$$\int \operatorname{sen}^n x \cdot \cos^m x \, dx$$

- Se ambas as potencias forem impares Voce pode separar qualquer um dos dois Mas voce vai preferir separar o de exp. menor

- Aquele que tiver expoente ímpar, você separa e usa a relação trigonométrica fundamental

$$(\operatorname{sen}^2 x + \cos^2 x = 1)$$

- Se ambas as potencias forem pares, aí temos que usar as seguintes relações:

$$\operatorname{sen}^2(x) = \frac{1 - \cos(2x)}{2} \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\operatorname{sen} x \cdot \cos x = \frac{\operatorname{sen}(2x)}{2}$$

Ex 1.: $\displaystyle\int \cos^2(x)\,dx = \int \frac{1+\cos(2x)}{2}\,dx$

$$= \frac{1}{2}\int (1+\cos(2x))\,dx$$

$$= \frac{1}{2}\left[\int 1\,dx + \int \cos(2x)\,dx\right]$$

$$= \frac{1}{2}\left[x + \frac{\operatorname{sen}(2x)}{2}\right] + C.$$

Ex. 2.: $\displaystyle\int \operatorname{sen}^2 x\,dx = \dots = \frac{1}{2}\left[x - \frac{\operatorname{sen}(2x)}{2}\right] + C$ (VERIFIQUE)

Ex. 3.: Tentativa 1: $\operatorname{sen}^2(x) = \dfrac{1-\cos(2x)}{2}$; $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

e) $\displaystyle\int \operatorname{sen}^2(x)\cos^2(x)\,dx$

$(A - B)\cdot(A+B) = A^2 - B^2$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)dx = \frac{1}{4}\int \left(1 - \cos^2(2x)\right)dx$$

$$= \frac{1}{4}\int \underbrace{\operatorname{sen}^2(2x)}\,dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2}\,dx$$

Relação: $\operatorname{sen}^2(2x) = \dfrac{1-\cos(4x)}{2}$

$$= \frac{1}{8}\left[\int 1\,dx - \int \cos(4x)\,dx\right] = \frac{1}{8}\left[x - \frac{\operatorname{sen}(4x)}{4}\right] + C.$$

opção 2: $\displaystyle\int \operatorname{sen}^2 x \cos^2 x\,dx = \int (\operatorname{sen} x \cos x)^2\,dx$

$$= \int \left(\frac{\operatorname{sen}(2x)}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \operatorname{sen}^2(2x)\, dx$$

$$= \frac{1}{4}\int \frac{1-\cos(4x)}{2}\, dx$$

$$= \frac{1}{8}\left[x - \frac{\operatorname{sen}(4x)}{4}\right] + C$$

d) $\int \operatorname{sen}^3(x)\cos^{21}(x)\,dx$

Opção 1:

$$\int \operatorname{sen}^2 x \cdot \cos^{21} x \cdot \underbrace{\operatorname{sen} x\, dx}_{-du}$$

$u = \cos x$

$$= \int \operatorname{sen}^2 x \cdot u^{21} \cdot (-du)$$

$1 - \cos^2 x = 1 - u^2$

$$= \int (1 - u^2) \cdot u^{21} \cdot (-du)$$

$$= -\int (u^{21} - u^{23})\, du$$

$$= -\left[\frac{u^{22}}{22} - \frac{u^{24}}{24} \right] + C$$

$$= -\left[\frac{\cos^{22}(x)}{22} - \frac{\cos^{24}(x)}{24} \right] + C$$

Opção 2:

$$\int \operatorname{sen}^3 x \cdot \cos^{20} x \cdot \underbrace{\cos x\, dx}_{du}$$

$u = \operatorname{sen} x$

$$= \int u^3 \cdot \cos^{20} x \cdot du$$

$(\cos^2 x)^{10} = (1 - \operatorname{sen}^2 x)^{10}$

$= (1 - u^2)^{10}$

$$= \int u^3 \cdot (1 - u^2)^{10} \cdot du$$

(DIFÍCIL!)

Relações de prostaférese: *transformação de produto de senos e cossenos em somas*

$$2\,\text{sen}\,A\,\cos B = \text{sen}(A-B) + sen(A+B).$$
$$2\,\text{sen}\,A\,\text{sen}\,B = \cos(A-B) - \cos(A+B).$$
$$2\,\cos A\,\cos B = \cos(A-B) + \cos(A+B).$$

a) $\int \underbrace{\text{sen}(4x)}_{A}\,\underbrace{\cos(x)}_{B}\,dx$

arg. dif.

Usando:

$$\text{sen}\,A\,\cos B = \frac{\text{sen}(A-B) + sen(A+B)}{2}$$

Obs.: Reescrevendo

$$\text{sen}(4x)\cos(x) = \frac{1}{2}\left[\text{sen}(4x-x) + \text{sen}(4x+x)\right]$$
$$= \frac{1}{2}\left[\text{sen}(3x) + \text{sen}(5x)\right]$$

Na integral:

$$\int \text{sen}(4x)\cos x\,dx = \frac{1}{2}\left(\int \text{sen}(3x) + \text{sen}(5x)\,dx\right)$$
$$= \frac{1}{2}\left(\int \text{sen}(3x)\,dx + \int \text{sen}(5x)\,dx\right)$$
$$= \frac{1}{2}\left[-\frac{\cos(3x)}{3} - \frac{\cos(5x)}{5}\right] + C$$

b) $\int \text{sen}(4x)\cos(2x)\cos(3x)\,dx$

I) $2\,\text{sen}\,A\,\cos B = \text{sen}\,(A - B) + \text{sen}\,(A + B).$

II) $2\,\text{sen}\,A\,\text{sen}\,B = \cos\,(A - B) - \cos\,(A + B).$

III) $2\,\cos A\,\cos B = \cos\,(A - B) + \cos\,(A + B).$

Obs.: Usando a fórmula (I):

$$\text{sen}(4x)\cdot\cos(2x) = \frac{\text{sen}(4x-2x) + \text{sen}(4x+2x)}{2}$$

$$= \frac{\text{sen}(2x) + \text{sen}(6x)}{2}$$

Temos:

$$\left(\text{sen}(4x)\cos(2x)\right)\cdot\cos(3x) = \left(\frac{\text{sen}(2x) + \text{sen}(6x)}{2}\right)\cdot\cos(3x)$$

$$= \frac{1}{2}\cdot\left[\text{sen}(2x)\cdot\cos(3x) + \text{sen}(6x)\cos(3x)\right]$$

$$= \frac{1}{2}\left[\frac{\text{sen}(2x-3x) + \text{sen}(2x+3x)}{2} + \frac{\text{sen}(6x-3x) + \text{sen}(6x+3x)}{2}\right]$$

$$= \frac{1}{4}\left[\text{sen}(-x) + \text{sen}(5x) + \text{sen}(3x) + \text{sen}(9x)\right]$$

Daí:

$$\int \text{sen}(4x)\cos(2x)\cdot\cos(3x)\,dx = \frac{1}{4}\left[-\frac{\cos(-x)}{-1} - \frac{\cos(5x)}{5} + \right.$$

$$\left. - \frac{\cos(3x)}{3} - \frac{\cos(9x)}{9}\right] + C$$

$$= \frac{1}{4}\left[\cos(-x) - \frac{1}{5}\cos(5x) - \frac{1}{3}\cos(3x) - \frac{1}{9}\cos(9x)\right] + C.$$

e) $\int \text{sen}(2x)\cos^2(3x)\,dx$

$2\,\text{sen}\,A\cos B = \text{sen}\,(A-B) + \text{sen}\,(A+B).$
$2\,\text{sen}\,A\,\text{sen}\,B = \cos\,(A-B) - \cos\,(A+B).$
$2\cos A\cos B = \cos\,(A-B) + \cos\,(A+B).$

Ideia 1: $\int \left(\text{sen}(2x)\cdot\cos(3x)\right)\cos(3x)\,dx$

Ideia 2: $\int \text{sen}(2x)\cdot\left[\frac{1+\cos(6x)}{2}\right]dx$

$$= \int\left(\frac{\text{sen}(2x)}{2} + \frac{\text{sen}(2x)\cdot\cos(6x)}{2}\right)dx$$

$$= \frac{1}{2}\left[\int \text{sen}(2x)\,dx + \frac{1}{2}\int \text{sen}(-4x) + \text{sen}(8x)\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{\cos(2x)}{2} + \frac{1}{2}\left[-\frac{\cos(-4x)}{-4} - \frac{\cos(8x)}{8}\right]\right] + C$$

$$= -\frac{1}{4}\cos(2x) + \frac{1}{4}\left(\frac{\cos(-4x)}{4} - \frac{1}{8}\cos(8x)\right) + C.$$

b) $\int \underbrace{\text{arcsen}(x)}_{A}\, dx$ P: x^0

$\overset{\leftarrow u}{\text{L A P T E}}$ ($\xrightarrow{dv}$)

$u = \text{arcsen}\,x$, $\quad dv = 1\,dx \quad \Rightarrow \begin{cases} du = \dfrac{1}{\sqrt{1-x^2}}\,dx \\ v = x \end{cases}$

Fórmula da Vaca:

$$\int u\,dv = u.v - \int v\,du$$

$$\Rightarrow \int \text{arcsen}\,x\,dx = (\text{arcsen}\,x).x - \underbrace{\int \frac{x}{\sqrt{1-x^2}}\,dx}_{(*)}$$

(*): $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = \int (1-x^2)^{-1/2}.\,\underbrace{x\,dx}$

$t = 1-x^2 \longrightarrow dt = -2x\,dx \longrightarrow x\,dx = \dfrac{dt}{-2}$

$$= \int t^{-1/2}.\frac{dt}{-2} = -\frac{1}{2}\int t^{-1/2}\,dt$$

$$= -\frac{1}{2}.\frac{t^{(-1/2+1)}}{(1/2+1)} = -\frac{1}{2}.\frac{t^{1/2}}{1/2} = -\sqrt{t}$$

$$= -\sqrt{1-x^2}.$$

voltando:

$$\Rightarrow \int \text{arcsen}\,x\,dx = (\text{arcsen}\,x).x - \underbrace{\int \frac{x}{\sqrt{1-x^2}}\,dx}_{(*)}$$

$$\Rightarrow \int \text{arcsen}\,x\,dx = (\text{arcsen}\,x).x + \sqrt{1-x^2} + C.$$

d) $\displaystyle\int \frac{1}{\sqrt{x^2+16}}\,dx$

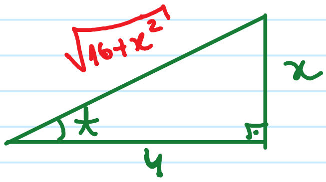
$$\|$$

$$\underbrace{\int (x^2+16)^{-1/2}\,dx}$$

substituição simples não funciona!

SUGESTÕES DE SUBSTITUIÇÕES:

Raíz:	Subst.	Relação:	Resultado:
$\sqrt{1-x^2}$	$x=\text{sen}\,t$	$1-\text{sen}^2t=\cos^2t$	$\sqrt{1-x^2}=\cos t$
$\sqrt{1+x^2}$	$x=\text{tg}\,t$	$1+\text{tg}^2t=\sec^2t$	$\sqrt{1+x^2}=\sec t$
$\sqrt{x^2-1}$	$x=\sec t$	$\sec^2t-1=\text{tg}^2t$	$\sqrt{x^2-1}=\text{tg}\,t$
$\sqrt{a^2-x^2}$	$x=a\,\text{sen}\,t$	$a^2-a^2\text{sen}^2t=a^2\cos^2t$	$\sqrt{a^2-x^2}=a\cos t$
$\sqrt{a^2+x^2}$	$x=a\,\text{tg}\,t$	$a^2+a^2\text{tg}^2t=a^2\sec^2t$	$\sqrt{a^2+x^2}=a\sec t$
$\sqrt{x^2-a^2}$	$x=a\sec t$	$a^2\sec^2t-a^2=a^2\text{tg}^2t$	$\sqrt{x^2-a^2}=a\,\text{tg}\,t$.



* Subst. trigonométrica: $x = 4\,\text{tg}(t) \longrightarrow dx = 4\sec^2t\,dt$

$\text{tg}(t) = \dfrac{x}{4}$

d) $$\int \frac{1}{\sqrt{x^2+16}}\,dx = \int \frac{1}{\sqrt{(4\,\text{tg}\,t)^2+16}}\cdot 4\sec^2t\,dt$$

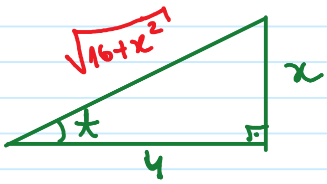
$$= \int \frac{1}{\sqrt{\underbrace{16\,\text{tg}^2t+16}}}\cdot 4\sec^2t\,dt$$

$$16.(\text{tg}^2t+1) = 16\sec^2t$$

$$= \int \frac{1}{\sqrt{16\sec^2t}}\cdot 4\sec^2t\,dt$$

$$= \int \frac{1}{\cancel{4}.\cancel{\sec t}}\cdot \cancel{4}\sec\cancel{^2}t\,dt$$

$$= \int \sec(t)\,dt$$



$$= \ln|\sec t + \text{tg}\,t| + C$$

$$= \ln\left|\frac{\sqrt{16+x^2}}{4} + \frac{x}{4}\right| + C.$$

* $\sec t = \dfrac{1}{\cos t} = \dfrac{1}{\left(\frac{C.A}{hip.}\right)} = \dfrac{hip}{C.A} = \dfrac{\sqrt{16+x^2}}{4}$

f) $\int \frac{x}{\underbrace{\sqrt{8-2x-x^2}}_{\text{completar quadrado!}}}\,dx$

$a^2 - x^2$

$a^2 + x^2$

$\underbrace{-(x^2+2x-8)}$ $\quad x^2 - a^2$

Tentar comparar $8-2x-x^2$ com o quadrado perfeito:

$$A^2 + 2AB + B^2 = (A+B)^2$$

$$x^2 + 2x.1 + 1 = (x+1)^2$$

preciso!

SACADA:

$$-(x^2+2x-8) = -\left((x^2+2x+1)-1-8\right)$$

$$= -\left((x+1)^2 - 9\right)$$

$$= -(x+1)^2 + 9$$

$$= 9 - (x+1)^2$$

f) $\int \frac{x}{\sqrt{8-2x-x^2}}\,dx = \int \frac{x}{\sqrt{9-(x+1)^2}}\,dx$

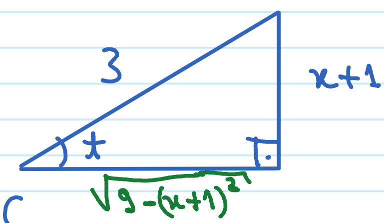
Ideia: $\boxed{x+1 = 3\,\text{sen}\,t} \Rightarrow dx = 3\cos t\,dt$

$\text{sen}\,t = \frac{x+1}{3}$ → CAT. OPOSTO, 3 → Hipotenusa.

$$9 - 9\,\text{sen}^2 t = 9(1-\text{sen}^2 t) = 9\cos^2 t$$

$$= \int \frac{(3\,\text{sen}\,t - 1)}{\cancel{\sqrt{9\cos^2 t}}} \cdot \cancel{3\cos t}\,dt = \int (3\,\text{sen}\,t - 1)\,dt$$

$$= -3\cos t - t + C.$$



$$= -\cancel{3}.\frac{\sqrt{9-(x+1)^2}}{\cancel{3}} - \text{arcsen}\left(\frac{x+1}{3}\right) + C$$

$$= -\sqrt{9-(x+1)^2} - \text{arcsen}\left(\frac{x+1}{3}\right) + C$$

Cuidado: $\text{sen}\,t = \left(\frac{x+1}{3}\right) \Rightarrow t = \text{arcsen}\left(\frac{x+1}{3}\right)$

Se o ângulo tiver sozinho, no caso do t, use a função Trigonometrica inversa (arco)

Desafio: $\int t^3 . e^{-t^2} \, dt$

Spoiler: NÃO É PRA USAR INTEGRAÇÃO POR PARTES DE PRIMEIRA!

ALUNO EMOCIONADO:	NIVEL AVANÇADO:
$u = t^3$, $dv = e^{-t^2}$ $du = 3t^2 \, dt$, $v = ?????$	João: $u = t^2$. $dv = te^{-t^2} dt$

NIVEL ALUNO DO WESLLEY:
''viu função composta? Substituição primeiro!
Para enxugar a integral! "

$$\int \underline{t^3} . e^{(-t^2)} \, dt$$

$$y = -t^2 \; ; \quad dy = -2t . dt$$

$$\int \underline{t^3} . e^{(-t^2)} \, dt = \int t^2 . e^{\overset{y}{(-t^2)}} . \underbrace{t \; dt}_{dy/-2} = \int -y . e^y . \frac{dy}{-2}$$

penetra!
$t^2 = -y$

$$= \frac{1}{2} \int y . e^y \, dy$$

Produto de funções **simples de familias diferentes**
É o metodo de integração por partes (LAPTE)

$$u = y, \quad dv = e^y dy \Rightarrow du = 1 \, dy, \quad v = e^y$$

Daí:

$$= \frac{1}{2} \left[ye^y - \int e^y . dy \right] = \frac{1}{2} \left[ye^y - e^y \right] + C.$$

$y = -t^2$

$$= \boxed{\frac{1}{2} \left[(-t^2) . e^{-t^2} - e^{-t^2} \right] + C.}$$