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# Term Structure Analysis with Big Data: One-Step Estimation Using Bond Prices

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ABSTRACT

Nearly all studies that analyze the term structure of interest rates take a *two-step* approach. First, actual bond prices are summarized by interpolated synthetic zero-coupon yields, and second, some of these yields are used as the source data for further empirical examination. In contrast, we consider the advantages of a *one-step* approach that directly analyzes the universe of bond prices. To illustrate the feasibility and desirability of the one-step approach, we compare arbitrage-free dynamic term structure models estimated using both approaches. We also provide a simulation study showing that a one-step approach can extract the information in large panels of bond prices and avoid any arbitrary noise introduced from a first-stage interpolation of yields.


## 1. Introduction

Most term structure analysis takes a two-step approach when investigating the pricing of fixed-income securities. The first step constructs constant-maturity zero-coupon yields from the universe of coupon bond prices. The second step uses some of these fitted synthetic yields as an input to estimate a dynamic term structure model (DTSM) (see Dai and Singleton (2000), Duffee (2002) among many others). This two-step process was developed decades ago because of the computational burden of working directly with large data sets of actual bond prices. While the two-step approach is convenient, there has been remarkably little attention paid to the first step and its possible influence on the estimated DTSM. Instead, the creation of synthetic yields is completely taken for granted and outsourced to a handful of researchers. This lack of attention to the underlying bond data – the foundational underpinnings for empirical DTSMs – has continued despite documented challenges and problems in constructing synthetic zero-coupon yields — as described in, for instance, Bliss (1997), Gürkaynak et al. (2007, 2010), and Steeley (2008). Even more concerning, some researchers such as Dai et al. (2004) and Fontaine and Garcia (2012) have argued that using synthetic interpolated yields can erase interesting dynamics by excessively smoothing bond price variation and may even introduce unnecessary measurement errors.

The contribution of the present paper is to show that the initial step of constructing synthetic zero-coupon yields is not completely innocuous and, more importantly, can be avoided. In particular, advances in computing power allow researchers to work directly with the big data universe of coupon bonds. Specifically, we elucidate a one-step approach that estimates a DTSM directly on bond prices. We illustrate the advantages of this alternative by comparing the same DTSM when estimated by one- and two-step approaches—both on actual and simulated samples of coupon bond prices.

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Our empirical case study for comparing the one- and two-step estimation approaches is the Canadian government bond market between January 2000 and April 2016. The Canadian market is a good laboratory for our analysis for several reasons. First, Canadian bonds face no appreciable credit risk during our sample and do not attract the same large and variable liquidity and safety premiums that affect the pricing of U.S. Treasuries.[1] Second, during this period, Canadian bond yields spent relatively little time near the zero lower bound – again compared to the U.S. – which simplifies our analysis. Finally, the number of Canadian government bonds is representative of sovereign bond markets in many developed countries. In total, our Canadian sample for the estimation of DTSMs based on the one-step approach contains end-of-month prices on 105 bonds. The corresponding data for the two-step approach follows the existing literature and uses a limited number of synthetic zero-coupon yields. We consider two sources for these synthetic yields. The first data set is produced by the Bank of Canada and described in Bolder et al. (2004).[2] We construct the second data set of synthetic yields by estimating the flexible parametric discount function of Svensson (1995) month-by-month on the same panel of coupon bonds as used for the one-step approach.[3] The differences between these two data sets of synthetic zero-coupon yields are generally small for maturities within the one- to twenty-year maturity range, but the differences may easily exceed ten basis points outside this maturity range where fewer coupon bonds are available. This observation provides suggestive evidence that the various curve-fitting techniques may induce non-negligible measurement errors in synthetic yields.

Our benchmark DTSM for estimation is the arbitrage-free Nelson–Siegel (AFNS) model of Christensen et al. (2011), which is a Gaussian affine model where a level, slope, and curvature factor explain the evolution of the yield curve. We estimate this DTSM on Canadian bond prices using both the one- and two-step approaches. We find that the one-step approach gives a substantially closer fit to the underlying coupon bond price data than the conventional two-step approach. For the AFNS model, the fit to market prices of coupon bonds may deteriorate by as much as 41% when going from a one- to a two-step approach. This poorer fit may add considerable noise to predicted bond prices from a DTSM estimated with a two-step approach. We also find that the parameters determining the functional form between bond yields and the factors (i.e., the risk-neutral parameters) are the ones most affected by the choice of estimation approach.

To complement these empirical estimates, we also explore the finite-sample properties of the one- and two-step approaches in a Monte Carlo study. A novel feature of this simulation experiment is to work at the level of coupon bonds and hence account for estimation uncertainty in the construction of synthetic zero-coupon yields within the two-step approach. The main insight from this Monte Carlo study is that DTSMs may be estimated more reliably by using observed bond prices instead of synthetic zero-coupon yields. Although these synthetic yields are estimated quite accurately with well-established curve-fitting techniques, we nevertheless find that seemingly negligible errors in these synthetic yields do affect the estimated parameters in a DTSM. In particular, the risk-neutral parameters are estimated with smaller biases and greater precision using a one-step rather than a two-step approach.

Finally, to showcase the suitability, tractability, and general applicability of the one-step approach, we forecast three-month and ten-year Canadian bond yields out of sample. In addition to the AFNS model, we also include a nonlinear DTSM that enforces the zero lower bound and a five-factor model to get an even tighter fit of long-term Canadian bonds than from just three factors. We find that the one-step approach has superior forecast accuracy for the models considered. Moreover, the performance of the two-step approach can vary depending on which synthetic zero-coupon yields are used for estimating the models.

This paper is most closely related to the work of Fontaine and Garcia (2012) and Pancost (2018), which are among the few papers that also use the underlying data on coupon bond prices to estimate DTSMs. Fontaine and Garcia (2012) consider pairs of old and newly issued bonds within various maturity buckets to study the on-the-run liquidity premium in U.S. Treasuries. Pancost (2018) uses the full sample of U.S. Treasuries to show that the pricing errors on bonds are predictable over time and in the cross section, and that realized excess bond returns can be used to improve the estimation of the time series parameters in a DTSM. Importantly, neither Fontaine and Garcia (2012) nor Pancost (2018) compare their results to those obtained from a corresponding two-step approach, which is a key contribution of the present paper.[4]

The remainder of the paper is structured as follows. Section 2 describes the Canadian government bond data, while Section 3 summarizes the AFNS model and presents its estimation results on Canadian data. Section 4 is devoted to our Monte Carlo study, while Section 5 provides an out-of-sample forecasting exercise of Canadian bond yields. Section 6 concludes. Appendices available online contain additional details related to the paper.

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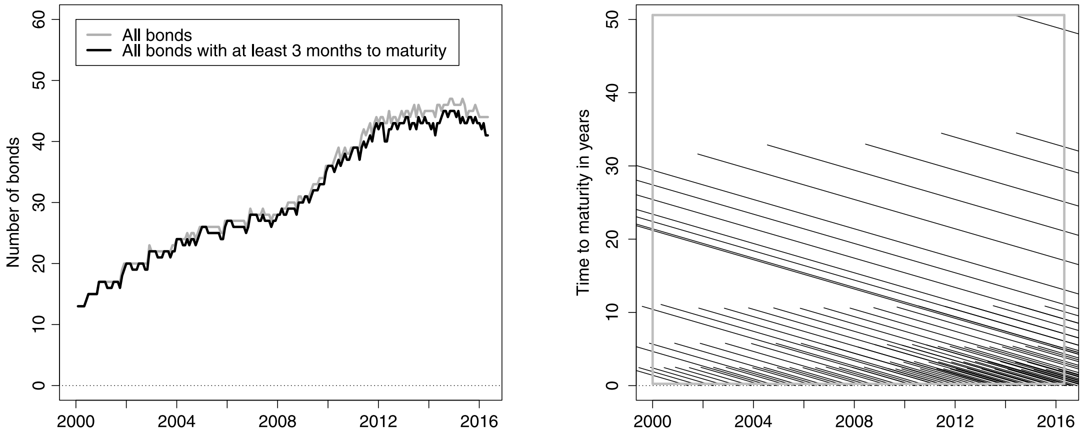
1 For example, in constructing their interpolated nominal U.S. Treasury yield curves, Gürkaynak et al. (2007) generally exclude the two most recently issued securities, i.e. the "on-the-run" and "first off-the-run" bonds, which often trade at a premium. A one-step approach could also exclude these bond prices or augment the DTSM of interest to accommodate bond-specific liquidity characteristics as in Fontaine and Garcia (2012) and Andreasen et al. (2018), but such extensions are not considered in the present paper.

2 See Diez de los Rios (2015) for an empirical application using these data.

3 This is the exact procedure used by Gürkaynak et al. (2007) for U.S. Treasuries and by researchers in many other countries. Alternative functional forms could be considered such as the cubic splines used by Steeley (2008), the hybrid combination of cubic splines and parametric functions advocated by Faria and Almeida (2018), or the optimally smooth spline yield curves derived from an exact bootstrap method based on the Moore–Penrose pseudoinverse developed by Filipović and Willems (2018).

4 Finlay and Wende (2012) estimate DTSMs directly in a one-step approach similar to ours using coupon bond prices in combination with a nonlinear Kalman filter. However, they focus on Australian data with a small number of bonds, which prevents them from assessing the efficiency of their method relative to that of conventional two-step approaches.




(a) Number of Canadian bonds (b) Maturity distribution of Canadian bonds

**Fig. 1. Description of the Canadian Bond Market**. Panel (a) shows the number of Canadian government bonds at each date. The solid gray line refers to the entire sample of bonds. The solid black line indicates the number of securities after eliminating bonds with less than three months to maturity. Panel (b) shows the maturity distribution of the full set of Canadian government bonds in our sample. The gray rectangle indicates the subsample used throughout the paper.

## 2. The Canadian bond market

This section describes the market for Canadian government bonds. We first describe the universe of Canadian bonds, which will be used for the one-step approach. Then we present two data sets of synthetic zero-coupon yields for the two-step approach.

### 2.1. The universe of government bonds

As of April 2016, the Canadian government bond market had a total outstanding notional amount of CAD 512.5 billion, which is equivalent to 25% of the gross domestic product in Canada. The Canadian government holds a AAA rating with a stable outlook by all major rating agencies, meaning that no correction for credit risk is required. Panel (a) of Fig. 1 shows that the number of coupon bonds grows gradually from about 15 bonds at the start of the sample to roughly 45 bonds in 2012, where it has remained until the end of our sample in 2016. The size of the Canadian market is representative of sovereign bond markets in several developed countries.[5]

The time-varying maturity distribution of all 105 bonds in our sample is illustrated in panel (b) of Fig. 1, where each security is represented by a downward-sloping line showing its remaining years to maturity at each date. The short end of the bond market has been densely populated with many two-year bonds. Five- and ten-year bonds were issued fairly regularly since the start of our sample. At the very long maturities, thirty-year bonds were issued approximately every three years, and a single fifty-year bond was issued in 2014.[6] All bond prices are represented by their clean mid-market price as provided by Bloomberg. Following Gürkaynak et al. (2007), securities with less than three months to maturity are excluded from our sample, as the implied yield on these securities often display erratic behavior.

### 2.2. Synthetic zero-coupon yields

The data for the two-step approach follows the existing literature, which represents the universe of bonds by a limited number of synthetic zero-coupon yields. We consider two sources for such synthetic yields. The first data set is produced by the Bank of Canada using the "Merrill Lynch exponential spline model" and is publicly available.[7] We construct the second data set by estimating the flexible discount function of Svensson (1995) month-by-month on the same panel of coupon bonds as used for the one-step approach (see online Appendix B). This technique is quite common worldwide as a procedure to create zero-coupon yields. For each data set, we extract synthetic yields with ten maturities of 0.25, 0.5, 1, 2, 3, 5, 7, 10, 20, and 30 years.

Table 1 reports summary statistics for the differences between the two data sets at various maturities. The mean absolute differences for yields in the one- to twenty-year maturity range are within six basis points and hence small.

5 Online Appendix A contains the corresponding details for France, Switzerland, and the U.K.

6 The contractual characteristics of all 105 bonds and the number of monthly observations for each bond are reported in online Appendix A.

7 See Bolder et al. (2004) for a description of the yield curve construction and the algorithm used to filter out "strange" observations.

**Table 1**
Comparing two data sets of synthetic zero-coupon yields.

| Maturity in months | Mean diff. | Mean abs. diff. | Max. abs. diff. | Correlation Levels | Correlation Diff. |
|---|---|---|---|---|---|
| 3 | 0.78 | 21.52 | 105.24 | 0.982 | 0.410 |
| 6 | −1.95 | 11.41 | 65.25 | 0.995 | 0.693 |
| 12 | −3.80 | 4.77 | 22.13 | 0.999 | 0.966 |
| 24 | −1.13 | 3.22 | 15.85 | 1.000 | 0.986 |
| 36 | 1.12 | 2.69 | 11.74 | 1.000 | 0.990 |
| 60 | 1.42 | 3.25 | 23.37 | 1.000 | 0.992 |
| 84 | −0.71 | 4.85 | 21.57 | 0.999 | 0.989 |
| 120 | −5.37 | 5.48 | 19.46 | 1.000 | 0.988 |
| 240 | 5.12 | 5.84 | 20.03 | 0.999 | 0.968 |
| 360 | −6.63 | 7.86 | 71.43 | 0.995 | 0.848 |

The table reports the summary statistics for the mean differences, the mean absolute differences, and the maximum absolute differences between synthetic Canadian zero-coupon yields from the Bank of Canada and our implementation of the Svensson (1995) discount function. These differences are reported in annual basis points. The last two columns report the correlations between the two yield series for each maturity in levels and first differences, respectively. The data series are monthly covering the period from January 31, 2000, to April 30, 2016.

Larger deviations emerge at the very short and long maturities with mean absolute differences at the six-month and thirty-year maturities of 11 and 8 basis points, respectively. Large maximum outlier differences are also evident. Non-negligible discrepancies are also evident in the correlations between the two data sets shown in the last two columns in Table 1. The correlations are clearly less than one at short and long maturities. To further illustrate these differences, panels (a) and (b) in Fig. 2 plot the six-month and thirty-year yields from the two data sets. There are notable differences at the start of the sample and when the short rate approaches the zero lower bound in 2009.

Another way to evaluate the magnitude of these differences is to re-visit two classic regressions. The first is due to Campbell and Shiller (1991), where realized returns are regressed on the slope of the yield curve. Panel (c) in Fig. 2 shows that the loadings in these regressions differ quite a bit at the short and long end of the yield curve but are almost identical in the five- to twenty-year maturity spectrum. The second regression is due to Fama (1976), where realized excess returns are regressed on the slope of the forward curve. Although most regression loadings in Panel (d) coincide closely, we do find substantial differences beyond the twenty-year maturity. Here, loadings increase monotonically for the Svensson (1995) yields but not when using the synthetic yields from the Bank of Canada. Importantly, though, these differences are not statistically significant across the two regressions, as the estimated regression loadings based on the Svensson (1995) yields are well within one standard deviation of the estimated coefficients from the Bank of Canada yields.

## 3. Empirical application

This section presents an empirical application of the one- and two-step approaches. We first present a benchmark DTSM, and then describe the econometric issues related to the one- and two-step estimation approaches. After discussion of the results, we consider several robustness checks as well.

### *3.1. A Gaussian DTSM*

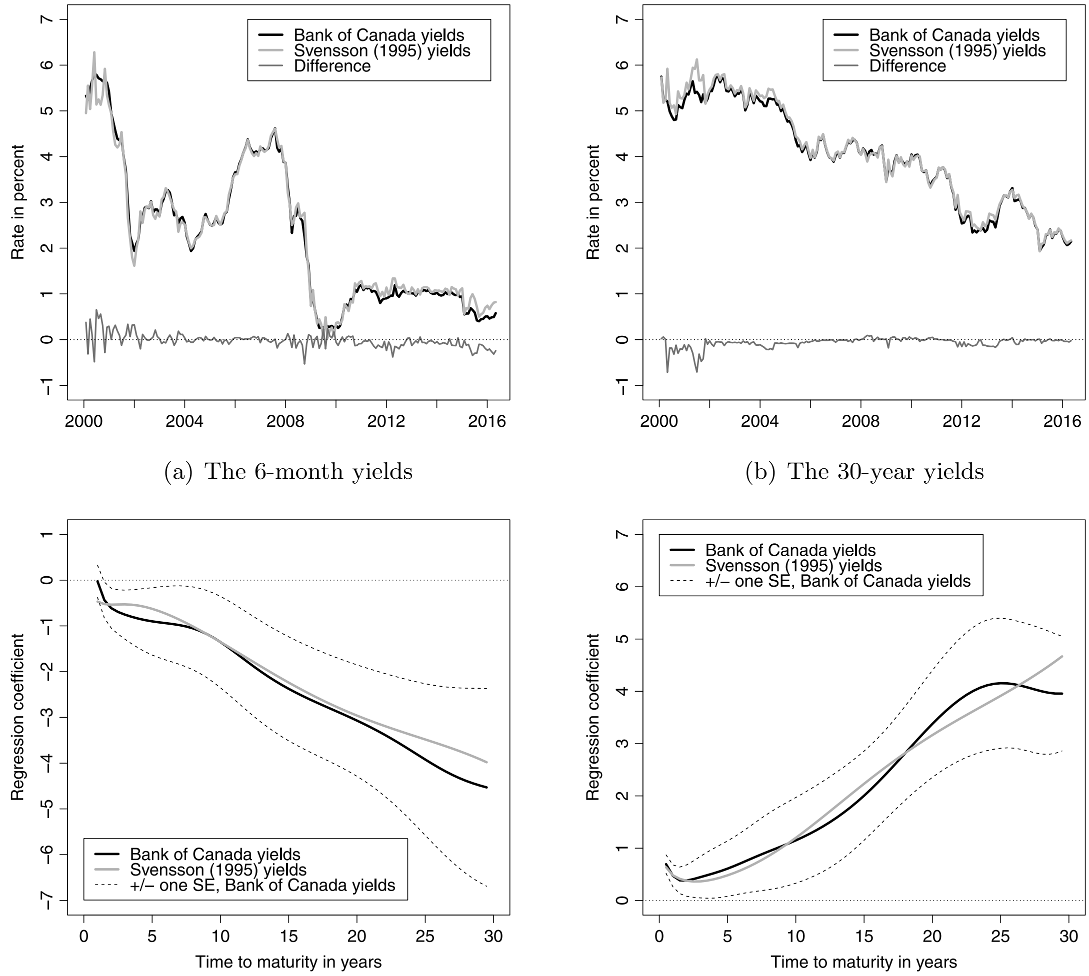
To model the bond market, we use the three-factor Gaussian DTSM of Christensen et al. (2011), which can be viewed as a restricted version of the affine DTSMs in Dai and Singleton (2000). In this arbitrage-free Nelson–Siegel (AFNS) model, the state vector is denoted by $X_t = (L_t, S_t, C_t)$, where $L_t$, $S_t$, and $C_t$ are the level, slope and curvature factors, respectively. The instantaneous risk-free rate is defined as $r_t = L_t + S_t$, and the risk-neutral (or $\mathbb{Q}$ -) dynamics of the state variables are given by

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \Sigma \begin{pmatrix} dW_t^{L,\mathbb{Q}} \\ dW_t^{S,\mathbb{Q}} \\ dW_t^{C,\mathbb{Q}} \end{pmatrix}. \tag{1}$$

Here, $dW^{i,\mathbb{Q}}$ for $i = \{L, S, C\}$ denotes independent Wiener processes and $\Sigma$ is a constant covariance matrix with dimensions $3 \times 3$.[8] The zero-coupon bond yield at maturity $\tau$ is

$$y(\tau; X_t) = L_t + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) C_t - \frac{A(\tau)}{\tau}, \tag{2}$$

8 As discussed in Christensen et al. (2011), the unit root in the level factor implies that the model is only free of arbitrage for bonds with a finite horizon. For our sample of Canadian bonds described in Section 2, and most other sovereign bond markets, this restriction is not binding and therefore of no practical relevance.




(a) The 6-month yields

(b) The 30-year yields

(c) Loadings in Campbell-Shiller regressions

(d) Loadings in forward rate regressions

**Fig. 2. Two Data Sets of Synthetic Zero-Coupon Yields: Key Differences**. Panel (a) shows the six-month synthetic yields from the Bank of Canada and our implementation of the Svensson (1995) discount function. Panel (b) shows the thirty-year synthetic yields from the Bank of Canada and our implementation of the Svensson (1995) discount function. Panel (c) shows $\delta_k$ from the regression $y_{t+h}(k-h) - y_t(k) = \alpha_k + \delta_k \frac{h}{k-h}(y_t(k) - y_t(h)) + \varepsilon_t(k)$ with $h = 6$ months, where $y_t(k)$ refers to the yield in period $t$ with $k$ months to maturity. Panel (d) shows $\theta(k)$ in the regression $xhpr_{t+h}(k) = \mu(k) + \theta_k x_t(k) + \nu_{t+h}(k)$ with $h = 6$ months, where $xhpr_{t+h}(k) \equiv hpr_{t+h}(k) - \frac{h}{12}y_t(h)$ is the excess holding period return and $hpr_{t+h}(k) \equiv -\frac{k-h}{12}y_{t+h}(k-h) + \frac{k}{12}y_t(k)$ is the holding period return. The variable $x_t(k)$ denotes the forward spread $f_t^{(k-h,k)} - \frac{h}{12}y_t(h)$, where $f_t^{(k-h,k)} \equiv \frac{k}{12}y_t(k) - \frac{k-h}{12}y_t(k-h)$ is the forward rate between time $t+k-h$ and $t+k$.

where $A(\tau)$ is a convexity term that adjusts the functional form in Nelson and Siegel (1987) to ensure absence of arbitrage (see Christensen et al. (2011)).

The model is closed by adopting the essentially affine specification for the market price of risk $\Gamma_t$ from Duffee (2002). That is, we let $\Gamma_t = \gamma^0 + \gamma^1 X_t$, where $\gamma^0 \in \mathbf{R}^3$ and $\gamma^1 \in \mathbf{R}^{3\times 3}$ contain unrestricted parameters. The physical (or $\mathbb{P}$-) dynamics of the three factors in the AFNS model are therefore

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{\mathbb{P}} & \kappa_{12}^{\mathbb{P}} & \kappa_{13}^{\mathbb{P}} \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} & \kappa_{23}^{\mathbb{P}} \\ \kappa_{31}^{\mathbb{P}} & \kappa_{32}^{\mathbb{P}} & \kappa_{33}^{\mathbb{P}} \end{pmatrix} \left( \begin{pmatrix} \theta_1^{\mathbb{P}} \\ \theta_2^{\mathbb{P}} \\ \theta_3^{\mathbb{P}} \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} \right) dt + \Sigma \begin{pmatrix} dW_t^{L,\mathbb{P}} \\ dW_t^{S,\mathbb{P}} \\ dW_t^{C,\mathbb{P}} \end{pmatrix}, \tag{3}$$

where $\kappa_{i,j}^{\mathbb{P}}$ and $\theta_i^{\mathbb{P}}$ are free parameters, subject to $X_t$ being stationary under the $\mathbb{P}$ -measure.

### 3.2. Estimation methodology for one- and two-step approaches

To describe the econometric implementation of the one-step approach, let $P_t^i(\tau, C)$ denote the price at time $t$ of the $i$th coupon bond, which matures at time $t+\tau$ and pays the coupon $C$ semi-annually at times $t_j$. In the absence of arbitrage, the clean price of this coupon bond must equal the discounted sum of all remaining payments, i.e.,

$$P_t^i(\tau, C) = \frac{C}{2}\frac{(t_1 - t)}{1/2}P_t^{zc}(t_1 - t) + \sum_{j=2}^{N}\frac{C}{2}P_t^{zc}(t_j - t) + P_t{}^{zc}(t_N - t), \tag{4}$$

where $t < t_1 < \cdots < t_N = \tau$. Here, $P_t^{zc}(\tau) = \exp\{-y(\tau; X_t)\tau\}$ denotes the price of the zero-coupon bond with $\tau$ years to maturity, and $y(\tau; X_t)$ is the zero-coupon yield from the DTSM.[9] The corresponding bond price in the data is denoted $P_t^{i,Data}(\tau, C)$. A preliminary analysis showed that the pricing errors $\varepsilon_t^i$ of the AFNS model tend to be larger for longer-term bonds. Therefore, we scale these errors by duration, so the measurement equation for the $i$th bond price in the one-step approach is given by

$$P_t^{i,Data}(\tau, C) = P_t^i(\tau, C) + D_t^{i,Data}(\tau, C)\varepsilon_t^i, \tag{5}$$

where we apply the model-free Macaulay duration $D_t^{i,Data}(\tau, C)$.[10] The pricing errors are assumed to be Gaussian, independent across time, and independent of the state innovations in Eq. (3), i.e. $\varepsilon_t^i \sim \mathcal{NID}\left(0, \sigma_\varepsilon^2\right)$. The state transition dynamics for $X_t$ under the $\mathbb{P}$-measure is given by Eq. (3).

The states $X_t$ are taken to be unobserved and must be estimated along with the model parameters $\psi$ from the panel of bond prices. The nonlinear relationship between $X_t$ and the price of a coupon bond $P_t^i(\tau, C)$ in Eq. (4) implies that our arbitrage-free DTSM cannot be estimated using the Kalman filter. Instead, the extended Kalman filter (EKF) is used to obtain an approximated log-likelihood function $L^{EKF}(\psi)$, which serves as the basis for estimating $\psi$ by quasi-maximum likelihood (QML), as described in further detail in the online Appendix C.

The econometric implementation of the two-step approach is well-known but summarized here for completeness. Let the synthetic zero-coupon yields in the data be denoted by $y_t^{Data}(\tau)$, and let $y(\tau, X_t)$ denote the corresponding yield from the DTSM. The measurement equation is

$$y_t^{Data}(\tau) = y(\tau, X_t) + \varepsilon_t(\tau),$$

for a selection of constant maturities yields, as indexed by $\tau$. Here, $\varepsilon_t(\tau) \sim \mathcal{NID}\left(0, \sigma_\varepsilon^2\right)$ and accounts for estimation errors in the construction of these synthetic yields and pricing errors in the DTSM. The state transition dynamics for $X_t$ under the $\mathbb{P}$-measure is given by Eq. (3). For the AFNS model, the zero-coupon yields are affine in $X_t$ as seen from Eq. (2), and all model parameters $\psi$ are therefore estimated by maximum likelihood (ML) using the Kalman filter.

### 3.3. Estimation results

The estimated model parameters in the AFNS model are reported in Table 2 when using the one- and two-step approaches. The conventional two-step approach is implemented on the two samples of synthetic yields discussed in Section 2.2 to explore whether the highlighted differences in the two data sets affect the estimated parameters. Hence, the one-step approach uses all available bond prices with maturities exceeding three months, whereas the two-step approach only uses the ten maturities selected in Table 1. In the interest of simplicity, we focus on the most parsimonious version of the AFNS model with independent factor dynamics. This restriction comes at practically no loss of generality for the reported results as the estimated factors and model fit are insensitive to omitting the off-diagonal terms in $\mathcal{K}^{\mathbb{P}}$ and $\Sigma$.[11]

We first note that the estimates of $\mathcal{K}^{\mathbb{P}}$ and $\theta^{\mathbb{P}}$ have sizeable standard errors in all three data sets, which is a well-known characteristic of estimating persistent autoregressive processes using a relatively short time span. The diagonal elements in $\Sigma$ and $\lambda$ are estimated much more accurately and reveal some notable differences. First, the volatility of the level factor $\sigma_{11}$ is 0.0071 in the two-step approach based on yields from Bank of Canada, but only 0.0052 in the one-step approach and in the two-step approach based on Svensson (1995) yields. Second, the volatility of the slope factor $\sigma_{22}$ is 0.0085 in the two-step approach using yields from the Bank of Canada, whereas we find $\sigma_{22} = 0.0103$ in the two other data sets. Finally, the decay parameter $\lambda$ is 0.375 in the one-step approach, 0.305 in the two-step approach based on Bank of Canada yields, and 0.451 in the two-step approach using Svensson (1995) yields. These findings reveal that the estimated parameters in a DTSM are more affected by the choice of synthetic yields data set than by the use of bond prices. Clearly, even small differences between synthetic yields of the same maturity can alter the estimation results.

9 The continuous-time formulation of our model makes the implementation of Eq. (4) straightforward. For discrete-time models with one period exceeding one day (say, a week or a month), standard interpolation schemes may be used to price the coupon payments related to the $i$th bond at time $t$. Note that time in Eq. (4) is measured in years, meaning that $t_1 - t$ must be divided by 0.5 to obtain the fraction of the semi-annual coupon payment $C/2$ that remains between time $t$ and $t_1$.

10 A similar scaling is used in Gürkaynak et al. (2007) and Hu et al. (2013) to estimate the static discount function of Svensson (1995). Eq. (5) is equivalent to dividing bond prices by duration to obtain a first-order approximation of the implied yield to maturity on a coupon bond. Using the exact yield to maturity in the estimation is computationally much more demanding because it requires solving a nonlinear fixed-point problem for every evaluation of the measurement equation, although we do discuss the results for this alternative specification in Section 3.4.

11 See for instance Christensen et al. (2011), who also show that this restricted model often does better at forecasting yields out of sample than the most flexible version of the AFNS model, where $\mathcal{K}^{\mathbb{P}}$ and $\Sigma$ are unrestricted.

**Table 2**
Parameter estimates in the AFNS model.

| Par. | One-step approach | | Two-step approach | | | |
|---|---|---|---|---|---|---|
| | | | Bank of Canada yields | | Svensson (1995) yields | |
| | Est | SE | Est | SE | Est | SE |
| $\kappa_{11}^{\mathbb{P}}$ | 0.1060 | 0.0763 | 0.2172 | 0.3086 | 0.0835 | 0.1327 |
| $\kappa_{22}^{\mathbb{P}}$ | 0.2157 | 0.1443 | 0.1839 | 0.1696 | 0.2982 | 0.1969 |
| $\kappa_{33}^{\mathbb{P}}$ | 0.7255 | 0.3649 | 0.4214 | 0.2675 | 0.3543 | 0.2301 |
| $\sigma_{11}$ | 0.0052 | 0.0001 | 0.0071 | 0.0001 | 0.0052 | 0.0001 |
| $\sigma_{22}$ | 0.0103 | 0.0010 | 0.0085 | 0.0005 | 0.0103 | 0.0004 |
| $\sigma_{33}$ | 0.0207 | 0.0015 | 0.0197 | 0.0013 | 0.0212 | 0.0013 |
| $\theta_1^{\mathbb{P}}$ | 0.0529 | 0.0034 | 0.0542 | 0.0111 | 0.0477 | 0.0143 |
| $\theta_2^{\mathbb{P}}$ | −0.0275 | 0.0093 | −0.0295 | 0.0136 | −0.0251 | 0.0088 |
| $\theta_3^{\mathbb{P}}$ | −0.0230 | 0.0060 | −0.0187 | 0.0129 | −0.0181 | 0.0156 |
| $\lambda$ | 0.3747 | 0.0105 | 0.3070 | 0.0047 | 0.4511 | 0.0051 |

This table reports the estimated parameters (Est) in the AFNS model with independent factors and their standard errors (SE) using either the one-step or two-step approach. The SE in the one-step approach are computed by pre- and post-multiplying the variance of the score by the inverse of the Hessian matrix, which we compute as outlined in Harvey (1989). The SE in the two-step approach are computed from the inverse of the variance of the score. The data are monthly and cover the period from January 31, 2000, to April 29, 2016.

**Table 3**
Summary statistics for pricing errors on coupon bonds.

| Maturity bucket in years | No. obs. | DTSM: The AFNS model | | | | | | Static model: The Svensson (1995) discount function | |
|---|---|---|---|---|---|---|---|---|---|
| | | One-step approach | | Two-step approach | | | | | |
| | | | | Bank of Canada yields | | Svensson (1995) yields | | | |
| | | Mean | RMSE | Mean | RMSE | Mean | RMSE | Mean | RMSE |
| 0–2 | 1472 | −0.09 | 5.75 | 2.40 | 7.87 | 0.33 | 9.81 | −1.08 | 8.83 |
| 2–4 | 1098 | 0.46 | 4.76 | 1.78 | 7.20 | 2.02 | 5.61 | 0.87 | 4.27 |
| 4–6 | 744 | −0.32 | 4.01 | 1.04 | 4.37 | −1.23 | 4.61 | 0.40 | 3.50 |
| 6–8 | 404 | −1.24 | 5.46 | 0.04 | 5.02 | −3.12 | 6.61 | −1.89 | 4.50 |
| 8–10 | 477 | −2.54 | 6.07 | −2.27 | 6.61 | −4.62 | 7.81 | −2.95 | 5.43 |
| 10–12 | 289 | −1.07 | 6.36 | −1.09 | 8.56 | −2.01 | 8.35 | −2.06 | 5.84 |
| 12–14 | 155 | 3.78 | 6.72 | 4.70 | 11.98 | 4.79 | 11.18 | 2.01 | 3.65 |
| 14–16 | 168 | 0.76 | 4.32 | −1.61 | 9.05 | 0.36 | 8.28 | 0.35 | 2.87 |
| 16–18 | 179 | 0.70 | 4.66 | −1.97 | 9.89 | 0.94 | 8.84 | 0.24 | 3.80 |
| 18–20 | 192 | 1.71 | 4.33 | 2.02 | 8.89 | 4.88 | 8.64 | 0.68 | 3.60 |
| 20–22 | 186 | 3.64 | 5.98 | 5.06 | 10.06 | 7.36 | 10.32 | 2.32 | 4.58 |
| 22–24 | 142 | 0.84 | 5.74 | 2.37 | 7.27 | 4.62 | 7.09 | 1.39 | 3.59 |
| 24–26 | 124 | 0.05 | 5.63 | 3.75 | 8.37 | 4.67 | 7.43 | 1.63 | 3.56 |
| 26–28 | 113 | −5.45 | 8.71 | 0.73 | 5.90 | 0.33 | 4.49 | −1.32 | 3.33 |
| 28< | 288 | −4.97 | 11.98 | 6.36 | 18.13 | 0.88 | 8.69 | −2.75 | 5.37 |
| All bonds | 6031 | −0.33 | 5.90 | 1.50 | 8.31 | 0.42 | 7.90 | −0.44 | 5.78 |

This table reports the mean pricing errors (Mean) and the root mean-squared pricing errors (RMSE) of the Canadian bond prices for the AFNS model with independent factors when evaluated at the filtered states. The AFNS model is estimated on three different data sets: (1) the universe of Canadian coupon bond prices, (2) zero-coupon yields constructed by the Bank of Canada, and (3) zero-coupon yields constructed from Canadian coupon bond prices using the Svensson (1995) discount function. The final two columns report the corresponding statistics when pricing coupon bonds using the estimated Svensson (1995) discount function. The pricing errors are reported in annual basis points and computed as the difference between the implied yield to maturity on the coupon bond and the model-implied yield to maturity on this bond. The data are monthly and cover the period from January 31, 2000, to April 29, 2016.

Table 3 evaluates the ability of the three estimated AFNS models to match coupon bond prices. The pricing errors are computed using the implied yield to maturity on each coupon bond to make the errors comparable across securities. That is, for the price on the $i$th coupon bond $P_t^i(\tau, C)$, we find the value of $y_t^i(\tau, C)$ that solves

$$P_t^i(\tau, C) = \frac{C}{2}\frac{(t_1 - t)}{1/2} e^{-y_t^i(\tau,C)(t_1-t)} + \sum_{j=2}^{N} \frac{C}{2} e^{-y_t^i(\tau,C)(t_j-t)} + e^{-y_t^i(\tau,C)\tau}. \tag{6}$$

For the model-implied estimate of this bond price $\hat{P}_t^i(\tau, C)$ we find the corresponding yield $\hat{y}_t^i(\tau, C)$ and report the pricing errors as $y_t^i(\tau, C) - \hat{y}_t^i(\tau, C)$. Table 3 shows that the two-step approach provides a fairly tight fit to the underlying coupon bond prices with an overall root mean squared error (RMSE) of 8.31 basis points for the Bank of Canada yields and 7.90 basis points for the Svensson (1995) yields. We emphasize that both the states and the model estimates in the AFNS model are here obtained from synthetic zero-coupon yields. Thus, the conventional two-step approach provides a fairly accurate

**Table 4**
Summary statistics for pricing errors on synthetic zero-coupon yields.

| Maturity in months | Panel A: Bank of Canada yields | | | | Panel B: Svensson (1995) yields | | | |
|---|---|---|---|---|---|---|---|---|
| | One-step approach | | Two-step approach | | One-step approach | | Two-step approach | |
| | Mean | RMSE | Mean | RMSE | Mean | RMSE | Mean | RMSE |
| 3 | −4.80 | 20.14 | 0.06 | 9.77 | −5.58 | 28.91 | −3.17 | 9.67 |
| 6 | −4.93 | 11.37 | −0.95 | 3.52 | −2.98 | 16.96 | −0.11 | 1.95 |
| 12 | −3.93 | 6.47 | −1.20 | 9.00 | −0.12 | 4.27 | 3.02 | 8.28 |
| 24 | −0.14 | 5.02 | 1.34 | 8.64 | 0.98 | 5.74 | 3.19 | 8.66 |
| 36 | 1.30 | 4.75 | 2.39 | 6.64 | 0.18 | 4.26 | 0.80 | 4.33 |
| 60 | 0.26 | 4.07 | 1.11 | 3.71 | −1.16 | 4.75 | −2.99 | 6.98 |
| 84 | −1.61 | 7.07 | −1.27 | 6.76 | −0.91 | 5.44 | −3.42 | 7.39 |
| 120 | −4.56 | 6.72 | −5.39 | 8.72 | 0.82 | 4.73 | −0.54 | 7.10 |
| 240 | 4.97 | 8.20 | 5.69 | 12.09 | −0.15 | 6.04 | 4.77 | 11.64 |
| 360 | −17.06 | 25.59 | −2.05 | 8.24 | −10.43 | 20.66 | −2.06 | 10.26 |
| All yields | −3.05 | 12.08 | −0.03 | 8.11 | −1.94 | 13.16 | −0.05 | 8.09 |

This table reports the mean pricing errors (Mean) and the root mean-squared pricing errors (RMSE) of the AFNS model with respect to synthetic zero-coupon yields from the Bank of Canada (in Panel A) and our implementation of Svensson (1995) yields (in Panel B). For the implementation of the two-step approach, we use Bank of Canada yields in Panel A and Svensson (1995) yields in Panel B of this table. All estimated versions of the AFNS model have independent factors and evaluate the fit at the filtered states. All numbers are measured in annual basis points. The data series are monthly covering the period from January 31, 2000, to April 30, 2016.

fit to the underlying coupon bond prices, although they only enter indirectly in the estimated AFNS model through the synthetic zero-coupon yields. However, the one-step approach delivers an even better fit to these coupon bonds with an overall RMSE of only 5.90 basis points. Thus, going from the one-step approach to the two-step approach gives a deterioration in overall RMSE of 41% and 34% when using the Bank of Canada yields and the Svensson (1995) yields, respectively. This shows that the first step in the conventional two-step approach may add sizable noise to the predicted bond prices from an estimated DTSM.

These results are benchmarked in the final two columns of Table 3 to the fit of the Svensson (1995) discount function. That is, we compute the predicted coupon bond prices using the synthetic Svensson (1995) yields and express these pricing errors in yield to maturity. As expected, the RMSEs for bonds with maturities exceeding two years are all smaller for the Svensson (1995) discount function than for any of the estimated AFNS models. However, the deterioration in fit for the estimated AFNS model based on the one-step approach is surprisingly small except for long-term bonds with more than 26 years to maturity. Indeed, for the zero to two-year maturity bucket, the one-step estimated AFNS model does *better* than the Svensson (1995) discount function (RMSE of 5.75 versus 8.83 basis points). Thus, for the construction of fitted yields, a one-step approach based on the AFNS model is about as accurate as fitting a flexible functional form like the Svensson (1995) curve to the data, which may be of independent interest to those wishing to construct a theoretically consistent synthetic zero-coupon yield curve.

When using the traditional two-step approach, the performance of DTSMs is normally evaluated by their ability to fit synthetic yields and not the underlying prices on coupon bonds. Therefore, we also consider the ability of the one-step estimated AFNS model to match the two samples of synthetic zero-coupon yields. Panel A in Table 4 evaluates the fit to the Bank of Canada synthetic zero-coupon yields from the one-step estimated AFNS model and from the two-step AFNS model estimated using the Bank of Canada yields. As expected, the estimated AFNS model from the two-step approach has a much tighter fit to these synthetic yields than the estimated AFNS model from the one-step approach, notably in the short and long end of the yield curve. The second panel in Table 4 shows that we find the same pattern when using the synthetic zero-coupon yields from the Svensson (1995) discount function. Thus, if synthetic zero-coupon yields are treated as "observed data", the econometrician would incorrectly prefer the estimated AFNS model from the two-step approach as the best representation of the Canadian bond market, although the estimated AFNS model from the one-step approach clearly provides the best fit to the observed coupon bond prices, as shown in Table 3.

### *3.4. Robustness of the one-step approach*

The online Appendices D to H scrutinize the robustness of the one-step approach to four implementation choices. First, scaling the pricing errors by duration in Eq. (5) gives nearly identical results to representing coupon bond prices by their implied yield to maturity in the measurement equation. Omitting the duration-scaling of the pricing errors in Eq. (5) gives substantially different results, but we show in the online Appendix D that this alternative specification of the measurement equation is rejected by the data. Second, using the more accurate unscented Kalman filter (UKF) instead of the EKF to build the quasi log-likelihood function gives nearly identical results to those reported in 3.3. These results are also robust to using the sequential regression approach of Andreasen and Christensen (2015), which imposes fewer restrictions on the pricing errors and the state innovations than assumed in the EKF. Third, using a fully efficient ML estimator in the one-step approach is also shown to give nearly identical estimates to those based on QML and the EKF. The considered ML estimator is here derived by double limit asymptotics, where both the time series dimension and the number of bonds tend to infinity, as this allows us to obtain a likelihood function without resorting to simulation-based procedures. Finally, the one-step approach is also robust to considering weekly or even daily data, instead of the standard monthly data frequency adopted throughout the paper. Thus, the one-step approach with duration-scaled pricing errors and a QML estimator based on the EKF appears quite robust. However, the QML estimator based on the UKF or the proposed ML estimator could be helpful for estimating DTSMs with stronger deviations from linearity and Gaussianity than present in the AFNS model, for instance in models with stochastic volatility.

## 4. Simulation study: Is one step better than two?

The preceding analysis has shown that the one- and two-step approaches give somewhat different estimates of DTSMs. But which approach gives the most accurate estimates? Here we answer this question by conducting a Monte Carlo study to analyze the finite-sample properties of estimating the AFNS model by the one- and two-step approaches. We first describe the formulation of the Monte Carlo study in Section 4.1 and then analyze the precision of the estimated synthetic zero-coupon yields from applying a Svensson (1995) yield curve in Section 4.2. The results for the estimated model parameters are reported in Section 4.3, while the accuracy of the filtered states and the standard yield curve decomposition are explored in Sections 4.4 and 4.5, respectively. Section 4.6 is devoted to the implementation of the two-step approach, where we explore how the number of synthetic yields and the adopted curve-fitting technique for these yields affect the estimates of DTSMs. Finally, Section 4.7 provides a brief summary of our main findings.

### *4.1. Setup for the Monte Carlo study*

Unlike previous simulation studies in the literature, our Monte Carlo study is formulated at the level of individual coupon bonds to account for estimation uncertainty in the construction of synthetic zero-coupon yields within the two-step approach. To get a representative data generating process for the Canadian bond market, we use the estimates of the AFNS model in the one-step approach from Table 2. Based on these parameters, we first simulate $N = 100$ samples for the three states at a monthly frequency for 196 months, which corresponds to the number of monthly observations in our Canadian sample.[12] These simulated sample paths for the states will be common across all exercises in the Monte Carlo study to facilitate the interpretation. The inputs for each of the two estimation approaches are then constructed as follows.

For the one-step approach, we use the simulated states to compute $N$ panels of coupon-bond prices that match those observed in the Canadian sample in terms of available bonds and their characteristics. These bond prices are computed using the bond price formula in Eq. (4) in combination with the zero-coupon yields in Eq. (2). We then add measurement errors $\varepsilon_t^i \sim \mathcal{NID}\left(0, \sigma_\varepsilon^2\right)$ to the simulated bond prices and scale these errors by the duration of the simulated bond for consistency with Eq. (5).[13]

For the two-step approach, we take these simulated panels of coupon bond prices as input to extract synthetic zero-coupon yields based on the Svensson (1995) yield curve. For consistency with the empirical estimation results presented in the previous sections, we extract synthetic zero-coupon yields with ten constant maturities, 0.25, 0.5, 1, 2, 3, 5, 7, 10, 20, and 30 years, which we use for implementation of the two-step approach in the Monte Carlo study. Given that the underlying bond prices are already contaminated with measurement errors, we do not add additional noise to these synthetic yields.

To study the role of the data quality, we consider two cases where the standard deviation of the measurement errors $\sigma_\varepsilon$ is either 1 or 10 basis points. The first case with $\sigma_\varepsilon = 1$ basis point represents an ideal setting with hardly any noise in bond prices and helps to isolate the effects of the curve fitting procedure in the two-step approach. The second case with $\sigma_\varepsilon = 10$ basis points is included to describe a more realistic setting, as we find $\sigma_\varepsilon = 7$ basis points in our Canadian sample when using the one-step approach.

### *4.2. Accuracy of synthetic yields*

We first consider the accuracy of the synthetic zero-coupon yields from the Svensson (1995) yield curve when estimated on the simulated coupon bond prices. That is, we compare the estimated synthetic yields to the *true* zero-coupon yields from the AFNS model without measurement errors.

With small measurement errors of $\sigma_\varepsilon = 1$ basis point, Table 5 shows that the mean errors are generally very close to zero within the one- to twenty-year maturity range but somewhat larger at the three- and six-month maturities (−4 and −2 basis points, respectively) and at the thirty-year maturity (−1.3 basis points). This means that the Svensson (1995) yield curve slightly overpredicts the level of the zero-coupon yields at the short and long end of the curve. The low mean absolute errors (MAE) of roughly 1 basis point show that yields within the one- to twenty-year maturity spectrum are estimated very accurately, whereas yields at the short and long end of the curve are estimated less precisely (MAEs of

12 We simulate from Eq. (3) using a standard Euler-discretization, i.e., $X_t^i = X_{t-1}^i + \kappa_{ii}^{\mathbb{P}}(\theta_i^{\mathbb{P}} - X_{t-1}^i)\Delta t + \sigma_{ii}\sqrt{\Delta t} z_t^i$, where $z_t^i \sim \mathcal{N}(0, 1)$ and $\Delta t = 0.0001$. The starting values $X_0^i$ are drawn from the unconditional distribution of $X_t$.

13 Note that we also use the same set of simulated samples of $\varepsilon_t^i$ throughout the Monte Carlo study to make the results as comparable as possible.

**Table 5**
Accuracy of estimated Svensson (1995) yields.

| Maturity in months | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
|---|---|---|---|---|
| | Mean | MAE | Mean | MAE |
| 3 | −4.09 | 6.79 | −3.00 | 12.48 |
| 6 | −2.37 | 3.89 | −1.74 | 7.84 |
| 12 | −0.32 | 1.05 | −0.24 | 3.58 |
| 24 | 0.78 | 1.42 | 0.52 | 3.52 |
| 36 | 0.43 | 0.99 | 0.24 | 3.12 |
| 60 | −0.48 | 1.05 | −0.34 | 3.13 |
| 84 | −0.67 | 1.26 | −0.37 | 3.30 |
| 120 | −0.34 | 0.94 | −0.14 | 2.88 |
| 240 | 0.64 | 1.46 | 0.27 | 3.76 |
| 360 | −1.32 | 3.09 | −0.90 | 9.34 |

The table reports the mean of the sampling distribution of the mean errors (Mean) and mean absolute errors (MAE) for each zero-coupon yield constructed using the Svensson (1995) discount function relative to the true zero-coupon yield implied by the AFNS model with simulated samples of length $T = 196$ and $N = 100$ repetitions. The mean is obtained by first computing the mean errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these means across the $N = 100$ simulated samples. Similarly, the MAE are obtained by first computing the mean absolute errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these absolute means across the $N = 100$ simulated samples. The true states are generated from the AFNS model as described in Section 4.1. All numbers are reported in annual basis points.

**Table 6**
Accuracy of the parameter estimates in the AFNS model.

| Par. | True value | One-step approach | | | | Two-step approach | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
| | | Mean bias | Std. dev. | Mean bias | Std. dev. | Mean bias | Std. dev. | Mean bias | Std. dev. |
| $\kappa_{11}^{\mathbb{P}}$ | **0.1060** | 0.2734 | 0.2660 | 0.2951 | 0.2659 | 0.2009 | 0.2114 | 0.3549 | 0.2489 |
| $\kappa_{22}^{\mathbb{P}}$ | **0.2157** | 0.2970 | 0.3022 | 0.2939 | 0.3002 | 0.3139 | 0.3069 | 0.4454 | 0.3883 |
| $\kappa_{33}^{\mathbb{P}}$ | **0.7255** | 0.1801 | 0.3973 | 0.2043 | 0.3706 | 0.4264 | 0.6491 | 0.5756 | 0.5329 |
| $\sigma_{11}$ | **0.0052** | 0.0000 | 0.0000 | 0.0000 | 0.0001 | −0.0005 | 0.0003 | −0.0004 | 0.0003 |
| $\sigma_{22}$ | **0.0103** | 0.0000 | 0.0005 | 0.0000 | 0.0006 | 0.0002 | 0.0006 | 0.0014 | 0.0009 |
| $\sigma_{33}$ | **0.0207** | −0.0001 | 0.0007 | 0.0001 | 0.0014 | 0.0035 | 0.0037 | 0.0058 | 0.0020 |
| $\theta_1^{\mathbb{P}}$ | **0.0529** | 0.0015 | 0.0088 | −0.0019 | 0.0086 | −0.0023 | 0.0086 | −0.0016 | 0.0087 |
| $\theta_2^{\mathbb{P}}$ | **−0.0275** | 0.0021 | 0.0108 | 0.0020 | 0.0108 | 0.0036 | 0.0112 | 0.0028 | 0.0110 |
| $\theta_3^{\mathbb{P}}$ | **−0.0230** | −0.0012 | 0.0098 | −0.0006 | 0.0064 | −0.0027 | 0.0062 | −0.0031 | 0.0062 |
| $\lambda$ | **0.3747** | 0.0000 | 0.0008 | −0.0004 | 0.0056 | 0.0490 | 0.0314 | 0.0423 | 0.0264 |

The table reports the mean estimate minus true value (Mean bias) and the standard deviation (Std. dev.) of the sampling distribution for each of the estimated parameters in the AFNS model when using QML in the one-step approach and ML in the two-step approach, where synthetic yields are generated with the Svensson (1995) yield curve, both with simulated samples of length $T = 196$ and $N = 100$ repetitions.

6.8 and 3.1 basis points for the three-month and thirty-year yields). This imprecision reflects the challenges of fitting the endpoints of a yield curve.

With larger measurement errors of $\sigma_\varepsilon = 10$ basis points, maturities between one and twenty years remain well approximated with mean errors close to zero. The precision of these yields in terms of the MAEs only decreases by a factor of three, which is substantially lower than the ten-fold increase in $\sigma_\varepsilon$. In most cases, the construction of the synthetic zero-coupon yields are able to correctly smooth out a large fraction of the idiosyncratic noise, $\varepsilon_t^i$, in the underlying bond prices, which generally leaves the measurement equation in the two-step approach with *smaller* errors than in the one-step approach. Accordingly, the synthetic zero-coupon yields from the Svensson (1995) yield curve appear to be very accurate in the present setting. Hence, for $\sigma_\varepsilon = 10$ basis points, the Monte Carlo results may favor the two-step approach, as the measurement equation here has reduced noise.

### *4.3. Estimated parameters from one- and two-step approaches*

Table 6 summarizes the outcome of the Monte Carlo study for the model parameters by reporting the mean and the standard deviation of each of the estimated coefficients in the AFNS model across simulations. We first note that both the one- and two-step approaches generate the familiar positive bias in the mean-reversion parameters $\{\kappa_{11}^{\mathbb{P}}, \kappa_{22}^{\mathbb{P}}, \kappa_{33}^{\mathbb{P}}\}$, as discussed in Bauer et al. (2012). For the persistence of the slope factor, $\kappa_{22}^{\mathbb{P}}$, and the curvature factor, $\kappa_{33}^{\mathbb{P}}$, we find that these biases are somewhat larger for the two-step approach. For instance, with $\sigma_\varepsilon = 10$ basis points, the bias in $\kappa_{22}^{\mathbb{P}}$ is 0.29 in the one-step approach but 0.45 in the two-step approach. The results are mixed for the level factor, as the one-step estimate of $\kappa_{11}^{\mathbb{P}}$ has a smaller bias with $\sigma_\varepsilon = 10$ basis points but not with $\sigma_\varepsilon = 1$ basis points.

**Table 7**
Accuracy of estimated states in the AFNS model.

| State variable | One-step approach | | | | Two-step approach | | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
| | Mean | MAE | Mean | MAE | Mean | MAE | Mean | MAE |
| $L_t$ | −0.41 | 1.16 | 0.21 | 5.51 | −8.93 | 10.80 | −3.05 | 9.66 |
| $S_t$ | 0.42 | 1.16 | −0.14 | 5.97 | 14.28 | 15.67 | 7.45 | 13.46 |
| $C_t$ | 0.16 | 2.51 | −0.58 | 19.07 | −21.70 | 32.33 | −25.06 | 42.62 |

The table reports the mean errors (Mean) and mean absolute errors (MAE) of each estimated state variable in the AFNS model when using QML in the one-step approach and ML in the two-step approach, where synthetic yields are generated with the Svensson (1995) yield curve, both with simulated samples of length $T = 196$ and $N = 100$ repetitions. The mean error is obtained by first computing the mean errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these means across the $N = 100$ simulated samples. Similarly, the MAE are obtained by first computing the mean absolute errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these absolute means across the $N = 100$ simulated samples. The true states are generated from the AFNS model as described in Section 4.1. All numbers are reported in basis points.

The estimates of the volatility parameters in $\Sigma$ are basically unbiased in the one-step approach and estimated with great precision—both with small and large measurement errors. The corresponding estimates in the two-step approach display small biases with $\sigma_\varepsilon = 1$ basis points, which generally increase with larger measurement errors. All elements in $\theta^{\mathbb{P}}$ are generally close to their true values, although a careful inspection of Table 6 reveals that the biases in $\theta^{\mathbb{P}}$ typically are smaller in the one-step approach compared with the two-step approach.

The estimates of the decay parameter $\lambda$ for the slope and curvature factor are centered exactly around its true value in the one-step approach and estimated with great precision—both with small and large measurement errors. For the two-step approach, we see small positive biases in the estimates of $\lambda$, and relatively imprecise estimates compared to the one-step approach. For instance, when $\sigma_\varepsilon = 10$ basis points, the standard deviation in the estimates of $\lambda$ are 0.0264 in the two-step approach, but only 0.0056 in the one-step approach. These differences in the estimates of $\lambda$ are of particular interest given the work of Björk and Christensen (1999), which shows that the static Nelson–Siegel and Svensson yield curves are inconsistent with no-arbitrage restrictions because the corresponding $\lambda$ parameter(s) in these static models may change across time. The biased estimate of $\lambda$ in the two-step approach implies that the curvature factor carries a greater weight on shorter-term yields and is less sensitive to longer-term yields relative to the true model. Since short-term yields are more volatile than long-term yields, this explains the positive bias in the two-step estimates of $\sigma_{33}$. Furthermore, as short-term yields also tend to be less persistent than long-term yields, this also explains the more severe upward bias in the two-step estimates of $\kappa^{\mathbb{P}}_{33}$.

### 4.4. Estimated states

For each simulated sample and its related set of estimated parameters, we next study the accuracy of the estimated states. Table 7 shows that the filtered states in the one-step approach are basically unbiased, as the mean errors with $\sigma_\varepsilon = 10$ basis points are 0.21, −0.14, and −0.58 basis points for the level, slope, and curvature factors, respectively. In contrast, the conventional two-step approach generates notable biases in the estimated states. Furthermore, these biases in the two-step approach are largely unrelated to the degree of noise in the bond prices, implying that these biases must originate from the use of the estimated synthetic zero-coupon yields.

To measure the efficiency of the filtered states, we compute the mean absolute errors in each simulated sample of $T = 196$ observations, which we report in Table 7 by averaging across the $N = 100$ simulations. The states in the one-step approach are estimated very accurately with MAE of 1 to 2 basis points in the ideal case with $\sigma_\varepsilon = 1$ basis point. For the more realistic setting where $\sigma_\varepsilon = 10$ basis points, we find somewhat larger MAE of 6, 6, and 19 basis points for the level, slope, and curvature factors, respectively. The filtered states in the two-step approach are estimated much less accurately partly due to a lower number of cross-sectional observations to represent the yield curve compared with the one-step approach. Also, the efficiency of the state estimates in the two-step approach are much less affected by increasing the noise in the bond prices. For instance, the MAE of the level and slope factors are basically unaffected by the value of $\sigma_\varepsilon$. This feature of the two-step approach reflects the fact that the synthetic zero-coupon yields are able to smooth out much of the noise in the bond prices and mitigate the effects of measurement errors.

### 4.5. Accuracy of yield decomposition

DTSMs are often applied to decompose the yield curve into expected future short rates and term premiums. We next explore whether the use of the one-step approach improves the precision of this decomposition compared to the conventional two-step approach. Hence, let the $\tau$-year term premium be defined as $TP_t(\tau) = y(\tau, X_t) - \frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s]ds$, where $\frac{1}{\tau}\int_t^{t+\tau} E_t^{\mathbb{P}}[r_s]ds$ denotes expected future short rates, which we compute as described in the online Appendix I.

For each simulated sample and its related set of estimated parameters and states, we next decompose the yield curve into expected future short rates and term premiums in Table 8. The mean errors in expected future short rates (EXP) are

**Table 8**
Accuracy of the yield decomposition in the AFNS model.

| Component | One-step approach | | | | Two-step approach | | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
| | Mean | MAE | Mean | MAE | Mean | MAE | Mean | MAE |
| Two-year yield | 0.09 | 0.24 | 0.08 | 2.04 | 0.66 | 1.23 | 0.77 | 2.80 |
| Five-year yield | 0.18 | 0.33 | 0.19 | 2.23 | −0.21 | 0.88 | −0.23 | 2.67 |
| Ten-year yield | 0.25 | 0.33 | 0.24 | 1.72 | −0.57 | 1.40 | −0.67 | 2.67 |
| Two-year EXP | −1.96 | 24.09 | −1.67 | 24.27 | −7.71 | 24.60 | −7.08 | 27.81 |
| Five-year EXP | −3.76 | 42.51 | −2.95 | 42.15 | −9.85 | 41.25 | −9.36 | 43.84 |
| Ten-year EXP | −5.20 | 59.37 | −3.64 | 58.49 | −11.55 | 56.81 | −11.02 | 58.67 |
| Two-year TP | 2.05 | 24.09 | 1.75 | 24.23 | 8.36 | 24.64 | 7.85 | 27.64 |
| Five-year TP | 3.95 | 42.52 | 3.13 | 42.33 | 9.64 | 41.36 | 9.13 | 44.16 |
| Ten-year TP | 5.44 | 59.38 | 3.88 | 58.53 | 10.98 | 56.85 | 10.34 | 58.72 |

The table reports the mean errors (Mean) and mean absolute errors (MAE) for each component of the yield curve decomposition into expected future short rates (EXP) and term premium (TP) at various maturities. The mean error for a given maturity is obtained by first computing the mean errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these means across the $N = 100$ simulated samples. Similarly, the MAE for a given maturity is obtained by first computing the mean absolute errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these absolute means across the $N = 100$ simulated samples. All errors are shown in basis points and defined as the true value minus the model-implied value. The parameter and state estimates in the AFNS model are obtained by QML in the one-step approach and by ML in the two-step approach, where zero-coupon synthetic yields are generated with the Svensson (1995) yield curve. The true yields and expected future short rates are generated from the AFNS model as described in Section 4.1.

somewhat closer to zero in the one-step approach. This finding seems consistent with the smaller biases in $\mathcal{K}^{\mathbb{P}}$ and in the filtered states for the one-step approach reported in Sections 4.3 and 4.4. The one-step approach also implies slightly lower mean errors for term premiums (TP) than the two-step approach. However, the MAE in Table 8 for expected future short rates and term premiums are very large and almost identical for both estimation approaches, meaning that the large estimation uncertainty clearly dominates the small improvement in mean errors for term premiums within the one-step approach.

Thus, standard yield curve decompositions do not benefit from the one-step approach. This is because the reported biases and large estimation uncertainty in term premiums originate from expected future short rates, and hence the estimated mean-reversion parameters in $\mathcal{K}^{\mathbb{P}}$, which are relatively insensitive to the number of cross-sectional observations used to represent the yield curve.

### *4.6. The implementation of the two-step approach*

Given the widespread use of the conventional two-step approach, it seems useful to explore whether its performance can be improved compared to Sections 4.3 and 4.4. The essential decisions for the econometrician in the two-step approach are the number of synthetic zero-coupon yields to include and how to extract these yields from the panel of coupon bonds. We next analyze how these choices affect the estimated DTSM.

#### *4.6.1. The number of synthetic yields*

Our implementation of the two-step approach has so far used ten synthetic zero-coupon yields to represent the yield curve—a typical number in the literature. But this choice may affect the performance of the two-step approach. In particular, it is likely that there may exist a trade-off within the two-step approach between bias and efficiency for the estimated parameters in a DTSM. A large number of synthetic yields may increase efficiency but at the potential cost of including some less precisely measured yields, which could bias the estimated DTSM coefficients. On the other hand, including only a few very accurately measured yields minimizes the risk of coefficient bias but at the potential cost of lower efficiency. We explore whether such a trade-off exists by implementing the two-step approach on two additional sets of simulated data samples that both are drawn from the *same* data generating process. The first is an extended sample of 31 synthetic yields with maturities of 0.5, 1, 2, …, 30 year. The second is a reduced sample of only six yields with maturities of 1, 2, 3, 5, 7, and 10 years, where we omit the imprecisely estimated yields at the short and long end of the thirty-year yield curve.[14]

The estimated coefficients in the AFNS model from this simulation exercise are reported in Table 9. We find somewhat surprisingly that the biases for $\kappa_{11}^{\mathbb{P}}$, $\kappa_{22}^{\mathbb{P}}$, $\sigma_{11}$, $\sigma_{22}$, and $\lambda$ are *smaller* in the extended sample compared to the standard sample (in Table 6) and the reduced sample. Still, it is hard to detect any efficiency gains from the extended sample for these parameters, except possibly for $\lambda$. Furthermore, we note that the estimates of $\kappa_{33}^{\mathbb{P}}$ and $\sigma_{33}$ have the largest biases in the extended sample, which also provides the most imprecise estimates of $\kappa_{33}^{\mathbb{P}}$ and $\sigma_{33}$. The performance of the reduced sample with six synthetic yields is very similar to what we found for the standard sample with ten synthetic yields,

[14] See Christensen and Krogstrup (2019) for an empirical application using this set of synthetic yield maturities.

**Table 9**
Accuracy of the parameter estimates: The number of synthetic yields.

| Par. | True value | Extended sample | | | | Reduced sample | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
| | | Mean bias | Std. dev. | Mean bias | Std. dev. | Mean bias | Std. dev. | Mean bias | Std. dev. |
| $\kappa_{11}^{\mathbb{P}}$ | **0.1060** | 0.2549 | 0.2245 | 0.2854 | 0.2397 | 0.2742 | 0.2145 | 0.4359 | 0.3017 |
| $\kappa_{22}^{\mathbb{P}}$ | **0.2157** | 0.2915 | 0.2961 | 0.3807 | 0.3518 | 0.3052 | 0.2959 | 0.3963 | 0.3488 |
| $\kappa_{33}^{\mathbb{P}}$ | **0.7255** | 2.5730 | 2.3665 | 1.7932 | 1.6718 | 0.2011 | 0.4196 | 0.5085 | 0.5102 |
| $\sigma_{11}$ | **0.0052** | −0.0004 | 0.0004 | −0.0001 | 0.0003 | −0.0002 | 0.0004 | 0.0009 | 0.0004 |
| $\sigma_{22}$ | **0.0103** | 0.0000 | 0.0005 | 0.0008 | 0.0007 | 0.0002 | 0.0005 | 0.0011 | 0.0007 |
| $\sigma_{33}$ | **0.0207** | 0.0192 | 0.0135 | 0.0144 | 0.0084 | −0.0012 | 0.0016 | 0.0028 | 0.0017 |
| $\theta_1^{\mathbb{P}}$ | **0.0529** | 0.0027 | 0.0065 | 0.0021 | 0.0082 | −0.0033 | 0.0084 | −0.0021 | 0.0086 |
| $\theta_2^{\mathbb{P}}$ | **−0.0275** | −0.0015 | 0.0118 | −0.0012 | 0.0115 | 0.0045 | 0.0107 | 0.0032 | 0.0108 |
| $\theta_3^{\mathbb{P}}$ | **−0.0230** | −0.0069 | 0.0097 | −0.0048 | 0.0083 | −0.0021 | 0.0060 | −0.0030 | 0.0059 |
| $\lambda$ | **0.3747** | 0.0026 | 0.0178 | −0.0133 | 0.0210 | 0.0671 | 0.0308 | 0.0516 | 0.0256 |

The table reports the mean estimate minus true value (Mean bias) and the standard deviation (Std. dev.) of the sampling distribution for each of the estimated parameters in the AFNS model when using ML in the two-step approach, with simulated samples of length $T = 196$ and $N = 100$ repetitions. The extended samples consists of the 31 synthetic zero-coupon yields with maturities of 0.5, 1, 2, ..., 30 year from the Svensson (1995) yield curve. The reduced sample consists of six synthetic zero-coupon yields with maturities of 1, 2, 3, 5, 7, and 10 years from the Svensson (1995) yield curve.

**Table 10**
Accuracy of estimated states: The number of synthetic yields.

| State variable | Extended sample | | | | Reduced sample | | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
| | Mean | MAE | Mean | MAE | Mean | MAE | Mean | MAE |
| $L_t$ | 40.27 | 41.36 | 34.44 | 34.78 | −19.59 | 20.33 | −7.31 | 14.52 |
| $S_t$ | −35.83 | 37.46 | −32.06 | 32.77 | 24.68 | 25.21 | 11.98 | 17.13 |
| $C_t$ | −65.82 | 67.28 | −43.46 | 54.12 | −14.52 | 31.40 | −23.80 | 43.47 |

The table reports mean errors (Mean) and mean absolute errors (MAE) of each estimated state variable in the AFNS model using ML in the two-step approach. The mean error is obtained by first computing the mean errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these means across the $N = 100$ simulated samples. Similarly, the MAE are obtained by first computing the mean absolute errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these absolute means across the $N = 100$ simulated samples. The true states are generated from the AFNS model as described in Section 4.1. The extended samples consists of the 31 synthetic zero-coupon yields with maturities of 0.5, 1, 2, ..., 30 year from the Svensson (1995) yield curve. The reduced sample consists of six synthetic zero-coupon yields with maturities of 1, 2, 3, 5, 7, and 10 years from the Svensson (1995) yield curve. All numbers are reported in basis points.

meaning that the reduced sample avoids the large biases we occasionally find in the extended sample (e.g. in $\kappa_{33}^{\mathbb{P}}$ and $\sigma_{33}$). For each simulated sample and its related set of estimated parameters, Table 10 shows the accuracy of the filtered states. The extended sample has larger positive biases for the level factor and larger negative biases for the slope and curvature factor. The biases in the reduced sample are much smaller than in the extended sample, which explains the lower MAE of the filtered states for the reduced sample compared with the extended sample.

We draw two conclusions from this exercise. First, there does not appear to be an obvious trade-off between bias and efficiency in the two-step approach when varying the number of synthetic zero-coupon yields in the estimation of the DTSM. Second, the current practice of using a relatively low number of synthetic yields (i.e. between six and ten) seems well justified, at least when the synthetic yields are extracted based on the parametric discount function in Svensson (1995).

*4.6.2. Synthetic Nelson and Siegel (1987) yields*

An obvious difference between the true zero-coupon yields from the AFNS model and those from the Svensson (1995) yield curve is that the latter allows for an extra "hump" at the long end of the yield curve compared to the AFNS model. Within our setting, this additional hump clearly seems redundant and we therefore explore the effects of omitting it when extracting synthetic zero-coupon yields from our simulated panels of coupon bond prices. That is, we consider the case where the synthetic yields are constructed using the parametric discount function in Nelson and Siegel (1987), which is described further in the online Appendix B. We continue to use the same ten constant yield maturities as in Section 4.3.

Table 11 reports the results for the estimated coefficients in the AFNS model from this simulation exercise, which we benchmark to the findings in Section 4.3 based on Svensson (1995) yields. We generally find that the estimated coefficients are adversely affected by using the more parsimonious specification of Nelson and Siegel (1987) to extract the synthetic

**Table 11**
Accuracy of the parameter estimates: Different synthetic yields.

| Par. | True value | Svensson (1995) yields | | | | Nelson and Siegel (1987) yields | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
| | | Mean bias | Std. dev. | Mean bias | Std. dev. | Mean bias | Std. dev. | Mean bias | Std. dev. |
| $\kappa_{11}^{\mathbb{P}}$ | **0.1060** | 0.2397 | 0.2114 | 0.2489 | 0.2186 | 0.0260 | 0.1009 | 0.0551 | 0.1195 |
| $\kappa_{22}^{\mathbb{P}}$ | **0.2157** | 0.3139 | 0.3069 | 0.4454 | 0.3883 | 0.3255 | 0.2813 | 0.4504 | 0.3739 |
| $\kappa_{33}^{\mathbb{P}}$ | **0.7255** | 0.4264 | 0.6491 | 0.5756 | 0.5329 | 2.7574 | 2.9780 | 1.2863 | 1.1920 |
| $\sigma_{11}$ | **0.0052** | −0.0005 | 0.0003 | −0.0004 | 0.0003 | −0.0023 | 0.0003 | −0.0019 | 0.0003 |
| $\sigma_{22}$ | **0.0103** | 0.0002 | 0.0006 | 0.0014 | 0.0009 | 0.0004 | 0.0005 | 0.0015 | 0.0008 |
| $\sigma_{33}$ | **0.0207** | 0.0035 | 0.0037 | 0.0058 | 0.0020 | 0.0210 | 0.0179 | 0.0125 | 0.0087 |
| $\theta_1^{\mathbb{P}}$ | **0.0529** | −0.0023 | 0.0086 | −0.0016 | 0.0087 | −0.0013 | 0.0073 | −0.0027 | 0.0079 |
| $\theta_2^{\mathbb{P}}$ | −**0.0275** | 0.0036 | 0.0112 | 0.0028 | 0.0110 | 0.0011 | 0.0127 | 0.0036 | 0.0114 |
| $\theta_3^{\mathbb{P}}$ | −**0.0230** | −0.0027 | 0.0062 | −0.0031 | 0.0062 | −0.0051 | 0.0110 | −0.0028 | 0.0074 |
| $\lambda$ | **0.3747** | 0.0490 | 0.0314 | 0.0423 | 0.0264 | 0.0694 | 0.0310 | 0.0778 | 0.0268 |

The table reports the mean estimate minus true value (Mean bias) and the standard deviation (Std. dev.) of the sampling distribution for each of the estimated parameters in the AFNS model when using ML in the two-step approach on synthetic yields from the Svensson (1995) and Nelson and Siegel (1987) yield curves. The true yields are generated from the AFNS model as described in Section 4.1, with simulated samples of length $T = 196$ and $N = 100$ repetitions. For both types of yields we use the same ten constant maturities, 0.25, 0.5, 1, 2, 3, 5, 7, 10, 20, and 30 years.

**Table 12**
Accuracy of estimated states: Different synthetic yields.

| State variable | Svensson (1995) yields | | | | Nelson and Siegel (1987) yields | | | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | | $\sigma_\varepsilon = 1$ basis point | | $\sigma_\varepsilon = 10$ basis points | |
| | Mean | MAE | Mean | MAE | Mean | MAE | Mean | MAE |
| $L_t$ | −8.93 | 10.80 | −3.05 | 9.66 | 6.69 | 39.44 | −10.69 | 22.45 |
| $S_t$ | 14.28 | 15.67 | 7.45 | 13.46 | −2.78 | 37.30 | 15.13 | 26.69 |
| $C_t$ | −21.70 | 32.33 | −25.06 | 42.62 | −43.72 | 74.43 | −24.83 | 56.51 |

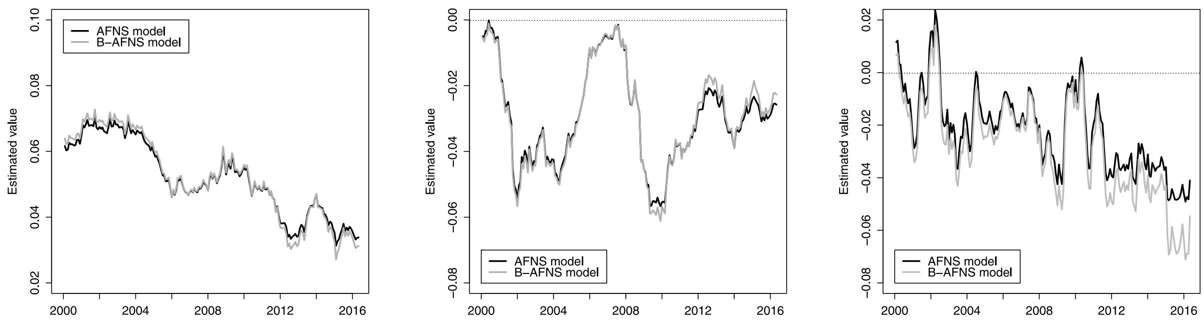
The table reports the mean errors (Mean) and mean absolute errors (MAE) of each estimated state variable in the AFNS model using ML in the two-step approach based on Svensson (1995) and Nelson and Siegel (1987) yields, each with the same ten maturities, 0.25, 0.5, 1, 2, 3, 5, 7, 10, 20, and 30 years. The mean error is obtained by first computing the mean errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these means across the $N = 100$ simulated samples. Similarly, the MAE are obtained by first computing the mean absolute errors in each of the simulated samples across the $T = 196$ observations, and we then report the average of these absolute means across the $N = 100$ simulated samples. The true states are generated from the AFNS model as described in Section 4.1. All numbers are reported in basis points.

yields. Most notably, the biases for $\kappa_{22}^{\mathbb{P}}$, $\sigma_{11}$, $\sigma_{22}$, $\sigma_{33}$, and $\lambda$ increase somewhat when using the Nelson and Siegel (1987) yields compared to the Svensson (1995) yields, whereas the opposite applies for $\kappa_{11}^{\mathbb{P}}$. Table 12 further shows that the filtered state estimates with the Nelson and Siegel (1987) yields are less efficient (as measured by MAE) compared to the Svensson (1995) yields, whereas the mean errors are more similar across the two specifications.

To understand why the use of Nelson and Siegel (1987) yields worsen the performance of the two-step approach, recall that yields in the AFNS model include the convexity-adjustment term $A(\tau)/\tau$ to ensure absence of arbitrage, which is not present in the Nelson and Siegel (1987) and Svensson (1995) specifications. This yield-adjustment term grows in size with maturity and becomes non-negligible for long-term bonds. Our results therefore suggest that the extra flexibility for long-term yields in the Svensson (1995) discount function is desirable in this context, because it allows us to capture this convexity-adjustment when estimating synthetic zero-coupon yields. Thus, restricting the parametric discount function for extracting synthetic zero-coupon yields does not improve performance, which supports the widespread use of the synthetic U.S. Treasury yields constructed by Gürkaynak et al. (2007, 2010).

### *4.7. Summarizing the main insights from the Monte Carlo study*

The main insight from this Monte Carlo study is that a DTSM may be estimated more reliably by using directly observed market prices on coupon bonds instead of synthetic zero-coupon yields. Although these synthetic zero-coupon yields are estimated very accurately with well-established curve-fitting techniques, we nevertheless find that seemingly negligible errors in these synthetic yields do affect the estimated parameters in a DTSM. In particular, all risk-neutral parameters are estimated with smaller biases and greater efficiency with a one-step approach compared with the conventional two-step approach. In large part, this improvement reflects the denser representation of the yield curve used in the one-step approach as well as the avoidance of estimation errors introduced by synthetic zero-coupon yields. We also find that parameters in the $\mathbb{P}$-dynamics benefit from a one-step approach, although these parameters are unrelated to




(a) $L_t$ (b) $S_t$ (c) $C_t$

**Fig. 3. Estimated States in the One-Step Approach**. This figure reports the filtered estimates of level, slope, and curvature in the AFNS and B-AFNS model. The data are monthly and cover the period from January 31, 2000, to April 29, 2016.

the $\mathbb{Q}$-dynamics with an essentially affine specification for the market prices of risk. This improvement therefore arises mainly because the states are estimated with lower biases and greater precision in the one-step approach than seen in the two-step approach.[15]

## 5. Forecasting bond yields

The previous analysis has shown that parameters and states in the AFNS model are estimated with smaller biases and greater efficiency by the one-step approach when compared to the two-step approach. The present section explores whether these advantages are sufficiently large to improve the ability of the AFNS model to forecast yields out of sample. To make this forecasting exercise more comprehensive, we include two additional models: a shadow-rate specification to accommodate the zero lower bound and a five-factor AFNS model to better fit long-term bonds. From a methodological perspective, these extensions also illustrate that the one-step approach is applicable to nonlinear DTSMs and to models with more than three states. As in the previous section, we benchmark the performance of the one-step approach to those from the two-step approach based on synthetic zero-coupon yields.

### 5.1. A shadow-rate model

Given the very low policy rates in many economies during the recent financial crisis and its aftermath, it has become popular to account for the zero lower bound (ZLB) in DTSMs. Although relative to other developed economies, Canadian short rates were close to zero for only a limited period in our sample (as seen from Fig. 2(a)), it is still possible that the ZLB may affect the shape and dynamics of the yield curve. To enforce the ZLB in the AFNS model, we introduce the shadow rate $s_t = L_t + S_t$ and let $r_t = \max\{0, s_t\}$, as in Christensen and Rudebusch (2015). All other aspects of this B-AFNS model remain as described above for the AFNS model.[16] The expression for zero-coupon yields in the B-AFNS model is not available in closed form but approximated numerically using the accurate method of Krippner (2013).[17]

Table 13 shows that all elements in $\mathcal{K}^{\mathbb{P}}$ and $\theta^{\mathbb{P}}$ in the B-AFNS model are estimated very imprecisely across the three data sets, which is similar to our finding for the AFNS model. The volatility parameters in $\Sigma$ are estimated much more precisely and are generally higher in the B-AFNS model when compared to the AFNS model. Fig. 3 shows that this difference is mainly explained by greater variability of the states after 2008, because the shadow-rate specification in the B-AFNS model allows the states to move more freely than seen in the AFNS model without violating the ZLB. We also find that $\lambda$ is estimated to be somewhat higher in all three data sets when accounting for the ZLB in comparison to the AFNS model. Similar to the pattern observed for the AFNS model, the estimate of $\lambda$ in the B-AFNS model using the one-step approach lies in between those from the two-step approach, as $\lambda$ is 0.392 in the one-step approach, 0.347 in the two-step approach based on Bank of Canada yields, and 0.475 in the two-step approach using Svensson (1995) yields.

Table 14 reports the pricing errors of the B-AFNS model for the underlying coupon bonds. For the one-step approach and both versions of the two-step approach, we find slightly smaller RMSEs in the B-AFNS model compared to the AFNS

15 Our current Monte Carlo study uses a small number of simulations ($N = 100$), unreported results using $N = 1000$ suggest that little changes by increasing $N$.

16 Following Kim and Singleton (2012), the prefix "B-" refers to a shadow-rate model in the spirit of Black (1995).

17 See also Christensen and Rudebusch (2015, 2016) for further details on this approximation and its accuracy.

**Table 13**
Estimated parameters in the B-AFNS model.

| Par. | One-step approach | | Two-step approach | | | |
|---|---|---|---|---|---|---|
| | | | Bank of Canada yields | | Svensson (1995) yields | |
| | Est | SE | Est | SE | Est | SE |
| $\kappa_{11}^{\mathbb{P}}$ | 0.1450 | 0.0891 | 0.1373 | 0.0669 | 0.0521 | 0.0468 |
| $\kappa_{22}^{\mathbb{P}}$ | 0.1066 | 0.0819 | 0.1074 | 0.0701 | 0.5166 | 0.1828 |
| $\kappa_{33}^{\mathbb{P}}$ | 0.4337 | 0.2897 | 0.3503 | 0.3573 | 0.2646 | 0.3935 |
| $\sigma_{11}$ | 0.0073 | 0.0002 | 0.0083 | 0.0003 | 0.0059 | 0.0002 |
| $\sigma_{22}$ | 0.0122 | 0.0011 | 0.0098 | 0.0008 | 0.0114 | 0.0011 |
| $\sigma_{33}$ | 0.0180 | 0.0021 | 0.0205 | 0.0021 | 0.0220 | 0.0026 |
| $\theta_1^{\mathbb{P}}$ | 0.0546 | 0.0034 | 0.0546 | 0.0115 | 0.0555 | 0.0242 |
| $\theta_2^{\mathbb{P}}$ | −0.0396 | 0.0087 | −0.0248 | 0.0135 | −0.0231 | 0.0046 |
| $\theta_3^{\mathbb{P}}$ | −0.0248 | 0.0117 | −0.0250 | 0.0135 | −0.0238 | 0.0219 |
| $\lambda$ | 0.3920 | 0.0123 | 0.3473 | 0.0135 | 0.4754 | 0.0149 |

This table reports the estimated parameters (Est) in the B-AFNS model with independent factors and their standard errors (SE) using either the one-step or the two-step approach. The SE are in all cases computed by pre- and post-multiplying the variance of the score by the inverse of the Hessian matrix, computed as outlined in Harvey (1989). The data are monthly and cover the period from January 31, 2000, to April 29, 2016.

**Table 14**
Summary statistics of bond fitted errors in the B-AFNS model.

| Maturity bucket in years | No. obs. | One-step approach | | Two-step approach | | | |
|---|---|---|---|---|---|---|---|
| | | | | Bank of Canada yields | | Svensson (1995) yields | |
| | | Mean | RMSE | Mean | RMSE | Mean | RMSE |
| 0–2 | 1472 | −0.41 | 5.51 | 2.91 | 7.84 | 0.70 | 9.54 |
| 2–4 | 1098 | 0.47 | 4.92 | 1.55 | 6.92 | 1.26 | 5.20 |
| 4–6 | 744 | −0.36 | 4.17 | 0.14 | 4.67 | −2.06 | 5.08 |
| 6–8 | 404 | −1.79 | 5.82 | −1.04 | 5.55 | −3.62 | 7.07 |
| 8–10 | 477 | −3.65 | 6.75 | −3.04 | 6.89 | −4.76 | 7.77 |
| 10–12 | 289 | −2.35 | 6.80 | −1.23 | 8.55 | −1.80 | 8.02 |
| 12–14 | 155 | 2.54 | 5.62 | 4.93 | 11.28 | 5.12 | 10.55 |
| 14–16 | 168 | −0.34 | 4.17 | −0.96 | 8.66 | 0.89 | 7.90 |
| 16–18 | 179 | −0.30 | 4.79 | −1.68 | 9.93 | 1.13 | 8.80 |
| 18–20 | 192 | 0.94 | 3.93 | 1.89 | 8.33 | 4.66 | 8.26 |
| 20–22 | 186 | 3.31 | 5.38 | 5.44 | 9.58 | 7.43 | 10.07 |
| 22–24 | 142 | 1.49 | 5.38 | 2.90 | 6.82 | 4.74 | 6.89 |
| 24–26 | 124 | 1.45 | 5.40 | 4.16 | 8.20 | 4.78 | 7.37 |
| 26–28 | 113 | −2.98 | 6.88 | 1.14 | 5.66 | 0.43 | 4.45 |
| 28< | 288 | −2.59 | 9.24 | 1.16 | 5.95 | −1.71 | 4.59 |
| All yields | 6031 | −0.52 | 5.62 | 1.15 | 7.34 | 0.15 | 7.57 |

This table reports the mean pricing errors (Mean) and the root mean-squared pricing errors (RMSE) of the Canadian bond prices for the B-AFNS model with independent factors. The pricing errors are reported in annual basis points and computed as the difference between the implied yield on the coupon bond and the model-implied yield on this bond. The data are monthly and cover the period from January 31, 2000, to April 29, 2016.

model. For instance, the overall RMSE falls by 5% from 5.90 to 5.62 basis points in the one-step approach. Thus, accounting for the ZLB does not materially improve the ability of the AFNS model to match Canadian coupon bond prices.

### *5.2. A five-factor model*

The main motivation of Gürkaynak et al. (2007) to prefer the Svensson (1995) curve over the simpler specification in Nelson and Siegel (1987) is that the Svensson (1995) curve allows for an additional hump that helps fit U.S. bond yields beyond the ten- to fifteen-year maturity spectrum. The AFNS model may potentially also benefit from additional dynamics to fit long-term Canadian bond prices, as its loadings for the slope and curvature factor decay to zero as maturity approaches infinity. This often implies (for reasonable values of $\lambda$) that only the level factor in the AFNS model can be used to fit long-term bonds, which may at times be insufficient as noted in Christensen et al. (2011).

To explore whether the performance of the AFNS model on our Canadian sample may be improved further, we consider the arbitrage-free generalized Nelson–Siegel (AFGNS) model of Christensen et al. (2009), which includes an additional pair of slope and curvature factors that help to fit long-term bonds. In this AFGNS model, the short rate is $r_t = L_t + S_t + \tilde{S}_t$,

**Table 15**
Parameter estimates in the AFGNS model.

| Par. | One-step approach | | Two-step approach | | | |
|---|---|---|---|---|---|---|
| | | | Bank of Canada yields | | Svensson (1995) yields | |
| | Est | SE | Est | SE | Est | SE |
| $\kappa_{11}^{\mathbb{P}}$ | 0.0453 | 0.0484 | 0.3656 | 0.4340 | 0.1365 | 0.1194 |
| $\kappa_{22}^{\mathbb{P}}$ | 0.1835 | 0.1418 | 0.6233 | 0.3229 | 0.5279 | 0.2508 |
| $\kappa_{33}^{\mathbb{P}}$ | 0.2015 | 0.1973 | 0.1214 | 0.1790 | 1.1160 | 0.3580 |
| $\kappa_{44}^{\mathbb{P}}$ | 0.7371 | 0.3144 | 0.9582 | 0.4657 | 1.1599 | 0.3570 |
| $\kappa_{55}^{\mathbb{P}}$ | 0.1970 | 0.1255 | 0.3950 | 0.2917 | 0.1464 | 0.1347 |
| $\sigma_{11}$ | 0.0031 | 0.0006 | 0.0092 | 0.0007 | 0.0050 | 0.0002 |
| $\sigma_{22}$ | 0.0125 | 0.0010 | 0.0125 | 0.0009 | 0.0140 | 0.0009 |
| $\sigma_{33}$ | 0.0106 | 0.0009 | 0.0093 | 0.0009 | 0.0150 | 0.0015 |
| $\sigma_{44}$ | 0.0237 | 0.0019 | 0.0209 | 0.0013 | 0.0359 | 0.0021 |
| $\sigma_{55}$ | 0.0200 | 0.0014 | 0.0215 | 0.0033 | 0.0189 | 0.0013 |
| $\theta_1^{\mathbb{P}}$ | 0.0500 | 0.0053 | 0.0982 | 0.0110 | 0.0580 | 0.0091 |
| $\theta_2^{\mathbb{P}}$ | 0.0183 | 0.0172 | −0.0040 | 0.0087 | 0.0069 | 0.0099 |
| $\theta_3^{\mathbb{P}}$ | −0.0452 | 0.0140 | −0.0691 | 0.0219 | −0.0456 | 0.0045 |
| $\theta_4^{\mathbb{P}}$ | 0.0064 | 0.0090 | −0.0041 | 0.0071 | 0.0061 | 0.0093 |
| $\theta_5^{\mathbb{P}}$ | 0.0486 | 0.0185 | −0.0318 | 0.0229 | 0.0252 | 0.0329 |
| $\lambda$ | 0.6416 | 0.0280 | 1.3699 | 0.0297 | 0.9290 | 0.0102 |
| $\tilde{\lambda}$ | 0.1166 | 0.0084 | 0.0786 | 0.0039 | 0.1185 | 0.0026 |

This table reports the estimated parameters (Est) in the AFGNS model with independent factors and their standard errors (SE) using either the one-step or the two-step approach. The SE in the one-step approach are computed by pre- and post-multiplying the variance of the score by the inverse of the Hessian matrix, computed as outlined in Harvey (1989). The SE in the two-step approach are computed from the inverse of the variance of the score. The data are monthly and cover the period from January 31, 2000, to April 29, 2016.

where $\tilde{S}_t$ is an additional (long-term) slope factor. The state dynamics under the risk-neutral $\mathbb{Q}$-measure is

$$\begin{pmatrix} dL_t \\ dS_t \\ d\tilde{S}_t \\ dC_t \\ d\tilde{C}_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -\lambda & 0 \\ 0 & 0 & \tilde{\lambda} & 0 & -\tilde{\lambda} \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \tilde{\lambda} \end{pmatrix} \left[ \begin{pmatrix} \theta_1^{\mathbb{Q}} \\ \theta_2^{\mathbb{Q}} \\ \theta_3^{\mathbb{Q}} \\ \theta_4^{\mathbb{Q}} \\ \theta_5^{\mathbb{Q}} \end{pmatrix} - \begin{pmatrix} L_t \\ S_t \\ \tilde{S}_t \\ C_t \\ \tilde{C}_t \end{pmatrix} \right] dt + \Sigma d\tilde{W}_t^{\mathbb{Q}},$$

where $\lambda > \tilde{\lambda} > 0$ and $\tilde{C}_t$ is an additional (long-term) curvature factor. Zero-coupon yields are then given by

$$\begin{aligned} y(t,T) = {} & L_t + \frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)} S_t + \left[ \frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)} - e^{-\lambda(T-t)} \right] C_t \\ & + \frac{1-e^{-\tilde{\lambda}(T-t)}}{\tilde{\lambda}(T-t)} \tilde{S}_t + \left[ \frac{1-e^{-\tilde{\lambda}(T-t)}}{\tilde{\lambda}(T-t)} - e^{-\tilde{\lambda}(T-t)} \right] \tilde{C}_t - \frac{\tilde{A}(t,T)}{T-t}, \end{aligned}$$

where the yield-adjustment term $\tilde{A}(t,T)$ is derived in Christensen et al. (2009). The $\mathbb{P}$-dynamics for this five-factor model is obtained in a standard fashion by adopting an essential affine specification for the market price of risk, as in Section 3.1.

The estimation results for the AFGNS model are reported in Table 15—again, limited to independent factor dynamics under the $\mathbb{P}$-measure. The decay parameter $\lambda$ is estimated to be somewhat larger than in the AFNS model, because $S_t$ and $C_t$ no longer have to fit long-term bonds. The very low estimate of the second decay parameter $\tilde{\lambda}$ implies that the additional factors $\tilde{S}_t$ and $\tilde{C}_t$ greatly assist the level factor in matching the long end of the Canadian yield curve.

Table 16 reports the pricing errors of the AFGNS model for the underlying coupon bonds, and we clearly see that all three versions of this five-factor model provide a closer fit to nearly all bonds when compared with the AFNS model. This is highligthed in Table 16 by $\Delta$RMSE, which shows the difference in RMSE between the AFGNS model and the AFNS model for the one-step approach and for the two implementations of the two-step approach. For the one-step approach, we see large improvements in the RMSEs for long-term bonds, but also in the zero to two-year and two- to four-year maturity buckets, which both contain a large number of bonds. As a result, the overall RMSE within the one-step approach drops from 5.90 basis points in the AFNS model to just 4.51 basis points in this extended model, which corresponds to a 24% reduction in the size of the in-sample fitted errors. This also means that the AFGNS model clearly provides a *better* overall fit to the coupon bond prices than the Svensson (1995) discount function with an overall RMSE of 5.78 basis points. Given this satisfying performance of the AFGNS model, its zero-coupon yields may thus be used as another and slightly more accurate representation of the Canadian yield curve than the zero-coupon yields from the Svensson (1995) discount function.

**Table 16**
Summary statistics of bond fitted errors in the AFGNS model.

| Maturity bucket in years | No. obs. | One-step approach | | | Two-step approach | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | Bank of Canada yields | | | Svensson (1995) yields | | |
| | | Mean | RMSE | ΔRMSE | Mean | RMSE | ΔRMSE | Mean | RMSE | ΔRMSE |
| 0–2 | 1472 | −0.20 | 4.86 | −0.88 | 2.63 | 5.56 | −2.31 | −0.73 | 9.01 | −0.80 |
| 2–4 | 1098 | 0.36 | 3.82 | −0.92 | −0.93 | 4.49 | −2.71 | 0.33 | 4.28 | −1.33 |
| 4–6 | 744 | 0.77 | 3.77 | −0.08 | −0.03 | 3.29 | −1.08 | 0.06 | 3.91 | −0.70 |
| 6–8 | 404 | −0.78 | 3.88 | −1.59 | 1.54 | 3.96 | −1.06 | −1.47 | 4.32 | −2.29 |
| 8–10 | 477 | −2.39 | 5.33 | −0.74 | 0.91 | 5.00 | −1.61 | −2.28 | 5.31 | −2.50 |
| 10–12 | 289 | −2.01 | 6.45 | 0.25 | 2.14 | 7.98 | −0.58 | −1.68 | 6.00 | −2.35 |
| 12–14 | 155 | 1.87 | 4.01 | −2.71 | 7.45 | 10.56 | −1.42 | 2.59 | 4.86 | −6.32 |
| 14–16 | 168 | 0.03 | 2.45 | −1.87 | 1.71 | 6.36 | −2.69 | 0.61 | 3.04 | −5.24 |
| 16–18 | 179 | −0.21 | 3.10 | −1.56 | −1.62 | 7.70 | −2.19 | −0.06 | 4.13 | −4.71 |
| 18–20 | 192 | 0.71 | 3.77 | −0.56 | −0.53 | 5.92 | −2.97 | 0.44 | 4.41 | −4.23 |
| 20–22 | 186 | 2.39 | 5.10 | 0.13 | 1.17 | 4.74 | −5.32 | 2.47 | 4.69 | −5.63 |
| 22–24 | 142 | 1.82 | 5.31 | 0.84 | −0.92 | 4.51 | −2.76 | 0.99 | 3.67 | −3.42 |
| 24–26 | 124 | 1.26 | 4.48 | −0.53 | 0.16 | 5.54 | −2.83 | 1.44 | 3.66 | −3.77 |
| 26–28 | 113 | −1.21 | 4.51 | −3.85 | −1.51 | 4.71 | −1.19 | −1.56 | 3.51 | −0.98 |
| 28< | 288 | −1.88 | 5.32 | −6.59 | 11.59 | 32.98 | 14.85 | 0.33 | 7.82 | −0.87 |
| All bonds | 6031 | −0.13 | 4.51 | −1.28 | 1.46 | 8.93 | 0.62 | −0.26 | 6.05 | −1.85 |

This table reports the mean pricing errors (Mean) and the root mean-squared pricing errors (RMSE) of Canadian coupon bond prices for the AFGNS model with independent factors. The table also reports the difference in RMSE (ΔRMSE) between the AFGNS model and the AFNS model within the one-step approach and each of the two implementations of the two-step approach. All pricing errors are reported in annual basis points and computed as the difference between the implied yield on the coupon bond and the model-implied yield on this bond. The data are monthly and cover the period from January 31, 2000, to April 29, 2016.

### *5.3. Forecast exercise and results*

We structure the forecast exercise to match the Consensus Forecasts, which is a monthly survey of professional forecasters. Those survey participants submit forecasts of the three-month Treasury bill rate and the ten-year government bond yield for the next three and twelve months. To measure the realization of the three-month Treasury bill rate, we linearly interpolate between two Treasury bills whose remaining times to maturity constitute the tightest bracket around the three-month maturity point. For the ten-year government yield, we first note that the Bank of Canada (like the U.S. Treasury) tends to issue new bonds as close to par as possible, and we therefore interpret the survey participants as forecasting the ten-year par-coupon yield. The Bank of Canada only issues new ten-year bonds roughly once a year, and we therefore exploit our finding in Section 5.2 and use an estimated version of the AFGNS model to compute accurate realizations of the ten-year par yield. All three DTSMs in this forecasting study are estimated by the one-step approach and by the two-step approach using synthetic zero-coupon yields from the Bank of Canada, the Svensson (1995) discount function, and the Nelson and Siegel (1987) yield curve. To get a reasonable handle on the persistence of the states in the three models, we begin the forecast analysis in December 2006. Further details related to the implementation of the forecast study are provided in the online Appendix J.

Starting with the three-month ahead forecasts of the three-month yield in Table 17, the one-step approach does clearly better than the two-step approach using Svensson (1995) yields and Nelson and Siegel (1987) yields for all three models. However, we see the opposite pattern when the two-step approach is implemented using the Bank of Canada yields. The one-step approach continues to do well for the three-month yield when forecasting twelve months ahead, and it delivers the most accurate predictions in the AFNS and B-AFNS model. We also find that the forecasts from the AFNS model are generally not improved by accounting for the ZLB via the B-AFNS model, which is likely explained by the relatively brief period that the Canadian short rate stayed at the ZLB during our sample. Instead, the forecasts from the AFNS model are greatly improved by using the more flexible AFGNS model, but only when using the one-step approach and the two-step approach based on Bank of Canada yields, which both outperform the Consensus Forecasts when forecasting twelve months ahead.

The corresponding forecasts for the ten-year bond yield are summarized in Table 18. We once again find that the one-step approach delivers competitive forecasts when compared to the two-step approach, where the performance depends crucially on the chosen set of synthetic yields. That is, for the AFNS and B-AFNS model, we obtain the best results in the two-step approach by using the Nelson and Siegel (1987) yields, whereas we get the best results for the AFGNS model when using Bank of Canada yields. The one-step approach is in this sense more robust, as it either gives the best forecasts or is close to the best performing approach for all three models. We also note that the flexible five-factor AFGNS model also delivers the best forecasts for the ten-year bond yield among the three DTSMs, but only when using the one-step approach and the two-step approach based on Bank of Canada yields.

Thus, the one-step approach can also be used to generate competitive out-of-sample forecasts when compared to surveys and the conventional two-step approach.

**Table 17**
Summary statistics of three-month yield forecast errors.

| Model | Three-month forecasts | | | Twelve-month forecasts | | |
|---|---|---|---|---|---|---|
| | Mean | RMSE | MAE | Mean | RMSE | MAE |
| Consensus Forecasts | −18.47 | 41.50 | 24.18 | −84.97 | 122.25 | 85.07 |
| AFNS model: | | | | | | |
| *One-step approach* | | | | | | |
| | −34.50 | 55.27 | 39.68 | −79.52 | 115.80 | 89.08 |
| *Two-step approach* | | | | | | |
| Bank of Canada yields | −33.36 | 52.52 | 36.57 | −91.75 | 126.53 | 96.11 |
| Svensson (1995) yields | −37.15 | 56.87 | 43.32 | −83.10 | 115.90 | 88.01 |
| Nelson and Siegel (1987) yields | −43.76 | 63.02 | 47.74 | −81.90 | 116.92 | 84.79 |
| B-AFNS model: | | | | | | |
| *One-step approach* | | | | | | |
| | −36.87 | 55.80 | 40.03 | −85.34 | 117.83 | 91.56 |
| *Two-step approach* | | | | | | |
| Bank of Canada yields | −33.52 | 52.63 | 36.59 | −90.66 | 128.83 | 96.17 |
| Svensson (1995) yields | −42.66 | 58.49 | 44.95 | −96.84 | 121.68 | 96.95 |
| Nelson and Siegel (1987) yields | −55.96 | 71.21 | 58.50 | −108.40 | 135.47 | 109.94 |
| AFGNS model: | | | | | | |
| *One-step approach* | | | | | | |
| | −30.46 | 51.44 | 33.96 | −72.42 | 107.72 | 77.87 |
| *Two-step approach* | | | | | | |
| Bank of Canada yields | −20.71 | 43.71 | 28.83 | −61.21 | 88.19 | 64.12 |
| Svensson (1995) yields | −48.28 | 68.40 | 55.72 | −106.33 | 131.49 | 109.39 |
| Nelson and Siegel (1987) yields | −56.85 | 75.06 | 61.04 | −105.63 | 129.74 | 105.63 |

This table reports the mean forecasting errors (Mean), the root mean squared forecasting errors (RMSE), and the mean absolute forecasting errors (MAE). All forecasts are computed from DTSMs that are estimated recursively with diagonal matrices for $\mathcal{K}^P$ and $\Sigma$. The forecast errors are reported as the true value minus the model-implied prediction, and all numbers are reported in annual basis points.

**Table 18**
Summary statistics of ten-year yield forecast errors.

| Model | Three-month forecasts | | | Twelve-month forecasts | | |
|---|---|---|---|---|---|---|
| | Mean | RMSE | MAE | Mean | RMSE | MAE |
| Consensus Forecasts | −15.00 | 47.34 | 38.62 | −78.69 | 101.77 | 87.44 |
| AFNS model: | | | | | | |
| *One-step approach* | | | | | | |
| | −24.29 | 43.26 | 35.45 | −70.27 | 86.41 | 72.85 |
| *Two-step approach* | | | | | | |
| Bank of Canada yields, | −29.34 | 46.88 | 38.54 | −86.63 | 100.09 | 87.73 |
| Svensson (1995) yields | −23.68 | 43.00 | 35.12 | −68.94 | 86.21 | 73.51 |
| Nelson and Siegel (1987) yields | −21.34 | 41.02 | 33.20 | −60.05 | 79.39 | 67.83 |
| B-AFNS model: | | | | | | |
| *One-step approach* | | | | | | |
| | −22.84 | 42.50 | 34.46 | −67.65 | 84.65 | 71.19 |
| *Two-step approach* | | | | | | |
| Bank of Canada yields | −26.56 | 45.40 | 37.11 | −81.41 | 97.07 | 84.45 |
| Svensson (1995) yields | −22.70 | 42.62 | 34.32 | −68.92 | 88.57 | 75.49 |
| Nelson and Siegel (1987) yields | −23.57 | 43.71 | 34.83 | −67.09 | 89.09 | 74.97 |
| AFGNS model: | | | | | | |
| *One-step approach* | | | | | | |
| | −19.04 | 41.86 | 34.52 | −62.34 | 81.71 | 68.57 |
| *Two-step approach* | | | | | | |
| Bank of Canada yields | −14.02 | 41.42 | 33.33 | −58.80 | 84.23 | 67.50 |
| Svensson (1995) yields | −29.68 | 47.78 | 39.33 | −82.96 | 96.96 | 84.49 |
| Nelson and Siegel (1987) yields | −28.44 | 45.40 | 37.32 | −74.73 | 88.73 | 76.32 |

This table reports the mean forecasting errors (Mean), the root mean squared forecasting errors (RMSE), and the mean absolute forecasting errors (MAE). All forecasts are computed from DTSMs that are estimated recursively with diagonal matrices for $\mathcal{K}^P$ and $\Sigma$. The forecast errors are reported as the true value minus the model-implied prediction, and all numbers are reported in annual basis points.

## 6. Conclusion

This paper demonstrates the advantages of estimating DTSMs directly on actual bond prices as opposed to the usual approach of using a limited number of synthetic zero-coupon yields. For our real-world empirical comparison, we find that seemingly small differences between two data sets of synthetic yields can affect the estimated parameters in a DTSM. Furthermore, we find that a one-step estimation of a DTSM gives a substantially closer fit to actual coupon bond prices than a two-step approach. Accordingly, the use of synthetic yields in the conventional two-step approach may add some non-negligible noise to the predicted bond prices from an estimated DTSM.

We also explore the finite-sample properties of the one- and two-step approaches in a Monte Carlo study. A novel feature of this simulation exercise is that it is formulated at the level of individual coupon bond prices, so we can account for estimation uncertainty in the construction of the synthetic zero-coupon yields. A key insight from this Monte Carlo study is that the risk-neutral parameters in DTSMs are estimated with smaller biases and greater efficiency with the one-step approach.

There are likely additional advantages to estimating DTSMs directly on coupon bond prices. For example, DTSMs could be augmented with bond-specific characteristics to assess liquidity premiums, as in Fontaine and Garcia (2012) and Andreasen et al. (2018), or simply to get a better estimate of the term premium, as in Christensen and Rudebusch (2019). Similarly, one could expand the work of Pancost (2018) and use the one-step approach to determine the individual bonds that trade cheap or at a premium relative to the overall market and use this information for portfolio management, arbitrage trading, or market surveillance. We leave these and other applications for future research.

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## Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jeconom.2019.04.019.

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