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Volatility has to be rough

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Under power-law blow-up of the short ATM skew, volatility must be rough in a viable market for the underlying asset

1. Introduction

It has been almost two decades since a power law of volatility skew in option markets was reported (Carr and Wu 2003, Lee 2004, Fouque *et al.* 2003). Denoting by $\sigma_{\mathrm{BS}}(k,\theta)$ the Black–Scholes implied volatility with log moneyness k and time to maturity $\theta > 0$, the power law can be formulated as

$$\frac{\sigma_{\mathrm{BS}}(k,\theta) - \sigma_{\mathrm{BS}}(k',\theta)}{k - k'} \propto \theta^{H-1/2}$$

for $k \approx 0$ and $k' \approx 0$, with $H \approx 0$, when $\theta \approx 0$. It describes an asymptotic blow-up ($\theta \to 0$) and so is not a fully verifiable condition from market data. Nevertheless, the observed term structure of the at-the-money (ATM, that is, $k \approx 0$) skew in equity markets is known to be well-approximated by a power-law curve that has only 2 degrees of freedom. A curve without blow-up can approximate the observed term structure but with many more parameters. It is now well known that classical local stochastic volatility models, where volatility is modelled as a diffusion, are not consistent with the power law, while some rough volatility models are so (Alòs *et al.* 2007, Fukasawa 2011, Bayer *et al.* 2016, Fukasawa 2017, Garnier and Solna 2017, Forde and Zhang 2017, Guennoun *et al.* 2018, Jacquier *et al.* 2018, El Euch *et al.* 2019, Alòs and Shiraya 2019, Bayer *et al.* 2019) as well as stable-type discontinuous price models (Carr and Wu 2003, Friz *et al.* 2018, Figueroa-López and Ólafsson 2016, Forde *et al.* 2020). To be precise, at a fixed time point, a local volatility model can reproduce the power law with a singular local volatility function (Pigato 2019). Since the singularity has to be around the ATM, the power law that is stable in time requires, if any, a pathological local volatility function which is singular everywhere (or recalibrations, as is common in the financial practice). In any case, the volatility cannot be seen as a diffusion. The present article extends the preceding works and shows that there is an arbitrage opportunity if volatility is not rough given an option market with volatility skew obeying the power law, under the assumption that the asset price is a positive continuous Itô semimartingale.

A related work is Bergomi (2009), where an arbitrage strategy was constructed to exploit the difference between the observed and theoretical Skew Stickiness Ratios (SSR) under a classical stochastic volatility dynamics. The SSR is related to the constant of proportionality (not the rate of blow-up) in the power law (see remark 2.10). Our arbitrage strategy is simpler, valid beyond the classical local stochastic volatility framework (with a proof of the validity), and exploits the difference between the rates of blow-up in the power law. On the other hand, our principal aim is to highlight the inconsistency of non-rough modelling with the power-law skew. We assume an ideal option market where we can purchase any number of units of any vanilla at any countably many time points prior to the maturity of the vanilla without any transaction costs, and we do not pursue here how to implement the arbitrage strategy effectively in reality.

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In section 2, we give an asymptotic expansion of short-dated at-the-money implied volatility, which is a refinement of the results in Fukasawa (2017). Both the result and proof are much simpler than in Fukasawa (2017) thanks to choosing the square root of the averaged forward variance as the leading term of the expansion, as in El Euch *et al.* (2019) and Bergomi and Guyon (2012). Also, the adoption of the forward variance framework (Bergomi 2005, Bergomi and Guyon 2012, Bayer *et al.* 2016, El Euch *et al.* 2019) justifies not considering a time consistency issue treated in Fukasawa (2017). In section 3, we first introduce the notion of *H-power law of negative volatility skew*, which is a mathematical formulation of the empirically observed power law blow-up for a fixed maturity. A consequence from section 2 is that this property is satisfied by rough volatility models with the leverage effect, and so it is compatible with arbitrage-free dynamics. Then we show there is an arbitrage opportunity under this property if volatility is not rough. Some concluding remarks are given in section 4. All the proofs are given in appendices. Throughout the paper, interest rates are assumed to be zero for brevity.

2. An asymptotic expansion

Denote by S the underlying asset price process of vanilla options. We regard it as an adapted process on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}\}_{t\geq 0})$ with $\mathcal{F}_0$ consisting of the null sets. We assume that there is a probability measure Q which is equivalent to P and such that the price of each vanilla option at time 0 is the expectation of its payoff under Q. Denote by $\sigma_{\mathrm{BS}}(k,\theta)$ the Black–Scholes implied volatility as in Introduction at time $t = 0$.

Theorem 2.1 *Suppose that S is a positive continuous martingale under Q with the quadratic variation $\langle \log S \rangle$ being absolutely continuous. Let*

$$V_t = \frac{\mathrm{d}}{\mathrm{d}t}\langle \log S \rangle_t, \quad v(t) = E[V_t],$$

where E is the expectation under Q, and assume that $v(t)$ is positive and continuous at $t = 0$. If there exists $H \in (0, 1/2]$ such that

$$\frac{1}{\theta^H}\left(\frac{V_\theta}{v(\theta)} - 1\right)$$

is uniformly integrable and

$$\left(\frac{1}{\sqrt{\theta}}\left(\frac{S_\theta}{S_0} - 1\right), \frac{1}{\theta^H}\left(\frac{V_\theta}{v(\theta)} - 1\right)\right)$$

converges in law to a two dimensional random variable (ξ, η) as $\theta \to 0$ under Q, then

$$\sigma_{\mathrm{BS}}\left(z\sqrt{\theta}, \theta\right) = \sqrt{\bar{v}(\theta)}(1 + \alpha(z)\theta^H) + o(\theta^H), \text{ as } \theta \to 0$$

uniformly in z on compact sets of $\mathbb{R}$, where

$$\bar{v}(\theta) = \frac{1}{\theta}\int_0^\theta v(t)\,\mathrm{d}t,$$

$$\alpha(z) = \frac{1}{2}\int_0^1\int_{-\infty}^{\infty} u^H A(u,w,z)\phi(w)\mathrm{d}w\mathrm{d}u,$$

$$A(u,w,z) = E\left[\eta \,\middle|\, \xi = z\sqrt{u} + w\sqrt{v(0)(1-u)}\right],$$

and ϕ is the standard normal density.

Remark 2.1 $\sqrt{E[\eta|\xi = x]}$ is a renormalized limit of the Dupire local volatility.

Remark 2.2 The function v is called the forward variance curve (at time 0).

Remark 2.3 The integrated forward variance is the fair strike of the variance swap. Under this continuous framework, it is a model-free quantity determined by vanilla option prices. The leading term $\sqrt{\bar{v}(\theta)}$ of the expansion with $\theta = 30$ days corresponds to the VIX (VIX White Paper 2019, Gatheral 2006).

Corollary 2.1 *Under the condition of theorem 2.1, if $(\xi, \eta) \sim \mathcal{N}(0, \Sigma)$ with covariance matrix $\Sigma = [\Sigma_{ij}]$, then $\Sigma_{11} = v(0)$, $E[\eta|\xi = x] = x\Sigma_{12}/\Sigma_{11}$, and*

$$\alpha(z) = \frac{\Sigma_{12}}{2v(0)}\frac{z}{H + 3/2}.$$

In particular, a power law of volatility skew follows: for $\zeta \neq z$,

$$\frac{\sigma_{\mathrm{BS}}\left(z\sqrt{\theta}, \theta\right) - \sigma_{\mathrm{BS}}\left(\zeta\sqrt{\theta}, \theta\right)}{z\sqrt{\theta} - \zeta\sqrt{\theta}} \sim \frac{\Sigma_{12}}{\sqrt{v(0)}(2H+3)}\theta^{H-1/2}.$$

Remark 2.4 If V is an integrable Itô semimartingale, then $v(\theta) = V_0 + O(\theta)$, and Theorem VIII.3.8 of Jacod and Shiryaev (2002) verifies the assumptions of corollary 2.1 with $H = 1/2$ and

$$(\xi, \eta) \sim \mathcal{N}(0, \Sigma),$$

$$\Sigma_{11} = \frac{\mathrm{d}}{\mathrm{d}t}\langle \log S\rangle_t\Big|_{t=0} = v(0), \quad \Sigma_{12} = \frac{\mathrm{d}}{\mathrm{d}t}\langle \log S, \log V\rangle_t\Big|_{t=0}.$$

In particular, for a local volatility model $V_t = \sigma(S_t, t)^2$ with a smooth function σ, we have $v(0) = \sigma(S_0, 0)^2$ and $\Sigma_{12} = 2S_0\sigma(S_0, 0)\partial_S\sigma(S_0, 0)$. Thus we conclude the so-called 1/2 rule:

$$\frac{\sigma_{\mathrm{BS}}\left(z\sqrt{\theta}, \theta\right) - \sigma_{\mathrm{BS}}\left(\zeta\sqrt{\theta}, \theta\right)}{z\sqrt{\theta} - \zeta\sqrt{\theta}} \sim \frac{1}{2}S_0\partial_S\sigma(S_0, 0).$$

Remark 2.5 The martingale property of the rough Bergomi model

$$S_t = S_0\exp\left(\int_0^t \sqrt{V_s}\left[\rho\mathrm{d}W_s + \sqrt{1-\rho^2}\mathrm{d}W_s^\perp\right] - \frac{1}{2}\int_0^t V_s\mathrm{d}s\right),$$

$$V_t = v(t)\exp\left(\int_0^t k(t,s)\mathrm{d}W_s - \frac{1}{2}\int_0^t k(t,s)^2\mathrm{d}s\right)$$

with $k(t,s) = \beta|t-s|^{H-1/2}$, $\beta > 0$ was shown by Gassiat (2019) for $\rho \in [-1,0]$. Using that

$$\left(H+\frac{1}{2}\right)\theta^{-H-1/2}\int_0^\theta k(\theta,s)\mathrm{d}s \to \beta,$$
$$2H\theta^{-2H}\int_0^\theta k(\theta,s)^2\mathrm{d}s \to \beta^2 > 0$$

as $\theta \to 0$, the assumptions of corollary 2.1 are verified with $\Sigma_{12} = \sqrt{v(0)}\rho\beta/(H+1/2)$, and therefore we have a power law of volatility skew

$$\frac{\sigma_{\mathrm{BS}}\left(z\sqrt{\theta},\theta\right) - \sigma_{\mathrm{BS}}\left(\zeta\sqrt{\theta},\theta\right)}{z\sqrt{\theta}-\zeta\sqrt{\theta}} \sim \frac{\rho\beta}{(H+1/2)(2H+3)}\theta^{H-1/2}.$$

This small-time asymptotic power-law skew formula has been already shown in Fukasawa (2017) for essentially the same model and is consistent with the older results (Alòs *et al.* 2007) (a Malliavin approach) and (Fukasawa 2011, Bergomi and Guyon 2012, Bayer *et al.* 2016) (small volatility-of-volatility expansions). When $|\rho| < 1$, a second-order small-time expansion is available in El Euch *et al.* (2019) as well. Our approach here is more abstract than those previous ones and so, in particular we can extend this result to $k(t,s) = |t-s|^{H-1/2}b_1(t-s) + b_2(t,s)$, where b_1 and b_2 are locally bounded functions with $\lim_{\theta\to 0} b_1(\theta) = \beta$. Further, an application to the affine forward variance models (Gatheral and Keller-Ressel 2019)

$$V_\theta = v(\theta) + \int_0^\theta k(\theta,s)\sqrt{V_s}\mathrm{d}W_s$$
$$\approx v(\theta) + \sqrt{V_0}\int_0^\theta k(\theta,s)\mathrm{d}W_s$$

is straightforward. In particular, a power-law skew under the rough Heston model (El Euch and Rosenbaum 2018) follows. Note, however, also that a second-order expansion under the rough Heston model has been already given in Forde *et al.* (2020) via an expansion of the characteristic function.

Remark 2.6 The model-free implied leverage is defined by Fukasawa (2014) as the normalized difference of the gamma and variance swap fair strikes :

$$\lambda(\theta) = \frac{1}{E[\langle \log S\rangle_\theta]}E\left[\int_0^\theta\left(\frac{S_t}{S_0}-1\right)\mathrm{d}\langle\log S\rangle_t\right]$$
$$= \frac{\sqrt{\theta}}{\bar{v}(\theta)}\int_0^1 E[X_u^\theta V_{\theta u}]\mathrm{d}u,$$

where X^θ is defined as (A1) in appendix. Under a slightly stronger assumption than in corollary 2.1, namely,

$$E\left[\frac{1}{\theta^{H+1/2}}\left(\frac{S_\theta}{S_0}-1\right)\left(\frac{V_\theta}{v(\theta)}-1\right)\right] \to \Sigma_{12},$$

we have

$$\theta^{-H}E[X_u^\theta V_{\theta u}]$$
$$= E\left[X_u^\theta\theta^{-H}\left(V_{\theta u}-v(\theta u)\right)\right] \to u^{H+1/2}v(0)\Sigma_{12}$$

uniformly in $u \in [0,1]$ and so, a model-free representation of the slope

$$\frac{\Sigma_{12}}{\sqrt{v(0)}(2H+3)}\theta^{H-1/2} \sim \frac{\lambda(\theta)}{2\theta\sqrt{\bar{v}(\theta)}}.$$

Under a perturbation framework, including both the fast mean-reverting and the small vol-of-vol asymptotics, this model-free representation was shown in Fukasawa (2014). This verifies also some computations in Neuberger (2009), where the difference between the gamma and variance swap fair strikes is called the *slope*. It is also related to a model-free representation in Bergomi and Guyon (2012) using a covariance function instead of λ under the small vol-of-vol asymptotics.

Remark 2.7 Here we comment more on abstract (model-free) relations between the skew and the covariances. Corollary 2.1 relates the volatility skew to the asymptotic covariance between the spot price and its spot variance process. Similarly, the skew was related to the covariance between the log spot price and its integrated variance in Fukasawa (2011) under a perturbation framework including both the fast mean-reverting and the small vol-of-vol asymptotics. Under the latter asymptotics, with the forward variance framework, a second-order expansion was given in Bergomi and Guyon (2012) in terms of the variance and covariance functions. A non-asymptotic relation of the skew to the covariation between the price and the forward variance processes can be seen in Alòs (2012).

Remark 2.8 Our small-time asymptotics does not fall in the general perturbation framework of Fukasawa (2011). In fact, the limit of the perturbation models is the Black–Scholes model, while the limit in the small-time asymptotics is the Bachelier model as will be clearly seen in the proof of theorem 2.1 in appendix.

Remark 2.9 Corollary 2.1 has been recently further extended in Bayer *et al.* (2020) to the case where the convergence rate is a regularly varying function of θ.

Remark 2.10 The short-dated limit of the Skew Stickiness Ratio (SSR) introduced in Bergomi (2009) is given by

$$R = \lim_{\theta\to 0}\frac{E\left[\left(\sqrt{V_\theta}-\sqrt{V_0}\right)\log\frac{S_\theta}{S_0}\right]}{\frac{\partial\sigma_{\mathrm{BS}}}{\partial k}(0,\theta)E\left[\left(\log\frac{S_\theta}{S_0}\right)^2\right]}.$$

In light of corollary 2.1, a formal computation gives

$$R = \lim_{\theta\to 0}\frac{\frac{\sqrt{v(0)}}{2}E\left[\left(\frac{V_\theta}{v(\theta)}-1\right)\left(\frac{S_\theta}{S_0}-1\right)\right]}{\frac{\partial\sigma_{\mathrm{BS}}}{\partial k}(0,\theta)v(0)\theta}$$
$$= \lim_{\theta\to 0}\frac{\Sigma_{12}\theta^{H-1/2}}{2\frac{\partial\sigma_{\mathrm{BS}}}{\partial k}(0,\theta)\sqrt{v(0)}} = H + \frac{3}{2}$$

that supports some computations in Bergomi (2009). A time series plot of the SSR is given in Bergomi (2009) that indicates $H \approx 0$, consistently with the power law. The arbitrage strategy of Bergomi (2009) was designed to make its P&L equal to $2-R$, where the value 2 is the SSR when $H=1/2$.

3. An arbitrage opportunity

In the previous section, we considered the implied volatility at time $t=0$ and varied the maturity θ. Here, we fix a maturity $T>0$ instead and consider the implied volatility at time $\tau<T$. The short-dated asymptotics corresponds to $\tau\uparrow T$. We assume a hypothetical option market where call and put options with the underlying asset S and maturity T are traded at any time $\tau<T$ and for any strike price $K>0$. Denote by $\sigma_{\mathrm{BS},\tau}(K)$ the market implied volatility for the strike price K at time τ. For $H\in(0,1/2)$, we say *the H-power law of negative volatility skew holds* if there exist adapted processes σ_τ and α_τ such that

$$\liminf_{\tau\uparrow T}\sigma_\tau>0,\quad \limsup_{\tau\uparrow T}\sigma_\tau<\infty,$$
$$\liminf_{\tau\uparrow T}\alpha_\tau>-\infty,\ \limsup_{\tau\uparrow T}\alpha_\tau<0$$

and for any positive adapted process K_τ with $|K_\tau/S_\tau-1|=O(\sqrt{T-\tau})$,

$$\sigma_{\mathrm{BS},\tau}(K_\tau)=\sigma_\tau+(T-\tau)^{H-1/2}\alpha_\tau\log\frac{K_\tau}{S_\tau}+o((T-\tau)^H)\text{ as }\tau\uparrow T.$$

Note that negative volatility skew is typically observed in equity option markets. What is essential in the following is that α_τ does not change its sign.

The H-power law with negative volatility skew is compatible with arbitrage-free dynamics. In fact, a consequence from the previous section, in particular, remark 2.5, is that rough volatility models with the leverage effect satisfy this property. For example, if the log spot variance process is a fractional Ornstein–Uhlenbeck process :

$$\log V_t=\int_{-\infty}^t g(t-s)\mathrm{d}W_s,$$
$$g(x)=\beta x^{H-1/2}-\beta\lambda e^{-\lambda x}\int_0^x s^{H-1/2}e^{\lambda y}\mathrm{d}y,$$

where W is a standard Brownian motion with $\mathrm{d}\langle\log S,W\rangle_t=\rho\sqrt{V_t}\mathrm{d}t$, then a translation of the result in the previous section yields the above H-power law with σ_τ being the square root of the average forward variance and

$$\alpha_\tau=\frac{\rho\beta}{(H+1/2)(2H+3)},$$

which is negative when $\beta>0$ and $\rho<0$.

Now we give a lemma that tells about the magnitude of the Black–Scholes delta hedging error for ATM options when volatility is Hölder continuous.

Lemma 3.1 *Suppose that S is a positive continuous semimartingale with $\langle\log S\rangle$ being absolutely continuous. Let*

$$V_t=\frac{\mathrm{d}}{\mathrm{d}t}\langle\log S\rangle_t$$

and assume that V is positive and H_0-Hölder continuous with $H_0\in(0,1/2]$ almost surely on $[0,T]$, that is,

$$\sup_{0\le s<t\le T}\frac{|V_t-V_s|}{|t-s|^{H_0}}<\infty,\text{ a.s.}$$

Then, for any positive adapted process K_τ, as $\tau\uparrow T$,

$$\left(S_T-\frac{S_\tau^2}{K_\tau}\right)_+=c_{\mathrm{BS}}(S_\tau,T-\tau)+\int_\tau^T\frac{\partial c_{\mathrm{BS}}}{\partial S}(S_t,T-t)\,\mathrm{d}S_t+O((T-\tau)^{H_0+1/2}),$$
$$(K_\tau-S_T)_+=p_{\mathrm{BS}}(S_\tau,T-\tau)+\int_\tau^T\frac{\partial p_{\mathrm{BS}}}{\partial S}(S_t,T-t)\,\mathrm{d}S_t+O((T-\tau)^{H_0+1/2}),$$

and

$$(K_\tau-S_T)_+-\frac{K_\tau}{S_\tau}\left(S_T-\frac{S_\tau^2}{K_\tau}\right)_+=\int_\tau^T\left(\frac{\partial p_{\mathrm{BS}}}{\partial S}(S_t,T-t)-\frac{K_\tau}{S_\tau}\frac{\partial c_{\mathrm{BS}}}{\partial S}(S_t,T-t)\right)\mathrm{d}S_t+O((T-\tau)^{H_0+1/2})$$

almost surely, where $c_{\mathrm{BS}}(S,\theta)$ (resp. $p_{\mathrm{BS}}(S,\theta)$) is the Black–Scholes price of the call (resp. put) option with the underlying asset price S, time to maturity θ, strike price S_τ^2/K_τ (resp. K_τ), and volatility parameter $\sqrt{V_\tau}$.

Now we construct building blocks of our arbitrage strategy. Let $\tau_n=T-1/n$ and choose K_{τ_n} so that $|K_{\tau_n}/S_{\tau_n}-1|=O(n^{-1/2})$ and

$$\limsup_{n\to\infty}\sqrt{n}\log\frac{K_{\tau_n}}{S_{\tau_n}}<0.$$

Denote by Π^n the P&L of one unit short of the put option with strike K_{τ_n} and K_{τ_n}/S_{τ_n} unit long of the call option with strike $S_{\tau_n}^2/K_{\tau_n}$ with the Black–Scholes delta hedging :

$$\Pi^n=P_{\tau_n}(K_{\tau_n})-\frac{K_{\tau_n}}{S_{\tau_n}}C_{\tau_n}\left(\frac{S_{\tau_n}^2}{K_{\tau_n}}\right)-(K_{\tau_n}-S_T)_+ +\frac{K_{\tau_n}}{S_{\tau_n}}\left(S_T-\frac{S_{\tau_n}^2}{K_{\tau_n}}\right)_+ +\int_{\tau_n}^T\left(\frac{\partial p_{\mathrm{BS}}}{\partial S}(S_t,T-t)-\frac{K_{\tau_n}}{S_{\tau_n}}\frac{\partial c_{\mathrm{BS}}}{\partial S}(S_t,T-t)\right)\mathrm{d}S_t,$$

where $C_\tau(K)$ and $P_\tau(K)$ are respectively the market price of call and put options with strike K at time τ, and c_{BS} and p_{BS} are as in lemma 3.1 with $\tau=\tau_n$.

THEOREM 3.1 *Suppose the H-power law of negative volatility skew holds. Under the condition of lemma* 3.1 *with* $H_0 > H$,

$$\sum_{n=1}^{\infty} n^{H-1/2}\Pi^n = \infty, \text{a.s..}$$

The idea behind theorem 3.1 is simple. If the volatility is H_0-Hölder continuous, the Black–Scholes delta hedging error of the specific option portfolio in the nth building block is only of $O(n^{-H_0-1/2})$ almost surely by lemma 3.1. The Black–Scholes price of the portfolio is zero due to the put-call symmetry (Carr and Lee 2009), and the assumed power law of volatility skew implies the market price of the portfolio of $O(n^{-H-1/2})$. That

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

while

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+H_0-H}} < \infty$$

enables us to make an almost sure infinite profit.

The implication of theorem 3.1 is that in a viable market, the volatility cannot have a better Hölder regularity than H, that is, it has to be rough.

4. Concluding remarks

REMARK 4.1 This paper concludes rough volatility as a consequence of the power law in option markets. The origin of the power law can be explained by a financial practice convention. In FX option markets, the convention is to quote prices in terms of the implied volatility and tends to quote the same implied volatility for the same value of the Black–Scholes delta. Since the delta is approximately a function of $k/\sqrt{\theta}$, this convention makes $\sigma_{\mathrm{BS}}(z\sqrt{\theta}, \theta)$ approximately independent of θ, which is nothing but the H-power law with $H = 0$. The origin of this convention is not clear. Naively one may attribute it to the traditional financial engineering that perceives the risk of a position only via its delta. There is, however, much room for discussion and further research.

REMARK 4.2 The volatility is indeed statistically estimated to be rough; see Fukasawa *et al.* (2019).

REMARK 4.3 Refining an estimate by Lee (2004), a model-free bound of volatility skew

$$\left|\frac{\partial \sigma_{\mathrm{BS}}}{\partial k}(0,\theta)\right| \le \sqrt{\frac{\pi}{2\theta}}$$

is given in Fukasawa (2010) and shown to be sharp in Pigato (2019). This extreme skew corresponds to the H-power law with $H = 0$. The H-power law with $H < 0$ results in the violation of the above model-free bound for small θ, hence provides a butterfly arbitrage opportunity when θ is small. Note that $\sqrt{\theta}\sigma_{\mathrm{BS}}(k,\theta)$ is the implied total volatility. It can be intuitively understood that an excessive slope of the total volatility contradicts the convexity of the vanilla prices.

REMARK 4.4 Volatility with regularity $H = 0$ can be understood as a Gaussian multiplicative chaos. It is, however, an open question whether there exists a continuous-time model with both the regularity of $H = 0$ and the nondegenerate conditional skewness that is necessary to recover the power law of volatility skew stably in time.

REMARK 4.5 Derivations of rough volatility as a scaling limit of Hawkes-type market micro structure models are given in Jaisson and Rosenbaum (2016),El Euch *et al.* (2018),Jusselin and Rosenbaum (2019). In Jaisson and Rosenbaum (2016) and El Euch *et al.* (2018), a heavy-tailed nearly unstable self-exciting kernel of order flow is the source of the rough volatility. In Jusselin and Rosenbaum (2019), such a heavy-tailed kernel is derived via Tauberian theorems by assuming the existence of a scaling limit of market impact functions.

REMARK 4.6 An inspection of the proof of lemma 3.1 reveals that the Hölder regularity of volatility only around the maturity T does matter. Therefore a more precise statement of our finding is that the volatility has to be rough near the maturities of options. The volatility has to be rough everywhere under a hypothetical framework where vanilla options are traded for any strike prices around at-the-money and any maturities. Note also that our study does not apply to any stock price or index whose options are not traded.

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Appendix A. Proof of theorem 2.1

First we recall that for a sequence of functions f_n on $\mathbb{R}$ and a continuous function f on $\mathbb{R}$, the uniform convergence on compact sets $f_n \to f$ is equivalent to that $\lim_{n\to\infty} f_n(z_n) = f(\lim_{n\to\infty} z_n)$ holds for any converging sequence z_n on $\mathbb{R}$. Therefore we consider an arbitrary log moneyness $k(\theta)$ with $\theta^{-1/2}k(\theta)$ converging to some value $z \in \mathbb{R}$ as $\theta \to 0$. The continuity of $z \mapsto \alpha(z)$ is shown below.

Step 1 [An expansion of a rescaled put option price]. Denote

$$X_u^\theta = \frac{1}{\sqrt{\theta}}\left(\frac{S_{\theta u}}{S_0} - 1\right) \tag{A1}$$

for $u \in [0,1]$. Note that X^θ is a martingale and with

$$\mathrm{d}\langle X^\theta\rangle_u = \left(\frac{S_{\theta u}}{S_0}\right)^2 V_{\theta u}\mathrm{d}u = (1+\sqrt{\theta}X_u^\theta)^2 V_{\theta u}\mathrm{d}u.$$

A rescaled put option price can be expressed as

$$\frac{E[(S_0 e^{k(\theta)} - S_\theta)_+]}{S_0\sqrt{\theta}} = E[(\Delta - X_1^\theta)_+], \quad \Delta = \frac{e^{k(\theta)}-1}{\sqrt{\theta}}. \tag{A2}$$

Consider the Bachelier pricing equation with time-dependent variance

$$\frac{\partial p}{\partial u}(x,u) + \frac{1}{2}v(\theta u)\frac{\partial^2 p}{\partial x^2}(x,u) = 0, \quad p(x,1) = (\Delta - x)_+.$$

The solution and its derivatives are given explicitly :

$$\begin{aligned} p(x,u) &= (\Delta - x)\Phi\left(\frac{\Delta - x}{\sqrt{w(1)-w(u)}}\right) \\ &\quad + \sqrt{w(1)-w(u)}\phi\left(\frac{\Delta - x}{\sqrt{w(1)-w(u)}}\right), \\ \frac{\partial p}{\partial x}(x,u) &= -\Phi\left(\frac{\Delta - x}{\sqrt{w(1)-w(u)}}\right), \\ \frac{\partial^2 p}{\partial x^2}(x,u) &= \frac{1}{\sqrt{w(1)-w(u)}}\phi\left(\frac{\Delta - x}{\sqrt{w(1)-w(u)}}\right), \end{aligned}$$

where Φ and ϕ are respectively the standard normal distribution function and the density, and

$$w(u) = \frac{1}{\theta}\int_0^{\theta u} v(t)\mathrm{d}t.$$

Since the process X^θ takes values on the interval $[-\theta^{-1/2}, \infty)$ and the function $p(x,u)$ is bounded on $[-\theta^{-1/2},\infty)\times[0,1]$ for each $\theta > 0$, Itô's formula, with the aid of a localization argument, gives that

$$\begin{aligned} &E[(\Delta - X_1^\theta)_+] \\ &\quad = E[p(X_1^\theta, 1)] \\ &\quad = p(0,0) + \frac{1}{2}E\left[\int_0^1 \frac{\partial^2 p}{\partial x^2}(X_u^\theta, u)((1+\sqrt{\theta}X_u^\theta)^2 V_{\theta u} - v(\theta u))\mathrm{d}u\right]. \end{aligned} \tag{A3}$$

By the assumption,

$$\left(X_u^\theta, \theta^{-H}(V_{\theta u} - v(\theta u))\right) \to (X_u, Y_u) := (\sqrt{u}\xi, u^H v(0)\eta).$$

We have $\xi \sim \mathcal{N}(0, v(0))$ by the martingale central limit theorem. Since

$$\frac{\partial^2 p}{\partial x^2}(x,u) \to \frac{1}{\sqrt{v(0)(1-u)}}\phi\left(\frac{z-x}{\sqrt{v(0)(1-u)}}\right)$$

as $\theta \to 0$, we have

$$\frac{\partial^2 p}{\partial x^2}(X_u^\theta, u) \to \frac{1}{\sqrt{v(0)(1-u)}}\phi\left(\frac{z-X_u}{\sqrt{v(0)(1-u)}}\right)$$

in law for each $u \in [0,1)$. For any polynomial q, there exists a constant $C > 0$ such that

$$\left|q(x)\frac{\partial^2 p}{\partial x^2}(x,u)\right| \le \frac{C}{\sqrt{1-u}}. \tag{A4}$$

Therefore, the dominated convergence theorem gives that

$$\int_0^1 E\left[\frac{\partial^2 p}{\partial x^2}(X_u^\theta, u)X_u^\theta V_{\theta u}\right]\mathrm{d}u$$

$$\to v(0)\int_0^1 E\left[\frac{1}{\sqrt{v(0)(1-u)}}\phi\left(\frac{z-X_u}{\sqrt{v(0)(1-u)}}\right)X_u\right]\mathrm{d}u$$
$$=\frac{z\sqrt{v(0)}}{2}\phi\left(\frac{z}{\sqrt{v(0)}}\right)$$

and that

$$\theta^{-H}\int_0^1 E\left[\frac{\partial^2 p}{\partial x^2}(X_u^\theta,u)(V_{\theta u}-v(\theta u))\right]\mathrm{d}u$$
$$\to \int_0^1 E\left[\frac{1}{\sqrt{v(0)(1-u)}}\phi\left(\frac{z-X_u}{\sqrt{v(0)(1-u)}}\right)Y_u\right]\mathrm{d}u$$
$$=2\alpha(z)\sqrt{v(0)}\phi\left(\frac{z}{\sqrt{v(0)}}\right).$$

Here we have used the assumed uniform integrability. It ensures in particular the integrability of η, and so another application of the dominated convergence theorem gives that

$$z\mapsto \int_0^1 E\left[\frac{1}{\sqrt{v(0)(1-u)}}\phi\left(\frac{z-X_u}{\sqrt{v(0)(1-u)}}\right)Y_u\right]\mathrm{d}u$$

is continuous. This implies the continuity of $z\mapsto\alpha(z)$. From (A2) and (A3), we have then that

$$\begin{aligned}&\frac{E[(S_0e^{k(\theta)}-S_\theta)_+]}{S_0\sqrt{\theta}}\\ &=p(0,0)+\alpha(z)\sqrt{v(0)}\phi\left(\frac{z}{\sqrt{v(0)}}\right)\theta^H\\ &\quad+\frac{z\sqrt{v(0)}}{2}\phi\left(\frac{z}{\sqrt{v(0)}}\right)\sqrt{\theta}+o(\theta^H)\\ &=\Delta\Phi\left(\frac{\Delta}{\sqrt{\bar v(\theta)}}\right)\\ &\quad+\sqrt{\bar v(\theta)}\phi\left(\frac{\Delta}{\sqrt{\bar v(\theta)}}\right)\left(1+\alpha(z)\theta^H+\frac{z}{2}\sqrt{\theta}\right)+o(\theta^H).\end{aligned}\qquad\text{(A5)}$$

Step 2 [A comparison with the Black–Scholes model]. The Black–Scholes model $\sqrt{V_\theta}\equiv\sigma$, the volatility parameter, satisfies the assumption with $H=1/2$ and $\eta=0$. Therefore, (A5) gives

$$\begin{aligned}&\frac{P_{\mathrm{BS}}(S_0e^{k(\theta)},\theta,\sigma)}{S_0\sqrt{\theta}}\\ &\quad=\Delta\Phi\left(\frac{\Delta}{\sigma}\right)+\sigma\phi\left(\frac{\Delta}{\sigma}\right)\left(1+\frac{z}{2}\sqrt{\theta}\right)+o(\theta^{1/2}),\end{aligned}\qquad\text{(A6)}$$

where $P_{\mathrm{BS}}(K,\theta,\sigma)$ is the Black–Scholes price of put option with strike K, time to maturity θ and volatility parameter σ. By the Taylor expansion,

$$\begin{aligned}&\frac{P_{\mathrm{BS}}(S_0e^{k(\theta)},\theta,\sigma+a\theta^H)}{S_0\sqrt{\theta}}\\ &\quad=\Delta\Phi\left(\frac{\Delta}{\sigma}\right)+\sigma\phi\left(\frac{\Delta}{\sigma}\right)\left(1+\frac{z}{2}\sqrt{\theta}+\frac{a}{\sigma}\theta^H\right)+o(\theta^H).\end{aligned}$$

We can equate this and (A5) by setting

$$\sigma=\sqrt{\bar v(\theta)},\quad a=\sigma\alpha(z),$$

which implies the result.

Appendix B. Proof of lemma 3.1

Since the Black–Scholes prices c_{BS} and p_{BS} satisfy the Black–Scholes equation

$$\frac{\partial c_{\mathrm{BS}}}{\partial\theta}=\frac{1}{2}V_\tau S^2\frac{\partial^2 c_{\mathrm{BS}}}{\partial S^2},\quad \frac{\partial p_{\mathrm{BS}}}{\partial\theta}=\frac{1}{2}V_\tau S^2\frac{\partial^2 p_{\mathrm{BS}}}{\partial S^2}$$

with

$$c_{\mathrm{BS}}(S,0)=\left(S-\frac{S_\tau^2}{K_\tau}\right)_+,\quad p_{\mathrm{BS}}(S,0)=(K_\tau-S)_+,$$

Itô's formula gives

$$\begin{aligned}\left(S_T-\frac{S_\tau^2}{K_\tau}\right)_+&=c_{\mathrm{BS}}(S_\tau,T-\tau)+\int_\tau^T\frac{\partial c_{\mathrm{BS}}}{\partial S}(S_t,T-t)\mathrm{d}S_t\\ &\quad+\frac{1}{2}\int_\tau^T(V_t-V_\tau)S_t^2\frac{\partial^2 c_{\mathrm{BS}}}{\partial S^2}(S_t,T-t)\mathrm{d}t,\\ (K_\tau-S_T)_+&=p_{\mathrm{BS}}(S_\tau,T-\tau)+\int_\tau^T\frac{\partial p_{\mathrm{BS}}}{\partial S}(S_t,T-t)\mathrm{d}S_t\\ &\quad+\frac{1}{2}\int_\tau^T(V_t-V_\tau)S_t^2\frac{\partial^2 p_{\mathrm{BS}}}{\partial S^2}(S_t,T-t)\mathrm{d}t.\end{aligned}$$

Since $|V_t-V_\tau|\le C|t-\tau|^{H_0}$ for some finite random variable C by the assumption and

$$\begin{aligned}&\left|\frac{\partial^2 c_{\mathrm{BS}}}{\partial S^2}(S_t,T-t)\right|\\ &\quad\vee\left|\frac{\partial^2 p_{\mathrm{BS}}}{\partial S^2}(S_t,T-t)\right|\le\frac{1}{\sqrt{2\pi V_\tau(T-t)}\inf_{t\in[\tau,T]}S_t},\end{aligned}$$

we obtain the first two equations. The last equation follows from the first two with aid of the put-call symmetry (Carr and Lee 2009):

$$p_{\mathrm{BS}}(S_\tau,T-\tau)=\frac{K_\tau}{S_\tau}c_{\mathrm{BS}}(S_\tau,T-\tau).$$

Appendix C. Proof of theorem 3.1

Let

$$Z_n=\sqrt{n}\log\frac{K_{\tau_n}}{S_{\tau_n}}.$$

Then, $\liminf_{n\to\infty}Z_n>-\infty$, $\limsup_{n\to\infty}Z_n<0$ and

$$\begin{aligned}\sigma_{\mathrm{BS},\tau_n}(K_{\tau_n})&=\sigma_{\tau_n}+n^{-H}\alpha_{\tau_n}Z_n+o(n^{-H}),\\ \sigma_{\mathrm{BS},\tau_n}(S_{\tau_n}^2/K_{\tau_n})&=\sigma_{\tau_n}-n^{-H}\alpha_{\tau_n}Z_n+o(n^{-H})\end{aligned}$$

by the assumed power law. The Taylor expansion of the Black–Scholes price with respect to the volatility parameter gives

$$P_{\tau_n}(K_{\tau_n})=p_{\mathrm{BS}}+\frac{\partial p_{\mathrm{BS}}}{\partial\sigma}n^{-H}\alpha_{\tau_n}Z_n+o(n^{-H})$$

and

$$C_{\tau_n}(S_{\tau_n}^2/K_{\tau_n})=c_{\mathrm{BS}}-\frac{\partial c_{\mathrm{BS}}}{\partial\sigma}n^{-H}\alpha_{\tau_n}Z_n+o(n^{-H}),$$

where p_{BS} and c_{BS} are the Black–Scholes prices with volatility parameter σ_{τ_n} of, respectively, put option with strike K_{τ_n} and call option with strike $S_{\tau_n}^2/K_{\tau_n}$. By the put-call symmetry (Carr and Lee 2009) of the Black–Scholes prices,

$$P_{\tau_n}(K_{\tau_n})-\frac{K_{\tau_n}}{S_{\tau_n}}C_{\tau_n}(S_{\tau_n}^2/K_{\tau_n})$$

$$= \left(\frac{\partial p_{\mathrm{BS}}}{\partial \sigma} + \frac{K_{\tau_n}}{S_{\tau_n}} \frac{\partial c_{\mathrm{BS}}}{\partial \sigma} \right) n^{-H} \alpha_{\tau_n} Z_n + o(n^{-H}).$$

Note that $\liminf_{n\to\infty} Z_n > -\infty$ ensures

$$\liminf_{n\to\infty} \sqrt{n} \left(\frac{\partial p_{\mathrm{BS}}}{\partial \sigma} + \frac{K_{\tau_n}}{S_{\tau_n}} \frac{\partial c_{\mathrm{BS}}}{\partial \sigma} \right) > 0.$$

Further, we have $\liminf_{n\to\infty} \alpha_{\tau_n} Z_n > 0$ and so,

$$\sum_{n=1}^{\infty} n^{H-1/2} \left(P_{\tau_n}(K_{\tau_n}) - \frac{K_{\tau_n}}{S_{\tau_n}} C_{\tau_n}(S_{\tau_n}^2 / K_{\tau_n}) \right) = \infty.$$

The result then follows from lemma 3.1.