

Kai Wong (704 451 679) Mathematics 151b Homework 4

1a. Set  $y(t_{i+1}) = y(t_i) + h[a f(t_{i+1}, y(t_{i+1})) + b f(t_{i+1}, y(t_{i+1})) + c f(t_{i+1}, y(t_{i+1})) + d f(t_{i+1}, y(t_{i+1}))]$   
 $= y(t_i) + ah y'(t_i) + bh y'(t_{i+1}) + ch y'(t_{i+1}) + dh y'(t_{i+1}) \quad (1)$

Taylor expand both sides:

$$\begin{aligned} y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{6} y'''(t_i) + \frac{h^4}{24} y^{(4)}(t_i) + o(h^5) \\ = y(t_i) + ah y'(t_i) + bh (y'(t_i) - h y''(t_i) + \frac{h^2}{2} y'''(t_i) - \frac{h^3}{6} y^{(4)}(t_i) + o(h^4)) \\ + ch (y'(t_i) - 2h y''(t_i) + 2h^2 y'''(t_i) - \frac{4}{3} h^3 y^{(4)}(t_i) + o(h^4)) \\ + dh (y'(t_i) - 3h y''(t_i) + \frac{3}{2} h^2 y'''(t_i) - \frac{1}{2} h^3 y^{(4)}(t_i) + o(h^4)) \\ = y(t_i) + (a+b+c+d) h y'(t_i) + (-b-2c-3d) h^2 y''(t_i) + (\frac{1}{2}b+2c+\frac{3}{2}d) h^3 y'''(t_i) + (-\frac{1}{6}b-\frac{4}{3}c-\frac{9}{2}d) h^4 y^{(4)}(t_i) + o(h^5) \\ \Rightarrow 1 = a+b+c+d \\ \frac{1}{2} = -b-2c-3d \\ \frac{1}{6} = \frac{1}{2}b+2c+\frac{3}{2}d \\ \frac{1}{24} = \frac{1}{6}b-\frac{4}{3}c-\frac{9}{2}d \end{aligned}$$

solving, we get  $a = \frac{55}{24}, b = -\frac{59}{24}, c = \frac{37}{24}, d = -\frac{3}{8}$

Substitute back to (1):

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + h \left[ \frac{55}{24} y'(t_i) - \frac{59}{24} y'(t_{i+1}) + \frac{37}{24} y'(t_{i+1}) - \frac{3}{8} y'(t_{i+1}) \right] + o(h^5) \\ \Rightarrow W_{i+1} &= W_i + \frac{h}{24} [55 f(t_i, W_i) - 59 f(t_{i+1}, W_{i+1}) + 37 f(t_{i+1}, W_{i+1}) - 9 f(t_{i+1}, W_{i+1})] \end{aligned}$$

The order of the local truncation error is  $\tau(h) = O(h^4)$  ✗

2a. Set  $y(t_{i+1}) = y(t_i) + a f(t_{i+1}, y_{i+1}) + b f(t_i, y_i)$   
 $= y(t_i) + a y'(t_{i+1}) + b y'(t_i) \quad (1)$

Taylor expand both sides:

$$\begin{aligned} y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) + o(h^3) \\ = y(t_i) + a (y'(t_i) + h y''(t_i) + o(h^2)) + b y'(t_i) \\ = y(t_i) + (a+b) y'(t_i) + ah y''(t_i) + o(h^2) \\ \Rightarrow a+b=h, a=\frac{h}{2} \Rightarrow b=\frac{h}{2} \end{aligned}$$

Hence, we have  $W_{i+1} = W_i + \frac{h}{2} [f(t_{i+1}, W_{i+1}) + f(t_i, W_i)]$  ✗

2b. Using Midpoint method, we have:  $W_{i+1,p} = W_i + h f(t_i + \frac{h}{2}, W_i + \frac{h}{2} f(t_i, W_i))$

Substitute  $W_{i+1,p}$  back into implicit method derived in (a):

$$W_{i+1} = W_i + \frac{h}{2} [f(t_{i+1}, W_{i+1,p}) + f(t_i, W_i)] \quad \text{✗}$$

2c. Let  $u_1(t) = y(t)$  and  $u_2(t) = y'(t) \Rightarrow \begin{cases} \textcircled{1} u_1'(t) = u_2(t) \\ \textcircled{2} u_2'(t) = 6e^{-t} + 4u_1(t) \end{cases}$  with  $u_1(0) = 0, u_2(0) = 0 \Rightarrow W_{1,0} = 0, W_{2,0} = 0$

$$K_{1,1} = hf_1(t_0, W_{1,0}, W_{2,0}) = hW_{2,0} = 0, K_{1,2} = hf_2(t_0, W_{1,0}, W_{2,0}) = h(6e^{-t_0} + 4W_{1,0}) = 6h, K_{2,1} = hf_1(t_0 + \frac{h}{2}, W_{1,0} + \frac{1}{2}K_{1,1}, W_{2,0} + \frac{1}{2}K_{2,1}) = h(3h) = 3h^2$$

$$\text{So, } y(0.1) \approx W_{1,p} = W_{1,0} + K_{1,1} = 3(0.1)^2, K_{2,p} = h(6e^{-0.05} + 4(W_{1,0} + \frac{1}{2}K_{1,1})) = h6e^{-0.05} \Rightarrow W_{2,p} = W_{2,0} + K_{2,1} = K_{2,1}$$

$$\Rightarrow y(0.1) \approx W_{1,1} = W_{1,0} + \frac{1}{2}[hf_1(t_0+h, W_{1,p}, W_{2,p}) + K_{1,1}] = 3h^2 e^{-0.05} \dots y(1.0) \approx W_{1,10} = W_{1,9} + \frac{1}{2}[hf_1(t_9+h, W_{1,p}, W_{2,p}) + K_{1,1}] \approx 3.161777 \quad \text{✗}$$

2d. Exact value:  $y(1) = 3.161772$

$$\text{Order of convergence } p = \log_2 \left[ \frac{W_{110}^{h=0.1} - y(1)}{W_{110}^{h=0.05} - y(1)} \right] \approx 2$$

This PECE method is not better than the explicit Midpoint method in terms of order of accuracy, but because it is an implicit method it is more stable than the explicit midpoint method, meaning we can perform the iteration using larger values of 'h' without losing as much accuracy. (requires more computation however).

3. Rewriting Adams-Moulton 3-step implicit method:

$$\begin{aligned} W_{i+1} &= W_i + \frac{h}{24} [9y'(t_{i+1}) + 19y'(t_i) - 5y'(t_{i-1}) + y'(t_{i-2})] \\ &= W_i + \frac{h}{24} [9W_{i+1}g(t_{i+1}) + 19W_i g(t_i) - 5W_{i-1}g(t_{i-1}) + W_{i-2}g(t_{i-2})] \end{aligned}$$

$$\Rightarrow W_{i+1} \left( 1 - \frac{9h}{24} g(t_{i+1}) \right) = \frac{h}{24} [19W_i g(t_i) - 5W_{i-1} g(t_{i-1}) + W_{i-2} g(t_{i-2})]$$

$$W_{i+1} \left( \frac{24 - 9hg(t_{i+1})}{24} \right) = \frac{h}{24} [19W_i g(t_i) - 5W_{i-1} g(t_{i-1}) + W_{i-2} g(t_{i-2})]$$

$$\Rightarrow W_{i+1} = \frac{h}{24 - 9hg(t_{i+1})} [19W_i g(t_i) - 5W_{i-1} g(t_{i-1}) + W_{i-2} g(t_{i-2})] \quad \times$$

What could go wrong?

This method may give an undefined approximation at a given  $t_i$  when  $24 - 9hg(t_{i+1}) = 0$

$\Rightarrow$  This happens when  $g(t_i) = \frac{24}{9h}$  which is possible for some smooth function  $g$ .  $\times$