Finance Research Letters 30 (2019) 44–50

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

Optimization of multi-period portfolio model after fitting best distribution

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ARTICLE INFO

Keywords:
Portfolio Optimization
Probability risk measure
Multi-period
Kernel distribution estimator

MSC:
91G10
91G70

ABSTRACT

In this paper, we use a multi-periodic portfolio selection algorithm to maximum the investor wealth using probabilistic risk measure. We use ASX100 stock data from 2015 to 2017 with 36 periods, 100 stocks and 725 days. Then we examine and use T-student, stable and kernel distributions to improve and optimize the multi-period portfolio optimization model. Kolmogorov-Smirnov test indicates that these distributions fit the experimental data better in comparison with normal distribution. Furthermore, kernel density estimator is the best density function to fit returns.

1. Introduction

The portfolio theory and the selection of the optimal stock of portfolios after the first attempts by Markowitz (1952) have always been one of the attractive areas. The stock portfolio selection or portfolio optimization involves designing an appropriate optimization model and choosing suitable criteria for stock selection. The first criteria used by Markowitz in the traditional portfolio model, was expected returns and portfolio return variance. However, the reviews of variance turned into semi-variance by Markowitz (1959). The term risk has undergone many changes in recent decades, and it has been introduced for different criteria in different situations. Righi and Borenstein (2018) compare risk measures regarding performance of optimal portfolio strategies. Chen (2005) presented a multi-period optimization model using value-at-risk as a risk measure. Wei and Ye (2007) introduced a multi-period variance model under the control of bankruptcy risk. Liu et al. (2013) developed a multi-period portfolio optimization model in terms of variance as a risk measure in a fuzzy environment. Liu and Zhang (2015) use fuzzy semi-variance as a risk measure and for each goal, consider the degree of investor satisfaction and consider the multi-objective model which be transformed into a single-objective model using solving a genetic algorithm. Sun et al. (2015) presented a flexible portfolio selection method as a bi-criteria optimization problem. Their model maximizes the expected portfolio return and minimizes the maximum individual risk of the assets in the portfolio. Moreover, Sun et al. (2016) develop the previous model using multiperiodic variables and probabilistic risk measure.

The probability density function using in Sun et al. (2015, 2016) was Gaussian. In this paper, we examine and use T-student, stable and kernel distribution functions to improve and optimize the multi-period portfolio optimization model. In financial matters, especially when an investor is supposed to invest a large sum, the accuracy of optimizing the portfolio and making the answers closer to reality are important for any financial analyst. Empirical evidences have shown that in a portfolio selection, in particular when a large sum of investments is involved, assuming the distribution of the output to be normal in normal terms could result a significant error in expected return (Sun et al., 2016). By observing the kernel distribution density diagram and because of the dispersion of data

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https://doi.org/10.1016/j.frl.2019.03.027
Received 9 January 2019; Received in revised form 17 February 2019; Accepted 20 March 2019
Available online 26 March 2019
1544-6123/

in the tail, we find out that the use of the kernel estimator provides a much more reliable and precise answer. We also realized that using the kernel estimator could provide a more precise answer. Finally, we present a model that maximizes the expected portfolio return and minimizes the maximum individual risk of the assets in the portfolio. In our procedure we use probability risk measure related to the one introduced by Sun et al. (2015)

2. Preliminaries

Let S_j, $j = 1, \ldots, N$ be risky assets and the investor decides to invest in with initial wealth M_0. Let T be the period of portfolio. Also x_{tj} is the percentage of wealth at the end of period $(t-1)$th invested in asset S_j at the beginning of period t. With assumption of total fund of each period as the risk assets of the next period, we have

$$\sum_{j=1}^{N} x_{tj}=1, \quad t=1,\ldots, T, \quad x_t = [x_{t1},\ldots, x_{tN}]^T \in \mathbb{R}^N. \tag{1}$$

$$x_{tj} \geq 0 \quad t=1,\ldots, T, \; j=1,\ldots, N. \tag{2}$$

The rate of return of asset S_j with period t is defined as R_{tj} with location parameter r_{tj} and scale parameter σ_{tj}. If R_{tj} for some t and j has a distribution with finite mean and variance, it is obvious that r_{tj} is its mean and σ_{tj} is its standard deviation and we estimate them by method of moment estimation.

$$R_t = [R_{t1},\ldots, R_{tN}]^T, \; E(R_t) = r_t = [r_{t1},\ldots, r_{tN}]^T. \tag{3}$$

Otherwise, for example in stable distribution when $\alpha < 1$ or $\alpha \neq 2$, in order to estimate r_{tj}'s and σ_{tj}'s respectively, we use well-known estimators used in MATLAB software box.

We assume that $R_{tj}'s$ are independent.

Total wealth of the investor at the end of period t equals to

$$V_t = V_{t-1}(1+R_t^T x_t), \quad t=1,\ldots, T \tag{4}$$

where $V_0 = M_0$.

For multi-period portfolio optimization, the probabilistic risk measure we will apply is

$$\min_{1\leq t\leq T} \min_{1\leq j\leq N} F_{tj}(x_{tj}), \tag{5}$$

where $F_{tj}(x_{tj}) = Pr\{|R_{tj}x_{tj} - r_{tj}x_{tj}| \leq \theta\hat{\sigma}_{tj}\}$.

In order to be more precise, instead of using the average risk of the portfolio with the same coefficient for all assets (ε), the risk of each participant's assets ($\hat{\sigma}_{tj}$'s) is used. Similarly, θ is a constant adjusting the risk level. Sun et al. (2016)

Eventually, the final target equation has been

$$\max_{x}\left(\min_{1\leq t\leq T} \min_{1\leq j\leq N} F_{tj}(x_{tj}) \;, \; E(V_T)\right). \tag{6}$$

Assume that the investor desires to maximize the terminal wealth and at the same time minimizes the risk. It is easy to show that the portfolio selection problem is formulated as follows

$$\max_{x}\left(\min_{1\leq t\leq T} \min_{1\leq j\leq N} F_{tj}(x_{tj}) \;, \; \sum_{t=1}^{T}\sum_{j=1}^{N} x_{tj}r_{tj}\right), \tag{7}$$

$$\text{s. t.} \quad \sum_{j=1}^{N} x_{tj}=1, \quad t=1,\ldots, T, \quad x_t = [x_{t1},\ldots, x_{tN}]^T \in \mathbb{R}^N \;,$$

$$x_{tj} \geq 0 \quad t=1,\ldots, T \qquad j=1,\ldots, N.$$

With the same argument as Sun et al. (2016), the optimization problem is simplified as follows

$$\max_{x} \sum_{t=1}^{T}\sum_{j=1}^{N} x_{tj}r_{tj} \tag{8}$$

$$\text{s. t.} \quad \sum_{j=1}^{N} x_{tj}=1, \quad t=1,\ldots, T, \quad x_t = [x_{t1},\ldots, x_{tN}]^T \in \mathbb{R}^N,$$

$$0 \leq x_{tj} \leq U_{tj} \qquad t=1,\ldots, T, \qquad j=1,\ldots, N,$$

where

$$U_{tj}=\min\{1, F_{tj}^{-1}(y)\}, \tag{9}$$

y is an arbitrary but fixed real number between 0 and 1 and F_{tj}^{-1} is the inverse function of F_{tj}.

The optimal solution for every period k, $k=1,\ldots,\ T$ according to Theorem 3.2 of Sun et al. (2016), is given below

Theorem 2.1. *For any k, $k=1,\ldots,\ T$, let the assets are arranged in such a way that $r_{k1} \geq r_{k2} \geq \cdots \geq r_{kN}$. Then there exists an integer $n \leq N$ such that*

$$\sum_{j=1}^{n-1} U_{kj} < 1 \ \text{and} \ \sum_{j=1}^{n} U_{kj} \geq 1.$$

Let

$$x^*_{kj} = \begin{cases} U_{kj}, & j=1,\ldots,\ n-1, \\ 1-\sum_{i=1}^{n-1} U_{ki}, & j=n, \\ 0, & j > n, \end{cases} \tag{10}$$

for which U_{kj} is the same as defined in the (9). Then the optimal solution to Problem (8) is $[x^*_1,\ \ldots,x^*_N]$ with $x^*_k = [x^*_{k1},\ldots,x^*_{kN}]^T$.

What we do here is solving the optimization problem (8) in which the distribution of returns is stable, T-student or kernel.

One method of estimation used in this work is Kernel density estimation (KDE), which is a non-parametric way to estimate the probability density function of a random variable. In this method, a continuous curve (kernel) is drawn in all single points of data, and then all of these curves are added together to obtain a smooth density estimate. The Kernel density estimator is defined by the following equation:

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right), \tag{11}$$

where $x_1,\ \ldots,x_n$ are the data observing from a distribution with unknown density f, K is the kernel smoothing function and h is called bandwidth Rosenblatt (1956).

In the next section, we will give a general description of how to solve the optimization problem (8).

3. Optimization problem solving

It is assumed in Sun et al. (2016) that the distribution of returns is normal, but here using the Kolmogorov-Smirnov test, we show that in most cases it is more reasonable to use another distribution.

Compare these distribution's results in terms of their closeness to reality, solutions and speed of running program is bring in the next section. Numerical results from the utilization of the desired distributions in optimization problem are also be presented.

The optimization problem (8) has been solved by programming in MATLAB software. For this purpose, various estimators and Monte Carlo method have been used.

In the case that the distribution of asset returns is considered as T-student (μ, σ), we estimate μ, the mean of T-student distribution, to calculate r_{tj}'s, $j = 1,\ \ldots,n,\quad t = 1,\ \ldots,T$, and to compute $\hat{\sigma}_{tj}$'s we use the following

$$\hat{\sigma}_{tj} = \begin{cases} \sqrt{\frac{\nu}{\nu - 2}}\sigma, & \nu > 2 \\ \sigma & \nu \leq 2, \end{cases} \tag{12}$$

where σ is the standard deviation of T-student distribution fitted to jth asset's data in period t and ν is degree of freedom of that.

Another of distribution by which we fit the return's movement, is stable distribution. In this case, we use numerical methods to find F_{tj}^{-1} and estimate r_{tj} and σ_{tj}. If Stable(α, β, σ, μ) is fitted to jth asset returns at period t, $j = 1,\ \ldots,n,\quad t = 1,\ \ldots,T$, we use $\hat{r}_{tj} = \mu$ and $\hat{\sigma}_{tj} = \sigma$.

The next problem is choosing suitable values for y and θ. In Sun et al. (2016), it is assumed that the investor is risk averse and therefore for y the values are from 0.95 to 0.8. We allow the investor to take a higher risk, which would then have higher expected returns, so y could have a value less than 0.8.

The difficulty part in solving problem (8) is the calculation of $F_{tj}^{-1}(y)$ in equation (9), especially when the distribution is not Gaussian. To overcome this difficulty, we proceed as

$$\begin{aligned} F_{tj}(x_{tj}) = \ & Pr\{|R_{tj} - r_{tj}| x_{tj} \leq \theta\hat{\sigma}_{tj}\} \\ & Pr\left\{R_{tj} \leq \frac{\theta\hat{\sigma}_{tj}}{x_{tj}} + r_{tj}\right\} - Pr\left\{R_{tj} \leq -\frac{\theta\hat{\sigma}_{tj}}{x_{tj}} + r_{tj}\right\} \\ & \widetilde{F}_{tj}\left(\frac{\theta\hat{\sigma}_{tj}}{x_{tj}}\right) - \widetilde{F}_{tj}\left(-\frac{\theta\hat{\sigma}_{tj}}{x_{tj}}\right), \end{aligned} \tag{13}$$

in which $\tilde{F}_{tj}$ is the distribution function that we choose to fit the return of the jth asset in period t.

For every asset return in every period, after finding the distribution function with the appropriate parameters, $\tilde{F}_{tj}$, we construct the F_{tj} function and then obtain its inverse in y with programming in MATLAB software.

Table 1
Expected portfolio wealth for different θ and y in **Normal** Distribution.

	y=0.9	y=0.8	y=0.7	y=0.6	y=0.5
θ=0.01	0.716	0.826	0.961	1.034	1.064
θ=0.1	1.241	1.277	1.310	1.345	1.385
θ=0.5	1.532	1.591	1.651	1.704	1.769
θ=1	1.710	1.787	1.874	**1.891**	**1.891**
θ=2	**1.891**	**1.891**	**1.891**	**1.891**	**1.891**

Table 2
Expected portfolio wealth for different θ and y in ***T*-student** Distribution.

	y=0.9	y=0.8	y=0.7	y=0.6	y=0.5
θ=0.01	0.839	0.924	1.005	1.035	1.059
θ=0.1	1.180	1.208	1.228	1.246	1.267
θ=0.5	1.341	1.370	1.395	1.414	1.422
θ=1	1.407	1.438	1.452	1.451	**1.446**
θ=2	**1.446**	**1.446**	**1.446**	**1.446**	**1.446**

Table 3
Expected portfolio wealth for different θ and y in **Stable** Distribution.

	y=0.9	y=0.8	y=0.7	y=0.6	y=0.5
θ=0.01	0.757	0.792	0.841	0.896	0.975
θ=0.1	1.139	1.163	1.183	1.203	1.225
θ=0.5	1.262	1.288	1.301	1.316	1.329
θ=1	1.301	1.320	1.322	1.329	**1.334**
θ=2	1.313	1.318	1.317	1.317	1.317

In order to verify the accuracy of the numerical solutions, we assume $F_{tj}^{-1}(y) = \hat{x}_{tj}$ and calculate $\left[\tilde{\boldsymbol{F}}_{tj}\left(\frac{\theta\hat{\sigma}_{tj}}{x_{tj}}\right) - \tilde{\boldsymbol{F}}_{tj}\left(-\frac{\theta\hat{\sigma}_{tj}}{x_{tj}}\right)\right] - y$. We observe that almost always this difference is zero, which indicates the high accuracy of the numerical solutions.

Another issue that has been considered, is determining the number of periods for the multi-period optimization problem. In the next section, we will show that for the fixed time interval (three years of work), the more the number of periods is planned for, the higher final value of the portfolio.

4. Numerical result

In this section, we provide portfolio selection method using multi-periodic variables and probability of risk management. Available data is the daily stock of ASX100 and 723 days from 2015 to 2017. For evaluation of data, we assign initial wealth of $M_0 = 1$ and $N = 100$ stocks are chosen to evaluate for period $T = 36$ months. With changing of risk, we adjust parameter θ and bound of probability carry out using different probability distribution.

Tables 1 through 4 show the final value of the capital portfolio by using normal, T-student, stable and kernel estimator distributions respectively, in which the investor invests a unit of equity over 100 assets over a period of 36 months. In these tables, θ is the risk adjustment parameter and y is the lower bound of the probabilistic constraint. By changing θ or y, the investor will be able to make different portfolios with different risk levels. Whatever the less, the portfolio can be more varied while for a lower value of θ, the hardening of the probabilistic constrains is reduced. As θ increases and y decreases, returns also increase, and as we look at the x-vectors we see that as θ increases, the number of assets in the portfolio is less but better (with more return). On the other hand, when y decreases, the risk value and the expected returns increase.

The other result we get from these tables is that in $\theta = 0.01$ we have the least terminal wealth and sometimes we have no profit. Moreover, by increasing in risk adjustment, the increment in total wealth has shown. By comparing Tables 1–4, we observe that the portfolio terminal wealth due to stable distribution is the least. Overly, kernel distribution have the best results. The question now is which investor to choose which portfolio of capital (by what y and θ). From December 29, 2014 to December 31, 2017, the market index return is 10.52%, which according to the tables when $\theta = 0.01$, in almost all scenarios the return of the portfolio is lower than this so the investor, although not to be risk averse, would prefer to doesn't choose them. If investor is less risk averse, he or she will choose a portfolio with more θ and less y. If the investor will accept reasonable risk, the final value of the portfolio for him / her can be maximum at which the normal distribution or kernel is used to be 1.891, if the distribution is T-student, 1.446 and when the distribution is stable 1.334, in other word, the method of using the probabilistic risk measure to optimize the portfolio is better in

Table 4
Expected portfolio wealth for different θ and y in **Kernel** Distribution.

	y=0.9	y=0.8	y=0.7	y=0.6	y=0.5
θ=0.01	0.738	0.906	1.039	1.081	1.113
θ=0.1	1.293	1.347	1.390	1.429	1.468
θ=0.5	1.646	1.694	1.736	1.772	1.816
θ=1	1.772	1.825	1.882	**1.891**	**1.891**
θ=2	**1.891**	**1.891**	**1.891**	**1.891**	**1.891**

Table 5
Time consumed to run the program by MATLAB software ($N = 100$, $T = 36$, $\theta = 1\, and\, y = 0.8$).

Distribution	Normal	T-student	Stable	Kernel
Time consuming	12.491 s	230.753 s	**945.112** s	27.079 s

Table 6
The percent of acceptance in the Kolmogorov-Smirnov test for the desired distributions and periods.

No. of Period	1	3	6	12	36
Distribution					
Normal	**00.00%**	**00.67%**	**02.33%**	**06.00%**	**12.69%**
T-student	32.00%	35.67%	36.33%	31.33%	20.14%
stable	10.00%	19.33%	19.33%	22.08%	25.56%
Kernel	**42.00%**	**39.33%**	**38.50%**	**38.17%**	**40.19%**
—	16.00%	05.00%	03.5%	02.42%	01.42%

dealing with returns fitted with kernel or normal distribution. Result related to Kernel estimator is close to the normal one because the kernel function (K) which has been used, is normal density.

In addition, the speed of computing by MATLAB software is also presented in Table 5 and 6.

In this review, we used multi-period portfolio selection algorithm to find the expected portfolio and reaching to maximum investor wealth.

The probability distribution histogram of 63th asset in 34th period is shown in Fig. 1, moreover difference between density functions is significant. Furthermore, Fig. 2 shows the cumulative probability function of data from asset 63 in period 34. Results show ignorable difference between distributions of empirical data and the other distribution functions. Almost none of the returns reject the null hypothesis at the 5% significance level ($\alpha = 0.05$), the data comes from the assumed distribution; in other words, almost all returns are coincided with the selected distributions. The most rejection and the least one has been shown as bold contexts.

Furthermore, it is important for us to choose a distribution that will cover more output. By using the Kolmogorov-Smirnov test, we

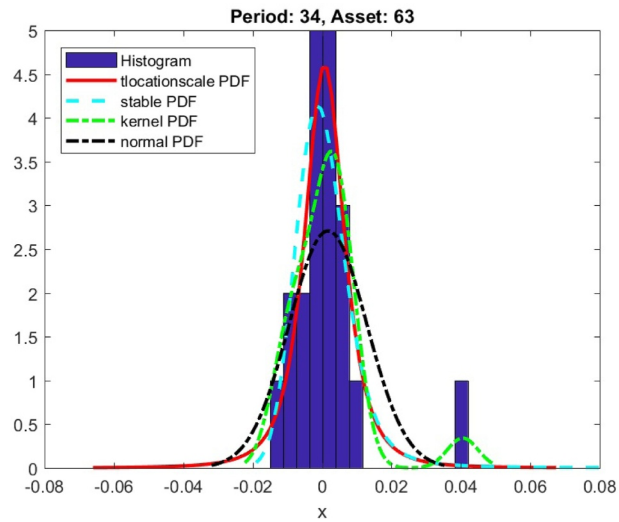

Fig. 1. Histogram of data and different densities.

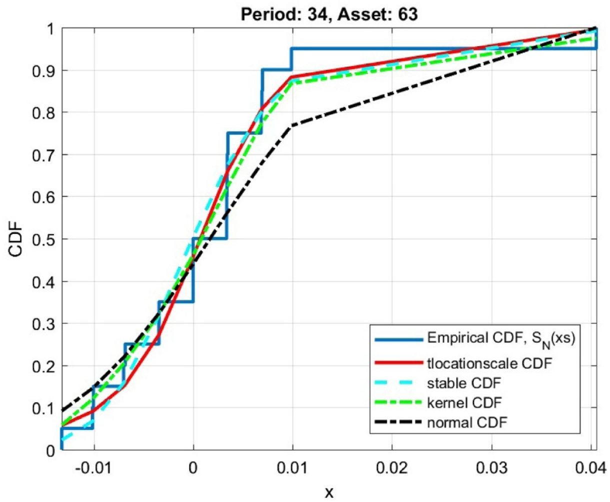

Fig. 2. The plot of cumulative probability distributions.

Table 7
Results of the Kolmogorov-Smirnov test to indicate the frequency of the appropriateness of each distribution for the return asset's returns ($N = 100$, $T = 36$, $\theta = 1\,and\,y = 0.8$).

Distribution	Normal	T-student	Stable	Kernel
No. of acceptance	3490	3527	3529	3547
No. of rejection	**110**	73	71	**53**

find that the kernel distribution is the best and the normal distribution is the most inelastic. Table 7 shows the results of this test, in which the first row shows the number of assets that the distribution is suitable for fitting their returns, and the second row indicates that the distribution is not appropriate for fitting this number of asset returns.

Table 8 shows the variation of expected portfolio returns for desired distributions. We carry out using period ($T = 1, ...,36$) for $\theta = 1$ and $y = 0.8$. The results show that increasing of periods cause the portfolio returned valuably. This increase is almost linear and provides increment in total wealth.

5. Conclusion

In this paper, we change in the main definition of the probabilistic risk measure given by Sun et al. (2015) for greater precision, such that for each asset, the estimate of standard deviation of itself is used to calibrate the risk instead of using the average standard deviation of portfolio for all assets. Also, the probability density function in Sun et al. (2015, 2016) is considered Gaussian probability function which we examine and use T-student, stable and kernel distribution function to develop and optimize the multi-period portfolio optimization model. Results show that T-student distribution fitness provides the most wealth in all periods. However, kernel density estimator results closer fitness to reality. At the end, we show that with the participating assets and the same time periods, the more the number of periods in the optimization problem, the greater the value of the portfolio will be. Future research can eliminate the condition of the independence of returns in different periods, since empirical evidences show that sometimes such a hypothesis is not reasonable and returns of the asset in different periods are dependent.

Table 8
Expected portfolio values for desired distributions and periods ($\theta = 1$ *and* $y = 0.8$).

Distribution	Normal	T-student	Stable	Kernel
Number of Period				
1	1.001	1.001	1.001	1.001
3	1.011	1.010	1.009	1.012
6	1.042	1.023	1.022	1.046
12	1.116	1.088	1.071	1.124
36	1.787	1.438	1.322	1.825

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.frl.2019.03.027.

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