2b $\vee$ ¬2b: That is the Multiplicative Identity

Proof School Math Burst 2022

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Identifying an isomorphism between a Boolean algebra and a substructure of a power set algebra

Definition (Boolean algebra)

A **Boolean algebra** is a set B with two binary operations $\wedge$ ("and") and $\vee$ ("or"), a unary operation $\neg$ ("not"), and two special elements 0 and 1 such that for all a, b, c $\in$ B, the following properties hold:

- Commutativity: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
- Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$
- Distributivity: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- Identities: $0 \vee a = a$ and $1 \wedge a = a$
- Complements: $a \vee \neg a = 1$ and $a \wedge \neg a = 0$

Examples

Consider a nonempty set X, and let P(X) be the power set of X, i.e. the set of all the subsets of X. Define

- $\wedge$ to be intersections ($\bigcap$)
- $\vee$ to be unions ($\bigcup$)
- $\neg$ to be complement (c)
- 0 to be the empty set ($\varnothing$)
- 1 to be the full set X

Then (P(X), $\bigcap$, $\bigcup$, c, $\varnothing$, X) is a Boolean algebra.

Another example is the infinite binary strings (sequences of 0s and 1s).

Initial Results

For all a, b $\in$ B:

- $a \wedge a = a$ and $a \vee a = a$
- $1 \vee a = 1$ and $0 \wedge a = 0$
- $a \vee b = 1$ and $a \wedge b = 0 \Rightarrow b = \neg a$
- $\neg(\neg a) = a$
- $\neg(a \vee b) = \neg a \wedge \neg b$ and $\neg(a \wedge b) = \neg a \vee \neg b$

Not every Boolean algebra isomorphic to a power set algebra

- Consider the set of infinite sequences of 0s and 1s such that there are only finitely many 0s or finitely many 1s
- This is countably infinite

Definition (atom & atomic)

Given a Boolean algebra B, we can define a **partial ordering** on B by defining $a \leqslant b$ if and only if $a \vee b = b$. (Equivalently, $a \leqslant b$ if $a \wedge b = a$).

- An **atom** is a minimal non-zero element
- A Boolean algebra is **atomic** if no two distinct elements are greater than or equal to the same set of atoms
  - This means that every element can be built from a unique set of atoms using $\vee$

Proof of the finite case

- Not all Boolean algebras are atomic & not all Boolean algebras contain atoms
- All finite Boolean algebras contain at least one atom & are atomic

Every finite Boolean algebra is isomorphic the power set of the set of atoms

- This is true because each element is $\geqslant$ to a unique set of atoms
- Any set A of atoms is finite, so $\bigvee A$ is an element which is $\geqslant$ to the atoms in A

Definition (Boolean ring)

A **ring** consists of a set with the binary operations +, -, and $\cdot$.

- A **Boolean ring** is a ring where $x \cdot x = x$ for all ring elements x

There is an equivalence between Boolean algebras and Boolean rings

- $a + b = (a \wedge \neg b) \vee (b \wedge \neg a) = (a \vee b) \wedge \neg(a \wedge b)$
- $a \cdot b = a \wedge b$
- $a \vee b = a + b + (a \cdot b)$
- $a \wedge b = a \cdot b$
- $\neg a = 1 + a$

Results about Boolean rings

- Every Boolean ring is commutative and has characteristic 2
- Every Boolean ring has zero divisors
- An ideal is prime if and only if it is maximal
- If I is a finitely generated ideal of a Boolean ring, I is principal
- In any infinite Boolean ring, there exist ideals which are not finitely generated

Correspondence between algebras & rings

- An ideal I in a Boolean ring B corresponds to a nonempty proper subset of B which is downwards closed
- A subring corresponds to a subset that is closed under union, intersection, and complement, and has the same 0 and 1

Proving the isomorphism

Every Boolean algebra embeds into a power set algebra—it is isomorphic to a substructure of a power set algebra

- Given any Boolean algebra B, B embeds into the power set of the set of its maximal ideals
- An element $b \in B$ is sent to the set of maximal ideals that do not contain it
- Every ideal contains 0 but not 1, so 0 is sent to the empty set and 1 is sent to the complete set of all maximal ideals

$b \mapsto$ {maximal ideals not containing b}

- For every distinct pair of elements a, b $\in$ B, one of them is not greater than the other one. Assume $a \ngeq b$. Then $a \vee \neg b$ is contained in some maximal ideal I. I contains a but not b, so it distinguishes between a and b.
- This means that every element in B is sent to a different set of ideals, so this is an embedding

Further areas of research

All Boolean algebras are substructures of power set algebras. But what is missing?

- Do these missing elements look like gaps, holes, or something else?
- Are maximal ideals more special than just "half of the algebra"?
- The complement of a maximal ideal is an ultrafilter. How do those behave in an arbitrary Boolean ring, especially non-principal ones?