1. [COMBINATORIAL PROOFS] [12]

Recall (from lectures) that we have seen two types of combinatorial proofs (for proving identities):
(i) establishing a bijection between two sets (one counted by LHS, and the other counted by RHS), or
(ii) using a double counting argument (that is, proving that LHS and RHS both count the same set).

The goal of this exercise is to prove some more identities using combinatorial arguments.

(a) Give a bijective proof to show that, for all $k, n \in \mathbb{N}$, where $0 \leq k \leq n$, the following holds:

$$\binom{n}{k} = \binom{n}{n-k}$$

Response: Warm up exercise. NO response required.

(b) Give a combinatorial proof to show that, for all $n \in \mathbb{N}$, the following holds:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Response: Consider two sets A and B containing **n** elements each. Let us label the elements of A as $1_A, 2_A, 3_A, ..., n_A$ and the elements of B as $1_B, 2_B, 3_B, ..., n_B$. Now, let us choose n elements in totality from the sets A and B together.

The first method will be to choose k elements from the set A and $n-k$ elements from the set B where k ranges from 0 to n. Then, we will sum up the number of ways we obtain for every k from 0 to n in order to obtain the total number of ways of choosing n elements from the two sets.

There are $\binom{n}{k}$ ways to choose k elements from the first set.

There are $\binom{n}{n-k} = \binom{n}{k}$ ways to choose $n-k$ elements form the second set.

We will now multiply the number of ways of either action to obtain the total number of ways of choosing k elements as $\binom{n}{k}^2$ and sum it to get the LHS.

The second method will be to join the two sets into one big set $A \dot{\bigcup} B$. This set has $2n$ elements and the number of ways of choosing n elements from this is $\binom{2n}{n}$ which forms the RHS. □

(c) Give a combinatorial proof to show that, for all $n \in \mathbb{N}$, the following holds:

$$\sum_{j=0}^{n} j\binom{n}{j} = n \cdot 2^{n-1}$$

Response: Let us count the number of ways we can pick a subset from a set of n elements and then picking one element from this subset,

One way would be to first pick a subset of j elements, of which there are $\binom{n}{j}$ ways and then picking an element from this, of which there are j ways. Thus the total number of ways would be the LHS.

In order to understand the RHS, let us first choose one of the n elements and take that as the element from the subset we are going to choose. From the remaining n-1 elements, there are 2^{n-1} ways to form a set. Multiplying these two together, we get the total number of ways to be equal to $n \cdot 2^{n-1}$, which is the RHS.

Since we are performing the same action of choosing a subset and then choosing an element from the subset in both cases, the number of ways must be the same thus proving the identity. □