4. [An application of the Principle of Inclusion-Exclusion] [12]

We started Module-3 by counting the number of one-to-one functions between two finite sets.

In this exercise, we will count the number of onto functions — using the principle of inclusion-exclusion. As we did in the case of derangements (in lectures), it will be useful to first count the number of functions that are **not** onto, and then subtract that from the total number of functions.

Throughout this exercise, let A and B denote two finite sets with $|A| = m$ and $|B| = n$ where $n \leq m$.

(a) Prove that the total number of functions from A to B is n^m.

Response: There are n choices for the functional value of each of the m elements present in the domain in the function. The total number of functions will thus be n^m.

(b) Use the Principle of Inclusion-Exclusion (often abbreviated to PIE) to prove that the total number of functions from A to B that are **not** onto is:

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)^m$$

Explain clearly how and where PIE is being used.

Response: Consider the set of non-onto functions as the set N. Let us give an arbitrary ranking to the various elements in the set B, which is possible, since the set B has a finite cardinality. Consider A_i to be the set of functions having the co-domain B - $\{i$th element of B$\}$. Clearly, the union of all the sets A_i is the set N.

The intersection of any k of the $A_i's$ will be a set of functions having some $n-k$ elements in the co-domain (We remove k elements. This intersection will have $(n-k)^m$ elements. There are $\binom{n}{k}$ such intersections.

Let us apply the principle of inclusion exclusion.

$$|\bigcup_{i=1}^{n} A_i| = \Sigma_{k=1}^{n} |(-1)^{k-1} \Sigma_{\binom{I \epsilon \{1,2,..,n\}}{k}} || \bigcap_{i \epsilon I} A_i |||$$

From the previous argument, the k^{th} term in the RHS will be $(-1)^{k-1}\binom{n}{k}(n-k)^m$.

Hence, our required LHS would be the sum of all such terms which is

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)^m$$

(c) Use parts (a) and (b), and some easy manipulations, to prove that the total number of onto functions from A to B is:

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n-k)^m$$

Response: Using part (b), the number of non-onto functions is

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)^m$$

. Using part (a), the totoal number of functions is n^m. However, note that this can be written as $\binom{n}{0}(n-0)^m$.

The total number of onto functions summed with the total number of onto functions give the total number of functions as they form a disjoint set.

Hence,

$$N_{onto} = \binom{n}{0}(n-0)^m - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m \tag{1}$$

$$= \binom{n}{0}(n-0)^m + \sum_{k=1}^{n}(-1)^{k}\binom{n}{k}(n-k)^m \tag{2}$$

$$= \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(n-k)^m \tag{3}$$
