# Math 620: HW 5, Permutations

Due on Wednesday, September 30, 2015

*Boynton 10:00*

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Section 2.3: 1, 3, 4(d), 5, 8, 13, 15

**Exercise 2.3.1: Consider the following permutations in $S_7$**

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \text{ and } \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

Compute the following products.

**(a)**

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 6 & 7 & 4 & 1 & 5 \end{pmatrix}$$

**(b)**

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 4 & 7 & 6 & 2 & 3 \end{pmatrix}$$

**(c)**

$$\tau^2\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 7 & 3 & 6 & 1 & 5 \end{pmatrix}$$

**(d)**

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 1 & 4 & 3 & 5 & 7 \end{pmatrix}$$

**(e)**

$$\sigma\tau\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 7 & 6 & 4 & 5 \end{pmatrix}$$

**(f)**

$$\tau^{-1}\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 7 & 6 & 4 & 5 & 2 & 3 \end{pmatrix}$$

**Exercise 2.3.3: Write the following permutation as a product of disjoint cycles and as a product of transpositions. Construct its associated diagram, find its inverse, and find its order.**

Let

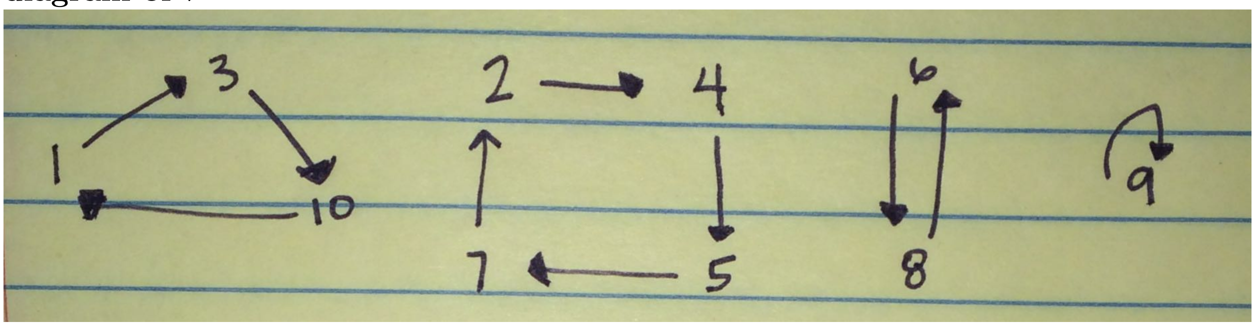
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix}$$

$\tau$ **as product of disjoint cycles:** $\tau = (1\ 3\ 10)\ (2\ 4\ 5\ 7)\ (6\ 8)$

Thus, by proposition 2.3.8, **the order of** $\tau$ is $\text{lcm}(3, 4, 2) = 12$.

$\tau$ **as product of transpositions:** $\tau = (1\ 3)\ (3\ 10)\ (2\ 4)\ (4\ 5)\ (5\ 7)\ (6\ 8)$

**diagram of** $\tau$




$$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 7 & 1 & 2 & 4 & 6 & 5 & 6 & 9 & 3 \end{pmatrix}$$

From above, **the order of** $\tau$ 12.

**Exercise 2.3.4(d): Find the order of the permutation**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 4 & 9 & 6 & 5 & 2 & 3 & 1 & 7 \end{pmatrix}$$

$\tau$ **as product of disjoint cycles:** $\tau = (1\ 8)\ (2\ 4\ 6)\ (3\ 9\ 7)$

Thus, by proposition 2.3.8, **the order of** $\tau$ is $\text{lcm}(2, 3, 3) = 6$.

**Exercise 2.3.5: Let $3 \leq m \leq n$. Calculate $\sigma\tau^{-1}$ for the cycles $\sigma = (1, 2, \ldots, m-1)$ and $\tau = (1, 2, \ldots, m-1, m)$ in $S_n$.**

To better understand $\sigma, \tau$, write:

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & m-2 & m-1 \\ 2 & 3 & \ldots & m-1 & 1 \end{pmatrix} \text{ and } \quad \tau = \begin{pmatrix} 1 & 2 & \ldots & m-1 & m \\ 2 & 3 & \ldots & m & 1 \end{pmatrix}.$$

Then

$$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & \ldots & m-1 & m \\ m & 1 & 2 & \ldots & m-2 & m-1 \end{pmatrix}.$$

Now, if we apply $\sigma$ to $\tau^{-1}$, we have

$$\sigma\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & \ldots & m-1 & m \\ m & 2 & 3 & \ldots & m-1 & 1 \end{pmatrix}.$$

Thus, $\sigma\tau^{-1} = (1, 2, \ldots, m-1)(1, m, m-1, m-2, m-3, \ldots 3, 2) = (1, m)$

**Exercise 2.3.8: Count the permutations $\sigma$ in $S_6$ that satisfy the conditions $\sigma(1) = 2$ and $\sigma(2) = 3$.**

If $\sigma(1) = 2$ and $\sigma(2) = 3$, then $\sigma$ has the form

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & k_3 & k_4 & k_5 & k_6 \end{pmatrix}, \text{ where } k_i \in S_n, 3 \leq i \leq 6, k_i \neq 2, 3.$$

If we start by choosing $k_3$, since $k_3 \neq 2, 3$, $k_3$ can be $1, 4, 5, 6$ so there are 4 options for $k_3$. Then, similarly, there are 3 choices for $k_4$, 2 choices for $k_5$, and 1 choice for $k_6$. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations that satisfy the given conditions.

**Exercise 2.3.13: Let $\tau \in S_n$ be the cycle $(1, 2, 3, \ldots, k)$ of length $k$, where $k \le n$. (a) Prove that if $\sigma \in S_n$, then $\sigma\tau\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))$. Thus $\sigma\tau\sigma^{-1}$ is a cycle of length $k$.**

*Proof.* Let $\sigma \in S_n$ and consider. To prove $\sigma\tau\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))$, we will first show $\sigma\tau = (\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma$ and then we will multiply this equation by $\sigma^{-1}$ on the right to obtain the desired equality. To prove these functions are equivalent, we must show for any $x \in S_n$, $\sigma\tau(x) = (\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma(x)$. We will consider two cases:

**Case 1:** $x \notin \{1, 2, \ldots, k\}$

If $x \notin \{1, 2, \ldots, k\}$, then $\tau$ will fix $x$ so that $\tau(x) = x$. Thus, $\sigma\tau(x) = \sigma(x)$. To calculate $(\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma(x)$, we need to determine what $(\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$ does to $\sigma(x)$. The cycle $(\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$ only acts on elements of the form $\sigma(a)$ where $a \in \{1, 2, \ldots, k\}$. Since $\sigma(x)$ is not of this form, $(\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$ will fix $\sigma(x)$ so that $(\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma(x) = \sigma(x)$. Thus, when $x \notin \{1, 2, \ldots, k\}$, $\sigma\tau(x) = (\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma(x)$.

**Case 2:** $x \in \{1, 2, \ldots, k\}$

Suppose $x \in \{1, 2, \ldots, k\}$. Then, $\sigma\tau(x) = \sigma(x + 1)$ since $x \in \{1, 2, \ldots, k\}$. In this case, the cycle $(\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$ acts on $\sigma(x)$ since $x \in \{1, 2, \ldots, k\}$. The cycle $(\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$ maps $\sigma(x)$ to $\sigma(x+1)$. Hence $(\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma(x) = \sigma(x+1)$ and we've shown when $x \in \{1, 2, \ldots, k\}$, $\sigma\tau(x) = (\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma(x)$.

Since $\sigma\tau(x) = (\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma(x)$ for all $x \in S_n$, we can multiply on the right by $\sigma^{-1}$ to obtain $\sigma\tau\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma\sigma^{-1}$. Composition of functions is associative so $(\sigma(1), \sigma(2), \ldots, \sigma(k))\sigma\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))(\sigma\sigma^{-1}) = (\sigma(1), \sigma(2), \ldots, \sigma(k))(1) = (\sigma(1), \sigma(2), \ldots, \sigma(k))$. Thus, $\sigma\tau\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))$. Additionally, $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ has length $k$, so $\sigma\tau\sigma^{-1}$ is a cycle of length $k$.

$\square$

**2.2.13 (b) Let $\rho$ be any cycle of length $k$. Prove that there exists a permutation $\sigma \in S_n$ such that $\sigma\tau\sigma^{-1} = \rho$.**

*Proof.* Consider $\rho = (a_1, a_2, \ldots, a_k)$, a cycle of length $k$ in $S_n$ and define $\sigma \in S_n$ by $\sigma(j) = a_j$ for $1 \leq j \leq k$. Let $\tau = (1, 2, 3, \ldots, k)$. Then, from part (a), $\sigma\tau\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k)) = (a_1, a_2, \ldots, a_k) = \rho$. Thus for any $\rho \in S_n$ we can find a $\sigma \in S_n$ such that $\sigma\tau\sigma^{-1} = \rho$. □

**Exercise 2.3.15: For $\alpha, \beta \in S_n$, let $\alpha \sim \beta$ if there exists $\sigma \in S_n$, such that $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation.**

*Proof.* **(reflexive)** Consider any $\alpha \in S_n$. Then $\alpha\alpha\alpha^{-1} = \alpha(\alpha\alpha^{-1}) = \alpha(1) = \alpha$ so $\alpha \sim \alpha$. Let $\sigma_i = \sigma^{-1}$.

**(symmetric)** Let $\alpha, \beta \in S_n$ and $\alpha \sim \beta$. Then, there exists $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$. Let $\gamma = \sigma^{-1}$, then $\gamma \in S_n$ and $\gamma^{-1} = \sigma$. Substituting $\gamma$ for $\sigma$ in the equation $\sigma\alpha\sigma^{-1} = \beta$, we have $\gamma^{-1}\alpha\gamma = \beta$. Now, multiply the left by $\gamma$ and the right by $\gamma^{-1}$ to obtain $\gamma\gamma^{-1}\alpha\gamma\gamma^{-1} = \gamma\beta\gamma^{-1}$. By associativity of composition of functions, $\gamma\gamma^{-1}\alpha\gamma\gamma^{-1} = (\gamma\gamma^{-1})\alpha(\gamma\gamma^{-1}) = (1)\alpha(1) = \alpha$. Therefore, $\gamma\beta\gamma^{-1} = \alpha$. Thus, $\beta \sim \alpha$.

**(transitive)** Suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. Then, there exists $\sigma_1, \sigma_2 \in S_n$ such that $\sigma_1\alpha\sigma_1^{-1} = \beta$ and $\sigma_2\beta\sigma_2^{-1} = \gamma$. Substitute $\beta = \sigma_1\alpha\sigma_1^{-1}$ into the equation $\sigma_2\beta\sigma_2^{-1} = \gamma$ to obtain $\sigma_2(\sigma_1\alpha\sigma_1^{-1})\sigma_2^{-1} = \gamma$. Composition of permutations is associative, so $\sigma_2(\sigma_1\alpha\sigma_1^{-1})\sigma_2^{-1} = (\sigma_2\sigma_1)\alpha(\sigma_1^{-1}\sigma_2^{-1})$. From page 81 of Beachy, we know $(\sigma_2\sigma_1)^{-1} = \sigma_1^{-1}\sigma_2^{-1}$. Thus $(\sigma_2\sigma_1)\alpha(\sigma_1^{-1}\sigma_2^{-1}) = (\sigma_2\sigma_1)\alpha(\sigma_2\sigma_1)^{-1}$. Also, the composition of two permutations in $S_n$ is a permutation in $S_n$, so $\sigma_2\sigma_1 \in S_n$. Hence,

$$(\sigma_2\sigma_1)\alpha(\sigma_2\sigma_1)^{-1} = \gamma \text{ implies } \alpha \sim \gamma.$$

□