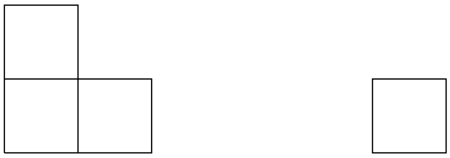
5. [Using and solving Recurrences: yet another Tiling Problem] [12]

In this exercise, we our goal is to compute the number of tilings[2] of a $(2 \times n)$-grid for all $n \in \mathbb{N} - \{0\}$ using two types of tiles: L-shaped tiles (drawn below on the left) and box-shaped (1×1) tiles (drawn below on the right):



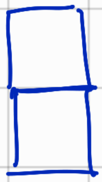
(a) For convenience, we denote by T_{n-1} the number of tilings of a $(2 \times n)$-grid — using L-shaped and box-shaped tiles. Compute T_0, T_1 & T_2, and explain your answers clearly — with drawings (if required).

Response: We obtain T_0 as *1*, T_1 as *5* and T_2 as *11*. There is exactly one way to tile a (2x1) and that too only with the (1x1) tile as the L shaped tile is too big to fit. A (2x2) tile cane be tiled with one L shaped tile and one (1x1) tile. There are 4 ways each corresponding to a 90 degree rotation of the L shaped tile and placing the 1x1 tile. The ways of tiling are illustrated in the figure attached.

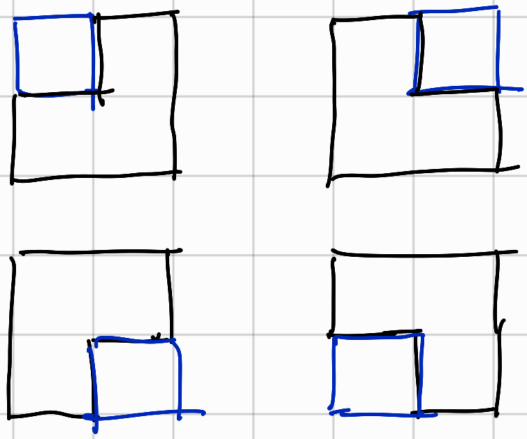
[2]As per examples discussed in lectures, tiling means: "no two tiles should overlap" and "each square of the grid should be covered by a tile".

DEMONSTRATION OF TILINGS:

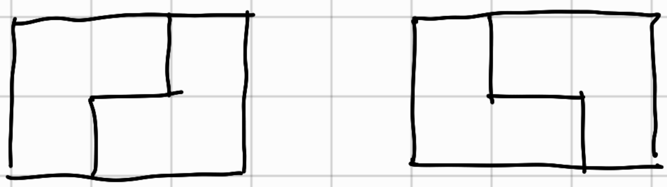
T_0
(1 way).



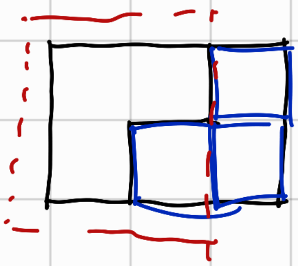
T_1
(4 ways)



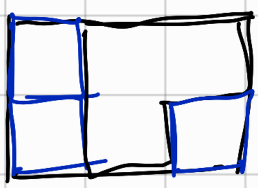
T_3
(11 ways)



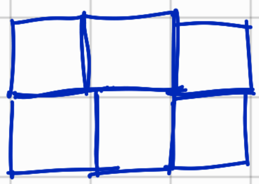
(from T_2)



× 4 (There are 4 ways to tile red box)



× 4



(b) Write down a recurrence for T_n, and explain clearly why this recurrence is correct.

Response: Let us say we we have to make a $(2xn)$ tiling, where $n > 3$, and let us suppose that we want to make this tiling from smaller tilings. I claim there are *exactly* three ways to do this.

i. Start with a (2x(n-1)) tiling, and then add two (1x1) tiles.

ii. Start with a (2xn-2) tiling, and then add one L shaped tile and one (1x1) tile, of which there are two ways. Observe that placing 4 (1x1) tiles is covered in option (a). This is the only way which is not counted in the (2xn-1) case. In other ways, we need to make a (2x2) tile without making a (2x1) tile, (on the left). From the previous diagram, it is clear that the described method is the only possibility.

iii. Start with a (2xn-3) tiling, and then add two L shaped tiles of which there are 2 ways. This is the only way way which starts with a (2xn-3) tile and is not counted in the (2xn-2) or the (2xn-1) case. In other words, we need to make a (2x3) tiling without forming a (2x2) tiling or a (2x1) tiling (, starting from the left). From the previous diagram, it is clear that the described method is the only possibility.

The recursion will be therefore

$$T_n = T_{n-1} + 4 * T_{n-2} + 2 * T_{n-3}$$

(c) Solve the recurrence obtained in part (b) — using the initial conditions from part (a) — and write down a closed form formula for T_n. Explain the steps followed clearly.

Response: Start with the characteristic equation

$$x^3 - x^2 - 4x - 2 = 0$$

The roots of this equation turn out to be (-1), (1+$\sqrt{3}$) and (1-$\sqrt{3}$). The general solution is of the form $c_0(-1)^n + c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$. Putting in the boundary conditions, we get

$$\begin{aligned} c_0 + c_1 + c_2 &= 1 \\ -c_0 + c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) &= 4 \\ c_0 + c_1(1+\sqrt{3})^2 + c_2(1-\sqrt{3})^2 &= 11 \end{aligned}$$

We obtain $c_0 = -1$, $c_1 = 1 + \frac{1}{\sqrt{3}}$, and $c_2 = 1 - \frac{1}{\sqrt{3}}$. The closed form for T_n is $T_n = (-1)^{n+1} + (1 + \frac{1}{\sqrt{3}})(1+\sqrt{3})^n + (1 - \frac{1}{\sqrt{3}})(1-\sqrt{3})^n$.