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# Optimal and equilibrium execution strategies with generalized price impact

MASAMITSU OHNISHI†‡ and MAKOTO SHIMOSHIMIZU*†§

†Graduate School of Economics, Osaka University, Osaka 560-0043, Japan
‡Center for Mathematical Modeling and Data Science, Osaka University, Osaka 560-8531, Japan
§Japan Society for the Promotion of Science (JSPS) Research Fellow

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This paper examines the execution problems of large traders with a generalized price impact. Constructing two related models in a discrete-time setting, we solve these problems by applying the backward induction method of dynamic programming. In the first problem, we formulate the expected utility maximization problem of a single large trader as a Markov decision process and derive an optimal execution strategy. Then, in the second model, we formulate the expected utility maximization problem of two large traders as a Markov game and derive an equilibrium execution strategy at a Markov perfect equilibrium. Both of these two models enable us to investigate how the execution strategies and trade performances of a large trader are affected by the existence of other traders. Moreover, we find that these optimal and equilibrium execution strategies become deterministic when the total execution volumes of non-large traders are deterministic. We also show, by some numerical examples, the comparative statics results with respect to several problem parameters.

*Keywords*: Large trader; Market impact; Dynamic programming; Backward induction; Optimal execution; Markov perfect equilibrium

*JEL Classification*: C73, G12

## 1. Introduction

In security market analysis, there is a growing awareness among academic researchers or practitioners that some kind of institutional traders called ‘large traders’ cause a ‘price impact’ through their own trades. A life insurance company, trust company, or a company which manages pension fund exhibit the typical examples of such traders of great importance. Large traders recognize these price impacts as ‘liquidity risk.’ They can reduce the liquidity risk by splitting their order into small sizes over the course of the trading epoch. Conversely, submitting small pieces of the order gradually may expose them to the price risk. Consequently, when large traders allocate large orders to (small) pieces, they have to pay attention to two distinct facets; the liquidity risk which arises owing to the large orders they submit and the price risk which corresponds to the price fluctuations in the future.

Various ways to trade are available to a preponderance of trading markets since the structure of trading systems diverges in different directions. As an example in a wide variety of electric trading platforms, ‘algorithmic trading’ has emerged in recent years and the so-called high-frequency trading (HFT) with computer systems, which typifies algorithmic trading, significantly influences the financial markets. The development of trading systems facilitates an increasing number of studies encompassing such fields as market impact modeling, or the optimal execution problem.

In this paper, we investigate execution problems pertaining to the interaction among large traders and smaller traders from a theoretical point of view. Bertsimas and Lo (1998) were at the forefront of investigations of this field, which address the optimization problem of minimizing the expected execution cost in a discrete-time framework via a dynamic programming approach. This analysis identifies the optimal execution volume as an equally divided volume throughout the trading epochs. Notwithstanding a valuable insight into the execution problem, their model disregards any attitudes toward risk. Subsequently, Almgren and Chriss (2000) derive an optimal execution strategy by considering both the execution cost and the volatility risk, which entails an analysis with a mean–variance approach. As for Kuno and Ohnishi (2015) and Kuno *et al.* (2017), they construct models with the residual effect of the price impact, i.e. the transient price impact which dissipates over the trading time window. These papers

*Corresponding author. Email: shimosshi1q94@gmail.com
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solve the optimization problem of maximizing the expected utility payoff from the final wealth at the maturity to derive an optimal execution strategy.

The trend of existing papers, including those mentioned above, discuss the behavior of a large trader (or an institutional trader) which has dominated the research field of execution problems in recent years. This research, however, does not incorporate into their models the existence of traders other than large traders, whom we call the 'trading crowd' following Huberman and Stanzl (2004). Only a few existing researches concerned with execution problems have thoroughly investigated the price impact model with a trading crowd. As Potters and Bouchaud (2003) show, small trades have statistically larger impacts on the price by far than that of large trades in a relative sense. These results imply that one should take into account the price impact caused by the trading crowd when constructing a price impact model. Cartea and Jaimungal (2016) incorporates the price impact caused by other traders into the construction of the midprice process by describing the market order-flow through a general Markov process and they derive a closed-form strategy for a large trader. They show that the optimal execution strategies are different from Almgren and Chriss (2000) when noise-traders cause price impact and coincide with Almgren and Chriss (2000) when noise-traders do not affect the midprice. This analysis is based on the assumption that the price impact is decomposed into temporary and permanent, and is not transient. Our model considers the transient price impact through the 'residual effect' of past execution (caused by both a large trader and noise-traders) on the risky asset price. This setting enables us to analyze how residual effects influence the execution strategy of a large trader. The effect of the price impact caused by trading crowd on the execution price determines the generalized price impact in our model.

Moreover, a multitude of large traders ordinarily exist in a real market. Nevertheless, most of the prior studies deal with the optimal execution problem for a single large trader model. The following example underlines the above fact; consider a security market where multiple institutional traders or brokers manage the trading execution ordered by their clients. The clients can split their whole order into some blocks to submit their orders on different institutional traders or brokers. Then, these institutional traders or brokers execute their orders in the same market to make a profit. Each of their orders is so large that the submissions of the other institutional traders or brokers can affect the execution price, unlike the case that only a single large trader exists in the market. This situation requires consideration of the interaction of more than one large trader in the same security market.

Only a few papers delve into the interaction between more than one large trader; examples are Schöneborn (2008), Schied and Zhang (2018), Luo and Schied (2018), to mention only a few related papers. Schöneborn (2008) analyzes the interaction of two large traders on their execution strategies, which inspired the following two works. In Schied and Zhang (2018), they formulate what they call a market impact game model (as a static strategic game model). This study discovers some features of a Nash equilibrium strategy, proving that a unique Nash equilibrium exists in a class of static and deterministic strategies in explicit form. They also find, via a rather direct method, that the equilibrium is also a Nash equilibrium in a broader class of dynamic strategies. Subsequently, Luo and Schied (2018) extend the above model to an $n$-large trader model and construct cost minimization problems in terms of the mean–variance and expected utility maximization problems of the $n$-large traders. An important result of their analysis is that a Nash equilibrium exists in each problem which is also in explicit form and is unique for the former problem. They also show that the Bachelier price model renders the Nash equilibrium obtained from each problem identical, when the price is composed of a Brownian motion as a term expressing the volatility of the stock price. These studies are noteworthy since they theoretically highlight the interaction of execution strategies among multiple large traders which is novel in this research field.

This paper explores two execution strategies in the following two models: a single-large-trader Markov decision model and a two-large-trader Markov game model. These large traders maximize their expected utility payoff from his/her final wealth at the maturity. The methods by which we derive these strategies are the backward induction procedure of dynamic programming based on the principle of optimality, which is equivalent to those introduced in Kuno and Ohnishi (2015) and Kuno *et al.* (2017). The price impact caused by the trading crowd is embedded in the construction of a generalized price impact model, leading to a similar but different model than the previous research. Under appropriate model settings, our investigation shows that there exists an optimal execution strategy in the first model and an equilibrium execution strategy at a Markov perfect equilibrium in the second model. Our contribution to the field of the execution problem is that in general, both optimal and equilibrium execution strategies are not necessarily static nor deterministic. These execution strategies become deterministic when the execution volume of the trading crowd is static and deterministic. The contents presented at Ohnishi and Shimoshimizu (2018a) and Ohnishi and Shimoshimizu (2008b) form a basis for this paper. Further, we referred to Guéant (2016), Kuno *et al.* (2018), Kunou and Ohnishi (2010) and Ohnishi and Shimoshimizu (2019) in writing this paper.

The paper proceeds as follows. In Section 2, we construct a generalized price impact model and solve the maximization problem of the expected utility of a risk-averse large trader with Constant Absolute Risk Aversion (CARA) type utility (or negative exponential utility) from the wealth at the maturity. This leads to an optimal execution strategy. In Section 4, we construct a Markov game model of two-large traders with a similar model setting to Section 2, from which we derive an equilibrium execution strategy at a Markov perfect equilibrium. Sections 3 and 5 give numerical examples of each model. Finally, Section 6 concludes.

## 2. Price impact model with non-large traders effects

In a discrete time framework $t \in \{1, \ldots, T, T+1\}(T \in \mathbb{Z}_+ := \{1, 2, \ldots\})$, we assume that there exists one large trader in a trading market. The large trader plans to purchase $Q$ volume of one risky asset by the time $T+1$. We also suppose that he/she has the CARA utility function with the absolute risk aversion rate $R > 0$.

### 2.1. Market model

We assume that $q_t (\in \mathbb{R})$ represents a large amount of orders submitted by the large trader at time $t \in \{1, \ldots, T\}$. Then, $\overline{Q}_t$ denotes the number of shares remained for the large trader at time $t \in \{1, \ldots, T, T+1\}$. The positive and negative $\overline{Q}_t$ stand for the acquisition and liquidation of the risky asset, respectively. This leads to a similar setup for a selling problem. From this assumption, we have $\overline{Q}_1 = Q$ and

$$\overline{Q}_{t+1} = \overline{Q}_t - q_t, \quad t = 1, \ldots, T. \tag{1}$$

The market price (or quoted price) of the risky asset at time $t \in \{1, \ldots, T, T+1\}$ is represented by $p_t$. Since the large trader submits a large amount of orders, the execution price becomes not $p_t$ but $\widehat{p}_t$ with the additive execution cost. Submitting one unit of (large) order at time $t \in \{1, \ldots, T\}$ causes the instantaneous price impact denoted as $\lambda_t$. The trading volume of trading crowd also has an impact on the execution price. $\kappa_t$ represents the price impact per unit at time $t \in \{1, \ldots, T\}$ which stems from the submission of trading crowd. We denote the total execution volume of trading crowd at time $t \in \{1, \ldots, T\}$ by a sequence of random variables $v_t$, which follows a normal distribution with mean $\mu_t^v$ and variance $(\sigma_t^v)^2$ for each time $t \in \{1, \ldots, T\}$, that is,

$$v_t \sim N\left(\mu_t^v, (\sigma_t^v)^2\right), \quad t = 1, \ldots, T. \tag{2}$$

Throughout this paper, we assume that the buy-trade and sell-trade of a large trader induce the same (instantaneous) price impact, although it would be different in a real market. This assumption is, however, justified from the statistical analysis of market data in Cartea and Jaimungal (2016). In this work, they estimate the permanent and temporary price impacts by conducting a robust linear regression of price changes on net order-flow. This estimation and the relevant statistics obtained by using several stock market data reveal that the linear assumption of the price impact is compatible with the real stock market and that the price impacts caused by both buy and sell trades are same from the statistical point of view.

From these facts, we define the execution price in the form of the linear price impact model as follows:

$$\widehat{p}_t = p_t + (\lambda_t q_t + \kappa_t v_t), \quad t = 1, \ldots, T. \tag{3}$$

In the following, we discuss how the price impact caused by both large trader and trading crowd can be formulated. The graphical image of the temporary, permanent, and transient price impact are depicted below.

As shown in figure 1, the permanent and temporary price impacts are assumed to be proportional in our model. Assume $(\lambda_t q_t + \kappa_t v_t)\alpha_t$ denotes the temporary price impact at time $t \in \{1, \ldots, T\}$. Then, we formulate the residual price impact of past price $r_t$ at time $t \in \{1, \ldots, T, T+1\}$ by means of a decay kernel function $G(t)$ of time $t \in \{1, \ldots, T, T+1\}$.

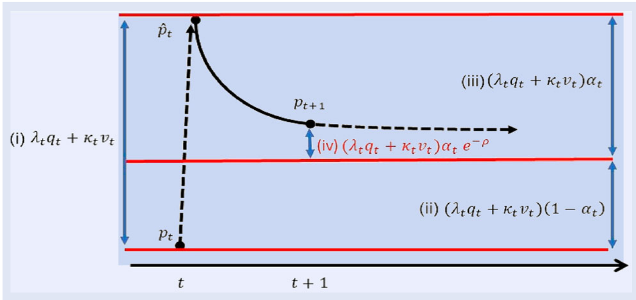


Figure 1. Graphical depiction of the price impact. (i) Instantaneous price impact. (ii) Permanent price impact. (iii) Temporary price impact. (iv) Transient price impact.

In our model, the transient price impact is defined with an exponantial decay kernel,

$$G(t) := \mathrm{e}^{-\rho t}, \quad t = 1, \ldots, T, T+1. \tag{4}$$

With a deterministic price reversion rate $\alpha_t$ ($\in [0, 1]$) and deterministic resilience speed $\rho$ ($\in [0, \infty)$), the dynamics of the residual effect of past price impact $r_t$ is given as follows:

$$\begin{aligned}
r_1 &= 0; \\
r_{t+1} &= \sum_{k=1}^{t} (\lambda_k q_k + \kappa_k v_k)\, \alpha_k \mathrm{e}^{-\rho((t+1)-k)} \\
&= \mathrm{e}^{-\rho} \sum_{k=1}^{t-1} (\lambda_k q_k + \kappa_k v_k)\, \alpha_k \mathrm{e}^{-\rho(t-k)} + (\lambda_t q_t + \kappa_t v_t)\, \alpha_t \mathrm{e}^{-\rho} \\
&= [r_t + (\lambda_t q_t + \kappa_t v_t)\, \alpha_t]\, \mathrm{e}^{-\rho}, \quad t = 1, \ldots, T.
\end{aligned} \tag{5}$$

Equation (5) shows the recursiveness of the residual effect, i.e. $r_{t+1}$ depends on only $r_t$ and the transient price impact $(\lambda_t q_t + \kappa_t v_t)\alpha_t \mathrm{e}^{-\rho}$, which indicates that $r_t$ has a Markov property in this settings. The Markov property of this residual effect arises thanks to the assumption of the exponantial decay kernel.

Some public news or information about the economic situation affect the price. Therefore, we define the independent random variables $\varepsilon_t$ at time $t \in \{1, \ldots, T\}$ as the effect of the public news/information about the economic situation between $t$ and $t+1$, and assume that $\varepsilon_t$ follows a normal distribution with mean $\mu_t^\varepsilon$ and variance $(\sigma_t^\varepsilon)^2$, i.e.

$$\varepsilon_t \sim N\left(\mu_t^\varepsilon, (\sigma_t^\varepsilon)^2\right), \quad t = 1, \ldots, T. \tag{6}$$

We suppose in the following that the two stochastic processes, $v_t$ and $\varepsilon_t$ for $t \in \{1, \ldots, T\}$, are mutually independent. However, we can derive similar results without this assumption (that is, if they follow a bivariate normal distribution).

The construction of the 'fundamental price' at time $t \in \{1, \ldots, T\}$, denoted by $p_t^f$, must be carefully considered. From the fact that the residual effect of the past execution disappears over the course of the trading horizon, we define $p_t - r_t$ as the fundamental price of the risky asset. By the definition of $\varepsilon_t$ and the assumption of permanent price impact $(1 - \alpha_t)$, we can set the fundamental price $p_t^f := p_t - r_t$ with a permanent price impact $(\lambda_k q_k + \kappa_k v_k)(1-\alpha_t)$ as follows:

$$\begin{aligned} p_{t+1}^f &= p_{t+1} - r_{t+1} \\ &= p_t - r_t + (\lambda_t q_t + \kappa_t v_t)(1-\alpha_t) + \varepsilon_t \\ &= p_t^f + (\lambda_t q_t + \kappa_t v_t)(1-\alpha_t) + \varepsilon_t, \quad t = 1, \ldots, T. \end{aligned} \tag{7}$$

As this relation shows, the permanent part of the price impact caused by the large trader and trading crowd and the public news or information about an economic situation are assumed to affect the fundamental price. This assumption also reveals that the permanent price impact may give a non-zero trend to the fundamental price, even if the mean of $\varepsilon_t$ is zero for all $t \in \{1, \ldots, T\}$. According to equations (3), (5), and (7), the dynamics of market price or the relation between $p_{t+1}$ and $p_t$ for $t \in \{1, \ldots, T\}$ are described as

$$\begin{aligned} p_{t+1} &= p_t + (r_{t+1} - r_t) + (\lambda_t q_t + \kappa_t v_t)(1-\alpha_t) + \varepsilon_t \\ &= p_t - (1 - \mathrm{e}^{-\rho}) r_t + (\lambda_t q_t + \kappa_t v_t)\{\alpha_t \mathrm{e}^{-\rho} \\ &\quad + (1-\alpha_t)\} + \varepsilon_t, \quad t = 1, \ldots, T. \end{aligned} \tag{8}$$

Remark 2.1 In this context, $(\lambda_t q_t + \kappa_t v_t)(1-\alpha_t)$, $(\lambda_t q_t + \kappa_t v_t)\alpha_t$, and $(\lambda_t q_t + \kappa_t v_t)\alpha_t \mathrm{e}^{-\rho}$ represent the permanent impact, temporary impact, and transient impact, respectively. Moreover, if $\rho \to \infty$, the residual effect of past price impact becomes zero for all $t \in \{1, \ldots, T\}$ since $r_1 = 0$ and from equation (5)

$$\lim_{\rho\to\infty} r_{t+1} = \lim_{\rho\to\infty} [r_t + (\lambda_t q_t + \kappa_t v_t)\alpha_t]\mathrm{e}^{-\rho} = 0, \quad t = 1, \ldots, T, \tag{9}$$

and therefore,

$$\begin{aligned} p_{t+1} &= p_t - (1 - \mathrm{e}^{-\rho}) r_t + (\lambda_t q_t + \kappa_t v_t)\{\alpha_t \mathrm{e}^{-\rho} \\ &\quad + (1-\alpha_t)\} + \varepsilon_t, \\ &= p_t + (\lambda_t q_t + \kappa_t v_t)(1-\alpha_t) + \varepsilon_t, \quad t = 1, \ldots, T, \end{aligned} \tag{10}$$

that is, we have a permanent impact model. Also, if $\alpha_t = 1$, the model is reduced to a transient impact model. Also, if $\kappa_t = 0$, the model is reduced to Kuno *et al.* (2017).

From the definition of the execution price, the wealth process $w_t$ satisfies

$$\begin{aligned} w_{t+1} &= w_t - \widehat{p}_t q_t \\ &= w_t - \{p_t + (\lambda_t q_t + \kappa_t v_t)\} q_t, \quad t = 1, \ldots, T. \end{aligned} \tag{11}$$

### 2.2. *Formulation as a Markov decision process*

In this subsection, we formulate the large trader's execution problem as a discrete-time Markov decision process. Time elapses as $1, \ldots, T, T+1$. The state of the process at time $t \in \{1, \ldots, T, T+1\}$ is a 4 -tuple, and is denoted as

$$s_t = (w_t, p_t, \overline{Q}_t, r_t) \in \mathbb{R}^4 =: S. \tag{12}$$

For $t \in \{1, \ldots, T\}$, an allowable action chosen at a state $s_t$ is an execution volume $q_t \in \mathbb{R} =: A$ so that the set $A$ of admissible actions is independent of the current state $s_t$.

When an action $q_t$ is chosen in a state $s_t$ at time $t \in \{1, \ldots, T\}$, a transition to a next state

$$s_{t+1} = (w_{t+1}, p_{t+1}, \overline{Q}_{t+1}, r_{t+1}) \in S \tag{13}$$

occurs according to the law of motion precisely described in the previous subsection which is symbolically denoted by a (Borel measurable) system dynamics function $h_t$ $(: S \times A \times (\mathbb{R} \times \mathbb{R}) \longrightarrow S)$:

$$s_{t+1} = h_t(s_t, q_t, (v_t, \varepsilon_t)), \quad t = 1, \ldots, T. \tag{14}$$

A utility payoff (or reward) arises only in a terminal state $s_{T+1}$ at the end of horizon $T+1$ as

$$g_{T+1}(s_{T+1}) := \begin{cases} -\exp\{-R w_{T+1}\} & \text{if } \overline{Q}_{T+1} = 0; \\ -\infty & \text{if } \overline{Q}_{T+1} \neq 0, \end{cases} \tag{15}$$

where $R > 0$ represents the risk aversion rate. The term $-\infty$ means a hard constraint enforcing the large trader to execute all of the remaining volume $\overline{Q}_T$ at the maturity $T$, that is, $q_T = \overline{Q}_T$.

If we define a (history-independent) one-stage decision rule $f_t$ at time $t \in \{1, \ldots, T\}$ by a Borel measurable map from a state $s_t \in S = \mathbb{R}^4$ to an action

$$q_t = f_t(s_t) \in A = \mathbb{R}, \tag{16}$$

then a Markov execution strategy $\pi$ is defined as a sequence of one-stage decision rules

$$\pi := (f_1, \ldots, f_t, \ldots, f_T). \tag{17}$$

We denote the set of all Markov execution strategies as $\Pi_{\mathrm{M}}$. Further, for $t \in \{1, \ldots, T\}$, we define the sub execution strategy after time $t$ of a Markov execution strategy $\pi = (f_1, \ldots, f_t, \ldots, f_T) \in \Pi_{\mathrm{M}}$ as

$$\pi_t := (f_t, \ldots, f_T), \tag{18}$$

and the entire set of $\pi_t$ as $\Pi_{\mathrm{M},t}$.

By definition (15), the value function under an execution strategy $\pi$ becomes an expected utility payoff arising from the terminal wealth $w_{T+1}$ of the large trader with the absolute risk aversion $R$:

$$\begin{aligned} V_1^\pi[s_1] = \mathbb{E}_1^\pi\Big[g_{T+1}(s_{T+1})\Big| s_1\Big] = \mathbb{E}_1^\pi\Big[-\exp\Big\{-R w_{T+1}\Big\} \cdot \\ 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty) \cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}\Big| s_1\Big], \end{aligned} \tag{19}$$

where $1_A$ is the indicator function of an event $A$ and, for $t \in \{1, \ldots, T\}$, $\mathbb{E}_t^\pi$ is a conditional expectation given a condition at time $t$ under $\pi$.

Then, for $t \in \{1,\dots,T,T+1\}$ and $s_t \in S$, we further let

$$V_t^\pi\big[s_t\big] = \mathbb{E}_t^\pi\Big[g_{T+1}(s_{T+1})\Big|s_t\Big] = \mathbb{E}_t^\pi\Big[-\exp\Big\{-Rw_{T+1}\Big\}\cdot \times 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty)\cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}\Big|s_t\Big] \quad (20)$$

be the expected utility payoff at time $t$ under the strategy $\pi$. It is noted that the expected utility payoff $V_t^\pi[s_t]$ depends on the Markov execution policy $\pi = (f_1,\dots,f_t,\dots,f_T)$ only through the sub execution policy $\pi_t := (f_t,\dots,f_T)$ after time $t$.

Now, we define the optimal value function as follows:

$$V_t\big[s_t\big] = \sup_{\pi\in\Pi_M} V_t^\pi\big[s_t\big], \quad s_t \in S, \quad t = 1,\dots,T,T+1. \quad (21)$$

From the principle of optimality, the optimality equation (Bellman equation, or dynamic programming equation) becomes

$$V_t\big[s_t\big] = \sup_{q_t\in\mathbb{R}} \mathbb{E}\Big[V_{t+1}\big[h_t\left(s_t,q_t,(v_t,\varepsilon_t)\right)\big]\Big|s_t\Big], \quad s_t \in S, \quad t = 1,\dots,T. \quad (22)$$

### 2.3. Dynamics of the optimal execution

The optimal dynamic execution strategy $\pi$ is acquired by solving the above equation (22) backwardly in time $t$ from maturity $T$. We obtain the following theorem.

THEOREM 2.1 (Optimal Value Function and Optimal Execution Strategy)

(i) *The optimal execution volume at time $t \in \{1,\dots,T,T+1\}$, denoted as $q_t^*$, becomes an affine function of the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect $r_t$ at time t. That is,*

$$q_t^* = f_t(s_t) = a_t + b_t\overline{Q}_t + c_t r_t, \quad t = 1,\dots,T. \quad (23)$$

(ii) *The optimal value function $V_t[s_t]$ at time $t \in \{1,\dots,T,T+1\}$ is represented as follows:*

$$V_t\big[s_t\big] = -\exp\Big\{-R\Big[w_t - p_t\overline{Q}_t + G_t\overline{Q}_t^2 + H_t\overline{Q}_t + I_t\overline{Q}_t r_t + J_t r_t^2 + L_t r_t + Z_t\Big]\Big\}, \quad (24)$$

*where $a_t, b_t, c_t$; $G_t, H_t, I_t, J_t, L_t, Z_t$ for $t \in \{1,\dots,T,T+1\}$ are deterministic functions of time t which are dependent on the problem parameters, and can be computed backwardly in time t from maturity T.*

See the proof of this theorem in Appendix 1.

From the theorem above, the optimal execution volume $q_t^*$ depends on the state $s_t = (w_t,p_t,\overline{Q}_t,r_t)$ of the decision process only through the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect $r_t$, not through the wealth $w_t$ or market price $p_t$. In addition, from the definition of residual effect $r_t$, the optimal execution volume $q_t$ includes a nondeterministic term (random variable) thorough $v_t$ in $r_t$, which indicates that $v_t$ affects the optimal execution strategies. Therefore, we have the following facts.

COROLLARY 1 *If the orders of trading crowd $v_t$ for $t \in \{1,\dots,T\}$ are deterministic, the optimal execution volumes $q_t^*$ at time $t \in \{1,\dots,T\}$ also become deterministic functions of time in a class of the static (and non-randomized) execution strategy.*

### 2.4. In the case with target close order

In this subsection, we consider a model with closing price. The time framework $t \in \{1,\dots,T,T+1\}$ is same in the model mentioned above. However, we add an assumption that a large trader can execute his/her remaining execution volume at time $T+1$, $\overline{Q}_{T+1}$, with closing price $p_{T+1}$. We further assume that the trading at time $T+1$ impose the large trader to pay the additive cost $\chi_{T+1}$ per unit of the remaining volume.

According to the above settings, the value function at the maturity becomes

$$V_{T+1}\big[s_{T+1}\big] = -\exp\Big\{-R\Big[w_{T+1} - (p_{T+1} + \chi_{T+1}\overline{Q}_{T+1})\overline{Q}_{T+1}\Big]\Big\}, \quad (25)$$

and thereby, the optimal value function at time $T$ is calculated as follows:

$$\begin{aligned} V_T\big[s_T\big] &= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[V_{T+1}\big[w_{T+1},p_{T+1},\overline{Q}_{T+1},r_{T+1}\big]\Big|w_T,p_T,\overline{Q}_T,r_T\Big] \\ &= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-R\Big[w_{T+1} - (p_{T+1}+\chi_{T+1}\overline{Q}_{T+1})\overline{Q}_{T+1}\Big]\Big\}\Big|w_T,p_T,\overline{Q}_T,r_T\Big] \\ &= \sup_{q_T\in\mathbb{R}} -\exp\Big\{-R\Big[-A_T q_T^2 + (B_T\overline{Q}_T + C_T R_T + D_T)q_T + w_T - p_T\overline{Q}_T \\ &\quad + \Big(-\chi_{T+1} - \tfrac{1}{2}R\kappa_T^2(\alpha^T)^2(\sigma_T^v)^2 - \tfrac{1}{2}R(\sigma_T^\varepsilon)^2\Big)\overline{Q}_T^2 \\ &\quad + \big\{-\kappa_T\alpha^T\mu_T^v - \mu_T^\varepsilon\big\}\overline{Q}_T + (1-\mathrm{e}^{-\rho})\overline{Q}_T r_T\Big]\Big\}, \end{aligned} \quad (26)$$

where $\alpha^T := \alpha_T\mathrm{e}^{-\rho} + (1-\alpha_T)$, and

$$\begin{aligned} A_T &:= (1-\alpha^T)\lambda_T + \chi_{T+1} + \frac{1}{2}R\kappa_T^2(1-\alpha^T)^2(\sigma_T^v)^2 + \frac{1}{2}R(\sigma_T^\varepsilon)^2; \\ B_T &:= -\lambda_T\alpha^T + 2\chi_{T+1} - R\kappa_T^2(1-\alpha^T)\alpha^T(\sigma_T^v)^2 + R(\sigma_T^\varepsilon)^2; \\ C_T &:= -(1-\mathrm{e}^{-\rho}); \quad D_T := -\kappa_T(1-\alpha^T)\mu_T^v + \mu_T^\varepsilon. \end{aligned} \quad (27)$$

We can derive the optimal execution volume satisfying equation (26) by obtaining the optimal execution volume $q_t^*$ which attains the maximum of

$$\begin{aligned} K_T\big[q_T\big] &:= -A_T q_T^2 + (B_T\overline{Q}_T + C_T r_T + D_T v_T + F_T)q_T \\ &\quad + w_T - p_T\overline{Q}_T \\ &\quad + \Big(-\chi_{T+1} - \tfrac{1}{2}R\kappa_T^2(\alpha^T)^2(\sigma_T^v)^2 - \tfrac{1}{2}R(\sigma_T^\varepsilon)^2\Big)\overline{Q}_T^2 \\ &\quad + \big\{-\kappa_T\alpha^T\mu_T^v - \mu_T^\varepsilon\big\}\overline{Q}_T + (1-\mathrm{e}^{-\rho})\overline{Q}_T r_T, \end{aligned} \quad (28)$$

since equation (28) is a quadratic function with a negative coefficient of second-order term with respect to $q_T$, and thereby a concave function with respect to $q_T$, which leads to the concavity of equation (26) with respect to $q_T$. Therefore, by completing the square of $K_T[q_T]$ with respect to $q_T$, we obtain the optimal execution volume at time $t = T$:

$$q_T^* =: f(s_T) = \frac{B_T\overline{Q}_T + C_T r_T + D_T}{2A_T} =: a_T + b_T\overline{Q}_T + c_T r_T, \tag{29}$$

where

$$a_T := \frac{D_T}{2A_T}; \quad b_T := \frac{B_T}{2A_T}; \quad c_T := \frac{C_T}{2A_T},$$

and by substituting this into equation (26) the optimal value function becomes

$$V_T\big[s_T\big] = -\exp\Big\{ -R\Big[w_T - p_T\overline{Q}_T + G_T\overline{Q}_T^2 + H_T\overline{Q}_T + I_T\overline{Q}_T r_T + J_T r_T^2 + L_T r_T + Z_T\Big]\Big\}, \tag{30}$$

where

$$\begin{aligned}
G_T &:= -\chi_{T+1} - \frac{1}{2}R\kappa_T^2(\alpha^T)^2(\sigma_T^v)^2 - \frac{1}{2}R(\sigma_T^\varepsilon)^2 + \frac{B_T^2}{4A_T};\\
H_T &:= -\kappa_T\alpha^T\mu_T^v - \mu_T^\varepsilon + \frac{B_T D_T}{2A_T};\\
I_T &:= (1-\mathrm{e}^{-\rho}) + \frac{B_T C_T}{2A_T}; \quad J_T := \frac{C_T^2}{4A_T};\\
L_T &:= \frac{C_T D_T}{2A_T}; \quad Z_T := \frac{D_T^2}{4A_T}.
\end{aligned} \tag{31}$$

For $t \in \{T-1, \ldots, 1\}$, we can recursively derive the optimal execution volume and optimal value function at each time by a similar derivation which we use to obtain the optimal execution volume in the last subsection for time $t \in \{T-2, \ldots, 1\}$.

## 3. Numerical examples

In this section, we illustrate some numerical examples and show some properties of the optimal execution strategies derived in the above section. The maturity is set as $T = 10$, and the large trader plans to execute the volume $Q = 100\,000$ within the time horizon $\{1, \ldots, T\}$ at the beginning. We assume the time homogeneity of the time-dependent parameters $\mu_t^v, \sigma_t^v, \mu_t^\varepsilon, \sigma_t^\varepsilon, \alpha_t, \lambda_t, \kappa_t$ for simplicity. To obtain the explicit form of the optimal execution volume, we impose in the following an assumption that the price impact of the trading crowd is deterministic, i.e. $\sigma_t^v = 0$, for all $t \in \{1, \ldots, T\}$. The benchmark values are set as follows:

$$\begin{aligned}
&\mu_t^v \equiv 0; \quad \sigma_t^v \equiv 0; \quad \mu_t^\varepsilon \equiv 0; \quad \sigma_t^\varepsilon \equiv 0.02; \quad \alpha_t \equiv 0.5;\\
&\lambda_t \equiv 0.001; \quad \kappa_t \equiv 0.005; \quad \rho \equiv 0.1; \quad R = 0.001,\\
&\forall t = 1, \ldots, T.
\end{aligned} \tag{32}$$

We also set $r_1 = 0$ since there exist no residual effects of the past price at time $t = 1$.

### 3.1. A round trip trading

For a sequence $\boldsymbol{q} := (q_1, \ldots, q_T) \in \mathbb{R}^T$, a static (and non-randomized) execution strategy $\pi = (f_1, \ldots, f_T) \in \Pi_M$ defined by $f_t(s_t) = q_t$ for any $s_t \in S = \mathbb{R}^4$ is called a round trip trading schedule if it satisfies the following condition:

$$\sum_{t=1}^{T} q_t = 0. \tag{33}$$

In particular, zero-trade schedule is the (trivial) round trip trading schedule defined by $\mathbf{0} = (0, \ldots, 0)$.

If the initial execution volume $\overline{Q}_1 = Q = 0$, the zero-trade schedule $(\pi =)\ \mathbf{0}$ obviously satisfies the hard terminal constraint $\overline{Q}_{T+1} = 0$ and results in a final wealth $w_{T+1} = w_1$ with certainty. Furthermore, with respect to our expected utility criterion, if we have $V_1^{\mathbf{0}}[s_1] < V_1^{\boldsymbol{q}}[s_1]$ for some round trip trading schedule $\boldsymbol{q}$, then, by Jensen's inequality, we obtain

$$\begin{aligned}
-\exp\{-Rw_1\} &= V_1^{\mathbf{0}}[s_1] < V_1^{\boldsymbol{q}}[s_1]\\
&= \mathbb{E}_1^{\boldsymbol{q}}\left[-\exp\{-Rw_{T+1}\}|\, s_1\right] \le -\exp\left\{-R\,\mathbb{E}_1^{\boldsymbol{q}}[w_{T+1}|s_1]\right\},
\end{aligned} \tag{34}$$

which implies $w_1 < \mathbb{E}_1^{\boldsymbol{q}}[w_{T+1}|s_1]$.

The following three cases represent a round trip trade where the trading crowd has no impact, positive impact, and negative impact on avarage, i.e. $\overline{Q}_1 = Q = \overline{Q}_{T+1} = 0$: $\mu_t^v = 0$, $\mu_t^v \equiv 100$, $\mu_t^v \equiv -100$, for all $t \in \{1, \ldots, T\}$, respectively.

From figure 2, when $\mu_t^v \neq 0$ the large trader can increase the expected utility of the final wealth by starting from $Q = 0$ through a round trip trading.

### 3.2. The effect of risk aversion

Here we examine how a large trader with different risk aversion executes his/her order through the following three cases: $R = 0.001, R = 0.5$, and $R = 1$.

As figure 3 illustrates, the more risk-averse the large trader is, the faster he or she executes. This is because the more risk-averse trader tends to avoid the price risk of the fluctuation in the future as possible, which is consistent with intuitive understanding.

### 3.3. The effect of $\alpha_t$

Next, we see how the difference of a price reversion rate $\alpha_t \in [0, 1]$ affects the execution volume of the three cases: $\alpha_t = 0.01, \alpha_t = 0.5$, and $\alpha_t = 1$.

Figure 4 shows that the larger $\alpha_t$ leads to the U-shaped execution volumes, the shape of which resembles the intraday execution curve of a real stock market. This indicates that the faster price reverses incurres more executions at the beginning and at the end of the trading epoch and less in the middle of the trading.

### 3.4. The effect of $\sigma_t^\varepsilon$

We demonstrate the following three cases in order to illustrate how $\sigma_t^\varepsilon$ has an effect on the execution strategy: $\sigma_t^\varepsilon = 0.02, \sigma_t^\varepsilon = 0.5$, and $\sigma_t^\varepsilon = 1$.

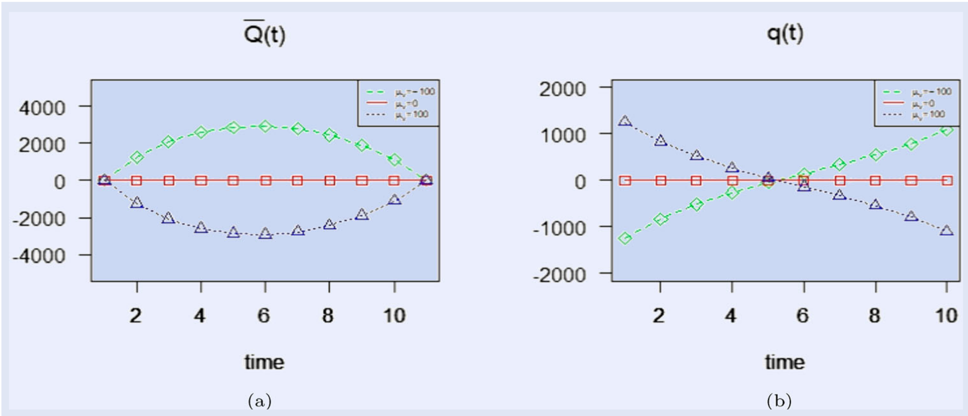


Figure 2. The effect of $\mu_t^v$. (a) Remaining execution volume $\overline{Q}_t (t = 1, \ldots, T)$. (b) Execution volume $q_t (t = 1, \ldots, T)$.

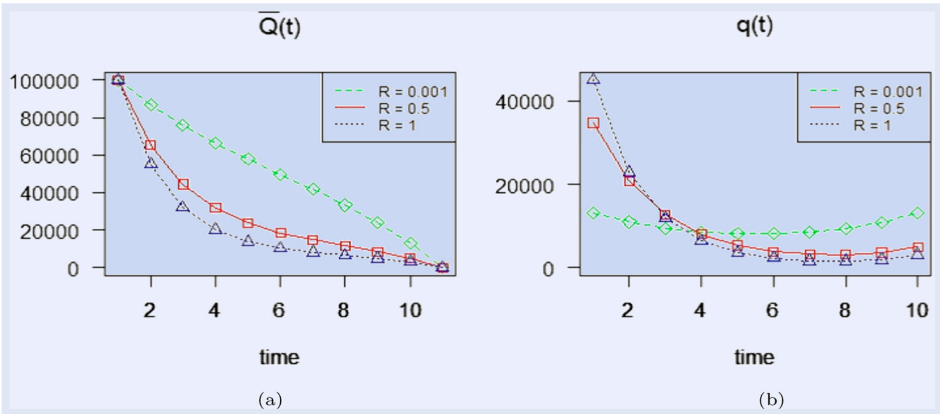


Figure 3. The effect of risk aversion. (a) Remaining execution volume $\overline{Q}_t (t = 1, \ldots, T)$. (b) Execution volume $q_t (t = 1, \ldots, T)$.

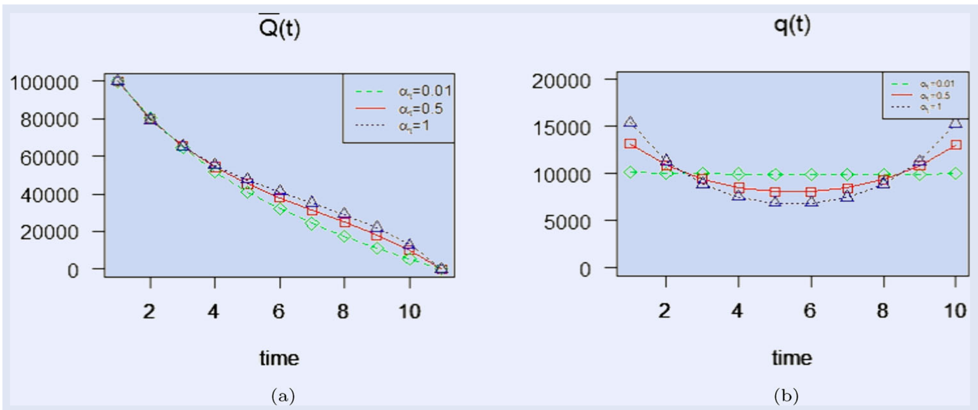


Figure 4. The effect of $\alpha_t$. (a) Remaining execution volume $\overline{Q}_t (t = 1, \ldots, T)$. (b) Execution volume $q_t (t = 1, \ldots, T)$.

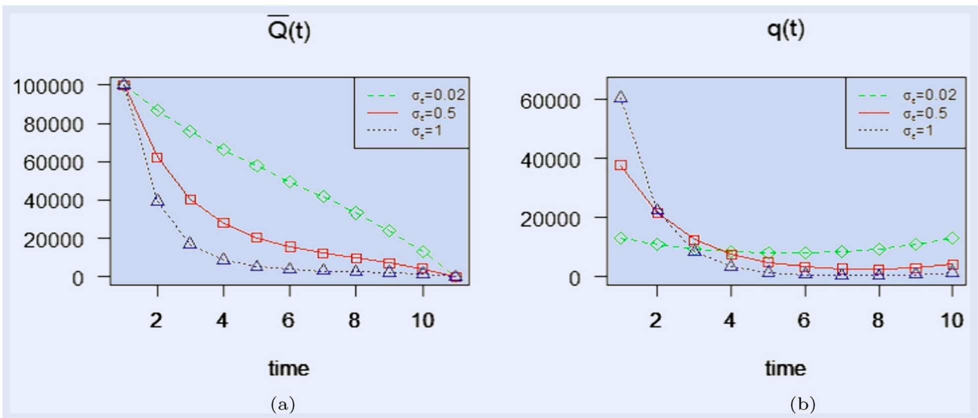


Figure 5. The effect of $\sigma_t^\varepsilon$. (a) Remaining execution volume $\overline{Q}_t (t = 1, \ldots, T)$. (b) Execution volume $q_t (t = 1, \ldots, T)$.

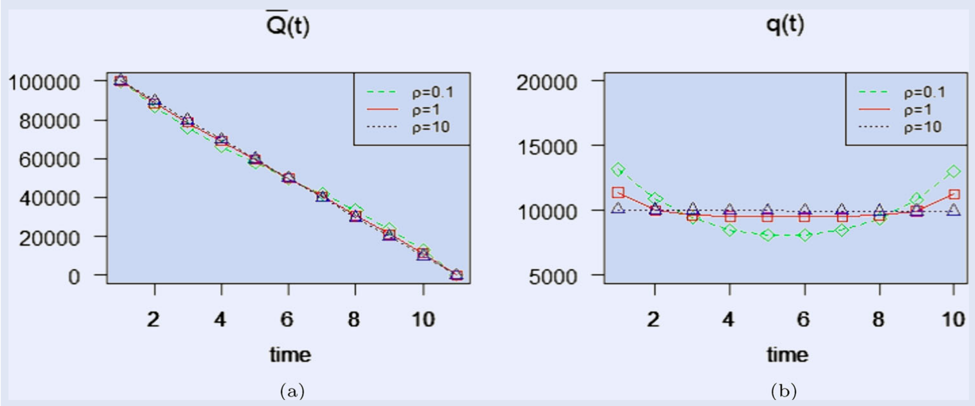


Figure 6. The effect of resilience speed. (a) Remaining execution volume $\overline{Q}_t(t = 1, \ldots, T)$. (b) Execution volume $q_t(t = 1, \ldots, T)$.

As illustrated in figure 5, if $\sigma_t^\varepsilon$ is large, the large trader executes much faster. From these kinds of phenomena, we can intuitively interpret that the large trader takes into account the possibility that the market price of the asset might increase suddenly in the future since there exist much higher possibilities that the future price of a risky asset fluctuates if the variance of the effect of the public news is large.

### 3.5. *The effect of resilience speed*

As the final part of this section, we demonstrate how the degree of the resilience speed affects the way a large trader executes his/her order through the following three cases: $\rho = 0.1, \rho = 1$, and $\rho = 10$.

Figure 6 shows that as the resilience speed increases, the curvature of trajectory $q_t$ becomes smaller. The larger $\rho$ causes the residual effect on the price to decrease, as the definition of $r_t$ clarifies this decreasing property of $r_t$ with respect to $\rho$. When $\rho \to \infty$, the residual effect of past price impact becomes zero for all $t \in \{1, \ldots, T\}$, as shown in Remark 2.1. Thus, if $\rho \to \infty$, the optimal execution volume becomes an affine function of the remaining execution volume $\overline{Q}_t$ as follows:

$$q_t^* = f_t(s_t) = a_t + b_t\overline{Q}_t + c_t r_t = a_t + b_t\overline{Q}_t, \quad t = 1, \ldots, T. \tag{35}$$

Therefore, in this case, $q_t^*$ becomes a linear difference equation which is deterministic on time $t \in \{1, \ldots, T\}$, and hinges on the state $s_t = (w_t, p_t, \overline{Q}_t, r_t)$ only through the remaining execution volume $\overline{Q}_t$. These facts infer that this deterministic optimal execution strategy also satisfies the optimality in a more general class of non-static execution strategy if the price impact is permanent.

## 4. Two- large trader stochastic game model

In this section, we suppose that there exist two large traders in a trading market, denoted by $i = 1, 2$ who intend to purchase $Q^i$ $(\in \mathbb{R})$ units of one risky asset in the finite time interval $t \in \{1, \ldots, T, T+1\}$.

### 4.1. *Assumption of the model*

Their inventories (or remaining execution volumes) $\overline{Q}_t^i$ for $i = 1, 2$ at time $t \in \{1, \ldots, T, T+1\}$ satisfy

$$\overline{Q}_{t+1}^i = \overline{Q}_t^i - q_t^i, \quad t = 1, \ldots, T, \quad i = 1, 2, \tag{36}$$

as is the case with equation (1).

For simplicity, we assume that the two large traders cause same price impact per unit $\lambda_t$ and we set the execution price with a linear price impact as follows:

$$\widehat{p}_t = p_t + \lambda_t(q_t^1 + q_t^2) + \kappa_t v_t, \quad t = 1, \ldots, T, \tag{37}$$

where $v_t \sim N(\mu_t^v, (\sigma_t^v)^2)$ for $t \in \{1, \ldots, T\}$ represents the aggregate quantity of the volume submitted by trading crowd at time $t$. The generalization of the price impact caused by each large trader, that is, the dependency of $\lambda_t$ on $i$ is not what we want to explore and it leads to a substantially tedious extension. This dependence will not contribute to additional intriguing results or will not have significant influences on the execution strategies we obtain in the following. We henceforth conduct the subsequent formulation with $\lambda_t$ which is independent of each large trader $i$ from these reasons. Referring to the proposed formulation of equation (5), the cumulative residual effect is defined as

$$r_1 := 0, \quad r_{t+1} := \left[r_t + \{\lambda_t(q_t^1 + q_t^2) + \kappa_t v_t\}\alpha_t\right] \mathrm{e}^{-\rho}, \\ t = 1, \ldots, T, \tag{38}$$

which also satisfies the Markov property as explained in Section 2. These settings yield the following dynamics of the market price process:

$$p_{t+1} = p_t - (1 - \mathrm{e}^{-\rho})r_t + \{\lambda_t(q_t^1 + q_t^2) + \kappa_t v_t\} \\ \times \{\alpha_t \mathrm{e}^{-\rho} + (1 - \alpha_t)\} + \varepsilon_t, t = 1, \ldots, T, \tag{39}$$

where $\varepsilon_t \sim N(\mu_t^\varepsilon, (\sigma_t^\varepsilon)^2)$ for $t \in \{1, \ldots, T\}$ represents the effect of the public news/information on the (risky asset) price between time $t$ and $t + 1$ (, which is the same assumption in Section 2).

Let $w_t^i$ for $i = 1, 2$ denote the wealth processes of the two large traders. The dynamics of $w_t^i$ becomes

$$w_{t+1}^i = w_t^i - \widehat{p}_t q_t^i, \quad t = 1, \ldots, T, \quad i = 1, 2, \tag{40}$$

which is similar to equation (11).

### 4.2. Formulation as a Markov game model

We formulate, in the following, the value function of each large trader $i = 1, 2$ to employ the backward induction method of dynamic programming, which is based on, and thereby similar to, the method we have used in Section 2. The state $s_t$ at time $t \in \{1, \ldots, T, T+1\}$ is denoted as 6-tuple:

$$s_t := ((w_t^1, w_t^2), p_t, (\overline{Q}_t^1, \overline{Q}_t^2), r_t) \in \mathbb{R}^6 =: S. \tag{41}$$

We define an action available for the large trader $i = 1, 2$ to choose at state $s_t$ as (an execution volume) $q_t^i \in A^i =: \mathbb{R}$ for $i = 1, 2$, assuming that the sets $A^1$ and $A^2$ of admissible actions do not depend on the current state $s_t$.

The dynamics of $s_t$, or the transition to a next state

$$s_{t+1} = ((w_{t+1}^1, w_{t+1}^2), p_{t+1}, (\overline{Q}_{t+1}^1, \overline{Q}_{t+1}^2), r_{t+1}) \in S, \tag{42}$$

after an action $q_t^i$ for $i = 1, 2$ is taken in a state $s_t$ at time $t \in \{1, \ldots, T, T+1\}$, is symbolically denoted (as in Section 2.2) by a (Borel measurable) system dynamics function $h_t$ $(: S \times (A^1 \times A^2) \times (\mathbb{R} \times \mathbb{R}) \longrightarrow S)$:

$$s_{t+1} = h_t(s_t, (q_t^1, q_t^2), (v_t, \varepsilon_t)). \tag{43}$$

This transition occurs based on the law of motion explained in Section 4.1.

Each large trader $i = 1, 2$ obtains his/her utility payoff only at the maturity $t = T + 1$, which is defined as

$$\begin{aligned} g_{T+1}^i(s_{T+1}) := &-\exp\left\{ - R^i w_{T+1}^i \cdot 1_{\{\overline{Q}_{T+1}^i = 0\}} \right\} \\ &+ (-\infty) \cdot 1_{\{\overline{Q}_{T+1}^i \neq 0\}}, \quad i = 1, 2. \end{aligned} \tag{44}$$

The expression $(-\infty) \cdot 1_{\{\overline{Q}_{T+1}^i \neq 0\}}$ corresponds to the constraint that the large traders $i = 1, 2$ must unwind all the volume $Q^i$ in the decision epochs $t \in \{1, \ldots, T\}$. $R^1$ $(> 0)$ and $R^2$ $(> 0)$ represent the absolute risk aversion rate of each large trader respectively. The types of large traders could be defined by

$$(w_1^i, Q^i, R^i), \quad i = 1, 2, \tag{45}$$

and these are assumed to be their common knowledge. In the real market, large traders have little access to this information of the counterpart. We can, however, consider a plausible explanation for the assumption of equation (45) from the viewpoint of game theoretic analysis. In this model, our focus is placed on how the existence of the other large trader influences the execution strategy in comparison with a single large trader's (optimal) execution problem. We formulate this Markov game model as a dynamic game of complete information. Therefore, the above (hypothesized) definition and assumption associated with the notion of common knowledge are legitimate so that the solution concept of a Nash equilibrium in a non-cooperative game is (rationally or ideally) applicable in this model. The formulation of a generalized model as a dynamic game of incomplete information requires further intricate analysis, which is left for our future research.

We define a (history-independent) decision rule $f_t^i$ for $i = 1, 2$ at time $t \in \{1, \ldots, T\}$ by a Borel measurable map from a state $s_t \in S$ to each action

$$q_t^i = f(s_t) \in A^i, \quad i = 1, 2, \tag{46}$$

and a Markov execution strategy $\pi^i$ of trader $i = 1, 2$ as a sequence of one-stage decision rules

$$\pi^i := (f_1^i, \ldots, f_t^i, \ldots, f_T^i), \quad i = 1, 2. \tag{47}$$

As in Section 2.2, $\Pi_{\mathrm{M}}^i$ for $i = 1, 2$ denote the set of all Markov execution strategies of each large trader. We also describe the sub Markov execution strategy after time $t$ of a Markov execution strategy $\pi^i$ as

$$\pi_t^i := (f_t^i, \ldots, f_T^i), \quad i = 1, 2, \tag{48}$$

and the entire set of $\pi_t^i$ for $i = 1, 2$ as $\Pi_{\mathrm{M},t}^i$. Hereafter, we consider only non-randomized Markov strategies in this paper.

An expected utility payoff arising from the terminal wealth $w_{T+1}^i$ with the absolute risk aversion $R^i (> 0)$ for $i = 1, 2$ constitutes the value function for a large trader $i = 1, 2$:

$$V_1^i(\pi^1, \pi^2)\big[s_1\big] := \mathbb{E}_1^{(\pi^1, \pi^2)}\Big[g_{T+1}^i(s_{T+1})\Big|s_1\Big], \quad i = 1, 2. \tag{49}$$

Furthermore, we define the expected utility payoff at time $t$ under the strategy $\pi_t^1, \pi_t^2$ as

$$\begin{aligned} V_t^i(\pi_t^1, \pi_t^2)\big[s_t\big] &:= \mathbb{E}_t^{(\pi_t^1, \pi_t^2)}\Big[g_{T+1}^i(s_{T+1})\Big|s_t\Big], \quad s_t \in S = \mathbb{R}^6, \\ t &= 1, \ldots, T, T+1, \quad i = 1, 2. \end{aligned} \tag{50}$$

What we seek here is an equilibrium execution strategy of these two large traders. First, we consider a Nash equilibrium. The definition of a Nash equilibrium in this model becomes as follows:

Definition 4.1 (Nash Equilibrium) $(\pi^{1*}, \pi^{2*}) \in \Pi_{\mathrm{M}}^1 \times \Pi_{\mathrm{M}}^2$ is a Nash equilibrium starting from a fixed initial state $s_1$ if and only if

$$V_1^1(\pi^{1*}, \pi^{2*})\big[s_1\big] \geq V_1^1(\pi^1, \pi^{2*})\big[s_1\big], \quad \forall \pi^1 \in \Pi_{\mathrm{M}}^1, \tag{51}$$

$$V_1^2(\pi^{1*}, \pi^{2*})\big[s_1\big] \geq V_1^2(\pi^{1*}, \pi^2)\big[s_1\big], \quad \forall \pi^2 \in \Pi_{\mathrm{M}}^2. \tag{52}$$

We can define a refinement of a Nash equilibrium of this model as the notion of a Markov perfect equilibrium:

Definition 4.2 (Markov Perfect Equilibrium) $(\pi^{1*}, \pi^{2*}) \in \Pi_{\mathrm{M}}^1 \times \Pi_{\mathrm{M}}^2$ is a Markov perfect equilibrium if and only if

$$V_t^1(\pi_t^{1*}, \pi_t^{2*})[s_t] \geq V_t^1(\pi_t^1, \pi_t^{2*})[s_t], \quad \forall \pi_t^1 \in \Pi_{\mathrm{M},t}^1,\ \forall s_t \in S,\ \forall t = 1, \dots, T; \tag{53}$$

$$V_t^2(\pi_t^{1*}, \pi_t^{2*})[s_t] \geq V_t^1(\pi_t^{1*}, \pi_t^2)[s_t], \quad \forall \pi_t^2 \in \Pi_{\mathrm{M},t}^2,\ \forall s_t \in S,\ \forall t = 1, \dots, T. \tag{54}$$

Based on the following One Stage [Step, Shot] Deviation Principle, we obtain an equilibrium execution strategy at a Markov perfect equilibrium by backward induction procedure of dynamic programming from time $T$ to 1.

$$\begin{aligned} &V_t^1(\pi_t^{1*}, \pi_t^{2*})[s_t] \\ &= \sup_{q_t^1 \in \mathbb{R}} \mathbb{E}\Big[ V_{t+1}^1(\pi_{t+1}^{1*}, \pi_{t+1}^{2*})\big[h_t(s_t, (q_t^1, f_t^{2*}(s_t)), (v_t, \varepsilon_t))\big] \Big| s_t \Big] \\ &= \mathbb{E}\Big[ V_{t+1}^1(\pi_{t+1}^{1*}, \pi_{t+1}^{2*})\big[h_t(s_t, (f_t^{1*}(s_t), f_t^{2*}(s_t)), (v_t, \varepsilon_t))\big] \Big| s_t \Big]; \end{aligned} \tag{55}$$

$$\begin{aligned} &V_t^2(\pi_t^{1*}, \pi_t^{2*})[s_t] \\ &= \sup_{q_t^2 \in \mathbb{R}} \mathbb{E}\Big[ V_{t+1}^2(\pi_{t+1}^{1*}, \pi_{t+1}^{2*})\big[h_t(s_t, (f_t^{1*}(s_t), q_t^2), (v_t, \varepsilon_t))\big] \Big| s_t \Big] \\ &= \mathbb{E}\Big[ V_{t+1}^2(\pi_{t+1}^{1*}, \pi_{t+1}^{2*})\big[h_t(s_t, (f_t^{1*}(s_t), f_t^{2*}(s_t)), (v_t, \varepsilon_t))\big] \Big| s_t \Big]. \end{aligned} \tag{56}$$

### 4.3. *Markov perfect equilibrium*

From the above setting, we obtain the following main theorem.

Theorem 4.1 (Markov Perfect Equilibrium) *There exists a Markov perfect equilibrium* $(\pi^{1*}, \pi^{2*}) \in \Pi_{\mathrm{M}}^1 \times \Pi_{\mathrm{M}}^2$ *at which for the large traders* $i = 1, 2$ *the following properties hold:*

(i) *The execution volume* $q_t^{i*}, i = 1, 2,$ *for* $t \in \{1, \dots, T\}$ *at the Markov perfect equilibrium are affine functions of each remaining execution volume* $\overline{Q}_t^1$ *and* $\overline{Q}_t^2$ *and the cumulative residual effect* $r_t$ *at time t, i.e.*

$$q_t^{i*} = f_t^i(s_t) = a_t^i + b_t^i \overline{Q}_t^i + c_t^i \overline{Q}_t^j + d_t^i r_t, \quad t = 1, \dots, T, \quad i, j = 1, 2, \quad i \neq j. \tag{57}$$

(ii) *The expected utility at the Markov perfect equilibrium* $(\pi^{1*}, \pi^{2*})$ *for the large traders* $i = 1, 2$ *in the subgame starting from the state* $s_t$ $(t \in \{1, \dots, T\})$ *have a functional form as follows:*

$$\begin{aligned} &V_t^i(\pi_t^{i*}, \pi_t^{j*})[s_t] \\ &= -\exp\Big\{ -R^i\Big[ w_t - p_t \overline{Q}_t^i - G_t^i (\overline{Q}_t^i)^2 - H_t^i \overline{Q}_t^i \\ &\quad + I_t^i r_t \overline{Q}_t^i + J_t^i r_t^2 + L_t^i r_t \\ &\quad + M_t^i (\overline{Q}_t^j)^2 + N_t^i \overline{Q}_t^j + X_t^i r_t \overline{Q}_t^j + Y_t^i \overline{Q}_t^i \overline{Q}_t^j + Z_t^i \Big] \Big\}, \quad i, j = 1, 2, \quad i \neq j. \end{aligned} \tag{58}$$

We provide the derivation of the recursive equations:

$$a_t^i, b_t^i, c_t^i, d_t^i, G_t^i, H_t^i, I_t^i, J_t^i, L_t^i, M_t^i, N_t^i, X_t^i, Y_t^i, Z_t^i, \quad t = 1, \dots, T, \quad i = 1, 2, \tag{59}$$

in Appendix 2.

This theorem states that a Markov perfect equilibrium strategy profile $(\pi^{1*}, \pi^{2*}) \in \Pi_{\mathrm{M}}^1 \times \Pi_{\mathrm{M}}^2$ in this model depends on the state $s_t = ((w_t^1, w_t^2), p_t, (\overline{Q}_t^1, \overline{Q}_t^2), r_t)$ of the game only through each remaining execution volume $\overline{Q}_t^1, \overline{Q}_t^2$ and the cumulative residual effect $r_t$, and not through the wealth of each large trader $w_t^1, w_t^2$, or market price $p_t$. As mentioned in Section 2.3, the equilibrium execution volume $q_t^{i*}$ includes a nondeterministic term (random variable) thorough $v_t$ in $r_t$. Therefore, a deterministic $v_t$ yields a deterministic equilibrium execution strategy.

Corollary 2 *If* $v_t$ *are deterministic functions of time* $t \in \{1, \dots, T\}$, *then equilibrium execution volumes,* $q_t^{i*}$, *for* $i = 1, 2$ *and* $t \in \{1, \dots, T\}$ *also become deterministic functions of time.*

These results are our contribution to the field of a market impact game and the different points compared with the existing research of equilibrium execution strategies such as Schied and Zhang (2018) or Luo and Schied (2018). Their works reveal that the equilibrium execution strategies are deterministic when minimizing the expected execution cost and considering a mean–variance optimization, that is, minimizing the mean–variance functional of trading costs. Moreover, they show that maximizing expected CARA utility of revenues defined by negative costs is consistent with mean–variance optimization over the class of deterministic strategies. As shown in our model, however, equilibrium execution strategies obtained by backward induction methods of dynamic programming are not always deterministic. It is mainly when the aggregate volume of orders submitted by trading crowd are deterministic that the equilibrium execution strategies become also deterministic.

## 5. Comparative statics

In this section, we illustrate some numerical examples and show some properties of the equilibrium execution strategies derived in Section 4. The maturity is set as $T = 10$, and large trader $i = 1, 2$ plans to execute the volume $Q^i = 100\,000$ within the time horizon $\{1, \dots, T\}$ at the beginning. We conduct the following comparative statics assuming the same situation in Section 3; time consistency of the time-dependent parameters $\mu_t^v, \sigma_t^v, \mu_t^\varepsilon, \sigma_t^\varepsilon, \alpha_t, \lambda_t, \kappa_t$. Here we further assume that there is no price impact caused by trading crowd, i.e. $\kappa_t = 0$, which is equivalent with setting $\mu_t^v$ and $\sigma_t^v$ as zero for all $t \in \{1, \dots, T\}$. This assumption yields the explicit form of equilibrium execution volumes at a Markov perfect equilibrium (, which indicates the deterministic equilibrium execution volumes), since $\sigma_t^v = 0$ means that the submission of the trading crowd becomes deterministic. The benchmark values are set as follows:

$$\begin{aligned}&\mu_t^\varepsilon \equiv 0; \quad \sigma_t^\varepsilon \equiv 0.02; \quad \alpha_t \equiv 0.5;\\ &\lambda_t \equiv 0.001; \quad \kappa_t \equiv 0; \quad \rho = 0.1; \quad R^i = 0.001,\\ &\forall t = 1, \ldots, T, \quad i = 1, 2.\end{aligned} \tag{60}$$

As considered in Section 3, the residual effect $r_t$ satisfies $r_1 = 0$ in the following comparative statics.

### 5.1. *In the case of symmetric large trader*

In this subsection, we deal with the case that the initial inventory and the risk aversion rate of each large trader are equal; $Q^1 = Q^2$, and $R^1 = R^2$. We set the same risk aversion rate as $R$ in this subsection.

**5.1.1. The effect of risk aversion.** First, we demonstrate the difference of a large trader's execution volume with different risk aversion rates in the following three cases: $R = 0.001, R = 0.5$, and $R = 1$.

Figure 7 shows that the large traders tend to execute faster if they are more risk-averse. The reason is identical to the one shown in Section 3.2.

**5.1.2. The effect of $\alpha_t$.** We examine the effect on the execution volume caused by the price reversion rate $\alpha_t$ through the following three cases: $\alpha_t = 0.01, \alpha_t = 0.5$, and $\alpha_t = 1$.

As depicted in figure 8, the large traders execute faster as $\alpha_t \to 0$. The reason for this is that the smaller $\alpha_t$ at time $t \in \{1, \ldots, T\}$ coincides with the smaller price recovery at the next time $t + 1$, which infers that the large part of the price impact remains, leading to a higher execution price at the next trading.

**5.1.3. The effect of $\sigma_t^\varepsilon$.** The next example describes how $\sigma_t^\varepsilon$ affects the execution volume through the following three cases: $\sigma_t^\varepsilon = 0.02, \sigma_t^\varepsilon = 0.5$, and $\sigma_t^\varepsilon = 1$.

Figure 9 illustrates that the larger $\sigma_t^\varepsilon$ makes large traders execute faster, which is the same result in Section 3.4, i.e. large traders are going to execute faster if $\sigma_t^\varepsilon$ is large since they are likely to avoid the risk of exposition to the higher price in the future.

**5.1.4. The effect of resilience speed.** To examine the effect of the resilience speed on the execution volume, we draw, in the following, the figures of the three cases: $\rho = 0.1, \rho = 1$, and $\rho = 10$ (figure 10).

Note that even if $v_t$ is not deterministic, i.e. $v_t \sim N(\mu_t^v, (\sigma_t^v)^2)$, when $\rho \to \infty$, the limiting equilibrium strategy becomes a deterministic one as stated below. Since $r_{t+1} \to 0$ as $\rho \to \infty$ from equation (5), the equilibrium execution volume at a Markov perfect equilibrium at time $t \in \{1, \ldots, T\}$ becomes an affine function of the remaining execution volumes of each large trader $\overline{Q}_t^1$ and $\overline{Q}_t^2$:

$$q_t^{i*} = f_t^i(s_t) = a_t^i + b_t^i \overline{Q}_t^i + c_t^i \overline{Q}_t^j, \quad t = 1, \ldots, T, i, j = 1, 2, \quad i \neq j, \tag{61}$$

when $\rho \to \infty$. Hence, as we have shown in Section 3, when the price impact is supposed to be permanent, $q_t^*$ follows a simultaneous difference equation which is a deterministic function on time $t \in \{1, \ldots, T\}$ and depends on the state $s_t = ((w_t^1, w_t^2), p_t, (\overline{Q}_t^1, \overline{Q}_t^2), r_t)$ only through the remaining execution volumes $\overline{Q}_t^1$ and $\overline{Q}_t^2$.

### 5.2. *In the case of asymmetric large trader*

In this subsection, we illustrate the situation where the position of each large trader at the beginning of the trading horizon or the risk aversion rate are different; $Q^1 \neq Q^2$ or $R^i \neq R^j$.

**5.2.1. The effect of $Q^i$.** We demonstrate the case that at the beginning one large trader $i$ plans to purchase $Q^i$ volumes of one risky asset which is not equal to (or less than) the quantity $Q^j$ to be bought by the large trader $j$. We present the three cases as follows: $Q^i = 100\,000$, $Q^i = 50\,000$, and $Q^i = 0$; whereas the remaining execution volume of the counterpart is fixed: $Q^j = 100\,000$.

Figure 11 illustrates that if the initial volume $Q^i$ is equal to 0, there exists a round trip trading for trader $i$, by which the large trader $i = 1, 2$ can increase the expected utility of the final wealth by starting from $Q^i = 0$. This round-trip strategy satisfies, by the Jensen's inequality,

$$\begin{aligned}-\exp\left\{-Rw_1^i\right\} &= V_1^i(\mathbf{0}, \pi_1^{j*})[s_1] \leq V_1^i(\pi_1^{1*}, \pi_1^{2*})[s_1]\\ &= \mathbb{E}_1^{(\pi_1^{1*},\pi_1^{2*})}\left[-\exp\{-R^i w_{T+1}^i\}|s_1\right]\\ &\leq -\exp\left\{-R^i\, \mathbb{E}_1^{(\pi_1^{1*},\pi_1^{2*})}\left[w_{T+1}^i|s_1\right]\right\},\end{aligned} \tag{62}$$

which implies $\mathbb{E}_1^{(\pi_1^{1*},\pi_1^{2*})}[w_{T+1}^i|s_1] \geq w_1^i$. Moreover, if $w_{T+1}^i$ is not constant under the probability measure $\mathbb{P}^{(\pi_1^{1*},\pi_1^{2*})}$, the strict inequality holds due to the strict concavity of the (negative) exponential function: $\mathbb{E}_1^{(\pi_1^{1*},\pi_1^{2*})}[w_{T+1}^i|s_1] > w_1^i$. When $Q^i$ is 50000, a round trip trading is included in the trajectory of the remaining execution volume. These results infer that a volume $Q^{i*}$ exists for the large trader $i$ satisfying the following condition: if the initial volume $Q^i$ is smaller than $Q^{i*}$, the execution strategy of the large trader $i$ becomes a round trip trading or partially includes a round trip trading.

**5.2.2. The effect of $R^j$.** We examine how the execution strategies are affected by the risk aversion of the opponent by illustrating the following three cases: $R^j = 0.001, R^j = 0.1$ and $R^j = 10$ while $R^i$ is always 0.1.

From figure 12, we confirm that the large trader $i$ executes less volume at the beginning and liquidates his/her last position more at the terminal as the risk aversion rate of the

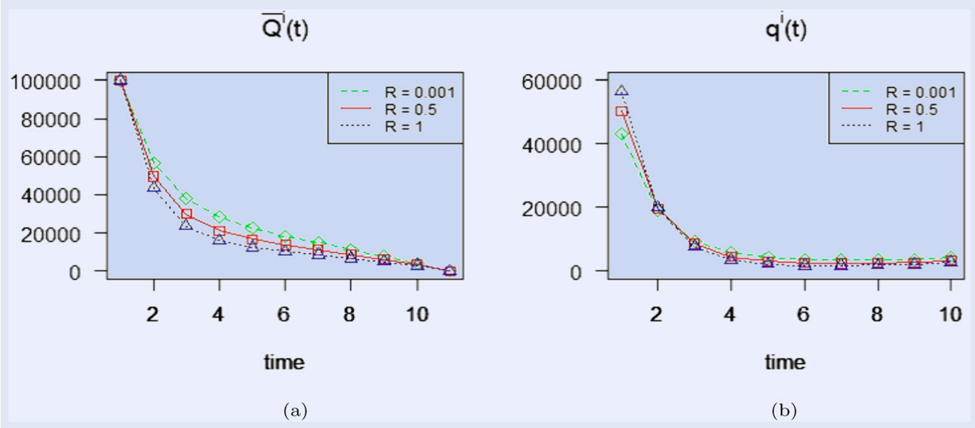


Figure 7. The effect of risk aversion. (a) Remaining execution volume $\overline{Q}_t^i (t = 1, \ldots, T)$. (b) Execution volume $q_t^i (t = 1, \ldots, T)$.

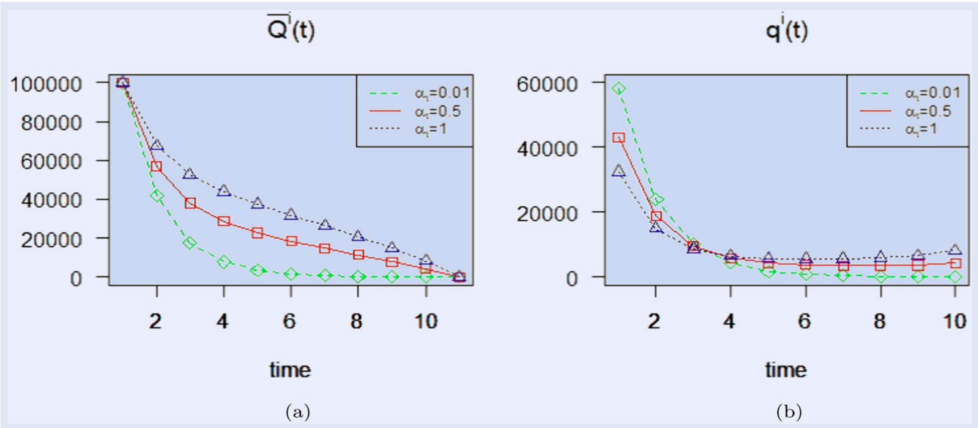


Figure 8. The effect of $\alpha_t$. (a) Remaining execution volume $\overline{Q}_t^i (t = 1, \ldots, T)$. (b) Execution volume $q_t^i (t = 1, \ldots, T)$.

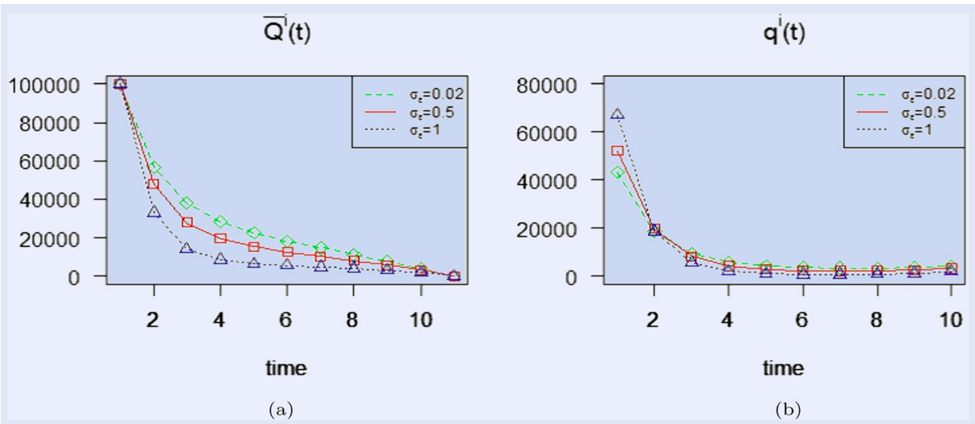


Figure 9. The effect of $\sigma_t^\varepsilon$. (a) Remaining execution volume $\overline{Q}_t^i (t = 1, \ldots, T)$. (b) Execution volume $q_t^i (t = 1, \ldots, T)$.

counterpart $j$ becomes larger. In the middle time of the trading epoch, the trajectory of the executed volume tends to increase, particularly seen when $R^j = 10$. These facts infer the following statements: as the counterpart is more risk-averse, the speed of the execution becomes slower at the beginning and the path of the execution volume of the large trader is gradually growing over the course of the execution process. Then, the large trader executes more volume at the end of the trading.

Proposition 5.1 *If the price impact* (*or quoted price*) *is not affected by the public news effect for all the time, that is,* $\mu_t^\varepsilon = 0$ *and* $\sigma_t^\varepsilon = 0$ *for* $t \in \{1, \ldots, T\}$, *the execution strategies of the large trader* $i$ *do not depend on the risk aversion rate of the counterpart* $R^j$. *Then, a unique strategy is determined by other determining factors and parameters.*

**5.2.3. The existence of sell trader.** In the following, we illustrate the case that one large trader $j$ sells the quantity $Q^j$. The quantity $Q^j$ to be sold is the same as the volume $Q^i$ to be bought by the large trader $i$. In mathematical expression, $Q^j = -Q^i = -100\,000$.

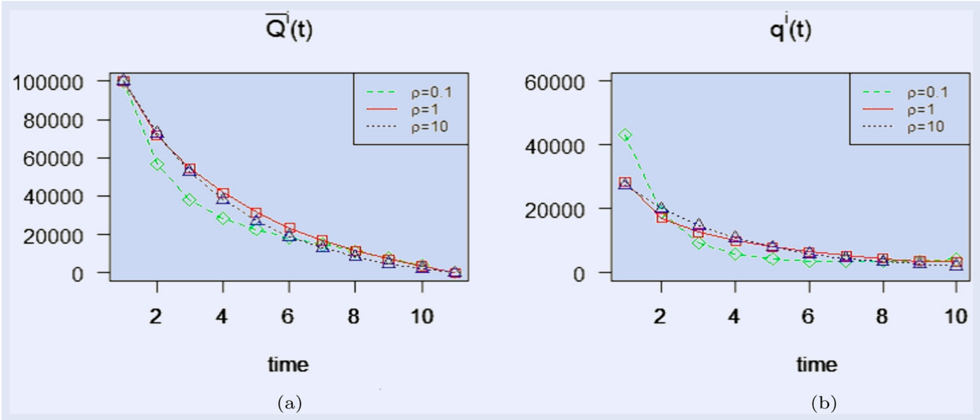


Figure 10. The effect of the resilience speed. (a) Remaining execution volume $\overline{Q}_t^i (t = 1, \ldots, T)$. (b) Execution volume $q_t^i (t = 1, \ldots, T)$.

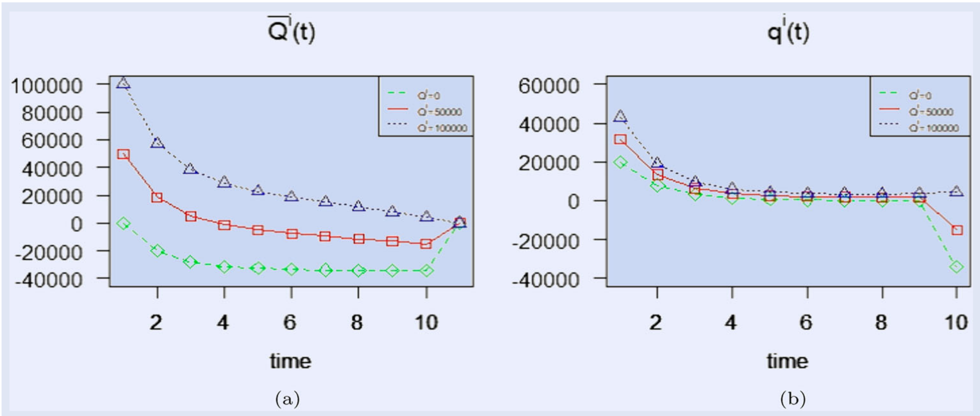


Figure 11. The effect of $Q^i$. (a) Remaining execution volume $\overline{Q}_t^i (t = 1, \ldots, T)$. (b) Execution volume $q_t^i (t = 1, \ldots, T)$.

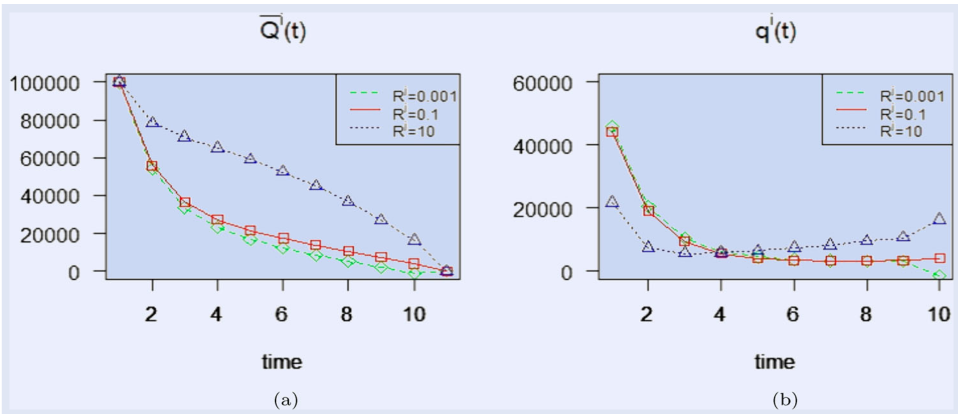


Figure 12. The effect of $R^j$. (a) Remaining execution volume $\overline{Q}_t^i (t = 1, \ldots, T)$. (b) Execution volume $q_t^i (t = 1, \ldots, T)$.

Figure 13 indicates both the buy- and sell-traders remain patient with the other side's execution with each other before the maturity $T$, liquidating all their remaining positions at the terminal. A close consideration reveals that they don't execute their whole order simultaneously at the beginning of the trade, although it leads to canceling out the impact of their execution on market price. The following reasoning illustrates the above insights. The sell-trader can earn if he/she executes the sell-transaction after the execution price goes up because of a buy-side execution than the simultaneous execution at the beginning. In contrast, the buy-side trader can purchase the risky asset at a lower price after the sell-trader unwinds his/her position, which saves execution costs. These examinations imply that both sides of the large traders can produce a profit by executing slowly, which causes them to liquidate their positions much measuredly.

### 5.3. Comparison of the two models

The following figures show how the executions differ between in the single-large trader model in Section 2 and the two-large

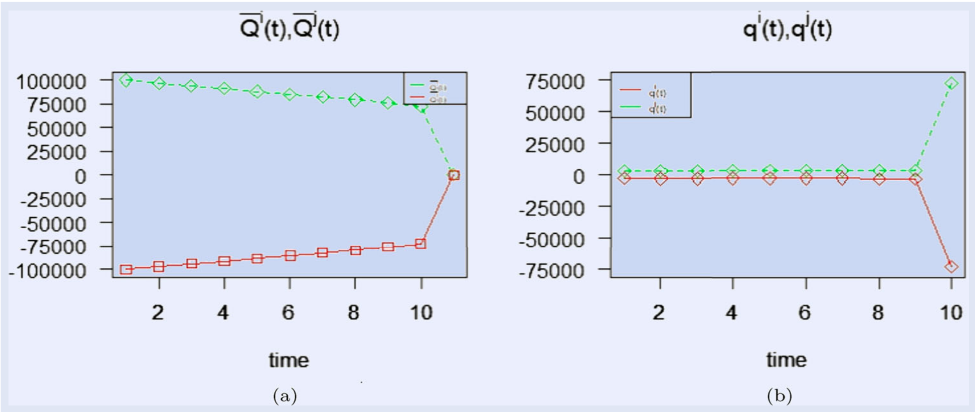


Figure 13. The existence of sell trader. (a) Remaining execution volume $\overline{Q}_t^i (t = 1, \ldots, T)$. (b) Execution volume $q_t^i (t = 1, \ldots, T)$.

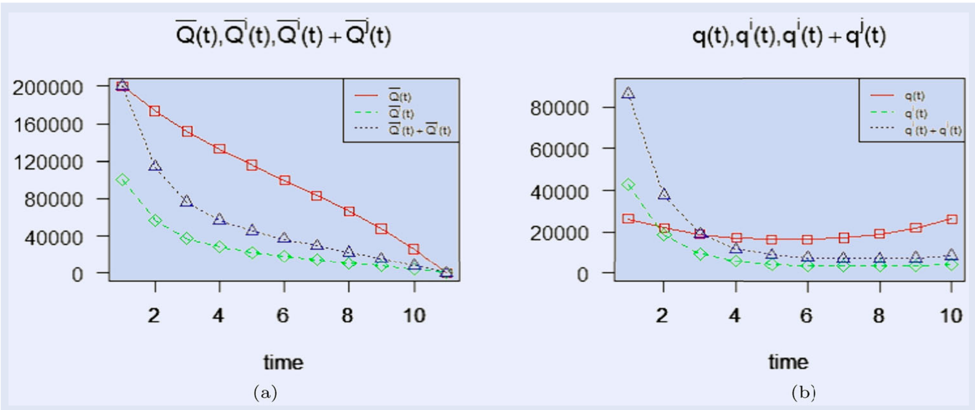


Figure 14. Comparison of the two models. (a) Remaining execution volumes $\overline{Q}_t, \overline{Q}_t^i$, and $\overline{Q}_t^1 + \overline{Q}_t^2 (t = 1, \ldots, T)$. (b) Execution volumes $q_t$, $q_t^i$, and $q_t^1 + q_t^2 (t = 1, \ldots, T)$.

trader model in Section 4. In order to compare the differences, the price impact coefficient of the trading crowd $\nu_t$ is set as 0, which is the same meaning as setting $\mu_t^\nu = 0$ and $\sigma_t^\nu = 0$ for $t \in \{1, \ldots, T\}$ in both models. Then, the volume of which a single large trader unwinds the position and the total volume of two large traders are deemed as same, i.e. $Q/2 = Q^i = Q^j = 100\,000$ (that is, $Q = Q^i + Q^j = 200\,000$ and $Q^i = Q^j$).

From figure 14, the single large trader of the two-large trader model executes each order not too fast over the trading time horizon at first glance. The initial position is, however, quite different, the second model of which is half the volume of the first model ($Q = 200\,000$ and $Q^i = 100\,000$). It is not overstated that the large trader of the latter model unwinds his/her position twice as fast as that of the former model. The interaction of the multiple large traders becomes a vital factor for the reason. The traders in the second model unwind their positions faster for fear that the price will be pushed up by the execution of the counterpart. This logic contrasts with the explanation in Section 5.2.2., where each large trader intends to acquire/liquidate their positions after the counterpart executes his/her order; if two large traders with equivalent initial inventories, either buying or selling, exist in a market, each large trader takes advantage of acquiring the risky asset at a lower price and liquidating at a higher price by executing faster than the counterpart.

## 6. Conclusion and future research

We have constructed, in a (finite) discrete-time framework, two models focusing on a single large trader in the first model and two large traders in the second model. The traders maximize the expected CARA utility arising from each large trader's wealth at the end of the trading epoch in a market with a trading crowd. By constructing a generalized price impact model, the backward induction method of dynamic programming permitted us to derive the optimal execution strategy in the first model and a Markov perfect equilibrium strategy in the second model. The most important result emerging from this research is as follows: the aggregate execution volume of a trading crowd has an impact on the execution of each large trader, yielding nondeterministic optimal and equilibrium execution strategies. All the intriguing results we have obtained from the numerical examples show how various kinds of situations can influence the execution strategy of a large trader in both models. This kind of work which is concerned with an execution problem through the backward induction procedure of dynamic programming will be explored in future work from a more in-depth and extensive perspective, which we expect will also give us a more illuminating insight into all the other problems left in this field of research as follows.

In the above models, we have assumed that that the price reversion rate and the resilience speed are deterministic. This assumption makes the fundamental price of the risky asset observable for large traders before the trading time. In a real market the fundamental value of a risky asset is, however, unobservable and uncertain. Therefore, we can use an incomplete state information model from control theory, which leads to an analysis of the more realistic situation in the marketplace. For example, an application of the (Kalman) filtering method to the modeling of fundamental price is seemingly novel in the research of market microstructure. Developing an incomplete state information model for either a single- or multiple-large traders will contribute to the studies of a trading market.

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**Disclosure statement**

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## Appendices

### Appendix 1. Proof of Theorem 2.1

We derive the optimal execution volume $q_t^*$ at time $t \in \{1, \ldots, T\}$ by backward induction method of dynamic programming from the maturity $T$. From the assumption that the large trader must unwind all the remainder of his/her position at time $t = T$,

$$\overline{Q}_{T+1} = \overline{Q}_T - q_T = 0, \tag{A1}$$

must hold, which yields $\overline{Q}_T = q_T$. Then, for $t = T$, with the relation of the moment-generating function of $v_t$:

$$\mathbb{E}\left[\exp\{R\kappa_T q_T v_T\}\right] = \exp\left\{R\kappa_T q_T \mu_T^v + \frac{1}{2}R^2\kappa_T^2 q_T^2 (\sigma_T^v)^2\right\}, \tag{A2}$$

Equation (22) (or Bellman equation) becomes

$$\begin{aligned}
V_T\big[s_T\big] &= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[V_{T+1}\big[s_{T+1}\big]\Big|s_T\Big] \\
&= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[V_{T+1}\big[w_{T+1}, p_{T+1}, \overline{Q}_{T+1}, r_{T+1}\big]\Big|w_T, p_T, \overline{Q}_T, r_T\Big] \\
&= \mathbb{E}\Big[-\exp\Big\{-Rw_{T+1}\Big\}\Big] \\
&= \mathbb{E}\Big[-\exp\Big\{-R\Big[w_T - \widehat{p}_T q_T\Big]\Big\}\Big] \\
&= -\exp\Big\{-R\Big[w_T - p_T\overline{Q}_T - \Big(\lambda_T + \frac{1}{2}R\kappa_T^2(\sigma_T^v)^2\Big)\overline{Q}_T^2 \\
&\qquad - \kappa_T\mu_T^v\overline{Q}_T\Big]\Big\} \\
&= -\exp\Big\{-R\Big[W_T - P_T\overline{Q}_T + G_T\overline{Q}_T^2 + H_T\overline{Q}_T\Big]\Big\},
\end{aligned} \tag{A3}$$

where

$$G_T := -\left(\lambda_T + \frac{1}{2}R\kappa_T^2(\sigma_T^v)^2\right)(< 0); \quad H_T := -\kappa_T\mu_T^v.$$

For $t = T - 1$, according to the equation (22) (or Bellman equation), we have

$$\begin{aligned}
&V_{T-1}\big[s_{T-1}\big] \\
&\quad = \sup_{q_{T-1}\in\mathbb{R}} \mathbb{E}\Big[V_T\big[s_T\big]\Big|s_{T-1}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{q_{T-1}\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-R\Big[w_T - p_T\overline{Q}_T + G_T\overline{Q}_T^2 + H_T\overline{Q}_T\Big] \\
&\quad \times \Big| w_{T-1}, p_{T-1}, \overline{Q}_{T-1}, r_{T-1}\Big] \\
&= \sup_{q_{T-1}\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-R\Big[w_{T-1} - \{p_{T-1} \\
&\quad + (\lambda_{T-1}q_{T-1} + \kappa_{T-1}v_{T-1})\}q_{T-1} \\
&\quad - \{p_{T-1} - (1-\mathrm{e}^{-\rho})r_{T-1} + (\lambda_{T-1}q_{T-1} + \kappa_{T-1}v_{T-1}) \\
&\quad \times \{\alpha_{T-1}\mathrm{e}^{-\rho} + (1-\alpha_{T-1})\} + \varepsilon_{T-1}\} \\
&\quad \times (\overline{Q}_{T-1} - q_{T-1}) + G_T(\overline{Q}_{T-1} - q_{T-1})^2 \\
&\quad + H_T(\overline{Q}_{T-1} - q_{T-1})\Big]\Big\}\Big| w_{T-1}, p_{T-1}, \overline{Q}_{T-1}, r_{T-1}\Big] \\
&= \sup_{q_{T-1}\in\mathbb{R}} -\exp\Big\{-R\Big[-A_{T-1}q_{T-1}^2 + (B_{T-1}\overline{Q}_{T-1} \\
&\quad + C_{T-1}r_{T-1} + D_{T-1})q_{T-1} \\
&\quad + w_{T-1} - p_{T-1}\overline{Q}_{T-1} \\
&\quad + \Big\{G_T - \frac{1}{2}R(\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 - \frac{1}{2}R(\sigma_{T-1}^\varepsilon)^2\Big\}\overline{Q}_{T-1}^2 \\
&\quad + \Big(H_T - \alpha^{T-1}\kappa_{T-1}\mu_{T-1}^v - \mu_{T-1}^\varepsilon\Big) \\
&\quad \times \overline{Q}_{T-1} + (1-\mathrm{e}^{-\rho})\overline{Q}_{T-1}r_{T-1}\Big]\Big\},
\end{aligned} \tag{A4}
$$

with the following relation:

$$\alpha^{T-1} := \alpha_{T-1}\mathrm{e}^{-\rho} + (1-\alpha_{T-1}); \tag{A5}$$

$$
\begin{aligned}
A_{T-1} &:= (1-\alpha^{T-1})\lambda_{T-1} - G_T + \frac{1}{2}R(1-\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 \\
&\quad + \frac{1}{2}R(\sigma_{T-1}^\varepsilon)^2;
\end{aligned} \tag{A6}
$$

$$
\begin{aligned}
B_{T-1} &:= -\alpha^{T-1}\lambda_{T-1} - 2G_T - R\alpha^{T-1}(1-\alpha^{T-1})\kappa_{T-1}^2(\sigma_{T-1}^v)^2 \\
&\quad + R(\sigma_{T-1}^\varepsilon)^2;
\end{aligned} \tag{A7}
$$

$$
\begin{aligned}
C_{T-1} &:= -(1-\mathrm{e}^{-\rho}); \\
D_{T-1} &:= -H_T - (1-\alpha^{T-1})\kappa_{T-1}\mu_{T-1}^v + \mu_{T-1}^\varepsilon.
\end{aligned} \tag{A8}
$$

Finding the optimal execution volume $q_{T-1}^*$ which attains the supremum of equation (A4) is equivalent to finding the one which yields the maximum of

$$
\begin{aligned}
K_{T-1}(q_{T-1}) &:= -A_{T-1}q_{T-1}^2 + (B_{T-1}\overline{Q}_{T-1} \\
&\quad + C_{T-1}r_{T-1} + D_{T-1})q_{T-1} \\
&\quad + w_{T-1} - p_{T-1}\overline{Q}_{T-1} \\
&\quad + \Big\{G_T - \frac{1}{2}R(\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 - \frac{1}{2}R(\sigma_{T-1}^\varepsilon)^2\Big\}\overline{Q}_{T-1}^2 \\
&\quad + \Big(H_T - \alpha^{T-1}\kappa_{T-1}\mu_{T-1}^v - \mu_{T-1}^\varepsilon\Big) \\
&\quad \overline{Q}_{T-1} + (1-\mathrm{e}^{-\rho})\overline{Q}_{T-1}r_{T-1},
\end{aligned} \tag{A9}
$$

since both equations (A4) and (A9) are concave functions with respect to $q_{T-1}$. Thus, by completing the square of $K_{T-1}(q_{T-1})$ with respect to $q_{T-1}$, we obtain the optimal execution volume $q_{T-1}^*$ as

$$
\begin{aligned}
q_{T-1}^* &= \frac{B_{T-1}\overline{Q}_{T-1} + C_{T-1}r_{T-1} + D_{T-1}}{2A_{T-1}} \\
&\quad \big(=: a_{T-1} + b_{T-1}\overline{Q}_{T-1} + c_{T-1}r_{T-1}\big).
\end{aligned} \tag{A10}
$$

Then, the optimal value function at time $T-1$ becomes a functional form as follows:

$$
\begin{aligned}
&V_{T-1}[s_{T-1}] \\
&= -\exp\Big\{-R\Big[w_{T-1} - p_{T-1}\overline{Q}_{T-1} \\
&\quad + \Big\{G_T - \frac{1}{2}R(\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 - \frac{1}{2}R(\sigma_{T-1}^\varepsilon)^2\Big\}\overline{Q}_{T-1}^2 \\
&\quad + \Big(H_T - \alpha^{T-1}\kappa_{T-1}\mu_{T-1}^v - \mu_{T-1}^\varepsilon\Big)\overline{Q}_{T-1} \\
&\quad + (1-\mathrm{e}^{-\rho})\overline{Q}_{T-1}r_{T-1} \\
&\quad + \frac{(B_{T-1}\overline{Q}_{T-1} + C_{T-1}r_{T-1} + D_{T-1})^2}{4A_{T-1}}\Big]\Big\} \\
&= -\exp\Big\{-R\Big[w_{T-1} - p_{T-1}\overline{Q}_{T-1} + G_{T-1}\overline{Q}_{T-1}^2 \\
&\quad + H_{T-1}\overline{Q}_{T-1} + I_{T-1}\overline{Q}_{T-1}r_{T-1} \\
&\quad + J_{T-1}r_{T-1}^2 + L_{T-1}r_{T-1} + Z_{T-1}\Big]\Big\},
\end{aligned} \tag{A11}
$$

where

$$
\begin{aligned}
G_{T-1} &:= G_T - \frac{1}{2}R(\alpha^{T-1})^2\kappa_{T-1}^2(\sigma_{T-1}^v)^2 - \frac{1}{2}R(\sigma_{T-1}^\varepsilon)^2 + \frac{B_{T-1}^2}{4A_{T-1}}; \\
H_{T-1} &:= H_T - \alpha^{T-1}\kappa_{T-1}\mu_{T-1}^v - \mu_{T-1}^\varepsilon + \frac{B_{T-1}D_{T-1}}{2A_{T-1}}; \\
I_{T-1} &:= (1-\mathrm{e}^{-\rho}) + \frac{B_{T-1}C_{T-1}}{2A_{T-1}}; \quad J_{T-1} := \frac{C_{T-1}^2}{4A_{T-1}}, \\
L_{T-1} &:= \frac{C_{T-1}D_{T-1}}{2A_{T-1}}, \quad Z_{T-1} := \frac{D_{T-1}^2}{4A_{T-1}}.
\end{aligned} \tag{A12}
$$

For $t \in \{T-2, \ldots, 1\}$, we can assume from the above results that the optimal value function has the following functional form at time $t+1$:

$$
\begin{aligned}
&V_{t+1}[s_{t+1}] \\
&= -\exp\Big\{-R\Big[w_{t+1} - p_{t+1}\overline{Q}_{t+1} + G_{t+1}\overline{Q}_{t+1}^2 + H_{t+1}\overline{Q}_{t+1} \\
&\quad + I_{t+1}\overline{Q}_{t+1}r_{t+1} + J_{t+1}r_{t+1}^2 + L_{t+1}r_{t+1} + Z_{t+1}\Big]\Big\}.
\end{aligned} \tag{A13}
$$

Then, we can obtain the following calculation by substituting the dynamics of $w_t, p_t, \overline{Q}_t, r_t$ into the equation above:

$$
\begin{aligned}
V_t[s_t] &= \sup_{q_t\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-R\Big[w_{t+1} - p_{t+1}\overline{Q}_{t+1} \\
&\quad + G_{t+1}\overline{Q}_{t+1}^2 + H_{t+1}\overline{Q}_{t+1} + I_{t+1}\overline{Q}_{t+1}R_{t+1} + J_{t+1}R_{t+1}^2 \\
&\quad + L_{t+1}R_{t+1} + Z_{t+1}\Big]\Big\}\Big| w_t, p_t, \overline{Q}_t, r_t,\Big] \\
&= \sup_{q_t\in\mathbb{R}} -\exp\Big\{-R\Big[-A_tq_t^2 + (B_t\overline{Q}_t + C_tr_t + D_t) \\
&\quad \times q_t + w_t - p_t\overline{Q}_t \\
&\quad + \Big[G_{t+1} - \frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\eta_t^2(\sigma_t^v)^2 - \frac{1}{2}R(\sigma_t^\varepsilon)^2\Big]\overline{Q}_t^2 \\
&\quad + \Big[H_{t+1} + \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\eta_t\mu_t^v \\
&\quad - \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\theta_t\phi_t(\sigma_t^v)^2 - \mu_t^\varepsilon\Big]\overline{Q}_t \\
&\quad + \Big[(1-\mathrm{e}^{-\rho}) + \mathrm{e}^{-\rho}I_{t+1} \\
&\quad - \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\eta_t\theta_t(\sigma_t^v)^2\Big]\overline{Q}_tr_t
\end{aligned}
$$

$$+\Big[e^{-2\rho}J_{t+1}-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\theta_t^2(\sigma_t^v)^2\Big]r_t^2$$
$$+\Big[e^{-\rho}L_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\theta_t\mu_t^v$$
$$-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\theta_t\phi_t(\sigma_t^v)^2\Big]r_t$$
$$+\Big[Z_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\phi_t\mu_t^v$$
$$-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\phi_t^2(\sigma_t^v)^2$$
$$+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\zeta_t(\mu_t^v)^2+x_t\Big]\Big]\Big\}, \tag{A14}$$

where

$$\alpha^t:=\alpha_t e^{-\rho}+(1-\alpha_t);\quad \zeta_t:=\kappa_t^2\alpha_t^2e^{-2\rho}J_{t+1};\delta_t$$
$$:=(\alpha^t-1)\kappa_t-\kappa_t\alpha_te^{-\rho}I_{t+1}+2\lambda_t\kappa_t\alpha_t^2e^{-2\rho}J_{t+1};$$
$$\eta_t:=-\kappa_t\alpha^t+\kappa_t\alpha_te^{-\rho}I_{t+1};\theta_t$$
$$:=2\kappa_t\alpha_te^{-\rho}J_{t+1};\quad \phi_t$$
$$:=\kappa_t\alpha_te^{-\rho}L_{t+1};\quad x_t$$
$$:=-\frac{1}{R}\log\frac{1}{\sqrt{1+2R\zeta_t(\sigma_t^v)^2}},$$

and

$$A_t:=(1-\alpha^t)\lambda_t-G_{t+1}+\lambda_t\alpha_te^{-\rho}I_{t+1}-\lambda_t^2\alpha_t^2e^{-2\rho}J_{t+1}$$
$$+\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\delta_t^2(\sigma_t^v)^2+\frac{1}{2}R(\sigma_t^\varepsilon)^2;$$
$$B_t:=-\lambda_t\alpha^t-2G_{t+1}+\lambda_t\alpha_te^{-\rho}I_{t+1}$$
$$-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\delta_t\eta_t(\sigma_t^v)^2+R(\sigma_t^\varepsilon)^2;$$
$$C_t:=-(1-e^{-\rho})-e^{-\rho}I_{t+1}+2\lambda_t\alpha_te^{-2\rho}J_{t+1} \tag{A15}$$
$$-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\delta_t\theta_t(\sigma_t^v)^2;$$
$$D_t:=-H_{t+1}+\lambda_t\alpha_te^{-\rho}L_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\delta_t\mu_t^v$$
$$-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\delta_t\phi_t(\sigma_t^v)^2+\mu_t^\varepsilon.$$

Here we have used the following relation:

$$\mathbb{E}\Big[\exp\Big\{-R\big[\delta_tq_t+\eta_t\overline{Q}_t+\theta_tR_t+\phi_t\big]v_t-R\zeta_tv_t^2\Big\}\Big]$$
$$=\exp\Big\{\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\Big[-R\big[\delta_tq_t+\eta_t\overline{Q}_t+\theta_tR_t+\phi_t\big]\mu_t^v$$
$$+R^2\big[\delta_tq_t+\eta_t\overline{Q}_t+\theta_tR_t+\phi_t\big]^2(\sigma_t^v)^2-R\zeta_t(\mu_t^v)^2\Big]$$
$$-Rx_t\Big\}.$$

To find the optimal execution volume $q_t^*$ at time $t\in\{T-2,\dots,1\}$ which satisfy (A14), we only have to calculate the same derivation at time $t=T-1$, that is, completing the square of

$$K_t(q_t):=-A_tq_t^2+(B_t\overline{Q}_t+C_tr_t+D_t)q_t+w_t-p_t\overline{Q}_t$$
$$+\Big[G_{t+1}-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\eta_t^2(\sigma_t^v)^2-\frac{1}{2}R(\sigma_t^\varepsilon)^2\Big]\overline{Q}_t^2$$
$$+\Big[H_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\eta_ta_t^v$$
$$-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\theta_t\phi_t(\sigma_t^v)^2-\mu_t^\varepsilon\Big]\overline{Q}_t$$
$$+\Big[(1-e^{-\rho})+e^{-\rho}I_{t+1}-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\eta_t\theta_t(\sigma_t^v)^2\Big]\overline{Q}_tr_t$$
$$+\Big[e^{-2\rho}J_{t+1}-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\theta_t^2(\sigma_t^v)^2\Big]r_t^2$$
$$+\Big[e^{-\rho}L_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\theta_t\mu_t^v-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}$$
$$\times R\theta_t\phi_t(\sigma_t^v)^2\Big]r_t+\Big[Z_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\phi_t\mu_t^v$$
$$-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\phi_t^2(\sigma_t^v)^2+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\zeta_t$$
$$\times(\mu_t^v)^2+x_t\Big], \tag{A16}$$

which yields the optimal execution volume $q_t^*$ at time $t\in\{T-2,\dots,1\}$:

$$q_t^*\left(:=f(s_t)=\frac{B_t\overline{Q}_t+C_tr_t+D_t}{2A_t}\right)$$
$$=a_t+b_t\overline{Q}_t+c_tr_t,\quad t=T-2,\dots,1. \tag{A17}$$

where

$$a_t:=\frac{D_t}{A_t},\quad b_t:=\frac{B_t}{A_t},\quad c_t:=\frac{C_t}{A_t}. \tag{A18}$$

Then, by inserting this into equation (A14), the optimal value function at time $t\in\{T-2,\dots,1\}$ has a functional form as follows:

$$V_t\big[s_t\big]=-\exp\Big\{-R\Big[w_t-p_t\overline{Q}_t$$
$$+\Big[G_{t+1}-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\eta_t^2(\sigma_t^v)^2-\frac{1}{2}R(\sigma_t^\varepsilon)^2\Big]\overline{Q}_t^2$$
$$+\Big[H_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\eta_ta_t^v$$
$$-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\theta_t\phi_t(\sigma_t^v)^2-\mu_t^\varepsilon\Big]\overline{Q}_t$$
$$+\Big[(1-e^{-\rho})+e^{-\rho}I_{t+1}-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\eta_t\theta_t(\sigma_t^v)^2\Big]\overline{Q}_tr_t$$
$$+\Big[e^{-2\rho}J_{t+1}-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\theta_t^2(\sigma_t^v)^2\Big]r_t^2$$
$$+\Big[e^{-\rho}L_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\theta_t\mu_t^v$$
$$-\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\theta_t\phi_t(\sigma_t^v)^2\Big]r_t+\Big[Z_{t+1}+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}$$
$$\phi_t\mu_t^v-\frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\phi_t^2(\sigma_t^v)^2$$
$$+\frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\zeta_t(\mu_t^v)^2+x_t\Big]+\frac{(B_t\overline{Q}_t+C_tr_t+D_t)^2}{4A_t}\Big]\Big\}$$
$$=-\exp\Big\{-R\Big[w_t-p_t\overline{Q}_t+G_t\overline{Q}_t^2+H_t\overline{Q}_t+I_t$$
$$\times\overline{Q}_tr_t+J_tr_t^2+L_tr_t+Z_t\Big]\Big\}, \tag{A19}$$

where

$$G_t := G_{t+1} - \frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}} R\eta_t^2(\sigma_t^v)^2 - \frac{1}{2}R(\sigma_t^\varepsilon)^2 + \frac{B_t^2}{4A_t};$$

$$H_t := H_{t+1} + \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\eta_t\mu_t^v - \frac{1}{1+2R\zeta_t(\sigma_t^v)^2} \times R\eta_t\phi_t(\sigma_t^v)^2 - \mu_t^\varepsilon + \frac{B_tD_t}{2A_t};$$

$$I_t := (1-\mathrm{e}^{-\rho}) + \mathrm{e}^{-\rho}I_{t+1} - \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}R\eta_t\theta_t(\sigma_t^v)^2 + \frac{B_tC_t}{2A_t};$$

$$J_t := \mathrm{e}^{-2\rho}J_{t+1} - \frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\theta_t^2(\sigma_t^v)^2 + \frac{C_t^2}{4A_t};$$

$$L_t := \mathrm{e}^{-\rho}L_{t+1} + \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\theta_t\mu_t^v - \frac{1}{1+2R\zeta_t(\sigma_t^v)^2} \times R\theta_t\phi_t(\sigma_t^v)^2 + \frac{C_tD_t}{2A_t};$$

$$Z_t := Z_{t+1} + \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\phi_t\mu_t^v - \frac{1}{2\{1+2R\zeta_t(\sigma_t^v)^2\}}R\phi_t^2(\sigma_t^v)^2 + \frac{1}{1+2R\zeta_t(\sigma_t^v)^2}\zeta_t(\mu_t^v)^2 + x_t + \frac{D_t^2}{4A_t}. \tag{A20}$$

## Appendix 2. Proof of Theorem 4.1

In the following, we derive an equilibrium execution strategy for each large trader at a Markov perfect equilibrium by backward induction method of dynamic programming from maturity $T$. The following proof describes the sketchy one of Theorem 4.1 for the case $\kappa_t \equiv 0$ or $v_t \equiv 0$, from the results of which we conduct the comparative statics shown in Section 5. We can derive the similar results if $\kappa_t \neq 0$ and $v_t$ satisfies $\sigma_t^v \neq 0$ for all $t \in \{1,\dots,T\}$. First, at time $t = T$, due to the (hard) constraint for each large trader to execute all their positions by the maturity,

$$\overline{Q}_{T+1}^i = \overline{Q}_T^i - q_T^i = 0, \quad i = 1,2, \tag{A21}$$

must be satisfied, which yields $\overline{Q}_T^i = q_T^i$ for $i = 1, 2$. Therefore,according to the One Stage [Step, Shot] Deviation Principle, the expected utility payoff of the large trader $i = 1, 2$ (or the value function of large trader $i$) at the maturity becomes

$$\begin{aligned}
V_T^i(\pi_T^{1*},\pi_T^{2*})\big[s_T\big] &= \sup_{q_T^i\in\mathbb{R}} \mathbb{E}\Big[V_{T+1}^i(\pi_{T+1}^{1*},\pi_{T+1}^{2*})\big[s_{T+1}\big]\Big|s_T\Big] \\
&= \sup_{q_T^i\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-R^iw_{T+1}^i\Big\}\Big|s_T\Big] \\
&= \sup_{q_T^i\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-R^i\Big[w_T^i - \{p_T+\lambda_T \times (q_T^i+q_T^j)\}q_T^i\Big]\Big\}\Big|s_T\Big] \qquad \text{(A1)}\\
&= -\exp\Big\{-R^i\Big[w_T^i - \{p_T+\lambda_T \times (\overline{Q}_T^i+\overline{Q}_T^j)\}\overline{Q}_T^i\Big]\Big\} \\
&= -\exp\Big\{-R^i\Big[w_T^i - p_T\overline{Q}_T^i - \lambda_T(\overline{Q}_T^i)^2 - \lambda_t\overline{Q}_T^i\overline{Q}_T^j\Big]\Big\}, \quad i,j=1,2, \quad i\neq j.
\end{aligned} \tag{A22}$$

For $t = T-1$, the value functions $V_{T-1}^i(\pi_{T-1}^{1*},\pi_{T-1}^{2*})[s_{T-1}]$ for $i = 1, 2$ become the following functional form:

$$\begin{aligned}
&V_{T-1}^i(\pi_{T-1}^{1*},\pi_{T-1}^{2*})\big[s_{T-1}\big] \\
&= \sup_{q_{T-1}^i\in\mathbb{R}} \mathbb{E}\Big[V_T^i(\pi_T^{1*},\pi_T^{2*})\big[s_T\big]\Big|s_{T-1}\Big] \\
&= \sup_{q_{T-1}^i\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-R^i\Big[w_T^i - p_T\overline{Q}_T^i - \lambda_T(\overline{Q}_T^i)^2 - \lambda_T\overline{Q}_T^i\overline{Q}_T^j\Big]\Big\}\Big|s_{T-1}\Big] \\
&= \sup_{q_{T-1}^i\in\mathbb{R}} -\exp\Big\{-R^i\{(1-\alpha^{T-1})\lambda_{T-1}+\lambda_T + \frac{1}{2}R^i(\sigma_{T-1}^\varepsilon)^2\}(q_{T-1}^i)^2 \\
&\quad + \{(-\lambda_{T-1}\alpha^{T-1}+2\lambda_T+R^i(\sigma_{T-1}^\varepsilon)^2)\overline{Q}_{T-1}^i + \lambda_T\overline{Q}_{T-1}^j - (1-\mathrm{e}^{-\rho})r_{T-1} \\
&\quad - \{(1-\alpha^{T-1})\lambda_{T-1}+\lambda_T\}q_{T-1}^j + \mu_{T-1}^\varepsilon\}q_{T-1}^i \\
&\quad + w_{T-1} - p_{T-1}\overline{Q}_{T-1}^i - (\lambda_T + \frac{1}{2}R^i(\sigma_{T-1}^\varepsilon)^2)(\overline{Q}_{T-1}^i)^2 \\
&\quad - \mu_{T-1}^\varepsilon\overline{Q}_{T-1}^i + (1-\mathrm{e}^{-\rho})r_{T-1}\overline{Q}_{T-1}^i \\
&\quad - \lambda_T\overline{Q}_{T-1}^i\overline{Q}_{T-1}^j + (\lambda_T - \lambda_{T-1}\alpha^{T-1})\overline{Q}_{T-1}^iq_{T-1}^j\Big\},
\end{aligned} \tag{A23}$$

where $\alpha^{T-1} := \alpha_{T-1}\mathrm{e}^{-\rho} + (1-\alpha_{T-1})$. Thus, we can obtain the execution volume of the large trader $i, j = 1, 2$ ($i \neq j$) for $t = T-1$ at the supremum of equation (A23) by completing the square of the quadratic equation in the negative exponential function above as follows:

$$q_{T-1}^{i*} = \frac{1}{2A_{T-1}^i}(B_{T-1}^i\overline{Q}_{T-1}^i + C_{T-1}^i\overline{Q}_{T-1}^j + D_{T-1}^ir_{T-1} + F_{T-1}^iq_{T-1}^j + S_{T-1}^i), \tag{A24}$$

where

$$\begin{aligned}
A_{T-1}^i &:= (1-\alpha^{T-1})\lambda_{T-1} + \lambda_T + \frac{1}{2}R^i(\sigma_{T-1}^\varepsilon)^2, \\
B_{T-1}^i &:= -\lambda_{T-1}\alpha^{T-1} + 2\lambda_T + R^i(\sigma_{T-1}^\varepsilon)^2, \\
C_{T-1}^i &:= \lambda_T, \quad D_{T-1}^i := -(1-\mathrm{e}^{-\rho}), \\
F_{T-1}^i &:= -\{(1-\mathrm{e}^{-\rho})\lambda_{T-1}+\lambda_T\}, \quad S_{T-1}^i := \mu_{T-1}^\varepsilon.
\end{aligned} \tag{A25}$$

Solving the simultaneous equation of equation (A24) with respect to $q_{T-1}^1$ and $q_{T-1}^2$ yields the execution volumes at a Markov perfect equilibrium:

$$\begin{aligned}
f_{T-1}^{i*}(s_{T-1}) &=: B_{T-1}^{i*}\overline{Q}_{T-1}^i + C_{T-1}^{i*}\overline{Q}_{T-1}^j + D_{T-1}^{i*}r_{T-1} + S_{T-1}^{i*} \\
&= a_{T-1}^i + b_{T-1}^i\overline{Q}_{T-1}^i + c_{T-1}^i\overline{Q}_{T-1}^j + d_{T-1}^ir_{T-1}, \\
&\quad i,j = 1,2, \quad i\neq j,
\end{aligned} \tag{A26}$$

where for $i, j = 1, 2, i \neq j$,,

$$\delta_{T-1}^i := 2A_{T-1}^i - \frac{F_{T-1}^iF_{T-1}^j}{2A_{T-1}^j},$$

$$B_{T-1}^{i*} := b_{T-1}^i = \frac{1}{\delta_{T-1}^i}\left(B_{T-1}^i + \frac{F_{T-1}^jC_{T-1}^i}{2A_{T-1}^i}\right),$$

$$C^{i*}_{T-1} := c^i_{T-1} = \frac{1}{\delta^i_{T-1}}\left(C^i_{T-1} + \frac{F^j_{T-1}B^i_{T-1}}{2A^i_{T-1}}\right),$$

$$D^{i*}_{T-1} := d^i_{T-1} = \frac{1}{\delta^i_{T-1}}\left(D^i_{T-1} + \frac{F^j_{T-1}D^i_{T-1}}{2A^i_{T-1}}\right), \quad S^{i*}_{T-1} := a^i_{T-1} = \frac{1}{\delta^i_{T-1}}\left(S^i_{T-1} + \frac{F^j_{T-1}S^i_{T-1}}{2A^i_{T-1}}\right).$$

Then, we obtain the following expected utility payoff for large trader $i = 1, 2, i \neq j$ at the Markov perfect equilibrium:

$$\begin{aligned} &V^i_{T-1}(\pi^{1*}_{T-1}, \pi^{2*}_{T-1})[s_{T-1}] \\ &= -\exp\Big\{ -R^i\Big[w^i_{T-1} - p_{T-1}\overline{Q}^i_{T-1} - G^i_{T-1}(\overline{Q}^i_{T-1})^2 - H^i_{T-1}\overline{Q}^i_{T-1} \\ &\quad + I^i_{T-1}r_{T-1}\overline{Q}^i_{T-1} + J^i_{T-1}r^2_{T-1} + L^i_{T-1}r_{T-1} + M^i_{T-1}(\overline{Q}^j_{T-1})^2 \\ &\quad + N^i_{T-1}\overline{Q}^j_{T-1} + X^i_{T-1}r_{T-1}\overline{Q}^j_{T-1} + Y^i_{T-1}\overline{Q}^i_{T-1}\overline{Q}^j_{T-1} + Z^i_{T-1}\Big]\Big\}, \end{aligned} \tag{A27}$$

where

$$\begin{aligned} &B^{i**}_{T-1} := B^i_{T-1} + F^i_{T-1}C^{j*}_{T-1}; \quad C^{i**}_{T-1} := C^i_{T-1} + F^i_{T-1}B^{j*}_{T-1}; \\ &D^{i**}_{T-1} := D^i_{T-1} + F^i_{T-1}D^{j*}_{T-1}; \quad S^{i**}_{T-1} := S^i_{T-1} + F^i_{T-1}S^{j*}_{T-1}; \\ &i, j = 1, 2, \quad i \neq j, \end{aligned} \tag{A28}$$

and for trader $i, j = 1, 2, i \neq j$,

$$\begin{aligned} &G^i_{T-1} := \lambda_T + \frac{1}{2}R^i(\sigma^\varepsilon_{T-1})^2 - (\lambda_T - \lambda_{T-1}\alpha^{T-1})C^{j*}_{T-1} - \frac{(B^{i**}_{T-1})^2}{4A^i_{T-1}}; \\ &H^i_{T-1} := \mu^\varepsilon_{T-1} - (\lambda_T - \lambda_{T-1}\alpha^{T-1})S^{j*}_{T-1} - \frac{B^{i**}_{T-1}S^{i**}_{T-1}}{2A^i_{T-1}}; \\ &I^i_{T-1} := (1 - e^{-\rho}) + (\lambda_T - \lambda_{T-1}\alpha^{T-1})D^{j*}_{T-1} + \frac{B^{i**}_{T-1}D^{i**}_{T-1}}{2A^i_{T-1}}; \\ &J^i_{T-1} := \frac{(D^{i**}_{T-1})^2}{4A^i_{T-1}}; \quad L^i_{T-1} := \frac{D^{i**}_{T-1}S^{i**}_{T-1}}{2A^i_{T-1}}, \quad M^i_{T-1} := \frac{(C^{i**}_{T-1})^2}{4A_{T-1}}; \\ &N^i_{T-1} := \frac{C^{i**}_{T-1}S^{i**}_{T-1}}{2A^i_{T-1}}; \quad X^i_{T-1} := \frac{C^{i**}_{T-1}D^{i**}_{T-1}}{2A^i_{T-1}}; \\ &Y^i_{T-1} := (\lambda_T - \lambda_{T-1}\alpha^{T-1})B^{j*}_{T-1} + \frac{B^{i**}_{T-1}C^{i**}_{T-1}}{2A^i_{T-1}} - \lambda_T; \\ &Z^i_{T-1} := \frac{(S^{i**}_{T-1})^2}{4A^i_{T-1}}. \end{aligned} \tag{A29}$$

Hereafter, we assume that the expected utility payoff functions of the large trader $i = 1, 2$ at time $t + 1$ take the following functional form:

$$\begin{aligned} &V^i_{t+1}(\pi^{1*}_{t+1}, \pi^{2*}_{t+1})[s_{t+1}] \\ &= -\exp\Big\{ -R^i\Big[w^i_{t+1} - p_{t+1}\overline{Q}^i_{t+1} - G^i_{t+1}(\overline{Q}^i_{t+1})^2 - H^i_{t+1}\overline{Q}^i_{t+1} \\ &\quad + I^i_{t+1}r_{t+1}\overline{Q}^i_{t+1} + J^i_{t+1}r^2_{t+1} + L^i_{t+1}r_{t+1} + M^i_{t+1}(\overline{Q}^j_{t+1})^2 \\ &\quad + N^i_{t+1}\overline{Q}^j_{t+1} + X^i_{t+1}r_{t+1}\overline{Q}^j_{t+1} + Y^i_{t+1}\overline{Q}^i_{t+1}\overline{Q}^j_{t+1} + Z^i_{t+1}\Big]\Big\}. \end{aligned} \tag{A30}$$

Then, $V^i_t(\pi^{1*}_t, \pi^{2*}_t)[s_t]$ for $i = 1, 2$ become

$$\begin{aligned} &V^i_t(\pi^{1*}_t, \pi^{2*}_t)[s_t] \\ &= \sup_{q^i_t \in \mathbb{R}} \mathbb{E}\Big[V^i_{t+1}(\pi^{1*}_{t+1}, \pi^{2*}_{t+1})[s_{t+1}]\Big| s_t\Big] \\ &= \sup_{q^i_t \in \mathbb{R}} -\exp\Big\{ -R^i\Big[ -A^i_t(q^i_t)^2 + \{B^i_t\overline{Q}^i_t + C^i_t\overline{Q}^j_t \\ &\quad + D^i_t r_t + F^i_t q^j_t + S^i_t\}q^i_t \\ &\quad + w^i_t - p_t\overline{Q}^i_t - (G^i_{T+1} + \frac{1}{2}R^i(\sigma^\varepsilon_t))(\overline{Q}^i_t)^2 - (H^i_{t+1} + \mu^\varepsilon_t)\overline{Q}^i_t \\ &\quad + \{(1 - e^{-\rho}) + I^i_{t+1}e^{-\rho}\}r_t\overline{Q}^i_t \\ &\quad + J^i_{t+1}e^{-2\rho}r^2_t + L^i_{t+1}e^{-\rho}r_t + M^i_{t+1}(\overline{Q}^j_t)^2 + N^i_{t+1}\overline{Q}^j_t \\ &\quad + X^i_{t+1}e^{-\rho}r_t\overline{Q}^j_t + Y_{t+1}\overline{Q}^i_t\overline{Q}^j_t + Z^i_{t+1} \\ &\quad + \big\{(-\lambda_t\alpha^t + I^i_{t+1}e^{-\rho}\lambda_t\alpha_t - Y^i_{t+1})\overline{Q}^i_t + (-2M^i_{t+1} \\ &\quad + Y^i_{t+1}e^{-\rho}\lambda_t\alpha_t)\overline{Q}^j_t \\ &\quad + (2J^i_{t+1}e^{-2\rho}\lambda_t\alpha_t - X^i_{t+1}e^{-\rho})r_t + (L^i_{t+1}e^{-\rho}\lambda_t\alpha_t - N^i_{t+1})\big\}q^j_t \\ &\quad + (J^i_{t+1}e^{-2\rho}\lambda^2_t\alpha^2_t + M^i_{t+1} - X^i_{t+1}e^{-\rho}\lambda_t\alpha_t)(q^j_t)^2\Big]\Big\}, \end{aligned} \tag{A31}$$

where for $i = 1, 2$

$$\begin{aligned} &A^i_t := (1 - \alpha^t)\lambda_t + G^i_{t+1} + L^i_{t+1}e^{-\rho}\lambda_t\alpha_t - J^i_{t+1}e^{-2\rho}\lambda^2_t\alpha^2_t + \frac{1}{2}R^i(\sigma^\varepsilon_t)^2; \\ &B^i_t := -\lambda_t\alpha^t + 2G^i_{t+1} + I^i_{t+1}e^{-\rho}\lambda_t\alpha_t + R^i(\sigma^\varepsilon_t)^2; \\ &C^i_t := X^i_{t+1}e^{-\rho}\lambda_t\alpha_t - Y^i_{t+1}; \\ &D^i_t := -(1 - e^{-\rho}) - I^i_{t+1}e^{-\rho} + 2J^i_{t+1}e^{-2\rho}\lambda_t\alpha_t; \\ &F^i_t := -(1 - \alpha^t)\lambda_t - I^i_{t+1}e^{-\rho}\lambda_t\alpha_t + 2J^i_{t+1}e^{-2\rho}\lambda^2_t\alpha^2_t - X^i_{t+1}e^{-\rho}\lambda_t\alpha_t + Y^i_{t+1}; \\ &S^i_t := H^i_{t+1} + L^i_{t+1}e^{-\rho}\lambda_t\alpha_t - \mu^\varepsilon_t. \end{aligned} \tag{A32}$$

Therefore, the execution volume of the large trader $i = 1, 2$ at time $t \in \{T - 2, \ldots, 1\}$ attaining the supremum of $V^i_t(\pi^1_t, \pi^2_t)[s_t]$ becomes like equation (A24) by the same derivation methods beforehand:

$$q^{i*}_t(s_t) = \frac{1}{2A^i_t}(B^i_t\overline{Q}^i_t + C^i_t\overline{Q}^j_t + D^i_t r_t + F^i_t q^j_t + S^i_t), \tag{A2}$$

$$i, j = 1, 2, \quad i \neq j. \tag{A33}$$

From equation (A33), we obtain the following execution volume at the Markov perfect equilibrium $q^{i*}_t$ for $i, j = 1, 2$ $(i \neq j)$:

$$f^{i*}_t(s_t) = B^{i*}_t\overline{Q}^i_t + C^{i*}_t\overline{Q}^j_t + D^{i*}_t r_t + S^{i*}_t = a^i_t + b^i_t\overline{Q}^i_t + c^i_t\overline{Q}^j_t + d^i_t r_t, \tag{A34}$$

where for $i, j = 1, 2, i \neq j$,

$$\begin{aligned} &\delta^i_t := 2A^i_t - \frac{F^i_tF^j_t}{2A^j_t}; \quad B^{i*}_t := b^i_t = \frac{1}{\delta^i_t}(B^i_t + \frac{F^j_tC^i_t}{2A^i_t}); \\ &C^{i*}_t := c^i_t = \frac{1}{\delta^i_t}(C^i_t + \frac{F^j_tB^i_t}{2A^i_t}); \\ &D^{i*}_t := d^i_t = \frac{1}{\delta^i_t}(D^i_t + \frac{F^j_tD^i_t}{2A^i_t}); \quad S^{i*}_t := a^i_t = \frac{1}{\delta^i_t}(S^i_t + \frac{F^j_tS^i_t}{2A^i_t}), \end{aligned} \tag{A35}$$

and the expected equilibrium payoff for large trader $i = 1, 2$ at the Markov Perfect equilibrium $(\pi^{1*}, \pi^{2*}) \in \Pi^1_M \times \Pi^2_M$ becomes

$$\begin{aligned} &V^i_t(\pi^{1*}_t, \pi^{2*}_t)[s_t] \\ &= -\exp\Big\{ -R^i\Big[w^i_t - p_t\overline{Q}^i_t - (G^i_{T+1} + \frac{1}{2}R_i(\sigma^\varepsilon_t)^2)(\overline{Q}^i_t)^2 \end{aligned}$$

$$
\begin{aligned}
&- (H^i_{t+1} + \mu^\varepsilon_t)\overline{Q}^i_t + \{(1-\mathrm{e}^{-\rho}) + I^i_{t+1}\mathrm{e}^{-\rho}\}r_t\overline{Q}^i_t \\
&+ J^i_{t+1}\mathrm{e}^{-2\rho}r_t^2 + L^i_{t+1}\mathrm{e}^{-\rho}r_t + M_{t+1}(\overline{Q}^j_t)^2 + N_{t+1}\overline{Q}^j_t \\
&+ X^i_{t+1}\mathrm{e}^{-\rho}r_t\overline{Q}^j_t + Y_{t+1}\overline{Q}^i_t\overline{Q}^j_t + Z^i_{t+1} \\
&+ \{(-\lambda_t\alpha^t + I^i_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t - Y^i_{t+1})\overline{Q}^i_t + (-2M^i_{t+1} \\
&+ Y_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t)\overline{Q}^j_t + (2J^i_{t+1}\mathrm{e}^{-2\rho}\lambda_t\alpha_t - X^i_{t+1}\mathrm{e}^{-\rho})r_t \\
&+ (L^i_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t - N^i_{t+1})\}q^{j*}_t + (J^i_{t+1}\mathrm{e}^{-2\rho}\lambda_t^2\alpha_t^2 + M^i_{t+1} \\
&- X^i_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t)(q^{j*}_t)^2 \\
&+ \frac{1}{4A^i_t}(B^{i**}_t\overline{Q}^i_t + C^{i**}_t\overline{Q}^j_t + D^{i**}_t r_t + S^{i**}_t)^2\Big]\Big\} \\
= &-\exp\Big\{-R^i\Big[w^i_t - p_t\overline{Q}^i_t - G_t(\overline{Q}^i_t)^2 - H_t\overline{Q}^i_t \\
&+ I_t r_t\overline{Q}^i_t + J_t r_t^2 + L_t r_t \\
&+ M^i_t(\overline{Q}^j_t)^2 + N^i_t\overline{Q}^j_t + X^i_t r_t\overline{Q}^j_t + Y^i_t\overline{Q}^i_t\overline{Q}^j_t + Z^i_t\Big]\Big\}, \qquad \text{(A36)}
\end{aligned}
$$

where

$$\psi^i_t := -\lambda_t\alpha^t + I^i_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t - Y^i_{t+1}; \qquad \text{(A37)}$$

$$\phi^i_t := -2M^i_{t+1} + Y^i_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t; \qquad \text{(A38)}$$

$$\theta^i_t := 2J^i_{t+1}\mathrm{e}^{-2\rho}\lambda_t\alpha_t - X^i_{t+1}\mathrm{e}^{-\rho};$$

$$\iota^i_t := L^i_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t - N^i_{t+1}; \qquad \text{(A39)}$$

$$\nu^i_t := J^i_{t+1}\mathrm{e}^{-2\rho}\lambda_t^2\alpha_t^2 + M^i_{t+1} - X^i_{t+1}\mathrm{e}^{-\rho}\lambda_t\alpha_t, \qquad \text{(A40)}$$

and

$$
\begin{aligned}
&B^{i**}_t := B^i_t + F^i_t C^{j*}_t; \quad C^{i**}_t := C^i_t + F^i_t B^{j*}_t; \\
&D^{i**}_t := D^i_t + F^i_t D^{j*}_t; \quad S^{i**}_t := S^i_t + F^i_t S^{j*}_t, \quad i,j = 1,2, \quad i \neq j, \qquad \text{(A41)}
\end{aligned}
$$

and

$$
\begin{aligned}
G^i_t &:= G^i_{t+1} + \frac{1}{2}R^i(\sigma^\varepsilon_t)^2 - \psi^i_t C^{j*}_t - \nu^i_t(B^{j*}_t)^2 - \frac{(B^{i**}_t)^2}{4A_t}; \\
H^i_t &:= H^i_{t+1} + \mu^\varepsilon_t - \psi^i_t S^{j*}_t - \iota^i_t C^{j*}_t - 2\nu^i_t C^{j*}_t S^{j*}_t - \frac{B^{i**}_t S^{i**}_t}{2A^i_t}; \\
I^i_t &:= I^i_{t+1}\mathrm{e}^{-\rho} + (1-\mathrm{e}^{-\rho}) + \psi^i_t D^{j*}_t + \theta^i_t C^{j*}_t + 2\nu^i_t C^{j*}_t D^{j*}_t \\
&\quad + \frac{B^{i**}_t D^{i**}_t}{2A^i_t}; \\
J^i_t &:= J^i_{t+1}\mathrm{e}^{-2\rho} + \theta^i_t D^{j*}_t + \nu^i_t(D^{j*}_t)^2 + \frac{(D^{i**}_t)^2}{4A^i_t}; \\
L^i_t &:= L^i_{t+1}\mathrm{e}^{-\rho} + \theta^i_t S^{j*}_t + \iota^i_t D^{j*}_t + 2\nu^i_t D^{j*}_t S^{j*}_t + \frac{D^{i**}_t S^{i**}_t}{2A^i_t}; \\
M^i_t &:= M^i_{t+1} + \phi^i_t B^{j*}_t + \nu^i_t(C^{j*}_t)^2 + \frac{(C^{i**}_t)^2}{4A^i_t}; \\
N^i_t &:= N^i_{t+1} + \phi^i_t S^{j*}_t + \iota^i_t B^{j*}_t + 2\nu^i_t B^{j*}_t S^{j*}_t + \frac{C^{i**}_t S^{i**}_t}{2A^i_t}; \\
X^i_t &:= X^i_{t+1}\mathrm{e}^{-\rho} + \phi^i_t D^{j*}_t + \theta^i_t B^{j*}_t + 2\nu^i_t B^{j*}_t D^{j*}_t + \frac{C^{i**}_t D^{i**}_t}{2A^i_t}; \\
Y^i_t &:= Y^i_{t+1} + \psi^i_t B^{j*}_t + \phi^i_t C^{j*}_t + 2\nu^i_t B^{j*}_t C^{j*}_t + \frac{B^{i**}_t C^{i**}_t}{2A^i_t}; \\
Z^i_t &:= Z^i_{t+1} + \iota^i_t S^{j*}_t + \nu^i_t(S^{j*}_t)^2 + \frac{(S^{i**}_t)^2}{4A^i_t}.
\end{aligned}
\qquad \text{(A42)}
$$