c) (4 points) $z^2 - z(x^2+y^2) + xy = 0$. Find $\frac{\partial z}{\partial x}$.

Answer: (a) $-\frac{x}{z}$. (b) $\frac{x}{r}$. (c) $\frac{2zx-y}{2z-(x^2+y^2)}$.

7. Consider the paraboloid $z = x^2 + y^2$ and the point $P = (2, -1, 5)$.

   a) (4 points) Find the tangent plane to the paraboloid at the point P.

   b) (2 points) Find the parametric equation of the normal line at P.

   c) (4 points) The plane that contains the normal line and the origin $(0,0,0)$ intersects the paraboloid along a curve. Find the parametric equation of this curve.

Answer: (a) $z - 5 = 4(x-2) - 2(y+1)$. (b) $(x, y, z) = (2+4t, -1-2t, 5-t)$. (c) The plane containing the normal line and the origin is $x + 2y = 0$. Therefore the curve is $(x, y, z) = (-2t, t, 5t^2)$.

8. Suppose $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (1, 1, -1)$ are two vectors.

   a) (3 points) Find the projection of $\mathbf{b}$ along $\mathbf{a}$.

   b) (7 points) Suppose $\mathbf{b}$ is rotated with $\mathbf{a}$ as the axis of rotation by 45^o or $\frac{\pi}{4}$ (with the sense of rotation determined by the right hand rule). Find the resulting vector.

Solution: The projection is given by

$$\frac{(\mathbf{a}.\mathbf{b})}{(\mathbf{a}.\mathbf{a})}\mathbf{a} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

If we write

$$\mathbf{b} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right),$$

only the second vector in the sum rotates. The magnitude of the second vector is $2\sqrt{2}/\sqrt{3}$.

We may calculate and find

$$\mathbf{a} \times \mathbf{b} = (-2, 2, 0).$$

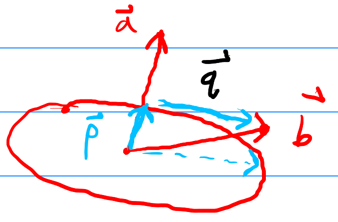
The magnitude of this vector is $2\sqrt{2}$. The magnitude of $\mathbf{c} = (-2/\sqrt{3}, 2/\sqrt{3}, 0)$ is therefore $2\sqrt{2}/\sqrt{3}$. The vector $\mathbf{c}$ is orthogonal to $\mathbf{a}$ as well as $\left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right)$ and is of the same magnitude as the latter.

Therefore, the vector obtained by rotating $\mathbf{b}$ is

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \frac{1}{\sqrt{2}}\left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right) + \frac{1}{\sqrt{2}}\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, 0\right).$$

Solution for 8B:

$\vec{a} = \langle 1,1,1\rangle \qquad \vec{b} = \langle 1,1,-1\rangle$



Normal plane to $\vec{a}$

$$\vec{p} = \text{proj}_a(b) = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\cdot\vec{a} = \frac{1}{3}\vec{a} = \left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)$$

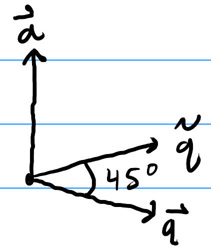
$$\vec{q} = \left(\tfrac{2}{3},\tfrac{2}{3},-\tfrac{4}{3}\right)$$

$\vec{b} = \vec{p} + \vec{q}$ ⟵ actually rotates

$\vec{p}$: stays fixed in place during rotation

$\Rightarrow$ If $\tilde{v}$ denotes the rotated version of v, then we have $\tilde{b} = \vec{p} + \tilde{q}$

So we need to calculate $\tilde{q}$



$\tilde{q}$ and $\vec{q}$ are both orthogonal to $\vec{a}$:

$\vec{q}\cdot\vec{a} = \tilde{q}\cdot\vec{a} = 0$

Angle between $\tilde{q}$ and $\vec{q} = 45°$:

$$\frac{\tilde{q}\cdot\vec{q}}{\|\tilde{q}\|\,\|\vec{q}\|} = \cos(45°) = \frac{\sqrt{2}}{2}$$

$\tilde{q} = (s,t,v) \qquad \tilde{q}\cdot\vec{a} = s+t+v = 0$

$$\|\vec{q}\| = \sqrt{\left(\tfrac{4}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2\cdot 2} = \frac{2\sqrt{6}}{3}$$

$$\tilde{q}\cdot\vec{q} = \frac{\sqrt{2}}{2}\cdot\frac{2\sqrt{6}}{3}\cdot\|\tilde{q}\|$$

But $\|\tilde{q}\| = \|q\|$ so

$$\tilde{q} \cdot \bar{q} = \frac{\sqrt{2}}{2} \frac{2\sqrt{6}}{3} \quad \frac{2\sqrt{6}}{3} = \frac{4\sqrt{2}}{3}$$

$$\| \quad \frac{2}{3}s + \frac{2}{3}t - \frac{4}{3}u \implies$$

$$s + t - 2u = 2\sqrt{2}$$

$$\underline{s + t + u \;\; = \;\; 0}$$

$$-3u = 2\sqrt{2}$$

$$u = -2\sqrt{2}/3$$

$$s + t = 2\sqrt{2}/3$$

$$\frac{8}{3} = \|\tilde{q}\|^2 = s^2 + t^2 + u^2 \Leftarrow$$

$$s^2 + t^2 = \frac{8}{3} - \left(\frac{2\sqrt{2}}{3}\right)^2$$

$$= \frac{8}{3} - \frac{8}{9} = \frac{16}{9}$$

$\longrightarrow$ $s + t = \dfrac{2\sqrt{2}}{3}$, $s^2 + t^2 = \dfrac{16}{9}$ $\rightarrow$ $\dfrac{8}{9} = (s+t)^2 = s^2 + t^2 + 2st$

$$2st = -\frac{8}{9} \Rightarrow st = \frac{-4}{9}$$

$\longrightarrow$ s and t are roots of

$$x^2 - \frac{2\sqrt{2}}{3}x - \frac{4}{9} = 0$$

(substitute st eq'n into s+t eq'n or vice versa)

Using quadratic formula:

$$s, t = \frac{\frac{2\sqrt{2}}{3} \pm \sqrt{\frac{8}{9} + \frac{16}{9}}}{2} = \frac{\sqrt{2}}{3} \pm \frac{\sqrt{6}}{3}$$

But do we have $(s,t) = \left(\dfrac{\sqrt{2}+\sqrt{6}}{3}, \dfrac{\sqrt{2}-\sqrt{6}}{3}\right)$

or $\left(\dfrac{\sqrt{2}-\sqrt{6}}{3}, \dfrac{\sqrt{2}+\sqrt{6}}{3}\right)$?

We need $\vec{q} \times \tilde{q}$ to be a positive multiple of $\vec{a}$! (by right hand rule)

$$\vec{q} \times \tilde{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & \frac{2}{3} & -\frac{4}{3} \\ s & t & u \end{vmatrix} = \left(\frac{2u+4t}{3}, -\frac{2u+4s}{3}, \frac{2t-2s}{3} \right)$$

$$\frac{2t-2s}{3} > 0 \implies t > s$$

We conclude $\tilde{q} = (s, t, u) = \left(\frac{\sqrt{2}-\sqrt{6}}{3}, \frac{\sqrt{2}+\sqrt{6}}{3}, \frac{-2\sqrt{2}}{3} \right)$

and thus $\tilde{b} = \vec{p} + \tilde{q}$

$$= \left(\frac{1+\sqrt{2}-\sqrt{6}}{3}, \frac{1+\sqrt{2}+\sqrt{6}}{3}, \frac{1-2\sqrt{2}}{3} \right)$$