Math 620: §3.8 Cosets, Normal Subgroups, and Factor Groups

Due on Monday, November 16, 2015

Boynton 10:00

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Exercise 3.8.4: For each of the subgroups $H = \{e, r^2\}$ and $K = \{e, s\}$ of D_4, list all left and right cosets.

($H = \{e, r^2\}$): Note $[D_4 : H] = |D_4|/|H| = 4$ and $r^2H = H$. Then,

$$\begin{aligned}
rH &= rr^2H = r^3H \\
sH &= sr^2H \\
srH &= srr^2H = sr^3H \\
\text{Therefore, } \mathcal{L}(H) &= \{H, rH, sH, srH\} \\
Hr^2 &= H \\
Hr &= Hr^2r = Hr^3 \\
Hs &= Hr^2s = Hsr^2 \\
Hsr &= Hr^2sr = Hsr^3 \\
\text{Therefore, } \mathcal{R}(H) &= \{H, Hr, Hs, Hsr\}
\end{aligned}$$

($K = \{e, s\}$): Note $[D_4 : K] = |D_4|/|K| = 4$ and $sK = K$. Then,

$$\begin{aligned}
rK &= rsK = sr^3K \\
r^2K &= r^2sK = sr^2K \\
srK &= srsK = r^3K \\
\text{Therefore, } \mathcal{L}(H) &= \{K, rK, r^2K, srK\} \\
Ks &= K \\
Kr &= Ksr \\
Ksr^2 &= Kssr^2 = Kr^2 \\
Ksr^3 &= Kssr^3 = Kr^3 \\
\text{Therefore, } \mathcal{R}(H) &= \{K, Kr, Ksr^2, Ksr^3\}
\end{aligned}$$

Exercise 3.8.6: Prove that if N is a normal subgroup of G, and H is any subgroup of G, then $H \cap N$ is a normal subgroup of H.

Proof. Assume N is a normal subgroup of G. Let H is any subgroup of G. By exercise 17 in section 3.2, $H \cap N$ is a subgroup of G. So $H \cap N \neq \emptyset$. Also $H \cap N \subseteq H$. Let $a, b \in H \cap N$. Since $H \cap N$ is a subgroup of G, corollary 3.2.3 implies $ab^{-1} \in H \cap N$ for all $a, b \in H \cap N$. Thus, $H \cap N$ is a subgroup of H. Next, consider any $a \in H$; then, $a \in G$. Let $x \in a(H \cap N)a^{-1}$. So, $x = ana^{-1}$ for some $n \in H \cap N$. Notice $a(H \cap N)a^{-1} = aHa^{-1} \cap aNa^{-1}$:

$$x = ana^{-1} \Leftrightarrow n \in H \text{ and } n \in N \Leftrightarrow ana^{-1} \in aHa^{-1} \text{ and } ana^{-1} \in aNa^{-1} \Leftrightarrow x \in aHa^{-1} \cap aNa^{-1}.$$

Since N is a normal subgroup of G, $aNa^{-1} \subset N$ which implies $x \in N$. Also, since $a \in H$, $aHa^{-1} \subseteq H$ so $x \in aHa^{-1}$ implies $x \in H$. Thus, $x \in H \cap N$. Hence $a(H \cap N)a^{-1} \subseteq H \cap N$ for all $a \in H$. Thus, $H \cap N$ is a normal subgroup of H by Theorem 13 of Boynton. □

Exercise 3.8.9: Let G be a finite group, and let n be a divisor of $|G|$. Show if H is the only subgroup of G of order n, then H must be normal in G.

Lemma 0.1. *For any subgroup H of G and $a \in G$, aHa^{-1} is a subgroup of G.*

Proof. Let $H \leq G$ and $a \in G$. Since $H \leq G$, H contains the identity element of G so $aea^{-1} \in aHa^{-1}$ and $aea^{-1} = aa^{-1} = e$. Thus, aHa^{-1} contains the identity element and is nonempty. Since $a \in G$ and $H \subseteq G$, $aHa^{-1} \subseteq G$. Consider $ah_1a^{-1}, ah_2a^{-1} \in H$. Then,

$(ah_1a^{-1})(ah_2a^{-1})^{-1} = ah_1a^{-1}(h_2a^{-1})^{-1}a^{-1} = ah_1a^{-1}ah_2^{-1}a^{-1} = ah_1h_2^{-1}a^{-1}$. Since $h_1h_2^{-1} \in H$,

$(ah_1a^{-1})(ah_2a^{-1})^{-1} \in aHa^{-1}$, corollary 3.2.3 implies $aHa^{-1} \leq G$. □

Proof. Let G be a finite group, and let n be a divisor of $|G|$. Assume H is the only subgroup of G of order n. By lemma 0.1 above, $aHa^{-1} \leq G$. Since $H, aHa^{-1} \leq G$ and G has finite order, H, aHa^{-1} must have finite order. Define $\phi_a : H \to aHa^{-1}$ where a is any fixed element of G and $\phi_a(x) = axa^{-1}$.

(well-defined) Since $H \leq G$, for any $x \in H$, $axa^{-1} \in aHa^{-1}$ and if $x_1 = x_2$ then $ax_1 = ax_2$ and so $ax_1a^{-1} = ax_2a^{-1}$.

(onto) Consider any $aha^{-1} \in aHa^{-1}$. Then, $h \in H$, so $\phi_a(h) = aha^{-1}$.

(1-1) Suppose $ax_1a^{-1} = ax_2a^{-1}$. Then, left and right cancellation imply $x_1 = x_2$.

Thus, ϕ_a is a bijection between two sets of finite order. Therefore, $|H| = |aHa^{-1}|$. Since H is the only subgroup of order n and $|aHa^{-1}| = n$ it must be the case that $aHa^{-1} = H$. Therefore by theorem 13(3) of Boynton, H must be normal in G. □

Exercise 3.8.10: Let N be a normal subgroup of index m in G. Show that $a^m \in N$ for all $a \in G$.

Proof. Let N be a normal subgroup of index m in G. By theorem 16 of Boynton, $|G/N| = m$. Using the identity element of G/N given in the proof of theorem 16 of Boynton, since $aN \in G/N$, we have $(aN)^m = e_{G/N} = eN = N$. By corollary 17, part a, of Boynton, $(aN)^m = a^mN$. Thus, $a^mN = N$ which implies $a^m \in N$ by corollary 4 of Boynton. □

Exercise 3.8.11: Let N be a normal subgroup of G. Show that the order of any coset aN in G/N is a divisor of $o(a)$, when $o(a)$ is finite.

Proof. Let N be a normal subgroup of G. Consider any coset aN in G/N with $o(a)$ finite. Suppose $o(a) = n$. By example 3.8.5 in Beachy, the order of aN is the smallest positive integer k such that $a^k \in N$. Notice $(aN)^n = a^nN = e_{G/N}$. Since G/N is a group, the order of aN must divide $n = o(a)$. □

Exercise 3.8.12: Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Proof. Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Consider $h \in H$ and $k \in K$. Since H, K are subgroups, $h^{-1} \in H$ and $k^{-1} \in K$. Also, H and K are normal, so for all $a \in G$, $aHa^{-1} \subset H$ and $aKa^{-1} \subset K$. Then, $H, K \subset G$, so $kh^{-1}k^{-1}, khk^{-1} \in H$ and $hk^{-1}h^{-1}, hkh^{-1} \in K$. By closure of H, K $hkh^{-1}, k^{-1} \in K$ implies $hkh^{-1}k^{-1} \in K$ and $kh^{-1}k^{-1}, h \in H$ implies $hkh^{-1}k^{-1} \in H$. Therefore $hkh^{-1}k^{-1} \in H \cap K$. Since $H \cap K = \langle e \rangle, hkh^{-1}k^{-1} = e$ and $hkh^{-1} = k$. Thus, $hk = kh$ for any $h \in H, k \in K$. $\square$

Exercise 3.8.13: Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains all elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

Proof. ($\Rightarrow$) Let N be a normal subgroup in G. Assume G/N is abelian. Then, for all $a, b \in G$, $aNbN = bNaN$ so $abN = baN$. Thus, $N = (ba)^{-1}abN = (ba)^{-1}NabN = abN(ba)^{-1}N = ab(ba)^{-1}N$. $N = ab(ba)^{-1}N$, so by corollary 4 of Boynton $ab(ba)^{-1} \in N$. Hence, $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

($\Leftarrow$) Let N be a normal subgroup of G. Assume N contains all elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$. Then, by corollary 4 of Boynton, $aba^{-1}b^{-1} = ab(ba)^{-1} \in N$ implies $ab(ba)^{-1}N = N$ and so $(ba)^{-1}N = (ab)^{-1}N$. By definition of inverse element given in proof of theorem 16 (Boynton), $(ba)^{-1}N = (baN)^{-1} = (abN)^{-1} = (ab)^{-1}N$. G/N is a group so the inverse of each element in G/N is unique, so $baN = abN$. Thus, $bNaN = aNbN$. $\square$

Exercise 3.8.14: Let N be a subgroup of the center of G. Show that if G/N is a cyclic group, then G must be abelian.

Proof. Let $N \leq Z(G)$. Then, $Z(G) \leq G$ implies $N \leq G$. Suppose G/N is a cyclic group. Then, $G/N = \langle gN \rangle$ for some $g \in G$. Let $a, b \in G$. Then, $aN = g^k N$ and $bN = g^h N$ for some $h, k \in \mathbb{Z}$. By theorem 3 part 1, $g^{-h}a, g^{-k}b \in N$. Thus, there exists $n_1, n_2 \in N$ such that $g^{-h}a = n_1$ and $g^{-k}b = n_2$. Since $n_1, n_2 \in Z(G)$ and by associativity in G,

$$ab = g^k n_1 g^h n_2 = g^k g^h n_1 n_2 = g^{k+h} n_1 n_2 = g^h g^k n_2 n_1 = g^h n_2 g^k n_1 = ba.$$

Thus, G is abelian. □

Exercise 3.8.17: Compute the factor group $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(2,2)\rangle$.

Note $N = \langle(2,2)\rangle = \{(2,2),(4,0),(0,2),(2,0),(4,2),(0,0)\}$. So $|N| = 6$ which implies $[\mathbb{Z}_6 \times \mathbb{Z}_4 : N] = \frac{24}{6} = 4$. Thus, $\mathbb{Z}_6 \times \mathbb{Z}_4/N$ will contain 4 elements. These are listed below.

$$N = \{(2,2),(4,0),(0,2),(2,0),(4,2),(0,0)\}$$

$$(1,1) + N = \{(3,3),(5,1),(1,3),(3,1),(5,3),(1,1))\}$$

$$(1,0) + N = \{(3,2),(5,0),(1,2),(3,0),(5,2),(1,0)\}$$

$$(0,1) + N = \{(2,3),(4,1),(0,3),(2,1),(4,3),(0,1)\}$$

Since $|\mathbb{Z}_6 \times \mathbb{Z}_4/N| = 4$ but does not contain an element of order 4, $\mathbb{Z}_6 \times \mathbb{Z}_4/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise 3.8.18: Compute the factor group $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3,2)\rangle$.

Note $N = \langle(3,2)\rangle = \{(3,2),(0,0)\}$. So $|N| = 2$ which implies

$[\mathbb{Z}_6 \times \mathbb{Z}_4 : N] = \dfrac{24}{2} = 12$. Thus, $\mathbb{Z}_6 \times \mathbb{Z}_4/N$ will contain 12 elements. These are listed below.

$$N = (3,2)+N = \{(3,2),(0,0)\} \qquad (1,1)+N = \{(4,3),(1,1)\}$$

$$(1,0)+N = \{(4,2),(1,0)\} \qquad (0,1)+N = \{(3,3),(0,1)\}$$

$$(2,0)+N = \{(5,2),(2,0)\} \qquad (0,2)+N = \{(3,0),(0,2)\}$$

$$(2,2)+N = \{(5,0),(2,2)\} \qquad (3,1)+N = \{(0,3),(3,1)\}$$

$$(2,3)+N = \{(5,1),(2,3)\} \qquad (1,2)+N = \{(4,0),(1,2)\}$$

$$(2,1)+N = \{(5,3),(2,1)\} \qquad (1,3)+N = \{(4,1),(1,3)\}$$

Notice $(2,3)+N$ has order 12 which implies $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3,2)\rangle$ is cyclic.

Then, $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3,2)\rangle \cong \mathbb{Z}_{12}$.

Exercise 3.8.20: Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle$ is an infinite cyclic group.

Proof. Define $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $\phi((a,b)) = a - b$. We will use ϕ and the fundamental homomorphism theorem to show $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle \cong \mathbb{Z}$. First, we will show ϕ is a group homomorphism. Addition in $\mathbb{Z}$ is closed, so ϕ satisfies WD1. ϕ also satisfies WD2: consider $(a_1,b_1),(a_2,b_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a_1,b_1) = (a_2,b_2)$. Then, $a_1 = a_2$ and $b_1 = b_2$. Subtracting the second equality from the first we obtain, $a_1 - b_1 = a_2 - b_2$ which implies $\phi((a_1,b_1)) = \phi((a_2,b_2))$. Thus, ϕ is a function. Now consider any $(a_1,b_1),(a_2,b_2) \in \mathbb{Z} \times \mathbb{Z}$, notice ϕ is a group homomorphism:

$$\phi((a_1,b_1)+(a_2,b_2)) = \phi((a_1+a_2,b_1+b_2)) = a_1+a_2-(b_1+b_2) = a_1-b_1+a_2-b_2 = \phi((a_1,b_1))+\phi((a_2,b_2)).$$

Finally, we will show $\ker\phi = \langle(1,1)\rangle$. Suppose $x \in \ker\phi$. Then, $x \in \mathbb{Z} \times \mathbb{Z}$, so $x = (a,b)$ for some $a,b \in \mathbb{Z}$. If $x \in \ker\phi$, then $\phi(x) = e_{\mathbb{Z}} = 0$. Thus, $\phi((a,b)) = 0$ so $a - b = 0$. Thus, if $(a,b) \in \ker\phi$, $a = b$. Note that $\langle(1,1)\rangle = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a = b\}$. Thus, $x = (a,b) \in \langle(1,1)\rangle$. Suppose $x \in \langle(1,1)\rangle$. Then $x = (a,b) \in \mathbb{Z} \times \mathbb{Z}$ with $a = b$ so $a - b = 0$. Therefore $\phi((a,b)) = 0$ which implies $x \in \ker\phi$. Hence, $\ker\phi = \langle(1,1)\rangle$.
Next, we will show ϕ is onto. Consider any $z \in \mathbb{Z}$. Then, $(z+1,1) \in \mathbb{Z} \times \mathbb{Z}$ and $\phi(z+1,1) = z+1-1 = z$. Thus, ϕ is onto and so $\phi(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z}$.
By, the fundamental homomorphism theorem, $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle \cong \phi(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z}$. $\mathbb{Z} = \langle 1 \rangle$ is an infinite cyclic group. By proposition 3.4.3, $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle$ must be an infinite cyclic group. □

Exercise 3.8.21: Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(2,2)\rangle$ is not a cyclic group.

Proof. Define $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}_2$ by $\phi((a,b)) = (a - b, b \pmod 2)$. We will use ϕ and the fundamental homomorphism theorem to show $(\mathbb{Z} \times \mathbb{Z})/\langle(2,2)\rangle \cong \mathbb{Z} \times \mathbb{Z}_2$. First, we will show ϕ is a group homomorphism. Addition in $\mathbb{Z}$ is closed and $b \pmod 2 \in \mathbb{Z}_2$, so ϕ satisfies WD1. ϕ also satisfies WD2: consider $(a_1, b_1), (a_2, b_2) \in \mathbb{Z} \times \mathbb{Z}_2$ such that $(a_1, b_1) = (a_2, b_2)$. Then, $a_1 = a_2$ and $b_1 = b_2$. Subtracting the second equality from the first we obtain, $a_1 - b_1 = a_2 - b_2$. If $b_1 = b_2$, $b_1 \pmod 2 = b_2 \pmod 2$. which implies $\phi((a_1, b_1)) = (a_1 - b_1, b_1 \pmod 2) = (a_2 - b_2, b_2 \pmod 2) = \phi((a_2, b_2))$. Thus, ϕ is a function. Now consider any $(a_1, b_1), (a_2, b_2) \in \mathbb{Z} \times \mathbb{Z}$, notice ϕ is a group homomorphism: $\phi((a_1, b_1) + (a_2, b_2)) = \phi((a_1 + a_2, b_1 + b_2)) = (a_1 + a_2 - (b_1 + b_2), (b_1 + b_2) \pmod 2) =$ $(a_1 - b_1 + a_2 - b_2, b_1 \pmod 2 + b_2 \pmod 2) = (a_1 - b_1, b_1 \pmod 2) + (a_2 - b_2, b_2 \pmod 2)$ $= \phi((a_1, b_1)) + \phi((a_2, b_2))$.

Finally, we will show $\ker\phi = \langle (2, 2) \rangle$. Suppose $(a, b) \in \ker\phi$. Then, $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $\phi((a, b)) = (0, 0)$. Thus, $a - b = 0$ and $b \pmod 2 = 0$. Thus, if $(a, b) \in \ker\phi$, $a = b$. And because $b \pmod 2 = 0$, both a, b must be multiples of 2. Note that $\langle (2, 2) \rangle = \{2(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a = b\}$. Thus, $(a, b) \in \langle (2, 2) \rangle$. Suppose $(a, b) \in \langle (2, 2) \rangle$. Then $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ with $a = b$ so $a - b = 0$. Also, $(a, b) \in \langle (2, 2) \rangle$ implies $b \pmod 2 = 0$. Therefore $\phi((a, b)) = (0, 0)$ which implies $(a, b) \in \ker\phi$. Hence, $\ker \phi = \langle (2, 2) \rangle$.

Next, we will show ϕ is onto. Consider any $(a, b) \in \mathbb{Z} \times \mathbb{Z}_2$. Note $b = 0, 1$. First consider $(a, 0)$. Notice $(a, 0) \in \mathbb{Z} \times \mathbb{Z}$ and $\phi(a, 0) = (a - 0, 0 \pmod 2) = (a, 0)$. Next, consider $(a, 1)$. Then, $(a + 1, 1) \in \mathbb{Z} \times \mathbb{Z}$ and $\phi(a + 1, 1) = (a + 1 - 1, 1 \pmod 2) = (a, 1)$. Thus, ϕ is onto and so $\phi(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2$.

By the fundamental homomorphism theorem, $(\mathbb{Z} \times \mathbb{Z}) / \langle (2, 2) \rangle \cong \phi(\mathbb{Z} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2$. We will show $\mathbb{Z} \times \mathbb{Z}_2$ is not a cyclic group. Suppose $\mathbb{Z} \times \mathbb{Z}_2$ is cyclic. Then there must be $(a, b) \in \mathbb{Z} \times \mathbb{Z}_2$ such that $\mathbb{Z} \times \mathbb{Z}_2 = \langle (a, b) \rangle$. Since $(1, 0) \in \mathbb{Z} \times \mathbb{Z}_2$, $(1, 0) = k(a, b)$ so $ka = 1$ and $kb = 0$ implies $b = 0$. Thus, the generator must have the form $(a, 0)$. But, $(1, 1)$ is also in $\mathbb{Z} \times \mathbb{Z}_2$. So, there must be some k such that $k(a, 0) = (1, 1)$. No such k exists. Thus, $\mathbb{Z} \times \mathbb{Z}_2$ is not cyclic. By proposition 3.4.3, $(\mathbb{Z} \times \mathbb{Z}) / \langle (2, 2) \rangle$ is not a cyclic group.

$\square$

Exercise 3.8.24: Let S be an infinite set. Let H be the set of all elements $\sigma \in$ Sym(S) such that $\sigma(x) = x$ for all but finitely many $x \in S$. Prove that the subgroup H is normal in Sym(S).

Proof. Let S be an infinite set. Let H be the set of all elements $\sigma \in \mathrm{Sym}(S)$ such that $\sigma(x) = x$ for all but finitely many $x \in S$. Consider any $\sigma \in H$ and let $A = \{a_1, a_2, \ldots, a_n\}$ be the set of $x \in S$ such that $\sigma(x) \neq x$. Note A is finite since $\sigma \in H$. Next, consider any $\tau \in \mathrm{Sym}(S)$ and let $\tau(A) = \{\tau(a_1), \tau(a_2), \ldots, \tau(a_n)\}$.

Claim: $\tau\sigma\tau^{-1}(x) = x$ if and only if $x \notin \{\tau(a_1), \tau(a_2), \ldots, \tau(a_n)\}$.

Suppose $x \in \{\tau(a_1), \tau(a_2), \ldots, \tau(a_n)\}$. Then, $x = \tau(a_j)$ and $\tau\sigma\tau^{-1}(\tau(a_j)) = \tau\sigma(a_j)$. Note $\sigma(a_j) = a_k$ and $k \neq j$ and so $\tau(a_k) \in \tau(A)$. Thus, $\tau\sigma\tau^{-1}(\tau(a_j)) = \tau(a_k)$. Since τ is one to one, $\tau(a_j) \neq \tau(a_k)$. Thus, if $x \in \{\tau(a_1), \tau(a_2), \ldots, \tau(a_n)\}$, $\tau\sigma\tau^{-1}$ moves x on the finite set $\tau(A)$.

Suppose $x \notin \{\tau(a_1), \tau(a_2), \ldots, \tau(a_n)\}$. Suppose $\tau\sigma\tau^{-1}(x) = y$ with $x \neq y$. Then, $\sigma\tau^{-1}(x) = \tau^{-1}(y)$. Since τ is one to one $\tau^{-1}(x) \neq \tau^{-1}(y)$ which implies σ moves $\tau^{-1}(x)$. Thus, $\tau^{-1}(x) \in A$ and so $\tau^{-1}(x) = a_j$. Thus, $x = \tau(a_j)$ implies $x \in \tau(A)$ which contradicts our assumption that $x \notin \{\tau(a_1), \tau(a_2), \ldots, \tau(a_n)\}$. Hence, $\tau\sigma\tau^{-1}$ must fix x when $x \notin \{\tau(a_1), \tau(a_2), \ldots, \tau(a_n)\}$.

Thus, $\tau\sigma\tau^{-1}$ fixes all x except when $x \in \tau(A)$ which implies $\tau\sigma\tau^{-1} \in H$. Hence H is normal in $\mathrm{Sym}(S)$.

$\square$