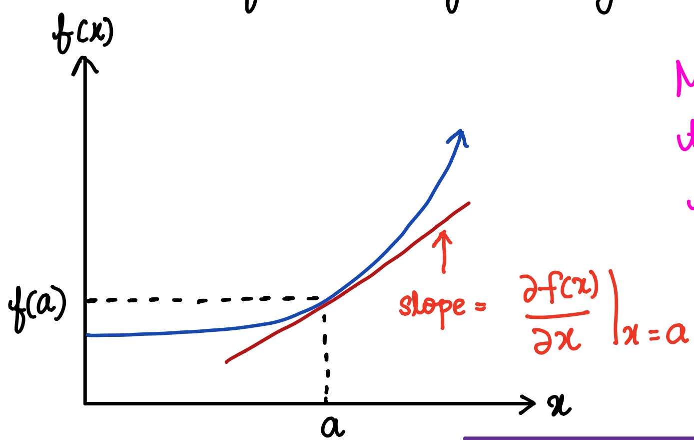
Extended Kalman filter

→ Linearizing a non linear function

In 2D, choose an operating point 'a' approximate the nonlinear function by a tagent line at the point.



Mathematically, we compute this linear approx using a first order Taylor's expansion.

$$\underbrace{\boxed{f(x) \approx f(a) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=a} (x-a)}}_{\text{first order terms}} + \frac{1}{2!} \left.\frac{\partial^2 f(x)}{\partial x}\right|_{x=a}$$

For the EKF, we choose the operating point to be our most recent state estimate our known input and zero noise.

Linearized Motion Model.

$$x_K = f_{K-1}(x_{K-1}, u_{K-1}, w_{K-1}) \approx f_{K-1}(\hat{x}_{K-1}, u_{K-1}, 0) + \underbrace{\left.\frac{\partial F_{K-1}}{\partial x_{K-1}}\right|_{\substack{\hat{x}_{K-1} \\ \hat{u}_{K-1}}, \; =0}}_{F_{K-1}} (x_{K-1} - \hat{x}_{K-1})$$

Motion model is linearized about posterior estimate of the previous state.

$$\underbrace{\left.\frac{\partial f_{K-1}}{\partial x_{K-1}}\right|_{\substack{\hat{x}_{K-1} \\ \hat{u}_{K-1}}, \; =0}}_{L_{K-1}} w_{K-1}$$

Linearized Measurement Model

Measurement Model is linearized based on the prediction of the current state based on the motion model.

$$y_k = h_k(x_k, v_k) \approx h_k(\check{x}_k, 0) + \underbrace{\left.\frac{\partial h_k}{\partial x_k}\right|_{\hat{x}_k, 0}}_{H_k} (\hat{x}_k - \check{x}_k) + \underbrace{\left.\frac{\partial h_k}{\partial v_k}\right|_{\hat{x}_k, 0}}_{M_k} v_k$$

We now have a linear system in state space
The matrices F, L, H, M are called the **Jacobian Matrices** of the System.

In vector calculus a jacobian matrix is the matrix of all first-order partial derivatives of a vector valued function.

Intuitively, the Jacobian matrix tells you how fast each output of the function is changing along each input direction.

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right] = \begin{bmatrix} \partial f_1/\partial x_1 & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

for example

$$f(x) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1^2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2x_1 & 0 \end{bmatrix}$$

Linearized motion model

$$x_k = f_{k-1}(\hat{x}_{k-1}, u_{k-1}, 0) + F_{k-1}(x_{k-1} - \hat{x}_{k-1}) + L_{k-1} w_{k-1}$$

Linearized measurement model.

$$y_k = h_k(\check{x}_k, 0) + H_k(x_k - \check{x}_k) + M_k v_k$$

Prediction

$$\check{x}_k = f_{k-1}(\hat{x}_{k-1}, u_{k-1}, 0)$$
$$\check{P}_k = F_{k-1} \hat{P}_{k-1} F_{k-1}^T + L_{k-1} Q_{k-1} L_{k-1}^T$$

Optimal gain

$$K_k = \check{P}_k H_k^T (H_k \check{P}_k H_k^T + M_k R_k M_k^T)^{-1}$$

Correction

$$\hat{x}_k = \check{x}_k + K_k(y_k - h_k(\check{x}_k, 0))$$
$$\hat{P}_k = (1 - K_k H_k)\check{P}_k$$

$\underbrace{\check{x}_k \rightarrow \text{Prediction} \qquad \hat{x}_k \rightarrow \text{corrected prediction}}_{\text{both at a time } k.}$

Limitations of EKF

1) Linearization error

i) How non-linear is the function.

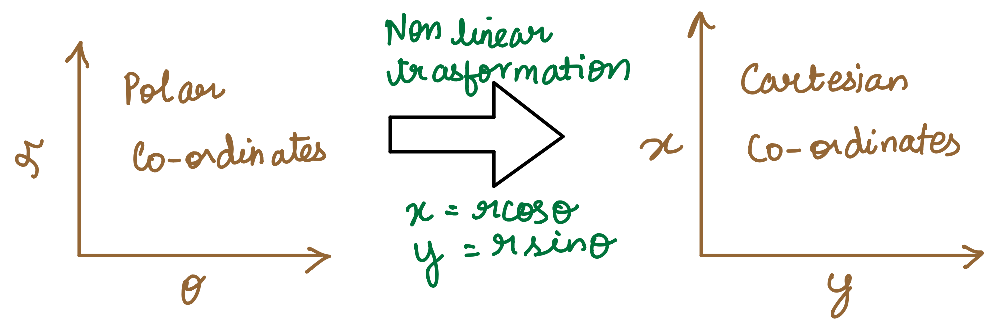
→ if the the non-linear function varies pretty slowly then a linear approximation will be a pretty good fit.

→ if the function varies too quickly then the linearization will do a bad job approximating the function.

ii) How far away from the operating point are you trying to do a linear approximation.

The further away from the operating point more divergence in the linear approximation.

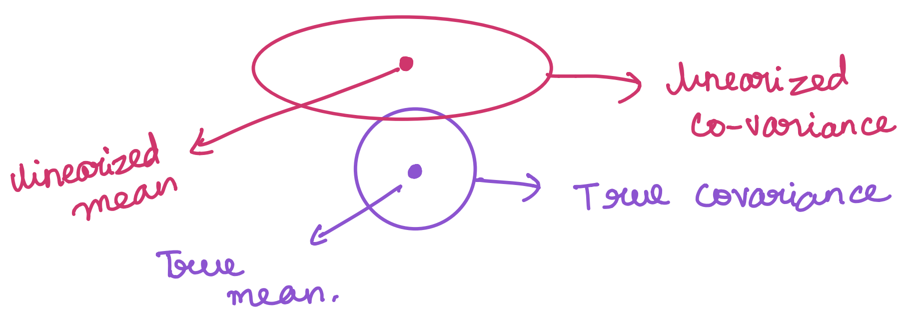
Lets look at an example of how linearization error affects the mean and variance of a random variable by a non linear function.



Linearized transformation

$$x \approx \bar{r}\cos\bar{\theta} + \cos\bar{\theta}\,(r-\bar{r}) - \bar{r}\sin\theta\,(\theta-\bar{\theta})$$

$$y \approx \bar{r}\sin\bar{\theta} + \sin\bar{\theta}\,(r-\bar{r}) + \bar{r}\cos\theta\,(\theta-\bar{\theta})$$



EKF is prone to linearization error

1) System dynamics is highly non-linear
2) sensor sampling time is slow relative to how fast the system evolves.

Two important consequences.

a) estimated mean can become very different to the true state.

b) estimated co-variance fails to capture the true uncertainity of the state

Huge problem of safety

Linearization error can cause the estimator to be overconfident about the wrong answer!!!!

→ Computing Jacobians is also a common soure of errors
→ What if the functions are non differentiable ???