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ORIGINAL ARTICLE

Periodic strategies in optimal execution with multiplicative price impact

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Abstract
We study the optimal execution problem with multiplicative price impact in algorithmic trading, when an agent holds an initial position of shares of a financial asset. The interselling decision times are modeled by the arrival times of a Poisson process. The criterion to be optimized consists in maximizing the expected net present value of the gains of the agent, and it is proved that an optimal strategy has a barrier form, depending only on the number of shares left and the level of the asset price.

KEYWORDS
multiplicative price impact, optimal execution problem, periodic stochastic control

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1 | INTRODUCTION

In this paper, we are interested in finding optimal execution strategies for a financial market impact model where transactions can have a permanent effect. The analysis of this problem has practical and mathematical motivations, and has been studied from different perspectives. Nowadays, the use of algorithmic trading to execute large book orders has raised important questions on the best way to execute a position, in order to decrease the negative effect on the shift of the asset price, and also obtain the best performance in terms of the criteria to be optimized. In general, the existence of optimal strategies cannot be guaranteed, and this clearly depends on the structure of the market model as well as on the parameters involved in its description.

In any market impact model, it is crucial to describe the way that order execution algorithms will be generated. Despite the analytical tractability of the classical continuous time models, these are unfortunately not implementable in practice. On the other hand, while models with discrete execution decision times are ideal, they lack analytical tractability, and numerical methods are required to solve them. Recently, with the aim of developing a more realistic yet analytically tractable model, random discrete execution times have been considered. Random observations have been suggested in the economic literature. See, for example, the discussion in the introduction of Schaefer and Szimayer (2018), motivated by *rational inattention* (see Sims, 2003) in the macroeconomics literature. See also the discussions given in Boyarchenko and Levendorskii (2015) and Lempa (2010) for real option problems with random intervention times, and Dupuis and Wang (2002) and Guo & Liu (2005) for applications to optimal stopping problems and Bermudan look-back option pricing. In this regard, an important motivation for considering the Poissonian interarrival model is its potential applications to approximate the constant interarrival time cases. It is known, in the mathematical finance literature, that randomization techniques (see, e.g., Carr, 1998) are efficient at approximating constant maturity problems with those with Erlang-distributed maturities. In particular, for short maturity cases, it is known empirically that accurate approximations can be obtained by simply replacing the constant with an exponential random variable; see Leung, Yamazaki, and Zhang (2015). We also remark that randomization works because the value of a constant barrier policy is a good approximation for the value of the optimal policy, in some stopping time problems (see, e.g., Broadie & Detemple, 1996).

In this paper, we propose a random clock, attached to the jumps of a Poisson process, for the times when the execution decisions will take place. This is a new instrument that may have advantages from the perspective of implementation, because the randomness provided by the random clock included in the execution strategy introduces an additional unpredictable structure to the strategies. The empirical justification of this model can be approached from the following perspective, related to market microstructural factors. It is well known that the dynamics and the volume of trades interact with the evolution of the market liquidity of an asset. The design of each portfolio is based on the information arriving from the order flow of buy and sell decisions of the other investors, but not on who is behind each decision. This suggests that strategic sequential trading to execute large book orders is “event based,” represented by the new information inferred regarding the value of the asset from the composition and existence of the trades of the other market participants. Thus, we are proposing that the arrival process is linked with market parameters, such as liquidity, volume, market depth, and order flows. Interestingly, this viewpoint reflects the fact, well understood in high-frequency trading, for instance, that time has a different meaning when we are carrying out an algorithmic trading strategy using cycles depending on the amount of information received, instead of measurements based on chronological time.

The benchmark models assume that either trading can be done in continuous time or there are constant intervals of time at which the portfolio is balanced. Neither of these has practical reasons for being sustainable, besides their analytical tractability, because investors are continually gaining information about the trading environment. In the model proposed in this paper, we allow a random clock of time in which new information is processed, based on the evolution of the main factors of the market. This is a good example of algorithmic trading that does not necessarily occur at a high frequency, but retains the possibility of executing a position with frequent sells when the parameter of the Poisson process is manipulated to do so. Of course, it also helps to include asynchronous transactions to hide as much as possible the strategy followed.

In this paper, we assume that the agent holds a large position and, as typically happens, we expect that any selling strategy will lead to a decrease of prices. When the agent is not active, the model adopted is a standard geometric Brownian motion with drift, following Guo and Zervos (2015). Another important element in the model is related with the manner of quantifying the revenues obtained by each selling strategy. In this regard, the criterion will be the net present value of the difference between the gains of the selling strategy and the associated transaction costs.

In contrast with the multiplicative impact model presented here, in the seminal papers of Almgren and Chriss (1999, 2000), and Almgren (2003), the execution strategies are assumed to be absolutely continuous functions of time, having a price impact acting additively. Bertsimas and Lo (2000) also made fundamental contributions considering a discrete random walk model. In our case, the strategies are described as Lebesgue–Stieltjes integrals with respect to the paths of the Poisson process, called *periodic strategies*, in analogy with the terminology used in insurance models when dividend payment decisions are taken; see, for instance, Avanzi, Cheung, Wong, and Woo (2013); Avanzi, Tu, and Wong (2014); Tan, Yang, Li, and Cheng (2016); and Pérez and Yamazaki (2017).

More recent contributions to the theory of optimal execution found in the literature include Huberman and Stanzl (2004); He and Mamaysky (2005); Gatheral, Schied, and Slynko (2012); Obizhaeva and Wang (2013); Almgren and Lorenz (2007); Engle and Ferstenberg (2007); Schied and Schöneborn (2009); Alfonsi, Fruth, and Schied (2008, 2010); Schied, Schöneborn, and Tehranchi (2010); Predoiu, Shaikhet, and Shreve (2011); and Løkka (2014).

In order to find an optimal strategy over the set of periodic strategies, we restrict our analysis to the set of periodic barrier strategies. This class of barrier strategies is very easy to implement, because the selling decisions are taken by observing whether the price of the stock lies above a certain fixed level F and the remaining number of shares. Then, the first step consists in finding the optimal barrier strategy that maximizes the performance criteria. This is done by solving the Hamilton–Jacobi–Bellman (HJB) equation associated to this problem, which allows us to obtain an explicit form of the value function for this restricted problem. Imposing a suitable smoothness condition on the value function, we obtain the explicit value of the barrier F_γ associated to the optimal strategy. This strategy can be described as follows: If the stock price is below a critical level F_γ at a selling time, then it is optimal not to sell any shares. However, if the stock price lies above the level F_γ when the random clock rings, it is optimal either to sell all available shares or liquidate a fraction of the position, which will, as a consequence, make the stock price decrease. A verification theorem is used to prove that the original problem of optimization within the periodic strategies can be solved implementing only barrier strategies.

The rest of this paper is organized as follows. In Section 2, we review the underlying model for the stock price with multiplicative price impact and provide the performance criterion, as well as the formulation of the optimal execution problem with periodic strategies. In Section 3, we obtain an explicit form for the solution of the HJB equation associated to the value function over the set of periodic barrier strategies. A verification theorem is provided (Theorem 3.5), giving an explicit form for the optimal (or ε-optimal) periodic strategy, under appropriate conditions on the parameters of the model. We defer the proofs of some technical lemmas to the Appendix.

2 | MARKET IMPACT MODEL

In this section, we describe the optimal execution model, based on Guo and Zervos (2015). Let us fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ satisfying the usual conditions and carrying a standard $(\mathcal{F}_t)$-Brownian motion W and an independent Poisson process N^γ. We consider an agent holding an initial position of y shares of a financial asset, which has to be sold so as to maximize the expected gains. The information available to the agent is enclosed in the filtration $\mathcal{F}_t$.

The trading strategies are denoted by the couple $(\xi^{\mathrm{s}}_t, \xi^{\mathrm{b}}_t)$, which represents the total number of shares that the investor has sold and bought up to time t, respectively. Then, the total number of shares held by the agent at time t is given by

$$Y_t := y - \xi^{\mathrm{s}}_t + \xi^{\mathrm{b}}_t \geq 0, \qquad \text{for } t \geq 0, \tag{1}$$

where ξ^{s}, ξ^{b} are $(\mathcal{F}_t)$-adapted increasing càglàd processes such that

$$\xi^{\mathrm{s}}_0 = \xi^{\mathrm{b}}_0 = 0, \quad \mathbb{E}\left[\mathrm{e}^{4\lambda\xi^{\mathrm{b}}_{t+}}\right] < \infty \quad \text{and} \quad \lim_{t\to\infty} Y_t = 0.$$

Although not reflected in $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$, there is the restriction that the agent cannot sell and buy shares at the same time. The set of admissible strategies satisfying the previous conditions is denoted by $\Xi(y)$.

The stock price observed by the agent, independently of the actions of the other market participants, is modeled by the geometric Brownian motion X^0 with drift

$$X^0_t = \mu X^0_t \mathrm{d}t + \sigma X^0_t \mathrm{d}W_t, \quad X^0_0 = x > 0, \tag{2}$$

where $\sigma \in \mathbb{R}$ and $\mu > 0$ are constants. Suppose that the agent implements the strategy $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \Xi(y)$. Hence, when the agent decides to sell or buy some number of shares of the asset at time t, we assume that there is an impact on the price, described by a multiplicative factor, namely, the resulting price X_t is assumed to have the form

$$X_t = X^0_t \exp\left\{-\lambda\left(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t\right)\right\}, \tag{3}$$

for some positive constant λ describing the permanent impact on the price, and X^0_t is the solution to (2). More specifically, following Guo and Zervos (2015), the controlled process dynamics can be described as the solution of the following stochastic differential equation:

$$dX_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t - \lambda\left(X_t \circ_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_t - X_t \circ_{\mathrm{b}} \mathrm{d}\xi^{\mathrm{b}}_t\right),$$

where

$$\begin{aligned} X_t \circ_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_t &= X_t \mathrm{d}(\xi^{\mathrm{s}})^{\mathrm{c}}_t + \frac{1}{\lambda} X_t\left(1 - \mathrm{e}^{-\lambda\Delta\xi^{\mathrm{s}}_t}\right) = X_t \mathrm{d}(\xi^{\mathrm{s}})^{\mathrm{c}}_t + X_t \int_0^{\Delta\xi^{\mathrm{s}}_t} \mathrm{e}^{-\lambda u}\mathrm{d}u, \\ X_t \circ_{\mathrm{b}} \mathrm{d}\xi^{\mathrm{b}}_t &= X_t \mathrm{d}(\xi^{\mathrm{b}})^{\mathrm{c}}_t + \frac{1}{\lambda} X_t\left(\mathrm{e}^{\lambda\Delta\xi^{\mathrm{b}}_t} - 1\right) = X_t \mathrm{d}(\xi^{\mathrm{b}})^{\mathrm{c}}_t + X_t \int_0^{\Delta\xi^{\mathrm{b}}_t} \mathrm{e}^{\lambda u}\mathrm{d}u, \end{aligned} \tag{4}$$

and the processes $(\xi^{\mathrm{s}})^{\mathrm{c}}$ and $(\xi^{\mathrm{b}})^{\mathrm{c}}$ are the continuous part of ξ^{s} and ξ^{b}, respectively. The pair (X_t, Y_t) is referred to as the state process associated to the strategy $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$.

One of the main differences between the model introduced by Guo and Zervos (2015) and the approach presented in this paper is in presenting a different framework to execute the initial position y. While Guo and Zervos (2015) assume that for $t \geq 0$ the agent should decide on the number of shares to sell or buy, in this paper, we assume that selling or buying can only occur at some (typically random) times, modeled by the jump times of an independent Poisson process $(N^\gamma_t : t \geq 0)$ with rate $\gamma > 0$. More precisely, the selling and buying strategies are given by

$$\xi^{\mathrm{s}}_t = \int_0^t \nu^{\mathrm{s}}_s \mathrm{d}N^\gamma_s \quad \text{and} \quad \xi^{\mathrm{b}}_t = \int_0^t \nu^{\mathrm{b}}_s \mathrm{d}N^\gamma_s, \tag{5}$$

where v_t^{s}, v_t^{b} are $\mathcal{F}_t$-adapted processes, representing the number of shares sold and bought at time t, respectively. As the agent cannot sell and buy at the same time, it is easy to see

$$\begin{cases} v_t^{\mathrm{s}} = 0, & \text{if } v_t^{\mathrm{b}} > 0, \\ v_t^{\mathrm{b}} = 0, & \text{if } v_t^{\mathrm{s}} > 0, \\ v_t^{\mathrm{s}} = v_t^{\mathrm{b}} = 0, & \text{otherwise.} \end{cases}$$

Within this context, selling/buying shares is necessarily done at discrete periods of time (there cannot be continuous selling and buying) because selling/buying decisions can only occur when the process N^γ has a jump. The set of selling/buying decision times is denoted by $\mathcal{T} = \{T_1, T_2, \dots\}$, and the quantities $T_k - T_{k-1}$, $k \geq 0$ are the inter selling–buying decision times, which are exponentially distributed with mean $1/\gamma$. The number of shares sold and bought at each decision time T_j is denoted by $v_{T_j}^{\mathrm{s}}$ or $v_{T_j}^{\mathrm{b}}$, respectively, with $\Theta^{\mathrm{s,b}} = \{(v_{T_1}^{\mathrm{s}}, v_{T_2}^{\mathrm{s}}, \dots), (v_{T_1}^{\mathrm{b}}, v_{T_2}^{\mathrm{b}}, \dots)\}$ representing a selling/buying strategy via (5); the subset of strategies $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \Xi(y)$, which can be represented as in (5) is denoted by $\mathcal{A}(y)$. For those strategies $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y)$, the operator defined in (4) can be written as

$$\begin{aligned} X_t \circ_{\mathrm{s}} \mathrm{d}\xi_t^{\mathrm{s}} &= \frac{1}{\lambda} X_t (1 - \mathrm{e}^{-\lambda v_t^{\mathrm{s}}}) \mathrm{d}N_t^\gamma = X_t \left(\int_0^{v_t^{\mathrm{s}}} \mathrm{e}^{-\lambda u} \mathrm{d}u \right) \mathrm{d}N_t^\gamma, \\ X_t \circ_{\mathrm{b}} \mathrm{d}\xi_t^{\mathrm{b}} &= \frac{1}{\lambda} X_t (\mathrm{e}^{\lambda v_t^{\mathrm{b}}} - 1) \mathrm{d}N_t^\gamma = X_t \left(\int_0^{v_t^{\mathrm{b}}} \mathrm{e}^{\lambda u} \mathrm{d}u \right) \mathrm{d}N_t^\gamma. \end{aligned} \tag{6}$$

Let $C_{\mathrm{s}}, C_{\mathrm{b}}$ be positive constants representing the transaction costs associated with the selling and buying of shares, respectively. Then, the gains associated with each strategy $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y)$ are given by

$$\int_0^\infty \left(X_t \circ_{\mathrm{s}} \mathrm{d}\xi_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}} \mathrm{d}\xi_t^{\mathrm{b}} - C_{\mathrm{s}} \mathrm{d}\xi_t^{\mathrm{s}} - C_{\mathrm{b}} \mathrm{d}\xi_t^{\mathrm{b}} \right),$$

and the agent's objective is to maximize the expected net present value of the gains,

$$J_{x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) := \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\delta t} \left(X_t \circ_{\mathrm{s}} \mathrm{d}\xi_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}} \mathrm{d}\xi_t^{\mathrm{b}} - C_{\mathrm{s}} \mathrm{d}\xi_t^{\mathrm{s}} - C_{\mathrm{b}} \mathrm{d}\xi_t^{\mathrm{b}} \right) \right], \tag{7}$$

over the set $\mathcal{A}(y)$. The parameter $\delta > 0$ is the discount factor, and we assume that $\delta > \mu$ in order to avoid arbitrage opportunities, as described in Guo and Zervos (2015, Proposition 3.4). Given an initial condition $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, we say that $(\xi^{\mathrm{s}*}, \xi^{\mathrm{b}*}) \in \mathcal{A}(y)$ is an optimal strategy if, and only if,

$$J_{x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \leq J_{x,y}(\xi^{\mathrm{s}*}, \xi^{\mathrm{b}*}), \text{ for all } (\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y).$$

The value function of this stochastic control problem is defined as

$$u_0(x, y) = \sup_{(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y)} J_{x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}). \tag{8}$$

2.1 | Regularity of the model

A remarkable property of our model is *regularity* (see, e.g., Gatheral & Schied, 2013), which is understood as the requirement that, first, the optimization problem (8) has an optimal solution and, second, as the initial position y is positive, it should be expected that the optimal execution strategy does not involve buying decisions along the time needed to liquidate the initial position. In the rest of this section, we elaborate about the second condition, while the first one will be treated in the next section.

Consider strategies where the agent only sells shares, that is, strategies of the form $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y)$ such that $\xi^{\mathrm{b}} \equiv 0$. The subset of strategies $(\xi^{\mathrm{s}}, 0) \in \mathcal{A}(y)$ is denoted by $\mathcal{A}^{\mathrm{s}}(y)$, whose elements are represented only by ξ^{s}, and the value function $u(x, y)$ for this problem is given by

$$u(x, y) := \sup_{(\xi^{\mathrm{s}},0)\in\mathcal{A}(y)} J_{x,y}(\xi^{\mathrm{s}}, 0) = \sup_{\xi^{\mathrm{s}}\in\mathcal{A}^{\mathrm{s}}(y)} \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\delta t} \left(X_t \circ_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}} - C_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_t \right) \right]. \tag{9}$$

It is clear that $u(x, y) \le u_0(x, y)$, because $\mathcal{A}^{\mathrm{s}}(y) \subset \mathcal{A}(y)$; the equality between these value functions is established in the next result.

Proposition 2.1. *Let u_0 and u be the value functions given in (8) and (9), respectively. Then $u = u_0$.*

In order to prove this result, we need to use a technical tool described in the next lemma, whose proof is presented in the Appendix.

Lemma 2.2. *For each $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y)$, there exists $\bar{\xi}^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}(y)$ such that $\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t \le \bar{\xi}^{\mathrm{s}}_t \le \xi^{\mathrm{s}}_t$ for all $t \ge 0$.*

Proof of Proposition 2.1. Recall that we only need to show $u_0(x, y) \le u(x, y)$. Let $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$ be in $\mathcal{A}(y)$ and consider the strategy $\bar{\xi}^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}(y)$ as in Lemma 2.2. Now, we consider the processes $X_t = X^0_t \exp\{-\lambda(\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t)\}$ and $\overline{X}_t = X^0_t \exp\{-\lambda\bar{\xi}^{\mathrm{s}}_t\}$, which satisfy the stochastic differential equations

$$\begin{aligned} dX_t &= \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t - \lambda(X_t \circ_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_t - X_t \circ_{\mathrm{b}} \mathrm{d}\xi^{\mathrm{b}}_t), \\ d\overline{X}_t &= \mu \overline{X}_t \mathrm{d}t + \sigma \overline{X}_t \mathrm{d}W_t - \lambda \overline{X}_t \circ_{\mathrm{s}} \mathrm{d}\bar{\xi}^{\mathrm{s}}_t, \end{aligned} \tag{10}$$

respectively, where $X_t \circ_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_t$, $X_t \circ_{\mathrm{b}} \mathrm{d}\xi^{\mathrm{b}}_t$, and $\overline{X}_t \circ_{\mathrm{s}} \mathrm{d}\bar{\xi}^{\mathrm{s}}_t$ are as in (6). Integrating by parts in $\mathrm{e}^{-\delta(t\wedge\bar{\tau}_m)} X_{t\wedge\bar{\tau}_m}$, where $\bar{\tau}_m := \inf\{t > 0 : X^0_t > m\}$, and using (10), it follows that

$$\begin{aligned} &\int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta t} \left(X_t \circ_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_t - X_t \circ_{\mathrm{b}} \mathrm{d}\xi^{\mathrm{b}}_t - C_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_t - C_{\mathrm{b}} \mathrm{d}\xi^{\mathrm{b}}_t \right) \\ &= \frac{1}{\lambda} \left(x - \mathrm{e}^{-\delta(t\wedge\bar{\tau}_m)} X^0_{t\wedge\bar{\tau}_m} \mathrm{e}^{-\lambda(\xi^{\mathrm{s}}_{t\wedge\bar{\tau}_m} - \xi^{\mathrm{b}}_{t\wedge\bar{\tau}_m})} \right) - \frac{\delta-\mu}{\lambda} \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} X^0_s \mathrm{e}^{-\lambda(\xi^{\mathrm{s}}_s - \xi^{\mathrm{b}}_s)} \mathrm{d}s \\ &\quad - \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} \left(C_{\mathrm{s}} \mathrm{d}\xi^{\mathrm{s}}_s + C_{\mathrm{b}} \mathrm{d}\xi^{\mathrm{b}}_s \right) + \frac{\sigma}{\lambda} \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} X^0_s \mathrm{e}^{-\lambda(\xi^{\mathrm{s}}_s - \xi^{\mathrm{b}}_s)} \mathrm{d}W_s \\ &\le \frac{1}{\lambda} \left(x - \mathrm{e}^{-\delta(t\wedge\bar{\tau}_m)} X^0_{t\wedge\bar{\tau}_m} \mathrm{e}^{-\lambda\bar{\xi}^{\mathrm{s}}_{t\wedge\bar{\tau}_m}} \right) - \frac{\delta-\mu}{\lambda} \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} X^0_s \mathrm{e}^{-\lambda\bar{\xi}^{\mathrm{s}}_s} \mathrm{d}s \\ &\quad - \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} C_{\mathrm{s}} \mathrm{d}\bar{\xi}^{\mathrm{s}}_s + \frac{\sigma}{\lambda} \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} X^0_s \mathrm{e}^{-\lambda(\xi^{\mathrm{s}}_s - \xi^{\mathrm{b}}_s)} \mathrm{d}W_s \\ &= \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} \left(\overline{X}_s \circ_s \mathrm{d}\bar{\xi}^{\mathrm{s}}_s - C_{\mathrm{s}} \mathrm{d}\bar{\xi}^{\mathrm{s}}_s \right) + \frac{\sigma}{\lambda} \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} X^0_s (\mathrm{e}^{-\lambda(\xi^{\mathrm{s}}_s - \xi^{\mathrm{b}}_s)} - \mathrm{e}^{-\lambda\bar{\xi}^{\mathrm{s}}_s}) \mathrm{d}W_s. \end{aligned} \tag{11}$$

Note that $\int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s} X_s^0 (\mathrm{e}^{-\lambda(\xi_s^{\mathrm{s}}-\xi_s^{\mathrm{b}})} - \mathrm{e}^{-\lambda\bar{\xi}_s^{\mathrm{s}}})\mathrm{d}B_s$ is a square-martingale (because by hypothesis $\mathbb{E}[\mathrm{e}^{4\lambda\xi^{\mathrm{b}}_{(t\wedge\bar{\tau}_m)+}}] < \infty$), whose expected value is zero; for more, see Guo and Zervos (2015, p. 293). Then, taking expected values in (11), one has

$$\mathbb{E}_x\left[\int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta t}\left(X_t \circ_{\mathrm{s}} \mathrm{d}\xi_t^{\mathrm{s}} - X_t \circ_{\mathrm{b}} \mathrm{d}\xi_t^{\mathrm{b}} - C_{\mathrm{s}}\mathrm{d}\xi_t^{\mathrm{s}} - C_{\mathrm{b}}\mathrm{d}\xi_t^{\mathrm{b}}\right)\right]$$
$$\leq \mathbb{E}_x\left[\int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s}(\overline{X}_s \circ_s \mathrm{d}\bar{\xi}_s^{\mathrm{s}} - C_{\mathrm{s}}\mathrm{d}\bar{\xi}_s^{\mathrm{s}})\right].$$

Now, letting $t, m \to \infty$ and using the Dominated Convergence Theorem, we get

$$J_{x,y}(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \leq J_{x,y}(\bar{\xi}^{\mathrm{s}}, 0) \leq u(x, y).$$

As this inequality holds for each $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y)$, we conclude that $u_0(x, y) \leq u(x, y)$, completing the proof of this result. □

3 | HJB EQUATIONS AND OPTIMAL EXECUTION

The remainder of this paper is devoted to the presentation of an optimal solution to the execution problem (9), paying particular attention to the structure of this strategy. Roughly, we look for optimal selling strategies with a simple structure, facilitating their practical implementation. One of the main results of the present paper is the existence of an optimal strategy $\hat{\xi}^{\mathrm{s}}$ that has a barrier form in the state space. A *barrier strategy* is described in terms of the remaining number of shares to be sold and the observed price at each period of time. These are compared with a mark, which was decided on from the beginning, and depends on a nonnegative constant F, from now on referred to as a *periodic barrier*.

More precisely, given a periodic barrier $F > 0$, the number of shares to be sold at the ith arrival time T_i of the Poisson process N^γ is

$$\nu_F(T_i) := Y_{T_i} \wedge \frac{1}{\lambda}(\ln X_{T_i} - \ln F)^+, \tag{12}$$

where (X_{T_i}, Y_{T_i}) is the position of the state process at the arrival time T_i and $(\ln X_{T_i} - \ln F)^+ := \max\{0, (\ln X_{T_i} - \ln F)\}$. This type of strategy is denoted by $\xi^{\mathrm{s},F}$, and the set of these strategies is defined denoted by $\mathcal{A}^{\mathrm{s}}_{\mathrm{b}}(y)$, which is clearly a subset of $\mathcal{A}^{\mathrm{s}}(y)$. The corresponding value function for this set of strategies is defined as

$$u_{\mathrm{b}}(x, y) = \sup_{\xi^{\mathrm{s},F} \in \mathcal{A}^{\mathrm{s}}_{\mathrm{b}}(y)} J_{x,y}(\xi^{\mathrm{s},F}, 0), \tag{13}$$

with $J_{x,y}$ as in (7).

In order to relate the value function u_{b} with u defined in (9), it is convenient to provide a brief description of the approach to be followed. In the first step, we will solve (13) using dynamic programming techniques. Noting that the periodic strategies are described by a single parameter, the crucial point consists in proving that there is an F_γ, defined below by (21), for which there exists a smooth solution v to the HJB equation (15) associated with u_{b}; see Proposition 3.2. The second step is to use F_γ and its associated periodic strategy $\xi^{\mathrm{s},F_\gamma}$ built from (12). In the last step, using the HJB equation associated with the original value function u described in (14), it is proved that v satisfies that equation, and that $\xi^{\mathrm{s},F_\gamma}$ is optimal for u within the set $\mathcal{A}^{\mathrm{s}}(y)$, concluding that $u_{\mathrm{b}} = u$; see the verification theorem 3.7.

Remark 3.1. By standard dynamic programming arguments, it is well known that the value functions u and u_{b} are associated to the HJB equations given by

$$\begin{cases} \mathcal{L}w(x,y) + \max\limits_{0\le l\le y}\{\gamma G(x,y,l;w)\} = 0, & \text{for all } x>0 \quad \text{and} \quad y>0, \\ w(x,y)=0, & \text{for all } x>0 \quad \text{and} \quad y=0, \end{cases} \tag{14}$$

$$\begin{cases} \mathcal{L}v(x,y) + \gamma G\left(x, y, \left[y \wedge \frac{1}{\lambda}\ln(x/F_\gamma)^+\right]; v\right) = 0, & \text{for all } x>0 \quad \text{and} \quad y>0, \\ v(x,y)=0, & \text{for all } x>0 \quad \text{and} \quad y=0, \end{cases} \tag{15}$$

where F_γ is a positive constant, which will be determined later on. Here, the operators $\mathcal{L}$ and G are defined by

$$\begin{aligned} \mathcal{L}f(x,y) &:= \frac{1}{2}\sigma^2x^2 f_{xx}(x,y) + \mu x f_x(x,y) - \delta f(x,y), \\ G(x,y,l;f) &:= f(x\mathrm{e}^{-\lambda l}, y-l) - f(x,y) + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda l})x - C_{\mathrm{s}} l. \end{aligned} \tag{16}$$

3.1 | Construction and regularization of the solution v to the HJB equation (15)

Observe that we can simplify the HJB equation (15) depending on the following three different scenarios:

(i) When $x < F_\gamma$, this restriction corresponds to the *waiting region* $\mathcal{W}$ because the price is too low for selling any shares to be optimal, and therefore (15) takes the form

$$\mathcal{L}v(x,y) = 0. \tag{17}$$

(ii) When $F_\gamma \le x < F_\gamma \mathrm{e}^{\lambda y}$, the agent takes an intermediate position of selling $\mathbb{Y}(x) := \frac{1}{\lambda}\ln(x/F_\gamma)$ assets. Now as $\mathrm{e}^{-\lambda \mathbb{Y}(x)} = \frac{F_\gamma}{x}$, (15) can be written as follows:

$$\mathcal{L}_\gamma v(x,y) + \gamma\left[v\big(F_\gamma, y-\mathbb{Y}(x)\big) + \frac{x-F_\gamma}{\lambda} - C_{\mathrm{s}}\mathbb{Y}(x)\right] = 0, \tag{18}$$

where

$$\mathcal{L}_\gamma v(x,y) := \frac{1}{2}\sigma^2x^2 v_{xx}(x,y) + \mu x v_x(x,y) - (\delta+\gamma)v(x,y).$$

(iii) When $x \ge F_\gamma \mathrm{e}^{\lambda y}$, we have that the asset price is sufficiently high that the decision is to execute the complete set of assets available, and then (15) reduces to

$$\mathcal{L}_\gamma v(x,y) + \gamma\left[\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda y})x - C_{\mathrm{s}} y\right] = 0. \tag{19}$$

We will now obtain explicit solutions for each of the three regions described in (20). In order to obtain an explicit form of the solution, we impose the condition that it has to be smooth at the boundary of each region. The proof of the following result is given in the Appendix.

Proposition 3.2. *The HJB equation (15) has a solution v, which belongs to* $\mathrm{C}^{2,1}(\mathbb{R}_+ \times \mathbb{R}_+)$*:*

$$v(x,y) = \begin{cases} 0, & \text{if } y=0 \quad \text{and} \quad x>0,\\ \dfrac{(F_\gamma - C_{\mathrm{s}})(1-\mathrm{e}^{-\lambda n y})x^n}{\lambda n F_\gamma^n}, & \text{if } y>0 \quad \text{and} \quad x<F_\gamma,\\ A_\gamma\left(\dfrac{x}{F_\gamma}\right)^{m_\gamma} - \dfrac{(F_\gamma - C_{\mathrm{s}})\mathrm{e}^{-\lambda n y}x^n}{\lambda n F_\gamma^n} & \\ \quad + \dfrac{\gamma x}{\lambda(\eta+\gamma)} - \dfrac{\gamma C_{\mathrm{s}} \ln x}{\lambda(\delta+\gamma)} + C_\gamma, & \text{if } y>0 \quad \text{and} \quad F_\gamma \le x < F_\gamma \mathrm{e}^{\lambda y},\\ \dfrac{A_\gamma(1-\mathrm{e}^{-\lambda m_\gamma y})x^{m_\gamma}}{F_\gamma^{m_\gamma}} + \dfrac{\gamma x(1-\mathrm{e}^{-\lambda y})}{\lambda(\eta+\gamma)} - \dfrac{\gamma C_{\mathrm{s}} y}{\delta+\gamma}, & \text{if } y>0 \quad \text{and} \quad x \ge F_\gamma \mathrm{e}^{\lambda y}, \end{cases} \tag{20}$$

where $\eta := \delta - \mu$,

$$F_\gamma := \frac{\dfrac{C_{\mathrm{s}}}{\delta+\gamma}\left(\delta - m_\gamma\left(\dfrac{\delta}{n} + \dfrac{\gamma b}{\delta+\gamma}\right)\right)}{\dfrac{\eta}{\eta+\gamma} - \dfrac{m_\gamma}{\delta+\gamma}\left(\dfrac{\delta}{n} - \dfrac{\gamma\mu}{\eta+\gamma}\right)}, \tag{21}$$

$$A_\gamma := \frac{F_\gamma}{\lambda(\delta+\gamma)}\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma}\right) - \frac{C_{\mathrm{s}}}{\lambda(\delta+\gamma)}\left(\frac{\delta}{n} + \frac{\gamma b}{\delta+\gamma}\right), \tag{22}$$

$$C_\gamma := \frac{\gamma(F_\gamma - C_{\mathrm{s}})}{\lambda n(\delta+\gamma)} + \frac{\gamma}{\lambda(\delta+\gamma)}\left(\frac{bC_{\mathrm{s}}}{\delta+\gamma} + C_{\mathrm{s}} \ln F_\gamma - F_\gamma\right), \tag{23}$$

and $b := \frac{1}{2}\sigma^2 - \mu$. *The constants* n *and* m_γ *are the positive and negative solutions to*

$$\frac{1}{2}\sigma^2 l^2 - bl - \delta = 0, \tag{24}$$

$$\frac{1}{2}\sigma^2 l^2 - bl - (\delta+\gamma) = 0, \tag{25}$$

respectively.

Remark 3.3. The fact that $\delta > \mu$ implies that $n > 1$, for all $\sigma \in \mathbb{R}$.

3.2 | Equivalence between the HJB equations

The rest of this paper is dedicated to verifying that the strategy given in (12), with barrier F_γ defined above in (21), is optimal within the set of strategies $\mathcal{A}^{\mathrm{s}}(y)$, and that the function v given in (20) satisfies the HJB equation (14). To this end, we need the following technical results, whose proofs are given in the Appendix in order to succinctly introduce here their main consequences.

Lemma 3.4. *Let a_γ be defined by*

$$a_\gamma := \frac{\frac{1}{\delta+\gamma}\left(\delta - m_\gamma\left(\frac{\delta}{n} + \frac{\gamma b}{\delta+\gamma}\right)\right)}{\frac{\eta}{\eta+\gamma} - \frac{m_\gamma}{\delta+\gamma}\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma}\right)}, \tag{26}$$

where $\gamma > 0$. Then, for each $\gamma > 0$, $1 < a_\gamma < \frac{n}{n-1}$, and it satisfies the following asymptotic limits:

$$\begin{cases} a_\gamma \to 1, & \text{when } \gamma \to 0, \\ a_\gamma \to \frac{n}{n-1}, & \text{when } \gamma \to \infty. \end{cases}$$

Proposition 3.5. *Let v be as in (20). Then, for each $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$,*

$$\max_{0\le l\le y} G(x, y, l; v) = G\left(x, y, \left[y \wedge \frac{1}{\lambda}\ln(x/F_\gamma)^+\right]; v\right), \tag{27}$$

with G defined as in (16).

Remark 3.6. Note that putting together Propositions 3.2 and 3.5, it follows immediately that v is a solution to the HJB equation (14).

The next result identifies the solution of the HJB equation (20) with the value function u, providing also an optimal strategy within the set $\mathcal{A}^{\mathrm{s}}(y)$.

Theorem 3.7 (Verification Theorem)**.** *Consider the periodic optimal execution problem formulated in Section 2 and the function v defined by (20). Then, v agrees with the value function u of the periodic stochastic control problem. In particular,*

$$u(x, y) = \sup_{\xi^{\mathrm{s}}\in\mathcal{A}^{\mathrm{s}}(y)} J_{x,y}(\xi^{\mathrm{s}}, 0) = v(x, y) \qquad \text{for all } (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Now, define the strategy $\xi_t^{\mathrm{s}*} = \int_0^t \nu_s^* dN_s^\gamma$ with

$$\nu_{T_i}^* = Y_{T_i}^* \wedge \frac{1}{\lambda}\left(\ln X_{T_i}^* - \ln F_\gamma\right)^+, \tag{28}$$

where F_γ is as in (21), $\mathcal{T} = \{T_i\}_{i=1}^\infty$ is the set of arrival times of the Poisson process N^γ, and (X^*, Y^*) is the state process associated with the liquidation strategy $\xi^{\mathrm{s}*}$. Then, the following statements hold:

- **(i)** *If $\mu - \frac{1}{2}\sigma^2 \ge 0$, then $\xi^{\mathrm{s}*}$ is an optimal periodic liquidation strategy.*
- **(ii)** *If $\mu - \frac{1}{2}\sigma^2 < 0$, then $\xi^{\mathrm{s}*}$ is not an optimal periodic liquidation strategy. So, if we define*

$$\xi_t^{\mathrm{s}*j} = \xi_t^{\mathrm{s}*} 1_{\{t\le j\}} + y 1_{\{j<t\}}, \qquad \text{for } t > 0 \quad \text{and} \quad j \ge 1, \tag{29}$$

then $\{\xi^{\mathrm{s}*j}\}_{j=1}^\infty$ is a sequence of ε-optimal periodic strategies.

Proof. Let us take $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$ an initial condition, $\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}(y)$, described by the following selling strategy: $\Theta = \{\nu_{T_1}, \nu_{T_2}, \dots\}$, where $\mathcal{T} = \{T_1, T_2, \dots\}$ is the set of selling times, and $(\tau_m)_{m\in\mathbb{N}}$ the sequence of stopping times defined by $\tau_m := \inf\{t > 0 : X_t > m\}$. Using the Itô–Tanaka–Meyer formula and the left continuity of the processes X and Y, we have

$$\mathrm{e}^{-\delta(t\wedge\tau_m)}v(X_{t\wedge\tau_m},Y_{t\wedge\tau_m}) = v(x,y) + \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}\mathcal{L}v(X_s,Y_s)\mathrm{d}s + M_{t\wedge\tau_m} + \sum_{0\le s\le t\wedge\tau_m} \mathrm{e}^{-\delta s}[v(X_{s+},Y_{s+}) - v(X_s,Y_s)],$$

where $M_{t\wedge\tau_m} := \sigma\int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}X_s w_x(X_s,Y_s)\mathrm{d}W_s$. On the other hand,

$$\begin{aligned}
&\sum_{0\le s\le t\wedge\tau_m} \mathrm{e}^{-\delta s}[v(X_{s+},Y_{s+}) - v(X_s,Y_s)]\\
&= \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}\left[v(X_{s-}\mathrm{e}^{-\lambda\nu_s},Y_{s-}-\nu_s) - v(X_{s-},Y_{s-})\right]\mathrm{d}N_s^\gamma\\
&= \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}G(X_{s-},Y_{s-},\nu_s;v)\mathrm{d}N_s^\gamma - \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\nu_s})X_{s-} - C_\mathrm{s}\nu_s\right]\mathrm{d}N_s^\gamma\\
&= H_{t\wedge\tau_m} + \int_0^{t\wedge\tau_m} \gamma\mathrm{e}^{-\delta s}G(X_{s-},Y_{s-},\nu_s;v)\mathrm{d}s - \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\nu_s})X_{s-} - C_\mathrm{s}\nu_s\right]\mathrm{d}N_s^\gamma,
\end{aligned}$$

with G as in (27), $H_{t\wedge\tau_m} := \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}G(X_{s-},Y_{s-},\nu_s;v)\mathrm{d}\widetilde{N}_s^\gamma$, and $\widetilde{N}^\gamma$ the compensated Poisson process. Hence, putting these pieces together and observing that

$$\mathcal{L}v(X_s,Y_s) + \gamma G(X_{s-},Y_{s-},\nu_s;v) \le \mathcal{L}v(X_s,Y_s) + \max_{0\le l\le y}\{\gamma G(X_{s-},Y_{s-},l;v)\} = 0,$$

we obtain

$$\begin{aligned}
\mathrm{e}^{-\delta(t\wedge\tau_m)}v(X_{t\wedge\tau_m},Y_{t\wedge\tau_m}) &= v(x,y) + M_{t\wedge\tau_m} + H_{t\wedge\tau_m}\\
&\quad - \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\nu_s})X_{s-} - C_\mathrm{s}\nu_s\right]\mathrm{d}N_s^\gamma\\
&\quad + \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}[\mathcal{L}v(X_s,Y_s) + \gamma G(X_{s-},Y_{s-},\nu_s;v)]\mathrm{d}s\\
&\le v(x,y) + M_{t\wedge\tau_m} + H_{t\wedge\tau_m}\\
&\quad - \int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\nu_s})X_{s-} - C_\mathrm{s}\nu_s\right]\mathrm{d}N_s^\gamma.
\end{aligned} \tag{30}$$

From (20) it is not difficult to see that there exists a positive constant K such that

$$|v(x,y)| \le K(1+x+y), \quad \text{for all } (x,y)\in\mathbb{R}_+\times\mathbb{R}_+.$$

Hence, it follows that the processes $(M_{t\wedge\tau_m}; t\ge 0)$ and $(H_{t\wedge\tau_m}; t\ge 0)$ are zero-mean $\mathbb{P}$-martingales. Then, taking expectations in (30), we obtain

$$\begin{aligned}
v(x,y) &\ge \mathbb{E}_x\left[\mathrm{e}^{-\delta(t\wedge\tau_m)}v(X_{t\wedge\tau_m},Y_{t\wedge\tau_m})\right]\\
&\quad + \mathbb{E}_x\left[\int_0^{t\wedge\tau_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda\nu_s})X_{s-} - C_\mathrm{s}\nu_s\right]\mathrm{d}N_s^\gamma\right].
\end{aligned} \tag{31}$$

Now, from the expression of the process X_t in (3) and recalling that $\delta > \mu$, we have that

$$\lim_{t,m\to\infty} \mathbb{E}_x[\mathrm{e}^{-\delta(t\wedge\tau_m)} v(X_{t\wedge\tau_m}, Y_{t\wedge\tau_m})] \leq \lim_{t,m\to\infty} \mathbb{E}_x[\mathrm{e}^{-\delta(t\wedge\tau_m)} K(1 + X_{t\wedge\tau_m} + Y_{t\wedge\tau_m})] \qquad (32)$$

$$\leq \lim_{t,m\to\infty} \mathbb{E}_x[\mathrm{e}^{-\delta(t\wedge\tau_m)} K(1 + X_{t\wedge\tau_m} + y)] = 0.$$

Using (1), we note that

$$\int_0^\infty \nu_s \mathrm{d}N_s^\gamma = \lim_{t\to\infty} \xi_t^{\mathrm{s}} \leq y. \qquad (33)$$

Then, letting $m, t \to \infty$ in (31), using (32), (33), and the Monotone Convergence Theorem, we obtain

$$v(x,y) \geq \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda\nu_s})X_{s-} - C_{\mathrm{s}}\nu_s\right]\mathrm{d}N_s^\gamma\right]$$

$$= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\delta s}\left[X_s \circ_{\mathrm{s}} \mathrm{d}\xi_s^{\mathrm{s}} - C_{\mathrm{s}}\mathrm{d}\xi_s^{\mathrm{s}}\right]\right].$$

Taking the maximum over all $\xi^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}(y)$, we conclude that $u(x,y) \leq v(x,y)$. Let (X^*, Y^*) be the state process associated with the liquidation strategy $\xi^{\mathrm{s}*}$, given by $\xi_t^{\mathrm{s}*} = \int_0^t \nu_s^* \mathrm{d}N_s^\gamma$, with ν^* as in (28). Note that ξ^{s} is admissible as long as $\lim_{t\to\infty} \xi_t^{\mathrm{s}} = y$. We can easily check, using (3), that this is indeed the case if, and only if, $\mu - \frac{1}{2}\sigma^2 \geq 0$, because $\limsup_{t\to\infty} X_t^0 = \infty$. Proceeding in a similar way as in (30), we get that

$$\mathrm{e}^{-\delta(t\wedge\bar{\tau}_m)} v\left(X^*_{t\wedge\bar{\tau}_m}, Y^*_{t\wedge\bar{\tau}_m}\right) = v(x,y) + M^*_{t\wedge\bar{\tau}_m} + H^*_{t\wedge\bar{\tau}_m}$$

$$- \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda\nu_s})X^*_{s-} - C_{\mathrm{s}}\nu_s\right]\mathrm{d}N_s^\gamma$$

$$+ \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s}\left[\mathcal{L}v\left(X_s^*, Y_s^*\right)ds + \gamma G\left(X^*_{s-}, Y^*_{s-}, \nu_s^*; v\right)\right]\mathrm{d}s,$$

where $\bar{\tau}_m := \inf\{t > 0 : X_t^* > m\}$. Now, from the construction of v, we know that it is the solution to (15). Therefore,

$$\int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s}\left[\mathcal{L}v\left(X_s^*, Y_s^*\right)ds + \gamma G\left(X^*_{s-}, Y^*_{s-}, \nu_s^*; v\right)\right]\mathrm{d}s = 0.$$

Hence,

$$\mathrm{e}^{-\delta(t\wedge\bar{\tau}_m)} v\left(X^*_{t\wedge\bar{\tau}_m}, Y^*_{t\wedge\bar{\tau}_m}\right) = v(x,y) + M^*_{t\wedge\bar{\tau}_m} + H^*_{t\wedge\bar{\tau}_m}$$

$$- \int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda\nu_s^*})X^*_{s-} - C_{\mathrm{s}}\nu_s^*\right]\mathrm{d}N_s^\gamma.$$

Now, taking expectations in the previous identity,

$$v(x,y) = \mathbb{E}_x\left[\mathrm{e}^{-\delta(t\wedge\bar{\tau}_m)} v\left(X^*_{t\wedge\tau_m}, Y^*_{t\wedge\bar{\tau}_m}\right)\right]$$

$$+ \mathbb{E}_x\left[\int_0^{t\wedge\bar{\tau}_m} \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda\nu_s^*})X^*_{s-} - C_{\mathrm{s}}\nu_s^*\right]\mathrm{d}N_s^\gamma\right]. \qquad (34)$$

Letting $m, t \to \infty$ in (34), and using (32), (33), and the Monotone Convergence Theorem, we get

$$v(x,y) \le \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\delta s}\left[\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda v_s*})X^*_{s-} - C_\mathrm{s} v^*_s\right]\mathrm{d}N^\gamma_s\right] = J_{x,y}(\xi^{\mathrm{s}*},0) \le u(x,y),$$

which implies the result. For the case when $\mu - \frac{1}{2}\sigma^2 < 0$, let us take $(\overline{X}^*, \overline{Y}^*)$ as the state process associated with the strategy $\xi^{\mathrm{s}*j}$ given by (29). We can check that $\xi^{\mathrm{s}*j}$ has payoff

$$J_{x,y}(\xi^{\mathrm{s}*j},0) = \mathbb{E}_x\left[\int_0^j \mathrm{e}^{-\delta t}\left[\overline{X}^*_t \circ_\mathrm{s} \mathrm{d}\xi^{\mathrm{s}*}_t - C_\mathrm{s}\mathrm{d}\xi^{\mathrm{s}*}_t\right]\right] + \frac{1}{\lambda}\mathbb{E}_x\left[\overline{X}^*_{\tilde{\tau}_j}\left[1-\mathrm{e}^{-\lambda(y-\overline{Y}^*_{\tilde{\tau}_j})}\right]\right], \tag{35}$$

where $\tilde{\tau}_j = \inf\{T_i > 0 : T_i > j\}$. Also, we see that (34) is satisfied if $t = j$, (X^*, Y^*) is replaced by $(\overline{X}^*, \overline{Y}^*)$, and τ_m with the stopping time $\bar{\tau}_m = \inf\{t > 0 : \overline{X}^*_t > m\}$. Then, letting $m \to \infty$ in (34), it follows that

$$\mathbb{E}_x\left[\int_0^j \mathrm{e}^{-\delta t}\left[\overline{X}^*_t \circ_\mathrm{s} \mathrm{d}\xi^{\mathrm{s}*}_t - C_\mathrm{s}\mathrm{d}\xi^{\mathrm{s}*}_t\right]\right] = v(x,y) - \mathbb{E}_x\left[\mathrm{e}^{-\delta j} v\left(\overline{X}^*_j, \overline{Y}^*_j\right)\right]. \tag{36}$$

Now, using (36) in (35),

$$J_{x,y}(\xi^{\mathrm{s}*j},0) = v(x,y) - \mathbb{E}_x\left[\mathrm{e}^{-\delta j} v\left(\overline{X}^*_j, \overline{Y}^*_j\right)\right] + \frac{1}{\lambda}\mathbb{E}_x\left[\overline{X}^*_{\tilde{\tau}_j}\left[1-\mathrm{e}^{-\lambda(y-\overline{Y}^*_{\tilde{\tau}_j})}\right]\right].$$

Therefore, noting that the right-hand side of this expression converges to $v(x,y)$ as $j \to \infty$ allows us to establish that $\{\xi^{\mathrm{s}*j}\}_{j=1}^\infty$ is a sequence of ε-optimal strategies. □

Remark 3.8. Note that, as a consequence of Proposition 3.2 and Lemma 3.4, we obtain that

$$\lim_{\gamma\to\infty} F_\gamma = \frac{nC_\mathrm{s}}{n-1} := F_\infty, \quad \lim_{\gamma\to\infty} A_\gamma = 0, \quad \text{and} \quad \lim_{\gamma\to\infty} C_\gamma = \frac{F_\infty}{\lambda n^2} + \frac{1}{\lambda}\left(C_\mathrm{s}\ln F_\infty - F_\infty\right).$$

Recall that $F_\gamma, C_\gamma, A_\gamma$ are given in (21)–(23), respectively. Hence, straightforward computations yield

$$\lim_{\gamma\to\infty} v(x,y) = \begin{cases} 0, & \text{if } y = 0 \text{ and } x > 0,\\ \dfrac{(1-\mathrm{e}^{-\lambda ny})x^n}{\lambda n^2 F_\infty^{n-1}}, & \text{if } y > 0 \text{ and } x < F_\infty,\\ \dfrac{F_\infty}{\lambda n^2}\left(1-\left(\dfrac{x\mathrm{e}^{-\lambda y}}{F_\infty}\right)^n\right) + \dfrac{x-F_\infty}{\lambda} - \dfrac{C_\mathrm{s}}{\lambda}\ln\dfrac{x}{F_\infty}, & \text{if } y > 0 \text{ and } F_\infty \le x < F_\infty \mathrm{e}^{\lambda y},\\ \dfrac{x(1-\mathrm{e}^{-\lambda y})}{\lambda} - C_\mathrm{s} y, & \text{if } y > 0 \text{ and } x \ge F_\infty \mathrm{e}^{\lambda y}. \end{cases}$$

These asymptotic limits allow us to recover the value function for the case of singular strategies for the optimal execution problem obtained by Guo and Zervos (2015) in Proposition 5.1.

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APPENDIX A: PROOFS OF SOME TECHNICAL RESULTS

Proof of Lemma 2.2. Let us take $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}}) \in \mathcal{A}(y)$ and $\Theta^{\mathrm{s,b}} = \{(\nu^{\mathrm{s}}_{T_1}, \nu^{\mathrm{s}}_{T_2}, \dots), (\nu^{\mathrm{b}}_{T_1}, \nu^{\mathrm{b}}_{T_2}, \dots)\}$ the selling/buying strategy associated with $(\xi^{\mathrm{s}}, \xi^{\mathrm{b}})$. Let $\mathcal{T}^{\mathrm{s}} \subset \mathcal{T}$ be the subset of decision times whose elements κ_i are given as follows:

$$\begin{aligned}\kappa_1 &:= \inf\{T_j \in \mathcal{T} : \nu^s_{T_j} > 0\},\\ \kappa_i &:= \inf\{T_j \in \mathcal{T} : T_j > \kappa_{i-1} \quad \text{and} \quad \nu^s_{T_j} > 0\}, \text{ for } i \in \{2, 3, \dots\}.\end{aligned}$$

From this, we see

$$\begin{cases} \nu^{\mathrm{b}}_{T_j} = 0 < \nu^{\mathrm{s}}_{T_j}, & \text{if } T_j = \kappa_i,\\ \nu^{\mathrm{s}}_{T_j} = 0 \le \nu^{\mathrm{b}}_{T_j}, & \text{if } \kappa_{i-1} < T_j < \kappa_i,\end{cases}$$

for $i \in \{1, 2, 3, \dots\}$, with $\kappa_0 = 0$. Let $\alpha_{i,j}$ be defined by

$$\alpha^j_i := \left(\sum_{\iota=i}^{j}\left(\nu^{\mathrm{s}}_{\kappa_\iota} - \sum_{k=1}^{\infty} \nu^{\mathrm{b}}_{T_k} 1_{\{\kappa_{\iota-1} < T_k \le \kappa_\iota\}}\right)\right)^+, \quad \text{with} \quad i, j \in \{1, 2, \dots\} \quad \text{and} \quad i \le j.$$

We construct $\bar{\nu}^{\mathrm{s}}$ as follows:

(i) If $T_j \notin \mathcal{T}^{\mathrm{s}}$, then $\bar{\nu}^{\mathrm{s}}_{T_j} = 0$.

(ii) If $T_j = \kappa_1$, then $\bar{\nu}^{\mathrm{s}}_{\kappa_1} := \alpha_{1,1}$.

(iii) If $T_j = \kappa_2$,

$$\bar{\nu}^{\mathrm{s}}_{\kappa_2} := \begin{cases} \alpha^2_2, & \text{if } \bar{\nu}^{\mathrm{s}}_{\kappa_1} > 0,\\ \alpha^2_1, & \text{if } \bar{\nu}^{\mathrm{s}}_{\kappa_1} = 0.\end{cases}$$

(iv) If $T_j = \kappa_3$,

$$\bar{v}^{\mathrm{s}}_{\kappa_3} := \begin{cases} \alpha^3_3, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_1} \geq 0 \quad \text{and} \quad \bar{v}^{\mathrm{s}}_{\kappa_2} > 0, \\ \alpha^3_2, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_1} > 0 \quad \text{and} \quad \bar{v}^{\mathrm{s}}_{\kappa_2} = 0, \\ \alpha^3_1, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_\rho} = 0, \quad \text{for } \rho \in \{1, 2\}. \end{cases}$$

(v) Recursively, if $T_j = \kappa_i$, with $i \in \{3, 4, \dots\}$,

$$\bar{v}^{\mathrm{s}}_{\kappa_i} := \begin{cases} \alpha^i_i, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_{i-1}} > 0 \quad \text{and} \quad \bar{v}^{\mathrm{s}}_{\kappa_\rho} \geq 0 \quad \text{for } \rho \in \{1, \dots, i-2\}, \\ \alpha^i_{i-1}, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_{i-1}} = 0,\ \bar{v}^{\mathrm{s}}_{\kappa_{i-2}} > 0 \quad \text{and} \quad \bar{v}^{\mathrm{s}}_{\kappa_\rho} \geq 0 \quad \text{for } \rho \in \{1, \dots, i-3\}, \\ \vdots & \\ \alpha^i_{i-(p-2)}, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_\rho} = 0 \quad \text{for } \rho \in \{i-(p-2), \dots, i-1\},\ \bar{v}^{\mathrm{s}}_{\kappa_{i-(p-1)}} > 0 \quad \text{and} \\ & \bar{v}^{\mathrm{s}}_{\kappa_\rho} \geq 0 \quad \text{for } \rho \in \{1, \dots, i-p\}, \\ \vdots & \\ \alpha^i_2, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_\rho} = 0 \quad \text{for } \rho \in \{2, \dots, i-1\}, \quad \text{and} \quad \bar{v}^{\mathrm{s}}_{\kappa_1} > 0, \\ \alpha^i_1, & \text{if } \bar{v}^{\mathrm{s}}_{\kappa_\rho} = 0, \quad \text{with} \quad \rho \in \{1, \dots, i-1\}. \end{cases}$$

Now, we take the selling strategy $\bar{\xi}^{\mathrm{s}}$ as

$$\bar{\xi}^{\mathrm{s}}_t := \int_0^t \bar{v}^{\mathrm{s}}_s \mathrm{d}N^\gamma_s = \sum_{j=1}^{\infty} \bar{v}^{\mathrm{s}}_{T_j} 1_{\{T_j \leq t\}}, \quad \text{for } t \geq 0,$$

and define $\overline{Y}_t := y - \bar{\xi}^{\mathrm{s}}_t$ for all $t \geq 0$. Note that $\bar{\xi}^{\mathrm{s}}_t = \xi^{\mathrm{s}}_t = 0$ when $0 \leq t < \kappa_1$. Then, $\overline{Y}_t = y - \bar{\xi}^{\mathrm{s}}_t \geq 0$ and

$$\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t = -\sum_{j=1}^{\infty} v^{\mathrm{b}}_{T_j} 1_{\{T_j \leq t\}} \leq 0 = \bar{\xi}^{\mathrm{s}}_t.$$

Let us take $t \in [\kappa_1, \kappa_2)$. If $\bar{v}^{\mathrm{s}}_{\kappa_1} > 0$, this means that the agent sold $\bar{v}^{\mathrm{s}}_{\kappa_1}$ shares, which are a fraction (or the total) of the agent's y initial shares. Otherwise, the agent only sold a fraction (or the total) of $\sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{0 < T_k \leq \kappa_1\}}$ and $\sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{0 < T_k \leq \kappa_1\}} - v^{\mathrm{s}}_{\kappa_1}$ are accumulated for the next occasion when the agent decides to sell. Then, $\overline{Y}_t = y - \bar{v}^{\mathrm{s}}_{\kappa_1} \geq 0$. On the other hand,

$$\xi^{\mathrm{s}}_t = v^{s}_{\kappa_1} \geq \left(v^{\mathrm{s}}_{\kappa_1} - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{0 < T_k \leq \kappa_1\}} \right)^+ = \bar{v}^{\mathrm{s}}_{\kappa_1} = \bar{\xi}^{\mathrm{s}}_t,$$

$$\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t = v^{\mathrm{s}}_{\kappa_1} - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{0 < T_k \leq \kappa_1\}} - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{\kappa_1 < T_k \leq t\}} \leq \bar{v}_{\kappa_1} = \bar{\xi}^{\mathrm{s}}_t.$$

Let us take $t \in [\kappa_2, \kappa_3)$. If $\bar{v}^{\mathrm{s}}_{\kappa_2} > 0$, in the same way as in the above case, we have that the agent sold $\bar{v}^{\mathrm{s}}_{\kappa_2}$ shares, which are a fraction (or the total) of $y - \bar{v}^{\mathrm{s}}_{\kappa_1}$. Otherwise, the agent only sold a fraction (or the total) of $\sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{\kappa_1 < T_k \leq \kappa_2\}}$, or, a fraction of $\sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{0 < T_k \leq \kappa_2\}} - v^{\mathrm{s}}_{\kappa_1}$. Then, $\sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{\kappa_1 < T_k \leq \kappa_2\}} - v^{\mathrm{s}}_{\kappa_2}$, or, $\sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{0 < T_k \leq \kappa_2\}} - (v^{\mathrm{s}}_{\kappa_1} + v^{\mathrm{s}}_{\kappa_2})$ are accumulated for the next occasion when the agent decides to sell. Then, $\overline{Y}_t = y - (\bar{v}^{\mathrm{s}}_{\kappa_1} + \bar{v}^{\mathrm{s}}_{\kappa_2}) \geq 0$. On the other hand,

$$\xi^{\mathrm{s}}_t = v^{\mathrm{s}}_{\kappa_1} + v^{\mathrm{s}}_{\kappa_2} \geq \bar{v}^{\mathrm{s}}_{\kappa_1} + \bar{v}^{\mathrm{s}}_{\kappa_2} = \bar{\xi}^{\mathrm{s}}_t,$$

$$\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t = v^{\mathrm{s}}_{\kappa_1} - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{0<T_k\leq\kappa_1\}}$$

$$+ v^{\mathrm{s}}_{\kappa_2} - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{\kappa_1<T_k\leq\kappa_2\}} - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{\kappa_2<T_k\leq t\}} \leq \bar{v}_{\kappa_1} + \bar{v}_{\kappa_2} = \bar{\xi}^{\mathrm{s}}_t.$$

Recursively, we can see that if $\kappa_{i-1} \leq t < \kappa_i$, with $i \in \{3, 4, \dots\}$, then $\overline{Y}_t = y - \sum_{\iota=1}^{i-1} \bar{v}^{\mathrm{s}}_{\kappa_\iota} \geq 0$ and

$$\xi^{\mathrm{s}}_t = \sum_{\iota=1}^{i-1} v^{s}_{\kappa_\iota} \geq \sum_{\iota=1}^{i-1} \bar{v}^{\mathrm{s}}_{\kappa_\iota} = \bar{\xi}^{\mathrm{s}}_t,$$

$$\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t = \sum_{\iota=1}^{i-1} \left(v^{\mathrm{s}}_{\kappa_\iota} - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{\kappa_{\iota-1}<T_k\leq\kappa_\iota\}} \right) - \sum_{k=1}^{\infty} v^{\mathrm{b}}_{T_k} 1_{\{\kappa_{i-1}<T_k\leq t\}} \leq \sum_{\iota=1}^{i-1} \bar{v}^{\mathrm{s}}_{\kappa_\iota} = \bar{\xi}^{\mathrm{s}}_t.$$

Therefore, by what was seen previously, we conclude $\bar{\xi}^{\mathrm{s}} \in \mathcal{A}^{\mathrm{s}}(y)$ and $\xi^{\mathrm{s}}_t - \xi^{\mathrm{b}}_t \leq \bar{\xi}^{\mathrm{s}}_t \leq \xi^{\mathrm{s}}_t$ for all $t \geq 0$. □

Proof of Proposition 3.2 (Construction of (20)). The proof of this result shall be given in two parts. In the first part, by smooth fit arguments, the function v which is a solution to the HJB equation (15) is constructed. In the second part, we prove that v is in $\mathrm{C}^{2,1}(\mathbb{R}_+ \times \mathbb{R}_+)$. Let $x < F_\gamma$ and consider Equation (17). In this case, the only solution that remains bounded as $x \downarrow 0$ is given by

$$v(x, y) = A_1(y)x^n, \tag{A.1}$$

where n is the positive solution to (24). In order to find the form of the function $A_1(y)$ that appears above, we study the behavior of the solution $v(x, y) = A_1(y)x^n$ along the boundary $x = F_\gamma$. Now, we look for a solution that is continuously differentiable at the boundary $x = F_\gamma$. Evaluating (A.1) on the left-hand side of the equality in (18), and recalling that $\mathbb{Y}(x) = \dfrac{1}{\lambda}\ln(x/F_\gamma)$, we obtain

$$\mathcal{L}_\gamma v(x, y) + \gamma \left[v(F_\gamma, y - \mathbb{Y}(x)) + \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda \mathbb{Y}(x)})x - C_{\mathrm{s}}\mathbb{Y}(x) \right]$$

$$= -\gamma A_1(y)x^n + \gamma A_1(y - \mathbb{Y}(x))F^n_\gamma + \frac{\gamma(x - F_\gamma)}{\lambda} - \gamma C_{\mathrm{s}}\mathbb{Y}(x) =: K(x, y).$$

By differentiating with respect to x, we get

$$K_x(x, y) = -\gamma n A_1(y)x^{n-1} - \gamma A_1'(y - \mathbb{Y}(x))\frac{F^n_\gamma}{\lambda x} + \frac{\gamma}{\lambda} - \gamma\frac{C_{\mathrm{s}}}{\lambda x}. \tag{A.2}$$

In order for the solution to be continuously differentiable at the boundary, we take $x = F_\gamma$ in (A.2), and note that

$$-\gamma n A_1(y)F^{n-1}_\gamma - \gamma A_1'(y)\frac{F^n_\gamma}{\lambda F_\gamma} + \frac{\gamma}{\lambda} - \gamma\frac{C_{\mathrm{s}}}{\lambda F_\gamma} = 0,$$

where the equality follows because (18) holds in $x = F_\gamma$. The above equation is equivalent to the following ordinary differential equation for A_1:

$$A_1'(y)F_\gamma^n = -\lambda n A_1(y)F_\gamma^n + F_\gamma - C_\mathrm{s}.$$

The solution of this equation is given by

$$A_1(y) = \frac{(F_\gamma - C_\mathrm{s})}{\lambda n F_\gamma^n}(1 - \mathrm{e}^{-\lambda n y}),$$

which implies that when $x < F_\gamma$, the solution to the HJB equation (15) is given by

$$v(x, y) = \frac{(F_\gamma - C_\mathrm{s})}{\lambda n F_\gamma^n}(1 - \mathrm{e}^{-\lambda n y})x^n. \tag{A.3}$$

Now we look for the solution of the HJB equation within the region $F \le x < F_\gamma \mathrm{e}^{\lambda y}$. As

$$v(F_\gamma -, y - \mathbb{Y}(x)) = \frac{(F_\gamma - C_\mathrm{s})}{\lambda n}\left(1 - \frac{x^n}{F_\gamma^n}\mathrm{e}^{-\lambda n y}\right),$$

Equation (18) can be rewritten as

$$\mathcal{L}_\gamma v(x, y) + \gamma\left[\frac{F_\gamma - C_\mathrm{s}}{\lambda n}\left(1 - \frac{x^n}{F_\gamma^n}\mathrm{e}^{-\lambda n y}\right) + \frac{x - F_\gamma}{\lambda} - C_\mathrm{s}\mathbb{Y}(x)\right] = 0. \tag{A.4}$$

In order to find the solution to this equation, we look first at the following set of equations:

$$\mathcal{L}_\gamma v_1(x, y) + \gamma\frac{(F_\gamma - C_\mathrm{s})}{\lambda n}\left(1 - \frac{x^n}{F_\gamma^n}\mathrm{e}^{-\lambda n y}\right) = 0,$$

$$\mathcal{L}_\gamma v_2(x, y) + \gamma\left[\frac{(x - F_\gamma)}{\lambda} - C_\mathrm{s}\mathbb{Y}(x)\right] = 0.$$

The solutions to the previous equations are

$$v_1(x, y) = -\frac{(F_\gamma - C_\mathrm{s})}{\lambda n F_\gamma^n}\mathrm{e}^{-\lambda n y}x^n + \frac{\gamma(F_\gamma - C_\mathrm{s})}{\lambda n(\delta + \gamma)},$$

$$v_2(x, y) = \frac{\gamma}{\lambda(\eta + \gamma)}x - \frac{\gamma C_\mathrm{s}}{\lambda(\delta + \gamma)}\ln x + C,$$

with $\eta = \delta - \mu$ and $C := \frac{\gamma}{\lambda(\delta + \gamma)}\left(\frac{bC_\mathrm{s}}{\delta + \gamma} + C_\mathrm{s}\ln F_\gamma - F\right)$. Hence, the solution to (A.4) that remains bounded for large values of γ is

$$v(x, y) = A_2(y)x^{m_\gamma} - \frac{(F_\gamma - C_\mathrm{s})}{\lambda n F_\gamma^n}\mathrm{e}^{-\lambda n y}x^n + \frac{\gamma}{\lambda(\eta + \gamma)}x - \frac{\gamma C_\mathrm{s}}{\lambda(\delta + \gamma)}\ln x + \frac{\gamma(F_\gamma - C_\mathrm{s})}{\lambda n(\delta + \gamma)} + C,$$

for some function $A_2 : \mathbb{R}_+ \longrightarrow \mathbb{R}$. Recall that m_γ is the negative solution to (25). As u satisfies $u(F_\gamma -, y) = u(F_\gamma +, y)$, we conclude that for each $F_\gamma \le x < F_\gamma \mathrm{e}^{\lambda y}$, the solution u to Equation (15) has the expression

$$v(x,y) = A_\gamma\left(\frac{x}{F_\gamma}\right)^{m_\gamma} - \frac{(F_\gamma - C_s)}{\lambda n F_\gamma^n} e^{-\lambda n y} x^n + \frac{\gamma}{\lambda(\eta+\gamma)}x - \frac{\gamma C_s}{\lambda(\delta+\gamma)}\ln x + \frac{\gamma(F_\gamma - C_s)}{\lambda n(\delta+\gamma)} + C, \tag{A.5}$$

where A_γ is as in (22). Finally, in order to obtain the value of the optimal barrier F_γ, we look for a solution v such that v_x is continuous at $x = F_\gamma$. As $v_x(F_\gamma -, y) = v_x(F_\gamma +, y)$, we get

$$F_\gamma - C_s = \frac{F_\gamma m_\gamma}{\delta+\gamma}\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma}\right) - \frac{C_s m_\gamma}{\delta+\gamma}\left(\frac{\delta}{n} + \frac{\gamma b}{\delta+\gamma}\right) + \frac{\gamma F_\gamma}{\eta+\gamma} - \frac{\gamma C_s}{\delta+\gamma}.$$

This implies that F_γ is given as in (21).

Now, let us find the general solution to (15) for the region $x \geq F_\gamma e^{\lambda y}$. We have that a particular solution to (19) is given by

$$\frac{\gamma}{\lambda(\eta+\gamma)}x(1 - e^{-\lambda y}) - \frac{\gamma}{\delta+\gamma}C_s y.$$

Then, the solution to Equation (19) that remains bounded for large values of γ is given by

$$v(x,y) = A_3(y)x^{m_\gamma} + \frac{\gamma}{\lambda(\eta+\gamma)}x(1 - e^{-\lambda y}) - \frac{\gamma}{\delta+\gamma}C_s y, \tag{A.6}$$

for some function $A_3(y) : \mathbb{R}_+ \longrightarrow \mathbb{R}$. Finally, to find expressions for the function $A_3(y)$ involved in (A.6), we ask that v be continuous at $x = F_\gamma e^{\lambda y}$. Then, as $v(F_\gamma e^{\lambda y}-, y) = v(F_\gamma e^{\lambda y}+, y)$, it is not difficult to check that

$$A_3(y) = \frac{A_\gamma(1 - e^{-\lambda m_\gamma y})}{F_\gamma^{m_\gamma}}.$$

Therefore, for each $x \geq F_\gamma e^{\lambda y}$, the solution u has the following expression:

$$v(x,y) = \frac{A_\gamma(1 - e^{-\lambda m_\gamma y})x^{m_\gamma}}{F_\gamma^{m_\gamma}} + \frac{\gamma(1 - e^{-\lambda y})x}{\lambda(\eta+\gamma)} - \frac{\gamma C_s y}{\delta+\gamma}. \tag{A.7}$$

From (A.3), (A.5), (A.7), and because $v(x, 0) = 0$, we conclude that the solution v to the HJB equation (15) is given by (20). □

Now, we shall proceed to verify that v, given in (20), belongs to $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^+)$.

Proof of Proposition 3.2 (Smooth of (20)). Note that by construction, it is sufficient to show that v is $C^{2,1}$ at $x = F_\gamma$ and $x = F_\gamma e^{\lambda y}$, respectively, because $v \in C^{2,1}((\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{A})$, where

$$\mathcal{A} := \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x = F_\gamma \quad \text{or} \quad y = F_\gamma e^{\lambda y}\}.$$

Smooth fit at the variable y. Using (20), it is easy to see that $v_y(F_\gamma-, y) = v_y(F_\gamma+, y)$, implying that v_y is continuous at $x = F_\gamma$. Calculating first derivative in (20), it can be verified that

$$\begin{cases} v_y(F_\gamma e^{\lambda y}-, y) = F_\gamma - C_s, \\ v_y(F_\gamma e^{\lambda y}+, y) = \lambda m_\gamma A_\gamma + \dfrac{\gamma F_\gamma}{\eta+\gamma} - \dfrac{\gamma C_s}{\delta+\gamma}. \end{cases} \tag{A.8}$$

From (21) and (22), it can be verified that

$$\lambda m_\gamma A_\gamma + \frac{\gamma F_\gamma}{\eta+\gamma} - \frac{\gamma C_s}{\delta+\gamma} - (F_\gamma - C_s) = 0. \tag{A.9}$$

Then, by (A.8) and (A.9), it follows that $v_y(F_\gamma e^{\lambda y}-, y) = F_\gamma - C_s = v_y(F_\gamma e^{\lambda y}+, y)$. Therefore, v_y is continuous at $x = F_\gamma e^{\lambda y}$.

Smooth fit at the variable x.

We will show that v_{xx} is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$. We will first verify that v_x is continuous at $x = F_\gamma e^{\lambda y}$. From (20), it follows that

$$v_x(x,y) = \begin{cases} \dfrac{A_\gamma m_\gamma x^{m_\gamma-1}}{F_\gamma^{m_\gamma}} - \dfrac{(F_\gamma - C_s)e^{-\lambda n y}x^{n-1}}{\lambda F_\gamma^n} + \dfrac{\gamma}{\lambda(\eta+\gamma)} - \dfrac{\gamma C_s}{\lambda(\gamma+\delta)x}, & \text{if } F_\gamma \le x < F_\gamma e^{\lambda y}, \\ \dfrac{A_\gamma m_\gamma (1-e^{-\lambda m_\gamma y})x^{m_\gamma-1}}{F_\gamma^{m_\gamma}} + \dfrac{\gamma(1-e^{-\lambda y})}{\lambda(\eta+\gamma)}, & \text{if } x \ge F_\gamma e^{\lambda y}. \end{cases}$$

Then, using (A.9), we get that

$$\begin{aligned} v_x(F_\gamma e^{\lambda y}+, y) &= \frac{A_\gamma m_\gamma (1-e^{-\lambda m_\gamma y})e^{\lambda y(m_\gamma-1)}}{F_\gamma} + \frac{\gamma(1-e^{-\lambda y})}{\lambda(\eta+\gamma)} \\ &= \frac{A_\gamma m_\gamma e^{\lambda y(m_\gamma-1)}}{F_\gamma} - \frac{e^{-\lambda y}}{\lambda F_\gamma}\left(\lambda A_\gamma m_\gamma + \frac{\gamma F_\gamma}{\eta+\gamma}\right) + \frac{\gamma}{\lambda(\eta+\gamma)} \\ &= \frac{A_\gamma m_\gamma e^{\lambda y(m_\gamma-1)}}{F_\gamma} - \frac{(F_\gamma - C_s)e^{-\lambda y}}{\lambda F_\gamma} + \frac{\gamma}{\lambda(\eta+\gamma)} - \frac{\gamma C_s e^{-\lambda y}}{\lambda F_\gamma(\gamma+\delta)} \\ &= v_x(F_\gamma e^{\lambda y}-, y). \end{aligned}$$

We now show that v_{xx} is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ using the fact that v_x is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$. As v_x is continuous at $x = F_\gamma$, from (17) and (18), we have

$$v_{xx}(F_\gamma-, y) = \frac{\delta v(F_\gamma, y) - \mu v_x(F_\gamma, y)}{\frac{1}{2}\sigma^2 F_\gamma^2} = v_{xx}(F_\gamma+, y). \tag{A.10}$$

If $x = F_\gamma e^{\lambda y}$, then, using (18) and (19), it follows that

$$\begin{aligned} v_{xx}(F_\gamma e^{\lambda y}-, y) &= \frac{1}{\sigma^2 F_\gamma^2}\left[2\left((\delta+\gamma)v(F_\gamma, y) - \mu F_\gamma v_x(F_\gamma, y) + \gamma\left(C_s y - \frac{F_\gamma}{\lambda}(1-e^{-\lambda y})\right)\right)\right] \\ &= v_{xx}(F_\gamma e^{\lambda y}+, y). \end{aligned} \tag{A.11}$$

Hence, by (A.10) and (A.11), we conclude that v_{xx} is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$. □

Proof of Lemma 3.4. First, recall that m_γ was defined as the negative solution of (25), and observe that $m_\gamma \underset{\gamma\to 0}{\longrightarrow} m_0$, where m_0 is the negative solution to (24). Letting $\gamma \to 0$ in (26), it is easy to see that $a_\gamma \underset{\gamma\to 0}{\longrightarrow} 1$. On the other hand, letting $\gamma \to \infty$ in (26), it can be verified that

$$\lim_{\gamma\to\infty} a_\gamma = \lim_{\gamma\to\infty} \frac{\dfrac{\delta}{m_\gamma} - \left(\dfrac{\delta}{n} + \dfrac{\gamma b}{\delta+\gamma}\right)}{\dfrac{\eta(\delta+\gamma)}{(\eta+\gamma)m_\gamma} - \left(\dfrac{\delta}{n} - \dfrac{\gamma\mu}{\eta+\gamma}\right)} = \frac{\delta+nb}{\delta-\mu n}. \tag{A.12}$$

As n is the positive solution to (24), we see

$$\delta + nb = \frac{1}{2}\sigma^2 n^2, \quad \text{and} \quad \delta - \mu n = \frac{1}{2}\sigma^2 n(n-1). \tag{A.13}$$

Therefore, from (A.12) and (A.13), we conclude that $a_\gamma \to \dfrac{n}{n-1}$ if $\gamma \to \infty$. Now, we shall prove that $1 < a_\gamma < \frac{n}{n-1}$ for all $\gamma > 0$. In order to prove this, we first note that by (A.13),

$$\begin{cases} \dfrac{\delta}{n} + \dfrac{\gamma b}{\delta+\gamma} = \dfrac{\gamma\sigma^2 n^2 + 2\delta^2}{2n(\delta+\gamma)} > 0, \\ \dfrac{\delta}{n} - \dfrac{\gamma\mu}{\eta+\gamma} = \dfrac{\gamma\sigma^2 n(n-1) + 2\delta\eta}{2n(\eta+\gamma)} > 0. \end{cases} \tag{A.14}$$

On the other hand, we have for each $\gamma > 0$,

$$\begin{aligned} &\frac{\delta}{\delta+\gamma} - \frac{\eta}{\eta+\gamma} - \frac{\gamma m_\gamma}{\delta+\gamma}\left(\frac{b}{\delta+\gamma} + \frac{\mu}{\eta+\gamma}\right) \\ &= \frac{\gamma\mu}{(\delta+\gamma)(\eta+\gamma)} - \frac{\gamma m_\gamma}{(\delta+\gamma)}\left(\frac{(\eta+\gamma)\sigma^2 + 2\mu^2}{2(\delta+\gamma)(\eta+\gamma)}\right) > 0, \end{aligned} \tag{A.15}$$

because $m_\gamma < 0$, with $\gamma > 0$. (A.14) and (A.15) imply

$$0 < \frac{\eta}{\eta+\gamma} - \frac{m_\gamma}{\delta+\gamma}\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma}\right) < \frac{\delta}{\delta+\gamma} - \frac{m_\gamma}{\delta+\gamma}\left(\frac{\delta}{n} + \frac{\gamma b}{\delta+\gamma}\right). \tag{A.16}$$

Therefore, using (26) and (A.16), it follows that $1 < a_\gamma$. In order to prove the remaining inequality, we just note that using (26) it is enough to show that

$$\frac{n\eta}{(n-1)(\eta+\gamma)} - \frac{\delta}{\delta+\gamma} - \frac{m_\gamma}{\delta+\gamma}\left(\frac{\delta}{n-1} - \frac{\delta}{n} - \frac{\gamma b}{\delta+\gamma} - \frac{\gamma\mu n}{(n-1)(\eta+\gamma)}\right) > 0. \tag{A.17}$$

Note that

$$\frac{n\eta}{(n-1)(\eta+\gamma)} - \frac{\delta}{\delta+\gamma} = \frac{\gamma(\delta-n\mu) + \delta\mu}{(n-1)(\delta+\gamma)(\eta+\gamma)} > 0, \tag{A.18}$$

because $\delta - n\mu > 0$. Similarly, using (A.13) and the fact that $\eta = \delta - \mu$, we obtain that

$$\frac{\delta}{n-1} - \frac{\delta}{n} - \frac{\gamma b}{\delta+\gamma} - \frac{\gamma\mu n}{(n-1)(\eta+\gamma)}$$

$$
\begin{aligned}
&= \frac{\delta}{n(n-1)} - \frac{\gamma b(n-1)(\eta+\gamma) + \gamma\mu n(\delta+\gamma)}{(n-1)(\delta+\gamma)(\eta+\gamma)} \\
&= \frac{\delta(\delta+\gamma)(\eta+\gamma) - n^2\gamma\mu(\delta+\gamma) - \gamma b n(n-1)(\eta+\gamma)}{n(n-1)(\delta+\gamma)(\eta+\gamma)} \\
&= \frac{\delta(\delta+\gamma)(\eta+\gamma) - \delta n\gamma\mu + \frac{1}{2}\sigma^2 n^2\gamma\mu(n-1) - \delta\gamma(\eta+\gamma)}{n(n-1)(\delta+\gamma)(\eta+\gamma)} \\
&= \frac{\delta^2\eta + \frac{\sigma^2}{2}n(n-1)(\delta+\gamma\mu)}{n(n-1)(\delta+\gamma)(\eta+\gamma)} > 0. \qquad \text{(A.19)}
\end{aligned}
$$

Therefore, using (A.18), (A.19), and the fact that $m_\gamma < 0$, we obtain (A.17) and hence $a_\gamma < \frac{n}{n-1}$. □

Proof of Proposition 3.5. In order to prove (27), it is enough to show that

$$
\begin{cases}
G_l(x,y,l;v) < 0, & \text{for } x < F_\gamma, \\
G_l(x,y,\mathbb{Y}(x);v) = 0 \quad \text{and} \quad G_{ll}(x,y,\mathbb{Y}(x);v) < 0, & \text{for } F_\gamma \le x < F_\gamma e^{\lambda y}, \\
G_l(x,y,l;v) > 0, & \text{for } x \ge F_\gamma e^{\lambda y},
\end{cases}
$$

where $\mathbb{Y}(x) = \dfrac{1}{\lambda}\ln(x/F_\gamma)$.

Maximum in the first zone

Let $x < F_\gamma$. Taking first derivatives in (20) and evaluating at the point $(xe^{-\lambda l}, y-l)$, we get that

$$
\begin{aligned}
v_x(xe^{-\lambda l}, y-l) &= \frac{F_\gamma - C_s}{\lambda F_\gamma^n}(e^{-\lambda nl} - e^{-\lambda ny})x^{n-1}e^{\lambda l}, \\
v_y(xe^{-\lambda l}, y-l) &= \frac{F_\gamma - C_s}{F_\gamma^n}e^{-\lambda ny}x^n.
\end{aligned}
$$

Then,

$$
\begin{aligned}
G_l(x,y,l;v) &= -\lambda v_x(xe^{-\lambda l}, y-l)xe^{-\lambda l} - v_y(xe^{-\lambda l}, y-l) + e^{-\lambda l}x - C_s \\
&= -\frac{F_\gamma - C_s}{F_\gamma^n}x^n e^{-\lambda nl} + e^{-\lambda l}x - C_s.
\end{aligned}
$$

Note that the above expression is negative if, and only if,

$$
\frac{e^{-\lambda l}x - C_s}{(xe^{-\lambda l})^n} < \frac{F_\gamma - C_s}{F_\gamma^n}. \qquad \text{(A.20)}
$$

Taking the first derivative with respect to x on the left-hand side of (A.20), we have

$$
\frac{\partial}{\partial x}\left(\frac{e^{-\lambda l}x - C_s}{(xe^{-\lambda l})^n}\right) = \frac{n-1}{x^{n+1}}\left(\frac{nC_s}{n-1}e^{\lambda l} - x\right)e^{\lambda l(n-1)}.
$$

By (21) and Lemma 3.4, we know that $x < F_\gamma e^{\lambda l} < \frac{nC_s}{n-1}e^{\lambda l}$. Then, $\frac{\partial}{\partial x}\left(\frac{e^{-\lambda l}x - C_s}{(xe^{-\lambda l})^n}\right) > 0$, which implies that $\frac{e^{-\lambda l}x - C_s}{(xe^{-\lambda l})^n}$ is nondecreasing with respect to x. Hence,

$$\frac{e^{-\lambda l}x - C_s}{(xe^{-\lambda l})^n} < \frac{e^{-\lambda l}F_\gamma - C_s}{(F_\gamma e^{-\lambda l})^n}, \text{ for each } x < F_\gamma.$$

Notice that if we verify that $\frac{e^{-\lambda l}F_\gamma - C_s}{(F_\gamma e^{-\lambda l})^n} < \frac{F_\gamma - C_s}{F_\gamma^n}$, we obtain (A.20), which is equivalent to

$$(e^{-\lambda l} - e^{-\lambda nl})a_\gamma < 1 - e^{-\lambda nl}, \tag{A.21}$$

because $F_\gamma = C_s a_\gamma$. Taking $l^* := \frac{\ln n}{\lambda(n-1)}$, it can be verified that

$$(e^{-\lambda l^*} - e^{-\lambda nl^*})a_\gamma = \max_l\{(e^{-\lambda l} - e^{-\lambda nl})a_\gamma\}.$$

As $a_\gamma < \frac{n}{n-1}$ and $(n+1)^n < n^n(n+1)$, with $n > 1$, we get that

$$(e^{-\lambda l^*} - e^{-\lambda nl^*})a_\gamma = \left(n^{-\frac{1}{n-1}} - n^{-\frac{n}{n-1}}\right)a_\gamma < 1 - n^{-\frac{n}{n-1}} = 1 - e^{-\lambda nl^*}. \tag{A.22}$$

This means that (A.21) is satisfied for any $l > l^*$. Now, if $l \le l^*$, we shall prove statement (A.21) by contradiction. Suppose that there exists $0 \neq l_1 \le l^*$ such that

$$\begin{aligned}&(e^{-\lambda l_1} - e^{-\lambda nl_1})a_\gamma = 1 - e^{-\lambda nl_1},\\ &(e^{-\lambda l} - e^{-\lambda nl})a_\gamma \ge 1 - e^{-\lambda nl}, \quad \text{for each } l \le l_1 \le l^*.\end{aligned} \tag{A.23}$$

Because

$$(e^{-\lambda l} - e^{-\lambda nl})a_\gamma \le (e^{-\lambda l_1} - e^{-\lambda nl_1})a_\gamma \le (e^{-\lambda l^*} - e^{-\lambda nl^*})a_\gamma,$$

we have that $(e^{-\lambda l^*} - e^{-\lambda nl^*})a_\gamma \ge 1 - e^{-\lambda nl^*}$, which is a contradiction to (A.22). If $l^* < l_1$ and satisfies (A.23), we have that

$$1 - e^{-\lambda l^* n} < 1 - e^{-\lambda l_1 n} = (e^{-\lambda l_1} - e^{-\lambda nl_1})a_\gamma < (e^{-\lambda l^*} - e^{-\lambda nl^*})a_\gamma,$$

which contradicts (A.22). Therefore, (A.21) is true for any l, which yields (A.20). We conclude that the maximum on the right-hand side of (27) is achieved at $l = 0$ when $x < F_\gamma$.

Maximum in the second zone. Let $F_\gamma \le x < F_\gamma e^{\lambda y}$. Taking first derivatives of v and evaluating $(F_\gamma, y - \mathbb{Y}(x))$ in them, it follows that

$$\begin{aligned}-\lambda F_\gamma v_x(F_\gamma, y - \mathbb{Y}(x)) &= -\frac{\delta m_\gamma(F_\gamma - C_s)}{n(\delta+\gamma)} + (F_\gamma - C_s)e^{-\lambda n(y-\mathbb{Y}(x))}\\ &\quad - \frac{\gamma F_\gamma}{\eta+\gamma}\left(1 - \frac{\mu m_\gamma}{\delta+\gamma}\right) + \frac{\gamma C_s}{\delta+\gamma}\left(1 + \frac{bm_\gamma}{\delta+\gamma}\right),\\ -v_y(F_\gamma, y - \mathbb{Y}(x)) &= -(F_\gamma - C_s)e^{-\lambda n(y-\mathbb{Y}(x))}.\end{aligned}$$

Then, recalling that $F_\gamma = C_\mathrm{s} a_\gamma$, where a_γ is given in (26), we get that

$$G_l(x, y, \mathbb{Y}(x); v) = a_\gamma C_\mathrm{s}\left(\frac{\eta}{\eta+\gamma} - \frac{m_\gamma}{\delta+\gamma}\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma}\right)\right) - \frac{C_\mathrm{s}}{\delta+\gamma}\left(\delta - m_\gamma\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right)\right) = 0.$$

Therefore, $l = \mathbb{Y}(x)$ is a critical point of $G(x, y, l; v)$ (recall that the definition of G was given in (16)). To verify that $l = \mathbb{Y}(x)$ is a maximum of $G(x, y, l; v)$, we need to see that

$$G_{ll}(x, y, \mathbb{Y}(x); v) < 0.$$

First, note that

$$\lambda^2 F_\gamma v_x(F_\gamma, y - \mathbb{Y}(x)) = \lambda^2 A_\gamma m_\gamma - \lambda(F_\gamma - C_\mathrm{s})\mathrm{e}^{-\lambda n(y-\mathbb{Y}(x))} + \frac{\lambda\gamma F_\gamma}{\eta+\gamma} - \frac{\lambda\gamma C_\mathrm{s}}{\delta+\gamma}. \tag{A.24}$$

Now, taking the second derivatives of v and evaluating $(F_\gamma, y - \mathbb{Y}(x))$ in them, it follows that

$$\begin{cases} \lambda^2 F_\gamma^2 v_{xx}(F_\gamma, y - \mathbb{Y}(x)) = \lambda^2 A_\gamma m_\gamma(m_\gamma - 1) - \lambda(F_\gamma - C_\mathrm{s})(n-1)\mathrm{e}^{-\lambda n(y-\mathbb{Y}(x))} + \dfrac{\lambda\gamma C_\mathrm{s}}{\delta+\gamma}, \\ 2\lambda F_\gamma v_{xy}(F_\gamma, y - \mathbb{Y}(x)) = 2\lambda n(F_\gamma - C_\mathrm{s})\mathrm{e}^{-\lambda n(y-\mathbb{Y}(x))}, \\ \quad v_{yy}(F_\gamma, y - \mathbb{Y}(x)) = -\lambda n(F_\gamma - C_\mathrm{s})\mathrm{e}^{-\lambda n(y-\mathbb{Y}(x))}. \end{cases} \tag{A.25}$$

By (A.24) and (A.25), we get that

$$\begin{aligned} G_{ll}(x, y, \mathbb{Y}(x); v) &= \lambda^2 F_\gamma^2 v_{xx}(F_\gamma, y - \mathbb{Y}(x)) + 2\lambda F_\gamma v_{xy}(F_\gamma, y - \mathbb{Y}(x)) \\ &\quad + \lambda^2 F_\gamma v_x(F_\gamma, y - \mathbb{Y}(x)) + v_{yy}(F_\gamma, y - \mathbb{Y}(x)) - \lambda F_\gamma \\ &= \frac{C_\mathrm{s}\lambda m_\gamma^2}{\delta+\gamma}\left(\frac{\delta a_\gamma}{n} - \frac{a_\gamma\gamma\mu}{\eta+\gamma} - \frac{\delta}{n} - \frac{b\gamma}{\delta+\gamma} - \frac{a_\gamma\eta(\delta+\gamma)}{m_\gamma^2(\eta+\gamma)}\right). \end{aligned} \tag{A.26}$$

To see that the above expression is negative, we only need to prove that

$$a_\gamma\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma} - \frac{\eta(\delta+\gamma)}{m_\gamma^2(\eta+\gamma)}\right) - \left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right) < 0,$$

which is equivalent to

$$a_\gamma\left(\frac{\delta}{n} - \frac{\mu\gamma}{\eta+\gamma}\right) - \left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right) < 0, \tag{A.27}$$

because $-a_\gamma \dfrac{\eta(\delta+\gamma)}{(\eta+\gamma)m_\gamma^2} < 0$. Verifying that

$$\delta\left(\frac{\delta}{n} - \frac{\mu\gamma}{\eta+\gamma}\right) < \frac{\eta(\delta+\gamma)}{\eta+\gamma}\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right), \tag{A.28}$$

and recalling that a_γ is given by (26), we obtain (A.27). We shall prove (A.28). Observe that

$$\delta\left(\frac{\delta}{n} - \frac{\mu\gamma}{\eta+\gamma}\right),$$

is nonincreasing with respect to $\gamma > 0$ and

$$\begin{cases} \delta\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma}\right)\uparrow \frac{\delta^2}{n}, & \text{when } \gamma \to 0, \\ \delta\left(\frac{\delta}{n} - \frac{\gamma\mu}{\eta+\gamma}\right)\downarrow \delta\left(\frac{\delta}{n} - \mu\right), & \text{when } \gamma \to \infty. \end{cases} \tag{A.29}$$

If $b > 0$, that is, $\frac{\sigma^2}{2} > \mu$, then

$$\frac{\eta(\delta+\gamma)}{\eta+\gamma}\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right), \tag{A.30}$$

is nondecreasing with respect to $\gamma > 0$ and

$$\begin{cases} \frac{\eta(\delta+\gamma)}{\eta+\gamma}\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right)\downarrow \frac{\delta^2}{n}, & \text{when } \gamma \to 0, \\ \frac{\eta(\delta+\gamma)}{\eta+\gamma}\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right)\uparrow \eta\left(\frac{\delta}{n} + b\right), & \text{when } \gamma \to \infty. \end{cases}$$

From this and by (A.29), there follows (A.28) and therefore we have that (A.26) is negative. If $b \leq 0$, that is, $\frac{\sigma^2}{2} \leq \mu$, it can be verified that (A.30) is nonincreasing with respect to $\gamma > 0$ and

$$\begin{cases} \frac{\eta(\delta+\gamma)}{\eta+\gamma}\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right)\uparrow \frac{\delta^2}{n}, & \text{when } \gamma \to 0, \\ \frac{\eta(\delta+\gamma)}{\eta+\gamma}\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right)\downarrow \eta\left(\frac{\delta}{n} + b\right), & \text{when } \gamma \to \infty. \end{cases} \tag{A.31}$$

Defining the function $h(\gamma)$ by

$$h(\gamma) := \delta\left(\frac{\delta}{n} - \frac{\mu\gamma}{\eta+\gamma}\right) - \frac{\eta(\delta+\gamma)}{\eta+\gamma}\left(\frac{\delta}{n} + \frac{b\gamma}{\delta+\gamma}\right), \tag{A.32}$$

we can see that

$$\begin{cases} h(\gamma) \to 0, & \text{when } \gamma \to 0, \\ h(\gamma) \to \delta\left(\frac{\delta}{n} - \mu\right) - \eta\left(\frac{\delta}{n} + b\right), & \text{when } \gamma \to \infty, \end{cases}$$

because (A.29) and (A.31) hold. In order to prove (A.28), it is enough to show that $h(\gamma)$ is nonincreasing and

$$\delta\left(\frac{\delta}{n} - \mu\right) - \eta\left(\frac{\delta}{n} + b\right) < 0. \tag{A.33}$$

As n is the positive solution to (24) and is bigger than one, it follows that

$$\frac{\delta}{\mu} = n + \frac{\sigma^2}{2\mu} n(n-1) > n, \tag{A.34}$$

which yields that $n\mu < \delta$. Then, applying this in (A.34), we have

$$\frac{\delta\mu}{n} = \mu^2 + \frac{\sigma^2\mu}{2}(n-1) < \frac{1}{2}\sigma^2\delta - \mu\left(\frac{1}{2}\sigma^2 - \mu\right), \tag{A.35}$$

which implies (A.33). Now, taking the first derivative of (A.32), we see that

$$h'(\gamma) = \frac{\eta}{(\eta+\gamma)^2}\left(-\delta\mu + \frac{\delta\mu}{n} - b\eta\right).$$

Using (A.35), it can be shown that $-\delta\mu + \dfrac{\delta\mu}{n} - b\eta < 0$. This implies that $h(\gamma)$ is a negative nonincreasing function. Therefore, (A.27) holds, and we have that (A.26) is negative. Thus, the maximum of the right-hand side of (27) is achieved at $l = \mathbb{Y}(x)$, when $F_\gamma \le x < F_\gamma e^{\lambda y}$.

Maximum in the third zone. Let $x \ge F_\gamma e^{\lambda y}$. Taking the first derivatives of v and evaluating $(xe^{-\lambda l}, y-l)$ in them, it follows that

$$-\lambda v_x(xe^{-\lambda l}, y-l)xe^{-\lambda l} = -\frac{\lambda m_\gamma A_\gamma x^{m_\gamma}(e^{-\lambda m_\gamma l} - e^{-\lambda m_\gamma y})}{F_\gamma^{m_\gamma}} - \frac{\gamma x(e^{-\lambda l} - e^{-\lambda y})}{\eta+\gamma},$$

$$-v_y(xe^{-\lambda l}, y-l) = -\frac{\lambda m_\gamma A_\gamma (xe^{-\lambda y})^{m_\gamma}}{F_\gamma^{m_\gamma}} - \frac{\gamma x e^{-\lambda y}}{\eta+\gamma} + \frac{\gamma C_s}{\delta+\gamma}.$$

Then,

$$\begin{aligned} G_l(x,y,l;v) &= -\lambda v_x(xe^{-\lambda l}, y-l)xe^{-\lambda l} - v_y(xe^{-\lambda l}, y-l) + e^{-\lambda l}x - C_s \\ &= -\frac{\lambda m_\gamma A_\gamma (xe^{-\lambda l})^{m_\gamma}}{F_\gamma^{m_\gamma}} + \frac{\eta x e^{-\lambda l}}{\eta+\gamma} - \frac{\delta C_s}{\delta+\gamma}. \end{aligned}$$

To see that the above expression is positive, it is equivalent to show that

$$\frac{\eta x e^{-\lambda l}}{\eta+\gamma} - \frac{m_\gamma \lambda A_\gamma (xe^{-\lambda l})^{m_\gamma}}{F_\gamma^{m_\gamma}} > \frac{\delta C_s}{\delta+\gamma}. \tag{A.36}$$

Observe that from (22) and (A.27), it can be verified that $A_\gamma < 0$. Then, it follows that

$$\frac{\eta x e^{-\lambda l}}{\eta+\gamma} \ge \frac{\eta F_\gamma e^{\lambda(y-l)}}{\eta+\gamma},$$

$$-\frac{m_\gamma \lambda A_\gamma (xe^{-\lambda l})^{m_\gamma}}{F_\gamma^{m_\gamma}} \ge -m_\gamma \lambda A_\gamma e^{\lambda m_\gamma (y-l)},$$

because $x \geq F_\gamma e^{\lambda y}$ and $m_\gamma < 0$. Then,

$$\frac{\eta x e^{-\lambda l}}{\eta + \gamma} - \frac{m_\gamma \lambda A_\gamma (x e^{-\lambda l})^{m_\gamma}}{F_\gamma^{m_\gamma}} > \frac{\eta F_\gamma e^{\lambda(y-l)}}{\eta + \gamma} - m_\gamma \lambda A_\gamma e^{\lambda m_\gamma (y-l)} =: g(l). \tag{A.37}$$

Note that $g(l)$ is nonincreasing with respect to l and from (21) and (22), we get that $g(y) = \dfrac{\delta C_s}{\delta + \gamma}$. Therefore, (A.37) implies (A.36). Thus, the maximum of the right-hand side of (27) is achieved at $l = y$, when $x \geq F_\gamma e^{\lambda y}$. □