3. [ANOTHER APPLICATION OF THE PIGEONHOLE PRINCIPLE: HANDSHAKES AT A PARTY] [12]

In this exercise, we will use the pigeonhole principle to prove the following.

Theorem 0.1. *Some $n \in \mathbb{N} - \{0, 1\}$ people attended the CSE@IIT-M Freshers' 2023 Party, and some of them shook hands with others*[1]*. There exist two partygoers — each of whom shook hands with an equal number of people.*

(a) Write down a theorem statement — using the language/terminology of graphs — that has the same meaning as Theorem 0.1.

Response:

Theorem 0.2. *Consider a complete graph K_n with $n \in \mathbb{N} - \{0, 1\}$ vertices. Each edge of the graph is either colored blue or pink. There exist two distinct vertices u and v such that number of blue edges adjacent to u is the same as the number of blue adjacent to v.*

(b) Prove the theorem you wrote in part (a) — using the Pigeonhole Principle (often abbreviated to PHP). In your proof, explain the use of PHP clearly — in particular, state the version of PHP you are using, and clearly state where and how it is applied.

Response: Let us take such a graph G. Note that there can be at most $n-1$ blue edges adjacent to a vertex. There are n different vertices. Let us associate a number p_i, the number of edges associated with the i^{th} vertex in the graph. Note that if we associate a vertex with 0 blue edges, then there can be no edges associated with n-1 blue edges. Hence there are at max n-1 choices for edges.

By the pigeon hole principle, we have to make n choices from $n-1$ options. This is analogous to putting N objects (choices) in $n-1$ boxes(options). Hence, there exists at least one box (option) corresponding to at least 2 objects (choices). Hence at least two choices have to be picking the same option, which means there exists at least two vertices adjacent to the same number of blue edges. □

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[1] At this party, **<u>no one</u>** shakes hands with themself!