Math 215 — Second Midterm

November 7, 2019

First 3 Letters of Last Name: ☐☐☐ UM Id#: ____________

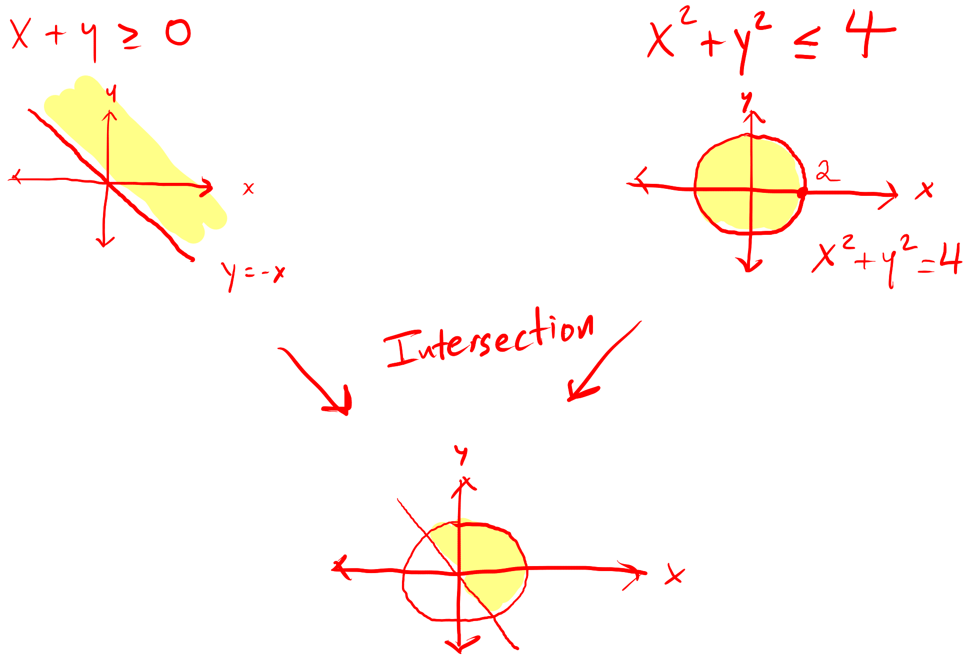
Instructor: ______________________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam, other than the formula sheet at the end of the exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. The true or false questions are the only questions that do not require you to show your work. For all other questions show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use no aids (e.g., calculators or notecards) on this exam.
7. **Turn off all cell phones**, remove all headphones, and **place any watch you are using on the desk in front of you**.

Problem	Points	Score
1	10	
2	12	
3	15	
4	10	
5	12	
6	15	
7	15	
8	11	
Total	100	

1. [10 points] Consider the function $f(x, y) = x - y$ on the region D in $\mathbb{R}^2$ defined by $x + y \geq 0$ and $x^2 + y^2 \leq 4$.

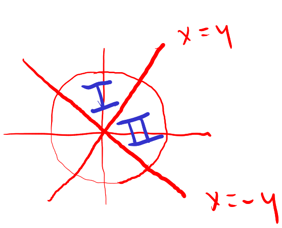
a. [5 points] Sketch the region D.



b. [5 points] Calculate the integral of f over D.

Symmetry:

Think of $I = \iint_D f(x,y)\,dx\,dy$ as what you get when you take all points (x,y) in D, plug them into f, and add up the results. Then:



If (a,b) is in region I, then (b,a) is in region II. Moreover $f(a,b) + f(b,a) = 0$. Pairing all points $\Rightarrow I = 0$.

2. [12 points] Find and classify the critical points for the function $h(x, y) = x^4 + y^3 - 6y - 2x^2$.

Derivation of classification:

Let (a,b) be a critical point of f so $\nabla f(a,b) = 0$.

Taylor expand f near $(a,b) =: \vec{p}$:

$$f(\vec{p} + \vec{h}) = f(\vec{p}) + \nabla f(\vec{p}) \cdot \vec{h} + \vec{h}^T \cdot \mathcal{H}_f(\vec{p}) \cdot \vec{h} + O(|\vec{h}|^3)$$

where $\vec{h} = (da, db)$ and $\mathcal{H}_f = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}$

So : $f(\vec{p} + \vec{h}) = f(\vec{p}) + f_{xx}\, da^2 + 2 f_{xy}\, da\, db + f_{yy}\, db^2 + \ldots$

Completing the square, assuming $f_{xx} \neq 0$, we get

$$f(\vec{p} + \vec{h}) = f(\vec{p}) + f_{xx} \left(\left(da + \frac{f_{xy}}{f_{xx}} db \right)^2 + y^2 \left(\frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}^2} \right) \right)$$

Set $D(x,y) = f_{xx} f_{yy} - f_{xy}^2$

If $D < 0 \Rightarrow$ second order term can be positive or negative
$\Rightarrow$ saddle point

If $D > 0$ and $f_{xx} > 0 \Rightarrow$ second order term always positive $\Rightarrow$ local min

If $D > 0$ and $f_{xx} < 0 \Rightarrow$ second order term always negative $\Rightarrow$ local max.

$$h(x,y) = x^4 + y^3 - 6y - 2x^2$$

$$\nabla h = (4x^3 - 4x,\ 3y^2 - 6)$$

$$\nabla h = 0 \implies \begin{matrix} 4x^3 - 4x = 0 \\ 3y^2 - 6 = 0 \end{matrix} \implies \begin{matrix} x = 0, \pm 1 \\ y = \pm\sqrt{2} \end{matrix}$$

(six solutions)

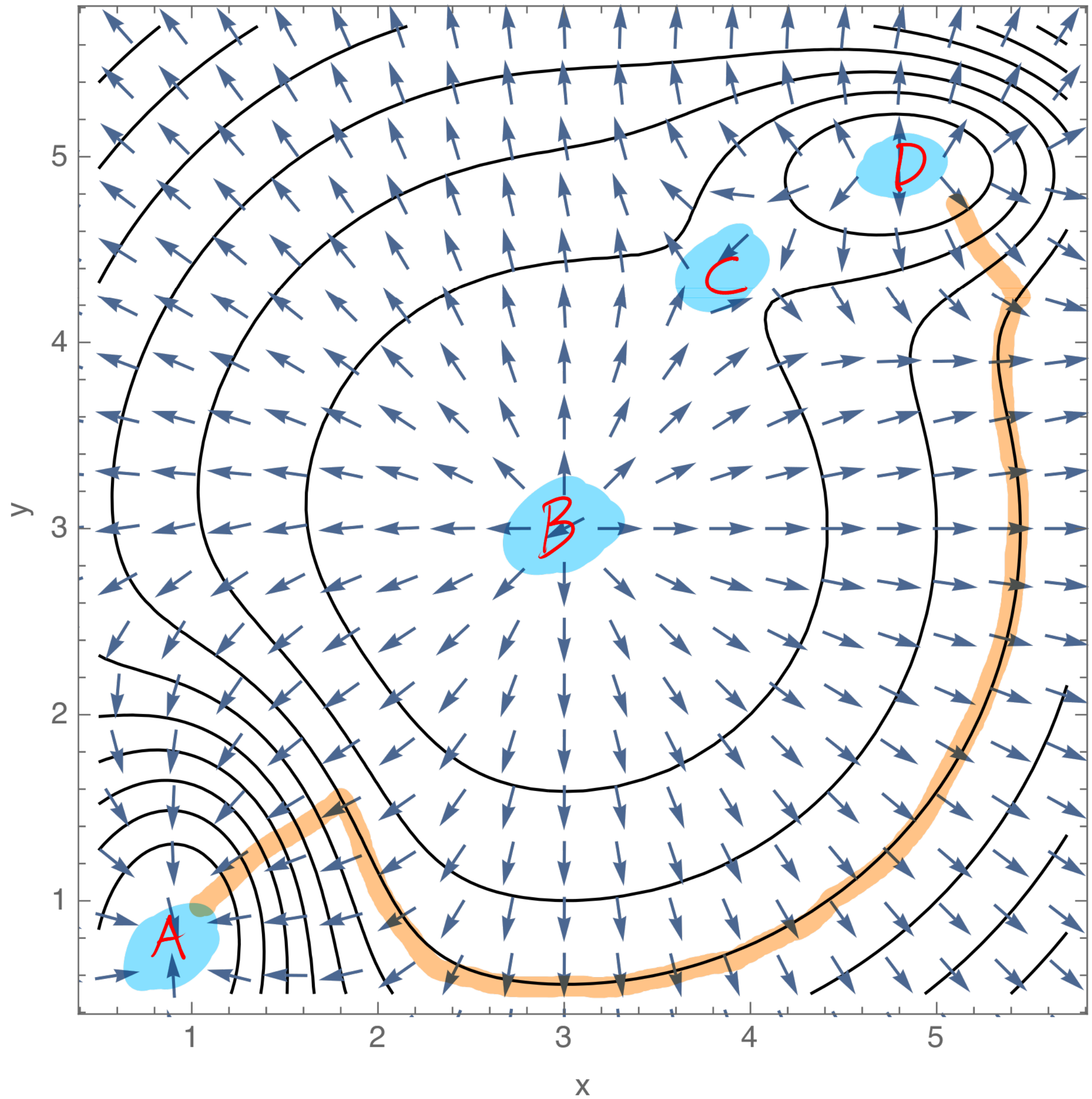
$$f_{xx} = 12x^2 - 4 \qquad f_{xy} = 0 \qquad f_{yy} = 6y$$

$$D(x,y) = 24y\,(3x^2 - 1)$$

x	y	$\mathrm{sgn}(D)$	$\mathrm{sgn}(f_{xx})$	Type
0	$\sqrt{2}$	$-$	$-$	saddle
0	$-\sqrt{2}$	$+$	$-$	local max
1	$\sqrt{2}$	$+$	$+$	local min
1	$-\sqrt{2}$	$-$	$+$	saddle
-1	$\sqrt{2}$	$+$	$+$	local min
-1	$-\sqrt{2}$	$-$	$+$	saddle

If $f_{xx} = f_{xy} = f_{yy} = 0$, then need to look at 3rd order terms (you won't have to do this)

3. [15 points] The graph below is a plot of some of the level curves of a function g in a rectangular region $R = [.4, 5.8] \times [.4, 5.8]$. Assume that as we move between adjacent level curves the value of g increases or decreases by exactly one. The arrows point in the direction of ∇g.

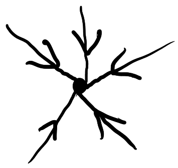


a. [9 points] Identify the approximate coordinates[1] of the critical points of g in R. For each critical point, indicate if it is a point where the function has a local maximum, a local minimum, or a saddle.

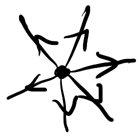
Imagine arrows represent flow of stream. Drop a leaf, where does it go?

[1]That is, within .1 in each coordinate direction.

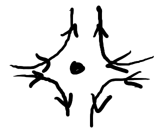
For gradient flow:



sink = local max



source = local min



Close call
= saddle

Near local mins/maxes, level curves form bullseyes.

A: sink $\to$ local max

B: source $\to$ local min

C: close call $\to$ saddle

D: source $\to$ local min

b. [4 points] Identify the approximate coordinates[2] of the points where the function g attains its global maximum and global minimum over the rectangle R.

Global max at unique local max A or on boundary; by inspection, A is at a higher "height".

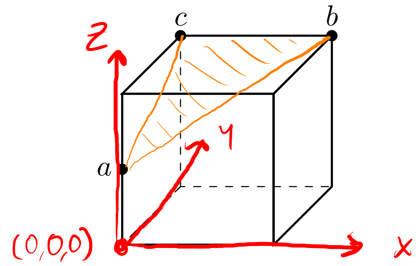
Global min at B, D, or boundary; by inspection D is at lower "height".

c. [2 points] If the value of g at its global minimum on R is between 23 and 24, then the value of the global maximum of g is between _______and _______.

Follow the contour curves (orange) and count up by 1 each time.

[2]That is, within .1 in each coordinate direction.

4. [10 points] The sides of the cube below have length six. The point a is at the midpoint of its edge. Let P be the plane that contains the points a, b, and c.



a. [8 points] Set up, but do not evaluate, an integral for the volume of that part of the cube that lies below the plane P.

$a = (0,0,3)$

$b = (6,6,6)$

$c = (0,6,6)$

Let $Ax + By + Cz + D = 0$ be the equation of the plane through a, b, c.

Then:

$$3C + D = 0$$
$$6A + 6B + 6C + D = 0$$
$$6B + 6C + D = 0$$

$\Rightarrow$

$A = 0$

$B = -\frac{C}{2}$

$C = C$

$D = -3C$

$\Longrightarrow$ Eq'n of plane is $-\frac{y}{2} + z - 3 = 0$

Thus the plane is the graph of $f(x,y) = 3 + \frac{y}{2}$

Volume of region beneath plane is therefore

$$\int_0^6 \int_0^6 f(x,y)\, dx\, dy = \int_0^6 \int_0^6 \left(3 + \frac{y}{2}\right) dx\, dy$$

b. [2 points] Calculate the volume of that part of the cube that lies below the plane P.

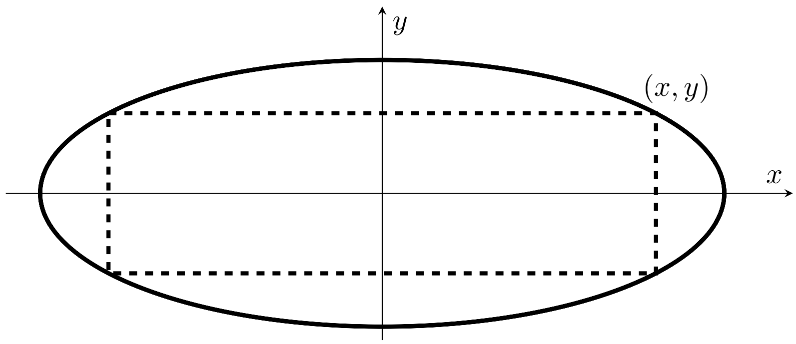
$$\int_0^6 \int_0^6 \left(3 + \frac{y}{2}\right) dx\, dy$$

$$= 6 \int_0^6 \left(3 + \frac{y}{2}\right) dy$$

$$= 6 \left[3y + \frac{y^2}{4} \right]\Big|_0^6 = 6\,(18 + 9)$$

$$= 162.$$

5. [12 points] Suppose a, b are positive constants.
In this problem we will consider rectangles that are inscribed in the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The sides of the inscribed rectangles are parallel to the coordinate axes, as in the figure below.



a. [10 points] Using the method of Lagrange multipliers, find the rectangle of largest area that can be inscribed in the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. What is its area?
No credit given for solutions that do not use the method of Lagrange multipliers.

Each rectangle inscribed in the ellipse is uniquely determined by coordinates of upper right corner (x,y). Area of rectangle w/ upper right corner at (x,y) is $4xy$. Thus, we want to maximize $f(x,y) = 4xy$ subject to constraint $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$\nabla f = (4y, 4x)$

$\nabla g = \left(\frac{2x}{a^2}, \frac{2y}{b^2}\right)$

$\nabla f = \lambda \nabla g \Rightarrow$

$4y = \lambda \frac{2x}{a^2}$

$4x = \lambda \frac{2y}{b^2}$

$$\lambda = 2a^2 \frac{y}{x} = 2b^2 \frac{x}{y} \Rightarrow (ay)^2 = (bx)^2$$

$$\Rightarrow ay = bx \quad (\text{since } x, y > 0)$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{a^2} + \left(\frac{b}{a}\right)^2 \frac{x^2}{b^2} = 2\frac{x^2}{a^2} \Rightarrow x = \frac{a}{\sqrt{2}}$$

$$y = \frac{b}{a} \cdot x = \frac{b}{\sqrt{2}}.$$

Thus, maximum area is $4xy = 4 \frac{a}{\sqrt{2}} \frac{b}{\sqrt{2}} = 2ab$.

b. [2 points] For the inscribed rectangle of maximum area, what are the coordinates of its vertex that lies in the first quadrant?

From the above argument, $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

6. [15 points] Suppose $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and consider the iterated double integral

$$\int_0^4 \int_{y^{1/2}}^2 g(x,y)dx\,dy.$$

In this integral, x is innermost and y is outermost.

a. [5 points] Sketch the region of integration.

y ranges from 0 to 4.

For fixed y, x-coords of points in region range from $\sqrt{y}$ to 2.

$x = \sqrt{y} \iff y = x^2$

b. [5 points] Rewrite the integral with y innermost and x outermost.

From picture: x ranges 0 to 2 and for fixed x, y ranges 0 to x^2 $\Longrightarrow$

$$\int_0^2 \int_0^{x^2} g(x,y)\,dy\,dx$$

c. [5 points] Evaluate the integral for $g(x,y) = \sin(x^3 - 1)$.

$$\int_0^2 \int_0^{x^2} \sin(x^3-1)\,dy\,dx = \int_0^2 \sin(x^3-1)\left[\int_0^{x^2} dy\right] dx$$

$$= \int_0^2 x^2 \sin(x^3-1)\,dx = \left.\frac{-\cos(x^3-1)}{3}\right|_0^2$$

$$= \frac{\cos(1) - \cos(7)}{3}$$

7. [15 points] Indicate if each of the following is true or false by circling the correct answer. No partial credit will be given.

a. [3 points] Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function and $(c,d) \in \mathbb{R}^2$. If $f_{xx}(c,d)f_{yy}(c,d) < [f_{xy}(c,d)]^2$ then f has a saddle point at (c,d).

Additionally need $\nabla f(c,d) = 0$

True False

b. [3 points] Suppose $h\colon \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. Suppose $(a,b) \in \mathbb{R}^2$ and C is the curve in $\mathbb{R}^2$ described by the equation $h(x,y) = h(a,b)$. If ℓ is the tangent line to C at (a,b), then $\nabla h(a,b)$ is perpendicular to ℓ.

$\nabla h(a,b)$ is always normal to level curve through (a,b)

True False

c. [3 points] If $f\colon [0,1] \to \mathbb{R}$ is continuous, then

$$\int_0^1 \int_0^1 f(t)f(s)\,ds\,dt = \left[\int_0^1 f(u)\,du\right]^2.$$

$$\int_0^1\int_0^1 f(s)f(t)\,ds\,dt = \int_0^1 f(t) \underbrace{\int_0^1 f(s)\,ds}_{\text{const. wrt } t}\,dt = \left(\int_0^1 f(s)\,ds\right)\left(\int_0^1 f(t)\,dt\right) \text{''} \left(\int_0^1 f(u)\,du\right)^2$$

True False

d. [3 points] Suppose $g\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $\mathbf{k} = \langle 0,0,1\rangle$. We have

$$D_{\mathbf{k}}g(x,y,z) = g_z(x,y,z).$$

$\frac{\partial g}{\partial z}$ is defined as the directional derivative in the z direction.

True False

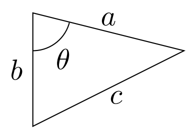
e. [3 points]

$$\int_{-1}^1 \int_0^1 e^{x^2+y^2}\sin(x)\,dy\,dx = 0.$$

$$\left(\underbrace{\int_{-1}^1 e^{x^2}\sin(x)\,dx}_{\text{odd function}}\right) \left(\int_0^1 e^{y^2}\,dy\right) \Rightarrow \int_{-1}^1 e^{x^2}\sin(x)\,dx = 0$$

True False

8. [11 points] Recall from your homework that the Law of Cosines states that for a triangle with sides of length a, b, and c we have $c^2 = a^2 + b^2 - 2ab\cos(\theta)$ where θ is the measure (in radians) of the angle opposite side c. Thus, the Law of Cosines implicitly defines θ as a function of the side lengths a, b, and c.



a. [3 points] Compute $\partial\theta/\partial c$.

$$\cos(\theta) = \frac{a^2 + b^2 - c^2}{2ab}$$

$$-\sin\theta \cdot \frac{\partial\theta}{\partial c} = -\frac{2c}{2ab} \Rightarrow \frac{\partial\theta}{\partial c} = \frac{c}{ab\sin(\theta)}$$

b. [3 points] Compute $\partial\theta/\partial a$.

$$a^2 + b^2 - c^2 = 2ab\cos(\theta)$$

$$2a = 2b\cos\theta + 2ab(-\sin\theta)\frac{\partial\theta}{\partial a} \Rightarrow \frac{\partial\theta}{\partial a} = \frac{b\cos\theta - a}{ab\sin\theta}$$

c. [5 points] Suppose the lengths of the sides of the triangle (measured in meters) are changing as a function of time (measured in seconds) according to the rules $a(t) = 3 + t$, $b(t) = 3$, and $c(t) = 3 + 2t$. What is $d\theta/dt$ at time $t = 1$?

$$2ab\cos\theta = a^2 + b^2 - c^2$$

$$2\dot{a}b\cos\theta + 2a\dot{b}\cos\theta - 2ab\sin(\theta)\,\dot{\theta} = 2a\dot{a} + 2b\dot{b} - 2c\dot{c}$$

At $t = 1$:

$a = 4$, $\dot{a} = 1$

$b = 3$, $\dot{b} = 0$

$c = 5$, $\dot{c} = 2$

$$\cos\theta = \frac{3^2 + 4^2 - 5^2}{2\cdot 3\cdot 4} = 0$$

$$\sin\theta = 1$$

$$0 + 0 - 2 \cdot 3 \cdot 4 \cdot 1 \cdot \dot{\theta}$$

$$= 2 \cdot 4 \cdot 1 + 2 \cdot 3 \cdot 0 - 2 \cdot 5 \cdot 2$$

$$-24\dot{\theta} = -12$$

$$\dot{\theta} = 1/2$$

This sheet will not be graded. Do not turn it in.

- $\sin^2(x) + \cos^2(x) = 1$, $\cos(2x) = \cos^2(x) - \sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/4) = \sqrt{2}/2$, $\sin(\pi/4) = \sqrt{2}/2$, $\cos(\pi/6) = \sqrt{3}/2$, $\sin(\pi/6) = 1/2$, $\cos(0) = 1$, $\sin(0) = 0$.
- $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$.
- Volume of the parallelepiped determined by the vectors $\mathbf{v_1} = \langle a, b, c\rangle$, $\mathbf{v_2} = \langle d, e, f\rangle$, and $\mathbf{v_3} = \langle g, h, i\rangle$ is $|\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})| =$ absolute value of $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$
- Distance from a point (a, b, c) to a plane $Ax + By + Cz + D = 0$ is $\frac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}$.
- The circumference of a circle of radius a is $2\pi a$.
- The area of a disk of radius a is πa^2.
- The volume of a right circular cylinder of radius a and height h is $\pi a^2 h$.
- The curvature of the curve given by the parametric equation $\mathbf{r}(t)$ is $\kappa(t) = \frac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{\mathbf{r}'(t)}$.
- Polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$.
- $\int \sin^2(u)\,du = \frac{u}{2} - \frac{\sin(2u)}{4} + C$ $\int \cos^2(u)\,du = \frac{u}{2} + \frac{\sin(2u)}{4} + C$
- $\int \ln(u)\,du = u\ln(u) - u + C$