Kai Wong (704 451 679) Mathematics 151b Homework 4

1a. Set $y(t_{i+1}) = y(t_i) + h[af(t_i, y(t_i)) + bf(t_{i-1}, y(t_{i-1})) + cf(t_{i-2}, y(t_{i-2})) + df(t_{i-3}, y(t_{i-3}))]$

$$= y(t_i) + ahy'(t_i) + bhy'(t_{i-1}) + chy'(t_{i-2}) + dhy'(t_{i-3}) \quad \text{—— (1)}$$

Taylor expand both sides:

$$y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t_i) + \frac{h^4}{24}y''''(t_i) + O(h^5)$$

$$= y(t_i) + ahy'(t_i) + bh\left(y'(t_i) - hy''(t_i) + \frac{h^2}{2}y'''(t_i) - \frac{h^3}{6}y''''(t_i) + O(h^4)\right)$$

$$+ ch\left(y'(t_i) - 2hy''(t_i) + 2h^2y'''(t_i) - \frac{4}{3}h^3y''''(t_i) + O(h^4)\right)$$

$$+ dh\left(y'(t_i) - 3hy''(t_i) + \frac{9}{2}h^2y'''(t_i) - \frac{9}{2}h^3y''''(t_i) + O(h^4)\right)$$

$$= y(t_i) + (a+b+c+d)hy'(t_i) + (-b-2c-3d)h^2y''(t_i) + \left(\frac{1}{2}b + 2c + \frac{9}{2}d\right)h^3y'''(t_i) + \left(-\frac{1}{6}b - \frac{4}{3}c - \frac{9}{2}d\right)h^4y''''(t_i) + O(h^5)$$

$$\Rightarrow 1 = a+b+c+d$$

$$\frac{1}{2} = -b-2c-3d$$

$$\frac{1}{6} = \frac{1}{2}b + 2c + \frac{9}{2}d$$

$$\frac{1}{24} = \frac{-1}{6}b - \frac{4}{3}c - \frac{9}{2}d$$

Solving, we get $a = \frac{55}{24}$, $b = -\frac{59}{24}$, $c = \frac{37}{24}$, $d = \frac{-3}{8}$

Substitute back to (1):

$$y(t_{i+1}) = y(t_i) + h\left[\frac{55}{24}y'(t_i) - \frac{59}{24}y'(t_{i-1}) + \frac{37}{24}y'(t_{i-2}) - \frac{3}{8}y'(t_{i-3})\right] + O(h^5)$$

$$\Rightarrow w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

The order of the local truncation error is $\tau(h) = O(h^4)$ ※

2a. Set $y(t_{i+1}) = y(t_i) + af(t_{i+1}, y_{i+1}) + bf(t_i, y_i)$

$$= y(t_i) + ay'(t_{i+1}) + by'(t_i) \quad \text{—— (1)}$$

Taylor expand both sides:

$$y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3)$$

$$= y(t_i) + a\left(y'(t_i) + hy''(t_i) + O(h^2)\right) + by'(t_i)$$

$$= y(t_i) + (a+b)y'(t_i) + ahy''(t_i) + aO(h^2)$$

$$\Rightarrow a+b = h, \quad a = \frac{h}{2} \Rightarrow b = \frac{h}{2}$$

Hence, we have $w_{i+1} = w_i + \frac{h}{2}[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)]$ ※

2b. Using Midpoint method, we have: $w_{(i+1)p} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i))$

Substitute $w_{(i+1)p}$ back into implicit method derived in (a):

$$w_{i+1} = w_i + \frac{h}{2}[f(t_{i+1}, w_{(i+1)p}) + f(t_i, w_i)] \text{ ※}$$

2c. Let $u_1(t) = y(t)$ and $u_2(t) = y'(t)$ $\Rightarrow$ ① $u_1'(t) = u_2(t)$ ② $u_2'(t) = 6e^{-t} + 4u_1(t)$ with $u_1(0) = 0$, $u_2(0) = 0$ $\Rightarrow w_{1,0} = 0$, $w_{2,0} = 0$

$K_{1,1} = hf_1(t_0, w_{1,0}, w_{2,0}) = hw_{2,0} = 0$, $K_{1,2} = hf_2(t_0, w_{1,0}, w_{2,0}) = h(6e^{-t_0} + 4w_{1,0}) = 6h$, $K_{2,1} = hf_1(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}K_{1,1}, w_{2,0} + \frac{1}{2}K_{1,2}) = h(3h) = 3h^2$

So, $y(0.1)_p \approx w_{1,1p} = w_{1,0} + K_{2,1} = 3(0.1)^2$. $K_{2,2} = h(6e^{-0.05} + 4(w_{1,0} + \frac{1}{2}K_{1,1})) = h6e^{-0.05}$ $\Rightarrow w_{2,1p} = w_{2,0} + K_{2,2} = K_{2,2}$

$\Rightarrow y(0.1) \approx w_{1,1} = w_{1,0} + \frac{1}{2}[hf_1(t_0 + h, w_{1,1p}, w_{2,1p}) + K_{1,1}] = 3h^2e^{-0.05}$ ..... $y(1.0) \approx w_{1,10} = w_{1,9} + \frac{1}{2}[hf_1(t_9 + h, w_{1,9p}, w_{2,9p}) + K_{9,1}] \approx 3.161777$ ※

2d. Exact value: $y(1) = 3.161772$

Order of convergence $p = \log_2 \left[\dfrac{w_{1,10}^{h=0.1} - y(1)}{w_{1,10}^{h=0.05} - y(1)}\right] \approx 2$

This PECE method is not better than the explicit midpoint method in terms of order of accuracy, but because it is an implicit method it is more stable than the explicit midpoint method, meaning we can perform the iterations using larger values of 'h' without losing as much accuracy. (requires more computations however).

3. Rewriting Adams-Moulton 3-step implicit method:

$$w_{i+1} = w_i + \frac{h}{24}\left[9y'(t_{i+1}) + 19y'(t_i) - 5y'(t_{i-1}) + y'(t_{i-2})\right]$$

$$= w_i + \frac{h}{24}\left[9w_{i+1}g(t_{i+1}) + 19w_i g(t_i) - 5w_{i-1}g(t_{i-1}) + w_{i-2}g(t_{i-2})\right]$$

$$\Rightarrow w_{i+1}\left(1 - \frac{9h}{24}g(t_{i+1})\right) = \frac{h}{24}\left[19w_i g(t_i) - 5w_{i-1}g(t_{i-1}) + w_{i-2}g(t_{i-2})\right]$$

$$w_{i+1}\left(\frac{24 - 9hg(t_{i+1})}{24}\right) = \frac{h}{24}\left[19w_i g(t_i) - 5w_{i-1}g(t_{i-1}) + w_{i-2}g(t_{i-2})\right]$$

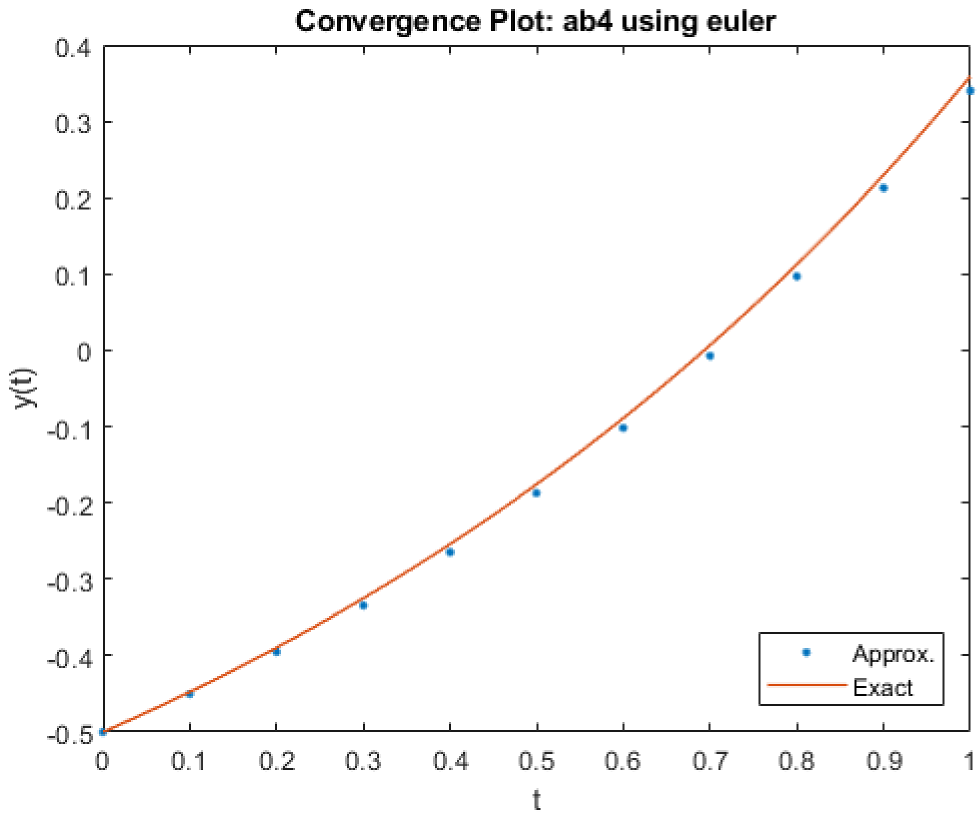
$$\Rightarrow w_{i+1} = \frac{h}{24 - 9hg(t_{i+1})}\left[19w_i g(t_i) - 5w_{i-1}g(t_{i-1}) + w_{i-2}g(t_{i-2})\right] \quad \#$$

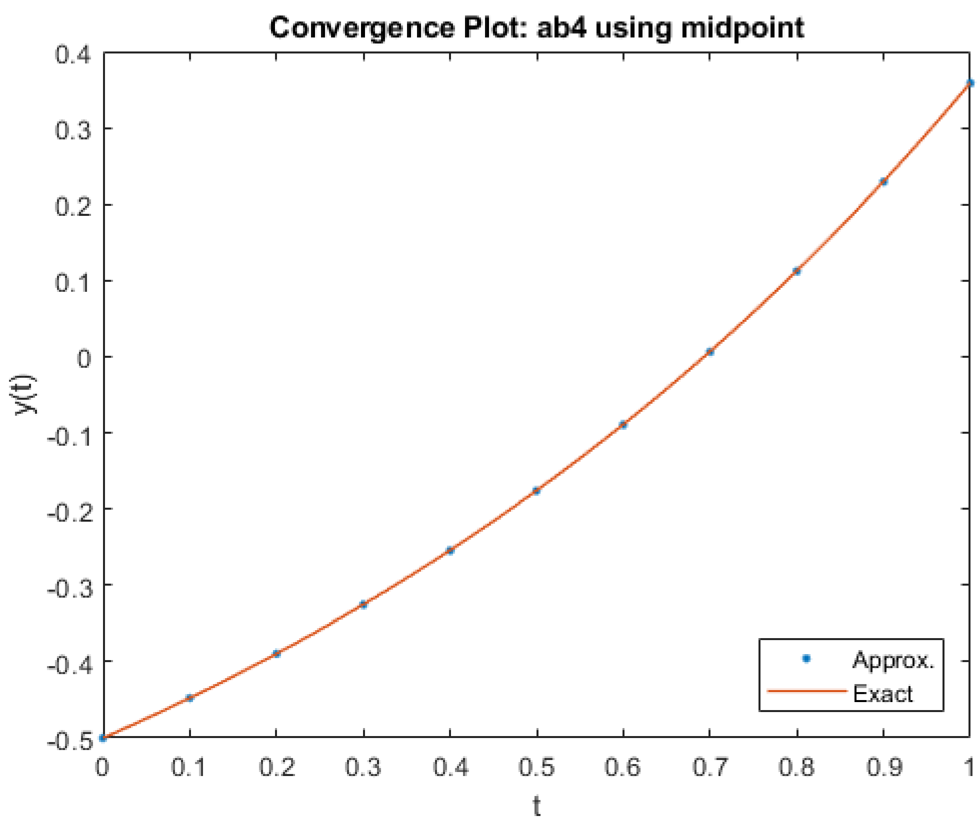
What could go wrong?

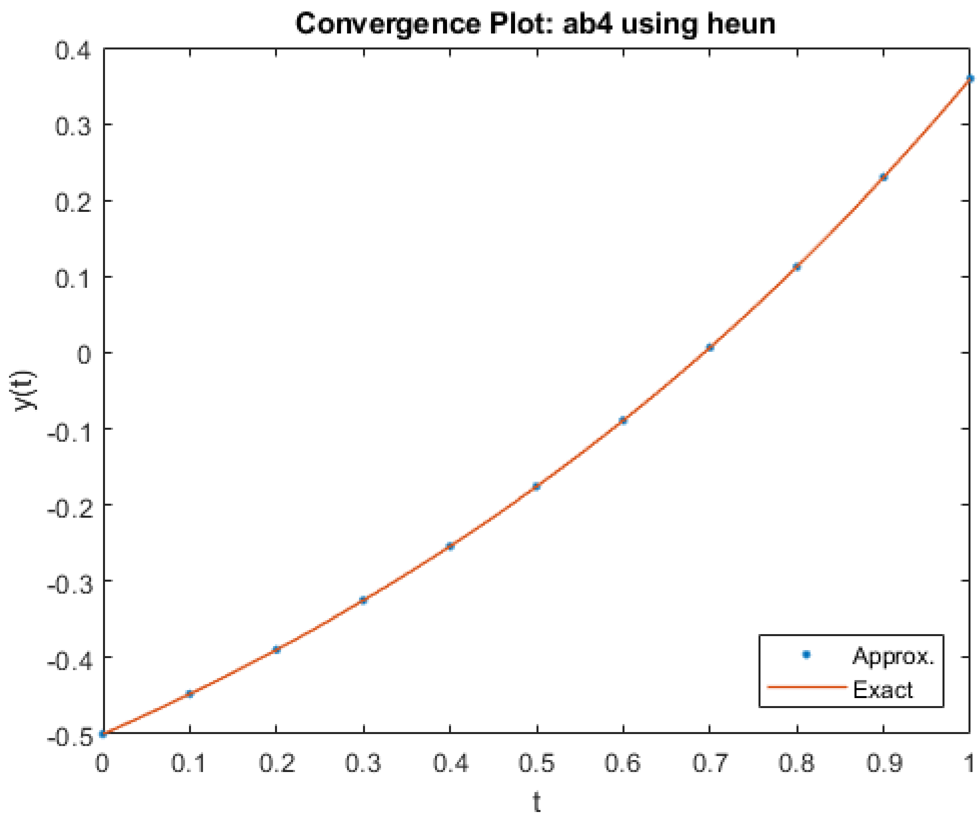
This method may give an undefined approximation at a given t_i when $24 - 9hg(t_{i+1}) = 0$

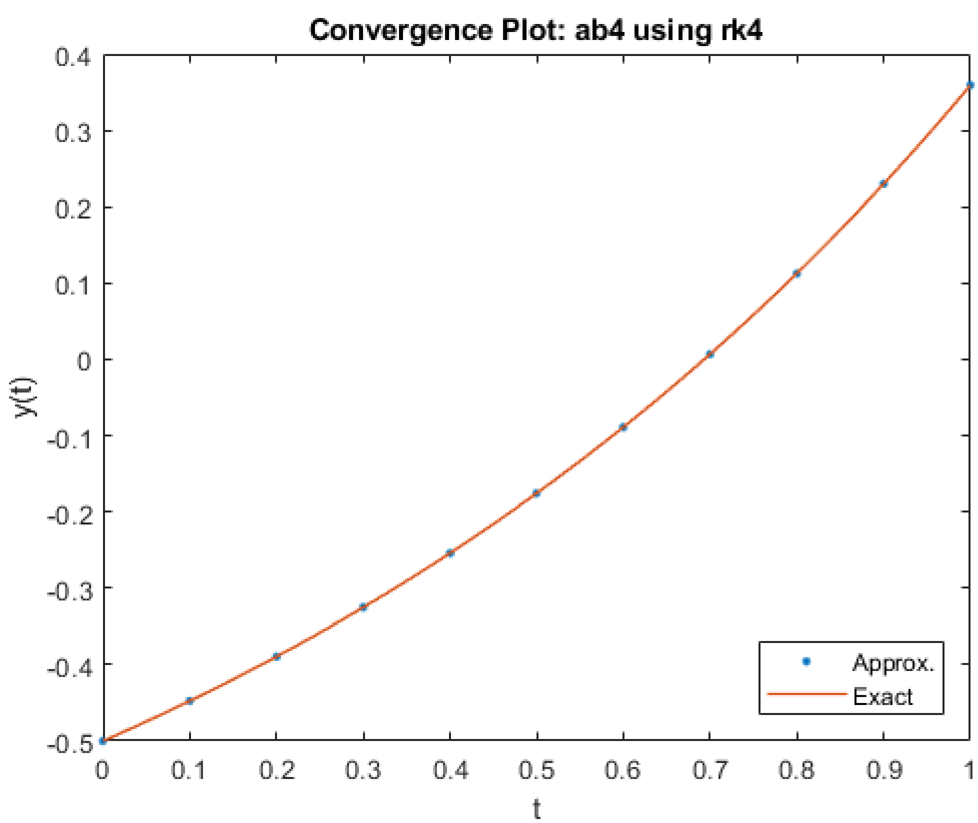
$\Rightarrow$ This happens when $g(t_i) = \frac{24}{9h}$ which is possible for some smooth function g. #

Question 1:


Convergence Plot: ab4 using euler
y(t)
t
Approx.
Exact
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Convergence Plot: ab4 using midpoint
y(t)
t
Approx.
Exact
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1





We see that the performance of ab4 improves as we use higher order methods to find the starting values. While it is less noticeable when using methods of second order and above, the improvement in ab4's approximation when going from a first order (euler) to second order (midpoint) method is apparent.

Question 2:

```
f = @(t,y) 1+y; % RHS of the ODE
h = .1; % step size
t = 0:h:5; % vector of time points
alpha = -1/2; % initial condition

exact = @(s) exp(s)/2-1; % exact solution
T = 0:.01:5; % fine grid for plotting exact solution

for i = 2:length(t)-1
    % predict using midpoint method
    wp = w(i) + h*f(t(i) + h/2, w(i) + h*f(t(i), w(i))/2);

    % correct using 1-step implicit method (trapezoidal rule)
    w(i+1) = w(i)+h/2*(f(t(i+1),wp)+f(t(i),w(i)));
end

figure(1)
plot(T,exact(T),t,w,'.')
```