LISTA III AVLC | Kailane Eduarda Felix da Silva | @ KEFS

① Para as matrizes A e B serem diagonalizáveis, ambas precisam ter autovetores distintos. Primeiro, calcularemos os autovalores de A:

Temos que os autovalores reais de A são as raízes reais do polinômio característico: $p(\lambda) = \det(A - \lambda I)$. Então:

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & a-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)\cdot(a-\lambda) = 0 \begin{cases} \lambda_1 = 1 \\ \lambda_2 = a \end{cases}$$

Substituindo para encontrar os autovetores:

$$\begin{bmatrix} 1-\lambda & 1 \\ 0 & a-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & a-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (a-1)y = 0$$

O autovetor associado à $y=1$ é $(1,0) \rightarrow v_1$

Para $\lambda = a$, temos:

$$\begin{bmatrix} 1-a & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (1-a)x + y = 0 \rightarrow y = (a-1)\cdot x$$

Para $v_1 \neq v_2$, $0 \neq (a-1)\cdot 1$ (quando $x=1$), então $a \neq 1$

Para a matriz B, a mesma lógica =>

$$\begin{vmatrix} 1-\lambda & a \\ 0 & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)^2 = 0 \rightarrow \lambda = 1$$

Substituindo:

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$, então $a = 0$

② Para achar os autovalores, usamos a relação: $\det(A-\lambda I)$ matriz identidade

a)

$$A-\lambda.I = \begin{bmatrix} -1 & -4 & 14 \\ 2 & -7 & 14 \\ 2 & -4 & 11 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & -4 & 14 \\ 2 & -7-\lambda & 14 \\ 2 & -4 & 11-\lambda \end{bmatrix}$$

Então, usando a Regra de Sarrus:

$$\begin{vmatrix} -1-\lambda & -4 & 14 \\ 2 & -7-\lambda & 14 \\ 2 & -4 & 11-\lambda \end{vmatrix} = 0 \qquad \begin{aligned} &-\lambda^3+3\lambda^2+45\lambda+81=0 \\ &-(\lambda+3)(\lambda+3)(\lambda-9)=0 \\ &\lambda_1=-3 \quad \lambda_2=9 \end{aligned}$$

* autovetores associados a $\gamma_1=-3$:

$$T(v)=\lambda(v) \rightarrow \begin{bmatrix} -1 & -4 & 14 \\ 2 & -7 & 14 \\ 2 & -4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -3\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} -x-4y+14z \\ 2x-7y+14z \\ 2x-4y+11z \end{bmatrix} = \begin{bmatrix} -3x \\ -3y \\ -3z \end{bmatrix}$$

encontramos o seguinte sistema linear:

$$\begin{cases} -x-4y+14z=-3x \\ 2x-7y+14z=-3y \\ 2x-4y+11z=-3z \end{cases} \rightarrow \begin{aligned} &2x-4y+14z=0 \rightarrow x=2y-7z \\ &(2y-7z,y,z)=y(2,1,0)+z(-7,0,1) \end{aligned}$$

* autovetores associados a $\gamma_2=9$:

$$T(v)=\lambda v \rightarrow \begin{bmatrix} -1 & -4 & 14 \\ 2 & -7 & 14 \\ 2 & -4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 9\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} -x-4y+14z \\ 2x-7y+14z \\ 2x-4y+11z \end{bmatrix} = \begin{bmatrix} 9x \\ 9y \\ 9z \end{bmatrix}$$

$$\begin{cases} -x-4y+14z=9x \\ 2x-7y+14z=9y \\ 2x-4y+11z=9z \end{cases} \rightarrow \begin{aligned} &-10x-4y+14z=0 \\ &2x-16y+14z=0 \rightarrow x=8y-14y+7x \rightarrow x=y \\ &2x-4y+2z=0 \rightarrow z=2y-x \rightarrow z=2y-y \end{aligned}$$

só que $x=y$, então $z=y$.

$$(y,y,y)=y(1,1,1)$$

Assim, os autovalores são

$\lambda_1=-3$ e $\lambda_2=9$

e os autovetores:

$v_1=(2,1,0)(-7,0,1)$

$v_2=(1,1,1)$

② b)

$$A-\lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \to (1-\lambda)(1-\lambda)(1-\lambda) = 0 \to \lambda = 1$$

autovalor

* autovetores associados a $\lambda = 1$

$$T(v) = \lambda v \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x+2y+3z \\ y+2z \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

encontramos o sistema:

$$\begin{cases} x+2y+3z = x \\ y+2z = y \\ z = z \end{cases} \to \begin{array}{l} y+2z = y \to z = 0 \\ x+2y+3z = x \to 2y+3z = 0 \to y = 0 \end{array}$$

$$(x,0,0) = x(1,0,0)$$

assim, o autovalor $\lambda = 1$

e seu autovetor associado: $v = (1,0,0)$

③ a)

$$A-\lambda I = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a-\lambda & 0 & 0 \\ 0 & b-\lambda & c \\ 0 & c & b-\lambda \end{bmatrix}$$

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & b-\lambda & c \\ 0 & c & b-\lambda \end{vmatrix} = 0 \qquad \begin{array}{l} (a-\lambda)(b-\lambda)^2 - c^2(a-\lambda) = 0 \\ (a-\lambda)\left((b-\lambda)^2 - c^2\right) = 0 \to \end{array}$$

$a - \lambda_1 = 0 \to a = \lambda_1$

$(b-\lambda)^2 = c^2$

$b - \lambda_2 = c \to \lambda_2 = b - c$

$\lambda_3 - b = c \to \lambda_3 = b + c$

* De fato, os autovalores são a, $b-c$ e $b+c$

③ b) encontrando o autovetor associado a $\lambda = a$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & b-a & c \\ 0 & c & b-a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

-> chegamos ao sistema:

$$\begin{cases} (b-a)y + cz = 0 \\ cy + (b-a)z = 0 \end{cases} \rightarrow \quad z = \frac{(a-b)y}{c} \qquad c \cdot y - \frac{(a-b)^2 y}{c} = 0$$

$y = 0 \rightarrow z = 0$

$v_1 = (x, 0, 0)$

$$y\left(c - \frac{(a-b)^2}{c}\right) = 0$$

* encontrando o autovetor associado a $\lambda = b-c$

$$\begin{bmatrix} a-b-c & 0 & 0 \\ 0 & -c & c \\ 0 & c & -c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

-> sistema:

$$\begin{cases} (a-b-c)x = 0 \\ c(z-y) = 0 \\ c(y-z) = 0 \end{cases} \rightarrow$$

solucionando-o, encontramos o vetor $v_2 = (0, y, y)$

* encontrando o autovetor associado a $\lambda = b+c$

$$\begin{bmatrix} a-b+c & 0 & 0 \\ 0 & b-b+c & c \\ 0 & c & b-b+c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x(a-b+c) = 0 \\ c(y+z) = 0 \\ c(y+z) = 0 \end{cases} \qquad \begin{aligned} x &= 0 \\ y &= -z \\ v_3 &= (0, y, -y) \end{aligned}$$

Então, a base de autovetores é:

$$\{(1,0,0), (0,1,1), (0,1,-1)\}$$

(4) axiomas do produto interno:

a)

$\langle (x,y,z),(x,y,z)\rangle = x^2 + 5y^2 + 2z^2 \geq 0$

quando $x = y = z = 0$.

$\langle \alpha v_1, v_2 \rangle = \alpha x_1 x_2 + 5\alpha y_1 y_2 + 2\alpha z_1 z_2 = \alpha(x_1x_2 + 5y_1y_2 + 2z_1z_2) = \alpha \langle v_1 v_2\rangle$

$\langle v_1 + v_2, v_3 \rangle = (x_1 + x_2)x_3 + 5(y_1+y_2)y_3 + 2(z_1+z_2)z_3 \rightarrow$

$= x_1x_3 + 5y_1y_3 + 2z_1z_3$

$= \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$

$\langle v_1, v_2\rangle = x_1x_2 + 5y_1y_2 + 2z_1z_2 = x_2x_1 + 5y_2y_1 + 2z_2z_1 = \langle v_1, v_2\rangle$

Conclusão: é, de fato, um produto interno.

b) aplicando Gram-Schmidt:

(1)

$v_1 = (1,0,0)$

$\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{1^2} = 1 \rightarrow e_1 = (1,0,0)$:0

(2)

$u_2 = (0,1,0) - \langle (1,0,0),(0,1,0)\rangle \dfrac{(1,0,0)}{\langle (1,0,0),(1,0,0)\rangle} = (0,1,0)$

$\|u_2\| = \sqrt{\langle u_2, u_2\rangle} = \sqrt{5}$

$\rightarrow e_2 = \dfrac{1}{\sqrt{5}} \cdot (0,1,0)$

(3)

$u_3 = (0,0,1) - \langle (0,0,1)(0,1,0)\rangle \dfrac{(0,1,0)}{\langle (0,1,0),(0,1,0)\rangle} - \langle (1,0,0),(0,0,1)\rangle \dfrac{(1,0,0)}{\langle (1,0,0),(1,0,0)\rangle}$

$u_3 = (0,0,1)$

$\|u_3\| = \sqrt{\langle u_3, u_3\rangle} = \sqrt{2} \rightarrow e_3 = \dfrac{1}{\sqrt{2}}(0,0,1)$

* Portanto, base $= \left\{(1,0,0), \dfrac{1}{\sqrt{5}}(0,1,0), \dfrac{1}{\sqrt{2}}(0,0,1)\right\}$

5

$v_1 = (1,1,0)$, $v_2 = (1,0,1)$, $v_3 = (0,2,0)$

1 Definindo $v'_1 = (1,1,0)$

Calculando v'_2 =>

$$v'_2 = v_2 - \frac{\langle v_2, v'_1 \rangle}{\langle v'_1, v'_1 \rangle} \cdot v'_1 = (1,0,1) - \frac{\langle (1,0,1),(1,1,0) \rangle}{\langle (1,1,0),(1,1,0) \rangle} (1,1,0)$$

$\langle (1,0,1),(1,1,0) \rangle = 1$

$\langle (1,1,0),(1,1,0) \rangle = 1+1 = 2$ $\quad \rightarrow \; v'_2 \; (1,0,1) - \frac{1}{2}(1,1,0) = \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$

Calculando v'_3 =>

$$v'_3 = v_3 - \frac{\langle (v_3, v'_2) \rangle}{\langle v'_2, v'_2 \rangle} \cdot v'_2 - \frac{\langle v_2, v'_1 \rangle}{\langle v'_1, v'_1 \rangle} \cdot v'_1$$

$$v'_3 = (0,2,0) - \frac{\langle (1/2, -1/2, 1),(0,2,0) \rangle}{\langle (1/2, -1/2, 1),(1/2, -1/2, 1) \rangle} \cdot \left(\frac{1}{2}, -\frac{1}{2}, 1\right) - \frac{\langle (1,0,1),(1,1,0) \rangle}{\langle (1,1,0),(1,1,0) \rangle} \cdot (1,1,0)$$

$$v'_3 = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Normalizando os vetores, temos que

$$\beta' = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right); \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{6}}{3}\right); \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) \right.$$