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From equity to default correlation with taxes

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For fixed income investment, the preponderant risk is the clustering of defaults in the portfolio. Accurate prediction of such clustering depends on the knowledge of default correlation. We develop models with exogenous debt and endogenous debt to predict default correlations from equity correlations based on a self-consistent structural framework. We also examine how taxes affect the prediction of default correlations based on the two models. The empirical analysis shows that the corporate taxes tend to decrease default correlations, while personal taxes could increase or decrease default correlations. Our default correlation model with exogenous debt does a better job of predicting default correlations for high quality bonds, while the one with endogenous debt predicts more accurately for lower rated bonds. Our studies not only theoretically improve the modeling of default correlation in the structural setting but also shed new light on various aspects of default correlations and thereby help financial practitioners price credit derivatives more accurately and formulate more effective strategies to manage default risk of credit portfolios.

Keywords: Default correlation; Equity correlation; Taxes; Structural model

JEL Classification: G1, G2

1. Introduction

Default correlation measures the extent to which two firms become insolvent simultaneously within a given time frame. When there are multiple risky bonds in a portfolio, a single default correlation between two firms or two assets can be extended to a default correlation matrix. The financial crisis of 2008 demonstrated how default correlations among underlying assets for asset backed securities led to massive defaults, resulting in billions of losses among financial institutions and investors. A recent study by Molins and Vives (2016) further reveals the complex nature of default correlations in their phase transition-like contribution to systematic risk, which implies that at a certain level, a very small shift in default correlation can trigger an avalanche-like event to the entire system, such as the collapse of credit portfolios or financial crisis. The new findings and the severe consequence of the financial crisis suggest that it is critical to understand how to model and estimate default correlations accurately.

Zhou (2001) pioneered the line of research of modeling default correlation in a structural setting. To make his model implementable, he made such assumptions as constant asset correlation across firms and asset correlation equal to equity correlation. However, the assumptions are theoretically unsound and may result in self-inconsistencies. For example, when a firm is close to default, equity and asset could be very different and thereby equity correlation can substantially deviate from asset correlation. De Servigny and Renault (2004) empirically show that default correlations estimated based on equity correlations that proxy for asset correlations are weakly related to historical default correlations, which suggests that equity correlation is not an appropriate proxy for asset correlation. Liu *et al.* (2015) attempt to tackle the problem with Zhou's model by proposing a hybrid approach to apply the dynamic trade-off model (Leland and Toft 1996, LT hereafter) to estimate asset correlation based on equity correlation and then input the model implied asset correlation to Zhou's (2001) default correlation model. Nevertheless, their hybrid approach is theoretically ad hoc due to inconsistencies between the LT and Zhou's models. For example, the former assumes that firms dynamically optimize their leverage ratios, while the latter assumes constant leverage ratios.† In addition,

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† The empirical results in Section 5.2 show that our self-consistent default correlation model with endogenous debt has noticeable improvement in predicting default correlations for lower rated bonds compared with the hybrid model proposed by Liu *et al.* (2015). For example, 10-year empirical default correlations between B rated bonds for 1970–1993 and 1990–2010 are 38% and 32%, respectively. The hybrid model predicts the two default correlations as 32.2% and 36.1%. However, the predictions of our model are more close

both Zhou's (2001) and Liu *et al*. (2015) models do not consider the tax effects on default correlation, while taxes play an important role in the default process.

In this study, we address the unsolved issues in the literature by advancing structural default correlation modeling with an entirely self-consistent framework inspired by works of Merton (1974) with exogenous debt, LT (1996) with endogenous debt and corporate taxes, and Liu *et al*. (2006) who further endogenizes personal taxes for single firm's credit spreads. We thus develop two intrinsically self-consistent structural models for default correlation—one with exogenous debt and the other with endogenous debt. Tax effects on default correlations are analyzed under the two frameworks. To our knowledge, our default correlation models with exogenous debt and with endogenous debt are the first attempt in the literature to explore the tax effects on default correlation. We also note that the estimated equity correlations are different when daily and monthly equity returns are applied. Since equity correlation is a critical input variable in our default correlation models, we examine how predicted default correlations could be different with daily and monthly equity returns.

Our empirical results show that our default correlation model with exogenous debt, easy to implement due to the closed-form solution, generally does a good job for high quality bonds. However, it does not capture the time trend of default correlations and underestimate default correlations between low quality bonds. In contrast, our default correlation model with endogenous debt is more realistic in that it allows the firm to endogenously optimize its leverage ratio based on the tradeoff between the bankruptcy cost and the benefit of tax shields. Another major improvement is modeling default as the stopping time event that recognizes default once firm value drops below certain endogenously determined threshold for the first time rather than examining their relation only at maturity. Hence, our default correlation model with endogenous debt can capture the time trend of default correlation and make better predictions for low quality bonds. However, the implementation of this model usually requires significant amount of numerical computation. In terms of the impacts of taxes on the prediction of default correlations, we find that corporate taxes tend to reduce default correlations, while personal taxes have opposing effects on default correlations and the overall effect depends on which factors dominate. Finally, we find that the predicted default correlations are significantly different as daily and monthly stock returns are applied. Our findings shed new light on various aspects of default correlations and thereby help financial practitioner price credit derivatives more accurately and formulate more effective strategies to manage default risk of credit portfolios.

The remainder of the paper is organized as follows. Section 2 discusses the tax effects on default correlations. Section 3 develops the default correlation model with exogenous debt and discusses the predicted default correlations. Section 4 develops the default correlation model with endogenous debt, discusses the model implementation, and conducts simulation. Section 5 discusses the empirical results based on the endogenous model. Finally, Section 6 concludes the paper.

to the empirical default correlations, which are 39.3% and 35.5%, respectively.

2. Tax Effects on Default Correlations

Since taxes play an important role in the default process, we take the first attempt in the literature to incorporate tax effects into our default correlation models derived in Sections 3 and 4. Moreover, tax effects on default correlations are not straightforward and the tax system, including corporate taxes and personal taxes, is complicated. Hence, in this section we illustrate how corporate and personal taxes affect default correlations.

2.1. The effect of corporate taxes on default correlations

The correlation coefficient ρ_S of equity returns R_i is given by

$$\rho_S = \frac{\text{cov}[R_i, R_j]}{\sqrt{\text{var}[R_i]\text{var}[R_j]}}, \tag{1}$$

where cov is covariance and var is variance. Note that the equity return, R_i, are based on the market value, which is after corporate tax. We use τ_C to denote the corporate tax rate and the before-tax return R_i^B is given by $R_i^B(1-\tau_C) = R_i$. Substituting the relation into equation (1), we obtain the correlation coefficient ρ_S of before-tax equity returns between firms i and j as

$$\begin{aligned}\rho_S &= \frac{\text{cov}[R_i^B(1-\tau_C), R_j^B(1-\tau_C)]}{\sqrt{\text{var}[R_i^B(1-\tau_C)]\text{var}[R_j^B(1-\tau_C)]}}\\ &= \frac{\text{cov}[R_i^B, R_j^B]}{\sqrt{\text{var}[R_i^B]\text{var}[R_j^B]}}.\end{aligned}$$

The tax rate drops out of the equation. Therefore, equity correlation is independent of the corporate taxes.

However, corporate taxes have complicated effects on default probability. Modigliani and Miller (M&M 1963) show that, under the absence of arbitrage opportunities and in the presence of corporate taxes, the value of the levered firm is

$$V = \int_0^\infty p(\omega)V(\omega)d\omega,$$

where $p(\omega)$ is the probability density at state w. V increases linearly with the level of debt, B, as shown by M&M:

$$V = V_u + \tau_C B, \tag{2}$$

where V_u is the value of the unlevered firm, and τ_C B is the value of tax shield.

The default probability can be written as $E(I_{V<B})$, where E is the expectation operator, $I_{\{u\}}$ is an indicator function with a value equal to one if u is true. The default probability decreases as the corporate tax rate increases because the value of tax shield ($\tau_C B$) increases. Default correlation can be expressed as

$$\rho_D = \frac{E(I_{\{V_1(T)<B_1, V_2(T)<B_2\}}) - E(I_{\{V_1(T)<B_1\}})E(I_{\{V_2(T)<B_2\}})}{\sqrt{\text{var}(I_{\{V_1(T)<B_1\}})\text{var}(I_{\{V_2(T)<B_2\}})}}. \tag{3}$$

PROPOSITION 1 *Default correlation is dependent on the corporate taxes.*

Proof: We prove it by contradiction. Assuming that default correlation is independent of the corporate taxes, we show this is impossible by a counter example. For simplicity, let $p = E(I_{\{V_1(T)<B_1\}}) = E(I_{\{V_2(T)<B_2\}})$. Since $\mathrm{var}(I_{\{V_1(T)<B_1\}}) = p(1-p)$, we rewrite equation (3) as

$$E(I_{\{V_1(T)<B_1, V_2(T)<B_2\}}) = p^2 + \rho_D p(1-p) \geq 0 \text{ or } \rho_D \geq -\frac{p}{1-p}.$$

The last inequality is just the consequence that the probability, $E(I_{\{V_i(T)<B_i, V_j(T)<B_j\}})$, cannot be less than 0. Assume $\rho_D^* = -p^*/(1-p^*)$ for a pair of firms with default probability $p*$ at which $E(I_{\{V_1(T)<B_1, V_2(T)<B_2\}}) = 0$. As the corporate tax rate decreases, the default probability increases, thus $(dp/d\tau_C) < 0$. Assuming a decrease of corporate tax rate $-\Delta\tau_C$, the default probability increases to $p + (dp/d\tau_C)\Delta\tau_C$ and the default probability ρ_{D} at $p + (dp/d\tau_C)\Delta\tau_C$ must satisfy

$$\rho_D \geq -\frac{p + (dp/d\tau_C)\Delta\tau_C}{1 - (p + (dp/d\tau_C)\Delta\tau_C)} > \rho_D^*,$$

which shows that default correlation cannot remain the same as corporate taxes change in this counter example. Hence, it is not true that default correlation is independent of the corporate taxes.

To quantify the corporate tax effects on default correlation, we develop a default correlation model with exogenous debt in Section 3. The model allows examining how the corporate tax rate affects default correlation given levels of debt for firms. The model also links equity correlation to asset correlation and comes up with a closed form solution for default correlations between a pair of firms. Our empirical results support our argument that corporate taxes tend to reduce default correlation.

2.2. *The effect of personal taxes on default correlations*

The personal taxes include ordinary income taxes on interests, dividends and capital appreciation if the asset is held less than a year, and capital gains taxes on capital appreciation if the asset is held longer than a year. The corporate taxes and different personal taxes affect investors' returns from equity and bond, and in turn, change the demand for and supply of equity and bond as well as the capital structure of firms.

To see the effects of personal taxes on default correlations, we may look at the overall gain G from debt's tax shields (see Miller 1977) as follows:

$$G = V_L - V_U = \left[1 - \frac{(1-\tau_C)(1-\tau_E)}{1-\tau_P}\right] D,$$

where V_U and V_L are unlevered and levered firm values; D is the book value of debt, and τ_C, τ_P, τ_E are corporate, personal income and effective equity tax rates, respectively.

First, the model shows that personal income tax τ_P on bond interest offsets the benefit of corporate tax shields. This counter effect suggests that personal income tax may increase default correlation. Second, effective equity tax rate τ_E is the weighted average of capital gains and income taxes.† An increase in τ_P is likely to increase τ_E, which offsets the effect of personal income tax, and hence we may expect a setback in default correlation. Since $\tau_E < \tau_P$, the overall effect of personal taxes is likely dominated by that of τ_P, i.e. increasing default correlation. Third, since the corporate tax shields are offset by the personal income taxes, a firm's endogenously optimized leverage could be reached at a lower level because the use of debt brings less net tax benefit. A lower leverage means lower default probability of a single firm and thereby lower joint default probability between two firms, which may imply lower default correlation. Fourth, if bankruptcy happens, equity and bond holders expect to receive capital loss rebate, which mitigate the negative impact of bankruptcy. This would adjust up the endogenously optimized leverage, leading to a higher default probability as well as a higher joint default probability, which may imply higher default correlation. The above analysis shows that personal taxes have opposing effects on default correlations and the overall impact depends on which forces dominate.

To examine the intertwined relation between personal taxes and default correlations, we develop a default correlation model with endogenous debt in Section 4. Using the first passage time methodology, our endogenous model takes into account of corporate and personal taxes and optimizes the capital structure of a firm by making trade-off between tax benefits and default costs. The default boundary is determined strategically by a firm to stop the business. Our endogenous model shows that personal taxes affect default correlations and the combination of personal and corporate taxes affect optimal capital structure. The simulation and empirical results support the complicated effects of personal taxes on default correlation, i.e. personal taxes could increase or decrease default correlations. The results echo the analysis here that the impact of personal taxes on default correlation relies on which forces dominate.

3. The Default Correlation Model with Exogenous Debt

3.1. *The model derivation*

Merton (1974) proposes the structural model in which equity is a financial claim on the underlying firm assets. Specifically, equity represents a European call option on the firm's asset with the strike price equal to the debt face value. In his original work, Merton assumes away taxes. To estimate the tax effects on default correlation, we modify Merton's model to capture the first-order effect of taxes on default correlation.

Merton also assumes that the face value of debt is exogenously given. He shows that the Modigliani-Miller theorem holds in the presence of bankruptcy, that is, the value of the firm with bankruptcy risk is invariant to its capital structure. To keep the spirit of the Merton's model, we assume that (1)

† The detailed form is explained in Section 4.1.

a firm's capital structure is exogenously given and is independent of taxes; (2) Taxes on gains and losses are symmetric, which means that a firm receives tax rebates once it has losses.

Denote V_1 and V_2 as total assets defined in equation (2) for firms 1 and 2, respectively, and the dynamics of V_1 and V_2 are given by the following stochastic process:

$$\frac{dV(t)}{V(t)} = \mu dt + \Sigma dW_t, \tag{4}$$

where $\ln V(t) = (\ln V_1, \ln V_2)', \mu = (\mu_1, \mu_2)'$, $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$, and $W_t = (W_{1,t}, W_{2,t})'$ are column vectors. μ_1 and μ_2 are instantaneous expected rates of return of firms per unit of time. $W_{1,t}$ and $W_{2,t}$ are independent standard Wiener processes for firms 1 and 2 that are correlated with correlation coefficient ρ, respectively. $\sum$ is the volatility of the processes.

Equity values S_1 and S_2 are call options on firm assets with maturity T and strike price B_i. That is,

$$\begin{aligned} S_i(V, B, t) &= e^{-r(T-t)} E(V_i(T) - B_i)^+ \\ &= V_i(t)N(d_{i,1}) - e^{-r(T-t)} B_i N(d_{i,2}) \end{aligned} \tag{5}$$

for $i = 1, 2$, and

$$d_{i,1} = \frac{\ln(V_i(t)/B_i) + (r + (\sigma_i{}^2/2))(T-t)}{\sigma_i\sqrt{T-t}} \tag{6}$$

$$d_{i,2} = d_{i,1} - \sigma_i\sqrt{T-t}. \tag{7}$$

If the firm's asset value $V_{i,}(T)$ at maturity is greater than the face value of debt B_i, the firm does not default and shareholders receive $V_i(T) - B_i$. However, if $V_i(T) < B_i$, the firm defaults on its debts and the equity value reduces to zero.

The average equity returns over time period *T-t* is given by†

$$\begin{aligned} R_i &= \frac{1}{T-t} \ln \frac{S_i(V, B, T)}{S_i(V, B, t)} \\ &= \frac{1}{T-t} [\ln S_i(V, B, T) - \ln S_i(V, B, t)] \end{aligned} \tag{8}$$

and thus the correlation coefficient between two equity returns (or equity correlation) is

$$\rho_S = \frac{\operatorname{cov}(\ln S_i(V, B, T), \ln S_j(V, B, T))}{\sqrt{\operatorname{var}(\ln S_i(V, B, T))\operatorname{var}(\ln S_j(V, B, T))}}. \tag{9}$$

The computations of variance and covariance of equity returns are given by (A1) and (A2) in Appendix A.

The default probability is given by

$$E(I_{V_i(T) > B_{i,T}}) = N(d_{i,2}). \tag{10}$$

The variance $\operatorname{var}(I_{\{V_i(T) > B_i\}})$ of zero-one distribution is given by

$$\operatorname{var}(I_{\{V_i(T) > B_i\}}) = E(I_{\{V_i(T) > B_i\}})[1 - E(I_{\{V_i(T) > B_i\}})]. \tag{11}$$

Finally, the expectation $E(I_{\{V_1(T) > B_1, V_2(T) > B_2\}})$ can be expressed in terms of the bivariate normal distribution function Ψ (see Appendix B).

$$\begin{aligned} E(I_{V_1(T) > B_{1,T}, V_2(T) > B_{2,T}}) = \Psi(\varsigma_1 + \rho\varsigma_2 + \frac{m_1 - \ln B_1}{\varsigma_1}, \rho\varsigma_1 \\ + \varsigma_2 + \frac{m_2 - \ln B_2}{\varsigma_2}), \end{aligned} \tag{12}$$

where $\varsigma_i^2 = \sigma_i^2(T-t)$ and

$$m_i = \ln V_i(t) + \left(r - \frac{\sigma_i{}^2}{2}\right)(T-t)_i,$$

which is the expected return of firm asset i ($i = 1, 2$).

Since both equity correlation ρ_S and default correlation ρ_D are functions of asset correlation, given equity correlation ρ_S, we can solve for asset correlation ρ and the default probability $N(d_{i,2})$ and then substitute ρ into equation (12) to compute $E(I_{\{V_{1,t} > B_1, V_{2,t} > B_2\}})$. Finally, we can substitute $E(I_{\{V_{1,t} > B_1, V_{2,t} > B_2\}})$ and $N(d_{i,2})$ into equation (3) to determine default correlation ρ_D. In this way, we establish the relation between equity correlation and default correlation directly without the information for asset correlation. That is, default correlation can be directly computed from equity correlation without *a priori* knowledge of asset correlation.

3.2. *Predictions of the default correlation model with exogenous debt*

We first obtain the empirical default correlations (see table 1) as the benchmark to examine the performance of the model predictions. The default correlations for five and ten years over the sample period of 1970–1993 are obtained from Lucas (1995) and that for ten years over the sample period of 1990–2010 are obtained from Liu *et al*. (2015). However, the empirical default correlations for five years over the sample period of 1990–2010 are not available in any existing work. Hence, we follow Lucas (1995) approach by using Moody's *Corporate Default Risk Service* database to estimate the missing data.

Next, we calibrate parameters in the model to estimate default correlations. A key variable is asset volatility σ that is unobservable. The way to pin down its value is to try different possible values until the model-generated default probability matches the observed default rate. This procedure is well explained in papers such as Huang and Huang (2012). To minimize estimation error, in the actual calibration we choose an asset volatility σ that minimizes the squared difference of log odds between the implied and observed default probabilities,

$$\sigma = \arg\min_{\sigma \geq 0} \sum_{i=1}^{10} [ln\, P_i(\sigma) - ln\, \overline{P_i}]^2, \tag{13}$$

where P_i is the model-implied default probability by year i, and $\overline{P_i}$ is the corresponding observed default rate (see table 4 Panel A). Other important input variables in the model are equity correlations and equity volatilities. Equity correlations are estimated using monthly stock returns and obtained from Liu *et al*. (2015) (see table 6 Panel B and table 9 Panel B). Equity volatilities are calculated as the standard deviations of

† The instantaneous return cannot be used here because it does not reflect the potential default of debt.

Table 1. Empirical default correlations for sample periods 1970–1993 and 1990–2010.

Panel A: Sample period 1970–1993										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0					1				
A	1	1				2	2			
Baa	1	1	0			1	1	0		
Ba	3	4	3	15		3	4	2	8	
B	4	6	7	25	29	8	9	6	17	38

Panel B: Sample period 1990–2010										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	1.1					1.8				
A	1.2	1.9				2.0	3.9			
Baa	1.6	2.7	2.9			2.6	5.0	6.3		
Ba	2.6	3.2	3.7	7.5		3.4	6.2	9.1	17.3	
B	3.6	4.5	5.5	11.5	19.1	4.1	10.8	11.4	20.2	32.0

Note: This table reports the empirical default correlation estimates for different credit ratings over time horizons of five and ten years. All estimates are in percentages. The empirical default correlations over the sample period of 1970–1993 are obtained from Lucas (1995) and that for ten years over the sample period of 1990–2010 are obtained from Liu *et al*. (2015). The five-year default correlations over the sample period of 1990–2010 are estimated based on Moody's *Corporate Default Risk Service* database by following the approach in Lucas (1995).

the equal-weighted stock returns in a particular rating group for each month. After the calibrated parameters and input variables are implemented in the model, default correlations are estimated based on equity correlations.

Table 2 reports the predicted default correlations for the sample periods of 1970–1993 and 1990–2010 as the corporate tax rate is assumed to be 33%.† We find that the default correlation model with exogenous debt generates a pattern of default correlations similar to that shown by the empirical default correlations. First, default correlations become stronger as ratings of both bonds decline. For example, the predicted default correlations between two Aa-rated bonds for five years over the two sample periods are 0, while that between two B-rated bonds increase to 14.7% and 11.7%, respectively. Second, given the rating for one bond, default correlation generally increases as the rating for another bond declines. For example, the predicted default correlations between two A-rated bonds for five years over both sample periods are 0.5% but increase to 2.2% and 1.5% between A- and B-rated bonds over the two sample periods, respectively.

To examine the corporate tax effects on default correlation, we also estimate the default correlations as the corporate tax rate is zero in table 3. The comparison of the results in tables 2 and 3 shows that the corporate taxes tend to reduce default correlations. Actually, the corporate tax rate has different impacts on default probability of firms with different leverages. An increase in the corporate tax rate affects a highly leveraged firm more than a less leveraged firm. This unequal effect reduces default correlation between a pair of firms. For example, for a pair of highly correlated firms, one firm becomes more likely to default than the other after an increase in the corporate tax rate, and then the pair of firm becomes less correlated.

Another potential issue related to the estimation of default correlation is the survivorship bias. As a key input variable in the default correlation model, equity correlations are computed based on the equity returns of the survived firms. To adjust for the survivorship bias, we replace R_i with $R_{i,V_1(T)>B_1}$ in equity correlation

$$\rho_S = \frac{\mathrm{cov}[R_{i,V_1(T)>B_1}, R_{j,V_2(T)>B_2}]}{\sqrt{\mathrm{var}[R_{i,V_1(T)>B_1}]\mathrm{var}[R_{j,V_2(T)>B_2}]}}.$$

In other words, we truncate the region of the firm's assets below the firm's debt from the variance and covariance calculations.

Table 3 reports the estimated default correlations adjusted for the survivorship bias versus the default correlations without the adjustments. We find that the adjustment reduces the estimated default correlations, while the magnitude of the effect is minor. Also, the adjustment affects default correlations for lower rating bonds more than that for higher rating bonds.

Overall, the default correlation model with exogenous debt by considering the effect of corporate taxes can do a fine job of predicting default correlations for investment grade bonds. For example, the model assuming the corporate tax rate of 33% predicts default correlations between two Aa-rated bonds and between A- and Baa-rated bonds for five years over the sample period of 1970–1993 are 0 and 1.1%, respectively, which are very comparable to their empirical benchmarks of 0 and 1%.

† Corporate tax rate in the United States averaged 32.58% from 1909 to 2019. The tax reached the highest rate of 52.80% in 1968 and a record low of 1% in 1910.

Table 2. Default correlations predicted by the default correlation model with exogenous debt (Sample periods: 1970–1993 and 1990–2010; the corporate tax rate: 33%).

Panel A: Sample period 1970–1993										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.0					1.4				
A	0.0	0.5				1.9	3.3			
Baa	0.4	1.1	1.3			2.8	4.8	4.7		
Ba	0.4	2.1	3.8	6.6		3.0	6.9	9.0	10.0	
B	0.5	2.2	4.6	9.6	14.7	3.8	7.8	10.6	12.6	16.4
Panel B: Sample period 1990–2010										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.0					0.9				
A	0.0	0.5				1.3	2.4			
Baa	0.2	0.7	1.0			2.0	3.6	3.9		
Ba	0.3	1.4	3.2	5.9		2.2	5.2	7.8	8.9	
B	0.4	1.5	3.5	7.7	11.7	2.5	5.2	8.3	10.2	13.2

Note: This table reports the predicted five- and ten-year default correlations based on the default correlation model with exogenous debt. The corporate tax rate is assumed to be 33%. The model inputs of equity correlations are estimated based on monthly stock returns and obtained from Liu *et al*. (2015) (see table 6 Panel B and table 9 Panel B). All numbers are in percentages.

However, the model tends to underestimate default correlations for non-investment grade bonds. For example, the predicted ten-year default correlations between two B-rated bonds over the sample periods of 1970–1993 and 1990–2010 are 16.4% and 13.2%, respectively, while the corresponding empirical default correlations are 38% and 32%. Moreover, the model cannot capture the effect of time horizon on default correlation well. For example, when the time horizon is five years, the model predicts that default correlations between B-rated bonds are 14.7% and 11.7% over the two sample periods. As the time horizon lengthens to ten years, the predicted B-rated default correlations only moderately increase to 16.4% and 13.2%, respectively. In contrast, the empirical default correlations show significant increases as time horizon gets longer. For instance, the B-rated empirical default correlations for five years are 29% and 19.1% over the two sample periods, but that increase to 38% and 32% for ten years. Even though we consider the tax effect and adjust for survivorship bias, the results do not have significant improvement.

A possible reason for the underestimation of default correlations between lower rated bonds by our default correlation model with exogenous debt is that equity value is estimated using European call options. As a result, default probability is evaluated only on the expiration date of the options, which allows the asset process of a bankrupted firm before the expiration date to continue evolving and possibly become solvent again on the expiration date. This setup could underestimate the default probability and lead to an increasing divergence between the predicted and the realized default correlations over longer time horizon. This problem should affect non-investment grade bonds more severely than investment grade bonds due to high default probabilities for non-investment grade bonds.

To improve the prediction of default correlations between non-investment grade bonds, we consider the first-passage-time model that allows bankruptcy to occur once a firm's assets hit its default boundary.† In the following section, we develop a default correlation model with endogenous debt to estimate default correlation using the first-passage-time method.

4. The Default Correlation Model with Endogenous Debt

4.1. Incorporating personal taxes into default correlation

The default correlation model with exogenous debt takes into account of corporate taxes and default risk, but ignores that firms tend to optimize their capital structures. Since personal taxes are investors' costs of investments in equity and debt, firms must compensate investors' personal tax liabilities.‡ Personal taxes thus enter firms' optimal capital structure decision and thereby affect default correlation between firms.

Our default correlation model with endogenous debt is inspired by the works of Leland-Toft (1996) and Liu *et al*. (2006). In our endogenous model, equity-debt ratio is optimized considering given corporate and personal taxes. In addition, the model estimates default as a stopping time event, that is, once the firm value is below the default

† In some cases, bankrupt firms do come back after bankruptcy reorganization. In practice, it is Chapter XI Bankruptcy practice. However, after the reorganization, the firm's capital structure differs from that before bankruptcy, so does the asset process. Here, we strictly refer to Chapter VII Bankruptcy practice.
‡ See Miller (1977).

Table 3. Default correlations predicted by the default correlation model with exogenous debt (Sample periods: 1970–1993 and 1990–2010; the corporate tax rate 0%).

Panel A: Sample period 1970–1993; No adjustment for survivorship bias										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.3					2.3				
A	0.5	1.2				3.1	4.9			
Baa	0.8	2.3	2.6			4.2	6.9	6.6		
Ba	0.9	3.6	6.3	9.5		4.3	9.1	11.4	11.9	
B	1.2	3.6	6.9	12.4	17.1	5.1	9.7	12.8	14.3	17.7
Adjusted for survivorship bias										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.3					2.2				
A	0.5	1.2				3.1	4.9			
Baa	0.8	2.3	2.6			4.2	6.8	6.6		
Ba	0.9	3.3	5.6	9.5		4.2	8.2	10.3	11.9	
B	1.0	2.9	5.4	11.8	15.6	4.1	7.7	10.0	13.6	16.1
Panel B: Sample period 1990–2010; No adjustment for survivorship bias										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.1					1.6				
A	0.3	0.8				2.2	3.7			
Baa	0.6	1.6	2.1			3.2	5.3	5.6		
Ba	0.7	2.7	5.4	8.5		3.2	7.0	10.0	10.7	
B	0.7	2.5	5.5	10.1	13.8	3.6	6.7	10.2	11.7	14.3
Adjusted for survivorship bias										
	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	0.1					1.6				
A	0.3	0.8				2.1	3.4			
Baa	0.6	1.6	2.1			3.1	5.2	5.6		
Ba	0.7	2.4	4.8	8.5		3.2	6.3	9.0	10.7	
B	0.6	2.0	4.2	9.6	12.6	2.8	5.2	7.9	11.1	13.1

Note: This table reports the predicted five- and ten-year default correlations by adjusting and not adjusting for survivorship bias. The corporate tax rate is assumed to be zero. Equity correlations are estimated based on monthly stock returns and obtained from Liu *et al*. (2015) (see table 6 Panel B and table 9 Panel B). All numbers are in percentages.

boundary that is the optimal value of the firm debt at which the firm chooses to default and remains default.

As with the LT model (1996), we assume firm's asset process V to follow a drifting geometric diffusion process:

$$\frac{dV(t)}{V(t)} = [\mu(V,t) - \delta]\,dt + \sigma\,dW_t, \tag{14}$$

where $\mu(V,t)$ is the expected rate of return on the firm's assets, δ is the payout ratio that is the proportion of the firm value paid to all shareholders, σ is the constant volatility of asset returns, and W is a standard Wiener process. The asset value V includes the net cash flows generated by the firm's activities.

Moreover, the default correlation model with endogenous debt explicitly takes into account of the corporate income tax (τ_C), the effective tax on equity returns (τ_E), the capital gains tax on equity (τ_{EC}), and the ordinary income tax (τ). Tax shields of debts are $(1-\tau_C)(1-\tau_E)-(1-\tau)$, which is the difference between the double taxation on equity and income tax on interests. Personal taxes tend to have an effect of offsetting the benefit of corporate tax shields. We define the equity process $S(V, V_B, T)$ as Liu *et al.* (2006) as follows:

$$S(V, V_B, T) = \frac{\begin{array}{c} V + (1 - ((1-\tau_C)(1-\tau_E)/1-\tau)) \\ (C/r)[1 - (V_B/V)^{a+z}] - \beta V_B (V_B/V)^{a+z} \\ -D(V, V_B, T) \end{array}}{1 - \tau_{EC}(V_B/V)^{a+z}}, \tag{15}$$

where the default boundary V_B can be determined through the smooth-pasting condition

$$\left.\frac{\partial S(V, V_B, T)}{\partial V}\right|_{V=V_B} = 0.$$

Since V_B no longer has an easy analytical form due to the existence of personal taxes, we use numerical method to find the solution in our model implementation. C is the annual coupon payment and the effective tax rate on equity returns τ_E is the weighted average of dividend and capital gains tax rates given by $\tau_E = (1-\delta)\alpha\tau + \delta\tau$, where the weight depends on the payout ratio δ, and $\tau_{EC} = \alpha\tau$.† Upon bankruptcy, shareholders receive a tax rebate from the government, which equals their investment loss times the capital gains tax rate. Parameters a and z are functions of asset volatility σ and interest rate r.

To develop the default correlation model with endogenous debt, we extend the single-firm credit risk model with taxes into a two-firm setting based on the work by Liu *et al.* (2006) who extended the LT model by incorporating personal taxes. The dynamics of V_1, V_2, $S(V_1)$, and $S(V_2)$ are specified by equations (14) and (15). Equity correlation ρ_S is given by equation (9). Asset returns $\Delta \ln V_1$ and $\Delta \ln V_2$ are correlated with a coefficient ρ, which can be estimated from equity correlation ρ_S.

The divergence between these correlation coefficients may increase as time horizon gets longer because leverage ratio and equity volatility can vary over time. In general, equity and debt do not necessarily move in tandem as can be seen from the options perspective where equity is a call option on a firm's underlying assets, and debt is a shorted put plus a constant. When asset volatility increases, the equity value tends to go up while the debt value can move in an opposite direction. When corporate taxes are considered, the debt-equity movements can be further complicated.‡ To incorporate the time horizon effect, we define the default status of the two firms over a given horizon T as

$$I_i(T) = \begin{cases} 1 & \text{if firm } i \text{ defaults by } T \\ 0 & \text{otherwise} \end{cases}, \tag{16}$$

and default correlation is defined as

$$\rho_D = \frac{E[I_1(T)I_2(T)] - E[I_1(T)]E[I_2(T)]}{\sqrt{\mathrm{var}[I_1(T)]\mathrm{var}[I_2(T)]}}. \tag{17}$$

Appendix C gives a detailed overview of the default correlation model with endogenous debt.

† For this definition of the effective equity tax rate, see Graham (2003).
‡ The effect of taxes on debt and equity is not linear because of differential treatments for capital gains tax, ordinary income tax, and tax rebate due to default losses.

4.2. *The model implementation*

The first step is the calibration by minimizing estimation error (see Huang and Huang 2012). We choose an asset volatility σ that minimizes the squared difference of log odds between the implied and observed default probabilities (see equation (13)). To value debt and equity with the model, we return to the risk-neutral measure by retaining asset return volatility σ and forcing equity premium to zero as required by the equivalent martingale measure approach. The risk-free rate and the payout ratio are set as 8% and 6% in simulation and the empirical estimation for the sample period of 1970–1993, which are in line with Huang and Huang (2012). For the sample period of 1990–2010, the risk-free rate and the payout ratio are set as 4% and 2% respectively, which are the averages for the S&P 500 companies over the same time period. The corporate tax rate τ_C is set at 35% and the bankruptcy loss ratio β is set at 20% of the ongoing concern value right before default based on the estimates in Andrade and Kaplan (1998).

Next, in Monte Carlo simulations of our model, for each iteration we generate a time-series sample path according to equation (14) with the starting asset value $V_i(0)$ normalized to 100, where $i = 1, 2$ denoting the two firms. For each random movement in $V_i(0)$ at time t, we apply equation (15) to obtain equity value $S_i(t)$. For the next random movement in $V_i(t + \Delta t)$ at time $t + \Delta t$, where $\Delta t = T/n$, we again use equation (15) to get $S_i(t + \Delta t)$ while keeping the coupon, principal and default boundary unchanged to recognize the fact that the stationary capital structure of the default correlation model with endogenous debt rules out any debt restructuring after the optimization is done. The procedure is repeated until we reach the horizon at $t = T$. This allows us to map out one sample path. For a second iteration, the same procedure is repeated to generate another sample path of $V_i(t)$ (and thereby $D_i(t)$ and $S_i(t)$ as well) for each firm.

We also introduce random variables Z_1 and Z_2 which are independent and have standard normal distributions. Then the asset return has the following return dynamics:

$$\Delta V_1 = V_1\left[\frac{\mu_1}{n} + \frac{\sigma_1}{\sqrt{n}} \times \Delta Z_1\right]$$
$$\Delta V_2 = V_2\left[\frac{\mu_2}{n} + \frac{\rho\sigma_2}{\sqrt{n}} \times \Delta Z_1 + \frac{\sqrt{1-\rho^2}\sigma_2}{\sqrt{n}} \times \Delta Z_2\right], \tag{18}$$

where n denotes the number of time intervals partitioned for each year and μ_i is the net drift rate. Volatility for each period is $\sigma_i/\sqrt{n}$ where σ_i is the annualized asset return volatility for firm i. For example, when the time interval is months, n is set to 12 and $\sigma_i/\sqrt{12}$ is the monthly asset return volatility. In each simulation, t is represented by the number of steps within the period $[0, t]$. To ensure obtaining a stable convergence in the Monte Carlo simulations, we generate 30 000 sample paths for each rating pair (e.g. Baa and Ba firms). Correlations of assets, equities and debts are calculated for each sample path and then we take their averages over all the sample paths, respectively.

4.3. *Predictions of the default correlation model with endogenous debt*

The simulation results first show how asset correlation and equity correlation are linked across different rating bonds over different time horizons. To obtain the results, we set the capital gains tax rate τ_{EC} to be the half of the ordinary income tax rateτ. Furthermore, we examine how taxes affect the linkages. Table 4 (Panels B and C) reports the simulated equity correlations for given the fixed asset correlation of 40%. First, we find that as bond rating declines and/or time horizon increases, equity correlation deviates downward from asset correlation further. This indicates that equity processes and asset processes become less correlated as bond rating declines and/or time horizon increases. The result suggests that generally equity correlation is not a good proxy for asset correlation. Second, Liu *et al*. (2006) show that the LT model by incorporating personal taxes could considerably improve the explanatory power for default spread. The effective personal tax rate is likely to be different from the statutory rates due to the tax clientele effect and the complexity of personal tax system (e.g. Graham 2003). Therefore, we further study the role of personal taxes in default correlation.† Specifically, we focus on the impact of personal taxes on the relation between equity correlation and asset correlation.‡ We find that in most cases, personal taxes increase equity correlation for given asset correlation. However, the impact is reverse in such cases as ten-year equity correlations between Aa bonds, between Aa and A bonds, and between A bonds. The reason is that personal taxes introduce two opposite effects. On one hand, they reduce the value of corporate tax shields and thereby decrease the optimal leverage, resulting in an increase in the model-implied equity correlation. On the other hand, personal taxes increase the model-implied asset volatility (see table 4 Panel A). Higher asset volatilities will tend to drive the two equity processes apart and hence reduces equity correlation. The overall effect of personal taxes on equity correlation depends on which effect dominates. Our result suggests that the impact of personal taxes on optimal leverage generally dominates that on asset volatility.

Next, the simulation results in table 5 show the model-implied default correlations based on the simulated equity correlations and other parameters in table 4. First, default correlations grow stronger as the bond rating deteriorates and time horizon becomes longer. Second, personal taxes have a mixed impact on default correlations. In most cases, default correlations estimated by considering personal taxes are higher than the counterparties without personal taxes. The impact of personal taxes on default correlations is strong for Aa and A rated bonds over 10-year horizon. For example, default correlation is 9.53% between Aa bonds without personal taxes, while it is almost doubled to 18.2% with personal taxes. In contrast, the impact of personal taxes in other cases is not very significant. For example, default correlation between Baa rated bonds over 5-year horizon is 15.56% without and 15.27% with personal taxes considered.

† We note that corporate taxes are progressive and the effective rate may likely be different from the statutory rates over our sample periods. However, for firms with a taxable income exceeding \$335,000, the federal corporate income (marginal) tax rate is 35% which can be a good approximation as a flat rate for taxable incomes exceeding \$1 million.

‡ We assume that the personal tax rate is 23% in the simulation and empirical studies because it represents the ordinary case in U.S. even under 2018 new tax bill.

5. Empirical Results based on the Default Correlation Model with Endogenous Debt

In this section, we will use empirical data to further test how well our endogenous model predicts default correlations and how taxes affect the model predictions.

5.1. *Default correlation predictions based on daily equity data*

Since equity correlation is a critical input variable in our endogenous model, it is important to estimate equity correlation appropriately. Empirically, equity correlation can be estimated by using daily, monthly, and other frequency of stock returns. We first study equity correlations based on daily and monthly stock returns and then examine how differently predicted default correlations fit empirical default correlations with the different equity correlations. The equity correlations based on monthly stock returns for the sample period of 1970–1993 are obtained from Liu *et al*. (2015). Following their approach, we estimate equity correlations based on daily stock returns over the same sample period. Table 6 shows that on average equity correlations estimated based on daily stock returns are substantially lower than that based on monthly stock returns. This discrepancy grows as bond ratings decline. For example, the discrepancy of Aa-Aa, A-A, Baa-Baa, Ba-Ba, and B-B is 9.32%, 11.85%, 11.50%, 12.72%, and 13.52%, respectively.§

Next, we proceed to investigate default correlations based on equity correlations estimated by stock returns with different frequency. Table 7 reports the model-predicted five-year default correlations based on daily stock returns with and without considering personal taxes over the sample period of 1970–1993. Since a critical step to predict default correlations is to calibrate asset correlations based on the estimated equity correlations, Panel A shows the calibrated asset correlations over the sample period. First, we find that asset correlations are stronger than equity correlations (shown in Panel A of table 7) regardless whether personal taxes are considered. Moreover, as bond rating declines, the discrepancy between asset correlations and equity correlations gets larger. For example, asset correlation between Aa rated bonds without considering personal tax effect is only 1.19% higher than its equity correlation. However, the discrepancy is as high as 4.98% for B rated bonds. Our empirical results echo our simulation findings in Section 4.3 that equity correlation is

§ Numerous studies in asset pricing have shown that market takes time to remove idiosyncratic noises and correct stock prices (e.g. Fama 1980, 1981; Handa *et al.* 1989, 1993). Hence, the stock returns are expected to show stronger correlation once the noises in each individual return are reduced. Not surprisingly, equity correlations based on the monthly stock returns are significantly higher than that based on the daily stock returns.

Table 4. Simulated equity correlations by the default correlation model with endogenous debt.

Panel A: Calibration parameters for the default correlation model with endogenous debt

		Aa	A	Baa	Ba	B
Historical cumulative default rates	5 year	0.32	0.62	1.97	11.85	28.38
	10 year	0.91	1.96	4.96	19.48	39.96
Equity premium		5.60	5.99	6.55	7.30	8.80
Implied asset volatility $\sigma(\tau = 0)$		3.43	3.81	9.51	23.25	42.89
Implied asset volatility $\sigma(\tau = 23\%)$		8.16	10.36	14.94	30.61	55.30

Panel B: $T = 5$ years

	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	37.09					38.04				
A	36.60	35.81				37.53	37.44			
Baa	35.43	34.90	34.70			37.26	37.04	36.62		
Ba	33.13	32.54	32.74	30.91		34.98	34.64	34.64	32.96	
B	29.55	28.94	29.37	28.23	26.24	31.84	31.84	31.57	30.40	28.37

Panel C: $T = 10$ years

	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	36.30					34.57				
A	35.41	34.40				33.81	33.28			
Baa	32.52	31.82	30.66			32.67	32.32	31.29		
Ba	27.79	27.11	26.97	24.00		28.80	28.25	27.77	25.29	
B	22.41	21.74	21.95	20.48	17.96	23.76	23.86	23.36	21.68	19.25

Note: This table reports the simulated equity correlations for given asset correlation of 40% based on the default correlation model with endogenous debt. Panel A reports the calibration parameters for the endogenous model. The historical cumulative default rates are from Fons (1994) and equity premium is from Bhandari (1988). Implied asset volatilities by assuming personal tax rate of zero and 23% are estimated based on the endogenous model, respectively. Panel B compares equity correlations by assuming the personal tax rate of zero with that by assuming personal tax rate of 23% over five-year horizon. Panel C reports the results for ten-year horizon. All numbers are in percentages.

Table 5. Simulated default correlations by the default correlation model with endogenous debt.

Panel A: $T = 5$ years

	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	10.69					9.19				
A	10.32	13.24				11.80	11.20			
Baa	9.59	15.60	15.56			10.68	13.91	15.72		
Ba	14.64	15.90	16.01	22.36		12.20	14.36	18.09	21.45	
B	9.86	14.31	18.00	21.89	23.66	12.08	12.26	14.87	21.63	22.17

Panel B: $T = 10$ years

	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	9.53					18.20				
A	10.94	13.18				19.64	20.45			
Baa	11.93	17.53	21.63			15.97	21.02	20.31		
Ba	13.57	14.88	19.45	23.55		17.21	21.24	22.85	25.07	
B	8.66	12.99	19.06	22.26	24.41	14.78	17.82	20.92	24.09	22.52

Note: This table reports the simulated default correlations by the default correlation model with endogenous debt based on the simulated equity correlations reported in table 4 for time horizons of five and ten years, respectively. The two scenarios with the personal tax rates of zero and 23% are compared. All numbers are in percentages.

Table 6. Equity return correlations estimated with stock return data (Sample period: 1970–1993).

Panel A: Equity return correlations based on daily stock return data					
	Aa	A	Baa	Ba	B
Aa	16.51 (0.0950)				
A	14.85 (0.0416)	13.95 (0.0184)			
Baa	13.54 (0.0335)	12.77 (0.0149)	11.71 (0.0133)		
Ba	11.89 (0.0458)	11.29 (0.0204)	10.29 (0.0181)	9.21 (0.0250)	
B	11.78 (0.0611)	11.30 (0.0271)	10.30 (0.0240)	9.31 (0.0335)	9.62 (0.0434)
Panel B: Equity return correlations based on monthly stock return data					
	Aa	A	Baa	Ba	B
Aa	25.83 (0.2234)				
A	25.05 (0.0713)	25.80 (0.0464)			
Baa	23.46 (0.0644)	24.32 (0.0295)	23.21 (0.0378)		
Ba	21.86 (0.0929)	23.13 (0.0424)	21.89 (0.0382)	21.93 (0.0771)	
B	21.24 (0.1213)	23.01 (0.0551)	21.75 (0.0494)	22.36 (0.0702)	23.14 (0.1287)

Note: This table reports estimated equity correlations by using stock return data obtained from CRSP over the sample period of 1970–1993. Panel A shows the average equity correlations and standard errors (in parentheses) estimated based on daily stock return data. Panel B shows the estimated results based on monthly stock return data and the data are obtained from Liu *et al*. (2015). All numbers are in percentages.

not a good proxy for asset correlation, especially for non-investment bonds. Second, for given equity correlations, asset correlations by considering personal taxes are lower than that without considering personal taxes. The tax effect is stronger for non-investment bonds too. For instance, asset correlation between Aa bonds with personal taxes is 0.4% lower than that without personal taxes. In contrast, the difference is tripled for B rated bonds, i.e. 1.2%. This suggests that personal taxes have weak to mild impacts on asset correlations for given equity correlations.

Panel B shows the five-year model-predicted default correlations based on daily equity returns. We find that personal taxes have nonlinear impacts on default correlations. That is, default correlations by considering personal taxes can be slightly higher than, equal to, or lower than its counterparties without considering personal taxes. However, the magnitude of the impacts of personal taxes on default correlations implies that personal tax effects may not be an important factor in default correlation prediction, while the effects do improve accuracy of predictions in some cases. For example, the predicted default correlations for ratings of Aa-Aa, Aa-Ba, and Baa-B by considering personal taxes fit the empirical data better than that without considering personal taxes.

5.2. *Default correlation predictions based on monthly equity data*

In this section, we examine if predicted default correlations can better fit empirical default correlations for non-investment grade bonds if monthly equity data are used. Panel A of table 8 reports the calibrated five-year asset correlations based on monthly stock returns. The relation between asset correlation and equity correlation and the impacts of personal taxes on asset correlation are similar to that based on the daily

Table 7. Calibrated asset correlations and predicted default correlations by the default correlation model with endogenous debt based on daily stock returns (Sample period: 1970–1993).

Panel A: Calibrated five-year asset correlations based on daily stock returns										
	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	17.7					17.3				
A	16.2	15.5				15.7	14.8			
Baa	15.2	14.6	13.4			14.5	13.7	12.7		
Ba	14.3	13.8	12.5	11.8		13.5	12.9	11.8	11.1	
B	15.8	15.5	15.3	13.1	14.6	14.7	14.1	14.2	12.1	13.4
Panel B: Predicted five-year default correlations based on daily stock returns										
	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	1.0					0.0				
A	1.0	3.0				2.0	3.0			
Baa	3.0	3.0	4.0			3.0	4.0	4.0		
Ba	4.0	4.0	4.0	6.0		3.0	4.0	4.0	6.0	
B	4.0	6.0	8.0	8.0	9.0	4.0	6.0	7.0	8.0	8.0

Note: Panel A reports the calibrated five-year asset correlations based on equity correlations estimated with daily stock returns (see table 6 Panel A) by assuming personal tax rate of zero and 23%, respectively. Panel B reports the predicted five-year default correlations with the input variables reported in Panel A by assuming personal tax rate of zero and 23%, respectively. All numbers are in percentages and the sample period is 1970–1993.

Table 8. Calibrated asset correlations and predicted default correlations by the default correlation model with endogenous debt based on monthly stock returns (Sample period: 1970–1993).

Panel A: Calibrated five-year asset correlations based on monthly stock returns

	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	27.6					27.1				
A	27.3	28. 7				26.4	27.4			
Baa	26.6	27.7	26.6			25.3	26.1	25.3		
Ba	26.0	27.9	26.0	28.1		24.5	26.1	24.6	26.4	
B	28.7	29.4	29.3	32.1	35.0	26.6	28.4	27.1	29.6	32.3

Panel B: Predicted five-year default correlations based on monthly stock returns

	$\tau = 0$					$\tau = 23\%$				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	2.0					3.0				
A	4.0	5.0				2.0	4.0			
Baa	6.0	6.0	9.0			6.0	7.0	8.0		
Ba	7.0	6.0	10.0	13.0		7.0	8.0	11.0	13.0	
B	8.0	8.0	13.0	18.0	22.0	8.0	9.0	11.0	17.0	20.0

Panel C: Calibrated ten-year asset correlations based on monthly stock returns and zero personal tax rate

	Aa	A	Baa	Ba	B
Aa	28.4				
A	28.2	29.9			
Baa	29.0	30.5	30.2		
Ba	29.9	33.6	31.7	36.3	
B	37.7	39.2	39.2	44.2	51.3

Panel D: Predicted ten-year default correlations based on monthly stock returns and zero personal tax rate

	Aa	A	Baa	Ba	B
Aa	2.3				
A	4.5	5.4			
Baa	7.1	6.3	9.8		
Ba	8.2	8.0	13.2	17.2	
B	8.3	10.1	14.9	22.4	39.3

Note: Panel A reports the calibrated five-year asset correlations based on equity correlations estimated with monthly stock returns (see table 6 Panel B) by assuming personal tax rate of zero and 23%, respectively. Panel B reports the predicted five-year default correlations based on the input variables reported in Panel A of table 4 and calibrated five-year asset correlations reported in Panel A by assuming personal tax rate of zero and 23%, respectively. Panel C reports the calibrated ten-year asset correlations based on equity correlations estimated with monthly stock return data by assuming zero personal tax rate. Panel D reports the predicted ten-year default correlations based on the input variables reported in Panel A of table 4 and calibrated ten-year asset correlations reported in Panel C by assuming personal tax rate of zero. All numbers are in percentages and the sample period is 1970–1993.

stock returns. However, the calibrated asset correlations based on monthly stock returns are much higher than that based on daily stock returns. The differences range from 9.8% to 20.4%. Obviously, higher asset correlations based on monthly stock returns are largely attributed to higher estimated equity correlations based on monthly stock returns.

Panel B reports the predicted five-year default correlations based on equity correlations estimated by monthly stock returns with and without considering personal taxes. We find that the predicted default correlations between non-investment grade bonds fit the empirical default correlations much better compared with the performance of the exogenous model and the endogenous model estimated with daily stock returns. For example, the empirical 5-year default correlation is 15% between Ba rated bonds over the sample period of 1970–1993. The corresponding prediction by the exogenous model is 6.6% and that by the endogenous model with daily returns is 6%, while that by the endogenous model with the monthly returns is 13%, which is remarkably close to the empirical benchmark.

As discussed in Section 3.2, the exogenous model cannot capture the time effect on default correlation well. Next, we extend the prediction to ten-year time horizon to examine whether the endogenous model can capture the time effect better. Panel D shows that as time horizon gets longer, the predicted default correlations become substantially larger,

Table 9. Default correlations predicted by the default correlation model with endogenous debt (Sample period: 1990–2010).

Panel A: Calibration parameters for the default correlation model with endogenous debt

		Aa	A	Baa	Ba	B
Historical cumulative default rates	5 year	0.07	0.38	1.17	12.61	28.98
	10 year	0.16	0.88	2.63	25.99	53.29
Equity premium		0.94	0.94	0.99	1.20	1.92
Implied asset volatility $\sigma(\tau = 0)$		2.8	3.2	3.7	10.7	28.4

Panel B: Equity return correlations based on monthly stock returns and zero personal tax rate

	Aa	A	Baa	Ba	B
Aa	21.18				
	(0.1608)				
A	20.17	20.28			
	(0.0575)	(0.0404)			
Baa	19.26	19.88	20.47		
	(0.0494)	(0.0240)	(0.0281)		
Ba	16.89	18.05	19.26	19.88	
	(0.0593)	(0.0288)	(0.0238)	(0.0398)	
B	13.96	15.16	16.99	18.31	18.70
	(0.0728)	(0.0357)	(0.0293)	(0.0346)	(0.0596)

Panel C: Predicted default correlations based on monthly stock returns and zero personal tax rate

	$T = 5$ years					$T = 10$ years				
	Aa	A	Baa	Ba	B	Aa	A	Baa	Ba	B
Aa	1.2					1.7				
A	1.6	2.2				2.6	3.9			
Baa	2.1	3.3	3.9			3.7	5.0	6.3		
Ba	3.3	4.6	6.0	12.9		5.1	7.6	11.3	24.6	
B	3.6	5.3	7.5	16.4	24.8	3.7	7.1	11.2	25.7	35.5

Note: This table reports the predicted default correlations by the default correlation model with endogenous debt over the sample period of 1990–2010. Panel A reports the calibration parameters for the model. The historical cumulative default rates and equity premium are from Liu *et al*. (2015). Implied asset volatility by assuming personal tax rate of zero is estimated based on the default correlation model with endogenous debt. Panel B reports the average equity correlations and standard errors (in parentheses) estimated based on monthly stock returns that are obtained from Liu *et al*. (2015). Panel C reports the predicted five- and ten-year default correlations based on the input variables reported in Panels A and B by assuming zero personal tax rate. All numbers are in percentages.

especially for non-investment grade bonds. For example, the five-year and ten-year predicted default correlations between B rated bonds by the exogenous model are 14.7% and 16.4%, respectively. However, the predictions by the endogenous model by using monthly stock returns are 22% and 39.3%, respectively. The latter is closer to the empirical default correlations, i.e. 29% and 38% respectively. This indicates that the endogenous model can capture time effect on default correlation much better than the exogenous model.

To examine the robustness of the endogenous default correlation model, we further extend our study to the sample period of 1990–2010. Panels A and B in table 9 report the calibrated parameters and equity correlations estimated by monthly stock returns. Panel C reports the predicted five- and ten-year default correlations based on monthly stock returns without considering personal taxes. The results show that the predicted default correlations capture the trend of empirical default correlations and the time effect quietly well. For example, the predicted five- and ten-year default correlations between Baa bonds are 3.9% and 6.3%, respectively, which are close to the empirical default correlations of 2.9% and 6.3% respectively. Again, the endogenous model predicts default correlations between non-investment grade bonds much better than the exogenous model. For example, the predicted ten-year default correlation by the endogenous model is 35.5%, which is pretty close to the empirical default correlation of 32%, while the prediction of the exogenous model of 13.2% is far away from the empirical benchmark.

Overall, the empirical results suggest that the endogenous model does a good job of predicting default correlations for non-investment grade bonds and over a longer time horizon. Personal taxes could increase or decrease default correlations, which supports the analysis in Section 2 that personal taxes have opposing effects on default correlations and the overall impact depends on which effects dominate. The frequency of stock returns is an important factor to affect default correlation prediction.

6. Conclusions

Default correlation is a critical component for risk management in areas such as fixed income portfolios, bank management, and insurance industry. Hence, how to accurately predict default correlations is a great concern for risk managers. The existing literature on structural modeling of default correlation has a few intrinsic inconsistencies. In this paper, we address these flaws by developing two default correlation models, one with exogenous debt and the other with endogenous debt, based on entirely self-consistent structural frameworks. Our models are also the first one to examine the tax effects on default correlations based on the observable equity data. Our work not only improves the existing default correlation models (e.g. Zhou 2001, Liu *et al*. 2015) theoretically, but provides insightful implications to risk managers as well. First, we find that neither the model with exogenous debt nor the model with endogenous debt can completely fit empirical default correlations. In general, our default correlation model with exogenous debt does a good job of predicting default correlations for high quality bonds, while it considerably underestimates default correlations for low quality bonds and cannot appropriately capture the time trend of default correlations. In contrast, our default correlation model with endogenous debt can do a good job of predicting default correlations for low quality bonds and captures the time trend well. Second, we find that the frequency of equity returns has a significant impact on the predicted default correlations and which frequency of equity data is used depends on risk managers' interest of time horizons. Finally, corporate income taxes tend to reduce default correlations, while personal taxes have opposing effects on default correlations and the overall impact depends on which effect dominates.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix A

Derivation of asset correlation from equity correlation for the default correlation model with exogenous debt

In this appendix, we present the results needed to calculate asset correlation from equity correlation based on the default correlation model with exogenous debt. Merton (1974) models equity S_i of firm i as a call options of the firm's assets $V_i$on the face value of the firm debt B_i. The dynamic process of V_i is defined in equation (4) and the equity value is given by the Black–Scholes option pricing model:

$$\begin{aligned}S_i(V,B,t) &= e^{-r(T-t)}E(V_i(T)-B_i)^+\\ &= e^{-r(T-t)}E\left\{\left(V_i(t)\exp\left[\left(r-\frac{\sigma_i{}^2}{2}\right)(T-t)\right.\right.\right.\\ &\quad \left.\left.\left.+\,\sigma_i(W_i(T)-W_i(t))\right]-B_i\right)^+\right\}\\ &= V_i(t)N(d_{i,1})-e^{-r(T-t)}B_iN(d_{i,2}),\end{aligned}$$

where E is the expectation operator under the risk-neutral probability, r is the risk-free interest rate that is constant, and T is maturity. The variance and covariance are given by

$$\mathrm{var}(\ln S_i(V,V_B,T)) = E([\ln S_i(V,V_B,T)]^2) - [E(\ln S_i(V,V_B,T))]^2 \tag{A1}$$

and

$$\begin{aligned}&\mathrm{cov}(\ln S_i(V,V_B,T),\ln S_j(V,V_B,T))\\ &\quad = E(\ln S_i(V,V_B,T)\times \ln S_j(V,V_B,T))\\ &\qquad - E(\ln S_i(V,V_B,T))E(\ln S_j(V,V_B,T))\end{aligned} \tag{A2}$$

under the condition $S_i(V,V_B,T)>0$. The above results show that the conditional expectation and the second moment of equity return can be computed as

$$E(\ln S_i(V,V_B,T)) = E\left\{\ln\left(V_i(t)\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right.\right.\right.$$
$$\left.\left.\left.+\,\sigma_i(W_i(T)-W_i(t))\right]-B_i\right)\right|$$
$$\left. V_i(T) > B \text{ and } V_j(T) > B_j\right\}$$

$$E([\ln S_i(V,V_B,T)]^2) = E\left\{\left[\ln\left(V_i(t)\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\,\sigma_i(W_i(T)-W_i(t))\right]-B_i\right)\right]^2\right|$$
$$\left. V_i(T) > B \text{ and} V_j(T) > B_j\right\}$$

and

$$E(\ln S_i(V,V_B,T)\times \ln S_j(V,V_B,T))$$
$$= E\left\{\ln\left(V_i(t)\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right.\right.\right.$$
$$\left.\left.+\,\sigma_i(W_i(T)-W_i(t))\right]-B_i\right)$$
$$\times \ln\left(V_j(t)\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right.\right.$$
$$\left.\left.\left.+\,\sigma_i(W_i(T)-W_i(t))\right]-B_i\right)\right|$$
$$\left. V_i(T) > B \text{ and } V_j(T) > B_j\right\}.$$

The condition $V_i(T) > B$, i.e $V_i(t)\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma_i(W_i(T)-W_i(t))\right] > B_i$ is equivalent to

$$M_i = \frac{(W_i(T)-W_i(t))}{\sqrt{T-t}} > -\frac{\left[\ln\frac{V_i(t)}{B_i}+\left(r-\frac{\sigma_i^2}{2}\right)(T-t)\right]}{\sigma_i\sqrt{T-t}} = -d_1,$$

where M_i has a standard normal distribution

$$\psi_{Mi} = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{M_i{}^2}{2}\right]$$

and we can rewrite the conditional expectations as

$$E(\ln S_i(V,V_B,T)) = \int_{-d_{i1}}^{\infty}\ln\left(V_i(t)\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right.\right.$$
$$\left.\left.+\,\sigma_i\sqrt{(\text{T - t})}M_i\right]-B_i\right)\psi_{Mi}dM_i$$

$$E([\ln S_i(V,V_B,T)]^2) = \int_{-d_{i1}}^{\infty}\left[\ln\left(V_i(t)\exp\left[\left(r-\frac{\sigma^2}{2}\right)(T-t)\right.\right.\right.$$
$$\left.\left.\left.+\,\sigma_i\sqrt{(\text{T - t})}M_i\right]-B_i\right)\right]^2\psi_{Mi}dM_i.$$

The calculation of $E(\ln S_i(V,V_B,T)\times\ln S_j(V,V_B,T))$ involves double integrals over a correlated bivariate normal distribution. Applying the change of variable theorem to the integral we can transform the integral into two independent random variables (see Appendix B for the transformation).

Appendix B

Derivation of default correlation from asset correlation for the default correlation model with exogenous debt

Given the results in Appendix A, we can derive asset correlation from equity correlation for the default correlation model with exogenous debt. The equity value of firm i is the call value on the underlying firm asset process $V_{i,t}$. Default correlation is defined by equation (3).

The option pricing model shows that

$$E(I_{V_i(T)>B_i}) = N(d_{i,2}).$$

Since $\ln V_i(T)$ is normally distributed (see Appendix A) and $V_i(T) > B_i$ is equivalent to $m_i > -\sigma_i\sqrt{T-t}Z_i$, where

$$m_i = \ln V_i(t) + \left(r-\frac{\sigma_i{}^2}{2}\right)(T-t),$$

then integrate the probability density function ψ_{ij} over. $\{\ln(V_{i,T}) > \ln(B_i), \ln(V_{j,T}) > \ln(B_j)\}$ to compute the joint default probability $E(I_{V_{i,T}>B_{i,T},V_{j,T}>B_{j,T}})$,

$$E(I_{V_{i,T}>B_{i,T},V_{j,T}>B_{j,T}}) = \int_{\ln(B_i)}^{\infty}\int_{\ln(B_j)}^{\infty}\psi_{ij}d\ln(V_{i,T})d\ln(V_{j,T}).$$

For simplicity we let $\varsigma_i^2 = \sigma_i^2(T-t)$ and $\mathrm{m} = (m_1, m_2)$ which are the variance and the expected value of $\ln V_i(T)$.

Under the conditions $\sigma_1\sigma_2 \neq 0$ and $|\rho| < 1$, we can write

$$\psi(\ln\mathrm{V}) = \frac{1}{2\pi\sqrt{\det\Sigma_{\mathrm{T}}}}\exp\left[-\frac{1}{2}((\ln\mathrm{V}\text{ - m})\Sigma_{\mathrm{T}}^{-1}(\ln\mathrm{V}\text{ - m}'))\right]$$

$$\Sigma\cdot\Sigma' = \begin{bmatrix}\varsigma_1^2 & \rho\varsigma_1\varsigma_2\\ \rho\varsigma_1\varsigma_2 & \varsigma_2^2\end{bmatrix}.$$

We transform the two correlated Brownian motion into independent Brownian motion by an affine transformation

$$\mathrm{U} = \ln\mathrm{V}\cdot\mathrm{F} + \nu,$$

where

$$\mathrm{U} = (U_1, U_2)$$
$$\mathrm{F} = \begin{bmatrix}\frac{1}{\varsigma_1} & -\frac{\rho}{\varsigma_1\sqrt{1-\rho^2}}\\ 0 & \frac{1}{\varsigma_2\sqrt{1-\rho^2}}\end{bmatrix}$$
$$\nu = (\nu_1,\nu_2) = -\mathrm{mF}.$$

U_1 and U_2 are independent and identically distributed standard normal random variables and

$$\ln\mathrm{V} = \mathrm{U}\cdot\mathrm{F}^{-1} + \mathrm{m}.$$

The Jacobian of the transformation is

$$J(\mathrm{U}) = \left|\frac{\partial\ln\mathrm{V}}{\partial\mathrm{U}}\right| = \varsigma_1\varsigma_2\sqrt{1-\rho^2}.$$

The region under the integration in determining the joint default probability above is bounded in ($\ln V_1$, $\ln V_2$) plane by

$$V_1 = B_1, \quad V_2 = B_2.$$

The image under the transformation in (U_1,U_2) plane is bounded by

$$U_1 = b_1$$
$$U_2 = -a_2 U_1 + b_2$$

$$b_1 = \frac{\ln B_1 - m_1}{\varsigma_1}, \quad a_2 = \frac{\rho}{\sqrt{1-\rho^2}}, \quad b_2 = \frac{\ln B_2 - m_2}{\varsigma_2\sqrt{1-\rho^2}}.$$

The change of variable theorem gives

$$E(I_{V_{1,T}>B_{1,T},V_{2,T}>B_{2,T}}) = \int_{b_1}^{\infty}\int_{a_1U_1+b_2}^{\infty} \phi(U_1)\phi(U_2)J(\mathrm{U})dU_1dU_2,$$

where φ is the standard normal probability density function. We can rewrite

$$E(I_{V_{1,T}>B_{1,T},V_{2,T}>B_{2,T}}) = \int_{b_1}^{\infty}\int_{a_2U_1+b_2}^{\infty} \phi(U_1-(\varsigma_1+\rho\varsigma_2))\phi \left(U_2 - \varsigma_2\sqrt{1-\rho^2}\right)dU_1dU_2.$$

Integrating with respective to U_2 yields

$$E(I_{V_{1,T}>B_{1,T},V_{2,T}>B_{2,T}}) = \int_{b_1}^{\infty} \phi(U_1-(\varsigma_1+\rho\varsigma_2))\phi \left(a_2U_1 - b_2 - \varsigma_2\sqrt{1-\rho^2}\right)dU_1dU_2.$$

The integral can be expressed in terms of the bivariate normal distribution function, (Chuang 1996)

$$E(I_{V_{1,T}>B_{1,T},V_{2,T}>B_{2,T}}) = \Psi\left(\varsigma_1 + \rho\varsigma_2 + \frac{m_1 - \ln B_1}{\varsigma_1}, \rho\varsigma_1 + \varsigma_2 + \frac{m_2 - \ln B_2}{\varsigma_2}\right).$$

Appendix C

The default correlation model with endogenous debt in correlated two-firm environment

In this appendix, we briefly explain the default correlation model with endogenous debt and its modified version by Liu *et al.* (2006) with taxes, which we further extend into a two-firm setting in Section 4 of this study.

The LT (1996) model assume that the firm's asset process V follows a drifting geometric diffusion process

$$\frac{dV(t)}{V(t)} = [\mu(V,t) - \delta]\,dt + \sigma\, dW_t,$$

where $\mu(V,t)$ is the expected rate of return on the firm's assets, δ is the payout ratio that is the proportion of the firm value paid to all security holders, σ is the constant volatility of asset returns, and W is a standard Wiener process. The asset value V includes the net cash flows generated by the firm's activities.

Then we assume an identical but levered firm issuing a risky debt d per unit time with t periods to maturity, a continuous constant coupon flow $c(t)$ and a principal $p(t)$. The firm remains solvent until the asset value V hits a default boundary V_B that is determined dynamically as shareholders maximize the equity value of the firm. Upon bankruptcy, bondholders receive a fraction $\chi = (1-\beta)$ of the asset value V_B, where β is the bankruptcy cost ratio and βV_B is loss due to bankruptcy. Further, we assume that r represents the continuous interest rate paid by a default-free asset. Investors pursue a buy-and-hold investment strategy and are subject to an ordinary income tax rate τ and a capital gains tax rate $\alpha\tau$ where $\alpha < 1$. Because state tax is a deduction against federal income tax, the effective ordinary income tax rate for corporate bond investors is $\tau = \tau_F + \tau_S(1-\tau_F)$, where τ_S and τ_F are state and federal tax rates, respectively. Therefore, under the risk-neutral valuation, the value of the debt, d, is given by

$$\begin{aligned} d(V,V_B,t) = &\int_0^t (1-\tau)c(t)e^{-rs}[1-F(s,V,V_B)]ds \\ &+ [p(t) - \alpha\tau(p(t) - d(V,V_B,t))]e^{-rt}[1-F(t,V,V_B)] \\ &+ \int_0^t [\chi V_B + \alpha\tau(d(V,V_B,t) - \chi V_B)] \\ &\times e^{-rs}f(s,V,V_B)ds, \end{aligned}$$

where $F(s,V,V_B)$ is the cumulative default probability up to time s, and $f(s,V,V_B)$ is the density of F with respect to time s. The first term on the right side is the discounted expected after-tax value of the coupon flow paid at time s. The second term represents the discounted expected after-tax repayment value of principal, where $\alpha\tau(p(t) - d)$ is the capital gains tax liability at maturity. With par bonds, this value is zero. The third term is the discounted expected residual value of the debt χV_B plus the tax rebate, $\alpha\tau(d - \chi V_B)$, from the investment loss if default occurs at $s \le t$. Solving for d, we get

$$d(V,V_B,t) = \frac{\begin{matrix}((1-\tau)c(t)/r) + e^{-rt}[(1-\alpha\tau)p(t) \\ -((1-\tau)c(t)/r)](1-F(t)) + [(1-\alpha\tau)\chi V_B \\ -((1-\tau)c(t)/r)]G(t)\end{matrix}}{1-\alpha\tau\,[e^{-rt}(1-F(t)) + G(t)]},$$

where $F(t)$ and $G(t)$ are given in Leland and Toft (1996). The total outstanding debt D is the integration of the debt flow $d(V,V_B,t)$ over T that is the maturity of newly issued debt:

$$D(V,V_B,t) = \int_{t=0}^{T} d(V,V_B,t)dt.$$

The integral can be carried out numerically. The tradeoff between the benefit of tax shields and bankruptcy costs suggests that there exists an endogenously determined bankruptcy threshold V_B that maximizes firm value. The equity value, as a function of default boundary V_B and asset value V, is given by

$$S(V,V_B,T) = \frac{\begin{matrix}V + (1-(1-\tau_C)(1-\tau_E)/(1-\tau))(C/r) \\ [1-(V_B/V)^{a+z}] - \beta V_B(V_B/V)^{a+z} - D(V,V_B,T)\end{matrix}}{1-\tau_{EC}(V_B/V)^{a+z}},$$

where C is the annual coupon payment, τ_C is the corporate income tax rate, τ_E is the effective tax rate on equity returns, and τ_{EC} is the capital gains tax rate on equity. Dividend income is taxed at the ordinary income tax rateτ, and capital gains are taxed at the capital gains rate τ_{EC}. The effective tax rate on equity returns τ_E is the weighted average of dividend and capital gains tax rates given by $\tau_E = (1-\delta)\alpha\tau + \delta\tau$, where the weight depends on the payout ratio δ, and $\tau_{EC} = \alpha\tau$. In the event of bankruptcy, equity holders receive a tax rebate from the government, which equals their investment loss times the capital gains tax rate. Parameters a and z are functions of asset volatility σ and interest rate r. Finally, the default boundary V_B can be determined through the smooth-pasting condition $\frac{\partial S(V,V_B,T)}{\partial V}\Big|_{V=V_B} = 0$. The solution for equity S (V, V_B, T) is what is used as the basis for developing the default correlation model in Section 4.