CS1200 — Discrete Mathematics for Computer Science — 2023 (Jan-May)

Assignment 4

Release Date: 09/06/2023 **Due Date: 17/06/2023 — 8:00 AM IST**

Collaborator (if any): *write/type the full name (or roll number) of your (at most one) collaborator; if none, then type NONE.*

Academic Integrity Statement: I, *write/type your full name here*, affirm that I have not given or received any **unauthorized** help (from any source: people, internet, etc.) on this assignment, and that I have written/typed each response on my own, and in my own words.

The marks for each problem (1, 2, 3, 4, 5) are fixed. However, the marks for each subproblem (1a, 1b, 2a, etc.) are tentative — they may be changed during marking if necessary.

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1. [COMBINATORIAL PROOFS] [12]

Recall (from lectures) that we have seen two types of combinatorial proofs (for proving identities):
(i) establishing a bijection between two sets (one counted by LHS, and the other counted by RHS), or
(ii) using a double counting argument (that is, proving that LHS and RHS both count the same set).

The goal of this exercise is to prove some more identities using combinatorial arguments.

(a) Give a bijective proof to show that, for all $k, n \in \mathbb{N}$, where $0 \leq k \leq n$, the following holds:

$$\binom{n}{k} = \binom{n}{n-k}$$

Response: Warm up exercise. NO response required.

(b) Give a combinatorial proof to show that, for all $n \in \mathbb{N}$, the following holds:

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Response: Consider two sets A and B containing **n** elements each. Let us label the elements of A as $1_A, 2_A, 3_A, ..., n_A$ and the elements of B as $1_B, 2_B, 3_B, ..., n_B$. Now, let us choose n elements in totality from the sets A and B together.

The first method will be to choose k elements from the set A and $n-k$ elements from the set B where k ranges from 0 to n. Then, we will sum up the number of ways we obtain for every k from 0 to n in order to obtain the total number of ways of choosing n elements from the two sets.

There are $\binom{n}{k}$ ways to choose k elements from the first set.

There are $\binom{n}{n-k} = \binom{n}{k}$ ways to choose $n-k$ elements form the second set.

We will now multiply the number of ways of either action to obtain the total number of ways of choosing k elements as $\binom{n}{k}^2$ and sum it to get the LHS.

The second method will be to join the two sets into one big set $A \dot{\bigcup} B$. This set has $2n$ elements and the number of ways of choosing n elements from this is $\binom{2n}{n}$ which forms the RHS. □

(c) Give a combinatorial proof to show that, for all $n \in \mathbb{N}$, the following holds:

$$\sum_{j=0}^{n} j\binom{n}{j} = n \cdot 2^{n-1}$$

Response: Let us count the number of ways we can pick a subset from a set of n elements and then picking one element from this subset,

One way would be to first pick a subset of j elements, of which there are $\binom{n}{j}$ ways and then picking an element from this, of which there are j ways. Thus the total number of ways would be the LHS.

In order to understand the RHS, let us first choose one of the n elements and take that as the element from the subset we are going to choose. From the remaining n-1 elements, there are 2^{n-1} ways to form a set. Multiplying these two together, we get the total number of ways to be equal to $n \cdot 2^{n-1}$, which is the RHS.

Since we are performing the same action of choosing a subset and then choosing an element from the subset in both cases, the number of ways must be the same thus proving the identity. □

2. [Placing Non-attacking Rooks on a Huge Chessboard] [12]

In your hostel is a 100×100 marble chessboard, and a large number of chess pieces.

(It is very beautiful, and very heavy, and you have no idea what to do with it! So, let's count something for fun!)

(a) Prove that the number of ways to place k non-attacking rooks on this chessboard is $k!\binom{100}{k}^2$.

Note: Two rooks attack each other if they are in the same row or column. We consider any two rooks to be indistinguishable, so we only care about which squares contain rooks and which squares do not. Hence the question is really asking about the number of ways to choose a set of k squares on the chessboard so that no two squares are in the same row or column.

Response: The strategy of counting the number of ways in this question is to choose k rows, and k columns and arrange rooks in them. There are $\binom{n}{k}$ ways of choosing k rows from n rows. After choosing rows, there are $\binom{n}{k}$ ways of choosing k columns from n columns. We now need to associate the rows and columns into pairs. We assume an arbitrary ranking of the columns. In this ranking, the first column has k rows it can be associated with, the second column has $k-1$ rows it can be related to and so on and the last column has 1 rows that it can be related to. Hence, by multiplying all these choices, there are $k!$ ways of associating rows with columns and in the process determining the points. Note that the rooks which are in different rows and different columns do not attack each other and attack each other otherwise.

(b) Prove that the number of ways to place k non-attacking rooks on the chessboard, with all rooks on white squares, is:

$$\sum_{i=0}^{k} i!\binom{50}{i}^2 (k-i)!\binom{50}{k-i}^2.$$

Hint: You may want to color the white squares.

Response: We assume that the rows and columns are each numbered from 1 to 100, rows from top to bottom and columns from left to right. (Assuming a fixed orientation of the board)

The key observation here is that rooks placed in white squares in rows having an odd row number can never attack rooks placed in white squares in rows having an even row number since they are in different columns.

Let us now look at placing rooks in even numbered rows without attacking each other. white squares in even rows are arranged so that in every row, there are 50 white squares and the nth white squares in each row are present in the same column. There are 50 such columns and 50 rows. This is identical to placing i rooks in a grid of side 50 such that the rooks are non-attacking. From part (a), there are $i!\binom{100}{i}^2$ of placing rooks in such a way

Placing the remaining $k-i$ rooks in odd columns is done in the exact same way. There are $(k-i)!\binom{100}{(k-i)}^2$ ways of doing so.

These two events are independent of each other, hence we multiply the number of ways to get the number of ways of placing i rooks, where i ranges from 0 to 50 in even columns as $i!\binom{100}{i}^2(k-i)!\binom{100}{(k-i)}^2$.

Now 'i' can take values from 0 to 50 and we have to sum over the number of ways we obtain for each 'i' in order to get the total number of ways of placing rooks. Hence the total number of ways of placing rooks comes out to be

$$\sum_{i=0}^{k} i!\binom{50}{i}^2 (k-i)!\binom{50}{k-i}^2$$

3. [ANOTHER APPLICATION OF THE PIGEONHOLE PRINCIPLE: HANDSHAKES AT A PARTY] [12]

In this exercise, we will use the pigeonhole principle to prove the following.

Theorem 0.1. *Some $n \in \mathbb{N} - \{0, 1\}$ people attended the CSE@IIT-M Freshers' 2023 Party, and some of them shook hands with others*[1]*. There exist two partygoers — each of whom shook hands with an equal number of people.*

(a) Write down a theorem statement — using the language/terminology of graphs — that has the same meaning as Theorem 0.1.

Response:

Theorem 0.2. *Consider a complete graph K_n with $n \in \mathbb{N} - \{0, 1\}$ vertices. Each edge of the graph is either colored blue or pink. There exist two distinct vertices u and v such that number of blue edges adjacent to u is the same as the number of blue adjacent to v.*

(b) Prove the theorem you wrote in part (a) — using the Pigeonhole Principle (often abbreviated to PHP). In your proof, explain the use of PHP clearly — in particular, state the version of PHP you are using, and clearly state where and how it is applied.

Response: Let us take such a graph G. Note that there can be at most $n - 1$ blue edges adjacent to a vertex. There are n different vertices. Let us associate a number p_i, the number of edges associated with the i^{th} vertex in the graph. Note that if we associate a vertex with 0 blue edges, then there can be no edges associated with n-1 blue edges. Hence there are at max n-1 choices for edges.

By the pigeon hole principle, we have to make n choices from $n - 1$ options. This is analogous to putting N objects (choices) in $n - 1$ boxes(options). Hence, there exists at least one box (option) corresponding to at least 2 objects (choices). Hence at least two choices have to be picking the same option, which means there exists at least two vertices adjacent to the same number of blue edges. □

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[1] At this party, **no one** shakes hands with themself!

4. [AN APPLICATION OF THE PRINCIPLE OF INCLUSION-EXCLUSION] [12]

We started Module-3 by counting the number of one-to-one functions between two finite sets.

In this exercise, we will count the number of onto functions — using the principle of inclusion-exclusion. As we did in the case of derangements (in lectures), it will be useful to first count the number of functions that are **not** onto, and then subtract that from the total number of functions.

Throughout this exercise, let A and B denote two finite sets with $|A| = m$ and $|B| = n$ where $n \leq m$.

(a) Prove that the total number of functions from A to B is n^m.

Response: There are n choices for the functional value of each of the m elements present in the domain in the function. The total number of functions will thus be n^m.

(b) Use the Principle of Inclusion-Exclusion (often abbreviated to PIE) to prove that the total number of functions from A to B that are **not** onto is:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m$$

Explain clearly how and where PIE is being used.

Response: Consider the set of non-onto functions as the set N. Let us give an arbitrary ranking to the various elements in the set B, which is possible, since the set B has a finite cardinality. Consider A_i to be the set of functions having the co-domain B - {ith element of B}. Clearly, the union of all the sets A_i is the set N.

The intersection of any k of the $A_i's$ will be a set of functions having some $n-k$ elements in the co-domain (We remove k elements. This intersection will have $(n-k)^m$ elements. There are $\binom{n}{k}$ such intersections.

Let us apply the principle of inclusion exclusion.

$$|\bigcup_{i=1}^{n} A_i| = \Sigma_{k=1}^{n}|(-1)^{k-1}\Sigma_{\binom{I\epsilon\{1,2,..,n\}}{k}}||\bigcap_{i\epsilon I} A_i|||$$

From the previous argument, the k^{th} term in the RHS will be $(-1)^{k-1}\binom{n}{k}(n-k)^m$.

Hence, our required LHS would be the sum of all such terms which is

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m$$

(c) Use parts (a) and (b), and some easy manipulations, to prove that the total number of onto functions from A to B is:

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(n-k)^m$$

Response: Using part (b), the number of non-onto functions is

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m$$

. Using part (a), the totoal number of functions is n^m. However, note that this can be written as $\binom{n}{0}(n-0)^m$.

The total number of onto functions summed with the total number of onto functions give the total number of functions as they form a disjoint set.

Hence,

$$N_{onto} = \binom{n}{0}(n-0)^m - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m \tag{1}$$

$$= \binom{n}{0}(n-0)^m + \sum_{k=1}^{n}(-1)^{k}\binom{n}{k}(n-k)^m \tag{2}$$

$$= \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(n-k)^m \tag{3}$$

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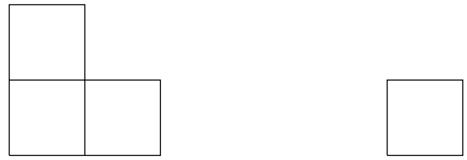
5. [USING AND SOLVING RECURRENCES: YET ANOTHER TILING PROBLEM] [12]

In this exercise, we our goal is to compute the number of tilings[2] of a $(2 \times n)$-grid for all $n \in \mathbb{N} - \{0\}$ using two types of tiles: L-shaped tiles (drawn below on the left) and box-shaped (1×1) tiles (drawn below on the right):



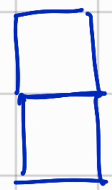
(a) For convenience, we denote by T_{n-1} the number of tilings of a $(2 \times n)$-grid — using L-shaped and box-shaped tiles. Compute T_0, T_1 & T_2, and explain your answers clearly — with drawings (if required).

Response: We obtain T_0 as *1*, T_1 as *5* and T_2 as *11*. There is exactly one way to tile a (2x1) and that too only with the (1x1) tile as the L shaped tile is too big to fit. A (2x2) tile cane be tiled with one L shaped tile and one (1x1) tile. There are 4 ways each corresponding to a 90 degree rotation of the L shaped tile and placing the 1x1 tile. The ways of tiling are illustrated in the figure attached.

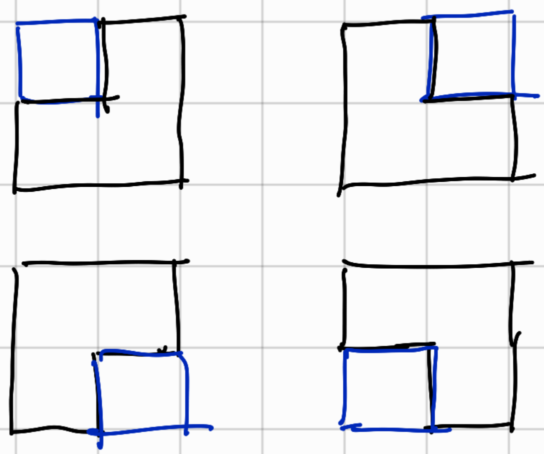
[2]As per examples discussed in lectures, tiling means: "no two tiles should overlap" and "each square of the grid should be covered by a tile".

DEMONSTRATION OF TILINGS:

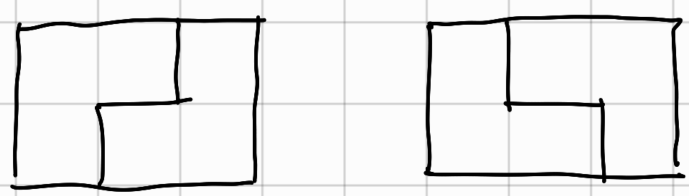
T_0
(1 way).



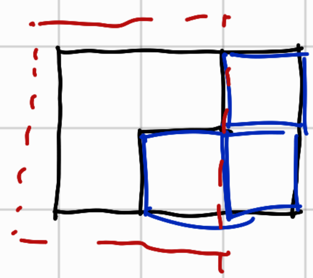
T_1
(4 ways)



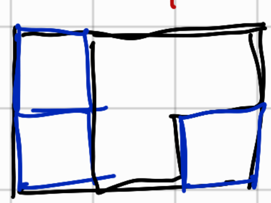
T_3
(11 ways)



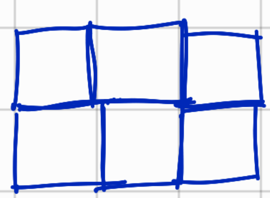
(from T_2)



$\times 4$ (There are 4 ways to tile red box)



$\times 4$



(b) Write down a recurrence for T_n, and explain clearly why this recurrence is correct.

Response: Let us say we we have to make a $(2xn)$ tiling, where $n > 3$, and let us suppose that we want to make this tiling from smaller tilings. I claim there are *exactly* three ways to do this.

i. Start with a (2x(n-1)) tiling, and then add two (1x1) tiles.

ii. Start with a (2xn-2) tiling, and then add one L shaped tile and one (1x1) tile, of which there are two ways. Observe that placing 4 (1x1) tiles is covered in option (a). This is the only way which is not counted in the (2xn-1) case. In other ways, we need to make a (2x2) tile without making a (2x1) tile, (on the left). From the previous diagram, it is clear that the described method is the only possibility.

iii. Start with a (2xn-3) tiling, and then add two L shaped tiles of which there are 2 ways. This is the only way way which starts with a (2xn-3) tile and is not counted in the (2xn-2) or the (2xn-1) case. In other words, we need to make a (2x3) tiling without forming a (2x2) tiling or a (2x1) tiling (, starting from the left). From the previous diagram, it is clear that the described method is the only possibility.

The recursion will be therefore

$$T_n = T_{n-1} + 4 * T_{n-2} + 2 * T_{n-3}$$

(c) Solve the recurrence obtained in part (b) — using the initial conditions from part (a) — and write down a closed form formula for T_n. Explain the steps followed clearly.

Response: Start with the characteristic equation

$$x^3 - x^2 - 4x - 2 = 0$$

The roots of this equation turn out to be (-1), (1+$\sqrt{3}$) and (1-$\sqrt{3}$). The general solution is of the form $c_0(-1)^n + c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$. Putting in the boundary conditions, we get

$$\begin{aligned} c_0 + c_1 + c_2 &= 1 \\ -c_0 + c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) &= 4 \\ c_0 + c_1(1+\sqrt{3})^2 + c_2(1-\sqrt{3})^2 &= 11 \end{aligned}$$

We obtain $c_0 = -1$, $c_1 = 1 + \frac{1}{\sqrt{3}}$, and $c_2 = 1 - \frac{1}{\sqrt{3}}$. The closed form for T_n is $T_n = (-1)^{n+1} + (1 + \frac{1}{\sqrt{3}})\frac{(1+\sqrt{3})^n}{2} + (1 - \frac{1}{\sqrt{3}})\frac{(1-\sqrt{3})^n}{2}$.