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Combining statistical intervals and market prices: The worst case state price distribution☆

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ABSTRACT

The paper shows how to combine (historical) statistical data and (current) market prices to form conservative trading strategies for options. This gives rise to a "worst case" state price distribution, which provides sharp price bounds for all convex European options. The paper provides for existence and computational algorithms under conditions which can be understood as "no arbitrage". The worst case distribution converges to a regular state price distribution if the number of traded options increases to span the space of possible option payouts.

1. Introduction

The pricing and hedging of options usually presupposes a known probability distribution P for the price S of the underlying security. When P is not known, one approach is to find a prediction set for, say, the volatility of S, and then to hedge in such a way that the option liability is covered whenever the prediction set is realized (Avellaneda et al., 1995; Lyons, 1995; Mykland, 2000, 2003a, Buff, 2002, Section 4.2, pp. 35–40). This procedure, however, fails to take account of the values of market traded options on the same security. This paper will show in the context of convex European options that such values can be incorporated in a uniform manner with the help of what we term a *worst case distribution*. The development is related to earlier work by Bergman et al. (1996), Frey and Sin (1999), Frey (2000) and Mykland (2003b).

The worst case distribution, therefore, has two inputs: (1) A prediction set for realized volatility, based, for example, on a time series analysis. One can, for example, fit an ARCH or GARCH model, or a time series based on estimated daily volatilities (from high frequency data). (2) Market traded European put and call options with the same expiry date for which one seeks to obtain the worst case distribution.

ARCH and GARCH type models go back to the seminal papers of Engle (1982) and Bollerslev (1986). There is a huge literature in this area, see, for example the surveys by Bollerslev et al. (1992, 1994), and Engle (1995). Time series based on estimated volatilities are explored in Andersen et al. (2003, 2005), Aït-Sahalia and Mancini (2008), Kang et al. (2010), Shephard and Sheppard (2010), and Ghysels and Sinko (2011). A simple example of how to incorporate a data analysis is given in Section 4.

The construction in this paper is exact and is valid for any number of market traded options. When the number of such options is large, however, we shall see in Section 8 that the worst case distribution converges to the state price distribution obtained by interpolating options. Meanwhile, if the statistical prediction interval is wide, we obtain in Section 5 that the

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contribution from the statistical interval translates into a sinusoidal shape for the worst case distribution between strike prices for market traded options.

It is conjectured that a worst case distribution will exist also with multiple and even large numbers of underlying securities, with the exception that the particularly simple form described in Section 2 will not hold.

The structure of the paper is as follows. The concept of a worst case distribution is introduced below in this section. Section 2 describes the structure of this distribution (and the accompanying trading strategy) for the case of one underlying security and multiple market traded options, culminating in the main result (Theorem 1, existence of the distribution and of an accompanying trading strategy). Section 3 discusses the relationship to implied volatilities. Section 4 gives a simple example of implementation in data. Section 6 discusses how to find the worst case distribution, in the form of three algorithms and a theoretical mathematical validation (Theorem 2). The trading strategy is further explained in Section 7. Most proofs are in the Appendix.

The relationship between historical data and options prices has previously also been discussed in terms of the relationship between realized and implied volatilities (Bollerslev et al., 2009; Zhang, 2012). For options with short horizon, the relationship is explored by Andersen et al. (2017). It also related to the so-called transport problem (see, e.g., Hobson (2010), Beiglböck et al. (2013), and Kallblad et al. (2017)).

To describe our results in this paper, we begin with the cast:
(i) The securities that are traded in the market:
• $S = (S_t)_{0 \leq t \leq T}$, the price process of a stock that pays no dividend.
• $\Lambda = (\Lambda_t)_{0 \leq t \leq T}$, the price of a zero coupon bond maturing at T, with value one dollar (or euro, or yuan, or krone).
• European call and put options maturing at T (see Section 2.1).
We can think of the value $S^*_t = S_t/\Lambda_t$ as the price of a forward contract on the stock S with maturity T. We shall assume that S^* is governed by an unknown probability P which belongs to a class $\mathcal{Q}$ of distributions. The main requirement on P is that S^* be an Ito process

$$dS^*_t = \mu_t S^*_t dt + \sigma_t S^*_t dW^*_t, \tag{1.1}$$

where (μ_t) and (σ_t) are random processes and (W^*_t) is a Wiener process.
(ii) A prediction bound on the volatility σ_t^2, in the form of a prediction interval I^{Ξ^+}

$$I^{\Xi^+} = \{(\sigma_t) : \int_0^T \sigma_u^2 du \leq \Xi^+\}. \tag{1.2}$$

(iii) A European payoff $g(S_T)$ to be made at time T, where g will mostly be taken to be convex.

The problem we wish to solve is the following. We look for a process (V_t) with two properties. V_t must be the value of a self financing dynamic portfolio in the market traded securities (see Section 2.2 for the definition of this concept). Also, V_T must cover the option liability if the prediction set is realized, *i.e.*,

$$V_T \geq g(S_T)\ P - \text{a.s. on } I^{\Xi^+},\ \text{ for all } P \in \mathcal{Q}. \tag{1.2a}$$

In particular, we wish to find the amount $V_0(g)$ which is the smallest starting value for such a self financing portfolio:

Definition. The quantity $V_0(g)$, provided it exists, will be called the conservative starting value for prediction set I^{Ξ^+} and payoff $g(S_T)$ at T.

A similar setup involving more general prediction sets and market traded securities is given in Mykland (2003a), which discusses the relevant concepts in some detail.

What is special about the development in the current paper is that we show the existence of a mapping from the prediction bound Ξ^+ to a cumulative distribution F^{Ξ^+} on S_T:

$$\Xi^+ \ \rightarrow\ F^{\Xi^+} \tag{1.3}$$

so that for all (non strictly) convex g which do not grow too fast, the conservative starting value for prediction set I^{Ξ^+} and payoff $g(S_T)$ at T is

$$V_0(g) = \Lambda_0 \int g(s) dF^{\Xi^+}(s). \tag{1.4}$$

In other words: there is *one* F^{Ξ^+} which can be taken as the worst case state price distribution for all convex payoffs $g(S_T)$. Convex options include calls and puts.

F^{Ξ^+} is what we call the *worst case distribution* for the market structure and prediction set described above. In consequence, since F^{Ξ^+} is independent of the payoff function g, one does not need to compute the value $V_0(g)$ for each g, but instead can find the distribution F^{Ξ^+}. Also, a distribution function is a more conceptual object. F^{Ξ^+} is a *state price distribution* in the sense used in finance, see, for example, Duffie (1996). (Note, however, that for non-convex g, the price is no longer conservative in the sense of (1.2a). In particular, for concave g, the inequality in (1.2a) is reversed. See also the discussion at the end of Section 9.)

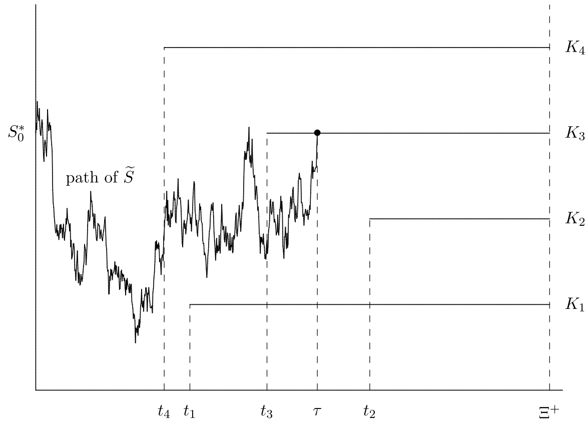

Fig. 1. Death of a stock price. Example of how the stopping time τ in Eq. (2.3) is formed. The quantities K_1 to K_4 are strike prices of market traded options, and Ξ^+ is the upper end of the prediction interval (1.2). $\widetilde{S}$ is a standardized stock price process following (2.1), and S_0^* is the inflated value of the actual stock price at time 0. The adjusted implied volatilities (the t_i's) are determined from market prices of options through Eq. (2.4).

2. Description and main theorem

2.1. The worst case distribution

For reasons of mathematical convenience, assume that all market traded options are European puts. This is no restriction on results, as puts and calls can be converted to each other via put–call parity, see Chapter 7.4 (pp. 174–175) of Hull (2003). In practice, one would want to take the most liquid of the put and the call to minimize the transaction cost. A put option with strike price K has value $(K - S_T)^+$ at maturity T. Its market price at time t will be denoted by P_t^K, and for the inflated (by the zero coupon bond) quantity, we use $P_t^{K*} = P_t^K/\Lambda_t$. A discussion of a formulation in terms of calls is given in Section 9.

We suppose that the market traded puts have strike prices $K_1, \ldots, K_q$.

To describe F^{Ξ^+}, introduce the standardized process

$$d\widetilde{S}_t = \widetilde{S}_t d\widetilde{W}_t, \quad \widetilde{S}_0 = S_0^*(= S_0/\Lambda_0), \tag{2.1}$$

where $\widetilde{W}$ is a standard Brownian motion. The corresponding probability distribution will be called $\widetilde{P}$. Then

$$F^{\Xi^+}(s) = \widetilde{P}(\widetilde{S}_{\tau\wedge\Xi^+} \le s), \tag{2.2}$$

where

$$\tau = \inf\{t : t \ge t_i \text{ and } \widetilde{S}_t = K_i \text{ for some } i,\ 1 \le i \le q\}, \tag{2.3}$$

and where the $t_1, \ldots, t_q$ are nonrandom, independent of Ξ^+, and the solution of

$$\widetilde{E}(K_i - \widetilde{S}_\tau)^+ = P_0^{K_i}/\Lambda_0 \quad 1 \le i \le q, \tag{2.4}$$

where the left hand side of (2.4) is taken as a function of the t_is through (2.3). A display illustrating how τ is formed is given in Fig. 1. Note that one can think of solving (2.4) in the t_i's, or in τ, subject to the specification (2.3).

Definition. The t_i's given by (2.1) and (2.3)–(2.4) will be called *adjusted (cumulative) implied volatilities*. (Compare to Section 3.1). The *worst case (state price) distribution* F^{Ξ^+} is then as given by (2.2).

Proposition 1. *If (2.3)–(2.4) have a solution, it is unique.*

This follows directly from Theorem 2 in Section 6. The case of no solution will be considered in Section 3.2. We shall suppose that the solution of (2.4) satisfies

$$\max_{i=1,\ldots,q} t_i \le \Xi^+, \tag{2.5}$$

without which there is "arbitrage" in a sense to be discussed also in Section 3.2.

2.2. Trading strategies: Theoretical considerations

We define the class $\mathcal{Q}$ given the initial inflated values S_0^* and $P_0^{K_i*}$, $1 \le i \le q$, as follows:

Definition. $\mathcal{Q}$ is a collection of distributions on the set of functions $\Omega = \mathbb{C}[0,T] \times \mathbb{D}[0,T]^q$, and $(S_t^*, P_t^{K_1*}, \ldots, P_t^{K_q*})$ is the coordinate process. Every $P \in \mathcal{Q}$ must satisfy (1.1), and the coordinate process must have the correct initial value $(S_0^*, P_0^{K_1*}, \ldots, P_0^{K_q*})$ with probability one. Also, for given P , the process (σ_t) must be bounded P-a.s. (but one does not need to know the bound). Finally, each P must be mutually absolutely continuous with a P^* under which the coordinate process is a martingale. The collection of such P^*s will be called $\mathcal{Q}^*$. ■

The requirement of equivalence to a "risk neutral measure" P^* is the most convenient way to avoid arbitrage opportunities in the market. Not only is S^* a martingale under P^*, but processes P^{K_i} can be found, with correct initial value $P_0^{K_i}$, such that P^{K_i*} is a martingale.

The filtration describing the market will be called $(\mathcal{F}_t)$ and can be any that is right continuous, and which makes the coordinate process adapted and martingales under all $P^* \in \mathcal{Q}^*$. This $(\mathcal{F}_t)$ can be either the smallest filtration with these properties, or anything bigger satisfying the same criteria, such as the "analytic completion" discussed in Mykland (2003a). The latter is particularly useful from a statistical perspective. A precise discussion of the conceptual issues involved can be found in Sections 2, 3.2 and 4 of this earlier paper. All processes are taken to be *càdlàg* and adapted to $(\mathcal{F}_t)$.

A self financing dynamic portfolio (V_t), with inflated value $V_t^* = V_t/\Lambda_t$, is defined as a process which, for any $P \in \mathcal{Q}$, can be represented by $V^* = H^* - D^*$, where D^* is non decreasing (D^* provides for the case where profit can be removed from the portfolio), and H^* is a stochastic integral with respect to S^* and the P^{K_i*}. Stochastic integrals are as defined in Chapter I.4d (pp. 46–51) of Jacod and Shiryaev (2003). The random variables $\{H_\lambda^{*-}\}$ must be uniformly integrable for all $P^* \in \mathcal{Q}^*$, where the λ describes the set of stopping times taking values in $[0,T]$.

We can confine ourselves to considering inflated quantities by virtue of numeraire invariance, see Chapter 6 of Duffie (1996). In the current situation where we inflate by the zero coupon bond Λ, F^{Ξ^+} and the option liabilities $g(S_T)$ and $(K-S_T)^+$ also only depend on inflated quantities, since $S_T = S_T^*$. All conditions, therefore, may be, and are, directly imposed on the inflated processes. The author learned this device from the paper of El Karoui et al. (1998).

The uniform integrability condition is intended to avoid doubling strategies, *cf.* Chapter 6.B of Duffie (1996), and p. 670 of Mykland (2000). For other discussions of self financing trading strategies, see Harrison and Kreps (1979), Harrison and Pliskà (1981), Chapter 6 of Duffie (1996), and, in the context of super-hedging, Cvitanić and Karatzas (1992, 1993), El Karoui and Quenez (1995), Eberlein and Jacod (1997), Karatzas (1996), Karatzas and Kou (1996), Kramkov (1996), and and Föllmer and Leukert (1999, 2000).

2.3. Form of the self financing portfolio

The trading strategy that starts with $V_0(g)$ only requires a static position in the market traded options. We explain this in the following.

At the outset of trading, suppose one takes a position of λ_i units in the put option with strike price K_i. We let this position be static, in the sense that we hold it without change until expiry at time T. The problem then changes to that of covering a liability of the form $h_\lambda(S_T)$, where

$$h_\lambda(s) = g(s) - \sum_{i=1}^{q} \lambda_i[(K_i - s)^+ - P_0^{K_i*}]. \tag{2.6}$$

This is since a loan of P_0^K dollars at time 0 requires a repayment of P_0^{K*} dollars at maturity.

The two problems are equivalent, and $V_0(h_\lambda) = V_0(g)$. It turns out, however, that there is one value of $\lambda = (\lambda_1, \ldots, \lambda_q)$ so that the dynamic hedge for liability $h_\lambda(S_T)$ only involves trading in the forward contract S^* (in other words, H^* is a stochastic integral over S^* only).

Practically, this is important because transaction costs are normally higher in the market traded options than in the forward contract S^* (which, if need be, can be created by securities S and Λ). We otherwise ignore the issue of trading cost in this paper.

2.4. The main result

Theorem 1. *Assume that (2.3)–(2.4) have a solution. Also suppose (2.5). Then there exists a mapping (1.3) so that for all (non strictly) convex g for which $|g(s)|$ increases no more than polynomially in s as $s \to \infty$, the conservative starting value $V_0(g)$ for prediction set I^{Ξ^+} and payoff $g(S_T)$ at T exists and is given by (1.4). F^{Ξ^+} is given by (2.2). There is a trading strategy that starts with value $V_0(g)$, and which only requires a static position in the market traded options, as described in Section 2.3.*

The theorem is proved in the Appendix. The general issue of how to compute the worst case distribution and its corresponding hedging strategy is discussed below in Sections 6 and 7. First, however, an interpretation of the t_is.

3. Implied volatilities, and arbitrage

Note first that the construction in Section 2.1 is a conversion of time to volatility scale. The t_i's can be seen as a form of implied volatilities. First, consider the case where this is exactly true.

3.1. Connection to implied volatility

The Black and Scholes (1973) and Merton (1973) form of the price at time 0 of a European put option with strike K is $BSMP(S_0, -\log \Lambda_0, \sigma^2 T)$, where

$$BSMP(S, R, \Xi) = K\exp(-R)\Phi(-d_2) - S\Phi(-d_1)$$

with

$$d_{1,2} = (\log(S/K) + R \pm \Xi/2)\,/\sqrt{\Xi},$$

and where the instantaneous volatility σ_t^2 is taken to be constant and equal to σ^2. Also, Φ is the standard normal cumulative distribution function.

Of course, the model underlying this formula may not be valid, and prices do not generally behave as if it were, see for example Hull (2003), but it is customary to invert the function to find so-called implied volatilities. We shall here do this on the cumulative scale.

Definition. The *(cumulative) implied volatility* at time zero for strike price K, Ξ_K, is defined by

$$BSMP(S_0, -\log \Lambda_0, \Xi_K) = P_0^K$$

It is also natural to call the t_i from Theorem 1 the *conditional (cumulative) implied volatility* at time zero for strike price K_i. For both objects, we omit "cumulative" unless this is not clear from the context. ■

The first connection to F^{Ξ^+} is now as follows.

Example 1. If all traded options have the same implied volatility:

$$\Xi_{K_1} = \cdots = \Xi_{K_q},$$

then

$$t_1 = \cdots = t_q = \Xi_{K_1}$$

This is easily seen from Theorem 1. In the more general case of unequal implied volatilities, $\text{argmin}_i(t_i) = \text{argmin}_i(\Xi_{K_i})$ (there can be several such i's, or course) and t_i and Ξ_{K_i} must coincide for these indices i. Also, one can see more generally from the convexity of the put payoff that $t_i \geq \Xi_{K_i}$.

3.2. The t_is and arbitrage

Arbitrage is the construction of a self financing strategy which makes a profit for some $P \in \mathcal{Q}$, and which does not lose money almost surely, for all $P \in \mathcal{Q}$. For issues related to the avoidance of doubling strategies, see Chapter 6 of Duffie (1996).

There are two ways that arbitrage can occur in our setting. One is if (2.5) is violated. The other is if the system (2.3)–(2.4) has no solution. The latter case is the most clear cut (with proof in the Appendix):

Proposition 2. *If (2.3)–(2.4) have no solution, then there is arbitrage.*

The former case is one of "statistical arbitrage", in the sense that the prediction interval I^{Ξ^+} has to be realized, otherwise a loss can occur.

Proposition 3. *Assume that (2.3)–(2.4) have a solution, but that $\max_{i=1,\ldots,q} t_i > \Xi^+$. Then there is a trading strategy which, provided I^{Ξ^+} is realized, yields a positive return of at least c^* at time T, almost surely for any $P \in \mathcal{Q}$. Here c^* is a positive constant independent of P.*

Proof of Proposition 3. Let $\mathcal{S}$ be the indices for which the t_is are smaller than Ξ^+, and let I be the index i corresponding to the smallest t_i strictly greater than Ξ^+. We shall be interested in earning money on the put payoff $g(s) = (K_I - s)^+$. The following argument remains valid if $\mathcal{S}$ is empty.

Note that the t_i, $i \in \mathcal{S}$, solve (2.3)–(2.4) for this index set. Let $V_0(g)$ be the price given by Theorem 1 based on hedging in S, Λ and the puts with strike K_i, $i \in \mathcal{S}$. Our claim is now that

$$V_0(g) < P_0^{K_I}. \tag{3.1}$$

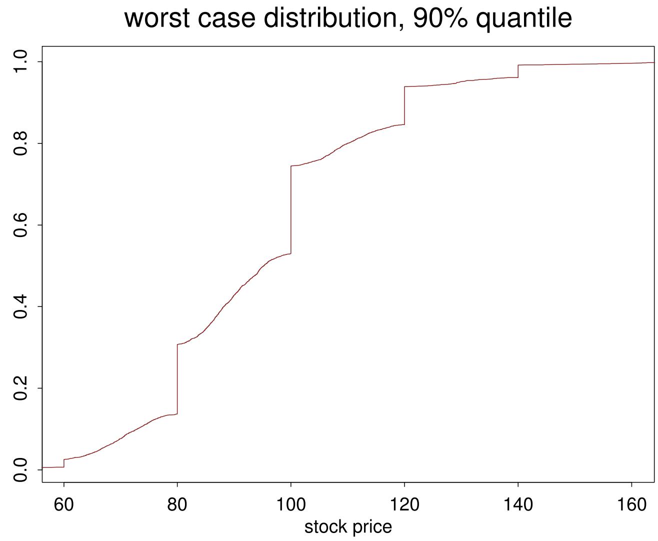

Fig. 2. Worst case distribution based on the 90% posterior quantile from Table 1, and incorporating traded options with strike prices 60, 80, 100, 120, and 140. The traded options were taken to have Black–Scholes implied volatility (see Section 3) equal to the 80% posterior quantile.

Thus, one can sell the option with payoff $g(S_T)$, start a self financing trading strategy with initial value $V_0(g)$, and be sure that the liability is covered so long as I^{Ξ^+} is realized. The profit is at least $c = P_0^{K_I} - V_0(g)$, if taken at time 0, or $c^* = c/\Lambda_0$ if taken at time T (any random additional profit has, of course, to be taken at time T).

To see (3.1), let τ be given on the form (2.3) based on t_i, $i \in \mathcal{S} \cup \{I\}$. Also let $K_- = \sup\{K_i < K_I,\ i \in \mathcal{S}\}$ and $K_+ = \inf\{K_i > K_I,\ i \in \mathcal{S}\}$. Let $C = \{\tau \geq \Xi^+$ and $\widetilde{S}_{\Xi^+} \in (K_-, K_+)\}$. It then follows that

$$P_0^{K_I} - V_0(g) = \widetilde{E}[g(\widetilde{S}_\tau) - g(\widetilde{S}_{\tau\wedge\Xi^+})] = \widetilde{E}[X],$$

where X is zero outside C, and otherwise positive. ■

4. An implementation with data

We give an example of how the worst case distribution can look. Table 1 provides (Bayesian) posterior quantiles for the square root of $\Xi = \int_0^T \sigma_t^2 dt$ for the S&P 500. The posterior distribution is based on work by Jacquier et al. (1994), which analyzes (among other series) the S&P 500 data recorded daily, and has also been used by Mykland (2003a), which provides further discussion. We emphasize that the prediction intervals can be combined with either frequentist, Bayesian, or fiducial inference.

Table 1
S&P 500: Posterior distribution of $\Xi = \int_0^T \sigma_t^2 dt$ for $T =$ one year.

Posterior coverage	50%	80%	90%	95%	99%
Upper end $\sqrt{\Xi^+}$ of posterior interval	.168	.187	.202	.217	.257

The posterior is conditional on $\log(\sigma_0^2)$ taking the value of the long run mean of $\log(\sigma^2)$.

We now assume that market traded options are available for a certain number of strike prices. For a given set of securities prices, a worst case distribution is given in Fig. 2, corresponding to the 90% posterior quantile. A comparison of the worst case distribution for different quantiles is given in Fig. 3.

We see in Figs. 2–3 that the worst case distribution F^{Ξ^+} has point mass at the strike prices (inflated by Λ_0) of the market traded options. These point masses are the contribution to F^{Ξ^+} of the information in the market traded options. Meanwhile, the (close to) sinusoidal shape of F^{Ξ^+} between the strike prices represents the contribution of the information in the statistical prediction interval $I^{\Xi^+} = \{(\sigma_t) : \int_0^T \sigma_u^2 du \leq \Xi^+\}$. The probability distribution F^{Ξ^+} thus has a decomposition into a continuous part (the econometric information) and a discrete part (due to the market information in the options).

Fig. 3 illustrates that for a short prediction interval ($\Xi^+ = .168$, in this case with posterior coverage 50%), the statistical information dominates, and F^{Ξ^+} is smooth. As the upper limit Ξ^+ of the prediction interval increases, the statistical

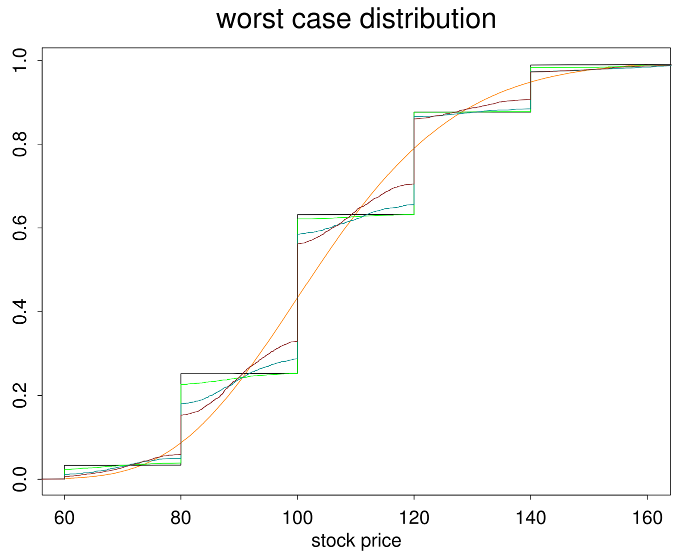

Fig. 3. Worst case distributions for the 80% (yellow curve), 90% (red), 95% (blue), 99% (green) and 100% (black) posterior quantiles from Tabled 1. Traded options and their values are as in Fig. 2. Note that at each of the strike prices K, more point mass is placed on K the higher the posterior coverage of the statistical interval. See further comments in the text.

information (the smooth part of F^{Ξ^+}) becomes less important, and more weight is placed on the discrete part (the information from the market traded options).

The shape of the continuous part of F^{Ξ^+} is asymptotically sinusoidal (as $\Xi^+ \to \infty$), cf. Section 5. On the other hand, if the strike prices K of the market traded options become dense in $(0, \infty)$, F^{Ξ^+} will be asymptotically fully dominated by the options information, and will tend to a regular state price distribution (Section 8). When this happens, the point mass part of F^{Ξ^+} will converge to a distribution that is often (but not necessarily) continuous, while the continuous part of F^{Ξ^+} will disappear.

5. Asymptotic form of the worst case distribution

As can be seen in Figs. 2–3, the shape of the continuous part of the distribution has a characteristic form. We here give the asymptotic form of this shape.

Proposition 4. *Suppose that (2.3)–(2.4) have a solution, and let (2.5) be satisfied. Then, for $1 \le i \le q-1$, and $K_i < s < K_{i+1}$, and as $\Xi^+ \to \infty$,*

$$\frac{d}{ds}\widetilde{P}(\widetilde{S}_{\tau\wedge\Xi^+} \le s \mid K_i < \widetilde{S}_{\tau\wedge\Xi^+} < K_{i+1}) = c_i s^{-3/2} \sin\left(\pi \frac{\log(s/K_i)}{\log(K_{i+1}/K_i)}\right) + o(1), \tag{5.1}$$

for each s, where the c_i are constants. Similarly, at the edges, both $d\widetilde{P}(\widetilde{S}_{\tau\wedge\Xi^+} \le s \mid \widetilde{S}_{t_1}$ and $\widetilde{S}_{\tau\wedge\Xi^+} < K_1)/ds$ and $d\widetilde{P}(\widetilde{S}_{\tau\wedge\Xi^+} \le s \mid \widetilde{S}_{t_1}$ and $\widetilde{S}_{\tau\wedge\Xi^+} > K_q)/ds$ have the form, for $I = 1$ or q,

$$c' s^{-3/2}(t-t_I)^{-1/2}\left|\phi\left(\frac{\log(s/\widetilde{S}_{t_I})}{(t-t_I)^{1/2}}\right) - \phi\left(\frac{\log(s/K_I) + \log(\widetilde{S}_{t_I}/K_I)}{(t-t_I)^{1/2}}\right)\right|$$

on the respective sets $\{0 < s < K_1, \widetilde{S}_{t_1} < K_1, t > t_1\}$ and $\{s > K_q, \widetilde{S}_{t_q} > K_q, t > t_q\}$, and are otherwise zero. Here, the proportionality constant depends on $\widetilde{S}_{t_1}$.

6. Finding $t_1, \ldots, t_q$: the general case

We here present algorithms for finding the conditional implied volatilities. Set $P_t^{K*} = P_t^K/\Lambda_t$. In other words, this is the inflated put price at t. We start by characterizing the output, and the algorithms are stated just after the theorem.

Theorem 2.

(a) *Algorithms 1 and 2 yield the same result.*

(b) (2.3)–(2.4) have a solution if and only if either algorithm does not return a "no solution" message.
(c) In the absence of a "no solution" message, the output of either algorithm is unique, and satisfies (2.3)–(2.4).
(d) If (2.3)–(2.4) have a solution, then it is unique, and is provided by either algorithm.

Obviously, one of statements (b) and (d) is redundant given (c), but it seemed to improve readability to have them both there. The result is proved in the Appendix. Recall that the case of no solution to (2.3)–(2.4) has been discussed in Proposition 2.

Before going to the two main algorithms, we give the core component of both as Algorithm 0. At the end of the section, we show some more details on how to compute Step 1 in the following.

Algorithm 0. We use an index set $\mathcal{S} \subseteq \{1, \ldots, q\}$, where index i corresponds to K_i, $P_0^{K_i*}$, and t_i. t_j's where j is not in $\mathcal{S}$ have either already been found, or are irrelevant. Also, a provisional version of τ is given.

(0) If, for any $i \in \mathcal{S}$, $\widetilde{E}(K_i - \widetilde{S}_\tau)^+ \leq P_0^{K_i*}$, then the algorithm terminates with a "no solution" message. Otherwise:
(1) For $i \in \mathcal{S}$, find t_i by:

$$\widetilde{E}(K_i - \widetilde{S}_{\tau \wedge t_i})^+ = P_0^{K_i*}.$$

(2) Remove all i corresponding to the smallest t_i from $\mathcal{S}$ (there can be ties between the is).
(3) Set

$$\tau = \inf\{t : t \geq t_i \text{ and } \widetilde{S}_\tau = K_i \text{ for } i \text{ not in } \mathcal{S}\}$$

The algorithms described in Theorem 2 are then given by

Algorithm 1. Finds $t_1, \ldots, t_q$ in accordance with Theorem 2. This is the loop version.
If any option value is non-positive, the algorithm terminates with a "no solution" message. Otherwise, set initial values: $\mathcal{S} = \{1, \ldots, q\}$ and $\tau = +\infty$. Then
Loop:
Go through Steps 0–3 in Algorithm 0
Repeat loop until $\mathcal{S}$ is empty (unless Step 0 has triggered early termination)

For aficionados of recursion, an alternative scheme would be the following.

Algorithm 2. Find $t_1, \ldots, t_q$ in accordance with Theorem 2. This is the recursive version.
For index sets $\mathcal{S} \subseteq \{1, \ldots, q\}$, and for stopping times τ, a functional $F = F(\mathcal{S}, \tau)$ is defined below. It returns either a solution, or detects its absence.
If any option value is non-positive, the algorithm terminates with a "no solution" message. Otherwise, the overall solution to the algorithm is $F(\{1, \ldots, q\}, \tau = +\infty)$.
Definition of F:
If $\mathcal{S}$ is the empty set, then $F(\mathcal{S}, \tau)$ returns no information. Otherwise
Carry out items 0–3 from Algorithm 0. Let $i_1 < \cdots < i_r$ be the indices picked out by Step 2. Then

(4) Define $\mathcal{S}_1 = \mathcal{S} \cap \{1, \ldots, i_1 - 1\}$, $\mathcal{S}_{r+1} = \mathcal{S} \cap \{i_r + 1, \ldots, q\}$, and, for $v = 2, \ldots, r$, $\mathcal{S}_v = \mathcal{S} \cap \{i_{v-1} + 1, \ldots, i_v - 1\}$
(5) $F(\mathcal{S}_v, \tau)$ returns the $\{t_j,\ j \in \mathcal{S}_v\}$ for all v. Hence $F(\mathcal{S}, \tau)$ returns $\{t_j,\ j \in \mathcal{S}\}$. If either of the $r + 1$ recursions returns a "no solution" message, then $F(\mathcal{S}, \tau)$ returns a "no solution" message.

A useful fact is the following.

Proposition 5. *In Algorithm 1, if index i is picked out from $\mathcal{S}$ in an earlier pass through the loop than j, then $t_i < t_j$. Similarly, in Algorithm 2, if index i is picked out from $\mathcal{S}$ at an earlier point in the recursion than j, then $t_i < t_j$.*

Remark 1. The calculation of t_i in Step 1 in Algorithm 0 can be implemented as follows. Let t_- be the largest previous value of t_j's selected by Step 3 earlier in the loop or the recursion, or set $t_- = 0$ if none has been selected. In view of Step 3 and of Proposition 5, and also of the requirement that traded options values be positive (so $\Xi_{k_i} > 0$), $t_i > t_-$. Let $a = \max\{j \text{ not in } \mathcal{S},\ j < i\}$ or $a = 0$ if this set is empty. Similarly, let $b = \min\{j \text{ not in } \mathcal{S},\ j > i\}$ or $b = q + 1$ if this set is empty. Take $K_0 = 0$ and $K_{q+1} = +\infty$. Note that, on the set $A = \{K_a \leq \widetilde{S}_{t_-} \leq K_b\} \cap \{\tau \geq t_-\}$, the preexisting τ is given by

$$\tau = \inf\{t \geq t_- \ :\ \widetilde{S}_t = K_a \text{ or } K_b\}.$$

It follows that t_i is given by by Step 1 via

$$\widetilde{E}(DO(S_{t_-}, t_-, t_i)I_A) = P_0^{K_i*} - \widetilde{E}(K_i - \widetilde{S}_\tau)^+ I_{A^c} \tag{6.1}$$

where, for $K_a \leq s \leq K_b$,

$$DO(s, t_-, t) = \widetilde{E}[(K_i - \widetilde{S}_{\tau \wedge t})^+ | S_{t_-} = s]$$

is the value at t_- of the double barrier put option with starting value s and cumulative volatility $t - t_-$ (from t_- onward). Since this function is known to be continuous and strictly increasing in the volatility, it follows that (6.1) has a unique solution on $(t_-, +\infty)$ unless the algorithm has been terminated in Step 0.

7. Finding the static hedge coefficients λ_i

A remaining issue is to define the λ_i's from (2.6). For given payoff function g, let μ be the *second derivative measure* associated with g. In other words, μ is a measure which satisfies $\mu([x, y)) = g'(y) - g'(x)$ for all x, y outside a countable set on the real line. μ exists since g is convex, see Karatzas and Shreve (1991), pp. 212–214, for details.

For any τ on the form (2.3),

$$\widetilde{E}g(\widetilde{S}_{\tau\wedge\Xi^+}) = \int \widetilde{E}(K - \widetilde{S}_{\tau\wedge\Xi^+})^+ \mu(dK). \tag{7.1}$$

$\widetilde{E}(K - \widetilde{S}_{\tau\wedge\Xi^+})^+$ is continuously differentiable in each t_i. Therefore so is $\widetilde{E}g(\widetilde{S}_{\tau\wedge\Xi^+})$. (The smoothness follows by the same argument as in Remark 1, and also using derivations akin to those of Section 2.8.C (pp. 97–100) of Karatzas and Shreve (1991), in this case modified with the help of Girsanov's Theorem.) As a consequence of the proof of Lemma 2 in the Appendix, one thus obtains the λ_is through

$$\frac{\partial \widetilde{E}g(\widetilde{S}_{\tau\wedge\Xi^+})}{\partial t_i} = \sum_{j=1}^{q} \lambda_j \frac{\partial \widetilde{E}(K_j - \widetilde{S}_\tau)^+}{\partial t_i}, \quad i = 1, \ldots, q, \tag{7.2}$$

when this expression is evaluated at the values t_i which solves Eqs. (2.3)–(2.4). Note, from Remark 1, that $\partial\widetilde{E}(K_j - \widetilde{S}_\tau)^+/\partial t_i$ is nonzero only when $t_j > t_i$. Hence, if one orders the indices so that $t_1 \le \cdots \le t_q$, the system of equations (7.2) involves a triangular matrix, and hence has a unique solution.

8. Behavior of the worst case distribution as the strike prices of traded options become dense

There is a substantial literature which considers the case where $P_0^{K,*}$ exists for all $K \in (0, \infty)$. In this case, a state price distribution can be derived for the relevant maturity, see, e.g., Breeden and Litzenberger (1978) and Aït-Sahalia and Lo (1998). If we call this distribution F^c (c for complete), one obtains that

$$F^c(K) = \frac{d}{dK} P_0^{K,*}.$$

We here show that the worst case distribution F^{Ξ^+} converges to F^c as the K becomes dense in the interval $(0, \infty)$. This means that the importance of the prediction interval (1.2) diminishes as the number of traded options increases.

Theorem 3. *Let $\mathcal{K}_q = \{K_{1,q}, \ldots, K_{q,q}\}$ be (non-nested) strike prices in $(0, \infty)$. For each q, assume that put options with strike prices $\mathcal{K}_q$ are traded, and that the worst case distribution $F_q^{\Xi^+}$ is formed on the basis of these and the prediction interval (1.2). Assume that the conditions of Theorem 1 are satisfied for each q.*

Assume that there is a state price distribution F^c (which is latent for finite q) in the sense that $P_0^{K_{l,q},*} = E_{F^c}(K_{l,q} - S)^+$, for $l = 1, \ldots, q$, and for each q, where S has distribution F^c. Assume that F^c is consistent with the interval (1.2), in the sense that for any bounded convex g, $E_{F^c} g(S) \le \widetilde{E}g(\widetilde{S}_{\Xi^+})$.

Finally, assume that, as $q \to \infty$, $K_{1,q} \to 0$, $K_{q,q} \to \infty$, and $\max_{2\le l\le q} \; K_{l,q} - K_{l-1,q} \to 0$. Then

$$F_q^{\Xi^+} \xrightarrow{\mathcal{L}} F^c \text{ as } q \to \infty,$$

in the sense of convergence in law (weak convergence) of distribution functions $F_q^{\Xi^+}$ to limiting distribution function F^c.

For properties of convergence in law, including metrizability, see Jacod and Shiryaev (2003), Chapter VI.3a (pp. 347–348), as well as Billingley (1968), Pollard (1984), and van der Vaart and Wellner (1996).

9. Other issues

Formulation in terms of call options. The results can equally well be put in terms of call options, but one then needs an additional *caveat*. We use in the proof of Theorem 1 that $(K_i - \widetilde{S}_{\tau\wedge t})^+$ is a uniformly integrable martingale from $\max t_i$ onward. For $(\widetilde{S}_{\tau\wedge t} - K_i)^+$ the same statement is true for $i < q$, but for $i = q$ it is not: $(\widetilde{S}_{\tau\wedge t} - K_q)^+$ is a martingale, but its limit at $t \to +\infty$ is zero. This can be remedied by requiring τ in (2.3) to be bounded by a constant c, which one can take to be Ξ^+, or $\max t_i$, or any other number greater than either. Algorithm 1 then uses the starting value $\tau = c$ rather than $\tau = +\infty$. The reason we preferred to avoid this formulation is that it does not make it clear that τ is, in fact, independent of Ξ^+. This problem does not arise when evaluating the expectation of $g(\widetilde{S}_{\tau\wedge\Xi^+})$, since Ξ^+ is an upper bound on the stopping time.

An alternative formulation in terms of calls would be to replace (2.4) by

$$\widetilde{E}(\widetilde{S}_\tau - K_i)^+ = C_0^{K_i}/\Lambda_0,$$

with the side condition that $\tau \leq t_q$ on the set $\{S_{t_q} \geq K_q\}$. This is again somewhat more inelegant than the formulation with puts, which is why we have stuck with Theorem 1 as it is.

Lower bounds for prices of convex options. There is no corresponding state price distribution. For a call or put option payoff $g(s)$ with strike price K, the stopping time τ which would minimize $\widetilde{E}g(\widetilde{S}_\tau)$ would concentrate on the set $\{\tau = \Xi^{*+}$ or $S_\tau = K$ or $S_\tau = K_i\}$. Thus the distribution would depend on the strike price. Also, if one also introduces a lower bound on $\int_0^T \sigma_t^2 dt$, this lower bound can also be effective.

Appendix. Proofs of results

A.1. Proofs of results outside Sections 5 and 8

Logical sequence of proofs. For ease of reference, the proofs are given in the order of appearance of the results in the main text. The results, however, depend on each other in a different logical sequence, and should be taken to be proved in the following order. Proposition 5 is proved from scratch. Then Theorem 2 is proved using Proposition 5, Proposition 1 is a direct corollary to Theorem 2, and Theorem 1 uses Proposition 1. Proposition 3 (proved in the main text) uses Theorem 1. Proposition 2 uses both Theorem 1 and the development in Theorem 2. Finally, Lemmas 1 and 2 are embedded in the proof of Theorem 1.

The proofs of Proposition 4 and Theorem 3 depend on the rest of the development, and are given in Appendix A.2.

Proof of Theorem 1. *Exit Λ, followed by bear.* As discussed in Section 2.2, we make use of the inflating by Λ_t to restate the problem in terms of finding a self financing strategy in S^* and the P^{K_i*}. First of all, the option liability $\eta = g(S_T)$ can be re-expressed as $\eta = g(S_T^*)$, since $\Lambda_T = 1$. Second, by numeraire invariance (see Duffie (1996), Chapter 6), V_t is a self financing portfolio in the securities S, Λ and the P^{K_i} if and only if $V_t^* = V_t/\Lambda_t$ is a self financing strategy in the forward contracts given by S^* and the P^{K_i*}. Since $V_T = V_T^*$, the liability η will by covered by V if and only if it is covered by V^*. It is enough, therefore, to prove Theorem 1 as if the Λ process were identically equal to 1. In other words, as if uninvested cash were stored in the mattress.

The function g can be taken to be bounded below. Without loss of generality, we can assume that the g function is bounded below. This is because, by convexity, there is a constant c_1 so that $g_1(s) = g(s) + c_1 s$ is bounded below. One can then hedge the liability $g(S_T)$ by instead hedging $g_1(S_T)$, and in addition take a static position of $-c_1$ units of security S.

Reformulation of the problem. Recall in the following that the reason for using risk-neutral measures when optimizing in this proof, is that the lowest starting value for a conservative trading strategy covering the liability $h_\lambda(S_T)$ is given by the supremum of $E^* h_\lambda(S_T)$, where P^* describes all risk neutral probabilities for which $P^*(I^{\Xi^+}) = 1$. This is a consequence of Theorem 2.1 (pp. 1418–19) in Mykland (2003a).

We shall consider related problems on the set $\Omega' = \mathbb{C}[0, T]$, with coordinate process (S_t^*), and pre-specified initial value S_0^*. We work with various collections of probabilities.

$\mathcal{R}^*_{\Xi+}$ is the set of probabilities so that S^* is a martingale satisfying (1.1), in particular,

$$dS_t^* = \sigma_t S_t^* dW_t^*, \tag{A.1}$$

and so that for given $P^* \in \mathcal{R}^*_{\Xi+}$ the process (σ_t) must be bounded with probability one. We also require $P^*(I^{\Xi^+}) = 1$ for all $P^* \in \mathcal{R}^*_{\Xi+}$. Also,

$$\mathcal{P}^*_{\Xi+} = \{\, P^* \in \mathcal{R}^*_{\Xi+} : \; E^*(K_i - S^*{}_T)^+ = P_0^{K_i*} \,\}.$$

Note that $\mathcal{P}^*_\infty$ extends to $\mathcal{Q}^*$ on Ω. Define

$$\underline{V}_0(g) = \sup_{P^* \in \mathcal{P}^*_{\Xi+}} E^* g(S_T^*). \tag{A.2}$$

and

$$\bar{V}_0(g; \lambda) = \sup_{P^* \in \mathcal{R}^*_{\Xi+}} E^* h_\lambda(S_T^*). \tag{A.3}$$

By the Dambis (1965)/Dubins and Schwartz (1965) time change,

$$\bar{V}_0(g; \lambda) = \sup_{0 \leq \tau \leq \Xi^+} \widetilde{E} h_\lambda(\widetilde{S}_\tau), \tag{A.4}$$

where τ describes all stopping times in the interval $[0, \Xi^+]$. Finally set

$$\bar{V}_0(g) = \inf_\lambda V_0(g; \lambda) \tag{A.5}$$

First, by a Lagrange argument, we get

Solution of (A.4)–(A.5), and equality to (A.2). Problem (A.4) can be solved using standard procedure for American options (see Karatzas (1988), Myneni (1992), and the references therein), which yield that the supremum is attained at a stopping time τ^*_λ. The American option argument makes use of the *Snell envelope* for h_λ, which reenters the discussion below:

$$\mathrm{SE}(s, \Xi; \lambda) = \sup_{\Xi \le \tau \le \Xi^+} \widetilde{E}(h_\lambda(\widetilde{S}_\tau) \mid S_\Xi = s).$$

By an argument similar to that of Theorem 3 of Mykland (2003b), $\tau^*_\lambda = \tau_\lambda \wedge \Xi^+$, where

$$\tau_\lambda = \inf\{t : t \ge t_i^\lambda \text{ and } \widetilde{S}_\tau = K_i \text{ for some } i,\ 1 \le i \le q\}.$$

Lemma 1. *Suppose that $g(s)$ is bounded below and increases no more than polynomially as $s \to \infty$. Also assume that (2.3)–(2.4) have a solution τ satisfying (2.5), Then there is such a τ so that $\bar{V}_0 = \widetilde{E}g(\widetilde{S}_{\tau\wedge\Xi^+})$.*

Proof of Lemma 1. Consider a sequence of λs so that $\bar{V}_0(g; \lambda)$ converges to $\bar{V}_0(g)$. Since the t_i^λs live in the compact set $[0, \Xi^+]^q$, there is a subsequence which is convergent in the t_i^λs. Call the relevant limit t_i, and define τ^* as the limit of the τ^*_λ as one passes through the subsequence. By uniform integrability, and since $\widetilde{S}$ is a continuous process, $\widetilde{E}g(\widetilde{S}_{\tau^*_\lambda}) \to \widetilde{E}g(\widetilde{S}_{\tau^*})$ and, for $i = 1, \ldots, q$, $\widetilde{E}(K_i - \widetilde{S}_{\tau^*_\lambda})^+ \to \widetilde{E}(K_i - \widetilde{S}_{\tau^*})^+$. Also, τ^* can be taken to be on the form $\tau \wedge \Xi^+$, where τ is on the form (2.3), again since $\widetilde{S}$ is a continuous process.

τ^* must satisfy (2.4), otherwise the infimum in (A.5) would not be finite. By the argument just underneath the statement of Lemma 1, τ^* can be replaced by τ for the purpose of satisfying (2.4). ■

If (2.3)–(2.5) have a solution, $\underline{V}_0(g) > -\infty$. This is because, since $(K_i - \widetilde{S}_{\tau\wedge t})^+$ is a uniformly integrable martingale from $\max t_i$ onward, the constraint (2.4) will also be satisfied if τ is replaced by $\tau \wedge \Xi^+$, provided $\max t_i \le \Xi^+$. Hence also $\underline{V}_0(g) \le \bar{V}_0(g)$. Lemma 1 then shows that $\underline{V}_0(g) = \bar{V}_0(g)$.

Connection to the worst case distribution, and the trading strategy. Now combine Lemma 1 with Proposition 1 to see that

$$\underline{V}_0(g) = \bar{V}_0(g) = \int g(s)dF^{\Xi^+}(s).$$

Also, observe that $\underline{V}_0(g)$ must be a lower bound for the starting value V_0^* of any self financing strategy (V_t^*) for which $V_T^* \ge g(S_T^*)$ on I^{Ξ^+}. This is because any strategy would have to be self financing and solvent under each $P^* \in \mathcal{P}^*_{\Xi^+}$, cf. the development in Mykland (2000, 2003a), and the literature on superhedging cited in the introduction to the former of these two papers, cf. also Section 2.2 in this paper.

Existence of a self financing strategy with initial value $\underline{V}_0(g) = \bar{V}_0(g)$. Theorem 1 will have been shown if V_0^* can be taken to be $\bar{V}_0(g)$. To do this, observe that

Lemma 2. *Under the assumptions of Lemma 1, there is a (finite) value of λ so that $\bar{V}_0(g) = \bar{V}_0(g; \lambda)$. This value is the unique solution of the system of equations (7.2).*

For this λ, the process $V_t^* = SE(S_t^*, \Xi^+ - \int_0^t \sigma_u^2 du, \lambda)$ satisfies our requirements for a self financing strategy in S^* that is solvent on I^{Ξ^+}, for all $P^* \in \mathcal{R}^*_\infty$, and hence for all $P \in \mathcal{Q}$. This is, again, by the Dambis/Dubins–Schwartz time change, and by the solution for the American payoff $h_\lambda(S_\tau)$ under the model (2.1).

Proof of Lemma 2. For a given values of λ_i, the values of t_i that minimize $\bar{V}_0(g; \lambda)$ must satisfy equations (7.2). Also, by the triangular matrix argument mentioned in Section 7, this system of equations *defines* the λ_i's from the minimizing

t_i's. When one takes the limit in Lemma 1, Eq. (7.2) therefore remains valid, by continuity of $\partial\widetilde{E}g(\widetilde{S}_{\tau\wedge\Xi^+})/\partial t_i$ and the $\partial\widetilde{E}(K_j-\widetilde{S}_\tau)^+/\partial t_i$. ■

This ends the proof of Theorem 1. ■

Proof of Proposition 2. If (2.3)–(2.4) have no solution, this means that there will be a pass though the loop in Algorithm 1 where step 0 returns a no solution message. Let $\mathcal{S}$ be the index set at this stage, and let $I \in \mathcal{S}$ be an index that causes the termination condition to be triggered. The arbitrage strategy is then constructed as in the proof of Proposition 3, in view of (A.10). ■

Proof of Proposition 5. We only show the loop case. The recursion case is similar.

Let t_- be as in Remark 1. In the first pass through the loop, Step 1 sets $t_i = \Xi_{K_i}$. This value will exceed $t_- = 0$.

We now proceed by induction, assuming that we have gone through n passes of the loop, $n \geq 1$. We need to show that all t_i found in Step 1 strictly exceed t_-.

If $t_i = t_-$, then, the index i would have been selected in the previous pass. If $t_i < t_-$, this means that there is a set $\{j_1,\ldots,j_r\}$ so that the t_{j_k} have already been picked out in earlier passes through the loop, and so that

$$t_i \leq t_{j_1} \leq \cdots \leq t_{j_r} = t_-, \tag{A.6}$$

where at least one of the inequalities is strict. Let l be the pass of the loop where t_{j_1} is picked out, and let τ' be the τ given by Step (3) in pass number $l-1$. (If $l=1$, then $\tau' = +\infty$).

Since $(K_i - \widetilde{S}_t)^+$ is a martingale for $t \geq t_i$, it follows that

$$\widetilde{E}(K_i-\widetilde{S}_{\tau\wedge t_i})^+|\mathcal{F}_{t_i}) = (K_i-\widetilde{S}_{\tau\wedge t_i})^+ = (K_i-\widetilde{S}_{\tau'\wedge t_i})^+. \tag{A.7}$$

Hence, by taking unconditional expectations, and using (A.6),

$$\widetilde{E}(K_i-\widetilde{S}_{\tau'\wedge t_i})^+ = P_0^{K_i*}.$$

It follows that index i would give rise to value t_i in Step 1 of the l's iteration of the loop.

Let $\mathcal{S}_n$ be the set of unselected indices at the start of iteration $n+1$, and set $\mathcal{S}_n^c = \{1,\ldots,q\} - \mathcal{S}$. There are two possibilities in (A.6): either, for some $k < r$,

$$t_i \leq t_{j_1} = \cdots = t_{j_k} < t_{j_{k+1}} \leq t_-, \tag{A.8}$$

or there is no such k, in which case

$$t_i < t_{j_1} = \cdots = t_{j_r} = t_-. \tag{A.9}$$

In the event of (A.8), Step 1 of pass l of the loop gives the values $t_{j_1} = \cdots = t_{j_k}$ for indices $\{j_1,\ldots,j_k\}$. Also, the indices in $\{j_{k+1},\ldots,j_r\}\cup(\mathcal{S}_n^c - \{i\})$ are rejected in Step 2. Hence, at most, $\{i,j_1,\ldots,j_k\}$ are selected in Step 2, and possibly only $\{i\}$. In the event of (A.9), the same reasoning applies. In any case, index i is picked out in iteration $l < n+1$, the current iteration number of the loop. Hence, again, if t_i were strictly smaller than t_-, it would already have been picked out by the loop in a previous step. ■

Proof of Theorem 2. (a) is trivial. We do the proof of the rest in steps (i)–(iii) below. (c) follows from (i) and (ii), (d) follows from (i) and (iii), (b) follows from (c) and (d) in the statement of the theorem.

(i) Algorithm 1, provided there is no termination in Step 0 at any point in the loop, provides a unique result $t_1,\ldots,t_q$. This follows from Proposition 5 and Remark 1.

(ii) Assume that Algorithm 1 has a solution $t_1,\ldots,t_q$. Then this solution satisfies (2.3)–(2.4).

To see this, let τ' be the τ from Step 3 in the loop where t_i was picked out from $\mathcal{S}$. We now use τ to denote the final product of Algorithm 1. (A.7) will remain valid, and integrating gives

$$\widetilde{E}(K_i-\widetilde{S}_\tau)^+ = P_0^{K_i*}.$$

which is what we needed to show.

(iii) Assume that (2.3)–(2.4) have a solution $t_1,\ldots,t_q$. Then this solution coincides with the output of Algorithm 1.

To see this, consider the order statistic $t_{(1)} \leq \cdots \leq t_{(q)}$. Obviously, if $t_{(1)} = t_i$ for some i, then this index i must be picked out in the first passage through the loop of Algorithm 1, and t_i must have the same value as the t_i picked out by the Algorithm. (If there are several t_i's with the same smallest value, the same applies.) Similarly, by induction, one assumes that $t_{(1)} \leq .. \leq t_{(j)}$ coincide with the results of Algorithm 1, and it is then easy to see that $t_{(j+1)}$ also coincides with the results of the Algorithm, in view of Proposition 5. Note that the termination condition in Step 0 cannot be triggered, since with the τ from the previous step,

$$\widetilde{E}(K_i-\widetilde{S}_{\tau\wedge t_i})^+ < \widetilde{E}(K_i-\widetilde{S}_\tau)^+. \quad ■ \tag{A.10}$$

A.2. Proofs for Sections 5 and 8

Proof of Proposition 4. Consider first the case $1 \leq i \leq q-1$. We show the result for the density f of $Y_t = \log \widetilde{S}_t = W_t - t/2$ given $\log K_i < Y_t < \log K_{i+1}$. The density of the proposition follows by a change of variable, and is $f(\log(s))/s$.

Following the discussion in Section 13 (pp. 330–332) of Karlin and Taylor (1981), and in particular equation (13.11), $f(y) = cm(y)\phi_1(y) + o(1)$, where c is a normalizing constant, and the other quantities are as defined by Karlin and Taylor. In particular, $m(y) = e^{-y}$, while the $\phi_n(y)$'s are solutions to the eigenvalue problem $L\phi_n = -\lambda_n \phi_n$, with $\phi_n(\log K_i) = \phi_n(\log K_{i+1}) = 0$. Here, L is the infinitesimal generator, in this case $L\phi = \frac{1}{2}\phi'' - \frac{1}{2}\phi'$. Obviously, for $\log(K_i) < y < \log(K_{i+1})$,

$$\phi_n(y) = c_n e^{\frac{1}{2}y} \sin\left(n\pi \frac{y - \log(K_1)}{\log(K_2) - \log(K_1)}\right) \text{ and } \lambda_n = \frac{1}{2}\left((n\pi)^2 + \frac{1}{8}\right),$$

where the c_n's are constants.

Meanwhile, for the lower edge, let $t > t_1$ and $x < \log K_1$). By Girsanov's Theorem,

$$\widetilde{P}(Y_t \leq y \mid Y_{t_1} = x \text{ and } \max_{t_1 < s < t} Y_s < \log K_1) = \frac{E\exp(-W_u)I(W_u \leq y - x \text{ and } M_u \leq \log K_1 - x)}{E\exp(-W_u)I(M_u \leq \log K_1 - x)}$$

where $u = t - t_1$, W is a standard Brownian motion, and M is the running maximum of W. Thus, in this case, for $y < \log K_1$, and using, say, formula (8.2) (p. 95) in Karatzas and Shreve (1991),

$$f(y) \propto u^{-1/2} \exp(-\frac{1}{2}(y-x)) \left[\phi\left(\frac{y-x}{u^{1/2}}\right) - \phi\left(\frac{y + x - 2\log K_1}{u^{1/2}}\right)\right].$$

This gives the result of the proposition. The derivation for the upper edge is similar. ■

Proof of Theorem 3. Following Theorem III.12 (p. 49) of Pollard (1984), it is enough to show that

$$\widetilde{E}g(\widetilde{S}_{\tau_q \wedge \Xi^+}) \to E_{F^c} g(S) \text{ as } q \to \infty, \tag{A.11}$$

for all functions g that are twice continuously differentiable, with bounded $|g|$, $|g'|$, and $|g''|$. It is further enough to show (A.11) for functions $g(x)$ that vanish for $x > L$, where L is an arbitrary positive constant. To wit, set $\widetilde{g}''(x) = g''(x)I\{x \leq L\}$, and note that $g(S) - \widetilde{g}(S) = \{g(S) - g(L) - (S - L)g'(L)\}I\{S > L\}$ so that, for E denoting expectation under $F_q^{\Xi^+}$ or F_c, $E|g(S) - \widetilde{g}(S)| \leq \max_x |g(x)| ES/L + \max_x |g(x)'| ES^{3/2}/L^{1/2} \leq \max_x |g(x)| \widetilde{E}(\widetilde{S}_{\Xi^+})/L + \max_x |g(x)'| \widetilde{E}(\widetilde{S}_{\Xi^+})^{3/2}/L^{1/2} \to 0$ as $L \to \infty$. By decomposing $\widetilde{g}''$ into positive and negative parts, it is further enough to show (A.11) for twice continuously differentiable and convex g, with bounded $|g|$, $|g'|$, and $|g''|$, that vanish for $x > L$, where L is an arbitrary constant.

Take such a g, and let Q be such that $K_{q,q} > L$ for $q > Q$. We now invoke (7.1), which yields

$$\widetilde{E}g(\widetilde{S}_{\tau \wedge \Xi^+}) = \int \widetilde{E}(K - \widetilde{S}_{\tau \wedge \Xi^+})^+ \mu(dK) \leq \int \widetilde{E}(K_q^* - \widetilde{S}_{\tau \wedge \Xi^+})^+ \mu(dK),$$

where $K_q^* = \min\{K_{l,q} \geq K\}$. Since the $P_0^{K_q^*}$ are traded, $\widetilde{E}(K_q^* - \widetilde{S}_{\tau \wedge \Xi^+})^+ = P_0^{K_q^*}$, whence

$$\widetilde{E}g(\widetilde{S}_{\tau \wedge \Xi^+}) \leq \int P_0^{K_q^*} \mu(dK) \to E_{F^c}(K - S)\mu(dK)$$

since μ has zero mass above L. The opposite inequality follows by replacing K_q^* by $K_{q,*} = \max\{K_{l,q} \leq K\}$ (or $= 0$ for $K < K_{1,q}$). This shows Theorem 3.

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