REAL ANALYSIS NOTEBOOK

K. GRAY

Contents

1. Section 1.2 (Folland): σ-algebras

1.1. Section 1.2: Definitions and Theorems.

Definition 1 (algebra)**.** An algebra A is a nonempty collection of subsets of a nonempty set X which is closed under finite unions and complements.

Example. If $X = \{1, 2, 3\}$ then the following are algebras:

- $\{\emptyset, X\}$,
- $\{\emptyset, \{1\}, \{2, 3\}, X\}$,
- $\{\emptyset, \{2\}, \{1, 3\}, X\}$,
- $\{\emptyset, \{3\}, \{1, 2\}, X\}$, and
- $P(X)$.

Any other subset of the power set is not an algebra.

Definition 2 (σ-algebra)**.** An algebra A is a σ-algebra if it is closed under countable unions.

Example. Any algebra over a finite set is a σ-algebra, as is the power set of the set X. More examples are found in exercises.

Definition 3 (co-countable sets)**.** a cocountable subset of a set X is a subset Y whose complement in X is a countable set. In other words, Y contains all but countably many elements of X. While the rational numbers are a countable subset of the reals, for example, the irrational numbers are a cocountable subset of the reals.

Definition 4 (σ-algebra of co-countable or countable sets)**.** The σ-algebra of co-countable or countable sets is the collection of subsets such that either the subset or its complement is countable.

Proposition 1. Given any nonempty collection $\mathcal{E}$ in the power set of X, there is a smallest σ-algebra that contains $\mathcal{E}$.

Proof. The idea of the proof is to show that an arbitrary intersection of σ-algebras is a σ-algebra. One then considers the collection of σ-algebras that contain $\mathcal{E}$, which is nonempty since the power set is such a set, and then take the intersection of all such sets. $\square$

Lemma 1. If a collection of sets $\mathcal{E}$ is contained in a σ-algebra $\mathcal{B}$ then the σ-algebra generated by $\mathcal{E}$ is contained in $\mathcal{B}$.

Definition 5 (σ-algebra generated by $\mathcal{E}$)**.** If $\mathcal{E}$ is a subset of $P(X)$ then the σ-algebra generated by $\mathcal{E}$ is the smallest σ-algebra that contains $\mathcal{E}$.

Definition 6 (Borel σ-algebra)**.** The Borel σ-algebra for a metric space X (the σ-algebra of Borel sets) is the σ-algebra generated by open sets.

Definition 7 (G_δ, F_σ)**.** A set in a metric space is a G_δ set if it is the countable intersection of open sets, it is an F_σ set if it the countable union of a closed set.

Example. $[0,1)$ is an F_σ set since $[0,1) = \cup[0, 1-\frac{1}{n}]$. Any singleton is a G_δ set, it is also an F_σ set.

Proposition 2. Each of the following sets generates the the Borel sets in $\mathbb{R}$:

- open intervals,
- closed intervals,
- half open intervals,
- open rays,
- closed rays.

Definition 8 (product σ-algebra)**.** If M_α is a indexed collection of σ-algebras on an indexed collections of sets X_α, then the product σ-algebra on the product space $X = \prod X_\alpha$ is the σ algebra generated by the inverse image of elements of M_α under the coordinate maps π_α.

The σ-algebra of subsets of $X \times Y$ generated by the semi-algebra $\mathcal{R}$ is called the product σ-algebra and is denoted by $\mathcal{A} \otimes \mathcal{B}$.

Example. If X_1, X_2 are the indexed set with associated σ-algebras M_1, M_2 then the inverse image of the coordinate maps are sets of the form $E_1 \times X_2$ and $X_1 \times F_2$, where $E_1 \in M_1$ and $F_2 \in M_2$. Notice that using intersections you get all sets of the form $E_1 \times F_2$.

Proposition 3. The product σ-algebra (in the case that the index set is countable) is generated by all products of things in the M_i's

Proposition 4. If each M_α is generated by a set E_α then the product sigma algebra is generated by elements of the E_α, rather than arbitrary elements of M_α.

Theorem 1. If $X_1, X_2, \cdots, X_n$ are metric spaces then the Borel sets on the product space is the product of the Borel sets on the individual spaces.

Proof. The proof uses the preceding proposition. $\square$

Definition 9 (elementary family)**.** An elementary family is a collection of subsets that contains the empty set, is closed under finite intersections, and the complement of any set in the collection is a finite disjoint union of elements of the collection.

Example. Any σ-algebra is an elementary family. In general the point of elementary families is starting with minimal information how does one build a σ-algebra, vs. knowing that there is a minimal σ-algebra.

Proposition 5. The finite disjoint union of members of an elementary family is an algebra.

1.2. **August 26 Group Assignment.** Note: solutions for the August 26th assignment were provided by Professor Duncan and expanded upon by author.

Let $\{A_k\}_{k=1}^\infty$ be a sequence of sets. We define

$$\limsup_{k\to\infty} A_k = \bigcap_{j=1}^\infty \left(\bigcup_{k=j}^\infty A_k \right) \qquad \text{and} \qquad \liminf_{k\to\infty} A_k = \bigcup_{j=1}^\infty \left(\bigcap_{k=j}^\infty A_k \right).$$

(1) Prove that $\limsup_{k\to\infty} A_k = \{x : x \in A_k \text{ for infinitely many } k\}$.

Proof. Let $E = \{x : x \in A_k \text{ for infinitely many } k\}$. To show $\limsup\limits_{k\to\infty} A_k = E$, we will show $\limsup\limits_{k\to\infty} A_k \subseteq E$ and $\limsup\limits_{k\to\infty} A_k \supseteq E$. First, let's show $\limsup\limits_{k\to\infty} A_k \subseteq E$. Let $x \in \limsup\limits_{k\to\infty} A_k$. Suppose for the purposes of contradiction that $x \notin E$. If $x \notin E$, then it is not the case that x is in infinitely many sets A_k. If x is not in infinitely many sets A_k, then there must exist some j_0 such that for all $k \geq j_0$, $x \notin A_k$. Then, $x \notin \bigcup_{k=j_0}^\infty A_k$. Thus, $x \notin \limsup\limits_{k\to\infty} A_k$ which contradicts our assumption that $x \in \limsup\limits_{k\to\infty} A_k$. Hence, if $x \in \limsup\limits_{k\to\infty} A_k$, $x \in E$.

$\square$

Prove that $\liminf_{k\to\infty} A_k = \{x : \text{ there is a } j \text{ such that } x \in A_l \text{ for all } l \geq j\}$.

Proof. If there is some j such that $x \in A_l$ for any $l \geq j$ then $x \in \cap_{j=k}^{\infty} A_k$ and hence $x \in \liminf A_k$. If on the other hand if for any j there is $k_0 > j$ such that $x \notin A_{k_0}$ then $x \notin \cap_{k=j}^{\infty} A_k$ for any j and hence $x \notin \liminf A_k$.

$\square$

(2) Prove that $\liminf A_k \subseteq \limsup A_k$.

Proof. If $x \in \liminf A_k$ then there is some j such that $x \in A_l$ for any $l \geq j$ then x is in infinitely many of the A_j and hence $x \in \limsup A_k$. $\square$

(3) Prove that $\limsup A_k = \liminf A_k = \cap A_k$ if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$.

Proof. We have that $\cup_{j=k}^{\infty} A_k = A_j$ since the sequence is nested. It follows that $\limsup A_k = \cap_{j=1}^{\infty} A_j \subseteq \liminf A_k$. $\square$

(4) Give examples of sequences of sets that satisfy the following :

i. $\limsup A_k = \emptyset$.
$X = \mathbb{N}$ and $A_k = \{k\}$.

ii. $\liminf A_k = \emptyset$.
$X = \mathbb{R}$ and $A_k = [k, k+1]$.

iii. $\liminf A_k \neq \emptyset$ and $\limsup A_k \neq \liminf A_k$.
Let $X = \mathbb{R}$ and let

$$A_k = \begin{cases} \{0,1\} & \text{if } k \text{ is even} \\ \{0\} & \text{otherwise} \end{cases}.$$

Then, $\limsup A_k = \{0,1\}$ and $\liminf A_k = \{0\}$.

1.3. **August 28 Group Assignment.**

(1) Show that the intersection of two σ-algebras is a σ-algebra.

Proof. Let $\mathcal{M}, \mathcal{A}$ be any σ-algebras on some set X. We will show $\mathcal{M} \cap \mathcal{A}$ is a σ-algebra. Because $\mathcal{M}$ and $\mathcal{A}$ are σ-algebras, $X \in \mathcal{M}, \mathcal{A}$ and $\emptyset \in \mathcal{M}, \mathcal{A}$. Therefore $X, \emptyset \in \mathcal{M} \cap \mathcal{A}$, so $\mathcal{M} \cap \mathcal{A}$ is nonempty. Consider some $E \in \mathcal{M} \cap \mathcal{A}$. Then $E \in \mathcal{M}$ and $E \in \mathcal{A}$. Since $\mathcal{M}, \mathcal{A}$ are σ-algebras, $E \in \mathcal{M}$ and $E \in \mathcal{A}$ imply $E^c \in \mathcal{M}$ and $E^c \in \mathcal{A}$. Thus, $E^c \in \mathcal{M} \cap \mathcal{A}$. Consider $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M} \cap \mathcal{A}$. Then, for all i, $E_i \in \mathcal{M}$ and $E_i \in \mathcal{A}$. $\mathcal{M}$ and $\mathcal{A}$ are closed under countable unions, so $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$ and $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$. Hence, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M} \cap \mathcal{A}$. Thus the intersection of any two σ-algebras is a σ-algebra. $\square$

(2) Show that an algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable increasing unions.

Proof. Assume $\mathcal{A}$ is a σ-algebra. Then, $\mathcal{A}$ is closed under countable unions so $\mathcal{A}$ must be closed under countable increasing unions.

Let $\mathcal{A}$ be an algebra on some set X that is closed under countable increasing unions. Then, consider a countable collection of sets in $\mathcal{A}$, namely $\{E_i\}_{i=1}^{\infty}$. To show $\mathcal{A}$ is a σ-algebra we must show $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$. Define $F_n = \bigcup_{j=1}^{n} E_j$. Since $\mathcal{A}$ is an algebra, $\mathcal{A}$ is closed under finite unions, so $F_n \in \mathcal{A}$ for all n. Further, $\{F_j\}_{j=1}^{\infty} \subset \mathcal{A}$. Additionally, notice $F_1 = E_1$, $F_2 = E_1 \cup E_2$, $F_3 = E_1 \cup E_2 \cup E_3, \cdots$ so $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$. Thus, $\{F_j\}_{j=1}^{\infty}$ is an increasing collection of sets. Because $\mathcal{A}$ is closed under countable increasing unions, $\bigcup_{j=1}^{\infty} F_j \in \mathcal{A}$. But $\bigcup_{j=1}^{\infty} F_j = \bigcup_{i=1}^{\infty} E_i$. Thus, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$ so we have shown $\mathcal{A}$ is a σ-algebra. $\square$

(3) Prove that the collection of countable or co-countable sets is a σ-algebra.

Proof. Let $\mathcal{A} = \{E \subset X : E \text{ is countable or } E^c \text{ is co-countable}\}$. We will show that $\mathcal{A}$ is a σ-algebra on X. Since $|\emptyset| = 0$, $\emptyset$ is countable which implies $\emptyset \in \mathcal{A}$. Also, $X^c = \emptyset$, so $X \in \mathcal{A}$. Consider any $A \in \mathcal{A}$. Then, either A is countable or A^c is countable. If A is countable, $(A^c)^c$ is countable, so $A^c \in A$. If A^c is countable, then $A^c \in \mathcal{A}$.

Now, consider $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$. To show $\mathcal{A}$ is a σ-algebra we must show $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. If for all $i \geq 1, A_i$ is countable, then $\bigcup_{i=1}^{\infty} A_i$ is countable since the countable union of countable sets is countable. Thus, since $\bigcup_{i=1}^{\infty} A_i$ is countable, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Suppose there exists a least one $A_k \in \mathcal{A}$ for that is not countable. Then, $(A_k)^c$ is countable. Also, $(\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty}(A_i)^c \subset (A_k)^c$. Since $(A_k)^c$ is countable, $(\bigcup_{i=1}^{\infty} A_i)^c$ must be countable. Thus, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

$\square$

(4) Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of sets in a σ-algebra, $\mathcal{A}$. Prove there exist disjoint sets $\{F_k\}_{k=1}^{\infty}$ in $\mathcal{A}$ such that $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} F_k$.

Proof. Consider a countable collection of sets in a σ-algebra $\mathcal{A}$, $\{A_k\}_{k=1}^{\infty}$. Let

$$F_k = A_k \backslash \left[\bigcup_{j=1}^{k-1} A_j \right]. \text{ So,}$$

$$F_1 = A_1, \quad F_2 = A_2 \backslash A_1, \quad F_3 = A_3 \backslash (A_1 \cup A_2) = A_3 \cap A_1^c \cap A_2^c, \quad \cdots, \quad F_k = A_k \bigcap \left[\bigcup_{j=1}^{k-1} A_j \right]^c.$$

Thus all F_k are disjoint and $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} F_k$. □

1.4. **August 31 Group Assignment.**

(1) Prove proposition 1.2 from Folland (p. 22): $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following: the open intervals: $\mathcal{E}_1 = \{(a, b) : a < b\}$, the closed intervals: $\mathcal{E}_2 = \{[a, b] : a < b\}$, the half-open intervals: $\mathcal{E}_3 = \{(a, b] : a < b\}$ or $\mathcal{E}_4 = \{[a, b) : a < b\}$, the open rays: $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$, the closed rays: $\mathcal{E}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Proof. Let $\mathcal{O}$ denote the collection of all open intervals in $\mathbb{R}$ so that $\mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$. For all $j, j \neq 3, 4$, the elements of $\mathcal{E}_j$ are Borel sets so $\mathcal{E}_j \subset \mathcal{B}_{\mathbb{R}}$ implies $\mathcal{M}(\mathcal{E}_j) \subset \mathcal{B}_{\mathbb{R}}$. So we will prove $\mathcal{M}(\mathcal{E}_j) \supset \mathcal{B}_{\mathbb{R}}$ for all $j \neq 3, 4$. It will suffice to show $(a, b) \in \mathcal{M}(\mathcal{E}_j)$ for any $a, b \in \mathbb{R}$ with $a < b$ since this implies $\mathcal{M}((a, b)) \subset \mathcal{M}(\mathcal{E}_j)$ and so $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_j)$ as desired.

$\mathcal{E}_1$ Every open set in $\mathbb{R}$ is at most a countable union of open intervals, so for $\mathcal{E}_1 = \{(a, b) : a < b\}$, $\mathcal{M}(\mathcal{E}_1) \supset \mathcal{B}_{\mathbb{R}}$.

$\mathcal{E}_2$ Prove the closed intervals: $\mathcal{E}_2 = \{[a, b] : a < b\}$, generate $\mathcal{B}_{\mathbb{R}}$. Notice

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n} \right]. \text{ Thus, } (a, b) \in \mathcal{M}(\mathcal{E}_2).$$

$\mathcal{E}_3$ Prove the $\mathcal{E}_3 = \{(a,b] : a < b\}$ generate $\mathcal{B}_{\mathbb{R}}$. Notice

$$(a,b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right]. \text{ Thus, } (a,b) \in \mathcal{M}(\mathcal{E}_3). \text{ So } \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_3)$$

Also, notice

$$(a,b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right). \text{ Thus, } (a,b] \in \mathcal{B}_{\mathbb{R}}. \text{ So } \mathcal{M}(\mathcal{E}_3) \subset \mathcal{B}_{\mathbb{R}}. \text{ Therefore, } \mathcal{M}(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}}.$$

$\mathcal{E}_4$ Prove $\mathcal{E}_4 = \{[a,b) : a < b\}$, Notice

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a - \frac{1}{n}, b\right). \text{ Thus, } (a,b) \in \mathcal{M}(\mathcal{E}_4. \text{ So } \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_4).$$

Also, notice

$$[a,b) = \bigcap_{n=1}^{\infty} \left(a + \frac{1}{n}, b\right). \text{ Thus, } [a,b) \in \mathcal{B}_{\mathbb{R}}. \text{ So } \mathcal{M}(\mathcal{E}_4) \subset \mathcal{B}_{\mathbb{R}}. \text{ Therefore, } \mathcal{M}(\mathcal{E}_4) = \mathcal{B}_{\mathbb{R}}.$$

$\mathcal{E}_5$ Prove the open rays: $\mathcal{E}_5 = \{(a,\infty) : a \in \mathbb{R}\}$ generates $\mathcal{B}_{\mathbb{R}}$. Notice

$$(a,b) = \bigcup_{n=1}^{\infty} \left(a + \frac{1}{n}, b - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} \left\{ \left(a + \frac{1}{n}, \infty\right) \cap \left(b - \frac{1}{n}, \infty\right)^c \right\}$$

implies $(a,b) \in \mathcal{M}(\mathcal{E}_5)$. Thus, $\mathcal{O} \subset \mathcal{M}(\mathcal{E}_5)$ which implies $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_5)$.

$\mathcal{E}_6$ Prove $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$ generates $\mathcal{B}_{\mathbb{R}}$. Notice

$$(a,b) = \bigcup_{n=1}^{\infty} \left(a + \frac{1}{n}, b - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} \left\{ \left(a + \frac{1}{n}, \infty\right) \cap \left(b - \frac{1}{n}, \infty\right)^c \right\}$$

implies $(a,b) \in \mathcal{M}(\mathcal{E}_6)$. Thus, $\mathcal{O} \subset \mathcal{M}(\mathcal{E}_6)$ which implies $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_5)$.

$\mathcal{E}_7$ Prove the closed rays $\mathcal{E}_7 = \{[a,\infty) : a \in \mathbb{R}\}$ generates $\mathcal{B}_{\mathbb{R}}$. Notice

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a - \frac{1}{n}, b + \frac{1}{n}\right) = \bigcup_{n=1}^{\infty} \left\{ \left[a - \frac{1}{n}, \infty\right) \cap \left[b + \frac{1}{n}, \infty\right)^c \right\}$$

implies $(a,b) \in \mathcal{M}(\mathcal{E}_7)$. Thus, $\mathcal{O} \subset \mathcal{M}(\mathcal{E}_7)$ which implies $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_7)$.

$\mathcal{E}_8$ Prove $\mathcal{E}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$ generates $\mathcal{B}_{\mathbb{R}}$. Notice

$$(a,b) = \bigcup_{n=1}^{\infty} \left(a + \frac{1}{n}, b - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} \left\{ \left(-\infty, b - \frac{1}{n}\right] \cap \left(-\infty, a + \frac{1}{n}\right]^c \right\}$$

implies $(a,b) \in \mathcal{M}(\mathcal{E}_8)$. Thus, $\mathcal{O} \subset \mathcal{M}(\mathcal{E}_8)$ which implies $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_8)$.

□

1.5. **September 2 Group Assignment.** For the following problems, let X be a nonempty set and let $E \subseteq \mathcal{P}(X)$. Note: I used the following resource on these problems: `http://math.stackexchange.com/questions/612266/on-sigma-algebra-generated-by-mathcale`.

(1) Let $\mathcal{F}$ denote the collection of countable subsets of E. If $F \in \mathcal{F}$ prove that the σ-algebra generated by F is contained in the σ-algebra generated by E.

Proof. Let $F \in \mathcal{F}$, then $F \subset E$. Thus, by lemma 1.1, $\mathcal{M}(F) \subset \mathcal{M}(E)$. □

(2) We will denote by M_F as the σ-algebra generated by F. Prove that $N := \bigcup_{F\in\mathcal{F}} M_F$ is a σ-algebra.

Proof. $F \in N$ so $N \neq \emptyset$. Suppose $Y \in N$. Then, $Y \in \mathcal{M}_F$. Since $\mathcal{M}_F$ is a σ-algebra, $Y^c \in \mathcal{M}_F \subset N$. Thus, $Y^c \in N$. Consider $\{Y_i\}_{i=1}^{\infty} \subset N$. Then, $Y_i \in M_{F_i}$ for some i. Consider $K = \bigcup_{i=1}^{\infty} F_i$. Notice K is a countable union of countable subsets of E, so K is a countable subset of E and so $K \subset \mathcal{F}$ and $K \in M_{\mathcal{F}} \subset N$. Then, for all F_i, $F_i \subset K$ so by lemma 1.1 in Rolland, $\mathcal{M}_{F_i} \subset \mathcal{M}_K \subset N$. Recall for all $Y_i \in \{Y_i\}_{i=1}^{\infty}$, there exist F_i such that $Y_i \in M_{F_i}$. Thus, for all i, $Y_i \in N$, so $\bigcup_{i=1}^{\infty} Y_i \subset N$. Hence, N is closed under countable unions. □

(3) Prove that E is contained in N and hence the σ-algebra generated by E is contained in N.

Proof. Let $x \in E$. Then, consider a countable sub collection of E that contains x, namely, S_x, such that $S_x = \bigcup_{x\in S\subset E} x$. Then, since S_x is a countable subset of E, $S_x \in \mathcal{F}$, so $S_x \subset N := \bigcup_{F\in\mathcal{F}} M_F$. Notice $\bigcup_{x\in E} S_x = E$. From part (2), we know N is a σ-algebra, so if $S_x \subset N$, $\bigcup_{x\in E} S_x \subset N$ and so $E \subset N$. Thus, $\mathcal{M}(E) \subset N$. □

(4) Conclude that N is the σ-algebra generated by E.

Proof. From (3), we know $\mathcal{M}(E) \subset N$. So, we will show $N \subset \mathcal{M}(E)$. From part (1), we know for any $F \in \mathcal{F}$, $\mathcal{M}(F) \subset \mathcal{M}(E)$. Thus, $\bigcup_{F\in\mathcal{F}} M_F \subset \mathcal{M}(E)$. So, $N \subset \mathcal{M}(E)$. Hence, $N = \mathcal{M}(E)$, so N is the σ-algebra generated by E. □

1.6. **September 4 Group Assignment.**

(1) Prove that any σ-algebra forms a monotone class.

Proof. Consider some σ-algebra $\mathcal{A}$. Then, $\mathcal{A}$ is non-empty, closed under complements, and closed under countable unions. Since $\mathcal{A}$ is closed under arbitrary countable unions, $\mathcal{A}$ is closed under increasing countable unions. Also, since $\bigcap_j E_j = \left(\bigcup_j (E_j)^c\right)^c$, $\mathcal{A}$ is closed under arbitrary countable intersections. Thus, $\mathcal{A}$ is closed under decreasing countable intersections and so $\mathcal{A}$ is a monotone class. □

(2) Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be a collection of monotone classes on a space X. Prove that $\bigcap_{\lambda\in\Lambda} E_\lambda$ is a monotone class on X.

Proof. Consider $\{Y_i\}_{i=1}^\infty \subset \bigcap_{\lambda\in\Lambda} E_\lambda$. Then, for all λ, $\{Y_i\}_{i=1}^\infty \subset E_\lambda$ so, $\bigcup_{i=1}^\infty Y_i \subset E_\lambda$ for all $\lambda \in \Lambda$. Thus, $\{E_\lambda\}_{\lambda\in\Lambda}$ is closed under countable increasing unions.
Similarly, for all λ, $\{Y_i\}_{i=1}^\infty \subset E_\lambda$ so, $\bigcap_{i=1}^\infty Y_i \subset E_\lambda$ for all $\lambda \in \Lambda$. Thus, $\{E_\lambda\}_{\lambda\in\Lambda}$ is closed under countable decreasing intersections. □

(3) Prove that given any subset of the power set of X there is a unique smallest monotone class containing the given subset.

Proof. Suppose there exist two smallest monotone classes M, N. From (2) we know $M \cap N$ must be a monotone class. Then, since M, N are the smallest monotone classes, $M \subset M \cap N$ and $N \subset M \cap N$. But, $M \cap N \subset M$ and $M \cap N \subset N$ which implies $M = N$ and also that $M \cap N$ is a smaller monotone class. Thus, there must exist a unique smallest monotone class. □

1.7. **September 9 Group Assignment.**

(1) Let M, N be σ-algebras on X, Y respectively. We define a rectangle to be a set $A \times B \subseteq X \times Y$ such that $A \in M$ and $B \in N$.

(a) Prove that the σ-algebra generated by the set of rectangles is equal to $M \otimes N$.

Proof. Let R denote the set of rectangles. Note $M \otimes N =$ set of all sets in $M \times Y \cup X \times$ set of all sets in N. We will show the σ-algebra generated by R, $\mathcal{R} = M \otimes N$. First, show $\mathcal{R} \subset M \otimes N$. For any $A \times B \in R, A \times B \subseteq X \times Y$ with $A \in M$ and $B \in N$. Then, $A \times B = (A \times Y) \cap (X \times B) \in M \otimes N$. Next, show $\mathcal{R} \supset M \otimes N$. Note $M \otimes N =$ set of all sets in $M \times Y \cup X \times$ set of all sets in N. So for any $M_i \times N_i \in M \otimes N$, $M_i \times N_i = (A \times Y) \cap (X \times B)$ for some $A \in M, B \in N$. Then, $M_i \times N_i = A \times B$ for some $A \in M$ and $B \in N$, so $M_i \times N_i \in \mathcal{R}$. $\square$

(2) Let M, N be σ-algebras on X, Y respectively. Given a set $E \subseteq X \otimes Y$ and a fixed $x \in X$, we define the x cross-section $E_x = \{y : (x, y) \in E\}$ and for a fixed $y \in Y$ we define the y cross-section $E^y = \{x : (x, y) \in E\}$

(a) Prove that if $E = A \times B$ is a rectangle, then

$$E_x = \begin{cases} B & x \in A \\ \emptyset & x \notin A \end{cases} \quad \text{and}$$

$$E^y = \begin{cases} A & y \in B \\ \emptyset & y \notin B \end{cases}$$

Proof. Let M, N be σ-algebras on X, Y respectively. Given a set $E \subseteq X \otimes Y$, with $E = A \times B$ and a fixed $x \in X$, $E_x = \{y : (x, y) \in E\}$. For all $(x, y) \in E$, $y \in B$. So, $E_x = B$ when $x \in A$. If $x \notin A$, then $(x, y) \notin E$ so $E_x = \emptyset$. Thus,

$$E_x = \begin{cases} B & x \in A \\ \emptyset & x \notin A \end{cases}$$

Next, for a fixed $y \in Y$, $E^y = \{x : (x, y) \in E\}$. For all $(x, y) \in E$, $x \in A$. So, $E^y = A$ when $y \in B$ and $(x, y) \in E$. If $y \notin B$, then, $(x, y) \notin E$ so $E^y = \emptyset$. Thus,

$$E^y = \begin{cases} A & y \in B \\ \emptyset & y \notin B \end{cases}$$

□

(b) Consider $M = N = B_{\mathbb{R}}$ and let $E = \{(x, y) : x < y\}$. Determine

$$E^{\frac{1}{3}}, E_{\frac{1}{3}}, E^0, E_1, \text{ and } E^{\frac{1}{2}}.$$

$$E^{\frac{1}{3}} = \left\{x : \left(x, \frac{1}{3}\right) \in E\right\}. \text{ This is the line } y = \frac{1}{3}.$$
$$E_{\frac{1}{3}} = \left\{y : \left(\frac{1}{3}, y\right) \in E\right\}. \text{ This is the line } x = \frac{1}{3}$$

$$E^0 = \{x : (x, 0) \in E\}. \text{ This is the line } y = 0$$

$$E_1 = \{y : (1, y) \in E\}. \text{ This is the line} x = 1$$

$$E^{\frac{1}{2}} = \left\{x : \left(x, \frac{1}{2}\right) \in E\right\}. \text{ This is the line } y = \frac{1}{2}.$$

(c) Prove that the set $\mathcal{R}$ consisting of $E \subseteq X \times Y$ such that $E_x \in N$ for all $x \in X$ and $E^y \in M$ for all $y \in Y$ contains all rectangles.

Proof. Consider $\mathcal{R} = \{E \subset X \times Y : E_x \in N \text{ for all } x \in X \text{ and } E^y \in M \text{ for all } y \in Y\}$. $\mathcal{R}$ contains all rectangles since all x from $M \times N$ are either from N or $\emptyset$ and all y are either from M or $\emptyset$. □

(d) Prove that $\mathcal{R}$ is a σ-algebra, and hence it contains all of $M \otimes N$.

Proof. Consider some $\{E_i\}_{i=1}^{\infty}$ with $E_i \subseteq X \times Y$ for all i. Then, $\bigcup_{i=1}^{\infty} E_i = (\cup E_i)_x \times (\cup E_i)^y$. By part (d), $E \in M \otimes N \subseteq \mathbb{R}$. So for all $E \in R$, $E_x \in N$ and $E_y \in M$. Notice $(\cup E_i)_x = \cup(E_i)_x$ and $(E_i)_x \in N$ implies $\cup(E_i)_x \in N$. N is an σ-algebra, so $(\cup E_i)_x \in N$. Similarly, $(\cup E_i)^y = \cup(E_i)^y$ and $(E_i)^y \in M$ implies $\cup(E_i)^y \in M$. M is an σ-algebra, so $(\cup E_i)^y \in M$. Thus, $\mathcal{R}$ is closed under countable unions. Also, for any $E \in \mathcal{R}$, $E_x \in N$ and $E^y \in M$.

Since M, N are σ-algebra $(E_x)^c = (E^c)_x$ and $(E^y)^c = (E^c)^y$ imply $(E^y)^c, (E_x)^c \in M, N$ respectively. Thus, $\mathcal{R}$ is closed under complements. So, $\mathcal{R}$ is a σ-algebra □

2. Section 1.3 (Folland): Measures

2.1. Section 1.3 Definitions and Theorems.

Definition 10 (measure)**.** Let X be a set equipped with a σ-algebra $\mathcal{M}$. A measure on $\mathcal{M}$ or on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

i. $\mu(\emptyset) = 0$,

ii. **(countable additivity)** if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$ then

$$\mu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \mu(E_j)$$

Definition 11 (measurable space)**.** If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{M})$ is called a measurable space and the sets in $\mathcal{M}$ are called measurable sets. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a measure space.

Definition 12 (finite measure)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space. μ is called finite when $\mu(X) < \infty$. Note for all $E \in \mathcal{M}$, $\mu(X) = \mu(E) + \mu(E^c)$, so for all $E \in \mathcal{M}$, $\mu(E) < \infty$.

Definition 13. ($\sigma-$finite measure) Let $(X, \mathcal{M}, \mu)$ be a measure space. μ is σ-finite if

$$X = \bigcup_1^\infty E_j \text{ where } \mu(E_j) < \infty \text{ for all } i, j.$$

Definition 14. ($\sigma-$finite set) Let $(X, \mathcal{M}, \mu)$ be a measure space. E is σ-finite for μ if

$$E = \bigcup_1^\infty E_j \text{ where } E_j \in \mathcal{M},\ \mu(E_j) < \infty \text{ for all } i, j.$$

Definition 15. (semi-finite measure) Let $(X, \mathcal{M}, \mu)$ be a measure space. μ is semi-finite if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$.

Example (measure)**.** Let $X \neq \emptyset$, $\mathcal{M} = \mathcal{P}(X)$ and $f : X \to [0, \infty]$. f determines a measure μ on $\mathcal{M}$ by the formula $\mu(E) = \sum_{x \in E} f(x)$.

(1) μ is semifinite iff $f(x) < \infty$ $\forall x \in X$

(2) μ is σ-finite iff μ is semifinite and $\{x : f(x) > 0\}$ is countable

(3) **(counting measure)** $f(x) = 1$ for all x, then μ is counting measure

(4) **(point mass, Dirac measure)** f defined by $f(x_0) = 1$ and $f(x) = 0$ for all $x \neq x_0$, then μ is the point mass or Dirac measure

Example (measure)**.** Let X be an uncountable set and let $\mathcal{M}$ be the σ-algebra of countable or co-countable sets. The function μ on $\mathcal{M}$ defined by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E is co-countable is a measure.

Example (finitely additive measure, not a measure)**.** Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$. Define $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. μ is a finitely additive measure but not a measure.

Theorem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) **(monotonicity)** If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$.

(2) **(subadditivity)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\cup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.

(3) **(continuity from below)** If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \dots$, then $\mu(\cup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

(4) **(continuity from above)** If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \supset E_2 \supset \dots$, then $\mu(\cap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

Definition 16 (null set)**.** If $(X, \mathcal{M}, \mu)$ is a measure space, $E \in \mathcal{M}$ is called null if $\mu(E) = 0$.

Remark 1. Any countable union of null sets is a null set.

Definition 17 (almost-everywhere (a.e.))**.** If a statement is true about points $x \in X$ except for x in some null set, then it is true almost everywhere, a.e.

Remark 2. If $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ by monotonicity provided that $F \in \mathcal{M}$. If $F \notin \mathcal{M}$ this may not be true.

Definition 18. (complete measure) A measure whose domain includes all subsets of null sets. This can always be achieved by enlarging the domain of μ as we can see in the theorem below.

Theorem 3. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then, $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

2.2. **September 11 Group Assignment.**

(1) If $\mu_1, \mu_2, \cdots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$, then

$$\sum_{i=1}^{n} a_i \mu_i \text{ is a measure on } (X, \mathcal{M}).$$

Proof. Since μ_i for $1 \le i \le n$ are measures, their range is $[0, \infty]$ so the sum of these measures will have range of $[0, \infty]$. Also, $(\sum_{i=1}^{n} a_i \mu_i)(\emptyset) = \sum_{i=1}^{n} a_i \mu_i(\emptyset) = \sum_{i=1}^{n} a_i \cdot 0 = 0$. Consider some disjoint collection of sets $\{A_j\}_{j=1}^{\infty}$. Then, by countable additivity of $\mu_i's$,

$$\left(\sum_{i=1}^{n} a_i \mu_i\right)\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{i=1}^{n} a_i \left(\mu_i \left(\bigcup_{j=1}^{\infty} A_j\right)\right) = \sum_{i=1}^{n} a_i \sum_{j=1}^{\infty} \mu_i(A_j) = \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_i \mu_i(A_j).$$

Hence, $\sum_{i=1}^{n} a_i \mu_i$ is countably additivity and so this is a measure on $(X, \mathcal{M})$. ☐

(2) If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Proof. Assume $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$. Notice $E = E \backslash F \cup (F \cap E)$ and $F = F \backslash E \cup (E \cap F)$. Then, $\mu(E) = \mu(E \backslash F \cup (F \cap E))$ and $\mu(F) = \mu(F \backslash E \cup (F \cap E))$. Also, $F \backslash E \cap (E \cap F) = \emptyset$ and $E \backslash F \cap (F \cap E) = \emptyset$. μ is finitely additive over disjoint sets, so $\mu(E) = \mu(E \backslash F) + \mu(F \cap E)$ and $\mu(F) = \mu(F \backslash E) + \mu(F \cap E)$. Thus,

$$\mu(E) + \mu(F) = \mu(E \backslash F) + \mu(F \cap E) + \mu(F \backslash E) + \mu(F \cap E) = (\mu(E \backslash F) + \mu(F \cap E) + \mu(F \backslash E)) + \mu(F \cap E).$$

Since $(E \backslash F) \cup (F \cap E) \cup (F \backslash E)$ is a disjoint union, $\mu((E \backslash F) \cup (F \cap E) \cup (F \backslash E)) = \mu(E \backslash F) + \mu(F \cap E) + \mu(F \backslash E)$. But, $(E \backslash F) \cup (F \cap E) \cup (F \backslash E) = E \cup F$. Hence, $\mu(E \cup F) = \mu(E \backslash F) + \mu(F \cap E) + \mu(F \backslash E)$ and so

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

□

(3) If $(X, \mathcal{M}, \mu)$ is a measure space and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for any $A \in \mathcal{M}$. Prove that μ_E is a measure on $\mathcal{M}$.

Proof. Fix some $E \in \mathcal{M}$. Define $\mu_E(A) = \mu(A \cap E)$ for any $A \in \mathcal{M}$. Notice $\emptyset \in \mathcal{M}$, so $\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0$. Next consider a collection of disjoint sets, $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$. Then $\{A_i \cap E\}_{i=1}^{\infty}$ is a collection of disjoint sets, so by countable additivity of μ, we have:

$$\mu_E\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i \cap E\right) = \sum_{i=1}^{\infty} \mu(A_i \cap E) = \sum_{i=1}^{\infty} \mu_E(A_i).$$

Thus, μ_E is a mesure on $\mathcal{M}$. □

2.3. **September 14 Group Assignment.** These solutions are from `https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=4&ved=0CDIQFjADahUKEwiVu-_PtM_IAhVBFz4KHamFDmM&url=http%3A%2F%2Fwww.math.psu.edu%2Fballif%2Fassignments%2FMath%2520501%2520Analysis%2FMATH501_HW2.tex&usg=AFQjCNHXa01zt_Ejvp67o5jcfLLtqmuV8w&bvm=bv.105454873,d.cWw&cad=rja` and `http://www.math.tamu.edu/~thomas.schlumprecht/hw3_math607_13c_sol.pdf`

(1) If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.

Proof. Recall that $\liminf_{j\to\infty} x_j = \lim_{j\to\infty}\left(\inf_{k\geq j} x_k\right)$. First, notice for all $n \in \mathbb{N}$, $\bigcap_{j\geq n} E_j \subseteq E_j$ so by monotonicity of μ, $\mu\left(\bigcap_{j\geq n} E_j\right) \leq \mu(E_j)$ for all $j \geq n$. Therefore, $\mu\left(\bigcap_{j\geq n} E_j\right) \leq \inf_{j\geq n} \mu(E_j)$. Thus,

$$\mu\left(\liminf_{j\to\infty} E_j\right) = \mu\left(\bigcup_{n=1}^{\infty}\bigcap_{j\geq n} E_j\right) = \lim_{j\to\infty}\mu\left(\bigcap_{j\geq n} E_j\right) \leq \lim_{n\to\infty}\inf_{j\geq n}\mu(E_j) = \liminf_{j\to\infty}\mu(E_j).$$

□

(2) If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ with $\mu(E_j) < \infty$, then $\mu(\limsup E_j) \geq \limsup \mu(E_j)$

Proof. Recall that $\limsup_{j\to\infty} x_j = \lim_{j\to\infty}\left(\sup_{k\geq j} x_k\right)$. We will show $\mu\left(\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_k\right) = \lim_{j\to\infty}\mu\left(\bigcup_{k=j}^{\infty} E_k\right)$.

Proof. Let $G_j = \bigcup_{k=j}^{\infty} E_k$. We have a decreasing chain of subsets $G_j \supset G_{j+1}$ for all j. Also, we know that $\mu(G_j) < \infty$. Hence, by continuity from above we know that $\mu(\bigcap_{j=1}^{\infty} G_j) = \lim\limits_{j\to\infty} \mu(G_j)$. $\square$

For each k we know by monotonicity that $\mu(E_k) \leq \mu(\bigcup_{k=j}^{\infty} E_k)$. Hence, $\sup\limits_{k>j} \mu(E_k) \leq \mu(\bigcup_{k=j}^{\infty} E_k)$. Now we take the limit as $j \to \infty$ to get

$$\lim_{j\to\infty} \mu\left(\bigcup_{k=j}^{\infty} E_k\right) = \mu\left(\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_k\right) \leq \lim_{j\to\infty}\left(\sup_{k\geq j} \mu(E_k)\right).$$

$\square$

(3) A finitely additive measure μ on $(X, \mathcal{M})$ is a measure if and only if it is continuous from below.

Proof. ($\Rightarrow$) Theorem 1.8(c) implies a measure is continuous from below.

($\Leftarrow$) Suppose μ is finitely additive and continuous from below. Consider $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ with $A_j \cap A_i = \emptyset$ for all $j \neq i$ and $A_j \subset A_i$ for all $j \leq i$. Then,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} A_n\right) && \text{since } \mu \text{ is continous from below}\\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n) && \mu \text{ is finitely additive}\\
&= \sum_{n=1}^{\infty} \mu(A_n)
\end{aligned}$$

Thus μ is countably additive so μ is a measure. $\square$

(4) A finitely additive measure μ on $(X, \mathcal{M})$ with $\mu(X) < \infty$ is a measure if and only if it is continuous from above.

Proof. ($\Rightarrow$) Theorem 1.8(d) implies a measure is continuous from above.

($\Leftarrow$) Suppose μ is finitely additive and continuous from above. If we show μ is continuous from below, (3) will imply μ is a measure. Let $\{A_j\}_{j=1}^{\infty} \subset \mathcal{M}$ such that $A_j \cap A_i = \emptyset$ for all $j \neq i$ and $A_j \subset A_i$ for all $j \leq i$. Note

$$\bigcap_{j=1}^{\infty} X \backslash A_j = \bigcap_{j=1}^{\infty} X \cap A_j^c = X \cap \bigcap_{j=1}^{\infty} A_j^c = X \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c = X \backslash \bigcup_{j=1}^{\infty} A_j$$

(1) Thus, $\bigcap_{j=1}^{\infty} X\backslash A_j = X\backslash \bigcup_{j=1}^{\infty} A_j$. Also, since $A_j \subset A_i$ for all $j \le i$, $X\backslash A_j \supset X\backslash A_i$ for all $j \le i$.

$$\begin{aligned}
\mu\left(\bigcap_{j=1}^{\infty} X\backslash A_j\right) &= \lim_{j\to\infty} \mu(X\backslash A_j) && \text{since } \mu \text{ is continuous from below} \\
&= \lim_{j\to\infty} (\mu(X) - \mu(A_j)) && \mu \text{ is finitely additive} \\
&= \mu(X) - \lim_{j\to\infty} \mu(A_j)
\end{aligned}$$

From (1) we have

$$\mu(X) - \lim_{j\to\infty} \mu(A_j) = \mu\left(X\backslash \bigcup_{j=1}^{\infty} A_j\right) = \mu(X) - \mu\left(\bigcup_{j=1}^{\infty} A_j\right).$$

Because $\mu(X) < \infty$, we can subtract $\mu(X)$ from both sides of the previous equation to obtain

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right).$$

Therefore, μ is continuous from above which by (3) implies μ is a measure.

□

2.4. **September 16 Group Assignment.**

(1) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Denote by $E\triangle F\colon = (E \cap F^c) \cup (E^c \cap F)$.

(a) If $E, F \in M$ and $\mu(E\triangle F) = 0$ then $\mu(E) = \mu(F)$.

Proof. Let $E, F \in M$ and $\mu(E\triangle F) = 0$. Then $E \subseteq F \cup (E\triangle F)$. By monotonicity of μ, $\mu(E) \le \mu(F \cup (E\triangle F)$. By subadditivity of μ, $\mu(F \cup (E\triangle F) \le \mu(F) + \mu(E\triangle F) = \mu(F)$. Thus, $\mu(E) \le \mu(F)$.

Similarly, $F \subseteq E \cup (E\triangle F)$ which, by monotonicity and subadditivity of μ implies $\mu(F) \le \mu(E \cup (E\triangle F) \le \mu(E) + \mu(E\triangle F) = \mu(E)$. Thus, $\mu(F) \le \mu(E)$.

Since $\mu(F) \le \mu(E)$ and $\mu(E) \le \mu(F)$, $\mu(F) = \mu(E)$. □

(b) Show that $E \sim F$ if $\mu(E\triangle F) = 0$ is an equivalence relation on $\mathcal{M}$.

Proof. **(reflexive)** For any set $E \in \mathcal{M}$, $\mu(E\triangle E) = \mu(\emptyset) = 0$, so $E \sim E$.

(symmetric) Let $E, F \in \mathcal{M}$ such that $E \sim F$. Then, $\mu(E\triangle F) = 0$. Since $E\triangle F = F\triangle E$,

$\mu(E\triangle F) = \mu(F\triangle E)$ so $F \sim E$.

(transitive) Let $E, F, G \in \mathcal{M}$ such that $E \sim F$ and $F \sim G$. Then, $\mu(E\triangle F) = 0$ and $\mu(F\triangle G) = 0$. Notice $G \subseteq F \cup (G\backslash F)$ so $G\backslash E \subseteq (F \cup (G\backslash F))\backslash E$. Also $(F \cup (G\backslash F))\backslash E = (F \cup (G\backslash F)) \cap E^c = (F \cap E^c) \cup (G\backslash F) \cap E^c$. Since $(G\backslash F) \cap E^c \subseteq G\backslash F$, $(F \cap E^c) \cup (G\backslash F) \cap E^c \subseteq (F \cap E^c) \cup (G\backslash F) = (F\backslash E) \cup (F\backslash F)$. Thus, $G\backslash E \subseteq (G\backslash F) \cup (F\backslash E)$ and so $G\backslash E \cup (E\backslash G) \subseteq (G\backslash F) \cup (F\backslash E) \cup (E\backslash G)$. By Venn diagram, we can see $(G\backslash F) \cup (F\backslash E) \cup (E\backslash G) = (E\triangle F) \cup (F\triangle G)$. Hence, $(E\triangle G) \subseteq (E\triangle F) \cup (F\triangle G)$. By monotonicity and subadditivity, $\mu(E\triangle G) \leq \mu((E\triangle F) \cup (F\triangle G)) \leq \mu(E\triangle F) + \mu(F\triangle G) = 0$. Therefore, $\mu(E\triangle G) = 0$ so $E \sim G$. □

(2) Prove that a σ-finite measure is semi-finite.

Proof. Let $(X, \mathcal{M}, \mu)$ be a measure space and μ be σ-finite. Then, $X = \bigcup_{j=1}^{\infty} X_j$ with $X_j \in \mathcal{M}$ and $\mu(X_j) < \infty$ for all j. If μ is finite, then μ is semi-finite, so suppose there exists some $E \in \mathcal{M}$ such that $\mu(E) = \infty$. Notice $E = E \cap X = E \cap \bigcup_{j=1}^{\infty} X_j = \bigcup_{j=1}^{\infty}(E \cap X_j)$ so by monotonicity and subadditivity of μ

$$\infty = \mu(E) \leq \mu\left(\bigcup_{j=1}^{\infty}(E \cap X_j)\right) \leq \sum_{j=1}^{\infty} \mu(E \cap X_j). \text{ Hence, } \sum_{j=1}^{\infty} \mu(E \cap X_j) = \infty.$$

Because $\sum_{j=1}^{\infty} \mu(E \cap X_j) = \infty$, there must exist some $k \in \mathbb{N}$ with $\mu(E \cap X_k) > 0$. Because μ is σ-finite, we know $\mu(X_k) < \infty$. Also, $E \cap X_k \subseteq X_k$, so

$$0 < \mu(E \cap X_k) \leq \mu(X_k) < \infty.$$

Thus, for any $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $X_k \in \mathcal{M}$ such that $E \cap X_k \subset E$ and $0 < \mu(E \cap X_k) < \infty$; μ is semi-finite.

`http://faculties.sbu.ac.ir/~shahrokhi/M-P.pdf` □

(3) If μ is a semi-finite measure and $\mu(E) = \infty$ prove that for any $C > 0$ there is $F \subseteq E$ with $C < \mu(F) < \infty$.

Proof. Assume μ is a semi-finite measure. Then, if $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < \infty$. Let $\mathcal{C} = \{F \subset E : \mu(E) < \infty\}$. Since μ is semi-finite, $\mathcal{C} \neq \emptyset$. Thus, we can consider $\alpha = \sup\{\mu(F) : F \in \mathcal{C}\}$. Suppose $\alpha < \infty$. Then, for all $n \geq 1$, there exist sets $F_n \in \mathcal{C}$ such that $\alpha \geq \mu(F_n) \geq \alpha - \frac{1}{n}$; further, $\lim_{n\to\infty} \mu(F_n) = \alpha$. Let $F = \bigcup_{k=1}^{n} F_n$ such that $\mu(F) = \alpha$. If $\alpha < \infty$, $\mu(F) < \infty$. Since $\mu(E) = \infty$, $\mu(F) < \infty$ implies $\mu(E \backslash F) = \infty$. Because μ is semi-finite, $\mu(E\backslash F) = \infty$ implies there exists some $F' \subset E\backslash F$ such that $0 < \mu(F') < \infty$. Since $F \subset F \cup F'$, $\mu(F) \leq \mu(F \cup F')$ so $\alpha \leq \mu(F \cup F')$. But, $0 < \mu(F') < \infty$ and $0 < \mu(F) < \infty$ implies $0 < \mu(F \cup F') < \infty$. So, $F \cup F' \in \mathcal{C}$ with $\alpha \leq \mu(F \cup F')$ which contradicts α as the supremum. Thus, $\alpha = \infty$.

Since $\alpha = \infty$, for all $C > 0$, there exists a $F \in \mathcal{C}$ such that $\mu(F) > C$. Thus, for any $C > 0$, there is $F \subset E$ with $C < \mu(F) < \infty$.

`http://faculties.sbu.ac.ir/~shahrokhi/M-P.pdf` □

3. Section 1.4 (Folland): Outer Measures

3.1. **Section 1.4 Definitions and Theorems.**

Remark 3. This section develops tools we will use to construct measures.

Definition 19. An outer measure on $X \neq \emptyset$ is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

$$\mu^*(\emptyset) = 0,$$

$$\mu^*(A) \leq \mu^*(B) \text{ if } A \subset B,$$

$$\mu^*\left(\bigcup_1^\infty A_j\right) \leq \sum_1^\infty \mu^*(A_j)$$

Proposition 6. Let $\mathcal{E} \subset \mathcal{P}(X)$ and define $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}, X \in \mathcal{E}, \rho(\emptyset) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf\left\{\sum_1^\infty \rho(E_j) : E_j \in \mathcal{E}, A \subset \bigcup_1^\infty E_j\right\}$$

Then, μ^* is an outer measure.

Definition 20 (μ^*-measurable)**.** A set $A \subset X$ is called μ^*-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X.$$

Remark 4. Note $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c$ is true form any A, E.

Theorem 4 (Carathéodory's Theorem)**.** If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Definition 21 (premeasure)**.** If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ will be called a premeasure if

(1) $\mu_0(\emptyset) = 0$

(2) if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\cup_1^\infty A_j \in \mathcal{A}$, then $\mu_0(\cup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$

Proposition 7. Let μ_0 be a premeasure on $\mathcal{A}$ and

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \mu_0(E_j) : E_j \in \mathcal{E}, A \subset \bigcup_1^\infty E_j \right\}, \text{ then}$$

(1) $\mu^*|\mathcal{A} = \mu_0$

(2) every set in $\mathcal{A}$ is μ^*-measurable

3.2. **September 18 Group Assignment.**

(1) Prove that (a,∞) is m^*-measurable for any a.

Proof. We will show for any set E, $m^*(E) = m^*(E\cap(a,\infty))+m^*(E\cap(-\infty,a])$. If $m^*(E) = \infty$, since $m^*((a,\infty)) = m^*((-\infty,a]) = \infty$, the above equality will hold, so assume $m^*(E) < \infty$. Notice for any set E, $E = (E\cap(a,\infty))\cup(E\cap(-\infty,a])$ so by subadditivity of m^*, we have

$$m^*(E) \leq m*(E\cap(a,\infty)) + m^*(E\cap(-\infty,a]).$$

Thus, we must show $m^*(E) \geq m^*(E\cap(a,\infty)) + m^*(E\cap(-\infty,a])$:

Let $\varepsilon > 0$. By definition of m^*, there exists

$$\bigcup_{i=1}^{\infty}(a_i,b_i) \text{ with } E \subset \bigcup_{i=1}^{\infty}(a_i,b_i) \text{ and } \sum_{i=1}^{\infty}(b_i-a_i) \leq m^*(E)+\varepsilon.$$

Define $(a_i,b_i)' = (a,\infty)\cap(a_i,b_i)$ and $(a_i,b_i)'' = (-\infty,a]\cap(a_i,b_i)$. Notice $((a,\infty)\cap(a_i,b_i))\cup((-\infty,a]\cap(a_i,b_i)) = (a_i,b_i)$. Then, $m^*(a_i,b_i) = b_i - a_i = m^*((a_i,b_i)') + m^*((a_i,b_i)'')$.

Since $E \subset (a_i,b_i)$ for any i and $E\cap(a,\infty) \subset (a_i,b_i)\cap(a,\infty) \subset \bigcup_{i=1}^{\infty}(a_i,b_i)'$. So, by monotonicity and subadditivity of m^*,

$$m^*(E\cap(a,\infty)) \leq m^*\left(\bigcup_{i=1}^{\infty}(a_i,b_i)'\right) \leq \sum_{i=1}^{\infty} m^*((a_i,b_i)').$$

Similarly, since $E \subset (a_i,b_i)$ for any i and $E\cap(-\infty,a] \subset (a_i,b_i)\cap(-\infty,a] \subset \bigcup_{i=1}^{\infty}(a_i,b_i)''$. So, by monotonicity and subadditivity of m^*,

$$m^*(E\cap(-\infty,a]) \leq m^*\left(\bigcup_{i=1}^{\infty}(a_i,b_i)''\right) \leq \sum_{i=1}^{\infty} m^*((a_i,b_i)''). \text{ Thus,}$$

$$m^*(E\cap(a,\infty)) + m^*(E\cap(-\infty,a]) \leq \sum_{i=1}^{\infty} m^*((a_i,b_i)') + \sum_{i=1}^{\infty} m^*((a_i,b_i)'') = \sum_{i=1}^{\infty}\big(m^*((a_i,b_i)') + m^*((a_i,b_i)'')\big)$$

$$= \sum_{i=1}^{\infty}(b_i-a_i) \leq m^*(E)+\varepsilon. \text{ Hence, } m^*(E) \geq m^*(E\cap(a,\infty)) + m^*(E\cap(-\infty,a]).$$

Thus, (a,∞) is m^*-measurable. ∎

(2) Prove that any Borel set is m^*-measurable.

Proof. In (1) we showed that (a, ∞) is m^*-measurable. From exercise 1 on August 31, we know that $\mathcal{M}(\{(a, \infty) : a \in \mathbb{R}\}) = \mathcal{B}_{\mathbb{R}}$. Since the collection of m^*-measurable sets is a σ-algebra, any Borel set is m^*-measurable. $\square$

(3) Prove that for any set E and any $\varepsilon > 0$ there is an open set A with $E \subseteq A$ such that $m^*(A) \leq m^*(E) + \varepsilon$.

Proof. Consider any set E. Then

$$m^*(E) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \colon \ E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}.$$

Then, for any $\varepsilon > 0$ there exists some $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $m^*(E) + \varepsilon \geq \sum_{i=1}^{\infty}(b_i - a_i)$ with $E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$. Let $A = \bigcup_{i=1}^{\infty}(a_i, b_i)$. Note A is an open set. Then, $m^*(A) \leq \sum_{i=1}^{\infty}(b_i - a_i)$. Hence, $m^*(E) + \varepsilon \geq m^*(A)$. $\square$

(4) Let E be a given set. Prove the following are equivalent. (a) E is m^*-measurable. (b) Given $\varepsilon > 0$ there is an open set $O \supset E$ such that $m^*(O \backslash E) < \varepsilon$. (c) Given $\varepsilon > 0$ there is an closed set $K \subset E$ such that $m^*(E \backslash K) < \varepsilon$. (d) There is a G_δ set G with $E \subseteq G$ and $m^*(G \backslash E) = 0$. (e) There is a F_σ set F with $F \subseteq E$ and $m^*(E \backslash F) = 0$.

(a $\Rightarrow$ b). Assume E is m^*-measurable. Suppose $m^*(E) < \infty$. Then, since $m^*(E) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \colon \ E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)\}$, given $\varepsilon > 0$, there exists some cover of $E, \bigcup_{i=1}^{\infty}(a_i, b_i)$ such that $\sum_{i=1}^{\infty}(b_i - a_i) < m^*(E) + \varepsilon$. Let $O = \bigcup_{i=1}^{\infty}(a_i, b_i)$. Then, O is open and by subadditivity of m^*, $m^*(O) = m^*(\bigcup_{i=1}^{\infty}(a_i, b_i)) \leq \sum_{i=1}^{\infty}(b_i - a_i)$. Thus, $m^*(O) < m^*(E) + \varepsilon$. Since $m^*(E) < \infty$, we can subtract $m^*(E)$ from both sides of the previous inequality to obtain, $m^*(O) - m^*(E) < \varepsilon$. Equivalently, $m^*(O \backslash E) < \varepsilon$.

Now, suppose $m^*(E) = \infty$. Then, $E = E \cap \mathbb{R}$ and $\mathbb{R} = \bigcup_{i=1}^{\infty}[-i, i]$ so $E = \bigcup_{i=1}^{\infty}[-i, i] \ \cap \ E$. For all i, let $E_i = [-i, i] \ \cap \ E$. Then, since $m^*([-i, i]) = 2i$, the measure of E_i must be finite. From the previous paragraph, if $E_i < \infty$ there exists some open set O_i with $O_i \supset E_i$

such that

$$m^*(O_i \backslash E_i) < \frac{\varepsilon}{2^i}.$$

Let $O = \bigcup_{i=1}^{\infty}(O_i)$. Then, O is open and since $O_i \supset E_i$, $\bigcup_{i=1}^{\infty}(O_i) \supset \bigcup_{i=1}^{\infty}(E_i)$. Therfore, $O \supset E$ and $O \backslash E = \bigcup_{i=1}^{\infty}(O_i \backslash E_i)$. (I think I need to prove the previous statement) Thus,

$$m^*(O \backslash E) = m^* \left(\bigcup_{i=1}^{\infty}(O_i \backslash E_i) \right) \leq \sum_{i=1}^{\infty} m^*(O_i \backslash E_i) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

$\square$

(b $\Rightarrow$ c). Assume given $\varepsilon > 0$ there is an open set $O \supset E$ such that $m^*(O \backslash E) < \varepsilon$. Then for E^c, given $\varepsilon > 0$ there is an open set $O \supset E^c$ such that $m^*(O \backslash E^c) < \varepsilon$. Let $K = O^c$. Then, K is closed. If $O \supset E^c$, $O^c \subset E^{c^c}$ which implies $K \subset E$. Also $O \cap E^c = O \cap E = E \backslash O^c = E \backslash K$, so $m^*(O \backslash E^c) < \varepsilon$ implies $m^*(E \backslash K) < \varepsilon$. $\square$

(b $\Rightarrow$ d). Assume given $\varepsilon > 0$ there is an open set $O \supset E$ such that $m^*(O \backslash E) < \varepsilon$. Pick $\varepsilon = \frac{1}{i}$. Then, there must exist some open set O_i such that $m^*(O_i \backslash E) < \frac{1}{i}$ for any i. Let $G = \cap_{i=1}^{\infty} O_i$. Then, G is the countable intersection of open sets, so G is a G_δ set. $E \subset O_i$ for all i implies $E \subset \cap_{i=1}^{\infty} O_i$ so $E \subset G$. Also notice $G \backslash E = (\cap_{i=1}^{\infty} O_i) \backslash E = (\cap_{i=1}^{\infty} O_i) \cap E^c = \cap_{i=1}^{\infty}(O_i \cap E^c) = \cap_{i=1}^{\infty}(O_i \backslash E)$; thus $G \backslash E = \cap_{i=1}^{\infty}(O_i \backslash E)$. Since $\cap_{i=1}^{\infty}(O_i \backslash E) \subset O_i \backslash E$ for any i, monotonicity of m^* implies $m^*(\cap_{i=1}^{\infty}(O_i \backslash E)) \leq m^*(O_i \backslash E)$. Hence $m^*(G \backslash E) = m^*(\cap_{i=1}^{\infty}(O_i \backslash E)) \leq m^*(O_i \backslash E) \leq \frac{1}{i}$ for any $i \geq 1$. Thus, $m^*(G \backslash E) = 0$. $\square$

(d $\Rightarrow$ a). Let E be any subset of $\mathbb{R}$. Assume there is a G_δ set G with $E \subseteq G$ and $m^*(G \backslash E) = 0$. From exercise 2 on September 18, we know all Borel sets are m^*-measurable, so G is m^*-measurable. Also, since $m^*(G \backslash E) = 0$, by exercise 2 on September 21, we know $G \backslash E$ is m^*-measurable. Because the collection of m^*-measurable sets forms a σ-algebra we know if $G \backslash E$ is m^*-measurable then $(G \backslash E)^c$ is m^*-measurable and also $G \cap (G \backslash E)^c$ is m^*-measurable. Notice

$$G \cap (G \backslash E)^c = G \cap (G \cap E^c)^c = G \cap (G^c \cup E) = (G \cap G^c) \cup (G \cap E) = \emptyset \cup (G \cap E) = G \cap E.$$

Since $E \subseteq G$, $G \cap E = E$. Therefore, E is m^*-measurable. $\square$

(c $\Rightarrow$ e). Assume given any $\varepsilon > 0$ there is an closed set $K \subset E$ such that $m^*(E \backslash K) < \varepsilon$. Pick $\varepsilon = \frac{1}{i}$. Then, there must exist some closed set K_i such that $m^*(E \backslash K_i) < \frac{1}{i}$ for any i. Let $F = \cup_{i=1}^{\infty} K_i$. Then, F is the countable union of closed sets, so F is a F_σ set. $K_i \subset E$ for all i implies $\cup_{i=1}^{\infty} K_i \subset E$ so $F \subset E$. Also notice $E \backslash F = E \backslash \cup_{i=1}^{\infty} K_i = E \cap (\cup_{i=1}^{\infty} K_i)^c = E \cap (\cap_{i=1}^{\infty} K_i^c) = \cap_{i=1}^{\infty}(E \cap K_i^c) = \cap_{i=1}^{\infty}(E \backslash K_i)$; thus $E \backslash F = \cap_{i=1}^{\infty}(E \backslash K_i)$. Since $\cap_{i=1}^{\infty}(E \backslash K_i) \subset E \backslash K_i$ for any i, monotonicity of m^* implies $m^*(\cap_{i=1}^{\infty}(E \backslash K_i)) \leq m^*(E \backslash K_i)$. Hence $m^*(E \backslash F) = m^*(\cap_{i=1}^{\infty}(E \backslash K_i)) \leq m^*(E \backslash K_i) \leq \frac{1}{i}$ for any $i \geq 1$. Thus, $m^*(E \backslash F) = 0$. $\square$

(e $\Rightarrow$ a). Let E be any subset of $\mathbb{R}$. Assume there is a F_σ set F with $F \subseteq E$ and $m^*(E \backslash F) = 0$. From exercise 2 on September 18, we know all Borel sets are m^*-measurable, so F is m^*-measurable. Also, since $m^*(E \backslash F) = 0$, by exercise 2 on September 21, we know $E \backslash F$ is m^*-measurable. Because the collection of m^*-measurable sets forms a σ-algebra we know if $E \backslash F$ is m^*-measurable then $(E \backslash F)^c$ is m^*-measurable and also F^c is m^*-measurable and $F^c \cap (E \backslash F)^c$ is m^*-measurable. Notice

$$F^c \cap (E \backslash F)^c = F^c \cap (E^c \cup F) = (F^c \cap E^c) \cup (F^c \cap F) = (F^c \cap E^c) \cup \emptyset = F^c \cap E^c.$$

Since $F \subseteq E$, $F^c \cap E^c = E^c$. Therefore, E^c is m^*-measurable. Since the collection of m^*-measurable sets forms a σ-algebra, E is m^*-measurable. $\square$

3.3. **September 21 Group Assignment.** Let $X = [0, 1)$ and m^* denote Lebesgue outer measure.

(1) Prove that $A \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$ then $m^*(A) = m^*(A + \lambda)$.

Lemma: For any set A, $A \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$ if and only if $A + \lambda \subseteq \bigcup_{i=1}^{\infty}(a_i + \lambda, b_i + \lambda)$.

Proof. Assume $A \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$. Then, let $a \in A + \lambda$. Then, $a - \lambda \in A$ implies there exists some i such that $a - \lambda \in (a_i, b_i)$. Equivalently, $a_i < a - \lambda < b_i$ and $a_i + \lambda < a < b_i + \lambda$. Thus, $a \in (a_i + \lambda, b_i + \lambda)$ for some i which implies $A + \lambda \subseteq \bigcup_{i=1}^{\infty}(a_i + \lambda, b_i + \lambda)$.

Assume $A + \lambda \subseteq \bigcup_{i=1}^{\infty}(a_i + \lambda, b_i + \lambda)$. Suppose $a \in A$. Then, $a + \lambda \in A + \lambda$ which implies there exists i such that $a + \lambda \in (a_i + \lambda, b_i + \lambda)$. Thus $a_i + \lambda \leq a + \lambda \leq b_i + \lambda$ and so $a_i \leq a \leq b_i$. Thus, $A \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$. $\square$

Prove that $A \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$ then $m^*(A) = m^*(A + \lambda)$.

Proof. Notice $m^*(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \colon\ A \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)\}$. From the previous lemma we know $A \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$ implies $A + \lambda \subseteq \bigcup_{i=1}^{\infty}(a_i + \lambda, b_i + \lambda)$. Thus, we can write $m^*(A+\lambda) = \inf\{\sum_{i=1}^{\infty}(b_i + \lambda - (a_i + \lambda)) \colon\ A + \lambda \subseteq \bigcup_{i=1}^{\infty}(a_i + \lambda, b_i + \lambda)\}$. Since $\sum_{i=1}^{\infty}(b_i + \lambda - (a_i + \lambda)) = \sum_{i=1}^{\infty}(b_i - a_i)$, $m^*(A) = m^*(A + \lambda)$. $\square$

(2) Prove that if $m^*(A) = 0$, then A is m^*-measurable.

Proof. Assume $m^*(A) = 0$. Then, for any set $B \subset \mathbb{R}$, $B \cap A \subseteq A$ and $B \cap A^c \subseteq B$. By monotonicity of m^*, $m^*(B \cap A) \leq m^*(A)$ and $m^*(B \cap A^c) \leq m^*(B)$. Thus,

$$m^*(B \cap A) + m^*(B \cap A^c) \leq m^*(A) + m^*(B) = 0 + m^*(B);$$

$$\text{equivalently, } m^*(B \cap A) + m^*(B \cap A^c) \leq m^*(B).$$

Notice $B = (B \cap A) \cup (B \cap A^c)$. Additionally, by subadditivity of m^*,

$$m^*(B) = m^*((B\cap A)\cup(B\cap A^c)) \leq m^*(B\cap A)+m^*(B\cap A^c) \text{ and so } m^*(B) \leq m^*(B\cap A)+m^*(B\cap A^c).$$

Hence for any $B \subset \mathbb{R}$, $m^*(B) = m^*(B \cap A) + m^*(B \cap A^c)$ which implies A is m^*-measurable.

$\square$

(3) Prove that if E is an m^*-measurable subset of X and $E \subseteq N$, where N is the non-measurable set in section 1.1, then $m^*(E) = 0$.

Proof. Assume E is an m^*-measurable subset of X and $E \subseteq N$, where N is described below. **(construction of N from §1.1)** Define an equivalence relation on $[0,1)$ by declaring that $x \sim y$ if and only if $x - y \in \mathbb{Q}$. By the axiom of choice, we can let N be a subset of $[0,1)$ that contains exactly one member of each equivalence class. Consider $R = \mathbb{Q} \cap [0,1)$ and for each $r \in R$, let $N_r = \{x + r : x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in N \cap [1 - r, 1)\}$ We showed in class that N_r is not measurable. By exercise 1 from September 21, m^* is translation invariant so N is not measurable. Next, consider $E \subseteq N$. Then, E contains one element or none from each of the equivalence classes defined above. Following the construction of N_r, let $E_r = \{x + r : x \in E \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in E \cap [1 - r, 1)\}$. Thus, for every $r \in R$, $E_r \subseteq N_r$. E_r is just E translated r units and then translated back into $[0,1)$ as needed, so from exercise 1, $m^*(E) = m^*(E_r)$. For any $r \in R$, $E_r \subset [0,1)$ so $\bigcup_{r \in R} E_r \subset [0,1)$. In class we showed for all $r, s \in R, r \neq s$ $N_r \cap N_s = \emptyset$, thus $\bigcup_{r \in R} E_r$ is a disjoint union. Therefore, $m^*(\bigcup_{r \in R} E_r) = \sum_{r \in R} m^*(E_r)$. Since $m^*(E_r) = m^*(E)$, $m^*(\bigcup_{r \in R} E_r) = \sum_{r \in R} m^*(E) = m^*(E) \sum_{r \in R}(1) = m^*(E) \cdot \infty$. By monotonicity of m^*, $m^*(\bigcup_{r \in R} E_r) \leq m^*([0,1)) = 1$. Thus $m^*(\bigcup_{r \in R} E_r) \leq 1$ and $m^*(\bigcup_{r \in R} E_r) = m^*(E) \cdot \infty$. If $m^*(E) > 0$, the previous equality would imply $m^*(\bigcup_{r \in R} E_r) = \infty$ which is false since $m^*(\bigcup_{r \in R} E_r) \leq 1$; thus $m^*(E) = 0$. $\square$

(4) If E is measurable and $m^*(E) > 0$, then E contains a non-measurable subset. (this is exercise 29 on pg 39 in Rolland)

Proof. Assume E is measurable with $m^*(E) > 0$. Without loss of generality, assume $E \subset [0,1)$. Assume that all subsets of E are measurable. Then, using the construction of N_r from

§1.1, $E = \cup_{r\in R}(E \cap N_r)$. Then, from (3), $m^*(E \cap N_r) = 0$ implies $E \cap N_r \subset E$ are measurable for all $r \in R$. Since $E = \cup_{r\in R}(E\cap N_r)$, $m^*(E) = m^*(\cup_{r\in R}(E\cap N_r))$. Since $E\cap N_{r_1}$ and $E\cap N_{r_2}$ are disjoint for any $r_1, r_2 \in R$, $m^*(\cup_{r\in R}(E\cap N_r)) = \sum_{r\in R} m^*(E\cap N_r) = \sum_{r\in R} 0 = 0$. Thus, $m^*(E) = 0$. But, $m^*(E) > 0$ so there must be some r for which $E \cap N_r$ is not measurable. Thus, $E \cap N_r \subseteq E$, so E contains a non-measurable subset.

`http://math.ucr.edu/~edwardb/Graduate%20Classes/Math%20209A/Homework%201.pdf`

`http://www.math.ttu.edu/~drager/Classes/01Fall/reals/ans2.pdf` □

3.4. **September 28 Group Assignment.**

(1) Let X be a set and A a proper subset of X. Let $\mathcal{E} = \{\emptyset, A\}$ and define $\mu : \mathcal{E} \to [0, \infty]$ by $\mu(\emptyset) = 0$ and $\mu(A) = 1$. (a) For any subset E of X prove that

$$\mu^*(E) = \begin{cases} 0 & E = \emptyset \\ 1 & \emptyset \neq E \subseteq A \\ \infty & \text{otherwise} \end{cases}$$

Proof. First, suppose $E = \emptyset$. Then,

$$\mu^*(E) = \mu^*(\emptyset) = \inf \left\{ \sum_{j=1}^{\infty} \mu(E_j) : E_j \in \mathcal{E} \text{ and } \emptyset \subset \bigcup_{j=1}^{\infty} E_j \right\}.$$

We can take $E_j = \emptyset$ for all j so that

$$\mu^*(\emptyset) = \inf \left\{ \sum_{j=1}^{\infty} \mu(\emptyset) : \emptyset \in \mathcal{E} \text{ and } \emptyset \subset \bigcup_{j=1}^{\infty} \emptyset \right\} = \inf \left\{ \sum_{j=1}^{\infty} 0 : \emptyset \in \mathcal{E} \text{ and } \emptyset \subset \bigcup_{j=1}^{\infty} \emptyset \right\} = 0.$$

Next, suppose $\emptyset \neq E \subseteq A$. Then,

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(E_j) : E_j \in \mathcal{E} \text{ and } E \subset \bigcup_{j=1}^{\infty} E_j \right\}.$$

Since $\mathcal{E}$ contains only $\emptyset, A$ and $E \not\subset \emptyset$, we must let $E_j = A$ for all j

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A) : A \in \mathcal{E} \text{ and } E \subset \bigcup_{j=1}^{\infty} A \right\} = \mu(A) = 1.$$

Finally, suppose $E \not\subset A$. Then $E \not\subset \emptyset$ and $E \not\subset A$ so $\{E_j \in \mathcal{E}$ and $E \subset \bigcup_{j=1}^{\infty} E_j\} = \emptyset$. Thus, $\mu^*(E) = \inf \emptyset = \infty$. $\square$

(b) Prove that the σ-algebra of μ^*-measurable sets is $\{\emptyset, A, A^c, X\}$.

Proof. First, we will show $\{\emptyset, A, A^c, X\}$ are μ^*-measurable sets. Let M be the σ-algebra of μ^*-measurable sets.

(Claim: $\emptyset$ is μ^*-measurable.) Consider any set $E \subset X$. Notice $\mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^c) = \mu^*(\emptyset) + \mu^*(E \cap X) = 0 + \mu^*(E)$. Thus, $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^c)$ so $\emptyset$ is μ^*-measurable. Since the set of μ^*-measurable sets is a σ-algebra, $\emptyset^c \in M$, so $X \in M$.

(Claim: A is μ^*-measurable.) Consider any set $E \subset X$. Then, either $E \subseteq A$ or $E \supset A$. Suppose $E \subseteq A$. Then, $\mu^*(E) = 1$. Since $E \subseteq A$, $E \cap A = E$ and $E \cap A^c = \emptyset$ so $\mu^*(E \cap A) = \mu^*(E)$ and $\mu^*(E \cap A^c) = \mu^*(\emptyset) = 0$. Thus, when $E \subseteq A$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Next, suppose $E \supset A$. Then, $\mu^*(E) = \infty$. Since $E \supset A$, $E \cap A = A$, $\mu^*(E \cap A) = \mu^*(A) = 1$. Also, and $E \cap A^c \not\subseteq A$ so $\mu^*(E \cap A^c) = \infty$. Thus, when $E \supset A$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Since the set of μ^*-measurable sets is a σ-algebra, if $A \in M$, $A^c \in M$. Now, suppose there exists $N \subset X$ such that $N \notin \{\emptyset, A, A^c, X\}$ but N is μ^*-measurable. Since $N \neq \emptyset$, $\mu^*(N) \neq 0$, so either $\mu^*(N) = 1$ or $\mu^*(N) = \infty$.

Consider $\mu^*(N) = 1$. Then, $N \subset A$. Since N is μ^*-measurable, for any set E in X, $\mu^*(E) = \mu^*(E \cap N) + \mu^*(E \cap N^c)$. This equality must hold if $E \subseteq A$ where $E \cap N \neq \emptyset$ and $E \neq N$. So, if $E \subseteq A$, $\mu^*(E) = 1$. Since $E \cap N \subseteq E \subseteq A$ and $E \cap N \neq \emptyset$, $\mu^*(E \cap N) = 1$. Similarly, $E \cap N^c \subseteq E \subseteq A$ and $E \cap N^c \neq \emptyset$, $\mu^*(E \cap N^c) = 1$. But, if $\mu^*(E) = \mu^*(E \cap N) + \mu^*(E \cap N^c)$, we have $1 = 1 + 1$ which is false. Thus, $\mu^*(N) \neq 1$.

Next, suppose $\mu^*(N) = \infty$. Thus, $N \not\subseteq A$. Suppose, however, that $N \cap A \neq \emptyset$. Since N is μ^*-measurable, for any set E in X, $\mu^*(E) = \mu^*(E \cap N) + \mu^*(E \cap N^c)$. This equality must hold if $E \subset A$ where $E \cap N = A \cap N$, $E \neq N$ and $E \cap N \neq \emptyset$. Thus, $E \subset A$ implies $\mu^*(E) = 1$; $\emptyset \neq E \cap N \subset E \subset A$ implies $\mu^*(E \cap N) = 1$; and $\emptyset \neq E \cap N^c \subset E \subset A$ implies $\mu^*(E \cap N^c) = 1$. Thus, if $\mu^*(E) = \mu^*(E \cap N) + \mu^*(E \cap N^c)$, $1 = 1 + 1$ which is false. Therefore, $\mu^*(N) \neq \infty$.

Thus, σ-algebra of μ^*-measurable sets is $\{\varnothing, A, A^c, X\}$. $\square$

(c) For any $\infty \geq a > 0$, define

$$\begin{aligned} \sigma(\emptyset) = &\quad 0 \\ \sigma(A) = &\quad 1 \\ \sigma(A^c) = &\quad a \\ \sigma(X) = &\quad 1 + a. \end{aligned}$$

Prove σ is a measure extending μ.

Proof. First, we will show σ is a measure. Notice $\sigma(\emptyset) = 0$. So we need only check that σ is countably additive under disjoint unions. Notice σ is countably additive under disjoint unions:

$$\begin{aligned} \sigma(\emptyset \cup A) = &\quad \sigma(A) = &\quad 1 = &\quad \sigma(\emptyset) + \sigma(A) \\ \sigma(\emptyset \cup A^c) = &\quad \sigma(A^c) = &\quad a = &\quad \sigma(\emptyset) + \sigma(A^c) \\ \sigma(\emptyset \cup X) = &\quad \sigma(X) = &\quad 1 + a = &\quad \sigma(\emptyset) + \sigma(S) \\ \sigma(A \cup A^c) = &\quad \sigma(X) = &\quad 1 + c = &\quad \sigma(A) + \sigma(A^c). \end{aligned}$$

Thus, σ is a measure. To show σ is a measure extending μ, we need only prove $\sigma = \mu$ when μ is defined. μ is defined only on $A, \emptyset$. So, we have:

$$\begin{aligned} \sigma(\emptyset) = &\quad 0 = &\quad \mu(\emptyset) \\ \sigma(A) = &\quad 1 = &\quad \mu(A). \end{aligned}$$

Thus, σ is a measure extending μ. $\square$

3.5. **September 30 Group Assignment.** This is exercise 23 on page 32 of Rolland.

(1) Let A be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$.

(a) Prove that A is an algebra on $\mathbb{Q}$.

Proof. Without loss of generality, assume $\left\{\bigcup_{i=1}^{n}(a_{i_j}, b_{i_j}] \cap \mathbb{Q}\right\}_{j=1}^{N}$ is a collection of disjoint sets so that for all i, j we have $a_1 < a_2 < a_3 < \cdots$, $b_1 < b_2 < b_3 < \cdots$, and $b_1 < a_2$, $b_2 < a_3, \cdots$. Then,

$$\bigcup_{j=1}^{N}\left(\bigcup_{i=1}^{n}(a_{i_j}, b_{i_j}] \cap \mathbb{Q}\right) = \bigcup_{j=1}^{N}\bigcup_{i=1}^{n}\left((a_{i_j}, b_{i_j}] \cap \mathbb{Q}\right)$$

This is a finite union of sets in A, so A is closed under finite unions. Also,

$$\left(\bigcup_{i=1}^{n}(a_{i_j}, b_{i_j}]\right)^c \cap \mathbb{Q} = \bigcap_{i=1}^{n}(a_{i_j}, b_{i_j}]^c \cap \mathbb{Q} = \left(\bigcap_{i=1}^{n}(-\infty, a_{i_j}] \cup (b_{i_j}, \infty)\right) \cap \mathbb{Q}.$$

Notice we can rewrite $\displaystyle\bigcap_{i=1}^{n}(-\infty, a_{i_j}] \cup (b_{i_j}, \infty) = (-\infty, a_1] \cup (b_1, a_2] \cup (b_2, a_3] \cup \cdots \cup (b_{n-1}, a_n] \cup (b_n, \infty)$.

The sets $(-\infty, a_1], (b_1, a_2], (b_2, a_3], \cdots, (b_{n-1}, a_n], (b_n, \infty)$ are in A. So by the previous rewrite and since A is closed under finite unions, A is closed under complements. $\square$

(b) Prove that the σ-algebra generated by A is the power set of $\mathbb{Q}$.

Proof. Consider any $q \in \mathbb{Q}$. Then, $q = \cap_{n=1}^{\infty}(q - \frac{1}{n}, q]$. Any power set of $\mathbb{Q}$ is a countable union of these sets. Thus, any power set of $\mathbb{Q}$ is in A. Any set in A is a set of rational numbers so it must be contained in the power set of $\mathbb{Q}$. $\square$

(c) Define $\mu_0$by $\mu_0(\emptyset) = 0$ and $\mu_0(E) = \infty$ if $E \neq \emptyset$. Prove μ_0 is a pre measure on A.

Proof. Since $\mu_0(\emptyset) = 0$ we need only show that μ_0 is additive over finite, disjoint unions. Consider $\{E_i\}_{i=1}^{n}$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$. Then, by definition of μ_0, $\mu_0\left(\bigcup_{i=1}^{n} E_i\right)$ is either 0 or ∞. Suppose $\mu_0\left(\bigcup_{i=1}^{n} E_i\right) = 0$. Then, $\bigcup_{i=1}^{n} E_i = \emptyset$ and so $E_i = \emptyset$ for all i. Thus, $\mu_0(E_i) = 0$ for all i and so $\sum_{i=1}^{n} \mu_0(E_i) = 0$. Hence, $\sum_{i=1}^{n} \mu_0(E_i) = \mu_0\left(\bigcup_{i=1}^{n} E_i\right)$. Suppose $\mu_0\left(\bigcup_{i=1}^{n} E_i\right) = \infty$. Then, $\bigcup_{i=1}^{n} E_i \neq \emptyset$ and so there must exist at least one $E_i \neq \emptyset$. Thus, $\mu_0(E_i) = \infty$ and so regardless of the other E_i's $\sum_{i=1}^{n} \mu_0(E_i) = \infty$. Hence, $\sum_{i=1}^{n} \mu_0(E_i) = \mu_0\left(\bigcup_{i=1}^{n} E_i\right)$. Thus, μ_0 is a pre-measure. $\square$

(d) Prove that there is more than one measure on $P(\mathbb{Q})$ whose restriction to A is μ_0.

Take μ_1 to be the measure induced by μ_0. Then $\mu_1(A) = \infty$ unless $A = \emptyset$. Let μ_2 be the counting measure. Now $\mu_1(\{1\}) = \infty > 1 = \mu_2(\{1\})$. However, $\mathcal{A}_0$ consists of $\emptyset$ and infinite sets. Therefore $\mu_1|_{\mathcal{A}_0} = \mu_2|_{\mathcal{A}_0}$.

(e) Why does this not contradict Theorem 1.14?

Because μ_0 is not σ-finite.

3.6. **October 2 Group Assignment.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces and let

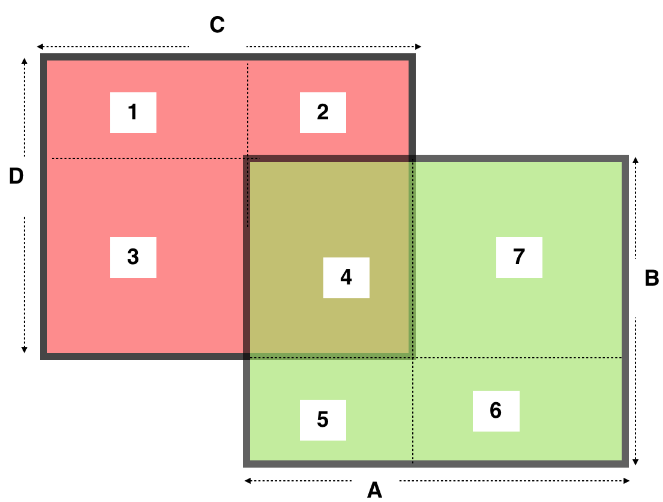
$$\mathcal{A} := \left\{ \bigcup_{i=1}^{n} A_i \times B_i \ : \ A_i \in \mathcal{M}, B_i \in \mathcal{N}, \text{ and } (A_i \times B_i) \cap (A_j \times B_j) = \varnothing \text{ if } i \neq j \right\}.$$

(1) Prove that $\mathcal{A}$ is an algebra.

Proof. Consider $\{R_i\}_{i=1}^{k} \subset \mathcal{A}$. Then, for all $1 \leq i \leq k$, $R_i = \bigcup_{i=1}^{n} A_i \times B_i \ : \ A_i \in \mathcal{M}, B_i \in \mathcal{N}$, and $(A_i \times B_i) \cap (A_j \times B_j) = \varnothing$ if $i \neq j$. We will show $\bigcup_{i=1}^{k} R_i \in \mathcal{A}$ inductively. Notice

$$R_i \cup R_j = \bigcup_{i=1}^{n} A_i \times B_i \cup \bigcup_{j=1}^{m} C_j \times D_j = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \times B_i \cup C_j \times D_j)$$

So, we will first consider the union of two rectangles, $A \times B$ and $C \times D$ and write $(A \times B) \cup (C \times D)$ as the union of disjoint rectangles:



.

Notice we can write $(A\times B)\cup(C\times D)$ as the union of the disjoint rectangles, labeled 1 to 7 in the figure above.

$$(A\times B)\cup(C\times D) = ((A\backslash C)\times(B\backslash D))\cup((A\backslash C^c)\times(B\backslash D))\cup((A\backslash C)\times(B\backslash D^c))\cup((A\backslash C^c)\times(B\backslash D^c))\cup$$

$$((A\backslash C^c)\times(D\backslash B))\ \cup\ ((C\backslash A)\times(D\backslash B))\ \cup\ ((C\backslash A)\times(B\backslash D^c))$$

From the figure, we can see that all these rectangles are disjoint; thus, $R_i \cup R_j \in \mathcal{A}$.

Assume that $\bigcup_{i=1}^n R_i \in \mathcal{A}$. Then, $\bigcup_{i=1}^n R_i \cup R_j = \bigcup_{i=1}^n (R_i \cup R_j)$. Our previous conclusion implies that $(R_i \cup R_j) \in \mathcal{A}$. So, since $\bigcup_{i=1}^n R_i \in \mathcal{A}$ and $(R_i \cup R_j) \in \mathcal{A}$, $\bigcup_{i=1}^{n+1}(R_i) \in \mathcal{A}$. Thus, $\bigcup_{i=1}^k R_i \in \mathcal{A}$.

Next, to show $R_i^c \in \mathcal{A}$ for any i, consider $(A\times B)\cap(C\times D)$ in the figure. Notice $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D) \in \mathcal{A}$. Thus, inductively, we can show $\bigcap_{i=1}^n R_i \in \mathcal{A}$. So, by inspection of figure, we can see

$$R_i^c = \left(\bigcup_{i=1}^n A_i \times B_i\right)^c = \bigcap_{i=1}^n (A_i \times B_i)^c = \bigcap_{i=1}^n \left((A_i^c \times B_i^c)\cup(A_i \times B_i^c)\cup(A_i^c \times B_i)\right).$$

Notice $(A_i^c \times B_i^c), (A_i \times B_i^c), (A_i^c \times B_i)$ are disjoint so $(A_i^c \times B_i^c)\cup(A_i \times B_i^c)\cup(A_i^c \times B_i) \in \mathcal{A}$. Thus, $\bigcap_{i=1}^n \left((A_i^c \times B_i^c)\cup(A_i \times B_i^c)\cup(A_i^c \times B_i)\right) \in \mathcal{A}$, so $R_i^c \in \mathcal{A}$ for any i.

Hence, $\mathcal{A}$ is an algebra. □

(2) Prove that if μ and ν are σ-finite then so is $\mu \times \nu$.

Proof. Assume μ and ν are σ-finite. Then, $X = \bigcup_{j=1}^\infty A_j$ with $\mu(A_j) < \infty$ for all j and $Y = \bigcup_{i=1}^\infty B_i$ with $\nu(B_i) < \infty$ for all i. Then, $X \times Y = \bigcup_{j=1}^\infty A_j \times \bigcup_{i=1}^\infty B_i = \bigcup_{j=1}^\infty \bigcup_{i=1}^\infty A_j \times B_i$. Since $\mu(A_j) < \infty$ and $\nu(B_i) < \infty$ for all i, j, $\mu(A_j)\nu(B_i) < \infty$ for all i, j. Therefore, $\mu \times \nu$ is σ-finte. □

4. Section 1.3 (Folland): Complete proof of Theorem 1.9

4.1. **October 5 Group Assignment.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Let $N = \{S \in M : \mu(S) = 0\}$ and $\overline{M} = \{E \cup F : E \in M$ and there is $S \in N$ such that $F \subseteq S\}$. Prove that $\overline{M}$ is a σ-algebra that contains M.

Proof. To show $\overline{M}$ is closed under countable unions, we will start by showing N is closed under countable unions. Consider a disjoint collection of sets, $\{E_i\}_{i=1}^{\infty} \subset N$. Then, for all i, $E_i \in M$ and $\mu(E_i) = 0$. Also, because μ is sub-additive and $E_i \cap E_j = \emptyset$ for all $i \neq j$, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} 0 = 0$. Since $E_i \in M$ for all i, $\bigcup_{i=1}^{\infty} E_i \in M$. Thus, $\bigcup_{i=1}^{\infty} E_i \in N$ and so N is closed under countable unions. Notice $\overline{M}$ is the union of sets from M and N. Thus, because M and N are closed under countable unions, $\overline{M}$ is closed under countable unions.

Next, we will show $\overline{M}$ is closed under complements. Consider $E \cup F \in \overline{M}$. Then, $E \in M$ and $F \in N$. Suppose $E \cap N = \emptyset$.

Lemma: $E \cup F = (E \cup N) \cap (N^c \cup F)$

Proof. First, we will show $E \cup F \subseteq (E \cup N) \cap (N^c \cup F)$. Let $x \in E \cup F$. Then, $x \in E$ or $x \in F$. Suppose $x \in E$. Then, $x \in E \cup N$. Also, since $E \cap N = \emptyset$, $x \in N^c$ so $x \in N^c \cup F$. Thus, $x \in (E \cup N) \cap (N^c \cup F)$. Suppose $x \in F$. Then, $x \in N^c \cup F$. Since $x \in F$ and $E \cup F \in \overline{M}$, $F \in N$ and there exists some $S \in N$ such that $F \subseteq S$. Thus, $x \in N$ and so $x \in E \cup N$. Therefore, $E \cup F \subseteq (E \cup N) \cap (N^c \cup F)$.

Next, we will show $E \cup F \supseteq (E \cup N) \cap (N^c \cup F)$. Let $x \in (E \cup N) \cap (N^c \cup F)$. Then, $x \in E \cup N$ and $x \in N^c \cup F$. If $x \in E$, then $x \in E \cup F$ and we have our desired relation. So, suppose $x \notin E$. Then, since $x \in E \cup N$, $x \in N$. Since $x \in N$, $x \notin N^c$. Since $x \in N^c \cup F$, this implies $x \in F$ so that $x \in E \cup F$. Thus, $E \cup F \supseteq (E \cup N) \cap (N^c \cup F)$. $\square$

By the lemma above, we can write $(E \cup F)^c = ((E \cup N) \cap (N^c \cup F))^c = (E \cup N)^c \cup (N^c \cup F)^c = (E \cup N)^c \cup (N \backslash F)$. M is closed under complements and countable unions, so $E \in M$ and $N \subset M$ implies $E \cup N \in M$ and $(E \cup N)^c \in M$. Also, $N \backslash F \subset N$. Thus, $(E \cup F)^c \in \overline{M}$.

Now, suppose $E \cap N \neq \emptyset$, and consider the following lemma:

Lemma: If $E \cap N \neq \emptyset$**,** $E \cup F = (E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E)$

Proof. First, we will show $E \cup F \subseteq (E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E)$. Let $x \in E \cup F$. Then, $x \in E$ or $x \in F$. Suppose $x \in E$. Then, $x \in E \cup N \backslash E$. Since $x \in E$, $x \in (N^c \cup E) = (N \backslash E)^c$. Thus, $x \in (E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E)$. Suppose $x \in F$, but $x \notin E$. Then, $x \in F \backslash E$, so $x \in (N \backslash E)^c \cup F \backslash E$. Since $x \in F$ and $E \cup F \in \overline{M}$, there exists some $S \in N$ such that $F \subseteq S$. Thus, $x \in N$. Since $x \notin E$, $x \in N \backslash E$ so $x \in E \cup N \backslash E$. Therefore, $E \cup F \subseteq (E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E)$.

Next, we will show $E \cup F \supseteq (E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E)$. Let $x \in (E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E)$. Then, $x \in (E \cup N \backslash E)$ and $x \in ((N \backslash E)^c \cup F \backslash E)$. If $x \in E$, then $x \in E \cup F$ and we have our desired relation. So, suppose $x \notin E$.

Then, since $x \in E \cup N \backslash E$, $x \in N \backslash E$ and so $x \in N$. Since $x \in N$, $x \notin N^c$ and so $x \notin (N \backslash E)^c$. Since $x \in (N \backslash E)^c \cup F \backslash E$, $x \in F \backslash E$ so that $x \in E$. Thus, $x \in E \cup F$.

Hence, $E \cup F \supseteq (E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E)$. □

By the lemma above, we can write $(E \cup F)^c = ((E \cup N \backslash E) \cap ((N \backslash E)^c \cup F \backslash E))^c = (E \cup N \backslash E)^c \cup ((N \backslash E)^c \cup F \backslash E)^c = (E \cup N \backslash E)^c \cup (N \cap (E \backslash F))$. M is closed under complements and countable unions and intersections, so $E \in M$ and $N \backslash E \subset M$ implies $E \cup N \backslash E \in M$ and $(E \cup N \backslash E)^c \in M$. Also, $N \cap (E \backslash F) \subset N$. Thus, $(E \cup F)^c \in \overline{M}$.

□

(2) If $E \cup F \in \overline{M}$ define $\overline{\mu}(E \cup F) = \mu(E)$. Prove that this function is well defined.

Proof. Assume $E^{'} \cup F^{'} = E^{''} \cup F^{''}$ with $E^{'} \cup F^{'}$ and $E^{''} \cup F^{''}$ in $\overline{M}$. Since $E^{'} \subseteq E^{'} \cup F^{'}$, $E^{'} \subseteq E^{''} \cup F^{''}$. Also, since $E^{''} \cup F^{''} \in \overline{M}$, there exists $S^{''} \in N$ such that $F^{''} \subseteq S^{''}$. Thus, $E^{'} \subseteq E^{''} \cup F^{''} \subseteq E^{''} \cup S^{''}$. By monotonicity of μ, $\mu(E^{'}) \leq \mu(E^{''} \cup F^{''}) \leq \mu(E^{''} \cup S^{''})$. Also, by subadditivity of μ, $\mu(E^{''} \cup S^{''}) \leq \mu(E^{''}) + \mu(S^{''})$. Since $S^{''} \in N$, $\mu(S^{''}) = 0$. Thus, $\mu(E^{'}) \leq \mu(E^{''})$.

Similarly, $E^{''} \subseteq E^{''} \cup F^{''}, so\ E^{''} \subseteq E^{'} \cup F^{'}$. Also, since $E^{'} \cup F^{'} \in \overline{M}$, there exists $S^{'} \in N$ such that $F^{'} \subseteq S^{'}$. Thus, $E^{''} \subseteq E^{'} \cup F^{'} \subseteq E^{'} \cup S^{'}$. By monotonicity of μ, $\mu(E^{''}) \leq \mu(E^{'} \cup F^{'}) \leq \mu(E^{'} \cup S^{'})$. Also, by subadditivity of μ, $\mu(E^{'} \cup S^{'}) \leq \mu(E^{'}) + \mu(S^{'})$. Since

$S^{'} \in N$, $\mu(S^{'}) = 0$. Thus, $\mu(E^{''}) \leq \mu(E^{'})$.

Hence $\mu(E^{'}) = \mu(E^{''})$ and by definition of $\overline{\mu}$, $\mu(E^{'}) = \overline{\mu}(E^{'} \cup F^{'})$ and $\mu(E^{''}) = \overline{\mu}(E^{''} \cup F^{''})$ which implies $\overline{\mu}(E^{'} \cup F^{'}) = \overline{\mu}(E^{''} \cup F^{''})$. Thus, $\overline{\mu}$ is well defined. □

(3) Prove that $\overline{\mu}$ is a measure on $\overline{M}$ and that the measure is complete.

Proof. We will prove $\overline{\mu}$ is a measure on $\overline{M}$. Note $\emptyset \in M$ and since $\mu(\emptyset) = 0$ $\emptyset \in N$. Thus, $\overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0$. Now, we will show $\overline{\mu}$ is countably additive. Consider a disjoint collection of sets in $\overline{M}$, $\{A_i\}_{i=1}^{\infty}$. Then, for all i, $A_i = E_i \cup F_i$ for some $E_i \in M$ and where $F_i \subseteq S_i$ for some $S_i \in N$. Then,

$$\overline{\mu}\left(\bigcup_{i=1}^{\infty} A_i\right) = \overline{\mu}\left(\bigcup_{i=1}^{\infty} E_i \cup F_i\right) = \overline{\mu}\left(\bigcup_{i=1}^{\infty} E_i \cup \bigcup_{i=1}^{\infty} F_i\right)$$

M and N are closed under countable unions, so $\bigcup_{i=1}^{\infty} E_i \in M$ and $\bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} S_i \in N$. By definition of $\overline{\mu}$, because $E_i^{\prime}s$ are disjoint and because μ is countably additive we can write

$$\overline{\mu}\left(\bigcup_{i=1}^{\infty} E_i \cup \bigcup_{i=1}^{\infty} F_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \overline{\mu}(E_i \cup F_i) = \sum_{i=1}^{\infty} \overline{\mu}(A_i).$$

Thus, $\overline{\mu}$ is countably sub-additive. □

Proof. Next, we will prove $\overline{\mu}$ is a complete measure on $\overline{M}$. To prove this we must show that the domain of $\overline{\mu}$ contains all subsets of null sets. Suppose $E \cup F$ is a $\overline{\mu}$-null set. Since $E \cup F$ is a $\overline{\mu}$-null set, $\overline{\mu}(E \cup F) = 0 = \mu(E)$. Thus, E is a μ-null set. Also, if $E \cup F \in \overline{M}$, then $F \subseteq S \in N$, so $0 \leq \mu(E \cup S) \leq \mu(E) + \mu(S) = 0$. Thus, $\mu(E \cup S) = 0$ so $E \cup S$ is a μ-null set. Consider any $A \subseteq E \cup F$. The subset relation is transitive, so $A \subseteq E \cup S$. Thus, A is a subset of some element in N, so since $\emptyset \in M$, $\emptyset \cup A \in N$, so we can write $\overline{\mu}(A) = \overline{\mu}(\emptyset \cup A) = \mu(\emptyset) = 0$. Thus, $A \in \overline{M}$.

**Sources used: `http://www.math.ubc.ca/~marcus/Math507420/Math507420_HW2_solns_2013.pdf` and `https://proofwiki.org/wiki/Completion_Theorem_(Measure_Spaces)` □

(4) Prove that if σ is a complete measure on $\overline{M}$ such that $\sigma|_M = \mu$, then $\sigma = \overline{\mu}$.

Proof. Let σ be a complete measure on $\overline{M}$ such that $\sigma|_M = \mu$. Consider any $E \cup F \in \overline{M}$. Then, $E \subset E \cup F \subset E \cup S$ for some $S \in N$. Thus, by monotonicity of σ, $\sigma(E) \leq \sigma(E \cup F) \leq \sigma(E \cup S)$. Since $E \in M$ and $S \in M$, $E \cup S \in M$ and we can write $\sigma(E \cup S) = \mu(E \cup S)$ and by subadditivity of μ, $\mu(E \cup S) \leq \mu(E) + \mu(S)$. $S \in N$, so $\mu(S) = 0$. Thus, $\sigma(E) \leq \sigma(E \cup F) \leq \sigma(E \cup S) = \mu(E \cup S) \leq \mu(E) = \sigma(E)$ since $E \in M$. Thus, $\sigma(E) \leq \sigma(E \cup F) \leq \sigma(E)$ so $\sigma(E \cup F) = \mu(E) = \overline{\mu}(E \cup F)$. Thus, $\sigma = \overline{\mu}$.

**Source used: `http://faculties.sbu.ac.ir/~shahrokhi/M-P.pdf` □

5. Section 1.5 (Folland): Borel Measures on the Real Line

5.1. **Section 1.5 Definitions and Theorems.**

Definition 22. A Borel measure is any measure μ defined on the σ-algebra of Borel sets.

Theorem 5. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b.

If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant.

Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$$

then F is increasing and right continuous and $\mu = \mu_F$.

Definition 23 ($\mathcal{B}_{\mathbb{R}}$)**.** Borel σ-algebra: generated by family of open sets in $\mathbb{R}$ (or closed, half-open, open rays, closed rays

Definition 24 (Lebesgue-Stieltjes measure)**.** For any $E \in \mathcal{M}_\mu$,

$$\mu(E) = \inf \left\{ \sum_1^\infty \mu((a_j, b_j)) : E \subset \bigcup_1^\infty (a_j, b_j) \right\}.$$

Theorem 6. If $E \in \mathcal{M}_\mu$, then

$$\mu(E) = \inf\{\mu(U) : U \supset E,\ U \text{ open}\} = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$$

Theorem 7.

$$E \subset \mathbb{R}. \quad E \in \mathcal{M}_\mu \iff E = V/N_1,\ V \text{ is } G_\delta, \mu(N_1) = 0 \iff E = H \cup N_2,\ H \text{ is } F_\sigma,\ \mu(N_2) = 0.$$

Definition 25 (Lebesgue measure)**.** Complete measure μ_F associated to the function $F(x) = x$ for which the measure of an interval is simply its length. Denoted m.

Remark 5. m is invariant under translations, simple behavior under dilations: $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$

Remark 6. Every singleton set in $\mathbb{R}$ has Lebesgue measure zero. Every countable set has Lebesgue measure zero.

5.2. **October 9 Group Assignment.**

(1) Let δ_i denote the Dirac measure concentrated on i, in other words:

$$\delta_i(E) = \begin{cases} 1 & i \in E \\ 0 & i \notin E \end{cases}$$

Find the distribution function for the measure $\sum_{i=1}^{10} i^2\delta_i$.

First notice that

$$1^2\delta_1(\{1\}) = 1$$
$$2^2\delta_2(\{2\}) = 4$$
$$3^2\delta_3(\{3\}) = 9$$
$$\vdots$$
$$10^2\delta_{10}(\{10\}) = 100$$

Also,

$$
\begin{aligned}
F(1) &= \sum_{i=1}^{10} i^2\delta_i((0,1]) = 1 \\
F(2) &= \sum_{i=1}^{10} i^2\delta_i((0,2]) = 1+4 = 5 \\
F(3) &= \sum_{i=1}^{10} i^2\delta_i((0,3]) = 1+4+9 = 14 \\
&\vdots \\
F(10) &= \sum_{i=1}^{10} i^2\delta_i((0,10]) = 1+4+9+\cdots+100
\end{aligned}
$$

Thus,

$$
F(E) = \begin{cases} \displaystyle\sum_{i=1}^{\lfloor x\rfloor} i^2 & x < 10 \\ \\ \displaystyle\sum_{i=1}^{10} i^2 & x \geq 10 \end{cases}
$$

(2) Let F be increasing and right continuous; let μ_F denote the associated measure.

$$\boxed{\text{Prove } \mu_F(\{a\}) = F(a) - F(a-)}$$

First, notice $\{a\} = \bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, a\right]$. So, $\mu_F(\{a\}) = \mu_F\left(\bigcap_{n=1}^{\infty}\left(a-\frac{1}{n}, a\right]\right)$.

Measures are continuous from above so, $\mu_F(\{a\}) = \mu_F\left(\bigcap_{n=1}^{\infty}\left(a-\frac{1}{n}, a\right]\right) = \lim_{n\to\infty}\mu_F\left(\left(a-\frac{1}{n}, a\right]\right)$

$$= \lim_{n\to\infty} F(a) - F\left(a - \frac{1}{n}\right) = F(a) - \lim_{n\to\infty} F\left(a - \frac{1}{n}\right) = F(a) - F(a-).$$

$$\boxed{\text{Prove } \ \mu_F((a,b)) = F(b-) - F(a)}$$

First, notice $(a,b)=\bigcup_{n=1}^{\infty}\left(a,b-\frac{1}{n}\right]$. So, $\mu_F((a,b))=\mu_F\left(\bigcup_{n=1}^{\infty}\left(a,b-\frac{1}{n}\right]\right)$. Measures are continuous

from below so, $\mu_F((a,b))=\mu_F\left(\bigcup_{n=1}^{\infty}\left(a,b-\frac{1}{n}\right]\right)=\lim_{n\to\infty}\mu_F\left(\left(a,b-\frac{1}{n}\right]\right)$

$$=\lim_{n\to\infty}\left(F\left(b-\frac{1}{n}\right)-F\left(a\right)\right)=\lim_{n\to\infty}\left(F\left(b-\frac{1}{n}\right)\right)-F\left(a\right)=F(b-)-F(a).$$

$$\boxed{\text{Prove }\ \mu_F([a,b))=F(b-)-F(a-)}$$

First, notice $[a,b)=\{a\}\cup(a,b)$ and $\{a\}\cap(a,b)=\emptyset$. μ_F is a pre-measure, so $\mu_F([a,b))=\mu_F(\{a\}\cup(a,b))$

$=\mu_F(\{a\})+\mu_F((a,b))$. From the two previous conclusions, we have

$$\mu_F(\{a\})+\mu_F((a,b))=F(a)-F(a-)+F(b-)-F(a)=F(b-)-F(a-).$$

$$\boxed{\text{Prove }\ \mu_F([a,b])=F(b)-F(a-)}$$

First, notice $[a,b]=\{b\}\cup[a,b)$ and $\{b\}\cap[a,b)=\emptyset$. μ_F is a pre-measure, so $\mu_F([a,b])=\mu_F(\{b\}\cup[a,b))$

$=\mu_F(\{b\})+\mu_F([a,b))$. From part 1 and 3 of this exercise, we have

$$\mu_F(\{b\})+\mu_F([a,b))=F(b)-F(b-)+F(b-)-F(a-)=F(b)-F(a-).$$

(3) Use the previous exercise to describe a Borel measure so that the four quantities given are different where $a=0$ and $b=1$.

Consider the function

$$F(x)=\begin{cases}1 & -\infty\le x<0\\ 3 & 0\le x<\frac{1}{2}\\ 9 & \frac{1}{2}\le x<1\\ 27 & 1\le x<\infty\end{cases}$$

If F is as defined above with $a = 0$, $b = 1$, then,

$$\begin{aligned}
F(a) - F(a-) = F(0) - F(0-) &= 3 - 1 &&= 2 \\
F(b-) - F(a) = F(1-) - F(0) &= 9 - 3 &&= 6 \\
F(b-) - F(a-) = F(1-) - F(0-) &= 9 - 1 &&= 8 \\
F(b) - F(a-) = F(1) - F(0-) &= 27 - 1 &&= 26
\end{aligned}.$$

5.3. **October 12 Group Assignment.** Exercise 32 from §1.5.

(1) Prove that

$$\prod_{j=1}^{\infty}(1 - a_j) > 0 \text{ if and only if } \sum_{j=1}^{\infty} a_j < \infty$$

Proof. To prove the desired statement, we will use the limit comparison test to show that $\sum_{j=1}^{\infty} a_j$ converges if and only if $\sum_{j=1}^{\infty} \log(1 - a_j)$ converges.

($\Leftarrow$) Suppose $\sum a_j$ converges. If we assume $a_j \to 0$,

$$\lim_{j\to\infty} \frac{a_j}{\log(1 - a_j)} = \lim_{x\to 0} \frac{x}{\log(1 - x)}.$$

$$\text{Applying L'Hospital's Rule, } \lim_{x\to 0} \frac{x}{\log(1 - x)} = \lim_{x\to 0} \frac{1}{\frac{-1}{1-x}} = -1.$$

Thus, if $a_j \to 0$, $\lim_{j\to\infty} \frac{a_j}{\log(1-a_j)} = -1$. By the limit comparison test,

$$\lim_{n\to\infty} \sum_{j=1}^{n} a_j \text{ converges if and only if } \lim_{n\to\infty} \sum_{j=1}^{n} \log(1 - a_j) \text{ converges. Thus,}$$

$$\text{since } \log(1 - a_j) < 0 \text{ for all } j. \sum_{j=1}^{\infty} \log(1 - a_j) > -\infty \text{ if and only if } \sum_{j=1}^{\infty} a_j < \infty$$

Since $\sum_{j=1}^{\infty} \log(1 - a_j) > -\infty$, let $\sum_{j=1}^{\infty} \log(1 - a_j) = k$ for $-\infty < k < 0$. Equivalently,

$$10^{\sum_{j=1}^{\infty} \log(1-a_j)} = 10^k \text{ implies } \prod_{j=1}^{\infty}(1 - a_j) = 10^k > 0.$$

($\Rightarrow$) Assume $\prod_{j=1}^{\infty}(1-a_j) > 0$. Then, let $\prod_{j=1}^{\infty}(1-a_j) = k$ for $k > 0$. Then,

$$\log\left(\prod_{j=1}^{\infty}(1-a_j)\right) = \sum_{j=1}^{\infty}\log(1-a_j) = \log(k) > -\infty \text{ implies } \lim_{n\to\infty}\sum_{j=1}^{n}\log(1-a_j) \text{ converges.}$$

Suppose $\lim_{j\to\infty}\log(1-a_j) = 0$. Since log is a continuous function, we can write $\log(1 - \lim_{j\to\infty} a_j) = 0$ which implies $\lim_{j\to\infty} a_j = 0$ Then, as shown above, the limit comparison test implies $\lim_{n\to\infty}\sum_{j=1}^{n} a_j$ converges so $\sum_{j=1}^{\infty} a_j < \infty$. $\square$

(2) Given $\beta \in (0,1)$, exhibit a sequence $\{a_j\}$ such that $\prod(1-a_j) = \beta$.

If $\prod(1-a_j) = \beta$, then $\ln\left(\prod(1-a_j)\right) = \ln\beta$ and $\sum\ln(1-a_j) = \ln\beta$. From part (1), we know that $\sum\ln(1-a_j)$ converges implies $\sum a_j$ converges. Also, $\ln\beta = \sum\frac{\ln\beta}{2^j}$ so $\sum\ln(1-a_j) = \sum\frac{\ln\beta}{2^j}$ implies

$$e^{\ln(1-a_j)} = e^{\frac{\ln\beta}{2^j}}. \text{ Equivalently, } 1-a_j = e^{\ln\beta^{-2^j}} = \beta^{-2^j}.$$

Thus, the sequence $a_j = 1-\beta^{-2^j}$ satisfies $\prod(1-a_j) = \beta$.

6. Section 2.1 (Folland): Measurable Functions

6.1. **Section 2.1 Definitions and Theorems.**

Remark 7. Any mapping $f : X \to Y$ between two sets induces a mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $f^{-1}(E) = \{x \in X : f(x) \in E\}$ which preserves unions, intersections, and complements. Thus, if $\mathcal{N}$ is a σ-algebra on Y, then $\{f^{-1}(E) : E \in \mathcal{N}\}$ is a σ-algebra on X

Definition 26 (measurable functions)**.** If $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ are measurable spaces, a mapping $f : X \to Y$ is called $(\mathcal{M},\mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Proposition 8. If $\mathcal{N}$ is generated by $\mathcal{E}$, then f is $(\mathcal{M},\mathcal{N})$-measurable iff $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$

Corollary 1. X, Y are topological spaces, every continuous $f : X \to Y$ is $(\mathcal{B}_x, \mathcal{B}_y)$-measurable.

Proposition 9. $(X,\mathcal{M})$ is a measure space and $f : X \to \mathbb{R}$, then

$$f \text{ is } \mathcal{M}-\text{ measurable } \iff f^{-1}((a,\infty)) \in \mathcal{M} \text{ for for all } a \in \mathbb{R} \iff f^{-1}([a,\infty)) \in \mathcal{M} \text{ for for all } a \in \mathbb{R}$$

$$\iff f^{-1}((-\infty,a)) \in \mathcal{M} \text{ for for all } a \in \mathbb{R} \iff f^{-1}((-\infty,a]) \in \mathcal{M} \text{ for for all } a \in \mathbb{R}$$

Definition 27 (simple functions)**.** A finite linear combination with complex coefficients of characteristic functions of sets in $\mathcal{M}$.

Definition 28 (standard representation of f)**.**

$$f = \sum_1^n z_j \chi_{E_j}, \text{ where } E_j = f^{-1}(\{z_j\}) \text{ and } \operatorname{range}(f) = \{z_1, \ldots, z_n\}.$$

6.2. **October 14 Group Assignment.**

(1) (exercise 8, §2.1): Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone then it is measurable.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. By Proposition 2.3 in Page 44, it suffices to show that for any $a \in \mathbb{R}$, we have $f^{-1}((a,\infty))$ is Borel measurable. WLOG, assume f is increasing. Let $x^{'} = \inf\{x : f(x) > a\}$

Case 1: Suppose $f(x^{'}) \leq a$**.** We will show $f^{-1}((a,\infty)) = (x^{'},\infty)$. First, show $f^{-1}((a,\infty)) \subseteq (x^{'},\infty)$. Let $x \in f^{-1}((a,\infty))$. Then, $f(x) > a$. Since $x^{'} = \inf\{x : f(x) > a\}$, $x^{'} < x$. Thus, $x \in (x^{'},\infty)$.

Next, show $f^{-1}((a,\infty)) \supseteq (x^{'},\infty)$. Let $x \in (x^{'},\infty)$. Then, $x > x^{'}$. Since $x^{'} = \inf\{x : f(x) > a\}$ and $x > x^{'}$, there exists some $x_0 \in \mathbb{R}$ such that $x > x_0 > x^{'}$ and $f(x_0) > a$. f is monotone, so $f(x) > f(x_0)$. Thus, $f(x) > a$ which implies $x \in f^{-1}((a,\infty))$.

Case 2: Suppose $f(x^{'}) > a$. We will show $f^{-1}((a,\infty)) = (x^{'},\infty)$. First, show $f^{-1}((a,\infty)) \subseteq (x^{'},\infty)$. Let $x \in f^{-1}((a,\infty))$. Then, $f(x) > a$. Since $x^{'} = \inf\{x : f(x) > a\}$, $x^{'} < x$. Thus, $x \in (x^{'},\infty)$.

Next, show $f^{-1}((a,\infty)) \supseteq (x^{'},\infty)$. Let $x \in (x^{'},\infty)$. Then, $x > x^{'}$. Since f is monotone, so $f(x) > f(x^{'}) > a$. Thus, $f(x) > a$ which implies $x \in f^{-1}((a,\infty))$.

Case 3: Suppose $f(x^{'}) = \infty$. We will show $f^{-1}((a,\infty)) = \emptyset$. If $f(x^{'}) = \infty$, $f(\inf\{x : f(x) > a\}) = \infty$ which implies $\inf\{x : f(x) > a\} = \infty$ so $\{x : f(x) > a\} = \emptyset$. Thus, $f^{-1}((a,\infty)) = \emptyset$.

Case 4: Suppose $f(x^{'}) = -\infty$. We will show $f^{-1}((a,\infty)) = \mathbb{R}$. If $f(x^{'}) = -\infty$,

$f(\inf\{x : f(x) > a\}) = -\infty$ which implies $\inf\{x : f(x) > a\} = -\infty$ so $\{x : f(x) > a\} = \mathbb{R}$. Thus, $f^{-1}((a,\infty)) = \mathbb{R}$.

Hence, for any $a \in \mathbb{R}$, we have $f^{-1}((a,\infty))$ is Borel measurable, so f is measurable. $\square$

(2) (exercise 5, §2.1): If $X = A \cup B$ with $A, B \in \mathcal{M}$, then a function f on X is measurable if and only if f is measurable on A and B.

Proof. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$. First, assume f is measurable on $X = A \cup B \in \mathcal{M}$. Then, for all $N \in \mathcal{N}$, $f^{-1}(N) \in \mathcal{M}$. Since $A, B \in \mathcal{M}$, $f^{-1}(N) \cap A \in \mathcal{M}$ and $f^{-1}(N) \cap B \in \mathcal{M}$ for all $N \in \mathcal{N}$. Thus, f is measurable on A and f is measurable on B.

Next, assume f is measurable on A and f is measurable on B. Then, for all $N \in \mathcal{N}$, $f^{-1}(N) \cap A \in \mathcal{M}$ and $f^{-1}(N) \cap B \in \mathcal{M}$. This implies $(f^{-1}(N) \cap A) \cup (f^{-1}(N) \cap B) \in \mathcal{M}$. Since $(f^{-1}(N) \cap A) \cup (f^{-1}(N) \cap B) = f^{-1}(N) \cap (A \cup B)$, f is measurable on $A \cup B$.

$\square$

`http://www.math.brown.edu/~rkenyon/teaching/2009/2210/Set3.pdf`

6.3. **October 16 Group Assignment.**

(1) (exercise 3, §2.1): If $\{f_n\}$ is a sequence of measurable functions then the set $\{x : \lim_{n\to\infty} f_n(x)$ exists$\}$ is measurable.

Proof. Assume $\{f_n\}$ is a sequence of measurable functions. Consider $\{x : \lim_{n\to\infty} f_n(x)$ exists$\}$. If $\lim_{n\to\infty} f_n(x)$ exists, $\limsup f_n(x) = \lim f_n(x)$. Thus,

$$\{x : \lim_{n\to\infty} f_n(x) \text{ exists}\} = \{x : \limsup f_n(x) = \lim f_n(x)\}.$$

By, proposition 2.7, $g_3(x) = \limsup f_n(x)$ and $g_4(x) = \lim f_n(x)$ are measurable functions. Define $g = g_3 - g_4$. Then, g is a measurable function. If $g(x) = 0$, then $x \in \{x : \lim_{n\to\infty} f_n(x)$ exists$\}$. Since g is measurable, $g^{-1}(\{0\}) = \{x : \lim_{n\to\infty} f_n(x)$ exists$\}$ is measurable. $\square$

(2) (exercise 6, §2.1): Show by example that there is an uncountable set A and for each $a \in A$ a measurable function f_a, but $\sup\{f_\alpha : \alpha \in A\}$ is not measurable.

Consider the set N_r constructed in section 1.1. Then N_r is an uncountable set and therefore not measurable. However, for every $r \in \mathbb{Q} \cap [0,1)$ and $x \in N$ (where N was defined as the subset of $[0,1)$ containing exactly one member of the equivalence classes defined by $x \sim y$ iff $x - y \in \mathbb{Q}$. Singletons are measurable, so, from page 46 of Folland, the indicator functions $\chi_{\{x+r\}}$ and $\chi_{\{x_r-1\}}$ are measurable for all $r \in \mathbb{Q} \cap [0,1)$ and $x \in N \cap [0, 1-r)$ or $x \in N \cap [1-r, 1)$. Notice $\sup\{\chi_{\{x+r\}}, \chi_{\{x+r-1\}} : r \in \mathbb{Q} \cap [0,1)$ and $x \in N \cap [0,1-r)$ or $x \in N \cap [1-r,1)\} = \chi_{N_r}$ But, χ_{N_r} is not measurable because N_r is not measurable.

7. Section 2.2 (Folland): Integration of Non-Negative Functions

7.1. **Section 2.2 Definitions and Theorems.** Fix a measure space $(X, \mathcal{M}, \mu)$.

Definition 29 (L^+)**.** The space of all measurable functions from X to $[0, \infty]$.

Definition 30. If ϕ is a simple function in L^+ with standard representation

$$\phi = \sum_1^n a_j \chi_{E_j}, \text{ then, } \int \phi \; d\mu = \sum_1^n a_j \mu(E_j).$$

Remark 8.

$$\int_A \phi d\mu = \int_A \phi = \int_A \phi(x) d\mu(x) = \int \phi \; \chi_A \; d\mu$$

Proposition 10. Let ϕ be simple functions in L^+, then $A \to \int_A \phi d\mu$ is a measure on $\mathcal{M}$.

Theorem 8 (monotone convergence theorem)**.** If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n$, then $\int f = \lim_{n\to\infty} \int f_n$

Theorem 9 (Fatou's Lemma)**.** If $\{f_n\}$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

Corollary 2. If $\{f_n\} \subset L^+$, $f \in L^+$, and $f_n \to f$ a.e., then $\int f \leq \liminf \int f_n$.

Proposition 11. If $f \in L^+$ and $\int f < \infty$, then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finte.

7.2. **October 19 Group Assignment.**

(1) Prove if φ and ψ are simple functions, then $\varphi + \psi$ and $\varphi \cdot \psi$ are simple functions.

Proof. Assume φ and ψ are simple functions. Then, $\varphi = \sum_{j=1}^{n} z_j \chi_{E_j}$ and $\psi = \sum_{i=1}^{m} \alpha_i \chi_{A_i}$. Then, φ and ψ are measurable. The sum of measurable functions is a measurable function, so $\varphi + \psi$ is measurable. Since φ and ψ are simple, their range is finite. Let $\text{ran}\psi = \{a_i\}_{i=1}^{n}$ and $\text{ran}\varphi = \{b_j\}_{j=1}^{m}$ The range of $\varphi + \psi \subseteq \bigcup_{j=1}^{m}(\{a_i\}_{i=1}^{n} + b_j)$. Thus, $\psi + \varphi$ is simple. Similarly, if φ, ψ are measurable, then $\varphi \cdot \psi$ is measurable. Also, the range of $\varphi \cdot \psi \subseteq \bigcup_{j=1}^{m}(\{a_i\}_{i=1}^{n} \cdot b_j)$ which is finite. Thus, $\varphi \cdot \psi$ is simple. □

`http://math.sfsu.edu/schuster/Assignment_08_03.pdf`

(2) Assume that $(X, \mathcal{M}, \mu)$ is complete. (a) If f is $\mathcal{M}$-measurable functions and $f = g$ μ almost everywhere, then g is $\mathcal{M}$-measurable.

Proof. Assume f is a $\mathcal{M}$-measurable function and $f = g$ μ almost everywhere. Define $A = \{x : f(x) = g(x)\}$ and $B = \{x : f(x) \neq g(x)\}$. Because f is measurable $f^{-1}((a, \infty)) \cap A \in \mathcal{M}$. Also, $f(x) = g(x)$ for all $x \in A$ so $f^{-1}((a, \infty)) \cap A = g^{-1}((a, \infty)) \cap A \in \mathcal{M}$. Since $g = f$ μ almost everywhere, $\mu(B) = 0$. Additionally, μ is complete, so $\mu(g^{-1}((a, \infty)) \cap B) \leq \mu(B) = 0$. Thus, $\mu(g^{-1}((a, \infty)) \cap B) = 0$ implies $g^{-1}((a, \infty)) \cap B \in \mathcal{M}$. Notice $X = A \cup B$, so by exercise 5 in §2.1, g is measurable. □

(b) If f_n is a sequence of $\mathcal{M}$-measurable functions such that $f_n \to f$ μ-almost everywhere, then f is $\mathcal{M}$-measurable.

Proof. Assume f_n is a sequence of $\mathcal{M}$-measurable functions such that $f_n \to f$ μ almost everywhere. Define $A = \{x : f_n(x) \to f(x)\}$ and $B = \{x : f_n(x) \not\to f(x)\}$. Because f_n are measurable $f_n^{-1}(N) \cap A \in \mathcal{M}$ for all $N \in \mathcal{B}_{\mathbb{R}}$. Also, $f_n(x) \to f(x)$ for all $x \in A$ so $\lim_{n\to\infty} f_n^{-1}(N) \cap A = f^{-1}(N) \cap A \in \mathcal{M}$. Since $f_n(x) \to f(x)$ μ almost everywhere, $\mu(B) = 0$. Additionally, μ is complete, so $\mu(f^{-1}(N) \cap B) \leq \mu(B) = 0$. Thus, $\mu(f^{-1}(N) \cap B) = 0$ implies $f^{-1}(N) \cap B \in \mathcal{M}$. Notice $X = A \cup B$, so by exercise 5 in §2.1, f is measurable. □

7.3. **October 21 Group Assignment.**

(1) If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Then, λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$.

Proof. First, we will show that λ is a measure. Notice

$$\lambda(\emptyset) = \int_{\emptyset} f \, d\mu = \int f\chi_{\emptyset} \, d\mu = \int f \cdot 0 \, d\mu = 0. \text{ Thus, } \lambda(\emptyset) = 0.$$

Next consider a disjoint collection of sets $\{E_j\}_{i=1}^{\infty} \subset \mathcal{M}$. Then,

$$\lambda\left(\bigcup_{j=1}^{\infty} E_i\right) = \int_{\bigcup_{j=1}^{\infty} E_i} f \, d\mu = \int f\chi_{\{\bigcup_{j=1}^{\infty} E_i\}} \, d\mu = \int f \sum_{i=1}^{\infty} \chi_{E_i} \, d\mu$$

$$= \sum_{i=1}^{\infty} \int f\chi_{E_i} d\mu = \sum_{i=1}^{\infty} \int_{E_i} f d\mu = \sum_{i=1}^{\infty} \lambda(E_i).$$

Therefore λ is countably sub-additive over disjoint unions; λ is a measure.

Next, by Theorem 2.10, $g \in L^+$ implies there is a sequence $\{g_j\}$ of simple functions such that $0 \leq g_1 \leq g_2 \leq \cdots \leq g$ such that $g_j \to g$ pointwise. Since g_j are simple functions, for all j we can write $g_j = \sum_{i=1}^{n} z_i\chi_{E_i}$ for $E_i = g_j^{-1}(\{z_i\})$ where $\text{range}(g_j) = \{z_1, z_2, \cdots, z_n\}$. Then, for all g_j we have

$$\begin{aligned}
\int g_j \, d\lambda &= \sum_{i=1}^{n} z_i\lambda(E_i) && \text{definition of integral of simple functions on p. 49} \\
&= \sum_{i=1}^{n} z_i \int_{E_i} f d\mu && \text{definition of } \lambda \\
&= \int_{E_i} \sum_{i=1}^{n} z_i f d\mu \\
&= \int \sum_{i=1}^{n} z_i\chi_{E_i} f d\mu \\
&= \int g_j f d\mu
\end{aligned}$$

Then, by the Monotone Convergence Theorem, since $g_j \to g$,

$$\int g \, d\lambda = \lim_{j\to\infty} \int g_j \, d\lambda = \lim_{j\to\infty} \int g_j f d\mu = \int \lim_{j\to\infty} g_j f d\mu = \int g f d\mu$$

$\square$

(2) If $f \in L^+$ and $\int f < \infty$ for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > \left(\int f\right) - \epsilon$.

Proof. Assume $f \in L^+$ and $\int f < \infty$. Since f is measurable and $\int f < \infty$, f is integrable and defined as

$$\int f \, d\mu = \sup\left\{\int \phi \, d\mu : 0 \le \phi \le f, \text{ where } \phi \text{ is simple}\right\}.$$

Thus, for any $\epsilon > 0$, there exists some simple function ϕ such that

$$\int f \, d\mu - \epsilon < \int \phi \, d\mu < \infty.$$

Because ϕ is a simple function we can write

$$\phi = \sum_{i=1}^{\infty} a_i \chi_{E_i} \text{ and so } \int \phi \, d\mu = \sum_{i=1}^{\infty} a_i \mu(E_i) < \infty.$$

If $\mu(E_k) = \infty$ for some k, then since $\sum_{i=1}^{\infty} a_i \mu(E_i) < \infty$, $a_k = 0$. So, rewrite $\phi = \sum_{i=1}^{\infty} \alpha_i \mu(B_i)$ with $\alpha_i \neq 0$ and $\mu(B_i) < \infty$. Since $\mu(B_i) < \infty$, $\mu(\bigcup_{i=1}^{\infty} B_i) < \infty$. Now, let $E = \bigcup_{i=1}^{\infty} B_i$, so $\mu(E) < \infty$. Since the $\phi = 0$ everywhere outside of E and because $\phi \le f$ we can write

$$\int \phi \, d\mu = \int_E \phi \, d\mu \le \int_E f \, d\mu.$$

$$\text{Therefore, } \int f \, d\mu - \epsilon < \int \phi \, d\mu \le \int_E f \, d\mu.$$

Thus, for every $\epsilon > 0$, there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and

$$\int_E f > \left(\int f\right) - \epsilon.$$

$\square$

7.4. October 23 Group Assignment.

(1) Suppose $\{f_n\}$ is a countable set of functions in L^+. If $f_n \to f$ pointwise and $\int f = \lim \int_E f_n$ for any $E \in \mathcal{M}$.

(2) Suppose $\{f_n\}$ is a countable set of functions in L^+. If $f_n \geq f_{n+1}$ for all n, $f_n \to f$ and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

Proof. Define $g_n = f_1 - f_n$. Then since $f_n \geq f_{n+1}$ for all n, g_n is an increasing sequence of functions. Since $f_1 \in L^+$ and $f_n \in L^+$, $g \in L^+$. Also, $g_n \to f_1 - f$. By the Monotone Convergence Theorem, $\int(f_1 - f) = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \int(f_1 - f_n)$. Thus,

$$\begin{aligned}
\lim_{n\to\infty} \int(f_1 - f_n) = &\quad \int(f_1 - f) \\
\lim_{n\to\infty} \int f_1 - \lim_{n\to\infty} f_n = &\quad \int(f_1 - f) \\
\int f_1 - \lim_{n\to\infty} f_n = &\quad \int f_1 - \int f \\
-\lim_{n\to\infty} f_n = &\quad -\int f \\
\lim_{n\to\infty} f_n = &\quad \int f.
\end{aligned}$$

$\square$

8. Section 2.3 (Folland): Integration of Complex Functions

8.1. Section 2.3 Definitions and Theorems.

Definition 31 (integrable)**.**

$$\int f = \int f^+ - \int f^-, \quad \text{If } \int f^+ \text{ and } \int f^- \text{ are both finite, we say that } f \text{ is integrable}$$

$$|f| = f^+ + f^- \text{ so } f \text{ is integrable iff } \int |f| < \infty$$

Definition 32 (L^1)**.** Space of complex-valued integrable functions

Proposition 12. If $f \in L^1$, then $\{x : f(x) \neq 0\}$ is σ-finite.

Definition 33 (convergence in L^1)**.** $f_n \to f$ iff $\int |f_n - f| \to 0$.

Theorem 10 (dominated convergence theorem)**.** Let $\{f_n\}$ be a sequence in L^1. If $f_n \to f$ a.e., and there exists a nonnegative $g \in L^1$ such that $|f_n| \leq g$ a.e. for all n, then

$$f \in L^1 \quad \text{and} \quad \int f = \lim_{n\to\infty} \int f_n$$

Theorem 11. If $f \in L^1(\mu)$ and $\epsilon > 0$, there is an integrable simple function $\phi = \sum a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$.

8.2. **October 26 Group Assignment.**

(1) Suppose $\{f_n\}_{n=1}^{\infty}$ is a countable sequence of functions in $L^1(\mu)$ and that $f_n \to f$ uniformly.

(a) If $\mu(X) < \infty$, prove that $f \in L^1(\mu)$ and $\int f_n \to \int f$.

Proof. Let $\epsilon > 0$ and $g = \max\{|f_1|, |f_2|, \ldots, |f_N|\} + 2\epsilon$. We will show $|f_n| \leq g$ for any n. If $n \leq N$, then $|f_n| \leq \max\{|f_1|, |f_2|, \ldots, |f_N|\} \leq g + 2\epsilon$ so suppose $n > N$. Since $f_n \to f$ uniformly, for any $\epsilon > 0$, there exists an $M \geq 0$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n > M$ and $x \in X$. Then, $|f_n| < |f| + \epsilon$. So, for $n = N$, we can write $|f| < |f_N| + \epsilon$. Thus,

$$|f_n| < |f| + \epsilon < |f_N| + \epsilon + \epsilon \leq g.$$

Since $\int |f_i| d\mu < \infty$ for any i and $\mu(X) < \infty$, let $\int |f_i| d\mu = C$ and $\mu(X) = K$, then

$$\int g d\mu = \int |f_i| d\mu + \int 2\epsilon d\mu = C + 2\epsilon\mu(X) = C + 2\epsilon K < \infty.$$

Thus, $g \in L^1$. Hence, by the dominated convergence theorem, $f \in L^1(\mu)$ and $\int f_n \to \int f$. $\square$

(b) Find an example to prove that the previous can be false if $\mu(X) = \infty$.

$$\text{Let } f_n = \frac{1}{2n}\chi_{[-n,n]}. \quad \text{Then, } \mu([-n,n]) = \infty. \text{ Note, } \int f_n = \frac{1}{2n}(2n) = 1.$$

Also, $f_n \to 0$ and $\int 0 = 0$. However $\int f_n = 1$ but $\int f = 0$ so 1(a) does not apply if $\mu(X) = \infty$.

(2)

$$\text{Let } f(x) = \begin{cases} x^{-\frac{1}{2}} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

If $\{r_n\}$ is an enumeration of the rationals, set

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n). \text{ Prove that } g \in L^1(m) \text{ but } g^2 \text{ is not.}$$

Proof.

$$\begin{aligned} \text{Notice } \int_{(r_j,1+r_j)} (x-r_1)^{-\frac{1}{2}} dm &= \int (x-r_1)^{-\frac{1}{2}} \chi_{(r_j,1+r_j)} dm \\ &= \int (x-r_1)^{-\frac{1}{2}} \lim_{n\to\infty} \chi_{(r_j+\frac{1}{n},1+r_j)} dm \\ &= \lim_{n\to\infty} \int (x-r_1)^{-\frac{1}{2}} \chi_{(r_j+\frac{1}{n},1+r_j)} dm \end{aligned}$$

$$\text{Note that } f_j = f(x-r_j)\chi_{(\frac{1}{n}+r_j,1+r_j)} = \begin{cases} (x-r_j)^{-\frac{1}{2}} & \text{if } r_j + \frac{1}{n} < x < 1 + r_j \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $f_j < \sqrt{n}$. So, by theorem 2.28, f_j is Lebesgue measurable and the Riemann integral is equal to the Lebesgue integral on an interval. Thus,

$$\lim_{n\to\infty}\int (x-r_1)^{-\frac{1}{2}}\chi_{(r_j+\frac{1}{n},1+r_j)}dm = \lim_{n\to\infty}\int\limits_{(r_j+\frac{1}{n},1+r_j)} (x-r_1)^{-\frac{1}{2}}dm$$
$$= \lim_{n\to\infty}\int\limits_{r_j+\frac{1}{n}}^{1+r_j} (x-r_1)^{-\frac{1}{2}}dx$$
$$= \int\limits_{r_j}^{1+r_j} (x-r_1)^{-\frac{1}{2}}dx$$
$$= 2$$

Thus, $f_j \in L^1$ and $2^{-j} \in L^1$ and so by theorem 2.25, $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x-r_n) = \sum_{n=1}^{\infty} 2^{-n} f_n \in L^1$. Also, by theorem 2.25,

$$\int gdm = \int\limits_{(r_j,r_j+1)} \sum_{j=1}^{\infty} 2^{-j} f_j dm = \sum_{j=1}^{\infty} 2^{-j} \int\limits_{(r_j,r_j+1)} f_j dm = \sum_{j=1}^{\infty} 2^{-j} 2 = 2. \text{ Thus, } g \in L^1.$$

□

Proof. Next, we will show $g^2 \notin L^1$. First, notice

$$g^2 \geq \sum_{n=1}^{\infty} 2^{-2n}(f(x-r_n))^2 \text{ and } (f(x-r_n))^2 = \begin{cases} \frac{1}{x-r_n} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

On $(r_n, 1+r_n)$, $\frac{1}{x-r_n} < 1$, so, by theorem 2.28,

$$\int\limits_{(r_n,1+r_n)} \frac{dm}{x-r_n} = \int\limits_{r_n}^{1+r_n} \frac{dx}{x-r_n} = \int\limits_{0}^{1} \frac{dx}{x} = \infty$$

Thus, $g^2 \notin L^1$. □

8.3. **October 28 Group Assignment.**

(1) Let $f \in L^1(m)$ and $F(x) = \int_{[-\infty,x]} f(t)\ dm(t)$ then F is continuous on $\mathbb{R}$.

Proof. Note $F(t) = \int_{[-\infty,x]} f(t)\,dm(t) = \int f(t)\chi_{[-\infty,x]}dm(t)$. F is continuous if $\lim_{x_n\to x} F(x_n) = F(x)$ for any $x \in \mathbb{R}$. So, consider a sequence $\{x_n\}_{n=1}^{\infty} \to x$. Let $f_n = f\chi_{[-\infty,x_n]}$ so f_n converges pointwise to $f\chi_{[-\infty,x]}$ and $f_n \in L^1$. Because $|f_n| \le |f|$ and $f \in L^1$, the dominated convergence theorem implies $\int f = \lim_{n\to\infty} \int f_n$. Thus,

$$\begin{aligned}
\lim_{n\to\infty} F(x_n) &= \lim_{n\to\infty} \int_{[-\infty,x_n]} f(t)\ dm(t) \\
&= \lim_{n\to\infty} \int f(t)\chi_{[-\infty,x_n]}\ dm(t) \\
&= \int f(t) \lim_{n\to\infty} \chi_{[-\infty,x_n]}\ dm(t) \\
&= \int f(t)\chi_{[-\infty,x]}\ dm(t) \\
&= \int_{[-\infty,x]} f(t)\ dm(t) \\
&= F(x).
\end{aligned}$$

`http://faculties.sbu.ac.ir/~shahrokhi/M-P.pdf` □

(2) Compute the following, justifying your calculations:

$$\text{(a)}\ \lim_{n\to\infty} \int_0^{\infty} \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\ dx$$

Using the binomial theorem, we have

$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{x}{n}\right)^k = 1+x+\frac{(n-1)x^2}{2n}+\sum_{k=3}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k$. Thus, $\left(1+\frac{x}{n}\right)^n \ge 1+x+\frac{(n-1)x^2}{2n}$.

If $n \ge 2$, $\frac{(n-1)}{2n} \ge \frac{1}{4}$. So, if $n \ge 2$, $\left(1+\frac{x}{n}\right)^n \ge 1+x+\frac{x^2}{4}$. Equivalently, $\left(1+\frac{x}{n}\right)^{-n} \le \frac{1}{1+x+\frac{x^2}{4}}$.

Therefore, $\left|\frac{\sin\left(\frac{x}{n}\right)}{\left(1+\frac{x}{n}\right)^n}\right| \le \frac{1}{1+x+\frac{x^2}{4}}$. $g(x) = \frac{1}{1+x+\frac{x^2}{4}}$ is Riemann integrable, so $g \in L^1$.

Thus, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_0^{\infty} \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\ dx = \int_0^{\infty} \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\ dx = 0$$

$$\text{(b)} \lim_{n\to\infty}\int_0^1 \left(1+nx^2\right)\left(1+x^2\right)^{-n}\, dx$$

Using the binomial theorem, we have

$$\left(1+x^2\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(x^2\right)^k = 1+nx^2+\sum_{k=2}^{n}\binom{n}{k}\left(x^2\right)^k. \text{ Thus, } \left(1+x^2\right)^n \geq 1+nx^2$$

$$\text{and so } \left(1+x^2\right)^{-n} \leq \frac{1}{1+nx^2} \text{ implies } \left|\left(1+nx^2\right)\left(1+x^2\right)^{-n}\right| \leq 1.\ g=1\in L^1.$$

Thus, by the dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^1 \left(1+nx^2\right)\left(1+x^2\right)^{-n}\, dx = \int_0^1 \lim_{n\to\infty}\left(1+nx^2\right)\left(1+x^2\right)^{-n}\, dx = 0$$

$$\text{(c)} \lim_{n\to\infty}\int_0^\infty n\sin\left(\frac{x}{n}\right)\left(x(1+x^2)\right)^{-1}\, dx$$

Note $\sin\left(\frac{x}{n}\right) \leq \frac{x}{n}$ on $[0,\infty)$ so

$$\left|n\sin\left(\frac{x}{n}\right)\left(x(1+x^2)\right)^{-1}\right| \leq n\frac{x}{n}\left(x(1+x^2)\right)^{-1} = \left(1+x^2\right)^{-1}.$$

Since $g=\left(1+x^2\right)^{-1}$ is Riemann integrable and bounded $g\in L^1$.

Thus, by the dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^\infty n\sin\left(\frac{x}{n}\right)\left(x(1+x^2)\right)^{-1}\, dx = \int_0^\infty \lim_{n\to\infty} n\sin\left(\frac{x}{n}\right)\left(x(1+x^2)\right)^{-1}\, dx = \int_0^\infty \left(1+x^2\right)^{-1}\, dx = \frac{\pi}{2}.$$

8.4. **November 2 Group Assignment.**

(1) Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0<a<b$. Verify the following:

$$\text{(a)} \sum_{n=1}^{\infty}\int_0^\infty |f_n(x)|\, dx = \infty$$

$$\sum_{n=1}^{\infty}\int_0^\infty |f_n(x)|\, dx \geq \sum_{n=1}^{\infty}\int_{\frac{1}{an}}^\infty \left|ae^{-nax}-be^{-nbx}\right|\, dx \geq \sum_{n=1}^{\infty}\left|\int_{\frac{1}{an}}^\infty ae^{-nax}-be^{-nbx}\right|\, dx.$$

$$\text{Note, } \int_{\frac{1}{an}}^{\infty} ae^{-nax} - be^{-nbx}\ dx = \lim_{k\to\infty}\left(-\frac{1}{ne^{nak}} + \frac{1}{ne^{nbk}}\right) + \frac{1}{ne^{na\frac{1}{an}}} - \frac{1}{ne^{nb\frac{1}{an}}} = 0 + \frac{1}{ne} - \frac{1}{ne^{\frac{b}{a}}}$$

$$\text{Thus, } \sum_{n=1}^{\infty}\int_0^{\infty} |f_n(x)|\ dx \geq \sum_{n=1}^{\infty}\left|\frac{1}{ne} - \frac{1}{ne^{\frac{b}{a}}}\right| = \left(e - e^{\frac{b}{a}}\right)\sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

$$\text{(b) } \sum_{n=1}^{\infty}\int_0^{\infty} f_n(x)\ dx = 0$$

$$\text{Note, } \int_{\frac{1}{an}}^{\infty} ae^{-nax} - be^{-nbx}\ dx = \lim_{k\to\infty}\left(-\frac{1}{ne^{nak}} + \frac{1}{ne^{nbk}}\right) + \frac{1}{ne^{na\cdot 0}} - \frac{1}{ne^{nb\cdot 0}} = 0 + \frac{1}{n} - \frac{1}{n}.$$

$$\text{Thus, } \sum_{n=1}^{\infty}\int_0^{\infty} f_n(x)\ dx = 0.$$

$$\text{(c) } \sum_{n=1}^{\infty} f_n(x) \in\ L^1\left([0,\infty), m)\right) \text{ and } \int_0^{\infty}\sum_{n=1}^{\infty} f_n(x)\ dx = \log\left(\frac{b}{a}\right)$$

$$\text{Notice } \sum_{n=1}^{\infty} f_n(x) = a\sum_{n=1}^{\infty}\left(\frac{1}{e^{ax}}\right)^n - b\sum_{n=1}^{\infty}\left(\frac{1}{e^{bx}}\right)^n = \frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}}. \text{ Thus,}$$

$$\int_0^{\infty}\sum_{n=1}^{\infty} f_n(x)\ dx = \int_0^{\infty}\frac{ae^{-ax}}{1-e^{-ax}} - \int_0^{\infty}\frac{be^{-bx}}{1-e^{-bx}} = \left[\ln\left|1-e^{-ax}\right| - \ln\left|1-e^{-bx}\right|\right]_0^{\infty} = \left[\ln\left|\frac{1-e^{-ax}}{1-e^{-bx}}\right|\right]_0^{\infty}.$$

$$\text{Additionally, } \left[\ln\left|\frac{1-e^{-ax}}{1-e^{-bx}}\right|\right]_0^{\infty}. = -\lim_{s\to 0}\ln\left|\frac{1-e^{-as}}{1-e^{-bs}}\right| = -\ln\left|\lim_{s\to 0}\frac{1-e^{-as}}{1-e^{-bs}}\right|$$

$$= -\ln\left|\lim_{s\to 0}\frac{ae^{-as}}{be^{-bs}}\right| = -\ln\left|\frac{a}{b}\right| = \ln\left(\frac{a}{b}\right)^{-1} = \ln\left(\frac{b}{a}\right)$$

Thus, $\int_0^{\infty}\sum_{n=1}^{\infty} f_n(x)\ dx < \infty$ so $\sum_{n=1}^{\infty} f_n(x)$ is measurable. Also, $\int_0^{\infty}\left|\sum_{n=1}^{\infty} f_n(x)\right|\ dx < \infty$. So, $\sum_{n=1}^{\infty} f_n(x) \in L^1$.

`http://faculties.sbu.ac.ir/~shahrokhi/M-P.pdf`

(b)

$$\sum\int_0^{\infty} f_n(x)dx = 0$$

Proof. For every $n \geq 1$,

$$\int_0^{\infty} f_n = \int_0^{\infty}\left(ae^{-nax} - be^{-nbx}\right)dx = \frac{-e^{-nax}}{n} + \frac{e^{-nbx}}{n}\Bigg|_0^{\infty} = 0.$$

$$\text{Thus, } \sum\int_0^{\infty} f_n(x)dx = 0.$$

$\square$

(3)

$$\sum f_n \in L^1([0,\infty)) \text{ and } \int_0^\infty \sum f_n(x)dx = \log(b/a).$$

Proof. Notice that $f = \sum f_n$ is the difference of two geometric series:

$$\sum f_n = a\sum (e^{ax})^{-n} - b\sum (e^{bx})^{-n}.$$

$$\text{Hence, } f_n(x) = \frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}} \text{ because } 0<a<b.$$

This is Riemann integrable as

$$\left[\ln\left|\frac{1-e^{-ax}}{1-e^{-bx}}\right|\right]_0^\infty = -\lim_{x\to 0}\left[\ln\left|\frac{1-e^{-ax}}{1-e^{-bx}}\right|\right] = -\left[\ln\left|\lim_{x\to 0}\frac{1-e^{-ax}}{1-e^{-bx}}\right|\right] = -\left[\ln\left|\lim_{x\to 0}\frac{ae^{-ax}}{be^{-bx}}\right|\right] = \ln\frac{b}{a}.$$

Because $-\infty < \int f < \infty$, we have $-\infty < \int f^+ - \int f^- < \infty$, so $\int |f| = \int f^+ + \int f^- < \infty$, and $f \in L^1([0,\infty))$. $\square$

9. Section 2.5 (Folland): Product Measures

9.1. **Section 2.5 Definitions and Theorems.** Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be measurable spaces.

Definition 34 (measurable rectangles)**.** A subset $E \subseteq X\times Y$ is called a measurable rectangle if $E = A\times B$ for some $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

Let $\mathcal{R}$ denote the class of all measurable rectangles. $\mathcal{R}$ is not, in general, a σ-algebra. It is a semi-algebra of subsets of $X\times Y$.

Remark 9.

$$(A\times B)\cap(E\times F) = (A\cap E)\times(B\cap F), \qquad (A\times B)^c = (X\times B^c)\cup(A^c\times B)$$

Thus, the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra.

Definition 35. $\mathcal{M}\otimes\mathbb{N}$ is the smallest σ-algebra generated by rectangles, $A\times B, A\in\mathcal{M}, B\in\mathcal{N}$.

Definition 36. $\mu\times\nu : \mathcal{M}\otimes\mathcal{N}\to[0,\infty]$. $(\mu\times\nu)(A\times B) = \mu(A)\nu(B)$.

Remark 10. If μ and ν are σ-finite, say

Definition 37. $E \in \mathcal{M} \otimes \mathcal{N}$. Fix x, $E^x = \{y : (x,y) \in E\} \subseteq Y$. Fix y, $E_y = \{x : (x,y) \in E\} \subseteq X$.

Definition 38. $f : X \times Y \to \mathbb{R}$. $f^x : Y \to \mathbb{R}$, $f^x(y) = f(x,y)$. $f_y : X \to \mathbb{R}$, $f_y(x) = f(x,y)$

Theorem 12. If f is $\mathcal{M} \otimes \mathcal{N}$ measurable, then f^x is $\mathcal{N}$ measurable and f_y is $\mathcal{M}$ measurable.

Remark 11. $f \in L^+$ implies f measurable and non-negative

Theorem 13 (Tonelli (positive functions))**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $f \in L^+$, then $g(x) = \int f_x d\nu \in L^+(\nu)$ and $h(y) = \int f^y d\mu \in L^+(\mu)$. Key point:

$$\int f d(\mu \times \nu) = \int \left(\int f(x,y) d\mu \right) d\nu = \int \left(\int f(x,y) d\nu \right) d\mu. \text{ So, } \int h(y) d\nu = \int g(x) d\mu$$

Theorem 14 (Fubini (integrable functions))**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $f \in L^1(\mu \times \nu$, then $f_x \in L^1(\nu)$ a.e. x and $f^y \in L^1(\mu)$ a.e. y, then

$$\begin{aligned} g(x) &= \int f_x d\nu \in L^1(\mu) \\ h(y) &= \int f^y d\mu \in L^1(\nu) \text{ and} \\ \int f d(\mu \times \nu) &= \int \left[\int f(x,y) d\mu \right] d\nu \\ &= \int \left[\int f(x,y) d\nu \right] d\mu \end{aligned}$$

9.2. **November 4 Group Assignment.**

(1) Let $X = Y = [0,1]$, $\mathcal{M} = \mathbb{N} = B_{[0,1]}$, μ is Lebesgue measure, and ν is the counting measure. If $D = \{(x,x) : x \in [0,1]\}$ then $\int\int \chi_D d\mu d\nu$, $\int\int \chi_D d\nu d\mu$, and $\int \chi_D d(\mu\times\nu)$ are all unequal. Verify this and explain why this does not contradict the Fubini-Tonelli Theorem.

Proof. First, calculate $\int\int \chi_D d\mu d\nu$, $\int\int \chi_D d\nu d\mu$:

$$\begin{aligned}
\int\int \chi_D d\mu d\nu &= \int\int (\chi_D)^y \, d\mu d\nu \\
&= \int \mu(D^y) d\nu \\
&= \int \mu(\{x\}) d\nu \\
&= \int 0 d\nu \\
&= 0 \\
\int\int \chi_D d\nu d\mu &= \int\int (\chi_D)_x \, d\nu d\mu \\
&= \int \nu(D^y) d\mu \\
&= \int \nu(\{x\}) d\mu \\
&= \int 1 d\mu \\
&= 1 \cdot \mu([0,1]) \\
&= 1.
\end{aligned}$$

Next, consider $\int \chi_D d(\mu \times \nu) = (\mu \times \nu)(D)$. By definition,

$$(\mu \times \nu)(D) = \inf \left\{ (\mu \times \nu) \left(\bigcup_1^\infty (A_i \times B_i) \right) : D \subseteq \bigcup_1^\infty (A_i \times B_i) \right\}.$$

Since $m([0,1]) = \infty$, $\mu \times \nu(D) \geq \infty$ because if it were less than ∞ we have some countable collection of rectangles $\{A_i \times B_i\}$ such that $D \subseteq \bigcup_1^\infty (A_i \times B_i)$ but $\cup B_i$ countable which contradicts $[0,1]$ uncountable.

This doesn't contradict the Fubini-Tonelli theorem because the counting measure is not σ-finite. $\square$

(2) Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$ and $\mu = \nu =$ counting measure. Define

$$f(m,n) = \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } m = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Prove that $\int |f| d(\mu \times \nu) = \infty$ and that $\int\int f d\mu d\nu, \int\int f d\nu d\mu$ exist and are unequal.

Proof. Let $D_1 = \{(m,n) \in \mathbb{N}^2 : m = n\}$, $D_2 = \{(m,n) \in \mathbb{N}^2 : m = n+1\}$ and $D = D_1 \cup D_2$.

Then, $f = \chi_{D_1} - \chi_{D_2}, |f| = \chi_D$. Notice
$$\begin{aligned} \int |f| d(\mu \times \nu) &= \int \chi_D d(\mu \times \nu) \\ &\geq \int \chi_{D_1} d(\mu \times \nu) \\ &= (\mu \times \nu)(D_1) \\ &= (\mu \times \nu) \bigcup \{(m,n) \in \mathbb{N}^2 : m = n\} \\ &= \sum (\mu \times \nu)\{(m,n) \in \mathbb{N}^2 : m = n\} \\ &= \sum \mu(m)\nu(n) \\ &= \sum 1 = \infty. \end{aligned}$$

$$\begin{aligned} \int f d\mu d\nu &= \int \chi_{D_1} - \chi_{D_2} d\mu d\nu \\ &= \int \mu(\chi_{D_1} - \chi_{D_2})^n \chi_{\mathbb{N}} d\nu \\ &= \int \chi_{\mathbb{N}}(\mu({D_1}^n) - \mu({D_2}^n)) d\nu \\ &= \int \chi_{\mathbb{N}}(\mu(\{n\}) - \mu(\{n\})) d\nu \\ &= \int \chi_{\mathbb{N}}(1-1) d\nu = 0 \end{aligned}$$

To calculate $\int f d\nu d\mu$, first note $D_{1_m} = \{m\}$ if $m = n$ and $\emptyset$ otherwise whereas $D_{2_m} = \{m\}$ if $m = n+1$ and $\emptyset$ otherwise. Hence, if $(m,n) \in D_2$, $m \neq 0$. Thus,

$$\begin{aligned}
\int f d\nu d\mu &= \int \chi_{D_1} - \chi_{D_2} d\nu d\mu \\
&= \int \mu(\chi_{D_1} - \chi_{D_2})^n \chi_{\mathbb{N}} d\nu \\
&= \int \chi_{\mathbb{N}} \nu(D_{1m}) - \chi_{\mathbb{N}/\{0\}} \nu(D_{2m}) d\mu \\
&= \int \chi_{\mathbb{N}} \nu(\{m\}) - \chi_{\mathbb{N}/\{0\}} \nu(\{m\}) d\mu \\
&= \int \chi_{\mathbb{N}} - \chi_{\mathbb{N}/\{0\}} d\nu \\
&= \int \chi_{\mathbb{N}/(\mathbb{N}/\{0\})} d\nu \\
&= \int \chi_{\{0\}} d\nu = \nu(\{0\} = 1
\end{aligned}$$

$\square$

9.3. **November 6 Group Assignment.**

(1) Let f and g be integrable functions on (X, M, μ) and (Y, N, ν) respectively. If we definite $h : X \times Y \to \mathbb{R}$ by $h(x, y) = f(x)g(y)$, prove that h is integrable and

$$\int hd(\mu \times \nu) = \int fd\mu \int gd\nu.$$

Proof. Suppose f, g are simple functions with $h = fg$ and $f = \sum a_i\chi_{A_i}$ and $g = \sum b_j\chi_{B_j}$. Thus, $h = \sum\sum a_ib_j\chi_{A_i\cap B_j}$ and

$$\int hd(\mu \times \nu) = \int \sum\sum a_ib_j\chi_{A_i\cap B_j}d(\mu \times \nu) = \sum\sum a_ib_j\mu(A_i)\nu(B_j)$$

$$= \sum a_i\mu(A_i)\sum b_j\nu(B_j) = \int fd\mu \int gd\nu.$$

□

(2) Let $f(x, y) = ye^{-(1+x^2)y^2}$. Use

$$\int_{[0,\infty)}\left(\int_{[0,\infty)} f(x, y)dx\right)dy = \int_{[0,\infty)}\left(\int_{[0,\infty)} f(x, y)dy\right)dx \text{ to prove } \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

By Tonelli, since $f(x, y) = ye^{-(1+x^2)y^2} \in L^1$, the following equality holds:

$$\int_{[0,\infty)}\left(\int_{[0,\infty)} f(x, y)dx\right)dy = \int_{[0,\infty)}\left(\int_{[0,\infty)} f(x, y)dy\right)dx.$$

$$\text{Also, } \int_{[0,\infty)} \left(\int_{[0,\infty)} ye^{-(1+x^2)y^2} dx \right) dy = \int_{[0,\infty)} ye^{-y^2} \left(\int_{[0,\infty)} e^{-x^2y^2} dx \right) dy. \text{ Let } u = xy, \text{ then } du = ydx$$
$$= \int_{[0,\infty)} e^{-y^2} \left(\int_{[0,\infty)} e^{-u^2} du \right) dy$$
$$= \int_{[0,\infty)} e^{-y^2} dy \left(\int_{[0,\infty)} e^{-u^2} du \right)$$
$$= \left(\int_{[0,\infty)} e^{-x^2} dx \right)^2.$$

$$\text{Let } u = y^2 \text{ then } du = 2ydy. \int_{[0,\infty)} \left(\int_{[0,\infty)} ye^{-(1+x^2)y^2} dy \right) dx = \frac{1}{2} \int_{[0,\infty)} \left(\int_{[0,\infty)} e^{-(1+x^2)u} du \right) dx$$
$$= \frac{1}{2} \int_{[0,\infty)} \left(-\frac{1}{1+x^2} e^{-(1+x^2)u} \right)_0^\infty dx$$
$$= \frac{1}{2} \int_{[0,\infty)} -\frac{1}{1+x^2} \left(\lim_{u\to\infty} e^{-(1+x^2)u} - e^0 \right) dx$$
$$= \frac{1}{2} \int_{[0,\infty)} \frac{1}{1+x^2} dx$$
$$= \frac{1}{2} \tan^{-1} x|_0^\infty$$
$$= \frac{1}{2} \left(\lim_{x\to\infty} \tan^{-1} x - \tan^{-1}(0) \right)$$
$$= \frac{1}{2} \frac{\pi}{2}$$

$$\text{Thus, } \left(\int_{[0,\infty)} e^{-x^2} dx \right)^2 = \frac{\pi}{4} \text{ and so } \int_{[0,\infty)} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

10. Section 2.4 (Folland): Modes of Convergence

10.1. Section 2.4 Definitions and Theorems.

Definition 39 (Rectangle)**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.

Definition 40 (1)**.**

$\{f_n\}_{n=1}^\infty$. $f_n \to f$ pointwise for $x \in X$ and $\varepsilon > 0$ there is $N(\varepsilon, x) \geq 0$. s.t. $|f_n(x) - f(x)| < \varepsilon$ when $n > N$

Definition 41 (2)**.**

$\{f_n\}_{n=1}^{\infty}$. $f_n \to f$ pointwise a.e. $x \in X$ and $\varepsilon > 0$ there is $N(\varepsilon, x) \geq 0$. s.t. $|f_n(x) - f(x)| < \varepsilon$ when $n > N$

Definition 42 (3)**.**

$\{f_n\}_{n=1}^{\infty}$. $f_n \to f$ uniformly for every $\varepsilon > 0$ there is $N(\varepsilon) \geq 0$. s.t. $|f_n(x) - f(x)| < \varepsilon$ when $n > N$

Definition 43 (4)**.**

$$\{f_n\}_{n=1}^{\infty}.\ f_n \to f \text{ in } L',\ \lim_{n\to\infty} \int |f_n - f| d\mu = 0$$

Definition 44 (5)**.**

$$f_n \to f \text{ in measure } \mu(\{x : |f_n(x) - f(x)| > \varepsilon\} \to 0 \text{ for all } \varepsilon > 0$$

Example.

$$f_x = \frac{\chi_{(0,n)}}{n}$$

converges uniformly, pointwise, pointwise a.e., in measure to 0. Not in L'.

Definition 45. We say that f_n converges to f uniformly if, for every $\epsilon > 0$, there exists N such that for every $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$ for every $x \in X$. The difference between uniform convergence and pointwise convergence is that with the former, the time N at which $f_n(x)$ must be permanently ϵ-close to $f(x)$ is not permitted to depend on x, but must instead be chosen uniformly in x. Uniform convergence implies pointwise convergence, but not conversely.

Let $\varepsilon > 0$. Choose N such that $\frac{1}{N} < \varepsilon$.

$$|f_n(x) - 0| = \left|\frac{1}{n}\chi_{(0,n)}\right| = \begin{cases} \frac{1}{n} & x < n \\ 0 & x \geq 0 \end{cases} \leq \frac{1}{n} < \varepsilon \text{ when } n > N$$

$$\int |f_n(x)| dm = \int \frac{1}{n}\chi_{(0,n)} dm = \frac{1}{n} n = 1$$

$f_n \to f$ in measure

$g_n = \chi_{(n,n+1)}$ converges pointwise to 0, pointwise a.e. to 0, but not uniformly - always equal to 1 w/e, not in L^1 not in measure

$$h_n = n\chi_{[0,\frac{1}{n}]}$$

converges pointwise a.e. to 0 and in measure to 0. but not pointwise, not uniform, not L^1

$3 \Rightarrow 1 \Rightarrow 2$; $4 \Rightarrow 5$

10.2. **November 9 Group Assignment.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Prove that $f_n \to f$ in measure iff for all $\varepsilon > 0$ there is N such that

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon \text{ for all } n \geq N.$$

Proof. The forward direction holds by definition of convergence in measure.

So, assume for all $\varepsilon > 0$ there is N such that

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon \text{ for all } n \geq N.$$

Then, for $\delta < \varepsilon, \{|f_n - f| \geq \varepsilon\} \subset \{|f_n - f| \geq \delta\}$. So, $\mu(\{|f_n - f| \geq \varepsilon\}) \leq \mu(\{|f_n - f| \geq \delta\})$.

Also, by assumption, there exists $N(\delta)$ such that $\mu(\{|f_n - f| \geq \delta\}) \leq \delta$ for all $n \geq N(\delta)$.

Thus, for all $0 < \delta < \varepsilon$ there exists $N(\delta)$ such that $\mu(\{|f_n - f| \geq \varepsilon\}) \leq \delta$ for all $n \geq N(\delta)$.

Hence, f_n converges to f in measure.

`http://math.stackexchange.com/questions/1426400/f-n-rightarrow-f-in-measure-iff-for-eve`

$\square$

(2) Suppose $f_n \to f$ in measure and $g_n \to g$ in measure. Prove that $f_n + g_n \to f + g$ in measure, and if $\mu(X) < \infty$ then $f_n g_n \to fg$ in measure.

Proof. Assume $f_n \to f$ in measure and $g_n \to g$ in measure. Then, for all $\varepsilon > 0, \delta > 0$ there are N_1, N_2 such that

$$\mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\}\right) < \frac{\delta}{2} \text{ for all } n \geq N_1 \text{ and } \mu\left(\left\{x : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}\right) < \frac{\delta}{2} \text{ for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, we have

$$\{x : |(f_n + g_n)(x) - (f+g)(x)| \geq \varepsilon\} \subset \left\{x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}$$

Thus, $\mu\left(\{|f_n + g_n - (f+g)| \geq \varepsilon\}\right) \leq \mu\left(\left\{|f_n - f| \geq \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{|g_n - g| \geq \frac{\varepsilon}{2}\right\}\right) \leq \delta.$

Hence, $f_n + g_n \to f + g$ in measure. □

Proof. Assume $f_n \to f$ in measure and $g_n \to g$ in measure. First, note

$$|f_n g_n - fg| = |f_n g_n - f_n g + f_n g - fg| \leq |g||f_n - f| + |f_n||g_n - g|$$

Because f_n, g_n converge to f, g, $f \neq \pm\infty$ and $g \neq \pm\infty$. Notice

$$\{x : |f(x)| = \infty\} = \bigcap_{n \geq 1} \{x : |f(x)| \geq n\}. \text{ Thus, } \mu\left(\bigcap_{n\geq 1}\{x : |f(x)| \geq n\}\right) = 0 = \mu\left(\bigcap_{n \geq 1}\{x : |g(x)| \geq n\}\right)$$

Then, for all $\varepsilon > 0, \delta > 0$ there are N_1, N_2 such that

$$\mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\}\right) < \frac{\delta}{2} \text{ for all } n \geq N_1 \text{ and } \mu\left(\left\{x : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}\right) < \frac{\delta}{2} \text{ for all } n \geq N_2.$$

□

`http://www.math.ntnu.no/~eugenia/Teaching/TMA4225/h06/probl8_sol.pdf`

`https://www.math.brown.edu/~rkenyon/teaching/2009/2210/Set3.pdf`

10.3. **November 13 Group Assignment.** Let $(X\mathcal{M}, \mu)$ be a measure space:

(1) If μ is σ-finite and $f_n \to f$ a.e. then there exists measurable sets $E_1, E_2, \ldots, \subseteq X$ such that $\mu((\cup E_i)^c) = 0$ and $f_n \to f$ on each E_j.

Proof. Assume μ is σ-finite and $f_n \to f$ a.e. Then, $X = \bigcup_1^\infty A_j$ where $\mu(A_j) < \infty$. Let $B_n = \cup_{j=1}^n A_j$. Since $\mu(A_j) < \infty$ for all j, $\mu(B_n) < \infty$ for any n. Also note that $\{B_n\}_{n=1}^\infty$ are nested sets. For every n, choose $E_n \subset B_n$ such that $\mu(E_n^c) < \frac{1}{n}$; note $f \to f$ a.e. on B_n and thus $f \to f$ a.e. on E_n. Thus, by Ergoroff's theorem, $f_n \to f$ uniformly on E_n. Also

$$\mu\left(\left(\bigcup E_j\right)^c\right) = \mu\left(\bigcap E_j^c\right) \leq \mu(E_n^c) < \frac{1}{n}$$

□

(2) Let μ be counting measure on $\mathbb{N}$. Prove that f_n converges uniformly to f if and only if f_n converges to f in measure.

Proof. Assume f_n converges to f in measure. Then, for any $\varepsilon > 0$ define $E_{\varepsilon,n} = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Since f_n converges to f in measure, $\mu(E_{\varepsilon,n}) \to 0$ as $n \to \infty$. Thus, $\mu(E_{\varepsilon,n}) = 0$ for all but finitely many n. Equivalently, $E_{\varepsilon,n} = \emptyset$ for all but finitely many n. Thus, there exists some N such that $\{x : |f_n(x) - f(x)| \geq \varepsilon\} = \emptyset$ for all $n > N$. Therefore f_n converges uniformly to f.

Assume f_n converges uniformly to f. Then, for any $\varepsilon > 0$, there exists some N such that for all $n \geq N$, $|f_n - f| < \varepsilon$. Equivalently, $\{x : |f_n(x) - f(x)| \geq \varepsilon\} = \emptyset$ for all $n > N$ Thus, $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$ and given $\delta > 0$ we can write $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \delta$. And so f_n converges to f in measure. □

11. Section 3.1 (Folland): Signed Measures

11.1. **Section 3.1 Definitions and Theorems.**

Definition 46 (signed measure)**.** A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

(1) $\nu(\emptyset) = 0$;

(2) ν assumes at most one of the values of $\pm\infty$

(3) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\cup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$. $\sum_1^\infty \nu(E_j)$ converges absolutely if $\nu(\cup_1^\infty E_j)$ is finite.

Remark 12. Every measure is a signed measure. For emphasis, measures may be referred to as positive measures.

Example. If μ_1, μ_2 are measures on $\mathcal{M}$ and at least on of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.

Example. If μ is a measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite, then $\nu(E) = \int_E f d\mu$ is a signed measure.

Proposition 13. Let ν be a signed measure on $(X, \mathcal{M})$.

(1) If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, then $\nu(\cup_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.

(2) If $\{E_j\}$ is an decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then $\nu(\cap_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.

Lemma 2. Any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive.

Theorem 15 (The Hahn Decomposition Theorem)**.** If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \emptyset$.

If P', N' is another such pair, then $P \triangle P' = N \triangle N'$ is null for ν.

Remark 13 (symmetric difference)**.** Recall the symmetric difference of two sets is the set of elements which are in either of the sets and not in their intersection.

$$A \triangle B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$$

.

Definition 47 (Hahn decomposition)**.** The decomposition $X = P \cup N$ of X as the disjoint union of a positive set and a negative se. Not unique - ν-null sets can be transferred from P to N or from N to P.

Definition 48 (mutually singular, ν is singular wrt μ, $\mu \perp \nu$)**.** There exist $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $E \cup F = X$, E is null for μ and F is null for ν.

Remark 14. Informally, mutual singularity means μ, ν live on disjoint sets.

Theorem 16 (Jordan decomposition theorem)**.** If ν is a signed measure, there exist positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Definition 49 (ν^+, ν^-)**.** ν^+ is the positive variation of ν and ν^- is the negative variation of ν

Definition 50 (total variation)**.** $|\nu| = \nu^+ + \nu^-$.

Definition 51 (integration wrt a signed measure)**.**

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-),$$

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-, \qquad f \in L^1(\nu)$$

11.2. **November 16 Group Assignment.**

(1) Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$ prove that $\nu(\cup E_i) = \lim \nu(E_i)$. Alternatively, if $\{F_j\}$ is a decreasing sequence in $\mathcal{M}$ with $\nu(F_1)$ finite, then $\nu(\cap F_j) = \lim \nu(F_j)$.

Proof. By the Hahn Decomposition Theorem, we can decompose $X = P \cup N$ where P is positive and N is negative so that $\nu(P) \geq 0$ and $\nu(N) \leq 0$. Then, ν is a measure on P and

$-\nu$ is a measure on N.

$$\begin{aligned}
\nu(\cup E_i) &= \nu\left(\bigcup(E_i \cap P) \cup \bigcup(E_i \cap N)\right) \\
&= \nu\left(\bigcup(E_i \cap P)\right) + \nu\left(\bigcup(E_i \cap N)\right) \\
&= \nu\left(\bigcup(E_i \cap P)\right) - (-\nu)\left(\bigcup(E_i \cap N)\right) \\
&= \lim \nu\left(E_i \cap P\right) - \lim(-\nu)\left(E_i \cap N\right) \\
&= \lim\left(\nu\left(E_i \cap P\right) - (-\nu)\left(E_i \cap N\right)\right) \\
&= \lim\left(\nu\left(E_i \cap P\right) - (-\nu)\left(E_i \cap N\right)\right) \\
&= \lim\left(\nu\left(E_i \cap P \cup E_i \cap N\right)\right) \\
&= \lim\left(\nu\left(E_i \cap P \cup E_i \cap N\right)\right) \\
&= \lim \nu\left(E_i\right)
\end{aligned}$$

Next, suppose $\{F_j\}$ is a decreasing sequence in $\mathcal{M}$ with $\nu(F_1)$ finite. Then,

$$\begin{aligned}
\nu(\cap F_i) &= \nu(\cap(F_i \cap P) \cup \cap(F_i \cap N)) \\
&= \nu(\cap(F_i \cap P)) + -\nu(\cap(F_i \cap N)) \\
&= \lim \nu(F_i \cap P) - \lim -\nu(F_i \cap N) \\
&= \lim\left(\nu(F_i \cap P) + \nu(F_i \cap N)\right) \\
&= \lim \nu\left((F_i \cap P) \cup (F_i \cap N)\right) \\
&= \lim \nu\left(F_i\right)
\end{aligned}$$

$\square$

(2) Let ν be a signed measure on $(X, \mathcal{M})$.

Since ν is a signed measure, the Hahn Decomposition Theorem implies there exists a positive set P and a negative set N with $X = P \cup N$.

(a) $L^1(\nu) = L^1(|\nu|)$.

Proof. Consider any $f \in L^1(\nu)$; so $\int |f| d\nu < \infty$. Then, since $\nu = \nu^+ - \nu^-$,

$$\int |f| d\nu = \int |f| d\nu^+ - \int |f| d\nu^- < \infty. \text{ Thus, } \int |f| d\nu^+ < \infty \text{ and } \int |f| d\nu^- < \infty.$$

$$\text{Since } \int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^-, \int |f| d|\nu| < \infty. \text{ Hence, } f \in L^1(|\nu|).$$

□

(b) If $f \in L^1(\nu)$, then $|\int f d\nu| \leq \int |f| d|\nu|$.

Proof. Consider any $f \in L^1(\nu)$; so $\int |f| d\nu < \infty$. Notice

$$\left| \int f d\nu \right| = \left| \int f d\nu^+ - \int f d\nu^- \right| \leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|.$$

□

(c) If $E \in \mathcal{M}$, then $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$.

Proof. By part b,

$$\left| \int_E f d\nu \right| \leq \int_E |f| d|\nu| \leq \int_E d|\nu| = |\nu|(E). \tag{i}$$

$$\text{Let } f(x) = \begin{cases} 1 & x \in P \\ -1 & x \in N \end{cases}$$

Then,

$$\begin{aligned}
\left|\int_E f d\nu\right| &= \left|\int_E f d\nu^+ - \int_E f d\nu^-\right| \\
&= \left|\int f\chi_E d\nu^+ - \int f\chi_E d\nu^-\right| \\
&= \left|\nu^+(E) - -\nu^-(E)\right| \\
&= \left|\nu^+(E) + \nu^-(E)\right| \\
&= \nu^+(E) + \nu^-(E) \\
&= |\nu|(E), \text{ so for the chosen } f
\end{aligned}$$

(ii)
$$\left|\int_E f d\nu\right| = |\nu|(E)$$

Thus, by i, ii,

$$\sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\} = |\nu|(E).$$

□

11.3. **November 18 Group Assignment.**

(1) If ν is a signed measure, E is $\nu-$null if and only if $|\nu|(E) = 0$.

Proof. ($\Rightarrow$) Since ν is a signed measure, we can write $X = P \cup N$. Assume E is $\nu-$null. Then, $\nu(E) = 0$. Then, $\nu(E \cap P) = 0$ and $\nu(E \cap N) = 0$. Thus, $\nu^+(E) = 0$ and $\nu^-(E) = 0$ and so $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Hence, $|\nu|(E) = 0$.

($\Leftarrow$) Assume $|\nu|(E) = 0$. Then, $\nu^+(E) + \nu^-(E) = 0$. Since ν^+, ν^- are positive, we have $\nu^+(E) = 0$ and $\nu^-(E) = 0$. Then, for any subset F in E so $F \cap P \subset E$ and $F \cap N \subset E$ so $\nu^+(F \cap P) = 0$ and $\nu^-(F \cap N) = 0$. Hence,

$$\nu(F) = \nu^+(F \cap P) - \nu^-(F \cap N) = 0 - 0 = 0.$$

$\square$

(2) If ν and σ are signed measures then $\nu \perp \sigma$ if and only if $\nu \perp |\sigma|$ if and only if $\nu^+ \perp \sigma$ and $\nu^- \perp \sigma$.

$$\textbf{Denote a: } \nu \perp \sigma, \textbf{ b: } \nu \perp |\sigma|, \textbf{ c: } \nu^+ \perp \sigma, \nu^- \perp \sigma$$

(a ⇒ b) Assume $\nu \perp \sigma$. Then, there exist E, F such that $X = E \cup F$, E is σ-null, and F is ν-null. Then, since E is σ-null for any $A \subseteq E$, $\sigma(A) = \sigma^+(A) - \sigma^-(A) = 0$. Thus, $\sigma^+(A) = 0$ and $\sigma^-(A) = 0$. Hence for any $A \subseteq E$, $|\sigma|(A) = \sigma^+(A) + \sigma^-(A) = 0$. Thus, E is $|\sigma|$-null. So there exist E, F such that $X = E \cup F$, E is $|\sigma|$-null, and F is ν-null. Hence $\nu \perp |\sigma|$.

(a ⇒ c) Assume $\nu \perp \sigma$. Then, there exist E, F such that $X = E \cup F$, E is σ-null, and F is ν-null. Since F is ν-null, for any $A \subseteq F$, $\nu(A) = 0$. Let $F = P \cup N$ then $P \subseteq F, N \subseteq F$, so $\nu(P) = 0 = \nu(N)$. Hence, for any $A \subseteq F$,

$$\nu^+(A) = \nu(A \cap P) = 0, \text{ and } \nu^-(A) = \nu(A \cap N) = 0.$$

Thus, F is ν^+-null and F is ν^--null. So, E, F satisfy the conditions for $\nu^+ \perp \sigma$ and $\nu^- \perp \sigma$.

(b ⇒ a) Assume $\nu \perp |\sigma|$. Then, there exist E, F such that $X = E \cup F$, E is ν-null, and F is $|\sigma|$-null. The, since F is $|\sigma|$-null, for any $A \subseteq F$, $|\sigma|(A) = 0$ and so $\sigma^+(A) + \sigma^-(A) = 0$. σ^+, σ^- are finite so $\sigma^+(A) + \sigma^-(A) = 0$ implies $\sigma^+(A) = 0 = \sigma^-(A)$. Thus, $\sigma(A) = \sigma^+(A) - \sigma^-(A) = 0$ so F is σ-null. Hence, $\nu \perp \sigma$.

(c ⇒ a)

Assume $\nu^+ \perp \sigma, \nu^- \perp \sigma$. Then, there exist E_1, F_1 such that $X = E_1 \cup F_1$, E_1 is σ-null, and F_1 is ν^+-null and there exists E_2, F_2 such that $X = E_2 \cup F_2$, E_2 is σ-null, and F_2 is ν^+-null. First, we will show $E = E_1 \cup E_2$ is σ-null. Consider any $A \subseteq E_1 \cup E_2$. Then, $A = A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$. Since E_1, E_2 are σ-null, $\sigma(A) = \sigma(A \cap E_1) + \sigma(A \cap E_2) = 0$. Thus, $E_1 \cup E_2$ is σ-null. Next, we will show $F = F_1 \cap F_2$ is ν-null. Consider any $A \subseteq F_1 \cap F_2$.

Then, $A \subseteq F_1$ and $A \subseteq F_2$ so $\nu^+(A) = 0$ and $\nu^-(A) = 0$. Thus, $\nu(A) = \nu^+(A) - \nu^-(A) = 0$. So, F is ν-null. Finally, notice

$$(E_1 \cup E_2)^c = F_1 \cap F_2. \text{ So, } (E_1 \cup E_2) \cup (F_1 \cap F_2) = X \text{ and } (E_1 \cup E_2) \cap (F_1 \cap F_2) = \emptyset$$

Hence, $\nu \perp \sigma$.

(3) Suppose $\nu(E) = \int_E f d\mu$ where μ is a positive measure and f is a $\mu-$integrable function (not necessarily positive). Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ.

Let $P = \{x : f(x) > 0\}$ and $N = \{x : f(x) \leq 0\}$. Note $P = f^{-1}(0, \infty)$ and $N = f^{-1}((-\infty, 0])$. Since f is μ measurable, P and N are measurable. Also,

$$\nu^+(E) = \nu(E \cap P) = \int_{E \cap P} f d\mu \text{ and } \nu^-(E) = \nu(E \cap N) = -\int_{E \cap N} f d\mu.$$

$$\text{Hence, } |\nu|(E) = \int_{E \cap P} f d\mu - \int_{E \cap N} f d\mu.$$

12. Section 3.2 (Folland): The Lebesgue-Radon-Nikodym-Theorem

12.1. **Section 3.2 Definitions and Theorems.**

Definition 52 (absolutely continuous)**.** ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. ν is absolutely continuous with respect to μ and write

$$\nu \ll \mu$$

Theorem 17 (Lebesgue-Radon-Nikodym Theorem)**.** ν is σ-finite signed measure on $(X, \mathcal{M})$

μ is σ-finite measure on $(X, \mathcal{M})$

then there are unique σ-finite signed measures, λ, ρ on $(X, \mathcal{M})$ s.t.

(1) $\lambda \perp \rho$

(2) $\rho \ll \lambda$

(3) $\nu = \lambda + \rho$

further there is $f \in L^1(\mu)$ s.t. $d\rho = f d\mu$ and f is unique μ-a.e.

sketch of LRN proof. Case 1: μ and ν are finite measures.

Step 1: Let $\mathcal{F} = \{f : X \to [0, \infty] : \int_E f d\mu \le \nu(E) \text{ for all } E \in \mathcal{M}\}$.

(1) $0 \in \mathcal{F}$ implies $\mathcal{F} \neq \emptyset$

(2) $f, g \in \mathcal{F}$ then $\max\{f, g\} \in \mathcal{F}$

Proof. . $A = \{x : f(x) > g(x)\}$. $\int_E \max\{f, g\} d\mu = \int_{E \cap A} f d\mu + \int_{E \setminus A} g d\mu \le \nu(E \cap A) + \nu(E \setminus A) = \nu(E)$ □

Step 2: Let $a = \sup\{\int_X f d\mu : f \in \mathcal{F}\}$. Then,

(1) $a \le \nu(X) < \infty$

(2) There is $f \in \mathcal{F}$ such that $\int_X f d\mu = a$

proof of (2). Choose $f_n \in \mathcal{F}$ such that $\int_{n \to \infty} \int_X f_n d\mu = a$.

If $g_n = \max\{f_1, f_2, \ldots, f_n\} \in \mathcal{F}$, then $f(x) = \lim_{n \to \infty} g_n(x)$.

$g_{n+1} \ge g_n$ and g_n are always bounded.

$$\int_X g_n d\mu \ge \int_X f_n d\mu. \text{ By MCT } \forall n, \int_X f_n d\mu \le \int_X f d\mu = a$$

$$\text{Since } g_n \in \mathcal{F}, \nu(E) \ge \lim \int_E g_n d\mu = \int_E f d\mu. \text{ Thus } f \in \mathcal{F}.$$

$$\text{So, } \int_X f d\mu \le a. \quad \int_X f_n d\mu \le \int_X g_n d\mu \le \int_X f d\mu \ \forall n. \text{ Thus, } \int_X f d\mu = a \text{ and } f \in L^1(\mu).$$

□

What we know: $\rho(E) = \int_E f \mu$. Any measure defined in this will satisfy $\rho \ll \mu$.

Now, prove parts (1) and (3).

Let $\lambda(E) = \int_E d\nu - \int f d\mu = \nu(E) - \rho(E)$. Thus, λ is a signed measure. And since ν, ρ, λ are finite, we have $\nu = \lambda + \rho$.

Next, we need to check $\lambda \perp \mu$. Suppose $\mu(E) > 0$. Then,

$$\lambda(E) = \nu(E) - \rho(E) = \nu(E) - \int_E f d\mu$$

Suppose you have an E with $\lambda(E) > 0$. Then, $\nu(E) > \int_E f d\mu$. Define a new function that is a little bigger than f on E by lemma. But, that would contradict $a = \sup$.

We would need to deal with case 2, 3... $\square$

Remark 15. For this course, you should at least remember the following theorems: dominated convergence, Lebesgue-Radon-Nikodym, Fubini-Tonelli.

Remark 16 (notation)**.** If ν is σ-finite and μ is σ-finite. Then, $\nu = \lambda + \rho$ with $\lambda \perp \mu$, $\rho \ll \mu$. This is the Lebesgue Radon Nikodym decomposition of ν wrt μ.

Definition 53 (LRN derivative of ν wrt μ)**.**

$$d\nu = f d\mu.\ f = \frac{d\nu}{d\mu}.\text{ Then, } \rho(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_E f d\mu$$

Definition 54. ν_1, ν_2, μ are σfinite signed measures.

$$f_{\nu_1+\nu_2} = \frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu} = f_{\nu_1} + f_{\nu_2}$$

Theorem 18. If $\nu \ll \mu$, $\mu \ll \sigma$, then $\nu \ll \sigma$.

$$\frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\sigma} = \frac{d\nu}{d\sigma} \text{ and } \nu(E) = \int_E \frac{d\nu}{d\sigma} d\sigma = \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\sigma} d\sigma$$

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu, \qquad \mu(E) = \int_E \frac{d\mu}{d\sigma} d\sigma$$

Remark 17. If $\nu \ll \mu, \mu \ll \nu$, then

$$\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu} = 1 \text{ a.e.}$$

Theorem 19. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then, $\nu \ll \mu$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Corollary 3. If $f \in L^1(\mu)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\int_E f d\mu| < \varepsilon$ whenever $\mu(E) < \delta$. Denote

$$\nu(E) = \int_E f d\mu : \quad d\nu = f d\mu.$$

12.2. **November 20 Group Assignment.**

(1) Let ν be a signed measure and μ be a measure on $(X, \mathcal{M})$. Prove the following are equivalent:

(a) $\nu \ll \mu$

(b) $|\nu| \ll \mu$

(c) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. **(a $\Rightarrow$ b,c)** Suppose $\nu \ll \mu$ and consider any $E \in \mathcal{M}$ with $\mu(E) = 0$. Then, $\nu(E) = 0$ by assumption. By Hahn Decomposition Theorem, we can write $E = P \cup N$. Since $P, N \subseteq E$ it must be the case that $\mu(P) = \mu(N) = 0$ and so by assumption $\nu(P) = \nu(N) = 0$. Then, $0 = \nu(P) = \nu(P \cap E) = \nu^+(E)$ and $0 = \nu(N) = \nu(N \cap E) = \nu^-(E)$. And so $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. Also, $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$ which implies $|\nu| \ll \mu$.

(b $\Rightarrow$ a) Suppose $|\nu| \ll \mu$ and consider any $E \in \mathcal{M}$ with $\mu(E) = 0$. Then, $|\nu|(E) = 0$ by assumption. So $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Since ν^+, ν^- are both finite and positive $\nu^+(E) + \nu^-(E) = 0$ implies $\nu^+(E) = \nu^-(E) = 0$. Thus, $\nu(E) = \nu^+(E) - \nu^-(E) = 0$ and $\nu \ll \mu$.

(c $\Rightarrow$ b) Suppose $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ and consider any $E \in \mathcal{M}$ with $\mu(E) = 0$. Then, $\nu^+(E) = 0$ and $\nu^-(E) = 0$ by assumption. Thus, $\nu(E) = \nu^+(E) - \nu^-(E) = 0$ so $\nu \ll \mu$.

$\square$

(2) Suppose $\{v_j\}$ is a sequence of positive measures and μ is a measure on $(X, \mathcal{M})$. Prove

(a) If $\nu_j \perp \mu$ for all j then $\sum \nu_j \perp \mu$.

Proof. Assume $\nu_j \perp \mu$ for all j. Then for each j there exists E_j, F_j such that $E_j \cup F_j = X$, $E_j \cap F_j = \emptyset$ with E_j ν_j-null and F_j μ-null. Notice $\cap E_i \subset E_j$ for any j, so $\nu_j(\cap E_i) = 0$. Also any subset A of $\cap E_i$ will be a subset of every E_j which implies $\nu_j(A) = 0$ for all j so

$\cap E_i$ is $\sum \nu_j$ null. Now, consider any $A \subseteq \cup F_j$ and write

$$A = \cup_j A_j = \cup_j \left(A \cap F_j / \cup_{i>j} F_i\right) \text{ so that } A_j \subseteq F_j.$$

Thus, $\mu(A) = \mu(\cup A_j) = \sum \mu(A_j) = \sum 0 = 0$ and so $\cup F_j$ is μ null. Since $E_j \cup F_j = X$, $E_j \cap F_j = \emptyset$, $\cap E_j \cup \cup F_j = X$ and $\cap E_j \cap \cup F_j = \emptyset$. Thus, $\sum \nu_j \perp \mu$. □

(b) If $\nu_j \ll \mu$ for all j then $\sum \nu_j \ll \mu$.

Proof. Assume $\nu_j \ll \mu$ for all j. Consider any $E \in \mathcal{M}$ with $\mu(E) = 0$. By assumption, $\nu_j(E) = 0$ for all j so $\sum \nu_j(E) = 0$. Thus, $\sum \nu_j \ll \mu$. □

12.3. **November 23 Group Assignment.**

(1) Suppose that μ_1 and ν_1 are σ-finite measures on $(X_1, \mathcal{M}_1)$ with $\nu_1 \ll \mu_1$. Similarly, assume that μ_2, ν_2 are σ-finite measures on $(X_2, \mathcal{M}_2)$ with $\nu_2 \ll \mu_2$. Prove that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

Proof. Suppose that μ_1 and ν_1 are σ-finite measures on $(X_1, \mathcal{M}_1)$ with $\nu_1 \ll \mu_1$. Similarly, assume that μ_2, ν_2 are σ-finite measures on $(X_2, \mathcal{M}_2)$ with $\nu_2 \ll \mu_2$. Assume $\mu_1 \times \mu_2(E) = 0$.

$$\begin{aligned}
\text{Then, } 0 = \mu_1 \times \mu_2(E) &= \int \chi_E d(\mu_1 \times \mu_2) \\
&= \int \int (\chi_E)_{x_2} d\mu_1 d\mu_2 \\
&= \int \mu_1 \left((\chi_E)_{x_2}\right) d\mu_2
\end{aligned}$$

By Fubini, if $\int \mu_1\left((\chi_E)_{x_2}\right) d\mu_2 = 0$, $\int \mu_2\left((\chi_E)^{x_1}\right) d\mu_1 = 0$ which imply $\mu_1\left((\chi_E)_{x_2}\right) = 0$ and $\mu_2\left((\chi_E)^{x_1}\right) = 0$. Then, by assumption, $\nu_1\left((\chi_E)_{x_2}\right) = 0$ and $\nu_2\left((\chi_E)^{x_1}\right) = 0$. Thus,

$$\begin{aligned}
\nu_1 \times \nu_2(E) &= \int \chi_E d\nu_1 \times \nu_2 \\
&= \int\int (\chi_E)^{x_1} d\nu_2 d\nu_1 \\
&= \int \nu_2\left((\chi_E)^{x_1}\right) d\nu_1 \\
&= \int 0 d\nu_1 \\
&= 0
\end{aligned}$$

Thus, $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$. $\square$

Given the same conditions as in (1), prove that $\dfrac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} = \dfrac{d\nu_1}{d\mu_1}$.

Proof. Suppose $E_1 \in \mathcal{M}_1$, $E_2 \in \mathcal{M}_2$, $x_1 \in X_1, x_2 \in X_2$. Then, we have

$$\frac{d\nu_1}{d\mu_1} = f_1(x_1), \nu_1(E_1) = \int_{E_1} f_1(x_1) d\mu_1 \quad \text{and} \quad \frac{d\nu_2}{d\mu_2} = f_2(x_2), \nu_2(E_2) = \int_{E_2} f_2(x_2) d\mu_2.$$

$$\begin{aligned}
\text{Then, } (\nu_1 \times \nu_2)(E_1 \times E_2) &= \nu_1(E_1)\nu_2(E_2) \\
&= \int_{E_1} f_1(x_1) d\mu_1 \int_{E_2} f_2(x_2) d\mu_2 \\
&= \int_{E_2}\int_{E_1} f_1(x_1) f_2(x_2) d\mu_1 d\mu_2 \\
&= \int_{E_1 \times E_2} f_1 f_2 d(\mu_1 \times \mu_2)
\end{aligned}$$

Since $f_1 f_2$ must be unique and $(\nu_1 \times \nu_2)(E_1 \times E_2) = \int_{E_1 \times E_2} f_1 f_2 d(\mu_1 \times \mu_2)$,

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} = f_1 f_2 = \frac{d\nu_1}{d\mu_1}\frac{d\nu_2}{d\mu_2}$$

$\square$

(2) Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure and $\mu =$ counting measure. Prove that $m \ll \mu$ but there is no f such that $dm = f d\mu$.

Proof. For $E \in \mathcal{M}$ suppose $\mu(E) = 0$. Then, $E = \emptyset$ and so $m(E) = 0$. Thus, $m \ll \mu$. Next, suppose there is an f such that $dm = f d\mu$. Let $a \in [0,1]$ and note $m(\{a\}) = 0$ and

$$m(\{a\}) = \int_{\{a\}} f d\mu = f(a)\mu(\{a\}) = f(a). \qquad \text{Hence, } f(a) = 0 \quad \forall\, a \in [0,1].$$

Also, notice $m([0,1]) = 1$ but

$$m([0,1]) = \int_{[0,1]} f d\mu = \int 0 d\mu = 0\mu([0,1]) = 0.$$

Thus, there is no f such that $dm = f d\mu$. $\square$

12.4. **November 25 Group Assignment.** (1) Suppose that μ and ν are σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = \frac{d\nu}{d\lambda}$ then prove that $0 \le f < 1$ μ-a.e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

Proof. Suppose that μ and ν are σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. First, we will show $\nu \ll \lambda, \mu \ll \lambda, \lambda \ll \mu$. Suppose $\lambda(E) = 0$. Then, $\lambda(E) = (\mu+\nu)(E) = \mu(E)+\nu(E) = 0$. Since μ, ν are positive, finite measures this implies $\mu(E) = \nu(E) = 0$. Thus, $\mu \ll \lambda$ and $\nu \ll \lambda$. Let

$$f = \frac{d\nu}{d\lambda}, \quad h = \frac{d\nu}{d\mu}, \quad j = \frac{d\lambda}{d\mu}, \quad g = \frac{d\mu}{d\lambda}.$$

$$\begin{aligned} \text{Then, } 1 = g + f \quad &\text{and} \quad 1 = g + gh \\ \frac{1}{g} &= 1 + h \\ h &= \frac{1}{g} - 1 \\ h &= \frac{1-g}{g} \\ &= \frac{f}{1-g} \end{aligned}$$

$$\text{Hence, } \frac{d\nu}{d\mu} = \frac{f}{1-f}.$$

To show $0 \leq f < 1$ μ-a.e., first note that $f + g = 1$. Suppose $f \geq 1$ on some A such that $\nu(A) > 0$. Then, $\lambda(A) > 0$ and

$$\nu(A) = \int_A f d\lambda < \int_A 0 d\lambda = 0, \text{ which is a contradiction.}$$

Therefore, $f > 0$. Next, suppose $f > 1$ on some A such that $\mu(A) > 0$. Then, $\lambda(A) > 0$ and

$$\nu(A) = \int_A f d\lambda \geq \int_A 1 d\lambda = \lambda(A) = \mu(A) + \nu(A) > \nu(A) \text{ which is a contradiction.}$$

Hence, $0 < f \leq 1$. ☐

13. Section 3.4 (Folland): Differentiation on Euclidean Space

13.1. Section 3.4 Definitions and Theorems.

Remark 18. From now on, we will focus on $(X, \mathcal{M}, \mu) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m)$.

Remark 19. $B(r, x) = \{y : ||x - y|| < r\}$, $m(B(r, x)) = k(n)\ \pi r^n$.

Our goal: We want $\nu \ll m$ so the RN theorem gives us $f dm = d\nu$ and

$$f = \frac{d\nu}{dm} \text{ we will make this precise: } \lim \frac{\nu(...)}{m(...)}$$

Lemma 3 (Vitali's Covering Lemma)**.** $\mathcal{C}$ is a collection of balls in $\mathbb{R}^n$ and $\mathcal{U} = \cup_{B \in \mathcal{C}} B$. If $c < m(\mathcal{U})$ there are $B_1, B_2, \ldots, B_k \in \mathcal{C}$ disjoint such that

$$\sum_{i=1}^{k} m(B_i) > \frac{c}{3^n}.$$

How to think about Vitali's theorem: there is some $\mathcal{U}$ and take a disjoint subset (maximized). However, you will miss stuff. The theorem says if you take all these subsets and blow them up by a factor of 3, then you are guaranteed to cover $\mathcal{U}$.

We need c because there may be unbounded stuff.

Remark 20. In other words, $\mathcal{C} = \{B(r,x)\}_{\{r,x\}\subset r,x}$

Vitali's lemma proof. Step 1: Find a compact set $K \subseteq \mathcal{U}$ with $m(K) > c$.

Step 2: Because K is compact there is $A_1, A_2, \ldots, A_m \in \mathcal{C}$ with $K \subseteq \cup_{j=1}^m A_j$. In other words, there exists a finite subcovering. Definition of compact: every cover has a finite subcover.

Step 3: Now we have finite but not disjoint. So start with $\{A_1, A_2, \ldots, A_m\}$ and choose the largest. Call it B_1. With what's left, chose the largest that doesn't hit B_1 (is disjoint from B_1) call B_2. If there is a tie, just pick one. With what's left... call it B_3...

We will end with $B_1, \ldots, B_n$ disjoint.

Step 4: Now we need to show

$$\sum_{i=1}^{k} m(B_i) > \frac{c}{3^n}.$$

Let $B_i = B(r_i, x_i)$. $A_j \subseteq B(3r_i, x_i)$ for some i. Assume $A_j \neq B_i$ for any i. Then $A_j \cap B_k \neq \emptyset$ for some k. Choose the smallest k for which $A_j \cap B_k \neq \emptyset$. See figure.

So $A_j \cap B_1 = \emptyset, A_j \cap B_2 = \emptyset, \ldots, A_j \cap B_{k_0 - 1} = \emptyset$ radius $A_j \leq$ radius for B_{k_0}. $x \in A_j \cap B_{k_0}$, $d(x_{k_0}, y) < r_{k_0}$

$d(y, x') < r_{k_0}$, and $d(x', z) < r_{k_0}$ for any $z \in A_j$. So $A_j \subseteq B(3r_i, x_i)$ for some i.

$c < m(K) \leq m(\cup_{k=1}^n (B(x_k, 3r_k)))$

$$K \subseteq \bigcup_{j=1}^{m} A_j \subseteq \bigcup_{k=1}^{n} B(x_k, 3r_k)$$

So, $m(\cup_{k=1}^n (B(x_k, 3r_k))) \leq \sum_{k=1}^n m(B(x_k, 3r_k)) = 3^n \sum_{k=1}^n B(x_k, r_k)$. $\square$

Definition 55. f is locally integrable if

$$\int_K |f| dm < \infty \text{ for any compact set } K \subseteq \mathbb{R}^n.$$

Let $L^1_{loc}(m)$. Let $f(x) \in L^1_{loc}$.

Example, average value function of f on $B(r,x)$:

$$(A_r f)(x) = \frac{1}{m(B(r,x))}\int_{B(r,x)} f(y)dm(y)$$

Example, Hardy-Littlewood Maximal Function: Pick a function f and fix x

$$(Hf)(x) = \sup_{r>0}(A_r|f|)(x) = \sup_{r>0}\frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)|dm(y)$$

Facts: (use dominated convergence theorem to prove)

1. $\lim_{r\to r_0}(A_r f)(x) = (A_{r_0} f)(x); \qquad \lim_{x\to x_0}(A_r f)(x) = (A_r f)(x_0)$

$$\text{Notice } \int_{B(r,x)} f(y)dm(y) = \int_{\mathbb{R}^n} \chi_{B(r,x)} f(y)dm(y)$$

2. $Hf(x)$ is measurable

3. There is $c > 0$ such that for all $f \in L^1$ and $\alpha > 0$

$$m(\{x : Hf(x) > \alpha\}) \le \frac{c}{\alpha}\int |f|dm$$

proof of 3. Let $E_\alpha = \{x : Hf(x) > \alpha\}$. Then, for all x, there is an $r_x > 0$ such that $(A_{r_x}|f|(x)) > \alpha$. Consider $\cup_{x\in E_\alpha} \supseteq E_\alpha$. Then, for any $c < m(E_\alpha)$,

$$c < 3^n \sum_{j=1}^{k} m(B_j) \text{ for some finite } B_1, \dots, B_j\text{. Then,}$$

$$\begin{aligned} c &< 3^n \sum_{j=1}^{k} m(B_j) \\ &\le \frac{3^n}{\alpha} \sum j = 1^k \int_{B_j} |f|dm \\ &\le \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f|dm \end{aligned}$$

Let $c \to m(E_\alpha)$. Read proof of differentiation theorem. He will hand out problem for next time. $\square$

Theorem 20.

$$f \in L^1_{loc} \Rightarrow \lim_{r \to 0} A_r f(x) = f(x) \quad \text{a. e.}$$

Proof. 1. True if the function is cts.

$$\frac{1}{m(B(r,x))} \int_{B(r,x)} f dm$$

Let $\varepsilon > 0$ there exists δ s.t. $y \in B(\delta, x)$ implies $f(x) - \varepsilon < f(y) < f(x) + \varepsilon$

2. Approximate f by cts function g.

3. (key step)

$$\limsup_{r \to 0} |A_r f(x) - f(x)|$$

If this is 0, then this is a limit, get rid of $||$.

Look at the set were $\limsup \neq 0$ and show this set has measure zero.

$$\begin{aligned}
\limsup_{r \to 0} |A_r f(x) - f(x)| &= \limsup_{r \to 0} |A_r(f-g)(x) + A_r g(x) - g(x) + g(x) - f(x)| \\
&\leq |H(f-g)(x)| + |0| + |(g-f)(x)|
\end{aligned}$$

4. $E_\alpha = \{x : \limsup |A_r f(x) - f(x)| > \alpha\}$.

$F_\alpha = \{x : |f - g| > \alpha\}$.

$E_\alpha \subseteq F_{\alpha/2} \cup \{x : H(f-g) > \frac{\alpha}{2}\}$

$$m(E_\alpha \leq m(F_{\alpha/2}) + m\left(\left\{x : H(f-g) > \frac{\alpha}{2}\right\}\right) \leq \frac{\epsilon}{2} + \frac{c\epsilon}{\frac{\alpha}{2}}$$

Thus, $m(E_\alpha) = 0 \;\forall\; \alpha$. $\square$

Definition 56 (Lebesgue set)**.**

$$L_f = \left\{x : \lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| dm(y) = 0\right\}$$

Corollary 4.

$$m(L_f^c) = 0$$

Remark 21 (notation)**.** $\{E_r\}_{r>0}$ is a collection of sets. Then, E_r shrinks nicely to x if

1. $E_r \subseteq B(r, x)$ for all r
2. There is a fixed α such that

$$m(E_r) > \alpha m(B(r, x))$$

Theorem 21 (Lebesgue Diffn Thm)**.** If $f \in L^1_{loc}$ and E_r shrinks nicely to $x \in L_f$, then

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0.$$

Definition 57. ν is a Borel measure on $\mathbb{R}^n$. ν is regular if

i. $\nu(K) < \infty$ for any compact K

ii. [follows from i.] $\nu(E) = \inf\{\nu(U) \ : \ U \text{ open } E \subseteq U\}$ for any Borel set E.

If ν is signed it is regular if $|\nu|$ is regular.

Theorem 22. If ν is a regular signed measure with LRN representation $d\nu = d\lambda + f dm$ and

$$f(x) = \lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} \text{ if } \{E_r\} \text{ shrinks nicely to } \{x\}.$$

13.2. **December 2, 4.**

(1) If $f \in L^1_{\text{loc}}$ and f is continuous at x, then x is in the Lebesgue set of f.

Proof. Assume $f \in L^1_{\text{loc}}$ and f is continuous at x. Then,

$$\lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} [f(y) - f(x)] dy = 0 \text{ a.e. } x; \ L_f = \{x : \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) = 0\}.$$

Since f is continuous at x for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Then, for all $r > 0$, $0 < r < \delta$ so

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) < \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} \epsilon dm(y) = \epsilon$$

Then, let $\epsilon \to 0$. $\square$

If E is a Borel set in $\mathbb{R}^2$ then the density of E at x is defined as

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(r,x))}{m(B(r,x))}.$$

(2) Show that $D_E(x) = 1$ for almost every $x \in E$, and that $D_E(x) = 0$ for almost every $y \notin E$.

(3) Show that $D_E(0) = \frac{1}{2}$ in the case that $E = [0, 1]$

(4) If λ and μ are positive Borel measures with $\lambda \perp \mu$ and $\lambda + \mu$ is regular, then λ and μ are regular.

Proof. Suppose there exists $\epsilon > 0$ such that $\lambda(U) > \lambda(E) + \epsilon$ for all $U \supseteq E$ where U is open.

$$\begin{aligned} (\lambda + \mu)(U) &= \lambda(U) + \mu(U) \\ &> \lambda(E) + \epsilon + \mu(E) \\ &= (\lambda + \mu)(E) + \epsilon. \end{aligned}$$

This contradicts $(\lambda + \mu)(E) = \inf\{(\lambda + \mu)(U) : E \subseteq U\}$.

The proof of μ regular follows similarly.

$\square$

14. Homework

14.1. **Assignment 1 (page 1).**

(1) Let $\mathcal{A}$ be an algebra of sets that is closed under countable increasing unions. Show that $\mathcal{A}$ is a σ-algebra.

Proof. Let $\mathcal{A}$ be an algebra on some set X that is closed under countable increasing unions. Then, consider a countable collection of sets in $\mathcal{A}$, namely $\{E_i\}_{i=1}^{\infty}$. To show $\mathcal{A}$ is a σ-algebra we must show $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$. Define $F_n = \bigcup_{j=1}^{n} E_j$. Since $\mathcal{A}$ is an algebra, $\mathcal{A}$ is closed under finite unions, so $F_n \in \mathcal{A}$ for all n. Further, $\{F_j\}_{j=1}^{\infty} \subset \mathcal{A}$. Additionally, notice $F_1 = E_1$, $F_2 = E_1 \cup E_2$, $F_3 = E_1 \cup E_2 \cup E_3, \cdots$ so $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$. Thus, $\{F_j\}_{j=1}^{\infty}$ is an increasing collection of sets. Because $\mathcal{A}$ is closed under countable increasing unions, $\bigcup_{j=1}^{\infty} F_j \in \mathcal{A}$. But $\bigcup_{j=1}^{\infty} F_j = \bigcup_{i=1}^{\infty} E_i$. Thus, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$ so we have shown $\mathcal{A}$ is a σ-algebra. $\square$

(2) Let $A \subset E \subset B$, where A, B are Lebesgue measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.

Proof. Let $A \subset E \subset B$, where A, B are Lebesgue measurable sets of finite measure. Assume $m(A) = m(B)$. Notice if $A \subset E \subset B$ set subtraction implies $\emptyset \subset E \backslash A \subset B \backslash A$. Since $m(A) = m(B)$ and $m(A), m(B)$ are finite we can write $m(B) - m(A) = 0$ which implies $m(B \backslash A) = 0$. Thus, $B \backslash A$ is a set of measure zero. Since Lebesgue measure is complete, if $E \backslash A$ is a subset of a set of measure zero, $E \backslash A$ must be measurable. $E = E \backslash A \cup A$, so $E \backslash A$ and A measurable imply E measurable since the collection of measurable sets is a σ-algebra. $\square$

(3) If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of Lebesgue measurable real-valued functions, prove that $f = \liminf f_n$ is Lebesgue measurable.

Proof. $\square$

(4) Let $f\colon \mathbb{R} \to [0, \infty)$ be Lebesgue measurable.

(a) Let $E_m = \{x \in \mathbb{R}\colon f(x) > 1/m\}$. Use the monotone convergence theorem to show

$$\lim_{m \to \infty} \int_{E_m} f dm = \int_{\mathbb{R}} f dm.$$

Proof. $\square$

(b) Prove that if $\int_{\mathbb{R}} f dm < \infty$, then for all $\varepsilon < \infty$, there exists $A \in \mathcal{B}_{\mathbb{R}}$ with $m(A) < \infty$ so that

$$\int_{\mathbb{R}} f dm < \int_A f dm + \varepsilon.$$

Proof. Assume $\int_{\mathbb{R}} f dm < \infty$. Then, $f \in L^+$ and

$$\int_{\mathbb{R}} f \ dm = \sup \left\{ \int_{\mathbb{R}} \phi \ dm : 0 \leq \phi \leq f, \ \phi \text{ simple} \right\}.$$

Thus, for any $\epsilon > 0$, we can find a simple function ϕ such that

$$\int_{\mathbb{R}} f \ dm < \int_{\mathbb{R}} \phi \ dm + \epsilon$$

Let

$$\phi = \sum_1^n a_j \chi_{E_j}. \text{ Then, } \int \phi \ dm = \sum_1^n a_j m(E_j).$$

Since $\int f \ dm$ is the supremum of all such integrals and $\int f \ dm < \infty$, $\int \phi \ dm < \infty$. Thus, if there exists some E_j with $m(E_j) = \infty$, the corresponding $a_j = 0$. So, consider only E_j with $m(E_j) \neq \infty$. Thus, by exercise 4 from September 18, for every E_j with $m(E_j) < \infty$ there exists some G_δ set, A_j, where $m(E_j) = m(A_j)$. Since $m(E_j) < \infty$, $m(A_j) < \infty$. Next, let

$$A = \bigcup_1^k A_j \quad E = \bigcup_1^k E_j$$

so A_j G_δ implies $A \in \mathcal{B}_R$. Also, $m(A_j) < \infty$ for all j implies $m(\cup_1^k A_j) = m(A) < \infty$. Additionally, since $a_j = 0$ whenever $m(E_j) = \infty$, we can write

$$\int_{\mathbb{R}} \phi \ dm = \sum_1^n a_j m(E_j) = \sum_1^k a_j m(E_j) = \sum_1^k a_j m(A_j) = \int_A \phi \ dm.$$

Then, since $\int_A f\ dm = \sup\{\int_A \phi\ dm : 0 \leq \phi \leq f, \phi \text{ simple }\}$, we have $\int_A \phi\ dm \leq \int_A f\ dm$.

Hence,

$$\int_{\mathbb{R}} f\ dm < \int_{\mathbb{R}} \phi\ dm + \epsilon = \int_A \phi\ dm + \epsilon \leq \int_A f\ dm + \epsilon.$$

□

(5) Let $\{f_n\}$ be a sequence of Lebesgue integrable functions that converge to f in L^1.

(a) Prove that $\{f_n\}$ converges to f in measure.

Proof. Assume $\{f_n\}$ is a sequence of Lebesgue integrable functions that converge to f in L^1. Then, $\int |f_n - f| \to 0$. Let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$ **see page 61, prop 2.29** □

(b) Give an example of a sequence $\{f_n\}$ and a function f such that $\{f_n\}$ converges to f in measure, but $\{f_n\}$ does not converge to f in L^1.

$f_n = n\chi_{[0,\frac{1}{n}]}$

$$\{x : n\chi_{[0,\frac{1}{n}]} \geq \varepsilon\} \subset \{x : n\chi_{[0,\frac{1}{n}]} > 0\}$$

$\mu(\{x : n\chi_{[0,\frac{1}{n}]} > 0\}) = \frac{1}{n}$

$$\int n\chi_{[0,\frac{1}{n}]} = n\mu\left(\left[0, \frac{1}{n}\right]\right) = 1$$

(6) Let f, g be Lebesgue integrable functions on $\mathbb{R}$. Prove that the function $F(x,y) = f(y)g(x-y)$ is Lebesgue integrable in $\mathbb{R}^2$.

Proof. Since f, g are Lebesgue integrable, fg is measurable. So, $\int |f| < \infty$ and $\int |g| < \infty$. Also, by Tonelli, $f(y)g(x-y) \in L^+$, and

$$\int |f(y)g(x-y)| d(m \times m)(x,y) = \int \int |f(y)g(x-y)| dm(x) dm(y).$$

Then, since $f(y)$ does not depend on x we can write

$$\int \int |f(y)g(x-y)| dm(x) dm(y) = \int |f(y)| \int |g(x-y)| dm(x) dm(y).$$

Also, Lebesgue integration is translation invariant so $\int g(x-y)$ is the same as $\int g(x)$. Thus,

$$\int |f(y)| \int |g(x-y)| dm(x) dm(y) = \int |f(y)| dm(y) \int |g(x)| dm(x).$$

$\square$

(7) Let μ_F be the Borel measure on $\mathbb{R}$ with distribution function

$$F(x) = \begin{cases} \arctan(x) & \text{if } x < 0, \\ x^2 + 1 & \text{if } \geq 0 \end{cases}$$

(a) Calculate $\mu_F([0,3))$ and $\mu_F((0,3))$.

By exercises from October 9, we have $\mu_F([0,3)) = F(3-) - F(0-) = 3^2 + 1 - \arctan(0) = 10$.

Also, $\mu_F((0,3)) = F(3-) - F(0) = 3^2 + 1 - (0^2 + 1) = 9$.

(b) If $\mu(E) = 0$ for every $E \in \mathbb{R}$ with $m(E) = 0$, then we write $\mu \ll m$ and say μ is absolutely continuous with respect to Lebesgue measure.

(c) Prove that μ_F is not absolutely continuous with respect to Lebesgue measure.

Proof. Notice

$$\mu_F(\{0\}) = \mu_F([0,3)/(0,3)) = \mu_F([0,3)) - \mu_F((0,3)) = 10 - 9 = 1$$

but $m(\{0\}) = 0$. This, $\mu_F \not\ll m$. $\square$

(8) Construct a family of Lebesgue measurable functions $\chi_t \colon \mathbb{R} \to \mathbb{R}$, $t \in \mathbb{R}$, with the property that $\chi = \sup_{t \in \mathbb{R}} \chi_t$ is not a Lebesgue measurable function. (You may assume without the proof that non-measurable sets exist.

(exercise 6, §2.1): Show by example that there is an uncountable set A and for each $a \in A$ a measurable function f_a, but $\sup\{f_\alpha : \alpha \in A\}$ is not measurable.

Consider the set N_r constructed in section 1.1. Then N_r is an uncountable set and therefore not measurable. However, for every $r \in \mathbb{Q} \cap [0,1)$ and $x \in N$ (where N was defined as the subset of $[0,1)$ containing exactly one member of the equivalence classes defined by $x \sim y$ iff

$x-y \in \mathbb{Q}$. Singletons are measurable, so, from page 46 of Folland, the indicator functions $\chi_{\{x+r\}}$ and $\chi_{\{x_r-1\}}$ are measurable for all $r \in \mathbb{Q} \cap [0,1)$ and $x \in N \cap [0, 1-r)$ or $x \in N \cap [1-r, 1)$. Notice $\sup\{\chi_{\{x+r\}}, \chi_{\{x+r-1\}} : r \in \mathbb{Q} \cap [0,1)$ and $x \in N \cap [0, 1-r)$ or $x \in N \cap [1-r, 1)\} = \chi_{N_r}$ But, χ_{N_r} is not measurable because N_r is not measurable.

(9) Show by way of an example that an open, dense set in $\mathbb{R}$ need not have infinite measure.

(10) Let f be a continuous function of bounded variation. Prove that $f = f_1 - f_2$ where both f_1, f_2 are monotonic and continuous.

14.2. **Assignment 2 (page 2).**

(1) Let f be real-valued.

(a) Give the definition of a measurable function.

(p. 43, Folland) If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, a mapping $F : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$. (b) If f is measurable, is f^2 measurable? Justify.

Yes. From group work, we know fg is measurable. So, ff is measurable.

(c) If f^2 is measurable, is f measurable. Justify.

use counterexample of $f = \chi_{N_r} - \chi_{N_r^c}$. Note f is not measurable but $f^2 = 1$ since $f = \pm 1$

(2) Let X be a countable set and μ a measure on X. Assume that for any $F \subseteq X$ there is $G \subseteq F$, $G \neq \emptyset$, with $\mu(G) < \infty$. Prove the following:

(a) For any $x \in X$, $\mu(\{x\}) < \infty$.

(b) μ is σ-finite.

Proof. Note $X = \cup_{x \in X} \{x\}$. By part a $\mu(\{x\}) < \infty$. Thus, X can be written as a countable union sets where each set has finite measure. Therefore, μ is σ-finite. □

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume $\{f_n\}$ is a sequence in $L^1(X, \mu)$ so that $f_n \to f$ pointwise, and there exists $M > 0$ so that for all n and all $x \in X$ the inequality $|f_n(x)| \leq M$ holds.

(a) If $\mu(X) < \infty$, show that $f \in L^1(X, \mu)$ and $\int f_n \to \int f$.

M dominated convergence theorem -

(b) Show by example that the above conclusion may fail if $\mu(X) = \infty$.

(4) Let μ^* be an outer measure on a set X. If $E \subseteq X$ satisfies $\mu^*(E) = 0$ prove that E is μ^*-measurable.

Proof. $\square$

(5) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function of bounded variation such that $f(0) = 0$.

(a) Give an example of such a function for with the identity

$$f(x) = \int_0^x f^{'}(x)dx \text{ fails to hold for a.e. } x \in [0, 1].$$

(b) For what type of functions f does the identity in (a) hold almost everywhere?

(6) (a) State the Monotone Convergence Theorem.

If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{j\to\infty} f_n$, then $\int f = \lim_{n\to\infty} \int f_n$.

(b) State Fatou's Lemma.

If $\{f_n\}$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

(c) Prove Fatou's Lemma using the Monotone Convergence Theorem.

Proof. (Proof from Folland, p. 52)

For each $k \geq 1$ we have $\inf_{n\geq k} f_n \leq f_j$ for $j \geq k$. Hence $\int \inf_{n\geq k} f_n \leq \int f_j$ for $j \geq k$. Therefore, $\int \inf_{n\geq k} f_n \leq \inf_{j\geq k} \int f_j$. Let $k \to \infty$ and apply the monotone convergence theorem to obtain:

$$\int (\liminf f_n) = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \liminf \int f_n.$$

$\square$

(7) Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } x < \frac{1}{4}, \\ x & \text{if } \frac{1}{4} \le x < 1, \\ x^2 + 1 & \text{if } x \ge 1, \end{cases}$$

Let μ_F be the Borel measure associated to F.

(a) Calculate $\mu_F((\frac{1}{4}, 1])$ and $\mu_F([\frac{1}{4}, 1])$.

By exercises from October 9, we know $\mu_F([a,b)) = F(b-) - F(a-) = \mu_F((a,b]) = F(b) - F(a)$.

Thus,

$$\mu_F\left(\left(\frac{1}{4}, 1\right]\right) = F(1) - F\left(\frac{1}{4}\right) = 1^2 + 1 - \frac{1}{4} = \frac{7}{4}.$$

$$\mu_F\left(\left[\frac{1}{4}, 1\right)\right) = F(1-) - F\left(\frac{1}{4}-\right) = 1 - 0 = 1.$$

(b) Calculate the Lebesgue derivative of μ_F.

To calculate, recall the Lebesgue-Radon-Nikodym representation given in Theorem 3.22, $d\mu_F = d\lambda + f\ dm$ where f is the Lebesgue derivative and since we are working in $\mathbb{R}$ we can write

$$f(x) = \lim_{r \to 0} \frac{\mu_F(B(r,x))}{m(B(r,x))} = \lim_{r \to 0} \frac{\mu_F((x-r, x+r))}{m((x-r, x+r))} = \lim_{r \to 0} \frac{F((x+r)-) - F(x-r)}{2r}.$$

Suppose $x < \frac{1}{4}$, then

$$\lim_{r \to 0} \frac{F((x+r)-) - F(x-r)}{2r} = \lim_{r \to 0} \frac{0 - 0}{2r} = 0.$$

Suppose $\frac{1}{4} < x < 1$, then

$$\lim_{r \to 0} \frac{F((x+r)-) - F(x-r)}{2r} = \lim_{r \to 0} \frac{x + r - (x - r)}{2r} = 1.$$

Suppose $x > 1$, then

$$\lim_{r \to 0} \frac{F((x+r)-) - F(x-r)}{2r} = \lim_{r \to 0} \frac{(x+r)^2 + 1 - ((x-r)^2 + 1)}{2r} = \lim_{r \to 0} \frac{4rx}{2r} = 2x.$$

Suppose $x = \frac{1}{4}$, then

$$\lim_{r\to 0} \frac{F\left(\left(\frac{1}{4}+r\right)-\right) - F\left(\frac{1}{4}-r\right)}{2r} = \lim_{r\to 0} \frac{\frac{1}{4}+r}{2r} = \infty \quad (r>0)$$

Suppose $x = 1$, then

$$\lim_{r\to 0} \frac{F\left((1+r)-\right) - F\left(1-r\right)}{2r} = \lim_{r\to 0} \frac{(1+r)^2+1-(1-r)}{2r} = \lim_{r\to 0} \frac{1+3r+r^2}{2r} = \infty \quad (r>0)$$

(c) Give the Lebesgue-Radon-Nikodym decomposition of μ_F with respect to m.

(8) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measures spaces and assume that $E \in \mathcal{M}\times\mathcal{N}$. Show that if $\mu\times\nu(E) = 0$ then $\nu(E_x) = \mu(E^x)$ for a.e. x and y.

Proof. □

14.3. **Assignment 3 (page 3).**

(1) Show that if $E_1 \cup E_2$ is Lebesgue measurable and $m(E_2) = 0$, then E_1 is Lebesgue measurable.

Proof. Suppose $E_1 \cup E_2$ is Lebesgue measurable and $m(E_2) = 0$. Since $m(E_2) = 0$, E_2 is Lebesgue measurable. Notice $(E_1 \cup E_2)/E_1 \subseteq E_2$. Since the subset of Lebesgue measurable sets are measurable, so $(E_1 \cup E_2)/E_1$ is measurable. Also, the complement and intersection of measurable sets is measurable, so $(E_1 \cup E_2) \cap ((E_1 \cup E_2)/E_1)^c$ is measurable. By inspection, we can see $E_1 = (E_1 \cup E_2) \cap ((E_1 \cup E_2)/E_1)^c$. Thus, E_1 is Lebesgue measurable. □

(2) Let μ_F be the Borel measure on $\mathbb{R}$ with distribution function F :

$$F(x) = \begin{cases} \arctan(x) + 5 & \text{if } x \geq 2, \\ x^2 - 2 & \text{if } 0 \leq x < 2, \\ e^x - 3 & \text{if } x < 0. \end{cases}$$

(a) Calculate $\mu_F(\mathbb{R}), \mu_F(\{2\})$, and $\mu_F((-\infty, 0))$.

By continuity from below, we have $\mu_F(\mathbb{R}) = \mu_F\left(\bigcup_{x\in\mathbb{N}}(-x, x]\right) = \lim_{x\to\infty}\mu_F((-x, x])$ so

$$\mu_F(\mathbb{R}) = \lim_{x\to\infty} F(x) - F(-x) = \lim_{x\to\infty}(\arctan x + 5) - \lim_{x\to\infty}(e^{-x} - 3) = \frac{\pi}{2} + 8.$$

$$\mu_F(\{2\}) = F(2) - F(2-) = \arctan(2) + 5 - (2^2 - 2) = \arctan(2) + 3.$$

$$\mu_F((-\infty,0)) = \mu_F\left(\bigcup_{x\in\mathbb{N}}\left(-x,-\frac{1}{x}\right]\right) = \lim_{x\to\infty}\mu_F\left(\left(-x,-\frac{1}{x}\right]\right) = \lim_{x\to\infty}F\left(-\frac{1}{x}\right) - \lim_{x\to\infty}F(-x)$$

$$= \lim_{x\to\infty}(e^{-\frac{1}{x}} - 3 - e^{-x} + 3) = 1.$$

(b) Calculate the Lebesgue derivative of μ_F.

To calculate, recall the Lebesgue-Radon-Nikodym representation given in Theorem 3.22, $d\mu_F = d\lambda + f\ dm$ where f is the Lebesgue derivative and since we are working in $\mathbb{R}$ we can write

$$f(x) = \lim_{r\to 0}\frac{\mu_F(B(r,x))}{m(B(r,x))} = \lim_{r\to 0}\frac{\mu_F((x-r,x+r))}{m((x-r,x+r))} = \lim_{r\to 0}\frac{F((x+r)-) - F(x-r)}{2r}.$$

Suppose $x < 0$, then

$$\lim_{r\to 0}\frac{F((x+r)-) - F(x-r)}{2r} = \lim_{r\to 0}\frac{e^{x+r} - e^{x-r}}{2r} = e^x\lim_{r\to 0}\frac{e^r - e^{-r}}{2r} = e^x.$$

Suppose $0 < x < 2$, then

$$\lim_{r\to 0}\frac{F((x+r)-) - F(x-r)}{2r} = \lim_{r\to 0}\frac{(x+r)^2 - 2 - (x-r)^2 + 2}{2r} = \lim_{r\to 0}\frac{4r}{2r} = 2.$$

Suppose $x > 2$, then

$$\lim_{r\to 0}\frac{F((x+r)-) - F(x-r)}{2r} = \lim_{r\to 0}\frac{\arctan(x+r) - \arctan(x-r)}{2r} = 0.$$

(c) Is μ_F absolutely continuous with respect to Lebesgue measure?

No. Recall $m(\{2\}) = 0$ but, by (a), $\mu_F(\{2\}) = \arctan(2) + 3 \neq 0$.

(3) Let $f \in L^1(X,\mu)$. Show that $\{x \in X : f(x) \neq 0\}$ is σ-finite.

Proof. To show A is σ-finite, we will first show $\bigcup_1^\infty A_n = A$ where $A_n = \{x : |f(x)| > \frac{1}{n}, n \in \mathbb{N}\}$.

Suppose $x \in A$. Then, $f(x) \neq 0$ so there must exist some $n \in \mathbb{N}$ such that $|f(x)| > \frac{1}{n}$. Thus, $\bigcup_1^\infty A_n \supset A$.

Suppose $x \in \bigcup_1^\infty A_n$. Then, there exists some $n \in \mathbb{N}$ such that $|f(x)| > \frac{1}{n}$. Thus, $f(x) \neq 0$ and so $x \in A$ and $\bigcup_1^\infty A_n \subset A$.

Next, we will show $\mu(A_n) \neq \infty$ for any n. Suppose there exists some n with $\mu(A_n) = \infty$. Then,

$$\int |f| \, d\mu \geq \int_{A_n} |f| \, d\mu \geq \int_{A_n} \frac{1}{n} \, d\mu = \frac{1}{n}\mu(A_n) = \infty.$$

However, $f \in L^1$, so $\int |F| < \infty$. Thus, $A = \bigcup_1^\infty A_n$ with $\mu(A_n) < \infty$ for all n so $\{x \in X : f(x) \neq 0\}$ is σ-finite.

$\square$

(4) Suppose $f \in L^1(\mathbb{R}, m)$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\cos(nx) \, dx = 0.$$

Hint: show this first for the characteristic function of an interval.

(5) Assume that for every $\epsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. Show that $f_n \to f$ a.e.

Proof. Assume that for every $\epsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. Let $A = \{x : f_n(x) \to f(x)\}$. To show $f_n \to f$ a.e., it suffices to show $\mu(A^c) = 0$. Suppose $\mu(A^c) \neq 0$. Then, $\mu(A^c) = k$ for some $k > 0$. Choose $\epsilon < k$. Then, there exists $E \subseteq X$ such that $\mu(E) < \epsilon < k$ and $f_n \to f$ on E^c. Since $f_n \to f$ on E^c and $A = \{x : f_n(x) \to f(x)\}$, $E^c \subseteq A$ and so $A^c \subseteq E$. Thus, $\mu(A^c) \leq \mu(E)$ and so $\mu(E) \geq k$. This is a contradiction to our assumption that $\mu(A^c) = k$, $k \neq 0$. Thus, $\mu(A^c) = 0$ and $f_n \to f$ a.e. $\square$

(6) For each $E \subseteq \mathbb{R}$, let $\mu(E) = \#(E \cap \mathbb{Z})$. Calculate

$$\int\int_{[2,\infty)\times[2,\infty)} (y-1)x^{-y} d(m(x) \times \mu(y)). \text{ What theorem did you use?}$$

We will use Fubini-Tonelli to write:

$$\begin{aligned}
\int\int_{[2,\infty)\times[2,\infty)} (y-1)x^{-y} d(m(x)\times\mu(y)) &= \int_{[2,\infty)}\int_{[2,\infty)} (y-1)x^{-y}\ dm(x)\ d\mu(y) \\
&= \int_{[2,\infty)} \frac{y-1}{-y+1}x^{1-y}\Big|_2^\infty\ d\mu(y) \\
&= \int_{[2,\infty)} \left(\lim_{x\to\infty}(-x^{1-y}) + 2^{1-y}\right)\ d\mu(y) \\
&= \int_{[2,\infty)} 2^{1-y}\ d\mu(y) \\
&= \sum_{n=2}^{\infty}\int_{(n,n+1)} 2^{y-1}\ d\mu(y) + \sum_{n=2}^{\infty}\int_{\{n\}} 2^{y-1}\ d\mu(y) \\
&= \sum_{n=2}^{\infty} 2^{n-1}\cdot\mu((n,n+1)) + \sum_{n=2}^{\infty}\int_{\{n\}} 2^{y-1}\ d\mu(y) \\
&= \sum_{n=2}^{\infty} 2^{n-1}\cdot 0 + \sum_{n=2}^{\infty}\mu(\{n\})2^{n-1} \\
&= \sum_{n=2}^{\infty} 2^{n-1} \\
&= 1.
\end{aligned}$$

(7) Let $F:\mathbb{R}\to\mathbb{R}$. Prove that there is a constant M such that $|F(x)-F(y)|\leq M|x-y|$ for every $x,y\in\mathbb{R}$ if and only if F is absolutely continuous and $|F^{'}|\leq M$ a.e.

Proof. ($\Rightarrow$) Assume that there is a constant M such that $|F(x)-F(y)|\leq M|x-y|$ for every $x,y\in\mathbb{R}$. Let $\varepsilon>0$. Choose $\delta=\frac{\varepsilon}{M}$. Then, for any collection $\{(x_1,y_1),\ldots(x_N,y_N)\}$ of disjoint intervals with $\sum_1^N|x_j-y_j|<\delta$ we have

$$\sum_1^N|F(x_j)-F(y_j)|\leq\sum_1^N M|x_j-y_j| = M\sum_1^N|x_j-y_j| < M\delta=\varepsilon.$$

Thus, F is absolutely continuous.

($\Leftarrow$) Suppose F is absolutely continuous and $|F^{'}|\leq M$ a.e. Then, by the Fundamental Theorem of Calculus for Lebesgue Integrals,

$$F(y) - F(x) = \int_x^y F'(t)dt$$

$$\begin{aligned}
\text{Thus, } F(y) - F(x) &= \int_x^y F'(t)dt \\
&\leq \int_x^y |F'(t)|dt \\
&\leq \int_x^y M dt \\
&= M|x-y|.
\end{aligned}$$

□

(8) A set $E \subseteq [0,1]$ has the property that there exists $0 < d < 1$ such that for every $(\alpha, \beta) \subset [0,1]$, $m(E \cap (\alpha, \beta)) > d(\beta - \alpha)$. Prove that $m(E) = 1$. Hint: Lebesgue's differentiation.

Proof. Since $m(E \cap (\alpha, \beta)) > d(\beta - \alpha)$,

$$\frac{m(E \cap (\alpha, \beta))}{d(\beta - \alpha)} = \frac{m(E \cap (\alpha, \beta))}{dm((\alpha, \beta))} > 1.$$

Consider $B(r, x_0)$ with $x_0 \in (\alpha, \beta) \subseteq (0,1)$, $x_0 = \frac{\alpha+\beta}{2}$ and $r = \frac{\beta-\alpha}{2}$. Then,

$$\begin{aligned}
d &< \frac{m(E \cap (\alpha, \beta))}{dm((\alpha, \beta))} \\
&= \frac{m(E \cap B(r, x_0))}{m(B(r, x_0))} \\
&= \frac{1}{m(B(r, x_0))} \int_{B(r,x_0)} \chi_E \, dm
\end{aligned}$$

$$\text{Therefore, } \lim_{r \to 0} \frac{1}{m(B(r, x_0))} \int_{B(r,x_0)} \chi_E \, dm > \lim_{r \to 0} d \text{ and so } \chi_E(x_0) > d \text{ a.e.}$$

Thus, $\chi_E(x_0) > 0$ a.e. which implies $x_0 \in E$ for almost every $x_0 \in (0,1)$ and so $E = (0,1)$ a.e. Since $m((0,1)) = 1$ it must be the case that $m(E) = 1$. □

14.4. **Assignment 4 (page 4).**

(1) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a measurable function, finite at every $x \in X$. Let $G_f = \{(x,t) \in X \times \mathbb{R} : t = f(x)\}$ be the graph of f. If μ is σ-finite, prove that G_f has measure zero with the product measure $\mu \times m$. Hint: use Fubini.

Proof. We will show for $G_f = \{(x,t) \in X \times \mathbb{R} : t = f(x)\}$, $(\mu \times m)(G_f) = 0$. Since μ is σ-finite we can use Fubini to rewrite $(\mu \times m)(G_f)$ as

$$(\mu \times m)(G_f) = \int_{X\times\mathbb{R}} \chi_{G_f} d(\mu \times m) = \int_X \int_{\mathbb{R}} \chi_{G_f} \; dm \; d\mu. \text{ Also, note}$$

$$(\chi_{G_f})_x = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{if } y \neq f(x) \end{cases} \quad \text{(eq. 1), and}$$

$$(G_f)_x = \{y\} \text{ where } y = f(x), \text{ so } m((G_f)_x) = m(\{y\}) = 0. \text{ (eq. 2)}$$

Thus, by eq. 1 and 2, we have

$$\begin{aligned}
(\mu \times m)(G_f) &= \int_X \int_{\mathbb{R}} \chi_{G_f} \; dm \; d\mu \\
&= \int_X \int_{\mathbb{R}} (\chi_{G_f})_x m(y) \mu(x) \\
&= \int_X m(G_{f_x}) \; d\mu(x) \\
&= \int_X 0 \; d\mu(x) \\
&= 0.
\end{aligned}$$

□

(2) Let $X = [0,1]$, $\mathcal{M} = B_{[0,1]}$, m the Lebesgue measure on $\mathcal{M}$ and μ the counting measure on $\mathcal{M}$. Show that $m \ll \mu$, but that there is no function f such that $dm = f d\mu$. Does this contradict the Radon-Nikodym theorem?

Proof. For $E \in \mathcal{M}$ suppose $\mu(E) = 0$. Then, since μ is the counting measure, E must not contain any elements and so $E = \emptyset$. Hence, $m(E) = 0$ and $m \ll \mu$.

Next, suppose there is an f such that $dm = f d\mu$. Let $a \in [0,1]$ and note $m(\{a\}) = 0$ and

$$m(\{a\}) = \int_{\{a\}} f d\mu = f(a)\mu(\{a\}) = f(a). \qquad \text{Hence, } f(a) = 0 \quad \forall\ a \in [0,1].$$

Also, notice $m([0,1]) = 1$ but

$$m([0,1]) = \int_{[0,1]} f d\mu = \int 0 d\mu = 0\mu([0,1]) = 0.$$

Thus, there is no f such that $dm = f d\mu$.

This does not contradict the Radon-Nikodym theorem because the counting measure is not σ-finite. $\square$

(3) (a) Let $f_n : [1, \infty) \to \mathbb{R}$ be a function defined by $f_n(x) = \frac{1}{x}\chi_{[n,\infty)}(x)$. Show that the sequence $\{f_n\}$ converges to zero uniformly on $[1, \infty)$.

Proof. Let $f_n : [1, \infty) \to \mathbb{R}$ be a function defined by $f_n(x) = \frac{1}{x}\chi_{[n,\infty)}(x)$. Note

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \geq n \\ 0 & \text{if } x < n \end{cases}.$$

Thus, when $f_n(x) = \frac{1}{x}$, $f_n(x) \leq \frac{1}{n}$. So, for any $\varepsilon > 0$ pick N such that $\varepsilon > \frac{1}{N}$. Then, for all $n \geq N$ and for all x, $|f_n(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. Hence, $\{f_n\}$ converges to zero uniformly on $[1, \infty)$. $\square$

(b) State Fatou's lemma.

If $\{f_n\}$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

(c) Apply Fatou's lemma to the sequence from part (a).

By corollary to Fatou's Lemma, since $f_n \to 0$

$$\int 0 = 0 \leq \liminf \int \frac{1}{x}\chi_{[n,\infty)}(x)\ dm = \int_{[n,\infty)} \frac{1}{x}\ dm.$$

(4) Let $f_n : X \to \mathbb{R}$ be measurable, bounded functions such that for every $n \in \mathbb{R}$, $x \in X$, $f_n(x) \geq f_{n+1}(x)$ and there is a measurable function $f : X \to \mathbb{R}$ such that $\lim f_n(x) = f(x)$ pointwise. If $\int f_k\ d\mu < \infty$ for some $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

(5) Prove the following: (a) If f is monotonic, then f is Lebesgue measurable.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. By Proposition 2.3 in Page 44, it suffices to show that for any $a \in \mathbb{R}$, we have $f^{-1}((a,\infty))$ is Borel measurable. WLOG, assume f is increasing. Let $x^{'} = \inf\{x : f(x) > a\}$

Case 1: Suppose $f(x^{'}) \leq a$**.** We will show $f^{-1}((a,\infty)) = (x^{'},\infty)$. First, show $f^{-1}((a,\infty)) \subseteq (x^{'},\infty)$. Let $x \in f^{-1}((a,\infty))$. Then, $f(x) > a$. Since $x^{'} = \inf\{x : f(x) > a\}$, $x^{'} < x$. Thus, $x \in (x^{'},\infty)$.

Next, show $f^{-1}((a,\infty)) \supseteq (x^{'},\infty)$. Let $x \in (x^{'},\infty)$. Then, $x > x^{'}$. Since $x^{'} = \inf\{x : f(x) > a\}$ and $x > x^{'}$, there exists some $x_0 \in \mathbb{R}$ such that $x > x_0 > x^{'}$ and $f(x_0) > a$. f is monotone, so $f(x) > f(x_0)$. Thus, $f(x) > a$ which implies $x \in f^{-1}((a,\infty))$.

Case 2: Suppose $f(x^{'}) > a$. We will show $f^{-1}((a,\infty)) = (x^{'},\infty)$. First, show $f^{-1}((a,\infty)) \subseteq (x^{'},\infty)$. Let $x \in f^{-1}((a,\infty))$. Then, $f(x) > a$. Since $x^{'} = \inf\{x : f(x) > a\}$, $x^{'} < x$. Thus, $x \in (x^{'},\infty)$.

Next, show $f^{-1}((a,\infty)) \supseteq (x^{'},\infty)$. Let $x \in (x^{'},\infty)$. Then, $x > x^{'}$. Since f is monotone, so $f(x) > f(x^{'}) > a$. Thus, $f(x) > a$ which implies $x \in f^{-1}((a,\infty))$.

Case 3: Suppose $f(x^{'}) = \infty$. We will show $f^{-1}((a,\infty)) = \emptyset$. If $f(x^{'}) = \infty$, $f(\inf\{x : f(x) > a\}) = \infty$ which implies $\inf\{x : f(x) > a\} = \infty$ so $\{x : f(x) > a\} = \emptyset$. Thus, $f^{-1}((a,\infty)) = \emptyset$.

Case 4: Suppose $f(x^{'}) = -\infty$. We will show $f^{-1}((a,\infty)) = \mathbb{R}$. If $f(x^{'}) = -\infty$, $f(\inf\{x :$

$f(x) > a\}) = -\infty$ which implies $\inf\{x : f(x) > a\} = -\infty$ so $\{x : f(x) > a\} = \mathbb{R}$. Thus, $f^{-1}((a,\infty)) = \mathbb{R}$.

Hence, for any $a \in \mathbb{R}$, we have $f^{-1}((a,\infty))$ is Borel measurable, so f is measurable. $\square$

(b) If f is continuous and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

Proof. Assume f is continuous and g is Lebesgue measurable. Note $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and $(g^{-1} \circ f^{-1})((a,\infty)) = g^{-1}(f^{-1}(a,\infty))$. Since f is continuous, $f^{-1}((a,\infty))$ must be an interval, so $f^{-1}((a,\infty)) \in \mathcal{B}_{\mathbb{R}}$. Then, since g is Lebesgue measurable, $g^{-1}(f^{-1}(a,\infty)) \in \mathcal{L}$. Hence, $f \circ g$ is Lebesgue measurable. $\square$

(c) If f is continuous and g is Lebesgue measurable, is $g \circ f$ Lebesgue measurable?

No. Consider the following counterexample:

(6) Let $f(x) = \int_0^\infty e^{-xt}(t^{-3}\sin^3(t))dt$. Show that (a) $f(x)$ is well-defined for each $x \in [0,\infty)$.

Proof. Consider any $x \in [0,\infty)$. Then,

$$|f(x)| \leq \int_0^\infty |e^{-xt}(t^{-3})\sin^3 t|dt \leq \int_0^\infty t^{-3} < \infty. \text{ So, } f(x) \text{ exists for any } x \in [0,\infty).$$

Next, suppose $x_1, x_2 \in [0,\infty)$ such that $x_1 = x_2$. Then,

$$\begin{aligned} f(x_1) - f(x_2) &= \int_0^\infty e^{-x_1 t}(t^{-3})\sin^3 t\ dt - \int_0^\infty e^{-x_2 t}(t^{-3})\sin^3 t\ dt \\ &= \int_0^\infty (e^{-x_1 t}(t^{-3})\sin^3 t - e^{-x_2 t}(t^{-3})\sin^3 t)\ dt \\ &= \int_0^\infty (e^{-x_1 t} - e^{-x_2 t})(t^{-3})\sin^3 t\ dt \\ &= 0. \end{aligned}$$

Hence, f is well-defined. $\square$

(b) $f(x)$ is continuous on $[0,\infty)$.

Proof. Let $x_n \to x$. Then, from part a, we know $|e^{-x_n t}(t^{-3})| \leq t^{-3}$, so by the dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \int_0^\infty e^{-x_n t}(t^{-3}) \sin^3 t dt \\
&= \int_0^\infty \lim_{n\to\infty} e^{-x_n t}(t^{-3}) \sin^3 t dt \\
&= \int_0^\infty e^{-xt}(t^{-3}) \sin^3 t dt \\
&= f(x).
\end{aligned}$$

Thus, $f(x)$ is continuous on $[0, \infty)$. $\square$

(7) Suppose that f is a continuous real-valued function of bounded variation on $[0, 1]$ and that for each $\epsilon \in (0, 1)$, f is absolutely continuous on $[\epsilon, 1]$. Must f necessarily be absolutely continuous on $[0, 1]$?

(8) Suppose $f \in L^1[0, 1]$ satisfies

$$\int_E |f| \leq (m(E))^2 \text{ for every measurable set } E \subseteq [0, 1].$$

Show that f is a.e. equal to zero. Use Lebesgue differentiation theorem.

(9) Prove or give a counterexample: every dense open subset of $(0, 1)$ has Lebesgue measure 1.

(10) Let $f \in L^1(\mathbb{R})$ such that $f(x) = 0$ for $|x| \geq 1$. Prove that f_n defined by $f_n(x) = f\left(x + \frac{1}{n}\right)$ converges to f in $L^1(\mathbb{R})$. Is the condition that $|x| \geq 1$ necessary?