Boolean Algebras

December 2022

Definition 1. A *Boolean ring* is a ring satisfying $x^2 = x$.

Definition 2. A *Boolean algebra* is a set $(B, \wedge, \vee, \neg, 0, 1)$ satisfying certain axioms.

- Associativity: $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- Commutativity: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
- Identity: $a \vee 0 = a$ and $a \wedge 1 = a$
- Distributivity: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- Complements: $a \vee \neg a = 1$ and $a \wedge \neg a = 0$

Example 3. Let X be a set and $\mathcal{P}(X)$ be the power set of X. We define $\wedge$ to be intersections and $\vee$ to be unions, $\neg$ to be complement, 0 to be the empty set and 1 to be X. Then $(\mathcal{P}(X), \cap, \cup, ^{\mathsf{c}}, \varnothing, X)$ is a Boolean algebra.

Definition 4. Given a Boolean algebra $\mathcal{B}$, we can define a partial ordering on $\mathcal{B}$ by defining $a \leq b$ to be true if and only if $a \vee b = b$. An *atom* under this partial ordering is a non-zero minimal element.

Definition 5. A Boolean algebra is *atomic* if no two distinct elements are $\geq$ to the same set of atoms.

Theorem 6. Suppose $\mathcal{B}$ is a Boolean algebra with atoms and b is an element with finitely many atoms below. If $\{a_1, a_2, \ldots, a_n\}$ are all the atoms less than or equal to b, then

$$b = a_1 \vee a_2 \vee \cdots \vee a_n.$$

Proof. Let $b' = a_1 \vee a_2 \vee \cdots \vee a_n$. It is easy to check that $b' \leq b$. Pick any $z \leq \neg b \wedge b'$. We have $z \leq \neg b$ and $z \leq b' \leq b$, so $z \leq \neg b \wedge b = 0$, so $z = 0$. Since $\neg b \wedge b' \leq \neg b \wedge b'$, $\neg b \wedge b' = 0$. □

This shows that finite Boolean algebras are atomic. It also shows that if every element has at least one atom below it, then the Boolean algebra is atomic.

Problem 7. Is every Boolean algebra atomic?

All finite Boolean algebras are atomic, and there are infinitely many atomic Boolean algebras. There exist Boolean algebras that are not atomic, such as the set of all repeating sequences in the set of natural numbers where each index corresponds to an element being in or not in the set (110110110110..., for example).

Problem 8. Can every Boolean algebra be realized as $\mathcal{P}(X)$ for some set X?

Every finite Boolean algebra can be realized as a power set. However, infinite ones cannot always be realized as a power set. The set of all sets in the power set of natural numbers that are either finite or cofinite, for example, is a Boolean algebra and converts to a Boolean ring, but it is not a power set: it is a subset of a power set.

Problem 9. There is an equivalence between Boolean algebras and Boolean rings. Define a Boolean algebra given a Boolean ring and vice versa such that these constructions are inverses of each other.

Proof. Given a Boolean algebra $(B, \wedge, \vee, \neg, 0, 1)$, define the ring operations on elements $a, b \in B$ as follows:

- Let $a + b = (a \wedge \neg b) \vee (b \wedge \neg a) = (a \vee b) \wedge \neg(a \wedge b)$.
- Let $ab = a \wedge b$.

Given a Boolean ring $(B, +, \cdot, 0, 1)$, define the Boolean operations on elements $a, b \in B$ as follows:

- Let $a \vee b = a + b + ab$.
- Let $a \wedge b = ab$.
- Let $\neg a = a + 1$.

Starting with a Boolean algebra, the derived ring satisfies $ab = a \wedge b$, $a + 1 = (a \wedge \neg 1) \vee (\neg a \wedge 1) = (a \wedge 0) \vee \neg a = 0 \vee \neg a = \neg a$, and

$$\begin{aligned}
&a + b + ab \\
&= ((a + b) \wedge \neg ab) \vee (\neg(a + b) \wedge ab) \\
&= ((a \vee b) \wedge \neg(a \wedge b) \wedge \neg(a \wedge b)) \vee ((\neg(a \wedge \neg b) \wedge \neg(b \wedge \neg a)) \wedge a \wedge b) \\
&= ((a \vee b) \wedge \neg(a \wedge b)) \vee ((\neg a \vee b) \wedge (\neg b \wedge a) \wedge a \wedge b) \\
&= ((a \vee b) \wedge \neg(a \wedge b)) \vee ((\neg a \vee b) \wedge a \wedge (\neg b \wedge a) \wedge b) \\
&= ((a \vee b) \wedge \neg(a \wedge b)) \vee (((\neg a \wedge a) \vee (b \wedge a)) \wedge ((\neg b \wedge b) \vee (\neg b \wedge a))) \\
&= ((a \vee b) \wedge \neg(a \wedge b)) \vee (((\neg a \wedge a) \vee (b \wedge a)) \wedge ((\neg b \wedge b) \vee (a \wedge b))) \\
&= ((a \vee b) \wedge \neg(a \wedge b)) \vee (a \wedge b \wedge a \wedge b) \\
&= ((a \vee b) \wedge \neg(a \wedge b)) \vee (a \wedge b) \\
&= (a \vee b \vee (a \wedge b)) \wedge ((a \wedge b) \vee \neg(a \wedge b)) \\
&= (a \vee b \vee (a \wedge b)) \wedge 1 \\
&= a \vee b \vee (a \wedge b) \\
&= (a \wedge 1) \vee (a \wedge b) \vee b \\
&= (a \wedge (1 \vee b)) \vee b \\
&= (a \wedge 1) \vee b \\
&= a \vee b,
\end{aligned}$$

so the derived algebra is the same as the original.

Starting with a Boolean ring, the derived algebra satisfies $a \wedge b = ab$ and

$$\begin{aligned}
&(a \wedge \neg b) \vee (b \wedge \neg a) \\
&= (a(b+1)) \vee (b(a+1)) \\
&= a(b+1) + b(a+1) + ab(a+1)(b+1) \\
&= ab + a + ba + b + abab + aba + abb + ab \\
&= a + b + 6ab \\
&= a + b \text{ (see Observation 11)},
\end{aligned}$$

so the derived ring is the same as the original. □

Question 10. What kind of properties must Boolean rings have?

Observation 11. Every Boolean ring has characteristic 2.

Theorem 12. Boolean rings are commutative.

Proof. We have

$$x + y = (x+y)^2 = x^2 + xy + yx + y^2 = x + y + xy + yx$$

This shows that

$$xy + yx = 0,$$

and since $yx = -yx$, we have $xy = yx$. □

Lemma 13. Boolean ring with more than two elements is not an integral domain.

Proof. Note that since $x^2 = x$, we have $x(x-1) = 0$. Thus if we have any element that's not 0 or 1, we have a zero divisor. □

Definition 14. Given an arbitrary ring $(R, +, \cdot)$, a subset I is an *ideal* of R if $(I, +)$ is a subgroup of $(R, +)$ and for every $r \in R$ and every $x \in I$, $rx \in I$.

- We don't need to distinguish between left and right ideals as Boolean rings are commutative.

Theorem 15. Let I be an ideal of a Boolean ring. If I is finitely generated, then I is principle.

Proof. Let R be a Boolean ring. Suppose $I = \{r_1x_1 + r_2x_2 + \cdots r_nx_n : r_i \in R\}$ for some $x_1, x_2 \cdots x_n$. We will induct on n.

Suppose $n = 2$. Let

$$g = x_1 + x_2 - x_1x_2$$

Then we have $x_1g = x_1, x_2g = x_2$. Thus

$$r_1x_1 + r_2x_2 = (r_1x_1 + r_2x_2)g,$$

so $I \subseteq (g)$. Also, $rg = rxg = (rg)x \in I$, thus $(g) \subseteq I$. This means that $(g) = I$ so g is the generator of I.

Now suppose $n > 2$ and g is the generator of $x_1, x_2 \cdots x_{n-1}$. Then $g' = x_n + g - gx_n$ is the generator of $(x_1, x_2 \cdots x_n)$, following a similar argument. □

Theorem 16. If R is a Boolean ring and I is an ideal of R, then I is prime iff it is maximal.

Proof. We know that maximal ideals are prime ideals in general, so we only need to show the other direction. Suppose I is not a maximal ideal. This means that there is a larger ideal X that contains it that doesn't contain any larger ideal that contains I. We can build descending chains of ideals, at least one of which must contain I and X. If there are two ideals that descend onto I, then we can take an element from X and an element from a different ideal that directly descends onto I and multiply them to get an element from I, making it not prime. We can do this because in a Boolean ring, the product of an element in one ideal and an element in another ideal must result in an element of both ideals. If there is only one chain of ideals descending onto I, then we can take a maximal element of I. If there is no pair of elements not in I that multiply to this element, that means that it has to be expressed as a multiple of itself and an element outside of the set, making it a prime element. This means that the only composition that results in this element is the 1 of R and itself. However, since I is contained within X, there must be a "larger" element of X. This means that this cannot be the case, and so there must be two elements that multiply to the maximal element. By contrapositivity, every prime ideal must be maximal. □

Theorem 17. Every Boolean algebra embeds into a power set algebra.

Proof. Let $(B, \vee, \wedge, 0, 1)$ be a Boolean algebra, and $(B, +, \cdot, 0, 1)$ its associated Boolean ring.
Let $\mathcal{I}$ be the set of maximal ideals in B, which by the previous theorem is the set of prime ideals in B.
Define $f : B \to \mathcal{P}(\mathcal{I})$ by $f(a) = \{I \in \mathcal{I} : a \notin I\}$. For any $a \neq b$, WLOG assume $a \not\geq b$. Then $a \vee \neg b \neq 1$, so there is an ideal generated by $a \vee \neg b$. This ideal must be contained in some maximal ideal I. We have $\neg b \leq a \vee \neg b \in I$, so $b \notin I$. Similarly, $a \leq a \vee \neg b \in I$. Therefore $I \in f(b)$ but $I \notin f(a)$, so f is injective.

Let $a, b \in B$.

- Suppose $I \notin f(a \vee b)$. Then $a \vee b \in I$. I is an ideal, so $a = a \wedge (a \vee b) \in I$ and $b = b \wedge (a \vee b) \in I$. Therefore $I \notin f(a)$ and $I \notin f(b)$, so $I \notin f(a) \cup f(b)$.
- Suppose $I \notin f(a) \cup f(b)$. Then $I \notin f(a)$ and $I \notin f(b)$, so $a, b \in I$. Therefore $a \vee b \in I$, so $I \notin f(a \vee b)$.

Therefore $I \in f(a \vee b)$ iff $I \in f(a) \cup f(b)$, so $f(a \vee b) = f(a) \cup f(b)$.

- Suppose $I \notin f(a \wedge b)$. Then $a \wedge b \in I$. I is a maximal ideal, so it is prime. Therefore $a \in I$ or $b \in I$, so $I \notin f(a)$ or $I \notin f(b)$. Therefore $I \notin f(a) \cap f(b)$.
- Suppose $I \notin f(a) \cap f(b)$. Then $I \notin f(a)$ or $I \notin f(b)$, so $a \in I$ or $b \in I$. In either case, $a \wedge b \in I$, so $I \notin f(a \wedge b)$.

Therefore $I \in f(a \wedge b)$ iff $f \in f(a) \cap f(b)$, so $f(a \wedge b) = f(a) \cap f(b)$.

- Suppose $I \in f(a)$. Then $a \notin I$. I is prime and $a \wedge \neg a = 0 \in I$, so $\neg a \in I$. Therefore $I \notin f(\neg a)$.
- Suppose $I \notin f(a)$. Then $a \in I$. Therefore $\neg a \notin I$, so $I \in f(\neg a)$.

Therefore $I \notin f(a)$ iff $I \in f(\neg a)$, so $f(\neg a) = \mathcal{I} \setminus f(a)$.

Therefore f is an embedding of B into $\mathcal{P}(\mathcal{I})$. □

Problem 18. What properties must Boolean algebras as posets have?

Problem 19. Under the correspondence between Boolean algebras and Boolean rings, what do various notions map to?

- An ideal of a Boolean ring would be like the ring, but missing some of the elements.
- This means that a prime ideal would be missing only one of the elements, as if it missed more, there could be two outside elements that multiplied to an object in it.
- This creates a subring, which must still be a Boolean ring. This corresponds to a subset that is closed under addition and multiplication, and must also be a Boolean algebra.
- If it is finite, then the algebra must be a power set of the maximal element.
- A prime ideal is only missing one element from the whole set.
- A subring corresponds to a subset of elements that is closed under union and intersection. This is not a Boolean algebra of the entire set, as it is not closed under complement. However, in relation to itself, if it is finite, it is isomorphic to a power set.
- An automorphism corresponds to several elements being changed into each other. This is clear, as an automorphism must be operation-preserving, and same-sized elements of a set are interchangeable with each other. This means the group of automorphisms is the permutation group of the amount of atoms in an atomic group.

Problem 20. Consider a ring in which every element satisfies $x^3 = x$. What can we say about these rings?

Observation 21. In any such ring, $8 = 2^3 = 2$, so the characteristic must be 6 (or 2 or 3).

Observation 22. For any x in a ring where $x^3 = x$, we have $0 = (x+1)^3 - (x+1) = x^3 + 3x^2 + 2x = 3x^2 + 3x$, so $3x^2 = 3x$. This gives us a Boolean ring, and this is closed under addition.

Observation 23. For any x in a ring where $x^3 = x$, we have $x^4 = x^2$, so all x^2 form a system closed under multiplication, but not necessarily under addition. If the ring is a composition of Z_2s and Z_3s, then we know that we must get a series of 0s and 1s as a result, since in a Z_3, $2^3 = 2$ and $2^2 = 1$. We can convert a 0 into an element of a set not existing, and a 1 into an element of a set existing. We can then have the union of two subsets a and b be referred to as $a+b-ab$, as $a+b+ab$ with $a, b = 1$ would result in $a + b + ab = 0$, when it should convert to 1. $a + b - ab$ handles this. Additionally, the complement of x should be redefined as $1 - x$. To see that these work, we only need to observe that every operation of 0s and 1s results the same as it would in a Boolean ring.

Question 24. Is every ring where $x^3 = x$ for all x a composition of Z_2s and Z_3s?

Observation 25. In a ring such that $x^3 = x$, the mod 2 parts must form a Boolean algebra. This is because the ring has characteristic 2, 3, or 6. In a characteristic 3 ring, it is clear that there is no element of characteristic 2. In a characteristic 6 ring, we know that $3x^2 = 3x$ for all x, and when writing this ring as a composition of a Z_3 extension and the same Z_2 extension, we get a Z_2 where $x^2 = x$ for all x, making a Boolean algebra. As a composition of Z_2s, we just convert it to a subset of a Z_6 by multiplying each component by 3. We can then apply the reasoning with the Z_6 ring where this is the case.

Question 26. What properties are there of the dual of a Boolean algebra?

Observation 27. We claim a Boolean algebra is isomorphic to its dual. This is because we can use the isomorphism $f(x) = \neg x$. Let $\wedge_d$, $\vee_d$, and $\neg_d$ represent operations in the dual Boolean algebra.

We have

$$f(x) \wedge_d f(y) = f(x) \vee f(y) = \neg x \vee \neg y = \neg(x \wedge y) = f(x \wedge y)$$

$$f(x) \vee_d f(y) = f(x) \wedge f(y) = (\neg x) \wedge (\neg y) = \neg(x \vee y) = f(x \vee y)$$

$$\neg_d f(x) = \neg f(x) = \neg\neg x = f(\neg x)$$

This shows that a Boolean algebra is isomorphic to its dual.

Observation 28. This means that if Boolean ring R corresponds to Boolean algebra A, and Boolean ring R' corresponds to Boolean algebra A', the dual of A, then R and R' are isomorphic. Moreover, the isomorphism is $f(x) = 1 - x$.

Definition 29. We define a Boolean algebra to be complete if every subset bounded below has a supremum.

Theorem 30. Every Boolean algebra has a unique completion up to isomorphism.

Proof. (sketch) Let B be a boolean algebra.

We will use a similar approach to complete the rationals with Dedekind cuts. We define a cut to be a closed downwards subset $C \subseteq B$.

We define

$$C_p = \{q : q \leq p\}$$

We say that a cut C is a *Legal Cut* if for every $p \notin C$, there exists $q \leq p$ such that $C \cap C_q = \{0\}$. It is easy to see that all C_p are legal cuts ($C_p \cap C_{\neg p} = \{0\}$).

Let B' be the set of all legal cuts on B. We order B' by inclusion. For all $C \in B'$, define $\overline{C} = \{p \in B : \forall q \leq p (C_q \cap C \neq \varnothing)\}$. Then $\overline{C}$ is the minimal legal cut that contains C. For $C_1, C_2 \in B'$, define $C_1 \vee C_2 = \overline{C_1 \cup C_2}$ and $C_1 \wedge C_2 = C_1 \cap C_2$. Finally we define $\neg C = \{p \in B : C_p \cap C = \varnothing\}$. This gives a complete boolean algebra.

Finally, B embeds into B' through the natural homomorphism $p \to C_p$. Thus, every boolean algebra has a completion.

Now we show that the completions are unique up to isomorphism. Let D_1, D_2 be two completions of B. Let $\pi_1 : B \to D_1$ and $\pi_2 : B \to D_2$ be the embbedment from B to D_1, D_2, respectively. Define $\phi : D_1 \to D_2$ by $\phi(c) = \bigvee_{D_2}\{\pi_2(q) : q \in B \text{ and } \pi_1(q) \leq c\}$. It is easy to check that ϕ an isomorphism. ∎