

# Clearing price distributions in call auctions

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We propose a model for price formation in financial markets based on the clearing of a standard call auction with random orders, and verify its validity for prediction of the daily closing price distribution statistically. The model considers random buy and sell orders, placed employing demand- and supply-side valuation distributions; an equilibrium equation then leads to a distribution for clearing price and transacted volume. Bid and ask volumes are left as free parameters, permitting possibly heavy-tailed or very skewed order flow conditions. In highly liquid auctions, the clearing price distribution converges to an asymptotically normal central limit, with mean and variance in terms of supply/demand-valuation distributions and order flow imbalance. By means of simulations, we illustrate the influence of variations in order flow and valuation distributions on price/volume, noting a distinction between high- and low-volume auction price variance. To verify the validity of the model statistically, we predict a year's worth of daily closing price distributions for five constituents of the Eurostoxx 50 index; Kolmogorov–Smirnov statistics and QQ-plots demonstrate with ample statistical significance that the model predicts closing price distributions accurately, and compares favourably with alternative methods of prediction.

**Keywords:** Call auction; Clearing price; Stochastic model; Price formation; Price impact; Closing price prediction

## 1. Introduction

In modern financial markets most securities are traded in continuous double auctions. During the trading day a sell/buy order for a price lower/higher than or equal to the best bid/ask price is immediately executed against the limit order book on the bid/ask side. If a sell/buy-order has a price higher/lower than the best bid/ask, it is added to the limit order book on the ask/bid side. To start and stop trading and determine daily opening and closing prices, standard call auctions are conducted for most securities. In these opening and closing auctions buy and sell orders are collected over a set interval in time, after which a *clearing price*  $X$  is determined to clear the maximal executable volume (Euronext 2019), transacting all against the price  $X$ .

A large part of the market microstructure literature focusses on detailed modelling of continuous double auctions and the limit order book. There are essentially two different lines of work: equilibrium models in which order arrival is governed by decisions of individual agents trying to maximize utility (see *e.g.* Parlour 1998, Foucault 1999, Goettler *et al.* 2005, Rosu 2009, Bressan and Facchi 2013, Bressan

and Wei 2016) and stochastic limit order book models in which order arrival is completely stochastic (see *e.g.* Luckock 2003, Smith *et al.* 2003, Cont *et al.* 2010, Abergel and Jedidi 2013, Cont and De Larrard 2013, Muni Toke 2015b, among many others). Some extensive studies of empirical properties of the limit order book can be found in Biais *et al.* (1995), Challet and Stinchcombe (2001), Bouchaud *et al.* (2002), Potters and Bouchaud (2003). The standard call auction has received less attention: Mendelson (1982) models a call auction in which all orders have size one and are uniformly distributed over some price interval, while buy and sell orders arrive *c.f.* a homogeneous Poisson process. The distribution of transacted volume is derived, together with the clearing price expectation. Technically, this paper is related to the work of Muni Toke (2015a), who gives the full solution of Mendelson's call auction model, deriving distributions for transacted volume, and lower/upper clearing prices, as well as asymptotic distributions in very liquid call auctions.

At the conceptual level, our approach is related to the seminal paper by Smith *et al.* (2003), who consider a statistical model for continuous double auctions assuming *i.i.d.* random order flow, modelled through independent, homogeneous Poisson processes for market orders, limit orders and cancellations with random order-prices from a single, uniform

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valuation distribution. Simulations, dimensional analysis and mean-field approximations then lead to predictions for price volatility, market depth, price-impact function, bid-ask spread and probability/time to fill a limit order.

In this paper we propose a model for price formation in financial markets with a *bid/ask equilibrium equation* at its core, that sets the *clearing price* such as to lead to maximal transacted volume, based on fixed numbers  $N_A, N_B$  of unit-sized sell and buy orders forming *i.i.d.* samples from distinct *valuation distributions*  $F_A$  and  $F_B$ . Due to the randomness in the orders, the equilibrium gives rise to a distribution for  $X|N_A, N_B$ , the clearing price conditional on  $N_A, N_B$ . The shape of the valuation distributions  $F_A, F_B$  and the distribution of the pair  $(N_A, N_B)$  remain unspecified; while the former models order density, the latter permits great freedom of modelling order flow conditions, including auctions in which extreme or skewed liquidity-conditions disturb equilibria and distort clearing prices. We derive closed-form expressions for distributions of clearing prices, jointly with transacted volumes.

Such mechanisms have direct application in the modelling of opening and closing auctions as demonstrated with data from intraday transactions to predict closing price distributions of several constituents of the Eurostoxx 50 index (roughly speaking, this index consists of the 50 main Eurozone companies) in section 6. Extending the argument more informally, we argue that the model applies also in *continuous* trading: if buy/sell orders are accrued over a period of time (and liquidity providers trade with a more-or-less neutral combined inventory) then, at the aggregate level, the detailed process of trading during the period can be interpreted as market-clearing at a price  $X$  with a distribution that depends on valuation distributions  $F_A, F_B$  and the distribution of the pair  $(N_A, N_B)$  that reflects order flow conditions during the interval. If liquidity providers do not trade neutrally, or if we take a limit order book into account, the equilibrium between newly accrued buy and sell orders is perturbed by so-called *excess liquidity*, which can be taken into account in full generality and lies at the heart of many interesting properties associated with real-world phenomena.

The remainder of this article is structured as follows. In section 2, the model is introduced, probability distributions for clearing price and volume are derived and several proposals for the order flow distributions are made. In section 3, we consider auctions in which the number of incoming orders is very large. Asymptotically the clearing price has a normal distribution, which implies that if we approximate continuous trading by a periodically cleared market, the resulting discrete price process follows a Brownian path. This is roughly in support of general pricing models based on the efficient market hypothesis, with mean and variance of the return distribution expressed in terms of the distributions of supply, demand and order flow. In section 4 we explore how changing supply and demand distributions affect the joint distribution of clearing price and transacted volume, leading to a distinction between two different types of auction price variance; one occurring when transacted volumes are high, the other one when these are low. In section 5 we study the model's perspective on the *price impact* of market orders. Remarkably, the model reproduces the

concave price impact functions observed empirically (Hasbrouck 1991, Lillo *et al.* 2003, Donier and Bonart 2015) and explained theoretically (Smith *et al.* 2003, Donier *et al.* 2015, Benzaquen and Bouchaud 2018). In section 6 the model is applied to estimate the distribution of the clearing price of a closing auction, based on the day's transaction data. For 5 (randomly selected) constituents of the Eurostoxx 50 index, it is shown that the model predicts the probability distribution of the closing price with precision, through assessment of QQ-plots and Kolmogorov–Smirnov statistics. For comparison, a more crude alternative method of estimation is assessed on the same basis. It is shown that the market clearing model provides significantly better estimates for clearing price distributions than this more straightforward method. Most important results are summarized in the concluding section 7. Proofs of the theoretical results of sections 2 and 3 as well as notation and conventions are collected in appendix 1.

## 2. Stochastic market clearing

In this section, we introduce the model and derive expressions for the distributions of central quantities in the clearing process.

### 2.1. Supply/demand equilibrium

Let us consider a standard call auction for a given asset. In the auction, buy and sell orders are matched to transact at a clearing price  $X$ , determined in such a way that the total transacted volume is maximal. Suppose that  $N_A$  sell orders are submitted, as well as  $N_B$  buy orders and that every order has equal size (set to one). We assume that participants on both sides of the market formulate their orders independently of each other, according to certain *valuation distributions*. That is, we model the ask prices as an *i.i.d.* sample  $(A_1, \dots, A_{N_A})$  from a *supply (or ask) distribution*  $F_A$  and the bid prices as an *i.i.d.* sample  $(B_1, \dots, B_{N_B})$  from a *demand (or bid) distribution*  $F_B$ . The interpretation of  $F_A$  is as follows: the probability that a randomly selected seller is willing to sell the asset for an ask price  $A \leq x$ , is given by  $F_A(x)$ , for all  $x \in \mathbb{R}$ . Similarly, if we randomly select a buyer, the probability that he is willing to buy the asset for a bid price  $B \leq x$  is given by  $F_B(x)$ . For reasons of technical feasibility, it is assumed that buyers and sellers formulate their quotes independently, *i.e.* bid- and ask-samples are independent *i.i.d.* samples.<sup>†</sup> Denote by  $\mathbb{F}_B$  and  $\mathbb{F}_A$  the empirical distribution functions associated with the bid- and ask-samples  $(B_1, \dots, B_{N_B})$  and  $(A_1, \dots, A_{N_A})$ , that is,

$$\mathbb{F}_B(x) = \frac{1}{N_B} \sum_{j=1}^{N_B} 1_{\{B_j \leq x\}}, \quad \mathbb{F}_A(x) = \frac{1}{N_A} \sum_{i=1}^{N_A} 1_{\{A_i \leq x\}}.$$

<sup>†</sup> Of course these independence assumptions are not realistic: especially when prices fluctuate a lot, it is likely that market participants on both sides of the market react on each other's decisions and hence their quotes are far from independent. However, we argue that, despite these simplifying assumptions, the model can still be interpreted as a reasonable description of price formation in auctions, as is confirmed by the results in section 6.

For every  $x \in \mathbb{R}$ , denote the number of submitted sell orders with a price less than or equal to  $x$  by  $\mathbb{D}_A(x)$  and the number of submitted buy orders with a price greater than  $x$  by  $\mathbb{D}_B(x)$ . As discussed above, the *clearing price*  $X$  is obtained by maximizing the total transacted volume. In terms of the above defined quantities, that implies  $X$  is defined as a solution of the *market clearing equation*  $\mathbb{D}_A(X) = \mathbb{D}_B(X)$ , or,

$$N_A \mathbb{F}_A(X) = N_B (1 - \mathbb{F}_B(X)), \quad (1)$$

which expresses that the transacted volume is maximized at (any) price  $X$  where the supply curve  $\mathbb{D}_A$  and the demand curve  $\mathbb{D}_B$  intersect. Consider the following definition.

**DEFINITION 2.1** For a given sell order sample  $(A_1, \dots, A_{N_A})$  from  $F_A$  and a buy order sample  $(B_1, \dots, B_{N_B})$  from  $F_B$ , the corresponding *clearing price*  $X$  is defined by

$$X = \inf\{x \in \mathbb{R} : \mathbb{D}_A(x) \geq \mathbb{D}_B(x)\}.$$

**REMARK 2.2** It should be noted that there are issues of existence and uniqueness of solutions to (1). Firstly, when the bid and ask-samples are such that,

$$B_1 \leq \dots \leq B_{N_B} < A_1 \leq \dots \leq A_{N_A}$$

there is *no solution* where  $\mathbb{D}_A$  and  $\mathbb{D}_B$  intersect. Secondly, it is possible that there is an interval  $[\underline{X}, \bar{X}]$  of possible

clearing prices for which  $\mathbb{D}_A = \mathbb{D}_B$ , ruining uniqueness. Both issues are addressed in Definition 2.1, much in the same way quantiles of a distribution are defined (see figure 1 for an illustration).

In subsequent subsections, closed-form expressions are provided for the probability distributions (conditional, given  $(N_A, N_B)$ ) of several important market quantities, like clearing price  $X$  and transacted volume  $V$ .

While this stochastic model of price formation is based on the mechanism of a call auction, the clearing price also has an interpretation for continuous trading. To appreciate the relation, the process of continuous bidding and transacting (with matching of orders as an instantaneous but momentary form of clearing) should be viewed in an aggregated form over an interval of time  $I$ . During any such interval the numbers of buyers and sellers must still be equal, and that is exactly what equation (1) expresses. Then, at the aggregate level, the detailed, step-by-step process of trading during the interval may be modelled equivalently (or in close approximation) as market clearing at a clearing price  $X$  associated with the interval  $I$ .

For both the auction and the continuous trading interpretations, the following applies: if  $F_A$ ,  $F_B$  and the distribution of  $(N_A, N_B)$  are chosen in an appropriate way, the clearing price  $X$  can be interpreted as a true, underlying price for the asset,

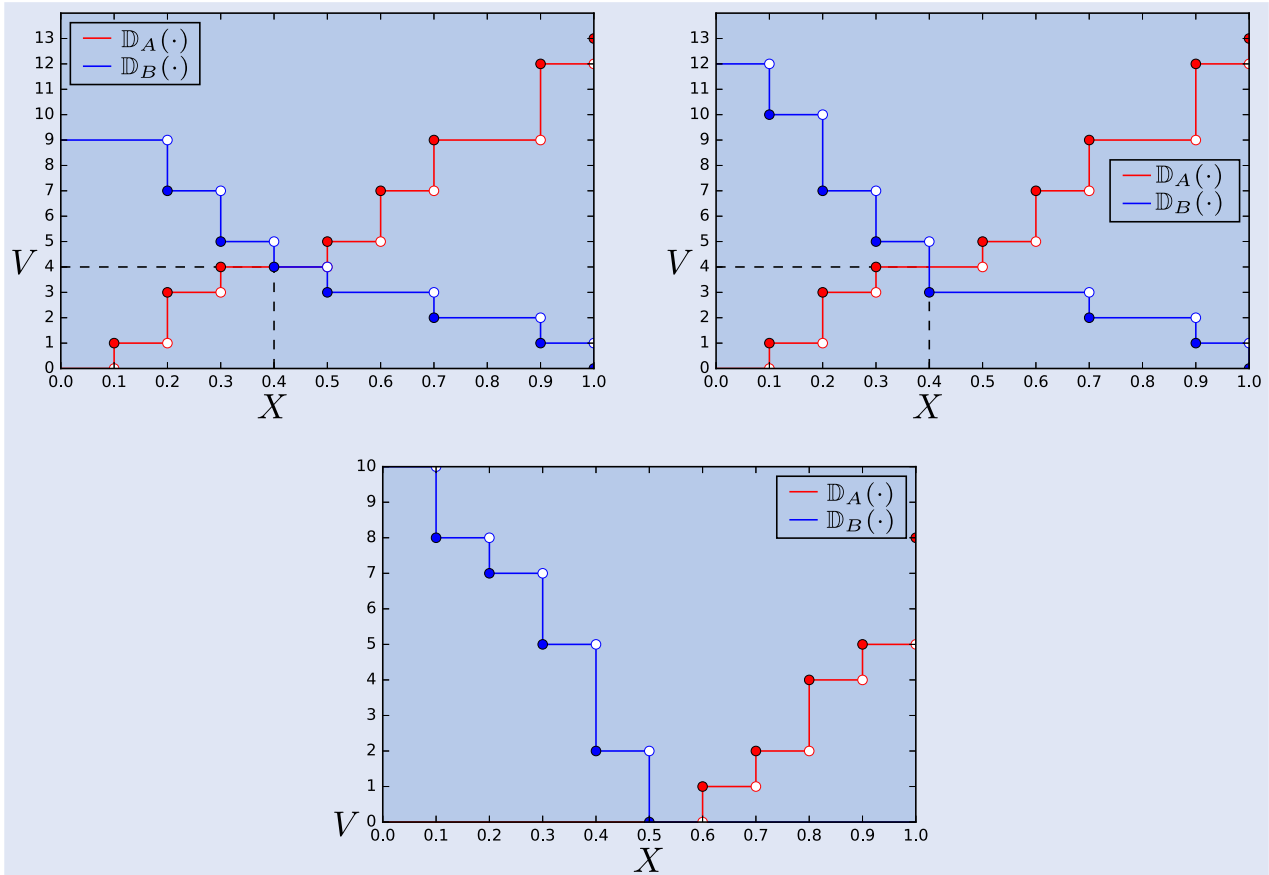


Figure 1. Three possible examples of the supply curve  $\mathbb{D}_A(\cdot)$  (the increasing (red) step function) and the demand curve  $\mathbb{D}_B(\cdot)$  (the decreasing (blue) step function). Left upper panel: a situation in which there is no unique point of intersection, note the position of  $X$  at the left of the interval where  $\mathbb{D}_A = \mathbb{D}_B$ . Right upper panel: a situation in which there is a unique intersection point, but  $\mathbb{D}_A(X) > \mathbb{D}_B(X)$ . Lower panel: A situation in which no transactions are possible, note the position of  $X$  at the highest placed buy order. Note also the position of the transacted volume ( $V$ ) after clearing (this quantity is defined later on).

associated not with any specific point in time but with the whole interval  $I$  (to relate such an *interval-price* to timed market prices, one may think of  $X$  loosely as the price *at a time  $T$  randomly sampled from  $I$* ). To justify the fixed distributions  $F_A, F_B$  and the independence assumptions on the order samples,  $I$  must not be too long due to possible non-stationarity but long enough statistically, aggregating a sufficiently large numbers of orders. Furthermore, the stochastic behaviour of  $\mathbb{F}_A$  and  $\mathbb{F}_B$  (that is, the randomness these quantities represent) must reflect the uncertainty in the incoming orders on the respective sides of the market during the time interval  $I$  with some accuracy. Similarly the distribution chosen for liquidity  $(N_A, N_B)$  must reflect the uncertainty in actual market liquidity conditions during the interval  $I$ . If these conditions are met, the model will provide an accurate reflection of the stochastic aspects of market clearing, and thereby, of price formation.

In the setting of continuous trading, it makes sense to measure time in terms of market events rather than physical time, in particular regarding the interval  $I$ . Combining with the interpretation of  $X$  as a true, underlying price for the interval  $I$ , we can fix  $N = N_A + N_B$  and interpret the resulting clearing price  $X$  as a true, underlying asset price associated with the interval spanned by the next  $N$  orders.

The unrestricted freedom in the choices for  $F_A, F_B$  and the distribution of  $(N_A, N_B)$  enables use of empirical fits for these distributions from previous intervals. It is also possible to make definite, default choices for these quantities: for instance, choosing independent Poisson distributions for  $N_A$  and  $N_B$  would correspond to the assumption of Poisson order flow, which is omnipresent in the literature (see, among many others, Smith *et al.* 2003, Cont *et al.* 2010, Abergel and Jedidi 2013, Cont and De Larrard 2013, Muni Toke 2015b for examples in context of continuous double auctions and Mendelson 1982, Muni Toke 2015a for examples in the standard call auction). In section 2.4 we consider further possible choices for the distribution of  $(N_A, N_B)$  and the model properties implied.

## 2.2. Distribution of clearing price and volume

In this subsection we derive the probability distributions of price and price-volume, resulting from the equilibrium equation (1), without and with a limit order book. We concentrate on the marginal distribution of the clearing price  $X$  only first, given by the following theorem (proved in the appendix).

**THEOREM 2.3 (Clearing price distribution)** *The distribution of the clearing price  $X$ , conditional on  $N_A$  and  $N_B$ , is given by,*

$$\begin{aligned} \mathbb{P}(X \leq x | N_A, N_B) &= \sum_{k=0}^{N_A} \sum_{l=0}^{N_B \wedge k} \binom{N_A}{k} F_A(x)^k (1 - F_A(x))^{N_A-k} \\ &\quad \times \binom{N_B}{l} (1 - F_B(x))^l F_B(x)^{N_B-l}. \end{aligned}$$

However, it is also possible to derive the joint distribution of clearing price and transacted volume, which is defined next.

**DEFINITION 2.4** The transacted volume  $V$  corresponding to the clearing price  $X$ , is defined by  $V = \mathbb{D}_A(X)$ .

**REMARK 2.5** The quantity  $V = \mathbb{D}_A(X)$  should be interpreted as the maximal number of orders that can be matched in clearing. In the context of a call auction, it is the total volume that is transacted. If  $F_A$  and  $F_B$  are continuous distributions, there is almost surely a unique point where  $\mathbb{D}_A$  and  $\mathbb{D}_B$  intersect, hence  $V = \mathbb{D}_A(X) = \mathbb{D}_B(X)$ . In the case of a discrete price-axis it is possible that  $\mathbb{D}_A(X) > \mathbb{D}_B(X)$ , which means that the volume  $\mathbb{D}_A(X)$  is not completely matched (see the upper right panel of figure 1). As a convention, we neglect such discretization effects and continue with Definition 2.4 (compare with the resolution to the ambiguity for  $X$ , as an infimum, see Remark 2.2).

In the next theorem (proved in the appendix), an explicit expression for the joint distribution of  $X$  and  $V$  is provided. It is assumed that the price-axis  $\mathcal{X}$  is a discrete set,  $\mathcal{X} := \{x_0, x_0 + \delta, \dots\}$ , where  $\delta$  is the *ticksize*.

**THEOREM 2.6 (Joint clearing price/transacted volume distribution)**

*The joint distribution of clearing price  $X$  and transacted volume  $V$ , conditional on  $N_A$  and  $N_B$ , is given by,*

$$\begin{aligned} \mathbb{P}(X \leq x, V \leq v | N_A, N_B) &= \sum_{u=0}^v \sum_{k=0}^u \sum_{l=0}^k \left[ \binom{N_B}{l} \binom{N_A}{k, u-k, N_A-u} \right. \\ &\quad \times (1 - F_B(x))^l F_B(x)^{N_B-l} F_A(x)^k (F_A(x + \delta) \\ &\quad \left. - F_A(x))^{u-k} (1 - F_A(x + \delta))^{N_A-u} \right] \\ &\quad + \sum_{y \in \mathcal{X}, y \leq x} \sum_{u=v+1}^{N_A} \sum_{k=0}^u \sum_{l=0}^k \left[ \binom{N_B}{l} \binom{N_A}{k, u-k, N_A-u} \right. \\ &\quad \times (1 - F_B(y))^l F_B(y)^{N_B-l} \\ &\quad \times F_A(y)^k (F_A(y + \delta) - F_A(y))^{u-k} (1 - F_A(y + \delta))^{N_A-u} \left. \right] \\ &\quad - \sum_{y \leq x} \sum_{l=0}^{N_B} \sum_{k=l \vee v+1}^{N_A} \binom{N_A}{k} F_A(y)^k (1 - F_A(y))^{N_A-k} \\ &\quad \times \binom{N_B}{l} F_B(y)^{N_B-l} (1 - F_B(y))^l. \end{aligned} \quad (2)$$

## 2.3. Excess liquidity

There are several variations possible on the definition of the clearing price  $X$  as given above: to start with, during continuous trading, exchanges often offer an open limit order book, which contains all visible limit orders on ask-side and bid-side. Denote by  $L_A(x)$  the total volume on the ask-side of the limit order book for a price less than or equal to  $x$ . Similarly, denote by  $L_B(x)$  the total volume on the bid-side of the limit order book for a price above  $x$ . Then Definition 2.1 of the



clearing price  $X$  is adapted to,

$$X = \inf\{x \in \mathbb{R} : \mathbb{D}_A(x) + L_A(x) \geq \mathbb{D}_B(x) + L_B(x)\},$$

corresponding to an adapted market clearing equation that takes the limit order book into account:

$$N_A \mathbb{F}_A(X) + L_A(X) = N_B(1 - \mathbb{F}_B(X)) + L_B(X), \quad (3)$$

Note that  $x \mapsto L_A(x)$  and  $x \mapsto L_B(x)$  are non-stochastic quantities and that for any  $x$ , either  $L_A(x)$  or  $L_B(x)$  is equal to zero (as, otherwise, the book could be cleared further by matching the overlapping orders).

To generalize, we include *excess liquidity* as any sort of liquidity that plays a role in the clearing process, but does not originate from the quoting process as described by  $\mathbb{F}_A$  and  $\mathbb{F}_B$ . As such, we view excess liquidity as an external influence.

**DEFINITION 2.7** If the clearing price  $X$  is defined by the equation,

$$N_A \mathbb{F}_A(X) = N_B(1 - \mathbb{F}_B(X)) + \Delta(X), \quad (4)$$

where  $\Delta : \mathcal{X} \rightarrow \mathbb{Z}$  is a right-continuous, non-increasing function, then  $\Delta$  is called the *excess liquidity*.

Excess liquidity takes the market out of the ‘pure’ equilibrium given by  $\mathbb{D}_A(X) = \mathbb{D}_B(X)$ . For example, inclusion of the limit order book is possible through  $\Delta(x) = L_B(x) - L_A(x)$ . Positive values of  $\Delta(x)$  correspond to an *excess demand* and negative values of  $\Delta(x)$  mean an *excess supply*. Another example of excess liquidity is the arrival of a market order. A sell market order of size  $\omega \in \mathbb{N}$  corresponds to the constant function  $\Delta = -\omega 1_{\mathcal{X}}$ , while a buy market order is described by the function  $\Delta = \omega 1_{\mathcal{X}}$ . Similarly, a buy limit order with limit price  $b$  can be described by  $\Delta = \omega 1_{[x_0, b]}$  and a sell limit order with limit price  $a$  by  $\Delta = -\omega 1_{[a, \infty)}$ .

Lemma A.1 can be re-derived with excess liquidity, in order to obtain the equivalence  $X \leq x \Leftrightarrow \mathbb{D}_A(x) \geq \mathbb{D}_B(x) + \Delta(x)$ . Exactly like in the proof of Theorem 2.3, this leads to the distribution of the clearing price, conditional on  $N_A$  and  $N_B$ , as stated in the next proposition.

**PROPOSITION 2.8** (Clearing price distribution in case of excess liquidity) *When excess liquidity  $x \mapsto \Delta(x)$  plays a role, the clearing price distribution conditional on  $N_A, N_B$ , is given by*

$$\begin{aligned} \mathbb{P}(X \leq x | N_A, N_B) &= \sum_{k=0}^{N_A} \sum_{l=0}^{U(k, x)} \binom{N_A}{k} F_A(x)^k (1 - F_A(x))^{N_A - k} \\ &\quad \times \binom{N_B}{l} (1 - F_B(x))^l F_B(x)^{N_B - l}, \end{aligned}$$

where  $U(k, x) = (k - \Delta(x)) \wedge N_B$ .

Note that the limit order book makes an appearance *only* in the summation bound, leaving the binomial character of the equilibrium distribution intact.

## 2.4. Order flow distributions

All the distributions derived in the previous subsections, are conditional on the pair  $(N_A, N_B)$ . In this subsection we discuss some possibilities for the distribution of  $(N_A, N_B)$  (the so-called *order flow distribution*) and their consequences for clearing price distributions. The common assumption in the (early) literature is what is called *Poisson order flow*: for continuous double auctions (Smith *et al.* 2003, Cont *et al.* 2010, Abergel and Jedidi 2013, Cont and De Larrard 2013, Muni Toke 2015b) and call auctions (Mendelson 1982, Muni Toke 2015a), Poisson order flow follows from assumed independent Poisson processes for the arrival of buy and sell orders. Here, we would take,

$$(N_A, N_B) \sim \text{Pois}(\mu_A T) \times \text{Pois}(\mu_B T),$$

for Poisson rates  $\mu_A, \mu_B$  and a given interval duration  $T$  to achieve the same.

However, in this setting it makes sense to consider more general models for order flow. Assume again that we consider an interval in which  $N$  new orders arrive. Fix  $N_A + N_B =: N \in \mathbb{N}$  and leave the distribution of  $N_A$  open for choice. A reasonable choice would be to choose  $N_A$  according to *binomial order flow*, i.e.

$$N_A \sim \text{Bin}(N, p),$$

for some  $p \in (0, 1)$  representing order flow imbalance. Taking,

$$(N_A, N_B) \sim \text{Pois}(\mu_A) \times \text{Pois}(\mu_B),$$

is equivalent to,

$$N = N_A + N_B \sim \text{Pois}(\mu_A + \mu_B), \quad N_A | N \sim \text{Bin}(N, p),$$

for  $p = \mu_A / (\mu_A + \mu_B)$ .

Both Poisson and binomial proposals express the conviction that *order flow imbalance*  $\alpha := N_A / N$  does not display great stochastic fluctuation and lies close to its expectation  $p$ , especially for greater values of  $N$  due to the central limit theorem. This makes it difficult to capture market phenomena that are due to fat tails in the order flow distribution, to describe more extreme, yet common market conditions. Hence our third proposal: we consider *beta order flow imbalance*,

$$N_A = \alpha N, \quad \alpha \sim \text{Beta}(\beta_1, \beta_2).$$

Choice of the parameters  $\beta_1, \beta_2$  permits great modelling freedom. For instance, if we expect the order flow on the bid- and ask-side to be roughly balanced, it is appropriate to set  $\beta_1 = \beta_2$ . If we expect the market to be out of balance (e.g. while trending), we may choose  $\beta_1 > \beta_2$  when we expect more supply than demand, and vice versa. Perhaps most interesting is the scale of the betas: if  $\beta_1, \beta_2 < 1$  we induce the fat tails not seen in Poisson or binomial order flow, while  $\beta_1, \beta_2 \gg 1$  will lower the variance and bring  $\alpha$  close to its expectation  $\beta_1 / (\beta_1 + \beta_2)$ .

To shed more light on the influence of the order flow distribution on the clearing price distribution, we consider a simple example. To focus on order flow, we make the trivial

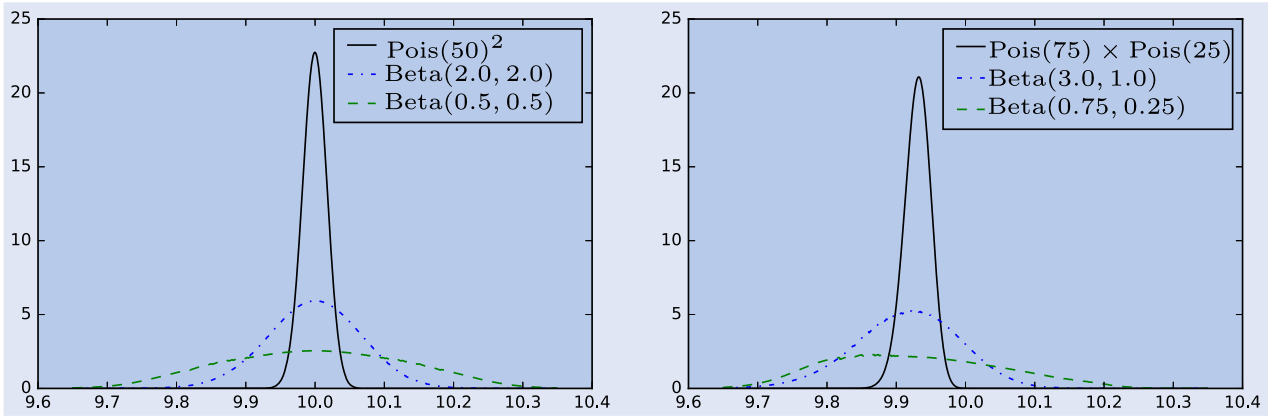


Figure 2. Clearing price density  $f_X$ , for  $F_A = F_B = \Phi_{10,0.1}$  and various balanced (left panel) and unbalanced (right panel) choices for the order flow distribution. The solid black line is the density for Poisson order flow, while the dashed blue line and the dashed green line correspond to beta order flow imbalance, for fixed  $N = 100$ .

choices for the other parameters:  $F_A(\cdot) = F_B(\cdot) = \Phi_{\mu,\sigma}(\cdot)$  for  $\mu = 10$  and  $\sigma = 0.1$ . To appreciate the effects of order flow on clearing price distributions, consider figure 2, the probability density  $f_X$  of the clearing price is plotted for various balanced (left panel) and unbalanced (right panel) choices of the order flow distribution. As expected,  $f_X$  centres around 10 in all balanced cases and around a lower location for the unbalanced cases. It is seen that Poisson order flow leaves little room for variation in the values of  $N_A$  and  $N_B$ , causing the density to peak relatively sharply. By contrast, beta order flow imbalance leads to more liquidity-driven uncertainty in the clearing price.

### 3. The high-liquidity limit

In this section, we provide the asymptotic clearing price distribution in limit of infinite liquidity. To be more precise, denote  $N = N_A + N_B$ , let  $N_A = \alpha N$ ,  $N_B = (1 - \alpha)N$  for some constant  $0 < \alpha < 1$  we refer to as *order flow imbalance* and consider the limit  $N \rightarrow \infty$ . We take a continuous price-axis  $\mathcal{X} = [x_0, \infty)$  and assume that the distribution functions  $F_A$  and  $F_B$  are strictly increasing, describing measures that are absolutely continuous with respect to the Lebesgue measure, with densities denoted  $f_A$  and  $f_B$ . Let  $X$  denote a solution to  $N_A F_A(X) = N_B(1 - F_B(X)) + \Delta(X)$ , with possibly non-zero excess liquidity  $\Delta$ . Denote by  $x_E$  the *real equilibrium price* which is the (non-random) price uniquely defined by the equilibrium equation,

$$\alpha F_A(x_E) = (1 - \alpha)(1 - F_B(x_E)). \quad (5)$$

According to the following theorem (the proof of which can be found in the appendix), the clearing price  $X$  is in the limit distributed according to a normal distribution centred on  $x_E$  with variance that depends on  $f_A$  and  $f_B$ .

**THEOREM 3.1** (High-liquidity clearing price distribution) *Let  $X$  be the clearing price in case of possible excess liquidity  $\Delta$ . Assume that  $F_A$  and  $F_B$  are strictly increasing and absolutely continuous with respect to the Lebesgue measure with densities  $f_A$  and  $f_B$ . Additionally, assume that excess liquidity scales*

*with  $N$  as  $\Delta(\cdot) = \sqrt{N}D(\cdot)$ , for some continuous and bounded function  $D : \mathcal{X} \rightarrow \mathbb{R}$ . Then, as  $N \rightarrow \infty$ ,*

$$\sqrt{N}(X - x_E) \xrightarrow{w} N(\mu(x_E), \sigma^2(x_E)), \quad (6)$$

*where the asymptotic mean and standard deviation are given by,*

$$\begin{aligned} \mu(x_E) &= \frac{D(x_E)}{\alpha f_A(x_E) + (1 - \alpha)f_B(x_E)}, \\ \sigma(x_E) &= \frac{\tau(x_E)}{\alpha f_A(x_E) + (1 - \alpha)f_B(x_E)}, \end{aligned}$$

*for*

$$\begin{aligned} \tau^2(x_E) &= \alpha F_A(x_E)(1 - F_A(x_E)) \\ &\quad + (1 - \alpha)F_B(x_E)(1 - F_B(x_E)), \end{aligned}$$

*and  $x_E$  is the real equilibrium price.*

Consider a standard call auction in which the number of orders collected is very large. The clearing price distribution is then closely concentrated around  $x_E$  and has a width proportional to  $1/\sqrt{N}$ . So the model confirms the intuition that large auctions lead to accurate price discovery and adds that this accuracy is inversely proportional to the square root of the number of orders. Non-zero excess liquidity of order  $\sqrt{N}$  biases  $X$  away from  $x_E$ , however, this bias is also proportional to  $1/\sqrt{N}$ . So the model says that in highly liquid auctions or markets, external influence in the form of excess liquidity  $\Delta$  must be of order larger than  $\sqrt{N}$  to force (the distribution of) the clearing price away from the real equilibrium price  $x_E$ . Furthermore, the shift caused by the excess liquidity is inversely proportional to a convex combination of  $f_A$  and  $f_B$ , hence price impact will be larger if the density of orders around the equilibrium price is low.

Next consider the case of continuous trading of a stock in an interval, during which supply and demand are described by the distributions  $F_A$  and  $F_B$ , and by order flow imbalance  $\alpha \in (0, 1)$ . Assume that the number of incoming orders during the interval is very large, so that the limit of Theorem 3.1

forms a good approximation for the clearing price distribution. In the absence of excess liquidity, the distribution of the clearing price associated with the interval is a sharply peaked normal distribution centred at the real equilibrium price. If we repeat this argument for consecutive intervals (possibly with changing  $F_A$ ,  $F_B$  and  $\alpha$ ) and approximate continuous trading by a periodically cleared market, the price process becomes a discrete Brownian path (possibly trending if we add excess liquidity). In many stochastic models for pricing, this type of stochastic process is postulated; by contrast, here, the Brownian path emerges from the central limit (in the form of Donsker's theorem, see the proof of Theorem 3.1) and the parameters of this Brownian path have an interpretation in terms of supply, demand and order flow imbalance.

As argued after Definition 2.7, the model invites the interpretation of the limit book as excess liquidity, in a market made around the equilibrium price  $x_E$  of many new orders. Think, for instance, of a situation where new orders originate from liquidity providers primarily; if the location of their equilibrium price distribution undergoes a small but quick jump (for example because of a sudden change in the price of a hedging index future), the result of Theorem 3.1 suggests that the limit book obstructs immediate market correction: if we consider a limit book of order greater than  $N^{1/2}$  over an interval of order  $N$  orders, the location of the clearing price distribution is expected to differ from the liquidity providers' new  $x_E$  on scales larger than  $N^{-1/2}$ . To re-phrase that slightly and more crudely, the model suggests that a limit book offering total liquidity of order  $L$  stabilizes the market price versus fluctuations in valuation distributions or order flow imbalance, if those fluctuations vary quickly enough to neutralize over a duration of order  $L^2$  (where time is measured in volume offered).<sup>†</sup>

Finally, note that the variance in (6) is not only dependent on  $\tau^2(x_E)$  in the way one might expect, but like the location in (6), it is inverse proportional to a liquidity-weighted convex combination of  $f_A$  and  $f_B$ , evaluated at the real equilibrium price. So the volatility of the Brownian path (as well as the influence of excess liquidity) goes down in ranges where orders are concentrated and goes up in ranges where orders are sparse. Consequently, the Brownian path has long occupation times in ranges where orders are dense.

**REMARK 3.2** Muni Toke (2015a) derives a normal asymptotic distribution of the clearing price in a similar setting, under the assumption of Poisson order flow (*i.e.*  $N \sim \text{Pois}(\lambda T)$ , where  $\lambda T \rightarrow \infty$ ,  $N_A \sim \text{Bin}(N, \alpha)$ ) and  $F_A = F_B = F$ . The Poisson order flow with  $\lambda T \rightarrow \infty$  represents a fixed randomization of the deterministic  $N \rightarrow \infty$  discussed here. However in the proof he firstly considers (in our notation) fixed  $N$  and  $N_A = \alpha N$  and finds an asymptotic normal distribution for  $X$ , with mean  $F^{-1}(1 - \alpha)$  and standard deviation  $\sqrt{\alpha(1 - \alpha)}f(F^{-1}(1 - \alpha))^{-1}$ . Setting  $F_A = F_B = F$  and  $D = 0$  in our result, the solution to (5) is  $x_E = F^{-1}(1 - \alpha)$ ,  $\tau^2(x_E) = \alpha(1 - \alpha)$  and  $\sigma(x_E) = \sqrt{\alpha(1 - \alpha)}f(x_E)^{-1}$ .

<sup>†</sup> Note that these  $\sqrt{N}$  scales originate from the central limit (more specifically, Donsker's theorem) and that is it a topic of further research to verify these exact scales empirically.

#### 4. Supply-demand distributions, price and volume

In Theorem 2.6 we derived the joint distribution of the clearing price  $X$  and the corresponding transacted volume  $V$ , given supply and demand distributions  $F_A$ ,  $F_B$  and volumes  $N_A$ ,  $N_B$ . In this subsection we explore the dependence of the distribution of  $(X, V)$  on  $F_A$  and  $F_B$ . We shall fix  $N_A$  and  $N_B$  as equal constants ( $N_A = N_B = 50$  in the examples below). It is also recalled that the distribution for  $(X, V)$  was derived in a setting with a discrete price axis  $\mathcal{X}$  with tick-size  $\delta > 0$  (below, we take  $\delta = 0.01$ ); normal distributions are discretized accordingly.

##### 4.1. Varying consensus between bid- and ask-side

The supply and demand valuation distributions  $F_A$ ,  $F_B$  express a difference of opinion concerning the valuation of the asset. We first consider how shifts of locations for  $F_A$ ,  $F_B$  influence the joint distribution of clearing price and volume.

We consider three different choices of the supply and demand distributions, denoted  $F_A$ ,  $F_B$ ,  $\bar{F}_A$ ,  $\bar{F}_B$  and  $\tilde{F}_A$ ,  $\tilde{F}_B$ :

$$\begin{aligned} F_A &= \Phi_{10.1,0.1}, F_B = \Phi_{9.9,0.1}, \\ \bar{F}_A &= \Phi_{10.05,0.1}, \bar{F}_B = \Phi_{9.95,0.1}, \\ \tilde{F}_A &= \tilde{F}_B = \Phi_{10,0.1}. \end{aligned} \quad (7)$$

The first case represents a relatively large difference between the locations of supply and demand distributions, while the second case represents a small difference, and the third complete consensus. In all three cases, the real equilibrium price is  $x_E = 10$ , however, as can be seen from the left panel of figure 3,

$$F_A(x_E) < \bar{F}_A(x_E) < \tilde{F}_A(x_E). \quad (8)$$

Figure 4 shows the distributions of price-volume in these three cases and suggests the following, intuitively reasonable mechanism: as the locations of supply and demand distributions diverge, marginally the transacted volume drops, while the width of the price marginal increases. Note that the location on the price-axis does not change, as all three  $(X, V)$ -distributions are centred around  $x_E = 10$ . Referring to Theorem 3.1, the result reflects the ordering expressed by (8): in the high-liquidity limit,  $X$  lies close to  $x_E$  and  $V = N_A \mathbb{F}_A(X)$  (respectively,  $\bar{V} = N_A \bar{\mathbb{F}}_A(X)$ ,  $\tilde{V} = N_A \tilde{\mathbb{F}}_A(X)$ ) lies close to  $N_A F_A(x_E)$  (respectively,  $N_A \bar{F}_A(x_E)$ ,  $N_A \tilde{F}_A(x_E)$ ). Similar arguments regarding the ordering of densities (see also the right panel of figure 3),

$$f_A(x_E) + f_B(x_E) < \bar{f}_A(x_E) + \bar{f}_B(x_E) < \tilde{f}_A(x_E) + \tilde{f}_B(x_E),$$

provide an asymptotic explanation for the observed increase in price uncertainty (*c.f.* the denominator of the variance in (A7); the numerator is bounded and plays no role here). To re-phrase and summarize: when consensus between bid- and ask-sides increases, transacted volume increases and price uncertainty decreases.

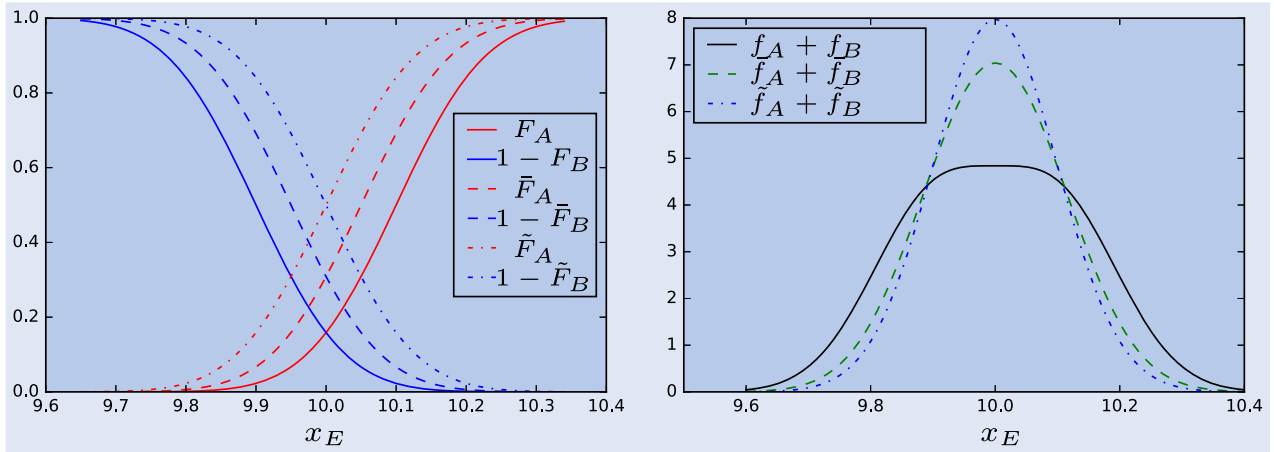


Figure 3. Left panel: distribution functions of supply/demand. Solid lines  $F_A, F_B$ ; dashed lines  $\tilde{F}_A, \tilde{F}_B$ ; dashed-dotted lines  $\tilde{\tilde{F}}_A, \tilde{\tilde{F}}_B$ . Right panel: sums of densities of supply/demand. Solid line  $f_A + f_B$ ; dashed line  $\tilde{f}_A + \tilde{f}_B$ ; dashed-dotted line  $\tilde{\tilde{f}}_A + \tilde{\tilde{f}}_B$ .

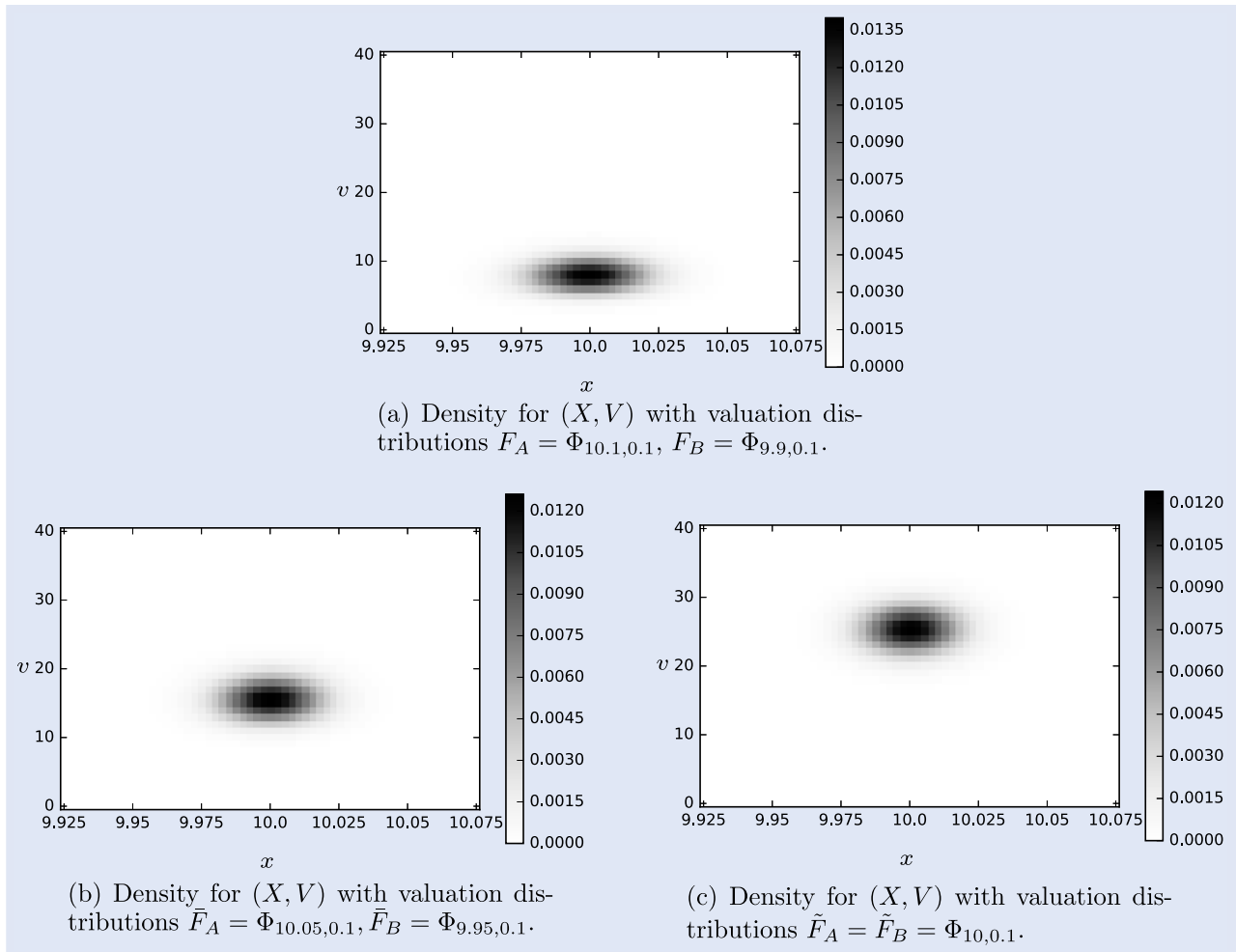


Figure 4. The influence of consensus between bid- and ask-side of the market on the distribution of  $(X, V)$ . Note: as the locations of supply and demand distributions diverge, transacted volume drops, while price uncertainty increases. (a) Density for  $(X, V)$  with valuation distributions  $\tilde{F}_A = \Phi_{10.05, 0.1}$ ,  $\tilde{F}_B = \Phi_{9.95, 0.1}$ . (b) Density for  $(X, V)$  with valuation distributions  $\tilde{\tilde{F}}_A = \tilde{\tilde{F}}_B = \Phi_{10, 0.1}$ .

#### 4.2. Increased uncertainty among market participants

Here we investigate the influence of valuation uncertainty among market participants on the distribution of clearing price and transacted volume: we consider three different choices of

the supply and demand distributions, denoted  $F_A, F_B, \tilde{F}_A, \tilde{F}_B, \tilde{\tilde{F}}_A, \tilde{\tilde{F}}_B$ ,

$$F_A = \Phi_{10.1, 0.1}, F_B = \Phi_{9.9, 0.1},$$



$$\begin{aligned}\bar{F}_A &= \Phi_{10.1,0.2}, \bar{F}_B = \Phi_{9.9,0.2}, \\ \tilde{F}_A &= \Phi_{10.1,0.3}, \tilde{F}_B = \Phi_{9.9,0.3}.\end{aligned}\quad (9)$$

The locations of supply and demand distributions are maintained, while their variances are increased, reflecting growing uncertainty in valuation among individual market participants. Again, in all three cases, the real equilibrium price is  $x_E = 10$  and (8) continues to hold (see the left panel of figure 5).

Panels (b), (c) and (d) of figure 6 show the distributions of price-volume in these three cases and suggests the following, reasonable-sounding (but incomplete, see below) rule: as the variance of the valuation distributions increases, marginally both transacted volume and price uncertainty increase. Note that the location on the price-axis does not change, as all three  $(X, V)$ -distributions have marginals centred around  $x_E = 10$ . The asymptotic argument for the observed ordering  $V < \bar{V} < \tilde{V}$  continues to hold. Note in the right panel of figure 5, however, that with locations and variances as chosen,

$$f_A(x_E) + f_B(x_E) > \bar{f}_A(x_E) + \bar{f}_B(x_E) > \tilde{f}_A(x_E) + \tilde{f}_B(x_E),$$

so that asymptotic variance of the clearing price *increases* when valuations become more widely spread (referring again to the variance in (6)).

But this explains only half of the mechanism that the model ascribes to the relation between valuation uncertainty and auction price variance. To appreciate the other half, consider a fourth different pair  $\underline{F}_A, \underline{F}_B$  of supply and demand distributions that reflects less valuation uncertainty among market participants, defined by

$$\underline{F}_A = \Phi_{10.1,0.075}, \quad \underline{F}_B = \Phi_{9.9,0.075}.$$

As can be seen from the right panel of figure 5, this choice of valuation distributions satisfies

$$f_{\underline{A}}(x_E) + f_{\underline{B}}(x_E) < f_A(x_E) + f_B(x_E),$$

which implies that the asymptotic variance of the clearing price also increases when we lower the variance of the valuation distributions. This is also confirmed by panel (a) of

figure 6, where the distribution of price-volume for  $\underline{F}_A, \underline{F}_B$  is shown. To explain this observed inversion, consider  $F_A$  and  $F_B$  that are two normal distributions of equal variance  $\sigma^2 > 0$ , located at  $\mu_1, \mu_2 \in \mathbb{R}$ . Reasoning again asymptotically, the denominator of the expression for the variance in (A7) equals,

$$f_A(x_E) + f_B(x_E) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \exp\left(-\frac{1}{2} \frac{(\mu_1 - \mu_2)^2}{\sigma^2}\right). \quad (10)$$

As a function of  $\sigma$ , (10) has a *maximum* at  $\sigma = \frac{1}{2}|\mu_1 - \mu_2|$  (see figure 7 for an example), which means that asymptotic variance of the clearing price is *minimal* at said level of valuation uncertainty  $\sigma$ . When  $\sigma$  rises above  $\frac{1}{2}|\mu_1 - \mu_2|$ , as in figure 6, panels (c) and (d), auction price variance increases; perhaps somewhat surprisingly, when  $\sigma$  decreases below  $\frac{1}{2}|\mu_1 - \mu_2|$ , as in figure 6 panel (a), auction price variance also increases. The heuristic reason for this inversion is as follows: when consensus between the bid- and ask-side of market is very low (large  $|\mu_1 - \mu_2|$ ) and valuation uncertainty among market participants is minimal (small  $\sigma$ ), orders around  $x_E$  are very scarce, so that clearing prices are based on *small numbers of matchable orders*, therefore displaying *high variance*; as the uncertainty in order prices on both sides increases, more orders appear around  $x_E$ , lowering the variance of the clearing price. The added valuation uncertainty ‘unlocks’ an otherwise *illiquid market*, in which buyers and sellers rarely cross. So in a market with illiquidity-driven price movements, raised valuation uncertainty aids accurate price discovery.

Combination with the previous subsection invites the following, intuitively reasonable conclusion: *observation of high levels of price variance can be driven by illiquidity or by valuation uncertainty among market participants; observation of the price and its fluctuations alone does not distinguish between those cases. To differentiate one must involve transacted volume, which is moderate when price variance is minimal, low in illiquid markets and high in markets with valuation uncertainty-driven price variance.*

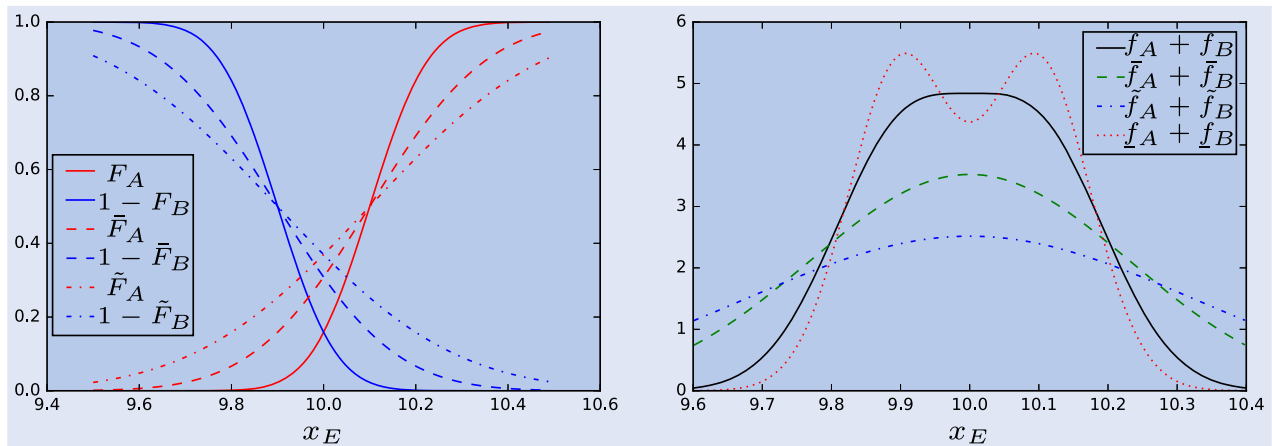


Figure 5. Left panel: distribution functions of supply/demand. Solid lines  $F_A, F_B$ ; dashed lines  $\bar{F}_A, \bar{F}_B$ ; dashed-dotted lines  $\tilde{F}_A, \tilde{F}_B$ . Right panel: sums of densities of supply/demand. Solid line  $f_A + f_B$ ; dashed line  $\bar{f}_A + \bar{f}_B$ ; dashed-dotted line  $\tilde{f}_A + \tilde{f}_B$ ; dotted line  $f_{\underline{A}} + f_{\underline{B}}$ .

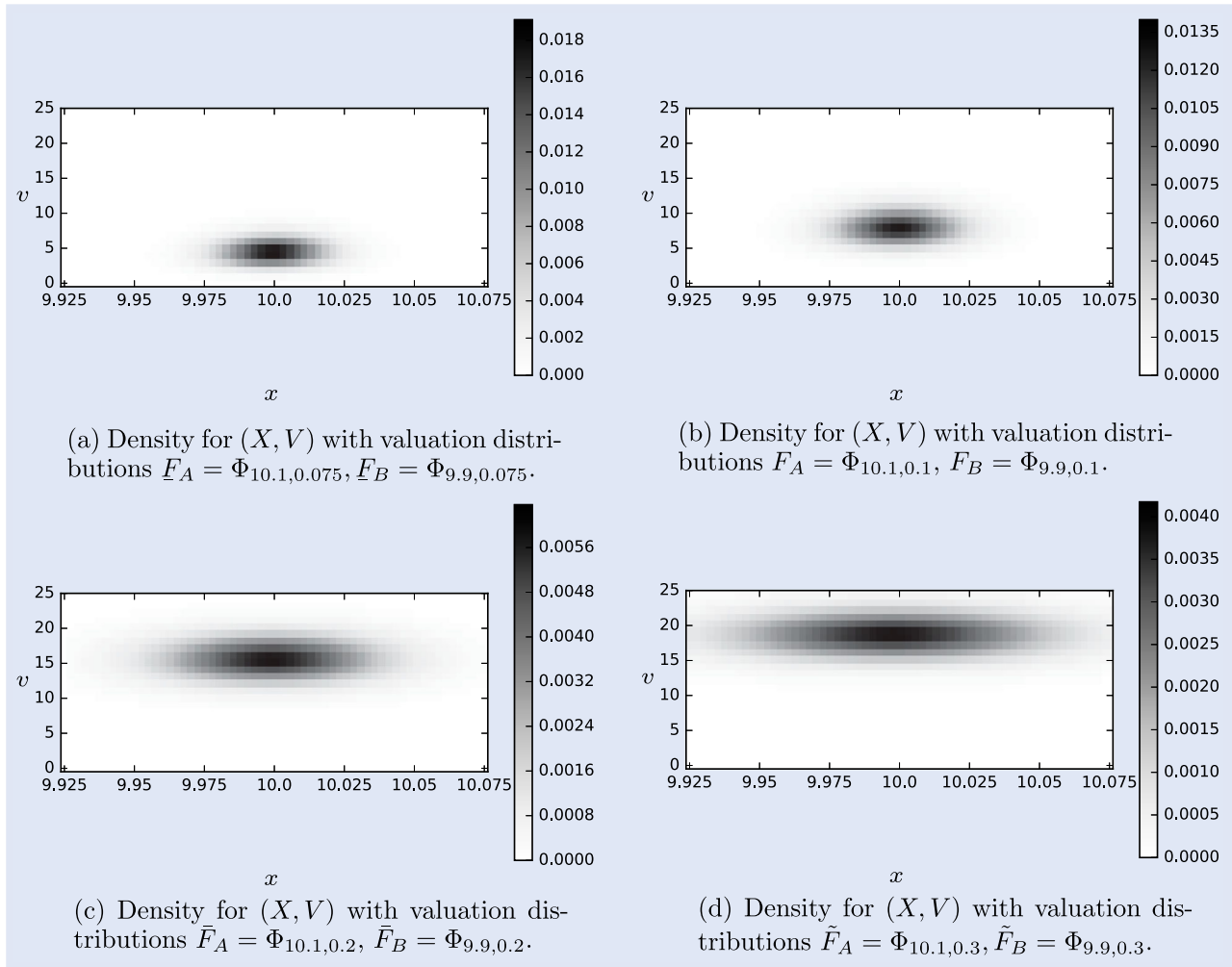


Figure 6. The influence of valuation uncertainty among market participants on the distribution of  $(X, V)$ . (a) Density for  $(X, V)$  with valuation distributions  $\underline{F}_A = \Phi_{10.1, 0.075}$ ,  $\underline{F}_B = \Phi_{9.9, 0.075}$ . (b) Density for  $(X, V)$  with valuation distributions  $F_A = \Phi_{10.1, 0.1}$ ,  $F_B = \Phi_{9.9, 0.1}$ . (c) Density for  $(X, V)$  with valuation distributions  $\bar{F}_A = \Phi_{10.1, 0.2}$ ,  $\bar{F}_B = \Phi_{9.9, 0.2}$ . (d) Density for  $(X, V)$  with valuation distributions  $\tilde{F}_A = \Phi_{10.1, 0.3}$ ,  $\tilde{F}_B = \Phi_{9.9, 0.3}$ .

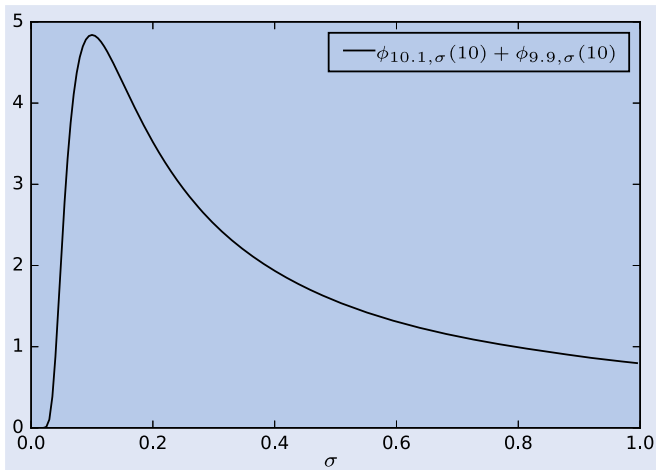


Figure 7.  $f_A(x_E) + f_B(x_E)$  as a function of  $\sigma$ , for  $f_A = \phi_{\mu_1, \sigma}$ ,  $f_B = \phi_{\mu_2, \sigma}$ , where  $\mu_1 = 10.1$ ,  $\mu_2 = 9.9$  and  $x_E = 10$ .

Note this function attains its maximum at  $\sigma = \frac{1}{2}|\mu_1 - \mu_2| = 0.1$ , implying that the asymptotic clearing price variance of equation (A7) attains its minimum at this  $\sigma$ .

## 5. Impact of market orders

In Definition 2.7 the clearing price in the presence of excess liquidity  $\Delta$  is defined and its distribution is provided in Proposition 2.8. Modelling the arrival of market orders as excess liquidity, this subsection compares clearing prices with and without market orders. Differences between clearing price distributions form the model's perspective on the *price impact of market orders*, a subject that has received quite some attention in the literature (see *e.g.* Hasbrouck 1991, Lillo *et al.* 2003, Smith *et al.* 2003, Donier and Bonart 2015, Donier *et al.* 2015, Benzaquen and Bouchaud 2018, and references therein).

Consider again the case that  $F_A = F_B = \Phi_{\mu, \sigma}$ , for  $\mu = 10$ ,  $\sigma = 0.1$  and  $(N_A, N_B) \sim \text{Pois}(50)^2$ . Departing from the case that this market is in equilibrium, next suppose that a market order of size  $|\omega|$  arrives: as in equation (4), we add an *excess liquidity* term to model this, in the form of constant functions  $\Delta(x) = \omega$ , where  $\omega > 0$  corresponds to a buy order and  $\omega < 0$  represents a sell order. In figure 8 the resulting clearing price distributions are plotted for various  $\omega$ . The common definition

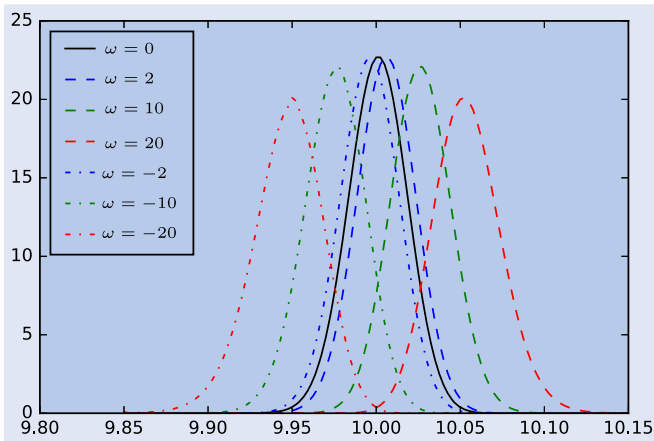


Figure 8. Clearing price densities, when market orders of sizes  $|\omega|$  are placed (negative  $\omega$  for sell orders, positive for buy orders). The supply and demand distributions are equal and normal,  $F_A = F_B = \Phi_{\mu, \sigma}$ , for  $\mu = 10$ ,  $\sigma = 0.1$ , while  $(N_A, N_B) \sim \text{Pois}(50)^2$ . Note that orders of size  $|\omega| = 2$  do not significantly influence the price distribution, but orders of sizes  $|\omega| = 10$  or 20 shift the clearing price distribution noticeably.

of the *price impact function*  $\delta p(\omega)$  is the size of the shift in market price when a market order of size  $\omega$  arrives. Empirical studies (see *e.g.* Hasbrouck 1991, Lillo *et al.* 2003, Donier and Bonart 2015, among many others) have shown that, in the situation of a continuous double auction, the price impact function is concave, and certain models confirm this concavity (see *e.g.* the seminal paper by Smith *et al.* (2003), or more recent work in this area (Donier *et al.* 2015, Benzaquen and Bouchaud 2018)). To consider the matter in our model, we define the price impact function  $\delta p(\omega)$  as the shift in *expectation* of  $X$  when a buy market order of size  $\omega > 0$  arrives. Figure 9 shows price impact functions for various supply and demand distributions that display the expected concavity. Furthermore, the picture shows that price impact becomes less concave as supply and demand distributions are shifted together, with the case  $F_A = F_B$  almost linear. This difference is explained by the number of orders that can be expected around  $\mathbb{E}X$ . In the case  $F_A = F_B$ ,  $\mathbb{E}X$  lies around  $x_E = 10$  and

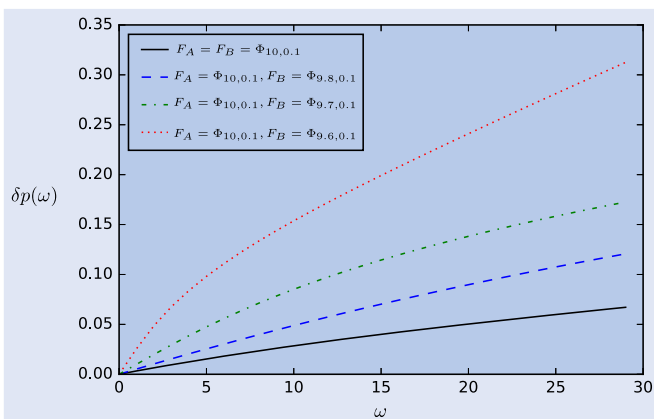


Figure 9. Price impact  $\delta p$  as a function of the size  $\omega$  of the market order, for various supply and demand distributions  $F_A, F_B$ . In all cases  $N = 100$ ,  $N_A = \alpha N$ ,  $\alpha \sim \text{Beta}(2, 2)$ .

all orders lie around 10. In cases where the locations of  $F_A$  and  $F_B$  differ,  $\mathbb{E}X$  lies between them, while buy orders concentrate around a lower price and sell orders around a higher one. In that situation fewer orders lie around  $\mathbb{E}X$  and consequently the clearing price is impacted more significantly in such regions; by contrast, in regions where orders are more concentrated, the clearing price is less easily moved. Comparing with Smith *et al.* (2003), their model produces an almost linear price impact function for a situation in which there is a large accumulation of orders near the market price and a very concave price impact function with lower levels of accumulation near the market price.

## 6. Prediction of the closing price distribution

For a quantitative model, a convincing statistical demonstration of applicability is ultimately the only possible proof of relevance. Below we perform this statistical exploration: we consider the statistical quality of the model's clearing price distributions in daily closing auctions for five (randomly selected) Eurostoxx 50 index constituents with the Kolmogorov–Smirnov goodness-of-fit test, and find that they explain the randomness in observed closing prices well. More specifically, we use a day's transactions to estimate clearing price distributions for daily closing auctions of five shares over the course of the trading year 2017. We assume that we have observed the market until 5 pm and then want to predict the closing price distribution.<sup>†</sup> To assess performance, we keep track of the estimated clearing price distribution functions, evaluated in the realized closing prices: if the estimates are accurate (and approximately independent), these probabilities form an approximate *i.i.d.* sample from the uniform distribution on  $[0, 1]$ . The match is assessed graphically, through QQ-plots, and tested with the Kolmogorov–Smirnov statistic. As a simple benchmark, the results are compared with results from a log-normal model.

### 6.1. Estimation of the closing price distribution

To obtain the daily estimator for the clearing price distribution, we first need estimators for the supply- and demand-distributions  $F_A$  and  $F_B$ . As we want to predict the closing price distribution  $F_X$  before the start of the closing auction itself, it is not an option to use quote data from the closing auction itself. Instead, intra-day transaction data is used: throughout the trading day, all transactions are recorded in a book that aggregates total volume traded for any price tick in the daily price range. In fact, two such books are kept, distinguished by the side of the market that initiated the trade. Half an

<sup>†</sup> The choice of the prediction time of 5 pm is not completely arbitrary. We have found empirically that around 90% of the closing prices falls within a range of 30bps of the last mid-price, and that the closing price is generally very close to the last mid-price. Hence, there is not much to predict when we wait until the closing auction starts, as the last mid-price is then more informative than our prediction. Of course we could start prediction already before 5 pm, but then the quality of the estimators will get worse, as less transaction data is observed.

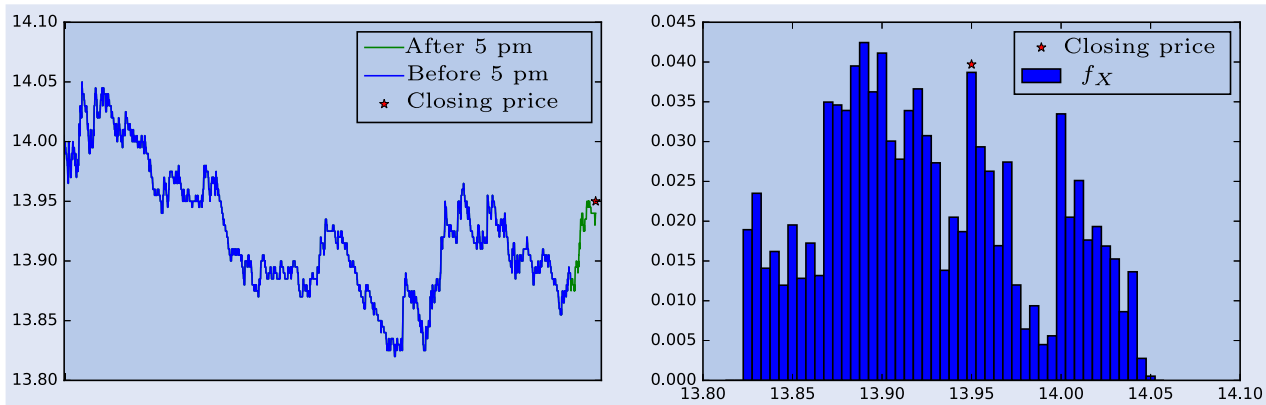


Figure 10. Estimation of the closing price distribution, based on daily transaction data. Left panel: ING stock price during a trading day in 2017. Right panel: the estimated distribution of the closing price  $X$ , where the distributions of supply and demand are taken to be the empirical estimators of  $F_A$  and  $F_B$  (based on the data until 5 pm) and where  $N = 100$ ,  $N_A = \alpha N$ ,  $\alpha \sim \text{Beta}(0.75, 0.75)$ .

hour before market close these books are normalized and converted into histogram-like estimators for the densities  $f_A$  and  $f_B$ . Expressed cumulatively, this leads to ‘empirical distributions functions’  $\hat{F}_A(\cdot)$  and  $\hat{F}_B(\cdot)$  that serve as estimators for  $F_A$  and  $F_B$ . Essentially we use a volume-weighted version of the day’s *transacted* orders to estimate market participants’ valuations. This leads to reasonable estimators, based on the idea that the intra-day valuations of market participants will be reflected in their valuations in the closing auction.

For any choice of  $N_A, N_B$ , these daily estimators can be used to estimate the distribution for  $X|N_A, N_B$ , that day’s clearing price given order flow  $N_A, N_B$ . Because  $\hat{F}_A(\cdot)$  and  $\hat{F}_B(\cdot)$  are supported on the range of prices visited that day, the clearing price distribution is supported on that range too. This causes a disadvantage of the proposed equilibrium model: regardless of the order flow, the model does not predict anything outside the daily price range and the estimator should be viewed as a ‘windowed’ or conditioned device, relevant only conditional on auction prices that fall inside the daily price range. To model order flow, we convolute with an order flow marginal in which  $N$  is fixed and  $N_A = \alpha N$ , with  $\alpha$  distributed according to a Beta-distribution. Because we have no reason to assume asymmetry, only the scale  $\beta > 0$  in  $\alpha \sim \text{Beta}(\beta, \beta)$  varies. To not exclude the possibility of fairly extreme, one-sided order flow (where  $N_A \gg N_B$  or vice versa with high probability), we keep  $\beta < 1$  (this is empirically verified, see Remark 6.2 below).

**DEFINITION 6.1** The daily *estimated clearing price distribution* is the distribution  $\hat{F}_X$  that results from Theorem 2.3 with empirical supply and demand  $F_A = \hat{F}_A$ ,  $F_B = \hat{F}_B$ , convoluted with the order flow distribution.

In all examples below, we choose  $N = 100$  and  $\beta = 0.75$  and note that these choices appear to work well for the five Eurostoxx 50 index constituents considered below.

**REMARK 6.2** It is important to note that the model is robust with respect to these choices. The choice of  $N$  is not very important: at first sight  $N$  has a very clear interpretation as the number of orders in the auction, however it should be noted that the size of orders is not modelled, hence changing  $N$  could also be interpreted as changing the order size. As long as  $N$  is taken sufficiently large to allow for enough

diversification in the orders (already for  $N > 50$ ) the choice of  $N$  does not really affect the result, as the large liquidity limit starts to do its work. † These claims are supported by figure A.1. The choice of  $\beta$  has more influence on the results, as it determines how many mass is shifted into the tails of the distribution. But still, the model is robust with respect to small variations in  $\beta$ , which is expressed by figure A.2. Estimation of the parameter  $\beta$  is also interesting: one can take the  $n$  previous closing auctions of the stock and compute the ratio  $\frac{N_A}{N_A + N_B}$ , i.e. the total volume of all sell orders in the closing auction, divided by the total volume of all orders. ‡ Then one obtains a sample of  $n$  order flow imbalances, which can be treated as an *i.i.d.* sample to find the parameter of the Beta-distribution, by computing moment estimators. We did this analysis for three of the five Eurostoxx 50 index constituents considered below. § For Airbus SE we found  $\hat{\beta} = 0.8057$ , for Engie SA we found  $\hat{\beta} = 0.8219$  and for Anheuser Busch Inbev NV we found  $\hat{\beta} = 0.6988$ , empirically justifying the observation that  $\beta$  should be picked around 0.75.

As an example, consider figure 10, an (arbitrarily selected) day’s trading in ING stocks and the estimate of the closing price distribution  $\hat{F}_X$  (based on  $F_A, F_B$  that are estimated from daily transaction data until 5 pm). Note the inhomogeneity of the estimated density. The following statistical analysis shows that this detailed shape with peak and troughs is informative for the realized closing price, meaning that the closing price is more likely to be realized on prices where the estimated density is higher, which is nicely illustrated in the above example

† There is also an easy way to estimate  $N$ : around 28% of the daily total transacted volume is transacted in the closing auction (in 2017, nowadays it is more), so one could take the total transacted volume until 5 pm and turn this into an estimator for  $N$  using this ratio. However, figure A.1 shows that this analysis is not worthwhile, as a different choice of  $N$  does not impact the results.

‡ In fact, the order book contains a lot of irrelevant volume far from the eventual closing price. This volume does not contribute to the determination of the closing price and should not be counted in the estimation. Instead, we only counted orders within ten levels of the closing price.

§ One needs full order book data to do the estimation, which is provided for the stocks traded on Euronext, but not for the German stocks Bayer AG and Deutsche Telekom AG.



by the closing price that realizes on one of the peaks in the estimated density.

## 6.2. Kolmogorov–Smirnov goodness-of-fit test

To assess performance, we predict closing price distributions for the circa 250 trading days in 2017. Because the model only concerns the daily trading range, we do not include those trading days on which the stock’s closing price lay *outside* the daily trading range. Typically, trading on such days is highly momentum-driven and is not well-represented by equilibrium models, at least, on daily or shorter time-scales. After removing the days where the price closed outside the daily trading range, this leads to samples of 200–230 trading days for the selected five stocks. As the valuation distributions  $F_A$ ,  $F_B$  differ from day to day, there is no straightforward way to assess the accuracy of the sample of estimated clearing price distributions  $\hat{F}_X$ . For that, we need a standard, distribution-free argument based on the observation that if  $X \sim F_X$ , then  $F_X(X) \sim U[0, 1]$ : if  $\hat{F}_X$  approximates  $F_X$  well on any trading day,  $\hat{F}_X(X)$  has a distribution approximating  $U[0, 1]$ . In our statistical experiment we have a sequence of predictions  $\hat{F}_i$  for closing prices  $X_i$  (assumed independent) with true marginal closing price distributions  $F_i$  (which are possibly very different as the day  $i$  varies). If the estimators  $\hat{F}_i$  approximate the  $F_i$  well, the resulting sequence of probabilities  $\hat{\xi}_i = \hat{F}_i(X_i)$  is distributed approximately as an *i.i.d.* sample from the uniform distribution on  $[0, 1]$ . Below, this degree of approximation is assessed graphically through QQ-plots and tested with statistical significance using the Kolmogorov–Smirnov(KS) statistic. This statistical assessment is not just a technically convenient choice, what is assessed in this way is highly relevant to daily market practice: good QQ-plots and KS-statistics indicate that clearing price distribution estimators provide an accurate picture of the relation between quoted price and probability of execution in the auction (conditional on a closing price inside the price range seen during the day). For example, from a trader’s perspective, the quantiles of the estimated clearing price distribution could give rise to a trading strategy that goes long/short when the market price lies in the low/high quantiles half an hour before the market closes. From an investor’s point of view, the  $p$ th percentile of the clearing price distribution answers the question at which price to quote in the auction to be for  $p\%$  sure that the order gets transacted.

To have a simple benchmark for comparison we also consider an alternative: we include a benchmark model that assumes that the daily log-return is normally distributed, with mean and variance estimated by (volume-weighted) average and variance of log-prices of transactions during the day. The resulting estimated closing price distribution  $\hat{F}_i$  is truncated to that day’s trading range. This leads to samples of 200–230 probabilities  $\hat{\xi}_i = \hat{F}_i(X_i)$ , subject to the same requirement of similarity to an *i.i.d.* sample from the  $U[0, 1]$ -distribution. The two samples  $\hat{\xi}_i$  (resulting from the market clearing model) and  $\hat{\xi}_i$  (resulting from the log-normal model) are assessed for uniformity by QQ-plots in figure 11.

Table 1 reports the associated KS-statistics and  $p$ -values. (Note that the KS-test does not fall within the standard

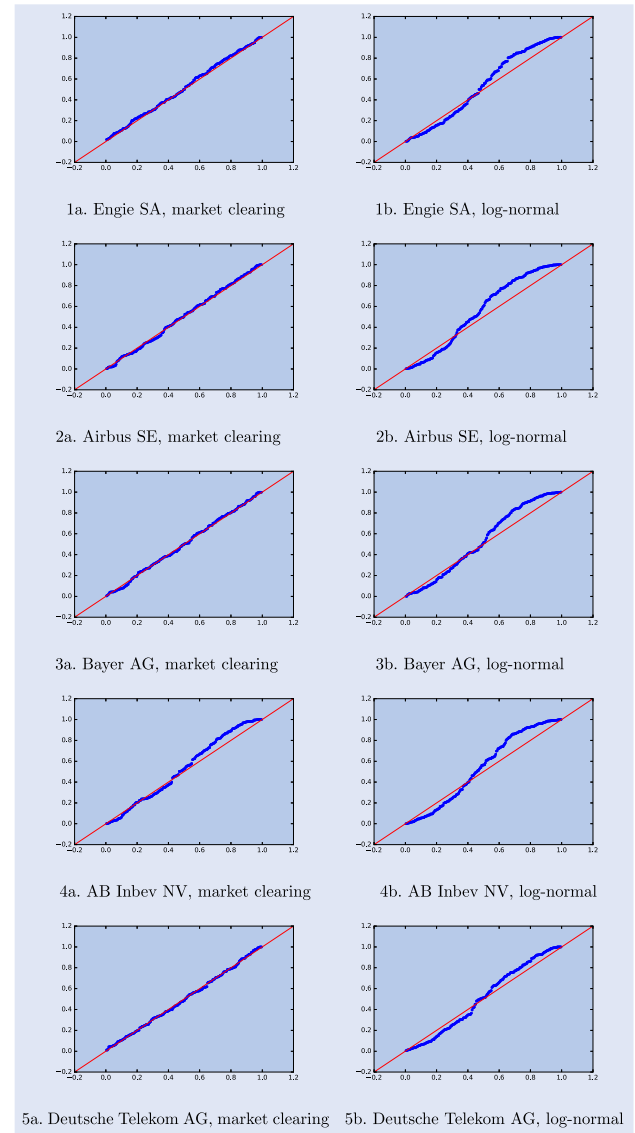


Figure 11. QQ-plots of the samples of probabilities that the clearing price is lower than the realized closing price (vertical axis) against theoretical  $U[0, 1]$ -quantiles(horizontal axis), for the market clearing model and the log-normal model, for 5 Eurostoxx 50 index constituents.

Neyman–Pearson framework of statistical testing, basically because one seeks to *confirm* the null-hypothesis. This changes the usual interpretation of  $p$ -values: if a model has a low  $p$ -value in this context, the hypothesis that it is correct is rejected based on the data with high statistical significance. By contrast, a model with a high  $p$ -value requires a high degree of relaxation of significance criteria before the correctness hypothesis is rejected based on the data.) The model of log-normal daily returns proves wholly inadequate as an explanation of the randomness observed in actual closing prices: only in the example of Deutsche Telekom is it possible to argue that (truncated) log-normal distributions for the daily returns form a prediction that is informative about closing prices at the distributional, predictive level. Furthermore, figure 11 shows that the log-normal model underestimates the tail of the closing price distributions in all five examples. By contrast, the QQ-plots for the market clearing model show a very decent



Table 1. Kolmogorov–Smirnov statistics and corresponding  $p$ -values for the samples of probabilities  $\hat{\xi}_i$  (resulting from the clearing model) and  $\tilde{\xi}_i$  (resulting from the log-normal model) for five (randomly selected) constituents of the Eurostoxx 50 index.

| Stock                      | Model           | KS-statistic | $p$ -value           |
|----------------------------|-----------------|--------------|----------------------|
| 1. Engie SA                | Market clearing | 0.0392       | 0.905                |
|                            | Log-normal      | 0.147        | $2.04 \cdot 10^{-4}$ |
| 2. Airbus SE               | Market clearing | 0.0320       | 0.988                |
|                            | Log-normal      | 0.164        | $3.76 \cdot 10^{-5}$ |
| 3. Bayer AG                | Market clearing | 0.0326       | 0.983                |
|                            | Log-normal      | 0.139        | $6.96 \cdot 10^{-4}$ |
| 4. Anheuser Busch Inbev NV | Market clearing | 0.104        | 0.0198               |
|                            | Log-normal      | 0.175        | $4.61 \cdot 10^{-6}$ |
| 5. Deutsche Telekom AG     | Market clearing | 0.0304       | 0.989                |
|                            | Log-normal      | 0.0837       | 0.0969               |

match for uniformity, indicating that the model is a good representation of the randomness in observed closing prices. Estimated clearing price distributions provide an accurate picture of the relation between quoted price and probability of execution (conditional on an auction price that falls within the daily trading range). This is confirmed by associated KS-statistics and their  $p$ -values in table 1: four out of five samples show exceptionally straight lines in their QQ-plots, confirmed by the exceptionally high  $p$ -values in table 1. The exception (recall, these five stocks have been selected randomly from the Eurostoxx 50 index) is the Anheuser Busch Inbev NV stock, with a KS-statistic that indicates evidence ( $p = 0.0198$ ) to reject the null-hypothesis and visual inspection through the QQ-plot reveals underestimation of the up-side tail. One could wonder how the market clearing model performs if the extreme beta distribution for order flow imbalance is replaced by Poisson order flow (this essentially corresponds to the call auction model of Muni Toke (2015a)). To investigate this option, we performed exactly the same analysis using the estimated clearing price distribution (as in Definition 6.1) to obtain a sample of probabilities, but now with Poisson order flow:  $(N_A, N_B) \sim \text{Pois}(50) \times \text{Pois}(50)$ . Figure 12 shows the corresponding QQ-plot for one of the stocks (Airbus SE), similar results are obtained for the other stocks. The Poisson order flow, expressing the conviction that the orderflow imbalance  $\alpha$  does not display great stochastic fluctuation around  $\alpha = \frac{1}{2}$ , leads to a clearing price distribution that extremely underestimates the tails (even worse than the log-normal model). It turns out that the extreme order flow distributions are necessary to capture the tails of closing price distributions, underlining the limitations of Poisson order flow.

We conclude that the detailed shape of estimated clearing price distributions from the market clearing model (with peaks and troughs as in the right panel of figure 10) is informative for the relation between the price of an order and the corresponding execution probability, while uni-modal shapes like those of the log-normal distribution are not. Furthermore, we conclude that Poisson order flow does not display enough stochastic fluctuation to capture the tails of the observed randomness in closing prices, emphasizing the relevance of extreme order flow distributions.

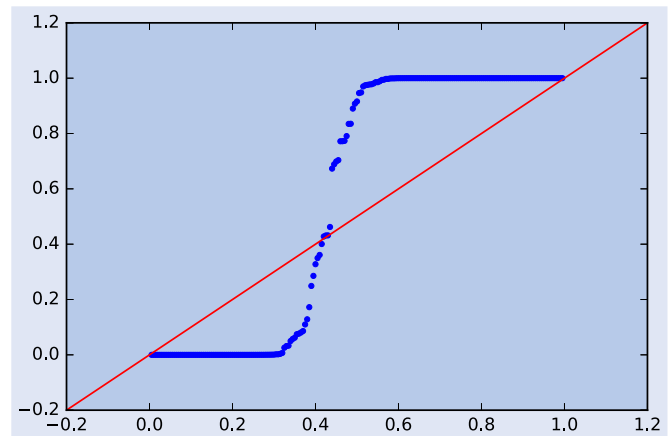


Figure 12. QQ-plot of the sample of probabilities that the clearing price is lower than the realized closing price, for Airbus SE (vertical axis), against theoretical  $U[0, 1]$ -quantiles (horizontal axis), for the market clearing model with  $(N_A, N_B) \sim \text{Pois}(50) \times \text{Pois}(50)$ .

## 7. Conclusions

In this article we propose a model for auction price distributions in standard call auctions based on a balance between two samples of random orders. The model assumes *i.i.d.* samples of buy- and sell-orders, placed following demand- and supply-side valuation distributions. An equilibrium equation (fixing the clearing price by requiring that the number of buyers equals the number of sellers) then leads to a distribution for clearing price and transacted volume. Bid- and ask-side volumes are left as free parameters (order flow); a choice for the distribution of these parameters (possibly heavy-tailed or very skewed) leads to distributions for clearing prices and transacted volumes, with or without a limit order book.

In the highly liquid auctions of section 3, the clearing price distribution converges to a normal central limit, with mean and variance in terms of supply/demand-valuation distributions and order flow imbalance. Most importantly, the variance of the limiting normal distribution at real equilibrium price  $x$  is inversely proportional to the density of orders around  $x$ . The interpretation is in regions on the price axis where price variance is suppressed due to density of orders.

In section 2.4, we consider the influence of order flow on clearing price distributions. Restriction to models involving Poisson or binomial assumptions concerning the amount of liquidity on offer is hard to justify. As confirmed empirically in section 6, extreme or skewed order flow conditions are equally important. Section 4 explores the influence of valuation distributions with some illustrative simulations: for example, bringing valuation distributions closer together increases transacted volume and decreases price variance. Closer inspection of the price/volume distribution reveals that there are two fundamentally different types of price variance, one driven by illiquidity and the other by valuation uncertainty among market participants. To differentiate, one must involve transacted volume, which is moderate when auction price variance is minimal, low in illiquid markets and high in markets with valuation uncertainty-driven price variance.

In section 5, we analyse the model's description of market impact. Remarkably, the model produces a concave price impact function, especially when the valuation distributions are widely separated, reflecting a market in which the consensus is low. This is in line with empirical results (Hasbrouck 1991, Lillo *et al.* 2003, Donier and Bonart 2015) and with the theoretical results of Smith *et al.* (2003).

To statistically verify the validity of the model and estimates of the daily closing price distributions in section 6, we predict a year's worth of daily closing-price distributions for five constituents of the Eurostoxx 50 index; Kolmogorov–Smirnov statistics and QQ-plots demonstrate with ample statistical significance that the model predicts closing price distributions accurately, and compares favourably with a simpler, log-normal, alternative method of prediction. We conclude that the model's predicted clearing price distributions explain the observed randomness in closing prices well, confirming that the proposed model provides a proper description of price formation in call auctions.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

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## Appendix 1. Notation and proofs

### A.1. Notation and conventions

We denote the multinomial coefficient for  $n \geq 3$  by

$$\binom{n}{k_1, \dots, k_n} := \frac{n!}{k_1! \dots k_n!}.$$

The binomial distribution with parameters  $n$  and  $p$  is denoted  $\text{Bin}(n, p)$ , the Poisson distribution with parameter  $\lambda$  is denoted by  $\text{Pois}(\lambda)$ , the uniform distribution on  $[0, 1]$  is denoted by  $U[0, 1]$  and the normal distribution with mean  $\mu$  and variance  $\sigma^2$  is denoted by  $N(\mu, \sigma^2)$  with cumulative distribution function  $\Phi_{\mu, \sigma}(\cdot)$ . Convergence in distribution is denoted  $\xrightarrow{w}$ . Let  $\mathcal{X} \subset \mathbb{R}$  be the price-axis which can be either discrete or continuous. The lowest possible price is denoted by  $x_0 := \inf \mathcal{X}$ . The valuation distributions for supply and demand prices, denoted  $F_A$  and  $F_B$ , are assumed to be distributions on the price-axis.

### A.2. Proofs

The expressions we derive for price and price-volume distributions hinge on the following two lemmas, which convert finding a solution to equation (1) into a question involving binomial distributions.

**LEMMA A.1** For any  $x \in \mathbb{R}$ , we have the equivalence:  $X \leq x \Leftrightarrow \mathbb{D}_A(x) \geq \mathbb{D}_B(x)$ .

*Proof* The left implication follows immediately from the definition of  $X$ , so suppose  $X \leq x$ . Note that  $x \mapsto \mathbb{D}_A(x)$  is non-decreasing and  $x \mapsto \mathbb{D}_B(x)$  is non-increasing. So the set  $\{y \in \mathbb{R} : \mathbb{D}_A(y) \geq \mathbb{D}_B(y)\}$  is of the form  $(a, \infty)$  or  $[a, \infty)$ , for some  $a \in \mathbb{R}$ . Through their definitions,  $\mathbb{D}_A$  and  $\mathbb{D}_B$  are right-continuous, so we can write,

$$\mathbb{D}_A(a) = \lim_{z \downarrow a} \mathbb{D}_A(z) \geq \lim_{z \downarrow a} \mathbb{D}_B(z) = \mathbb{D}_B(a).$$

Therefore  $\{y \in \mathbb{R} : \mathbb{D}_A(y) \geq \mathbb{D}_B(y)\} = [a, \infty)$ , which implies that  $a = \inf\{y \in \mathbb{R} : \mathbb{D}_A(y) \geq \mathbb{D}_B(y)\} \leq x$ . Hence  $x \in [a, \infty) = \{y \in \mathbb{R} : \mathbb{D}_A(y) \geq \mathbb{D}_B(y)\}$ , which proves the result. ■

The independence assumption for  $(A_1, \dots, A_{N_A})$  and  $(B_1, \dots, B_{N_B})$  directly implies the content of the following lemma.

LEMMA A.2 For every  $x \in \mathcal{X}$ ,  $(\mathbb{D}_A(x), \mathbb{D}_B(x))$  is a pair of independent, binomially distributed random variables,

$$(\mathbb{D}_A(x), \mathbb{D}_B(x)) \sim \text{Bin}(N_A, F_A(x)) \times \text{Bin}(N_B, 1 - F_B(x)). \quad (\text{A1})$$

These two lemmas imply the following explicit expression for the clearing price distribution in terms of the distributions of supply and demand,  $F_A$  and  $F_B$ , conditional on  $N_A$  and  $N_B$ .

THEOREM 2.3 (Clearing price distribution) The distribution of the clearing price  $X$ , conditional on  $N_A$  and  $N_B$ , is given by,

$$\begin{aligned} \mathbb{P}(X \leq x | N_A, N_B) \\ = \sum_{k=0}^{N_A} \sum_{l=0}^{N_B \wedge k} \binom{N_A}{k} F_A(x)^k (1 - F_A(x))^{N_A-k} \\ \times \binom{N_B}{l} (1 - F_B(x))^l F_B(x)^{N_B-l}. \end{aligned}$$

*Proof* From Lemma A.1 and the independence of  $\mathbb{D}_A(x)$  and  $\mathbb{D}_B(x)$  it follows that,

$$\begin{aligned} \mathbb{P}(X \leq x) &= \mathbb{P}(\mathbb{D}_A(x) \geq \mathbb{D}_B(x)) \\ &= \sum_{k=0}^{N_A} \mathbb{P}(\mathbb{D}_B(x) \leq k | \mathbb{D}_A(x) = k) \mathbb{P}(\mathbb{D}_A(x) = k) \\ &= \sum_{k=0}^{N_A} \sum_{l=0}^{N_B \wedge k} \mathbb{P}(\mathbb{D}_B(x) = l) \mathbb{P}(\mathbb{D}_A(x) = k), \end{aligned}$$

where conditioning on  $N_A, N_B$  has been omitted for ease of notation. The result follows from Lemma A.2. ■

Similarly, we derive the joint distribution of  $(X, V)$  from equation (1). Recall that the price-axis  $\mathcal{X}$  is a discrete set,  $\mathcal{X} := \{x_0, x_0 + \delta, \dots\}$ , for some  $\delta > 0$ .

THEOREM 2.4 (Joint clearing price/transaction volume distribution) The joint distribution of clearing price  $X$  and transacted volume  $V$ , conditional on  $N_A$  and  $N_B$ , is given by,

$$\begin{aligned} \mathbb{P}(X \leq x, V \leq v | N_A, N_B) \\ = \sum_{u=0}^v \sum_{k=0}^u \sum_{l=0}^k \left[ \binom{N_B}{l} \binom{N_A}{k, u-k, N_A-u} (1 - F_B(x))^l F_B(x)^{N_B-l} \right. \\ \times F_A(x)^k (F_A(x + \delta) - F_A(x))^{u-k} (1 - F_A(x + \delta))^{N_A-u} \Big] \\ + \sum_{y \in \mathcal{X}, y \leq x} \sum_{u=v+1}^{N_A} \sum_{k=0}^u \sum_{l=0}^k \left[ \binom{N_B}{l} \binom{N_A}{k, u-k, N_A-u} \right. \\ \times (1 - F_B(y))^l F_B(y)^{N_B-l} \\ \times F_A(y)^k (F_A(y + \delta) - F_A(y))^{u-k} (1 - F_A(y + \delta))^{N_A-u} \Big] \\ - \sum_{y \leq x} \sum_{l=0}^{N_B} \sum_{k=l \vee v+1}^{N_A} \binom{N_A}{k} F_A(y)^k (1 - F_A(y))^{N_A-k} \\ \times \binom{N_B}{l} F_B(y)^{N_B-l} (1 - F_B(y))^l. \quad (2) \end{aligned}$$

*Proof* In order to characterize the transacted volume  $V$  in a similar sense as the clearing price in Lemma A.1, define the generalized inverses  $\mathbb{D}_A^{-1}$  and  $\mathbb{D}_B^{-1}$  of  $\mathbb{D}_A$  and  $\mathbb{D}_B$  by

$$\mathbb{D}_A^{-1}(v) := \mathbb{F}_A^{-1}(v/N_A) = \inf\{x \geq 0 : \mathbb{D}_A(x) \geq v\},$$

$$\mathbb{D}_B^{-1}(v) := \mathbb{F}_B^{-1}(1 - v/N_B) = \inf\{x \geq 0 : \mathbb{D}_B(x) \leq v\},$$

where  $\mathbb{F}_A^{-1}, \mathbb{F}_B^{-1}$  are the generalized inverses of the empirical cumulative distribution functions  $\mathbb{F}_A, \mathbb{F}_B$  (for a distribution function  $F$ , its generalized inverse is defined as  $F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}$ , for  $p \in [0, 1]$ , (see e.g. Van der Vaart 1998, Chapter 21)). For a given distribution function  $F$ , its generalized inverse satisfies

$$F^{-1}(p) \leq x \Leftrightarrow p \leq F(x),$$

which implies

$$\mathbb{D}_A^{-1}(v) \leq x \Leftrightarrow v \leq \mathbb{D}_A(x), \quad \mathbb{D}_B^{-1}(v) \leq x \Leftrightarrow v \geq \mathbb{D}_B(x). \quad (\text{A2})$$

It follows from equation (A2) that  $V$  is characterized by the following equivalences.

$$\begin{aligned} V \leq v &\Leftrightarrow \mathbb{D}_A(X) < v + 1 \Leftrightarrow X < \mathbb{D}_A^{-1}(v + 1) \\ &\Leftrightarrow X \leq \mathbb{D}_A^{-1}(v + 1) - \delta, \end{aligned} \quad (\text{A3})$$

which leads to

$$X \leq x, V \leq v \Leftrightarrow X \leq \min(x, \mathbb{D}_A^{-1}(v + 1) - \delta).$$

So we can write

$$\begin{aligned} \mathbb{P}(X \leq x, V \leq v) &= \mathbb{P}(X \leq \min(x, \mathbb{D}_A^{-1}(v + 1) - \delta)) \\ &= \mathbb{P}(X \leq x, \mathbb{D}_A^{-1}(v + 1) - \delta > x) \\ &\quad + \mathbb{P}(X \leq \mathbb{D}_A^{-1}(v + 1) - \delta, \mathbb{D}_A^{-1}(v + 1) - \delta \leq x) \\ &= \mathbb{P}(X \leq x, \mathbb{D}_A^{-1}(v + 1) > x + \delta) + \\ &\quad \times \sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(X \leq y, \mathbb{D}_A^{-1}(v + 1) = y + \delta). \end{aligned} \quad (\text{A4})$$

Here, and in remainder of the proof, we have omitted the conditioning on  $N_A, N_B$  in the notation, for convenience. We start with the first term in this expression. From Lemma A.1 and (A2) it follows that

$$\begin{aligned} \mathbb{P}(X \leq x, \mathbb{D}_A^{-1}(v + 1) > x + \delta) &= \mathbb{P}(\mathbb{D}_A(x) \geq \mathbb{D}_B(x), \\ &\quad \times \mathbb{D}_A(x + \delta) < v + 1) \\ &= \sum_{u=0}^v \mathbb{P}(\mathbb{D}_A(x) \geq \mathbb{D}_B(x), \\ &\quad \times \mathbb{D}_A(x + \delta) = u), \end{aligned}$$

where, by independence of the bid- and ask-samples,

$$\begin{aligned} \mathbb{P}(\mathbb{D}_A(x) \geq \mathbb{D}_B(x), \mathbb{D}_A(x + \delta) = u) \\ = \sum_{k=0}^u \mathbb{P}(\mathbb{D}_B(x) \leq k) \mathbb{P}(\mathbb{D}_A(x) = k, \mathbb{D}_A(x + \delta) = u) \\ = \sum_{k=0}^u \sum_{l=0}^k \left[ \binom{N_B}{l} (1 - F_B(x))^l F_B(x)^{N_B-l} \right. \\ \times \frac{N_A!}{k!(u-k)!(N_A-u)!} F_A(x)^k \\ \times (F_A(x + \delta) - F_A(x))^{u-k} (1 - F_A(x + \delta))^{N_A-u} \Big], \end{aligned} \quad (\text{A5})$$

which gives the first term of the solution in equation (2).

Now we turn to the second term in equation (A4), for which we write

$$\sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(X \leq y, \mathbb{D}_A^{-1}(v + 1) = y + \delta)$$

$$\begin{aligned}
&= \sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(X \leq y, \mathbb{D}_A^{-1}(v+1) \leq y + \delta) \\
&\quad - \sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(X \leq y, \mathbb{D}_A^{-1}(v+1) \leq y) \\
&= \sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(\mathbb{D}_A(y) \geq \mathbb{D}_B(y), \mathbb{D}_A(y + \delta) \geq v + 1) \\
&\quad - \sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(\mathbb{D}_A(y) \geq \mathbb{D}_B(y), \mathbb{D}_A(y) \geq v + 1), \quad (\text{A6})
\end{aligned}$$

where the last line follows by Lemma A.1 and equation (A2). The first term of this expression equals

$$\begin{aligned}
&\sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(\mathbb{D}_A(y) \geq \mathbb{D}_B(y), \mathbb{D}_A(y + \delta) \geq v + 1) \\
&= \sum_{y \in \mathcal{X}, y \leq x} \sum_{u=v+1}^{N_A} \mathbb{P}(\mathbb{D}_A(y) \geq \mathbb{D}_B(y), \mathbb{D}_A(y + \delta) = u)
\end{aligned}$$

and its expression follows from equation (A5), by substituting  $y$  for  $x$ . This gives the second term of the solution in equation (2). Finally, consider the second term in equation (A6), which equals

$$\begin{aligned}
&\sum_{y \in \mathcal{X}, y \leq x} \mathbb{P}(\mathbb{D}_A(y) \geq \mathbb{D}_B(y), \mathbb{D}_A(y) \geq v + 1) \\
&= \sum_{y \in \mathcal{X}, y \leq x} \sum_{l=0}^{N_B} \mathbb{P}(\mathbb{D}_A(y) \geq \max(l, v + 1)) \mathbb{P}(\mathbb{D}_B(y) = l),
\end{aligned}$$

by independence of the bid- and ask-samples. Using the binomial distributions of  $\mathbb{D}_A(y)$  and  $\mathbb{D}_B(y)$  once more, we see that this equals the last term in the solution of equation (2), which concludes the proof. ■

**THEOREM 2.5 (High-liquidity clearing price distribution)** *Let  $X$  be the clearing price in case of possible excess liquidity  $\Delta$ . Assume that  $F_A$  and  $F_B$  are strictly increasing and absolutely continuous with respect to the Lebesgue measure with densities  $f_A$  and  $f_B$ . Additionally, assume that excess liquidity scales with  $N$  as  $\Delta(\cdot) = \sqrt{N}D(\cdot)$ , for some continuous and bounded function  $D : \mathcal{X} \rightarrow \mathbb{R}$ . Then, as  $N \rightarrow \infty$ ,*

$$\sqrt{N}(X - x_E) \xrightarrow{w} N(\mu(x_E), \sigma^2(x_E)), \quad (6)$$

where the asymptotic mean and standard deviation are given by,

$$\begin{aligned}
\mu(x_E) &= \frac{D(x_E)}{\alpha f_A(x_E) + (1 - \alpha)f_B(x_E)}, \\
\sigma(x_E) &= \frac{\tau(x_E)}{\alpha f_A(x_E) + (1 - \alpha)f_B(x_E)},
\end{aligned}$$

for

$$\tau^2(x_E) = \alpha F_A(x_E)(1 - F_A(x_E)) + (1 - \alpha)F_B(x_E)(1 - F_B(x_E)),$$

and  $x_E$  is the real equilibrium price.

*Proof* The assumption that  $F_A$  and  $F_B$  are continuous implies that the steps of  $\mathbb{D}_A$  and  $\mathbb{D}_B$  all have size 1, almost surely. So we have, almost surely,

$$N_A \mathbb{F}_A(X) = N_B(1 - \mathbb{F}_B(X)) + \Delta(X).$$

Combination with (5) yields,

$$N_A(\mathbb{F}_A(X) - F_A(x_E)) = -N_B(\mathbb{F}_B(X) - F_B(x_E)) + \Delta(X),$$

which, after introduction of  $F_A(X)$  and  $F_B(X)$ , reads,

$$\sqrt{\frac{N_A}{N_B}}(\mathbb{F}_A(X) - F_A(X)) + \sqrt{\frac{N_A}{N_B}}(F_A(X) - F_A(x_E))$$

$$= -\sqrt{\frac{N_B}{N_A}}(\mathbb{F}_B(X) - F_B(X)) - \sqrt{\frac{N_B}{N_A}}(F_B(X) - F_B(x_E)) \quad (21)$$

$$+ \frac{\Delta(X)}{\sqrt{N_A N_B}}. \quad (\text{A8})$$

Now denote

$$Z_{A,N_A}(x) = \sqrt{N_A}(\mathbb{F}_A(x) - F_A(x)),$$

$$Z_{B,N_B}(x) = \sqrt{N_B}(\mathbb{F}_B(x) - F_B(x)).$$

By Donsker's theorem (see *e.g.* Van der Vaart 1998, Theorem 19.3) and independence of the bid- and ask-samples, it holds that

$$\begin{aligned}
&(Z_{A,N_A}(x), Z_{B,N_B}(x)) \xrightarrow{w} N(0, F_A(x)(1 - F_A(x))) \\
&\quad \times N(0, F_B(x)(1 - F_B(x))),
\end{aligned}$$

as  $N_A, N_B \rightarrow \infty$ , uniformly over  $x \in \mathbb{R}$  (and hence for every random  $X$ ). Using  $N_A = \alpha N$ ,  $N_B = (1 - \alpha)N$  and  $D(x) = \Delta(x)/\sqrt{N}$ , we can rewrite (A8) as follows,

$$\begin{aligned}
&\sqrt{\frac{\alpha}{1 - \alpha}}(F_A(X) - F_A(x_E)) + \sqrt{\frac{1 - \alpha}{\alpha}}(F_B(X) - F_B(x_E)) \\
&= -\frac{1}{\sqrt{N(1 - \alpha)}}Z_{A,N_A}(X) - \frac{1}{\sqrt{\alpha N}}Z_{B,N_B}(X) + \frac{D(X)}{\sqrt{N(1 - \alpha)\alpha}}.
\end{aligned}$$

Hence, we obtain the following weak limit,

$$\begin{aligned}
&\frac{\sqrt{N}}{\tau(X)} \left( \alpha(F_A(X) - F_A(x_E)) + (1 - \alpha)(F_B(X) - F_B(x_E)) \right) \\
&\quad - \frac{D(X)}{\tau(X)} \xrightarrow{w} N(0, 1), \quad (\text{A9})
\end{aligned}$$

where the asymptotic variance  $\tau^2(X)$  is given by,

$$\tau^2(X) = \alpha F_A(X)(1 - F_A(X)) + (1 - \alpha)F_B(X)(1 - F_B(X)).$$

With the help of the distribution function  $F_R$ , defined by the convex combination,

$$F_R(\cdot) = \alpha F_A(\cdot) + (1 - \alpha)F_B(\cdot),$$

we rewrite equation (A9) as follows,

$$\frac{1}{\tau(X)} (\sqrt{N}(F_R(X) - F_R(x_E)) - D(X)) \xrightarrow{w} N(0, 1).$$

Since  $0 < \tau(X) < 1$  and  $D$  is bounded, we conclude that  $F_R(X)$  converges to  $F_R(x_E)$  in probability. The assumptions on  $F_A$  and  $F_B$  ensure that  $F_R$  has a Lebesgue density  $f_R$  and that  $F_R$  is invertible with continuous inverse  $F_R^{-1} : [0, 1] \rightarrow \mathbb{R}$ , so it follows that  $X$  converges to  $x_E$  in probability. By continuity it follows that  $\tau(X)$  converges in probability to  $\tau(x_E)$  and  $D(X)$  to  $D(x_E)$ . By Slutsky's Lemma (see *e.g.* Van der Vaart 1998, Lemma 2.8), we arrive at,

$$\sqrt{N}(F_R(X) - F_R(x_E)) \xrightarrow{w} N(D(x_E), \tau^2(x_E)).$$

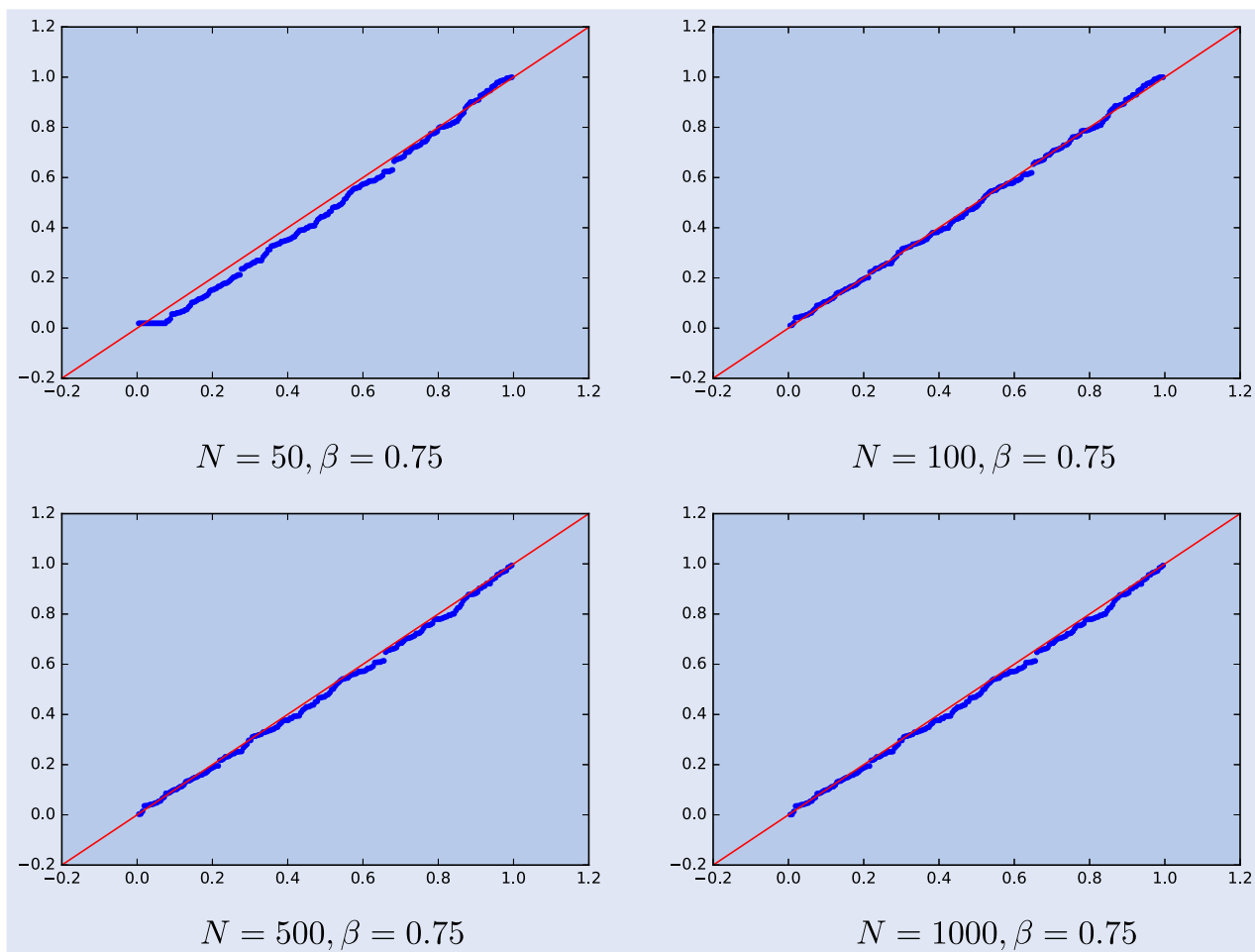


Figure A.1. How the choice of  $N$  affects the results of section 6, for the case of Deutsche Telekom AG (similar effects are observed for the other stocks). It is seen that the choice of  $N$  does not really impact the results, as long as  $N$  is sufficiently large ( $N > 50$ ). (a)  $N = 100, \beta = 0.75$ . (b)  $N = 500, \beta = 0.75$ .

The Delta-method (see *e.g.* Van der Vaart 1998, Theorem 3.1) then leads to,

$$\sqrt{N}(X - x_E) \xrightarrow{w} (F_R^{-1})'(F_R(x_E))N(D(x_E), \tau^2(x_E)),$$

where, according to the inverse function theorem,

$$(F_R^{-1})'(F_R(x_E)) = \frac{1}{f_R(x_E)}.$$

■



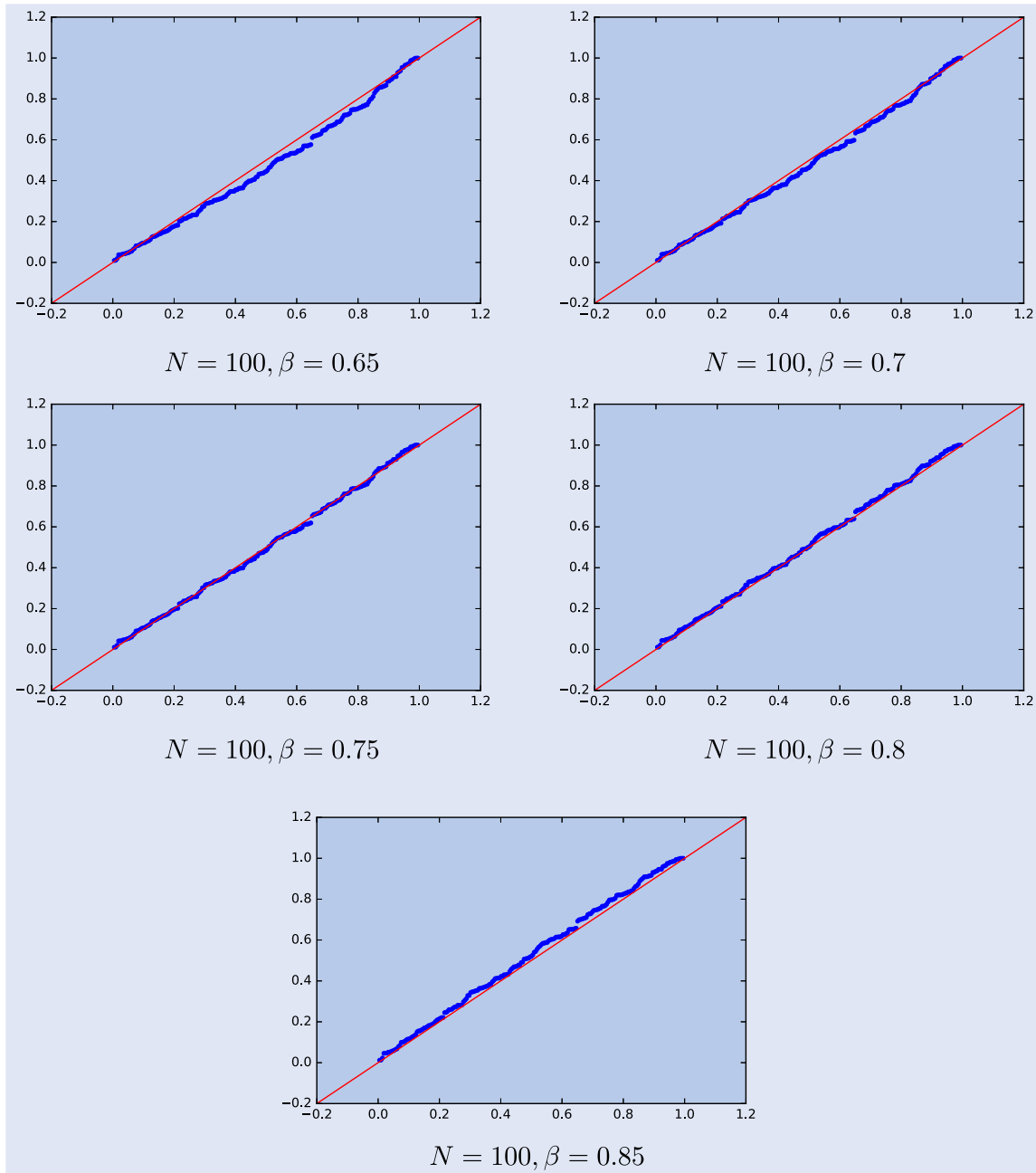


Figure A.2. How the choice of  $\beta$  affects the results reported in section 6, for the case of Deutsche Telekom AG (similar effects are observed for the other stocks). It is seen that the results are robust with respect to the choice of  $\beta \in (0.65, 0.85)$ . (a)  $N = 100, \beta = 0.65$ . (b)  $N = 100, \beta = 0.7$ . (c)  $N = 100, \beta = 0.75$ . (d)  $N = 100, \beta = 0.8$ . (e)  $N = 100, \beta = 0.85$ .