Quantitative Finance, 2020
Vol. 20, No. 11, 1795–1809, https://doi.org/10.1080/14697688.2020.1772497

Testing for jumps based on high-frequency data: a method exploiting microstructure noise

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(*Received 2 October 2018; accepted 15 May 2020; published online 2 September 2020*)

This paper tests for jumps of the price process based on noisy high-frequency data. Under the null hypothesis that the price process is continuous, the test statistic converges to a normal distribution, and under the alternative hypothesis that the price has jumps, the statistic converges to infinity. Compared with the test of Aït-Sahalia *et al.* [Testing for jumps in noisy high frequency data. *J. Econom.*, 2012, **168**(2), 207–222], our proposed statistic uses information on the microstructure noise, tends to infinity more rapidly under the alternative hypothesis and has a better power. A simulation confirms the theoretical results and an empirical study illustrates the practical application of the method.

Keywords: Testing for jumps; Price jumps; High-frequency data; Microstructure noise; Edgeworth expansion

JEL Classification: C22, C12

1. Introduction

The wide availability of financial asset price data at higher frequencies has prompted the development of methodologies to estimate some important quantities (for example, the integrated volatility) and tests of the specification of suitable models for these data. See for example Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2002) for estimating the integrated volatility, and Aït-Sahalia (2002) and Aït-Sahalia and Jacod (2009) for testing for jumps.

In this paper, we focus on testing for jumps, i.e. discontinuities of observed price processes. Several tests for jumps have been introduced by Aït-Sahalia (2002) (based on the transition function of the process), Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen and Shephard (2006) (based on bipower variations), Jiang and Oomen (2008) (based on the difference between arithmetic and logarithmic returns), Andersen *et al.* (2007) and Lee and Mykland (2008) (based on comparing standardized intraday returns to a threshold), and Aït-Sahalia and Jacod (2009) and Fan and Fan (2011) (based on power variations with different sampled frequencies). These methods only work in the noiseless high-frequency situation. For comparisons of the above and other methods, see Theodosiou and Zikes (2011) and Dumitru and Urga (2014). For some other spectrum tests of asset returns, see, among others, Aït-Sahalia and Jacod (2010), Aït-Sahalia and Jacod (2018), Aït-Sahalia and Jacod (2012), Kong *et al.* (2015) and Jing *et al.* (2012).

When we handle real high-frequency financial data, they are usually sampled with measurement errors, and have noise or microstructure noise due to measurement errors, bid–ask bounces, discreteness of price changes, and so on (Aït-Sahalia and Jacod 2014). There are many methods to handle the impact of microstructure noise, although most approaches in the literature focus mainly on estimating the integrated volatility. When the underlying observed price process is continuous, the methods include those of Zhang *et al.* (2005) (based on linear combination of realized volatilities obtained by subsampling), Barndorff-Nielsen *et al.* (2008) (based on linear combination of realized autocovariances), Jacod *et al.* (2009) and Podolskij and Vetter (2009b) (preaveraging), Xiu (2010) (quasi-maximum likelihood estimation), and Malliavin and Mancino (2009), Mancino and Sanfelici (2012) and Sanfelici *et al.* (2015) (Fourier analysis). When the underlying price process has jumps, the methods include those of Fan and Wang (2007) (based on wavelets and thresholds), Podolskij and Vetter (2009a) and Podolskij and Vetter (2009b) (based on preaveraging and bipower variations), and Jing *et al.* (2014) (based on preaveraging and thresholds).

Aït-Sahalia *et al.* (2012) tested for jumps using noisy high-frequency data. They proposed a robustification of the test statistic of Aït-Sahalia and Jacod (2009) based on the preaveraging method of Jacod *et al.* (2010). Aït-Sahalia and Jacod (2009) defined two pth-power variations with different

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sample intervals and used their ratio to test for jumps based on noiseless data. Aït-Sahalia *et al.* (2012) first constructed a robustified version of the pth-power variations and found that the ratio of robustified versions is still valid in testing for jumps in the noisy high-frequency situation. Under the null hypothesis that the process is continuous, the test statistic is asymptotically normal, and will tend to infinity at a rate $n^{1/4}$ under the alternative hypothesis when the process has jumps. Here, n is the number of observations on the unit time interval.

This paper again tests for jumps based on noisy high-frequency data. We use the preaveraging method to construct the average increments with a smaller window relative to the window of the standard preaveraging; then the jumps and noise parts will dominate in the average increments, and the continuous part will disappear asymptotically. Based on these increments, we use the two-scale method to build a test statistic. Under the null hypothesis that the process is continuous, our proposed test statistic is asymptotically normal, and tends to infinity more rapidly than that of Aït-Sahalia *et al.* (2012) under the alternative hypothesis that the process has jumps. That means our proposed test statistic has a better power than that of Aït-Sahalia *et al.* (2012).

The remainder of the paper is organized as follows. In section 2, we present the nonparametric model and the test statistic. Section 3 reports the Monte Carlo simulation results, and empirical results are presented in section 4. We conclude in section 5. The proofs of the main results are deferred to the appendix.

2. The test statistic

2.1. The setup

We suppose that the underlying process X is a semimartingale and that the following assumption on X holds. In financial econometrics, X is usually the log-price of a financial asset, and this assumption is standard; see for example Aït-Sahalia and Jacod (2014) and Jacod and Protter (2012).

Assumption 1 Suppose that X is defined in a probability space $(\Omega, \mathcal{F}, P)$ with the filtration $(\mathcal{F}_t)$ and is an Itô semimartingale of the form

$$X_t = X_0 + \int_0^t b_s\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}W_s + (\delta 1_{\{|\delta|\le 1\}}) \star (\underline{\mu} - \underline{\nu})_t + (\delta 1_{\{|\delta|>1\}}) \star \underline{\mu}_t, \tag{1}$$

where (W_t) is a standard Brownian motion and $\underline{\mu}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{E}$. Here, $\mathbb{E}$ is some Polish space, and the intensity measure or compensator of $\underline{\mu}$ is of the form $\underline{\nu}(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}t \otimes \lambda(\mathrm{d}z)$, where λ is a σ-finite measure on $\mathbb{E}$. We further assume that

(a) there is a localizing sequence (τ_n) of stopping times, for each n, a deterministic nonnegative function Υ_n on $\mathbb{E}$ such that $|\delta(\omega, t, z)| \wedge 1 \le \Upsilon_n(z)$ for all (ω, t, z) with $t \le \tau_n(\omega)$, and Υ_n satisfies $\int \Upsilon_n(z)^\beta \lambda(\mathrm{d}z) < \infty$ for some $\beta \le 2$;
(b) the adapted processes $(b_t), (\sigma_t)$ and the predictable process (δ_t) are locally bounded and càdlàg.

When $\delta \equiv 0$, X is continuous and has the form

$$X_t = X_0 + \int_0^t b_s\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}W_s. \tag{2}$$

In practice, high-frequency financial data are likely to be contaminated by microstructure noise induced by market microstructure effects such as measurement errors and the bid–ask spread. Instead of observing the log-price X, we observe the noisy data

$$Y_t = X_t + \epsilon_t, \tag{3}$$

where the noise variables $(\epsilon_t : t \ge 0)$ are zero-mean and mutually independent, and the noise process (ϵ_t) is independent of the underlying process X. Suppose the following assumption holds.

Assumption 2 All noise variables $(\epsilon_t : t \ge 0)$ are independent, and we have $E\epsilon_t = 0$, and that the variance $\alpha_t^2 = E(\epsilon_t^2)$ is a continuous deterministic finite function of t.

Suppose that the underlying process X is observed at equal distances $\Delta_n = 1/n$ over the interval $[0, T]$; i.e. we sample at the following discrete time points:

$$\{0 = t_0^n < t_1^n < t_2^n < \cdots < t_{[nT]}^n = [T/\Delta_n]\Delta_n\},$$

where $t_i^n = i\Delta_n$ and $[x]$ is the integer part of the number x. When we consider the asymptotic properties of the following statistic, we suppose that n goes to infinity. That means we are in the infill asymptotic situation. Note that the above assumptions on X and ϵ are usual and are often used in the literature; see for example Jing *et al.* (2011), Aït-Sahalia *et al.* (2012), and Aït-Sahalia and Jacod (2014).

2.2. The test statistic: a method exploiting microstructure noise

This paper tests for jumps from observations belonging to a fixed time interval $[0, T]$. Namely, we want to decide in which of the following two complementary sets the observed path falls:

$$\left.\begin{aligned} \Omega_T^{\mathrm{c}} &= \{\omega : t \to X_t(\omega) \text{ is continuous on } [0, T]\}, \\ \Omega_T^{\mathrm{j}} &= \{\omega : t \to X_t(\omega) \text{ is discontinuous on } [0, T]\}. \end{aligned}\right\} \tag{4}$$

When the null hypothesis holds, the observed path is continuous and falls in the set Ω_T^{c}. When the alternative hypothesis holds, the path is discontinuous with jumps and falls in Ω_T^{j}. Hence, the test problem is formally stated as

$$H_0 : \omega \in \Omega_T^{\mathrm{c}} \quad \text{vs.} \quad H_1 : \omega \in \Omega_T^{\mathrm{j}}. \tag{5}$$

To construct a test statistic that has better power than that of Aït-Sahalia *et al.* (2012), we use the idea of preaveraging. We define an integer

$$k_n = [cn^a], \quad a > 0, \quad c > 0, \tag{6}$$

which is the preaveraging window and is used to compute the local average increments. We choose a weight function g defined on $[0, 1]$ and satisfying the following assumption.

ASSUMPTION 3 g is continuous, piecewise C^1 with a piecewise Lipschitz derivative g', and $g(0) = g(1) = 0$, $\int_0^1 g(s)^2\,\mathrm{d}s > 0$.

We first define some additional numbers associated with the weight function g:

$$g_i^n = g(i/k_n), \quad g_i'^n = g_{i+1}^n - g_i^n, \quad \overline{g}(p) = \int_0^1 |g(s)|^p\,\mathrm{d}s,$$
$$\overline{g}'(p) = \int_0^1 |g'(s)|^p\,\mathrm{d}s, \quad p > 0. \tag{7}$$

For any process V, set

$$\overline{V}_i^n(k_n) = \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j}^n V = -\sum_{j=0}^{k_n-1} g_j'^n V_{i+j}^n, \tag{8}$$
$$\Delta_i^n V = V_i^n - V_{i-1}^n, \quad V_i^n = V_{i\Delta_n}.$$

In the case of $g(x) = 2(x \wedge (1-x))$, when $k_n = 2k_n'$ and k_n' is an integer, we have

$$\overline{V}_i^n(k_n) = \widetilde{V}_{(i+k_n')\Delta_n} - \widetilde{V}_{i\Delta_n}, \quad \widetilde{V}_{i\Delta_n} = \frac{1}{k_n'}\sum_{j=0}^{k_n'-1} V_{(i+j)\Delta_n},$$

where $\widetilde{V}_{i\Delta_n}$ is the mean of the process V in $[i\Delta_n, (i+k_n')\Delta_n]$. Note that $\overline{V}_i^n(k_n)$ is the local average increment of V based on the interval $[i\Delta_n, (i+k_n)\Delta_n]$. Some of the above notation follows that in Jacod *et al.* (2009).

We define the realized power variation and realized threshold power variation based on the preaveraging increments:

$$V_p(V, k_n)_t^n = \sum_{i=0}^{[t/\Delta_n]-k_n} |\overline{V}_i^n(k_n)|^p,$$
$$TV_p(V, k_n)_t^n = \sum_{i=0}^{[t/\Delta_n]-k_n} |\overline{V}_i^n(k_n)|^p 1_{|\overline{V}_i^n(k_n)|<u_n}, \quad p > 0, \tag{9}$$

where u_n is a truncation value to truncate the jumps of the underlying process. Let $u_n = \overline{c}n^{-\overline{a}}$, where $\overline{c}, \overline{a} > 0$; then we have the following limiting results of $V_p(V, k_n)_t^n$ and $TV_p(V, k_n)_t^n$.

THEOREM 2.1 *Suppose that Assumptions 1–3 hold and $E|\epsilon_t|^{2p+\gamma}$ is a bounded function of t for some $\gamma > 0$, and $0 < a < 1/2$.*

(a) *Assume $p < 2(1-a)/a$; then, as $n \longrightarrow \infty$,*

$$n^{pa/2-1} V_p(Y, k_n)_t^n \xrightarrow{P} m_p \left(\frac{\overline{g'}(2)}{c}\right)^{p/2} \int_0^t \alpha_s^p\,\mathrm{d}s, \tag{10}$$

where $m_p = E|N(0,1)|^p$ is the pth absolute moment of the standard normal distribution.

(b) *Assume $p > 2(1-a)/a$; then, as $n \longrightarrow \infty$,*

$$n^{-a} V_p(Y, k_n)_t^n \xrightarrow{P} c\overline{g}(p) \sum_{s \le t} |\Delta X_s|^p. \tag{11}$$

(c) *Assume $0 < \overline{a} < a/2$, if either $p \le 2$ or $p > 2$ with $\overline{a} > (pa + 2a - 2)/2(p - \beta)$; then, as $n \longrightarrow \infty$,*

$$n^{pa/2-1} TV_p(Y, k_n)_t^n \xrightarrow{P} m_p \left(\frac{\overline{g'}(2)}{c}\right)^{p/2} \int_0^t \alpha_s^p\,\mathrm{d}s. \tag{12}$$

REMARK 1 In the classical preaveraging method, when the process is a continuous semimartingale with jumps and noise, a is chosen to be 1/2, and the continuous semimartingale part, the jumps part and the noise part share the same magnitude in the local average increment term $\overline{Y}_i^n$. However, when $0 < a < 1/2$, the continuous semimartingale part will asymptotically disappear in the local average increments. Furthermore, when the power $p < 2(1-a)/a$, the noise part dominates in the term $\overline{Y}_i^n$ and determines the limiting result of the statistic $V_p(Y, k_n)_t^n$; when $p > 2(1-a)/a$, the jumps part determines the limiting result of $V_p(Y, k_n)_t^n$. When $\delta \equiv 0$, where the process X is continuous, the result (10) holds for all $p > 0$.

REMARK 2 For the case $a = 1/2$, Jing *et al.* (2014) analysed the asymptotic property of the statistic $TV_p(Y, k_n)_t^n$ and obtained a result similar to (12); i.e. the right limiting variable of (12) includes an additional term corresponding to the continuous part $\int_0^t \sigma_s\,\mathrm{d}W_s$ of the underlying process X.

Use the two-scale method, we construct our test statistic:

$$T_n = \frac{V_p(Y, k_n)_T^n}{V_p(Y, 2k_n)_T^n}. \tag{13}$$

By Theorem 2.1, we have the following result.

THEOREM 2.2 *Suppose that Assumptions 1–3 hold and $E|\epsilon_t|^{2p+\gamma}$ is a bounded function of t for some $\gamma > 0$. We further suppose that $0 < a < 1/2$, $p > 2(1-a)/a$ and $E\epsilon_t^2 > 0$ for $0 \le t \le T$. Then*

$$T_n \xrightarrow{P} \begin{cases} 2^{p/2} & \text{on the set } \Omega_T^{\mathrm{c}}, \\ \dfrac{1}{2} & \text{on the set } \Omega_T^{\mathrm{j}}. \end{cases}$$

REMARK 3 When the underlying process is continuous, the two terms $V_p(Y, k_n)_T^n$ and $V_p(Y, 2k_n)_T^n$ of T_n are asymptotically determined by the microstructure noise, since the moving window k_n is smaller than that of the classical preaveraging method. In other words, the noise information is exploited to construct our test statistic T_n, and the test statistic T_n has good power, which we will see from Theorem 2.3.

REMARK 4 When the noise disappears, i.e. $\epsilon_t \equiv 0$, by the proof of Theorem 1 of Podolskij and Vetter (2009a), suppose $p < 2$ or $p > 0$ for $J_t \equiv 0$, we can prove

$$n^{p(1-a)/2-1} V_p(Y, k_n)_t^n \xrightarrow{P} m_p\,(c\overline{g}(2))^{p/2} \int_0^t \sigma_s^p\,\mathrm{d}s \tag{14}$$

holds for $0 < a < 1$. Note that the result (11) also holds for $\epsilon_t \equiv 0$ and $p > 2$. Suppose $\epsilon_t \equiv 0$, $0 < a < 1$ and $p > 2$, then we have

$$T_n \xrightarrow{P} \begin{cases} 2^{-p/2} & \text{on the set } \Omega_T^{\mathrm{c}}, \\ \dfrac{1}{2} & \text{on the set } \Omega_T^{\mathrm{j}}. \end{cases}$$

In other words, when the infeasible situation $\epsilon_t \equiv 0$ happens, T_n also can discriminate the hypothesis problem (5).

To use the statistic T_n to test the problem (5), we need to construct the central limit theorem for T_n under the null hypothesis and determine the critical region. We need some additional assumptions on the weight function g and the noise ϵ.

Assumption 4 (1) g is continuous and piecewise C^2 with a piecewise Lipschitz derivative g''. $\inf_{x\in D_g} |g'(x)| > 0$ where D_g is the set of all differentiable points of function g.

(2) The noise ϵ satisfies the following conditions:

(i) ϵ_t is distributed symmetrically around zero for all $t \geq 0$, and we have

$$\sup_{|s|>s_0} \sup_{t\geq 0} |\chi_t(s)| < 1 \tag{15}$$

for all positive s_0, where χ_t denotes the characteristic function of ϵ_t, i.e. $\chi_t(s) = E\exp(\mathrm{i}s\epsilon_t)$ with $\mathrm{i}^2 = -1$.

(ii) The function α_t is Lipschitz continuous, and

$$0 < \inf_{t\geq 0} \alpha_t^2 \leq \sup_{t\geq 0} \alpha_t^2 < \infty,$$
$$\sup_{t\geq 0} E|\epsilon_t|^{4\vee(2\lceil p/2\rceil)+\gamma} < \infty \tag{16}$$

for some $\gamma > 0$, where $\lceil x\rceil$ is the ceiling function of x, i.e. the least integer greater than or equal to x.

When $(\epsilon_t, t \geq 0)$ are i.i.d., the condition (15) degenerates to $\sup_{|s|>s_0} |\chi(s)| < 1$ for all positive s_0, which is equivalent to the standard Cramér's condition $\limsup_{|s|\to\infty} |\chi(s)| < 1$, and the condition (ii) also holds provided that $E|\epsilon|^{4\vee(2\lceil p/2\rceil)+\gamma} < \infty$ holds. When one use an expansion of Edgeworth type, Cramér's condition is a standard assumption. A sufficient condition for Cramér's condition to hold for a random variable is that the distribution of this random variable has an absolutely continuous component with respect to Lebesgue measure. For more details of Cramér's condition and the asymptotic Edgeworth expansion, see Bhattacharya and Ranga Rao (1986) and Lahiri (2003). The Cramér's condition assumption with respect to the noise is also used in Podolskij and Vetter (2009a) to obtain the central limit theorem for a bipower-type statistic in a noisy setting. For the condition (ii), i.e. the Lipschitz continuity of α_t and the high-order moment condition (16), some similar conditions are used in Podolskij and Vetter (2009a) and Liu and Jing (2018).

To present the central limit theorem for T_n, we define some additional notation. Set

$$h(\rho, p) = \mathrm{Cov}(|H_1|^p, |H_2|^p), \tag{17}$$

where H_1 and H_2 are two standard normal random variables with the correlation $\rho = EH_1H_2$. Let

$$\left.\begin{aligned} \rho_{11}(s) &= \frac{\int_0^{1-s} g'(x)g'(x+s)\,\mathrm{d}x}{\overline{g}'(2)}, \\ \rho_{12}(s) &= \frac{\int_0^{1-s} g'(\frac{x}{2})g'(x+s)\,\mathrm{d}x}{\sqrt{2}\overline{g}'(2)}, \\ \rho_{21}(s) &= \frac{\int_0^{1\wedge(2-s)} g'(x)g'(\frac{x+s}{2})\,\mathrm{d}x}{\sqrt{2}\overline{g}'(2)}, \quad \rho_{22}(s) = \rho_{11}(s), \end{aligned}\right\} \tag{18}$$

where $\rho_{11}(s) = \rho_{12}(s) = 0$ for $s \notin [0,1]$ and $\rho_{21}(s) = 0$ for $s \notin [0,2]$. Note that $\rho_{lj}(s)$ is the limit of the correlation between $n^{a/2}\overline{\epsilon}_1^n(lk_n)$ and $n^{a/2}\overline{\epsilon}_{1+i}^n(jk_n)$ with $i/k_n \longrightarrow s$; i.e.

$$\begin{aligned} \rho_{lj}(s) &= \lim_{n\to+\infty} \rho_{lj}\left(\frac{i}{k_n}\right) \\ &=: \lim_{n\to+\infty} \mathrm{Cor}(n^{a/2}\overline{\epsilon}_1^n(lk_n), n^{a/2}\overline{\epsilon}_{1+i}^n(jk_n)), \quad l,j = 1,2. \end{aligned}$$

Then, we have the following limiting result.

Theorem 2.3 *Suppose that Assumptions* 1–4 *hold and that the function* $E|\epsilon_t|^{4\vee(2p)+\gamma}$ *is a bounded function for some* $\gamma > 0$. *Suppose* $1/3 < a < 1/2$ *and* $p > (1-a)/(1-2a)$. *Under the null hypothesis* H_0 *that* $\omega \in \Omega_T^{\mathrm{c}}$, *we have*

$$\frac{n^{(1-a)/2}(T_n - 2^{p/2})}{\Sigma_p} \xrightarrow{L} N(0,1), \tag{19}$$

where

$$\begin{aligned} \Sigma_p^2 &= c\frac{2^p\left[6\int_0^1 h(\rho_{11}(s),p)\,\mathrm{d}s - 2\left(\int_0^1 h(\rho_{12}(s),p)\,\mathrm{d}s + \int_0^2 h(\rho_{21}(s),p)\,\mathrm{d}s\right)\right]}{m_p^2} \\ &\quad\times \frac{\int_0^T \alpha_t^{2p}\,\mathrm{d}t}{\left(\int_0^T \alpha_t^p\,\mathrm{d}t\right)^2} \\ &=: c\Gamma_p\frac{\int_0^T \alpha_t^{2p}\,\mathrm{d}t}{\left(\int_0^T \alpha_t^p\,\mathrm{d}t\right)^2}, \end{aligned} \tag{20}$$

and Γ_p *is a constant dependent on the weight function* g *and the parameter* p.

Remark 5 Each $h(\rho, p)$ defined in (17) can be computed, but the calculations are rather complicated, except for some special cases. In fact, when p is an even integer, we have

$$\begin{aligned} &h(\rho,2) = 2\rho^2, \quad h(\rho,4) = 72\rho^2 + 24\rho^4, \\ &h(\rho,6) = 4050\rho^2 + 5400\rho^4 + 720\rho^6, \\ &h(\rho,8) = 352800\rho^2 + 1058400\rho^4 + 564480\rho^6 + 40320\rho^8. \end{aligned}$$

When $g(x) = x \wedge (1-x)$, by (18) and (20), we have

$$\rho_{11}(s) = (1-3s)1_{0\le s<0.5} + (s-1)1_{0.5\le s<1},$$

$$\rho_{12}(s) = \frac{1}{\sqrt{2}}(-s1_{0\le s<0.5} + (s-1)1_{0.5\le s<1}),$$

$$\rho_{21}(s) = \frac{1}{\sqrt{2}}(2s1_{0\le s<0.5} + 2(1-s)1_{0.5\le s<1} + (1-s)1_{1\le s<1.5} + (s-2)1_{1.5\le s<2}),$$

and

$$\Gamma_2 = 4, \quad \Gamma_4 = \frac{392}{5} = 78.4, \quad \Gamma_6 = \frac{5736}{5} = 1147.2,$$

$$\Gamma_8 = \frac{558816}{35} \approx 15966.17.$$

Set

$$\widehat{\Sigma_p^2} = c\Gamma_p \frac{nTV_{2p}(Y,k_n)_T^n m_p^2}{m_{2p}(TV_p(Y,k_n)_T^n)^2}; \tag{21}$$

then, by the result (12), we find that $\widehat{\Sigma_p^2}$ is a consistent estimator of the asymptotic variance Σ_p^2 provided that the conditions of the result (12) hold. Hence, we have the following corollary.

Corollary 2.4 *Suppose the conditions of Theorem* 2.3 *hold and* $\max\{(pa+2a-2)/2(p-\beta),(2pa+2a-2)/2(2p-\beta)\} < \overline{a} < a/2$; *then*

$$\widetilde{T}_n =: \frac{n^{(1-a)/2}(T_n - 2^{p/2})}{\sqrt{\widehat{\Sigma_p^2}}} \xrightarrow{L} N(0,1). \tag{22}$$

Proposition 2.5 *Suppose that Assumption* 1 *and condition* (1) *of Assumption* 4 *hold and that the noise variables are i.i.d. normal. Suppose* $1/5 < a < 1/2$ *and* $p > (1-a)/(1-2a)$. *Then, under the null hypothesis* H_0 *that* $\omega \in \Omega_T^c$, *the result* (19) *also holds with* $\Sigma_p^2 = c\Gamma_p/T$.

Remark 6 Aït-Sahalia *et al.* (2012) tested for jumps based on noisy high-frequency data and found that their test statistic is normal under the null hypothesis H_0 and goes to infinity at a rate $n^{1/4}$ under the alternative hypothesis H_1. By the condition $1/3 < a < 1/2$ of Theorem 2.3, we have $(1-a)/2 > 1/4$, which means that the rate $n^{(1-a)/2}$ of our test statistic is faster than that of Aït-Sahalia *et al.* (2012) and that the power of our test statistic is higher than that of Aït-Sahalia *et al.* (2012). If we suppose that the noise variables are i.i.d. normal, then, we can avoid some asymptotic approximations and obtain Proposition 2.5 with a better rate compared with Theorem 2.3. By Proposition 2.5, we obtain that the convergence rate will be improved to $n^{(1-a)/2} = n^{2/5-\gamma}$ for $\gamma > 0$ since $1/5 < a < 1/2$. The additional identical normal distribution information of the noise makes the test statistic T_n closer to its limit $2^{p/2}$ under the null hypothesis, hence, the convergence rate can be improved.

Remark 7 By Remark 5 and (21), the asymptotic variance Σ_p^2 can be directly estimated by $\widehat{\Sigma_p^2}$, the statistic $\widehat{\Sigma_p^2}$ is always positive, and further the standardized statistic $\widetilde{T}_n$ always exists. By Corollary 2.4, the feasible statistic $\widetilde{T}_n$ converges to a standard normal distribution under the null hypothesis, and to negative infinity under the alternative hypothesis. However, in Aït-Sahalia *et al.* (2012), to obtain the critical region of their test statistic, they have to estimate the asymptotic variance of their unstandardized statistic and the estimator is very complicated, which will lead to a negative asymptotic variance estimator in some situations. Further, the critical region of their test statistic cannot be obtained in some situations, which will be seen from the simulation results below.

We now construct the critical region. For a given asymptotic level $\alpha \in (0,1)$, we denote by z_α the corresponding quantile of the standard normal distribution $N(0,1)$, i.e. $P(H \le z_\alpha) = \alpha$, where H is $N(0,1)$. We define the critical region as

$$C_n = \left\{ T_n < 2^{p/2} + z_\alpha n^{(a-1)/2}\sqrt{\widehat{\Sigma_p^2}} \right\}.$$

Then we have the following result.

Theorem 2.6 *Suppose that Assumptions* 1–4 *hold and that the function* $E|\epsilon_t|^{4\vee(2p)+\gamma}$ *is a bounded function for some* $\gamma > 0$. *Suppose* $1/3 < a < 1/2, p > \max\{2(1-a)/a,(1-a)/(1-2a)\}$ *and* $\max\{(pa+2a-2)/2(p-\beta),(2pa+2a-2)/2(2p-\beta)\} < \overline{a} < a/2$. *Then we have*

$$\lim_{n\to\infty} P(C_n|\Omega_T^c) = \alpha$$

under the null hypothesis H_0 *that* $\omega \in \Omega_T^c$, *and*

$$\lim_{n\to\infty} P(C_n|\Omega_T^j) = 1$$

under the alternative hypothesis H_1 *that* $\omega \in \Omega_T^j$.

Note that the results of Theorem 2.6 do not hold when the noises disappear. In other words, Theorem 2.6 can not use to test the problem (5) under the infeasible situation $\epsilon_t \equiv 0$.

3. Monte Carlo simulations

In this section, we investigate the performance of our test statistic through simulations and compare it with the performance of the test statistic of Aït-Sahalia *et al.* (2012).

3.1. Simulation design

The log-price Y_t is generated from the Heston model (Heston 1993) with jumps and noise:

$$\left.\begin{aligned} Y_t &= X_t^c + J_t + \epsilon_t, \quad 0 \le t \le T,\\ \mathrm{d}X_t^c &= (\mu - \sigma_t^2/2)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t,\\ d\sigma_t^2 &= \kappa(\overline{\alpha} - \sigma_t^2)\,\mathrm{d}t + \nu\sigma_t\,\mathrm{d}\widetilde{W}_t, \quad E[\mathrm{d}W_t\,\mathrm{d}\widetilde{W}_t] = \rho\,\mathrm{d}t, \end{aligned}\right\} \tag{23}$$

where X_t^c is the continuous part with stochastic volatility, J_t is a pure jump process, ϵ_t is the additive noise, (W_t) and $(\widetilde{W}_t)$ are standard Brownian motions, and $T, \mu, \kappa, \overline{\alpha}, \nu$ and ρ are model parameters. We select the values of model parameters by taking both Jacod *et al.* (2009) and Aït-Sahalia *et al.* (2012) as references. Specifically, we set

$$(T, \mu, \kappa, \overline{\alpha}, \nu, \rho) = (5, 0.05/252, 5/252, 0.04/252, 0.05/252, -0.5).$$

Note that $T = 5$ corresponds to 5 days and that $\overline{\alpha} = 0.04/252$ corresponds to the case that the volatility σ of one year is about 20%.

We consider two settings for the additive noise:

$$\epsilon_t = \begin{cases} \sigma_\epsilon \epsilon_t^A & \text{(Gaussian noise with constant variance),} \\ \sigma_\epsilon \dfrac{\sigma_t}{\int_0^T \sigma_s \, ds/T} \epsilon_t^A & \text{(Gaussian noise with variable variance),} \end{cases} \tag{24}$$

where ϵ_t^A is a sequence variable sampled from an standard normal distribution and σ_ϵ is a positive constant reflecting the level of the noise. For the case of Gaussian noise with constant variance, σ_ϵ is the standard deviation of the noise. Besides the additive noise, we also consider the case of rounding error and the observed log-price given by $Z_t = \log(\tau \lfloor \exp(Y_t)/\tau \rfloor)$, where $\lfloor x \rfloor$ is the nearest integer to x and $\tau = 0.01$ is the rounding threshold value. Although our asymptotic theory does not cover the rounding error case, we perform some simulations to check the robustness. To further check the robustness, we present the simulation results for the two cases of σ_ϵ, i.e. the large-noise case $\sigma_\epsilon = 0.05$ and the small-noise case $\sigma_\epsilon = 0.025$.

The jump process J_t is simulated from a centered symmetric β-stable process with activity index $\beta = 0.5, 1, 1.5, 1.75$. To compare results across different activity levels, we scale J_t such that for each path, the realized quadratic variation of the jump process J_t equals σ_ϵ. When the jump activity index is large, the jump process J_t has many small jumps; when the jump activity index is small, J_t has a few big jumps. Note that the generation procedures for the noise and jump processes are similar to the procedure of Aït-Sahalia *et al.* (2012).

For the weight function, we choose $g(x) = x \wedge (1 - x), 0 \leq x \leq 1$. For each day, there are 6.5 transaction hours in the usual stock market and we sample the price process every second, which means that we have $n = 23\,400$ observed price data for each day. There are 5000 simulations in each experiment. For the tuning parameter $k_n = [cn^a]$, noting the condition $1/3 < a < 1/2$ of Theorem 2.3, we choose $a = 11/30$. To check the robustness of the tuning parameter c, we report the simulation results for $c = 0.9, 1, 1.1$. For the tuning parameter threshold value $u_n = \overline{c}n^{-\overline{a}}$, which is used to estimate the asymptotic variance (21), noting the condition $\overline{a} < a/2$, similar to the choice of Aït-Sahalia and Jacod (2009), we set

$$u_n = 5(\widehat{\sigma_\epsilon^2})^{1/2}(1/k_n)^{0.49}, \quad \widehat{\sigma_\epsilon^2} = \sum_{i=0}^{[nT]} |\Delta_i^n Y|^2/(2[nT]),$$

where the term $1/k_n$ is the level of $(\overline{\epsilon}_1^n(k_n))^2$, i.e. $E(\overline{\epsilon}_1^n(k_n))^2 = O(1/k_n)$. Note that $\widehat{\sigma_\epsilon^2}$ is a consistent estimator of the noise variance term $\int_0^T E\epsilon_t^2 \, dt/T$. Throughout the simulations and the empirical study, we choose $p = 4$. For the tuning parameters of the test of Aït-Sahalia *et al.* (2012), we retain their choices. In particular, we choose the averaging window of their test statistic to be 500, which means that we also use 500-second data to exclude the noise as in Aït-Sahalia *et al.* (2012).

3.2. Simulation results

Figure 1 gives the histogram and QQ plot of the standardized statistic $\widetilde{T}_n$, where the generating process follows (23) with $J \equiv 0$ and the tuning parameter $c = 1$. Figure 1 shows that

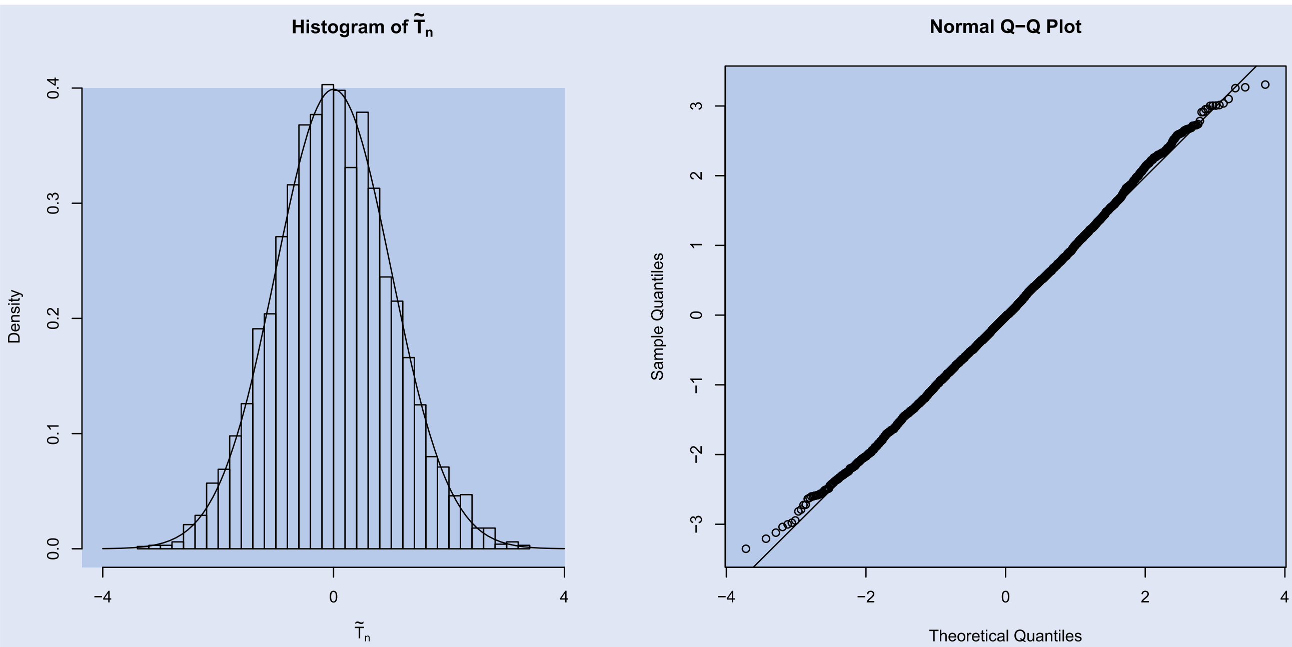

Figure 1. Histogram (left panel) and QQ plot (right panel) of the standardized test statistic $\widetilde{T}_n$. The tuning parameter $c = 1$.

Table 1. Finite sample size (%) of the test statistic $\widetilde{T}_n$ (22) and the test of Aït-Sahalia *et al.* (2012) (AJL) under the null hypothesis H_0 where the price process is continuous.

Additive noise	Size	No rounding $c = 0.9$	No rounding $c = 1$	No rounding $c = 1.1$	No rounding AJL	Rounding at 0.01 $c = 0.9$	Rounding at 0.01 $c = 1$	Rounding at 0.01 $c = 1.1$	Rounding at 0.01 AJL
$\sigma_\epsilon = 0.05$									
GC	10%	10.24%	10.28%	10.44%	4.48%	10.24%	10.24%	10.46%	4.57%
	5%	4.84%	4.74%	4.84%	2.92%	4.82%	4.74%	4.82%	2.97%
GT	10%	9.62%	10.06%	10.16%	4.71%	9.70%	10.06%	10.22%	4.78%
	5%	4.52%	4.80%	4.92%	3.38%	4.60%	4.82%	4.94%	3.28%
$\sigma_\epsilon = 0.025$									
GC	10%	11.44%	12.40%	12.48%	8.74%	11.44%	12.44%	12.80%	8.72%
	5%	5.22%	6.20%	6.80%	5.78%	5.26%	6.16%	6.78%	5.90%
GT	10%	11.70%	13.04%	14.28%	8.38%	11.70%	13.08%	14.28%	8.46%
	5%	5.92%	6.60%	6.80%	5.60%	5.92%	6.66%	6.80%	5.68%

Note: GC and GT denote Gaussian noise with constant and time-varying variance, respectively.

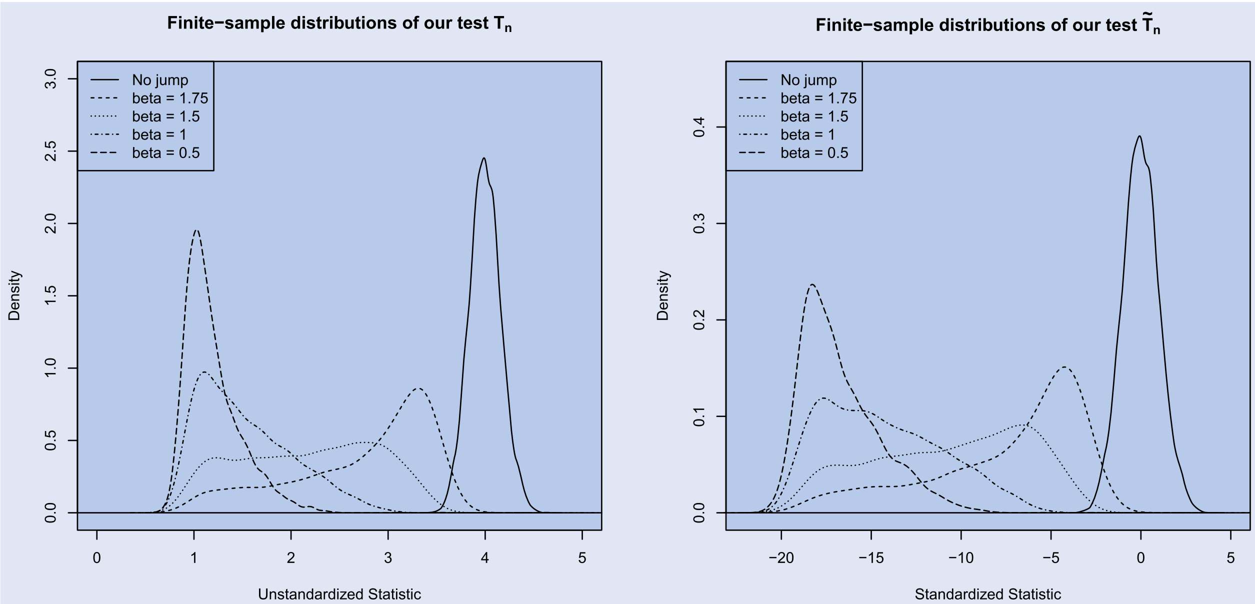

Figure 2. Finite-sample distributions of the unstandardized test statistic T_n (left panel) and the standardized test $\widetilde{T}_n$ (right panel). We plot the distribution of the model (23) with constant-variance Gaussian noise (one-second data). The solid line corresponds to the continuous case. For the discontinuous case, we plot the β-stable jump processes for $\beta = 1.75$ (short-dashed line), $\beta = 1.5$ (dotted line), $\beta = 1$ (dash-dotted line) and $\beta = 0.5$ (long-dashed line). In all cases, we set the tuning parameter $c = 1$. The considerable difference between the results of the continuous model and those of the discontinuous model suggests the great power of our test statistic.

the finite-sample distribution of $\widetilde{T}_n$ is very close to the theoretical normal distribution and verifies Theorem 2.3 under the null hypothesis H_0, which corresponds to the process being continuous. Table 1 presents the Monte Carlo sizes of our test statistic $\widetilde{T}_n$ and the test statistic of Aït-Sahalia *et al.* (2012) for different choices of the tuning parameter c and different noise situations. The results show that the finite sample sizes of our test statistic are all close to their nominal sizes and are robust to the choice of the tuning parameter c, and that the results are expected to become slightly worse when the noise size becomes smaller. However, for the test statistic of Aït-Sahalia *et al.* (2012), some paths of the generating process (23) cannot give the standardized test statistic, since the asymptotic variance estimators of the unstandardized statistic are negative, and there are some differences between the finite sample sizes and their nominal sizes.

Figure 2 presents the finite-sample distribution of the unstandardized test statistic T_n defined in (13). When the null hypothesis H_0 holds (solid line), i.e. there is no jump, the finite distribution is expected to be a standard normal distribution, as predicted by the asymptotic theory (Theorem 2.3). When the model has jumps, there is a big difference between the continuous model and the discontinuous model, which implies that the test statistic exhibits great power. Figure 2 shows that the smaller β of the β-stable jump process under the alternative hypothesis, the better separated the null hypothesis from the alternative hypothesis. Table 2 presents the power of our test statistic and that of the test of Aït-Sahalia *et al.* (2012), and the level is 5%. The results show that the rejection rate of our test is fairly robust to different tuning parameters c and to the various types of additive noise and is almost 100%. Since our test statistic tends to infinity faster than that of Aït-Sahalia

Table 2. Power comparison between our test and that of Aït-Sahalia *et al.* (2012) (AJL) for different jump activities β and different choices of c.

Additive noise	β	No rounding $c=0.9$	$c=1$	$c=1.1$	AJL	Rounding at 0.01 $c=0.9$	$c=1$	$c=1.1$	AJL
$\sigma_\epsilon=0.05$									
GC	1.75	97.86%	99.48%	99.88%	88.77%	97.86%	99.44%	99.88%	88.77%
	1.5	99.82%	100%	99.94%	99.14%	99.82%	100%	99.94%	99.14%
	1	100%	99.98%	100%	99.94%	100%	99.98%	100%	99.94%
	0.5	100%	99.96%	99.94%	99.89%	100%	99.96%	99.94%	99.89%
GT	1.75	97.68%	99.48%	99.84%	88.48%	97.70%	99.46%	99.84%	88.48%
	1.5	99.90%	99.96%	100%	99.02%	99.90%	99.96%	100%	99.02%
	1	100%	99.96%	99.94%	99.90%	100%	99.96%	99.94%	99.90%
	0.5	99.96%	99.98%	99.96%	99.89%	99.96%	99.98%	99.96%	99.89%
$\sigma_\epsilon=0.025$									
GC	1.75	99.98%	100%	99.98%	93.54%	99.98%	100%	99.98%	93.52%
	1.5	100%	100%	99.98%	99.64%	100%	100%	99.98%	99.66%
	1	99.98%	99.96%	99.96%	99.88%	99.98%	99.96%	99.96%	99.88%
	0.5	99.96%	100%	99.98%	99.96%	99.96%	100%	99.98%	99.96%
GT	1.75	100%	99.96%	99.96%	93.22%	100%	99.96%	99.96%	93.22%
	1.5	99.98%	100%	99.98%	99.56%	99.98%	100%	99.98%	99.56%
	1	99.98%	99.98%	99.98%	99.94%	99.98%	99.98%	99.98%	99.94%
	0.5	99.98%	100%	99.96%	99.94%	99.98%	100%	99.96%	99.94%

et al. (2012) under the alternative hypothesis, the power of our test is greater than that of the test of Aït-Sahalia *et al.* (2012), as predicted by Theorem 2.3. When $\beta=1.75$ and the generating process are rather close to the null hypothesis where the process is continuous, the powers are all relatively smaller compared to the other β cases.

Table 3. Summary statistics of the empirical results for a sample of 14 randomly selected stocks from the US stock markets in 2013. The summary results are computed for all 3528 ($=14\times 252$) stock-day pairs and different tuning parameters c.

	$c=0.9$	$c=1$	$c=1.1$
Mean of T_n	0.5710	0.5737	0.5724
Mean of $\widetilde{T}_n$	– 13.9814	– 12.9894	– 12.1386
Rejection rate (%)	98.75%	98.66%	98.66%

4. Real data analysis

In this section, we apply our method to test for jumps for a sample of randomly selected stocks from the US stock markets, namely Citigroup Inc. (tick symbol: C), Apple Inc. (tick symbol: AAPL), International Business Machines Corporation (tick symbol: IBM), General Motors Company (tick symbol: GM), Intel Corporation (tick symbol: INTC), Microsoft Corporation (tick symbol: MSFT), The Coca-Cola Company (tick symbol: KO), Wal-Mart Stores, Inc. (tick symbol: WMT), Boston Scientific Corporation (tick symbol: BSX), Celgene Corporation (tick symbol: CELG), Chevron Corporation (tick symbol: CVX), Johnson & Johnson (tick symbol: JNJ), Yahoo (tick symbol: YHOO), and Exxon Mobil Corporation (tick symbol: XOM). These are actively traded stocks on the markets. We study the second-by-second trade price data of these 14 stocks in the year 2013. There are 252 trading days in this year. We limit our interest to trade data during the usual exchange trading hours from 9:30 am to 4:00 pm. Also, overnight returns are excluded from our analysis. The one-second trade data are built on the tick-by-tick data by taking the closing price in each one-second interval as the price for that second. Then the previous tick interpolation (Zhang 2006) is applied to the one-second trade data sets to deal with those time stamps where prices are not available. Therefore, for each stock, we have a sample path that consists of $252\times 23\,400+1=5\,896\,801$ observed prices to analyze.

Table 3 provides the test statistic results for the 14 randomly selected stocks for different tuning parameters c. For the other tuning parameters, i.e. p, a and u_n, we select as in the simulation procedure. The second row reports the unstandardized test statistic T_n, and we can see that the mean value of T_n is 0.5737 for $c=1$, which is near to the center 0.5 of the alternative hypothesis discontinuous price. When the tuning parameter $c=1$, we reject the null hypothesis of a continuous path 98.66% at the 5% nominal size. When the tuning parameter c changes, the rejection rates are similarly high. We find statistical evidence for the presence of jumps in the price process.

5. Conclusion

We test for jumps of the log-price using noisy high-frequency financial data. When the sampling interval is very small, microstructure noise dominates. We use the preaveraging method to analyse a test problem. For the preaveraging window $k_n=[cn^a]$, we select a to be a smaller order ($a<1/2$) compared to the classical order $a=1/2$. Hence, the noise term and the jump term will dominate in the preaveraging increments. A power variation based on the preaveraging increments is constructed and the two-scale method is used to construct the test statistic. Under the null hypothesis when the path of the price process is continuous, the test statistic converges to a normal distribution. Under the alternative hypothesis when the price is discontinuous, the test statistic tends to infinity at a rate $n^{(1-a)/2}$, which is higher than the rate $n^{1/4}$ of Aït-Sahalia *et al.* (2012). That means that our proposed test statistic has better power than that of Aït-Sahalia *et al.* (2012). Simulation results confirm our asymptotic results and show that the proposed statistic enjoys a quite satisfying performance.

Acknowledgements

The authors thank the Editor, an Associate Editor, and an anonymous reviewer for their very extensive and constructive suggestions, which have helped to improve this paper significantly.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This research is partially supported by National Social Science Foundation of China (19BTJ035), Natural Science Foundation of China (11501503), the Qinglan Project of Jiangsu Province, the Natural Science Foundation of Jiangsu Province of China (BK20181417), and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (17KJA110001).

References

Aït-Sahalia, Y., Telling from discrete data whether the underlying continuous-time model is a diffusion. *J. Finance*, 2002, **57**(5), 2075–2112.

Aït-Sahalia, Y. and Jacod, J., Testing for jumps in a discretely observed process. *Ann. Stat.*, 2009, **37**(1), 184–222.

Aït-Sahalia, Y. and Jacod, J., Is Brownian motion necessary to model high frequency data?. *Ann. Stat.*, 2010, **38**(5), 3093–3128.

Aït-Sahalia, Y. and Jacod, J., Analyzing the spectrum of asset returns: Jump and volatility components in high frequency data. *J. Econ. Lit.*, 2012, **50**(4), 1007–1050.

Aït-Sahalia, Y. and Jacod, J., *High-Frequency Financial Econometrics*, 2014 (Princeton University Press: Princeton, NJ).

Aït-Sahalia, Y. and Jacod, J., Semimartingale: Itøor not? *Stoch. Process. Appl.*, 2018, **128**, 233–254.

Aït-Sahalia, Y., Jacod, J. and Li, J., Testing for jumps in noisy high frequency data. *J. Econom.*, 2012, **168**(2), 207–222.

Andersen, T.G. and Bollerslev, T., Answering the skeptics: Yes, standard volatility models do provide accurate forecasts. *Int. Econ. Rev.*, 1998, **39**(4), 885–905.

Andersen, T.G., Bollerslev, T. and Dobrev, D., No-arbitrage semimartingale restrictions for continuous-time volatility models subject to leverage effects, jumps and i.i.d. noise: Theory and testable distributional implications. *J. Econom.*, 2007, **138**(1), 125–180.

Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A. and Shephard, N., Designing realized kernels to measure the ex post variation of equity prices in the presence of noise. *Econometrica*, 2008, **76**(6), 1481–1536.

Barndorff-Nielsen, O.E. and Shephard, N., Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *J. R. Stat. Soc. Ser. B*, 2002, **64**(2), 253–280.

Barndorff-Nielsen, O.E. and Shephard, N., Power and bipower variation with stochastic volatility and jumps. *J. Financ. Econom.*, 2004, **2**(1), 1–37.

Barndorff-Nielsen, O.E. and Shephard, N., Econometrics of testing for jumps in financial economics using bipower variation. *J. Financ. Econom.*, 2006, **4**(1), 1–30.

Bhattacharya, R.N. and Ranga Rao, R., *Normal Approximation and Asymptotic Expansions*, 1986 (Wiley: New York).

Dumitru, A.M. and Urga, G., Identifying jumps in financial assets: A comparison between nonparametric jump tests. *J. Bus. Econ. Stat.*, 2014, **30**(2), 242–255.

Fan, Y. and Fan, J., Testing and detecting jumps based on a discretely observed process. *J. Econom.*, 2011, **164**(2), 331–344.

Fan, J. and Wang, Y., Multi-scale jump and volatility analysis for high-frequency financial data. *J. Amer. Statist. Assoc.*, 2007, **102**, 1349–1362.

Heston, S.L., A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.*, 1993, **6**(2), 327–343.

Jacod, J., Statistics and high frequency data. In *Statistical Methods for Stochastic Differential Equations*, edited by M. Kessler, A. Lindner and M. Sørensen, pp. 191–310, 2012 (CRC Press: London).

Jacod, J., Li, Y., Mykland, P.A., Podolskij, M. and Vetter, M., Microstructure noise in the continuous case: The pre-averaging approach. *Stoch. Process. Appl.*, 2009, **119**(7), 2249–2276.

Jacod, J., Podolskij, M. and Vetter, M., Limit theorems for moving averages of discretized processes plus noise. *Ann. Stat.*, 2010, **38**(3), 1478–1545.

Jacod, J. and Protter, P., *Discretization of Processes*, 2012 (Springer: Berlin, HL).

Jiang, G.J. and Oomen, R.C.A., Testing for jumps when asset prices are observed with noise—A 'swap variance' approach. *J. Econom.*, 2008, **144**(2), 352–370.

Jing, B.Y., Kong, X.B. and Liu, Z., Estimating the jump activity index under noisy observations using high-frequency data. *J. Amer. Statist. Assoc.*, 2011, **106**(494), 558–568.

Jing, B.Y., Kong, X.B. and Liu, Z., Modeling high-frequency financial data by pure jump processes. *Ann. Stat.*, 2012, **40**(2), 759–784.

Jing, B.Y., Liu, Z. and Kong, X.B., On the estimation of integrated volatility with jumps and microstructure noise. *J. Bus. Econ. Stat.*, 2014, **32**(3), 457–467.

Kong, X.B., Liu, Z. and Jing, B.Y., Testing for pure-jump processes for high-frequency data. *Ann. Stat.*, 2015, **43**(2), 847–877.

Lahiri, S.N., *Resampling Methods for Dependent Data*, 2003 (Springer: New York).

Lee, S.S. and Mykland, P.A., Jumps in financial markets: A new nonparametric test and jump dynamics. *Rev. Financ. Stud.*, 2008, **21**(6), 2535–2563.

Liu, G.Y. and Jing, B.Y., On estimation of hurst parameter under noisy observations. *J. Bus. Econ. Stat.*, 2018, **36**(3), 483–492.

Liu, G.Y., Zhang, L.X. and Wang, M., Estimation of the Hurst parameter in the simultaneous presence of jumps and noise. *Statistics*, 2018, **52**(5), 1156–1192.

Malliavin, P. and Mancino, M.E., A Fourier transform method for nonparametric estimation of multivariate volatility. *Ann. Stat.*, 2009, **37**(4), 1983–2010.

Mancino, M.E. and Sanfelici, S., Estimation of quarticity with high-frequency data. *Quant. Finance*, 2012, **12**(4), 607–622.

Podolskij, M. and Vetter, M., Bipower-type estimation in a noisy diffusion setting. *Stoch. Process. Appl.*, 2009a, **119**(9), 2803–2831.

Podolskij, M. and Vetter, M., Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps. *Bernoulli*, 2009b, **15**(3), 634–658.

Sanfelici, S., Curato, I.V. and Mancino, M.E., High-frequency volatility of volatility estimation free from spot volatility estimates. *Quant. Finance*, 2015, **15**(8), 1331–1345.
Theodosiou, M.G. and Zikes, F., A comprehensive comparison of nonparametric tests for jumps in asset prices. Working Paper, Imperial College London, 2011.
Xiu, D., Quasi-maximum likelihood estimation of volatility with high frequency data. *J. Econom.*, 2010, **159**, 235–250.
Zhang, L., Efficient estimation of stochastic volatility using noisy observations: A multi-scale approach. *Bernoulli*, 2006, **12**(6), 1019–1043.
Zhang, L., Mykland, P.A. and Aït-Sahalia, Y., A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *J. Amer. Statist. Assoc.*, 2005, **100**, 1394–1411.

Appendix. Proofs

In this appendix, we present the proofs of the main results and some supplementary theoretical results. We first define some notation. We use C to denote a constant that changes from line to line, and we use C_p if we want to emphasize that it depends on an additional parameter p. A standard localization procedure shows that the local boundedness of the processes b, σ, δ is replaced by the boundedness (uniformly in (ω, t)) of these processes; see for example Jacod (2012), Jacod and Protter (2012) and Aït-Sahalia and Jacod (2014). Hence, in the sequel, we use the strengthened version of Assumption 1 (strengthened Assumption 1), i.e. the processes b, σ, δ, X in Assumption 1 are assumed to be bounded, to prove the theoretical limit results. E_i^n denotes the conditional expectation with respect to the filtration $\mathcal{F}_{i\Delta_n}$.

We define the continuous part of X by X' and the discontinuous martingale part by X'':

$$X'_t = X_0 + \int_0^t b_s\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}W_s + (\delta 1_{\{|\delta|>1\}}) \star \underline{\mu}_t,$$
$$X''_t = X_t - X'_t = (\delta 1_{\{|\delta|\le 1\}}) \star (\underline{\mu} - \underline{\nu})_t.$$

We set, for $j = 1, 2$,

$$\xi_i^n(jk_n) = n^{a/2}\bar{\epsilon}_i^n(jk_n), \quad \eta_i^n(k_n) = |\xi_i^n(k_n)|^p - E|\xi_i^n(k_n)|^p,$$
$$\alpha_{i,n}^2(j) = E(\xi_i^n(jk_n))^2. \tag{A1}$$

For simplification, we write $\alpha_{i,n} = \alpha_{i,n}(1)$. Recalling that $u_n = \bar{c}n^{-\bar{a}}$, we then have the following lemma.

Lemma A.1 *Suppose Assumptions 2–3 and the strengthened Assumption 1 hold, and* $\sup_{t\ge 0} E|\epsilon_t|^{q+\gamma} < \infty$ *for $q > 0$ and some* $\gamma > 0$; *then we have*

$$E_i^n|\overline{X'}_i^n(k_n)|^q \le C(k_n\Delta_n)^{q/2}, \quad E_i^n|\overline{X''}_i^n(k_n)|^q \le C(k_n\Delta_n)^{1\wedge(q/2)},$$
$$E|\bar{\epsilon}_i^n(k_n)|^q \le C(k_n)^{-q/2}, \tag{A2}$$

$$E_i^n(|\overline{X''}_i^n(k_n)| \wedge u_n)^2 \le C(k_n\Delta_n)u_n^{2-\beta}, \tag{A3}$$

$$\alpha_{i,n}^2(j) = \alpha_{i\Delta_n}^2\overline{g'}(2)/(jc) + o(1), \quad j = 1, 2. \tag{A4}$$

Further, if α_t is Lipschitz continuous and condition (1) *of Assumption* 4 *holds, then*

$$|\alpha_{i,n}^2(j) - \alpha_{i\Delta_n}^2\overline{g'}(2)/(jc)| \le C\left(\frac{1}{k_n} \vee \frac{k_n}{n}\right), \quad j = 1, 2. \tag{A5}$$

Suppose $\sup_{t\ge 0} E|\epsilon_t|^{2p+\gamma} < \infty$ *for* $\gamma > 0$; *then we have for* $l, j = 1, 2$,

$$E\eta_{j_1}^n(lk_n)\eta_{j_2}^n(jk_n) = \begin{cases} \alpha_{j_1\Delta_n}^{2p}\left(\frac{\overline{g'}(2)}{\sqrt{lj}c}\right)^p h\left(\rho_{lj}\left(\frac{|j_1-j_2|}{(l\wedge j)k_n}\right), p\right) + o(1), \\ \quad |j_1 - j_2| \le (l \vee j)k_n \\ 0, \quad |j_1 - j_2| > (l\vee j)k_n. \end{cases} \tag{A6}$$

Proof By the inequality (3.73) of Jacod (2012), it is easy to prove the first two inequalities of (A2). A standard computation can prove the third inequality of (A2). For (A3), a similar method to that in Jing *et al.* (2014) or Liu *et al.* (2018), it follows from the inequality (3.75) of Jacod (2012) applied to the process X'' with the sampling interval $k_n\Delta_n$ and $\alpha_n = u_n/\sqrt{k_n\Delta_n}$. For (A4), by (8), we have

$$\alpha_{i,n}^2(j) = n^a \sum_{j_1=1}^{jk_n}\left(g\left(\frac{j_1}{jk_n}\right) - g\left(\frac{j_1-1}{jk_n}\right)\right)^2 \alpha_{(i+j_1)\Delta_n}^2;$$

hence, by the Riemann integral property and the continuity of α_t^2, we obtain (A4). We prove (A6) similarly.

For (A5), we prove only the case of $j = 1$. It is enough to show

$$\left|n^a\sum_{j_1=1}^{k_n}\left(g\left(\frac{j_1}{k_n}\right) - g\left(\frac{j_1-1}{k_n}\right)\right)^2(\alpha_{(i+j_1)\Delta_n}^2 - \alpha_{i\Delta_n}^2)\right| \le C\frac{k_n}{n}, \tag{A7}$$

$$\left|n^a\sum_{j_1=1}^{k_n}\left(g\left(\frac{j_1}{k_n}\right) - g\left(\frac{j_1-1}{k_n}\right)\right)^2\alpha_{i\Delta_n}^2 - \alpha_{i\Delta_n}^2\frac{\overline{g'}(2)}{c}\right| \le \frac{C}{k_n}. \tag{A8}$$

For (A8), since $g(j_1/k_n) - g((j_1-1)/k_n) = g'(j/k_n)/k_n + O(k_n^{-2})$, we obtain (A8) by the Riemann integral property. By Lipschitz continuity, we obtain (A7). ■

Proof of Theorem 2.1 *Step 1*. We prove the result (10). For $p \le 1$, by the triangle inequality, we have

$$\begin{aligned}&\left|n^{pa/2-1}V_p(Y,k_n)_t^n - m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t \alpha_s^p\,\mathrm{d}s\right| \\ &\le \left|n^{pa/2-1}V_p(Y,k_n)_t^n - n^{pa/2-1}V_p(\epsilon,k_n)_t^n\right| \\ &\quad + \left|n^{pa/2-1}V_p(\epsilon,k_n)_t^n - m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t\alpha_s^p\,\mathrm{d}s\right| \\ &\le n^{pa/2-1}V_p(X,k_n)_t^n \\ &\quad + \left|n^{pa/2-1}V_p(\epsilon,k_n)_t^n - m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t\alpha_s^p\,\mathrm{d}s\right|.\end{aligned}$$

Since $n^{pa/2-1}V_p(X,k_n)_t^n \le n^{pa/2-1}V_p(X',k_n)_t^n + n^{pa/2-1}V_p(X'',k_n)_t^n$, for the first term $n^{pa/2-1}V_p(X',k_n)_t^n$, by the result (A2) and $0 < a < 1/2$, we find that it converges in probability to zero as n goes to infinity. For the second term $n^{pa/2-1}V_p(X'',k_n)_t^n$, by the result (A2), the conditions $0 < a < 1/2$ and $p \le 1$ suggest that it converges in probability to zero as n goes to infinity. Then, it is enough to show that for $p \le 1$,

$$n^{pa/2-1}V_p(\epsilon,k_n)_t^n \xrightarrow{P} m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t\alpha_s^p\,\mathrm{d}s. \tag{A9}$$

For $p > 1$, by the Minkowski inequality, we have

$$\begin{aligned}&\left|\left(n^{pa/2-1}V_p(Y,k_n)_t^n\right)^{1/p} - \left(m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t\alpha_s^p\,\mathrm{d}s\right)^{1/p}\right| \\ &\le \left|\left(n^{pa/2-1}V_p(Y,k_n)_t^n\right)^{1/p} - \left(n^{pa/2-1}V_p(\epsilon,k_n)_t^n\right)^{1/p}\right|\end{aligned}$$

$$+\left|\left(n^{pa/2-1}V_p(\epsilon,k_n)_t^n\right)^{1/p}-\left(m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t\alpha_s^p\,\mathrm{d}s\right)^{1/p}\right|$$
$$\leq\left(n^{pa/2-1}V_p(X,k_n)_t^n\right)^{1/p}$$
$$+\left|\left(n^{pa/2-1}V_p(\epsilon,k_n)_t^n\right)^{1/p}-\left(m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t\alpha_s^p\,\mathrm{d}s\right)^{1/p}\right|.$$

For the first term $n^{pa/2-1}V_p(X,k_n)_t^n$, note that $V_p(X,k_n)_t^n\leq C_p(V_p(X',k_n)_t^n+V_p(X'',k_n)_t^n)$, similar to the case of $p\leq 1$, we find that it converges in probability to zero as $n\longrightarrow\infty$. Then, it is enough to show that (A9) holds for $p>1$.

We now prove the result (A9) for $p>0$. By (A1), we have

$$\begin{aligned}
&n^{pa/2-1}V_p(\epsilon,k_n)_t^n-m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\int_0^t\alpha_s^p\,\mathrm{d}s\\
&=\frac{1}{n}\sum_{i=0}^{[nt]-k_n}\eta_i^n(k_n)\\
&+\frac{1}{n}\sum_{i=0}^{[nt]-k_n}\left(E|\xi_i^n(k_n)|^p-m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\alpha_{i\Delta_n}^p\right)\\
&+m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\left(\frac{1}{n}\sum_{i=0}^{[nt]}\alpha_{i\Delta_n}^p-\int_0^t\alpha_s^p\,\mathrm{d}s\right)\\
&-\frac{1}{n}\sum_{i=[nt]-k_n+1}^{[nt]}m_p\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\alpha_{i\Delta_n}^p.
\end{aligned}\tag{A10}$$

Then, by the Riemann integral property, the last two terms converge to zero. Hence it is enough to show the first and second terms of (A10) converge to zero in probability. For the second term, by (A1) and (A4), we have

$$\begin{aligned}
E|\xi_i^n(k_n)|^p&=\alpha_{i,n}^pE\left|\frac{\xi_i^n(k_n)}{\alpha_{i,n}}\right|^p\\
&=\left(\left(\frac{\overline{g'}(2)}{c}\right)^{p/2}\alpha_{i\Delta_n}^p+o(1)\right)E\left|\frac{\xi_i^n(k_n)}{\alpha_{i,n}}\right|^p.
\end{aligned}\tag{A11}$$

Note that $\xi_i^n(k_n)/\alpha_{i,n}\xrightarrow{L}N(0,1),\ i=0,1,2,\ldots,[nt]-k_n$, and we have $E|\xi_i^n(k_n)/\alpha_{i,n}|^p=m_p+o(1)$. Hence, by (A11), we find that the second term of (A10) converges to zero. For the first term of (A10), by (A6), we have

$$E\left(\frac{1}{n}\sum_{i=0}^{[nt]-k_n}\eta_i^n(k_n)\right)^2=\frac{1}{n^2}\sum_{i,j=0}^{[nt]-k_n}E\eta_i^n(k_n)\eta_j^n(k_n)\leq C\frac{k_n}{n}\longrightarrow 0,$$

and we then find that the first term of (A10) converges to zero in probability. Thus, we prove the result (10) of Theorem 2.1.

Step 2. For (11), Theorem 3.2 of Jacod *et al.* (2010) gives this result for the case of $k_n=[cn^a]$ and $p>2(1-a)/a$ with $a=1/2$. In fact, the result (11) here is the case of $0<a<1/2$ for Theorem 3.2 of Jacod *et al.* (2010), and we can similarly prove (11).

Step 3. We prove the result (12). By (A9), noting that $n^{pa/2-1}\sum_{i=0}^{[nt]-k_n}|\overline{X'}_i^n(k_n)|^p\xrightarrow{P}0$, it is enough to show that

$$\begin{aligned}
&n^{pa/2-1}\sum_{i=0}^{[nt]-k_n}\varsigma_i^n\\
&=:n^{pa/2-1}\sum_{i=0}^{[nt]-k_n}\\
&\times\left(|\overline{Y}_i^n(k_n)|^p1_{|\overline{Y}_i^n(k_n)|<u_n}-|\overline{X'+\epsilon}_i^n(k_n)|^p\right)\xrightarrow{P}0,
\end{aligned}\tag{A12}$$

where $\varsigma_i^n=|\overline{Y}_i^n(k_n)|^p1_{|\overline{Y}_i^n(k_n)|<u_n}-|\overline{X'+\epsilon}_i^n(k_n)|^p$. We have the following inequality, for any $p>0,\gamma>0$:

$$\begin{aligned}
&\left||x+y|^p1_{|x+y|<\gamma}-|x|^p\right|\\
&\leq C_p\Big[|x|^p1_{|x|>\gamma/2}+|x|^p1_{|x|\leq\gamma/2,|x+y|\geq\gamma}\\
&+(|y|^p+1_{p>1}|x|^{p-1}|y|)1_{|x|\leq\gamma/2,|x+y|<\gamma}\Big]\\
&\leq C_p\left[\frac{|x|^{p+r_1}}{\gamma^{r_1}}+\frac{|x|^p|y|^{r_2}}{\gamma^{r_2}}+(|y|\wedge\gamma)^p+1_{p>1}|x|^{p-1}(|y|\wedge\gamma)\right],
\end{aligned}\tag{A13}$$

where r_1 and r_2 are two arbitrary positive constants.

For $p\leq 2$, by (A13), Lemma A.1 and Hölder inequality, we have

$$\begin{aligned}
&n^{pa/2}E_i^n\left|\varsigma_i^n\right|\\
&\leq C_pn^{pa/2}\Bigg[\frac{E_i^n|\overline{X'+\epsilon}_i^n(k_n)|^{p+r_1}}{u_n^{r_1}}\\
&+\frac{E_i^n|\overline{X'+\epsilon}_i^n(k_n)|^p|\overline{X''}_i^n(k_n)|}{u_n}+E_i^n(|\overline{X''}_i^n(k_n)|\wedge u_n)^p\\
&+1_{p>1}E_i^n|\overline{X'+\epsilon}_i^n(k_n)|^{p-1}(|\overline{X''}_i^n(k_n)|\wedge u_n)\Bigg]\\
&\leq C_pn^{pa/2}\Big[n^{-a(p+r_1)/2}n^{\bar{a}r_1}+n^{-ap/2}n^{(a-1)/2}n^{\bar{a}}+n^{(a-1)p/2}\\
&+n^{-a(p-1)/2}n^{(a-1)/2}\Big]\\
&=C_p\left[n^{(-a/2+\bar{a})r_1}+n^{(a-1)/2+\bar{a}}+n^{(2a-1)p/2}+n^{(2a-1)/2}\right],
\end{aligned}\tag{A14}$$

where $r_1<p/2$ and $r_2=1$. Then the conditions $0<a<1/2$ and $0<\bar{a}<a/2$ suggest that (A12) holds for $p\leq 2$. For $p>2$, we note that $E_i^n(|\overline{X''}_i^n(k_n)|\wedge u_n)^p\leq u_n^{p-2}E_i^n(|\overline{X''}_i^n(k_n)|\wedge u_n)^2$, and, similarly, we have

$$\begin{aligned}
&n^{pa/2}E_i^n\left|\varsigma_i^n\right|\\
&\leq C_pn^{pa/2}\Big[n^{-a(p+r_1)/2}n^{\bar{a}r_1}+n^{-ap/2}n^{(a-1)/2}n^{\bar{a}}\\
&+n^{(a-1)}n^{-\bar{a}(p-\beta)}+n^{-a(p-1)/2}n^{(a-1)/2}\Big]\\
&=C_p\Big[n^{(-a/2+\bar{a})r_1}+n^{(a-1)/2+\bar{a}}\\
&+n^{pa/2+a-1-\bar{a}(p-\beta)}+n^{(2a-1)/2}\Big],
\end{aligned}\tag{A15}$$

where $r_1<p/2$ and $r_2=1$. Then the conditions $0<a<1/2$ and $(pa+2a-1)/(2(p-\beta))<\bar{a}<a/2$ suggest that (A12) holds for $p>2$. Hence, we prove the result (A12) and obtain the result (12).

Thus, we have completed the proof of Theorem 2.1. ■

Proof of Theorem 2.2 This is easy, by Theorem 2.1. ■

Before we prove Theorem 2.3, we introduce an auxiliary result on Edgeworth-type expansions for triangular arrays of random variables $X_{n,j}$, where $X_{n,j}$ are independent for $1\leq j\leq n$ with zero mean, but not identically distributed. We consider the density for $S_n =: n^{-1/2}\sum_{j=1}^{n} X_{n,j}$; then, for some integer $s \geq 3$, when $E|X_{n,j}|^s < \infty, j = 1, \ldots, n$, for each n, we have the following density approximation of S_n:

$$\phi_n(x) = \left[1 + \sum_{i=0}^{s-2} n^{-i/2} p_i(x; \{\overline{\kappa}_{\nu,n}\})\right]\phi(x), \tag{A16}$$

where $p_i(x; \{\overline{\kappa}_{\nu,n}\})$ are polynomials, the expressions for which can be found in Lahiri (2003, p. 150), and $\phi(x)$ is the density function of the standard normal variable. Specifically,

$$p_1(x; \{\overline{\kappa}_{\nu,n}\}) = \tfrac{1}{6} H_3(x)\overline{\kappa}_{3,n},$$
$$p_2(x; \{\overline{\kappa}_{\nu,n}\}) = \tfrac{1}{72}\left(H_6(x)(\overline{\kappa}_{3,n})^2 + 3H_4(x)\overline{\kappa}_{4,n}\right),$$

where

$$H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3,$$
$$H_6(x) = x^6 - 15x^4 + 45x^2 - 15$$

are Hermite polynomials of order 3, 4, and 6, respectively, and $\overline{\kappa}_{m,n} = n^{-1}\sum_{j=1}^{n} \kappa_m(X_{n,j})$, where $\kappa_m(X)$ is the mth cumulant of the random variable X. Note that $\kappa_m(X)$ is well defined if $E|X|^m < \infty$. The coefficients of the polynomial $p_i(x; \{\overline{\kappa}_{\nu,n}\})$ include only the mth cumulant $\kappa_m(X_{n,j})$ for $1 \leq m \leq i+2$, and hence $p_i(x; \{\overline{\kappa}_{\nu,n}\})$ is well defined when $i \leq s-2$. We denote

$$a_n(s,\gamma) = \frac{1}{n}\sum_{j=1}^{n} E|X_{n,j}|^s 1_{|X_{n,j}|>\gamma\sqrt{n}}, \quad a_n(s) = a_n(s,1),$$

and define the integrated modulus of continuity of f with respect to ϕ as

$$\omega(\gamma; f, \phi) = \int \sup_y\{|f(x) - f(y)| : |x-y| \leq \gamma\}\phi(x)\,\mathrm{d}x.$$

The following lemma is a special case of Theorem 6.1 in Lahiri (2003); we present it here for convenience in proving Theorem 2.3.

Lemma A.2 *Let $X_{n,1}, \ldots, X_{n,n}$ be a sequence of independent random variables with zero mean and $n^{-1}\sum_{j=1}^{n} EX_{n,j}^2 = 1$ for each n. Suppose for some integer $s \geq 3$ that*

$$\overline{\rho}_{n,s} =: n^{-1}\sum_{j=1}^{n} E|X_{n,j}|^s < \infty. \tag{A17}$$

Let f be a Borel-measurable function satisfying

$$M_s(f) =: \sup_{x\in\mathbb{R}}(1 + |x|^{2[s/2]})^{-1}|f(x)| < \infty. \tag{A18}$$

Suppose

$$a_n(s, 2/3) \leq 2^{-(s+4)} n^{(s-2)/2}. \tag{A19}$$

Then, for any $\gamma \in (0,1)$, there exists a constant $C < \infty$ such that, with $v_n = n^{-(s-2)/2}$,

$$\begin{aligned}
&|Ef(S_n) - Ef(x)\phi_n(x)\,\mathrm{d}x| \\
&\leq CM_s(f)[v_n\widetilde{\rho}_{n,s} + (1 + \widetilde{\rho}_{n,s})((a_n(s) + v_n)v_n \\
&\quad + (v_n\overline{\rho}_{n,s})^2 + n^{(s+2)/2}(v_n\overline{\rho}_{n,s})^{2s+1}) \\
&\quad + (\eta_{n,\gamma}\gamma^{-2} + n^{s+2}\gamma^{-8}\exp(-\gamma^{-1}))] \\
&\quad + C(1 + \overline{\rho}_{n,s})\omega(2\gamma; f, \phi),
\end{aligned} \tag{A20}$$

where

$$\widetilde{\rho}_{n,s} = n^{-3/2}\sum_{j=1}^{n} E|X_{n,j}|^{s+1} 1_{|X_{n,j}|\leq\sqrt{n}},$$

$$\eta_{n,\gamma} = \sum_{1\leq j_1,\ldots,j_{s+2}\leq n} \times \sup\left\{\prod_{j\neq j_1,\ldots,j_{s+2}} |\chi_{n,j}(t)| : (16\overline{\rho}_{n,3})^{-1} \leq t \leq \gamma^{-4}\right\},$$

and $\chi_{n,j}(t) = |E\exp(\mathrm{i}tX_{n,j})| + 2P(|X_{n,j}| > \sqrt{n})$, where $E\exp(\mathrm{i}tX_{n,j})$ is the characteristic function of $X_{n,j}$.

Recalling that $\xi_1^n(jk_n) = n^{a/2}\overline{\epsilon}_1^n(jk_n) = \sum_{i=1}^{jk_n} n^{a/2}(g(i/jk_n) - g((i-1)/jk_n))\epsilon_i^n$, we have the following lemma.

Lemma A.3 *Suppose Assumption 4 holds and $\sup_{t\geq 0} E|\epsilon_t|^{4\vee(2p)+\gamma} < \infty$ for some $\gamma > 0$, if $1/3 < a < 1/2$; then*

$$n^{(-1-a)/2}\left(\sum_{i=0}^{[nT]-k_n} E|\xi_i^n(k_n)|^p - 2^{p/2}\sum_{i=0}^{[nT]-2k_n} E|\xi_i^n(2k_n)|^p\right) = o(1). \tag{A21}$$

Proof Note that $E|\xi_i^n(k_n)|^p < C$; then it is enough to show

$$n^{(-1-a)/2}\sum_{i=0}^{[nT]-2k_n}\left(E|\xi_i^n(k_n)|^p - 2^{p/2}E|\xi_i^n(2k_n)|^p\right) = o(1).$$

Recalling that $\alpha_{i,n}^2(j) = E(\xi_i^n(jk_n))^2$ and (A4), we have

$$\begin{aligned}
&E|\xi_i^n(k_n)|^p - 2^{p/2}E|\xi_i^n(2k_n)|^p \\
&= \alpha_{i,n}^p(1)\left(E\left|\frac{\xi_1^n(k_n)}{\alpha_{i,n}(1)}\right|^p - m_p\right) \\
&\quad - 2^{p/2}\alpha_{i,n}^p(2)\left(E\left|\frac{\xi_1^n(2k_n)}{\alpha_{i,n}(2)}\right|^p - m_p\right) \\
&\quad + m_p\left(\alpha_{i,n}^p(1) - 2^{p/2}\alpha_{i,n}^p(2)\right);
\end{aligned}$$

then the condition $1/3 < a < 1/2$ suggests that it is enough to show

$$\left|E\left|\frac{\xi_i^n(jk_n)}{\alpha_{i,n}(j)}\right|^p - m_p\right| \leq \frac{C}{k_n}, \quad j = 1, 2, \tag{A22}$$

$$\left|\alpha_{i,n}^p(1) - 2^{p/2}\alpha_{i,n}^p(2)\right| \leq \frac{C}{k_n}, \tag{A23}$$

for all $i = 0, 1, 2, \ldots, [nT] - 2k_n$. By (A5), we obtain the result (A23). We only prove the result (A22) for $j = 1$.

For the result (A22) with $j = 1$ and $i = 1$, now we will use Lemma A.2 to prove this result. Recalling (7), we set

$$X_{n,i} =: \frac{c^{1/2}n^a g_i'^n}{\alpha_{1,n}(1)}\epsilon_i^n, \quad 1 \leq i \leq k_n; \tag{A24}$$

then

$$\frac{\xi_1^n(k_n)}{\alpha_{1,n}(1)} = k_n^{-1/2}\sum_{i=1}^{k_n} X_{n,i}$$

and

$$EX_{n,i} = 0, \quad \sum_{i=1}^{k_n} EX_{n,i}^2/k_n = 1.$$

Letting $s = 4 \vee (2\lceil p/2 \rceil)$ in Lemma A.2, since $\sup_{t\geq 0} E|\epsilon_t|^s$ is bounded, we have

$$\begin{aligned}\overline{\rho}_{n,s} &= k_n^{-1}\sum_{i=1}^{k_n} E|X_{n,i}|^s \\ &= k_n^{-1}\sum_{i=1}^{k_n}\left|\frac{c^{-1/2}g'(\frac{i}{k_n})}{\alpha_{1,n}(1)}\right|^s E|\epsilon_i^n|^s + O(k_n^{-1}) \\ &\leq C\int_0^1 |g'(x)|^s\,\mathrm{d}x < \infty,\end{aligned} \tag{A25}$$

$$\widetilde{\rho}_{n,s} = k_n^{-3/2}\sum_{i=1}^{k_n} E|X_{n,i}|^{s+1}1_{|X_{n,i}|\leq k_n^{1/2}} \leq k_n^{-1}\sum_{i=1}^{k_n} E|X_{n,i}|^s < \infty;$$

hence the condition (A17) of Lemma A.2 holds. By $\sup_{t\geq 0} E|\epsilon_t|^{4\vee(2\lceil p/2\rceil)+\gamma} < +\infty$, Hölder inequality and Chebyshev inequality, we have

$$\begin{aligned}a_n(s,\gamma_1) &= k_n^{-1}\sum_{i=1}^{k_n} E|X_{n,i}|^s 1_{|X_{n,i}|>\gamma_1\sqrt{k_n}} \\ &\leq k_n^{-1}\sum_{i=1}^{k_n}(E|X_{n,i}|^{s+\gamma})^{s/(s+\gamma)}(E1_{|X_{n,i}|>\gamma_1\sqrt{k_n}})^{\gamma/(s+\gamma)} \\ &\leq Ck_n^{-1}\sum_{i=1}^{k_n}(E1_{|X_{n,i}|>\gamma_1\sqrt{k_n}})^{\gamma/(s+\gamma)} \\ &\leq Ck_n^{-1}\sum_{i=1}^{k_n}\left(\frac{E|X_{n,i}|^2}{\gamma_1^2 k_n}\right)^{\gamma/(s+\gamma)} \leq Ck_n^{-\gamma/(s+\gamma)} = o(1),\end{aligned} \tag{A26}$$

for any fixed $\gamma_1 > 0$, then $a_n(s) = a_n(s,1) = o(1)$. Note $s > 2$ and $n^{(s-2)/2} \to +\infty$, then (A26) suggests that the condition (A19) of Lemma A.2 holds. Let $f(x) = |x|^p, p > 0$; then $M_s(f) < \infty$ when $p \leq 2[s/2]$ and $\omega(\gamma; f, \phi) \leq C\gamma$. Hence, the conditions of Lemma A.2 all hold for this function $f(x) = |x|^p$ and the triangular array $\{X_{n,i} : 1 \leq i \leq k_n\}_{n\geq 1}$ defined by (A24). Furthermore, the conclusion (A20) of Lemma A.2 holds.

Now we analyze the right terms of the conclusion (A20). For the term $\overline{\rho}_{n,3}$, similarly to the proof of (A25), by Hölder inequality and the condition (16), we have $0 < C_1 < \overline{\rho}_{n,3} < C_2 < \infty$, where C_1, C_2 are two constants. Hence, $(16\overline{\rho}_{n,3})^{-1} > b > 0$ for some positive constant b. Since

$$\begin{aligned}|\chi_{n,j}(t)| &= \left|E\exp\left(it\frac{c^{1/2}n^a g_j'^n}{\alpha_{1,n}(1)}\epsilon_j^n\right)\right| + 2P(|X_{n,j}| > \sqrt{n}) \\ &\leq \left|\chi_{j\Delta_n}\left(t\frac{c^{1/2}n^a g_j'^n}{\alpha_{1,n}(1)}\right)\right| + 2\frac{EX_{n,j}^2}{n},\end{aligned} \tag{A27}$$

and noting that the limits of $c^{1/2}n^a g_j'^n/\alpha_{1,n}(1)$ and $EX_{n,j}^2$ exist as $n \to +\infty$; by the condition (1) of Assumption 4 and (16), there exist two positive constants c_1, c_2 such that for all $j = 1, \ldots, k_n$,

$$0 < c_1 < \left|\frac{c^{1/2}n^a g_j'^n}{\alpha_{1,n}(1)}\right| < c_2, \quad 0 < c_1 < EX_{n,j}^2 < c_2 \tag{A28}$$

for sufficiently large n. By (15) and (A28), let $s_0 = bc_1$, there exists a constant $0 < \theta_1 < 1$ such that

$$\sup_{|t|\geq(16\overline{\rho}_{n,3})^{-1}}\ \sup_{j\geq 1}\left|\chi_{j\Delta_n}\left(t\frac{c^{1/2}n^a g_j'^n}{\alpha_{1,n}(1)}\right)\right| < \theta_1 \tag{A29}$$

for sufficiently large n. Then, by (A27), (A28) and (A29), there exists a constant $0 < \theta_1 < \theta < 1$ such that

$$\sup_{|t|\geq(16\overline{\rho}_{n,3})^{-1}}\ \sup_{j\geq 1}|\chi_{n,j}(t)| < \theta$$

for sufficiently large n. Further, we have

$$\begin{aligned}\eta_{n,\gamma} &= \sum_{1\leq j_1,\ldots,j_{s+2}\leq k_n} \\ &\quad\times\sup\left\{\prod_{j\neq j_1,\ldots,j_{s+2}}|\chi_{n,j}(t)| : (16\overline{\rho}_{n,3})^{-1} \leq t \leq \gamma^{-4}\right\} \\ &\leq \sum_{1\leq j_1,\ldots,j_{s+2}\leq k_n}\theta^{k_n-s-2} \leq k_n^{s+2}\theta^{k_n-s-2},\end{aligned}$$

for sufficiently large n. Since $s = 4 \vee (2\lceil p/2\rceil)$, then $k_n^{-(s-2)/2} \leq k_n^{-1}$ and set $v_n = k_n^{-1}$, and choose $\gamma = \exp(-dk_n), 0 < d < -\ln(\theta)/2$; then, by Lemma A.2, we have

$$\begin{aligned}&\left|Ef\left(\frac{\xi_1^n(k_n)}{\alpha_{1,n}(1)}\right) - Ef(x)\phi_n(x), \mathrm{d}x\right| \\ &\quad\leq C[v_n + (o(1)v_n + (v_n)^2 + k_n^{(s+2)/2}(v_n)^{2s+1}) \\ &\qquad + (k_n^{s+2}\theta^{k_n-s-2}\exp(2dk_n) \\ &\qquad + k_n^{s+2}\exp(8dk_n)\exp(-\exp(dk_n))] + C\exp(-dk_n) \\ &\quad\leq Ck_n^{-1},\end{aligned} \tag{A30}$$

for sufficiently large n. By (A16) and (A30), since $H_3(x)f(x)$ is an odd function, we have

$$\begin{aligned}&\left|E\left|\frac{\xi_1^n(k_n)}{\alpha_{1,n}(1)}\right|^p - m_p\right| \\ &\quad\leq Ck_n^{-1} + \left|Ef(x)\phi_n(x)\,\mathrm{d}x - m_p\right| \\ &\quad= Ck_n^{-1} + \left|\int f(x)\left(\frac{k_n^{-1}}{72}\left(H_6(x)(\overline{\kappa}_{3,n})^2 + 3H_4(x)\overline{\kappa}_{4,n}\right)\right.\right. \\ &\qquad\left.\left. + \sum_{i=3}^{s-2}k_n^{-i/2}p_i(x;\{\overline{\kappa}_{\nu,n}\})\right)\phi(x)\,\mathrm{d}x\right| \\ &\quad\leq Ck_n^{-1} + Ck_n^{-1}((\overline{\kappa}_{3,n})^2 + \overline{\kappa}_{4,n}) \\ &\qquad + \left|\int f(x)\left(\sum_{i=3}^{s-2}k_n^{-i/2}p_i(x;\{\overline{\kappa}_{\nu,n}\})\right)\phi(x)\,\mathrm{d}x\right|,\end{aligned}$$

where

$$\begin{aligned}|\overline{\kappa}_{3,n}| &= \left|\frac{1}{k_n}\sum_{i=1}^{k_n}\kappa_3(X_{n,i})\right| \leq \frac{1}{k_n}\sum_{i=1}^{k_n}\left|\left(\frac{c^{1/2}n^a g_i'^n}{\alpha_{1,n}(1)}\right)^3\kappa_3(\epsilon_i^n)\right| \\ &< \infty, \quad |\overline{\kappa}_{4,n}| = \left|\frac{1}{k_n}\sum_{i=1}^{k_n}\kappa_4(X_{n,i})\right| < \infty.\end{aligned}$$

Since $p_i(x;\{\overline{\kappa}_{\nu,n}\})$ is well defined when $i \leq s-2$ and since $\sup_{t\geq 0}\kappa_m(\epsilon_t) < \infty$ for $m \leq s$, we obtain the result (A22) for $i = 1$. From the proof procedure above, note the conditions $\sup_{t\geq 0} E|\epsilon_t|^s < \infty$ and (15), we can see that the bound C in (A22) for $i = 1$ can be chosen to be a positive constant such that (A22) holds for $i = 0, 2, 3, \ldots, [nT] - 2k_n$. Hence, we prove (A22) for $i = 0, 1, \ldots, [nT] - 2k_n$.

Thus, we have completed the proof of Lemma A.3. ■

Proof of Theorem 2.3 *Step 1*. We prove the following asymptotic result:

$$n^{(1-a)/2}\begin{pmatrix} n^{pa/2-1}V_p(Y,k_n)_t^n - n^{-1}\sum_{i=0}^{[nt]-k_n} E|\xi_i^n(k_n)|^p \\ n^{pa/2-1}V_p(Y,2k_n)_t^n - n^{-1}\sum_{i=0}^{[nt]-2k_n} E|\xi_i^n(2k_n)|^p \end{pmatrix} \xrightarrow{L} \Phi\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{A31}$$

where

$$\Phi^{\mathrm{T}}\Phi = \frac{\int_0^t \alpha_s^{2p}\,\mathrm{d}s}{c^{p-1}}\overline{g'}(2)^p \times \begin{pmatrix} 2\int_0^1 h(\rho_{11}(s),p)\,\mathrm{d}s & \frac{1}{2^{p/2}}\left[\int_0^1 h(\rho_{12}(s),p)\,\mathrm{d}s + \int_0^2 h(\rho_{21}(s),p)\,\mathrm{d}s\right] \\ \frac{1}{2^{p/2}}\left[\int_0^1 h(\rho_{12}(s),p)\,\mathrm{d}s + \int_0^2 h(\rho_{21}(s),p)\,\mathrm{d}s\right] & \frac{1}{2^{p-2}}\int_0^1 h(\rho_{11}(s),p)\,\mathrm{d}s \end{pmatrix},$$

and U_1 and U_2 are two independent standard normal variables.

Since X is continuous, we write $X_t^{\mathrm{c}} = X_0 + \int_0^t b_s\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}W_s$; then $Y = X^{\mathrm{c}} + \epsilon$. We first prove

$$n^{(1-a)/2}\left|n^{pa/2-1}V_p(Y,k_n)_t^n - n^{pa/2-1}V_p(\epsilon,k_n)_t^n\right| \xrightarrow{P} 0.$$

Similar to the proof of the result (10) in Theorem 2.1, by the triangle inequality for $p \le 1$ and Minkowski inequality for $p > 1$, it is enough to show

$$n^{(1-a)/2}n^{pa/2-1}V_p(X^c,k_n)_t^n \xrightarrow{P} 0.$$

By (A2), we find that the above result holds for $p > (1-a)/(1-2a)$ and $0 < a < 1/2$. Hence, it is enough to show that the result (A31) holds when Y is replaced by ϵ.

To do this, we first let l_n be a positive integer and suppose that $l_n = [n^{1/6}]$; then $l_n \longrightarrow \infty$ as $n \longrightarrow \infty$. We denote by $J_n(l) = [([t/\Delta_n] - 2k_n)/(2(l_n+1)k_n)]$ the number of large blocks, and the qth block endpoints are

$$a_q(l) = 2q(l_n+1)k_n, \quad b_q(l) = a_q(l) + 2l_nk_n,$$

for $q = 0, 1, 2, \ldots, J_n(l)$. Setting

$$U(j)_t^n = n^{(1-a)/2}\frac{1}{n}\sum_{i=0}^{[nt]-jk_n}\left(|\xi_i^n(jk_n)|^p - E|\xi_i^n(jk_n)|^p\right) = n^{(-1-a)/2}\sum_{i=0}^{[nt]-jk_n}\eta_i^n(jk_n), \quad j = 1,2,$$

we have

$$U(j)_t^n = n^{(-1-a)/2}\sum_{q=0}^{J_n(l)}\sum_{i=a_q(l)}^{b_q(l)-1}\eta_i^n(jk_n) + n^{(-1-a)/2}\sum_{q=0}^{J_n(l)}\sum_{i=b_q(l)}^{a_{q+1}(l)-1}\eta_i^n(jk_n) + R(j)_t^n, \quad j = 1,2,$$

where

$$R(j)_t^n = n^{(-1-a)/2}\sum_{i=J_n(l)(2(l_n+1)k_n)}^{[t/\Delta_n]-jk_n} \times (|\xi_i^n(jk_n)|^p - E|\xi_i^n(jk_n)|^p), \quad j = 1,2.$$

It is not hard to find that the two terms $R(1)_t^n$ and $R(2)_t^n$ converge in probability to zero as n goes to infinity, provided that $0 < a < 1/2$. We set

$$\widetilde{U}(j)_q^n = n^{(-1-a)/2}\sum_{i=a_q(l)}^{b_q(l)-1}\eta_i^n(jk_n), \quad j = 1,2;$$

then it is enough to show

$$\sum_{q=0}^{J_n(l)}\begin{pmatrix}\widetilde{U}(1)_q^n \\ \widetilde{U}(2)_q^n\end{pmatrix} \xrightarrow{L} \Phi\begin{pmatrix} U_1 \\ U_2\end{pmatrix}, \tag{A32}$$

$$n^{(-1-a)/2}\sum_{q=0}^{J_n(l)}\sum_{i=b_q(l)}^{a_{q+1}(l)-1}\eta_i^n(jk_n) \xrightarrow{P} 0, \quad j = 1,2. \tag{A33}$$

For the case $j = 1$ of (A33), by the result (A6), we have

$$\begin{aligned} E&\left(n^{(-1-a)/2}\sum_{q=0}^{J_n(l)}\sum_{i=b_q(l)}^{a_{q+1}(l)-1}\eta_i^n(k_n)\right)^2 \\ &= n^{-1-a}\sum_{q=0}^{J_n(l)}E\left(\sum_{i=b_q(l)}^{a_{q+1}(l)-1}\eta_i^n(k_n)\right)^2 \\ &\le Cn^{-1-a}J_n(l) \times \left(\sum_{i,j=1}^{k_n} h\left(\rho_{11}\left(\frac{|i-j|}{k_n}\right),p\right) + k_n^2 o(1)\right) \\ &\le C\frac{1}{k_n^2 l_n}\left(k_n^2 + k_n^2 o(1)\right) \le \frac{C}{l_n} \longrightarrow 0 \end{aligned}$$

as n goes to infinity. Similarly, we obtain the case $j = 2$ of (A33).

For the result (A32), we note that $E\eta_i^n(jk_n) = 0$, by Theorem 2.2.13 of Jacod and Protter (2012), and it is enough to show

$$\sum_{q=0}^{J_n(l)}E\left(\widetilde{U}(1)_q^n\right)^2 \longrightarrow \frac{\int_0^t \alpha_s^{2p}\,\mathrm{d}s}{c^{p-1}}\overline{g'}(2)^p 2\int_0^1 h(\rho_{11}(s),p)\,\mathrm{d}s, \tag{A34}$$

$$\sum_{q=0}^{J_n(l)}E\left(\widetilde{U}(2)_q^n\right)^2 \longrightarrow \frac{\int_0^t \alpha_s^{2p}\,\mathrm{d}s}{c^{p-1}}\overline{g'}(2)^p \frac{1}{2^{p-2}}\int_0^1 h(\rho_{11}(s),p)\,\mathrm{d}s, \tag{A35}$$

$$\sum_{q=0}^{J_n(l)}E\left(\widetilde{U}(1)_q^n\widetilde{U}(2)_q^n\right) \longrightarrow \frac{\int_0^t \alpha_s^{2p}\,\mathrm{d}s}{c^{p-1}}\overline{g'}(2)^p\frac{1}{2^{p/2}} \times \left[\int_0^1 h(\rho_{12}(s),p)\,\mathrm{d}s + \int_0^2 h(\rho_{21}(s),p)\,\mathrm{d}s\right], \tag{A36}$$

$$\sum_{q=0}^{J_n(l)}E\left(\widetilde{U}(j)_q^n\right)^4 \longrightarrow 0, \quad j = 1,2, \tag{A37}$$

as n goes to infinity. For the result (A34), by (A6), we have

$$\begin{aligned} \sum_{q=0}^{J_n(l)}E&\left(n^{(-1-a)/2}\sum_{i=a_q(l)}^{b_q(l)-1}\eta_i^n(k_n)\right)^2 \\ &= n^{-1-a}\sum_{q=0}^{J_n(l)}E\left(\sum_{i=a_q(l)}^{b_q(l)-1}\eta_i^n(k_n)\right)^2 \end{aligned}$$

$$= n^{-1-a} \sum_{q=0}^{J_n(l)} \times \left(\left(\frac{\overline{g'}(2)}{c} \right)^p \alpha^{2p}_{a_q(l)\Delta_n} \right.$$

$$\left. \times \sum_{i,j=a_q(l)}^{b_q(l)-1} h\left(\rho_{11}\left(\frac{|i-j|}{k_n} \right), p \right) + (k_n)^2 o(1) \right)$$

$$= n^{-1-a} \sum_{q=0}^{J_n(l)} \times \left(\left(\frac{\overline{g'}(2)}{c} \right)^p \alpha^{2p}_{a_q(l)\Delta_n} \right.$$

$$\left. \times\ 2 l_n k_n \times 2 \sum_{i=1}^{k_n} h\left(\rho_{11}\left(\frac{i}{k_n} \right), p \right) + O(k_n^2) + (k_n)^2 o(1) \right)$$

$$\longrightarrow \frac{\int_0^t \alpha_s^{2p}\, \mathrm{d}s}{c^{p-1}} \overline{g'}(2)^p 2 \int_0^1 h(\rho_{11}(s), p)\, \mathrm{d}s$$

as n goes to infinity. We obtain the results (A35) and (A36) in a similar manner. For the case $j = 1$ of (A37), from $a < 1/2$ and $l_n = [n^{1/6}]$, we have

$$\sum_{q=0}^{J_n(l)} E\left(\widetilde{U}(1)^n_q \right)^4 = n^{-2(1+a)} \sum_{q=0}^{J_n(l)} \left(\sum_{i=a_q(l)}^{b_q(l)-1} \eta^n_i(k_n) \right)^4$$

$$\leq C n^{-2(1+a)} J_n(l)(l_n k_n)^4 \leq C n^{-1+a} l_n^3 \longrightarrow 0$$

as n goes to infinity. Similarly, we obtain the case $j = 2$ of (A37).

Step 2. We prove the result (19). We have

$$n^{(1-a)/2}(T_n - 2^{p/2})$$

$$= \frac{\begin{array}{c} n^{(-1-a)/2}[(n^{pa/2} V_p(Y, k_n)^n_T - \sum_{i=0}^{[nT]-k_n} E|\xi^n_i(k_n)|^p) \\ -2^{p/2}(n^{pa/2} V_p(Y, 2k_n)^n_T - \sum_{i=0}^{[nT]-2k_n} E|\xi^n_i(2k_n)|^p)] \end{array}}{n^{pa/2-1} V_p(Y, 2k_n)^n_T}$$

$$+ \frac{\begin{array}{c} n^{(-1-a)/2}(\sum_{i=0}^{[nT]-k_n} E|\xi^n_i(k_n)|^p \\ -2^{p/2} \sum_{i=0}^{[nT]-2k_n} E|\xi^n_i(2k_n)|^p) \end{array}}{n^{pa/2-1} V_p(Y, 2k_n)^n_T}. \tag{A38}$$

From the results (10) and (A31), it is not hard to find that the first term of (A38) converges weakly to the distribution $\Sigma_p N(0, 1)$. By Lemma A.3 and the condition $1/3 < a < 1/2$, we find that the second term of (A38) converges to zero. Thus, we complete the proof of Theorem 2.3. ■

Proof of Corollary 2.4 This follows easily from Theorems 2.3 and 2.1. ■

Proof of Proposition 2.5 Similarly to the proof of Theorem 2.3, it is enough to show that the result (A21) holds for $1/5 < a < 1/2$. When the noises ϵ_i are i.i.d. normal, for the result (A22), we have

$$E\left| \frac{\xi^n_i(jk_n)}{\alpha_{i,n}(j)} \right|^p - m_p = 0, \quad j = 1, 2, \quad i = 0, 1, 2, \ldots, [nt] - k_n.$$

Hence, it is enough to show

$$\left| \alpha^p_{i,n}(1) - 2^{p/2} \alpha^p_{i,n}(2) \right| \leq \frac{C}{k_n^2}. \tag{A39}$$

By the mean value theorem, it is enough to show that the result (A39) holds for $p = 2$. Letting $E\epsilon_t^2 = \sigma_\epsilon^2$, by (A4), we have

$$\alpha^2_{i,n}(1) - 2\alpha^2_{i,n}(2)$$

$$= n^a \sigma_\epsilon^2 \left[\sum_{i=1}^{k_n} \left(g\left(\frac{i}{k_n} \right) - g\left(\frac{i-1}{k_n} \right) \right)^2 \right.$$

$$\left. - 2 \sum_{i=1}^{2k_n} \left(g\left(\frac{i}{2k_n} \right) - g\left(\frac{i-1}{2k_n} \right) \right)^2 \right]$$

$$= -n^a \sigma_\epsilon^2 \sum_{i=1}^{k_n} \left(g\left(\frac{2i}{2k_n} \right) - 2g\left(\frac{2i-1}{2k_n} \right) + g\left(\frac{2(i-1)}{2k_n} \right) \right)^2$$

$$= -n^a \sigma_\epsilon^2 \sum_{i=1}^{k_n} \left(g''\left(\frac{i-1}{k_n} \right) \frac{1}{(2k_n)^2} + O(k_n^{-3}) \right)^2 = O(k_n^{-2}).$$

Hence we obtain the result (A39) and complete the proof of Proposition 2.5. ■

Proof of Theorem 2.6 This is easy using Theorems 2.3 and 2.1. ■