Math 620: HW 4, Equivalence Relations

Due on Wednesday, September 23, 2015

Boynton 10:00

Kailee Gray

Exercise 1: Let $A = \{1,2,3\}$. (a) Let $R = \{(1,1),(2,2),(3,3)\}$. Is R an equivalence relation on A?

(reflexive) This relation is reflexive since $(1,1),(2,2),(3,3) \in R$

(symmetric) Every element in R is of the form (a,a) so this relation is symmetric.

(transitive) Since every element in A is equivalent to itself, there do not exist elements of the form (a,b) and (b,c) in R with distinct $a, c \in C$. Thus the hypothesis of the conditional definition of transitivity is false, so this equivalence relation is transitive.

(b) Let $R = \{(1,1),(2,2),(1,2),(2,1)\}$. Is R an equivalence relation on A?

(reflexive) This relation is not reflexive since $(3,3) \notin R$.

(symmetric) Since $(1,2)$ and $(2,1)$ are in R, this relation is symmetric.

(transitive) This relation is transitive: $(1,2)$ and $(2,1)$ are in R, and we have $(1,1)$ and $(2,2)$ in R.

(c) Let $R = A \times A$. Then $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(2,3),(3,1),(3,2)\}$. Is R an equivalence relation on A?

(reflexive) This relation is reflexive since $(1,1),(2,2),(3,3) \in R$.

(symmetric) Because the following elements both appear in R, this relation is symmetric:

$$(1,2) \text{ and } (2,1)$$
$$(1,3) \text{ and } (3,1)$$
$$(2,3) \text{ and } (3,2)$$

(transitive) R contains (a,b) for any $a, b \in A$. So, if $(a,b),(b,c) \in R$, then $(a,c) \in R$ since R contains all possible (a,b). Thus,this relation is transitive.

Exercise 2: Suppose $\sim$ is a nonempty relation on a set A and that $\sim$ satisfies the symmetric property and the transitive property. Does it following that $\sim$ satisfies the reflexive property?

No. Define A and $\sim$ as defined in part (b) of exercise 1. We showed the given R satisfies the symmetric property and the transitive property but did not satisfy the reflexive property.

Exercise 3: Let $A = \mathbb{Z} \times \mathbb{N}$ and define a relation $\sim$ on A by $(a,b) \sim (r,s)$ if and only if $as = br$. (a) Prove that $\sim$ is an equivalence relation.

(reflexive) For any $(a,b) \in A$, we have $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ so $a = a$ and $ab = ab$. Since $ab = ba$, $(a,b) \sim (a,b)$. Therefore, this relation is reflexive.

(symmetric) Suppose $(a,b) \sim (r,s)$ for $(a,b),(r,s) \in A$. Then, $as = br$ and $br = as$. By commutative property of multiplication of real numbers, $br = as$ is equivalent to $rb = sa$ which implies $(r,s) \sim (a,b)$. Thus, this relation is symmetric.

(transitive) Suppose $(a,b) \sim (r,s)$ and $(r,s) \sim (m,n)$. Then, $as = br$ and $rn = sm$. Multiply both sides of $as = br$ to obtain $asn = brn$. Then, substitute $sm = rn$ to obtain $ans = bsm$. Note $s \neq 0$ because $s \in \mathbb{N}$. So divide by s to obtain $an = bm$. Hence $(a,b) \sim (m,n)$ and so this relation is transitive.

part (b) on next page...

(b): Find $\Phi : (A/\sim) \to \mathbb{Q}$ and prove Φ is well-defined, 1-1, and onto.

We can write the factor set

$$A/\sim = \{[(a,b)] \colon a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(a,b) = 1, a \neq 0\}. \quad \text{Define } \Phi : A/\sim \to \mathbb{Q} \text{ by } \Phi[(a,b)] = \frac{a}{b}.$$

(WD1) For every $[(a,b)] \in A/\sim$, $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, so $b \neq 0$ and $\frac{a}{b} \in \mathbb{Q}$.

(WD2) For any $[(a_1,b_1)] \in A/\sim$ and $[(a_2,b_2)] \in A/\sim$, if $[(a_1,b_1)] = [(a_2,b_2)]$, then $(a_1,b_1) \sim (a_2,b_2)$ implies $a_1b_2 = a_2b_1$. Thus, since $b_1, b_2 \neq 0$, $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. Thus, $\Phi[(a_1,b_1)] = \Phi[(a_2,b_2)]$ and so Φ is well-defined.

(1-1) Next, we will show Φ is one-to-one. Suppose $\Phi[(a_1,b_1)] = \Phi[(a_2,b_2)]$. Then, $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ implies $a_1b_2 = a_2b_1$. Thus, $(a_1,b_1) \sim (a_2,b_2)$ and so $[(a_1,b_1)] = [(a_2,b_2)]$.

(onto) Next, we will show Φ is onto. Consider any $q \in \mathbb{Q}$. By definition of rational numbers, q can be written as $\frac{a}{b}$ with $a, b \in \mathbb{Z}, b \neq 0$. If $q < 0$, then $a < 0$ or $b < 0$. If $a < 0$, then $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ as desired, so that $\Phi(\ [(a,b)]\ ) = \frac{a}{b} = q$. Suppose $b < 0$, then let $q = \frac{-a}{|b|}$ so that $-a \in \mathbb{Z}$ and $|b| \in \mathbb{N}$; thus, $\Phi(\ [(-a,|b|)]\ ) = q$. Therefore, Φ is onto.

Exercise 4(a): For a set A, equality is the smallest equivalence relation on A.

(b) *Proof.* Define R on A by $a \sim b$ if $a = b$. Note $\sim$ is an equivalence relation:
(reflexive) This relation is reflexive since for any $a \in A$, $a \sim a$.
(symmetric) Every element in R is of the form (a, a) so this relation is symmetric.
(transitive) Since every element in A is equivalent to itself, there do not exist elements of the form (a, b) and (b, c) in R with distinct $a, c \in A$. Thus the hypothesis of the conditional definition of transitivity is false, so this equivalence relation is transitive.
We will prove R is the smallest equivalence relation on A. Note R relates every element of A to itself and only itself, so $R =$diag(A). Consider any other equivalence relation on A, R'. Then, since R' is an equivalence relation, R' must be symmetric so diag$(A) \subseteq R'$. But, $R =$diag(A), so $R \subseteq R'$. Since R is contained in any equivalence relation on A, R is the smallest equivalence relation on A. $\square$

(c) For any set A, $R = A \times A$ is the largest equivalence relation on A.

(d) *Proof.* Notice R contains all possible (a, b) for all $a, b \in A$. Thus, for any equivalence relation R' on A, $R' \subseteq R$. Hence R is the largest equivalence relation on A. $\square$

Exercise 2.2.1: For each of the following functions find $f(S)$ and S/f and exhibit the one-to-one correspondence between them.

(a) $f : \mathbb{Z} \to \mathbb{C}$ **given by** $f(n) = i^n$ **for all** $n \in \mathbb{Z}$**.**

i raised to any n^{th} power ($n \in Z$) will be ± 1 or $\pm i$, so $f(\mathbb{Z}) = 1, i, -1, -i$

$$f([1]_4) = i, \qquad f([2]_4) = -1, \qquad f([3]_4) = -i, \qquad f([0]_4) = 1$$

Thus, $\mathbb{Z}/f = \{\mathbb{Z}_4\}$. Also, $\bar{f} : \mathbb{Z}/f \to f(\mathbb{Z})$ is defined by $\bar{f}([n]_4) = i^n$.

(b) $g : \mathbb{Z} \to \mathbb{Z}_{12}$ **given by** $g(n) = [8n]_{12}$ **for all** $n \in \mathbb{Z}$**.**

Notice $g\left([0]_{12}\right) = g([3]_{12}) = g([6]_{12}) = g([9]_{12}) = 0, \qquad g([1]_{12}) = g([4]_{12}) = g([7]_{12}) = g([10]_{12}) = 8,$ and $g([2]_{12}) = g([5]_{12}) = g([8]_{12}) = g([11]_{12}) = 4$.

Thus, $g(\mathbb{Z}) = \{0, 4, 8\}$. So let $[[0]_{12}] = \{[0]_{12}, [3]_{12}, [6]_{12}, [9]_{12}\}$, $[[1]_{12}] = \{[1]_{12}, [4]_{12}, [7]_{12}, [10]_{12}\}$, and $[[2]_{12}] = \{[2]_{12}, [5]_{12}, [8]_{12}, [11]_{12}\}$. Hence, $\mathbb{Z}/g = \{[[0]_{12}], [[1]_{12}], [[2]_{12}]\}$.

Also, $\bar{g} : \mathbb{Z}/g \to g(\mathbb{Z})$ is defined by $\bar{g}([[n]_{12}]) = [8n]_{12}$.

(c) $h : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ **given by** $h([x]_{12}) = [9x]_{12}$

Notice $h\left([0]_{12}\right) = h([4]_{12}) = h([8]_{12}) = 0, \qquad h([1]_{12}) = h([5]_{12}) = h([9]_{12}) = 9,$ $h([2]_{12}) = h([6]_{12}) = h([10]_{12}) = 6,$ and $h([3]_{12}) = h([7]_{12}) = h([11]_{12}) = 3$.

Thus, $h(\mathbb{Z}_{12}) = \{[0]_{12}, [9]_{12}, [6]_{12}, [3]_{12}\}$. So let $[[0]_{12}] = \{[0]_{12}, [4]_{12}, [8]_{12}\}$, $[[1]_{12}] = \{[1]_{12}, [5]_{12}, [9]_{12}\}$, $[[2]_{12}] = \{[2]_{12}, [6]_{12}, [10]_{12}\}$ and $[[3]_{12}] = \{[3]_{12}, [7]_{12}, [11]_{12}\}$.

Hence, $\mathbb{Z}/h = \{[[0]_{12}], [[1]_{12}], [[2]_{12}], [[3]_{12}]\}$.

Then, $\bar{h} : \mathbb{Z}/h \to h(\mathbb{Z}_{12})$ is defined by $\bar{h}([[x]_{12}]) = [9x]_{12}$.

Exercise 2.2.1 continued on next page.

Exercise 2.2.1 (d) $k : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ **given by** $k([x]_{12}) = [5x]_{12}$

$$\text{Notice } k\left([0]_{12}\right) 0, \quad k([1]_{12}) = 5, \quad = k([2]_{12}) = 10, \quad k([3]_{12}) = 3, \quad k([4]_{12}) = 8, \quad k([5]_{12}) = 1,$$
$$k([6]_{12}) = 6, \quad k([7]_{12}) = 11, \quad k([8]_{12}) = 4, \quad k([9]_{12}) = 9, \quad k([10]_{12}) = 2, \quad k([11]_{12}) = 7.$$
$$\text{Thus, } k(\mathbb{Z}_{12}) = \mathbb{Z}_{12}. \qquad \text{Hence, } \mathbb{Z}/k = \{[a]_{12} \text{ such that } a \in \mathbb{Z}_{12}.\}$$
$$\text{Then, } \bar{k} : \mathbb{Z}/h \to h(\mathbb{Z}_{12}) \text{ is defined by } \bar{k}([x]_{12}) = [5x]_{12}. \text{ In other words, } \bar{k} = k.$$

Exercise 2.2.3: Determine which of the three conditions of Definition 2.2.1 hold.

(a) For $a, b \in \mathbb{R}$**, define** $a \sim b$ **if** $a \leq b$**.**

(reflexive) This relation is reflexive since for any $a \in \mathbb{R}$, $a \leq a$, so $a \sim a$.

(symmetric) This relation is not symmetric: $1, 2 \in \mathbb{R}$ and $1 \leq 2$ implies $1 \sim 2$. But, $2 \not\leq 1$ so $2 \not\sim 1$.

(transitive) This equivalence relation is transitive: let $a, b, c \in \mathbb{R}$. Suppose $a \sim b$ and $b \sim c$. Then, $a \leq b$ and $b \leq c$ so $a \leq b \leq c$ implies $a \leq c$. Thus, $a \sim c$.

(b) For $a, b \in \mathbb{R}$**, define** $a \sim b$ **if** $a - b \in \mathbb{Q}$**.**

(reflexive) This relation is reflexive since for any $a \in \mathbb{R}$, $a - a = 0 \in \mathbb{Q}$, so $a \sim a$.

(symmetric) This relation is symmetric: Let $a, b \in \mathbb{R}$ and $a \sim b$. Then, $a - b = r$ for some $r \in \mathbb{Q}$. $a - b = r$ is equivalent to $-a + b = -r$ so $b - a = -r$. Multiplication of rational numbers is closed, so $r, -1 \in \mathbb{Q}$ implies $-r \in \mathbb{Q}$. Thus, $b \sim a$.

(transitive) This equivalence relation is transitive: let $a, b, c \in \mathbb{R}$. Suppose $a \sim b$ and $b \sim c$. Then, $a - b = r_1$ and $b - c = r_2$ for $r_1, r_2 \in \mathbb{Q}$. Add the last two equations together to obtain $a - b + b - c = r_1 + r_2$; equivalently $a - c = r_1 + r_2$. Rational numbers are closed under addition, so $r_1 + r_2 \in \mathbb{Q}$. Thus, $a \sim c$.

Exercise 2.2.3 (c) For $a, b \in \mathbb{R}$, define $a \sim b$ if $|a - b| \leq 1$.

(reflexive) This relation is reflexive since for any $a \in \mathbb{R}$, $|a - a| = |0| = 0 \leq 1$, so $a \sim a$.

(symmetric) This relation is symmetric: Let $a, b \in \mathbb{R}$ and $a \sim b$. Then, $|a - b| \leq 1$ which implies $-1 \leq a - b \leq 1$. Multiply this inequality by -1 to obtain $1 \geq -a + b \geq -1$. Equivalently, $-1 \leq b - a \leq 1$ so $|b - a| \leq 1$. Thus, $b \sim a$.

(transitive) This equivalence relation is not transitive: let $a = 2, b = 1, c = 0.5$. Then $a - b = 1$, $b - c = 0.5$, but $a - c = 1.5$. So when $a = 2, b = 1, c = 0.5$ $a \sim b$ and $b \sim c$ but $a \not\sim c$.

Exercise 7: Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(t) = (\cos t, \sin t)$ and define a relation $\sim$ on $\mathbb{R}$ by $r_1 \sim r_2$ if and only if $f(r_1) = f(r_2)$.

(a) Geometrically describe $f(\mathbb{R})$: $f(\mathbb{R})$ is the unit circle.

(b) Prove that $f^{-1}\{(1,0)\} = 2\pi\mathbb{Z} = [0]$ and $f^{-1}\{(-1,0)\} = \pi + 2\pi\mathbb{Z} = [\pi]$.

To find $f^{-1}\{(1,0)\}$, solve for t: $(\cos t, \sin t) = (1,0)$. Then, $\cos t = 1$ and $\sin t = 0$ only when $t = 2\pi k$ for some integer k. Thus, $f^{-1}\{(1,0)\} = 2\pi\mathbb{Z}$. Also, $f(0) = f(2\pi\mathbb{Z})$ so $0 \sim 2\pi\mathbb{Z}$ and we have $f^{-1}\{(1,0)\} = [0]$.

Next, consider $f^{-1}\{(-1,0)\}$ and solve for t: $(\cos t, \sin t) = (-1,0)$. Then, $\cos t = -1$ and $\sin t = 0$ only when $t = \pi(2k+1) = 2\pi k + \pi$ for some integer k. Thus, $f^{-1}\{(1,0)\} = 2\pi\mathbb{Z} + \pi$. Also, $f(\pi) = f(\pi + 2\pi\mathbb{Z})$ so $\pi \sim \pi + 2\pi\mathbb{Z}$ and we have $f^{-1}\{(-1,0)\} = [\pi]$.

(c) Suppose that $b \neq 0$ and $f^{-1}\{(a,b)\} = r + 2\pi\mathbb{Z} = [r]$. Find a formula for r in terms of a, b.

If $f^{-1}\{(a,b)\} = r + 2\pi\mathbb{Z}$, then $\cos(r + 2\pi\mathbb{Z}) = a$ and $\sin(r + 2\pi\mathbb{Z}) = b$. Thus $r + 2\pi\mathbb{Z} = \cos^{-1}(a)$ and $r + 2\pi\mathbb{Z} = \sin^{-1}(b)$. Adding these two equations together we obtain $r + 2\pi\mathbb{Z} + r + 2\pi\mathbb{Z} = \sin^{-1}(b) + \cos^{-1}(a)$, so

$$r = \frac{1}{2}\left(\cos^{-1}(a) + \sin^{-1}(b)\right) - 2\pi\mathbb{Z}$$

(d) Find a complete set of representatives for $\mathbb{R}/\sim$.

Notice $\cos(r + 2\pi\mathbb{Z}) = \cos(r)$ and $\sin(r + 2\pi\mathbb{Z}) = \sin(r)$. So, let $[r] = r + 2\pi\mathbb{Z}$ with $0 \leq r \leq 2\pi$. Then, $\mathbb{R}/\sim = \{[r]\}$

Exercise 8: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x + y$ and define a relation $\sim$ on $\mathbb{R}$ by $r_1 \sim r_2$ if and only if $f(r_1) = f(r_2)$.

(a) Prove that f is a surjective map.

Let $z \in \mathbb{R}$. We can let $x, y = z/2$ so that $x + y = z/2 + z/2 = z$. Thus for any $z \in \mathbb{R}$ there exists $x, y \in \mathbb{R}$ such that $f(x, y) = z$.

(b) Prove that $f^{-1}(\{5\}) = \{(x, y) \in \mathbb{R}^2 \colon y = 5 - x\} = [(0, 5)]$. Find three other representatives for the equivalence class $[(0, 5)]$.

To find $f^{-1}(\{5\})$, we need some $x, y \in \mathbb{R}$ such that $5 = x + y$. Equivalently, $y = 5 - x$. The set $\{(x, y) \in \mathbb{R}^2 \colon y = 5 - x\}$ contains all x, y such that $x + y = 5$, thus $f^{-1}(\{5\}) = \{(x, y) \in \mathbb{R}^2 \colon y = 5 - x\}$. Next, consider $[(0, 5)]$. Some $(x, y) \in \mathbb{R}^2$ is in $[(0, 5)]$ only if $f([(0, 5)]) = 5$. Notice $f([(0, 5)]) = f(0, 5) = 0 + 5 = 5$. Since $f(0, 5) = f([(0, 5)]) = 5$, $f^{-1}(\{5\}) = [(0, 5)]$. Since $2.5 + 2.5 = 5, 1 + 4 = 5$, and $-2 + 7 = 5$, $[(0, 5)] = [(2.5, 2.5)] = [(1, 4)] = [(-2, 7)]$.

(c) Prove that $f^{-1}(\{r\}) = \{(x, y) \in \mathbb{R}^2 \colon y = r - x\} = [(0, r)]$.

To find $f^{-1}(\{r\})$, we need some $x, y \in \mathbb{R}$ such that $r = x + y$. Equivalently, $y = r - x$. The set $\{(x, y) \in \mathbb{R}^2 \colon y = r - x\}$ contains all x, y such that $x + y = r$, thus $f^{-1}(\{r\}) = \{(x, y) \in \mathbb{R}^2 \colon y = r - x\}$. Next, consider $[(0, r)]$. Some $(x, y) \in \mathbb{R}^2$ is in $[(0, r)]$ only if $f([(0, r)]) = r$. Notice $f([(0, r)]) = f(0, r) = 0 + r = r$. Since $f(0, r) = f([(0, r)]) = r$, $f^{-1}(\{r\}) = [(0, r)]$.

(d) Find a complete set of representatives for $\mathbb{R}^2/\sim$.

$\mathbb{R}^2/\sim \; = \{[(0, r)]$ where $r \in \mathbb{R}\}$.

Exercise 9: Let $f : \mathbb{R}^\times \to \{\pm 1\}$ be given by $f(t) = \frac{t}{|t|}$ and define a relation $\sim$ on $\mathbb{R}$ by $r_1 \sim r_2$ if and only if $f(r_1) = f(r_2)$.

(a) Prove that f is a surjective map.

We must show there exists $t \in \mathbb{R}^\times$ such that $f(t) = 1$ and we must show there exists $t \in \mathbb{R}^\times$ such that $f(t) = -1$. If $t > 0$, $f(t) = \frac{t}{|t|} = \frac{t}{t} = 1$. If $t < 0$, $f(t) = \frac{t}{|t|} = \frac{t}{-t} = -1$. Thus this function is onto.

(b) Prove that $f^{-1}(\{1\}) = (0, \infty) = [r]$ where r is any positive real number.

To find $f^{-1}(\{1\})$, we need some $t \in \mathbb{R}^\times$ such that $\frac{t}{|t|} = 1$. Equivalently, $t = |t|$. This is true for any $t > 0$. Thus $f^{-1}(\{1\}) = (0, \infty)$. If $r_1 > 0, r_2 > 0$, $f(r_1) = f(r_2) = 1$ so $r_1 \sim r_2$. So, let $r > 0$, $f^{-1}(\{1\}) = [r]$.

(c) Prove that $f^{-1}(\{-1\}) = (-\infty, 0) = [r]$ where r is any negative real number.

To find $f^{-1}(\{-1\})$, we need some $t \in \mathbb{R}^\times$ such that $\frac{t}{|t|} = -1$. Equivalently, $-t = |t|$. This is true for any $t < 0$. Thus $f^{-1}(\{1\}) = (-\infty, 0)$. If $r_1 < 0, r_2 < 0$, $f(r_1) = f(r_2) = -1$ so $r_1 \sim r_2$. So, let $r < 0$, then $f^{-1}(\{-1\}) = [r]$.

(d) Find a complete set of representatives for $\mathbb{R}^\times / \sim$.

$\mathbb{R}^\times / \sim = \{[p], [n] \colon p \in \mathbb{R}^+, -n \in \mathbb{R}^+\}$

Exercise 10: Let $f : \mathbb{C}^\times \to \mathbb{R}$ be given by $f(a + bi) = \sqrt{a^2 + b^2}$ and define a relation $\sim$ on $\mathbb{R}$ by $r_1 \sim r_2$ if and only if $f(r_1) = f(r_2)$.

(a) Prove that $f(\mathbb{C}^\times) = (0, \infty)$.

For any $a + bi \in \mathbb{C}^\times$, $a + bi \neq 0$, so $a^2 + b^2 \neq 0$ and $\sqrt{a^2 + b^2} \neq 0$. Also, $a^2, b^2 \geq 0$ so $\sqrt{a^2 + b^2} \in \mathbb{R}$ and $\sqrt{a^2 + b^2} \geq 0$. Since $\sqrt{a^2 + b^2} \geq 0$ and $\sqrt{a^2 + b^2} \neq 0$, we have $f(a + bi) = \sqrt{a^2 + b^2} > 0$ for any $a + bi \in \mathbb{C}^\times$. Thus, $f(\mathbb{C}^\times) = (0, \infty)$.

(b) Prove that $[3 + 4i] = \{\alpha \in \mathbb{C} \colon |\alpha| = 5\} = f^{-1}(\{5\})$.

Because $f([3+4i]) = f(3+4i) = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, $f^{-1}(\{5\}) = f([3+4i])$. By definition of $\sim$, $a + bi \sim 3 + 4i$ if $f(a + bi) = f(3 + 4i)$. Since $f(3 + 4i) = 5$, $[3 + 4i]$ contains all $a + bi$ with $|a + bi| = 5$. Thus, $[3 + 4i] = \{\alpha \in \mathbb{C} \colon |\alpha| = 5\}$.

(c) Find a complete set of representatives for $\mathbb{C}^\times / \sim$.

For a fixed $r \in \mathbb{R}$, define $[\alpha] = \{$ any $a + bi \in \mathbb{C}^\times \colon |a + bi| = r, r \in \mathbb{R}\}$. Then, $\mathbb{C}^\times / \sim = \{[\alpha] \colon \alpha \in \mathbb{C}^\times\}$.