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Primal–dual quasi-Monte Carlo simulation with dimension reduction for pricing American options

JIANGMING XIANG * and XIAOQUN WANG

Department of Mathematical Sciences, Tsinghua University, Beijing 10084, People's Republic of China

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The pricing of American options is one of the most challenging problems in financial engineering due to the involved optimal stopping time problem, which can be solved by using dynamic programming (DP). But applying DP is not always practical, especially when the state space is high dimensional. However, the curse of dimensionality can be overcome by Monte Carlo (MC) simulation. We can get lower and upper bounds by MC to ensure that the true price falls into a valid confidence interval. During the recent decades, progress has been made in using MC simulation to obtain both the lower bound by least-squares Monte Carlo method (LSM) and the upper bound by duality approach. However, there are few works on pricing American options using quasi-Monte Carlo (QMC) methods, especially to compute the upper bound. For comparing the sample variances and standard errors in the numerical experiments, randomized QMC (RQMC) methods are usually used. In this paper, we propose to use RQMC to replace MC simulation to compute both the lower bound (by the LSM) and the upper bound (by the duality approach). Moreover, we propose to use dimension reduction techniques, such as the Brownian bridge, principal component analysis, linear transformation and the gradients based principle component analysis. We perform numerical experiments on American–Asian options and American max-call options under the Black–Scholes model and the variance gamma model, in which the options have the path-dependent feature or are written on multiple underlying assets. We find that RQMC in combination with dimension reduction techniques can significantly increase the efficiency in computing both the lower and upper bounds, resulting in better estimates and tighter confidence intervals of the true price than pure MC simulation.

Keywords: Simulation; Quasi-Monte Carlo methods; American option pricing; Least squares Monte Carlo; Duality approach

JEL Classification: C13, C15, G12

1. Introduction

Options have been playing an influential role in financial markets. The holder of an option has a right which can be exercised or abandoned to buy or sell the underlying assets at a specified strike price prior to or on a specified date. American options refer to options that the holder can execute at allowed exercise times during the validity period of the option, while European options refer to the options that can only be executed on the expiration date.

As early as 1900, Bachelier published the first article on option pricing. Since then, closed-form solutions have been derived for European options under many models, but no similar solutions have been found for American options, except in some trivial special cases. Thus the pricing of American-style options is one of the most challenging problems in practice and is typically accomplished using numerical methods.

In the 1970s, the derivation of the Black–Scholes formula promoted theoretical research in the field of option pricing. Later, more and more option pricing methods appeared. Two traditional methods for American option pricing are the finite difference method proposed by Brennan and Schwartz (1977) and the binomial lattice method proposed by Cox *et al.* (1979). Both of these can be classified as lattice-based methods. These methods behave particularly well for American options in the case of a single asset. However, as the number of the underlying assets increases, the computational burden increases exponentially because of the well-known curse of dimensionality.

Dynamic programming (DP) is a useful tool for the optimal stopping problems in American options pricing. Approximate DP methods have been developed to calculate approximations

*Corresponding author. Email: xjm15@mails.tsinghua.edu.cn

to the optimal value and strategy in DP problems over the years; see for instance Ben Ameur *et al.* (2002, 2007) and the references therein. In most methods, regression or machine learning techniques are used to parameterize the value function. For example, the value function can be estimated by a linear combination of a set of basis functions over the state space by least squares. But this is not always practical, especially when the state space is high dimensional.

In order to overcome the curse of dimensionality, the Monte Carlo (MC) method became a useful tool. MC simulation converges as $O(N^{-1/2})$ with respect to N randomly sampled paths, which is independent of the dimension. Boyle (1977) first used MC simulation to price options. Since then this convenient and easy-to-operate tool has become more and more popular compared with other numerical methods. However, because of the early exercise feature of American-style options, MC simulation was considered not to be suitable for pricing American options early on. Since MC simulation is a fundamentally forward induction technique, dealing with the backward induction of DP to determine the optimal stopping time by MC is a seemingly difficult problem. Tilley (1993) was the first to use MC simulation for pricing American options. A bundling approach is proposed for the one-dimensional problem. He bundled the paths with similar underlying asset prices into groups and determined the optimal exercise policies for each group. Barraquand and Martineau (1995) also proposed a stratification method that can be applied to high-dimensional problems.

Several numerical methods have been proposed since then, but the real breakthrough was the introduction of the regression-based MC methods for pricing American options. The basic idea is to compute the conditional estimates of the so-called continuation value approximately. Carrière (1996) used a sequential regression algorithm to determine the optimal exercise strategies by estimating a number of conditional expectations using nonparametric regression. Longstaff and Schwartz (2001) presented a least-squares MC method (LSM) that has been particularly popular due to its accuracy and simplicity. The main idea of the LSM approach is to estimate the continuation value by cross-sectional least-squares regression on a set of basis functions at each time step, proceeding backward from the expiry date. And the continuation value estimates obtained by the LSM approach yield feasible stopping policies. Tsitsiklis and Van Roy (2001) also proposed a similar regression-based method based on approximating the value function directly, instead of the conditional expectations, for pricing American options using MC simulation. The convergence of the approximation in the LSM method is analyzed in Clément *et al.* (2002), Stentoft (2004) and Zanger (2009). However, since the exercise policies given by LSM are not optimal, only a lower bound on the true prices of American options can be obtained. A drawback of LSM is the inability to see how close the estimate is to the true option value or how close the exercise strategy is to being optimal.

Broadie and Glasserman (1997, 2004) proposed the random tree method and the stochastic mesh method to generate both lower and upper bounds on American option prices that converge to the true option prices asymptotically. It is possible to construct a valid confidence interval using these lower and upper bounds. However, the computational complexity of the random tree method is exponential in the number of exercise time opportunities. And the stochastic mesh method is also computationally demanding.

The methods mentioned above estimate the continuation values of American options which are expressed as regression functions or estimate the value functions directly to make the earlier exercise decision in the DP framework. Kohler (2010) gave a review of these approaches.

The duality approach was proposed by Rogers (2002) and Haugh and Kogan (2004) independently. Since then the dual approach has become a new impulse for research on simulation methods for American-style options. By martingale theory, the optimal stopping time problem is reformulated in form of a minimization problem over a class of martingales or supermartingales and this leads to upper bounds on American option prices. The choice of the martingales or supermartingales has an important impact on the tightness of the upper bounds. Andersen and Broadie (2004) presented a practical way to use the duality approach to compute the upper bounds. However, the upper bounds are estimated using a nested simulation algorithm that is computationally demanding. Belomestny *et al.* (2009b) presented a generic non-nested MC procedure using the martingale representation theorem.

In numerical analysis, the quasi-Monte Carlo (QMC) methods, which are deterministic versions of MC, are used for numerical integration and for some other problems using, instead of random numbers, low-discrepancy sequences (also called quasi-random sequences) that are cleverly chosen. For comparing the sample variances and standard errors in the numerical experiments, randomized QMC (RQMC) methods are usually used. Unless otherwise specified, the QMC methods and estimates mentioned in this paper are RQMC methods and estimates. QMC methods have proved to be very useful in pricing European options, leading to much more accurate estimates than MC methods in many cases, even in high dimensions (see Paskov and Traub 1995, Joy *et al.* 1996, Glasserman 2003, L'Ecuyer 2009, Lemieux 2009 and the references given therein). One reasonable explanation for the success of QMC in pricing European options is that the effective dimensions of the these problems are typically small (see Caflisch *et al.* 1997, Wang and Fang 2003). This roughly means that the functions in financial problems depend mainly on a small number of variables, or are nearly sums of functions each depending on a small number of variables. They can be analyzed more precisely by examining the variance components of an ANOVA decomposition (see Sobol' 2001, L'Ecuyer and Munger 2012). To further improve QMC methods, several techniques have been proposed to reduce the effective dimensions, such as the Brownian bridge (see Moskowitz and Caflisch 1996, Caflisch *et al.* 1997), principal component analysis (see Acworth *et al.* 1998), linear transformation (see Imai and Tan 2006) and gradient based principle component analysis (see Xiao and Wang 2017). For QMC, there are also some more recent references (e.g. see L'Ecuyer 2009, Dick and Pillichshammer 2010).

Pricing American options using regression-based methods and the duality approach is based on MC simulation. One of the disadvantages of MC is its slow convergence, despite

its independence of dimension. Indeed, any method for pricing American options by MC simulation requires substantial computational effort. It is natural to ask whether QMC methods work well in regression-based methods and in the duality approach for pricing American options. Could QMC methods improve the accuracy of the estimates of lower and upper bounds in comparison to MC? Are dimension reduction techniques useful for pricing American options? Note that although a rich literature exists in QMC methods on pricing European options, there are few works on pricing American options, especially to compute the upper bound. Boyle *et al.* (2013) used QMC to improve the efficiency of stochastic mesh method. Chaudhary (2005) used QMC and the Brownian bridge to improve the LSM algorithm. Lemieux and La (2005), Wang (2007), and Dion and L'Ecuyer (2011) also combined LSM and QMC with some variance reduction techniques.

Dion and L'Ecuyer (2011) experimented with several types of strategies in the QMC setting, such as the Brownian bridge (BB), principal component analysis (PCA) and Array-RQMC, to compute the lower bound by LSM and the algorithm proposed by Tsitsiklis and Van Roy (2001) (denoted by TvR). The latter showed that RQMC provides a significant variance reduction compared with MC. When using TvR, the array-RQMC method brings a bigger improvement than BB and PCA, while when using LSM, BB and PCA perform better than array-RQMC. In Dion and L'Ecuyer (2011), the variance reduction factors in Asian–American call options and the American max-call option using BB and PCA with LSM match our results, and BB and PCA perform better than Array-RQMC. In our experiments, we compute the lower bound by LSM (not TvR). And RQMC provides a significant variance reduction compared with MC. Other types of sampling strategies such as LT and GPCA are also efficient in some cases. More importantly, we show that using QMC simulation with these strategies to compute the upper bound has advantages in reducing the variance in our experiments. Note that we use MC simulation in the so-called inner simulations when we compute the upper bound, since we found that there is no improvement in the standard errors by using the low discrepancy points instead of random numbers in the inner simulations.

In particular, research on using QMC to compute the upper bound by the duality approach is very rare. The applicability of QMC simulation in computing the upper bound on the price of American options has not been studied systematically. Couffignals (2010) made some efforts but did not see the benefits of using QMC for the upper bound. Clearly, a systematic investigation on the adoption of QMC for American options is necessary. In this paper, we adopt QMC simulation to compute the upper bound by the duality approach presented in Andersen and Broadie (2004). This method depends on computing the lower bound, so that we also compute the lower bound by LSM. As an alternative to dual upper bounds, one can consider the so-called consumption-based upper bounds that were introduced and studied in Belomestny and Milstein (2006) and Belomestny *et al.* (2009a). This new approach used for pricing American options is based on the fact that any American option is equivalent to a European one with a consumption process involved. It admits the construction of an upper bound on the true price using a lower bound by MC simulation.

Instead of the Black–Scholes model, the variance gamma (VG) model can be advantageous, since it allows for more flexibility in modeling skewness and kurtosis than the Brownian motion does. Madan and Seneta (1990) presented a symmetric version of the variance gamma process. Madan *et al.* (1998) extended the model to an asymmetric form and presented an analytical formula to price European-style options under the variance gamma model. Although analytical solutions are available for European-style options, numerical methods are required for many other options. Avramidis and L'Ecuyer (2006) developed efficient MC algorithms for pricing path-dependent options with the VG model. They also combined gamma bridge sampling methods as dimension reduction technique with randomized QMC. L'Ecuyer *et al.* (2008) applied the PCA method to a general Lévy process and found that in some cases it is effective in improving the QMC performance.

The contributions of this paper are to propose using QMC methods for computing the upper bounds of American option prices and to use dimension reduction methods for improving QMC. Traditionally, the upper bounds of American option prices are computed by MC methods (see for instance Broadie and Glasserman 1997, Rogers 2002, Andersen and Broadie 2004, Broadie and Glasserman 2004, Haugh and Kogan 2004, Belomestny *et al.* 2009b), and QMC methods are often used to price European options (see for instance Paskov and Traub 1995, Joy *et al.* 1996, Glasserman 2003, L'Ecuyer 2009, Lemieux 2009 and the references therein). In this paper, our main contribution can be summarized as follows:

- We adapt QMC simulation to compute the upper bound by the duality approach presented by Andersen and Broadie (2004), which as far as we know has not been done before. This approach depends on computing the lower bound, so we also compute the lower bound by LSM.
- We propose to use several dimension reduction techniques to improve QMC. It is the first time that the LT and GPCA methods, for instance, are used in the pricing of American options. We perform our experiments under both the Black–Scholes model and the variance gamma model.
- In our experiments, we find that QMC simulation in combination with these techniques can significantly increase the efficiency in computing both the lower and upper bounds, resulting in better estimates and tighter confidence intervals for the true price of American options than pure MC simulation.

This paper is organized as follows. In Section 2, we present a summary of LSM for computing the lower bounds and the duality approach for computing the upper bounds of American options. In Section 3, we discuss under the Black–Scholes model how to simulate the paths of the underlying assets using QMC methods with various dimension reduction techniques for a single asset and multiple assets. In Section 4, we perform some numerical experiments to demonstrate the practical advantage of QMC for computing the lower and upper bounds

of American options compared to MC simulation and show that extra improvements can be achieved by using dimension reduction strategies properly. In Section 5, we review several methods of path generation under the VG model for a single asset and also perform numerical experiments using QMC methods. Conclusions are presented in Section 6. The results of the numerical experiments are presented in the Appendix.

2. Computing lower and upper bounds for the prices of American options

2.1. The formulation of American option pricing problem

We introduce some notations and set up assumptions for American option pricing problem in this section. For the sake of simplicity, we focus on the Bermudan options that can be exercised on a fixed discrete set of dates $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. One may view the Bermudan options as an approximation to the American options as n tends to infinity. Let $\{X(t), t \geq 0\}$ be a $\mathbb{R}^d$-valued Markov process representing all information about the underlying assets or state variables of the model such as additional risk factors driving stochastic volatilities and interest rates, with fixed initial state $X(0)$. Let $r(t)$ denote the instantaneous risk-free interest rate and $\{B(t) = \mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s}, t \geq 0\}$ be the riskless account process. In particular, $B(t) = \mathrm{e}^{rt}$ in the case of a constant risk-free rate r. Let $h_t(\cdot)$ denote the payoff function to the option holder from exercising so that the holder will receive $h_t(X(t))$ if the option is exercised at time t. To simplify the notations, we write $X(t_i)$ as X_i, $B(t_i)$ as B_i, and $h_{t_i}(X(t_i))$ as $h_i(X_i)$.

Let $V_i(x)$ denote the value of a Bermudan option at time t_i in state x, given that the option is not exercised before t_i. The aim of pricing a Bermudan option consists of finding an optimal early exercise strategy and computing the expected payoff using this strategy under the risk-neutral measure, i.e. to compute

$$V_0(X_0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_0 \left[\frac{h_\tau(X_\tau)}{B_\tau} \right], \tag{1}$$

where $\mathcal{T}$ is the set of all stopping times with values in $\{0, 1, \ldots, n\}$. The notation $\mathbb{E}_i[\cdot]$ represents the expectation conditional on the information available until time t_i under the risk-neutral measure, i.e. $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_i]$, where $\mathcal{F} = \{\mathcal{F}_i, i = 0, 1, \ldots, n\}$ denotes the information filtration generated by X. Here we assume that the financial market is complete so that there exists a unique risk-neutral measure. We refer the problem (1) as the *primal problem* following Andersen and Broadie (2004) and Haugh and Kogan (2004).

In general, the option value at time t_i, $V_i(X_i)$, satisfies

$$\frac{V_i(X_i)}{B_i} = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}_i \left[\frac{h_\tau(X_\tau)}{B_\tau} \right], \tag{2}$$

where $\mathcal{T}_i$ is the set of all stopping times with values in $\{i, \ldots, n\}$. It can also be interpreted as the value of a Bermudan option newly issued at time t_i. The continuation value at time t_i is

$$C_i(X_i) = \sup_{\tau \in \mathcal{T}_{i+1}} \mathbb{E}_i \left[\frac{B_i h_\tau(X_\tau)}{B_\tau} \right], \tag{3}$$

which describes the value of holding the option and exercising it in the future. For time $t_n = T$, we define $C_n(x) = 0, x \in \mathbb{R}^d$, since there is no value after expiration.

The optimal stopping time is given by

$$\tau^* = \inf\{i \in \{0, 1, \ldots, n\} : C_i(X_i) \leq h_i(X_i)\}. \tag{4}$$

It is well defined since $C_n(x) = 0 \leq h_n(x)$ and it states that we should exercise the option as soon as the exercise value is at least as large as the value we get if we exercise it in the future.

Further, the continuation value at time t_i can be represented as

$$C_i(X_i) = \mathbb{E}_i \left[\frac{B_i}{B_{i+1}} V_{i+1}(X_{i+1}) \right], \tag{5}$$

and the value of the option at time t_i is then

$$V_i(X_i) = \max(h_i(X_i), C_i(X_i)). \tag{6}$$

Thus in theory the problem (1) can be solved by using the DP recursion

$$\begin{aligned} V_n(X_n) &= h_n(X_n), \\ V_i(X_i) &= \max(h_i(X_i), C_i(X_i)), \quad i = n-1, \ldots, 1, 0. \end{aligned} \tag{7}$$

Note that the equations (7) state that the option value at expiration is given by the payoff function $h_n(X_n)$, and at the ith exercise date t_i the option value is the maximum of the immediate exercise value $h_i(X_i)$ and the continuation value $C_i(X_i)$ which represents the expected value of holding the option at time t_i. Note that we usually exclude the current time 0 from the set of exercise opportunities. This is accommodated in (7) by setting $h_0(X_0) = 0$.

From (5) and (6), we know that the continuation values also satisfy a DP recursion

$$\begin{aligned} C_n(X_n) &= 0, \\ C_i(X_i) &= \mathbb{E}_i \left[\frac{B_i}{B_{i+1}} \max(h_{i+1}(X_{i+1}), C_{i+1}(X_{i+1})) \right], \\ & \qquad i = n-1, \ldots, 1, 0. \end{aligned} \tag{8}$$

Most methods for estimating $V_0(X_0)$ rely on approximating the recursions (7) and (8) in some way. In Ben Ameur *et al.* (2002), the recursions (7) turn to be more convenient because it makes the approximation much easier and faster to compute. However, as pointed out in Kohler (2010), in the recursion (7) the maximum occurs outside of the expectation and the value function is in general not differentiable as a consequence. In contrast, in the recursion (8) the maximum function is smoothed by taking the conditional expectation. Since the approximations for the integrals of smooth functions by QMC methods are generally more accurate, we would like to use recursion (8) to estimate the continuation values here. For the sake of convenience, in the following discussions and numerical experiments we consider options that cannot be exercised at time 0.

2.2. Constructing a lower bound by LSM

Several simulation methods have been proposed to estimate the value of the Bermudan option V_i or the continuation values C_i at time t_i from simulated paths using regression. We focus on the LSM algorithm proposed in Longstaff and Schwartz (2001), which estimates the continuation values by least-squares regression using the information in the simulated paths. The basic idea is to approximate the continuation values by a linear combination of known functions of the current state and use least-squares regression to estimate the best coefficients in this approximation.

Suppose that we have simulated N independent paths $\{P_j\}_{j=1}^N$ of the state vector $X : P_j = (X_1^j, \ldots, X_n^j), j = 1, \ldots, N$. The notation X_i^j represents the state of X at time t_i in path P_j. Processing backwards through time from the final expiration date $t_n = T$, the option value at t_n is $V_n(X_n^j) = h_n(X_n^j)$ and the continuation value is $C_n(X_n^j) = 0$ for the jth path. At any exercise opportunity t_i prior to t_n, we suppose that given the state x, the continuation value $C_i(x)$ at time t_i can be represented as

$$C_i(x) = \sum_{l=1}^{M} \beta_{i,l}\phi_l(x),$$

where $\phi_l(x) : \mathbb{R}^d \to \mathbb{R}, l = 1, \ldots, M$ are basis functions, M is the number of basis functions and $\beta_{i,l}$ are constants to be determined. We may express this in vector form

$$C_i(x) = \beta_i^T \phi(x),$$

where $\beta_i = (\beta_{i,1}, \ldots, \beta_{i,M})^T$ and $\phi(x) = (\phi_1(x), \ldots, \phi_M(x))^T$. Then the problem of finding the continuation values C_i reduces to determine the coefficients $\beta_{i,l}$. Based on the least-squares regression, the estimate of β_i is given by

$$\hat{\beta}_i = (\Psi^{\mathrm{T}}\Psi)^{-1}\Psi^{\mathrm{T}} Y_i,$$

where Ψ is an $N \times M$ matrix with the (j, l)-entry $\Psi_{j,l} = \phi_l(X_i^j)$ and

$$Y_i = \frac{B_i}{B_{i+1}}(V_{i+1}(X_{i+1}^1), \ldots, V_{i+1}(X_{i+1}^N))^T.$$

In practice, at each time t_i, we separate the simulated paths into two parts: in-the-money paths $P_i^{\mathrm{IN}} = \{P_j : h_i(X_i^j) > 0\}$ and out-of-money paths $P_i^{\mathrm{OUT}} = \{P_j\}_{j=1}^N \backslash P_i^{\mathrm{IN}}$. It is suggested in Longstaff and Schwartz (2001) to use only P_i^{IN} since the exercise decision is only relevant when the option is in-the-money.

Consequently, the estimate of the continuation value at time t_i is

$$\hat{C}_i(x) = \hat{\beta}_i^T \phi(x).$$

Then for $j = 1, \ldots, N$, set $\hat{V}_i(X_i^j) = \max(h_i(X_i^j), \hat{C}_i(X_i^j))$. Proceeding from time t_{n-1} to time t_1, the option value is approximated by

$$\hat{V}_0(X_0) = \frac{1}{NB_1}\sum_{j=1}^{N} \hat{V}_1(X_1^j),$$

or the maximum of this value and $h_0(X_0)$ if it is allowed to exercise at time 0.

We can determine an exercise policy using the sequence of estimated continuation values $\hat{C}_i$ for all exercise opportunities defined by the LSM procedure as

$$\hat{\tau} = \min\{i \in \{1, \ldots, n\} : \hat{C}_i(X_i) \le h_i(X_i)\}.$$

This is the first time when the immediate exercise value is at least as large as the estimated continuation value in correspondence with (4). And the holder will receive $h_{\hat{\tau}}(X_{\hat{\tau}})$ if the option is exercised at time $t_{\hat{\tau}}$. With this exercise policy, the option value at time 0 can be expressed as

$$\tilde{V}_0(X_0) = \mathbb{E}_0\left[\frac{h_{\hat{\tau}}(X_{\hat{\tau}})}{B_{\hat{\tau}}}\right].$$

Note that the estimate of the option value obtained by the LSM algorithm following the specific policy $\hat{\tau}$ is biased low since no policy is better than the optimal policy, i.e.

$$\tilde{V}_0(X_0) = \mathbb{E}_0\left[\frac{h_{\hat{\tau}}(X_{\hat{\tau}})}{B_{\hat{\tau}}}\right] \le \sup_{\tau\in\mathcal{T}}\left[\frac{h_\tau(X_\tau)}{B_\tau}\right] = V_0(X_0).$$

As a result, the LSM algorithm provides a negatively biased estimate of the true price of the Bermudan option. Note that any other algorithm that gives an exercise policy $\hat{\tau}$ can be used to compute a negatively biased estimate of the price of the Bermudan option.

The accuracy of the LSM estimate depends on the choice of the basis functions as well as the number of basis functions. Note that as the dimension of the underlying state X increases, the number of basis functions required could increase quickly. It is important to select basis functions carefully. As suggested in Kohler (2010), the simplest possibility is monomials, i.e.

$$\phi_l(u) = \phi_l(u_1, \ldots, u_d) = u_1^{s_{1,l}} u_2^{s_{2,l}} \cdots u_d^{s_{d,l}}$$

for $u \in \mathbb{R}^d$ and some nonnegative integers $s_{1,l}, \ldots, s_{d,l}$. For $d = 1$, it reduces to fitting a polynomial of a fixed degree (e.g. $M - 1$) to the data. For large d, the degree of the multinomial polynomial (e.g. defined by $s_{1,l} + \cdots + s_{d,l}$ or $\max_{j=1,\ldots,d,l=1,\ldots,M} s_{j,l}$) has to be chosen small in order to avoid a large number of basis functions, which may increase the computational burden and does not necessarily improve the results. In general, the basis functions must be differently tailored to different derivatives. Moreno and Navas (2003) investigated the effects of the choices of various polynomials and the number of basis functions in LSM algorithm. A new approach towards the choice of basis function was proposed in Belomestny *et al.* (2018). The main idea of the method is to reinforce standard linear regression algorithms in each backward induction step by adding new basis functions based on previously estimated continuation values. The convergence properties of the LSM estimate to the true price were analyzed in Clément *et al.* (2002), Stentoft (2004), and Zanger (2009). Due to its simplicity and accuracy, the LSM algorithm has become the most popular method for practitioners.

2.3. Constructing an upper bound by the duality approach

Although the LSM algorithm has been very successful, it yields a suboptimal exercise policy and thus only provides a

negatively biased estimate of the true price of the Bermudan option. A drawback of LSM algorithm is its inability to determine how far the obtained estimate is from the true option value. To the best of our knowledge, although there are some works on computing the lower bounds of American options by QMC-based LSM (see Chaudhary 2005, Boyle *et al.* 2013), there are very rare researches on using QMC to compute the upper bound by the duality approach. We aim to improve the efficiency by using QMC methods. This section focuses on the formulation of the duality approach.

As mentioned in introduction, the duality approach was developed to compute the upper bounds on American option prices (see Rogers 2002, Andersen and Broadie 2004, Haugh and Kogan 2004). This method reformulates the original problem of the American pricing task in form of a minimization problem over a class of martingales or supermartingales. By creating both the lower and upper bounds, it is possible to determine how far the approximation is from the true price.

Let $\pi = \{\pi_i, i = 0, 1, \ldots, n\}$ be an arbitrary adapted martingale with initial value $\pi_0 = 0$. Then we have

$$V_0(X_0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_0 \left[\frac{h_\tau(X_\tau)}{B_\tau} - \pi_\tau \right] \leq \mathbb{E}_0 \left[\max_{i \in \{1,\ldots,n\}} \left(\frac{h_i(X_i)}{B_i} - \pi_i \right) \right], \tag{9}$$

where the equality follows from the martingale property of π and the Optional Sampling Theorem, and the inequality follows from Jensen's inequality. Taking the infimum over adapted martingales with initial value 0 on the right-hand side of the inequality in (9), we have

$$V_0(X_0) \leq \inf_{\pi} \mathbb{E}_0 \left[\max_{i \in \{1,\ldots,n\}} \left(\frac{h_i(X_i)}{B_i} - \pi_i \right) \right]. \tag{10}$$

The minimization problem on the right-hand side of (10) is the *dual problem*. By the inequality in (9), any martingale π with initial value $\pi_0 = 0$ yields an upper bound. The choice of martingale can be arbitrary, but different choices affect the tightness of the upper bound.

Further, (10) holds with equality. Indeed, we can define

$$\Delta_i = \frac{V_i(X_i)}{B_i} - \mathbb{E}_{i-1} \left[\frac{V_i(X_i)}{B_i} \right], \quad i = 1, \ldots, n, \tag{11}$$

and set

$$\pi_i = \Delta_1 + \cdots + \Delta_i, \quad i = 1, \ldots, n, \tag{12}$$

with $\pi_0 = 0$. Obviously, $\mathbb{E}_{i-1}(\Delta_i) = 0$ and the process π is a martingale. The equality in (10) is achieved by this martingale π. In fact, this optimal martingale $\{\pi_i\}$ is the martingale part of the Doob–Meyer decomposition of the supermartingale $\{V_i(X_i)/B_i\}$ which can be shown to be the smallest supermartingale that dominates $\{h_i(X_i)/B_i\}$ on $i \in \{1, \ldots, n\}$, i.e. the Snell envelope of $\{h_i(X_i)/B_i\}$. However, since $\mathbb{E}_0[V_1(X_1)/B_1]$, which involves in the definition of Δ_1 in (11), is the option value, finding the optimal martingale π appears to be as difficult as the original option pricing problem.

One possible approach to avoid this dilemma is to find a suboptimal martingale $\hat{\pi}$ which is 'close' to the optimal one and then use

$$\mathbb{E}_0 \left[\max_{i \in \{1,\ldots,n\}} \left(\frac{h_i(X_i)}{B_i} - \hat{\pi}_i \right) \right] \tag{13}$$

as an upper bound of the price of the Bermudan option. The choice of the martingales can affect the tightness of the upper bound critically, it is a difficult problem to determine the optimal martingale.

Before proceeding to methods for finding suitable martingales, we discuss how to obtain an estimate of (13) based on having the knowledge of $\hat{\pi}$ in hand. We simulate N_0 independent paths starting from $X_0 : X_1^j, \ldots, X_n^j, j = 1, \ldots, N_0$, and compute $\max_{i \in \{1,\ldots,n\}}(h_i(X_i)/B_i - \hat{\pi}_i)$ for each path, then average the results to obtain the approximation of (13), i.e. the upper bound. This procedure is referred as the *outer simulation*.

Using (11), we can approximate π_i in (12) by estimating Δ_i. In general, we may construct the martingale by approximating the value functions (see Haugh and Kogan 2004) or stopping rules (see Andersen and Broadie 2004). We refer to Glasserman (2003) for a review. In our numerical experiments in Section 4 and Section 5, we use the method proposed in Andersen and Broadie (2004) which focuses on the approximation of the exercise policy. Here we review it briefly.

We assume that $\hat{C}_i$ is an approximation to the continuation value $C_i, i = 1, \ldots, n$, with $C_n = 0$. Then we can define a sequence of stopping times $\tau_1, \ldots, \tau_n$ as

$$\tau_i = \inf\{k = i, \ldots, n : \hat{C}_k(X_k) \leq h_k(X_k)\}, \quad i = 1, \ldots, n.$$

Note that τ_i denotes the first time at which an option that is newly issued at time t_i should be exercised according to the given exercise policy specified by the approximate continuation values $\hat{C}_i$. Define $L_i = \mathbb{E}_i[B_i h_{\tau_i}(X_{\tau_i})/B_{\tau_i}]$ as the value of the option at time t_i, following the chosen exercise policy from time t_i onward. Note that L_i is also seen to be the value of an option that is newly issued at time t_i and exercised according to the given exercise policy. Meanwhile we can use L_i as an approximation to the true value function V_i. This process $\{L_i\}$ can also be computed from any other feasible exercise policy.

Removing the assumption $\pi_0 = 0$, we still have

$$V_0(X_0) \leq \pi_0 + \mathbb{E}_0 \left[\max_{i \in \{1,\ldots,n\}} \left(\frac{h_i(X_i)}{B_i} - \pi_i \right) \right]. \tag{14}$$

Now we define the martingale π by $\pi_0 = L_0, \pi_1 = L_1/B_1$ and

$$\pi_k = \pi_{k-1} + \frac{L_k}{B_k} - \frac{L_{k-1}}{B_{k-1}} - l_{t_{k-1}} \mathbb{E}_{k-1} \left[\frac{L_k}{B_k} - \frac{L_{k-1}}{B_{k-1}} \right], \quad 2 \leq k \leq n, \tag{15}$$

where l_t is an adapted exercise indicator process that equals 1 if exercise should take place at time t and 0 otherwise. Then π can be easily checked to be a martingale. Since $\pi_0 = L_0$ is the lower bound of the option price which can be computed

by LSM algorithm, from (14) an upper bound of $V_0(X_0)$ is given by

$$L_0 + \mathbb{E}_0\left[\max_{i\in\{1,\dots,n\}}\left(\frac{h_i(X_i)}{B_i} - \pi_i\right)\right] := L_0 + \Delta_0, \tag{16}$$

where Δ_0 can be estimated by the following procedure.

Step 1. Simulate $\hat{N}_0$ paths of state variables from $X_0 : X_1^j, \dots, X_n^j, j = 1, \dots, \hat{N}_0$ and the corresponding riskless account process $B_1, \dots, B_n$.

Step 2. Check the exercise policy l_{t_k} which can be determined by LSM algorithm at each time step $k = 1, \dots, n$ for each path generated in Step 1. If $l_{t_k} = 0$ which means that holder should not exercise the option at time t_k, then follow the procedure in Step 2a, otherwise follow Step 2b.

(a) ($l_{t_k} = 0$). According to the definition of π_k in (15), we have

$$\pi_{k+1} = \pi_k + \frac{L_{k+1}}{B_{k+1}} - \frac{L_k}{B_k},$$

so we only need to estimate L_k/B_k for each k in this case. Since

$$\frac{L_k}{B_k} = \mathbb{E}_k\left[\frac{h_{\tau_k}(X_{\tau_k})}{B_{\tau_k}}\right], \tag{17}$$

we use an *inner simulation* (or *nested simulation*) to estimate L_k/B_k.

More precisely, we generate $\hat{N}_1$ subpaths $\{S_j^i, j > k, i = 1, \dots, \hat{N}_1\}$ starting from X_k ($S_k^i = X_k, i = 1, \dots, \hat{N}_1$). We can choose to stop the subpaths according to the exercise policy τ_k determined before and average $h_{\tau_k}(X_{\tau_k})/B_{\tau_k}$ over these subpaths, or generate the subpaths until expiration and determine the exercise policies $\tau_k^1, \dots, \tau_k^{\hat{N}_1}$ based on the information in the newly generated subpaths and use $\sum_{i=1}^{\hat{N}_1} h_{\tau_k^i}(S_{\tau_k^i}^i)/(\hat{N}_1 B_{\tau_k^i})$ to approximate L_k/B_k. In our numerical experiments, we use the latter.

(b) ($l_{t_k} = 1$). In this case, we have

$$\begin{aligned}\pi_{k+1} &= \pi_k + \frac{L_{k+1}}{B_{k+1}} - \frac{L_k}{B_k} - l_{t_k}\mathbb{E}_k\left[\frac{L_{k+1}}{B_{k+1}} - \frac{L_k}{B_k}\right] \\ &= \pi_k + \frac{L_{k+1}}{B_{k+1}} - \mathbb{E}_k\left[\frac{L_{k+1}}{B_{k+1}}\right].\end{aligned}$$

Since exercising the option is recommended, we can just set L_{k+1}/B_{k+1} equal to h_{k+1}/B_{k+1}. We can use a nested simulation to estimate

$$\begin{aligned}\mathbb{E}_k\left[\frac{L_{k+1}}{B_{k+1}}\right] &= \mathbb{E}_k\left[\mathbb{E}_{k+1}\left[\frac{h_{\tau_{k+1}}(X_{\tau_{k+1}})}{B_{\tau_{k+1}}}\right]\right] \\ &= \mathbb{E}_k\left[\frac{h_{\tau_{k+1}}(X_{\tau_{k+1}})}{B_{\tau_{k+1}}}\right].\end{aligned} \tag{18}$$

Again we generate $\hat{N}_2$ subpaths starting from X_k which are stopped at the first time $t \geq t_{k+1}$ such that $l_t = 1$. The exercise policy can be determined in the same way introduced in (a). Then we average the results over the subpaths.

Step 3. Set $\pi_1, \dots, \pi_n$ as in (15) and compute

$$\max_{i\in\{1,\dots,n\}}\left(\frac{h_i(X_i)}{B_i} - \pi_i\right)$$

for each path and average the results to estimate the quantity Δ_0 in (16). And the upper bound of $V_0(X_0)$ is given by $L_0 + \Delta_0$.

3. QMC simulation under the Black–Scholes model

We see that simulations play an important role in both the LSM algorithm and the duality approach. Traditionally, all these simulations are performed by MC. QMC methods are proved to be more efficient in pricing European options comparing with MC simulation. But there are few works on pricing American options, especially to compute the upper bound (see Chaudhary 2005, Couffignals 2010, Dion and L'Ecuyer 2011, Boyle *et al.* 2013). In this section, we describe briefly how to simulate the required paths of the underlying assets driven by Brownian motion by QMC simulation. There is a systematic description on the generation of Brownian motion by MC simulation in Glasserman (2003). However, simply replacing random numbers by low discrepancy points may lead to bad algorithm in QMC, since the performance of QMC is sensitive to the methods of path generation. It is important to optimize the use of low discrepancy points. Below, after introducing some backgrounds on QMC methods, we focus on the QMC simulation of paths for underlying asset prices in the cases of a single asset and multiple assets under the Black–Scholes model, with special attention on the possible different path generation methods.

3.1. *Low discrepancy sequences and QMC methods*

QMC methods are deterministic versions of MC methods. The basic idea of QMC methods is to replace the underlying random source of random numbers in MC methods with low discrepancy points or quasi-random numbers, which are more uniformly distributed over the unit cube $[0, 1]^d$.

Consider a set of N points in the d-dimensional unite cube $\mathbf{P} = \{\mathbf{x}_i \in [0, 1)^d, i = 1, \dots, N\}$. The uniformity of the point set $\mathbf{P}$ is measured by (L_∞-star) discrepancy defined as

$$D(\mathbf{P}) = \sup_E \left|\frac{A(E, N)}{N} - \lambda(E)\right|,$$

where the supremum is taken over all subsets of $[0, 1)^d$ of the form $E = [0, x_1) \times \cdots \times [0, x_d)$, $A(E, N)$ denotes the number of points of $\mathbf{P}$ contained in E, and λ is the Lebesgue measure. There are some other definitions of discrepancies such as the extreme discrepancy and L_2-discrepancy (see Niederreiter 1992). Traditionally, if the first N points of a sequence satisfy

$$D(\mathbf{P}) \leq c(d)\frac{(\log N)^d}{N}, \quad \forall\, N \geq 1,$$

where $c(d)$ is relevant to d, the sequence is called as a *low discrepancy sequence*.

For a multivariate integration problem

$$\int_{[0,1)^d} f(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

we approximate the integral by $\sum_{i=1}^{N} f(\mathbf{x}_i)/N$. The error can be bounded by the Koksma–Hlawka inequality (see Niederreiter 1992)

$$\left| \int_{[0,1)^d} f(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i) \right| \le D(\mathbf{P}) V_{\mathrm{HK}}(f),$$

where $V_{\mathrm{HK}}(f)$ is the variation of the function f in the sense of Hardy and Krause. This indicates that for functions with bounded variation, the convergence order of QMC for multivariate numerical integration is $O(N^{-1}(\log N)^d)$, which is asymptotically much better than $O(N^{-1/2})$ of MC. Thus for fixed d and sufficiently large N, QMC is superior to MC.

The core in QMC methods is to find point sets or sequences with discrepancies as small as possible. A substantial body of work has been devoted to constructing low discrepancy points sets or sequences (see Niederreiter 1992). On the other hand, when d is large, $N^{-1}(\log N)^d$ is substantially larger than $N^{-1/2}$ unless N is huge. Therefore, it was widely believed that QMC methods should not be used for high-dimensional integration. But since mid-90s of last century, there have been numerous works on the applications of QMC methods in finance, and great success of QMC methods for a variety of high-dimensional problems has been demonstrated (see Paskov and Traub 1995, Joy *et al.* 1996). However, we point out the following.

- First, most works on applications of QMC in finance problems focused on the pricing of European-style derivatives, which can be expressed in terms of multivariate integration. But for American option pricing problems, it does not fit the framework of multivariate integration. The applicability of QMC methods for pricing American options is unclear and should be verified both theoretically and empirically. Chaudhary (2005) and Boyle *et al.* (2013) made some efforts to use QMC methods for pricing American options.
- Second, the convergence of QMC for multivariate integration depends on the dimension of the problems. In high dimensions, the superiority of low discrepancy points over random numbers mainly derived from the projections associated with earlier dimensions and lower-dimensional projections (see L'Ecuyer 2009, L'Ecuyer and Munger 2016). The latter dimension projections and high-order projections of low discrepancy points may be no more uniform than those of random points on the average, unless the number of points is huge (see Morokoff and Caflisch 1994, Wang and Sloan 2008). Thus it is important to reduce the effective dimensionality of the finance problems, especially in the case of large number of time steps or multiple underlying assets. Many dimension reduction techniques have been proposed (see Moskowitz and Caflisch 1996, Caflisch *et al.* 1997, Acworth *et al.* 1998, Imai and Tan 2006, Xiao and Wang 2017).
- Third, simply replacing random numbers in existing MC methods by low discrepancy points may lead to inefficient or wrong algorithm, so additional design in QMC setting is necessary. We hope that the better distribution property of low discrepancy points in low-order projections can lead to more accuracy estimates by suitably designing the algorithms.

This paper aims to use numerical experiments to demonstrate that it is reasonable to use QMC simulation in combination with dimension reduction techniques in computing both the lower and upper bounds of American options.

3.2. *QMC simulation for single asset under the Black–Scholes model*

We first consider the case of a single underlying asset. In the Black–Scholes model, we assume that under the risk-neutral measure the price of the underlying asset follows the geometric Brownian motion

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \tag{19}$$

where r is the risk-free interest rate, σ is the volatility of the underlying asset, and $\{W_t\}$ is the one-dimensional standard Brownian motion. The analytical solution to the stochastic differential equation (19) is

$$S_t = S_0 \exp\left(\left(r - \sigma^2/2\right)t + \sigma W_t\right),$$

so simulating the asset prices reduces to simulating the Brownian motion W_t. We suppose that the asset prices are sampled at equally spaced times $t_j = j\Delta t$, $j = 1, \ldots, n$ with $\Delta t = T/n$ from current time to the expiration date T. By the property of the Brownian motion, the vector $(W_{t_1}, \ldots, W_{t_n})^T$ is normally distributed with mean zero and covariance matrix C, whose (i, j)-entry is given by $C_{ij} = \min(t_i, t_j)$. Thus the Brownian motion $\{W_t\}$ can be generated as

$$(W_{t_1}, \ldots, W_{t_n})^T = A(z_1, \ldots, z_n)^T, \quad (z_1, \ldots, z_n)^T \sim N(\mathbf{0}, \mathbf{I}_n), \tag{20}$$

where A is any matrix (called generating matrix) satisfying $AA^{\mathrm{T}} = C$ and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Here we briefly describe three traditional constructions.

- The *standard construction* (**STD**) generates the Brownian motion sequentially (see Glasserman 2003): given $W_0 = 0$, we have

  $$W_{t_j} = W_{t_{j-1}} + \sqrt{\Delta t} z_j, \quad z_j \sim N(0, 1),\ j = 1, \ldots, n.$$

  The generating matrix A in (20) corresponding to the standard construction is the Cholesky decomposition of the covariance matrix C.
- The *Brownian bridge construction* (**BB**) (see Moskowitz and Caflisch 1996, Caflisch *et al.* 1997)

is based on the Brownian bridge formula. More precisely, given a past value W_{t_i} and a future value W_{t_k}, the value W_{t_j} with $t_i < t_j < t_k$ can be computed according to

$$W_{t_j} = (1-\gamma)W_{t_i} + \gamma W_{t_k} + \sqrt{\gamma(1-\gamma)(k-i)\Delta t}z, \quad z \sim N(0,1),$$

where $\gamma = (j-i)/(k-i)$. Specially, if the number of time steps is a power of 2, given $W_0 = 0$, the Brownian motion can be simulated at times in order $T, T/2, T/4, 3T/4, \ldots$ The generating matrix A^{BB} associated with the BB construction can also be obtained (see Wang 2006).

- The *principle component analysis construction* (**PCA**) (see Acworth *et al.* 1998) takes the generating matrix as

$$A^{\mathrm{PCA}} = U\Lambda^{1/2},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix of eigenvalues of the covariance matrix C arranged in a non-increasing order and the columns of U are the corresponding eigenvectors of unit-length.

In the numerical experiments in Section 4, we compare these constructions with two relatively new methods, namely, the *linear transformation construction* (**LT**) and the *gradients based principle component analysis construction* (**GPCA**). The descriptions of these two constructions can be found in Imai and Tan (2006) and Xiao and Wang (2017), respectively, both aiming to reduce the effective dimensions by taking into account of the knowledge of the payoff functions. Here we use these two constructions by using the due date as the option exercise time to sample the paths.

In QMC methods, if we use the matrix formulation (20) for path generation, we use the representative point of the n-dimensional standard normal variable

$$(z_1, \ldots, z_n)^T = (\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_n))^T,$$

where $\Phi^{-1}(x)$ is the inverse function of the cumulative distribution function of standard normal variable and $\mathbf{x} = (x_1, \ldots, x_n)^T \in [0,1]^n$ is some low discrepancy point over the n-dimensional unit cube. Thus one point from n-dimensional unit cube generates one path of the asset prices. We can sequentially use the coordinates of $(z_1, \ldots, z_n)^T$ to generate one path of W_t to avoid matrix operation in (20) in the standard and BB constructions. All constructions are equivalent in probabilistic sense in MC simulation. However, their effects in QMC can be quite different due to the special distribution properties of low discrepancy points.

3.3. *QMC simulation for multiple assets under the Black–Scholes model*

Now we consider the case of multiple assets which is more complex than the single asset case. We refer to Wang (2009) for a review. Similarly, we assume that under the risk-neutral measure the assets follow the multi-dimensional geometric Brownian motion and satisfy the stochastic differential equations

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = (r-\delta_i)\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_i(t), \quad i = 1, \ldots, m, \tag{21}$$

where r is the risk-free interest rate, δ_i is the dividend yield, σ_i is the volatility for the ith asset, and $\{\mathbf{B}(t) = (B_1(t), \ldots, B_m(t))^T\}$ is an m-dimensional Brownian motion with each B_i being a standard one-dimensional Brownian motion and the instantaneous correlation of B_i and B_j being ρ_{ij}. The Brownian motion $\mathbf{B}$ is abbreviated as $\mathbf{B} \sim \mathrm{BM}(\mathbf{0}, \Sigma)$, where the (i,j)-entry of Σ is $\Sigma_{ij} = \rho_{ij}$.

By solving (21), we have

$$S_i(t_j) = S_i(0)\exp\left(\left(r - \delta_i - \sigma_i^2/2\right)t_j + \sigma_i B_i(t_j)\right), \quad j = 1, \ldots, n;\ i = 1, \ldots, m.$$

Thus we only need to simulate Brownian motion $\mathbf{B}$ at times $t_1, t_2, \ldots, t_n$ to simulate the paths of the asset prices.

We arrange the values of $\mathbf{B}$ at times $t_1, t_2, \ldots, t_n$ in the following form:

$$\mathcal{Y} := (B_1(t_1), \ldots, B_m(t_1), B_1(t_2), \ldots, B_m(t_2), \ldots, B_1(t_n), \ldots, B_m(t_n))^T. \tag{22}$$

For simulating Brownian motion $\mathbf{B}$, we only need to simulate the mn-dimensional vector $\mathcal{Y}$ that is normally distributed with mean zero and the covariance matrix given by $(C \otimes \Sigma)$, where the matrix C is defined in Section 3.2 and $\otimes$ denotes the Kronecker product.

Suppose that $\mathcal{U}$ is an $mn \times mn$ matrix satisfying $\mathcal{U}\mathcal{U}^{\mathrm{T}} = (C \otimes \Sigma)$. Then the mn-dimensional vector $\mathcal{Y}$ defined in (22) can be generated as

$$\mathcal{Y} = \mathcal{U}\mathcal{Z}, \quad \mathcal{Z} \sim N(0, \mathbf{I}_{mn}). \tag{23}$$

Given the covariance matrix $(C \otimes \Sigma)$ of $\mathcal{Y}$, we have two obvious but computationally expensive constructions: direct decomposition of $(C \otimes \Sigma)$ by Cholesky or PCA method which can be called as one-step Cholesky decomposition or one-step PCA decomposition. However, due to the special structure of the covariance matrix $(C \otimes \Sigma)$, we may use two-stage procedures that are computationally cheaper.

Let A and D be the decomposition matrices of C and Σ, respectively, satisfying

$$AA^{\mathrm{T}} = C, \quad DD^{\mathrm{T}} = \Sigma.$$

Then it is easy to check that $(A \otimes D)$ is a decomposition matrix of $(C \otimes \Sigma)$, satisfying

$$(A \otimes D)(A \otimes D)^{\mathrm{T}} = (C \otimes \Sigma).$$

Thus the vector $\mathcal{Y}$ can be generated as

$$\mathcal{Y} = (A \otimes D)\mathcal{Z}, \quad \mathcal{Z} \sim N(\mathbf{0}, \mathbf{I}_{mn}). \tag{24}$$

Therefore, for generating $\mathcal{Y}$ we may need to get decompositions for the matrices C and Σ. For example, we may use

Cholesky, BB, PCA, LT or GPCA method for the matrix C like in the single asset case, and use Cholesky or PCA method for the matrix Σ.

In practice, we may avoid the matrix product operations in (24). Since the covariance matrix of $(W_i(t_1), \ldots, W_i(t_n))^T$ is the matrix C, the decomposition $C = AA^{\mathrm{T}}$ corresponds to the generation of the standard one-dimensional Brownian motion

$$(W_i(t_1), \ldots, W_i(t_n))^T = A(z_{i,1}, \ldots, z_{i,n})^T, \quad i = 1, \ldots, m, \tag{25}$$

where $z_{i,1}, \ldots, z_{i,n}$ are independent standard normals. This is the dimension-by-dimension construction by fixing dimensions.

The decomposition $\Sigma = DD^{\mathrm{T}}$ corresponds to the way of generating the Brownian motion $\mathbf{B}$ by a linear transformation from the standard Brownian motion $\mathbf{W}$

$$(B_1(t_j), \ldots, B_m(t_j))^T = D(W_1(t_j), \ldots, W_m(t_j))^T, \quad j = 1, \ldots, n. \tag{26}$$

In QMC context, if we use matrix formulation (23) or (24) for path generation, we simply use the transformed point

$$\mathcal{Z} = (z_1, \ldots, z_{mn})^T = (\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_{mn}))^T,$$

where $\mathbf{x} = (x_1, \ldots, x_{mn})^T \in [0,1]^{mn}$ is a low discrepancy point in mn-dimensional unit cube. If we use the dimension-by-dimension construction (25), we first form an $m \times n$ matrix G from the components of the point $\mathcal{Z}$ as

$$G = \begin{pmatrix} z_1 & z_{m+1} & \cdots & z_{(n-1)m+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_m & z_{2m} & \cdots & z_{nm} \end{pmatrix}.$$

Then we use the ith row of the matrix G in (25) for the ith dimension $W_i(t)$ to guarantee the algebraical equivalence of (25) and (26) with the matrix representation (24). This algebraical equivalence is mentioned in Wang (2009). In the second numerical experiment in Section 4, we use this dimension-by-dimension construction.

4. Numerical experiments under the Black–Scholes model

In this section, we perform numerical experiments under the Black–Scholes model on American–Bermudan–Asian option as single asset option and Bermudan max-call option as multiple assets option. For simplicity, the risk-free interest rate is assumed to be constant, which indicates that $B_t = \mathrm{e}^{rt}$. We focus on the effects of using QMC methods to replace MC simulation, in combination with several dimension reduction techniques, in computing the lower bounds by LSM algorithm and the upper bounds by the duality approach. The results of the numerical experiments are presented in Appendix.

4.1. *American–Bermudan–Asian call option under the Black–Scholes model*

Under the Black–Scholes model, we assume that the underlying asset follows the geometric Brownian motion (19). We consider a call option on the average price of a stock over some time horizon, where the option can be exercised at any discrete time which are assumed to be equally spaced after some initial lockout period. The option has a Bermudan and American exercise feature and it is also an Asian option since it is a path-dependent option depending on an average of the asset prices. This American–Bermudan–Asian option was studied in Longstaff and Schwartz (2001).

We denote the final expiration date as T. The option can be exercised after the first 3 months since time 0 (called the initial lockout period) by payment of the strike price K. In other words, the option can be exercised at any discrete times $t_j = j\Delta t$, $j = 1, \ldots, n$, $\Delta t = T/n$ after the initial lockout period and the payoff that the option holder receives is $h_t = \max(0, A_t - K)$, where A_t denotes the arithmetic average of the underlying stock price during the period 3 months prior to the valuation date (i.e. time 0), to the exercise time t. In particular, A_0 is the initial average of the underlying stock price during the period from 3 months prior to time 0 to the valuation date.

For using the LSM method to compute the lower bound, the basis functions should be chosen carefully. Consider the weighted Laguerre polynomials as in Longstaff and Schwartz (2001)

$$L_n(X) = \exp(-X/2)\frac{\mathrm{e}^X}{n!}\frac{\mathrm{d}^n}{\mathrm{d}X^n}\left(X^n \mathrm{e}^{-X}\right).$$

Longstaff and Schwartz (2001) used a constant, the first two Laguerre polynomials evaluated at the stock price, the first two Laguerre polynomials evaluated at the average stock price, and the cross products of these Laguerre polynomials up to third-order terms. Here we choose a set of 10 basis functions slightly different. Namely, we choose the first three weighted Laguerre polynomials evaluated at the stock price $L_0(S_t), L_1(S_t), L_2(S_t)$ as well as the average stock price $L_0(A_t), L_1(A_t), L_2(A_t)$ and the cross products of the weighted Laguerre polynomials as $L_0(S_t)L_0(A_t), L_0(S_t)L_1(A_t), L_1(S_t)L_0(A_t), L_1(S_t)L_1(A_t)$.

For estimating the lower bound by LSM algorithm, we generate $N = 4096$ paths by STD, BB, PCA, LT, or GPCA construction mentioned in Section 3.2.

For estimating the upper bound by the duality approach, we generate $\hat{N}_0 = 256$ paths in the outer simulation mentioned in Section 2.3 by STD, BB, PCA, LT, or GPCA construction. To reduce the computational burden of the nested simulations, we generate $\hat{N}_1 = \hat{N}_2 = 64$ subpaths at each step in the inner simulations (17) and (18). The exercise policies obtained by LSM algorithm in computing the lower bound can be used in the duality approach to determine the martingales required. But here we choose to generate subpaths to the expiration date on each step and define the exercise policies using the LSM method with the newly generated subpaths.

The whole procedure is replicated 50 times to compute the estimators and the standard errors. In randomized QMC, N scrambled n-dimensional Sobol' points are used in each

replication to generate paths for computing the lower bound and $\hat{N}_0$ scrambled Sobol' points are used in the outer simulations. Each path is generated by one scrambled Sobol' point. We use MC simulation in the inner simulations, since we found that there is no improvement in the standard errors by using the low discrepancy points instead of random numbers in the inner simulations. The parameters are $K = 100, \sigma = 20\%, T = 2, r = 6\%, n = 200$. The results (lower bounds, upper bounds, and their standard errors) are presented in table A1 for different pairs (A_0, S_0) and the standard errors are given in the parentheses.

A valid approximate confidence interval can be obtained as follows. Suppose that the averages of the lower and upper bounds computed in all replications are $\hat{L}$ and $\hat{U}$, and the corresponding standard errors are e_L and e_U. Then the 95% confidence interval of the estimators of the lower bounds is roughly

$$\left[\hat{L} - 1.96e_L, \hat{L} + 1.96e_L\right],$$

while the 95% confidence interval of the estimators of the upper bounds is roughly

$$\left[\hat{U} - 1.96e_U, \hat{U} + 1.96e_U\right].$$

Thus the 95% confidence intervals of the price is roughly

$$\left[\hat{L} - 1.96e_L, \hat{U} + 1.96e_U\right].$$

The 95% confidence intervals are presented in table A2 for different pairs (A_0, S_0) and their lengths are given in the parentheses. We observe the following.

- The QMC methods perform very well in computing both the lower bounds by LSM and the upper bounds by the duality approach. The QMC estimates of the lower and upper bounds are remarkably better than MC estimates, giving much smaller standard errors (table A1) and much tighter confidence intervals (table A2).
- QMC with BB, PCA, LT and GPCA constructions reduces the standard errors of lower bounds and upper bounds. The variance reduction factors with respect to MC estimates can be more than 100 in computing the lower bounds and about 50 in computing the upper bounds using BB, PCA, LT, GPCA constructions, among which PCA, LT and GPCA have similar performance, all better than BB. Note that the variance reduction factors depend on the number of sampling points (see L'Ecuyer and Munger 2016, for example).
- Lower initial underlying stock price S_0 and lower initial average of the underlying stock price A_0 lead to smaller standard error and tighter confidence interval.
- The length of the 95% confidence interval using STD construction in the QMC method is about 70% of the length in MC. And the length of the 95% confidence interval using QMC with LT, PCA, and GPCA constructions can be shorter, about 50% of the length in MC.
- The performance of QMC methods is better in computing the lower bounds than the upper bounds. The variance reduction factors in computing the lower bounds can be two to five times as large as that in computing the upper bounds.

4.2. *American–Bermudan max-call option under the Black–Scholes model*

The second example is the pricing of American–Bermudan max-call option used as a test example in Broadie and Glasserman (1997), Rogers (2002), Andersen and Broadie (2004), and Haugh and Kogan (2004). We assume that the risk-neutral dynamics of m assets follow the correlated geometric Brownian motion (21). For simplicity, we take the dividend yield $\delta_i = \delta$, the volatility $\sigma_i = \sigma$ and correlation $\rho_{ij} = \rho$ for all $i, j = 1, \ldots, m$ and $i \neq j$. The option expires at time T and can be exercised at equally spaced times $t_i = iT/n, i = 1, \ldots, n$. The payoff function of the American–Bermudan m-max call at time t is $h_t(S_1, \ldots, S_m) = \max(0, \max(S_1(t), \ldots, S_m(t)) - K)$.

To compute the lower bounds by LSM algorithm, we choose 10 basis functions consisting of a constant, the largest and the second largest assets prices, three polynomials of degree 2 (i.e. the square of the largest and second largest assets prices and the product of the two) and four polynomials of degree 3 (i.e. the cube of the largest and second largest assets prices, the product of the square of the largest assets prices and second largest assets prices, and the product of the largest assets prices and the square of the second largest assets prices). Here we generate $N = 4096$ paths by dimension-by-dimension construction methods (25) and (26). More precisely, as introduced in Section 3.3, we use one-step PCA decomposition for $C \otimes \Sigma$, and for two-stage procedures, we use Cholesky, BB, PCA, LT, and GPCA constructions for the matrix C, and use Cholesky or PCA decomposition for Σ.

The upper bounds are computed by the duality approach. The paths generated in computing the lower bounds are also used as the outer simulation in the duality approach, i.e. $\hat{N}_0 = 4096$. We also generate $\hat{N}_1 = \hat{N}_2 = 64$ subpaths at each step in the inner simulation (17) and (18), and the exercise policies are determined using the same method as for the first experiment.

For MC, we use two-stage procedures by Cholesky decomposition for both C and Σ. In randomized QMC, scrambled Sobol' points are used in each replication to generate paths for computing the lower bound and scrambled Sobol' points are also used in the outer simulations while MC simulation is used in the inner simulations. The whole procedure is replicated 100 times to estimate the standard errors.

The parameters are as follows: risk-free interest rate $r = 5\%$, dividend yield $\delta = 10\%$, volatility $\sigma = 20\%$, expiration time $T = 3$, strike price $K = 100$. The initial stock price is $\mathbf{S}(0) = (S_0, \ldots, S_0)^T$ with $S_0 = 90, 100$, or 110. The number of time steps is $n = 9$, i.e. the option is exercisable three times per year. The number of assets is $m = 2, 3$, or 5. The correlation is $\rho = 0$ or 0.5. Since for $\rho = 0$ Cholesky decomposition for Σ is equivalent with PCA decomposition, we only

present the results (lower bounds, upper bounds, and their corresponding standard errors) by Cholesky decomposition here (see table A3). And for $\rho = 0.5$ we present the results by Cholesky decomposition (see table A4) and PCA decomposition for Σ (see table A5). The standard errors are given in the parentheses. We observe the following.

- The QMC methods work well in computing both the lower bounds by the LSM and the upper bounds by the duality approach. The QMC estimates of the lower and upper bounds are better than MC estimates, giving smaller standard errors. The variance reduction factors with respect to MC estimates can be about 5–10 in computing the lower bounds and about 5 in computing the upper bounds. The performance of QMC methods is better in computing the lower bounds than the upper bounds.
- In QMC methods, BB, PCA, and one-step PCA decomposition have advantages on giving consistently smaller standard errors, among which one-step PCA decomposition is computationally expensive and the other two are cheaper. Their performance is not as good as that in the first example, which may due to the absence of the path-dependent feature of the option. Note that in this example, LT and GPCA methods fail to give a better estimate.
- Different decomposition methods of Σ and different correlation ρ have small influence on the standard errors.
- Lower initial underlying stock price S_0 and smaller number of assets m lead to smaller standard errors.

5. QMC simulation under the VG model

5.1. QMC simulation for single asset under the VG model

A variance gamma process (see Madan and Seneta 1990, Madan *et al.* 1998) can be defined with three parameters θ, σ, ν by

$$\begin{aligned} X &= \{X(t), t \geq 0\} = \{X(t; \theta, \sigma, \nu), t \geq 0\} \\ &= \{B(G(t; 1, \nu), \theta, \sigma), t \geq 0\}, \end{aligned}$$

where $B = \{B(t), t \geq 0\} = \{B(t; \theta, \sigma), t \geq 0\}$ is the Brownian motion with drift θ and volatility σ and $G = \{G(t), t \geq 0\} = \{G(t; \mu, \nu), t \geq 0\}$ is the gamma process independent of B with drift μ and volatility ν. In other words, the variance gamma process can be seen as a Brownian motion subjected to a random time change which follows a gamma process. The increasing property of the gamma process ensures that the time change makes sense.

The risk-neutral asset price process is modeled as

$$S(t) = S(0) \exp\{(w + r - q)t + X(t)\}, \tag{27}$$

replacing Brownian motion in the original Black–Scholes model (19) by the VG process. Here r is the risk-free interest rate, q is the dividend yield and $w = \log(1 - \theta\nu - \sigma^2\nu/2)/\nu$ is defined by $\mathbb{E}[\exp(X(t))] = \exp(-wt)$ to ensure that the discounted price process is a martingale under the pricing measure. Indeed, w can be computed directly under the assumption $(\theta + \sigma^2/2)\nu < 1$ which ensures $\mathbb{E}[S(t)] < \infty$ for all $t > 0$. To simulate a path of the asset price, it suffices to generate a path of X and transform it by the definition (27). We now focus on several constructions for the VG process.

Since the VG process can be obtained by subjecting the Brownian motion to a random time change obeying a gamma process with $\mu = 1$, we can use the constructions for the Brownian motion described in Section 3.2 after sampling the gamma process. This algorithm can be called as Brownian-gamma sampling (BGS) (see Avramidis and L'Ecuyer 2006).

Another algorithm known as difference-of-gammas sampling (DGS) relies on the observation that the VG process can be represented as the difference of two independent gamma process (see Madan *et al.* 1998) as $X(t) = \Gamma^+(t) - \Gamma^-(t)$, where Γ^+ and Γ^- are independent gamma processes with parameters (μ_p, ν_p) and (μ_m, ν_m) respectively. Here $\mu_p = (\sqrt{\theta^2 + 2\sigma^2/\nu} + \theta)/2$, $\mu_m = (\sqrt{\theta^2 + 2\sigma^2/\nu} - \theta)/2$, $\nu_p = \mu_p^2\nu$ and $\nu_m = \mu_m^2\nu$. So we can simulate X by simulating these two gamma processes respectively. However, DGS algorithm may need to generate more beta variates than BGS so that DGS often leads to lower computation efficiency due to the expensive simulation of beta variates based on the inverse transform method, compared to generating normal or gamma variates. But with equally spaced times, L'Ecuyer and Simard (2006) proposes a fast algorithm for computing the inverse symmetrical beta distribution.

Since we need to sample the gamma process in both the BGS and DGS algorithms, we briefly describe several sampling methods for gamma process with parameter (μ, ν) at discrete times $0 = t_0 < t_1 < \cdots < t_n$.

- The *sequential sampling (Seq)* for gamma process G uses the independent increments feature. Let $G(0) = 0$, we have $G(t_i) = G(t_{i-1}) + \Delta_i$, where $\Delta_i \sim \text{Gamma}((t_i - t_{i-1})\mu^2/\nu, \nu/\mu)$, $i = 1, \ldots, n$. Here $\text{Gamma}(\alpha, \beta)$ denotes the gamma distribution with the probability density function

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \mathrm{e}^{-x/\beta}, \quad x, \alpha, \beta > 0.$$

  After generating the gamma process G by sequential sampling, we can generate the VG process X with $X(t) = B(G(t))$ using the standard construction for the Brownian motion B with the time steps replacing by the gamma process generating by the BGS algorithm.
- The *bridge sampling (BS)* (see Avramidis and L'Ecuyer 2006) for gamma process relies on the fact that given $G(t_1)$ and $G(t_2)$ and $0 \leq t_1 < t < t_2$, the conditional distribution of $G(t)$ is the same as $G(t_1) + (G(t_2) - G(t_1))Y$, where $Y \sim \text{Beta}((t - t_1)\mu^2/\nu, (t_2 - t)\mu^2/\nu)$. Here $\text{Beta}(\alpha, \beta)$ denotes the beta distribution with the probability density

function

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x},$$
$$0 < x < 1;\ \alpha,\beta > 0.$$

Thus we can first sample all increments of G by bridge sampling in the same order as for the Brownian bridge construction and then sample the Brownian motion B by BB construction, or sample them in pairs in the BGS algorithm. The pseudocode of the second approach is given in Avramidis and L'Ecuyer (2006).

- The *PCA sampling* for gamma process can be done in two ways. Both ways use PCA constructions to simulate a Brownian motion and transform its increments into uniform variates which are used to sample gamma process (see L'Ecuyer *et al.* 2008). In more details, for the first way, we simulate a Brownian motion $\{W(t), t \geq 0\}$ with mean zero and variance $\sigma^2 = \nu$ at $0 = t_0 < t_1 < \cdots < t_n$ by the PCA construction in Section 3.2. Then we transform the increments of W into independent random variates which have uniform distribution on $[0,1]$ by

$$V_j = \Phi\left(\frac{W(t_j) - W(t_{j-1})}{\sigma\sqrt{t_j - t_{j-1}}}\right), \quad j = 1,\ldots,n,$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Finally, we sample the increments of the gamma process as $G(t_j) - G(t_{j-1}) = G_j^{-1}(V_j)$, where $G_j(\cdot)$ is the cumulative gamma distribution function of the jth increment, for $j = 1,\ldots,n$. This method is called *PCA sampling with sequential transformation (PCAS)*.
Another way is called *PCA sampling with bridge transformation (PCAB)*. We first simulate the Brownian motion W by the PCA construction as in the PCAS method. Let $G_{t_1,g_1,t_2,g_2,t}$ denote the distribution function of $G(t)$ conditional on $\{G(t_1) = g_1, G(t_2) = g_2\}$ described in the bridge sampling method above and $\Phi_{t_1,w_1,t_2,w_2,t}$ denote the normal distribution function of $W(t)$ conditional on $\{W(t_1) = w_1, W(t_2) = w_2\}$. We start by sampling $G(t_n) = G_{t_n}^{-1}(\Phi(W(t_n/(\sigma t_n^{1/2}))))$, where $G_{t_n}(\cdot)$ denotes the gamma distribution function of $G(t_n)$, then let

$$G(t_{n/2}) = G^{-1}_{t_0,0,t_n,G(t_n),t_{n/2}}\left(\Phi_{t_0,0,t_n,W(t_n),t_{n/2}}\left(W\left(t_{n/2}\right)\right)\right),$$
$$G(t_{n/4}) = G^{-1}_{t_0,0,t_{n/2},G(t_{n/2}),t_{n/4}}\left(\Phi_{t_0,0,t_{n/2},W(t_{n/2}),t_{n/4}}\left(W\left(t_{n/4}\right)\right)\right),$$
$$G(t_{3n/4}) = G^{-1}_{t_{n/2},G(t_{n/2}),t_n,G(t_n),t_{3n/4}}\left(\Phi_{t_{n/2},W(t_{n/2}),t_n,W(t_n),t_{3n/4}}\left(W\left(t_{3n/4}\right)\right)\right),$$

and so on, in the same order as for the BB construction.

Thus we can simulate the gamma process by PCAS or PCAB at the n given times $t_1,\ldots,t_n$ and then simulate the Brownian motion B by PCA construction at the n times $G(t_1),\ldots,G(t_n)$ specified by the gamma process in the BGS algorithm.

As for the DGS algorithm, we can generate the gamma processes Γ^+ and Γ^- by the sampling methods described above and then use their difference as X.

5.2. *American–Bermudan–Asian call option under the VG model*

We consider an American–Bermudan–Asian call option under the VG model. The underlying asset price follows (27) and the payoff function of the option is $h_t = \max(0, A_t - K)$, where A_t denotes the arithmetic average of the underlying stock price during the valuation date, i.e. time 0, to the exercise time t (without initial lockout period). We use the same set of basis functions as in the first example.

We generate $N = 4096$ paths to estimate the lower bound using LSM algorithm and $\hat{N}_0 = 256$ paths in the outer simulation for the duality approach by different sampling methods for the VG process mentioned in Section 5.1. In more details, we generate the VG progress by sequential sampling (Seq), bridge sampling (BS), PCA sampling with sequential transformation (PCAS) and PCA sampling with bridge transformation (PCAB) in both Brownian-gamma sampling (BGS) and difference-of-gammas sampling (DGS) algorithms. $\hat{N}_1 = \hat{N}_2 = 64$ subpaths are generated at each step in the inner simulation. The exercise policies are determined in the same way as in the first experiment. The whole procedure is replicated 100 times to estimate the standard errors. In randomized QMC, scrambled Sobol' points are used in each replication to generate paths for computing the lower bound and the outer simulations while MC simulation is used in the inner simulations.

The parameters we used are borrowed from Avramidis and L'Ecuyer (2006), namely, $K = 100, \theta = -0.2859, \sigma = 0.1927, \nu = 0.2505, r = 0.0548$, and $q = 0$. We assume that the option is exercisable eight times per year which means the option can be exercised every other month and a half. The results (lower bounds, upper bounds, and their corresponding standard errors) for different pairs (T, S_0) are presented in table A6 in Appendix and the standard errors are given in the parentheses. We observe the following.

- QMC methods perform well in the VG model. The QMC estimates of the lower and upper bounds have smaller standard errors than MC estimates. The improvements of QMC in computing the lower bounds by LSM are more significant than in computing the upper bounds by the duality approach. The variance reduction factors with respect to MC estimates can be about 10–100 in computing lower bounds and 5–20 in computing the upper bounds.
- Lower initial underlying stock price S_0 and smaller expiration time T lead to smaller standard errors.
- As for different methods for path constructions, bridge sampling and PCAB algorithm have advantage on giving consistently smaller standard errors.

We can still see that the improvements in the lower bounds lead to the improvements in the upper bounds. The DGS method is better than the BGS method with respect to the standard errors. However, although generally DGS gives the maximal variance reduction, BGS often leads to higher computation efficiency due to the expensive simulation of beta variates based on the inverse transform method, compared to generating normal or gamma variates.

6. Conclusion

MC simulation is a powerful method for pricing American options when the options are written on multiple underlying assets or have a path dependent feature. Primal–dual representations of American option pricing problems allow both the lower and upper bounds to be estimated by simulation. A sound implementation of MC simulation is to use the LSM algorithm to estimate the lower bounds and use the duality approach to estimate the upper bounds. Although a rich literature exists on QMC methods for pricing European options, there are few works on pricing American options, especially on computing the upper bound.

In this paper, we review the primal–dual representation of the American option pricing problem, the LSM algorithm and the duality approach. We describe methods to simulate the paths of the underlying assets driven by Brownian motion or the gamma process using QMC simulation in the single asset and multiple assets cases. We preform numerical experiments on the Black–Scholes and VG models, and show that in computing the lower and upper bounds, QMC methods combined with suitable dimension reduction give much better estimates than MC methods and that the performance of QMC methods is significantly better than MC estimates. In the Black–Scholes model with a single underlying asset, PCA, LT, and GPCA constructions have advantage in reducing the variance. We thus get much tighter confidence intervals by using (randomized) QMC simulation. For multiple underlying assets, one-step PCA, BB, and PCA decompositions have an advantage in stably giving smaller standard errors. In the VG model, dimension reduction techniques such as bridge sampling and the PCAB algorithm significantly reduce the standard errors of the lower and upper bounds. The DGS method is better than the BGS method in this respect.

This paper focuses on numerical illustration of QMC methods in pricing American-style options. As future research, it would be desirable to establish the asymptotic convergence rate of the QMC estimators for American options. Different dimension reduction techniques also need theoretical support to figure out the applicable circumstances and conditions.

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ORCID

Jiangming Xiang http://orcid.org/0000-0002-4199-1205

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Appendix. Results of the numerical experiments

Table A1. The lower and upper bounds of the prices of American–Bermudan–Asian call options.

		Lower bound						Upper bound					
		MC	Sobol'					MC	Sobol'				
A_0	S_0	STD	STD	BB	PCA	LT	GPCA	STD	STD	BB	PCA	LT	GPCA
90	90	3.333	3.329	3.332	3.336	3.335	3.334	3.359	3.354	3.358	3.359	3.360	3.360
		(.0141)	(.0101)	(.0013)	(**.0007**)	(**.0007**)	(.0008)	(.0146)	(.0101)	(.0023)	(.0022)	(**.0018**)	(.0021)
100	90	3.732	3.736	3.725	3.725	3.726	3.725	3.776	3.781	3.772	3.771	3.774	3.769
		(.0139)	(.0110)	(.0018)	(**.0010**)	(.0011)	(.0012)	(.0150)	(.0121)	(.0030)	(.0033)	(.0031)	(**.0026**)
110	90	4.205	4.219	4.207	4.213	4.211	4.212	4.289	4.301	4.282	4.282	4.281	4.290
		(.0163)	(.0101)	(.0031)	(**.0016**)	(.0020)	(.0022)	(.0187)	(.0108)	(.0047)	(.0039)	(**.0038**)	(.0051)
90	100	7.874	7.916	7.911	7.912	7.914	7.914	7.926	7.964	7.957	7.958	7.959	7.955
		(.0261)	(.0115)	(.0025)	(.0013)	(**.0012**)	(.0013)	(.0260)	(.0125)	(.0039)	(.0029)	(**.0027**)	(.0030)
100	100	8.650	8.674	8.685	8.685	8.685	8.684	8.726	8.748	8.754	8.747	8.748	8.748
		(.0231)	(.0095)	(.0023)	(**.0018**)	(.0020)	(.0020)	(.0238)	(.0101)	(.0043)	(.0034)	(**.0033**)	(.0038)
110	100	9.820	9.847	9.824	9.831	9.833	9.832	9.945	9.961	9.939	9.934	9.935	9.939
		(.0183)	(.0113)	(.0043)	(.0043)	(**.0039**)	(.0049)	(.0201)	(.0123)	(.0073)	(.0059)	(**.0058**)	(.0064)
90	110	14.540	14.545	14.545	14.546	14.549	14.544	14.607	14.603	14.612	14.595	14.604	14.599
		(.0284)	(.0131)	(.0025)	(.0024)	(**.0019**)	(.0023)	(.0290)	(.0140)	(.0042)	(.0033)	(**.0032**)	(.0033)
100	110	15.668	15.716	15.721	15.722	15.721	15.720	15.767	15.808	15.811	15.798	15.798	15.805
		(.0326)	(.0104)	(.0036)	(.0030)	(**.0025**)	(.0028)	(.0324)	(.0116)	(.0059)	(**.0046**)	(**.0046**)	(.0049)
110	110	17.372	17.356	17.369	17.365	17.365	17.374	17.491	17.469	17.469	17.469	17.460	17.474
		(.0301)	(.0115)	(.0050)	(.0043)	(.0040)	(**.0035**)	(.0304)	(.0113)	(.0070)	(.0058)	(**.0055**)	(.0060)

Note: $N = 4096$ paths are used to estimate the lower bound by LSM. For the duality approach, $\hat{N}_0 = 256$ paths are used in the outer simulations and $\hat{N}_1 = \hat{N}_2 = 64$ subpaths are used in the inner simulations. The whole procedure is replicated 50 times to estimate the standard errors. In QMC, scrambled Sobol' points are used in each replication to generate paths for computing the lower bound and in the outer simulations in the computation of upper bounds. The parameters are $K = 100, \sigma = 20\%, T = 2, r = 6\%, n = 200$. The standard errors of these estimates are given in the parentheses.

Table A2. The 95% confidence intervals of the prices of American–Bermudan–Asian call options.

		MC	Sobol'				
A_0	S_0	STD	STD	BB	PCA	LT	GPCA
90	90	[3.306, 3.388]	[3.309, 3.374]	[3.329, 3.363]	[3.334, 3.364]	[3.333, 3.363]	[3.333, 3.364]
		(.0823)	(.0645)	(.0339)	(**.0295**)	(.0300)	(.0310)
100	90	[3.705, 3.805]	[3.714, 3.804]	[3.721, 3.778]	[3.723, 3.778]	[3.723, 3.780]	[3.723, 3.775]
		(.1003)	(.0900)	(.0563)	(.0546)	(.0563)	(**.0517**)
110	90	[4.173, 4.326]	[4.199, 4.322]	[4.201, 4.292]	[4.209, 4.289]	[4.208, 4.289]	[4.207, 4.300]
		(.1528)	(.1234)	(.0905)	(**.0801**)	(.0813)	(.0924)
90	100	[7.823, 7.977]	[7.893, 7.988]	[7.906, 7.964]	[7.910, 7.964]	[7.912, 7.964]	[7.911, 7.961]
		(.1539)	(.0950)	(.0584)	(.0541)	(.0521)	(**.0496**)
100	100	[8.605, 8.773]	[8.656, 8.768]	[8.680, 8.763]	[8.682, 8.754]	[8.681, 8.754]	[8.680, 8.755]
		(.1681)	(.1117)	(.0822)	(**.0723**)	(.0730)	(.0753)
110	100	[9.785, 9.984]	[9.825, 9.985]	[9.815, 9.954]	[9.822, 9.945]	[9.825, 9.946]	[9.822, 9.952]
		(.1999)	(.1604)	(.1383)	(.1230)	(**.1209**)	(.1293)
90	110	[14.484, 14.664]	[14.520, 14.630]	[14.540, 14.621]	[14.542, 14.601]	[14.545, 14.610]	[14.540, 14.606]
		(.1797)	(.1103)	(.0808)	(**.0593**)	(.0650)	(.0663)
100	110	[15.604, 15.831]	[15.696, 15.830]	[15.714, 15.823]	[15.716, 15.807]	[15.716, 15.807]	[15.715, 15.814]
		(.2267)	(.1346)	(.1091)	(**.0908**)	(.0911)	(.0997)
110	110	[17.313, 17.551]	[17.333, 17.492]	[17.359, 17.483]	[17.356, 17.480]	[17.357, 17.471]	[17.367, 17.486]
		(.2373)	(.1582)	(.1237)	(.1237)	(**.1140**)	(.1184)

Note: The lengths of these confidence intervals are given in the parentheses. The parameters are the same as in table A1.

Table A3. Comparisons for American–Bermudan max-call options for $\rho = 0$.

		MC	Sobol'					
S_0	m	TwoStage	OneStepPCA	STD	BB	PCA	LT	GPCA
				Lower bound				
90	2	8.129	8.114	8.105	8.106	8.119	8.122	8.126
		(.0204)	(**.0065**)	(.0093)	(.0070)	(.0066)	(.0072)	(.0074)
100	2	13.975	13.951	13.941	13.940	13.923	13.929	13.937
		(.0240)	(.0092)	(.0119)	(**.0086**)	(.0090)	(.0101)	(.0109)
110	2	21.463	21.366	21.391	21.368	21.348	21.396	21.394
		(.0275)	(.0117)	(.0122)	(.0105)	(**.0104**)	(.0111)	(.0127)
90	3	11.337	11.320	11.346	11.320	11.321	11.340	11.349
		(.0242)	(**.0079**)	(.0126)	(.0087)	(.0082)	(.0101)	(.0096)
100	3	18.744	18.726	18.719	18.736	18.718	18.718	18.753
		(.0260)	(**.0097**)	(.0132)	(.0098)	(.0108)	(.0135)	(.0118)
110	3	27.610	27.592	27.600	27.595	27.587	27.554	27.562
		(.0276)	(.0140)	(.0160)	(.0137)	(**.0124**)	(.0164)	(.0165)
90	5	16.704	16.692	16.690	16.696	16.679	16.699	16.703
		(.0255)	(**.0099**)	(.0155)	(.0105)	(.0117)	(.0156)	(.0144)
100	5	26.201	26.183	26.165	26.164	26.164	26.162	26.186
		(.0295)	(.0142)	(.0168)	(**.0138**)	(.0140)	(.0162)	(.0162)
110	5	36.839	36.816	36.772	36.792	36.784	36.782	36.806
		(.0343)	(**.0143**)	(.0210)	(.0161)	(.0153)	(.0182)	(.0172)
				Upper bound				
90	2	8.326	8.308	8.300	8.301	8.312	8.316	8.319
		(.0197)	(**.0064**)	(.0091)	(.0067)	(.0067)	(.0073)	(.0074)
100	2	14.319	14.292	14.287	14.277	14.263	14.269	14.280
		(.0235)	(.0093)	(.0118)	(**.0087**)	(**.0087**)	(.0102)	(.0108)
110	2	21.981	21.884	21.905	21.892	21.864	21.915	21.911
		(.0271)	(.0115)	(.0120)	(**.0102**)	(.0104)	(.0108)	(.0127)
90	3	11.575	11.559	11.584	11.556	11.558	11.580	11.588
		(.0236)	(**.0078**)	(.0126)	(.0092)	(.0081)	(.0104)	(.0098)
100	3	19.131	19.112	19.101	19.120	19.103	19.099	19.138
		(.0253)	(**.0098**)	(.0127)	(.0102)	(.0107)	(.0136)	(.0124)
110	3	28.147	28.126	28.135	28.127	28.120	28.086	28.099
		(.0266)	(.0138)	(.0162)	(.0139)	(**.0123**)	(.0159)	(.0168)
90	5	16.989	16.973	16.974	16.975	16.958	16.978	16.986
		(.0250)	(**.0098**)	(.0151)	(.0106)	(.0112)	(.0152)	(.0147)
100	5	26.604	26.581	26.565	26.562	26.565	26.565	26.586
		(.0292)	(.0142)	(.0166)	(**.0137**)	(**.0137**)	(.0158)	(.0166)
110	5	37.344	37.315	37.274	37.282	37.282	37.284	37.304
		(.0332)	(**.0143**)	(.0208)	(.0164)	(.0152)	(.0183)	(.0176)

Note: $N = 4096$ paths are used to estimate the lower bound by LSM. $\hat{N}_0 = 4096$ paths are used in the outer simulations and $\hat{N}_1 = \hat{N}_2 = 64$ subpaths are used in the inner simulations to estimate the upper bound by the duality approach. The whole procedure is replicated 100 times to estimate the standard errors. For two-stage procedures, we use Cholesky decomposition for Σ. In QMC, scrambled Sobol' points are used in each replication to generate paths for computing the lower bound and in the outer simulations in the computation of upper bounds. The parameters are $K = 100, \delta = 10\%, \sigma = 20\%, T = 3, r = 5\%, \rho = 0$ (the time interval $[0, T]$ is discretized into nine steps). The standard errors of these estimates are given in the parentheses.

Table A4. Comparisons for American–Bermudan max-call options for $\rho = 0.5$ using Cholesky decomposition for Σ.

		MC	Sobol'					
S_0	m	TwoStage	OneStepPCA	STD	BB	PCA	LT	GPCA
				Lower bound				
90	2	7.168	7.164	7.166	7.166	7.143	7.176	7.169
		(.0177)	(**.0070**)	(.0089)	(.0071)	(**.0070**)	(.0071)	(.0082)
100	2	12.282	12.218	12.239	12.210	12.216	12.239	12.237
		(.0242)	(.0090)	(.0108)	(.0091)	(**.0086**)	(.0102)	(.0094)
110	2	18.872	18.799	18.813	18.803	18.813	18.819	18.843
		(.0288)	(.0125)	(.0136)	(**.0091**)	(.0099)	(.0108)	(.0130)
90	3	9.262	9.217	9.222	9.203	9.203	9.222	9.231
		(.0216)	(.0081)	(.0101)	(.0087)	(**.0079**)	(.0092)	(.0105)
100	3	15.273	15.183	15.206	15.178	15.188	15.165	15.188
		(.0253)	(**.0103**)	(.0116)	(.0115)	(.0112)	(.0123)	(.0132)
110	3	22.585	22.616	22.606	22.619	22.604	22.595	22.618
		(.0330)	(.0142)	(.0135)	(.0146)	(**.0119**)	(.0166)	(.0143)
90	5	12.226	12.215	12.243	12.217	12.231	12.228	12.209
		(.0261)	(.0110)	(.0144)	(**.0103**)	(.0119)	(.0114)	(.0102)
100	5	19.389	19.370	19.364	19.391	19.357	19.372	19.414
		(.0319)	(.0138)	(.0159)	(**.0118**)	(**.0118**)	(.0125)	(.0148)
110	5	27.851	27.831	27.851	27.845	27.806	27.822	27.828
		(.0355)	(.0166)	(.0186)	(**.0155**)	(.0167)	(**.0155**)	(.0199)
				Upper bound				
90	2	7.349	7.345	7.347	7.347	7.321	7.355	7.349
		(.0173)	(.0071)	(.0091)	(.0074)	(**.0068**)	(.0073)	(.0080)
100	2	12.617	12.545	12.566	12.535	12.542	12.561	12.565
		(.0232)	(.0097)	(.0099)	(.0094)	(**.0086**)	(.0102)	(.0096)
110	2	19.403	19.325	19.338	19.330	19.342	19.339	19.372
		(.0285)	(.0118)	(.0126)	(**.0096**)	(.0106)	(.0108)	(.0128)
90	3	9.478	9.424	9.432	9.415	9.413	9.430	9.439
		(.0211)	(.0079)	(.0093)	(.0084)	(**.0078**)	(.0092)	(.0107)
100	3	15.636	15.543	15.561	15.536	15.549	15.524	15.546
		(.0251)	(**.0102**)	(.0118)	(.0114)	(.0103)	(.0119)	(.0139)
110	3	23.145	23.156	23.145	23.165	23.150	23.136	23.163
		(.0315)	(.0137)	(.0134)	(.0145)	(**.0124**)	(.0163)	(.0146)
90	5	12.472	12.455	12.481	12.453	12.468	12.468	12.444
		(.0256)	(.0110)	(.0141)	(.0105)	(.0122)	(.0114)	(**.0104**)
100	5	19.773	19.750	19.747	19.778	19.739	19.748	19.792
		(.0312)	(.0140)	(.0160)	(**.0117**)	(.0118)	(.0122)	(.0149)
110	5	28.396	28.374	28.399	28.378	28.356	28.364	28.370
		(.0350)	(.0169)	(.0176)	(**.0155**)	(.0174)	(.0159)	(.0196)

Note: $N = 4096$ paths are used to estimate the lower bound by LSM. $\hat{N}_0 = 4096$ paths are used in the outer simulations and $\hat{N}_1 = \hat{N}_2 = 64$ subpaths are used in the inner simulations to estimate the upper bound by the duality approach. The whole procedure is replicated 100 times to estimate the standard errors. For two-stage procedures, we use Cholesky decomposition for Σ. In QMC, scrambled Sobol' points are used in each replication to generate paths for computing the lower bound and in the outer simulations in the computation of upper bounds. The parameters are $K = 100, \delta = 10\%, \sigma = 20\%, T = 3, r = 5\%$, $\rho = 0.5$ (the time interval $[0, T]$ is discretized into nine steps). The standard errors of these estimates are given in the parentheses.

Table A5. Comparisons for American–Bermudan max-call options for $\rho = 0.5$ using PCA decomposition for Σ.

		MC	Sobol'					
S_0	m	TwoStage	OneStepPCA	STD	BB	PCA	LT	GPCA
				Lower bound				
90	2	7.209	7.145	7.176	7.156	7.171	7.153	7.157
		(.0168)	(**.0058**)	(.0076)	(.0070)	(.0072)	(.0070)	(.0074)
100	2	12.274	12.211	12.231	12.213	12.224	12.228	12.252
		(.0236)	(.0096)	(.0093)	(.0095)	(**.0080**)	(.0099)	(.0111)
110	2	18.789	18.809	18.804	18.804	18.812	18.798	18.844
		(.0257)	(.0118)	(.0117)	(**.0094**)	(.0105)	(.0109)	(.0131)
90	3	9.214	9.207	9.220	9.214	9.207	9.221	9.213
		(.0198)	(.0089)	(.0115)	(.0087)	(**.0079**)	(.0103)	(.0094)
100	3	15.188	15.204	15.176	15.185	15.195	15.221	15.198
		(.0278)	(.0119)	(.0117)	(.0114)	(**.0103**)	(.0115)	(.0116)
110	3	22.635	22.613	22.599	22.615	22.605	22.622	22.644
		(.0348)	(**.0134**)	(.0155)	(.0144)	(.0153)	(.0148)	(.0139)
90	5	12.276	12.220	12.250	12.232	12.218	12.231	12.214
		(.0264)	(.0113)	(.0133)	(.0111)	(**.0101**)	(.0109)	(.0107)
100	5	19.373	19.364	19.389	19.366	19.386	19.395	19.399
		(.0315)	(**.0120**)	(.0152)	(.0132)	(.0153)	(.0147)	(.0164)
110	5	27.886	27.827	27.846	27.819	27.838	27.840	27.832
		(.0380)	(**.0155**)	(.0212)	(**.0155**)	(.0182)	(.0166)	(.0161)
				Upper bound				
90	2	7.393	7.326	7.358	7.333	7.347	7.334	7.340
		(.0167)	(**.0059**)	(.0077)	(.0066)	(.0073)	(.0068)	(.0075)
100	2	12.602	12.541	12.559	12.539	12.546	12.554	12.582
		(.0229)	(.0096)	(.0090)	(.0097)	(**.0081**)	(.0098)	(.0110)
110	2	19.326	19.328	19.328	19.328	19.342	19.317	19.372
		(.0253)	(.0115)	(.0112)	(**.0091**)	(.0102)	(.0106)	(.0135)
90	3	9.426	9.415	9.429	9.422	9.420	9.429	9.427
		(.0196)	(.0088)	(.0114)	(.0087)	(**.0084**)	(.0104)	(.0092)
100	3	15.553	15.561	15.534	15.542	15.560	15.582	15.553
		(.0270)	(.0116)	(.0115)	(.0117)	(**.0109**)	(.0124)	(.0118)
110	3	23.194	23.157	23.143	23.159	23.146	23.166	23.180
		(.0328)	(**.0130**)	(.0155)	(.0145)	(.0153)	(.0145)	(.0144)
90	5	12.509	12.458	12.493	12.472	12.457	12.468	12.451
		(.0257)	(.0112)	(.0131)	(.0110)	(**.0100**)	(.0111)	(.0110)
100	5	19.754	19.743	19.771	19.743	19.765	19.778	19.780
		(.0309)	(**.0120**)	(.0155)	(.0133)	(.0156)	(.0149)	(.0169)
110	5	28.437	28.370	28.387	28.360	28.380	28.387	28.376
		(.0364)	(**.0151**)	(.0206)	(.0157)	(.0175)	(.0174)	(.0158)

Note: $N = 4096$ paths are used to estimate the lower bound by LSM. $\hat{N}_0 = 4096$ paths are used in the outer simulations and $\hat{N}_1 = \hat{N}_2 = 64$ subpaths are used in the inner simulations to estimate the upper bound by the duality approach. The whole procedure is replicated 100 times to estimate the standard errors. For two-stage procedures, we use PCA decomposition for Σ. In QMC, scrambled Sobol' points are used in each replication to generate paths for computing the lower bound and in the outer simulations in the computation of upper bounds. The parameters are $K = 100, \delta = 10\%, \sigma = 20\%, T = 3, r = 5\%, \rho = 0.5$ (the time interval $[0, T]$ is discretized into nine steps). The standard errors of these estimates are given in the parentheses.

Table A6. The lower and upper bounds of the prices of American–Bermudan–Asian call options under the VG model.

			Sobol'							
		MC	BGS				DGS			
T	S_0	Seq	Seq	Bridge	PCAS	PCAB	Seq	Bridge	PCAS	PCAB
					Lower bound					
1	90	2.108	2.105	2.103	2.102	2.101	2.102	2.102	2.102	2.101
		(.0058)	(.0020)	(**.0008**)	(.0012)	(.0012)	(.0016)	(.0010)	(.0022)	(.0009)
1	100	7.527	7.530	7.532	7.536	7.529	7.532	7.533	7.534	7.533
		(.0125)	(.0020)	(.0014)	(.0026)	(.0025)	(.0018)	(**.0013**)	(.0029)	(.0015)
1	110	15.844	15.858	15.853	15.850	15.852	15.853	15.859	15.853	15.855
		(.0168)	(.0026)	(**.0017**)	(.0043)	(.0040)	(.0019)	(.0019)	(.0039)	(.0022)
2	90	4.999	4.995	4.998	5.000	4.998	5.005	5.002	4.999	5.002
		(.0150)	(.0044)	(.0016)	(.0029)	(.0026)	(.0043)	(.0017)	(.0050)	(**.0012**)
2	100	11.303	11.292	11.288	11.285	11.285	11.290	11.289	11.289	11.289
		(.0190)	(.0048)	(.0025)	(.0045)	(.0041)	(.0042)	(.0019)	(.0050)	(**.0018**)
2	110	19.767	19.762	19.761	19.761	19.755	19.757	19.753	19.753	19.758
		(.0265)	(.0061)	(.0037)	(.0060)	(.0054)	(.0052)	(.0033)	(.0063)	(**.0032**)
					Upper bound					
1	90	2.165	2.157	2.158	2.154	2.154	2.162	2.154	2.157	2.150
		(.0065)	(.0030)	(.0026)	(.0028)	(.0026)	(.0036)	(.0026)	(.0036)	(**.0023**)
1	100	7.641	7.641	7.644	7.645	7.647	7.647	7.644	7.650	7.646
		(.0131)	(.0037)	(**.0032**)	(.0040)	(.0039)	(.0038)	(.0034)	(.0044)	(.0036)
1	110	16.004	16.019	16.012	15.999	16.012	16.015	16.025	16.018	16.024
		(.0172)	(.0045)	(.0045)	(.0054)	(.0053)	(.0047)	(**.0042**)	(.0059)	(.0051)
2	90	5.078	5.064	5.070	5.071	5.079	5.082	5.077	5.076	5.073
		(.0163)	(.0058)	(.0041)	(.0046)	(.0052)	(.0062)	(.0047)	(.0064)	(**.0040**)
2	100	11.361	11.355	11.348	11.347	11.348	11.361	11.345	11.350	11.343
		(.0196)	(.0060)	(.0044)	(.0053)	(.0054)	(.0060)	(**.0033**)	(.0060)	(**.0033**)
2	110	19.837	19.832	19.831	19.836	19.829	19.831	19.824	19.828	19.826
		(.0270)	(.0063)	(.0051)	(.0071)	(.0065)	(.0061)	(.0050)	(.0077)	(**.0042**)

Note: $N = 4096$ paths are used to estimate the lower bound by LSM. For the duality approach, $\hat{N}_0 = 256$ paths are used in the outer simulations and $\hat{N}_1 = \hat{N}_2 = 64$ subpaths are used in the inner simulations. The whole procedure is replicated 100 times to estimate the standard errors. In QMC, scrambled Sobol' points are used in each replication to generate paths for computing the lower bound and in the outer simulations in the computation of upper bounds. The parameters are $K = 100, \theta = -0.2859, \sigma = 0.1927, \nu = 0.2505, r = 0.0548, q = 0$ (the option is exercisable eight times per year). The standard errors of these estimates are given in the parentheses.