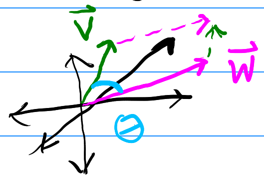
Math 215 Exam 1 Review

Fall 2019 Exam 1:

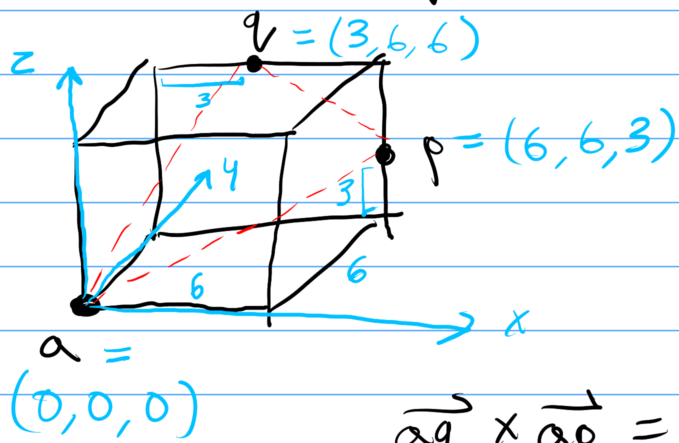
1. Relevant formulas:



Area of parallelogram spanned by $\vec{v}$ and $\vec{w}$ $= \|\vec{v} \times \vec{w}\|$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$

a)



$\vec{aq} = \langle 3,6,6 \rangle$
$\vec{ap} = \langle 6,6,3 \rangle$

$$\vec{aq} \times \vec{ap} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 6 & 6 \\ 6 & 6 & 3 \end{vmatrix}$$

$$= -18\hat{i} + 27\hat{j} - 18\hat{k}$$

$$\|\vec{aq} \times \vec{ap}\|^2 = 18^2 + 27^2 + 18^2$$
$$= 9^2(4+9+4) = 9^2 \cdot 17$$
$$\|\vec{aq} \times \vec{ap}\| = 9\sqrt{17}$$

Area of triangle $= \frac{1}{2}$(area of parallelogram)
$= 9\sqrt{17}/2$

b)

$$\|\vec{aq}\| = \|\vec{ap}\| = \sqrt{6^2+6^2+3^2} = 3\sqrt{4+4+1} = 9$$

$$\vec{aq} \cdot \vec{ap} = 3 \cdot 6 + 6 \cdot 6 + 3 \cdot 6 = 72$$

$$\cos\theta = 72/9 \cdot 9 = 8/9$$

2.

a) Is the statement true for <u>any</u> choice of $\vec{u}, \vec{v}$?

Try $\vec{u} = \langle 1,0 \rangle$, $\vec{v} = \langle 0,1 \rangle$

$\|\vec{u}\| = \|\vec{v}\| = 1$, $\|\vec{u}+\vec{v}\| = \sqrt{2}$

$\sqrt{2} \neq 1+1$ so this is a counterexample.

[FALSE]

b) Recall $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

For any vector $\vec{x}$, $\vec{x}$ and $-\vec{x}$ have the same norm.

[TRUE]

c) $\vec{q}(t) = (x(t), y(t), z(t))$

$$\frac{d}{dt}\|\vec{q}(t)\| = \frac{d}{dt}\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{2\|q\|} \cdot (2x\dot{x} + 2y\dot{y} + 2z\dot{z})$$

$$= \frac{\vec{q} \cdot \vec{q}\,'}{\|\vec{q}\|} = \frac{\|\vec{q}\| \, \|\vec{q}\,'\| \cos\theta}{\|\vec{q}\|}$$

$$= \|\vec{q}\,'\| \cos\theta$$ (θ = angle between $\vec{q}$ and $\vec{q}\,'$)

This only equals $\|\vec{q}\,'\|$ if $\cos\theta = 1$

$\Leftrightarrow$ $\vec{q}$ and $\vec{q}\,'$ are parallel.

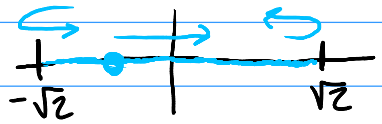
[FALSE]

d) Different parametrizations = different journeys along same curve.

Tangent vector = velocity $\Rightarrow$ You can go fast or slow! [FALSE]

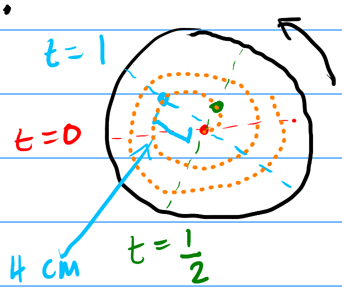
e) Key part of wording: $\vec{u}$ and $\vec{v}$ can be <u>any</u> unit vectors.

So if we choose $\vec{u} = \langle 1,0\rangle$ $\vec{v} = \langle -1,0\rangle$ then the curve $\vec{r}(t) = \cos(t)\,\vec{u} + \sin(t)\,\vec{v}$ always stays on x-axis $\Rightarrow$ not a circle!



[FALSE]

3.



a) Ladybug reaches edge of disk when it has traveled 50 cm in its own frame of reference.

Speed = 4 cm/s $\Rightarrow$ $50/4 = 12.5\,s$

b) Relevant formula:

Arclength of parametric curve $\gamma(t)$ from $t=a$ to $t=b$ is

$$\int_a^b \|\gamma'(t)\|\,dt$$

$$\ell'(t) = \langle 4\cos(\omega t) - 4\omega t\sin(\omega t),\ 4\sin(\omega t) + 4\omega t\cos(\omega t)\rangle$$

$$\begin{cases} c = \cos(\omega t) \\ s = \sin(\omega t) \end{cases} \qquad (c^2+s^2=1)$$

$$\|\ell'(t)\|^2 = (4c - 4\omega t s)^2 + (4s + 4\omega t c)^2$$

$$= 16c^2 - 32\,\omega t\,cs + 16\omega^2t^2s^2$$

$$\underline{+\,16s^2 + 32\,\omega t\,cs + 16\omega^2t^2c^2}$$

$$16 \qquad + \qquad 16\omega^2t^2$$

Thus in the first 6 secs, bug travels

$$\int_0^6 \sqrt{16+16\omega^2 t^2}\, dt = 4\int_0^6 \sqrt{1+\omega^2 t^2}\, dt \quad \text{cm.}$$

d) Let $r(T)$ be position of ladybug T secs after losing footing.

We know:

$$r(0) = \ell(6) = \langle 24, 0, 0\rangle$$
$$\dot{r}(0) = \ell'(6) = \langle 4, 24\omega, 0\rangle$$
$$\ddot{r}(T) = \langle 0, 0, T/10\rangle$$

Integrate once:

$$\dot{r}(T) = \langle C_1, C_2, \tfrac{T^2}{20} + C_3\rangle$$
$$\dot{r}(0) = \langle C_1, C_2, C_3\rangle$$
$$\Rightarrow C_1 = 4, \quad C_2 = 24\omega, \quad C_3 = 0$$

Integrate again:

$$\dot{r}(t) = \langle 4, 24\omega, T^2/20\rangle$$
$$r(t) = \langle 4t + D_1, 24\omega T + D_2, \tfrac{T^3}{60} + D_3\rangle$$

$$r(0) = \langle D_1, D_2, D_3\rangle$$
$$\Rightarrow D_1 = 24, \quad D_2 = 0, \quad D_3 = 0$$

$$r(T) = \langle 4T + 24, 24\omega T, \tfrac{T^3}{60}\rangle$$

4. Start w the easiest:

$(x^2+y^2)^{1/2} = \|\langle x, y\rangle\|$ is radially symmetric

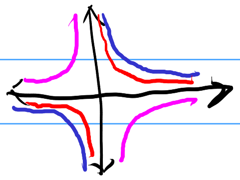
$\rightarrow$ level curves are circles $\rightarrow$ (v)

$$\sin(x+y) = A \Rightarrow x+y = \arcsin(A) + 2\pi k$$
$$\text{OR} = \pi - \arcsin(A) + 2\pi k$$

Each level curve is a family of slope -1 lines $\rightarrow$ (iv)

$\sin(xy) = A \implies xy = \arcsin(A) + 2\pi k$

$\text{OR} = \pi - \arcsin(A) + 2\pi k$



Each level curve is a family of hyperbolas

$\rightarrow$ (ix)

$\sin(x)\sin(y) = A$

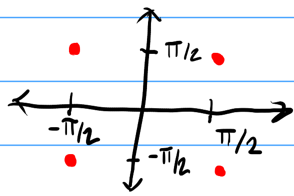
For $A=1 \implies \sin(x) = \sin(y) = 1$ (since $\sin x \le 1$)

OR $\sin(x) = \sin(y) = -1$

$\implies x = \frac{\pi}{2} + 2\pi k, \quad y = \frac{\pi}{2} + 2\pi k$

OR $x = \frac{3\pi}{2} + 2\pi k, \; y = \frac{3\pi}{2} + 2\pi k$

So the level curve is just a bunch of points:



Since $\sin(x)\sin(y) \le 1$, these points are "peaks" of the function $\rightarrow$ only picture matching this structure is (viii)

$\sin(x) + \sin(y)$ again has peak at $(\frac{\pi}{2}, \frac{\pi}{2})$ and valley at $(-\pi/2, -\pi/2)$ but no extrema at $(-\pi/2, \pi/2)$ or $(\pi/2, -\pi/2) \rightarrow$ (vii)

$\cos(x)/(x^2+y^2+1) = A \implies x^2+y^2+1 - \frac{\cos(x)}{A} = 0$

If $x \approx 0$, $\cos(x) \approx 1 - \frac{x^2}{2}$

$x^2+y^2+1 - \frac{1}{A} + \frac{x^2}{2A} = 0 \implies x^2\left(1+\frac{1}{2A}\right) + y^2 = \frac{1}{A} - 1$

For x, y small, the level curves are approximately ellipses stretched in y-direction $\rightarrow$ (i)

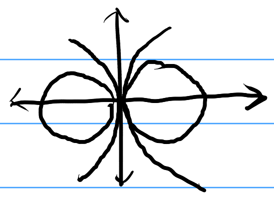
$\sin(x)/(x^2+y^2+1) = A \quad \rightarrow \quad x^2+y^2+1 = \frac{\sin(x)}{A}$

If $x \approx 0$, $\sin(x) \approx x$

$$x^2 - \frac{x}{A} + y^2 + 1 = 0$$

$$\left(x - \frac{1}{2A}\right)^2 + y^2 = \frac{1}{4A^2} - 1 \quad \leftarrow$$

circle w/ center at $\left(\frac{1}{2A}, 0\right)$ and radius $\approx \left|\frac{1}{2A}\right|$



Approx. level curves all pass through origin $\rightarrow$ (vi)

$(1-x^2)(1-y^2) = A \quad \rightarrow$ if (x,y) on level curve, so are $(-x, y)$, $(-x,-y)$, $(x,-y)$

Only remaining picture with this symmetry $\rightarrow$ (iii)

$e^x \cos(y) \rightarrow$ (ii) by elimination

5.

a) Relevant formula: if (a,b,c) is on the graph $z = f(x,y,z)$, the tangent plane at (a,b,c) has formula

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z-c) = 0$$

$$g_x(x,y) = -e^{y-x} \qquad g_x(3,3) = -1$$
$$g_y(x,y) = e^{y-x} \qquad g_y(3,3) = 1$$

Tangent plane is
$$-(x-3) + (y-3) - (z-1) = 0$$
$$\Rightarrow \quad x - y + z = 1$$

b) Relevant formula:
Linearization of $f(x,y)$ at (a,b) is the linear part of Taylor series:

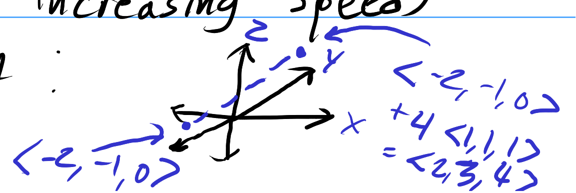
$$f(x,y) = f(a,b) + \nabla f_{(a,b)} \cdot (x-a, y-b)$$
$$+ \text{ higher order terms}$$
$$L_f(x,y) = f(a,b) + \nabla f_{(a,b)} \cdot (x-a, y-b)$$
$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$L_g = 1 - (x-3) + (y-3)$$
$$= -x + y + 1$$
(note: this is what you get if you solve for z in eq'n of tangent plane)

c) $g(2.9, 3.1) \approx L_g(2.9, 3.1) = -2.9 + 3.1 + 1$
$$= 1.2$$

7.

a) $r_1(t) = \langle t^2-2, t^2-1, t^2 \rangle$
$$= t^2 \langle 1,1,1 \rangle + \langle -2,-1,0 \rangle$$
Start at $\langle -2,-1,0 \rangle$ and go straight in direction $\langle 1,1,1 \rangle$ (w/ increasing speed)
C_1 is a line segment:

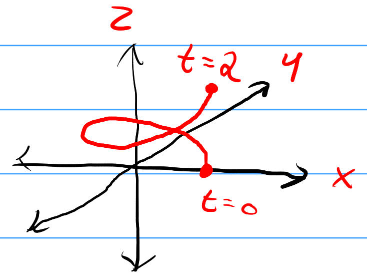


$r_2(t) = \langle \cos(\pi t), \sin(\pi t), t \rangle$

Projection onto x-y plane is circle.

As t ranges 0 to 2, z-coord increases:

C_2 is a helix



b) The curves intersect if there are times s_1, s_2 such that $r_1(s_1) = r_2(s_2)$. Try $s = s_1 = s_2$

$\Rightarrow \langle s^2-2, s^2-1, s^2 \rangle = \langle \cos(\pi s), \sin(\pi s), s \rangle$

Equating z-coords: if such an s exists then $s^2 = s \Rightarrow s = 0, 1$

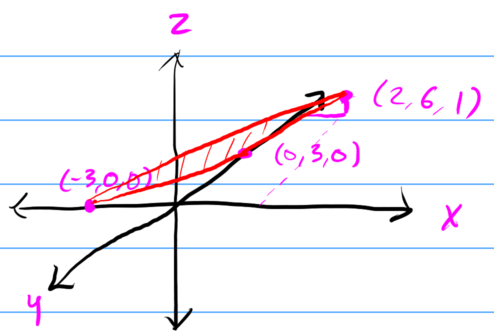
$r_1(0) = \langle -2, -1, 0 \rangle \neq \langle 1, 0, 0 \rangle = r_2(0)$

but $r_1(1) = \langle -1, 0, 1 \rangle = \langle -1, 0, 1 \rangle = r_2(1)$

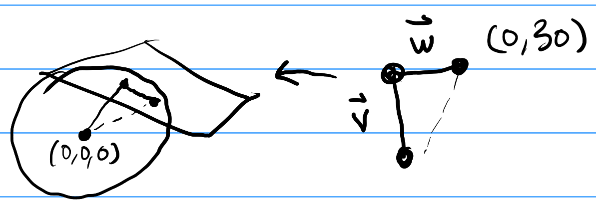
So the curves do intersect at time $s = 1$.

c) Since we found $s_1 = s_2 = 1$ above, the particles in fact collide at time 1

8.



Let $\vec{v}$ be the point on the plane at which the sphere is tangent.



$\vec{v} + \vec{w} = (0,3,0)$

$\vec{w} = (0,3,0) - \vec{v}$

But $\vec{v}$ and $\vec{w}$ are perpendicular, so

$\vec{v} \cdot ((0,3,0) - \vec{v}) = 0$

Similarly, $\vec{v} \cdot ((-3,0,0) - \vec{v}) = 0$

$\vec{v} \cdot ((2,6,1) - \vec{v}) = 0$

Let $\vec{v} = (a,b,c)$ and $\|\vec{v}\| = r$.

Then the eqns give:

$$3b - r^2 = 0 \qquad b = r^2/3$$

$$-3a - r^2 = 0 \qquad a = -r^2/3$$

$$2a + 6b + c - r^2 = 0 \qquad c = r^2 + \frac{2r^2}{3} - \frac{6r^2}{3}$$

$$= -r^2/3$$

$$\vec{v} = \langle a,b,c \rangle = r^2 \langle -1/3, 1/3, -1/3 \rangle$$

But $r = \|\vec{v}\| = r^2 \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}} = r^2/\sqrt{3}$

$\implies r = \sqrt{3}$

Thus the eq'n of the sphere is

$$x^2 + y^2 + z^2 = r^2 = 3.$$