Received: 1 December 2017 | Accepted: 1 October 2018

DOI: 10.1111/mafi.12207

WILEY MATHEMATICAL FINANCE An International Journal of Mathematics, Statistics and Financial Economics

ORIGINAL ARTICLE

A rational asset pricing model for premiums and discounts on closed-end funds: The bubble theory

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Abstract

This paper provides a new explanation for closed-end fund (CEF) discounts and premiums using the local martingale theory of asset price bubbles. This is a rational asset pricing model that is shown to be consistent with the existing empirical evidence on CEF discounts/premiums. Additional testable implications of the model are derived, which await subsequent research for their resolution. This bubble theory also applies equally well to understanding discounts and premiums on exchange traded funds.

KEYWORDS

asset price bubbles, closed-end funds, ETFs, no arbitrage

JEL CLASSIFICATION:

D53, G12, G14

1 | INTRODUCTION

Closed-end funds (CEFs) are investment companies issued under the Investment Company Act of 1940. A CEF invests in a collection of assets, often having liabilities to partially finance the purchase. The remaining capital is obtained when it issues equity shares via an initial public offering. These equity shares trade in a secondary market, an exchange, and can be bought or sold by investors. The market value of the asset pool less the liabilities is called the CEF's *net asset value* (NAV). It is well documented that a fund's NAV regularly differs from the market price of the fund's shares.[1] A negative difference is called a *discount*, and a positive difference is called a *premium*. Although discounts are the norm, premiums are not uncommon (e.g., Cherkes, 2012, states that on average premiums exist for one third of seasoned CEFs).

In a frictionless and competitive asset pricing model, premiums and discounts are (wrongly) believed to represent arbitrage opportunities.[2] This is because if there is a premium, one can short the CEF, buy the underlying asset pool, and hold the position until liquidation. Conversely, if there is a discount, one reverses the previous trading strategy. Thus, the existence of premiums and discounts on CEFs is viewed as a contradiction to market efficiency and rational asset pricing models more generally.

"The flotation and subsequent behavior of closed-end fund shares therefore represents a challenge to the hypothesis that investors behave rationally and markets function efficiently. Closed-end funds provide apparent evidence of market inefficiency, violations of standard asset pricing models, and exceptions to such fundamental principles of corporate finance as the Law of the Conservation of Value or the Modigliani-Miller propositions (Dimson & Minio-Kozerski, 1999, p. 2)."

Existing explanations for CEF premiums and discounts consistent with rational asset pricing models and market efficiency are based on including various market frictions/imperfections into the model. In this regard, proposed explanations for discounts/premiums include: (a) transaction costs and trading constraints create an interval around zero where premiums/discounts can lie, (b) NAVs are too large due to the present value of unrealized tax liabilities and/or marking-to-market not capturing a discount due to illiquidities, (c) the present value of management fees, including over- and underperformance of the management team, and (d) premiums paid for overcoming market segmentations (see Cherkes, 2012; Dimson & Minio-Kozerski, 1999; Ross, 2005, for excellent reviews). An alternative behavioral explanation has been advanced, called the investor sentiment theory, that CEF discounts and premiums are caused by irrational investors (see Lee, Shleifer, & Thaler, 1991). Unfortunately, none of these explanations appear to be consistent with all of the known evidence.

"Malkiel (1977, p. 847) provides the first careful analysis of the possible causes of discounts, finds all of them lacking in explanatory power, … Malkiel's list of possible reasons for discounts [which includes bookkeeping procedures, managerial fees, managerial skills, and unrealized capital appreciation (Malkiel, 1977, p. 847)], along with factors such as investors' irrationality and market segmentation, is still being discussed 35 years later, and no consensus has been reached regarding the source of discounts (Cherkes, 2012, p. 433)."

The purpose of this paper is to provide a new explanation of CEF discounts and premiums based on the local martingale theory of asset price bubbles, which is a rational asset pricing model. Simply stated, in the absence of market frictions/imperfections, discounts and premiums occur due to the existence of price bubbles in the securities underlying the CEF and in the fund shares themselves. Asset price bubbles are known to be consistent with frictionless, competitive, and arbitrage-free markets (see Protter, 2013, for a review). This rational model is easily extended to include the market frictions/imperfections previously mentioned with respect to CEFs. Furthermore, the existence of asset price bubbles is also known to be consistent with equilibrium in rational asset pricing models (see Jarrow, 2015, for a review of the equilibrium literature on bubbles). Hence, the CEF discounts/premiums do not necessarily imply market inefficiency (see Jarrow & Larsson, 2012, 2015, for the relation between equilibrium and market efficiency).

Turning to the empirical evidence, we argue that this bubble theory is consistent with the known patterns in historical CEF discounts and premiums. Two patterns not explicable by the investor sentiment theory (see Lee et al., 1991) or market imperfection arguments, but which are shown to be consistent with asset price bubbles, are why bond funds have smaller discounts than equity funds (see Abraham, Elan, & Marcus, 1993), and why UK CEF discounts exist, despite the fact that they are largely held by institutions (see Dimson & Minio-Kozerski, 1999, p. 25). In addition, additional empirical implications of this bubble theory are generated that can be tested to determine its truth or falsity. Such an empirical investigation awaits subsequent research.

Interestingly, this theory also applies equally well to premiums and discounts on exchange traded funds (ETFs), which are even more anomalous given the redemption and creation process, not available with CEFs (see Ben-David, Franzoni, & Moussawi, 2012; Engle & Sarkar, 2006, for some evidence in this regard). The bubble theory of discounts/premiums presented herein is also consistent with this evidence.

An outline of this paper is as follows. Sections 2 and 3 provide the bubble theory of CEF discounts/premiums for frictionless markets and markets with frictions, respectively. Section 4 discusses the empirical evidence with respect to CEF discounts/premiums, Section 5 has a brief discussion of ETFs, and Section 6 concludes.

2 | THEORY (NO FRICTIONS)

This section presents the bubble theory for CEF discounts/premiums to show that discounts/premiums can exist in a rational asset pricing model without transaction costs or trading constraints. We use the standard continuous time, continuous trading model on a finite horizon $t \in [0, T]$. Because the setup is standard, we will be brief in our discussion of the various components, leaving a more detailed explanation to Jarrow and Protter (2008).

2.1 | The setup

Given is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} \equiv (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ satisfying the usual conditions (see Protter, 2005) where Ω is the set of states, $\mathcal{F}$ is the $\sigma-$ algebra of events, $\mathbb{F}$ is a filtration, and $\mathbb{P}$ is the statistical probability measure. We assume that markets are frictionless, that is, there are no transaction costs or trading constraints.

Traded are a money market account (mma), a collection of risky assets, and CEF shares. The mma is assumed[3] to have a market price equal to 1 for all $t \in [0, T]$. Denote the time t market prices of the risky assets as $S_j(t) \ge 0$ for $j = 1, \dots, N$ and denote the time t market price of the CEF shares as $F_t \ge 0$. All of these price processes are assumed to be semimartingales adapted to $\mathbb{F}$. We assume that none of the risky assets or the CEF have any cash flows.

A *trading strategy* is a vector of adapted stochastic processes representing the number of units of the mma $n_0(t)$, the risky assets $n_1(t), \dots, n_N(t)$, and the CEF shares $n_{N+1}(t)$. The trading strategy's time $t \in [0, T]$ value process is

$$V_t = n_0(t) + \sum_{j=1}^{N} n_j(t) S_j(t) + n_{N+1}(t) F_t. \tag{1}$$

Assuming that $n_0(t)$ is an optional process and $n_1(t), \dots, n_N(t), n_{N+1}(t)$ are predictable processes all of which are integrable with respect to the integrators $(S_1(t), \dots, S_N(t), F_t)$, a trading strategy with initial value

$$V_0 = n_0(0) + \sum_{j=1}^{N} n_j(0) S_j(0) + n_{N+1}(0) F_0 \tag{2}$$

is said to be *self-financing* if for all $t \in [0, T]$,

$$V_t = V_0 + \sum_{j=1}^{N} \int_0^t n_j(s) d S_j(s) + \int_0^t n_{N+1}(s) d F_s. \tag{3}$$

A s.f.t.s. is *admissible* if there exists a $c \in \mathbb{R}$ such that $V_t \geq c$ for all $t \in [0, T]$. This admissibility condition is included to remove doubling strategies from the market. Such a restriction is necessary in continuous time and continuous trading models.

The market is said to satisfy no free lunch with vanishing risk (*NFLVR*) if there are no *FLVR*'s. A *FLVR* is either (a) an admissible s.f.t.s. $(n_0(t), n_1(t), \ldots, n_N(t), n_{N+1}(t))$ with value process V_t where $V_0 = 0$ such that $V_T \geq 0$ and $\mathbb{P}(V_T > 0) > 0$, or (b) a sequence of zero initial investment admissible s.f.t.s.'s $(n_0^i(t), n_1^i(t), \ldots, n_N^i(t), n_{N+1}^i(t))_{i=1}^{\infty}$ with lower admissibility bounds c^i, value processes V_t^i where $V_0^i = 0$, and a $\mathcal{F}_T$ measurable random variable $f \geq 0$ with $\mathbb{P}(f > 0) > 0$ such that $c^i \to 0$ and $V_T^i \to f$ in probability.

The first fundamental theorem of asset pricing states the market satisfies

$$NFVLR \iff \exists\, \mathbb{Q} \in \mathfrak{M}_\sigma, \tag{4}$$

where $\mathfrak{M}_\sigma$ is the set of equivalent sigma martingale measures. An equivalent sigma martingale measure $\mathbb{Q} \in \mathfrak{M}_\sigma$ is a probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ and such that $(S_1(t), \ldots, S_N(t), F_t)$ are $\mathbb{Q}$ sigma martingales. See Protter (2005) for the definition of a sigma martingale. Note that if processes have continuous paths, or alternatively are bounded below, then sigma martingales are local martingales. Because the risky assets and CEF shares are assumed to be nonnegative; in this case, $\mathfrak{M}_\sigma$ is equal to the set of local martingale measures.

Assumption (NFLVR). There exists a $\mathbb{Q} \in \mathfrak{M}_\sigma$.

Assuming the market satisfies NFLVR is a weaker assumption than assuming that the markets are in an economic equilibrium, which requires additional and more restrictive assumptions on investor preferences, endowments, asset supplies, and market clearing conditions. It is known, however, that rational asset price bubbles do exist in incomplete markets that are frictionless and competitive (as considered here), see Loewenstein and Willard (2000) and Jarrow (2017, 2018). Therefore, our model is consistent with, although more general than, these equilibrium constructs. As a consequence of this assumption, our model also allows for market prices that are in an economic disequilibrium.

To define an asset price bubble, we need to choose a local martingale measure $\mathbb{Q} \in \mathfrak{M}_\sigma$. The choice is unique if the market is complete. We do not assume that the market described above is complete; hence, there can exist an infinite number of local martingale measures. However, if enough derivative securities trade so that the expanded market, including the traded derivatives, is complete (see Jacod & Protter, 2010), then the local martingale measure $\mathbb{Q} \in \mathfrak{M}_\sigma$ is again uniquely determined. We call such a local martingale measure, the measure "chosen by the market." We assume that the expanded market is complete, and fix the unique $\mathbb{Q} \in \mathfrak{M}_\sigma$ chosen by the market.

A risky asset's time t *fundamental value* and the CEF's time t fundamental value are defined by

$$\mathbb{E}_t^{\mathbb{Q}}(S_j(T)) \quad \textit{for } j = 1, \ldots, N, \quad \textit{and} \quad \mathbb{E}_t^{\mathbb{Q}}(F_T), \tag{5}$$

respectively, where $\mathbb{E}_t^{\mathbb{Q}}(\cdot)$ is the time t conditional expectation with respect to $\mathcal{F}_t$ under the probability $\mathbb{Q} \in \mathfrak{M}_\sigma$. This conditional expectation represents the present value of the liquidating cash flow to the risky asset and CEF shares. As is well known, the local martingale measure $\mathbb{Q} \in \mathfrak{M}_\sigma$ contains an adjustment for risk. Using the mma as the numeraire implies that this cash flow is discounted to time t.

This definition of an asset's fundamental value is consistent with the definition used in the classical economics literature, see Loewenstein and Willard (2000) and Jarrow, Protter, and Shimbo (2010) and references therein for the classical literature. It is equal to the replication value of the asset if markets are complete. In an incomplete market, as noted above, there is nonuniqueness of the local martingale measure used to define the fundamental value. Our rule for selecting a unique element in the set of local martingale measures was discussed above.

In an incomplete market, an alternative definition of the fundamental value has been proposed in the literature that is the superreplication value of the asset, see Loewenstein and Willard (2013) and Herdegen and Schweizer (2016). The fundamental value used in our paper can be shown to be greater than or equal to the superreplication value. We do not use this alternative definition below.

The *asset price bubbles* are defined by

$$\beta_t^j = S_j(t) - \mathbb{E}_t^{\mathbb{Q}}(S_j(T)) \geq 0 \quad \textit{for } j = 1, \ldots, N, \textit{ and} \tag{6}$$

$$\beta_t^f = F_t - \mathbb{E}_t^{\mathbb{Q}}(F_T) \geq 0, \tag{7}$$

with $\beta_T^j = 0$ and $\beta_T^f = 0$.

We note that only allowing investors to use admissible trading strategies is imposing a trading constraint on the trading strategy's value. It is a constraint on the magnitude of a short position a trader can generate. It is well known in the economics literature that trading constraints can induce asset price bubbles in otherwise well-functioning markets (see Jarrow, 2015, for more discussion of this point). This is the case in this market setting as well. From the local martingale theory of bubbles, the following properties of asset price bubbles are known:

1. In an NFLVR market, bubbles can exist if the risky asset's price process is unbounded above.
2. Bubbles burst on or before time T.
3. Bubbles are always nonnegative.

As discussed above, property 1 holds because the s.f.t.s. to exploit a bubble is not admissible. The s.f.t.s. is to short the risky asset with a bubble, invest the proceeds in a mma, and buy the risky asset back at the liquidation date when the bubble bursts with the proceeds from the mma. This s.f.t.s. is not admissible, and hence cannot be used to exploit asset price bubbles.

The following example illustrates a price process satisfying properties 1–3.

Example 2.1 (**CEF Share Price Process with a Bubble**)**.** This example gives a price process for the CEF's shares with a price bubble in an NFLVR market. The price process selected has stochastic volatility, implying an incomplete market, and as noted previously, allows for the existence of a price bubble. Here, all processes have continuous sample paths.

Let B^1 and W be Brownian motions with correlation coefficient ρ.

Let the CEF's share price process F be given by

$$\frac{dF_t}{F_t} = v_t dB_t^1; \quad F_0 = 1, \tag{8}$$

$$dv_t = \sigma(v_t)dW_t + b(v_t)dt; \quad v_0 = 1. \tag{9}$$

Here, F is a positive price process and v is a stochastic volatility. Notice that this is under the local martingale measure $\mathbb{Q}$ because F is a positive local martingale.

We assume that σ and b are C^1 Lipschitz continuous functions that satisfy:

$$\sigma(0) = 0 \quad \sigma(x) > 0 \quad \text{if } x > 0, \tag{10}$$

$$b(0) \geq 0 \quad b(x) \leq C(1+x) \quad \text{for some constant } C, \tag{11}$$

and

$$\liminf_{x\to\infty} \frac{\rho x\sigma(x) + b(x)}{\phi(x)} > 0,$$

where $\phi(x)$ is an increasing, positive, smooth function that satisfies

$$\int_a^\infty \frac{1}{\phi(x)} ds < \infty,$$

where $a > 0$ is a constant.

By Lions and Musiela (2007), this implies that F is not a martingale but a supermartingale and a strict local martingale. Thus, F is a strictly positive strict local martingale with stochastic volatility process ν. This completes the example.

2.2 | The model

Consider a CEF that invests in the risky securities $j = 1, \dots, N$ using an admissible s.f.t.s. $(n_1(t), \dots, n_N(t))$ with value process V_t satisfying $\mathbb{E}_t^{\mathbb{Q}}(V_T) < \infty$ for all t. This represents the CEF's asset pool. The time *t marked-to-market valu*e of the CEF's assets is given by

$$A_t = \sum_{j=1}^N n_j(t) S_j(t). \tag{12}$$

Let the market price of the CEF's liabilities be denoted by $L_t \geq 0$. To be consistent with an NFLVR market, we assume that L_t is nonnegative, $\mathcal{F}_t$ adapted, and a $\mathbb{Q}$ local martingale with $\mathbb{E}_t^{\mathbb{Q}}(L_T) < \infty$.

We have that

$$E_t = A_t - L_t \geq 0$$

is the CEF's *NAV*. Equity is nonnegative due to limited liability. We note that this nonnegative condition implicitly gives the value of the equity after bankruptcy proceedings. Prior to bankruptcy, $A_t - L_t$ could be negative. The bankruptcy proceedings transfers the losses from the equity to the liability holders. This implies, of course, that under this condition, L_t depends on A_t.

At the liquidation date T, by construction, the CEF's NAV is equal to the market price of the CEF's shares, which is equal to the liquidation value, that is,

$$F_T = E_T = A_T - L_T \geq 0. \tag{13}$$

The fundamental value of the NAV is given by $\mathbb{E}_t^{\mathbb{Q}}(E_T)$ and the NAV's bubble is defined by

$$\beta_t^e = E_t - \mathbb{E}_t^{\mathbb{Q}}(E_T) \geq 0 \tag{14}$$

with $\beta_T^e = 0$.

From the above, we obtain the following lemma characterizing the NAV's bubble.

Lemma 2.2 (**NAV Bubble**).

$$\beta_t^e = \sum_{j=1}^N n_j(t) S_j(t) - \mathbb{E}_t^{\mathbb{Q}}\left(\sum_{j=1}^N \int_0^T n_j(s) dS_j(s)\right) - \left(L_t - \mathbb{E}_t^{\mathbb{Q}}(L_T)\right) \geq 0.$$

Proof. $E_t = A_t - L_t = \sum_{j=1}^{N} n_j(t)S_j(t) - L_t$.

Taking conditional expectations of the time T value yields

$$\mathbb{E}_t^{\mathbb{Q}}(E_T) = \mathbb{E}_t^{\mathbb{Q}}\left(\sum_{j=1}^{N}\int_0^T n_j(s)dS_j(s)\right) - \mathbb{E}_t^{\mathbb{Q}}(L_T)$$

Using the definition of β_t^e completes the proof. □

This lemma shows that the NAV's bubble is generated by any bubbles in the underlying assets over the life of the CEF, less any bubbles in the market value of the liabilities. If the market value of the liabilities L_t is bounded above for all t, then L_t is a $\mathbb{Q}$ martingale,[4] and the liabilities have no bubbles. This is the usual case because the total payoffs to the liabilities are bounded by the promised payments to the fixed income securities and it is often reasonable to assume, as we do here, that interest rates are nonnegative[5]. In this circumstance, the previous result simplifies to

$$\beta_t^e = \sum_{j=1}^{N} n_j(t)S_j(t) - \mathbb{E}_t^{\mathbb{Q}}\left(\sum_{j=1}^{N}\int_0^T n_j(s)dS_j(s)\right) \geq 0.$$

The CEF's *discount/premium* D_t is defined to be the difference between the market price of the CEF's shares and the NAV, that is,

$$D_t \equiv F_t - E_t. \tag{15}$$

It is a premium if it is strictly positive and it is a discount if it is strictly negative.

This observation, along with expression (13), trivially generates that the discount at liquidation satisfies

$$D_T = F_T - E_T = 0. \tag{16}$$

As shown, at the liquidation date T, the discount disappears in an NFLVR and frictionless market.

Example 2.3 (**NAV Discount/Premium Process in an NFLVR Market**)**.** This example gives an NAV discount/premium process that is nonzero in a NFLVR and frictionless market. It augments the example of the CEF shares in Example 2.1 above to include an NAV evolution that also has stochastic volatility, a Brownian shock that is distinct from that underlying the CEF shares, and its own price bubble. More formally, consider the setup and the price process for the CEF shares F from Example 2.1 above.

Let B^2 and V be another two Brownian motions independent of B^1 and W.

Let the NAV value E satisfy the equation

$$\frac{dE_t}{E_t} = f(E_t, \eta_t)dB_t^2; \quad E_0 = 1, \tag{17}$$

$$d\eta_t = s(\eta_t)dV_t + g(\eta_t)dt; \quad \eta_0 = 1, \tag{18}$$

where the volatility process η is a new stochastic volatility. Assume analogous conditions to those imposed on F in Example 2.1 so that E is a strict $\mathbb{Q}$ local martingale. Further, we also assume that $E_T = F_T$ so that $D_T = 0$.

Next, consider the NAV discount/premium $D = F - E$. It is a strict local martingale. Note that $D_0 = 0$; therefore, D must assume both positive and negative values because

$$
\begin{aligned}
[D, D]_t &= [F - E, F - E]_t = [F, F]_t + [E, E]_t - 2[F, E]_t \\
&= \int_0^t F_s^2 \nu_s^2 ds + \int_0^t E_s^2 f(E_s, \eta_s)^2 ds - 2\rho \int_0^t F_s E_s \nu_s f(E_s \eta_s) ds \\
&\neq 0.
\end{aligned}
$$

Indeed, a continuous local martingale such as D that starts at $D_0 = 0$ cannot be strictly nonnegative, nor strictly nonpositive, unless it is identically zero. But if that were to be the case, the quadratic variation process $[D, D]$ would also be identically zero, and it is not.

Because both F and E are strict local martingales, they both exhibit price bubbles. This observation is exploited below. This completes the example.

2.3 | Results

This section characterizes a CEF's discounts/premiums in an NFLVR and frictionless market.

Theorem 2.4 (**CEF Discounts/Premiums**).

$$D_t = \beta_t^f - \beta_t^e, \tag{19}$$

where $D_T = 0$. Moreover, D_t can be any of positive, negative, or equal to 0.

Proof. $D_t = F_t - E_t$. Noting that $E_t = \beta_t^e + \mathbb{E}_t^{\mathbb{Q}}(E_T)$, $F_t = \beta_t^f + \mathbb{E}_t^{\mathbb{Q}}(F_T)$, and $\mathbb{E}_t^{\mathbb{Q}}(E_T) = \mathbb{E}_t^{\mathbb{Q}}(F_T)$ completes the proof. □

This theorem characterizes a CEF's discount/premium as the difference between two asset price bubbles: that in the NAV and that in the CEF's shares. Given that bubbles exist in an NFLVR and frictionless market, we see that a CEF's discount/premium can exist in an NFLVR and frictionless market. This contradicts a commonly held belief in the CEF literature, as noted in the introduction, that CEV discounts/premiums imply the existence of arbitrage opportunities.

Indeed, we remark that given the longevity of the existence of CEFs, it seems a priori unlikely that they could routinely offer arbitrage opportunities. Our work here simple provides an explanation as to why they, in fact, do not provide such arbitrage opportunities.

3 | THEORY (FRICTIONS)

This section extends the bubble theory of CEF discounts/premiums to include market frictions. There are two types of market frictions: (a) transaction costs and (b) trading constraints. We consider both of these frictions below.

3.1 | The model

For a market with frictions, we add the following assumptions.

Assumption (**NFLVR with Frictions**).

$$\text{There exists a } \mathbb{Q} \in \mathfrak{M}_\sigma.$$

First, when there are transaction costs, the value process for an admissible s.f.t.s. is reduced by the costs of trading. Hence, given that the existence of a $\mathbb{Q} \in \mathfrak{M}_\sigma$ implies NFLVR, adding transaction costs does not introduce FLVRs into the market (see Cetin, Jarrow, & Protter, 2004, for a formal proof of this assertion). Second, when there are trading constraints, the set of admissible s.f.t.s. is reduced by the constraints. Here again, given that the existence of a $\mathbb{Q} \in \mathfrak{M}_\sigma$ implies NFLVR, then the existence of a $\mathbb{Q} \in \mathfrak{M}_\sigma$ implies that there is no NFLVR for the constrained set of admissible s.f.t.s. as well. Combined, this assumption implies that there are no constrained NFLVR admissible s.f.t.s. in a market with frictions.[6]

Trading constraints will impact the CEF's discount/premiums through the existence and magnitude of the relevant price bubbles. Indeed, trading constraints can create bubbles that would not otherwise exist in a frictionless market (see Pulido, 2016, for the case of short sale constraints). To incorporate transaction costs, however, we need to add the following assumption.

Assumption (Transaction Costs).

1. There exists an $\mathcal{F}_T$ measurable random variable δ such that

$$D_T = -\delta, \tag{20}$$

   where $\mathbb{E}_t^{\mathbb{Q}}[|\delta|] < \infty$.

2. At time t,

$$D_t = F_t - E_t - \mathbb{E}_t^{\mathbb{Q}}[\delta]. \tag{21}$$

This assumption is a "reduced-form" method of including the impact of transaction costs and market frictions on the liquidation value of the CEF. It is imposed purposefully to simplify the model. This reduced-form methodology has a long history in the commodity derivatives pricing literature, see Schwartz (1997).

We note that, in part 1 of the assumption, the random variable δ represents the cumulative costs incurred in the trading of F_t less the cumulative costs of creating E_t over $[0, T]$. This includes trading costs due to illiquidities as in Cetin et al. (2004) and costs in the running of a CEF including management fees, unrealized tax liabilities, and agency costs. It does not include the existence of positive alphas,[7] or superior management performance, because these would imply the contradiction of NFLVR (see Jarrow & Protter, 2013). These cumulative costs δ can be positive or negative because they represent the difference in the cumulative costs of trading F_t and creating E_t. In part 2 of this assumption, the NAV discount is assumed to be that of a frictionless market, plus the present value of the cumulative costs. The value of the cumulative costs could be inconsistent with NFLVR because they do not trade. To exclude this situation, we assume that the present value of the cumulative costs is consistent with NFLVR and given by the conditional expectation using the local martingale measure.

3.2 | Results

Given the model, we can now characterize the CEF's discounts/premiums in a market with constrained NFLVR and frictions.

Theorem 3.1 (CEF Discounts/Premiums).

$$D_t = \beta_t^f - \beta_t^e - \mathbb{E}_t^{\mathbb{Q}}[\delta], \tag{22}$$

where $D_T = -\delta$. Note that D_t can be any of positive, negative, or zero.

Proof. $D_t = F_t - E_t - \mathbb{E}_t^{\mathbb{Q}}[\delta]$. Noting that $E_t = \beta_t^e + \mathbb{E}_t^{\mathbb{Q}}(E_T)$, $F_t = \beta_t^f + \mathbb{E}_t^{\mathbb{Q}}(F_T)$, and $\mathbb{E}_t^{\mathbb{Q}}(E_T) = \mathbb{E}_t^{\mathbb{Q}}(F_T)$ completes the proof. □

This theorem characterizes a CEF's discount as being the difference between the asset price bubbles in the CEF's shares and the NAV less the present value of the cumulative costs of running the CEF. It is important to note that asset price bubbles can exist in a rational asset pricing model equilibrium with trading constraints (see Hugonnier, 2012). Because equilibrium precludes constrained FLVR, we see that CEF discounts/premiums do not imply the existence of a constrained FLVR. In addition, the existence of CEF discounts/premiums in a rational equilibrium implies that the market is efficient (see Jarrow & Larsson, 2012, 2015). Again, this insight is in contradiction to a common belief held in the literature, as discussed in Section 1, that CEF discounts/premiums are inconsistent with the existence of an efficient market.

Corollary 3.2 (Bounded CEF Values). *If $E_t \le K_e$ and $F_t \le K_f$ for all t where $K_e, K_f > 0$ are constants, then*

$$D_t = -\mathbb{E}_t^{\mathbb{Q}}[\delta].$$

Proof. We have that E_t is a $\mathbb{Q}$ local martingale. Given the definition of a local martingale, for a localizing sequence $\tau_n \to T$, we have $\mathbb{E}_t^{\mathbb{Q}}(E_{T\wedge\tau_n}) = E_{t\wedge\tau_n}$. Taking the limit of both sides of this expression and using dominated convergence gives that E_t is a $\mathbb{Q}$ martingale. But, because $F_T = E_T$, this implies that F_t is a $\mathbb{Q}$ martingale as well. This implies that $\beta_t^e = \beta_t^f = 0$. □

This corollary applies to bond CEFs because the asset pool, consisting of fixed income securities, has CEF values bounded above for all *t* because the promised payments on the bonds at liquidation are bounded. This corollary will prove useful later in the paper when discussing the empirical evidence with respect to bond CEFs.

Corollary 3.3 (Existence of Bubbles). 1. *No bubbles in both F_t and E_t imply $D_t = -\mathbb{E}_t^{\mathbb{Q}}[\delta]$ for all t a.e. $\mathbb{P}$, that is, D_t is a martingale.*

If $\mathbb{E}_t^{\mathbb{Q}}[\delta] > 0$, then

2. *A premium ($D_t > 0$) implies that F_t has a bubble, that is, $\beta_t^f > 0$. A bubble in E_t is ambiguous.*
3. *A discount ($D_t < 0$) is ambiguous with respect to whether either E_t or F_t has a bubble.*

Proof. 1. No bubbles means $\beta_t^f = 0$ and $\beta_t^e = 0$ for all *t* a.e. $\mathbb{P}$. The assumption about D_t completes the proof.

2. $D_t > 0$ implies $\beta_t^f - \beta_t^e - \mathbb{E}_t^{\mathbb{Q}}[\delta] > 0$ or $\beta_t^f > \beta_t^e + \mathbb{E}_t^{\mathbb{Q}}[\delta] \ge 0$.
3. $D_t < 0$ implies $\beta_t^f - \beta_t^e - \mathbb{E}_t^{\mathbb{Q}}[\delta] < 0$. □

There is some evidence that the present value of the cumulative transaction costs is positive, $\mathbb{E}_t^{\mathbb{Q}}[\delta] > 0$, due to management fees and unrealized tax overhangs (see Cherkes, 2012, p. 435, 437). Given this evidence, as shown in this corollary, the existence of CEF premiums is indirect evidence supporting the existence of asset price bubbles. Interestingly, given the rise in CEF premiums in recent markets, this is evidence consistent with the aggregate market experiencing a price bubble (see Wall Street Journal, 2016). The regularity and persistence of CEF premiums also support the regularity and persistence of asset price bubbles. There is direct evidence, albeit limited, on the existence of asset price bubbles and their regularities (see Jarrow, Kchia, & Protter, 2011, and the more recent and improved results of Obayashi, Protter, & Yang, 2017).

4 | EMPIRICAL EVIDENCE CEFs

Lee et al. (1991) identify four properties of CEFs which any theory needs to explain: (a) CEFs start with a premium, (b) CEFs move quickly to a discount, which is the usual case, (c) discounts fluctuate over time and can often be premiums, and (d) when a fund terminates, discounts narrow or disappear. A fifth pattern also appears in their empirical investigation: (e) the existence of positive cross-sectional correlations across discounts.

When embedded in an equilibrium model where traders have heterogeneous beliefs (see Jarrow, 2017), the local martingale theory of bubbles is consistent with all five of these patterns because bubbles exist when traders view the retrade value of an asset as exceeding the value of buying and holding the asset until liquidation. Because bubbles depend on investors' beliefs, which change according to Bayes' law and the realization of information, properties (a), (b), and (e) are easily understood. Property (c) requires theories of bubble birth (see Biagini, Follmer, & Nedelcu, 2014; Jarrow et al., 2010), and property (d) follows by expression (22) above.

Other properties of CEF discounts/premiums consistent with bubbles, after the inclusion of market frictions/imperfections as discussed above, are the observations that: (f) discounts depend on the level of market frictions (see Pontiff, 1996), (g) discounts are negatively correlated with future stock returns and discounts exhibit mean reversion (see Pontiff, 1995), and (h) CEFs with discounts earn positive abnormal returns (see Pontiff, 1995). Because the $\mathrm{var}^{\mathbb{P}}(F_t)$ is not equal to the $\mathrm{var}^{\mathbb{P}}(E_t)$ due to expression (21), bubbles are also consistent with the fact that: (i) $\mathrm{var}^{\mathbb{P}}(F_t) > \mathrm{var}^{\mathbb{P}}(E_t)$ (see Pontiff, 1997).

Patterns in discounts, which the investor sentiment theory cannot easily explain, include: (j) differences in discounts across otherwise similar CEFs (in terms of their asset pool) (see Cherkes, 2012), (k) why bond funds have smaller discounts than equity funds (see Abraham et al., 1993), and why UK CEF discounts exist, despite the fact that they are largely held by institutions (see Dimson & Minio-Kozerski, 1999, p. 25). Not surprisingly, bubbles are consistent with all of these observations, for example, for property (k) with respect to bond, funds use Corollary 3.2 above and the fact that $D_t = -\mathbb{E}_t^{\mathbb{Q}}[\delta] < 0$.

Although the investor sentiment and bubbles theory are similar in that they both attribute CEF discounts/premiums to investor beliefs, the existence of bubbles imposes fewer restrictions on the evolution of the market price process than does the investor sentiment theory. To verify the bubbles theory and distinguish it from investor sentiment, new testable implications are generated by the following expression:

$$D_t = \beta_t^f - \beta_t^e - \mathbb{E}_t^{\mathbb{Q}}[\delta]. \tag{23}$$

To use this equation to test the bubble theory of CEFs, the right side of this expression needs to be estimated independently of the left side. The newly developed tools for estimating asset price bubbles can be used in this regard (see Obayashi et al., 2017). This verification is left for subsequent research.

5 | ETFs

This section briefly discusses the application of the previous bubble theory of discounts and premiums to ETFs. As is well known, ETFs are very similar to CEFs, the only exception being that ETFs allow redemption and creation of unit shares (see Ben-David et al., 2012, for a discussion of the redemption and creation process). This creation and redemption process is not possible for CEFs. Nonetheless, this creation and redemption process does not change the modeling of ETFs relative to CEFs, and consequently, the bubble theory of discounts/premiums as constructed above applies unchanged to ETFs.

With respect to the empirical evidence on ETF premiums/discounts, it is less extensive than that for ETFs. Two important papers in this regard are Engle and Sarkar (2006) and Ben-David et al. (2012). Both papers document the existence of premiums and discounts on ETFs that vary across time and are autocorrelated. Discounts/premiums also appear to be smaller than those of CEFs, due to the redemption and creation process, which provides another admissible s.f.t.s. to exploit discounts/premiums. As with CEFs, the bubble theory of discounts/premiums is consistent with these properties of ETF discounts/premiums. Interestingly, ETFs have recently been linked to increasing volatility especially as regards to the infamous flash crash, but also in general (see Guo & Leung, 2015; Leung & Sircar, 2015; Menkveld & Yueshen, 2018). Such high volatility is associated with financial bubbles, lending indirect evidence to their possible presence. Validating the bubble theory of ETF discounts/premiums also awaits subsequent research.

6 | CONCLUSION

This paper provides a new rational asset pricing theory for CEFs and ETFs discounts and premiums, which is consistent with the empirical evidence. It is based on the local martingale theory of asset price bubbles. New testable implications of this theory are provided, which await subsequent research.

ACKNOWLEDGEMENTS

Helpful comments from the editor and referees are gratefully acknowledged. Philip Protter was supported in part by NSF grant DMS 1612758.

ENDNOTES

[1] See https://www.fidelity.com/learning-center/investment-products/closed-end-funds/relative-discounts-premiums.

[2] The word “wrongly” follows because this self-financing trading strategy (s.f.t.s.) is not admissible because its value process is unbounded below.

[3] This assumption is with some loss in generality because it implies that the set of integrable value processes generated by the set of s.f.t.s.'s (defined below) can differ from those in a market where this assumption is not true.

[4] See the proof to Corollary 3.2 below.

[5] With unbounded negative interest rates, L_t may not be bounded above for all t.

[6] Hence, this assumption is a sufficient condition for no constrained NFLVR. It is not a necessary condition.

[7] An alpha is the difference between the expected return on an asset and its “equilibrium” or “arbitrage-free” expected return. Positive alphas are an indication of an abnormal expected return.

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How to cite this article: Jarrow R, Protter P. A rational asset pricing model for premiums and discounts on closed-end funds: The bubble theory. *Mathematical Finance*. 2019;29:1157–1170. https://doi.org/10.1111/mafi.12207