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Adjusting covariance matrix for risk management

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The covariance matrix of asset returns can change drastically and generate huge losses in portfolio value under extreme conditions such as market interventions and financial crises. Estimation of the covariance matrix under a chaotic market is often a call to action in risk management. Nowadays, stress testing has become a standard procedure for many financial institutions to estimate the capital requirement for their portfolio holdings under various stress scenarios. A possible stress scenario is to adjust the covariance matrix to mimic the situation under an underlying stress event. It is reasonable that when some covariances are altered, other covariances should vary as well. Recently, Ng *et al.* proposed a unified approach to determine a proper correlation matrix which reflects the subjective views of correlations. However, this approach requires matrix vectorization and hence it is not computationally efficient for high dimensional matrices. Besides, it only adjusts correlations, but it is well known that high correlations often go together with high standard deviations during a crisis period. To address these limitations, we propose a Bayesian approach to covariance matrix adjustment by incorporating subjective views of covariances. Our approach is computationally efficient and can be applied to high dimensional matrices.

Keywords: Covariance matrix; Subjective view; IFRS 9; Stress testing

JEL Classification: C, G

1. Introduction

A covariance matrix plays a very important role in many financial applications such as portfolio selection, option pricing and risk management. It is well known that under extreme conditions such as market interventions and financial crises, variances and correlations of asset returns in a portfolio can jump to very high levels and generate huge losses in the portfolio value (Longin and Solnik 2001). Another example is that under some events like Brexit, the covariances between macroeconomic variables and systematic risk factors for counterparty asset returns can change a lot and generate a big credit portfolio loss to financial institutions. Adjusting the covariance matrix to meet an extreme market condition is often a call to action in stress testing for financial risk management and provision modeling for the International Financial Reporting Standard (IFRS) 9.

Stress testing in the banking industry is an analysis conducted under unfavorable economic scenarios designed to determine whether a bank has sufficient capital to withstand the impact of adverse developments. In the early 1990s, large international banks began to use internal stress testing, i.e. performing themselves stress testing for internal self-assessment. In 1996, the Basel Capital Accord (Basel I) (BCBS 1988) was amended to require banks and investment firms to conduct stress testing to determine their ability to respond to market events (BCBS 1996). In 2004, Basel II (BCBS 2004) introduced a requirement for credit risk stress testing. Stress testing was implemented and became more widespread after the 2007–2009 global financial crisis because the crisis left many banks and financial institutions severely under-capitalized, which stress testing aims to prevent. It has been observed that the covariance matrix of some asset returns or risk factors are very different in normal and crisis periods. Therefore, in designing the stress scenario in a stress testing exercise, risk managers adjust the covariance matrix to reflect the relationship among the risk factors according to their beliefs on covariances under the stress scenario.

In the wake of 2007–2009 global financial crisis, the International Accounting Standards Board (IASB) in cooperation with the Financial Accounting Standards Board (FASB) launched a project to address the weakness of International Accounting Standard (IAS) 39. This had been adopted as an international standard for accounting for financial assets and liabilities in financial statements since 2001. Finally, in July 2014, IASB (IASB 2014) finalized and published the new IFRS 9 methodology which should

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be implemented by 1st of January 2018. Under Basel II (BCBS 2004) and Basel III (BCBS 2010, 2011), risk managers in banks taking the Internal Ratings-Based (IRB) approach would make use of different methods to develop statistical models to estimate the probability of default (PD) (Merton 1974, Czado and Pflüger 2008, Crook and Bellotti 2010, Chen *et al.* 2011, Grimshaw and Alexander 2011, Jovan and Ahčan 2017). As compared with the requirements of Basel II and Basel III, one of the significant changes in IFRS 9 is the provision that modeling must incorporate forward looking and robust macroeconomic scenarios. In order to meet this requirement, some consultants and risk managers suggested adjusting the correlation matrix of the systematic factors and macroeconomic variables according to their belief in covariances during the crisis period in order to select potential risk drivers for PD model development (Levy 2008, Huang *et al.* 2012, Pospisil *et al.* 2013, Hong *et al.* 2016, Levy *et al.* 2017, Levy and Zhang 2018).

The main difficulty in adjusting a covariance matrix is that such an adjustment may result in an improper matrix in the sense that the resulting 'covariance matrix' violates the requirement of positive semidefiniteness (PSD). Some key risk factors (core risk factors) are often identified for a hypothetical scenario. The earlier approach to adjust the covariance matrix $\boldsymbol{\Sigma}$ among risk factors is to explicitly specify the shock on core risk factors and leave other risk factors (peripheral risk factors) unchanged. By decomposing $\boldsymbol{\Sigma}$ as $\boldsymbol{\Sigma} = \boldsymbol{DRD}$, where $\boldsymbol{D}$ is a diagonal matrix consisting of standard deviations and $\boldsymbol{R}$ is the correlation matrix, the variances and correlations can be stressed separately. It is easy to stress the variances of the core risk factors by adjusting diagonal entries of $\boldsymbol{D}$ but stressing correlations is not straightforward. Such a problem is known as correlation stress testing which has been studied by many researchers (Finger 1997, Qi and Sun 2010, Breuer and Csiszár 2013, So *et al.* 2013, Ng *et al.* 2014, Yu *et al.* 2014, Packham and Woebbeking 2019). A common approach to correlation stress testing is the nearest correlation matrix approach, which first constructs a so-called target matrix $\boldsymbol{R_T}$ by setting the specified correlations (core correlations) to the desired stressed levels and leaving the unspecified correlations (peripheral correlations) unchanged. If $\boldsymbol{R_T}$ is found to be improper, it is replaced by the proper correlation matrix nearest to $\boldsymbol{R_T}$ based on the a given matrix distance norm such as the Frobenius norm $\|\boldsymbol{R} - \boldsymbol{R_T}\|_F = \sqrt{\mathrm{tr}[(\boldsymbol{R} - \boldsymbol{R_T})(\boldsymbol{R} - \boldsymbol{R_T})^\top]}$, where $\mathrm{tr}(\boldsymbol{X})$ is the trace of matrix $\boldsymbol{X}$ (Higham 2002, Qi and Sun 2006, Grubišić and Pietersz 2007, Borsdorf and Higham 2010).

It is well known that empirical correlations are strongly correlated over time. For example, the multivariate GARCH and dynamic conditional correlation GARCH (DCC-GARCH) models (Engle 2002, Engle and Kroner 1995) assume the conditional covariance matrix of asset returns to be a deterministic function of their past returns so that conditional correlations are time varying. The factor models proposed by Meucci (2009) help explain the co-movement of correlations. The partial correlation network model proposed by Torri *et al.* (2018) makes use of the fact that partial correlations can be computed from the inverse of the covariance matrix. These models shed some light on the reason why correlations move together. Therefore, it is expected that if core correlations are explicitly adjusted, peripheral correlations should also be changed accordingly rather than assumed to be unchanged.

Inspired by the Black–Litterman approach (introduced in Black and Litterman (1992); extensions are given in Meucci (2009) and Giacometti and Mignacca (2010)) which incorporate subjective views on the expected returns. Ng *et al.* (2014) proposed a unified approach to determine a proper correlation matrix in which the stress impact on the core correlations is transmitted to the peripheral correlations through the dependence structure of empirical correlations. This can be viewed as a two-step procedure: first constructing a target matrix in a data-driven manner such that the peripheral correlations change accordingly, and then regularizing the target matrix based on the Mahalanobis norm that reasonably reflects the dependence structure of the empirical correlations. We refer to this approach as the 'Ng method'. However, the Ng method requires vectorizing the correlation matrix and may become computationally demanding for high dimensions, say 20 or more. Also, notice that high correlations often go hand in hand with high standard deviations during a crisis period. It is natural to extend the methodology to adjust the covariance matrix by stressing both variances and correlations together.

To tackle these problems, in this paper we propose a Bayesian approach to adjust the covariance matrix that reflects a subjective view of covariances. Under this framework, the peripheral covariances are adjusted according to the dependence structure of empirical covariances. Instead of vectorizing the matrix, our framework will adopt the symmetric matrix-variate normal distribution to specify prior beliefs about the covariance matrix. A new covariance matrix is constructed by maximizing the posterior density. The proposed method can be viewed as a two-step procedure: first constructing a target matrix in a data-driven manner such that the peripheral covariances change accordingly, and then regularizing the target matrix based on a matrix norm that reasonably reflects the dependence structure of the empirical covariances.

The rest of this paper is organized as follows. Section 2 introduces our Bayesian method for covariance matrix adjustment. Section 3 applies our methodology for covariance matrix adjustment under IFRS 9 and for stress testing an international stock portfolio. The empirical results demonstrate that our proposed method is computationally efficient. We conclude in Section 4.

2. A Bayesian approach for covariance matrix adjustment

In this paper, we propose a Bayesian approach for adjusting the covariance matrix. The basic idea is that we first specify the prior distribution of the covariance matrix to be a symmetric matrix-variate normal distribution, truncated over the set of all proper covariance matrices. Given a matrix of subjective views on the covariance matrix which follows another symmetric matrix-variate normal distribution, we can then derive the posterior distribution of the covariance matrix. Finally, the adjusted covariance matrix is obtained by using the maximum a posterior estimation method. In the following, we will go through the details of this new approach.

2.1. *Prior of covariance matrix*

Suppose that $\boldsymbol{\Sigma} = (\sigma_{ij})_{n\times n}$ denotes the $n \times n$ covariance matrix to be estimated. To specify a prior distribution for $\boldsymbol{\Sigma}$, a common choice is the inverse Wishart distribution as it is a conjugate prior for the covariance matrix of a multivariate normal distribution (Evans 1965, Chen 1979). However, this assumes that both the prior mean and prior variance of each σ_{ii} must be proportional to a pre-specified scale parameter, see Eaton (2007). This choice of prior cannot allow us to specify separately different prior views on the location for each covariance and the dependence among the covariances. Therefore, a more flexible prior distribution is needed to separately specify the location and the level of dependence for $\boldsymbol{\Sigma}$.

In this paper, we assume that $\boldsymbol{\Sigma}$ follows a truncated symmetric matrix-variate normal distribution with mean matrix $\boldsymbol{W} = (w_{ij})_{n\times n}$ and symmetric positive definite scale matrix $\boldsymbol{\Psi} = (\psi_{ij})_{n\times n}$ and its probability density function is given by

$$f(\boldsymbol{\Sigma}) \propto \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}-\boldsymbol{W})\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}-\boldsymbol{W})\right]\right\} \times I\{\boldsymbol{\Sigma} \in R_{PSD}\}, \tag{1}$$

where $\mathrm{tr}(\boldsymbol{X})$ is the trace of matrix $\boldsymbol{X}$, $I\{\cdot\}$ is an indicator function and R_{PSD} is the set of all proper $n \times n$ covariance matrices, see Gupta and Nagar (1999) for more details of the distribution. We denote this distribution by $SN_n(\boldsymbol{W}, \boldsymbol{\Psi})$. The following proposition and corollary show its relation to the normal distribution and some moment properties (see Section 2.5 of Gupta and Nagar 1999).

PROPOSITION 2.1 *Let* $\mathrm{vech}(\boldsymbol{A})$ *be the column vector comprising all upper triangular entries of a square matrix* $\boldsymbol{A}$, *taken columnwise. If a* $n \times n$ *random symmetric matrix* $\boldsymbol{X}$ *follows* $SN_n(\boldsymbol{M}, \boldsymbol{\Psi})$, *then* $\mathrm{vech}(\boldsymbol{X})$ *follows a multivariate normal distribution with mean vector* $\mathrm{vech}(\boldsymbol{M})$ *and covariance matrix* $\boldsymbol{D}_n^+(\boldsymbol{\Psi}\otimes\boldsymbol{\Psi})(\boldsymbol{D}_n^+)^\top$, *where* $\otimes$ *is the Kronecker product,* $\boldsymbol{D}_n$ *is the duplication matrix (i.e.* $\boldsymbol{D}_n\mathrm{vech}(\boldsymbol{A}) = \mathrm{vec}(\boldsymbol{A})$) *and* $\boldsymbol{D}_n^+$ *is the generalized inverse of* $\boldsymbol{D}_n$ *(i.e.* $\boldsymbol{D}_n^+ = (\boldsymbol{D}_n^\top\boldsymbol{D}_n)^{-1}\boldsymbol{D}_n^\top$).

COROLLARY 2.2 *Given* $\boldsymbol{X} \sim SN_n(\boldsymbol{M}, \boldsymbol{\Psi})$, *we have*

(i) $E(\boldsymbol{X}) = \boldsymbol{M}$;
(ii) $\mathrm{Cov}(x_{ij}, x_{kl}) = \frac{1}{2}(\psi_{ik}\psi_{jl} + \psi_{jk}\psi_{il})$;
(iii) $\boldsymbol{AXA}^\top \sim SN_m(\boldsymbol{AMA}^\top, \boldsymbol{A\Psi A}^\top)$ *for any* $m \times n$ *matrix* $\boldsymbol{A}$.

Using the above proposition, it can be shown that $\mathrm{vech}(\boldsymbol{\Sigma})$ follows a truncated multivariate normal distribution with mean vector $\boldsymbol{\mu} = \mathrm{vech}(\boldsymbol{W})$ and covariance matrix $\boldsymbol{D}_n^+(\boldsymbol{\Psi}\otimes\boldsymbol{\Psi})(\boldsymbol{D}_n^+)^\top$. Also under $SN_n(\boldsymbol{W}, \boldsymbol{\Psi})$, $\mathrm{Cov}(\sigma_{ij}, \sigma_{kl}) = \frac{1}{2}(\psi_{ik}\psi_{jl} + \psi_{jk}\psi_{il})$ and hence the scale matrix $\boldsymbol{\Psi}$ governs the dependence among the covariances.

Consider a simple example of two asset returns whose covariance matrix $\boldsymbol{\Sigma} = \left[\begin{smallmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{smallmatrix}\right]$ follows a truncated $SN_2(\boldsymbol{W}, \boldsymbol{\Psi})$ where $\boldsymbol{W} = \left[\begin{smallmatrix}w_{11} & w_{12}\\ w_{12} & w_{22}\end{smallmatrix}\right]$ and $\boldsymbol{\Psi} = \left[\begin{smallmatrix}\psi_{11} & \psi_{12}\\ \psi_{12} & \psi_{22}\end{smallmatrix}\right]$. This is equivalent to say that $\mathrm{vech}(\boldsymbol{\Sigma}) = [\sigma_{11}\ \ \sigma_{12}\ \ \sigma_{22}]^\top$ follows a truncated multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{K}$:

$$\boldsymbol{\mu} = \mathrm{vech}(\boldsymbol{W}) = \begin{bmatrix} w_{11} & w_{12} & w_{22}\end{bmatrix}^\top$$

and

$$\boldsymbol{K} = \boldsymbol{D}_2^+(\boldsymbol{\Psi}\otimes\boldsymbol{\Psi})(\boldsymbol{D}_2^+)^\top = \begin{bmatrix} \psi_{11}^2 & \psi_{11}\psi_{12} & \psi_{12}^2 \\ \psi_{11}\psi_{12} & \frac{1}{2}(\psi_{11}\psi_{22}+\psi_{12}^2) & \psi_{12}\psi_{22} \\ \psi_{12}^2 & \psi_{12}\psi_{22} & \psi_{22}^2 \end{bmatrix}.$$

It can be seen that ψ_{12} governs the dependency structure among σ_{11}, σ_{12} and σ_{22}. If $\boldsymbol{\Psi} = \left[\begin{smallmatrix}1 & 0\\ 0 & 2\end{smallmatrix}\right]$, then $\boldsymbol{K} = \left[\begin{smallmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{smallmatrix}\right]$. If $\boldsymbol{\Psi} = \left[\begin{smallmatrix}1 & 0.5\\ 0.5 & 2\end{smallmatrix}\right]$, then $\boldsymbol{K} = \left[\begin{smallmatrix}1 & 0.5 & 0.25\\ 0.5 & 1.125 & 1\\ 0.25 & 1 & 4\end{smallmatrix}\right]$.

The parameters $\boldsymbol{W}$ and $\boldsymbol{\Psi}$ can be specified according to expert opinion or estimated empirically. In this paper, we estimate $\boldsymbol{W}$ and $\boldsymbol{\Psi}$ based on the covariance matrices in non-overlapping subperiods using the Maximum Likelihood Estimation (MLE) method, see Appendix 1 for details.

2.2. *Subjective view*

Suppose that there is a subjective view on the covariance matrix of m linear combinations of the n risk factors, denoted by an $m \times m$ matrix $\boldsymbol{V} = (v_{ij})_{m\times m}$. Let $\boldsymbol{P} = [\boldsymbol{p}_1\ \ \boldsymbol{p}_2\ \ \cdots\ \ \boldsymbol{p}_m]^\top$ be a $m \times n$ matrix representing the m linear combinations. Therefore, $\boldsymbol{P\Sigma P}^\top$ is the covariance matrix of m linear combinations of the n risk factors and the matrix $\boldsymbol{V}$ is our subjective view on the matrix $\boldsymbol{P\Sigma P}^\top$. For example, some research may lead to a subjective view on the covariances among the first $m = 3$ risk factors, then we have

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\end{bmatrix},$$

$$\boldsymbol{P\Sigma P}^\top = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} \quad \text{and}$$

$$\boldsymbol{V} = \begin{bmatrix} v_{11} & v_{12} & v_{13}\\ v_{21} & v_{22} & v_{23}\\ v_{31} & v_{32} & v_{33}\end{bmatrix},$$

where $v_{i,j}$ is the view on $\sigma_{i,j}$. Note that the subjective view is not necessarily confined to m pre-specified risk factors. The general subjective view is referred to a set of m linear combinations of the n risk factors, and hence all n risk factors could be involved, depending on the actual specification of the m linear combinations.

From Corollary 2.2(iii), if $\boldsymbol{X} \sim SN_n(\boldsymbol{\Sigma}, \boldsymbol{\Psi})$, then $\boldsymbol{PXP}^\top \sim SN_m(\boldsymbol{P\Sigma P}^\top, \boldsymbol{P\Psi P}^\top)$. It is therefore natural to assume that

$$\boldsymbol{V} \mid \boldsymbol{\Sigma} \sim SN_m(\boldsymbol{P\Sigma P}^\top, \boldsymbol{\Phi}). \tag{2}$$

It can be seen from Corollary 2.2(i) that the conditional expectation of $\boldsymbol{V} \mid \boldsymbol{\Sigma}$ is $\boldsymbol{P\Sigma P}^\top$.

Here $\boldsymbol{\Phi}$ is used to specify the confidence level of the subjective view. We follow Ng *et al.* (2014) to set

$$\boldsymbol{\Phi} = \sqrt{\frac{\tau}{1-\tau}\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^{\top}},$$

where $0 \le \tau < 1$ is a parameter controlling the uncertainty of view, and $\tau = 0$ implies certainty views. Using Corollary 2.2(ii), it can be seen that

$$\begin{aligned}\mathrm{Cov}(v_{ij}, v_{kl} \mid \boldsymbol{\Sigma}) = \frac{\tau}{2(1-\tau)}[(\boldsymbol{p}_i^{\top}\boldsymbol{\Psi}\boldsymbol{p}_k)(\boldsymbol{p}_j^{\top}\boldsymbol{\Psi}\boldsymbol{p}_l) \\ + (\boldsymbol{p}_j^{\top}\boldsymbol{\Psi}\boldsymbol{p}_k)(\boldsymbol{p}_i^{\top}\boldsymbol{\Psi}\boldsymbol{p}_l)].\end{aligned}$$

This specification can help simplify the posterior distribution of the covariance matrix of $\mathrm{vech}(\boldsymbol{V}) \mid \mathrm{vech}(\boldsymbol{\Sigma})$, see Appendix 2 for the details. Of course, other specification for $\boldsymbol{\Phi}$ could also be considered.

2.3. *Posterior of covariance matrix*

Using Bayes' theorem, it can be shown that the posterior density of $\boldsymbol{\Sigma}$ is

$$\begin{aligned} f(\boldsymbol{\Sigma} \mid \boldsymbol{V}) \propto \exp\left\{-\frac{1}{2}\mathrm{tr}[\tilde{\boldsymbol{\Psi}}_1^{-1}(\boldsymbol{\Sigma} - \tilde{\boldsymbol{W}})\tilde{\boldsymbol{\Psi}}_2^{-1}(\boldsymbol{\Sigma} - \tilde{\boldsymbol{W}})]\right\} \\ \times I\{\boldsymbol{\Sigma} \in R_{PSD}\}, \qquad (3)\end{aligned}$$

where

$$\begin{aligned}\tilde{\boldsymbol{W}} &= \boldsymbol{W} + (1-\tau)[\boldsymbol{\Psi}\boldsymbol{P}^{\top}(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^{\top})^{-1}](\boldsymbol{V} - \boldsymbol{P}\boldsymbol{W}\boldsymbol{P}^{\top}) \\ &\quad \times [\boldsymbol{\Psi}\boldsymbol{P}^{\top}(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^{\top})^{-1}]^{\top} \\ \tilde{\boldsymbol{\Psi}}_1 &= \boldsymbol{\Psi} - (1-\tau)^{1/2}\boldsymbol{\Psi}\boldsymbol{P}^{\top}(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^{\top})^{-1}\boldsymbol{P}\boldsymbol{\Psi} \\ \tilde{\boldsymbol{\Psi}}_2 &= \boldsymbol{\Psi} + (1-\tau)^{1/2}\boldsymbol{\Psi}\boldsymbol{P}^{\top}(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^{\top})^{-1}\boldsymbol{P}\boldsymbol{\Psi}. \qquad (4)\end{aligned}$$

See Appendix 2 for the derivation of the posterior density in (3).

Following the maximum a posteriori estimation method, the new covariance matrix $\hat{\boldsymbol{\Sigma}}$ is obtained by maximizing the posterior density (3). If $\tilde{\boldsymbol{W}} \in R_{PSD}$, the posterior density (3) is maximized at $\hat{\boldsymbol{\Sigma}} = \tilde{\boldsymbol{W}}$. Otherwise, we need to find the positive semidefinite matrix $\hat{\boldsymbol{\Sigma}}$ which maximizes the posterior density (3).

Proposition 2.3 *Maximizing the posterior density of* $\boldsymbol{\Sigma}$ *in* (3) *is equivalent to minimizing the weighted sum of two distances*:

$$\tau d_1(\boldsymbol{\Sigma}, \tilde{\boldsymbol{W}}) + (1-\tau)d_2(\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^{\top}, \boldsymbol{P}\tilde{\boldsymbol{W}}\boldsymbol{P}^{\top}), \qquad (5)$$

subject to the constraint $\boldsymbol{\Sigma} \in R_{PSD}$. *Here,* $d_1(\boldsymbol{\Sigma}, \tilde{\boldsymbol{W}}) = \|\boldsymbol{\Psi}^{-1/2}(\boldsymbol{\Sigma} - \tilde{\boldsymbol{W}})\boldsymbol{\Psi}^{-1/2}\|_F^2$ *and*

$$\begin{aligned} d_2(\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^{\top}, \boldsymbol{P}\tilde{\boldsymbol{W}}\boldsymbol{P}^{\top}) \\ = \|(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^{\top})^{-1/2}(\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^{\top} - \boldsymbol{P}\tilde{\boldsymbol{W}}\boldsymbol{P}^{\top})(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^{\top})^{-1/2}\|_F^2.\end{aligned}$$

The proof is given in Appendix 3. Note that $d_1(\boldsymbol{\Sigma}, \tilde{\boldsymbol{W}})$ measures the overall distance between $\boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{W}}$ while $d_2(\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^{\top}, \boldsymbol{P}\tilde{\boldsymbol{W}}\boldsymbol{P}^{\top})$ measures the distance between the expected subjective view $\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^{\top}$ and its posterior mode $\boldsymbol{P}\tilde{\boldsymbol{W}}\boldsymbol{P}^{\top}$. If τ is small, then it will rely more on the distance defined on the subjective view. Therefore, the proposed method can be viewed as a two-step procedure. First, we construct a target matrix $\tilde{\boldsymbol{W}}$ in which the covariances change according to the dependences among the empirical covariances. In case that $\tilde{\boldsymbol{W}}$ is PSD, the optimal proper covariance matrix $\hat{\boldsymbol{\Sigma}}$ is $\tilde{\boldsymbol{W}}$. Otherwise, we determine $\hat{\boldsymbol{\Sigma}}$ by minimizing the distance function (5). It is expected that as long as the covariances of the risk factors are dependent of each other, our proposed method aims to have different degrees of impact on the correlation with those excluded risk factors (if any), depending on how strong the dependency among the covariances of the risk factors.

Remark Our proposed method is an extension to the one proposed in Ng *et al.* (2014) and it has two major contributions. First, the Ng method is designed to adjust the correlation matrix instead of the covariance matrix. However, the high correlations often go hand in hand with high standard deviations during the crisis period and therefore it is natural to extend the methodology to adjust the covariance matrix by stressing both variances and correlations together. Another contribution of this proposed method is the improvement on the complexity of the problem. Under the Ng method, in order to adjust a $n \times n$ correlation matrix $\boldsymbol{Y}$, we need to firstly vectorize this matrix into a $(n(n-1)/2) \times 1$ vector y which comprises all upper triangular entries of $\boldsymbol{Y}$ except those on its main diagonal. Ng *et al.* (2014) assumed that the prior distribution of this vector y is a truncated multivariate normal distribution with a $(n(n-1)/2) \times 1$ mean vector on all pairwise correlations and a $(n(n-1)/2) \times (n(n-1)/2)$ matrix governing the dependence among the correlations as the model parameters. Hence, the Ng method has $n(n-1)(n^2-n+6)/8 = \mathcal{O}(n^4)$ parameters. However, in order to adjust an $n \times n$ covariance matrix $\boldsymbol{\Sigma}$, our proposed model has $n^2 + n = \mathcal{O}(n^2)$ parameters only. For example, if $n = 20$, the Ng method has 18,335 parameters but our proposed method has 420 parameters only. This implies that our new method is more capable to handle higher dimensional covariance matrix.

3. Empirical studies

In this section, we first consider a toy example to illustrate the idea of our proposed method. We then consider two stress testing examples on the portfolio of MSCI Developed Market Country indices in order to evaluate the performance of our proposed method.

3.1. *A toy example on IFRS9*

In accordance to the requirements of IFRS 9, banks have to consider forward looking and robust macroeconomic scenarios to measure the expected credit losses of their loan portfolios. An important step of such risk management process is to make use of a correlation matrix of the systematic factors and macroeconomic variables to select an appropriate

Table 1. Sample covariance matrices for daily log returns of the 24 assets during the observation period (lower triangular) and the crisis period (upper triangular) with the covariances for PIGS bolded.

Asset	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	
	4.82	**4.57**	**5.27**	**5.59**	3.36	1.80	0.10	2.93	4.10	2.21	2.37	2.48	3.60	0.85	3.99	1.63	1.67	4.53	3.76	2.97	3.55	4.60	0.72	3.96	1
		5.32	**5.12**	**6.02**	3.75	1.95	0.27	3.24	4.57	2.33	2.61	2.69	3.99	1.01	4.29	1.71	1.79	5.08	4.03	3.24	3.88	5.03	0.78	4.32	2
1. Portugal	**1.26**		**10.84**	**6.32**	3.75	1.88	0.26	3.10	4.73	2.37	2.67	2.92	4.13	1.33	4.63	1.64	2.28	5.15	4.12	3.58	4.28	5.01	1.32	4.79	3
2. Italy	**0.83**	**1.62**		**7.78**	4.28	2.20	0.13	3.67	5.32	2.63	2.92	2.93	4.60	1.14	4.94	1.85	2.03	5.86	4.72	3.59	4.47	5.86	0.90	4.78	4
3. Greece	**0.70**	**0.81**	**2.83**		3.00	1.40	0.16	2.52	3.47	1.88	2.08	2.15	3.07	0.80	3.32	1.38	1.40	3.80	3.06	2.54	2.90	3.85	0.64	3.45	5
4. Spain	**0.95**	**1.39**	**0.87**	**1.88**		1.60	-0.06	1.25	1.69	1.44	0.92	1.03	1.50	0.38	1.75	0.53	0.68	1.90	1.46	1.31	1.40	1.82	0.35	1.78	6
5. Germany	0.91	1.45	0.87	1.53	2.24		1.42	0.21	0.24	0.07	0.18	0.94	0.22	0.63	0.05	0.49	0.49	0.36	0.13	0.32	0.24	0.37	0.16	0.26	7
6. USA	0.25	0.54	0.24	0.58	0.82	1.25		2.42	3.03	1.66	1.79	2.08	2.69	0.76	2.92	1.35	1.31	3.30	2.67	2.22	2.51	3.38	0.50	3.05	8
7. Japan	0.35	0.31	0.57	0.33	0.37	0.10	2.03		4.22	2.18	2.46	2.63	3.70	0.98	3.96	1.63	1.72	4.58	3.72	3.03	3.52	4.63	0.77	4.09	9
8. UK	0.61	1.01	0.62	1.07	1.21	0.50	0.31	1.27		2.00	1.34	1.71	2.01	0.58	2.28	0.96	1.12	2.50	1.95	1.81	1.88	2.35	0.51	2.48	10
9. France	0.86	1.40	0.82	1.49	1.66	0.64	0.38	1.18	1.76		1.70	1.61	2.18	0.56	2.36	1.01	1.07	2.62	2.19	1.83	2.14	2.77	0.46	2.49	11
10. Canada	0.46	0.68	0.45	0.73	0.91	0.86	0.29	0.62	0.80	1.54		3.46	2.39	1.30	2.60	1.78	1.82	3.15	2.26	2.21	2.28	2.86	0.79	3.19	12
11. Switzerland	0.69	1.05	0.71	1.11	1.23	0.44	0.34	0.92	1.15	0.55	1.31		3.39	0.88	3.60	1.46	1.58	4.03	3.34	2.76	3.13	4.19	0.70	3.74	13
12. Australia	0.43	0.44	0.65	0.48	0.48	0.11	0.74	0.43	0.49	0.38	0.41	1.36		1.09	0.88	0.73	0.97	1.21	0.83	0.82	0.80	1.00	0.29	1.04	14
13. Netherlands	0.80	1.34	0.83	1.42	1.61	0.60	0.37	1.19	1.55	0.71	1.17	0.50	1.81		4.40	1.66	1.79	4.48	3.57	3.16	3.48	4.57	0.82	4.28	15
14. Hong Kong	0.35	0.48	0.64	0.52	0.62	0.23	0.84	0.54	0.57	0.39	0.41	0.81	0.58	2.39		1.78	1.00	2.01	1.41	1.38	1.35	1.76	0.39	1.90	16
15. Sweden	0.98	1.44	0.99	1.55	1.76	0.69	0.62	1.27	1.70	0.95	1.22	0.69	1.58	0.84	3.03		1.77	2.03	1.52	1.55	1.49	1.97	0.51	2.00	17
16. New Zealand	0.38	0.36	0.52	0.38	0.38	0.04	0.51	0.32	0.36	0.23	0.30	0.88	0.41	0.59	0.50	1.76		5.85	4.10	3.52	3.87	5.03	0.89	4.58	18
17. Singapore	0.37	0.50	0.67	0.56	0.61	0.25	0.88	0.53	0.59	0.42	0.44	0.78	0.60	1.40	0.86	0.61	2.20		3.55	2.77	3.19	4.23	0.63	3.66	19
18. Austria	0.61	0.68	0.72	0.76	0.77	0.16	0.35	0.57	0.72	0.35	0.63	0.49	0.69	0.42	0.83	0.41	0.45	1.34		2.97	2.80	3.48	0.75	3.48	20
19. Belgium	0.74	1.08	0.81	1.15	1.27	0.47	0.35	0.93	1.24	0.55	1.03	0.49	1.29	0.44	1.21	0.39	0.48	0.73	1.59		3.93	3.92	0.77	3.58	21
20. Denmark	0.69	0.85	0.77	0.93	0.99	0.29	0.38	0.72	0.95	0.49	0.78	0.53	0.94	0.50	1.16	0.38	0.54	0.71	0.86	1.47		6.67	0.82	4.60	22
21. Finland	1.10	1.64	1.12	1.83	2.06	0.85	0.69	1.47	2.02	1.16	1.34	0.71	1.88	1.07	2.89	0.53	1.02	0.80	1.26	1.23	5.73		1.07	1.01	23
22. Ireland	0.67	0.83	0.84	0.88	0.95	0.30	0.42	0.80	0.93	0.43	0.78	0.59	0.94	0.50	1.07	0.44	0.53	0.73	0.88	0.80	1.16	1.66		5.19	24
23. Israel	0.37	0.63	0.56	0.64	0.88	0.71	0.33	0.58	0.76	0.74	0.47	0.36	0.69	0.56	1.01	0.25	0.56	0.30	0.45	0.53	1.30	0.43	2.33		
24. Norway	0.70	0.95	0.94	1.00	1.08	0.36	0.49	0.85	1.06	0.65	0.85	0.71	1.06	0.72	1.34	0.52	0.73	0.88	0.89	0.97	1.48	0.89	0.64	2.09	

set of macroeconomic variables for developing the PD model. Risk managers of some financial institutions like Moody's Analytics attempted to effectively incorporate their unique views regarding the correlations of these variables under a severe economic scenario to adjust the correlation matrix. If the adjusted matrix is not PSD, the risk managers adopted the Frobenius correlation method (i.e. finding the nearest proper correlation matrix under the Frobenius norm) to find a PSD matrix.

In this toy example, the following three variables are considered:

- RML: Delinquency rate on residential mortgages in United States
- UNE: Unemployment rate
- GDP: Real GDP (annual rate)

and the covariance matrix of these variables is assumed to be

$$\boldsymbol{W} = \boldsymbol{DRD} = \begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 2.0 \end{bmatrix} \begin{bmatrix} 1 & 0.7 & -0.8 \\ 0.7 & 1 & -0.4 \\ -0.8 & -0.4 & 1 \end{bmatrix} \times \begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 2.0 \end{bmatrix} = \begin{bmatrix} 3.24 & 1.64 & -2.88 \\ 1.64 & 1.69 & -1.04 \\ -2.88 & -1.04 & 4.00 \end{bmatrix}.$$

The choice of these figures makes reference to the data retrieved from the Board of Governors of the Federal Reserve System (https://www.federalreserve.gov/data.htm).

Suppose we know that under financial crisis, the correlation between RML and UNE (core correlations) will increase to a high value such as 0.95 and their standard deviations will drop significantly. That is, it is believed that under financial crisis the delinquency rate of residential mortgage is highly correlated with the unemployment rate and the standard deviations of delinquency rate and unemployment rate get much smaller as they might be fairly concentrated at a high level. Suppose that the subjective view on the covariance matrix of the first two variables (RML, UNE) is given as

$$\boldsymbol{V} = \begin{bmatrix} 0.25 & 0.10 \\ 0.10 & 0.04 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 0.95 \\ 0.95 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Note that simply replacing the upper left 2×2 submatrix of $\boldsymbol{W}$ by $\boldsymbol{V}$ results in an improper covariance matrix:

$$\boldsymbol{\Sigma}_T = \begin{bmatrix} 0.25 & 0.10 & -2.88 \\ 0.10 & 0.04 & -1.04 \\ -2.88 & -1.04 & 4.00 \end{bmatrix}.$$

In fact, even if $\boldsymbol{\Sigma}_T$ were PSD, it would be inappropriate to keep peripheral covariances unchanged because there may exist significant relationships between core covariances and some peripheral covariances.

3.1.1. Frobenius correlation method. Using the Frobenius correlation method to adjust the correlation matrix $\boldsymbol{R}$, we first construct the target correlation matrix $\boldsymbol{R}_T$ by changing the core correlations in $\boldsymbol{R}$ to 0.95:

$$\boldsymbol{R}_T = \begin{bmatrix} 1 & 0.95 & -0.80 \\ 0.95 & 1 & -0.40 \\ -0.80 & -0.40 & 1 \end{bmatrix}.$$

Since such $\boldsymbol{R}_T$ is not PSD, we obtain the optimal correlation matrix, $\hat{\boldsymbol{R}}_F$, which is nearest to $\boldsymbol{R}_T$ based on the Frobenius norm:

$$\hat{\boldsymbol{R}}_F = \begin{bmatrix} 1 & 0.90 & -0.77 \\ 0.90 & 1 & -0.43 \\ -0.77 & -0.43 & 1 \end{bmatrix}.$$

Note that the Frobenius correlation method does not consider the dependence of covariances and the adjustment is only finding the proper correlation matrix nearest to $\boldsymbol{R}_T$ based on the Frobenius norm.

So the adjusted covariance matrix based on the Frobenius correlation method is constructed by using $\hat{\boldsymbol{\Sigma}}_F = \boldsymbol{D}_a \hat{\boldsymbol{R}}_F \boldsymbol{D}_a$ where $\boldsymbol{D}_a$ is the diagonal matrix of the standard deviations with those for RML and UNE dropped to 0.5 and 0.2

Table 2. Sample standard deviations and sample correlations (upper triangular entries for crisis period, lower triangular entries for observation period) for daily log return of Country Indices for PIGS during the observation period and crisis period.

	Sample standard deviation			Sample correlation			
Asset	Observation period	Crisis period	Asset	Portugal	Italy	Greece	Spain
Portugal	1.12	2.20	Portugal	1	0.90	0.73	0.91
Italy	1.27	2.31	Italy	0.58	1	0.67	0.94
Greece	1.68	3.29	Greece	0.37	0.38	1	0.69
Spain	1.37	2.79	Spain	0.61	0.79	0.38	1

Table 3. Correlations between selected sample covariances $\sigma_{i,j}$ with values greater than 0.80 bolded.

	$\sigma_{1,1}$	$\sigma_{1,2}$	$\sigma_{1,3}$	$\sigma_{1,4}$	$\sigma_{2,2}$	$\sigma_{2,3}$	$\sigma_{2,4}$	$\sigma_{3,3}$	$\sigma_{3,4}$	$\sigma_{4,4}$
$\sigma_{2,10}$	0.80	0.79	0.74	**0.85**	**0.85**	0.73	**0.90**	0.42	0.77	**0.85**
$\sigma_{6,7}$	0.09	0.16	0.15	0.12	0.19	0.13	0.15	0.07	0.12	0.12
$\sigma_{9,9}$	0.60	0.54	0.42	0.66	0.70	0.38	**0.84**	0.19	0.45	**0.86**

Table 4. The new covariance matrix obtained by the proposed approach (upper triangular) and the sample covariance matrix during crisis period (lower triangular) with the covariances for PIGS bolded.

Asset	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	
	3.68	**3.59**	**4.60**	**3.88**	3.52	1.26	1.47	2.54	3.48	1.95	2.70	1.82	3.44	2.16	4.20	1.52	2.02	2.25	2.87	2.68	4.94	2.60	1.88	3.18	1
		4.73	**5.23**	**4.52**	4.34	1.63	1.62	3.12	4.26	2.33	3.25	2.03	4.22	2.52	5.01	1.66	2.33	2.51	3.41	3.07	5.86	3.00	2.31	3.72	2
1	**4.82**		**8.28**	**5.62**	5.12	1.90	2.21	3.73	5.06	2.88	3.93	2.72	5.08	3.36	6.15	2.20	3.14	3.27	4.19	3.90	7.27	3.81	2.92	4.75	3
2	**4.57**	**5.32**		**5.48**	4.62	1.77	1.67	3.33	4.58	2.52	3.50	2.15	4.55	2.67	5.37	1.75	2.55	2.75	3.70	3.31	6.41	3.20	2.44	3.97	4
3	**5.27**	**5.12**	**10.84**		4.98	1.86	1.59	3.22	4.40	2.49	3.34	1.98	4.37	2.55	5.16	1.61	2.37	2.53	3.52	3.11	6.10	3.02	2.48	3.72	5
4	**5.59**	**6.02**	**6.32**	**7.78**		1.65	0.57	1.26	1.68	1.47	1.25	0.68	1.65	0.96	1.99	0.52	0.92	0.83	1.33	1.10	2.39	1.09	1.32	1.38	6
5	3.36	3.75	3.75	4.28	3.00		2.53	1.20	1.60	0.98	1.27	1.36	1.59	1.65	2.11	1.01	1.60	1.09	1.33	1.28	2.45	1.29	1.01	1.60	7
6	1.80	1.95	1.88	2.20	1.40	1.60		2.75	3.19	1.78	2.46	1.53	3.21	1.95	3.76	1.22	1.82	1.86	2.57	2.27	4.44	2.32	1.75	2.79	8
7	0.10	0.27	0.26	0.13	0.16	-0.06	1.42		4.49	2.38	3.26	1.99	4.30	2.50	5.09	1.60	2.35	2.48	3.48	3.07	6.06	3.00	2.36	3.70	9
8	2.93	3.24	3.10	3.67	2.52	1.25	0.21	2.42		2.46	1.76	1.23	2.31	1.49	2.90	0.93	1.42	1.36	1.86	1.70	3.49	1.61	1.65	2.16	10
9	4.10	4.57	4.73	5.32	3.47	1.69	0.24	3.03	4.22		2.93	1.55	3.29	1.89	3.83	1.24	1.79	1.98	2.76	2.41	4.45	2.37	1.70	2.87	11
10	2.21	2.33	2.37	2.63	1.88	1.44	0.07	1.66	2.18	2.00		2.13	2.01	1.81	2.52	1.50	1.68	1.41	1.71	1.66	2.90	1.67	1.20	2.09	12
11	2.37	2.61	2.67	2.92	2.08	0.92	0.18	1.79	2.46	1.34	1.70		4.60	2.52	5.01	1.65	2.37	2.47	3.55	3.07	5.96	3.02	2.30	3.72	13
12	2.48	2.69	2.92	2.93	2.15	1.03	0.94	2.08	2.63	1.71	1.61	3.46		3.70	3.21	1.40	2.58	1.62	2.00	1.96	3.88	1.90	1.66	2.52	14
13	3.60	3.99	4.13	4.60	3.07	1.50	0.22	2.69	3.70	2.01	2.18	2.39	3.39		7.24	2.00	3.02	2.99	3.99	3.77	7.89	3.62	2.97	4.59	15
14	0.85	1.01	1.33	1.14	0.80	0.38	0.63	0.76	0.98	0.58	0.56	1.30	0.88	1.09		2.27	1.34	1.15	1.38	1.30	2.32	1.32	0.95	1.65	16
15	3.99	4.29	4.63	4.94	3.32	1.75	0.05	2.92	3.96	2.28	2.36	2.60	3.60	0.88	4.40		3.27	1.54	1.91	1.86	3.59	1.82	1.56	2.36	17
16	1.63	1.71	1.64	1.85	1.38	0.53	0.49	1.35	1.63	0.96	1.01	1.78	1.46	0.73	1.66	1.78		2.44	2.16	2.05	3.38	2.02	1.31	2.53	18
17	1.67	1.79	2.28	2.03	1.40	0.68	0.49	1.31	1.72	1.12	1.07	1.82	1.58	0.97	1.79	1.00	1.77		3.42	2.59	4.57	2.56	1.76	3.04	19
18	4.53	5.08	5.15	5.86	3.80	1.90	0.36	3.30	4.58	2.50	2.62	3.15	4.03	1.21	4.48	2.01	2.03	5.85		3.09	4.34	2.37	1.75	2.98	20
19	3.76	4.03	4.12	4.72	3.06	1.46	0.13	2.67	3.72	1.95	2.19	2.26	3.34	0.83	3.57	1.41	1.52	4.10	3.55		11.70	4.20	3.63	5.36	21
20	2.97	3.24	3.58	3.59	2.54	1.31	0.32	2.22	3.03	1.81	1.83	2.21	2.76	0.82	3.16	1.38	1.55	3.52	2.77	2.97		3.18	1.62	2.83	22
21	3.55	3.88	4.28	4.47	2.90	1.40	0.24	2.51	3.52	1.88	2.14	2.28	3.13	0.80	3.48	1.35	1.49	3.87	3.19	2.80	3.93		3.24	2.15	23
22	4.60	5.03	5.01	5.86	3.85	1.82	0.37	3.38	4.63	2.35	2.77	2.86	4.19	1.00	4.57	1.76	1.97	5.03	4.23	3.48	3.92	6.67		4.57	24
23	0.72	0.78	1.32	0.90	0.64	0.35	0.16	0.50	0.77	0.51	0.46	0.79	0.70	0.29	0.82	0.39	0.51	0.89	0.63	0.75	0.77	0.82	1.07		
24	3.96	4.32	4.79	4.78	3.45	1.78	0.26	3.05	4.09	2.48	2.49	3.19	3.74	1.04	4.28	1.90	2.00	4.58	3.66	3.48	3.58	4.60	1.01	5.19	

Table 5. The new covariance matrices obtained by the Ng method (Ng *et al.* 2014) (upper triangular) and the Frobenius correlation method (lower triangular).

Asset	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	
	5.04	**4.48**	**5.35**	**4.89**	2.41	0.78	1.18	1.83	2.23	1.29	1.93	1.43	2.21	1.42	2.86	1.08	1.32	1.85	2.17	2.00	3.40	1.89	1.15	1.99	1
		6.49	**6.09**	**6.01**	3.30	1.36	0.87	2.49	3.11	1.80	2.53	1.48	3.08	1.60	3.62	1.17	1.57	2.18	2.81	2.41	4.15	2.33	1.54	2.48	2
1	**5.04**		**11.34**	**6.55**	3.40	1.23	1.05	2.36	3.09	1.95	2.51	1.97	2.97	2.14	3.86	1.78	2.17	2.69	3.00	2.95	4.83	2.86	2.19	3.20	3
2	**5.11**	**6.49**		**7.51**	3.40	1.46	0.90	2.56	3.25	1.83	2.53	1.47	3.16	1.65	3.78	1.15	1.64	2.42	2.79	2.44	4.64	2.48	1.64	2.52	4
3	**6.86**	**7.08**	**11.34**		2.24	0.92	0.47	1.46	1.77	1.04	1.48	0.75	1.74	0.90	2.10	0.63	0.92	1.20	1.63	1.34	2.50	1.39	1.02	1.38	5
4	**5.50**	**6.70**	**7.67**	**7.51**		1.25	0.04	0.56	0.79	0.91	0.61	0.09	0.69	0.23	0.94	0.04	0.23	0.38	0.62	0.50	1.15	0.55	0.73	0.43	6
5	1.82	2.86	1.83	3.01	2.24		2.03	0.37	0.47	0.21	0.45	0.96	0.41	1.00	0.62	0.62	0.95	0.46	0.55	0.51	0.60	0.43	0.34	0.55	7
6	0.51	1.07	0.50	1.15	0.82	1.25		1.27	1.40	0.73	1.11	0.65	1.38	0.72	1.60	0.52	0.70	1.00	1.25	1.04	1.83	1.11	0.61	1.20	8
7	0.69	0.65	1.11	0.68	0.37	0.10	2.03		1.76	0.95	1.41	0.76	1.68	0.83	2.05	0.57	0.85	1.14	1.54	1.26	2.44	1.30	0.83	1.37	9
8	1.23	2.00	1.29	2.11	1.21	0.50	0.31	1.27		1.54	0.79	0.44	0.81	0.43	1.21	0.30	0.40	0.61	0.81	0.67	1.50	0.66	0.81	0.83	10
9	1.73	2.70	1.78	2.89	1.67	0.64	0.37	1.19	1.76		1.31	0.63	1.36	0.69	1.66	0.45	0.68	0.93	1.23	1.06	1.84	0.97	0.64	1.07	11
10	0.91	1.35	0.91	1.45	0.91	0.86	0.29	0.62	0.80	1.54		1.36	0.74	1.13	0.96	1.03	0.98	0.75	0.86	0.81	1.06	0.74	0.49	0.98	12
11	1.39	2.07	1.46	2.19	1.23	0.44	0.34	0.92	1.16	0.55	1.31		1.81	0.87	1.94	0.56	0.82	1.09	1.55	1.25	2.22	1.24	0.73	1.22	13
12	0.86	0.90	1.27	0.98	0.48	0.11	0.74	0.43	0.49	0.38	0.41	1.36		2.39	1.15	0.90	1.63	0.84	0.89	0.85	1.34	0.76	0.69	1.16	14
13	1.61	2.62	1.74	2.80	1.61	0.60	0.36	1.19	1.56	0.71	1.17	0.50	1.81		3.03	0.81	1.16	1.41	1.75	1.56	3.17	1.63	1.07	1.77	15
14	0.71	0.98	1.26	1.05	0.62	0.23	0.84	0.54	0.57	0.39	0.41	0.81	0.58	2.39		1.76	0.88	0.59	0.64	0.67	0.83	0.54	0.30	0.68	16
15	1.96	2.85	2.03	3.08	1.76	0.70	0.62	1.27	1.71	0.95	1.22	0.69	1.59	0.84	3.03		2.20	0.85	0.84	0.81	1.18	0.81	0.75	1.15	17
16	0.75	0.72	1.03	0.77	0.38	0.04	0.51	0.32	0.36	0.23	0.29	0.88	0.41	0.59	0.50	1.76		1.34	1.08	1.02	1.83	1.05	0.74	1.35	18
17	0.75	1.00	1.33	1.13	0.60	0.25	0.88	0.53	0.59	0.42	0.44	0.78	0.60	1.40	0.86	0.61	2.20		1.59	1.19	2.02	1.17	0.73	1.24	19
18	1.22	1.36	1.44	1.52	0.77	0.16	0.35	0.57	0.72	0.35	0.63	0.49	0.69	0.42	0.83	0.41	0.45	1.34		1.47	1.82	1.08	0.82	1.27	20
19	1.47	2.16	1.62	2.30	1.27	0.47	0.35	0.93	1.24	0.55	1.03	0.49	1.29	0.44	1.21	0.39	0.48	0.73	1.59		5.73	1.89	1.46	2.05	21
20	1.38	1.71	1.54	1.85	0.99	0.29	0.38	0.72	0.95	0.49	0.78	0.53	0.94	0.50	1.16	0.38	0.54	0.71	0.86	1.47		1.66	0.67	1.19	22
21	2.21	3.26	2.28	3.64	2.06	0.85	0.68	1.48	2.02	1.16	1.34	0.71	1.88	1.07	2.89	0.53	1.02	0.80	1.26	1.23	5.73		2.33	0.81	23
22	1.33	1.68	1.65	1.79	0.95	0.30	0.42	0.80	0.93	0.42	0.78	0.59	0.93	0.50	1.07	0.44	0.54	0.73	0.88	0.80	1.16	1.66		2.09	24
23	0.73	1.28	1.10	1.30	0.88	0.71	0.33	0.58	0.76	0.74	0.47	0.36	0.69	0.56	1.01	0.25	0.56	0.30	0.45	0.53	1.30	0.43	2.33		
24	1.40	1.91	1.86	2.01	1.07	0.36	0.49	0.85	1.05	0.65	0.85	0.71	1.06	0.72	1.34	0.52	0.73	0.88	0.89	0.97	1.48	0.89	0.64	2.09	

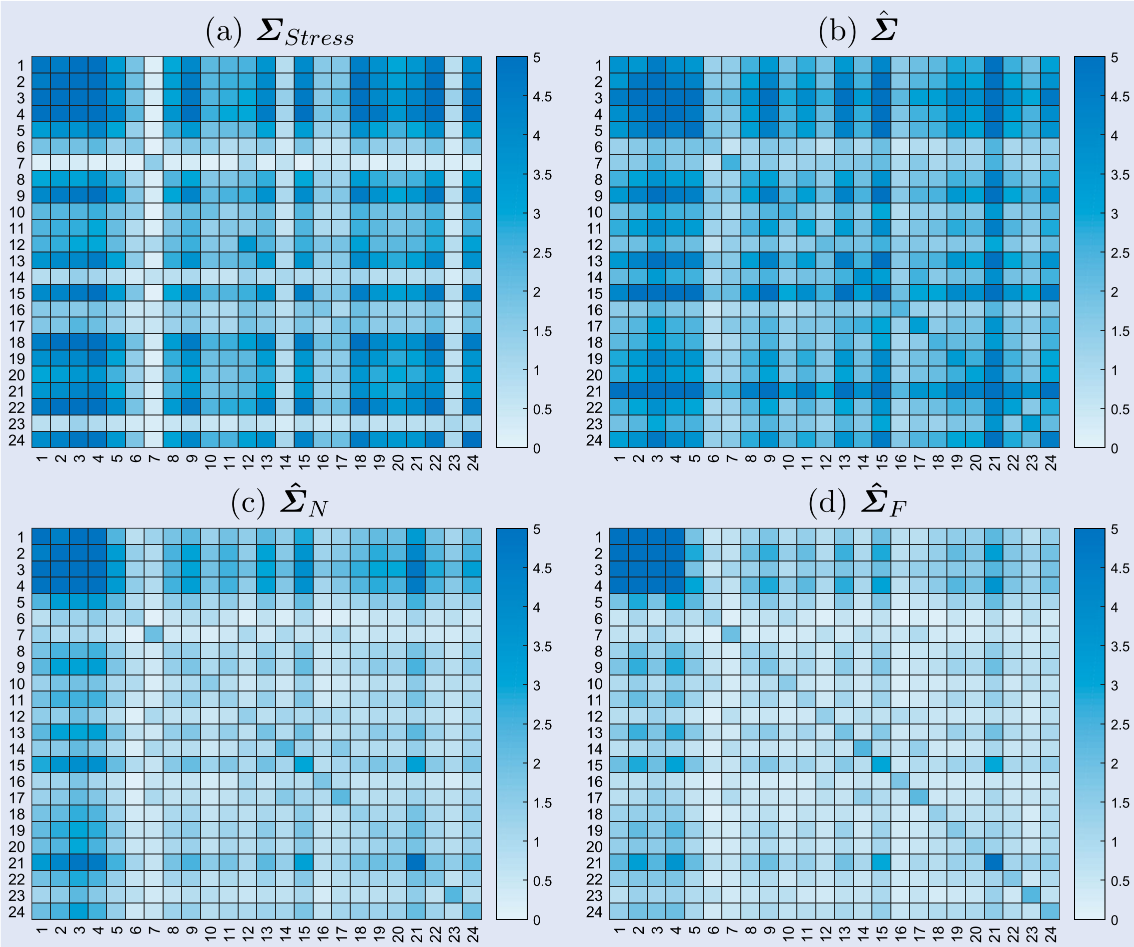

Figure 1. Heatmaps of $\boldsymbol{\Sigma}_{Stress}$, $\hat{\boldsymbol{\Sigma}}$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$, where $\boldsymbol{\Sigma}_{Stress}$ is the sample covariance matrix during crisis period, $\hat{\boldsymbol{\Sigma}}$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$ are the new covariance matrices obtained by the proposed approach, Ng method and Frobenius correlation method respectively.

respectively:

$$\hat{\boldsymbol{\Sigma}}_F = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 2.0 \end{bmatrix} \begin{bmatrix} 1 & 0.90 & -0.77 \\ 0.90 & 1 & -0.43 \\ -0.77 & -0.43 & 1 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 2.0 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.09 & -0.77 \\ 0.09 & 0.04 & -0.17 \\ -0.77 & -0.17 & 4.00 \end{bmatrix}.$$

It seems unreasonable to see that after adjustment, the peripheral variance σ_{33} remains unchanged even though the variances σ_{11} and σ_{22} dropped significantly.

3.1.2. Our new method. Assume that the covariance matrix $\boldsymbol{\Sigma}$ of the three variables {RML, UNE, GDP} follows a truncated $SN_3(\boldsymbol{W}, \boldsymbol{\Psi})$ with

$$\boldsymbol{\Psi} = \begin{bmatrix} 1.44 & 0.82 & -1.37 \\ 0.82 & 0.72 & -0.66 \\ -1.37 & -0.66 & 1.44 \end{bmatrix}$$

or equivalently that $\boldsymbol{y} = \text{vech}(\boldsymbol{\Sigma}) = [\sigma_{11} \;\; \sigma_{12} \;\; \sigma_{22} \;\; \sigma_{13} \;\; \sigma_{23} \;\; \sigma_{33}]^\top$ follows a truncated multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{K}$:

$$\boldsymbol{\mu} = \text{vech}(\boldsymbol{W}) = [3.24 \quad 1.64 \quad 1.69 \quad -2.88 \quad -1.04 \quad 4]^\top$$

and

$$\boldsymbol{K} = \boldsymbol{D}_n^+(\boldsymbol{\Psi} \otimes \boldsymbol{\Psi})(\boldsymbol{D}_n^+)^\top = \begin{bmatrix} 2.07 & 1.18 & 0.67 & -1.97 & -1.12 & 1.87 \\ 1.18 & 0.85 & 0.59 & -1.04 & -0.76 & 0.91 \\ 0.67 & 0.59 & 0.52 & -0.54 & -0.48 & 0.44 \\ -1.97 & -1.04 & -0.54 & 1.97 & 1.04 & -1.97 \\ -1.12 & -0.76 & -0.48 & 1.04 & 0.74 & -0.95 \\ 1.87 & 0.91 & 0.44 & -1.97 & -0.95 & 2.07 \end{bmatrix}.$$

The choice of these figures makes reference to the data retrieved from the Board of Governors of the Federal Reserve System (https://www.federalreserve.gov/data.htm). The entries of $\boldsymbol{\Psi}$ and $\boldsymbol{K}$ are about the dependencies among the σ_{ij}'s. For example, the correlation between σ_{11} and σ_{13} is $-1.97/\sqrt{(2.07)(1.97)} = -0.98$. Therefore, it is expected that when σ_{11} drops significantly, σ_{13} should increase. Similarly, since the correlation between σ_{11} and σ_{33} is

$1.87/\sqrt{(2.07)(2.07)} = 0.90$, it is expected that σ_{33} tends to drop when σ_{11} drops significantly. In this case, the resultant matrix $\hat{\boldsymbol{\Sigma}}_F$ with σ_{33} unchanged is obviously inappropriate.

Applying our method with $\tau = 0.01$, we obtain the adjusted covariance matrix $\hat{\boldsymbol{\Sigma}}$ by maximizing the posterior density (3):

$$\hat{\boldsymbol{\Sigma}} = \tilde{\boldsymbol{W}} = \begin{bmatrix} 0.28 & 0.11 & -0.003 \\ 0.11 & 0.06 & 0.08 \\ -0.003 & 0.08 & 1.05 \end{bmatrix}$$
$$= \begin{bmatrix} 0.53 & 0 & 0 \\ 0 & 0.24 & 0 \\ 0 & 0 & 1.02 \end{bmatrix} \begin{bmatrix} 1 & 0.88 & -0.01 \\ 0.88 & 1 & 0.33 \\ -0.01 & 0.33 & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 0.53 & 0 & 0 \\ 0 & 0.24 & 0 \\ 0 & 0 & 1.02 \end{bmatrix},$$

which is already PSD, though symmetric matrix-variate normal distribution does not guarantee to generate positive semidefinite matrices. It can be seen that the adjustment on the peripheral covariances σ_{13} and σ_{33} meets the above expectation that σ_{13} increases and σ_{33} decreases. The adjustment meets our expectation because our new method considers the dependence between covariances.

3.2. *Stress testing examples on a portfolio of MSCI developed market country indices*

3.2.1. Stressing the covariances among PIGS. Consider a portfolio of 24 MSCI Developed Market Country indices with their names listed in table 1. Suppose that we perform stress testing on the portfolio against the 2010 European sovereign debt crisis in Portugal, Italy, Greece and Spain (PIGS). It is suspected that the correlations among the equity indices of these four countries will increase to a high value (say 0.9) and their standard deviations will be doubled but we have no idea about the changes on the other covariances under such market condition.

To obtain some idea about the changes of covariances from the normal market environment to severe market environment, we compare the sample covariance matrices for the daily log returns of these 24 country indices during two periods, as shown in table 1. The lower triangular part, $\boldsymbol{\Sigma}_{Normal}$, gives the covariances from 1 January 1998 to 31 December 2007 (the observation period), which represents the long-run covariance matrix, and the upper triangular part, $\boldsymbol{\Sigma}_{Stress}$, gives those from 1 January 2010 to 30 June 2010 (the crisis period), during which the global financial market was influenced by the 2010 European sovereign debt crisis. The crisis period is chosen following Ng *et al.* (2014) and Yu *et al.* (2014). The covariance matrix $\boldsymbol{\Sigma}_{Stress}$ can be regarded as a proxy of the true covariance matrix under this scenario. Table 2 shows the summary statistics of daily log returns of the four country indices (PIGS) during the observation period and the crisis period. It can be seen that as compared with the observation period, the sample standard deviations of the four country index returns were more or less doubled and their sample correlations were increased significantly during the crisis period.

For each non-overlapping period of three months during the observation period, we compute the sample covariance matrix of the daily log returns on the 24 country indices, resulting in a total of 40 sample covariance matrices. Based on these covariance matrices, we estimate the parameters $\boldsymbol{W}$ and $\boldsymbol{\Psi}$ of the prior distribution for $\boldsymbol{\Sigma}$ using the MLE method stated in Appendix 1. Table 3 shows the correlations between some selected covariances. It can be seen that some covariances are highly correlated. For example, $\sigma_{2,10}$ (the covariance between Italy and Canada index returns) is highly correlated with $\sigma_{2,2}$ (the variance of Italy index returns), $\sigma_{2,4}$ (the covariance between Italy and Spain index returns), etc.

Note that there is a user-specified parameter $0 \le \tau < 1$ which represents the level of uncertainty of the subjective views relative to the data. For example, $\tau = 0$ implies certainty views. In forming stress testing scenarios, τ should be set as small as possible to ensure the core covariances are adjusted close to the values implied by the views. However, τ cannot be too small in practice as we need to have sufficient room for the covariance matrix to accommodate PSD. As we tend to be more uncertain about the subjective views when we have more assets, it is natural to select τ based on the number n of assets under study and our experience suggests setting $\tau/(1-\tau)$ proportional to n.

According to the stress testing example for $n = 10$ assets studied in Ng *et al.* (2014), we purpose a formula $\tau = n/(n + 42.6)$ for consideration. This formula just gives a

Table 6. Frobenius norms between covariance matrices, where $\boldsymbol{\Sigma}_{Stress}$ and $\boldsymbol{\Sigma}_{Normal}$ are the sample covariance matrices during the crisis period and the observation period respectively, $\hat{\boldsymbol{\Sigma}}_F$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}$ are the new covariance matrices obtained by the Frobenius correlation method, the Ng method (Ng *et al.* 2014) and the proposed method respectively.

	$\boldsymbol{\Sigma}_{Normal}$	$\hat{\boldsymbol{\Sigma}}_F$	$\hat{\boldsymbol{\Sigma}}_N$	$\hat{\boldsymbol{\Sigma}}$
$\boldsymbol{\Sigma}_{Stress}$	50.5	40.7	32.9	29.5

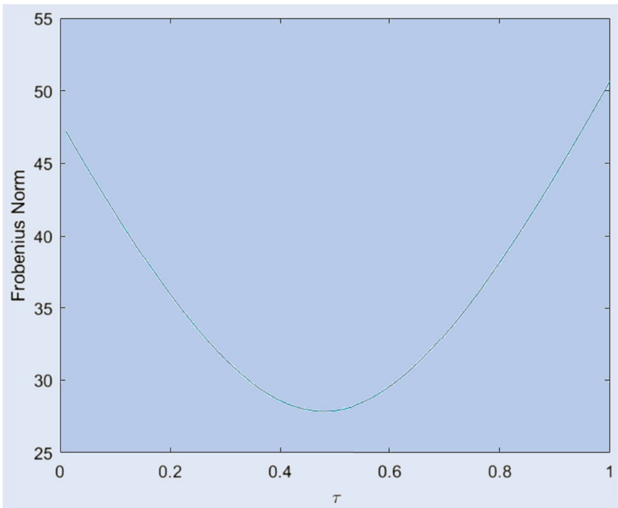

Figure 2. Plot of Frobenius norm $\|\hat{\boldsymbol{\Sigma}}(\tau) - \boldsymbol{\Sigma}_{Stress}\|_F$ against τ, where $\hat{\boldsymbol{\Sigma}}(\tau)$ is the new covariance matrix obtained by the proposed approach at different values of τ and $\boldsymbol{\Sigma}_{Stress}$ is the sample covariance matrix during crisis period.

Table 7. Sample covariance matrices for daily log returns of the 23 assets during the observation period (lower triangular) and the post-voting period (upper triangular).

Asset	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
	0.68	**0.50**	**0.36**	0.60	0.57	0.52	0.60	0.59	0.62	0.57	0.65	0.73	0.64	0.43	0.59	0.19	0.41	0.32	0.27	0.22	0.03	0.17	0.20	1
		1.06	**0.59**	0.61	0.65	0.60	0.58	0.54	0.64	0.61	0.75	0.81	0.73	0.55	0.58	0.20	0.39	0.24	0.39	0.14	0.09	0.28	0.16	2
1. UK	**2.04**		**1.27**	0.46	0.52	0.45	0.49	0.34	0.49	0.44	0.65	0.51	0.54	0.40	0.54	0.16	0.34	0.22	0.30	0.08	0.06	0.12	0.16	3
2. Belgium	**1.65**	**2.43**		0.78	0.77	0.66	0.73	0.78	0.77	0.70	0.81	0.95	0.86	0.52	0.70	0.22	0.42	0.32	0.31	0.24	0.05	0.24	0.23	4
3. Denmark	**1.47**	**1.58**	**2.24**		0.87	0.69	0.76	0.74	0.80	0.72	0.79	0.94	0.83	0.54	0.69	0.25	0.39	0.37	0.28	0.21	0.04	0.30	0.21	5
4. France	2.04	2.08	1.81	2.73		0.67	0.65	0.61	0.67	0.63	0.73	0.81	0.74	0.47	0.59	0.23	0.36	0.27	0.32	0.19	0.01	0.24	0.16	6
5. Germany	1.98	2.01	1.76	2.55	2.96		0.92	0.73	0.85	0.72	0.81	0.88	0.79	0.54	0.76	0.23	0.43	0.41	0.33	0.25	0.13	0.25	0.25	7
6. Netherlands	1.92	2.02	1.70	2.41	2.36	2.54		1.52	0.80	0.65	0.75	1.16	0.96	0.42	0.79	0.23	0.48	0.38	0.14	0.35	0.06	0.09	0.28	8
7. Sweden	2.19	2.17	2.10	2.76	2.72	2.53	4.08		1.13	0.73	0.82	0.92	0.80	0.54	0.73	0.26	0.42	0.43	0.26	0.27	0.11	0.37	0.32	9
8. Austria	1.73	1.87	1.81	2.12	2.06	1.96	2.33	3.22		1.14	0.67	0.81	0.74	0.55	0.56	0.29	0.36	0.19	0.19	0.08	-0.03	0.38	0.12	10
9. Finland	2.19	2.04	2.01	2.83	2.77	2.57	3.59	2.14	5.64		1.43	1.01	0.99	0.57	0.95	0.22	0.64	0.46	0.64	0.24	0.16	0.11	0.38	11
10. Ireland	1.77	1.89	1.71	2.07	1.96	1.97	2.28	2.13	2.19	3.45		1.81	1.31	0.58	0.80	0.32	0.52	0.41	0.41	0.30	0.12	0.22	0.29	12
11. Portugal	1.34	1.47	1.40	1.75	1.69	1.59	1.88	1.74	1.89	1.55	2.05		1.28	0.56	0.79	0.23	0.51	0.36	0.48	0.32	0.17	0.20	0.32	13
12. Italy	1.94	2.01	1.77	2.54	2.47	2.33	2.62	2.18	2.64	2.04	1.83	2.92		0.57	0.47	0.18	0.31	0.22	0.26	0.14	0.06	0.25	0.16	14
13. Spain	1.97	2.07	1.80	2.58	2.48	2.36	2.65	2.23	2.73	2.09	1.93	2.66	3.15		1.39	0.10	0.59	0.55	0.45	0.32	0.24	0.12	0.42	15
14. Switzerland	1.44	1.51	1.33	1.76	1.75	1.68	1.88	1.51	1.86	1.48	1.22	1.72	1.74	1.67		0.32	0.24	0.08	0.05	0.00	-0.03	0.13	0.07	16
15. Norway	2.01	1.97	2.07	2.37	2.31	2.21	2.80	2.48	2.65	2.22	1.75	2.33	2.30	1.67	3.85		0.65	0.35	0.32	0.17	0.21	0.11	0.28	17
16. USA	0.92	0.89	0.69	1.12	1.25	1.04	1.22	0.82	1.23	0.91	0.64	1.06	1.08	0.74	1.01	1.72		0.99	0.58	0.38	0.49	0.19	0.38	18
17. Canada	1.31	1.15	1.15	1.53	1.57	1.38	1.74	1.35	1.77	1.24	1.06	1.48	1.49	1.04	1.72	1.36	2.30		1.46	0.22	0.49	0.03	0.33	19
18. Australia	1.17	1.11	1.25	1.29	1.23	1.21	1.51	1.49	1.43	1.32	1.09	1.27	1.28	0.95	1.70	0.40	0.99	2.42		0.60	0.17	0.06	0.28	20
19. New Zealand	0.86	0.82	0.90	0.95	0.91	0.91	1.10	1.12	1.05	0.99	0.85	0.96	0.95	0.69	1.26	0.25	0.69	1.47	2.15		1.11	-0.06	0.27	21
20. HK	0.82	0.71	0.79	0.85	0.87	0.81	1.09	0.91	1.18	0.81	0.65	0.79	0.80	0.59	1.11	0.37	0.67	1.32	0.81	2.49		1.07	0.00	22
21. Japan	0.45	0.44	0.56	0.52	0.49	0.48	0.70	0.64	0.74	0.56	0.47	0.48	0.48	0.45	0.70	0.08	0.42	1.16	0.74	1.03	2.21		0.56	23
22. Israel	0.80	0.71	0.79	1.01	1.07	0.90	1.24	0.75	1.36	0.79	0.64	0.93	0.90	0.64	1.03	0.68	0.85	0.65	0.46	0.64	0.37	2.17		
23. Singapore	0.98	0.94	0.99	1.08	1.09	1.02	1.35	1.14	1.38	1.05	0.82	1.03	1.07	0.76	1.37	0.49	0.81	1.38	0.94	1.61	0.99	0.68	2.39	

Table 8. Sample standard deviations and sample correlations (upper triangular entries for the post-voting period, lower triangular entries for observation period) for daily log returns of Country Indices for the UK, Belgium and Denmark during the observation period and post-voting period.

	Sample standard deviation			Sample correlation		
Asset	Observation period	Post-voting period	Asset	UK	Belgium	Denmark
1. UK	1.43	0.82	1. UK	1.00	0.60	0.39
2. Belgium	1.56	1.03	2. Belgium	0.74	1.00	0.51
3. Denmark	1.50	1.13	3. Denmark	0.69	0.68	1.00

guideline for choosing the value of the parameter τ. If the risk managers have stronger confidence on their belief, they can set τ as a value smaller than the value of τ obtained from this formula. Applying this formula to our example ($n = 24$) here, we obtain $\tau = 0.36$, and using it in (4) gives the resulting matrix $\hat{\boldsymbol{\Sigma}} = \tilde{\boldsymbol{W}}$ which is found to be PSD. The upper part of table 4 shows the matrix $\hat{\boldsymbol{\Sigma}}$. It can be seen from table 4 that the covariances for PIGS in $\hat{\boldsymbol{\Sigma}}$ are close to those in the crisis period. Also, some peripheral covariances increase significantly as compared with those in the observation period shown in table 1. For instance, as seen from tables 1 and 4, the covariance between Italy and Canada index returns ($\sigma_{2,10}$) estimated by our model has increased significantly which can be explained by the strong positive correlations between $\sigma_{2,10}$ and some variances and covariances like $\sigma_{2,2}$ (the variance of Italy index returns) and $\sigma_{2,4}$ (the covariance between Italy and Spain index returns), as evidenced in table 3. Therefore, when $\sigma_{2,2}$ and $\sigma_{2,4}$ increase significantly, $\sigma_{2,10}$ will probably have a high value. Another example is that the covariance between USA and Japan index returns ($\sigma_{6,7}$) estimated by our model does not increase significantly, and this can be explained by the correlations between $\sigma_{6,7}$ and the covariances among PIGS (see table 3). Although we did not include France in the subjective views, the variance of France index returns ($\sigma_{9,9}$) estimated by our model increases significantly. This can be explained by the strong positive correlations between $\sigma_{9,9}$ and $\{\sigma_{2,4}, \sigma_{4,4}\}$. In fact, this result also points out that it is important to incorporate the potential dependence between variances and correlations.

Here, we also compare the performance of the proposed method with that of existing methods. Following Kupiec (1998), the traditional way of adjusting the covariance matrix is to first decompose $\boldsymbol{\Sigma}_{Normal}$ as $\boldsymbol{\Sigma}_{Normal} = \boldsymbol{DRD}$, where $\boldsymbol{R}$ and $\boldsymbol{D}$ are the correlation matrix and diagonal matrix of standard deviations respectively, and then to adjust the standard deviations and correlations separately. The specified standard deviations would be adjusted to targeted values whereas the correlation matrix would be adjusted by some correlation matrix adjustment methods such as the Frobenius correlation method and the Ng method (Ng *et al.* 2014). Under the Frobenius correlation method, we first construct the target correlation matrix $\boldsymbol{R}_T$ by changing the core correlations in $\boldsymbol{R}$ to 0.9. Since such $\boldsymbol{R}_T$ is not PSD, we obtain the optimal correlation matrix, $\hat{\boldsymbol{R}}_F$, which is nearest to $\boldsymbol{R}_T$ based on the Frobenius norm. Finally, we can estimate the stressed covariance matrix $\hat{\boldsymbol{\Sigma}}_F = \boldsymbol{D}_a\hat{\boldsymbol{R}}_F\boldsymbol{D}_a$ where $\boldsymbol{D}_a$ is the diagonal matrix of the standard deviations with those for PIGS being doubled. Under the Ng method (Ng *et al.* 2014), we first construct the target matrix $\tilde{\boldsymbol{R}}_N$ in which the correlations change according to the dependence structure of the empirical correlations. If $\tilde{\boldsymbol{R}}_N$ is not PSD, then Ng *et al.* (2014) suggested finding the matrix $\hat{\boldsymbol{R}}_N$ which minimizes the Mahalanobis distance $\|\boldsymbol{Y} - \tilde{\boldsymbol{R}}_N\|^2_{\tilde{\Sigma}}$ subject to the constraint that $\boldsymbol{Y}$ is PSD. The resulting stressed covariance matrix under the Ng method becomes $\hat{\boldsymbol{\Sigma}}_N = \boldsymbol{D}_a\hat{\boldsymbol{R}}_N\boldsymbol{D}_a$. Table 5 shows the stressed covariance matrices $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$. Figure 1 shows the heatmaps of the sample covariance matrix during crisis period ($\boldsymbol{\Sigma}_{Stress}$) and the new covariance matrices $\hat{\boldsymbol{\Sigma}}$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$, which are the new covariance matrices obtained by the proposed approach, the Ng method and Frobenius correlation method respectively. The heatmaps support that the covariance matrix derived by our proposed method is more similar to $\boldsymbol{\Sigma}_{Stress}$ as compared with the covariance matrices constructed by the Ng method and Frobenius correlation method. We measure the difference between each upper triangular entry of the new covariance matrix with the associated entry of the sample covariance matrix during crisis period ($\boldsymbol{\Sigma}_{Stress}$) and evaluate if the covariances of our proposed method are closest to the covariances of $\boldsymbol{\Sigma}_{Stress}$. Among all 288 entries, 178 entries of $\hat{\boldsymbol{\Sigma}}$ are closest to the associated entries in $\boldsymbol{\Sigma}_{Stress}$, 69 entries of $\hat{\boldsymbol{\Sigma}}_N$ are closest to the associated entries in $\boldsymbol{\Sigma}_{Stress}$ and 41 entries of $\hat{\boldsymbol{\Sigma}}_F$ are closest to the associated entries in $\boldsymbol{\Sigma}_{Stress}$.

To evaluate the performance of various covariance matrices in approximating the covariance matrix during the crisis period, we follow Ng *et al.* (2014) and Rebonato and Jäckel (2000) and report in table 6 the Frobenius norms between different stressed covariance matrices and $\boldsymbol{\Sigma}_{Stress}$. It can be seen that our proposed method $\hat{\boldsymbol{\Sigma}}$ provides the best covariance matrix adjustment. To study the sensitivity of our proposed method with respect to τ, we consider different values of τ and compute the distance between the resultant matrix $\hat{\boldsymbol{\Sigma}}(\tau)$ with $\boldsymbol{\Sigma}_{Stress}$ using the Frobenius norm, see figure 2. We can see that the distance between $\hat{\boldsymbol{\Sigma}}(\tau)$ and $\boldsymbol{\Sigma}_{Stress}$ becomes large when τ is too small because a very small τ moves the target matrix far from PSD, and thus regularizing it moves $\hat{\boldsymbol{\Sigma}}(\tau)$ farther from $\boldsymbol{\Sigma}_{Stress}$. The optimal covariance matrix $\hat{\boldsymbol{\Sigma}}(\tau)$ is the closest to $\boldsymbol{\Sigma}_{Stress}$ when τ is around 0.4 to 0.6, and this supports the above-mentioned choice of τ.

3.2.2. Stressing the covariances between the UK, Belgium and Denmark due to Brexit. Brexit (a portmanteau of 'British' and 'exit') is the withdrawal of the United Kingdom (UK) from the European Union (EU). From the 1990s, opposition to further European integration came mainly from the right. On 23 January 2013, Prime Minister David Cameron announced that a Conservative government would hold an in-or-out referendum on EU membership before the end of 2017,

Table 9. The new covariance matrix obtained by the proposed approach (upper triangular) and the sample covariance matrix during the post-voting period (lower triangular).

Asset	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
	1.04	**0.75**	**0.69**	0.97	0.98	0.88	1.06	0.78	1.08	0.83	0.61	0.91	0.91	0.69	0.92	0.37	0.56	0.50	0.39	0.35	0.21	0.44	0.47	1
		1.24	**0.74**	1.02	1.03	0.97	1.05	0.87	1.02	0.87	0.71	0.97	0.99	0.76	0.89	0.35	0.50	0.47	0.39	0.27	0.24	0.38	0.43	2
1	**0.68**		**1.15**	0.85	0.88	0.79	1.02	0.73	1.04	0.77	0.59	0.79	0.80	0.64	0.87	0.26	0.45	0.43	0.34	0.28	0.23	0.40	0.44	3
2	**0.50**	**1.06**		1.54	1.45	1.25	1.49	1.01	1.61	1.00	0.90	1.38	1.39	0.92	1.12	0.52	0.72	0.52	0.41	0.33	0.24	0.60	0.50	4
3	**0.36**	**0.59**	**1.27**		1.95	1.29	1.55	1.04	1.64	0.97	0.90	1.39	1.38	0.98	1.16	0.70	0.81	0.51	0.41	0.38	0.23	0.68	0.55	5
4	0.60	0.61	0.46	0.78		1.42	1.30	0.89	1.40	0.93	0.77	1.20	1.20	0.87	1.02	0.45	0.60	0.47	0.40	0.30	0.21	0.52	0.47	6
5	0.57	0.65	0.52	0.77	0.87		2.73	1.13	2.30	1.13	0.95	1.38	1.38	0.98	1.43	0.59	0.86	0.65	0.50	0.51	0.38	0.79	0.71	7
6	0.52	0.60	0.45	0.66	0.69	0.67		2.07	1.02	1.08	0.87	1.07	1.10	0.72	1.20	0.29	0.56	0.66	0.54	0.37	0.32	0.34	0.56	8
7	0.60	0.58	0.49	0.73	0.76	0.65	0.92		4.40	1.11	1.02	1.46	1.52	1.00	1.38	0.61	0.92	0.63	0.49	0.63	0.44	0.93	0.78	9
8	0.59	0.54	0.34	0.78	0.74	0.61	0.73	1.52		2.45	0.76	0.98	0.99	0.72	1.06	0.37	0.51	0.59	0.49	0.32	0.29	0.42	0.51	10
9	0.62	0.64	0.49	0.77	0.80	0.67	0.85	0.80	1.13		1.38	0.97	1.05	0.62	0.78	0.23	0.46	0.46	0.41	0.24	0.23	0.32	0.38	11
10	0.57	0.61	0.44	0.70	0.72	0.63	0.72	0.65	0.73	1.14		1.78	1.48	0.90	1.08	0.48	0.68	0.48	0.42	0.26	0.19	0.52	0.45	12
11	0.65	0.75	0.65	0.81	0.79	0.73	0.81	0.75	0.82	0.67	1.43		1.95	0.90	1.03	0.49	0.67	0.48	0.40	0.26	0.19	0.49	0.47	13
12	0.73	0.81	0.51	0.95	0.94	0.81	0.88	1.16	0.92	0.81	1.01	1.81		1.08	0.78	0.32	0.47	0.39	0.31	0.22	0.25	0.36	0.34	14
13	0.64	0.73	0.54	0.86	0.83	0.74	0.79	0.96	0.80	0.74	0.99	1.31	1.28		2.42	0.42	0.82	0.76	0.61	0.50	0.34	0.56	0.71	15
14	0.43	0.55	0.40	0.52	0.54	0.47	0.54	0.42	0.54	0.55	0.57	0.58	0.56	0.57		1.42	0.96	0.04	0.00	0.11	-0.05	0.49	0.21	16
15	0.59	0.58	0.54	0.70	0.69	0.59	0.76	0.79	0.73	0.56	0.95	0.80	0.79	0.47	1.39		1.71	0.41	0.28	0.28	0.20	0.55	0.40	17
16	0.19	0.20	0.16	0.22	0.25	0.23	0.23	0.23	0.26	0.29	0.22	0.32	0.23	0.18	0.10	0.32		1.81	1.05	0.92	0.91	0.33	0.95	18
17	0.41	0.39	0.34	0.42	0.39	0.36	0.43	0.48	0.42	0.36	0.64	0.52	0.51	0.31	0.59	0.24	0.65		1.85	0.53	0.57	0.24	0.64	19
18	0.32	0.24	0.22	0.32	0.37	0.27	0.41	0.38	0.43	0.19	0.46	0.41	0.36	0.22	0.55	0.08	0.35	0.99		2.24	0.88	0.43	1.33	20
19	0.27	0.39	0.30	0.31	0.28	0.32	0.33	0.14	0.26	0.19	0.64	0.41	0.48	0.26	0.45	0.05	0.32	0.58	1.46		2.12	0.24	0.83	21
20	0.22	0.14	0.08	0.24	0.21	0.19	0.25	0.35	0.27	0.08	0.24	0.30	0.32	0.14	0.32	0.00	0.17	0.38	0.22	0.60		2.01	0.46	22
21	0.03	0.09	0.06	0.05	0.04	0.01	0.13	0.06	0.11	-0.03	0.16	0.12	0.17	0.06	0.24	-0.03	0.21	0.49	0.49	0.17	1.11		2.09	23
22	0.17	0.28	0.12	0.24	0.30	0.24	0.25	0.09	0.37	0.38	0.11	0.22	0.20	0.25	0.12	0.13	0.11	0.19	0.03	0.06	-0.06	1.07		
23	0.20	0.16	0.16	0.23	0.21	0.16	0.25	0.28	0.32	0.12	0.38	0.29	0.32	0.16	0.42	0.07	0.28	0.38	0.33	0.28	0.27	0.00	0.56	

Table 10. The new covariance matrices obtained by the Ng method (Ng *et al.* 2014) (upper triangular) and the Frobenius correlation method (lower triangular).

Asset	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
	0.50	**0.32**	**0.30**	0.91	0.87	0.86	0.93	0.66	0.91	0.72	0.56	0.84	0.86	0.62	0.84	0.40	0.55	0.45	0.33	0.36	0.21	0.36	0.39	1
		0.61	**0.33**	0.89	0.85	0.88	0.85	0.78	0.72	0.79	0.62	0.84	0.89	0.66	0.81	0.32	0.42	0.43	0.31	0.25	0.19	0.25	0.31	2
1	**0.50**		**0.56**	0.79	0.77	0.74	0.87	0.76	0.78	0.72	0.61	0.75	0.78	0.57	0.89	0.24	0.45	0.48	0.32	0.30	0.23	0.32	0.34	3
2	**0.30**	**0.61**		2.73	2.41	2.25	2.48	1.73	2.49	1.70	1.58	2.37	2.43	1.57	2.08	0.94	1.35	1.04	0.73	0.73	0.48	0.92	0.84	4
3	**0.28**	**0.31**	**0.56**		2.96	2.18	2.41	1.68	2.37	1.59	1.50	2.26	2.29	1.59	2.02	1.09	1.37	0.98	0.70	0.75	0.44	0.98	0.84	5
4	0.99	1.03	0.89	2.73		2.54	2.20	1.63	2.22	1.68	1.39	2.13	2.18	1.53	2.00	0.86	1.17	1.00	0.76	0.69	0.46	0.80	0.78	6
5	0.98	1.00	0.88	2.56	2.96		4.08	1.79	3.31	1.77	1.59	2.28	2.35	1.57	2.37	0.96	1.45	1.20	0.85	0.97	0.71	1.15	1.08	7
6	0.93	0.99	0.84	2.43	2.36	2.54		3.22	1.41	1.77	1.44	1.75	1.83	1.24	2.05	0.51	0.94	1.13	0.85	0.65	0.50	0.49	0.76	8
7	1.08	1.08	1.04	2.76	2.73	2.53	4.08		5.64	1.59	1.49	2.18	2.32	1.48	2.18	0.94	1.45	1.08	0.81	1.13	0.78	1.23	1.16	9
8	0.85	0.93	0.90	2.12	2.06	1.97	2.34	3.22		3.45	1.30	1.65	1.70	1.25	1.84	0.61	0.89	1.12	0.81	0.63	0.51	0.59	0.75	10
9	1.09	1.02	1.01	2.83	2.76	2.57	3.59	2.14	5.64		2.05	1.62	1.74	1.06	1.48	0.46	0.88	0.88	0.67	0.47	0.41	0.52	0.56	11
10	0.87	0.93	0.84	2.08	1.97	1.99	2.29	2.13	2.19	3.45		2.92	2.48	1.55	2.01	0.87	1.23	0.99	0.75	0.63	0.42	0.82	0.76	12
11	0.66	0.73	0.70	1.75	1.69	1.59	1.88	1.74	1.89	1.56	2.05		3.15	1.59	2.02	0.90	1.28	1.04	0.73	0.65	0.41	0.76	0.80	13
12	0.96	1.00	0.89	2.54	2.47	2.33	2.63	2.19	2.64	2.04	1.83	2.92		1.67	1.44	0.59	0.81	0.74	0.52	0.46	0.38	0.51	0.51	14
13	0.97	1.03	0.90	2.58	2.48	2.36	2.66	2.24	2.73	2.09	1.93	2.66	3.15		3.85	0.78	1.35	1.37	1.00	0.92	0.65	0.89	1.01	15
14	0.71	0.74	0.66	1.77	1.76	1.70	1.88	1.51	1.86	1.49	1.23	1.73	1.75	1.67		1.72	1.28	0.26	0.16	0.31	0.08	0.73	0.39	16
15	0.98	0.98	1.03	2.38	2.31	2.23	2.80	2.49	2.65	2.23	1.75	2.33	2.30	1.68	3.85		2.30	0.78	0.50	0.57	0.41	0.85	0.65	17
16	0.46	0.44	0.34	1.12	1.25	1.04	1.22	0.82	1.23	0.91	0.64	1.06	1.08	0.74	1.01	1.72		2.42	1.36	1.10	1.01	0.51	1.09	18
17	0.65	0.57	0.57	1.53	1.57	1.38	1.74	1.35	1.77	1.24	1.06	1.48	1.49	1.04	1.72	1.36	2.30		2.15	0.67	0.66	0.36	0.75	19
18	0.58	0.55	0.62	1.30	1.23	1.21	1.51	1.49	1.43	1.33	1.09	1.27	1.28	0.95	1.70	0.40	0.99	2.42		2.49	0.93	0.56	1.47	20
19	0.43	0.41	0.45	0.95	0.91	0.91	1.10	1.12	1.05	0.99	0.85	0.96	0.95	0.69	1.26	0.25	0.69	1.47	2.15		2.21	0.35	0.92	21
20	0.40	0.35	0.39	0.85	0.87	0.81	1.09	0.91	1.18	0.81	0.65	0.79	0.80	0.59	1.11	0.37	0.67	1.32	0.81	2.49		2.17	0.57	22
21	0.23	0.22	0.28	0.52	0.49	0.48	0.70	0.64	0.74	0.55	0.47	0.48	0.48	0.45	0.70	0.08	0.42	1.16	0.74	1.03	2.21		2.39	23
22	0.40	0.36	0.40	1.01	1.07	0.90	1.24	0.75	1.36	0.79	0.64	0.93	0.90	0.64	1.03	0.68	0.85	0.65	0.46	0.64	0.37	2.17		
23	0.48	0.47	0.49	1.09	1.09	1.03	1.35	1.15	1.38	1.05	0.82	1.03	1.07	0.76	1.37	0.49	0.82	1.38	0.94	1.61	0.99	0.68	2.39	

on a renegotiated package, if elected in the general election on 7 May 2015. Following a referendum held in the UK on 23 June 2016 in which 51.9 percent of those voting supported leaving the EU, the UK government invoked Article 50 of the Treaty on European Union, starting a two-year process which was due to conclude with the UK withdrawing on 29 March 2019. Up to the end of October 2019, that deadline has been extended three times and is currently 31 January 2020. It was expected that some countries will be affected seriously by Brexit and their economic relationship with the UK will be weaken after Brexit. According to a Deloitte's report, the impact of Brexit on Belgium will be significant because the UK is one of Belgium's top 5 most important export partners. In year 2016, the total of Belgian exports to the UK represent almost nine percent of the total exports from Belgium (an amount of EUR 31.99 billion), whereas the UK accounts for nearly five percent of total Belgian imports (a total amount of EUR 16.06 billion). Denmark is another country which is expected to be affected seriously by the Brexit. Due to the historical and geographical factors, Denmark always sees the UK as her best friend in the EU. When the UK and Denmark joined the European Community in 1973, the UK was Denmark's largest trading partner. For both Belgium and Denmark, the value of total merchandise trade is expected to drop because of the increased tax on imports of goods after Brexit. Therefore, it is expected that the correlation between their equity indices with the UK equity index would be reduced after Brexit.

Here, we pick 1 January 1998 to 31 December 2012 as the observation period because Cameron announced in January 2013 that a Conservative government would hold an in-or-out referendum on EU membership and we pick 1 July 2016 to 31 December 2016 as the crisis period because the referendum was held in June 2016. We consider a portfolio of 23 MSCI Developed Market Country indices with their names listed in table 7. Note that we do not include Greece equity index here because the equity index provider MSCI moved Greece to the category of emerging markets since November 2013. Suppose that as of early 2013, research findings revealed that after voting, the correlations among the equity index returns of the UK, Belgium and Denmark would drop to a low value (say 0.5) and their standard deviations would be halved.

Table 7 displays the sample covariance matrices for the daily log returns of these $n = 23$ country indices during the two periods. The lower triangular part, $\boldsymbol{\Sigma}_{Normal}$, gives the covariances from 1 January 1998 to 31 December 2012 (the observation period), and the upper triangular part, $\boldsymbol{\Sigma}_{Stress}$, gives those from 1 July 2016 to 31 December 2016 (the post-voting period). The covariance matrix $\boldsymbol{\Sigma}_{Stress}$ can be regarded as a proxy of the true covariance matrix under this scenario. From table 8, as compared with the observation period, the sample standard deviations of the three country indices were

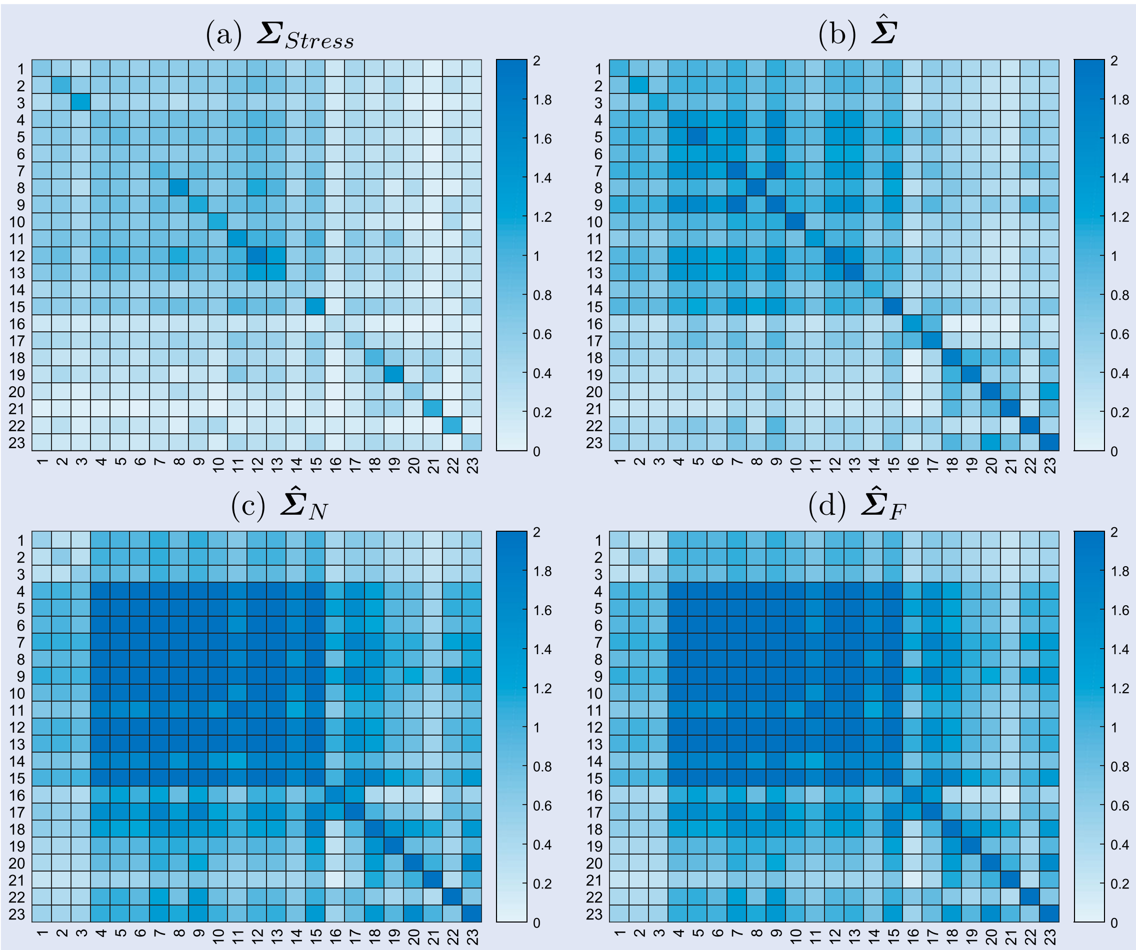

Figure 3. Heatmaps of $\boldsymbol{\Sigma}_{Stress}$, $\hat{\boldsymbol{\Sigma}}$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$, where $\boldsymbol{\Sigma}_{Stress}$ is the sample covariance matrix during crisis period, $\hat{\boldsymbol{\Sigma}}$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$ are the new covariance matrices obtained by the proposed approach, Ng method and Frobenius correlation method respectively.

Table 11. Frobenius norms between covariance matrices, where $\boldsymbol{\Sigma}_{Stress}$ and $\boldsymbol{\Sigma}_{Normal}$ are the sample covariance matrices during the post-voting period and the observation period respectively, $\hat{\boldsymbol{\Sigma}}_F$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}$ are the new covariance matrices obtained by the Frobenius correlation method, Ng method (Ng *et al.* 2014) and the proposed method respectively.

	$\boldsymbol{\Sigma}_{Normal}$	$\hat{\boldsymbol{\Sigma}}_F$	$\hat{\boldsymbol{\Sigma}}_N$	$\hat{\boldsymbol{\Sigma}}$
$\boldsymbol{\Sigma}_{Stress}$	26.0	23.1	19.0	9.5

more or less halved and the sample correlations of the three country indices were fairly dropped during the post-voting period.

For each non-overlapping period of three months during the observation period, we compute the sample covariance matrix of the daily log returns on the $n = 23$ country indices, resulting in a total of 60 sample covariance matrices. We estimate $\boldsymbol{W}$ and $\boldsymbol{\Psi}$ based on these covariance matrices in non-overlapping subperiods using the MLE method. Applying the formula $\tau = n/(n + 42.6)$, we obtain $\tau = 0.35$ for $n = 23$, and using it in (4) gives the resulting matrix $\hat{\boldsymbol{\Sigma}} = \tilde{\boldsymbol{W}}$ which is found to be PSD. The upper part of table 9 shows the matrix $\hat{\boldsymbol{\Sigma}}$. It can be seen that the covariances among the UK, Belgium and Denmark in $\hat{\boldsymbol{\Sigma}}$ are very close to the subjective views, and some peripheral covariances decrease significantly as well. Table 10 shows the stressed covariance matrices $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$. Figure 3 shows the heatmaps of the sample covariance matrix during the post-voting period ($\boldsymbol{\Sigma}_{Stress}$) and the new covariance matrices $\hat{\boldsymbol{\Sigma}}$, $\hat{\boldsymbol{\Sigma}}_N$ and $\hat{\boldsymbol{\Sigma}}_F$, which are the new covariance matrices obtained by the proposed approach, the Ng method and Frobenius correlation method respectively. The heatmaps support that the covariance matrix derived by our proposed method is more similar to $\boldsymbol{\Sigma}_{Stress}$ as compared with the covariance matrices constructed by the Ng method and Frobenius correlation method.

We measure the difference between each upper triangular entry of the new covariance matrix with the associated entry of the sample covariance matrix during the post-voting period ($\boldsymbol{\Sigma}_{Stress}$) and evaluate if the covariances of our proposed method are closest to the covariances of $\boldsymbol{\Sigma}_{Stress}$. Among all 275 entries, 221 entries of $\hat{\boldsymbol{\Sigma}}$ are closest to the associated entries in $\boldsymbol{\Sigma}_{Stress}$, 51 entries of $\hat{\boldsymbol{\Sigma}}_N$ are closest to the associated entries in $\boldsymbol{\Sigma}_{Stress}$ and 3 entries of $\hat{\boldsymbol{\Sigma}}_F$ are closest to the entries in $\boldsymbol{\Sigma}_{Stress}$. We report in table 11 the Frobenius norms between different stressed covariance matrices and $\boldsymbol{\Sigma}_{Stress}$ to evaluate the performance of various covariance matrices in approximating the covariance matrix during the post-voting period. It can be seen that our proposed method $\hat{\boldsymbol{\Sigma}}$ provides the best covariance matrix adjustment.

4. Conclusion

Under extreme market conditions such as market interventions and financial crises, covariance matrices behave differently from the long-run covariance matrix. Adjusting a covariance matrix to evaluate the potential impact of changes in the variances and correlations is necessary in risk management. Recently, Ng *et al.* (2014) discovered that empirical correlations may be strongly correlated. When some correlations are explicitly adjusted (core correlations), the others left unspecified (peripheral correlations) should vary accordingly. They proposed a unified approach to construct a proper correlation matrix in which the stress impact on the core correlations is transmitted to the peripheral correlations through the dependence structure of the empirical correlations. Very often, high correlations go hand in hand with high standard deviations during a crisis period. In this paper, we propose a framework for covariance matrix adjustment such that the peripheral covariances are adjusted according to the dependence structure of empirical covariances. A new covariance matrix is constructed by maximizing the posterior density. Unlike Ng *et al.* (2014), our method does not require matrix vectorization and hence, it is more computationally efficient, even for moderately high dimensions, say 20 or more. The empirical studies demonstrate that our proposed method improves the results of covariance stress testing.

It would be of interest to extend this proposed framework to adjust the mean and the covariance matrix of asset returns by incorporating a subjective view on them for future research. Furthermore, it could be explored as to whether the proposed approach might need to impose certain matrix structure when the dimension is large, say 100 or more.

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Appendices

Appendix 1. Maximum likelihood estimation of the parameters W and Ψ

Here, we describe the procedure of estimating the parameters $\boldsymbol{W}$ and $\boldsymbol{\Psi}$ via the maximum likelihood method. Consider a random sample of T observed covariance matrices $\{\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_T\}$, each follow a truncated symmetric matrix-variate normal distribution with mean matrix $\boldsymbol{W}$ and scale matrix $\boldsymbol{\Psi}$. The likelihood function is

$$L(\boldsymbol{W}, \boldsymbol{\Psi}; \boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_T) \propto |\boldsymbol{\Psi}|^{-T(n+1)/2} \times \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \mathrm{tr}[\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}_t - \boldsymbol{W})\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}_t - \boldsymbol{W})]\right\}.$$

The log-likelihood function is

$$l(\boldsymbol{W}, \boldsymbol{\Psi}; \boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_T) = -\frac{T(n+1)}{2}\log|\boldsymbol{\Psi}| - \frac{1}{2}\sum_{t=1}^{T} \mathrm{tr}[\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}_t - \boldsymbol{W})\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}_t - \boldsymbol{W})] + C,$$

where C is a constant. Setting its first derivatives to zero, then we have

$$\hat{\boldsymbol{W}} = \bar{\boldsymbol{\Sigma}} \equiv \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\Sigma}_t,$$

$$\hat{\boldsymbol{\Psi}} = \frac{2}{T(n+1)}\sum_{t=1}^{T}(\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}})\hat{\boldsymbol{\Psi}}^{-1}(\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}}).$$

Note that both sides of the last equation involve $\hat{\boldsymbol{\Psi}}$ and it can be determined using the following algorithm:

$$\text{Initial}: \boldsymbol{\Psi}_+ = \hat{\boldsymbol{\Psi}}_0 (\text{e.g.} \hat{\boldsymbol{\Psi}}_0 = \boldsymbol{I}_n, \text{that is a } n \times n \text{ identify matrix.})$$

$$\text{Repeat}: \boldsymbol{\Psi}_* = \boldsymbol{\Psi}_+$$

$$\boldsymbol{\Psi}_+ = \frac{2}{T(n+1)} \sum_{t=1}^{\top} (\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}}) \boldsymbol{\Psi}_*^{-1} (\boldsymbol{\Sigma}_t - \bar{\boldsymbol{\Sigma}})$$

$$\text{until} \|\boldsymbol{\Psi}_+ - \boldsymbol{\Psi}_*\|_2 \le \epsilon (\text{e.g. set } \epsilon = 0.000001).$$

Appendix 2. Derivation of the posterior density of $\boldsymbol{\Sigma}$

Denote $\boldsymbol{s} = \text{vech}(\boldsymbol{\Sigma}), \boldsymbol{v} = \text{vech}(\boldsymbol{V}), \boldsymbol{\mu} = \text{vech}(\boldsymbol{W}), \boldsymbol{K} = \boldsymbol{D}_n^+(\boldsymbol{\Psi} \otimes \boldsymbol{\Psi}) (\boldsymbol{D}_n^+)^\top, \boldsymbol{N} = \frac{1}{2}n(n+1)$, and $\boldsymbol{M} = \frac{1}{2}m(m+1)$ where $\boldsymbol{D}_n$ is a duplication matrix (i.e. $\boldsymbol{D}_n\text{vech}(\boldsymbol{A}) = \text{vec}(\boldsymbol{A})$) and $\boldsymbol{D}_n^+$ is the generalized inverse of $\boldsymbol{D}_n$ (that is, $\boldsymbol{D}_n^+ = (\boldsymbol{D}_n^\top \boldsymbol{D}_n)^{-1}\boldsymbol{D}_n^\top$), see Turkington (2002) for the details of the duplication matrix.

Since a symmetric matrix-variate normal distribution can be viewed as a multivariate normal distribution after matrix vectorization, vectorizing $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ in (1) and (2), it can be seen that $\boldsymbol{s} = \text{vech}(\boldsymbol{\Sigma})$ follows a truncated multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{K}$ whose density is given by:

$$f(\boldsymbol{s}) \propto |\boldsymbol{K}|^{-1/2} \exp\left[-\frac{1}{2}(\boldsymbol{s}-\boldsymbol{\mu})^\top \boldsymbol{K}^{-1}(\boldsymbol{s}-\boldsymbol{\mu})\right] \times I\{\text{vech}^{-1}(\boldsymbol{s}) \in R_{PSD}\}, \tag{A1}$$

and $\boldsymbol{v} \mid \boldsymbol{s}$ follows a multivariate normal distribution with the following density:

$$\boldsymbol{v} \mid \boldsymbol{s} \sim N_M(\text{vech}(\boldsymbol{P\Sigma P}^\top), \boldsymbol{D}_m^+(\boldsymbol{\Phi} \otimes \boldsymbol{\Phi})(\boldsymbol{D}_m^+)^\top).$$

Note that the mean of $\boldsymbol{v} \mid \boldsymbol{s}$ can be simplified as

$$\begin{aligned}\text{vech}(\boldsymbol{P\Sigma P}^\top) &= \boldsymbol{L}_m\text{vec}(\boldsymbol{P\Sigma P}^\top) = \boldsymbol{L}_m(\boldsymbol{P}\otimes\boldsymbol{P})\text{vec}(\boldsymbol{\Sigma}) \\ &= \boldsymbol{L}_m(\boldsymbol{P}\otimes\boldsymbol{P})\boldsymbol{D}_n\text{vech}(\boldsymbol{\Sigma}) = \tilde{\boldsymbol{P}}\boldsymbol{s},\end{aligned}$$

where $\boldsymbol{L}_m$ is the elimination matrix and $\tilde{\boldsymbol{P}} = \boldsymbol{L}_m(\boldsymbol{P}\otimes\boldsymbol{P})\boldsymbol{D}_n$, see Turkington (2002) for the details of the elimination matrix.

For ease of implementation, we follow Ng *et al.* (2014) to set

$$\boldsymbol{\Phi} = \sqrt{\frac{\tau}{1-\tau}}\boldsymbol{P\Psi P}^\top,$$

where $0 \le \tau < 1$ is a parameter controlling the uncertainty of view, and $\tau = 0$ implies certainty views. Then the covariance matrix of $\boldsymbol{v} \mid \boldsymbol{s}$ can be simplified as

$$\begin{aligned}\boldsymbol{D}_m^+(\boldsymbol{\Phi}\otimes\boldsymbol{\Phi})(\boldsymbol{D}_m^+)^\top &= \frac{\tau}{1-\tau}\boldsymbol{D}_m^+[(\boldsymbol{P\Psi P}^\top)\otimes(\boldsymbol{P\Psi P}^\top)](\boldsymbol{D}_m^+)^\top \\ &= \frac{\tau}{1-\tau}[\boldsymbol{L}_m(\boldsymbol{P}\otimes\boldsymbol{P})\boldsymbol{D}_n]\boldsymbol{K}[\boldsymbol{L}_m(\boldsymbol{P}\otimes\boldsymbol{P})\boldsymbol{D}_n]^\top \\ &= \frac{\tau}{1-\tau}\tilde{\boldsymbol{P}}\boldsymbol{K}\tilde{\boldsymbol{P}}^\top.\end{aligned}$$

Therefore, we have

$$\boldsymbol{v} \mid \boldsymbol{s} \sim N_M(\tilde{\boldsymbol{P}}\boldsymbol{s}, \boldsymbol{\Omega}), \tag{A2}$$

where $\tilde{\boldsymbol{P}} = \boldsymbol{L}_m(\boldsymbol{P}\otimes\boldsymbol{P})\boldsymbol{D}_n$ and $\boldsymbol{\Omega} = (\tau/(1-\tau))\tilde{\boldsymbol{P}}\boldsymbol{K}\tilde{\boldsymbol{P}}^\top$.

Applying Bayes theorem to (A1) and (A2), the posterior density of $\boldsymbol{s} \mid \boldsymbol{v}$ is

$$f(\boldsymbol{s} \mid \boldsymbol{v}) \propto \exp\left[-\frac{1}{2}(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})^\top \tilde{\boldsymbol{K}}^{-1}(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})\right] \times I\{\text{vech}^{-1}(\boldsymbol{s}) \in R_{PSD}\}, \tag{A3}$$

where

$$\begin{aligned}\tilde{\boldsymbol{\mu}} &= \boldsymbol{\mu} + (1-\tau)\boldsymbol{L}_n\{[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}]\otimes[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}]\} \\ &\quad \times \boldsymbol{D}_m[\boldsymbol{v} - \boldsymbol{L}_m(\boldsymbol{P}\otimes\boldsymbol{P})\boldsymbol{D}_n\boldsymbol{\mu}] \\ &= \text{vech}\{\boldsymbol{W} + (1-\tau)[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}](\boldsymbol{V} - \boldsymbol{PWP}^\top) \\ &\quad \times [\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}]^\top\},\end{aligned}$$

and

$$\begin{aligned}\tilde{\boldsymbol{K}} &= \boldsymbol{K} - (1-\tau)\boldsymbol{K}\tilde{\boldsymbol{P}}^\top(\tilde{\boldsymbol{P}}\boldsymbol{K}\tilde{\boldsymbol{P}}^\top)^{-1}\tilde{\boldsymbol{P}}\boldsymbol{K} \\ &= \boldsymbol{D}_n^+\{[\boldsymbol{\Psi} - (1-\tau)^{1/2}\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}] \\ &\quad \otimes[\boldsymbol{\Psi} + (1-\tau)^{1/2}\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}]\}(\boldsymbol{D}_n^+)^\top.\end{aligned} \tag{A4}$$

Therefore, we can see that $\boldsymbol{\Sigma} \mid \boldsymbol{V}$ follows a truncated symmetric matrix-variate normal distribution with density given below:

$$f(\boldsymbol{\Sigma} \mid \boldsymbol{V}) \propto \exp\left\{-\frac{1}{2}\text{tr}[\tilde{\boldsymbol{\Psi}}_1^{-1}(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\tilde{\boldsymbol{\Psi}}_2{}^{-1}(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})]\} I\{\boldsymbol{\Sigma} \in R_{PSD}\right\},$$

where

$$\begin{aligned}\tilde{\boldsymbol{W}} &= \boldsymbol{W} + (1-\tau)[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}] \\ &\quad \times (\boldsymbol{V} - \boldsymbol{PWP}^\top)[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}]^\top, \\ \tilde{\boldsymbol{\Psi}}_1 &= \boldsymbol{\Psi} - (1-\tau)^{1/2}\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}, \\ \tilde{\boldsymbol{\Psi}}_2 &= \boldsymbol{\Psi} + (1-\tau)^{1/2}\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}.\end{aligned}$$

Appendix 3. Derivation of the distance measure

The evaluation of the distance measure $(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})^\top\tilde{\boldsymbol{K}}^{-1}(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})$ needs the inverse of $\tilde{\boldsymbol{K}}$. Note that $\tilde{\boldsymbol{K}}$ can be written as

$$\begin{aligned}\tilde{\boldsymbol{K}} &= \boldsymbol{D}_n^+\{(\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}) - (1-\tau)\{[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}] \\ &\quad \otimes[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}]\}\}(\boldsymbol{D}_n^+)^\top = \boldsymbol{D}_n^+\boldsymbol{B}(\boldsymbol{D}_n^+)^\top,\end{aligned}$$

where $\boldsymbol{B} = (\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}) - (1-\tau)\{[\boldsymbol{\Psi P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}]\otimes[\boldsymbol{\Psi P}^\top (\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P\Psi}]\}$. Therefore, in order to derive the inverse of $\tilde{\boldsymbol{K}}$, we firstly derive the inverse of $\boldsymbol{B}$ as follows:

$$\begin{aligned}\boldsymbol{B}^{-1} &= \{(\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}) + (\tau-1)[(\boldsymbol{\Psi P}^\top)\otimes(\boldsymbol{\Psi P}^\top)] \\ &\quad \times[(\boldsymbol{P\Psi P}^\top)^{-1}\otimes(\boldsymbol{P\Psi P}^\top)^{-1}][(\boldsymbol{P\Psi})\otimes(\boldsymbol{P\Psi})]\}^{-1} \\ &= (\boldsymbol{\Psi}\otimes\boldsymbol{\Psi})^{-1} - (\boldsymbol{\Psi}\otimes\boldsymbol{\Psi})^{-1}(\tau-1)[(\boldsymbol{\Psi P}^\top)\otimes(\boldsymbol{\Psi P}^\top)] \\ &\quad \times\{[(\boldsymbol{P\Psi P}^\top)^{-1}\otimes(\boldsymbol{P\Psi P}^\top)^{-1}]^{-1} \\ &\quad + [(\boldsymbol{P\Psi})\otimes(\boldsymbol{P\Psi})](\boldsymbol{\Psi}\otimes\boldsymbol{\Psi})^{-1} \\ &\quad \times(\tau-1)[(\boldsymbol{\Psi P}^\top)\otimes(\boldsymbol{\Psi P}^\top)]\}^{-1} \\ &\quad \times[(\boldsymbol{P\Psi})\otimes(\boldsymbol{P\Psi})](\boldsymbol{\Psi}\otimes\boldsymbol{\Psi})^{-1} \\ &= (\boldsymbol{\Psi}^{-1}\otimes\boldsymbol{\Psi}^{-1}) \\ &\quad + (1-\tau)(\boldsymbol{P}^\top\otimes\boldsymbol{P}^\top)\{[(\boldsymbol{P\Psi P}^\top)\otimes(\boldsymbol{P\Psi P}^\top)] \\ &\quad + (\tau-1)[(\boldsymbol{P\Psi P}^\top)\otimes(\boldsymbol{P\Psi P}^\top)]\}^{-1}(\boldsymbol{P}\otimes\boldsymbol{P}) \\ &= (\boldsymbol{\Psi}^{-1}\otimes\boldsymbol{\Psi}^{-1}) + \left(\frac{1-\tau}{\tau}\right)\{[\boldsymbol{P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P}] \\ &\quad \otimes[\boldsymbol{P}^\top(\boldsymbol{P\Psi P}^\top)^{-1}\boldsymbol{P}]\}.\end{aligned}$$

Therefore, $\tilde{\boldsymbol{K}}^{-1} = \boldsymbol{D}_n^\top\{(\boldsymbol{\Psi}^{-1}\otimes\boldsymbol{\Psi}^{-1}) + ((1-\tau)/\tau)\{[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}]\otimes[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}]\}\}\boldsymbol{D}_n$. Substituting this into the distance measure $(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})^\top\tilde{\boldsymbol{K}}^{-1}(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})$ gives

$$
\begin{aligned}
&(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})^\top\tilde{\boldsymbol{K}}^{-1}(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})\\
&\quad= [\boldsymbol{D}_n(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})]^\top\{(\boldsymbol{\Psi}^{-1}\otimes\boldsymbol{\Psi}^{-1}) + \left(\frac{1-\tau}{\tau}\right)\\
&\qquad\times\{[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}]\\
&\qquad\otimes[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}]\}\}[\boldsymbol{D}_n(\boldsymbol{s}-\tilde{\boldsymbol{\mu}})]\\
&\quad= \mathrm{vec}^\top(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\mathrm{vec}[\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\boldsymbol{\Psi}^{-1}]\\
&\qquad+\left(\frac{1-\tau}{\tau}\right)\mathrm{vec}^\top(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\mathrm{vec}\{[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}]\\
&\qquad\times(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}]\}\\
&\quad= \mathrm{tr}[\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\boldsymbol{\Psi}^{-1}(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})]\\
&\qquad+\left(\frac{1-\tau}{\tau}\right)\mathrm{tr}\{[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}](\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\\
&\qquad\times[\boldsymbol{P}^\top(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1}\boldsymbol{P}](\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\}\\
&\quad= \|\boldsymbol{\Psi}^{-1/2}(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\boldsymbol{\Psi}^{-1/2}\|_F^2 + \left(\frac{1-\tau}{\tau}\right)\|\\
&\qquad\times(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1/2}(\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^\top-\boldsymbol{P}\tilde{\boldsymbol{W}}\boldsymbol{P}^\top)(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1/2}\|_F^2\\
&\quad\propto \tau\|\boldsymbol{\Psi}^{-1/2}(\boldsymbol{\Sigma}-\tilde{\boldsymbol{W}})\boldsymbol{\Psi}^{-1/2}\|_F^2 + (1-\tau)\|(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1/2}\\
&\qquad\times\{\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^\top-[\tau\boldsymbol{P}\boldsymbol{W}\boldsymbol{P}^\top+(1-\tau)\boldsymbol{V}]\}(\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{P}^\top)^{-1/2}\|_F^2.
\end{aligned}
$$

Finally, the result follows by noting from (4), $\boldsymbol{P}\tilde{\boldsymbol{W}}\boldsymbol{P}^\top = \tau\boldsymbol{P}\boldsymbol{W}\boldsymbol{P}^\top + (1-\tau)\boldsymbol{V}$.