Fall 2017 Exam 2 Problem 4 Solution:

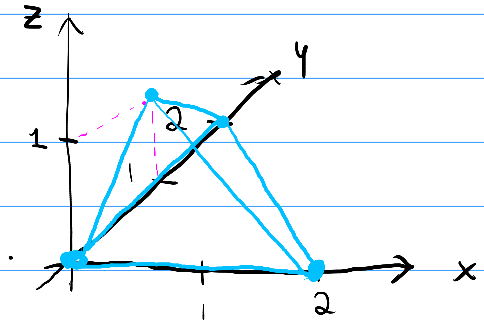
Each vertex of the tetrahedron is the intersection of three of the planes:

V1 $\quad x=0,\ z=0,\ y=z \implies (0,0,0)$

V2 $\quad x=0,\ z=0,\ x+y+z=2 \implies (0,2,0)$

V3 $\quad x=0,\ y=z,\ x+y+z=2 \implies (0,1,1)$

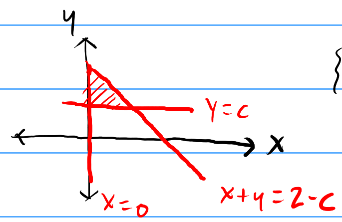
V4 $\quad z=0,\ y=z,\ x+y+z=2 \implies (2,0,0)$



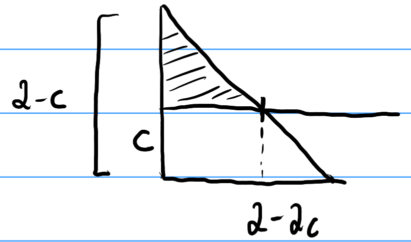
Consider the $z=c$ cross-section. $(0 \le c \le 1)$

This is a triangle with edges given by

$$x=0$$
$$y=c$$
$$x+y=2-c$$



$\{z=c\}$



Area of $z=c$ cross section is

$$\frac{(2-2c)^2}{2}$$

Thus volume of tetrahedron is:

$$\int_0^1 \frac{(2-2c)^2}{2}\,dc = 2\int_0^1 (1-c)^2\,dc$$

$$= 2\left[-\frac{(1-c)^3}{3}\right]\Bigg|_0^1$$

$$= \frac{2}{3}$$