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We develop novel mispricing of markets under asymmetric information and jumps for informed and uninformed investors, called m-Double Poisson markets, driven by independent Double Poisson processes. In the special case $m = 1$, called the Double Poisson pure-jump Lévy market, both types of investors hold the same optimal portfolio and expected utility, and hence, the informed investor has no utility advantage over the uninformed. For the general market, instantaneous centralized moments of returns are used to compute optimal portfolios and utilities. The mean, variance, skewness and kurtosis of instantaneous returns are reported using jump amplitudes and frequencies.

Keywords: Mispricing; Asymmetric information; m-Double Poisson markets; Optimal portfolio; Instantaneous centralized moments of returns

JEL Classification: G11

1. Introduction

Beginning with the seminal work of Markowitz (1952), the optimal allocation of capital to a portfolio of risky assets with the objective to maximize expected return while minimizing risk has been a very important issue in finance. These asset pricing and portfolio selection problems are much easier to study in efficient markets, where it is assumed that asset prices fully reflect all publicly available information. In this framework, markets are symmetric in that all investors have the same amount of information to build trading strategies and, thus, inefficiencies leading to abnormal profits do not exist.

However, one of the most influential findings in finance over the last few decades was how the most dearly held notion of market efficiency failed to be true. This new paradigm gradually took hold due to compelling evidence of strong unexpected price volatility in stock, bond, currency and real estate markets. There were a plethora of empirical studies that confirmed the existence of many market anomalies including excess volatility caused by investor overreaction and underreaction, fashions and fads (mispricing). Consequently, asset pricing and portfolio selection are best studied in an inefficient market framework, where investors have asymmetric information.

'Fundamental Analysis' is built on the foundation that an asset has a fair or fundamental value. Based on the law of one price, fundamental analysis follows the principle that assets, such as stocks, have a fundamental or fair value and traders can earn abnormal profits from asset price signals that indicate a departure of the market price from its fundamental or fair value. Mispricing is the difference between the fair (fundamental) value and the market price of the asset. The market price most surely reverts to its fair value when there is mispricing, otherwise, there would be persistent arbitrage opportunities. That is, divergence from fair value is more likely to decrease than increase.

Bartram and Grinblatt (2018) develop an unorthodox (non-parametric) model to estimate the peer-implied fair/fundamental value of the stock price of a firm using a linear function of accounting variables. They employ a cross-sectional regression that uses 28 pieces of accounting data from balance sheets, income statements and cash flow statements reported by a large number of firms. Mispricing is measured as the difference between the fair value and market value of the firm at the end of each month. The authors define a mispricing signal as this difference from the market price and show that this signal predicts risk-adjusted abnormal returns of 4% to 10% per annum, which is economically significant for both large and small-cap firms. They posit that convergence of market prices to their peer-implied fair value is the most likely source of the profitability resulting from this trading strategy and that the rate of convergence gradually diminishes to zero over the subsequent 34 months. In other words, the authors show that mispricing can be used to generate profitable trading strategies based on fundamental

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analysis, and in particular, a simple agnostic (free of stylized models) fundamental analysis works.

In another recent study, Zhang and Yan (2018) discuss modeling fundamental analysis for portfolio selection in the continuous-time framework. They derive a closed-form appraisal ratio and an information ratio for the informed investors who can observe both the fundamental values and the mispricings of the underlying securities. Their results suggest that investors can use fundamental analysis to pick up securities with a more volatile mispricing, a less volatile fundamental, a higher mean-reverting speed, and a larger dividend.

Kelly and Ljungqvist (2012) show that information asymmetry has a substantial effect on asset prices and investors' demands which affects assets through a liquidity channel. Furthermore, they attest that asset pricing models under asymmetric information rely on a noisy rational expectation equilibrium in which prices partially reveal the better-informed investors' information due to randomness in the asset's supply. In their study, Easley and O'Hara (2004) show that an increase in information asymmetry leads to a fall in share prices and a reduction in the uninformed investors' demand for the risky asset. Evidently, asymmetric information plays an important role in asset pricing and thus understanding the relationship between mispricing and asymmetric information has gained much attention in the finance literature lately.

Most recent mispricing models under asymmetric information are continuous-time. These models assume that there are two types of investors: informed and uninformed. The informed investor, such as a hedge fund or large financial institution with internal research capabilities, trades because of the possession of non-public informational advantage, whereas the uninformed investor, for example, a liquidity trader, observes market prices only, and hence, does not have this advantage. Moreover, these models assume that the informed investor knows both the asset's fair value and the market price, and partially reveals information to the uninformed investor through trades.

Buckley *et al.* (2012, 2014, 2015, 2016), Buckley and Long (2015) and Buckley and Perera (2019) develop continuous-time models of an asset (stock) under mispricing and asymmetric information in a non-Gaussian economy/market driven by Lévy processes with jumps, where investors have logarithmic and power preferences. They obtain random optimal portfolios and maximum expected utilities for both investors, including asymptotic excess utilities. These portfolios contain excess stock holdings, relative to the Merton optimal for each investor, that are driven by the first and second moments of returns. In contrast, Wang (1993), Guasoni (2006), and Vayanos and Wang (2012) develop purely continuous-time models without jumps.†

Although it is well-known that asset prices jump, based on the physical constraint of financial markets, there can only be a finite number of trades, and hence, a finite number of jumps in any time period. Thus, it is not unreasonable to model markets as having finite activity. Kirch and Runggaldier (2004) study a hedging problem in both complete and incomplete markets where the stock price follows a geometric Poisson process. This Poisson process is a linear combination of two independent Poisson processes with different intensities that may or may not be known. In this framework, prices change by discrete amounts at random times, which is an accurate reflection of how real markets work. The deterministic optimal hedging strategy is realized by minimizing the expected value of a convex loss function. To capture price evolution in real markets, we also allow a stock price to evolve as a geometric Poisson process, but add drift and the diffusion of a Brownian motion. Thus, in essence, our price process is really a product of Geometric Poisson and Geometric Brownian motions. While Kirch and Runggaldier only use two independent Poisson processes, we employ m pairs of independent Poisson processes that are linked by jump amplitudes, which better capture real prices. We only assume that jump intensities are known by investors but Kirch and Runggaldier also consider the case of unknown intensities via Bayesian updating. In contrast to their hedging problem in complete and incomplete markets, we seek and find the random optimal portfolio strategy of each investor by maximizing the expected utility of terminal wealth of investors under asymmetric information but in complete markets.

The mispriced asymmetric Kou Lévy market, first introduced in Buckley *et al.* (2014), is an example of a market with a finite number of jumps per time period (finite activity), while jump sizes are controlled by an asymmetric double exponential distribution. To the best of our knowledge, the m-Double Poisson market presented in this paper is the second finite activity model available in the extant literature. Although real markets are, by necessity, finite activity, most models assume infinite jump activity. Consequently, most recent models of mispricing under asymmetric information are driven by Lévy processes with infinite activity. One example is the Variance Gamma market (e.g. Buckley *et al.* 2016). In contrast, our model is designed to be of finite activity and is driven by m pairs of independent Poisson processes (called Skellam processes), where the upward and downward jump amplitudes are tied in a specific manner. In particular, each upward jump has a counterpart which is a fixed downward jump and no two pairs have the same jump amplitudes.

It is a widely accepted fact that jumps in asset prices lead to higher moments which are essential inputs for portfolio allocation and modeling returns. Thus, portfolio allocation must naturally incorporate higher moments. Cvitanic *et al.* (2008) study a risky asset driven by a pure-jump Lévy process with non-trivial higher moments and compute the optimal portfolio strategy of an investor with CRRA utility. They use higher moments, called instantaneous centralized moments of returns (ICMR), to approximate optimal portfolios. They also report the wealth loss from ignoring higher moments. This omission can lead to significant over-investment in the risky asset especially when volatility is high. Several authors propose advanced optimal portfolio methods that address the empirical evidence of higher moments in returns. For example, Harvey *et al.* (2010) incorporate skewness and higher moments in portfolio selection using a Bayesian framework and the

† Other useful studies are Shiller (1981), Summers (1986), Brunnermeier (2001), Biais *et al.* (2010), and Serrano-Padial (2012), which only consider discrete-time models of mispricing under asymmetric information, but with no jumps included.

normal distribution, to jointly model multivariate returns and parameter uncertainty.‡

In this paper, our model is a special case of the general model in Buckley *et al.* (2014), namely a pure-jump Lévy market with focus on the Double Poisson and m-Double Poisson markets. In these markets, we are able to provide explicit formulas and numerical approximations for the optimal portfolios. We build and analyze two novel finite activity models of mispricing under asymmetric information. The first is called the Double Poisson model, which is driven by two linked independent Poisson processes (Skellam)-one with a positive jump-amplitude, and the other with a dependent negative jump-amplitude. The jump-amplitudes are fixed but linked by a simple formula. The second or general model, called the m-Double Poisson market, is driven by m independent Double Poisson processes, each with distinct but linked upward and downward jump-amplitudes, and is hence able to better simulate the actual evolution of asset prices. Because the asymmetric optimal portfolio is closely linked to the symmetric optimal, our study focuses on the deterministic symmetric optimal portfolio, which provides the main ingredient for our asymmetric optimal portfolios. We provide an example of a pure-jump market driven by a Double Poisson process, which is a market with no diffusion coefficient and show that the optimal portfolios for each investor is equal to a non-random portfolio, and hence, has no volatility; in contrast, asymmetric optimal portfolios are random, and hence, volatile. We then consider a market which has a diffusive component in addition to a pure-jump component driven by an m-Double Poisson process. We use instantaneous centralized moments of returns (ICMRs) and linear iteration to approximate optimal portfolios and maximum expected utility for each investor. We also examine mean, variance, skewness and kurtosis of returns, using higher moments (ICMRs).

The rest of the paper is organized as follows. The model is briefly reviewed and the power utility case is outlined in Section 2. The pure-jump Double Poisson model is introduced in Section 3. We provide the optimal portfolio and utility for each investor and show that the informed has no utility advantage. The Double Poisson jump-diffusion market is introduced in Section 4, where an explicit formula is given for the optimal portfolio in the symmetric market. Section 5 gives details of the m-Double Poisson jump-diffusion market, including instantaneous centralized moments of returns and their effects on mean, variance, skewness and kurtosis of the return process. Finally, we connect the asymmetric optimal portfolio with its symmetric counterpart, and provide a quadratic approximation, as well as an exact solution. The section ends with a numerical example of a 4-Double Poisson market. Concluding remarks are given in Section 6. Proofs are in the appendix.

‡Studies on the importance of higher moments in financial applications include De Athayde and Flores (2004), Jondeau and Rockinger (2006), Baksi and Madan (2006), Barberis and Huang (2008), Chabi-Yo (2008), Martin (2013), Yang and Hung (2010), and Conrad *et al.* (2013).

2. The model

To make the paper self-contained, we begin by briefly reviewing the model presented in Buckley *et al.* (2014). The economy consist of a risky asset S, called stock and a money market account $\mathbf{A}$ that earns a constant risk-free interest r. The stock has log-returns dynamics:

$$\mathrm{d}(\log S_t) = \left(\mu_t - \frac{1}{2}\sigma_t^2\right)\mathrm{d}t + \sigma_t\,\mathrm{d}Y_t + \mathrm{d}X_t, \quad t \in [0, T], \tag{1}$$

$$Y_t = pW_t + qU_t, \quad p^2 + q^2 = 1, \quad p \geq 0,\ q \geq 0, \tag{2}$$

$$\mathrm{d}U_t = -\lambda U_t\,\mathrm{d}t + \mathrm{d}B_t, \quad \lambda > 0,\ U_0 = 0, \tag{3}$$

$$X_t = \int_0^t \int_{\mathbf{R}\setminus\{0\}} xN(\mathrm{d}s, \mathrm{d}x), \tag{4}$$

where W and B are independent standard Brownian motions independent of X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while mispricing is represented by $U = (U_t)$, a mean-reverting *Ornstein–Uhlenbeck* process with rate λ, which is a measure of mispricing. λ is a proxy for mispricing because the mean-reverting process has a half-life of $H = \ln 2/\lambda$, which is half the time taken for the mispricing to return to 0. This means that when λ is small, it takes a long time for mispricing to return to zero, compared to when it is large, in which case, mispricing does not last long, because H is small.

The stock return has three components. The first component is $\mu_t - \frac{1}{2}\sigma_t^2$, which is continuous and deterministic and is represented by the drift of the stock price process. The second component of return is Y_t which is continuous but random. It is expected to be mean zero and is a linear combination of the mean reverting mispricing process U_t and the independent Brownian motion driving the stock price process. The mispricing process U_t is stationary and expected to be mean zero. For example, when there is no mispricing, it only contributes continuous return as a Geometric Brownian Motion. The third component is X_t, which is random, discontinuous, and well-defined if $\int_{\mathbb{R}\setminus\{0\}} |x|\nu(\mathrm{d}x) < \infty$, where $\nu(\mathrm{d}x)$ is the characteristic measure of the Poisson random measure $N(\mathrm{d}t, \mathrm{d}x)$. It is driven by the jumps in the stock price only, and is independent of the Brownian motion and mispricing process. The characteristic measure $\nu(\cdot)$ is a proxy for the jump intensity of stock price. If there are no jumps, this is zero, and the discontinuous component of return vanishes. In the m-Double Poison model, X_t is the sum of m independent Poisson processes, while Y_t is a combination of standard Brownian motion and mispricing, normalized to have a mean of 0 and a variance of 1 over any unit interval.

It is assumed that there are two classes of investors, classified as informed investor and uninformed investor. The informed investor, indexed by $i = 1$, observes both the asset price and its fundamental value through the mispricing process U. The information flow (filtration) of the informed investor is defined by

$$\mathcal{H}_t^1 \triangleq \sigma\,(W_s, B_s, X_s : s \leq t) \vee \sigma(\mathcal{N}),$$

where $\mathcal{N} = \{D \subset \Omega : \exists A \in \mathcal{F}, D \subset A, \mathbf{P}(A) = 0\}$. The uninformed investor, indexed by $i = 0$, observes only the asset price and has no direct knowledge of the mispricing process U. Its filtration is defined by

$$\mathcal{H}_t^0 = \mathcal{F}_t^0 \vee \sigma(X_u : u \le t), \quad t \in [0, T],$$

where $\mathcal{F}_t^0 = \sigma(Y_s : s \le t)$, and X and Y are defined by equations (2)–(4). It is obvious that $\mathcal{F}_t^0 \subset \mathcal{H}_t^0 \subset \mathcal{H}_t^1 \subset \mathcal{F}$.

Both investors have logarithmic utility and the stock has percentage returns:

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}Y_t + \int_{\mathbf{R}\setminus\{0\}} (e^x - 1) N(\mathrm{d}t, \mathrm{d}x), \quad t \in [0, T]. \tag{5}$$

In the mispriced asymmetric information framework, although each investor observes the same stock price S, its dynamics depends on the information set or filtration of the observer. Consequently, the percentage returns dynamics for the ith investor becomes

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu_t^i \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t^i + \int_{\mathbf{R}\setminus\{0\}} (e^x - 1) N(\mathrm{d}t, \mathrm{d}x), \tag{6}$$

where $i \in \{0, 1\}$, B^i are $\mathcal{H}^i$-adapted standard Brownian motions, $\mu_t^i = \mu_t + \upsilon_t^i \sigma_t$, and

$$\begin{aligned} \upsilon_t^0 &= -\lambda \int_0^t e^{-\lambda(t-u)} (1 + \gamma(u)) \,\mathrm{d}B_u^0 \\ \upsilon_t^1 &= -\lambda q U_t = -\lambda q \int_0^t e^{-\lambda(t-u)} \,\mathrm{d}B_u^1 \\ \gamma(u) &= \frac{1 - p^2}{1 + p \tanh(\lambda p u)} - 1, \\ p^2 + q^2 &= 1, \quad p \ge 0, \ q \ge 0. \end{aligned} \tag{7}$$

Note that υ_t^i is the number of standard deviation that the random mean return/drift in equation (6) is away from the deterministic mean return μ_t in the stock price model given by equation (5).

The wealth process for the investor is $V^{i,\pi,w} : [0, T] \longrightarrow \mathbf{R}$, where $V_t^{i,\pi,w}$ is the value of the portfolio consisting of the stock and bond at time t, when $\pi_t \equiv \pi_t^i$ is invested in the stock, and the remainder, $1 - \pi_t^i$, is invested in the bank account and $w > 0$ is the initial wealth. The wealth process, V^i, satisfies the following budget constraint

$$\begin{aligned} \frac{\mathrm{d}V_t^i}{V_{t-}^i} &= (\pi_t^i \sigma_t \theta_t^i + r_t) \,\mathrm{d}t + \pi_t^i \sigma_t \,\mathrm{d}B_t^i \\ &\quad + \int_{\mathbf{R}\setminus\{0\}} \pi_t^i (e^x - 1) N(\mathrm{d}t, \mathrm{d}x), \end{aligned} \tag{8}$$

where $\theta^i = (\mu^i - r)/\sigma$ is the random Sharpe ratio and V_{t-}^i is the wealth immediately before a jump occurs in the stock price at time t. Subject to the budget constraint in equation (8), each investor maximizes logarithmic utility from terminal wealth by holding an optimal portfolio of stock given by

$$\pi^{i,*} = \arg\max_{\pi \in \mathcal{A}_w} \mathbf{E} \log \widetilde{V}_T^{i,\pi}, \tag{9}$$

and the maximum expected utility from terminal wealth is

$$u(w) = \mathbf{E} \log \widetilde{V}_T^{(i,\pi^{i,*},w)}, \tag{10}$$

where $\widetilde{V}_t^i = V_t^i \exp(-\int_0^t r_s \,\mathrm{d}s)$ is wealth discounted by the risk-free rate, $\mathcal{A}_w \equiv \mathcal{A}_w(i)$ is a suitable set of admissible or qualified portfolios. It is the largest collection of 'good' portfolios under which the optimization can be done.

Buckley *et al.* (2014) show that to maximize the expected logarithmic utility from terminal wealth, an investor must hold an optimal portfolio given by the following theorem.

THEOREM 2.1 *Let $\pi \in [0, 1]$ and $G(\pi) = \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi(e^x - 1))\nu(\mathrm{d}x)$ be the instantaneous centralized moment generating function, where $\nu(\cdot)$ is the characteristic measure of the Poisson random measure $N(\mathrm{d}t, \mathrm{d}x)$. If $\int_{\mathbf{R}\setminus\{0\}} (e^{\pm x} - 1)^2 \nu(\mathrm{d}x) < \infty$, then*

1. *$G''(\pi) < 0$. That is, G is strictly concave on $[0, 1]$.*
2. *Let $i \in \{0, 1\}$. If the ith investor has logarithmic utility, there is a unique optimal portfolio $\pi^{*,i}$, for the stock with dynamics (1), given by*

$$\pi_t^{*,i} = \frac{\theta_t^i}{\sigma_t} + \frac{G'(\pi_t^{*,i})}{\sigma_t^2}, \tag{11}$$

provided $\pi^{*,i} \subset \mathcal{A}_w$, the admissible set, and $\theta_t^i = (\mu_t^i - r_t)/\sigma_t$ is the Sharpe ratio of the asset relative to the ith investor, and w is the initial wealth/endowment of the investors.

REMARK 1 $1 + \pi(e^x - 1)$ is the return on the portfolio when a log-jump of x occurs in the the stock price. A log-jump of size x yields a percentage return of $e^x - 1$. $G(\cdot)$ is a moment generating function. It generates the instantaneous centralized moments of returns for the stock.

REMARK 2 We have discussed some example on the jump process which satisfies the integrability condition $\int_{\mathbf{R}\setminus\{0\}} (e^{\pm x} - 1)^2 \nu(\mathrm{d}x) < \infty$ in Buckley *et al.* (2014). For Kou's jump-diffusion model, the jump sizes follow an asymmetric double-exponential distribution with Lévy density of the form

$$\nu(x) = \kappa [p_u \kappa_+ e^{-\kappa_+ x} 1_{\{x>0\}} + (1 - p_u) \kappa_- e^{-\kappa_- |x|} 1_{\{x<0\}}] \,\mathrm{d}x.$$

We can easily verify that the integrability condition is satisfied when $\kappa_+ > 2$ and $\kappa_- > 2$

REMARK 3 In Theorem 2.1, we assume that $\pi \in [0, 1]$. As discussed in Buckley *et al.* (2014), this restriction is enforced to ensure that the second derivative of the partial objective function $G(\pi)$ exists, which then leads to the existence of an optimal portfolio. If the characteristic measure $\nu(\mathrm{d}x)$ is such that the second derivative of $G(\pi)$ exists in a wider interval $[a, b]$ with $a < 0$ and $b > 1$ (e.g. this is true when jumps are driven by linear combination of Poisson processes), then the restriction $\pi \in [0,1]$ (no short selling and borrowing) is not required.

Buckley *et al.* (2015) link π^*, the deterministic optimal portfolio in the symmetric market ($q = 0$) to $\pi^{*,i}$, the random optimal portfolio in the asymmetric market with dynamic given in equation (6) by the following proposition.

PROPOSITION 2.2 *Let $i \in \{0,1\}$ and $T > 0$, be the the investment horizon. Assume that G is restricted to $[0, 1]$ and $\int_{\mathbf{R}\setminus\{0\}}(e^{\pm x}-1)^k \nu(\mathrm{d}x) < \infty$ for some integer $k \geq 2$.*

(1) *There exists a process η^i between π^* and $\pi^{*,i}$, such that for all $t \in [0,T]$*

$$\pi_t^{*,i} = \pi_t^* + \frac{\upsilon_t^i \sigma_t}{\sigma_t^2 + |G''(\eta_t^i)|}. \tag{12}$$

(2) *Let $M_2 = \int_{\mathbf{R}\setminus\{0\}}(e^x-1)^2 \nu(\mathrm{d}x)$. Under quadratic approximation of G, we have*

$$\pi_t^{*,i} \approx \pi_t^* + \frac{\upsilon_t^i \sigma_t}{\sigma_t^2 + M_2}. \tag{13}$$

2.1. Power utility

We shall briefly discuss the optimal portfolio of investors under CRRA or power utility. Note that the dynamics of the wealth process V_t^1 for informed investor is still Markovian while the dynamics of the wealth process V_t^0 for uninformed investor is non-Markovian (depending on the whole path of the stock price S_t). The classical HJB methodology is still valid in determining the optimal portfolio for informed investor but would fail for finding the optimal portfolio of uninformed investor. All the details will be carried out in our ongoing research project. Hereby we just give an outline on how to obtain the optimal portfolio for informed investor.

Let

$$\Phi(x) = \frac{x^\gamma}{\gamma}, \quad 0 < \gamma < 1 \tag{14}$$

be the power utility function. Our objective is to maximize the expected power utility from the terminal wealth V_T^1. The value function of the informed investor is given by

$$\begin{aligned}&\phi(t,V,U) = \max_\pi \mathbf{E}[\Phi(V_T^1) \mid V_t^1 = V, U_t = U],\\&(t,V,U) \in [0,T] \times \mathbf{R}_+ \times \mathbf{R},\end{aligned} \tag{15}$$

where U_t is the mispricing process given in equation (3). By applying Ito's formula for jump-diffusion processes and the Hamilton–Jacobi–Bellman (HJB) methodology, we get the following HJB equation

$$\begin{aligned}&\phi_t + \max_\pi \Big\{(\pi\sigma\theta^1 + r)V\phi_V - \lambda U\phi_U\\&+\frac{1}{2}(\pi\sigma V)^2\phi_{VV} + \frac{1}{2}\phi_{UU} + q\pi\sigma V\phi_{VU}\\&+\int_{\mathbf{R}\setminus\{0\}}[\phi(t, V + V\pi(e^x-1), U) - \phi(t,V,U)]\nu(\mathrm{d}x)\Big\}\\&= 0\end{aligned} \tag{16}$$

with terminal condition $\phi(T,V,U) = V^\gamma/\gamma$ for $(V,U) \in \mathbf{R}_+ \times \mathbf{R}$. We assume that $\phi(t,V,U)$ has the following form

$$\phi(t,V,U) = \frac{V^\gamma}{\gamma}\cdot\psi(t,U). \tag{17}$$

Then by substituting $\phi(t,V,U)$ into (16), we get the following PDE

$$\begin{aligned}&\psi_t + \max_\pi \Big\{\gamma(\pi\sigma\theta^1 + r)\psi - \lambda U\psi_U + \frac{1}{2}\gamma(\gamma-1)(\pi\sigma)^2\psi\\&+\frac{1}{2}\psi_{UU} + \gamma q\pi\sigma\psi_U\\&+\psi\int_{\mathbf{R}\setminus\{0\}}[(1+\pi(e^x-1))^\gamma - 1]\nu(\mathrm{d}x)\Big\} = 0\end{aligned} \tag{18}$$

with terminal condition $\psi(T,U) = 1$. By basic calculation in the maximization of equation (18), we find that the optimal portfolio π^* satisfies the following equation

$$\begin{aligned}\pi^* = &\frac{1}{(1-\gamma)\sigma^2}\Big[\mu^1 - r + q\sigma\frac{\psi_U}{\psi}\\&+\int_{\mathbf{R}\setminus\{0\}}(1+\pi^*(e^x-1))^{\gamma-1}(e^x-1)\nu(\mathrm{d}x)\Big].\end{aligned} \tag{19}$$

When $\gamma = 0$ (the logarithmic utility case), the solution to equation (18) is $\psi(t,U) = 1$ so that we can recover the optimal portfolio for informed investor as indicated in Theorem 2.1 since the extra term $q\sigma(\psi_U/\psi)$ in (19) becomes zero. Furthermore, when there is no jump ($\nu(\mathrm{d}x) = 0$) and no asymmetric information ($q = 0$), we obtain the usual Merton optimal portfolio

$$\pi_t^{\mathrm{Mer}} = \frac{\mu_t - r_t}{(1-\gamma)\sigma_t^2}.$$

REMARK 4 Zariphopoulou (2001) studied the optimal portfolio problem for one-dimensional investment diffusion model with stochastic volatility. When the utility function is of the separable CRRA type, she derived reduced form solutions for the value function and optimal portfolios which satisfy the Hamilton–Jacobi–Bellman equation. By using a power transformation, the value function becomes the solution to a linear parabolic equation and the explicit formula of the optimal portfolio is provided. Pham (2002) discussed the optimal investment problem for a multi-dimensional diffusion model with stochastic volatility. The classical HJB methodology is employed to characterize the value function which satsfies a nonlinear HJB equation. Note that the power transformation used in Zariphopoulou (2001) is no longer valid in the general multiple dimensional setting. Instead, by assuming CRRA power utility function and applying logarithmic transformation, the nonlinear HJB equation is reduced to a semilinear equation involving quadratic growth of the derivative term. The existence of a smooth solution and an optimal portfolio is established.

REMARK 5 For a detailed discussion on stochastic control of jump diffusions with applications to optimal portfolios in finance, we refer to Øksendal and Sulem (2009) and references therein. We now highlight three interesting papers on this topic. Liu *et al.* (2003) studied the optimal portfolio problem for jump-diffusion investment model with stochastic volatility and power utility. They provided analytical solution to the underlying HJB equation and showed that both the jump in price and volatility are important in determining the optimal portfolio for investors. Cvitanic *et al.* (2008) proposed an investment model driven by Lévy process with the jump arrival intensity depending on a state variable. By using HJB methodology, they discussed the optimal portfolio of an investor with CRRA utility and the sensitivity of the investment in the risky asset to the higher moments. Aït-Sahalia *et al.* (2009) studied portfolio choice problem for an investment model with jumps. They used the orthogonal decomposition tool to construct the optimal portfolio in closed form under power utility as well as other types of utility functions. They showed that the investor should focus on controlling the portfolio's exposure and diversification effects to jump risk.

Remark 6 For data analysis purposes, we focus on logarithm utility only, since we can transform a power utility to a logarithmic utility. That is, the log of a power utility is a log utility. Most importantly, the critical role of the instantaneous centralized moments of returns (ICMRs) in approximating optimal portfolios becomes very clear under logarithmic utility.

3. The pure-jump Lévy market

Financial researchers generally accept that asset prices and returns have jump components, and possibly, a diffusive component. However, many researchers now believe that asset prices and returns evolve by jumps only. Motivated by this, we study a toy market that has no diffusive component, and therefore, evolves by jumps only. This is an example of a *pure-jump Lévy market*. In this case, the market is driven only by a pure-jump Lévy process X, and no Brownian motion/diffusion is involved. As usual, our market has two assets. There is a bank account $\mathbf{A}$ earning deterministic risk-free compounded interest rate r. There is also a single risky asset, called stock, with price S which has log-returns dynamics for the ith investor, $i \in \{0, 1\}$, given by:

$$d(\log S_t) = \mu_t^i \, dt + \int_{\mathbf{R}\setminus\{0\}} xN(dt, dx), \tag{20}$$

where μ_t^i is the continuous returns on the stock for the ith investor. The stock has percentage return:

$$\frac{dS_t}{S_{t-}} = \mu_t^i \, dt + \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)N(dt, dx), \tag{21}$$

where μ_t^i is the continuous component of the total stock appreciation rate $b = \mu^i + M_1$, and $M_1 = \int_{\mathbb{R}\setminus\{0\}} (e^x - 1)\nu(dx)$ is the first instantaneous centralized moment of returns. Note that the percentage returns has two components: one continuous and locally deterministic (μ_t^i); one is discontinuous and driven by the Poisson random measure N. In fact, when $\sigma_t = 0$, we have $\mu_t^i = \mu_t$ for $i \in \{0, 1\}$, which leads to the same model for both investors.

What are Pure-Jump Models? Consider the asymmetric diffusion market:

$$d(\log S_t) = \left(\mu_t^i - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t \, dB_t + \int_{\mathbf{R}\setminus\{0\}} xN(dt, dx), \tag{22}$$

with percentage return dynamics

$$\frac{dS_t}{S_{t-}} = \mu_t^i \, dt + \sigma_t \, dB_t + \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)N(dt, dx). \tag{23}$$

Pure-jump models result from jump-diffusion models when the contribution from the diffusive volatility component σ_t is negligible relative to the total volatility $\sigma_{\text{Tot}} = \sqrt{\sigma_t^2 + M_2}$; that is, when $\sigma_t \approx 0$. Thus a pure-Jump market (PJM) is a diffusive market with $\sigma_t \approx 0$ for all t. Explicitly,

$$\begin{aligned} d(\log S_t) &= \mu_t^i \, dt + \int_{\mathbf{R}\setminus\{0\}} xN(dt, dx) \\ &= \lim_{\sigma_t \to 0} \left(\left(\mu_t^i - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t \, dB_t \right. \\ &\quad \left. + \int_{\mathbf{R}\setminus\{0\}} xN(dt, dx) \right), \end{aligned}$$

and

$$\mu_t^i = \mu_t + v_t^i \sigma_t = \mu_t + v_t^i \sigma_t \longrightarrow \mu_t.$$

The optimal portfolio for the stock with dynamic of equation (20) is therefore the limiting optimal portfolio of the stock with dynamic of equation (22) as $\sigma_t \to 0$. Since $\mu_t^i \longrightarrow \mu_t$ as $\sigma_t \longrightarrow 0$, then the dynamic for both investors is

$$\begin{aligned} d(\log S_t) &= \mu_t \, dt + \int_{\mathbf{R}\setminus\{0\}} xN(dt, dx) \\ &= \lim_{\sigma_t \to 0} \left(\left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t \, dB_t \right. \\ &\quad \left. + \int_{\mathbf{R}\setminus\{0\}} xN(dt, dx) \right). \end{aligned} \tag{24}$$

We have the following result.

Proposition 3.1 *Let* $G(\pi) = \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi(e^x - 1))\nu(dx)$ *be the (instantaneous centralized) moment generating function, where* $\nu(\cdot)$ *is the characteristic measure of the Poisson random measure* $N(dt, dx)$ *and* $\pi \in [0, 1]$. *If* $\int_{\mathbf{R}\setminus\{0\}} (e^{\pm x} - 1)^2 \nu(dx) < \infty$, *then for the pure-jump Lévy market in equation (20), the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth is the same for both informed and uninformed investors. It is completely deterministic and given by* π_t, *where*

$$G'(\pi_t) = r_t - \mu_t, \quad \pi_t \in [0, 1], \; t \in [0, T]. \tag{25}$$

Let w *be the initial investment of investors. The maximum expected logarithmic utility from terminal wealth for investors* *are equal, and is given by:*

$$u(w) = \log(w) + \int_0^T \left(\pi_t(\mu_t - r_t) + \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi(e^x - 1))\nu(\mathrm{d}x)\right) \mathrm{d}t, \tag{26}$$

where $1 + \pi(e^x - 1)$ *is the return on the portfolio when a log-jump of x occurs in the the stock price.*

REMARK 7 Note that the informed investor has no excess utility in this case, and hence, no utility advantage. If the solution to the equation $G'(\pi_t) = r_t - \mu_t$ falls outside the range [0, 1], the optimal portfolio is obtained by truncating the solution as follows: Take $\pi_t = 1$ if $\pi_t > 1$, since no borrowing is allowed; and take $\pi_t = 0$ if $\pi_t < 0$, since no short-selling is allowed.

We also have the following:

PROPOSITION 3.2 *There exists a deterministic process* η_t *between 0 and* π_t, *such that*

$$\pi_t = \frac{\mu_t - r_t + M_1}{|G''(\eta_t)|} = \frac{b_t - r_t}{|G''(\eta_t)|}, \tag{27}$$

where the total return on the stock is $b_t = \mu_t + M_1$, *and* $M_1 = \int_{\mathbf{R}\setminus\{0\}}(e^x - 1)\nu(\mathrm{d}x)$ *is the return due to the jumps. Moreover, under quadratic approximation of* $G(\cdot)$, *we have*

$$\pi_t \approx \frac{\mu_t - r_t + M_1}{M_2} = \frac{b_t - r_t}{M_2}. \tag{28}$$

3.1. The Double Poisson market

We now give an example of a pure-jump market driven by two independently-linked Poisson processes. This is called the *Double Poisson Market*. Consider a stock that has price dynamic:

$$\mathrm{d}(\log S_t) = \mu_t\, \mathrm{d}t + \mathrm{d}X_t, \quad t \in [0, T], \tag{29}$$

where

$$X_t = \alpha_u N^u(t) + \alpha_d N^d(t), \tag{30}$$

with

$$\alpha_u \in (0, \log\Delta), \quad \alpha_d = \log(\Delta - e^{\alpha_u}), \quad \Delta \in (1, 2].$$

For illustration purposes only, we fix $\Delta = 2$, which represents an upward jump of at most 100%. This is similar to the constraint in the double-exponential Kou market. Note that α_d is always *negative*, while α_u is always *positive*. X is called the *Double Poisson* Lévy process with parameters $\alpha_u, \alpha_d, \lambda_u, \lambda_d$, where N^u and N^d are independent Poisson processes with intensities λ_u and λ_d, respectively. In this model, N^u controls the upward jumps which have log-amplitude α_u, while N^d controls the downward jumps, which have log-amplitude α_d. As usual, μ is the continuous component of total returns. We denote a Double Poisson process by $\Pi(1)$. The Lévy measure for the Double Poisson process is:

$$\nu(\mathrm{d}x) \equiv \nu_{\Pi(1)}(\mathrm{d}x) = \lambda_u \delta_{\alpha_u}(\mathrm{d}x) + \lambda_d \delta_{\alpha_d}(\mathrm{d}x), \tag{31}$$

where

$$0 < \lambda_d \le \lambda_u < 1.$$

$\delta_a(\cdot)$ is the Dirac measure on $\mathcal{B}(\mathbf{R} - \{0\})$, with $\delta_a(A) = 1$, if $a \in A$, and 0 otherwise.

3.2. Maximization of expected logarithmic utility

Let $V^\pi = (V_t^\pi) \equiv V$ be the wealth process corresponding to the portfolio process $\pi = (\pi_t)_{t\in[0,T]}$, where $V_0^\pi = w > 0$, is the initial capital investment or wealth. As in Buckley *et al.* (2014, 2016), assume the existence of an *admissible set*, $\mathcal{A}_w$ from which qualified portfolios are chosen. The wealth process $V \equiv V^\pi$ satisfies the dynamics:

$$\mathrm{d}V_t = (1 - \pi_t) r_t V_{t-}\, \mathrm{d}t + \frac{\pi_t V_{t-}}{S_{t-}}\, \mathrm{d}S_t. \tag{32}$$

PROPOSITION 3.3 *Let* π *be the deterministic optimal portfolio that maximizes the expected logarithmic utility from terminal wealth in the symmetric Double Poisson market with parameters* μ, r, σ^2, λ_u, λ_d, α_u, α_d. *Let* $A_u = e^{\alpha_u} - 1$ *and* $a = 1/A_u$. *Then a > 1 and the admissible set is* $\mathcal{A}_w = (-a, a)$, *where w is the initial wealth of the investors.*

The optimal portfolio is

$$\pi = \begin{cases} \pi_+ & \text{if } \mu > r \\ \dfrac{\lambda_u - \lambda_d}{\lambda_u + \lambda_d} a & \text{if } \mu = r \\ \pi_- & \text{if } \mu < r \end{cases} \tag{33}$$

where

$$\pi_\pm = -\left(\frac{\lambda_u + \lambda_d}{2(\mu - r)}\right) \pm \sqrt{\left(\frac{\lambda_u + \lambda_d}{2(\mu - r)}\right)^2 + \frac{(\lambda_u - \lambda_d)a}{\mu - r} + a^2}, \tag{34}$$

provided $\pi \in \mathcal{A}_w = (-a, a)$.

The maximum expected logarithmic utility from terminal wealth is

$$u(w) = \log(w) + \int_0^T \left[\pi_t(\mu_t - r_t) + \lambda_d \log\left(1 + \frac{\pi_t}{a}\right) + \lambda_u \log\left(1 - \frac{\pi_t}{a}\right)\right] \mathrm{d}t. \tag{35}$$

REMARK 8 If the optimal portfolio falls outside the admissible set $\mathcal{A}_w = (-a, a)$, we simple take the optimal to be the endpoint of the admissible set closest to this optimal. In other words, we truncate the solution as follows: $\pi_t = a$, if $\pi_t > a$; and $\pi_t = -a$, if $\pi_t < -a$.

REMARK 9 If $\lambda_d = 0$, we have no downward jumps, and $G(\pi)$ is well defined if $\pi > -a = -(e^{\alpha_u} - 1)^{-1}$. The admissible set $\mathcal{A}_w$, may be as large as $(-a, \infty)$. The objective function reduces to $f(\pi) = \pi(\mu - r) + \lambda_u \log(1 + \pi\, (e^{\alpha_u} - 1))$, with

$$f'(\pi) = \mu - r + \frac{\lambda_u}{a + \pi} = 0 \quad \text{iff} \frac{\lambda_u}{a + \pi} = r - \mu.$$

This has a solution only if $r - \mu > 0$, since $a + \pi$ is positive. If $r \le \mu$, we have no optimal. If $r - \mu > 0$, the optimal portfolio is $\pi = -a + \lambda_u/(r-\mu)$, with maximum expected utility

$$u(w) = \log w + \left[a(r-\mu) - \lambda_u + \lambda_u \log\left(\frac{\lambda_u}{a(r-\mu)}\right)\right] T,$$

if the market coefficients μ_t and r_t are constants. Note that, unlike the models in Buckley *et al.* (2012, 2014, 2015, 2016), short-selling and borrowing are permitted in this framework because the admissible set is $\mathcal{A}_w = (-a, a)$, where $a > 1$.

4. The jump-diffusion market driven by the Double Poisson process

The Double Poisson jump-diffusion market consists of a single stock S and a bank account $\mathbf{A}$, which earns the deterministic risk-free interest r. The risky asset (stock) has log-return dynamics:

$$\mathrm{d}(\log S_t) = (\mu_t - \tfrac{1}{2}\sigma_t^2)\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t + \mathrm{d}X_t, \tag{36}$$

where $X_t = \int_{\mathbf{R}\setminus\{0\}} x N(t, \mathrm{d}x)$ is a $\Pi(1)$ process introduced in Section 3.1, called the *Double Poisson* Lévy process with parameters α_u, α_d, λ_u, and λ_d. X is also called a Skellam process, which is the difference of two independent Poisson processes. Applying Itô's formula to equation (36) yields the percentage returns dynamics:

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t + \int_{\mathbf{R}\setminus\{0\}} (e^x - 1) N(\mathrm{d}t, \mathrm{d}x). \tag{37}$$

The total return on the stock is $b_t = \mu_t + M_1$, where

$$\begin{aligned} M_1 &= \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)\nu(\mathrm{d}x) \\ &= (\lambda_u - \lambda_d)(e^{\alpha_u} - 1) = (\lambda_u - \lambda_d)A_u, \end{aligned} \tag{38}$$

and

$$A \equiv A_u = e^{\alpha_u} - 1 \tag{39}$$

is the upward jump size. Note that since $\alpha_u \in (0, \log 2)$, then

$$0 < A_u < 1. \tag{40}$$

4.1. *Maximization of logarithmic utility from terminal wealth*

Because of the relatively simple structure of the Lévy measure for the Double Poisson process, we are able to explicitly compute $G(\cdot)$, the moment generating function for the instantaneous returns presented in Buckley *et al.* (2014, 2016). The other models do not allow for this, hence approximation methods are required. Note that

$$\begin{aligned} G(\alpha) &= \int_{\mathbf{R}\setminus\{0\}} \log(1 + \alpha(e^x - 1))\nu_{\Pi(1)}(\mathrm{d}x) \\ &= \int_{\mathbf{R}\setminus\{0\}} \log(1 + \alpha(e^x - 1))(\lambda_u \delta_{\alpha_u}(\mathrm{d}x) + \lambda_d \delta_{\alpha_d}(\mathrm{d}x)) \\ &= \lambda_u \log(1 + \alpha(e^{\alpha_u} - 1)) + \lambda_d \log(1 + \alpha(e^{\alpha_d} - 1)). \end{aligned} \tag{41}$$

Since $\alpha_d = \log(2 - e^{\alpha_u})$, $A_u = e^{\alpha_u} - 1$ and $a = 1/A_u$, then it is easy to show that

$$G'(\alpha) = \frac{\lambda_u}{a + \alpha} - \frac{\lambda_d}{a - \alpha}, \tag{42}$$

and

$$G''(\alpha) = -\frac{\lambda_u}{(a+\alpha)^2} - \frac{\lambda_d}{(a-\alpha)^2}, \tag{43}$$

where $a = 1/A_u > 1$ is the inverse upward jump size. Since $\alpha \in (-a, a)$, $G(\alpha)$ is well defined. In this case, we take the admissible set to be $\mathcal{A}_w = (-a, a)$. Recall that there is no asymmetric information in the symmetric market. Therefore, the optimal portfolio π that maximizes logarithmic utility from terminal wealth for the symmetric Double Poisson market is given by

$$\pi = g(\pi) = \frac{\mu - r}{\sigma^2} + \frac{1}{\sigma^2}\left[\frac{\lambda_u}{a + \pi} - \frac{\lambda_d}{a - \pi}\right], \tag{44}$$

where $\pi \in (-a, a)$.

Remark 10 It follows directly from equation (19) that the optimal portfolio π that maximizes *power utility* from terminal wealth for the symmetric Double Poisson market is given by

$$\begin{aligned} \pi = g(\pi) &= \frac{\mu - r}{(1-\gamma)\sigma^2} + \frac{1}{(1-\gamma)\sigma^2} \\ &\times \left[\frac{\lambda_u}{(a+\pi)^{1-\gamma}} - \frac{\lambda_d}{(a-\pi)^{1-\gamma}}\right], \end{aligned} \tag{45}$$

where $\pi \in (-a, a)$ with $a > 1$ and $1 - \gamma > 0$, $1 - \gamma \neq 1$, is the coefficient of relative risk aversion. In this case, the first derivative of the moment generating function is:

$$G'(\pi) = \frac{\lambda_u}{(a+\pi)^{1-\gamma}} - \frac{\lambda_d}{(a-\pi)^{1-\gamma}}.$$

4.2. *Analytic solution of optimal portfolio*

We now give an analytic formula for the optimal portfolio in the symmetric Double Poisson market. It is easy to show that the optimal portfolio π in equation (44) is the unique root of the following cubic equation

$$\pi^3 + b\pi^2 + c\pi + d = 0, \tag{46}$$

where

$$b = -\pi_{\mathrm{Mer}} = -\frac{(\mu - r)}{\sigma^2}, \quad c = -(\tilde{\lambda}_u + \tilde{\lambda}_d + a^2),$$
$$d = -a(\tilde{\lambda}_u + \tilde{\lambda}_d + ab), \tag{47}$$

and

$$\tilde{\lambda}_u = \frac{\lambda_u}{\sigma^2}, \quad \tilde{\lambda}_d = \frac{\lambda_d}{\sigma^2}, \quad a > 1. \tag{48}$$

REMARK 11 The Merton optimal, $\pi_{\mathrm{Mer}} = (\mu - r)/\sigma^2$, is the optimal portfolio that obtains when the market is completely continuous (see Merton 1971). That is, when no jumps exist in the market, and the process driving the stock price is continuous geometric Brownian motion. The Merton optimal is a useful starting point in estimating optimal portfolios when jumps exist. We take it as the first approximate optimal in Newton's method or linear iteration when the optimal portfolio is being approximated in the more general jump model.

PROPOSITION 4.1 *The optimal portfolio π, that maximizes logarithmic utility from terminal wealth for investors in the symmetric Double Poisson market is*

$$\pi = -\frac{\left(b + \sqrt[3]{\frac{L+M}{2}} + \sqrt[3]{\frac{L-M}{2}}\right)}{3} = \frac{\left(\pi_{\mathrm{Mer}} - \sqrt[3]{\frac{L+M}{2}} - \sqrt[3]{\frac{L-M}{2}}\right)}{3}. \tag{49}$$

a is the inverse average upward jump-amplitude, b, c, d are given by equation (47), *and*

$$L = 2b^3 - 9bc + 27d,$$
$$K = b^2 - 3c,$$
$$M = \sqrt{L^2 - 4K^3}.$$

REMARK 12 Observe that the optimal portfolio is the fixed point of $g(\cdot)$, whence it may be obtained by iteration from the sequence:

$$\pi_{(n+1)} = g(\pi_{(n)}). \tag{50}$$

Because this cubic equation may admit up to three distinct solutions, whenever there are multiple roots, the optimal portfolio is the portfolio in the admissible set $(-a, a)$, which yields the highest expected utility. Moreover, if the optimal solution falls outside the range $(-a, a)$, we take the optimal portfolio as a truncated root as follows: Take $\pi_t = a$, if $\pi_t > a$; and take $\pi_t = -a$, if $\pi_t < -a$. Note that equation (44) or (46) is cubic, and therefore, it can also be easily solved using Newton's method or linear iteration.

5. The *m*-Double Poisson market

The Double Poisson market allows only one upward or downward jump of fixed amplitude in the stock price. This is a bit restrictive because real markets allow jumps of different sizes. Using the intuition gained from the Double Poisson model, we now make the model more realistic by augmentation. That is, by allowing a finite number of jumps of distinct amplitudes, each having a different frequency. We now introduce this general model, which contains the Double Poisson model as a special case. The m-Double Poisson jump diffusion market consist of a single stock S and a bank account $\mathbf{A}$, which earns the risk-free interest r. The risky asset (stock) has log-return dynamics:

$$\mathrm{d}(\log S_t) = \left(\mu_t - \frac{1}{2}\sigma_t^2\right)\mathrm{d}t + \sigma_t\,\mathrm{d}B_t + \mathrm{d}X_t, \tag{51}$$

$$X_t = \sum_{i=1}^{m} X_t^i, \tag{52}$$

$$X_t^i = \alpha_{u_i} N^{u_i}(t) + \alpha_{d_i} N^{d_i}(t), \quad i = 1, 2, \ldots, m, \tag{53}$$

where X_t^i is a Double Poisson process satisfying

$$\alpha_{u_i} \in (0, \log \Delta_i), \quad \alpha_{d_i} = \log(\Delta_i - e^{\alpha_{u_i}}),$$
$$\Delta_i \in (1, 2], \quad i = 1, 2, \ldots, m. \tag{54}$$

X is called the *m-Double Poisson* Lévy process with parameters α_{u_i}, α_{d_i}, λ_{u_i}, λ_{d_i}, where N^{u_i} and N^{d_i} are independent Poisson processes with intensities λ_{u_i} and λ_{d_i} respectively, with

$$0 < \lambda_{d_i} \le \lambda_{u_i} < 1, \quad i = 1, 2, \ldots, m. \tag{55}$$

We denote the m-Double Poisson process by $\Pi(m)$. For simplicity, we will fix the maximum upward log-jump amplitude for each independent pair to be $\Delta_i = 2,\ i = 1, \ldots, m$. As in other models, B is standard Brownian motion; σ and $\mu - \frac{1}{2}\sigma^2$ are deterministic continuous components of total volatility and log-returns, respectively.

5.1. *The intensities of* $\boldsymbol{\Pi(m)}$

We expect smaller jumps to occur more frequently than larger jumps. In addition, we expect upward jumps to occur more frequently than downward jumps (large downward jumps represent market crashes, etc). Consequently, we have the following restrictions on this model:

$$1 > \lambda_{u_1} > \lambda_{u_2} >, \ldots, > \lambda_{u_m} > 0; \tag{56}$$

$$0 < \lambda_{d_1} < \lambda_{d_2} <, \ldots, < \lambda_{d_m} < 1; \tag{57}$$

$$\lambda_{d_1} \le \lambda_{u_1}, \lambda_{d_2} \le \lambda_{u_2}, \ldots, \lambda_{d_m} \le \lambda_{u_m}. \tag{58}$$

We therefore expect the following constraints among jump amplitudes:

$$\alpha_{u_1} < \alpha_{u_2} <, \ldots, < \alpha_{u_m} \quad \text{and} \quad \alpha_{d_1} < \alpha_{d_2} <, \ldots, < \alpha_{d_m}.$$

In other words, for downward jumps, we have $|\alpha_{d_1}| > |\alpha_{d_2}| >, \ldots, > |\alpha_{d_m}|$, ensuring that jumps with smaller absolute sizes have higher frequencies, similar to those with upward jumps. The requirement that $0 < \lambda_{d_i} \le \lambda_{u_i} < 1$, ensures that the Poisson processes do not interrupt the continuous Geometric Brownian motion component of the stock's trajectory too often. For example, we could use $\lambda_{u_i} \le \frac{1}{10}$ (one large upward jump every 10 time units) and $\lambda_{d_i} \le \frac{1}{20}$ (one large downward jump every 20 time units) since the greater the amplitude of the jump, the lower its frequency.

5.2. The Lévy measure of $\Pi(m)$

The Lévy measure for the Double Poisson process

$$X_t^i = \alpha_{u_i} N^{u_i}(t) + \alpha_{d_i} N^{d_i}(t),$$

is

$$\nu_i(\mathrm{d}x) = \lambda_{u_i}\delta_{\alpha_{u_i}}(\mathrm{d}x) + \lambda_{d_i}\delta_{\alpha_{d_i}}(\mathrm{d}x), \quad i = 1, 2, \dots, m, \tag{59}$$

where $\delta(\cdot)$ is the Dirac measure on $\mathcal{B}(\mathbf{R} - \{0\})$. It follows easily that the Lévy measure for the m-Double Poisson process: $X_t = \sum_{i=1}^m [\alpha_{u_i} N^{u_i}(t) + \alpha_{d_i} N^{d_i}(t)]$, is

$$\begin{aligned} \nu(\mathrm{d}x) \equiv \nu_{\Pi(m)}(\mathrm{d}x) &\triangleq \sum_{i=1}^m \nu_i(\mathrm{d}x) \\ &= \sum_{i=1}^m [\lambda_{u_i}\delta_{\alpha_{u_i}}(\mathrm{d}x) + \lambda_{d_i}\delta_{\alpha_{d_i}}(\mathrm{d}x)]. \end{aligned} \tag{60}$$

Let $N(t, A)$ be the Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R} - \{0\})$ that counts the jumps of X in the time interval $(0, t)$. Then we can express X and its Lévy measure ν, respectively, as $X_t = \int_{\mathbf{R}\setminus\{0\}} xN(t, \mathrm{d}x)$ and $\nu(\mathrm{d}x) = \mathbf{E}[N(1, \mathrm{d}x)]$. The first instantaneous centralized moment of return is

$$\begin{aligned} M_1 &= \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)\nu(\mathrm{d}x) \\ &= \sum_{i=1}^m \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)\nu_i(\mathrm{d}x) = \sum_{i=1}^m M_1(i), \end{aligned}$$

where $M_1(i)$, the first instantaneous centralized return of the ith Double Poisson, is

$$\begin{aligned} M_1(i) &= \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)\nu_i(\mathrm{d}x) \\ &= \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)(\lambda_{u_i}\delta_{\alpha_{u_i}}(\mathrm{d}x) + \lambda_{d_i}\delta_{\alpha_{d_i}}(\mathrm{d}x)) \\ &= \lambda_{u_i}(e^{\alpha_{u_i}} - 1) + \lambda_{d_i}(e^{\alpha_{d_i}} - 1) \\ &= \lambda_{u_i}A_i - \lambda_{d_i}A_i = (\lambda_{u_i} - \lambda_{d_i})A_i, \end{aligned}$$

where the upward jump size of the ith double component is $A_i = e^{\alpha_{u_i}} - 1$. Thus

$$M_1 = \sum_{i=1}^m (\lambda_{u_i} - \lambda_{d_i})A_i. \tag{61}$$

By applying Itô's formula to the log-returns equation (51), we get the percentage returns dynamics:

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t + \int_{R\setminus\{0\}} (e^x - 1)N(\mathrm{d}t, \mathrm{d}x). \tag{62}$$

The total returns on the stock is

$$b_t = \mu_t + M_1 = \mu_t + \sum_{i=1}^m (\lambda_{u_i} - \lambda_{d_i})A_i, \tag{63}$$

which is the sum of the continuous expected return and expected return from each of the Double Poisson jump processes.

5.3. The $G(\cdot)$ function for the $\Pi(m)$ model

We can easily compute G, the moment generating function for instantaneous centralized returns given in Theorem 2.1 because of the relatively simple structure of the Lévy measure for this market.

PROPOSITION 5.1 *Let $\nu \equiv \nu_{\Pi(m)}$ be the Lévy measure for the m-Double Poisson process. Then for all $\alpha \in (-a, a)$, the moment generating function of instantaneous centralized returns is*

$$G(\alpha) = \log[\Pi_{i=1}^m (1 + \alpha A_i)^{\lambda_{u_i}} (1 - \alpha A_i)^{\lambda_{d_i}}], \tag{64}$$

where $A_i = e^{\alpha_{u_i}} - 1$ with $0 < A_i < 1$, $a_i = 1/A_i$ for $i = 1, \dots, m$, and $a = \min_{i=1,\dots,m}\{a_i\} > 1$.

REMARK 13 Unlike the models in Buckley *et al.* (2014, 2016), this model allows for borrowing ($\pi > 1$) and short selling ($\pi < 0$), since $G(\pi)$ exists in a symmetric admissible set $\mathcal{A}_w = (-a, a)$, where $a > 1$.

5.4. Maximization of expected utility from terminal wealth

Let π be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth at time $T > 0$, with $w > 0$ in initial investment. Unlike most of the other models, such as Kou (2002, 2007) and Variance Gamma model (e.g. Madan and Seneta 1990, Buckley *et al.* 2016), we will allow for the possibility of short-selling ($\pi < 0$) by borrowing stocks, selling them, and investing the proceeds in the bank account. We also allow for the possibility of borrowing money at the risk-free interest rate, to buy stocks ($\pi > 1$). The relaxing of the assumption $\pi \in [0, 1]$, is possible for this, and hence, the Double Poisson model, because the admissible set is $\mathcal{A}_w = (-a, a)$, where $a > 1$. Thus $G(\alpha)$, the moment generating function, exists in the interval $(-a, a) \supset (-1, 1)$. The optimal portfolio for the symmetric market now follows.

PROPOSITION 5.2 *Let the stock be driven by a symmetric m-Double Poisson process with parameters m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, $i = 1, 2, \dots, m$. The optimal portfolio π can be solved by applying Newton's method to the equation*

$\pi = (\mu - r)/\sigma^2 + G'(\pi)/\sigma^2$, *where*

$$G'(\pi) = \sigma^2 \sum_{i=1}^{m} \left[\frac{\tilde{\lambda}_{u_i}}{a_i + \pi} - \frac{\tilde{\lambda}_{d_i}}{a_i - \pi} \right] \tag{65}$$

and

$$G''(\pi) = -\sigma^2 \sum_{i=1}^{m} \left[\frac{\tilde{\lambda}_{u_i}}{(a_i + \pi)^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i - \pi)^2} \right]. \tag{66}$$

We give an equivalent form of Proposition 5.2 as follows. For each $i = 1, 2, \ldots, m$, define functions:

$$g,\ g_i : (-a,\ a) \to \mathbf{R},$$

by the prescription

$$g_i(\alpha) = \frac{\tilde{\lambda}_{u_i}}{a_i + \alpha} - \frac{\tilde{\lambda}_{d_i}}{a_i - \alpha}, \tag{67}$$

$$g(\alpha) = \pi_{\mathrm{Mer}} + \sum_{i=1}^{m} g_i(\alpha), \tag{68}$$

$$\pi_{\mathrm{Mer}} = \frac{\mu - r}{\sigma^2}. \tag{69}$$

As in the case of the Double Poisson market, this sequence will converge if $|g'(\alpha)| < 1$, for all $\alpha \in (-a, a)$. We have an analogous result for the m-Double Poisson market.

Proposition 5.3 *Let π be the optimal portfolio for the m-Double Poisson market. The sequence $\{\pi_n\}$, defined by*

$$\pi_{n+1} = g(\pi_n), \quad \pi_0 = \pi_{\mathrm{Mer}}$$

converges to π, if $g'(\alpha) > -1$ for all $\alpha \in (-a,\ a)$.

We have the following corollary which gives an equivalent condition in terms of the market parameters.

Corollary 5.4 *If*

$$\sum_{i=1}^{m} \left[\frac{\tilde{\lambda}_{u_i}}{(a_i + \alpha)^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i - \alpha)^2} \right] < 1, \tag{70}$$

for all $\alpha \in (-a, a)$, then $\pi_n \to \pi$ as $n \to \infty$.

If no short-selling or borrowing from the bank account is allowed, then α is restricted to the interval [0, 1]. Thus, if

$$\max_{\alpha \in [0,1]} \sum_{i=1}^{m} \left[\frac{\tilde{\lambda}_{u_i}}{(a_i + \alpha)^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i - \alpha)^2} \right] < 1, \tag{71}$$

we get convergence. This leads to the following:

Proposition 5.5 *Let $\alpha \in [-1,\ 1]$ in the m-Double Poisson market, then*

$$|g'(\alpha)| < \frac{2m\lambda_{u_1}}{\sigma^2 (a - 1)^2}. \tag{72}$$

Moreover, if $\lambda_{u_1} < \sigma^2 (a-1)^2 / 2m$ then $\pi_n \to \pi$, as $n \to \infty$. Equivalently, if $\lambda_{u_1} < \sigma^2 (1 - A)^2 / 2mA^2$, we have convergence, where $A = 1/a$.

5.5. Instantaneous centralized moments of returns for $\mathbf{\Pi}(m)$ market

The kth instantaneous centralized moment of returns for the m-Double Poisson market is

$$M_k^{(m)} \equiv M_k = \int_{\mathbf{R} \setminus \{0\}} (e^x - 1)^k \nu_{\Pi(m)}(\mathrm{d}x). \tag{73}$$

We have the following result.

Proposition 5.6 *Let M_k be the kth instantaneous centralized moment of returns for any Lévy market with dynamic in equation (53). The total instantaneous variance at time $t \in [0, T]$ is*

$$\mathrm{VAR} = \sigma_t^2 + M_2. \tag{74}$$

Moreover, the Skewness and excess Kurtosis of the instantaneous return at time t are respectively:

$$\mathrm{SKEW} = \frac{M_3}{(\sigma_t^2 + M_2)^{3/2}}. \tag{75}$$

$$\mathrm{KURT} - 3 = \frac{M_4}{(\sigma_t^2 + M_2)^2}. \tag{76}$$

Remark 14 Note that when $\mathrm{d}S_t/S_t$ is driven only by Brownian motion with drift, it has zero skewness and a Kurtosis of 3. Therefore skewness and excess kurtosis in the model comes only from the jump component. This result is general and applies to any model with diffusive and jump components.

We now give the M_ks in terms of market parameters.

Proposition 5.7 *Let $M_k^{(m)} \equiv M_k$ be the kth centralized moment of the instantaneous returns for a stock in a m-Double Poisson market with parameters $m,\ \mu,\ r,\ \sigma^2,\ \lambda_{u_i},\ \lambda_{d_i},\ \alpha_{u_i},\ \alpha_{d_i}$ for $i = 1, 2, \ldots, m$. Then*

$$\begin{aligned} M_k &= \sum_{i=1}^{m} M_k(i), \\ M_k(i) &= (\lambda_{u_i} + (-1)^k \lambda_{d_i}) A_i^k. \end{aligned} \tag{77}$$

where $A_i = e^{\alpha_{u_i}} - 1$ is the upward jump size of the ith Double Poisson process X_t^i for $i = 1, 2, \ldots, m$.

A direct consequence of the last result is:

Corollary 5.8 *Let $M_k^{(m)} \equiv M_k$ be the kth centralized moment of the instantaneous return for a stock in a m-Double Poisson market with parameters $m,\ \mu,\ r,\ \sigma^2,\ \lambda_{u_i},\ \lambda_{d_i},\ \alpha_{u_i},\ \alpha_{d_i},\ i = 1, 2, \ldots, m$. Then*

$$M_{2k} = \sum_{i=1}^{m} (\lambda_{u_i} + \lambda_{d_i}) A_i^{2k}, \tag{78}$$

$$M_{2k+1} = \sum_{i=1}^{m} (\lambda_{u_i} - \lambda_{d_i}) A_i^{2k+1}, \tag{79}$$

where A_i is the upward jump size in the ith Double Poisson process X_t^i. Moreover, the Variance, Skewness, and Excess

Kurtosis of the instantaneous return $\mathrm{d}S_t/S_t$, *are respectively*

$$\text{VARIANCE} = \sigma_t^2 + \sum_{i=1}^{m} \lambda_i^+ A_i^2, \tag{80}$$

$$\text{SKEWNESS} = \frac{\sum_{i=1}^m \lambda_i^- A_i^3}{(\sigma_t^2 + \sum_{i=1}^m \lambda_i^+ A_i^2)^{3/2}}, \tag{81}$$

$$\text{KURT} - 3 = \frac{\sum_{i=1}^m \lambda_i^+ A_i^4}{(\sigma_t^2 + \sum_{i=1}^m \lambda_i^+ A_i^2)^2}, \tag{82}$$

where

$$\lambda_i^+ \triangleq \lambda_{u_i} + \lambda_{d_i} \quad \text{and} \quad \lambda_i^- \triangleq \lambda_{u_i} - \lambda_{d_i}. \tag{83}$$

REMARK 15 Observe from equation (81) that since $\lambda_{u_i} \geq \lambda_{d_i}$, we always have zero or positive skewness. In fact, if $\lambda_{u_i} > \lambda_{d_i}$ for some $i \leq m$, we have positive skewness, and the returns are skewed to the right. In addition, from equation (82) $\lambda_{u_i} + \lambda_{d_i} > 0$ implies that $KURT - 3 > 0$, namely the stock return always has excess kurtosis.

PROPOSITION 5.9 *For the m-Double Poisson market with parameters* m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, *for* $i = 1, 2, \ldots, m$, *we have*

$$M_k \leq \frac{2m\,\lambda_{u_1}}{a^k} \leq \frac{2m}{a^k}. \tag{84}$$

Moreover, as $k \longrightarrow \infty$

$$M_k \longrightarrow 0. \tag{85}$$

REMARK 16 Since $1 > \lambda_{u_i} \geq \lambda_{d_i}$, we see that $\lambda_{u_i} + (-1)^k \lambda_{d_i} \geq 0$. Thus for each $k \leq m$,

$$0 \leq M_k < \frac{2m}{a^k}.$$

However, if we relax the condition that $\lambda_{u_i} \geq \lambda_{d_i}$ then equation (84) still holds.

5.6. *Optimal portfolios for asymmetric m-Double Poisson market*

We now give the relationship between the symmetric and asymmetric optimal portfolios. As with the other models, we give the optimal portfolios for investors in the asymmetric Double Poisson market.

PROPOSITION 5.10 *Let* π *be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth in the symmetric m-Double Poisson market with parameters* m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, *for* $i = 1, 2, \ldots, m$. *Let* π^i *be the optimal portfolio for the ith investor in the asymmetric market. For each* $t \in [0, T]$, *there exists* η_t^i *between* π *and* π^i *such that*

$$\pi_t^i = \pi_t + v_t^i \frac{\sigma_t}{\sigma_t^2 + |G''(\eta_t^{(i)})|}, \tag{86}$$

provided $\pi^i \in (-a, a)$, *where* $A_i = e^{\alpha_{u_i}} - 1$, $a_i = 1/A_i$, $a = \min_{i=1,\ldots,m}\{a_i\} > 1$,

$$\upsilon_t^0 = -\lambda \int_0^t e^{-\lambda(t-u)}(1 + \gamma(u))\,\mathrm{d}B_u^0,$$

$$\upsilon_t^1 = -\lambda q \int_0^t e^{-\lambda(t-u)}\,\mathrm{d}B_u^1,$$

$$\gamma(u) = \frac{1 - p^2}{1 + p\tanh(\lambda p u)} - 1, \quad p^2 + q^2 = 1, \quad p \geq 0,\ q \geq 0,$$

and

$$G''(\alpha) = -\sum_{i=1}^{m} \left[\frac{\lambda_{u_i}}{(a_i + \alpha)^2} + \frac{\lambda_{d_i}}{(a_i - \alpha)^2} \right].$$

In particular, under quadratic approximation of G, *we have*

$$\pi_t^i \approx \pi_t + v_t^i \left(\frac{\sigma_t}{\sigma_t^2 + \sum_{i=1}^m (\lambda_{u_i} + \lambda_{d_i})\, A_i^2} \right). \tag{87}$$

We now give the asymmetric optimal portfolio directly, without approximation.

THEOREM 5.11 *Let* π^i *be the optimal portfolio for the ith investor that maximizes the expected logarithmic utility from terminal wealth in the asymmetric m-Double Poisson market with parameters* m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, *for* $i = 1, 2, \ldots, m$. *Then*,

$$\pi_t^i = \frac{\mu_t - r_t + \upsilon_t^i \sigma_t + G'(\pi_t^i)}{\sigma_t^2}, \quad t \in [0, T], \tag{88}$$

provided $\pi_t^i \in (-a, a)$, *where* υ_t^0, υ_t^1 *are given by equation* (7), $A_i = e^{\alpha_{u_i}} - 1$, $a_i = 1/A_i$, $a = \min_{i=1,\ldots,m}\{a_i\} > 1$, *and*

$$G'(\alpha) = \sum_{i=1}^{m} \left[\frac{\lambda_{u_i}}{(a_i + \alpha)} - \frac{\lambda_{d_i}}{(a_i - \alpha)} \right].$$

REMARK 17 Equation (88) can be solved either by Newton's method or linear iteration. If the solution to equation (88) falls outside the admissible set $(-a, a)$, we take the optimal portfolio as boundary point of the admissible set closest to solution satisfying the equation, which we call the truncated root. For example, if the optimal solution falls outside the range $(-a, a)$, we take the optimal portfolio as a truncated root as follows: Take $\pi_t = a$, if $\pi_t > a$; and take $\pi_t = -a$, if $\pi_t < -a$. The same caveat applies for the approximate optimal solution presented in Proposition 5.10.

5.7. *A numerical example*

EXAMPLE 1 (Optimal Portfolio of Symmetric 4-Double Poisson Market)

Input parameters: m, μ, r, σ^2, λ_{u_i}, λ_{d_i}, α_{u_i}, α_{d_i}, where $i = 1, 2, 3, \ldots, m$.

Set $m = 4$, $\pi_{\text{Mer}} = \frac{\mu - r}{\sigma^2}$, $\tilde{\lambda}_{u_i} = \frac{\lambda_{u_i}}{\sigma^2}$, $\tilde{\lambda}_{d_i} = \frac{\lambda_{d_i}}{\sigma^2}$, $A_i = e^{\alpha_{u_i}} - 1$, $a_i = \frac{1}{A_i}$.

Let $a = \min_{1 \leq i \leq 4}\{a_i\}$. For $\pi \in (-a,\ a)$, set

$$g(\pi) \equiv g^{(4)}(\pi) = \pi_{\text{Mer}} + \sum_{i=1}^{4} \left(\frac{\tilde{\lambda}_{u_i}}{a_i + \pi} - \frac{\tilde{\lambda}_{d_i}}{a_i - \pi} \right) \tag{89}$$

Table 1. Optimal portfolios in the symmetric market.

Stock	r	μ	σ	π_{mer}	π^*	A_{u_1}	A_{u_2}	A_{u_3}	A_{u_4}
1	.0036	.056616	.2428	.90	.899982	.1	.3	.5	.7
2	.0243	.074392	.2428	.10	.101731	.1	.3	.5	.7
3	.0546	.095889	.2428	.70	.700101	.1	.3	.5	.7
4	.0082	.040606	.2428	.55	.550260	.1	.3	.5	.7

1. Set error $\epsilon = 0.5 \times 10^{-d}$, where $d \in \{4, 5, 6, 7\}$
2. Set $\pi_0 = \pi_{\text{Mer}}$
3. Generate a sequence $\{\pi_n\}$ by the prescription $\pi_{n+1} = g(\pi_n)$
4. Stop if $|\pi_{n+1} - \pi_n| \leq \epsilon$ and take optimal to be $\pi \approx \pi_{n+1}$
5. Otherwise, set $n = n + 1$, and go to 3

Lévy Density: $\nu_{\Pi(4)}(\mathrm{d}x) = \sum_{i=1}^{4} \lambda_{u_i}(\delta_{\alpha_{u_i}}(\mathrm{d}x) + \lambda_{d_i}\delta_{\alpha_{d_i}}(\mathrm{d}x))$

ICMR: $M_k = \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)^k \nu_{\Pi(4)}(\mathrm{d}x)$

$M_1 = 0.072548$, $M_2 = 0.07496$, $M_3 = 0.015297$, $M_4 = 0.025064$, $M_5 = 0.005556$

Input Parameters:

$\lambda_{u_1} = \frac{1}{5}$, $\lambda_{u_2} = \frac{1}{10}$, $\lambda_{u_3} = \frac{1}{15}$, $\lambda_{u_4} = \frac{1}{20}$; $\lambda_{d_4} = \frac{1}{25}$, $\lambda_{d_3} = \frac{1}{30}$, $\lambda_{d_2} = \frac{1}{35}$, $\lambda_{d_1} = \frac{1}{40}$

$\alpha_{u_1} = \ln(1.1)$, $\alpha_{u_2} = \ln(1.3)$, $\alpha_{u_3} = \ln(1.5)$, $\alpha_{u_4} = \ln(1.7)$

Admissible Set: $\mathcal{A}_w = (-1.4282,\ 1.4286)$

Table 1 displays equal deterministic optimal portfolios (π^*) for both investors having logarithmic utility in the symmetric 4-Double Poisson for toy stocks with different risk-free rates, expected returns and volatility. Borrowing and short-selling are allowed because the admissible set is $(-1.4282,\ 1.4286)$. Upward jump amplitudes are 0.1, 0.3, 0.5 and 0.7, with corresponding frequencies $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{15}$, and $\frac{1}{20}$. Downward jump amplitudes are -0.1, -0.3, -0.5 and -0.7, with corresponding frequencies $\frac{1}{25}$, $\frac{1}{30}$, $\frac{1}{35}$, and $\frac{1}{40}$. The Merton optimal obtains when no jump exists.

Trajectories of Approximate Asymmetric Optimal Portfolios

By Proposition 5.10, under quadratic approximation, the approximate random optimal portfolios under λ level of mispricing and $100q^2\%$ of asymmetric information for the 4-Double Poisson market are

$$\pi_t^{*,i} = \pi^* + \left(\frac{\sigma}{\sigma^2 + M_2}\right) \upsilon_t^i = \pi^* + 1.81313318\upsilon_t^i, \quad (90)$$

where $i = 0,\ 1,\ p = \sqrt{1 - q^2}$, and υ_t^0, υ_t^1 are defined in Proposition 5.10. Trajectories of υ_t^0 and υ_t^1 can be computed numerically at time points t, ranging from 0 to 1, in steps of 0.1. Asymmetric information (q^2), ranges from 0 to 1, in steps of 25%.

Trajectories of Exact Asymmetric Optimal Portfolios

One may also calculate the *random* asymmetric optimal portfolios directly using Theorem 5.11 along with linear iteration or Newton's method applied to the equation (Figures 1 and 2)

$$\pi_t^i = \frac{\mu_t - r_t + \upsilon_t^i \sigma_t + G'(\pi_t^i)}{\sigma_t^2}, \qquad t \in [0, T].$$

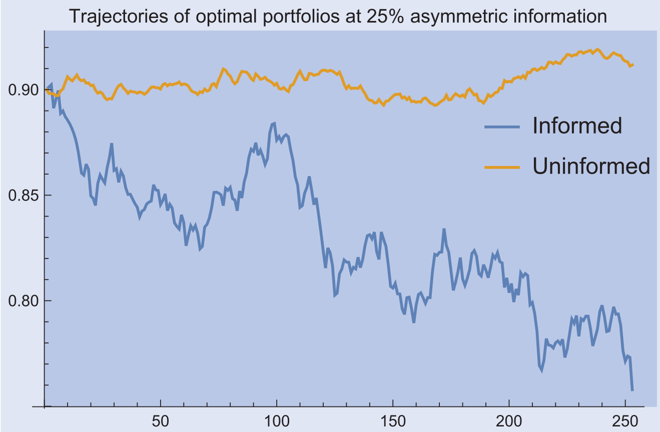

Figure 1. The plots show trajectories of optimal portfolios at 25% asymmetric information in the 4-Double Poisson market as a function of investment horizon (252 trading days) and asymmetric information for Stock 1, with $\pi^* = 0.8998$, $q^2 = 0.25$, $\lambda = 0.2$, $r = 0.0036$, $\mu = .056616$, $\sigma = 0.2428$.

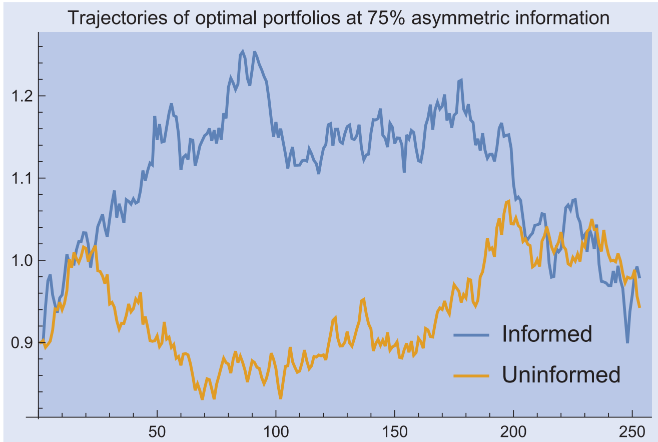

Figure 2. The plots show trajectories of optimal portfolios at q^2 =75% asymmetric information in the 4-Double Poisson market as a function of investment horizon (252 trading days) and asymmetric information for Stock 1 with the same parameters.

6. Conclusion

We present a novel finite activity model of mispricing under asymmetric information called the m-Double Poisson market, which is driven by m independent pairs of independent Poisson processes (Skellam processes). Each pair consists of two independent processes—one with a positive jump-amplitude and the other with negative jump-amplitude. The jump-amplitudes are fixed but linked by a simple formula. This model offers a realistic alternative to simulating actual evolution of asset prices, since only a finite number of jumps actually occur in asset prices during any time period.

We also study the Double Poisson pure-jump Lévy market, where no diffusion component exists, and show that both investors hold the same deterministic portfolio to achieve maximum expected utility from terminal wealth. In this case, the informed investor has no utility advantage over the uninformed. We then consider a more sophisticated market which has a diffusive component in addition to a pure-jump component driven by m-Double Poisson processes.

Under asymmetric information, the optimal portfolio is equal to the deterministic optimal portfolio of the symmetric market plus noise, and therefore, can be estimated from the symmetric optimal portfolio. Because the asymmetric optimal portfolio is closely linked to the symmetric optimal portfolio, our study focuses on the deterministic symmetric optimal portfolio, which provides the main ingredient for the asymmetric portfolios. We estimate asymmetric optimal portfolios using Newton's method and instantaneous centralized moments of returns (ICMR) to approximate optimal portfolios and maximum expected utility for each investor. Jump-amplitudes and frequencies are employed to report mean, variance, skewness and kurtosis of instantaneous returns. Unlike the extant models (e.g. Guasoni 2006, Buckley *et al.* 2014, 2016), the m-Double Poisson model allows for short selling and borrowing, and is therefore potentially richer in applications.

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References

Aït-Sahalia, Y., Cacho-Diaz, J. and Hurd, T.R., Portfolio choice with jumps: A closed-form solution. *Ann. Appl. Probab.*, 2009, **19**(2), 556–584.

Applebaum, D., *Lévy Processes and Stochastic Calculus*, 2004 (Cambridge University Press: Cambridge).

Baksi, G. and Madan, D., A theory of volatility spread. *Manage. Sci.*, 2006, **52**, 1945–1956.

Barberis, N. and Huang, M., Stocks as lotteries: The implications of probability weighting for security prices. *Am. Econ. Rev.*, 2008, **98**(5), 2066–2100.

Bartram, S. and Grinblatt, M., Agnostic fundamental analysis works. *J. Financ. Econ.*, 2018, **128**(1), 125–147.

Biais, B., Bossaerts, P. and Spatt, C., Equilibrium asset pricing and portfolio choice under asymmetric information. *Rev. Financ. Stud.*, 2010, **23**(4), 1503–1543.

Brunnermeier, M., *Asset Pricing Under Asymmetric Information: Bubbles, Crashes, Technical Analysis, and Herding*, 2001 (Oxford University Press: Oxford).

Buckley, W.S. and Long, H., A discontinuous mispricing model under asymmetric information. *Eur. J. Oper. Res.*, 2015, **243**(3), 944–955.

Buckley, W. and Perera, S., Optimal demand in a mispriced asymmetric Carr-Geman-Madan-Yor (CGMY) economy. *Ann. Finance*, 2019, **15**(3), 337–368.

Buckley, W.S., Brown, G.O. and Marshall, M., A mispricing model of stocks under asymmetric information. *Eur. J. Oper. Res.*, 2012, **221**(3), 584–592.

Buckley, W., Long, H. and Perera, S., A jump model for fads in asset prices under asymmetric information. *Eur. J. Oper. Res.*, 2014, **236**, 200–208.

Buckley, W., Long, H. and Perera, S., Link between asymmetric and symmetric optimal portfolios in fads markets. *Math. Finance Lett.*, 2015, **6**, 1–12.

Buckley, W., Long, H. and Marshall, M., Numerical approximation of optimal portfolios in mispriced asymmetric Lévy markets. *Eur. J. Oper. Res.*, 2016, **252**, 676–686.

Chabi-Yo, F., Conditioning information and variance bounds on pricing kernels with higher-order moments: Theory and evidence. *Rev. Financ. Stud.*, 2008, **21**, 181–231.

Conrad, J., Dittmar, R.F. and Ghysels, E., Ex ante skewness and expected stock returns. *J. Finance*, 2013, **68**(1), 85–124.

Cvitanic, J., Polimenis, V. and Zapatero, F., Optimal portfolio allocation with higher moments. *Ann. Finance*, 2008, **4**(1), 1–28.

De Athayde, G.M. and Flores Jr., R.G., Finding a maximum skewness portfolio: A general solution to three-moments portfolio choice. *J. Econ. Dyn. Control*, 2004, **28**, 1335–1352.

Easley, D. and O'Hara, M., Information and the cost of capital. *J. Finance*, 2004, **59**, 1553–1583.

Guasoni, P., Asymmetric information in fads models. *Finance Stoch.*, 2006, **10**, 159–177.

Harvey, C.R., Liechy, J.C., Liechy, M.W. and Müller, P., Portfolio selection with higher moments. *Quant. Finance*, 2010, **10**(5), 469–485.

Jondeau, E. and Rockinger, M., Optimal portfolio allocation under higher moments. *Eur. Financial Manag.*, 2006, **12**, 29–55.

Kelly, B. and Ljungqvist, A., Testing asymmetric information asset pricing models. *Rev. Financ. Stud.*, 2012, **25**(5), 1366–1413.

Kirch, M. and Runggaldier, W., Efficient hedging when asset prices follow a geometric Poisson process with unknown intensities. *SIAM J. Control Optim.*, 2004, **43**, 1174–1195.

Kou, S., A jump-diffusion model for option pricing. *Manage. Sci.*, 2002, **48**, 1086–1101.

Kou, S., Jump-diffusion models for asset pricing in financial engineering. In *Handbooks in Operations Research and Management Science: Financial Engineering*, edited by J. Birge and V. Linetsky, Vol. 15, pp. 73–116, 2007 (Elsevier: New York).

Liu, J., Longstaff, F.A. and Pan, J., Dynamic asset allocation with event risk. *J. Finance*, 2003, **58**, 231–259.

Madan, D. and Seneta, E., The VG model for share market returns. *J. Bus.*, 1990, **63**, 511–524.

Markowitz, H., Portfolio selection. *J. Finance*, 1952, **7**, 77–91.

Martin, I.W.R., Consumption-based asset pricing with higher cumulants. *Rev. Econ. Stud.*, 2013, **80**(2), 745–773.

Merton, R.C., Optimum consumption and portfolio rules in a continuous-time model. *J. Econ. Theory*, 1971, **3**, 373–413.

Øksendal, B. and Sulem, A., *Applied Stochastic Control of Jump Diffusions*, 2009 (Springer-Verlag: Berlin, Germany).

Pham, H., Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints. *Appl. Math. Optim.*, 2002, **46**, 55–78.

Protter, P., *Stochastic Integration and Differential Equations*, 2004 (Springer Verlag: New York).

Serrano-Padial, R., Naive traders and mispricing in prediction markets. *J. Econ. Theory*, 2012, **147**, 1882–1912.

Shiller, R., Do stock prices move too much to be justified by subsequent changes in dividends? *Am. Econ. Rev.*, 1981, **71**, 421–436.

Summers, L., Does the stock market rationally reflect fundamental value? *J. Finance*, 1986, **41**, 591–601.

Vayanos, D. and Wang, J., Liquidity and asset returns under asymmetric information and imperfect competition. *Rev. Financ. Stud.*, 2012, **25**, 1339–1365.
Wang, J., A model of intertemporal asset prices under asymmetric information. *Rev. Econ. Stud.*, 1993, **60**, 249–282.
Yang, C.W. and Hung, K., A generalized Markowitz portfolio selection model with higher moments. *Int. Rev. Account. Bank. Finance*, 2010, **2**, 1–7.
Zariphopoulou, T., A solution approach to valuation with unhedgeable risks. *Finance Stoch.*, 2001, **5**, 61–82.
Zhang, H. and Yan, C., Modelling fundamental analysis in portfolio selection. *Quant. Finance*, 2018, **18**(8), 1315–1326.

Appendix

All proofs are provided in this section.

Proof of Proposition 3.1 By Theorem 2.1, equation (24) yields optimal portfolio π given by $\sigma_t^2\pi = \mu_t - r_t + G'(\pi)$. When $\sigma_t = 0$, we get $0 = \mu_t - r_t + G'(\pi)$. Thus the optimal portfolio for both investors is the deterministic portfolio $\pi_t^i = \pi_t$, where $G'(\pi_t) = r_t - \mu_t$. Since both optimal portfolios are equal, the informed investor has no excess utility. ■

Proof of Proposition 3.2 Let $G(\pi) = \int_{\mathbf{R}\setminus\{0\}} \log(1+\pi(e^x-1))\, v(dx)$. Then $M_1 = G'(0) = \int_{\mathbf{R}\setminus\{0\}}(e^x-1)v(\mathrm{d}x)$. Since π is optimal, it obeys the equation $G'(\pi_t) = r_t - \mu_t$. By the Mean Value theorem, there exists $\eta_t \in (0, \pi_t)$ such that $\pi_t G''(\eta_t) = G'(\pi_t) - G'(0) = r_t - \mu_t - M_1$. Since $G''(\pi) = -\int_{\mathbb{R}\setminus\{0\}}((e^x-1)^2/[1+\pi(e^x-1)]^2)\, v(\mathrm{d}x) < 0$ for all π, then $\pi_t = (\mu_t - r_t + M_1)/-G''(\eta_t) = (\mu_t - r_t + M_1)/|G''(\eta_t)| = (b_t - r_t)/|G''(\eta_t)|$. Under quadratic approximation of G, $G''(\eta_t) \approx G''(0) = -M_2$, and the result follows. ■

Proof of Proposition 3.3 The optimal portfolio is deterministic because the input parameters $(\mu, r, \sigma^2, \lambda_u, \lambda_d, \alpha_u, \alpha_d)$ are all deterministic functions or constants (see Merton 1971, or alternatively the Hamilton-Jacobi-Bellman (HJB) methodology). Let $N(t, A)$ be the Poisson random measure that counts the jumps of X defined by equation (30) in the set $A \in \mathbf{B}(\mathbf{R} - \{0\})$ in the time interval $(0, t)$. Set $J_t = \int_0^t \int_{\mathbf{R}\setminus\{0\}}(e^x - 1)N(\mathrm{d}s, \mathrm{d}x)$. From equation (32), we have

$$
\begin{aligned}
\frac{\mathrm{d}V_t}{V_{t-}} &= (1-\pi_t)r_t\,\mathrm{d}t + \frac{\pi_t\,\mathrm{d}S_t}{S_{t-}} = (1-\pi_t)r_t\,\mathrm{d}t + \pi_t(\mu_t\,\mathrm{d}t + \mathrm{d}J_t)\\
&= (\pi_t(\mu_t - r_t) + r_t)\,\mathrm{d}t + \pi_t\,\mathrm{d}J_t\\
&= (r_t + \pi_t(\mu_t - r_t))\,\mathrm{d}t + \int_{\mathbf{R}\setminus\{0\}} \pi_t(e^x-1)N(\mathrm{d}t, \mathrm{d}x),
\end{aligned}
$$

which has solution (e.g. Applebaum 2004, Protter 2004)

$$
\begin{aligned}
V_t = V_0 \exp\left(\int_0^t r_s\,\mathrm{d}s + \int_0^t \pi_s(\mu_s - r_s)\,\mathrm{d}s\right)\\
\times \Pi_{0\le s\le t}(1 + \pi_s(e^{\Delta X_s} - 1)).
\end{aligned}
$$

Let the discounted wealth process be $\widetilde{V}_t \triangleq V_t \exp(-\int_0^t r_s\,\mathrm{d}s)$. Then

$$
\widetilde{V}_t = V_0 \exp\left(\int_0^t \pi_s(\mu_s - r_s)\,\mathrm{d}s\right) \Pi_{0\le s\le t}(1 + \pi_s(e^{\Delta X_s} - 1)).
$$

The logarithmic utility of terminal wealth is

$$
\begin{aligned}
u(\widetilde{V}_T) = \log \widetilde{V}_T &= \log w + \int_0^T \pi_s(\mu_s - r_s)\,\mathrm{d}s\\
&\quad + \sum_{0\le s\le T} \log(1 + \pi_s(e^{\Delta X_s} - 1))\\
&= \log w + \int_0^T \pi_t(\mu_t - r_t)\,\mathrm{d}t\\
&\quad + \int_0^T \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi_t(e^x - 1))N(\mathrm{d}t, \mathrm{d}x).
\end{aligned}
$$

The expected logarithmic utility from terminal wealth is

$$
\begin{aligned}
\mathbf{E}(\log \widetilde{V}_T) &= \log w + \mathbf{E}\int_0^T \pi_t(\mu_t - r_t)\,\mathrm{d}t\\
&\quad + \int_0^T \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi_t(e^x - 1))\mathbf{E}N(\mathrm{d}t, \mathrm{d}x)\\
&= \log w + \int_0^T \Bigg[\pi_t(\mu_t - r_t)\\
&\quad + \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi_t(e^x - 1))v(\mathrm{d}x)\Bigg]\mathrm{d}t,
\end{aligned}
$$

since all market coefficients are deterministic. Define a function $f : \mathcal{A}_w \to \mathbf{R}$ by the prescription

$$
\begin{aligned}
f(\pi) &= \pi(\mu - r) + \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi(e^x - 1))v(\mathrm{d}x)\\
&= \pi(\mu - r) + G(\pi).
\end{aligned}
$$

Then

$$
\mathbf{E}(\log \widetilde{V}_T) = \log w + \int_0^T f(\pi_s)\,\mathrm{d}s,
$$

and

$$
\begin{aligned}
u(w) = \max_{\pi_s\in\mathcal{A}_w} \mathbf{E}(\log \widetilde{V}_T) &= \log w + \max_{\pi_s\in\mathcal{A}_w} \int_0^T f(\pi_s)\,\mathrm{d}s\\
&= \log w + \int_0^T \max_{\pi_s\in\mathcal{A}_w} f(\pi_s)\,\mathrm{d}s,
\end{aligned}
$$

where π maximizes the objective function $f(\pi)$. Thus

$$
\begin{aligned}
f(\pi) &= \pi\,(\mu - r) + G(\pi)\\
&= \pi(\mu - r) + \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi(e^x - 1))v(\mathrm{d}x)\\
&= \pi(\mu - r) + \int_{\mathbf{R}\setminus\{0\}} \log(1 + \pi(e^x - 1))\\
&\quad \times (\lambda_u\delta_{\alpha_u}(\mathrm{d}x) + \lambda_d\delta_{\alpha_d}(\mathrm{d}x))\\
&= \pi(\mu - r) + \lambda_u \log(1 + \pi(e^{\alpha_u} - 1))\\
&\quad + \lambda_d \log(1 + \pi(e^{\alpha_d} - 1))\\
&= \pi(\mu - r) + \lambda_u \log(1 + \pi(e^{\alpha_u} - 1))\\
&\quad + \lambda_d \log(1 + \pi(1 - e^{\alpha_u})).
\end{aligned}
$$

π falls in the admissible set $\mathcal{A}_w$, if $G(\pi)$ is well-defined. We therefore insist that

$$
1 + \pi(e^{\alpha_u} - 1) > 0, \quad 1 + \pi(1 - e^{\alpha_u}) > 0.
$$

Let $0 < A_u = e^{\alpha_u} - 1 < 1$ and $a = 1/A_u$. Then we have $a > 1$ and $-a < \pi < a$. We can take the admissible set as $\mathcal{A}_w = (-a, a)$, and optimize $f(\pi)$ over this set, which clearly contains [-1, 1]. Since f is strictly concave, it admits a unique maximum π, where $f'(\pi) = 0$. Now

$$f'(\pi) = \mu - r + \frac{\lambda_u(e^{\alpha_u} - 1)}{1 + \pi(e^{\alpha_u} - 1)} + \frac{\lambda_d(1 - e^{\alpha_u})}{1 + \pi(1 - e^{\alpha_u})} = 0,$$

is equivalent to the quadratic equation $f'(\pi) = \mu - r + \lambda_u/(a + \pi) - \lambda_d/(a - \pi) = 0$. This has solution:

$$\pi = \pi_\pm = -\left(\frac{\lambda_u + \lambda_d}{2(\mu - r)}\right) \pm \sqrt{\left(\frac{\lambda_u + \lambda_d}{2(\mu - r)}\right)^2 + \frac{(\lambda_u - \lambda_d)a}{\mu - r} + a^2},$$

provided $\pi \in \mathcal{A}_w = (-a, a)$. Specifically, we have

$$\pi = \begin{cases} \pi_+ & \text{if } \mu > r \\ \dfrac{\lambda_u - \lambda_d}{\lambda_u + \lambda_d} a & \text{if } \mu = r \\ \pi_- & \text{if } \mu < r. \end{cases}$$

The maximum expected logarithmic utility is

$$\begin{aligned} u(w) &= \log w + \int_0^T f(\pi_t)\,\mathrm{d}t = \log(w) \\ &\quad + \int_0^T (\pi_t(\mu_t - r_t) + G(\pi_t))\,\mathrm{d}t \\ &= \log(w) + \int_0^T \Big[\pi_t(\mu_t - r_t) + \lambda_d \log\left(1 + \frac{\pi_t}{a}\right) \\ &\quad + \lambda_u \log\left(1 - \frac{\pi_t}{a}\right)\Big]\,\mathrm{d}t. \end{aligned}$$

■

Proof of Proposition 4.1 This is the standard unique root of a cubic equation with coefficients 1, b, c and d, and can be found on the internet. ■

Proof of Proposition 5.1 The moment generating function of instantaneous centralized returns is

$$\begin{aligned} G(\alpha) &= \int_{\mathbf{R}\setminus\{0\}} \log(1 + \alpha(e^x - 1))\nu(\mathrm{d}x) \\ &= \sum_{i=1}^m \int_{\mathbf{R}\setminus\{0\}} \log(1 + \alpha(e^x - 1))\nu_i(\mathrm{d}x) \\ &= \sum_{i=1}^m [\lambda_{u_i} \log(1 + \alpha(e^{\alpha_{u_i}} - 1)) + \lambda_{d_i} \log(1 + \alpha(e^{\alpha_{d_i}} - 1))] \\ &= \sum_{i=1}^m [\lambda_{u_i} \log(1 + \alpha(e^{\alpha_{u_i}} - 1)) + \lambda_{d_i} \log(1 - \alpha(e^{\alpha_{u_i}} - 1))] \\ &= \sum_{i=1}^m [\lambda_{u_i} \log(1 + \alpha A_i) + \lambda_{d_i} \log(1 - \alpha A_i)] \\ &= \sum_{i=1}^m \log[(1 + \alpha A_i)^{\lambda_{u_i}} (1 - \alpha A_i)^{\lambda_{d_i}}], \end{aligned}$$

whence, provided the product exists,

$$G(\alpha) = \log[\Pi_{i=1}^m (1 + \alpha A_i)^{\lambda_{u_i}} (1 - \alpha A_i)^{\lambda_{d_i}}].$$

Clearly $G(\alpha)$ exists, if for each $i = 1, \ldots, m$, $1 + \alpha A_i > 0$, $1 - \alpha A_i > 0$, which is equivalent to $a_i + \alpha > 0$, $a_i - \alpha > 0$. Thus $\alpha \in (-a_i, a_i)$ and $a_i > 1$, since $0 < A_i < 1$. Note that $a \triangleq \min_{i=1,\ldots,m}\{a_i\} > 1$. Therefore, $G(\alpha)$ exists for all $\alpha \in (-a, a)$. ■

Proof of Proposition 5.2 By Theorem 2.1, the optimal portfolio is given by $\pi = (\mu - r)/\sigma^2 + G'(\pi)/\sigma^2$, provided $G'(\pi)$ exists. Since $G(\alpha)$ exists for all $\alpha \in (-a, a)$, where $a = min\{a_i\}$, then $G(\alpha)$ exists within said interval. From the proof of Proposition 5.1, we have

$$G(\alpha) = \sum_{i=1}^m [\lambda_{u_i} \log(1 + \alpha A_i) + \lambda_{d_i} \log(1 - \alpha A_i)],$$

$$G'(\alpha) = \sum_{i=1}^m \left[\frac{\lambda_{u_i} A_i}{1 + \alpha A_i} - \frac{\lambda_{d_i} A_i}{1 - \alpha A_i}\right] = \sum_{i=1}^m \left[\frac{\lambda_{u_i}}{\frac{1}{A_i} + \alpha} - \frac{\lambda_{d_i}}{\frac{1}{A_i} - \alpha}\right],$$

which yields

$$\frac{G'(\alpha)}{\sigma^2} = \sum_{i=1}^m \left[\frac{\frac{\lambda_{u_i}}{\sigma^2}}{a_i + \alpha} - \frac{\frac{\lambda_{d_i}}{\sigma^2}}{a_i - \alpha}\right] = \sum_{i=1}^m \left[\frac{\tilde{\lambda}_{u_i}}{a_i + \alpha} - \frac{\tilde{\lambda}_{d_i}}{a_i - \alpha}\right],$$

and

$$\frac{G''(\alpha)}{\sigma^2} = -\sum_{i=1}^m \left[\frac{\tilde{\lambda}_{u_i}}{(a_i + \alpha)^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i - \alpha)^2}\right]. \qquad \text{(A1)}$$

Thus

$$\pi = \frac{\mu - r}{\sigma^2} + \sum_{i=1}^m \left(\frac{\tilde{\lambda}_{u_i}}{a_i + \pi} - \frac{\tilde{\lambda}_{d_i}}{a_i - \pi}\right) \triangleq g^{(m)}(\pi) \equiv g(\pi),$$

provided $\pi \in (-a,\, a)$. ■

Proof of Proposition 5.3 Convergence is assured if $-1 < g'(\alpha) < 1$. From equation (A1), we have for all $\alpha \in (-a,\, a)$, where $a = \min_{1\le i\le m}\{a_i\}$, that

$$g'(\alpha) = \frac{G''(\alpha)}{\sigma^2} = -\sum_{i=1}^m \left[\frac{\tilde{\lambda}_{u_i}}{(a_i + \alpha)^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i - \alpha)^2}\right],$$

which is strictly negative. Thus for all $\alpha \in (-a, a)$, $g'(\alpha) < 0$. This reduces the convergence condition to $g'(\alpha) > -1$. ■

Proof of Corollary 5.4 This follows directly from Proposition 5.3, equation (70) and the fact that $g'(\alpha) > -1$. ■

Proof of Proposition 5.5 Assume that $\alpha \in [0,\, 1]$. Then $a_i \geq a$, where $a = \min_{1\le i\le m}\{a_i\} > 1$. Note that

$$g_i(\alpha) = \frac{\tilde{\lambda}_{u_i}}{a_i + \alpha} - \frac{\tilde{\lambda}_{d_i}}{a_i - \alpha}.$$

Then

$$\begin{aligned} |g_i'(\alpha)| &= \frac{\tilde{\lambda}_{u_i}}{(a_i + \alpha)^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i - \alpha)^2} \le \frac{\tilde{\lambda}_{u_i}}{a_i^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i - \alpha)^2} \\ &\le \frac{\tilde{\lambda}_{u_i}}{a^2} + \frac{\tilde{\lambda}_{d_i}}{(a - \alpha)^2} \le \frac{\tilde{\lambda}_{u_i}}{(a - 1)^2} + \frac{\tilde{\lambda}_{d_i}}{(a - 1)^2} \\ &\le \frac{2\tilde{\lambda}_{u_i}}{(a - 1)^2} \le \frac{2\tilde{\lambda}_{u_1}}{(a - 1)^2}. \end{aligned} \qquad \text{(A3)}$$

Thus

$$|g'(\alpha)| = \left|\sum_{i=1}^m g_i'(\alpha)\right| \le \sum_{i=1}^m |g_i'(\alpha)| \le \sum_{i=1}^m \frac{2\tilde{\lambda}_{u_1}}{(a - 1)^2} = \frac{2m\tilde{\lambda}_{u_1}}{(a - 1)^2}.$$

By symmetry, if $\alpha \in [-1, 0]$, then $-\alpha \in [0, 1]$, and by replacing α by $\bar{\alpha} = -\alpha$, we get

$$|g'(\alpha)| = \sum_{i=1}^m \left[\frac{\tilde{\lambda}_{u_i}}{(a_i - \bar{\alpha})^2} + \frac{\tilde{\lambda}_{d_i}}{(a_i + \bar{\alpha})^2}\right] \le \frac{2m\tilde{\lambda}_{u_1}}{(a - 1)^2}.$$

Therefore, if $\alpha \in [-1, 1]$

$$|g'(\alpha)| < \frac{2m\lambda_{u_1}}{\sigma^2(a - 1)^2}.$$

Convergence of the sequence is assured if $|g'(\alpha)| < 1$, for $\alpha \in [-1, 1]$. This holds, if

$$\frac{2m\lambda_{u_1}}{\sigma^2(a - 1)^2} < 1, \quad \text{or} \quad \lambda_{u_1} < \frac{\sigma^2(a - 1)^2}{2m}. \qquad ■$$

Proof of Proposition 5.6 The results follow from the fact that the total volatility at time t for the stock price with percentage returns $\mathrm{d}S_t/S_{t-}$ given by equation (62), is $\sigma_t^2 + M_2$. ■

Proof of Proposition 5.7 We have

$$
\begin{aligned}
M_k(i) &= \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)^k (\lambda_{u_i}\delta_{\alpha_{u_i}}(\mathrm{d}x) + \lambda_{d_i}\delta_{\alpha_{d_i}}(\mathrm{d}x)) \\
&= \lambda_{u_i}(e^{\alpha_{u_i}} - 1)^k + \lambda_{d_i}(e^{\alpha_{d_i}} - 1)^k \\
&= \lambda_{u_i}(e^{\alpha_{u_i}} - 1)^k + \lambda_{d_i}(1 - e^{\alpha_{u_i}})^k = (\lambda_{u_i} + (-1)^k\lambda_{d_i})A_i^k.
\end{aligned}
$$

Since $\nu_{\Pi(m)}(\mathrm{d}x) = \sum_{i=1}^m \nu_i(\mathrm{d}x)$, then

$$
\begin{aligned}
M_k &= \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)^k \nu_{\Pi(m)}(\mathrm{d}x) = \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)^k (\sum_{i=1}^m \nu_i(\mathrm{d}x)) \\
&= \int_{\mathbf{R}\setminus\{0\}} \sum_{i=1}^m (e^x - 1)^k \nu_i(\mathrm{d}x) = \sum_{i=1}^m \int_{\mathbf{R}\setminus\{0\}} (e^x - 1)^k \nu_i(\mathrm{d}x) \\
&= \sum_{i=1}^m M_k(i).
\end{aligned}
$$

■

Proof of Proposition 5.9. From Proposition 5.7, $A_i \in (0, 1)$ and $M_k = \sum_{i=1}^m (\lambda_{u_i} + (-1)^k\lambda_{d_i})A_i^k$. Thus

$$
|M_k| \leq \sum_{i=1}^m (\lambda_{u_i} + \lambda_{d_i})A_i^k \leq \sum_{i=1}^m 2\lambda_{u_i}A_i^k \leq 2\lambda_{u_1} \sum_{i=1}^m A_i^k.
$$

But $A_i = 1/a_i$ and $a = \min\{a_i\} > 1$. Thus $A_i = 1/a_i \leq 1/a$, for all $i = 1, 2, \ldots, m.$. Hence

$$
|M_k| < 2\lambda_{u_1} \sum_{i=1}^m \frac{1}{a^k} = \frac{2m\lambda_{u_1}}{a^k},
$$

from which $M_k \longrightarrow 0$ when $k \longrightarrow \infty$. ■

Proof of Proposition 5.10 The result follows from Theorem 2.1 and Proposition 2.2. Under quadratic approximation $G''(\pi) = -M_2 = -\sum_{i=1}^m (\lambda_{u_i} + \lambda_{d_i}) A_i^2$, as given in Corollary 5.8. ■

Proof of Theorem 5.11 The result follows directly from Theorem 2.1 and equation (65) in Proposition 5.2. ■