2 \dot{q}_t q_t$.

We isolate liquidation errors whose expected value is null from those that on average either finish the liquidation before the time horizon T (negative liquidation error) or after it (positive liquidation error).

DEFINITION 2.8 We say that the admissible inventory trajectory q in \mathcal{Q}^{0, x_0} has an unbiased liquidation error if $\mathbb{E}[q_T] = x_T$. We say that q has a biased liquidation error if instead $\mathbb{E}[q_T] \neq x_T$.

By extension we say that a liquidation strategy is unbiased if its inventory trajectory has unbiased liquidation error, and we say that it is biased if it is not unbiased.

The next proposition shows that the classical optimisation problem corresponding to linear temporary market impact with quadratic inventory cost produces in general optimal liquidation strategies with biased liquidation error. In other words, if the optimal inventory trajectory in this framework happens to have unbiased liquidation error, this unbiasedness is not robust with respect to the values of the model parameters c_1, c_2 and c_5 : independently changing these values will disrupt the expected value of the inventory at T , turning it into a biased termination.

The solution to the optimisation problem is derived from Belak *et al.* (2018). Notice that the statement is general with respect to the distributional assumption of the price process, which is only assumed to be a totally square integrable semimartingale.

PROPOSITION 2.9 Let the price process S be a totally square integrable special semimartingale. Consider the minimisation problem

$$\inf \left\{ \mathbb{E} \left[\int_0^T F(t, S_t, q_t, \dot{q}_t) dt + c_5^2 (q_T - x_T)^2 - q_T S_T \right] : q \in \mathcal{Q}^{0, x_0} \right\}, \quad (13)$$

where the Lagrangian F is $F(t, S_t, q_t, \dot{q}_t) = \dot{q}_t S_t + c_1^2 \dot{q}_t^2 + c_2^2 q_t^2$. Let \hat{q} be the minimiser for (13), and let $M :=$

$\{(c_1, c_2, c_5) \in \mathbb{R}_+^3 : \mathbb{E}[\hat{q}_T] = x_T\}$. Then, M is included in a manifold of dimension 2.

Proof Let c_3 be the ratio $c_3 = c_2/c_1$ between the coefficient c_2 of risk aversion and the coefficient c_1 of linear temporary market impact. Let c_6 be the ratio $c_6 = c_5/c_1$ between the coefficient c_5 of penalisation of outstanding inventory at time T and the coefficient c_1 of linear temporary market impact. Define the functions φ and Φ as follows:

$$\begin{aligned} \varphi(t) &:= c_3 \cosh(c_3 t) + c_6^2 \sinh(c_3 t), \quad t \geq 0, \\ \Phi(s, t) &:= \frac{\varphi(T-t)}{\varphi(T-s)}, \quad 0 \leq s \leq t \leq T. \end{aligned} \quad (14)$$

Let $v(t)$ be the following conditional expectation at time t :

$$v(t) := \mathbb{E}_t \left[\frac{1}{2c_1^2} \int_t^T \Phi(t, r) dS_r \right] = \mathbb{E}_t \left[\frac{1}{2c_1^2} \int_t^T \Phi(t, r) dA_r \right], \quad (15)$$

where S is the price process with canonical decomposition $S_t = S_0 + A_t + M_t$. Then, Belak *et al.* (2018, theorem 3.1) proves that the optimal inventory trajectory that solves the minimisation problem in equation (13) is

$$\hat{q}_t = \Phi(0, t)x_0 + \int_0^t \Phi(s, t)v(s) ds. \quad (16)$$

This minimiser produces unbiased liquidation errors only if

$$\frac{\varphi(0)}{\varphi(T)}x_0 = \frac{1}{2c_1^2} \int_0^T \Phi(t, T) \mathbb{E} \left[\int_t^T \Phi(t, r) dA_r \right] dt.$$

Consider φ and Φ as functions of (c_1, c_2, c_5) . Let $f : \mathbb{R}_+^3 \rightarrow \mathbb{R}$ be defined as

$$\begin{aligned} f(c_1, c_2, c_5) &= \frac{\varphi(0)}{\varphi(T)}x_0 \\ &\quad - \frac{1}{2c_1^2} \int_0^T \Phi(t, T) \mathbb{E} \left[\int_t^T \Phi(t, r) dA_r \right] dt. \end{aligned}$$

Then, 0 is a regular value of f and $M \subset f^{-1}(0)$. ■

REMARK 2.10 In the spirit of proposition 2.2, we remark that the solution \hat{q} to the minimisation problem in equation (13) is static if the drift of the price process is deterministic, in particular if the price process is a martingale.

REMARK 2.11 When the price process is a martingale, the optimal inventory trajectory of equation (16) is such that the terminal value is

$$q_T = \frac{c_3 x_0}{c_3 \cosh(c_3 T) + c_6^2 \sinh(c_3 T)}.$$

In this case then, the optimal inventory trajectory will always finish with a positive inventory left to liquidate after the initially fixed time horizon T of the liquidation.

REMARK 2.12 A liquidation strategy that is unbiased for any choice of c_1 and c_2 is obtained from equation (16) only in the

limit as $c_5 \uparrow \infty$, which yields the fuel-constrained solution

$$\begin{aligned} q_t &= R(0, t)x_0 - \frac{1}{2c_1^2} \int_0^t R(s, t) S_s ds \\ &\quad - \frac{1}{2c_1^2} \int_0^t R(s, t) \int_s^T \mathbb{E}_s[S_r] \partial_r R(s, r) dr ds, \end{aligned}$$

where $R(s, t) = \sinh(c_3(T-t))/\sinh(c_3(T-s))$. This says that the inventory trajectory in equation (16) has unbiased liquidation error only in the degenerate case of deterministic terminal inventory.

3. Good trade executions

Let S be a price process as defined in definition 2.1. Let the class \mathcal{R} of inventory rates and the class \mathcal{Q}^{0, x_0} of admissible inventory trajectories be as defined in equations (2) and (3) respectively.

From the class \mathcal{Q}^{0, x_0} of admissible inventory trajectories, we isolate the class of unbiased admissible inventory trajectories. An unbiased admissible inventory trajectory is defined as an $(\mathcal{F}_t)_t$ -adapted process with absolutely continuous paths, with deterministic initial value x_0 , expected terminal value x_T , and such that its derivative is in \mathcal{R} . More precisely, we define the space \mathcal{U}^{0, x_0} of unbiased admissible inventory trajectories as

$$\mathcal{U}^{0, x_0} := \{(q_t)_{t \in [0, T]} \in \mathcal{Q}^{0, x_0} : \mathbb{E}[q_T] = x_T\}.$$

The constraint $\mathbb{E}[q_T] = x_T$ relaxes the fuel constraint $q_T = x_T$ used in the definition of $\mathcal{Q}_{\text{fuel}}^{0, x_0}$. Recall that, without loss of generality, the liquidation target x_T is set equal to 0; nonetheless, we do not suppress it from our equations because this makes the formulae easier to interpret (see remarks 3.13 and 3.15).

We consider the following minimisation problem over the class of unbiased admissible inventory trajectories:

$$\inf \left\{ \int_0^T F(t, S_t, q_t, \dot{q}_t) dt : q \in \mathcal{U}^{0, x_0} \right\}, \quad (17)$$

where $F = F(t, x_1, x_2, x_3) : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a space-differentiable Caratheodory function.† We use the symbol J to denote the map $q \mapsto \int_0^T F(t, S_t, q_t, \dot{q}_t) dt$, for q in \mathcal{U}^{0, x_0} .

ASSUMPTION 3.1 Let $F = F(t, x_1, x_2, x_3) : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be the Lagrangian in the minimisation problem (17). It is assumed that F is a space-differentiable Caratheodory function, and that the function $L = L(t, x_1, x_2, x_3) := F(t, x_1, x_2, x_3) - x_1 x_3$ is such that: (1) L is in the Sobolev space $W^{1,4}((0, T) \times K)$ for all compact subsets K of \mathbb{R}^3 ; (2) for almost every t in $(0, T)$, $(\partial_{x_2} L)^2(t, 0, x, 0) + (\partial_{x_3} L)^2(t, 0, 0, x) = 0$ only if $x = 0$; (3) the functions $t \mapsto 1/L(t, 0, 1, 0)$ and $t \mapsto 1/L(t, 0, 0, 1)$ are non-negative and square-integrable over $(0, T)$.

Assumption 3.1 is used to associate the Lagrangian F in equation (17) with a weight function on the Sobolev space $W^{1,4}(0, T)$ and with a weight function on \mathcal{U}^{0, x_0} .

† See definitions A.2 and A.3.

DEFINITION 3.2 Let $F = F(t, x_1, x_2, x_3) : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a space-differentiable Caratheodory function. Let F satisfy assumption 3.1. Let η be in $W^{1,4}(0, T)$. Then, the pathwise F -weight $|\eta|_F$ of η is defined by the equation

$$|\eta|_F^2 = \int_0^T (\ell_2^2(t, \eta_t) + \ell_3^2(t, \dot{\eta}_t)) dt, \quad (18)$$

where

$$\ell_2(t, x) := \frac{\partial_{x_2} L(t, 0, x, 0)}{2\sqrt{L(t, 0, 1, 0)}}, \quad \ell_3(t, x) := \frac{\partial_{x_3} L(t, 0, 0, x)}{2\sqrt{L(t, 0, 0, 1)}}.$$

REMARK 3.3 Let $F : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a space-differentiable Caratheodory function, and define $L(t, x) := F(t, x) - x_1 x_3$. Assume that L satisfies points (1) and (2) in assumption 3.1. Assume that $t \mapsto 1/L(t, 0, 0, 1)$ is non-negative and square-integrable. If $\partial_{x_2} L(t, 0, x, 0) = 0$ for all t and all x , then we drop the requirement that $t \mapsto 1/L(t, 0, 1, 0)$ is non-negative and square-integrable and we understand equation (18) with the convention that $\ell_2 \equiv 0$.

DEFINITION 3.4 Let F be as in definition 3.2. Let η be in \mathcal{U}^{0, x_0} . Then, the F -weight $\|\eta\|_F$ of η is defined by the following equation:

$$\|\eta\|_F^2 = \mathbb{E} [|\eta|_F^2], \quad (19)$$

where $|\eta|_F$ is the random variable $\omega \mapsto |\eta(\omega)|_F$, where $\{\eta(\omega) : \omega \in \Omega\}$ are the paths of η and, for every ω in Ω , $|\eta(\omega)|_F$ is the pathwise F -weight of $t \mapsto \eta_t(\omega)$.

Every square-integrable random variable ξ with $\mathbb{E}[\xi] = x_T$ identifies a subclass of trajectories in \mathcal{U}^{0, x_0} with specified terminal (random) variable. More precisely, for every ξ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\xi] = x_T$ we define

$$\mathcal{U}^{0, x_0}(\xi) := \{q \in \mathcal{U}^{0, x_0} : \mathbb{P}(q_T = \xi) = 1\}.$$

The class $\mathcal{U}^{0, x_0}(\xi)$ with $\xi \equiv x_T$ is the class of inventory trajectories that commit no liquidation error, namely $\mathcal{U}^{0, x_0}(x_T) = \mathcal{Q}_{\text{fuel}}^{0, x_0}$.

DEFINITION 3.5 ('Optimal execution of terminal variable ξ ') Let ξ be in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\xi] = x_T$. We say that q in \mathcal{U}^{0, x_0} is the optimal execution of terminal variable ξ if q minimises $\eta \mapsto J(\eta)$ over $\mathcal{U}^{0, x_0}(\xi)$, namely if $\mathbb{P}(q_T = \xi) = 1$ and for all η in $\mathcal{U}^{0, x_0}(\xi)$ it holds

$$\int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt \geq \int_0^T F(t, S_t, q_t, \dot{q}_t) dt$$

with probability one.

For every $q \in \mathcal{U}^{0, x_0}$, we trivially have that $q \in \mathcal{U}^{0, x_0}(q_T)$. We can define a 'tubular' neighbourhood of $\mathcal{U}^{0, x_0}(q_T)$ by looking at those trajectories η in \mathcal{U}^{0, x_0} such that the $L^2(\mathbb{P})$ -norm of the difference $\eta_T - q_T$ between terminal values is controlled by the F -weight of the difference $\eta - q$. More precisely, for q in \mathcal{U}^{0, x_0} and $C \geq 0$ we set

$$\mathcal{U}^{0, x_0}(q, C) := \{\eta \in \mathcal{U}^{0, x_0} : \|\eta_T - q_T\|_{L^2(\mathbb{P})} \leq C \|\eta - q\|_F^2\}. \quad (20)$$

This captures the idea of η_T not being too far from the terminal value q_T given that the trajectory η has kept close to q in the time window $0 \leq t < T$.

Also, we define a pathwise analogous to the tubular neighbourhood of equation (20). Given a non-negative ξ in $L^2(\mathbb{P})$ we define

$$\mathcal{U}_{\text{pw}}^{0, x_0}(q, \xi) := \left\{ \eta \in \mathcal{U}^{0, x_0} : |\eta_T - q_T| \leq \xi |\eta - q|_F^2 \right\}, \quad (21)$$

where $|\eta_T - q_T|$ is the absolute value of the difference between the values of η and of q at time T , and $|\eta - q|_F$ is the pathwise F -weight of the difference $\eta - q$.

REMARK 3.6 Notice that both $\mathcal{U}^{0, x_0}(q, C)$ and $\mathcal{U}_{\text{pw}}^{0, x_0}(q, \xi)$ depend on the Lagrangian F . Nonetheless, we omit this dependence from the notation, and the symbols for these tubular neighbourhoods do not carry reference to F .

DEFINITION 3.7 ('Good trade execution') We say that q in \mathcal{U}^{0, x_0} is a (C, ξ) -good trade execution for the minimisation in equation (17) if there exist $\xi \in L_+^2(\mathbb{P})$ and $C \geq 0$ such that

- (1) for all η in $\mathcal{U}^{0, x_0}(q, C)$ it holds

$$\mathbb{E} \left[\int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt \right] \geq \mathbb{E} \left[\int_0^T F(t, S_t, q_t, \dot{q}_t) dt \right];$$

- (2) for all η in $\mathcal{U}_{\text{pw}}^{0, x_0}(q, \xi)$ it holds

$$\int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt \geq \int_0^T F(t, S_t, q_t, \dot{q}_t) dt,$$

with probability one.

When we emphasise the path $t \mapsto q_t$ of a good trade execution, we use interchangeably the term *good inventory trajectory*.

REMARK 3.8 A good trade execution is in particular an optimal execution of its own terminal variable: if q is as in definition 3.7, then q is an optimal execution of terminal variable q_T as defined in definition 3.5. In other words, a (C, ξ) -good trade execution is in particular a $(0, 0)$ -good trade execution.

3.1. Quadratic inventory cost

In this section (section 3.1), we consider the following Lagrangian F :

$$F(t, S, q, r) := rS + c_1^2 r^2 + c_2^2 q^2, \quad (22)$$

where $c_1 > 0$ is a coefficient of market impact, and $c_2 \geq 0$ is a coefficient of risk aversion. For future reference, we set $c_3 := c_2/c_1$. Notice that in fact F does not depend on t . We study the problem in (17) with F as in equation (22).

REMARK 3.9 The Lagrangian F in equation (22) represents risk-adjusted revenues from trade where the market impact is temporary and linear, and the risk criterion is quadratic inventory cost. This aligns to common modelling choices such as those in Lehalle and Neuman (2019) and in Belak

et al. (2018). However, our relaxation of the fuel constraint entails that the inventory is sought in \mathcal{U}^{0,x_0} ; we do not modify the objective function as is instead common in the studies of optimal dynamic liquidation strategies, where the terms of terminal asset position $q_T S_T$ and of terminal inventory cost $c_5^2(q_T - x_T)^2$ are usually added to the function that describes revenues from trade (see beginning of section 2.2).

REMARK 3.10 The Lagrangian F in equation (22) is the same as the Lagrangian in equation (11). However, the optimisation in equation (17) is pathwise and hence it differs from the classical optimisation of expected risk-adjusted revenues used in equation (5). For this reason, the martingale cancellation exploited in the proof of proposition 2.2 is not applicable to the present case: we will be able to produce a non-static solution also in the case where the price process has deterministic drift (in particular, where the price process is a martingale).

LEMMA 3.11 Let F be as in equation (22). Then, F satisfies assumption 3.1. Moreover, the pathwise F -weight is a seminorm on $W^{1,4}(0, T)$, and the F -weight is a seminorm on \mathcal{U}^{0,x_0} . If $c_2 > 0$, then these seminorms are norms.

Proof As for the requirements in assumption 3.1, we only notice that the case $c_2 = 0$ is covered in remark 3.3. As for the second part of the claim, we apply lemma A.1 from appendix to see that the pathwise F -weight is a seminorm. The fact that the F -weight is a seminorm follows from the fact that the pathwise F -weight is a seminorm. Finally, $c_2 > 0$ guarantees that $|\eta - \tilde{\eta}|_F > 0$ if $\eta \neq \tilde{\eta}$. ■

We denote the pathwise seminorm induced by the pathwise F -weight by $|\cdot|_{c_1, c_2}$. More precisely, we set

$$|\eta|_{c_1, c_2}^2 := \int_0^T (c_2^2 \eta_t^2 + c_1^2 \dot{\eta}_t^2) dt, \quad (23)$$

for η in $W^{1,4}(0, T)$. Moreover, we denote the seminorm on \mathcal{U}^{0,x_0} induced by the F -weight by $\|\cdot\|_{c_1, c_2}$.

3.1.1. Closed-form formula.

PROPOSITION 3.12 Let F be as in equation (22). Let c_3 be the ratio of the coefficients of risk aversion and of market impact, namely $c_3 := c_2/c_1$. Let α be the function $\alpha(t) = 1 - \sinh(c_3(T-t))/\sinh(c_3T)$, and let K be the constant

$$K = \frac{1}{2c_1^2 \sinh(c_3T)} \int_0^T \cosh(c_3(T-u)) \mathbb{E}[S_u] du.$$

For $0 \leq t \leq T$, define

$$\begin{aligned} q_t &:= (1 - \alpha(t))x_0 + \alpha(t)x_T \\ &\quad - \frac{1}{2c_1^2} \int_0^t \cosh(c_3(t-u)) S_u du \\ &\quad + K \sinh(c_3t). \end{aligned} \quad (24)$$

Then, $(q_t)_{t \in [0, T]}$ is a (C, ξ) -good trade execution. The constant C is explicitly given by the formula

$$C^{-1} = c_3 \int_0^T \sinh(c_3(T-u)) \text{Var}^{\frac{1}{2}}(S_u) du;$$

the random variable ξ is explicitly given by the formula

$$\begin{aligned} \xi^{-1} &= \left| 2c_1 c_2 \frac{x_T - x_0}{\sinh(c_3T)} - c_3 \int_0^T \sinh(c_3(T-t)) S_t dt \right. \\ &\quad \left. + c_3 \frac{\cosh(c_3T)}{\sinh(c_3T)} \int_0^T \cosh(c_3(T-t)) \mathbb{E}[S_t] dt \right|. \end{aligned}$$

REMARK 3.13 The structure of the solution q in equation (24) is threefold: a time-dependent convex combination between initial inventory x_0 and liquidation target x_T appears on the first line; a dynamic response to the actual price trajectory appears on the second line; an adjustment for the terminal constraint $\mathbb{E}[q_T] = x_T$ appears on the third line.

If in the integral appearing on the second line of equation (24), we replace the fundamental price S_u with its expected value $\mathbb{E}[S_u]$, then the inventory trajectory q is turned into the optimal static one, i.e. into the minimiser of $\mathbb{E}[J(\eta)]$ over all η in $\mathcal{Q}_{\text{static}}^{0,x_0}$. See corollary A.7 in Appendix.

Instead, if in the definition of the constant K appearing on the third line of equation (24) we replace $\mathbb{E}[S_u]$ with S_u , then the inventory trajectory q is turned into the optimal a-posteriori one, i.e. into the minimiser of $J(\eta)$ over all η in $\mathcal{Q}_{\text{pw}}^{0,x_0}$. This is an immediate application of proposition A.6 in Appendix.

Proof of proposition 3.12 Let q be as in equation (24). The fact that q is in \mathcal{U}^{0,x_0} is apparent. Let $f_t := 2c_1^2 \dot{q}_t + S_t$ and notice that f is absolutely continuous with derivative

$$\dot{f}_t = 2c_1^2 q_t. \quad (25)$$

Let η be in \mathcal{U}^{0,x_0} . We write e for the difference $e := \eta - q$, and we observe that $e_0 = 0$ and $\mathbb{E}e_T = 0$. Then, we have

$$J(\eta) - J(q) = \int_0^T \left[f_t \dot{e}_t + 2c_1^2 q_t e_t \right] dt + \int_0^T (c_1^2 \dot{e}^2 + c_2^2 e^2) dt.$$

The second integral on the right-hand side is $|e|_{c_1, c_2}^2$. Using integration-by-parts, we see that in fact $J(\eta) - J(q) = f_T e_T + |e|_{c_1, c_2}^2$, because of equation (25). Therefore, the difference $J(\eta) - J(q)$ is non-negative if

$$|e_T| \leq \frac{|e|_{c_1, c_2}^2}{|f_T|}.$$

This gives $\xi = 1/|f_T|$.

Second, consider the expected difference $\mathbb{E}J(\eta) - \mathbb{E}J(q) = \mathbb{E}[f_T e_T] + \|e\|_{c_1, c_2}^2$. We can estimate

$$\mathbb{E}[f_T e_T] \leq \text{Var}^{\frac{1}{2}}(f_T) \|e_T\|_{L^2(\mathbb{P})},$$

because $\mathbb{E}e_T = 0$. Moreover,

$$\text{Var}^{\frac{1}{2}}(f_T) \leq c_3 \int_0^T \sinh(c_3(T-t)) \text{Var}^{\frac{1}{2}}(S_t) dt.$$

Therefore, the expected difference $\mathbb{E}J(\eta) - \mathbb{E}J(q)$ is non-negative if

$$\|e_T\|_{L^2(\mathbb{P})} \leq \frac{\|e\|_{c_1, c_2}^2}{c_3 \int_0^T \sinh(c_3(T-t)) \text{Var}^{\frac{1}{2}}(S_t) dt}.$$

This gives the constant C in the statement and concludes the proof. \blacksquare

REMARK 3.14 We remark that the good inventory trajectory of equation (24) is written without assuming a particular SDE dynamics for the price evolution. In particular, proposition 3.12 applies to the case in which the price process is modelled as a fractional Brownian motion, or as the sum of a possibly discontinuous semimartingale and a fractional Brownian motion. Moreover, the good inventory trajectory is robust, in the sense that it retains its optimality when one price process S is replaced by another price process \tilde{S} with $\mathbb{E}[S_t] = \mathbb{E}[\tilde{S}_t]$ for all t .

REMARK 3.15 Given q as in equation (24), we can compute

$$\begin{aligned} \|q_T - x_T\|_{L^2(\mathbb{P})} &= \frac{1}{2c_1^2} \mathbb{E}^{\frac{1}{2}} \left[\left(\int_0^T \cosh(c_3(T-t)) (S_t - \mathbb{E}[S_t]) dt \right)^2 \right] \end{aligned}$$

and estimate

$$\text{Var}(q_T) \leq \frac{T}{4c_1^4} \int_0^T \cosh^2(c_3(T-t)) \text{Var}(S_t) dt.$$

We therefore remark the following two facts. First, the smaller $\int \text{Var}(S_t) dt$ is, the more precise the good execution q of proposition 3.12 is. Second, the square of the coefficient c_1 of linear market impact is inversely proportional to the standard deviation of q_T , and thus the precision with which the good execution q of equation (24) gets to its liquidation target x_T increases when the strategy itself can exert more influence on the execution price.

REMARK 3.16 Unbiased admissible inventory trajectories q have been defined as absolutely continuous stochastic processes on $[0, T]$ such that $\mathbb{E}[q_T] = x_T$. This has meant that the constant K in equation (24) has been chosen to minimise $\mathbb{E}[(q_T - x_T)^2]$. We can give two alternatives to this minimisation:

(1) Choose K in such a way to minimise

$$\mathbb{E} \left[\int_{t_0}^T (q_t - x_T)^2 dt \right],$$

for some $0 \leq t_0 < T$. The symbol $\int_{t_0}^T dt$ stands for the mean $\frac{1}{T-t_0} \int_{t_0}^T dt$. This yields

$$K = \frac{1}{2c_1^2 \int_{t_0}^T \sinh^2(c_3 t) dt} \int_{t_0}^T \sinh(c_3 t) \psi(t) dt,$$

where $\psi(t) = \int_0^t \cosh(c_3(t-u)) \mathbb{E}[S_u] du - (1 - \alpha(t))(x_0 - x_T)$. Notice that this tends to the former choice when $t_0 \uparrow T$.

(2) Choose K in such a way to minimise

$$\mathbb{E} \left[\left(\int_{t_0}^T q_t dt - x_T \right)^2 \right],$$

for some $0 \leq t_0 < T$. This yields

$$K = \frac{c_3(T-t_0)}{2c_1^2 (\cosh(c_3 T) - \cosh(c_3 t_0))} \int_{t_0}^T \psi(t) dt,$$

with ψ as above.

Notice, however, that the alternative choices for the constant K make the corresponding inventory trajectory fall out of the set \mathcal{U}^{0, x_0} .

3.1.2. Characterisation via Euler–Lagrange equation.

The differential equation in (25) is the linchpin on which the derivation of the good inventory trajectory is based. This equation is the Euler–Lagrange equation associated with the functional J . Equation (25) is a random ordinary differential equation, where differentiation is possible because the price process is cancelled out in the sum $2c_1^2 \dot{q}_t + S_t$. Such a cancellation allows to circumvent the need of an integration with respect to the price process. However, this integration is possible; Appendix presents the theory of the Euler–Lagrange equation in the presence of a (rough) price path. The motivation for this theory comes from the fact that equation (25) can be rewritten as

$$\begin{aligned} dq_t &= r_t dt \\ dr_t &= c_3^2 (q_t - x_T) dt - dS_t / 2c_1^2. \end{aligned} \tag{26}$$

We interpret the system in equation (26) as a random Young differential equation. This equation is useful in simulations because it avoids the computation of the integrals in equation (24) and the evaluation of hyperbolic functions. Moreover, since we use Young integration to integrate with respect to the price process S , the system in equation (26) has a pathwise meaning and thus it also makes sense in the practical implementation of the trading strategy, where the price process S is replaced by the single price path observed during the liquidation.

In this paragraph, we apply the theory of Appendix in order to characterise the good trade execution of proposition 3.12 in terms of an initial value problem for the dynamics in (26).

We start by noticing that the Lagrangian F in equation (22) satisfies assumption A.4 in Appendix. Equation (26) is equation (A12) with F given by (22). Moreover, by casting definition A.11 to the case of equation (26), we have:

DEFINITION 3.17 Let q be in $\mathcal{U}^{0, \mathbf{x}_0}$. We say that q solves equation (26) if for all ω in Ω , all η in $C_0^\infty(0, T)$ and all $0 \leq s \leq t \leq T$ the following holds:

$$\begin{aligned} \int_s^t \dot{\eta}_u \, dq_u(\omega) \\ = \eta_t \dot{q}_t(\omega) - \eta_s \dot{q}_s(\omega) \\ - c_3^2 \int_s^t \eta_u (q_u(\omega) - \mathbf{x}_T) \, du + \frac{1}{2c_1^2} \int_s^t \eta_u \, dS_u(\omega). \end{aligned} \quad (27)$$

We remark that definition 3.17 is pathwise: a scenario ω in Ω could be fixed and the definition would still make sense. The integral on the left-hand side of equation (27) is well defined because for all ω in Ω the path $\dot{q}(\omega)$ is in $L^2[0, T]$. Similarly, the first integral on the right-hand side has a path-wise meaning and it is well defined because $q(\omega)$ is in $L^2[0, T]$ for all ω in Ω . Third, the integral $\int \eta dS$ on the right-hand side of equation (27) is the Young integral introduced in lemma A.10.

LEMMA 3.18 Let q and \tilde{q} be two solutions to equation (26). If $c_3 \neq 0$, then there exist constants K_1 and K_2 such that

$$q_t - \tilde{q}_t = K_1 e^{c_3 t} + K_2 e^{-c_3 t};$$

if $c_3 = 0$, then there exist constants K_1 and K_2 such that

$$q_t - \tilde{q}_t = K_1 + K_2 t.$$

In particular, in both cases the difference between q and \tilde{q} is deterministic.

Proof Let η be arbitrary in $C_0^\infty(0, T)$. By definition 3.17 we have that

$$\begin{aligned} \int_0^t \dot{\eta}_u \, dq_u &= \eta_t \dot{q}_t - c_3^2 \int_0^t \eta_u (q_u - \mathbf{x}_T) \, du + \frac{1}{2c_1^2} \int_0^t \eta_u \, dS_u; \\ \int_0^t \dot{\eta}_u \, d\tilde{q}_u &= \eta_t \dot{\tilde{q}}_t - c_3^2 \int_0^t \eta_u (\tilde{q}_u - \mathbf{x}_T) \, du + \frac{1}{2c_1^2} \int_0^t \eta_u \, dS_u. \end{aligned}$$

Let $\epsilon_t := q_t - \tilde{q}_t$. Subtract one line from the other and obtain that the function

$$t \mapsto \int_0^t \dot{\eta}_u \dot{\epsilon}_u \, du + c_3^2 \int_0^t \eta_u \epsilon_u \, du - \eta_t \dot{\epsilon}_t$$

is constantly null. The first two summands are differentiable in t and hence the third summand $\eta_t \dot{\epsilon}_t$ is differentiable too. Since η is arbitrary, $\dot{\epsilon}$ is differentiable in $(0, T)$. Differentiating $t \mapsto \eta_t \dot{\epsilon}_t$, we obtain

$$\dot{\eta}_t \dot{\epsilon}_t + c_3^2 \eta_t \epsilon_t - \dot{\eta}_t \dot{\epsilon}_t - \eta_t \ddot{\epsilon}_t = 0.$$

Hence $\ddot{\epsilon}_t = c_3^2 \epsilon_t$, proving the lemma. \blacksquare

LEMMA 3.19 The solution q to equation (26) with constraint

$$\begin{aligned} q_0 &= \mathbf{x}_0 \\ \mathbb{E}[q_T] &= \mathbf{x}_T \end{aligned} \quad (28)$$

is unique.

Proof Let q and \tilde{q} be two solutions to equation (26) satisfying the constraints in (28). Assume $c_3 \neq 0$. Then, by lemma 3.18 it must be

$$q_t - \tilde{q}_t = K_1 e^{c_3 t} + K_2 e^{-c_3 t},$$

for constants K_1 and K_2 . Therefore

$$\begin{aligned} q_0 - \tilde{q}_0 &= K_1 + K_2 = 0 \\ \mathbb{E}q_T - \mathbb{E}\tilde{q}_T &= K_1 e^{c_3 T} + K_2 e^{-c_3 T} = 0 \end{aligned}$$

Solving for K_1 and K_2 we find $K_1 = K_2 = 0$.

The case $c_3 = 0$ is analogous. \blacksquare

Having established uniqueness of the solution to equation (26) with constraint (28), we link equation (26) to the good trade execution of proposition 3.12. This link is established as an application of lemma A.12 from Appendix.

LEMMA 3.20 Let q be in $\mathcal{U}^{0, \mathbf{x}_0}$. Then, the following are equivalent

- (1) the inventory trajectory q solves the Euler–Lagrange equation (26);
- (2) the function $f_t := 2c_1^2 \dot{q}_t + S_t$ is absolutely continuous with derivative

$$\dot{f}_t = 2c_2^2 q_t.$$

In particular, the (C, ξ) -good trade execution in proposition 3.12 solves the Euler–Lagrange equation (26).

We are finally in the position to prove the main result of this paragraph.

PROPOSITION 3.21 (‘Characterisation of good trade execution via Euler–Lagrange equation’) The good trade execution for the minimisation of (17) with the Lagrangian F as in equation (22) is characterised as the solution to the Euler–Lagrange equation in (26) with initialisation

$$\begin{aligned} q_0 &= \mathbf{x}_0 \\ r_0 &= -S_0/(2c_1^2) + c_3 \sinh^{-1}(c_3 T) \\ &\times \left[(\mathbf{x}_T - \mathbf{x}_0) \cosh(c_3 T) + \tilde{K} \right], \end{aligned} \quad (29)$$

where

$$\begin{aligned} \tilde{K} &= \frac{\cosh(c_3 T)}{2c_1^2} \int_0^T \cosh(c_3 t) \mathbb{E}[S_t] \, dt \\ &- \frac{\sinh(c_3 T)}{2c_1^2} \int_0^T \sinh(c_3 t) \mathbb{E}[S_t] \, dt. \end{aligned}$$

In particular, the good trade execution in proposition 3.12 is the only good trade execution for the minimisation (17) with the Lagrangian F as in equation (22). \blacksquare

REMARK 3.22 Proposition 3.12 gives a characterisation of the good trade execution in terms of an initial value problem that is easily simulated. This is the practical relevance of the characterisation. We will rely on the initial value problem (26) with initial conditions (29) in our numerical experiments in section 4.

Proof First we examine the following two implications.

- (1) *A good trade execution solves equation (26).* Let \tilde{q} be a good trade execution for the minimisation of (17) with the Lagrangian F as in equation (22). Then in particular, for every ω in Ω it holds

$$\tilde{q}(\omega) = \operatorname{argmin} \left\{ J(\eta) : \eta \in \tilde{q}(\omega) + W_0^{1,2}(0, T) \right\}.$$

Therefore, by proposition A.13, we have that $\tilde{q}(\omega)$ solves the equation

$$\begin{aligned} d\tilde{q}_t(\omega) &= \tilde{r}_t(\omega) dt \\ d\tilde{r}_t(\omega) &= c_3^2 \tilde{q}_t(\omega) dt - dS_t(\omega)/2c_1^2. \end{aligned}$$

- (2) *A solution to equation (26) is a good trade execution.* Assume that q solves the Euler–Lagrange equation in (26). Then, by lemma 3.20 we have that $f_t(\omega) := 2c_1^2 \dot{q}_t(\omega) + S_t(\omega)$ is absolutely continuous in t for all ω in Ω and its time derivative is $\dot{f}_t(\omega) = 2c_2^2 q_t(\omega)$. Hence, for all ω in Ω and all η in $q(\omega) + W_0^{1,2}(0, T)$ we have

$$\begin{aligned} J(\eta) - J(q(\omega)) &= \int_0^T (2c_2^2 q_t(\omega) - \dot{f}_t(\omega)) e_t dt \\ &\quad + \int_0^T (c_1^2 \dot{e}_t^2 + c_2^2 e_t^2) dt, \end{aligned}$$

where $e = \eta - q(\omega)$. The first summand on the right hand side is null and thus $J(\eta) \geq J(q(\omega))$. This shows that q is a $(0, 0)$ -good trade execution.

In view of these two implications and of lemma 3.19, it only remains to show that the initialisations in equation (29) are equivalent to the constraints in (28).

Consider an equation of the form

$$dY_t = (AY_t + B) dt + D dS_t,$$

where the unknown Y is in \mathbb{R}^2 , the matrix A is in $\mathbb{R}^{2 \times 2}$ and B and D are two-dimensional real vectors. Equation (26) is of this form with the choice $Y = (q, r)^\top$ and

$$A = \begin{pmatrix} 0 & 1 \\ c_3^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -c_3^2 x_T \end{pmatrix} = 0, \quad D = \begin{pmatrix} 0 \\ -\frac{1}{2c_1^2} \end{pmatrix}.$$

Assume first that $t \mapsto \mathbb{E}[S_t]$ is differentiable. Then, if we set $\mu_t = \mathbb{E}[Y_t]$ we have that μ solves the ordinary differential equation

$$\dot{\mu}_t = A\mu_t + \varphi_t D, \quad (30)$$

where $\varphi_t = d\mathbb{E}[S_t]/dt$. This ordinary differential equation has a two-dimensional space of solutions; hence we can exploit these two degrees of freedom to adjust for the constraints $q_0 = \mu_0 = x_0$ and $\mu_T = x_T = 0$.

Define the function $e_1 = e_1(t)$ as

$$\begin{aligned} e_1(t) &= e_1(0) + \mathbb{E}[S_t] \sinh(c_3 t) / (2c_1 c_2) \\ &\quad - \frac{1}{2c_1^2} \int_0^t \cosh(c_3 u) \mathbb{E}[S_u] du. \end{aligned}$$

Define the function $e_2 = e_2(t)$ as

$$\begin{aligned} e_2(t) &= e_2(0) + \left(S_0 - \cosh(c_3 t) \mathbb{E}[S_t] \right) / (2c_1 c_2) \\ &\quad + \frac{1}{2c_1^2} \int_0^t \sinh(c_3 u) \mathbb{E}[S_u] du. \end{aligned}$$

The general solution to equation (30) is

$$\begin{aligned} \mu_t^{(1)} &= e_1(t) \cosh(c_3 t) + e_2(t) \sinh(c_3 t) \\ \mu_t^{(2)} &= c_3 e_1(t) \sinh(c_3 t) + c_3 e_2(t) \cosh(c_3 t). \end{aligned}$$

The constraints $q_0 = \mu_0 = x_0$ and $\mu_T = x_T = 0$ impose the choices

$$\begin{aligned} e_1(0) &= x_0 \\ e_2(0) &= R + \sinh^{-1}(c_3 T) \left[x_T - e_1(T) \cosh(c_3 T) \right], \end{aligned}$$

where

$$R = \frac{\cosh(c_3 T) \mathbb{E}[S_T] - S_0}{2c_1 c_2} - \frac{1}{2c_1^2} \int_0^T \sinh(c_3 u) \mathbb{E}[S_u] du.$$

Hence, the constraints $q_0 = \mu_0 = x_0$ and $\mu_T = x_T = 0$ are translated into the initialisation in the statement.

The case where the map $t \mapsto \mathbb{E}[S_t]$ is not differentiable is handled via a standard approximation argument. ■

3.2. Alternative risk criteria

We present two alternatives to the risk criterion used in section 3.1. This means that we modify the third summand in the Lagrangian of equation (22), and we study the minimisation problem with such a modified Lagrangian. The first alternative (section 3.2.1) preserves the same structure but increases the weight of the coefficient c_2 of risk aversion linearly in time. The second alternative (section 3.2.2) is instead inspired by the value-at-risk for geometric Brownian motion used in Gatheral and Schied (2011).

3.2.1. Linearly time-dependent coefficient of risk aversion. The third summand in the Lagrangian F of equation (22) accounts for the risk aversion. So far this term has been taken constant in the time variable t . We now propose a linear t -dependence, with higher risk aversion for t closer to the liquidation horizon T . More precisely, we consider the Lagrangian

$$F(t, S, q, r) := rS + c_1^2 r^2 + c_2^2 t q^2, \quad (31)$$

where $c_1 > 0$ is a coefficient of market impact and $c_2 \geq 0$ is a coefficient of risk aversion. For future reference, we set $c_3 := c_2/c_1$. We study the minimisation problem in (17) with F as in equation (31).

LEMMA 3.23 *Let F be as in equation (31). Then, F satisfies assumption 3.1. Moreover, the pathwise F -weight is a seminorm on $W^{1,2}(0, T)$, and the F -weight is a seminorm on \mathcal{U}^{0, x_0} . If $c_2 > 0$, then these seminorms are norms.*

We denote the pathwise seminorm induced by the pathwise F -weight by $|\cdot|_{c_1, c_2 \sqrt{t}}$. More precisely, we set

$$|\eta|_{c_1, c_2 \sqrt{t}}^2 := \int_0^T (c_2^2 t \eta_t^2 + c_1^2 \dot{\eta}_t^2) dt, \quad (32)$$

for η in $W^{1,2}(0, T)$. Moreover, we denote the seminorm on \mathcal{U}^{0, x_0} induced by the F -weight by $\|\cdot\|_{c_1, c_2 \sqrt{t}}$.

We proceed with statements analogous to those in sections 3.1.1 and 3.1.2, namely: in proposition 3.24 we give a closed-form formula for a good trade execution in the case of the Lagrangian F of (31); then, in proposition 3.25 we show that in fact such a good trade execution is unique, and we characterise it as the solution of a random Young differential equation. All the arguments are straightforward adaptations from those presented above, and thus we omit the proofs.

PROPOSITION 3.24 *Let the Lagrangian F be as in equation (31). Let Ai and Bi be the first and second Airy's functions, namely the two independent solutions to the second order linear ordinary differential equation $u''(t) - tu(t) = 0$. Define the functions α , β , ϕ as follows:*

$$\begin{aligned} \alpha(t) &= \text{Ai}(c_3^{2/3} t), \\ \beta(t) &= \text{Bi}(c_3^{2/3} t), \\ \phi(t) &= \frac{1}{2c_1^2} \int_0^t \alpha^{-2}(s) \int_0^s \alpha(u) dS_u ds, \end{aligned}$$

where the innermost integral in the definition of ϕ is the Young integral introduced in Appendix. Define the constants c_A and c_B as follows:

$$\begin{aligned} c_A &= \frac{\beta(T)x_0 - \alpha(T)\beta(0)\mathbb{E}\phi(T)}{\alpha(0)\beta(T) - \alpha(T)\beta(0)}, \\ c_B &= \frac{\alpha(T)}{\alpha(0)\beta(T) - \alpha(T)\beta(0)} (\alpha(0)\mathbb{E}\phi(T) - x_0). \end{aligned}$$

For $0 \leq t \leq T$, define

$$q_t := c_A \alpha(t) + c_B \beta(t) - \alpha(t)\phi(t). \quad (33)$$

Then, $(q_t)_{t \in [0, T]}$ is a (C, ξ) -good trade execution, where

$$\begin{aligned} C &= 1/\|S_T - 2c_1^2 \dot{\alpha}\phi(T)\|_{L^2(\mathbb{P})}, \\ \xi &= 1/|S_T + 2c_1^2 c_A \dot{\alpha}(T) + 2c_1^2 c_B \dot{\beta}(T) - 2c_1^2 \dot{\alpha}\phi(T)|, \end{aligned}$$

where the symbol $\dot{\alpha}\phi$ denotes the time derivative of the product function $t \mapsto \alpha(t)\phi(t)$.

The Lagrangian F in equation (31) satisfies assumption A.4 from Appendix. Equation (A12) in the present case reads

$$\begin{aligned} dq_t &= r_t dt \\ dr_t &= c_3^2 t q_t dt - dS_t/2c_1^2. \end{aligned} \quad (34)$$

We show that the good trade execution in proposition 3.24 is characterised as the unique solution to the random Young differential equation (34).

PROPOSITION 3.25 *The good trade execution of proposition 3.24 is characterised as the unique solution to the random Young differential equation (34) with initialisation*

$$\begin{aligned} q_0 &= x_0 \\ r_0 &= e_A(0)\alpha_0 + e_B(0)\beta_0, \end{aligned}$$

where α and β are as in proposition 3.24, and

$$\begin{aligned} e_A(0) &= \frac{\beta_T x_0 - \beta_0 \tilde{K}}{\alpha_0 \beta_T - \alpha_T \beta_0}, \quad e_B(0) = \frac{\alpha_0 \tilde{K} - \alpha_T x_0}{\alpha_0 \beta_T - \alpha_T \beta_0}, \\ \tilde{K} &= \frac{S_0}{2c_1^2 W_0} (\alpha_T \beta_0 - \alpha_0 \beta_T) \\ &\quad + \frac{1}{2c_1^2} \int_0^T \mathbb{E}[S_u] \frac{d}{du} \left(\frac{\alpha_T \beta_u - \beta_T \alpha_u}{W_u} \right) du, \\ W_t &= \alpha_t \dot{\beta}_t - \dot{\alpha}_t \beta_t. \end{aligned}$$

In particular, the good trade execution in proposition 3.24 is the only good trade execution for the minimisation (17) with the Lagrangian F as in equation (31).

3.2.2. VaR-inspired risk criterion. Gatheral and Schied (2011) model the fundamental price S as a geometric Brownian motion and they adopt the value-at-risk as measure of risk aversion. This means penalising instantaneous revenues from trade by subtracting a term proportional to $q_t S_t$ at every time t . Inspired by their modelling choices, we now consider the Lagrangian

$$F(t, S, q, r) := rS + c_1^2 r^2 + c_2^2 qS, \quad (35)$$

where $c_1 > 0$ is a coefficient of market impact and $c_2 \geq 0$ is a coefficient of risk aversion. Notice that in fact F does not depend on t . We study the minimisation problem in (17) with F as in equation (35).

LEMMA 3.26 *Let F be as in equation (35). Then, F satisfies assumption 3.1. Moreover, the pathwise F -weight is a seminorm on $W^{1,2}(0, T)$, and the F -weight is a seminorm on \mathcal{U}^{0, x_0} .*

Proof Assumption 3.1 is understood as per Remark 3.3, namely we drop the requirement on $t \mapsto 1/L(t, 0, 1, 0)$ because $\partial_{x_3} L(\cdot, 0, \cdot, 0) \equiv 0$. ■

We denote the pathwise seminorm induced by the pathwise F -weight by $|\cdot|_{c_1}$. More precisely, we set

$$|\eta|_{c_1}^2 := \int_0^T c_1^2 \dot{\eta}_t^2 dt, \quad (36)$$

for η in $W^{1,2}(0, T)$. Moreover, we denote the seminorm on \mathcal{U}^{0, x_0} induced by the F -weight by $\|\cdot\|_{c_1}$.

We proceed with statements analogous to those in sections 3.1.1 and 3.1.2, namely: in proposition 3.27 we give a closed-form formula for a good trade execution in the case of the Lagrangian F in equation (35); then, in proposition 3.30 we show that such a good trade execution is unique, and we characterise it as the solution of a random Young differential equation. All the arguments are straightforward adaptations from those presented above, and thus we omit the proofs.

PROPOSITION 3.27 *Let F be as in equation (35). Let K be the constant*

$$K = \frac{1}{2c_1^2 T} \int_0^T \left(\mathbb{E}[S_s] - c_2^2 \int_0^s \mathbb{E}[S_u] du \right) ds.$$

For $0 \leq t \leq T$, define

$$\begin{aligned} q_t = & \left(1 - \frac{t}{T}\right) x_0 + \frac{t}{T} x_T \\ & - \frac{1}{2c_1^2} \int_0^t \left(S_s - c_2^2 \int_0^s S_u du \right) ds \\ & + Kt \end{aligned} \quad (37)$$

Then, $(q_t)_{t \in [0, T]}$ is a (C, ξ) -good trade execution. The random variable ξ is explicitly given by the formula

$$\xi^{-1} = \left| \frac{2c_1^2}{T} (x_T - x_0) + 2c_1^2 K + c_2^2 \int_0^T S_t dt \right|$$

and the constant C is explicitly given by the formula $C^{-1} = \|\xi^{-1}\|_{L^2(\mathbb{P})}$.

REMARK 3.28 The good trade execution in equation (37) has the same structure of the one in equation (24), namely: a time-dependent convex combination of x_0 and x_T (first line), a dynamic response to the realisation of the price path (second line), and an adjustment for the constraint $\mathbb{E}[q_T] = x_T$ (third line). Moreover, notice that

$$\lim_{c_2 \downarrow 0} \frac{\sinh\left(\frac{c_2}{c_1}(T-t)\right)}{\sinh\left(\frac{c_2}{c_1}T\right)} = 1 - \lim_{c_2 \downarrow 0} \frac{\sinh\left(\frac{c_2}{c_1}t\right)}{\sinh\left(\frac{c_2}{c_1}T\right)} = 1 - \frac{t}{T},$$

so that the good trade execution in equation (37) and the good trade execution in equation (24) agree when the risk aversion vanishes, i.e. in the limit as $c_2 \downarrow 0$.

REMARK 3.29 Of the good trade execution in equation (37), we can compute

$$\begin{aligned} \|q_T - x_T\|_{L^2(\mathbb{P})} = & \frac{1}{2c_1^2} \mathbb{E}^{\frac{1}{2}} \left[\left(\int_0^T \left\{ c_2^2 \int_0^t (S_u - \mathbb{E}[S_u]) du \right. \right. \right. \\ & \left. \left. \left. - (S_t - \mathbb{E}[S_t]) \right\} dt \right)^2 \right] \end{aligned}$$

and estimate

$$\text{Var}(q_T) \leq \frac{T}{2c_1^4} \int_0^T \left(c_2^4 t \int_0^t \text{Var}(S_u) du + \text{Var}(S_t) \right) dt.$$

Therefore, the two facts presented in Remark 3.15 also hold for the good trade execution of proposition 3.27.

The Lagrangian F in equation (35) satisfies assumption A.4 from Appendix. With this F , equation (A12) reads

$$\begin{aligned} dq_t &= r_t dt \\ dr_t &= c_3^2 t S_t dt / 2 - dS_t / 2c_1^2, \end{aligned} \quad (38)$$

where $c_3 := c_2/c_1$. We now characterise the good trade execution in proposition 3.27 as the unique solution to the random Young differential equation (38). Observe that equation (38) is solved by direct integration and this simplifies several calculations compared to the cases of sections 3.1.2 and 3.2.1.

PROPOSITION 3.30 *The good trade execution of proposition 3.27 is characterised as the unique solution to the random Young differential equation (38) with initialisation*

$$\begin{aligned} q_0 &= x_0 \\ r_0 &= \frac{x_T - x_0}{T} + \frac{1}{2c_1^2 T} \int_0^T \left(\mathbb{E}[S_t] - S_0 - c_2^2 \int_0^t \mathbb{E}[S_u] du \right) dt. \end{aligned}$$

In particular, the good trade execution in proposition 3.27 is the only good trade execution for the minimisation (17) with F as in equation (35).

4. Applications

In this section, we give two applications of the trading schedule proposed in proposition 3.12.

Because of the reliance on the expected trajectory $t \mapsto \mathbb{E}[S_t]$ of the fundamental price, the use case of our framework is one where such an expected trajectory can serve as a reliable forecast for the price. This means that an implicit mean-reversion is assumed, and we will be choosing our price processes accordingly.

4.1. INTC shares: high-frequency mean-reverting jump diffusion

We consider the liquidation of a large portfolio of shares; the whole liquidation happens during intraday trading hours and on a single limit order book. We model the fundamental price S on the mid-price of the order book, according to the following high-frequency mean-reverting jump-diffusion model:

$$S_t = \exp(m(t) + Y(t) + N(t)), \quad (39)$$

where $m(t)$ is \mathfrak{F}_0 -measurable, continuous and of finite variation, $Y(t)$ is the Ornstein–Uhlenbeck process $dY(t) =$

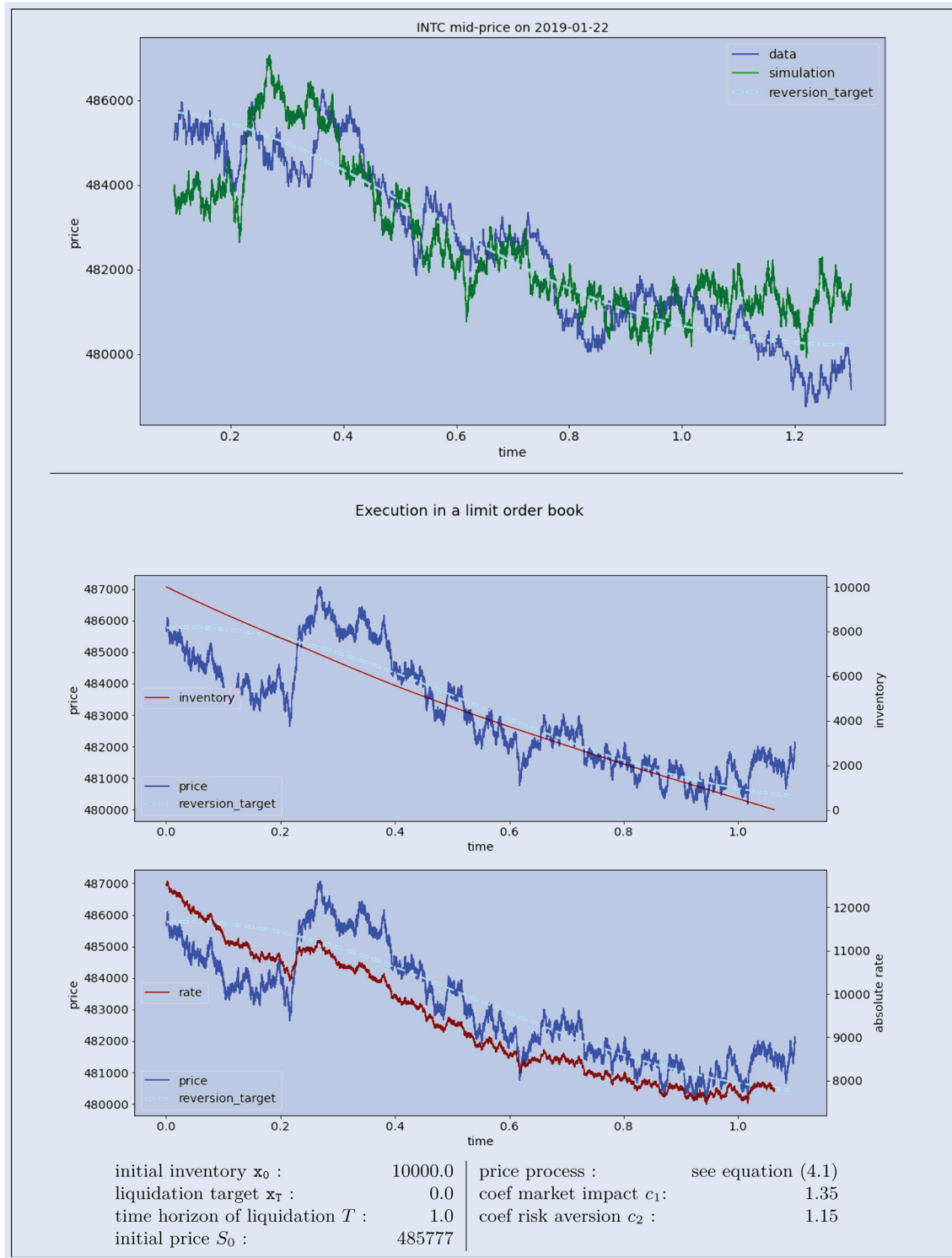


Figure 2. Upper quadrant: Historical intraday mid-price of INTC on 2019-01-22 (10am–3pm) and a sample path of the process in equation (39). Lower quadrant: Inventory trajectory and rate for the liquidation referring to the sampled price path.

$-\alpha Y(t) dt + \sigma dW(t)$, and $N(t)$ is a compound Poisson process independent from $Y(t)$ and with i.i.d marks symmetrically distributed around zero. Therefore, the expected price trajectory is $\mathbb{E}[S_t] = \exp(m(t))$, and it represents the reversion target. The liquidator acts adopting this as her price forecast during the execution.

As an example, we calibrate our model to the high-frequency NASDAQ order book data available for the trading of INTC on 22 January 2019. The dataset is

provided by LOBSTER (<https://lobsterdata.com/>). The calibration procedure is straightforward: we first extrapolate the mean-reversion target from the raw data, we then estimate the Ornstein–Uhlenbeck factor from the difference between the log-price and the logarithm of the mean-reversion target, and we finally calibrate the point process on the data points $\{Y(t_i) : |Y(t_i) - Y(t_{i-1}) \exp(-\alpha(t_i - t_{i-1}))| > k\sigma\sqrt{1 - \exp(-2\alpha(t_i - t_{i-1}))}/\alpha\}$ that are displaced beyond k standard deviations from the expected value. In the interest

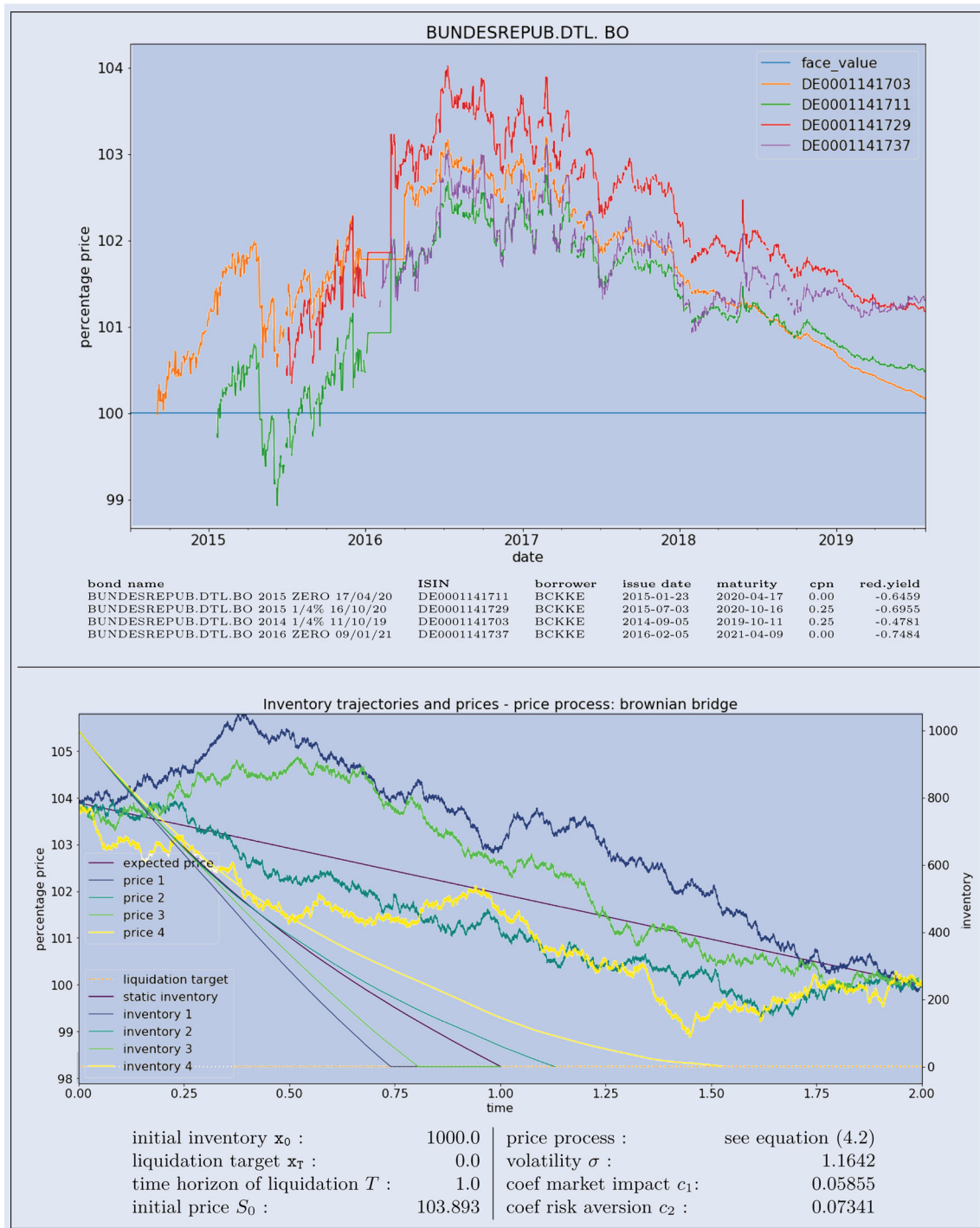


Figure 3. Upper quadrant: Historical prices of bonds with negative yields. Lower quadrant: Good trade executions in the context of a Brownian bridge.

of conciseness, we omit the details of this calibration procedure. The so-calibrated price process is displayed in the upper quadrant of figure 2, together with the original data stream.

The lower quadrant in figure 2 reports the inventory trajectory and its rate of execution computed as a solution to the Cauchy problem of equation (26) with initialisation (29). We remark two aspects observable from the picture. The first aspect is that the liquidation terminates after the initially decided horizon $T = 1.0$ for the liquidation. This is because the realisation of the price path dwells for most of

the time window of the liquidation below its expected value. Consequently, the liquidator decreases the rate of liquidation, in order to prevent her own price impact from exacerbating her trading cost even further. The trade-off between a limited exposure to market volatility and a parsimonious rate is resolved in favour of the latter. The second aspect is the reaction to the price jump that happens around $t = 0.21$. The jump is upward and hence favourable to the liquidation. Consequently, the liquidator reacts by increasing the rate of execution to exploit this.

4.2. 5Y government bonds: Brownian bridge

As a second application, we consider an entirely different time scale from the one of intraday INTC stock price, and in fact different from the time scale at which models of trade executions are usually applied. In this sense, the application is at the boundary between trade execution and dynamic portfolio management.

We consider the sale of 5Y government bonds motivated by market conditions whereby bond yields are negative. Historically, this was observed in 2016 for German bunds and such a phenomenon reappeared in March 2019. Four examples are reported in the upper quadrant of figure 3, where we show the prices of 5Y German bunds with maturities October 2019, April 2020, October 2020, and April 2021.

The reason for negative yields was twofold. On the one hand, central banks launched programmes to stimulate the economy by cutting interest rates and by injecting liquidity via the so-called quantitative easing. On the other hand, in volatile economic regimes risk-averse investors tend to move their capital to safe investments such as government bonds (a phenomenon known as flight-to-quality), and this exerted upward pressure on bond prices.

We consider an investor who wishes to unwind a long position on a negative yielding bond before incurring into the sure loss at the bond's maturity. The sale is done gradually in time, on the one hand because the investor does not want to suddenly lose all the liquidity associated with the bonds, and on the other hand because the investor is mindful of the market impact that an abrupt sale would incur into.

In the upper quadrant of figure 3 we observe that, after the period of overprice and as the maturity approaches, the trajectories converge on a downward slope towards the face value, which pins the trajectories at maturity. We base our modelling choices on this observation and, in line with a tradition that dates back to the Seventies (Boyce 1970), we adopt a Brownian bridge as the price process. More precisely, the price process S is modelled as

$$dS_t = \frac{V - S_t}{T - t} dt + \sigma dW_t, \quad (40)$$

where V is the face value of the bond, W is a standard one-dimensional Brownian motion, and σ is the volatility coefficient. In this model for the price process, the expected price path is the line segment from the decision price x_0 at time $t = 0$ to the face value at maturity.

The second quadrant in figure 3 shows good trade executions in this setting. The inventory trajectories are simulated following the dynamics in equation (26) with initialisation (29).

The way in which the good trade executions react to market scenarios is indicative of the dynamic adjustment of inventory trajectories during the liquidation. In particular, in scenarios #1 and #3 the liquidation is faster than the static solution; in this way it exploits favourable market conditions. Notice that this is advantageous especially in scenario #3, where a faster liquidation means that the liquidator concentrates her sale before the price plunges below its expected trend. On the contrary, in scenario #4, the good inventory trajectory is less

steep than the static solution, because unfavourable market conditions recommend to parsimoniously impact the price.

5. Conclusion

In this paper, we examined the mathematical models of optimal trade execution with respect to two properties: non-static trajectories and unbiased liquidation errors. Non-static trajectories are those that react to the actual realisation of the price path during the execution, rather than being based only on assumed distributional properties of this price. Second, a liquidation error is said to be unbiased if its expectation is zero, entailing that the expected value of the terminal inventory coincides with the execution target.

We introduced our proposal for execution strategies, which enjoy both properties. In particular, in order to have non-static solutions even when the fundamental price is modelled as a martingale, we considered the minimisation of trading costs from a *pathwise* perspective, rather than the minimisation of *expected* trading costs.

We considered three risk criteria. The first criterion is the classical quadratic inventory cost; the second is a time-dependent modification of the first; the third was inspired by the value-at-risk employed in Gatheral and Schied (2011). For all of them, we derived explicit closed-form formulae of our inventory trajectories. Furthermore, we characterised them through initial value problems that allow to easily implement our strategies in practice. We demonstrated this through two applications, one on the liquidation of INTC shares, the other on the liquidation of German bunds.

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Appendix. Euler–Lagrange equations with price paths

All the statements in this appendix are pathwise, hence we leave probability out of the picture. In particular, the symbol S will denote a (possibly discontinuous) single path of finite p -variation, for some $p \geq 1$. We consider the following functions spaces. The space $C_0^\infty(0, T)$ is the space of smooth functions with compact support in the open interval $(0, T)$. The Sobolev space $W^{1,2}(0, T)$ is the space of absolutely continuous functions u on the closed interval $[0, T]$ such that u and its derivative \dot{u} are square integrable over $[0, T]$. The Sobolev space $W^{1,2}(0, T)$ is endowed with the norm

$$\|u\|_{W^{1,2}}^2 = \int_0^T (u_t^2 + \dot{u}_t^2) dt.$$

The space $W_0^{1,2}(0, T)$ is defined as the closure of $C_0^\infty(0, T)$ in $W^{1,2}(0, T)$.

LEMMA A.1 *Let f and g be non-negative bounded measurable functions defined on $(0, T)$. Then,*

$$\|u\| := \left(\int_0^T (u^2(t)f(t) + \dot{u}^2(t)g(t)) dt \right)^{\frac{1}{2}}$$

is a seminorm on $W^{1,2}(0, T)$.

Proof Notice that for u in $W^{1,2}(0, T)$ we can write $\|u\|^2 = \|u\|_f^2 + \|\dot{u}\|_g^2$, where $\|\dot{u}\|_f$ is the L^2 norm associated with the measure $f(t) dt$, and $\|\dot{u}\|_g$ is the L^2 norm associated with the measure $g(t) dt$. Therefore, the triangle inequality is proved by the following chain of inequalities:

$$\begin{aligned} \|u + v\| &\leq \left((\|u\|_f + \|v\|_f)^2 + (\|\dot{u}\|_g + \|\dot{v}\|_g)^2 \right)^{\frac{1}{2}} \\ &\leq \left(\|u\|_f^2 + \|\dot{u}\|_g^2 \right)^{\frac{1}{2}} + \left(\|v\|_f^2 + \|\dot{v}\|_g^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where on the first line we used Minkowski inequality in L_f^2 and L_g^2 , and on the second line we applied Minkowski inequality in ℓ^2 . ■

DEFINITION A.2 (‘Caratheodory function’) Let U be an open set in \mathbb{R}^n . The function $F : U \times \mathbb{R}^d \rightarrow \mathbb{R}$ is called Caratheodory function if: (1) for almost every t in U , the map $x \mapsto F(t, x)$ is continuous; (2) for every x in \mathbb{R}^d , the function $t \mapsto F(t, x)$ is Lebesgue-measurable.

DEFINITION A.3 Let U be an open subset of \mathbb{R}_+ . Let $F : U \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Caratheodory function. We say that F is space-differentiable if for all t in U the map $x \mapsto F(t, x)$ is in $C^1(\mathbb{R}^d)$.

In this appendix, we consider Lagrangians F for which the following assumption holds.

ASSUMPTION A.4 Let $F : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a Caratheodory function. It is assumed that

$$F(t, x_1, x_2, x_3) = x_1 x_3 + L(t, x_1, x_2, x_3), \quad (\text{A1})$$

where L is a space-differentiable Caratheodory function on $(0, T) \times \mathbb{R}^3$ such that there exist a function α in $L^1(0, T)$ and a constant $\beta \geq 0$ such that

$$\begin{aligned} &|L(t, x_1, x_2, x_3)|, |\partial_{x_2} L(t, x_1, x_2, x_3)|, |\partial_{x_3} L(t, x_1, x_2, x_3)| \\ &\leq \alpha(t) + \beta(x_1^2 + x_2^2 + x_3^2), \end{aligned} \quad (\text{A2})$$

for all t in $(0, T)$ and all x_1, x_2, x_3 in \mathbb{R} .

The space variables $x = (x_1, x_2, x_3)$ are alternatively labelled as $(x_1, x_2, x_3) = (S, q, r)$. Hence, we interpret x_1 as the placeholder for the variable S denoting the fundamental price, we interpret x_2 as the placeholder for the variable q denoting the inventory, and we interpret x_3 as the placeholder for the variable r denoting the rate of execution. In the following, we adopt these letters for the space variables; hence, in particular $\partial_q L$ denotes the derivative of L with respect to the variable x_2 , and $\partial_r L$ denotes the derivative of L with respect to the variable x_3 .

REMARK A.5 When L does not depend on x_1 , the decomposition in equation (A1) has the form of the decomposition in equation (6). However, we do not restrict to that case in this appendix. In particular, our treatment encompasses the case of the functional used in Gatheral and Schied (2011), which we discuss in section 3.2.2.

Let $S = (S_t)_{0 \leq t \leq T}$ be a path on $[0, T]$ and assume that S is of finite p -variation for some $p \geq 1$. Let F be a Caratheodory function satisfying assumption A.4. Let J be the functional

$$J(\eta) := \int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt, \quad \eta \in W^{1,2}(0, T). \quad (\text{A3})$$

Let q_0 be in $W^{1,2}(0, T)$ and consider the affine space

$$q_0 + W_0^{1,2}(0, T) = \{q_0 + u : u \in W_0^{1,2}(0, T)\}.$$

We study the minimisation problem

$$\inf \{J(\eta) : \eta \in q_0 + W_0^{1,2}(0, T)\}. \quad (\text{A4})$$

PROPOSITION A.6 ('Weak form of Euler–Lagrange equation') *Let F be as in assumption A.4. Let $L(t, x_1, x_2, x_3) := F(t, x_1, x_2, x_3) - x_1 x_3$. Assume that q is a minimiser for (A4). Then, for all ψ in $C_0^\infty(0, T)$ it holds*

$$\int_0^T (S_t + \partial_r L(t, S_t, q_t, \dot{q}_t)) \dot{\psi}_t dt = - \int_0^T \psi_t \partial_q L(t, S_t, q_t, \dot{q}_t) dt. \quad (\text{A5})$$

Conversely, if $(x_2, x_3) \mapsto F(t, x_1, x_2, x_3)$ is convex for all t and all x_1 , then (A5) is a sufficient condition on $q \in q_0 + W_0^{1,2}(0, T)$ for being a minimiser for (A4).

Proof For ψ in $C_0^\infty(0, T)$ it holds

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^T [L(t, S_t, q_t + \epsilon \psi_t, \dot{q}_t + \epsilon \dot{\psi}_t) - L(t, S_t, q_t, \dot{q}_t)] dt \\ &= \int_0^T [\psi_t \partial_q L(t, S_t, q_t, \dot{q}_t) + \dot{\psi}_t \partial_r L(t, S_t, q_t, \dot{q}_t)] dt, \end{aligned}$$

owing to assumption (A2) on the growth of L and its space derivatives. For all ψ in $C_0^\infty(0, T)$ and all $\epsilon > 0$ we have that

$$J(q + \epsilon \psi) \geq J(q),$$

and thus necessity of (A5) is established.

Conversely, assume that $(x_2, x_3) \mapsto F(t, x_1, x_2, x_3)$ is convex for all t and all x_1 . Assume that q in $q_0 + W_0^{1,2}(0, T)$ satisfies equation (A5). Let η be arbitrary in $q_0 + W_0^{1,2}(0, T)$, and let t be arbitrary in $[0, T]$. By convexity it holds

$$\begin{aligned} F(t, S_t, \eta_t, \dot{\eta}_t) &\geq F(t, S_t, q_t, \dot{q}_t) + \partial_q L(t, S_t, q_t, \dot{q}_t)(\eta_t - q_t) \\ &\quad + [S_t + \partial_r L(t, S_t, q_t, \dot{q}_t)](\dot{\eta}_t - \dot{q}_t). \end{aligned}$$

Notice that $\eta - q$ is in $W_0^{1,2}(0, T)$ and thus it is approximated by ψ in $C_0^\infty(0, T)$. Therefore, if we integrate the latter inequality in dt and use (A5), we conclude $\int_0^T F(t, S_t, \eta_t, \dot{\eta}_t) dt \geq \int_0^T F(t, S_t, q_t, \dot{q}_t) dt$. ■

As an application of the sufficient condition for optimality established in proposition A.6, we derive the optimal static inventory trajectory in the case of linear temporary market impact and quadratic inventory cost.

COROLLARY A.7 *Let S be a price process as defined in definition 2.1. Let x_0, c_1 and c_2 be positive real numbers, and let $c_3 := c_2/c_1$. Let α and K be as in proposition 3.12. For $0 \leq t \leq T$ define*

$$\begin{aligned} q_t &:= (1 - \alpha(t))x_0 + \alpha(t)x_T \\ &\quad - \frac{1}{2c_1^2} \int_0^t \cosh(c_3(t-u)) \mathbb{E}[S_u] du + K \sinh(c_3 t). \end{aligned}$$

Then, the function $t \mapsto q_t$ is a minimiser for

$$\inf \left\{ \mathbb{E} \left[\int_0^T (S_t \dot{\eta}_t + c_1^2 \dot{\eta}_t^2 + c_2^2 \eta_t^2) dt \right] : \eta \in \mathcal{Q}_{\text{static}}^{0, x_0} \right\}.$$

Proof Let

$$F(x_1, x_2, x_3) := x_1 x_3 + c_1^2 x_3^2 + c_2^2 x_2^2, \quad (\text{A6})$$

and notice that $(x_2, x_3) \mapsto F(x_1, x_2, x_3)$ is convex for all x_1 . For all η in $\mathcal{Q}_{\text{static}}^{0, x_0}$ it holds

$$\mathbb{E} \left[\int_0^T (S_t \dot{\eta}_t + c_1^2 \dot{\eta}_t^2 + c_2^2 \eta_t^2) dt \right] = \int_0^T F(\mathbb{E}[S_t], \eta_t, \dot{\eta}_t) dt.$$

Let q be as in the statement. We have that

$$\frac{d}{dt} (2c_1^2 \dot{q}_t + \mathbb{E}[S_t]) = 2c_2^2 q_t.$$

Hence, an application of the integration-by-parts formula shows that q satisfies (A5) with F as in (A6) and S_t replaced by $\mathbb{E}[S_t]$. ■

Equation (A5) is the weak form of the Euler–Lagrange equation. In the next paragraph, we introduce the pathwise integration with respect to the path S . This will allow to move from the condition in equation (A5) to the stronger formulation of the Euler–Lagrange equation.

Young integration and integration-by-parts formula

Let $[0, T]$ be the closed time interval from time zero to the time horizon $T > 0$. We consider partitions π of $[0, T]$. A partition π is simultaneously considered as the finite collection of points and as the finite collection of adjacent subintervals. Given a partition π of $[0, T]$ and a time instant t in $[0, T]$, we adopt the following notational convention:

$$t' := \inf\{u \in \pi : u > t\}. \quad (\text{A7})$$

The mesh-size $|\pi|$ of the partition π is defined as

$$|\pi| := \sup\{|u' - u| : u \in \pi\}.$$

Let f be a function on $[0, T]$. If s and t are in $[0, T]$, we let $f_{s,t} := f_t - f_s$ denote the increment of f from s to t . The resulting two-parameter function $f = f_{s,t}$ is additive, in that for all $s \leq u \leq t$ it holds $f_{s,u} + f_{u,t} = f_{s,t}$. For more general two-parameter functions defined on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ we can relax the additivity and consider the following property.

DEFINITION A.8 Let $f = f(s, t)$ be a function on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. We say that f is super-additive if for all $s \leq u \leq t$ it holds

$$f(s, u) + f(u, t) \leq f(s, t).$$

LEMMA A.9 *Let f be a super-additive function on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ such that it is null and uniformly continuous on the diagonal, i.e. $f(t, t) \equiv 0$ and*

$$\limsup_{\epsilon \downarrow 0} \{f(s, t) : |t - s| \leq \epsilon\} = 0. \quad (\text{A8})$$

Let g be a super-additive function on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. Let $0 < a \leq 1$ and $0 < b < 1$ be exponents such that $a + b > 1$. Then,

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} f^a g^b(u, u') = 0,$$

where the limit is taken along arbitrary sequences of partitions of $[0, T]$ with mesh-size tending to zero.

Proof Let $0 < \epsilon < a$ be such that $a - \epsilon + b = 1$. Then,

$$\begin{aligned} \sum_{u \in \pi} f^a g^b(u, u') &\leq \left(\sup_{u \in \pi} f^\epsilon(u, u') \right) \cdot \sum_{u \in \pi} f^{a-\epsilon} g^b(u, u') \\ &\leq \left(\sup_{u \in \pi} f^\epsilon(u, u') \right) \\ &\quad \cdot \left(\sum_{u \in \pi} f(u, u') \right)^{a-\epsilon} \left(\sum_{u \in \pi} g(u, u') \right)^b \\ &\leq \left(\sup_{u \in \pi} f^\epsilon(u, u') \right) \cdot f^{a-\epsilon}(0, T) g^b(0, T). \end{aligned}$$

On the second line we have applied Hölder inequality and on the third line we have used super-additivity. We conclude by recalling the uniform continuity of equation (A8). ■

LEMMA A.10 *Let η be an absolutely continuous path on the closed interval $[0, T]$. Let S be a path on $[0, T]$ of finite p -variation for some*

$p \geq 1$. Then, for all $0 \leq s \leq t \leq T$ the limit

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \eta_u S_{u,u'}$$

exists and is the same along any sequence of partitions of $[s, t]$. Such a limit defines the Young integral

$$\int_s^t \eta_u dS_u.$$

Moreover, the following integration-by-parts formula holds

$$\int_s^t \eta_u dS_u + \int_s^t S_u d\eta_u = \eta_t S_t - \eta_s S_s,$$

where the second integral on the left hand side is the Stieltjes integral of S against η .

Proof Let π be an arbitrary partition of $[s, t]$. For all u in π we have

$$\eta_u S_{u,u'} + S_u \eta_{u,u'} = \eta_{u'} S_{u'} - \eta_u S_u - \eta_{u,u'} S_{u,u'}. \quad (\text{A9})$$

Consider first the right-hand side of equation (A9). Consider the first two summands on the right-hand side of equation (A9). If we sum over u in π we have the telescopic sum

$$\sum_{u \in \pi} (\eta_{u'} S_{u'} - \eta_u S_u) = \eta_t S_t - \eta_s S_s,$$

and this does not depend on the partition π of $[s, t]$. The third summand on the right-hand side of equation (A9) can be bound as follows:

$$|\eta_{u,u'} S_{u,u'}| \leq \|\eta\|_{1-\text{var},[u,u']} \|S\|_{p-\text{var},[u,u']}.$$

Hence, by lemma A.9, we have

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \eta_{u,u'} S_{u,u'} = 0.$$

Here, we have applied lemma A.9 with $f(s, t) = \|\eta\|_{1-\text{var},[s,t]}$, $g(s, t) = \|S\|_{p-\text{var},[s,t]}^p$, $a = 1$, and $b = 1/p$. Therefore, if we apply the operator $\lim_{|\pi| \downarrow 0} \sum_{u \in \pi}$ to the right-hand side of equation (A9), we obtain

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} (\eta_{u'} S_{u'} - \eta_u S_u - \eta_{u,u'} S_{u,u'}) = \eta_t S_t - \eta_s S_s.$$

Consider now the left-hand side of equation (A9). The limit

$$\lim_{|\pi| \downarrow 0} \sum_{u \in \pi} S_u \eta_{u,u'}$$

is the Stieltjes integral $\int_s^t S_u d\eta_u$ and thus converges and is the same along any sequence of partitions with vanishing meshsize. Therefore, if we apply the operator $\lim_{|\pi| \downarrow 0} \sum_{u \in \pi}$ to both sides of equation (A9) we must have that the limit in equation (A10) exists and is equal to

$$\eta_t S_t - \eta_s S_s - \int_s^t S_u d\eta_u.$$

This concludes the proof of the lemma. \blacksquare

Strong form of Euler–Lagrange equation

Relying on the Young integral introduced in the previous paragraph, we now formulate the strong form of the Euler–Lagrange equation associated with the minimisation problem in equation (A4).

DEFINITION A.11 Let c be a constant. Let S be a path on $[0, T]$ and assume that S is of finite p -variation for some $p \geq 1$. Let $b = b(t, x)$ be a Caratheodory function on $(0, T) \times \mathbb{R}^3$ such that

$$|b(t, x)| \leq \alpha(t) + \beta |x|^2,$$

for some integrable function α in $L^1(0, T)$ and some constant $\beta \geq 0$. We say that the function q in $W^{1,2}(0, T)$ solves the equation

$$\begin{aligned} dq_t &= r_t dt \\ dr_t &= b(t, S_t, q_t, \dot{q}_t) dt - c dS, \end{aligned} \quad (\text{A10})$$

if for all η in $C_0^\infty(0, T)$ and all $0 \leq s \leq t \leq T$ it holds

$$\int_s^t \dot{\eta}_u dq_u = \eta_t \dot{q}_t - \eta_s \dot{q}_s - \int_s^t \eta_u b(u, q_u, \dot{q}_u) du + c \int_s^t \eta_u dS_u, \quad (\text{A11})$$

where the last integral on the right hand side is the Young integral introduced in lemma A.10.

LEMMA A.12 Assume the setting of definition A.11. Then, q solves equation (A10) if and only if the function $f_t := \dot{q}_t + cS_t$ is absolutely continuous with derivative

$$\dot{f}_t = b(t, S_t, q_t, \dot{q}_t).$$

Proof Apply the integration-by-parts established in lemma A.10 to formula (A11), and observe that—for all η in $C_0^\infty(0, T)$ —formula (A11) is equivalent to

$$\int_0^T (\dot{q}_t + cS_t) \dot{\eta}_t dt = - \int_0^T \eta_t b(t, S_t, q_t, \dot{q}_t) dt.$$

\blacksquare

PROPOSITION A.13 (‘Strong form of Euler–Lagrange equation’) Let S be a path on $[0, T]$ and assume that S is of finite p -variation for some $p \geq 1$. Consider a Lagrangian F satisfying assumption A.4. Assume that the space-differentiable Caratheodory function $L(t, x) := F(t, x) - x_1 x_3$ is in $C^2([0, T] \times \mathbb{R}^3)$ and such that

$$\partial_{x_3} L(t, x_1, x_2, x_3) = \frac{1}{c} x_3 + \ell(t, x_2),$$

for some non-zero constant c in \mathbb{R} and some function ℓ in $C^1([0, T] \times \mathbb{R})$.[†] Assume that q is a minimiser for (A4). Then, q solves the Euler–Lagrange equation

$$\begin{aligned} dq_t &= r_t dt \\ dr_t &= \left(\partial_q L(t, S_t, q_t, \dot{q}_t) - \partial_{t,r}^2 L(t, S_t, q_t, \dot{q}_t) \right. \\ &\quad \left. - \dot{q}_t \partial_{q,r}^2 L(t, S_t, q_t, \dot{q}_t) \right) c dt - c dS_t, \end{aligned} \quad (\text{A12})$$

in the sense of definition A.11.

Proof Under the stated assumptions on the form of the partial derivative $\partial_r L$, the condition in equation (A5) reads

$$\int_0^T \left[S_t + \frac{1}{c} \dot{q}_t + \ell(t, q_t) \right] \dot{\psi}_t dt = - \int_0^T \psi_t \partial_q L(t, S_t, q_t, \dot{q}_t) dt.$$

This is equivalent to

$$\begin{aligned} \int_0^T \left[cS_t + \dot{q}_t \right] \dot{\psi}_t dt &= -c \int_0^T \left[\partial_q L(t, S_t, q_t, \dot{q}_t) \right. \\ &\quad \left. - \partial_t \ell(t, q_t) - \dot{q}_t \partial_{q,\ell}(t, q_t) \right] \psi_t dt. \end{aligned}$$

Notice that $\partial_t \ell = \partial_{t,r}^2 L$ and $\partial_{q,\ell} = \partial_{q,r}^2 L$. Hence, lemma A.12 concludes the proof. \blacksquare

[†] This assumption on the form of the partial derivative $\partial_r L$ is satisfied in all three examples that we consider in the paper, namely by the Lagrangians $F(t, S, q, r) = rS + L(t, S, q, r)$ in equations (22), (31) and (35).