2. [Placing Non-attacking Rooks on a Huge Chessboard] [12]

In your hostel is a 100×100 marble chessboard, and a large number of chess pieces.

(It is very beautiful, and very heavy, and you have no idea what to do with it! So, let's count something for fun!)

(a) Prove that the number of ways to place k non-attacking rooks on this chessboard is $k!\binom{100}{k}^2$.

Note: Two rooks attack each other if they are in the same row or column. We consider any two rooks to be indistinguishable, so we only care about which squares contain rooks and which squares do not. Hence the question is really asking about the number of ways to choose a set of k squares on the chessboard so that no two squares are in the same row or column.

Response: The strategy of counting the number of ways in this question is to choose k rows, and k columns and arrange rooks in them. There are $\binom{n}{k}$ ways of choosing k rows from n rows. After choosing rows, there are $\binom{n}{k}$ ways of choosing k columns from n columns. We now need to associate the rows and columns into pairs. We assume an arbitrary ranking of the columns. In this ranking, the first column has k rows it can be associated with, the second column has $k-1$ rows it can be related to and so on and the last column has 1 rows that it can be related to. Hence, by multiplying all these choices, there are $k!$ ways of associating rows with columns and in the process determining the points. Note that the rooks which are in different rows and different columns do not attack each other and attack each other otherwise.

(b) Prove that the number of ways to place k non-attacking rooks on the chessboard, with all rooks on white squares, is:

$$\sum_{i=0}^{k} i!\binom{50}{i}^2 (k-i)!\binom{50}{k-i}^2.$$

Hint: You may want to color the white squares.

Response: We assume that the rows and columns are each numbered from 1 to 100, rows from top to bottom and columns from left to right. (Assuming a fixed orientation of the board)

The key observation here is that rooks placed in white squares in rows having an odd row number can never attack rooks placed in white squares in rows having an even row number since they are in different columns.

Let us now look at placing rooks in even numbered rows without attacking each other. white squares in even rows are arranged so that in every row, there are 50 white squares and the nth white squares in each row are present in the same column. There are 50 such columns and 50 rows. This is identical to placing i rooks in a grid of side 50 such that the rooks are non-attacking. From part (a), there are $i!\binom{100}{i}^2$ of placing rooks in such a way

Placing the remaining $k-i$ rooks in odd columns is done in the exact same way. There are $(k-i)!\binom{100}{(k-i)}^2$ ways of doing so.

These two events are independent of each other, hence we multiply the number of ways to get the number of ways of placing i rooks, where i ranges from 0 to 50 in even columns as $i!\binom{100}{i}^2(k-i)!\binom{100}{(k-i)}^2$.

Now 'i' can take values from 0 to 50 and we have to sum over the number of ways we obtain for each 'i' in order to get the total number of ways of placing rooks. Hence the total number of ways of placing rooks comes out to be

$$\sum_{i=0}^{k} i!\binom{50}{i}^2 (k-i)!\binom{50}{k-i}^2$$