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Trading algorithms with learning in latent alpha models

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Abstract
Alpha signals for statistical arbitrage strategies are often driven by latent factors. This paper analyzes how to optimally trade with latent factors that cause prices to jump and diffuse. Moreover, we account for the effect of the trader's actions on quoted prices and the prices they receive from trading. Under fairly general assumptions, we demonstrate how the trader can learn the posterior distribution over the latent states, and explicitly solve the latent optimal trading problem. We provide a verification theorem, and a methodology for calibrating the model by deriving a variation of the expectation–maximization algorithm. To illustrate the efficacy of the optimal strategy, we demonstrate its performance through simulations and compare it to strategies that ignore learning in the latent factors. We also provide calibration results for a particular model using Intel Corporation stock as an example.

KEYWORDS
algorithmic trading, latent alpha, machine learning, partial information, statistical arbitrage, stochastic control

JEL CLASSIFICATION:
G11, C61, C40

1 | INTRODUCTION

The phrase “All models are wrong, but some are useful” (Box, 1978) rings true across all areas in finance, and intraday trading is no exception. If an investor wishes to efficiently trade assets, she must use a strategy that can anticipate the asset's price trajectory while simultaneously being mindful of the flaws in her model, as well as the costs borne from transaction fees and her own impact on prices. With all of the complexities in intraday markets, it is no surprise that strategies differ substantially based on what assumptions are made about asset price dynamics. Trading with an incorrect model can be very costly to an investor, and therefore being able to mitigate model risk is valuable.

The availability of information at very high frequencies can help a trader partially overcome the problem of model selection. The information provided from realized trajectories of the asset price and the incoming flow of orders of other traders allows her to infer which model best fits the observed data and, in turn, she may use it to predict future movements in asset prices. Ideally, the trader should be able to incorporate this information in an on-line manner. In other words, the trader should be continuously updating her model as she observes new information, keeping in mind that the market may switch between a number of regimes over the course of the trading period. Furthermore, the trader would like to have some means of incorporating a-priori knowledge about markets into her trading strategy before beginning to trade.

This paper studies the optimal trading strategy for a single asset when there are latent alpha components to the asset price dynamics, and where the trader uses price information to learn about the latent factor. Prices can diffuse as well as jump. The trader's goal is to optimally trade subject to this model uncertainty, and end the trading horizon with zero inventory. By treating the trader's problem as a continuous time control problem where information is partially obscured, we succeed in obtaining a closed form strategy, up to the computation of an expectation that is specific to the trader's prior assumptions on the model dynamics. The optimal trading strategy we find can be computed with ease for a wide variety of models, and we demonstrate its performance by comparing, in simulation, with approaches that that do not make use of learning.

Early works on partial information include Detemple (1986, 1991), who studies optimal technology investment problems (where the states that drive production are obfuscated by Gaussian noise); Gennotte (1986), who studies the optimal portfolio allocation problem when returns are hidden but satisfy an Ornstein–Uhlenbeck process; Dothan and Feldman (1986), who analyze a production and exchange economy with a single unobservable source of nondiversifiable risk; Karatzas and Xue (1991), who study utility maximization under partial observations; Bäuerle and Rieder (2005), Bäuerle and Rieder (2007), and Frey, Gabih, and Wunderlich (2012), who study model uncertainty in the context of portfolio optimization and the optimal allocation of assets; and Papanicolaou (2019), who studies an optimal portfolio allocation problem where the drift of the assets are latent Ito diffusions.

There are a few recent papers on partial information that are related to this study. Ekstrom and Vaicenavicius (2016) investigate the optimal timing problem associated with liquidating a single unit of an asset when the asset price is a geometric Brownian motion with random (unobserved) drift. Colaneri, Eksi, Frey, and Szölgyenyi (2018) study the optimal liquidation problem when the asset midprice is driven by a Poisson random measure with unknown mean measure. Gârleanu and Pedersen (2013) study the optimal trading strategy for maximizing the discounted, and penalized, future expected excess returns in a discrete-time, infinite-time horizon problem. In their model, prices contain an unpredictable martingale component, and an independent stationary (visible) predictable component—the alpha component. Bismuth, Guéant and Pu (2016) study models in which the drift of the asset price process is a latent random variable in an optimal portfolio selection setting, for an investor who seeks to maximize a Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) objective function, as well as in the cases of optimal liquidation in an Almgren–Chriss like setting.

The approach we take differs in several ways from the extant literature, but the two key generalizations are: (a) we account for quite general latent factors that drive the drift and jump components in the asset's midprice; and (b) we include both temporary and permanent impact that the agent's trading has on the market.

The structure of the remainder of this paper is as follows. Section 2 outlines our modeling assumptions, as well as providing the optimization problem with partial information that the trader wishes to solve. Section 3 provides the filter that the trader uses to make proper inference on the underlying model driving the data she is observing. Section 4 shows that the original optimization problem presented in Section 2 can be simplified to an optimization problem with complete information using the filter presented in Section 3. Section 5 shows how to solve the reduced optimization problem from Section 4 and verifies that the resulting strategy indeed solves the original optimization problem. Finally, Section 6 provides some numerical examples by applying the theory to a few specific models, and compares the resulting strategy, using simulations, to an alternative that does not learn from price dynamics.

2 | THE MODEL

We work on the filtered probability space $(\Omega, \mathcal{G} = \{\mathcal{G}_t, 0 \le t \le T\}, \mathbb{P})$, where $T > 0$, and finite, is some fixed time horizon. The filtration $\mathcal{G}$ is the natural one generated by the paths of the unimpacted asset midprice process $F = (F_t)_{t\in[0,T]}$, the counting processes for the number of buy and sell market orders (MOs) that cause price changes, denoted $N^+ = (N_t^+)_{t\in[0,T]}$ and $N^- = (N_t^-)_{t\in[0,T]}$, and a latent process $\Theta = (\Theta_t)_{t\in[0,T]}$. The exact nature of these processes will be provided in more detail in the remainder of the section.

The trader's optimization problem is to decide on a dynamic trading strategy to buy/sell an asset over the course of a trading horizon to maximize some performance criteria. We assume the trader executes orders continuously at a (controlled) rate denoted by $\nu = (\nu_t)_{t\in[0,T]}$. The trader's inventory, given some strategy ν, is denoted $Q^\nu = (Q_t^\nu)_{t\in[0,T]}$, with the initial condition $Q_0^\nu = \mathfrak{N}$. $\mathfrak{N}$ may be zero, positive (a long position), or negative (a short position)—and, hence, the inventory at time t can be written as

$$Q_t^\nu = \mathfrak{N} + \int_0^t \nu_u \, du. \tag{1}$$

The above can be interpreted as the investor purchasing $\nu_t\, dt$ shares over the period $[t, t + dt)$. A positive (negative) value for ν_t represents the trader buying (selling) the asset. The rate at which the investor buys or sells the asset affects prices through two mechanisms. First, a temporary price impact, which is effectively a transaction cost that increases with increasing trading rate. Second, a permanent impact, which incorporates the fact that when there are excess buy orders, prices move up, and excess sell orders, prices move down.

We further assume that other market participants also have a permanent impact on the asset midprice through their own buy and sell MOs. To model this, we let $N^\pm$ be doubly stochastic Poisson processes with respective intensity processes $\lambda^\pm = (\lambda_t^\pm)_{t\in[0,T]}$, which count the number of MOs that cause prices to move. In the remainder of the paper, we write $\boldsymbol{N} = (N_t^+, N_t^-)_{t\in[0,T]}$.

2.1 | Asset midprice dynamics

To model the permanent price impact of trades, we define two processes $S = (S_t)_{t\in[0,T]}$ and $F = (F_t)_{t\in[0,T]}$ to represent the asset midprice and the asset midprice without the trader's impact, respectively. As shown by Cartea and Jaimungal (2016), intraday permanent price impact (over short time scales) is well approximated by a linear model. Hence, we write

$$S_t^\nu = F_t + \beta \int_0^t \nu_u \, du, \tag{2}$$

where $\beta > 0$ controls the strength of the trader's impact on the asset midprice. Alternatively, one could write this as a pure jump model

$$S_t^\nu = F_t + \beta \left(\mathcal{M}_t^{+,\nu} - \mathcal{M}_t^{-,\nu} \right),$$

where $(\mathcal{M}_t^{+,\nu}, \mathcal{M}_t^{-,\nu})_{t\in[0,T]}$ are controlled doubly stochastic Poisson processes with $\mathbb{P}$-intensities $\gamma_t^+ = \nu_t \mathbb{1}_{\nu_t>0}$ and $\gamma_t^- = -\nu_t \mathbb{1}_{\nu_t<0}$, respectively. The results will be identical to that obtained using the continuous model above.

We assume the investor does not have complete knowledge of the dynamics of the asset midprice, nor the rates of arrival of MOs. This uncertainty is modeled by assuming there is a latent continuous time Markov chain $\Theta = (\Theta_t)_{t\in[0,T]}$ (with $\Theta_t \in \{\theta_j\}_{j\in\mathfrak{J}}$ and $\mathfrak{J} = \{1, 2, \dots, J\}$), which modulates the dynamics of state variables, but is not observable by the trader. The latent process Θ is assumed to have a known generator matrix[1] $\boldsymbol{C}$ and the trader places a prior $\pi_0^j = \mathbb{P}[\Theta_0 = \theta_j]$, $j \in \mathfrak{J}$, on the initial state of the latent process, all estimated, for example, by the EM algorithm (see Section 7.1 for details).

Conditional on a path of Θ, the unaffected midprice F is assumed to satisfy the Stochastic Differential Equation (SDE)

$$dF_t = A(t, F_t, \boldsymbol{N}_t; \Theta_t)\, dt + b \left(dN_t^+ - dN_t^- \right) + \sigma \, dW_t, \qquad F_0 = F,$$

where $N^\pm$ have $\mathbb{P}$-intensities

$$\lambda_t^\pm = \sum_{j\in\mathfrak{J}} \mathbb{1}_{\{\Theta_t=\theta_j\}} \lambda_t^{\pm,j}, \tag{3}$$

and $W = (W_t)_{t\in[0,T]}$ is a $\mathbb{P}$-Brownian motion. Moreover, we assume that each of the $\{\lambda^{\pm,j}\}_{j\in\mathfrak{J}}$ are $\mathcal{F}$-adapted processes, where $\mathcal{F} \subseteq \mathcal{G}$ is the natural filtration generated by the paths of the processes F (note that $\boldsymbol{N}$ can be inferred from this filtration, and strategies are therefore also adapted to the paths of $\boldsymbol{N}$). Furthermore, we assume $(\Theta_t, F_t, \lambda_t, \boldsymbol{N}_t)_{t\in[0,T]}$ is a $\mathcal{G}$-adapted Markov process, where $\lambda = (\{\lambda_t^{+,j}, \lambda_t^{-,j}\}_{j\in\mathfrak{J}})_{t\in[0,T]}$. The Markov assumption will, after modifying the problem to deal with partial information, allow a dynamic programming principle (DPP) and result in a dynamic programming equation (DPE). We assume that either (a) $\sigma > 0$ or (b) $\sigma = 0$ and $A := 0$ to prevent cases where the model is driven by a counting process but also has a continuous drift. In case (b), the asset price may indeed drift, but the drift will be due to imbalance in intensities, so that prices remain on a discrete grid. To compress notation, we define the process $A = (A_t)_{t\in[0,T]}$, where $A_t := A(t, F_t, \boldsymbol{N}_t; \Theta_t)$ as well as the processes $A^j = (A_t^j)_{t\in[0,T]}$, where $A_t^j := A(t, F_t, \boldsymbol{N}_t; \theta_j)$ for each $j \in \mathfrak{J}$. Finally, we make the technical assumption that

$$\mathbb{E}\left[\int_0^T \left(A_u\right)^2 + \left(\lambda_u^+\right)^2 + \left(\lambda_u^-\right)^2 \, du \right] < \infty. \tag{4}$$

This class of intensity models contains, among many others, deterministic intensities, shot-noise processes, and cross-exciting Hawkes processes with finite-dimensional Markov representations[2], all modulated by the latent factor(s). We provide some explicit examples in Section 6 where we also conduct numerical experiments.

The random variable Θ_t indexes the J possible models for the asset's drift and the rates at which other market participants' MOs arrive. Because Θ is (potentially) stochastic, it may change over time, hence, so will the underlying model. Furthermore, because Θ is invisible to the investor, to make intelligent trading decisions, the investor must infer from observations what is the current (and future) underlying model driving asset prices.

2.2 | Cash process

The price at which the trader either buys or sells each unit of the asset will be denoted as $\hat{S}^\nu = (\hat{S}^\nu_t)_{t\in[0,T]}$. Because there is limited liquidity at the best bid or ask price (the touch), the investor must "walk the book" starting at the bid (ask) and buy (sell) her assets at higher (lower) prices as she increases the size of each of her MOs. For tractability, and as Frei and Westray (2015) (among others) note, a linear model for this "temporary price impact" fits the data well, and adding in concavity, while empirically more accurate, does not improve the R^2 beyond 5%. Hence, here we adopt a linear temporary price impact model and write the execution price as

$$\hat{S}_t = S_t + a\,\nu_t, \tag{5}$$

where $a > 0$ controls the asset's liquidity, and hence the impact of trades.

The investor's cash process, that is, the accumulated funds from trading for some fixed strategy ν, is denoted $X^\nu = (X^\nu_t)_{t\in[0,T]}$, and is given by

$$X^\nu_t = X_0 - \int_0^t \nu_u\,\hat{S}^\nu_u\,du. \tag{6}$$

2.3 | Objective criterion

Over the course of the trading window $t \in [0,T]$, the trader wishes to find a trading strategy $\nu \in \mathcal{A}$ that maximizes the objective criterion

$$\mathbb{E}\left[X^\nu_T + Q^\nu_T\left(S^\nu_T - \alpha\,Q^\nu_T\right) - \phi\int_0^T \left(Q^\nu_u\right)^2 du\right], \tag{7}$$

where $\mathcal{A}$ is the set of admissible trading strategies, here consisting of the collection of all $\mathcal{F}$-predictable processes such that $\mathbb{E}[\int_0^T \nu_u^2\,du] < +\infty$.

The objective criterion (7) consists of three different parts. The first is X^ν_T, which represents the amount of cash the trader has accumulated from her trading over the period $[0,T)$. Next is the amount of cash received from liquidating all remaining exposure Q^ν_T at the end of the trading horizon. The value (per share) of liquidating these shares is penalized by an amount $\alpha\,Q^\nu_T$, where $\alpha \geq 0$. The amount $\alpha\,Q^\nu_T$ represents the liquidity penalty taken by the trader if she chooses to sell or buy an amount of assets Q^ν_T all at once. We eventually take the limit $\alpha \to \infty$ to ensure that the trader ends with zero inventory. The last term $-\phi\int_0^T (Q^\nu_u)^2\,du$ represents a running penalty that penalizes the trader for having a nonzero inventory throughout the trading horizon, and allows her to control her exposure. This penalty can also be interpreted as the quadratic variation of the book value of the traders position (ignoring jumps in the asset price), or can be seen as stemming from model uncertainty as shown in Cartea, Donnelly, and Jaimungal (2017).

Note that we take trading strategies to be $\mathcal{F}$-predictable. $\mathcal{F}$-predictability ensures that the trader does not have access to any information regarding the path of the process Θ_t, which governs the model driving the asset midprice drift and the intensities of $\boldsymbol{N}_t$. As well, $\mathcal{F}$-predictability prevents the trader for foreseeing a jump occurring at the same instant in time—in other words her decisions are based on the left limits of F, and hence also $\boldsymbol{N}$. Because admissible controls are $\mathcal{F}$-predictable, and not $\mathcal{G}$-predictable (the full filtration), maximizing (7) is a control problem with partial information.

Solving control problems with partial information is very difficult to do directly, because most tools that are used to work with the case of complete information no longer work. The former requires an indirect approach in which, first, we find an alternate $\mathcal{F}$-adapted representation for the dynamics of the state variable process, and second, we extend the state variable process, so that it becomes Markov when using Markov controls. The key step in this approach is to find the best guess for Θ_t conditional on the reduced filtration available at that time.

3 | FILTERING

Because the investor cannot observe Θ_t, she wishes to formulate a best guess for its value. The best possible guess for the distribution of Θ_t will be the distribution of Θ_t conditional on the information accumulated up until that time. Therefore, she wishes to compute

$$\pi_t^j = \mathbb{E}\left[\mathbb{1}_{\{\Theta_t=\theta_j\}} \,\middle|\, \mathcal{F}_t \right], \quad \forall j \in \mathfrak{J}.$$

The filter process $\boldsymbol{\pi} = (\{\pi_t^j\}_{j\in\mathfrak{J}})_{t\in[0,T]}$ is $\mathcal{F}$–adapted with initial condition $\boldsymbol{\pi}^0 = \{\pi_0\}_{j\in\mathfrak{J}}$. It represents the posterior latent state distribution (given all information accumulated by the investor up until t).

Theorem 3.1. *Let us assume that the Novikov condition*

$$\mathbb{E}\left[\exp\left\{\int_0^T \left(A_u\right)^2 + \left(\lambda_u^+\right)^2 + \left(\lambda_u^-\right)^2 \, du\right\}\right] < \infty \tag{8}$$

holds. Then the filter $\boldsymbol{\pi}$ admits a representation with components

$$\pi_t^i = \Lambda_t^i / \sum_{j=1}^J \Lambda_t^j., \tag{9}$$

where $\boldsymbol{\Lambda} = (\{\Lambda_t^j\}_{j\in\mathfrak{J}})_{t\in[0,T]}$. If $\sigma > 0$, for each $i \in \mathfrak{J}$, Λ_t^i solves the SDE

$$\begin{aligned}\frac{d\Lambda_t^i}{\Lambda_{t-}^i} &= \sigma^{-2} A_{t-}^i \left(dF_t - b\left(dN_t^+ - dN_t^-\right)\right) \\ &\quad + \left(\lambda_{t-}^{+,i} - 1\right)\left(dN_t^+ - dt\right) + \left(\lambda_{t-}^{-,i} - 1\right)\left(dN_t^- - dt\right) + \sum_{j\in\mathfrak{J}} \left(\frac{\Lambda_{t-}^j}{\Lambda_{t-}^i}\right) C_{i,j}\, dt\end{aligned} \tag{10}$$

with initial condition $\boldsymbol{\Lambda}_0 = \boldsymbol{\pi}_0$. If $\sigma = 0$ and $A_t := 0$, for each $i \in \mathfrak{J}$, Λ_t^i solves the SDE

$$\frac{d\Lambda_t^i}{\Lambda_{t-}^i} = \left(\lambda_{t-}^{+,i} - 1\right)\left(dN_t^+ - dt\right) + \left(\lambda_{t-}^{-,i} - 1\right)\left(dN_t^- - dt\right) + \sum_{j\in\mathfrak{J}} \left(\frac{\Lambda_{t-}^j}{\Lambda_{t-}^i}\right) C_{i,j}\, dt, \tag{11}$$

with the same initial condition.

Proof. See Appendix A. ∎

The process $\boldsymbol{\Lambda}$ admits a simple closed form solution when $\boldsymbol{C} = \boldsymbol{0}$. This case corresponds to when the latent regimes are constant over the trading period $[0, T]$ —in others words, the case of parameter uncertainty, but the model does not switch between regimes throughout the trading horizon. When $\boldsymbol{C} \neq \boldsymbol{0}$, solutions to the filter can be approximated reasonably well for most purposes by using methods outlined in George, Zhang, and Liu (2004), which will be discussed further in Section 6.

An SDE also exists for the normalized version of the filter $\boldsymbol{\pi}_t$, however, for simplicity, we keep track of the processes $\boldsymbol{\Lambda}$, and define the function (with a slight abuse of notation) $\pi^j : \mathbb{R}_+^J \mapsto [0, 1]$ via

$$\pi^j(\boldsymbol{\Upsilon}) = \Upsilon^j / \sum_{i=1}^{J} \Upsilon^i, \qquad \forall \boldsymbol{\Upsilon} \in \mathbb{R}_+^J, \tag{12}$$

so that $\pi_t^j = \pi^j(\boldsymbol{\Lambda}_t)$. This choice of mapping $\boldsymbol{\Lambda}$ into $\boldsymbol{\pi}$ guarantees that $\sum_{j=1}^J \pi_t^j = 1$, even when numerically approximating (10).

4 | $\mathcal{F}$-DYNAMICS PROJECTION

In this section, we show there exists an $\mathcal{F}$-adapted representation for the price dynamics, and the intensity processes. The sequence of arguments resemble those found in (Bäuerle and Rieder, 2007, section 3) adapted to the case where the observable process contains both jump and diffusive terms.

First, define the $\mathcal{G}$-adapted martingales $\boldsymbol{M} = (M_t^+, M_t^-)_{t \in [0,T]}$ to be the compensated versions of the Poisson processes $\boldsymbol{N}$, that is,

$$M_t^{\pm} = N_t^{\pm} - \int_0^t \lambda_u^{\pm} \, du. \tag{13}$$

The theorem below provides the necessary ingredients to provide the $\mathcal{F}$-adapted representations of the state processes.

Theorem 4.1. *If $\sigma > 0$, define the processes $\widehat{W} = (\widehat{W}_t)_{t \in [0,T]}$, $\widehat{\boldsymbol{M}} = (\widehat{M}_t^+, \widehat{M}_t^-)_{t \in [0,T]}$ by the following relations*

$$\widehat{W}_t = W_t + \sigma^{-1} \int_0^t \left(A_u - \widehat{A}_u \right) du, \tag{14a}$$

$$\widehat{M}_t^{\pm} = M_t^{\pm} + \int_0^t \left(\lambda_u^{\pm} - \widehat{\lambda}_t^{\pm} \right) du, \tag{14b}$$

where $\widehat{A} = (\widehat{A}_t)_{t \in [0,T]}$ and $\widehat{\lambda}^{\pm} = (\widehat{\lambda}_t^{\pm})_{t \in [0,T]}$ are the filtered drift and intensities, defined as $\widehat{A}_t := \mathbb{E}[A_t | \mathcal{F}_t]$ and $\widehat{\lambda}_t^{\pm} := \mathbb{E}[\lambda_t^{\pm} | \mathcal{F}_t]$. Then,

(A) *the process $\widehat{W}$ is an $\mathcal{F}$-adapted $\mathbb{P}$-Brownian motion;*

(B) *the process $\widehat{\boldsymbol{M}}$ is an $\mathcal{F}$-adapted $\mathbb{P}$-martingale;*

(C) $[\widehat{W}, \widehat{M}^{\pm}]_t = 0$ *and* $[\widehat{M}^+, \widehat{M}^-]_t = 0$, *$\mathbb{P}$-almost surely; and*

(D) *$N^{\pm}$ are $\mathcal{F}$-adapted doubly stochastic Poisson processes with $\mathbb{P}$-intensities $\widehat{\lambda}^{\pm}$*

If $\sigma = 0$ and $A := 0$, define $\widehat{\boldsymbol{M}} = (\widehat{M}_t^+, \widehat{M}_t^-)_{t \in [0,T]}$ as in (14b). Then, (B) and (D) hold and $[\widehat{M}^+, \widehat{M}^-]_t = 0$, $\mathbb{P}$-almost surely.

Proof. See Appendix B. □

Theorem 4.1 tells us that $N^{\pm}$, in addition to being viewed as a $\mathcal{G}$-adapted doubly stochastic Poisson process with $\mathbb{P}$-intensity of $\lambda^{\pm}$, can be viewed as an $\mathcal{F}$-adapted doubly stochastic process with $\mathbb{P}$-intensity $\hat{\lambda}^{\pm}$. That is, $N^{\pm}$ is a doubly stochastic Poisson process with respect to both the $\mathcal{F}$ and $\mathcal{G}$ filtrations, but with differing intensities.

Theorem 4.1 allows us to represent the dynamics of F in their $\mathcal{F}$-predictable form as

$$dF_t = \left(\hat{A}_t + b\left(\hat{\lambda}_t^+ - \hat{\lambda}_t^-\right)\right) dt + b\left(d\widehat{M}_t^+ - d\widehat{M}_t^-\right) + \sigma\, d\widehat{W}_t. \tag{15}$$

Let us also note that because $A_t = \sum_{j\in\mathfrak{J}} \mathbb{1}_{\{\Theta_t=\theta_j\}} A_t^j$ and $\lambda_t^{\pm} = \sum_{j\in\mathfrak{J}} \mathbb{1}_{\{\Theta_t=\theta_j\}} \lambda_t^{j,\pm}$, because $\{A^j : j \in \mathfrak{J}\}$ are $\mathcal{F}$-adapted, we may take a conditional expectation with respect to $\mathcal{F}_t$ to yield that $\hat{A}_t = \sum_{j\in\mathfrak{J}} \pi_t^j A_t^j$ and $\hat{\lambda}_t^{\pm} = \sum_{j\in\mathfrak{J}} \pi_t^j \lambda_t^{j,\pm}$. Therefore, we may define the functions, $\hat{A} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{Z}_+^2 \times \mathbb{R}_+^J \mapsto \mathbb{R}$ and $\hat{\lambda}^{\pm} : \mathbb{R}_+^{2J} \times \mathbb{R}_+^J \mapsto \mathbb{R}_+$ as

$$\hat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) := \sum_{j\in\mathfrak{J}} \pi^j(\boldsymbol{\Lambda})\, A(t, F, \boldsymbol{N}, \theta_j) \quad \text{and} \quad \hat{\lambda}^{\pm}(\lambda, \boldsymbol{\Lambda}) := \sum_{j\in\mathfrak{J}} \pi^j(\boldsymbol{\Lambda})\, \lambda^{\pm,j}, \tag{16}$$

so that $\hat{A}_t = \hat{A}(t, F_t, \boldsymbol{N}_t, \boldsymbol{\Lambda}_t)$ and $\hat{\lambda}_t^{\pm} = \hat{\lambda}^{\pm}(\lambda_t, \boldsymbol{\Lambda}_t)$.

Hence, the collection of processes $(F, \boldsymbol{N}, \lambda, \boldsymbol{\Lambda})$ are $\mathcal{F}$-adapted. The optimal control problem corresponding to maximizing (7), within the admissible set, can therefore be regarded as a problem with complete information with respect to the extended state variable process $(S^\nu, F, \boldsymbol{N}, X^\nu, Q^\nu, \lambda, \boldsymbol{\Lambda})$. The joint dynamics of this state process are all $\mathcal{F}$-adapted and do not depend on the process Θ. Therefore, the dynamics of the extended state process are completely visible to the investor, which reduces the control problem with partial information, in which we did not know the dynamics of the state variables, into a control problem with full information.

In the next section, we solve this control problem by using the fact that the extended state variable dynamics are $\mathcal{F}$-adapted for each $\nu \in \mathcal{A}$. Hence, the DPP can be applied to the optimization problem (7) and we derive a DPE for the new problem.

5 | SOLVING THE DYNAMIC PROGRAMMING PROBLEM

5.1 | The dynamic programming equation

Using the definitions for S_t^ν and Q_t^ν in (2) and (1), we can write S_t^ν as

$$S_t^\nu = F_t + \beta\left(Q_t^\nu - \mathfrak{N}\right), \tag{17}$$

as well we can write

$$dX_t^\nu = -\nu_t\left(F_t + \beta\left(Q_t^\nu - \mathfrak{N}\right) - a\,\nu_t\right) dt, \tag{18}$$

which allows X^ν to be defined independently of S^ν. Hence, the trader's objective criterion (7) becomes

$$\mathbb{E}\left[X_T^\nu + Q_T^\nu\left(F_T + \beta\left(Q_T^\nu - \mathfrak{N}\right) - \alpha\, Q_T^\nu\right) - \phi \int_0^T \left(Q_u^\nu\right)^2 du\right]. \tag{19}$$

With X given by (18), the trader's objective function does not depend on the value of the process S^ν. For the remainder of this section, we will use the above definition for the trader's objective criterion.

To optimize the objective criterion (19), we use the fact that $\forall\, \nu \in \mathcal{A}$, the $(3\boldsymbol{J}+5)$-dimensional state variable process $\boldsymbol{Z}^\nu = (F, \boldsymbol{N}, X^\nu, Q^\nu, \lambda, \boldsymbol{\Lambda})$ is $\mathcal{F}$-adapted and, hence, has dynamics visible to the trader. First, let us define the functional

$$H^\nu(t, \boldsymbol{Z}) = \mathop{\mathbb{E}}_{t,\boldsymbol{Z}} \left[X_T^\nu + Q_T^\nu \left(F_T + \beta \left(Q_T^\nu - \mathfrak{N} \right) - \alpha Q_T^\nu \right) - \phi \int_t^T \left(Q_u^\nu \right)^2 du \right], \tag{20}$$

and the value function

$$H(t, \boldsymbol{Z}) = \sup_{\nu \in \mathcal{A}} H^\nu(t, \boldsymbol{Z}), \tag{21}$$

where we use $\mathop{\mathbb{E}}_{t,\boldsymbol{Z}}[\,\cdot\,]$ to represent the expected value given the initial conditions $\boldsymbol{Z}_{t-}^\nu = \boldsymbol{Z} = (F, \boldsymbol{N}, X, Q, \lambda, \boldsymbol{\Lambda}) \in \mathcal{D}$, where $\mathcal{D} = \mathbb{R} \times \mathbb{Z}_+^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^{2J} \times R_+^J$. The definition of H^ν implies that $H^\nu(0, \boldsymbol{Z}_0)$, where $\boldsymbol{Z}_0 = (F, \boldsymbol{0}, X, \mathfrak{N}, \lambda, \boldsymbol{\pi}_0)$, is the objective criterion defined in Equation (19). Furthermore, a control $\nu^\star \in \mathcal{A}$ is optimal and solves the optimization problem described in Section 2.3 if it satisfies

$$H^{\nu^\star}(0, \boldsymbol{Z}_0) = H(0, \boldsymbol{Z}_0). \tag{22}$$

Given the $\mathcal{F}$-adapted version of the dynamics of the state variables, for any Markov admissible control $\nu \in \mathcal{A}$, there exists some function $g : \mathbb{R}_+ \times \mathcal{D}$, such that $\nu_t = g(t, \boldsymbol{Z}_t^\nu)$. For such controls, the function H must satisfy the DPP and the DPE (see, e.g., Pham, 2009, Chapter 3) applies. The DPE for our specific problem suggests that H satisfies the Partial Differential Equation (PDE)

$$\begin{cases} -\phi\, q^2 + \sup_{\nu \in \mathbb{R}} \left\{ (\partial_t + \mathcal{L}^\nu)\, H(t, \boldsymbol{Z}) \right\} = 0, \\ H(T, \boldsymbol{Z}) = X + Q\,(F + \beta\,(Q - \mathfrak{N}) - \alpha\, Q), \end{cases} \tag{23}$$

where $\mathcal{L}^\nu$ is the infinitesimal generator for the state process $\boldsymbol{Z}^\nu$ using the predictable representation for the dynamics of F and the intensity of $\boldsymbol{N}$, given a fixed control ν. Furthermore, the operator $\mathcal{L}^\nu$ acts on functions $f : \mathbb{R}_+ \times \mathcal{D} \mapsto \mathbb{R}$, once differentiable in t, twice differentiable in $F, \lambda, \boldsymbol{\Lambda}$ and all (componentwise) cross-derivatives, and once differentiable in X, Q, as follows

$$\mathcal{L}^\nu f = \nu\, \partial_Q f - \nu\,(F + \beta\,(Q - \mathfrak{N}) + a\,\nu)\, \partial_X f + \bar{\mathcal{L}} f,$$

where $\bar{\mathcal{L}}$ is the infinitesimal generator of the process $(F, \boldsymbol{N}, \lambda, \boldsymbol{\Lambda})$ using its $\mathcal{F}$-predictable representation, which is independent of the control ν. This portion of the generator can be fairly generic because we have not specified the precise nature of the dynamics of the intensity processes—which is the impetus for separating this portion of the generator.

5.2 | Dimensional reduction

The DPE (23) can be simplified by introducing the ansatz

$$H(t, \boldsymbol{Z}) = X + Q\ (F + \beta(Q - \mathfrak{N})) + h(t, \boldsymbol{\ell}(\boldsymbol{Z})),$$

where for $\boldsymbol{Z} = (F, \boldsymbol{N}, X, Q, \lambda, \boldsymbol{\Lambda}) \in \mathcal{D}$, we write $\boldsymbol{\ell}(\boldsymbol{Z}) = (F, \boldsymbol{N}, Q, \lambda, \boldsymbol{\Lambda}) \in \mathbb{R} \times \mathbb{Z}_+^2 \times \mathbb{R} \times \mathbb{R}_+^{2J} \times \mathbb{R}^J$. The PDE (23) then simplifies significantly to a PDE for h,

$$
\begin{cases}
0 = -\phi\, Q^2 + \left(\partial_t + \bar{\mathcal{L}}\right) h(t, \boldsymbol{\ell}) + Q\, \left(\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b \left(\widehat{\lambda}^+(\lambda, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\lambda, \boldsymbol{\Lambda})\right)\right) \\
\quad + \sup\limits_{\nu \in \mathbb{R}} \left\{ (\beta\, Q + \partial_Q h)\, \nu - a\, \nu^2 \right\} \\
h(T, \boldsymbol{\ell}) = -\alpha\, Q^2,
\end{cases}
\tag{24}
$$

where the functions $\widehat{A}$ and $\widehat{\lambda}^{\pm}$ are defined in Equation (16). This PDE implies that the feedback control for this problem should be

$$
\nu^\star(t, \boldsymbol{Z}) = \tfrac{1}{2a} \left(\beta\, Q + \partial_Q h(t, \boldsymbol{\ell}(\boldsymbol{Z})) \right). \tag{25}
$$

In other words, the second line of the PDE (24) attains its supremum at $\nu^\star$ defined above.

5.3 | Solving the DPE

The ansatz provided above permits us to indeed find a solution to the PDE (23), which is presented in the following proposition:

Proposition 5.1 (Candidate Solution). *The PDE (23), admits the classical solution H*

$$
H(t, \boldsymbol{Z}) = X + Q\, \left(F + \beta\, (Q - \mathfrak{N})\right) + h_0(t, \chi(\boldsymbol{Z})) + Q\, h_1(t, \chi(\boldsymbol{Z})) + Q^2\, h_2(t),
$$

where $\chi(\boldsymbol{Z}) = (F, \boldsymbol{N}, \lambda, \boldsymbol{\Lambda})$. Let, $\mathbb{E}_{t,\chi}[\,\cdot\,]$ denote expectation conditional on the initial conditions $(F_{t^-}, \boldsymbol{N}_{t^-}, \lambda_{t^-}, \boldsymbol{\Lambda}_{t^-}) = \chi$, and define the constants $\gamma = \sqrt{\phi/a}$ and $\zeta = \frac{\alpha - \frac{1}{2}\beta + a\gamma}{\alpha - \frac{1}{2}\beta - a\gamma}$. We have that

(i) *If $\alpha - \frac{1}{2}\beta \neq \sqrt{a\phi}$, then*

$$
h_2(t) = -a\,\gamma \left(\frac{\zeta\, e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}} \right) + \tfrac{1}{2}\beta, \tag{26a}
$$

$$
h_1(t, \chi) = \int_t^T \mathbb{E}_{t,\chi} \left[\widehat{A}_u + b\, \left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^- \right) \right] \left(\frac{\zeta\, e^{\gamma(T-u)} - e^{-\gamma(T-u)}}{\zeta\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}} \right) du, \tag{26b}
$$

$$
h_0(t, \chi) = \tfrac{1}{4a} \mathbb{E}_{t,\chi} \left[\int_t^T \left(h_1(u, \chi_u) \right)^2 du \right]. \tag{26c}
$$

(ii) *If $\alpha - \frac{1}{2}\beta = \sqrt{a\phi}$, then*

$$
h_2(t) = -a\,\gamma + \tfrac{1}{2}\beta, \tag{27a}
$$

$$
h_1(t, \chi) = \int_t^T \mathbb{E}_{t,\chi} \left[\widehat{A}_u + b\, (\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-) \right] e^{-\gamma(u-t)} du, \tag{27b}
$$

$$
h_0(t, \chi) = \tfrac{1}{4a} \mathbb{E}_{t,\chi} \left[\int_t^T \left(h_1(u, \chi_u) \right)^2 du \right], \tag{27c}
$$

where $\chi_t = \chi(\boldsymbol{Z}_t)$.

Proof. See Appendix C.1. □

For the remainder of the paper, we will concern ourselves with the case where $\alpha - \frac{1}{2}\beta > \sqrt{a\phi}$, because in most applications the trader wishes to completely liquidate by the end of the trading horizon, and so $\alpha \gg 1$, while $\sqrt{a\phi}$ is comparatively small.

The above proposition and Equation (25) suggest that the optimal trading speed the investor should employ is

$$\nu_t^\star = \frac{1}{2a}\left(2\,h_2(t) + \beta\right) Q_t^{\nu^\star} + \frac{1}{2a}\,h_1(t, \chi_t). \tag{28}$$

This optimal trading strategy is a combination of two terms (a) the classical Almgren–Chriss (AC) liquidation strategy represented by $\frac{1}{2a}(2h_2(t) + \beta)\,Q_t^{\nu^\star}$; and (b) a term that adjusts the strategy based on expected future midprice movements, represented by $\frac{1}{2a}\,h_1(t, \chi_t)$. From the representation of h_1 in (26b) (or (27b)), this latter term is the weighted average of the expected future drift of the asset's midprice. Therefore if, based on her current information, the trader believes that the asset midprice drift will remain largely positive for the remainder of the trading period, she will buy more of the asset relative to the AC strategy. This is reasonable, because she knows she will be able to sell the asset at a higher price once asset prices have risen. The exact opposite occurs when she expects the asset price drift to remain mostly negative over the rest of the trading period.

The result in (28) illustrates how the investor uses the filter $\boldsymbol{\pi}_t$ for the posterior probability of what latent state is currently prevailing, to consistently update her strategy based on her predictions of the future path of the asset midprice. Moreover, the solution here closely resembles the result obtained by Cartea and Jaimungal (2016), however, it explicitly incorporates latent information and jumps in the asset price.

Computing the expectation appearing in h_1 directly is not easy. There is, however, an alternate representation of this expectation. For any $u \geq t$, we have

$$\mathop{\mathbb{E}}_{t,\chi}\left[\widehat{A}_u + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-\right)\right] = \sum_{j \in \mathfrak{J}} \pi^j(\boldsymbol{\Lambda}) \mathop{\mathbb{E}}_{t,\chi,\theta_j}\left[A_u + b\left(\lambda_u^+ - \lambda_u^-\right)\right], \tag{29}$$

where $\mathop{\mathbb{E}}_{t,\chi,\theta_j}[\,\cdot\,]$ denotes expectation conditioning on the initial condition $(F_{t^-}, \boldsymbol{N}_{t^-}, \lambda_{t^-}, \boldsymbol{\Lambda}_{t^-}) = \chi$ and $\Theta_t = \theta_j$. The alternative form in the right hand side (rhs) above is almost always easier to compute than a direct computation of the left hand side (lhs).

Next, we provide a verification theorem showing that the candidate solution in Proposition 5.1 is exactly equal to the value function H defined in Equation (21).

Theorem 5.2 (Verification Theorem). *Suppose that h is the solution to the PDE (24), and that* $\alpha - \frac{1}{2}\beta \neq a\gamma$. *Let* $\widehat{H}(t, \boldsymbol{Z}) = X + Q\,(F + \beta(Q - \mathfrak{N})) + h(t, \boldsymbol{\ell}(\boldsymbol{Z}))$, *where* $\boldsymbol{\ell}(\boldsymbol{Z}) = (F, \boldsymbol{N}, Q, \lambda, \boldsymbol{\Lambda})$.

Then $\widehat{H}$ *is equal to the value function* H *defined in (21). Furthermore, the control*

$$\nu_t^\star = \frac{1}{2a}\left(2\,h_2(t) + \beta\right) Q_t^{\nu^\star} + \frac{1}{2a}\,h_1(t, \boldsymbol{\ell}(\boldsymbol{Z}_t)) \tag{30}$$

is optimal and satisfies

$$H(t, \boldsymbol{Z}) = H^{\nu^\star}(t, \boldsymbol{Z}). \tag{31}$$

Proof. See Appendix C.2. □

The theorem above guarantees that the control provided above indeed solves the optimization problem presented in Section 2.3. In retrospect, the optimal control to the trader's optimization problem with partial information is a Markov control. The key steps were to introduce the predictable representation for the dynamics of the process F, and to extend the original state process to include the unnormalized posterior distribution $\boldsymbol{\Lambda}$ of the latent states Θ.

5.4 | Zero terminal inventory

A useful limiting case is when the trader is forced to eliminate her market exposure before time T. This corresponds to taking the limit $\alpha \to \infty$ and the resulting optimal control simplifies to

$$\lim_{\alpha\to\infty} \nu_t^\star = -\gamma \coth(\gamma(T-t))\, Q_t^{\nu^\star} + \frac{1}{2a}\sum_{j\in\mathfrak{J}} \pi^j(\boldsymbol{\Lambda}_t)\int_t^T \mathop{\mathbb{E}}_{t,\chi_t,\theta_j}\left[A_u + b\left(\lambda_u^+ - \lambda_u^-\right)\right]\left(\frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))}\right) du. \tag{32}$$

A second interesting case is to additionally take the limit of no running inventory penalty, in which case the optimal strategy results in

$$\lim_{\phi\to 0}\lim_{\alpha\to\infty} \nu_t^\star = -\frac{1}{T-t}\, Q_t^{\nu^\star} + \frac{1}{2a}\sum_{j=1}^{J} \pi^j(\boldsymbol{\Lambda}_t)\int_t^T \mathop{\mathbb{E}}_{t,\chi_t,\theta_j}\left[A_u + b\left(\lambda_u^+ - \lambda_u^-\right)\right]\left(\frac{T-u}{T-t}\right) du. \tag{33}$$

This strategy corresponds to a time-weighted average price strategy plus an adjustment for the weighted expected future drift of the asset's midprice.

All of the expressions above for the optimal control can be computed in closed form for a large variety of models. In the next section, we provide two explicit, and useful, examples together with numerical experiments to illustrate the strategies dynamic behavior.

6 | NUMERICAL EXAMPLES

In this section, we will carry out some numerical experiments to test the performance of the optimal trading algorithms developed in Section 5. The examples show how the optimal trading performs using situations for two model setups.

6.1 | Mean-reverting diffusion

This section investigates the case where the trader wishes to liquidate her inventory before some specified time T. The asset price is assumed to be a pure diffusive Ornstein–Uhlenbeck process—alternatively, one can think of this midprice as the number of long-short position in a pairs trading strategy. The trader knows the volatility and rate of mean reversion, but does not know the level at which prices revert to. In this example, the mean-reversion level will remain constant over the course of the trading period $[0, T]$. More specifically, we assume that the asset midprice in USD has the dynamics

$$dF_t = \kappa(\Theta - F_t)\,dt + \sigma\, dW_t, \tag{34}$$

TABLE 1 The parameters in the OU model. All of the time-sensitive parameters are defined on an hourly scale

$\mathfrak{N} = 10^4$,	$F_0 = \$5$,	$\sigma = 0.15$,	$\beta = \$10^{-3}$,
$\kappa = 2$,	$a = \$10^{-5}$,	$\phi = 2 \times 10^{-5}$	

where Θ is a random variable taking values in the set $\{\theta_j\}_{j\in\mathfrak{J}}$ with probabilities $\{\pi_0^j\}_{j\in\mathfrak{J}}$. It remains constant over time but its value is hidden from the trader. This model does not contain any jumps so we can ignore the variables $\boldsymbol{N}$ and λ.

As mentioned in Section 3, there exists an exact closed form for the filter when Θ_t is constant in time. For the regime switching Ornstein-Uhlenbeck (OU) model in (34), the exact solution for the unnormalized filter is

$$\Lambda_t^j = \pi_0^j \exp\left\{\sigma^{-2}\left(\int_0^t \kappa(\theta_j - F_u)\, dF_u - \tfrac{1}{2}v\int_0^t \kappa^2(\theta_j - F_u)^2\, du\right)\right\}, \quad \forall j \in \mathfrak{J}. \tag{35}$$

Because, in practice, F is observed only discretely, the integrals above are approximated using the appropriate Riemann sums. The more frequently the trader observes F, the more accurate the filter will be.

The solution to the optimal control when $\alpha \to \infty$ can be computed exactly as

$$\begin{aligned} v^\star(t, F, Q, \boldsymbol{\Lambda}) &= -\gamma \coth(\gamma(T-t))\, Q \\ &\quad + \sum_{j=1}^{J} \pi^j(\boldsymbol{\Lambda})\, \kappa\,(F - \theta_j) \int_t^T e^{-\kappa(u-t)} \frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))} du. \end{aligned}$$

For the simulations, here, we assume there are two possible values the asset price mean-reverts to, so that $J = 2$ and we set $\theta_1 = \$4.85$ and $\theta_2 = \$5.15$. Furthermore, we assume the investor has an equal prior on the two possibilities, so that $\pi_0^1 = \pi_0^2 = 0.5$. The remaining parameters used in the simulation are provided in Table 1.

When simulating sample paths, we generate paths using $\Theta = \$5.15$. The trader will need to detect this value as she observes the price path.

Figure 1 shows the results of the simulation. The top right panel contains a heat map of the posterior probability of the two models. It shows, as time advances the trader on average will detect that the true rate of mean reversion is $\theta_2 = \$5.15$. Moreover, by the end of the trading period, she is on average at least 97% confident that model 2 is the true model governing asset prices. The top left graph in Figure 1 shows a heat-map of the trading speed for the investor, where the dashed line represents the classical AC strategy. The dotted-and-dashed line represents the median of the traders's strategy. The heat-map shows how the trader adjusts her positions in a manner consistent with her predictions: as the investor discovers that $\Theta = \$5.15$, she expects the asset price to rise over time. Because of this she slows down her rate of liquidation initially, so that she can sell her asset at a higher price toward the end of the trading period. She then must speed up trading toward the end in order to unwind her position. The bottom left panel shows the histogram of the excess return per share of the optimal control over the AC control, where the excess return is defined as $\frac{X_T^{v^\star} - X_T^{v^{AC}}}{X_T^{v^{AC}}} \times 10^4$ and $X_T^{v^{AC}}$ is the total cash the trader earns using the AC liquidation strategy. As the histogram shows, the filtered strategy outperforms the AC strategy during at least 73% of the simulations. Finally, the bottom right panel shows the trader's liquidation value per share over the trading period.

Figure 2 displays sample paths of the asset price and the filter. The top left plot demonstrates how the trader quickly detects the correct model based on the asset trajectory. In this simulation, the asset price

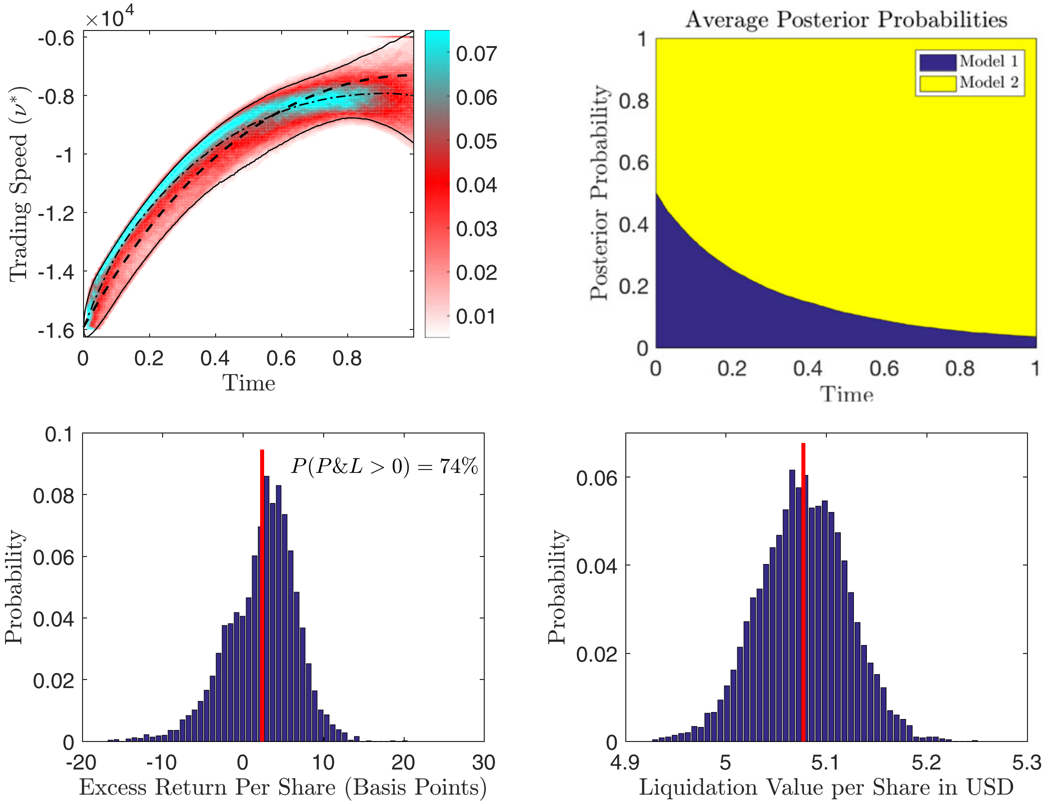

FIGURE 1 Simulation results with an Ornstein–Uhlenbeck process [Color figure can be viewed at wileyonlinelibrary.com]

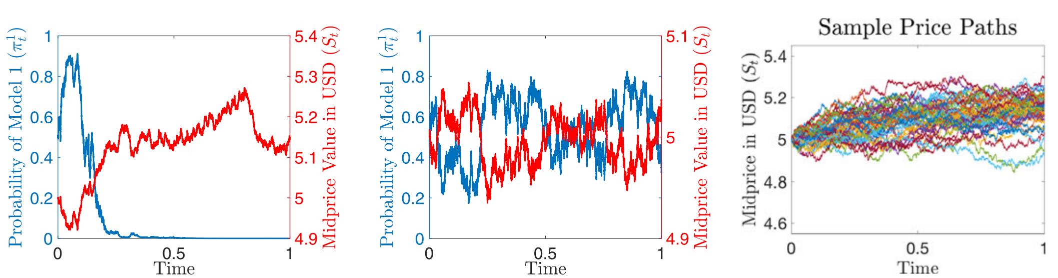

FIGURE 2 Sample simulation paths with an Ornstein–Uhlenbeck process [Color figure can be viewed at wileyonlinelibrary.com]

initially drops, but then increases consistently. The posterior probability of model 1 adjusts accordingly, and initially rises, but then quickly drops and remains low. In the simulated path in the middle panel, the path of the midprice fluctuates around \$5 over the entire time period. The trader's estimate for the posteriori probability of model 1 varies according to the price movements she is observing. The resulting strategy induced by the filter fluctuating is an advantage to the trader, because the fluctuating filter more accurately reflects the actual behavior of the asset price path—as opposed to being certain that the true model is model 2. Finally, the bottom panel of the figure shows a collection of 40 sample midprice path trajectories.

TABLE 2 The parameters in the pure jump mean-reverting model. All of the time-sensitive parameters are defined on an hourly scale

$\mathfrak{N} = 0,$	$F_0 = \$5,$	$b = \$0.01$	$\phi = 3 \times 10^{-6},$	$\boldsymbol{C} = \begin{bmatrix} -10 & 10 \\ 10 & -10 \end{bmatrix},$
$\kappa = 1077,$	$\mu = 481,$	$a = \$10^{-5},$	$\beta = \$10^{-3}$	

6.2 | Mean-reverting pure jump process

In this section, we investigate the case where the trader begins with no inventory and aims to gain profits from her alpha model through the use of a round-trip trading strategy. The asset price is assumed to be completely driven by the MO flow, so that there is no diffusion or drift in the unaffected midprice. We assume the asset price mean-reverts to some unknown level Θ_t, which the trader must detect. More specifically, the asset midprice in USD satisfies the SDE

$$dF_t = b\left(dN_t^+ - dN_t^-\right), \tag{36}$$

where N_t^+ and N_t^- are doubly stochastic Poisson processes with intensities λ_t^+ and λ_t^- defined by

$$\lambda_t^+ = \mu + \kappa\left(\Theta_t - F_t\right)_+ \quad \text{and} \quad \lambda_t^- = \mu + \kappa\left(\Theta_t - F_t\right)_-, \tag{37}$$

where $(x)_+$ and $(x)_-$ denote the positive and negative parts of x, respectively.

We assume Θ_t is a Markov chain with generator matrix $\boldsymbol{C}$, specified in Table 2. The filter for Θ_t cannot be computed explicitly, but it may be approximated via a Euler–Maruyama scheme of the SDE for the logarithm of the filter (see the SDE in Theorem 3.1). The resulting approximation for the value of the filter, given that the values of $\boldsymbol{N}$ have been observed at times $\{t_k\}_{k=1}^K$, where $t_0 = 0$ and $t_K = T$ is obtained via the recursive formula

$$\Lambda_{t_0}^j = \pi_0^j, \tag{38a}$$

and

$$\Lambda_{t_{k+1}}^j = \Lambda_{t_k}^j \exp\left\{2\left(1 - \mu - \tfrac{\kappa}{2}\left|\theta_j - F_{t_k}\right|\right)\Delta_{k+1} + \sum_{i=1}^{J}\left(\frac{\Lambda_{t_k}^i}{\Lambda_{t_k}^j}\right)C_{j,i}\,\Delta_{k+1}\right\}$$
$$\times(\mu + \kappa(\theta_j - F_{t_k})_+)^{\Delta N_{t_{k+1}}^+} \times \left(\mu + \kappa\left(\theta_j - F_{t_k}\right)_-\right)^{\Delta N_{t_{k+1}}^-}, \tag{38b}$$

$\forall k \geq 1$, where $\Delta N_{t_k}^{\pm} = N_{t_k}^{\pm} - N_{t_{k-1}}^{\pm}$ and $\Delta_k = t_k = t_{k-1}$. Alternatively, the trader's filter for Θ may be found using the forward equations, which are discussed in more detail in Appendix D as well as in Section 7.1. The forward equation approach is recommended if the time steps Δ_k are relatively large, which introduces inaccuracies into the above Euler–Maruyama approximation to the solution of the filtering SDE of Theorem 3.1.

We assume the trader has a 1-hr trading horizon, and completely unwinds her positions by the end of the trading period. The optimal control in this setup can be found in closed form. Let us define the constant $\kappa^\star = b\,\kappa$ and the $J \times J$ matrix $\boldsymbol{C}^\star = \boldsymbol{C} + \kappa^\star \boldsymbol{I}_{(J\times J)}$, where $\boldsymbol{I}_{(J\times J)}$ represents the $J \times J$ identity matrix. Let us also define the functions $\boldsymbol{\Psi}_1 : [0,T] \times \mathbb{R}^{J\times J} \to \mathbb{R}^{J\times J}$, $\boldsymbol{\Psi}_2 : [0,T] \times \mathbb{R}^{J\times J} \to$

$\mathbb{R}^{J\times J}$, $\psi_1 : [0,T]\times\mathbb{R}\to\mathbb{R}$, and $\psi_2 : [0,T]\times\mathbb{R}\to\mathbb{R}$, where

$$\begin{aligned}\boldsymbol{\Psi}_1(\tau,\,\boldsymbol{Y}) &= (e^{\tau\gamma}-e^{-\tau\gamma})^{-1}\,e^{\tau\boldsymbol{Y}}\left(\boldsymbol{\Psi}_2(\tau,\,\gamma\,\boldsymbol{I}_{(J\times J)}-\boldsymbol{Y})+\boldsymbol{\Psi}_2(\tau,\,-\gamma\,\boldsymbol{I}_{(J\times J)}-\boldsymbol{Y})\right),\\ \boldsymbol{\Psi}_2(\tau,\,\boldsymbol{Y}) &= \boldsymbol{Y}^{-1}\left(e^{\tau\boldsymbol{Y}}-\boldsymbol{I}_{(J\times J)}\right),\end{aligned}$$

and where ψ_1 and ψ_2 are the scalar versions of the functions above, defined as

$$\begin{aligned}\psi_1(\tau,\,y) &= (e^{\tau\gamma}-e^{-\tau\gamma})^{-1}\,e^{\tau y}\left(\psi_2(\tau,\,\gamma-y)+\psi_2(\tau,\,-\gamma-y)\right), \qquad \text{and}\\ \psi_2(\tau,\,y) &= y^{-1}\,(e^{\tau y}-1)\,.\end{aligned}$$

The optimal trading speed for this setup, letting $\alpha\uparrow\infty$ is

$$\begin{aligned}\nu^\star(t,F,Q,\boldsymbol{\Lambda}) = -\gamma\coth(\gamma\,(T-t))\,Q&\\ +\kappa^\star\Big[-F\,\psi_1(T-t,\,\kappa^\star)+\boldsymbol{\pi}^\intercal(\boldsymbol{\Lambda})\,\Psi_1(T-t,\,\boldsymbol{C})\,\boldsymbol{\theta}&\\ -\kappa^\star\boldsymbol{\pi}^\intercal(\boldsymbol{\Lambda})\,\boldsymbol{C}^{\star^{-1}}\left(\Psi_1(T-t,\,\boldsymbol{C}^\star)-\psi_1(T-t,\,\kappa^\star)\boldsymbol{I}_{(J\times J)}\right)\boldsymbol{\theta}\Big]&,\end{aligned}$$

where in the above, $\boldsymbol{\theta}=(\theta_j)_{j\in\mathfrak{J}}$ is a column vector containing all of the possible values that Θ_t can take.

In the numerical experiments, we assume two possible latent states for Θ_t: θ_1 = \$4.9 and θ_2 = \$5.1, and that the investor has an uninformed prior: $\pi_0^1=\pi_0^2=0.5$. The remaining parameters are provided in Table 2. Note that we assume the latent process's generator matrix is symmetric so the trader has no a priori preference for the asset's price trajectory. The parameters for μ, κ and $\boldsymbol{C}$ are all taken to be in the same range as the calibrated parameters for the two-state model fit to Intel Corporation (INTC) stock data. These calibrated parameters are found in Appendix E, and are discussed in more detail in Section 7.1. The parameters found in Appendix E are on a per-second scale and those here are on a per-hour scale, so we must multiply them by 3,600 to get those found in Table 2. For this simulation we fix a path for the latent process, so that the value of Θ_t stays at \$5.1 over the period $t\in[0,0.5]$ after which it jumps down to \$4.9 and remains there until the end of the trading period. This setup will put the trader at a disadvantage. With the generator matrix defined in Table 2, the trader expects the latent process to jump an average of 10 times during the hour, whereas the path we fix for Θ jumps only once. Figure 3 shows the performance of the trader over the course of the hour. This implies that the trader will be acting based on the assumption that the latent process is on average 10 times more active than it is in the simulation.

The top right portion of Figure 3 demonstrates that on average, the filter detects the jump from Θ_t = \$5.1 to Θ_t = \$4.9. Furthermore, the bottom right panel shows the trader made a positive profit in 100% of simulated scenarios. The left panels show heat maps of the trading speed (top left) and inventory (bottom left), respectively. Just as the strategy was constructed, the agent unwinds inventory by the end of the trading horizon. Moreover, she attains a short position initially, because she detects that Θ_t = \$5.1 in the first half of the strategy, but expects the regime to switch causing a drop in asset price.

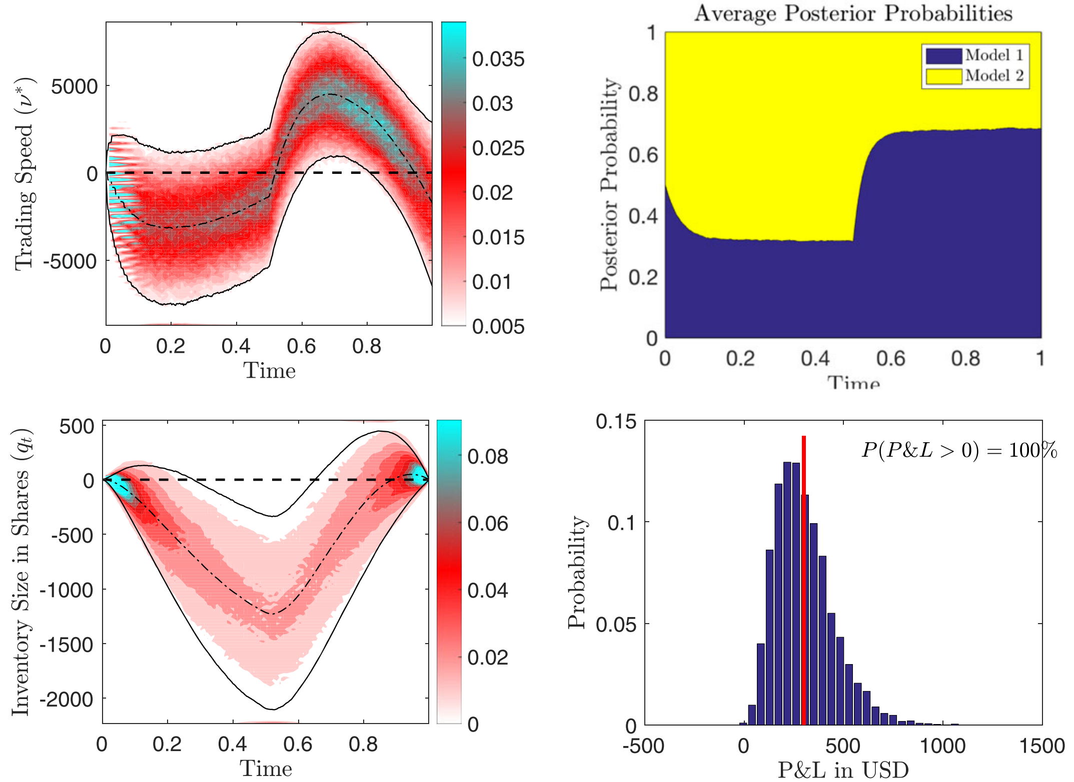

FIGURE 3 Simulation results with a pure jump mean-reverting process [Color figure can be viewed at wileyonlinelibrary.com]

7 | MODEL CALIBRATION

This section shows how to calibrate the model in Section 2 to market data using maximum likelihood estimation through the expectation–maximization (EM) algorithm. This procedure will simultaneously enable us to classify hidden states within historical data and determine what kind of dynamics governs the behavior of the midprice and order flow within each of the hidden states. The EM algorithm is based on Dempster, Laird, and Rubin (1977) and the Baum–Welch algorithm (Baum, Petrie, Soules, & Weiss, 1970). Once the general procedure is described, we apply it to calibrate a generalized version of the pure jump mean-reverting model in Section 6.2.

7.1 | The EM algorithm

This section presents the discretized version of the asset and market dynamics in Section 2 and its associated calibration algorithm. We assume we observed D independent paths (e.g., from several different days of trading within the same trading hour) of the process $(F, \boldsymbol{N}, \lambda)$ at discrete, uniformly spaced, times $\mathcal{T} = \{t_k = \Delta t \times k : k = 0, \dots, K\}$, where $\Delta t > 0$ is the time interval in between observations.

Define the discrete time processes $Y_k = (F_{t_k}, \boldsymbol{N}_{t_k}, \lambda_{t_k})$ and $Z_k = \Theta_{t_k}$ for each $t_k \in \mathcal{T}$. Given the assumptions in Section 2, $(Y, Z) = \{(Y_k, Z_k)\}_{k=0\dots K}$ is a Markov process. Furthermore, the transition

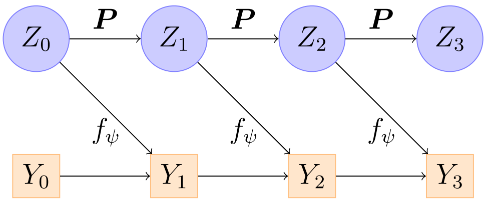

FIGURE 4 Directed graphical representation of the price and order flow model [Color figure can be viewed at wileyonlinelibrary.com]

matrix of the process Z_k and the distribution of Z_0 are known and can be written as

$$\boldsymbol{P} = \left[\mathbb{P}\left(Z_k = \theta_j \mid Z_{k-1} = \theta_i\right)\right]_{i,j=1}^{J} = e^{\Delta t\, \boldsymbol{C}}, \qquad \text{and} \tag{39}$$

$$\boldsymbol{\pi}_0 = \left[\mathbb{P}\left(Z_0 = \theta_i\right)\right]_{i=1}^{J} = \left(\pi_0^i\right)_{i=1}^{J}. \tag{40}$$

Moreover, by the definitions of the processes F, N^+, N^-, and Θ, the processes Y and Z inherit the conditional independence structure depicted in the directed graph shown in Figure 4. It is important to note that here we allow for dependence between the visible states Y even when conditioned on the latent states (i.e., there are connections between subsequent Y). This differs from the usual Hidden Markov Model (HMM) setup, where Y are conditionally independent when conditioned on Z.

More specifically, we have

$$\mathbb{P}\left(Y_k \in \mathcal{B}, Z_k = \theta_i \mid Y_{k-1}, Z_{k-1}\right) = \mathbb{P}\left(Y_k \in \mathcal{B} \mid Y_{k-1}, Z_{k-1}\right)\mathbb{P}\left(Z_k \mid Z_{k-1}\right), \tag{41}$$

for each $k = 1 \dots K$ and any Borel measurable set $\mathcal{B}$. Let us also assume the transition density for Y_k conditional on (Y_{k-1}, Z_{k-1}) is known for each $k = 1, \dots, K$, and that given a set of suitable parameters ψ, we can write

$$\mathbb{P}\left(Y_k \in dy \mid Y_{k-1} = y_0, Z_{k-1} = \theta_i\right) = f_\psi(t_k, y; t_{k-1}, y_0, \theta_i)\, \boldsymbol{\mu}(dy), \tag{42}$$

where for each fixed $(y_0, \theta_j) \in \mathbb{R}^{2J+3} \times \{\theta_j\}_{j\in\mathfrak{J}}$, and fixed times $0 \le t_{k-1} < t_k \le T$, f_ψ is a probability density function, and where $\boldsymbol{\mu}$ is the Lebesgue measure. In machine learning language, the function f_ψ is often called the *emission probability*. It may be found by solving the appropriate Kolmogorov/Fokker–Planck forward equation. If we consider $f_\psi^i(t, y) = f_\psi(t, y; s, y_0, \theta_i)$ for fixed values of y_0 and for $0 \le s \le t \le T$, then f_ψ^i is the solution to the PDE

$$\begin{cases} \left(\partial_t + \sum_{j\in\mathfrak{J}} \left(e^{t\,\boldsymbol{C}}\right)_{i,j} \widehat{\mathcal{L}}_j^{\dagger}\right) f_\psi^i(t, y) = 0, \\ f_\psi^i(s, y) = \delta(y - y_0), \end{cases} \tag{43}$$

where $\delta(y)$ is the dirac delta function, and the operator $\widehat{\mathcal{L}}_j^{\dagger}$ is the adjoint infinitesimal generator of the process $Y_t = (F_t, \boldsymbol{N}_t, \lambda_t)$ conditional on the event $\{\omega : Z_t(\omega) = \theta_j, \forall t \in [0, T]\}$, and where $(e^{t\,\boldsymbol{C}})_{i,j}$ is element (i, j) of the matrix exponential of $t\,\boldsymbol{C}$. In practice, it is possible to use any number of approximations to obtain the emission probability density.

In the remainder of this section, we use the notation Y_k^d and Z_k^d to denote the processes Y and Z, observed at time t_k in the dth independently observed path. We also introduce the notation $\mathcal{Y}_{m:n}^d \triangleq$

$\bigcap_{m\le k\le n}\{\omega : Y_k^d \in dy_k^d\}$, where each $\{\omega : Y_k^d \in dy_k^d\}$, represents the event in which we observe the visible process Y_k^d having taken the value y_k^d.

The objective is to find the set of parameters $\Gamma = (\pi_0, \boldsymbol{P}, \psi) \in \mathfrak{G}$, which maximize the likelihood of having made observations $\{y_k^d\}_{k,d=1}^{K,D}$ of the process Y, where $\mathfrak{G}$ is the set of allowable parameters, defined as

$$\begin{aligned}\mathfrak{G} = &\left\{\boldsymbol{\pi} = (\pi_i)_{t\in[0,T]}^J \in \mathbb{R}^J \;:\; \mathbf{1}_J^\intercal \boldsymbol{\pi} = 1, \pi_i \in [0,1]\right\} \\ &\times\left\{\boldsymbol{P} = (P_{i,j})_{t\in[0,T]}^J \in \mathbb{R}^{J\times J} \;:\; \mathbf{1}_J^\intercal \boldsymbol{P} = \mathbf{1}_J, P_{i,j} \in [0,1]\right\} \times \mathfrak{G}_\psi,\end{aligned} \tag{44}$$

where $\mathbf{1}_J$ is a column vector of ones, and $\mathfrak{G}_\psi$ is the set on which we restrict ψ. The first two sets in the definition of $\mathfrak{G}$ ensure the entries of $\boldsymbol{\pi}_0$ sum to one, and $\boldsymbol{P}$ is a valid Markov chain transition matrix, respectively.

Due to the presence of the unobserved (latent) states Z, the number of terms in the log-likelihood of the discrete observations of the process Y grows exponentially with the number of observations. It is, therefore, necessary to use the EM algorithm that provides a sequence of improving parameter estimates $\{\Gamma_k\}_{k=1}^\infty$. Each term in the sequence has larger likelihood than the previous.

The EM algorithm proceeds as follows. Begin with an initial guess (e.g., as estimated from an equivalent model with no latent states) for the model parameters, $\Gamma_0 \in \mathfrak{G}$. Generate a recursive sequence of parameters $\{\Gamma_n\}_{n=1}^\infty$, defined by the relationship

$$\Gamma_{n+1} = \arg\sup_{\Gamma\in\mathfrak{G}} \mathbb{E}^{\mathbb{P}^{\Gamma_n}}\left[\log L^\Gamma \,\middle|\, \{\mathcal{Y}_{0:K}^d\}_{d=1}^D\right], \tag{45}$$

where $\log L^\Gamma$ is the joint log-likelihood of having observed D independent paths of the process (Y, Z) given the parameter set Γ, and where $\mathbb{P}^{\Gamma_n}$ is the probability measure conditional on the dynamics of (Y, Z) having parameters Γ_n. The joint log-likelihood for our model yields the decomposition

$$\begin{aligned}\log L^\Gamma = &\sum_{d=1}^D\sum_{i=1}^J \log\left(\pi_0^i\right) \mathbb{1}_{\left\{Z_0^d=\theta_i\right\}} \\ &+\sum_{d=1}^D\sum_{k=0}^{K-2}\sum_{i,j=1}^J \log\left(P_{i,j}\right) \mathbb{1}_{\left\{Z_k^d=\theta_i,\, Z_{k+1}^d=\theta_j\right\}} \\ &+\sum_{d=1}^D\sum_{k=0}^{K-1}\sum_{i=1}^J \log\left(f_\psi\left(t_{k+1}, y_{k+1}^d; t_k, y_k^d, \theta_i\right)\right) \mathbb{1}_{\left\{Z_k^d=\theta_i\right\}}.\end{aligned} \tag{46}$$

We then have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{\Gamma_n}}\left[\log L^\Gamma \mid \mathcal{Y}_{0:K}^1, \dots, \mathcal{Y}_{0:K}^D\right] = &\sum_{d=1}^D\sum_{i=1}^J \log\left(\pi_0^i\right) \gamma_0^{i,d} \\ &+\sum_{d=1}^D\sum_{k=0}^{K-2}\sum_{i,j=1}^J \log\left(P_{i,j}\right) \xi_k^{i,j,d} \\ &+\sum_{d=1}^D\sum_{k=0}^{K-1}\sum_{i=1}^J \log\left(f_\psi\left(t_{k+1}, y_{k+1}^d; t_k, y_k^d, \theta_i\right)\right) \gamma_k^{i,d},\end{aligned} \tag{47}$$

where the *smoother* $\gamma_k^{i,d}$ and *two-slice marginal* $\xi_k^{i,j,d}$ are defined as

$$\gamma_k^{i,d} = \mathbb{P}^{\Gamma_n}\left(Z_k^d = \theta_i \mid \mathcal{Y}_{0:K}^d\right) \quad \text{and} \quad \xi_k^{i,j,d} = \mathbb{P}^{\Gamma_n}\left(Z_k^d = \theta_i, Z_{k+1}^d = \theta_j \mid \mathcal{Y}_{0:K}^d\right). \tag{48}$$

These coefficients can be computed by using the *forward–backward algorithm*, which, for our model, is provided in Appendix D. Next, because each line of Equation (47) depends only on one of $\boldsymbol{\pi}_0$, $\boldsymbol{P}$, and ψ, the updated estimates for the parameters can be obtained independently. The resulting update rules (which maximize (47)) are

$$\pi_0^{j\star} = \frac{1}{D}\sum_{d=1}^{D} \gamma_0^{j,d}, \qquad j \in \mathfrak{J}, \tag{49a}$$

$$P_{i,j}^{\star} = \frac{\sum_{d=1}^{D}\sum_{k=0}^{K-2} \xi_k^{i,j,d}}{\sum_{d=1}^{D}\sum_{k=0}^{K-2}\sum_{j=1}^{J} \xi_k^{i,j,d}}, \qquad i,j \in \mathfrak{J}, \tag{49b}$$

$$\psi^{\star} = \arg\max_{\psi\in\mathfrak{G}_\psi} \left\{ \sum_{d=1}^{D}\sum_{k=0}^{K-1}\sum_{i=1}^{J} \log\left(f_\psi\left(t_{k+1}, y_{k+1}^d; t_k, y_k^d, \theta_i\right)\right)\gamma_k^{i,d} \right\}. \tag{49c}$$

Hence, $\Gamma_{n+1} = (\boldsymbol{\pi}_0^\star, \boldsymbol{P}^\star, \psi^\star)$. The updated estimates for the emission probabilities (49c), may in some models be analytically tractable (e.g., Gaussian mixtures, or discrete distributions), while in others one may have to resort to numerical optimization schemes.

More details on the maximization of Equation (45) and of various other related computational concerns are outlined in Rabiner (1989). The one subtle difference between the model presented in this section and the classical Hidden Markov Model, such as the one found in Rabiner (1989) is that the visible process Y exhibits temporal correlation even when conditioning on the path of the hidden process Z.

7.2 | Mean-reverting pure jump model

In this section, we show how to implement the EM algorithm for the model presented in Section 6.2. Assuming that the observations are frequent enough to observe one or no jumps in the midprice, we adopt a censored version of the model. As in Section 6.2, we assume the increments of the midprice F_t in the interval $[t_n, t_{n+1})$ satisfy

$$F_{t_{n+1}} - F_{t_n} = b\left(\Delta N_{t_n}^+ - \Delta N_{t_n}^-\right), \tag{50}$$

where each of the $\Delta N_t^\pm \in \{0,1\}$ are censored, conditionally independent Poisson random variables with respective stochastic rate parameters $\lambda_t^+\Delta t$ and $\lambda_t^+\Delta t$, where

$$\lambda_t^\pm = \sum_{j=1}^{J} \lambda_t^{\pm,j}\mathbb{1}_{\left\{\Theta_t^j=\theta_j\right\}}, \tag{51}$$

and we assume $\lambda_t^{\pm,j}$ have constant paths over each observation period (this can easily be relaxed). Moreover,

$$\lambda_{t_n}^{+,j} = \mu_j + \kappa_j(\theta_j - F_{t_n})_+ \quad \text{and} \quad \lambda_{t_n}^{-,j} = \mu_j + \kappa_j(\theta_j - F_{t_n})_-. \tag{52}$$

This generalizes the model in Section 6.2 as all parameters μ, θ and κ may vary according to the state of the hidden process Θ_t. The emission probabilities have parameters $\psi = \{\mu_j\, \kappa_j, \theta_j\}_{j=1}^{J}$, which represent (within each state) the base noise level, the mean reversion rate, and the mean reversion level of the midprice, respectively. With these ingredients, we can now state the emission probability as follows:

$$
\begin{aligned}
&f_\psi(t_{k+1}, Y_{k+1}; t_k, Y_k, \theta_j)\\
&= \begin{cases}
e^{-\Delta t\, \lambda_{t_k}^{-,j}} (1 - e^{-\Delta t\, \lambda_{t_k}^{+,j}}), & \text{if } F_{t_{n+1}} > F_{t_k}\\
e^{-\Delta t\, \lambda_{t_k}^{+,j}} (1 - e^{-\Delta t\, \lambda_{t_k}^{-,j}}), & \text{if } F_{t_{n+1}} < F_{t_k}\\
e^{-\Delta t \left(\lambda_{t_k}^{+,j} + \lambda_t^{-,j}\right)} + (1 - e^{-\Delta t\, \lambda_{t_k}^{+,j}}) \left(1 - e^{-\Delta t\, \lambda_{t_k}^{-,j}}\right), & \text{otherwise.}
\end{cases}
\end{aligned} \tag{53}
$$

Armed with this expression, we can apply the EM algorithm from Section 7.1 to obtain parameter estimates. In this model, we cannot obtain explicit updates for ψ from (49c), and instead use a numerical optimization.

7.3 | Example fit to INTC stock data

In this section, we fit the model presented in Section 7.2 to intraday price data. We fit this model to the midprice of INTC stock on the NASDAQ exchange taken at each second between the hours of 10:00 and 11:00 on each business day during 2014. To normalize these data between days, we subtract the price at 10:00 on the same day from each data point, so that we are fitting the model from Section 7.2 to the price change in INTC since 10:00 on each day. Moreover, we assume that price paths are independent across days. From this data set, we set $\Delta t = 1$, so that all of the parameter estimates can be interpreted on a per-second scale. In Appendix E, we show the parameter estimates for the model with 1–6 latent states. All of these parameter estimates were obtained using the EM algorithm described in Section 7.1 applied to the censored pure jump model in Section 7.2.

An important part of calibrating models with latent states is to estimate the possible number of states that a model should have. To choose an "optimal" number of latent states we use two information criteria: the Bayesian Information Criterion (BIC), and the integrated completed likelihood (ICL). For the purposes of our paper, we use an approximation to the ICL, which can be found in Biernacki, Celeux, and Govaert (2000). The definitions of both of these criteria and some related computational details can be found in Appendix E. It is well known that ICL typically underestimates the true number of latent states, whereas the BIC typically overestimates the number of latent states. For INTC, BIC, and ICL estimate 5 and 1 latent states, respectively. This implies that the true number of states should be somewhere within that range.

Recall that μ_i, κ_i, and θ_i can be interpreted as the noise level, the mean-reversion rate, and the mean-reversion level, respectively. We find that the mean-reversion level does not vary much (all less than 3¢ in absolute value) in comparison to realized daily stock price movements, which can range anywhere in between −50 and 50¢. This indicates the daily asset prices typically mean-revert back to their initial values no matter the state of the latent process. The distinction between states occurs in the strength of the mean-reversion through κ_i, and the base noise level μ_i.

The estimates for the transition probabilities shown in Appendix E indicate there is persistence in the latent process regardless of the number of latent states. This implies the trader is able to detect market states and to act on them before the states switch again. From the estimated generator matrices in Appendix E, the mean time spent in a given state can range from a few seconds to over 800 s, depending on the number of allowed latent states.

8 | CONCLUSION

In this paper, we solved a finite horizon optimal trading problem in which the midprice of the asset contains a latent alpha component stemming from a diffusive drift as well as a pure jump component. We obtain the solution in closed form, up to the computation of an expectation, which depends on the class of potential models. The optimal trading speed is found to be a combination of the classical AC trading strategy plus a modulating term that incorporates the trader's estimate of the latent factors and its forecast. The form of the solution is similar in spirit to the results in Cartea and Jaimungal (2016) where the authors have an alpha component stemming from MO flow, but in that work viewed as visible. We presented two examples one where the trader wishes to completely liquidate a large position (the optimal execution problem) and the other where the trader uses the latent states to generate a statistical arbitrage trading strategy. Both examples show that there is significant value in incorporating latent states. Finally, we presented a method for obtaining parameter estimates from data using the EM algorithm and applied it to an example stock.

There are many potential future directions left open for investigation. One direction that we have already investigated is to generalize the analysis to trading multiple assets, as well as incorporating multiple latent factors. In this work, the trader is assumed to execute trades continuously and uses MOs, which walk the limit order book and hence obtain a temporary price impact. It would be interesting to analyze the case of executing MOs at discrete times, and hence recast the problem as an impulse control problem with latent alpha factors. Along similar lines, the agent may wish to use limit orders to squeeze even more profits out of the strategy. Combining market and limit orders, along the lines of Cartea and Jaimungal (2015) and Huitema (2013), but including latent alpha factors would also be quite interesting.

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ENDNOTES

[1] The generator matrix $\boldsymbol{C} \in \mathbb{R}^{J\times J}$ of a $\boldsymbol{J}$-state continuous time Markov chain Θ has nondiagonal entries $\boldsymbol{C}_{i,j} \geq 0$ if $i \neq j$ and diagonal entries $\boldsymbol{C}_{i,i} = -\sum_{j\neq i} \boldsymbol{C}_{i,j}$. $\boldsymbol{C}$ is defined so that $\mathbb{P}(\Theta_t = \theta_j | \Theta_0 = \theta_i) = (e^{t\boldsymbol{C}})_{i,j}$, where $(e^{t\boldsymbol{C}})_{i,j}$ is element (i, j) of the matrix exponential of $t\boldsymbol{C}$.

[2] To achieve this, we may extend λ_t to include the state variables necessary for the model to be Markov.

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APPENDIX A: PROOF OF THEOREM 3.1

Proof. We present here the proof of the case where $\sigma > 0$. The case where $\sigma = 0$ and $A := 0$ can be derived in the same fashion by excluding all of the diffusive terms.

Due to the Novikov Condition (8), we may define the measure $\mathbb{Q}$ through the Radon–Nikodym derivative

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = \exp\left\{-\sigma^{-1}\int_0^t A_{u-}\,dW_u - \frac{\sigma^{-2}}{2}\int_0^t \left(A_{u-}\right)^2 du\right\}$$
$$\times\exp\left\{\int_0^t \left(\lambda^+_{u-} - 1\right) du - \int_0^t \log\left(\lambda^+_{u-}\right) dN^+_u\right\}$$
$$\times\exp\left\{\int_0^t \left(\lambda^-_{u-} - 1\right) du - \int_0^t \log\left(\lambda^-_{u-}\right) dN^-_u\right\},$$

which is defined so that under the measure $\mathbb{Q}$, the process $\sigma^{-1}(F_t - b(N_t^+ - N_t^-))$ is a Brownian motion and both N_t^+ and N_t^- have intensity process equal to 1. This measure is deliberately chosen, so that F_t, N_t^+ and N_t^- are $\mathbb{Q}$-independent of Θ_t. Additionally, the inverse of this Radon–Nikodym derivative is

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_t = \zeta_t = \exp\left\{\sigma^{-2}\int_0^t A_{u-}(dF_u - b\,(dN_t^+ - dN_t^-)) - \frac{\sigma^{-2}}{2}\int_0^t \left(A_{u-}\right)^2 du\right\}$$
$$\times\exp\left\{\int_0^t \left(1 - \lambda^+_{u-}\right) du + \int_0^t \log\left(\lambda^+_{u-}\right) dN^+_u\right\} \tag{A.1}$$
$$\times\exp\left\{\int_0^t \left(1 - \lambda^-_{u-}\right) du + \int_0^t \log\left(\lambda^-_{u-}\right) dN^-_u\right\}.$$

Using these last two expressions, we can re-write the filter in terms of $\mathbb{Q}$ expected values to obtain

$$\pi_t^j = \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\Theta_t=\theta_j\}}\zeta_t \mid \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\zeta_t \mid \mathcal{F}_t\right]} \tag{A.2}$$

$$= \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\Theta_t=\theta_j\}}\zeta_t \mid \mathcal{F}_t\right]}{\sum_{i=1}^J \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\Theta_t=\theta_i\}}\zeta_t \mid \mathcal{F}_t\right]} \tag{A.3}$$

$$= \frac{\Lambda_t^j}{\sum_{i=1}^J \Lambda_t^i}. \tag{A.4}$$

Next, we will attempt to find an SDE for each Λ_t^j term. This can be done by first defining the process $\delta_t^j = \mathbb{1}_{\{\Theta_t=\theta_j\}}$, which satisfies the SDE

$$d\delta_t^j = \sum_{i=1}^{J} \delta_{t-}^i C_{j,i}\, dt + d\tilde{\mathcal{M}}_t^j,$$

under the measure $\mathbb{Q}$, where $\tilde{\mathcal{M}}_t^j$ is a square-integrable, $\mathcal{G}_t$-adapted, $\mathbb{Q}$-martingale and $\boldsymbol{C}$ is the generator matrix for Θ_t. All that is left to compute the dynamics of the Λ_t^j is to figure out the dynamics of $\zeta_t\delta_t^j$ and then take the appropriate $\mathbb{Q}$ expected value while conditioning on $\mathcal{F}_t$. The process $\delta_t^j\zeta_t$ satisfies the SDE

$$\begin{aligned}
d\left(\zeta_t\delta_t^j\right) &= \left(\zeta_{t-}\delta_{t-}^j\right)\Bigg(\sigma^{-2}A_{t-}^j\ \left(dF_t - b\left(dN_t^+ - dN_t^-\right)\right) \\
&\quad + \left(\lambda_{t-}^+ - 1\right)\left(dN_t^+ - dt\right) + \left(\lambda_{t-}^- - 1\right)\left(dN_t^- - dt\right)\Bigg) \\
&\quad + \sum_{i=1}^{J} \zeta_{t-}\delta_{t-}^i C_{j,i}\, dt + d\mathcal{M}_t^j \\
&= \left(\zeta_{t-}\delta_{t-}^j\right)\Bigg(\sigma^{-2}A_{t-}^j\ \left(dF_t - b\left(dN_t^+ - dN_t^-\right)\right) \\
&\quad + \left(\lambda_{t-}^{+,j} - 1\right)\left(dN_t^+ - dt\right) + \left(\lambda_{t-}^{-,j} - 1\right)\left(dN_t^- - dt\right)\Bigg) \\
&\quad + \sum_{i=1}^{J} \zeta_{t-}\delta_{t-}^i C_{j,i}\, dt + d\mathcal{M}_t^j,
\end{aligned}$$

where $\mathcal{M}_t^j$ is another square-integrable, $\mathcal{G}_t$-adapted, $\mathbb{Q}$-martingale.

Now, we rewrite the expression for Λ_t^j as the expected value of a stochastic integral

$$\Lambda_t^j = \mathbb{E}^{\mathbb{Q}}\left[\delta_t^j\zeta_t \mid \mathcal{F}_t\right] \tag{A.5}$$

$$\begin{aligned}
= \mathbb{E}^{\mathbb{Q}}\Bigg[&\left(\zeta_0\delta_0^j\right) + \int_0^t \left(\zeta_{u-}\delta_{u-}^j\right)\left(\sigma^{-2}A_{u-}^j\ \left(dF_u - b\left(dN_u^+ - dN_u^-\right)\right)\right) \\
&+ \int_0^t \left(\zeta_{u-}\delta_{u-}^j\right)\left(\lambda_{u-}^{+,j} - 1\right)\left(dN_u^+ - du\right) + \int_0^t \left(\zeta_{u-}\delta_{u-}^j\right)\left(\lambda_{u-}^{-,j} - 1\right)\left(dN_u^- - du\right) \\
&+ \int_0^t \left(\sum_{i=1}^{J} \zeta_{u-}\delta_{u-}^i C_{j,i}\, du + d\mathcal{M}_u^j\right)\Bigg|\mathcal{F}_t\Bigg].
\end{aligned} \tag{A.6}$$

At this point, we will need conditions guaranteeing that we can exchange the order of integration, in the expression above. First, by the definition of Λ_t^j,

$$\mathbb{E}^{\mathbb{Q}}\left[\zeta_0\delta_0^j \mid \mathcal{F}_t\right] = \pi_0^j, \tag{A.7}$$

which allows us to replace the first term in (A.6). We can use the fact that $F_t - F_0 - b(N_t^+ - N_t^-) = W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion to write

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left(\zeta_{u-}\delta_{u-}^j\right)\sigma^{-2}A_{u-}^j\, dW_u^{\mathbb{Q}} \mid \mathcal{F}_t\right]. \tag{A.8}$$

Applying Wong and Hajek (1985, Chapter VI, lemma 3.2), whose conditions are met with the bound (8), we can exchange the order of integration in the above term to get

$$\int_0^t \Lambda_{u-}^j \sigma^{-2} A_{u-}^j\, dW_u^{\mathbb{Q}}. \tag{A.9}$$

For the third and fourth terms, let us note that if we let $\mathcal{U}_t^{\pm} = \{u \in [0,t] : N_u^{\pm} > N_{u-}^{\pm}\}$, then by condition (8) we know $\mathcal{U}^{\pm}$ is almost surely finite and $\mathcal{F}_t$-measurable. Therefore,

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left(\zeta_{u-}\delta_{u-}^j\right)\left(\lambda_{u-}^{\pm,j}-1\right) dN_u^{\pm} \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{u\in\mathcal{U}_t^{\pm}} \left(\zeta_{u-}\delta_{u-}^j\right)\left(\lambda_{u-}^{\pm,j}-1\right)(N_u^{\pm}-N_{u-}^{\pm}) \mid \mathcal{F}_t\right] \tag{A.10}$$

$$= \sum_{u\in\mathcal{U}_t^{\pm}} \Lambda_{u-}^j \left(\lambda_{u-}^{\pm,j}-1\right)\left(N_u^{\pm}-N_{u-}^{\pm}\right) \tag{A.11}$$

$$= \int_0^t \Lambda_{u-}^j \left(\lambda_{u-}^{\pm,j}-1\right) dN_t^{\pm}. \tag{A.12}$$

We can apply Fubini's theorem on the remaining Riemann integral because the integrands are square integrable to exchange the order of integration and get

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left(\sum_{i=1}^J \zeta_{u-}\delta_{u-}^i C_{j,i} - \left(\zeta_{u-}\delta_{u-}^j\right)\left(\lambda_{u-}^{+,j}+\lambda_{u-}^{-,j}-2\right)\right) du \mid \mathcal{F}_t\right]$$
$$= \int_0^t \left(\sum_{i=1}^J \Lambda_{u-}^i C_{j,i} - \Lambda_{u-}^j(\lambda_{u-}^{+,j}+\lambda_{u-}^{-,j}-2)\right) du.$$

By using the last steps to exchange the order of integration for each part, we can let the martingale $(\mathcal{M}_t^j)$ portion vanish to obtain

$$\Lambda_t^j = \pi_0^j + \int_0^t \Lambda_{u-}^j \left(\sigma^{-2}A_{u-}^j\left(dF_u - b\left(dN_u^+ - dN_u^-\right)\right)\right)$$
$$+ \int_0^t \Lambda_{u-}^j\left(\lambda_{u-}^{+,j}-1\right)\left(dN_u^+ - du\right) + \int_0^t \Lambda_{u-}^j\left(\lambda_{u-}^{i,j}-1\right)\left(dN_u^- - du\right) + \sum_{i=1}^J \int_0^t \Lambda_{u-}^i C_{j,i} du,$$

which is the desired result. □

APPENDIX B: PROOF OF THEOREM 4.1

Proof. First assuming that $\sigma > 0$, we shall prove the claims of Theorem 4.1 in order. First of all, it is clear in the definition of the process $\widehat{W}_t$ that it is a $\mathbb{P}$-almost-surely continuous process satisfying $[\widehat{W}]_t = t$ because it is the sum of a $\mathbb{P}$-Brownian motion and a process of finite variation. Moreover, by the definition of the process F_t, we can write W_t as

$$W_t = \sigma^{-1}\left((F_t - F_0) - \int_0^t A_u\,du - b\left(N_t^+ - N_t^-\right)\right). \tag{B.1}$$

Hence, we can insert this last formula into the definition of $\widehat{W}_t$ to yield,

$$\widehat{W}_t = \sigma^{-1}\left((F_t - F_0) - \int_0^t \widehat{A}_u\,du - b\left(N_t^+ - N_t^-\right)\right), \tag{B.2}$$

which demonstrates that the process $\widehat{W}_t$ is $\mathcal{F}_t$-adapted. Next, we will show that $\widehat{W}_t$ is a $\mathbb{P}$-martingale with respect to the filtration $\mathcal{F}_t$. By taking the conditional expectation of $\widehat{W}_{t+h}$ for $h \geq 0$ and by using the properties of W, we get

$$\begin{aligned}
\mathbb{E}\left[\widehat{W}_{t+h} \mid \mathcal{F}_t\right] &= \widehat{W}_t + \mathbb{E}\left[\widehat{W}_{t+h} - \widehat{W}_t \mid \mathcal{F}_t\right] \\
&= \widehat{W}_t + \sigma^{-1}\mathbb{E}\left[\int_t^{t+h}\left(dF_u - \widehat{A}_u\,du - b\left(dN_u^+ - dN_u^-\right)\right) \mid \mathcal{F}_t\right] \\
&= \widehat{W}_t - \sigma^{-1}\mathbb{E}\left[\int_t^{t+h}\left(\widehat{A}_u - A_u\right)du \mid \mathcal{F}_t\right] \\
&= \widehat{W}_t - \sigma^{-1}\mathbb{E}\left[\int_t^{t+h}\mathbb{E}\left[\widehat{A}_u - A_u \mid \mathcal{F}_u\right]du \mid \mathcal{F}_t\right] \\
&= \widehat{W}_t,
\end{aligned}$$

where in the above, the use of Fubini's theorem is allowed due to Equation (4). We have shown that $\widehat{W}$ is a $\mathbb{P}$-a.s. continuous martingale with quadratic variation equal to t. Therefore, by the Lévy characterization of Brownian motion, the process $\widehat{W}$ is an $\mathcal{F}_t$–adapted $\mathbb{P}$-Brownian motion.

Next we need to verify the claims made about the processes $\widehat{M}_t^{\pm}$. By the definitions of $\widehat{M}_t^{\pm}$ and of $M_t^{\pm}$,

$$\widehat{M}_t^{\pm} = N_t^{\pm} - \int_0^t \widehat{\lambda}_u^{\pm}du, \tag{B.3}$$

where the shorthand $\widehat{\lambda}_t^{\pm}$ is described after Equation (15). Because the processes $\widehat{\lambda}_t^{\pm}$ are $\mathcal{F}_t$-adapted, we get that $\widehat{M}_t^{\pm}$ must also be $\mathcal{F}_t$-adapted processes. The processes $\widehat{M}_t^{\pm}$ are $\mathcal{F}_t$-martingales because for any $h > 0$,

$$\mathbb{E}\left[\widehat{M}_{t+h}^{\pm} \mid \mathcal{F}_t\right] = \widehat{M}_t^{\pm} + \mathbb{E}\left[M_{t+h}^{\pm} - M_t^{\pm} + \int_t^{t+h}\left(\lambda_u^{\pm} - \widehat{\lambda}_u^{\pm}\right)du \mid \mathcal{F}_t\right] \tag{B.4}$$

$$= \widehat{M}_t^{\pm} + \mathbb{E}\left[\int_t^{t+h}\mathbb{E}\left[(\lambda_u^{\pm} - \widehat{\lambda}_u^{\pm}) \mid \mathcal{F}_u\right]du \mid \mathcal{F}_t\right] \tag{B.5}$$

$$= \widehat{M}_t^{\pm}. \tag{B.6}$$

By the definition of $\widehat{M}^{\pm}$ in Equation (B.3), $\widehat{M}^{\pm}$ is the sum of a process with an almost-surely finite number of jumps in the interval $[0, T]$ and a process of finite variation. From its definition, $\widehat{W}$ is the sum of a Brownian motion and a process of finite variation. The last two remarks imply that $[\widehat{W}, \widehat{M}^{\pm}] = 0$. Finally, because $dM_t^{\pm} = dN_t^{\pm} - \widehat{\lambda}_t^{\pm} dt$, and $d[N^+, N^-]_t = 0$ almost surely, we get that $[\widehat{M}^+, \widehat{M}^-]_t = 0$ almost surely.

Because $N_t^{\pm}$ are a counting processes and $N_t^{\pm} - \int_0^t \widehat{\lambda}_u \, du$ are $\mathcal{F}$-adapted $\mathbb{P}$ martingales, by Watanabe's characterization theorem, $N_t^{\pm}$ must be $\mathcal{F}$–adapted doubly stochastic Poisson processes with respective $\mathbb{P}$-intensities $\widehat{\lambda}^{\pm}$, demonstrating claim (D).

If $\sigma = 0$ and $A := 0$ the results concerning $\widehat{M}^{\pm}$ and $N^{\pm}$ still hold, and thus the statements (***B***) and (***D***) are true. By the same logic we also find that $[\widehat{M}^+, \widehat{M}^-]_t = 0$ almost surely. □

APPENDIX C: PROOFS RELATED TO THE DPE

C.1 Proof of Proposition 5.1

Proof. Let us begin with the PDE (23) for $H(t, \boldsymbol{Z})$, where $\boldsymbol{Z} = (F, \boldsymbol{N}, X, Q, \lambda, \boldsymbol{\Lambda})$,

$$\begin{cases} 0 = -\phi Q^2 + \sup_{\nu \in \mathbb{R}} \left\{ (\partial_t + \bar{\mathcal{L}}) H + \nu \, \partial_Q H - \nu (F + \beta (Q - \mathfrak{N}) + a\nu) \partial_X H \right\} \\ H(T, \boldsymbol{Z}) = X + Q(F + \beta (Q - \mathfrak{N}) - \alpha Q) \end{cases}. \tag{C.1}$$

Because the term inside of the curly brackets is quadratic in ν, we can complete the square and simplify the supremum expression. This yields

$$0 = -\phi Q^2 + (\partial_t + \bar{\mathcal{L}}) H + \frac{1}{4a} \frac{\left(\partial_Q H - (F + \beta (Q - \mathfrak{N})) \, \partial_X H\right)^2}{\partial_X H}. \tag{C.2}$$

Next, we can insert the ansatz

$$H(t, \boldsymbol{Z}) = X + Q\,(F + \beta\,(Q - \mathfrak{N})) + h(t, \boldsymbol{\ell}(\boldsymbol{Z})) \tag{C.3}$$

where $\boldsymbol{\ell}(\boldsymbol{Z}) = (F, \boldsymbol{N}, Q, \lambda, \boldsymbol{\Lambda})$, into the last PDE to yield another PDE in terms of h,

$$\begin{cases} 0 = -\phi Q^2 + (\partial_t + \bar{\mathcal{L}}) h + Q \left(\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\lambda, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\lambda, \boldsymbol{\Lambda}) \right) + \frac{1}{4a} \left(\beta Q + \partial_Q h \right)^2 \\ h(T, \boldsymbol{\ell}(\boldsymbol{Z})) = -\alpha Q^2 \end{cases}. \tag{C.4}$$

In the above equation, the term $\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\lambda, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\lambda, \boldsymbol{\Lambda}))$ appears in the PDE due to the mean drift of the process F, found in Equation (15). If we assume that h is quadratic in the variable Q, so that

$$h(t, \boldsymbol{\ell}(\boldsymbol{Z})) = h_0(t, \chi(\boldsymbol{Z})) + Q\, h_1(t, \chi(\boldsymbol{Z})) + Q^2\, h_2(t), \tag{C.5}$$

where $\chi(\boldsymbol{Z}) = (F, \boldsymbol{N}, \lambda, \boldsymbol{\Lambda})$, then the PDE further simplifies down to

$$\begin{cases} 0 = \left\{ (\partial_t + \bar{\mathcal{L}})h_0 + \frac{1}{4a} h_1^2 \right\} \\ \quad + Q \left\{ (\partial_t + \bar{\mathcal{L}})h_1 + \left(\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\lambda, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\lambda, \boldsymbol{\Lambda}) \right) + \frac{1}{2a}(\beta + 2h_2)h_1 \right\} \\ \quad + Q^2 \left\{ \partial_t h_2 - \phi + \frac{1}{4a} \left(\beta + 2h_2\right)^2 \right\} \\ h(T, \boldsymbol{\ell}(\boldsymbol{Z})) = -\alpha Q^2 \end{cases}. \tag{C.6}$$

The above PDE must be satisfied for all values of $Q \in \mathbb{R}$. Because h_0 h_1 and h_2 are independent of Q, each of the terms inside curly brackets in (C.6) must be equal to 0 independently of Q. This yields the system of PDEs for h_0, h_1, and h_2

$$\begin{cases} 0 = \partial_t h_0 + \frac{1}{4a} h_1^2 \\ h_0(T, \chi(\boldsymbol{Z})) = 0 \end{cases} \tag{C.7}$$

$$\begin{cases} (\partial_t + \bar{\mathcal{L}})h_1 + \left(\widehat{A}(t, F, \boldsymbol{N}, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\lambda, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\lambda, \boldsymbol{\Lambda}) \right) + \frac{1}{2a}(\beta + 2h_2)h_1 \\ h_1(T, \chi(\boldsymbol{Z})) = 0 \end{cases} \tag{C.8}$$

$$\begin{cases} \partial_t h_2 - \phi + \frac{1}{4a} \left(\beta + 2h_2\right)^2 = 0 \\ h_2(T) = -\alpha, \end{cases} \tag{C.9}$$

which, due to their dependence on one another, can be solved in the order $h_2 \to h_1 \to h_0$. The Ordinary Differential Equation (ODE) for h_2 is a standard Riccati-type ODE, which admits the unique solution defined in the statement of the proposition. Next, the PDE for h_1 is linear and depends only on the solution for h_2. Therefore, we can use the Feynman–Kac formula to write the solution PDE (C.8) as

$$h_1(t, \chi(\boldsymbol{Z})) = \mathop{\mathbb{E}}_{t,\chi(Z)} \left[\int_t^T \left(\widehat{A}_u + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^- \right) \right) e^{\frac{1}{2a} \int_t^u (\beta + 2h_2(\tau))\, d\tau}\, du \right], \tag{C.10}$$

where the $\widehat{A}_u$ and the $\widehat{\lambda}_u^{\pm}$ are defined in Section 4. When plugging in the solution for h_2, we get the exact form presented in the statement of the proposition. Furthermore, condition (4) and the fact that the term $e^{\frac{1}{2a} \int_t^u (\beta + 2h_2(\tau))\, d\tau}$ is bounded for all $0 \le t \le u \le T$ allow us to use Fubini's theorem to yield

$$h_1(t, \chi(\boldsymbol{Z})) = \int_t^T \mathop{\mathbb{E}}_{t,\chi(Z)} \left[\widehat{A}_u + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^- \right) \right] e^{\frac{1}{2a} \int_t^u (\beta + 2h_2(\tau))\, d\tau}\, du, \tag{C.11}$$

which in combination with the solution for h_2, gives us the form presented in the statement of the proposition.

Finally, the PDE for h_0 is also linear, and we can therefore use the Feynman–Kac formula once more for a representation of the solution. This representation gives us the expression for h_0 that is present in the statement of the proposition. We can also guarantee that the solution for h_0 provided in Proposition 5.1 is bounded because $\mathbb{E}[\int_0^T (h_{1,u})^2 du] < \infty$. This bound is shown in Equations C.16–C.19 in the proof of Theorem 5.2. □

C.2 Proof of Theorem 5.2

Proof. Showing the control $\nu^\star$ is admissible.

The candidate optimal control $\nu^\star$ is defined as

$$\nu_t^\star = \frac{1}{2a}\left(Q_t^{\nu^\star}\left(\beta + 2h_2(t)\right) + h_1(t, \chi(\boldsymbol{Z}_{t-}))\right), \tag{C.12}$$

where $\chi(\boldsymbol{Z}) = (F, \boldsymbol{N}, \lambda, \boldsymbol{\Lambda})$, and $\boldsymbol{Z} = (F, \boldsymbol{N}, X, Q, \lambda, \boldsymbol{\Lambda})$. It is clear from the definition above that the control is $\mathcal{F}$-adapted, because it is a continuous function of $\mathcal{F}$-adapted processes. To guarantee that the control $\nu_t^\star$ is admissible, we must show that

$$\mathbb{E}\left[\int_0^T \left(\nu_u^\star\right)^2 du\right] < \infty. \tag{C.13}$$

By expanding the expression for $(\nu_u^\star)^2$ and by using Young's inequality twice, we can write an upper bound for $(\nu_u^\star)^2$ as

$$\left(\nu_u^\star\right)^2 \le \left(\frac{1}{2a^2}\right)\left(\left(Q_u^{\nu^\star}\right)^2 + (\beta + 2h_2(u))^2 + (h_{1,u})^2\right), \tag{C.14}$$

where $h_{1,u} = h_1(u, \chi(\boldsymbol{Z}_{u-}))$. This last inequality shows that Equation (C.13) holds if each of $\mathbb{E}[\int_0^T (\beta + 2h_2(u))^2 du]$, $\mathbb{E}[\int_0^T (Q_u^{\nu^\star})^2 du]$, and $\mathbb{E}[\int_0^T (h_{1,u})^2 du]$ are bounded.

Using the definition of h_2 in Proposition 5.1, we can integrate the first term directly to obtain

$$\mathbb{E}\left[\int_0^T (\beta + 2h_2(u))^2 du\right] = a^2\gamma^2\left(T + \frac{2}{\gamma}\right)\left(\frac{1}{1 - \zeta' e^{2T\gamma}} - \frac{1}{1 - \zeta'}\right), \tag{C.15}$$

where $\zeta' = \frac{\alpha - \frac{1}{2}\beta}{a\gamma}$. This last expression is bounded because $\alpha - \frac{1}{2}\beta \neq a\gamma$.

Next, we can use the definition of $h_{1,u}$ provided in Proposition 5.1 to write

$$\mathbb{E}\left[\int_0^T (h_{1,t})^2 dt\right] = \frac{1}{16a^2}\mathbb{E}\left[\int_0^T \left(\int_t^T \mathop{\mathbb{E}}_{t,\chi_t}\left[\widehat{A}_u + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-\right)\right]\left(\frac{\zeta e^{\gamma(T-u)} - e^{-\gamma(T-u)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}\right) du\right)^2 dt\right], \tag{C.16}$$

where $\chi_t = \chi(Z_t)$. Now if we notice that because $\gamma \ge 0$, $(\frac{\zeta e^{\gamma(T-u)} - e^{-\gamma(T-u)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}})^2 \le 1$ and that

$$\mathop{\mathbb{E}}_{t,\chi_t}\left[\widehat{A}_u + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-\right)\right] = \mathop{\mathbb{E}}_{t,\chi_t}\left[A_u + b\left(\lambda_u^+ - \lambda_u^-\right)\right], \tag{C.17}$$

then we can apply Jensen's inequality and Fubini's theorem, followed by Young's inequality to obtain

$$\mathbb{E}\left[\int_0^T (h_{1,t})^2 dt\right] \le \frac{1}{4a^2}\int_0^T \int_t^T \mathbb{E}\left[A_u^2 + b^2\left(\left(\lambda_u^+\right)^2 + (\lambda_u^-)^2\right)\right] du\, dt \tag{C.18}$$

$$\le \frac{T}{4a^2}\int_0^T \mathbb{E}\left[A_u^2 + b^2\left(\left(\lambda_u^+\right)^2 + (\lambda_u^-)^2\right)\right] du. \tag{C.19}$$

By the condition of Equation (4), this last term is bounded.

By the definition of $Q_t^{\nu^\star}$ and of $\nu^\star$, we have that

$$dQ_t^{\nu^\star} = \frac{1}{2a}\left(Q_t^{\nu^\star}\left(\beta + 2h_2(t)\right) + h_{1,t}\right) dt, Q_0^{\nu^\star} = \mathfrak{N}. \tag{C.20}$$

The above SDE has the solution

$$Q_t^{\nu^\star} = \mathfrak{N} + \frac{1}{2a}\int_0^t h_{1,u}\left(\frac{\zeta e^{\gamma(T-u)} - e^{-\gamma(T-u)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}\right) du. \tag{C.21}$$

By using Young's inequality and Jensen's inequality again, and by using the fact that $(\frac{\zeta e^{\gamma(T-u)}-e^{-\gamma(T-u)}}{\zeta e^{\gamma(T-t)}-e^{-\gamma(T-t)}})^2 \leq 1$, then we can write

$$\left(Q_t^{\nu^\star}\right)^2 \leq \frac{1}{a}\left(\mathfrak{N}^2 + \int_0^t (h_{1,u})^2 du\right). \tag{C.22}$$

Now by taking the expectation and the integral of this last expression, we get

$$\mathbb{E}\left[\int_0^T \left(Q_u^{\nu^\star}\right)^2 du\right] \leq \frac{1}{a}\left(T\,\mathfrak{N}^2 + \mathbb{E}\left[\int_0^T \int_0^t (h_{1,u})^2 du\, dt\right]\right) \tag{C.23}$$

$$\leq \frac{1}{a}\left(T\,\mathfrak{N}^2 + T\mathbb{E}\left[\int_0^T (h_{1,u})^2 du\right]\right). \tag{C.24}$$

Because the term $\mathbb{E}[\int_0^T (h_{1,u})^2 du]$ has already been shown to be bounded, we can conclude that $\mathbb{E}[\int_0^T (Q_u^{\nu^\star})^2\, du] < \infty$.

$\nu_t^\star$ is $\mathcal{F}_t$-adapted and satisfies $\mathbb{E}[\int_0^T (\nu_u^\star)^2 du] < \infty$, therefore it is an admissible control.

Showing $H \leq \widehat{H}$. By applying Itô's lemma to the function $\widehat{H}(t,\boldsymbol{Z}) = X + Q(F + \beta(Q - \mathfrak{N})) + h(t, \boldsymbol{\ell}(\boldsymbol{Z}))$ with an arbitrary control $\nu_t \in \mathcal{A}$ and the $\mathcal{F}_t$-predictable dynamics, we get

$$\widehat{H}_T = \widehat{H}(t,\boldsymbol{Z}) + \int_t^T \left\{Q_u^\nu\left(\widehat{A}_u + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-\right)\right) - a\nu_u^2 + (\beta + \partial_Q h_u)\nu_u + \left(\partial_t + \bar{\mathcal{L}}\right) h_u\right\} du$$

$$+ \int_t^T \eta_u^W\, d\widehat{W}_u + \int_t^T \eta_u^+\, d\widehat{M}_u^+ + \int_t^T \eta_u^-\, d\widehat{M}_u^-,$$

where in the above, we use the notation $\widehat{H}_t = \widehat{H}(t, \boldsymbol{Z}_t)$, $h_t = h(t, \boldsymbol{Z}_t)$ and we let η_u^W, η_u^+ and η_u^- be the square-integrable $\mathcal{F}_t$-predictable processes obtained by the martingale representation theorem.

By taking the conditional expected value of both sides, the martingale portions vanish and we are left with

$$\mathop{\mathbb{E}}_{t,\boldsymbol{Z}}\left[\widehat{H}_T\right] = \widehat{H}(t,\boldsymbol{Z})$$

$$+ \mathop{\mathbb{E}}_{t,\boldsymbol{Z}}\left[\int_t^T \left\{Q_u^\nu\left(\widehat{A}_u + b\left(\widehat{\lambda}_u^+ - \widehat{\lambda}_u^-\right)\right) - a\nu_u^2 + (\beta + \partial_Q h_u)\nu_u + \left(\partial_t + \bar{\mathcal{L}}\right) h_u\right\} du\right]. \tag{C.25}$$

From the PDE (24), we get that for all $\nu \in \mathbb{R}$,

$$0 \geq -\phi Q^2 + Q\left(\widehat{A}(t, F, \boldsymbol{N}_t, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\lambda, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\lambda, \boldsymbol{\Lambda})\right) + (\beta Q + \partial_Q h)\nu - a\nu^2 + \left(\partial_t + \bar{\mathcal{L}}\right) h. \tag{C.26}$$

Therefore, by plugging in the above, as well as the boundary condition for $\widehat{H}_T$,

$$\mathop{\mathbb{E}}_{t,Z}\left[\widehat{H}_T - \phi\int_t^T \left(Q_u^\nu\right)^2 du\right] = \mathop{\mathbb{E}}_{t,Z}\left[X_T^\nu + Q_t^\nu\left(F_T + \beta\left(Q_T^\nu - \mathfrak{N}\right) - \alpha Q_T^\nu\right) - \phi\int_t^T \left(Q_u^\nu\right)^2 du\right]$$
$$\leq \widehat{H}(t, \boldsymbol{Z}). \tag{C.27}$$

By the definition of $H^\nu(t, \boldsymbol{Z})$ in Equation (20), and because the above holds for an arbitrary $\nu_t \in \mathcal{A}$, we obtain

$$H(t, \boldsymbol{Z}) \leq \widehat{H}(t, \boldsymbol{Z}). \tag{C.28}$$

Showing $H \geq H^{\nu^\star} \geq \widehat{H}$. Next let us note that if we let $\nu^\star = \frac{\beta Q + \partial_Q h}{2a}$, then by Equation (24), $\forall \varepsilon > 0$,

$$-\varepsilon < -\phi Q^2 + Q\left(\widehat{A}(t, F, \boldsymbol{N}_t, \boldsymbol{\Lambda}) + b(\widehat{\lambda}^+(\lambda, \boldsymbol{\Lambda}) - \widehat{\lambda}^-(\lambda, \boldsymbol{\Lambda})\right) + (\beta Q + \partial_Q h)\nu^\star - a{\nu^\star}^2 + \left(\partial_t + \bar{\mathcal{L}}\right) h. \tag{C.29}$$

Using this last inequality with Equation (C.25) and the definition of H^ν gives

$$H^{\nu^\star}(t, \boldsymbol{Z}) \geq \mathop{\mathbb{E}}_t\left[X_T^{\nu^\star} + Q_t^{\nu^\star}\left(S_T^{\nu^\star} - \alpha Q_T^{\nu^\star}\right) - \phi\int_t^T Q_u^{\nu^\star} du\right] - \varepsilon$$
$$> \widehat{H}(t, \boldsymbol{Z}),$$

Because $H \geq H^\nu, \forall \nu \in \mathcal{A}$, we get

$$H(t, \boldsymbol{Z}) \geq H^{\nu^\star}(t, \boldsymbol{Z}) \geq \widehat{H}(t, \boldsymbol{Z}). \tag{C.30}$$

Therefore, we obtain the desired result that

$$H = H^{\nu^\star} = \widehat{H}. \tag{C.31}$$

□

APPENDIX D: DERIVATION OF FORWARD–BACKWARD ALGORITHM

This section provides further details on the forward–backward algorithm, which allows computation of the smoother and two-slice marginal. The forward–backward algorithm presented here differs from what is usually found in the literature due to the fact that Y may take continuous values, and that the process Y is not conditionally independent in the usual way. As Figure 4 shows, there is also dependence between Y even when conditioned on Z.

D.1 Recursive discrete filter

To begin, define the sequence $\{\alpha_n^{j,d}\}_{n=0}^{K-1}$ for each $j = 1 \dots J$ as $\alpha_n^{j,d} = \mathbb{P}(Z_n^d = \theta_j \mid \mathcal{Y}_{0:n}^d)$—the so-called *forward-filter*. These filters satisfy a recursive relationship, which we establish below. First note

$$\alpha_0^{j,d} = \pi_0^j. \tag{D.1}$$

Next, we can derive the recursion structure for this sequence by using applying Bayes' rule. Starting with the definition,

$$\alpha_n^{j,d} = \mathbb{P}\left(Z_n^d = \theta_j \mid \mathcal{Y}_{0:n}^d\right) = \frac{\mathbb{P}\left(Z_n^d = \theta_j, \mathcal{Y}_{0:n}^d\right)}{\sum_{i=1}^J \mathbb{P}\left(Z_n^d = \theta_i, \mathcal{Y}_{0:n}^d\right)}. \tag{D.2}$$

The numerator can be written recursively as

$$\mathbb{P}\left(Z_n^d = \theta_j, \mathcal{Y}_{0:n}^d\right) = \sum_{i=1}^J \mathbb{P}\left(Z_n^d = \theta_j, Z_{n-1}^d = \theta_i, \mathcal{Y}_{0:n}^d\right) \tag{D.3a}$$

$$= \sum_{i=1}^J \mathbb{P}\left(Z_n^d = \theta_j, Y_n^d \in dy_n^d \mid Z_{n-1}^d = \theta_i, \mathcal{Y}_{0:n-1}^d\right) \mathbb{P}\left(Z_{n-1}^d = \theta_i, \mathcal{Y}_{0:n-1}^d\right) \tag{D.3b}$$

$$= \sum_{i=1}^J \mathbb{P}\left(Z_n^d = \theta_j, Y_n^d \in dy_n^d \mid Z_{n-1}^d = \theta_i, Y_{n-1}^d \in dy_{n-1}^d\right) \mathbb{P}\left(\mathcal{Y}_{0:n-1}^d\right) \alpha_{n-1}^{i,d} \tag{D.3c}$$

$$= \sum_{i=1}^J \mathbb{P}\left(Y_n^d \in dy_n^d \mid Z_{n-1}^d = \theta_j, Y_{n-1}^d \in dy_{n-1}^d\right)$$

$$\times \mathbb{P}\left(Z_n^d = \theta_j \mid Z_{n-1}^d = \theta_i, Y_{n-1}^d \in dy_{n-1}^d\right) \mathbb{P}\left(\mathcal{Y}_{0:n-1}^d\right) \alpha_{n-1}^{i,d} \tag{D.3d}$$

$$= \mathbb{P}\left(\mathcal{Y}_{0:n}^d\right) \sum_{i=1}^J \boldsymbol{P}_{i,j}\, f_\psi\left(t_n\, y_n^d; t_{n-1}, \theta_i, y_{n-1}^d\right)\, d\boldsymbol{\mu}\left(y_n^d\right)\, \alpha_{n-1}^{i,d}. \tag{D.3e}$$

Therefore, by using the above result in Equation (D.2) and by canceling $\mathbb{P}(\mathcal{Y}_{0:n-1}^d)$ and $d\boldsymbol{\mu}(y_n^d)$ terms appearing in the numerator and in the denominator, we obtain

$$\alpha_n^{j,d} = \frac{\hat{\alpha}_n^{j,d}}{c_n^d}, \tag{D.4}$$

where

$$\hat{\alpha}_n^{j,d} = \sum_{i=1}^J \boldsymbol{P}_{i,j}\, f_\psi\left(t_n\, y_n^d; t_{n-1}, \theta_i, y_{n-1}^d\right)\, \alpha_{n-1}^{i,d}, \qquad \text{and} \qquad c_n^d = \sum_{j=1}^J \hat{\alpha}_n^{j,d}. \tag{D.5}$$

The normalization factor c_n^d has the additional property that $c_n^d\, \boldsymbol{\mu}(dy_n^d) = \mathbb{P}(Y_n^d \in dy_n^d \mid \mathcal{Y}_{0:n-1}^d)$. This can be seen by using the definition of the $\alpha_n^{j,d}$ and making use of the Markov property of (Y, Z) as follows:

$$c_n^d\, \boldsymbol{\mu}\left(dy_n^d\right) = \sum_{j=1}^J \sum_{i=1}^J \boldsymbol{P}_{i,j}\, f_\psi\left(t_n, y_n^d; t_{n-1}, \theta_j, y_{n-1}^d\right)\, d\boldsymbol{\mu}\left(y_n^d\right)\, \alpha_{n-1}^{i,d} \tag{D.6a}$$

$$= \sum_{j=1}^J \sum_{i=1}^J \mathbb{P}\left(Z_n^d = \theta_j, Y_n^d \in dy_n^d \mid Z_{n-1}^d = \theta_i, Y_{n-1}^d \in dy_{n-1}^d\right) \mathbb{P}\left(Z_{n-1}^d = \theta_i \mid \mathcal{Y}_{0:n-1}^d\right) \tag{D.6b}$$

$$= \sum_{j=1}^J \sum_{i=1}^J \mathbb{P}\left(Z_n^d = \theta_j, Y_n^d \in dy_n^d \mid Z_{n-1}^d = \theta_i, \mathcal{Y}_{0:n-1}^d\right) \mathbb{P}\left(Z_{n-1}^d = \theta_i \mid \mathcal{Y}_{0:n}^d\right) \tag{D.6c}$$

$$= \sum_{j=1}^{J} \sum_{i=1}^{J} \mathbb{P}\left(Z_n^d = \theta_j, Z_{n-1}^d = \theta_i, Y_n^d \in dy_n^d \mid \mathcal{Y}_{0:n-1}^d\right) \tag{D.6d}$$

$$= \mathbb{P}\left(Y_n^d \in dy_n^d \mid \mathcal{Y}_{0:n-1}^d\right). \tag{D.6e}$$

D.2 Recursive backward discrete filter

Here, we derive the recursion for the *backward-filter* $\{\beta_n^{j,d}\}_{n=0}^{K-1}$ for each $j = 1 \dots J$, defined as

$$\beta_n^{j,d} = \frac{\mathbb{P}\left(\mathcal{Y}_{n+1:K}^d \mid Z_n^d = \theta_j, Y_n^d \in dy_n^d\right)}{\mathbb{P}\left(\mathcal{Y}_{n+1:K}^d \mid \mathcal{Y}_{0:n}^d\right)}. \tag{D.7}$$

Just as with the forward-filter, the backward-filter can be obtained recursively. First note that

$$\beta_n^{j,d} = \frac{\mathbb{P}\left(\mathcal{Y}_{n+1:K}^d \mid Z_n^d = \theta_j, Y_n^d \in dy_n^d\right)}{\mathbb{P}\left(\mathcal{Y}_{n+1:K}^d \mid \mathcal{Y}_{0:n}^d\right)} = \frac{\mathbb{P}\left(\mathcal{Y}_{n+1:K}^d \mid Z_n^d = \theta_j, Y_n^d \in dy_n^d\right)}{\sum_{i=1}^{J} \mathbb{P}\left(\mathcal{Y}_{n+1:K}^d \mid Z_n^d = \theta_i, Y_n^d \in dy_n^d\right) \alpha_n^i}, \tag{D.8}$$

which can be computed at time $n = K - 1$ as

$$\beta_n^{j,d} = \frac{f_\psi\left(t_K, y_K^d; t_{K-1}, \theta_j, y_{K-1}^d\right)}{\sum_{i=1}^{J} f_\psi\left(t_K, y_K^d; t_{K-1}, \theta_i, y_{K-1}^d\right) \alpha_{K-1}^i}. \tag{D.9}$$

Continuing with expression for the numerator in Equation (D.8), we find that

$$\mathbb{P}\left(\mathcal{Y}_{n+1:K}^d \mid Z_n^d = \theta_j, Y_n^d \in dy_n^d\right) \tag{D.10a}$$

$$= \sum_{i=1}^{J} \mathbb{P}\left(Z_{n+1}^d = \theta_i, \mathcal{Y}_{n+1:K}^d \mid Z_n^d = \theta_j, Y_n^d \in dy_n^d\right) \tag{D.10b}$$

$$\begin{aligned} = \sum_{i=1}^{J} &\mathbb{P}\left(\mathcal{Y}_{n+2:K}^d \mid Z_{n+1}^d = \theta_i, Z_n^d = \theta_j, \mathcal{Y}_{n:n+1}^d\right) \\ &\times \mathbb{P}\left(Z_{n+1}^d = \theta_i, Y_{n+1}^d \in dy_{n+1}^d \mid Z_n^d = \theta_j, Y_n^d \in dy_n^d\right) \end{aligned} \tag{D.10c}$$

$$\begin{aligned} = \sum_{i=1}^{J} &\mathbb{P}\left(\mathcal{Y}_{n+2:K}^d \mid Z_{n+1}^d = \theta_i, Y_{n+1}^d \in dy_{n+1}^d\right) \\ &\times \boldsymbol{P}_{j,i}\, f_\psi\left(t_{n+1}, y_{n+1}^d; t_n, \theta_j, y_n^d\right)\, d\boldsymbol{\mu}\left(y_{n+1}^d\right) \end{aligned} \tag{D.10d}$$

$$= \mathbb{P}\left(\mathcal{Y}_{n+2:K}^d \mid \mathcal{Y}_{0:n+1}^d\right) f_\psi\left(t_{n+1}, y_{n+1}^d; t_n, \theta_j, y_n^d\right)\, d\boldsymbol{\mu}\left(y_{n+1}^d\right) \sum_{i=1}^{J} \beta_{n+1}^{i,d} \boldsymbol{P}_{j,i}. \tag{D.10e}$$

Plugging this last result back into Equation (D.8), and canceling $d\boldsymbol{\mu}(y^d_{n+2})$ and $\mathbb{P}(\mathcal{Y}^d_{n+3:K} \mid \mathcal{Y}^d_{0:n+2})$ terms, we obtain

$$\beta^{j,d}_n = \frac{\hat{\beta}^{j,d}_n}{\sum_{i=1}^J \hat{\beta}^{i,d}_n \alpha^{i,d}_n}, \tag{D.11}$$

where

$$\hat{\beta}^{j,d}_n = f_\psi\left(t_{n+1}, y^d_{n+1}; t_n, \theta_j, y^d_n\right) \sum_{i=1}^J \beta^{i,d}_{n+1} \boldsymbol{P}_{j,i}. \tag{D.12}$$

Furthermore, because (Y, Z) is a Markov process, by the Markov property

$$\beta^{j,k}_n = \boldsymbol{P}\left(\mathcal{Y}^d_{n+1:K} \mid Z^d_n = \theta_j, \mathcal{Y}_{0:n}\right) = \boldsymbol{P}\left(\mathcal{Y}^d_{n+1:K} \mid Z^d_n = \theta_j, Y^d_n \in dy^d_n\right), \tag{D.13}$$

a fact that will be used a number of times in the next part.

D.3 Expressions for the discrete smoother

The main objective of this section is to compute the smoother and two-slice marginal, $\{\gamma^{j,d}_n\}_{n=0}^{K-1}$ and $\{\xi^{i,j,d}_n\}_{n=0}^{K-2}$. For convenience, we repeat their definition here

$$\gamma^{j,d}_n = \mathbb{P}\left(Z^d_n = \theta_j \mid \mathcal{Y}^d_{0:K}\right), \qquad \text{and} \qquad \xi^{i,j,d}_n = \mathbb{P}\left(Z^d_n = \theta_i, Z^d_{n+1} = \theta_j \mid \mathcal{Y}^d_{0:K}\right)$$

for all allowed values of n, and for each $i, j = 1 \ldots J$.

To this end, note that

$$\gamma^{j,d}_n = \mathbb{P}\left(Z^d_n = \theta_j \mid \mathcal{Y}^d_{0:K}\right) = \frac{\mathbb{P}\left(Z^d_n = \theta_j, \mathcal{Y}^d_{0:K}\right)}{\mathbb{P}\left(\mathcal{Y}^d_{0:K}\right)} \tag{D.14a}$$

$$= \frac{\mathbb{P}\left(\mathcal{Y}^d_{n+1:K} \mid Z^d_n = \theta_j, \mathcal{Y}^d_{0:n}\right) \mathbb{P}\left(Z^d_n = \theta_j \mid \mathcal{Y}^d_{0:n}\right)}{\mathbb{P}\left(\mathcal{Y}^d_{n+1:K} \mid \mathcal{Y}^d_{0:n}\right)} \tag{D.14b}$$

$$= \alpha^{j,d}_n\, \beta^{j,d}_n. \tag{D.14c}$$

Next,

$$\xi^{i,j}_n = \mathbb{P}\left(Z^d_n = \theta_i, Z^d_{n+1} = \theta_j \mid \mathcal{Y}^d_{0:K}\right) \tag{D.15a}$$

$$= \frac{\mathbb{P}\left(Z^d_{n+1} = \theta_j, \mathcal{Y}^d_{n+1:K} \mid Z^d_n = \theta_i, \mathcal{Y}^d_{0:n}\right) \mathbb{P}\left(Z^d_n = \theta_i \mid \mathcal{Y}^d_{0:n}\right)}{\mathbb{P}\left(\mathcal{Y}^d_{n+1:K} \mid \mathcal{Y}^d_{0:n}\right)} \tag{D.15b}$$

$$= \alpha^{i,d}_n \left(\frac{\mathbb{P}\left(\mathcal{Y}^d_{n+2:K} \mid Z^d_{n+1} = \theta_j, \mathcal{Y}^d_{0:n+1}\right) \mathbb{P}\left(Y^d_{n+1} \in dy^d_{n+1}, Z^d_{n+1} = \theta_j \mid Z^d_n = \theta_i, \mathcal{Y}^d_{0:n}\right)}{\mathbb{P}\left(\mathcal{Y}^d_{n+2:K} \mid \mathcal{Y}^d_{0:n+1}\right) \mathbb{P}\left(Y^d_{n+1} \in dy^d_{n+1} \mid \mathcal{Y}^d_{0:n}\right)} \right) \tag{D.15c}$$

$$= \frac{\alpha^{i,d}_n\, \beta^{j,d}_{n+1}}{c_{n+1}\, d\boldsymbol{\mu}(y^d_{n+1})} \mathbb{P}\left(Y^d_{n+1} \in dy^d_{n+1}, Z^d_{n+1} = \theta_j \mid Z^d_n = \theta_i, \mathcal{Y}^d_{0:n}\right) \tag{D.15d}$$

$$= \frac{\alpha_n^{i,d}\, \beta_{n+1}^{j,d}}{c_{n+1}^d} \boldsymbol{P}_{i,j}\, f_\psi \left(t_{n+1}, y_{n+1}^d; t_n, \theta_i, y_n^d \right). \tag{D.15e}$$

The relationships between (γ, ξ), and (α, β) are applied when performing the E-step in the EM algorithm described in Section 7.1.

The natural ordering of computation proceeds by first computing $\{\alpha_n^{j,d}\}_{n=0}^{K-1}$, $\{c_n^d\}_{n=0}^{K-1}$ and $\{\beta_n^{j,d}\}_{n=0}^{K-1}$ in order, and then using the results to compute γ and ξ.

APPENDIX E: CALIBRATION TO INTC STOCK RETURNS

This section contains the results for the truncated pure jump model described in Section 7.2, calibrated to per-second prices on INTC stocks. The calibrated parameters are displayed below for the models with 1–6 latent states. As mentioned in Section 7.3, we use the BIC and ICL criterion to determine the "optimal" number of latent states. The BIC is defined as

$$BIC = \log L^\star - \frac{\nu_M}{2} \log (K \times D), \tag{E.1}$$

where $\log L^\star$ is the value of the maximized log-likelihood for a given model, ν_J is the number of parameters present in the model, and recall that D represents the number of observation days, and K the number of observations within a day (assumed equal across days).

As discussed in Section 7.1, the log-likelihood cannot be computed directly. Instead, we use the forward–backward algorithm in Section Appendix D to compute it. The log-likelihood for a given model, using the notation of Section 7.1, can be computed as

$$\log L = \sum_{d=1}^{D} \sum_{k=0}^{K-1} \log c_k^d. \tag{E.2}$$

The approximation to the ICL of Biernacki et al. (2000) for our model can be computed directly as

$$ICL = \sum_{d=1}^{D} \sum_{k=0}^{K-1} \log f_{\psi^\star} \left(t_{k+1}, y_{k+1}^d; t_k, y_k^d, \hat{Z}_k^d \right) - \frac{\nu_M}{2} \log (K \times D), \tag{E.3}$$

where ν_M is again the number of parameters present in the model. $\psi^\star$ are the parameters appearing in the transition density function f_ψ, which maximize the model's log-likelihood. $\hat{Z}^d$ is the most likely path of Z^d conditional on $\mathcal{Y}_{0:K}^d$, as computed by the Viterbi algorithm, using the parameters that maximize the model's likelihood.

Tables 1, 2 and A1–A6 record the calibrated parameters for the mean-reverting pure jump model presented in Section 7.2. Each table contains the calibrated parameters using the EM algorithm for the number of possible states for the latent process ranging from 1 to 6. Each of the rows in the tables below are ordered by the base noise level μ_i. Along with the parameters μ_i, κ_i, and θ_i, we also include the initial probability of the latent random variable starting in each of the given states (π_0^i) as well as the generator matrix for the latent process, which is computed as $\boldsymbol{C} = \log \boldsymbol{P}$.

TABLE A1 Mean-reverting pure jump model calibration results for INTC and $J = 1$

One latent state		
μ_i	κ_i	θ_i
0.0334	0.0748	0.0200

TABLE A2 Mean-reverting pure jump model calibration results for INTC and $J = 2$

Two latent states						
					$C_{i,j}$ State j	
State i	π_0^i	μ_i	κ_i	θ_i	1	2
1	0.7969	0.0899	0.0897	0.0200	−0.00792	0.00792
2	0.2031	0.0183	0.0100	0.0201	0.00245	−0.00245

TABLE A3 Mean-reverting pure jump model calibration results for INTC and $J = 3$

Three latent states							
					$C_{i,j}$ State j		
State i	π_0^i	μ_i	κ_i	θ_i	1	2	3
1	0.4483	0.1727	0.0950	0.0100	−0.0147	0.0147	0.0000
2	0.4510	0.0447	0.0288	0.0000	0.0020	−0.0037	0.0017
3	0.1006	0.0139	0.0117	0.0200	0.0000	0.0012	−0.0012

TABLE A4 Mean-reverting pure jump model calibration results for INTC and $J = 4$

Four latent states								
					$C_{i,j}$ State j			
State i	π_0^i	μ_i	κ_i	θ_i	1	2	3	4
1	0.3431	0.2457	0.1348	−0.0213	−0.0226	0.0225	0.0001	0.0000
2	0.2875	0.0658	0.0292	0.0100	0.0031	−0.0057	0.0026	0.0000
3	0.3368	0.0305	0.0176	0.0000	0.0000	0.0009	−0.0023	0.0014
4	0.0326	0.0100	0.0079	0.0000	0.0000	0.0000	0.0015	−0.0015

TABLE A5 Mean-reverting pure jump model calibration results for INTC and $J = 5$

Five latent states									
					$C_{i,j}$ State j				
State i	π_0^i	μ_i	κ_i	θ_i	1	2	3	4	5
1	0.1145	0.2255	0.1805	−0.0201	−0.0040	0.0000	0.0040	0.0000	0.0000
2	0.5979	0.1373	0.0463	−0.0095	0.0000	−0.0835	0.0786	0.0049	0.0000
3	0.0000	0.0376	0.0188	0.0301	0.0003	0.0325	−0.0328	0.0000	0.0000
4	0.2663	0.0269	0.0149	0.0222	0.0000	0.0006	0.0000	−0.0020	0.0014
5	0.0213	0.0084	0.0065	−0.0004	0.0000	0.0000	0.0000	0.0019	−0.0019

TABLE A6 Mean-reverting pure jump model calibration results for INTC and $J = 6$

Six latent states										
					$C_{i,j}$					
					State j					
State i	π_0^i	μ_i	κ_i	θ_i	**1**	**2**	**3**	**4**	**5**	**6**
1	0.5678	0.3206	0.0563	2.7090	−0.4276	0.0003	0.2485	0.1405	0.0331	0.0051
2	0.0906	0.3080	0.2608	−0.0800	0.0000	−0.0103	0.0103	0.0000	0.0000	0.0000
3	0.0054	0.1078	0.0359	−0.0137	0.0000	0.0012	−0.0144	0.0131	0.0001	0.0000
4	0.1291	0.0405	0.0176	−0.0021	0.0000	0.0000	0.0046	−0.0055	0.0009	0.0000
5	0.1915	0.0260	0.0143	0.0238	0.0000	0.0000	0.0003	0.0000	−0.0025	0.0022
6	0.0157	0.0074	0.0054	0.0161	0.0000	0.0000	0.0000	0.0000	0.0029	−0.0030