August 26 In Class Exercises Note: these solutions were provided by Professor Duncan and expanded upon by author.

Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of sets. We define $\limsup_{k\to\infty} A_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} A_k\right) and \liminf_{k\to\infty} A_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} A_k\right).$

enumerate

P rove that $\limsup A_k = \{x : x \in A_k for infinitely many k\} and that \liminf A_k = \{x : there is a j such that x \in A_l for all l \geq j\}$.

proof Notice that if x is in infinitely many of the A_k then given any j there is $k_0 \geq j$ such that $x \in A_{k_0}$ and hence $x \in \cup_{j=1}^{\infty} A_j$ so that $x \in \limsup A_k$. If x is in only finitely many of the A_j then there is some j_0 such that $x \notin A_k$ for any $k \geq j_0$. Then $x \notin \cup_{i=j_0}^{\infty} A_i$ so that $x \notin \limsup A_k$.

If there is some j such that $x \in A_l$ for any $l \geq j$ then $x \in \cap_{j=k}^{\infty} A_k$ and hence $x \in \liminf A_k$. If on the other hand if for any j there is $k_0 > j$ such that $x \notin A_{k_0}$ then $x \notin \cap_{k=j}^{\infty} A_k$ for any j and hence $x \notin \liminf A_k$.