CS6240 — Structural Graph Theory — 2022 (Jul-Nov)

Assignment-0[1]

Release Date: 30/07/2021 **Due Date: 09/08/2021 — 11:00 PM IST**

This is a warm-up assignment. Its purpose is to help you evaluate how well you are prepared for this course, and to help you refresh/revisit some basic concepts and proof techniques. You need <u>not</u> submit this assignment. However, if you choose to submit it <u>by the due date</u>, we are more than willing to mark it and to provide feedback.

Please reach out to us, or post on Piazza, if the meaning/expectation is unclear for any problem/subproblem. For Problems 1 and 2 combined, we recommend solving all of the subproblems in sequence, and not using 'latter' subproblems to solve 'former' subproblems.

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1. Paths, Trees and Forests [17]

   Recall that a *forest* is an acyclic[2] graph, a *tree* is a forest that is connected, and a *leaf* is any vertex whose degree equals one.

   (a) Prove the following statement: 'Each nontrivial[3] tree has at least two leaves'. [3]

   Response: LaTeX your response here.

   (b) Prove that a tree has at most two leaves if and only if it is (isomorphic to) a path. (This result may be viewed as a *structural characterization/description* of all trees with at most two leaves.) [3]

   Response: LaTeX your response here.

   (c) [4] Using part (a), establish[5] an induction tool/technique for trees. (You may find this induction tool/technique useful for some of the subsequent problems.) [1]

   Response: LaTeX your response here.

   (d) A *uv-path* is a path whose ends are the vertices u and v. Prove that a loopless graph G is a tree if and only if there exists a unique uv-path between each pair of distinct vertices u and v. [4]

   Response: LaTeX your response here.

   (e) For a graph G, we use $\Delta(G)$ to denote the maximum degree (i.e., the maximum degree of any vertex among all vertices of G). Prove that every tree T has at least $\Delta(T)$ leaves. Establish a structural characterization of all trees T that have precisely $\Delta(T)$ leaves. [6]

   Response: LaTeX your response here.

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2. Cut-edges and Spanning Trees [13]

   For a graph G, we use $c(G)$ to denote the number of (connected) components of G. An edge e is a *cut-edge*[6] of a graph G if $c(G - e) > c(G)$.

   (a) Prove that if e is a cut-edge of a graph G then $c(G - e) = c(G) + 1$. [1]

   Response: LaTeX your response here.

[1] This assignment does **<u>not</u>** contribute towards your final grade! I am a big fan of using footnotes! So don't ignore footnotes.

[2] A graph is *acyclic* if it does not have any subgraph that is (isomorphic to) a cycle.

[3] A graph is nontrivial if it has at least two vertices.

[4] This part may sound vague to you. The vagueness is deliberate. You may contact the instructor for an explanation/hint.

[5] Establish, in this context, means 'propose/describe and provide a proof for the same'.

[6] In many sources, cut-edges are called *bridges*. However, we will **<u>not</u>** use this terminology.

(b) Prove that an edge e of a graph G is a cut-edge if and only if e does not participate in any cycle of G. [3]

Response: LaTeX your response here.

(c) Deduce from part (b) that a graph is a forest if and only if each of its edges is a cut-edge. [3]

Response: LaTeX your response here.

(d) A subgraph T of a graph G is a *spanning tree of* G if (i) T is a tree and (ii)[7] $V(T) = V(G)$. Prove that a graph is connected if and only if it has a spanning tree. [3]

Response: LaTeX your response here.

(e) Prove that a connected graph G is a tree if and only if its size is precisely one less than its order (i.e., $m(G) = n(G) - 1$). [3]

Response: LaTeX your response here.

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3. [8] Self-complementary Graphs [15]

The *complement* $\overline{G}$ of a simple graph G is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. In other words, $E(\overline{G}) := \{uv : uv \notin E(G)\}$. A simple graph is *self-complementary* if it is isomorphic to its complement.

(a) Show that each of P_4 and C_5 is self-complementary (where P_4 denotes the path on 4 vertices). [2]

Response: LaTeX your response here.

(b) Prove that if G is a self-complementary graph then $n(G) \equiv 0, 1 \pmod 4$. [3]

Response: LaTeX your response here.

(c) Let G be a self-complementary graph and let P be a path of length three disjoint from G. Form a new graph H from $G \cup P$ by joining each end of P with each vertex of G. Show that H is self-complementary. [2]

Response: LaTeX your response here.

(d) Deduce that there exists a self-complementary graph of order n if and only if $n \equiv 0, 1 \pmod 4$. [3]

Response: LaTeX your response here.

(e) Prove that every self-complementary graph is connected. [2]

Response: LaTeX your response here.

(f) Every self-complementary graph on $4k + 1$ vertices has a vertex of degree $2k$. [3]

Response: LaTeX your response here.

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[7] In other words, T spans all of the vertices of G.

[8] For this problem only, you may assume that all graphs are simple.