# Computer Graphics - CS402

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## Homework # 3

1. Prove that a $y$-reflection (reflection about the $x$-axis) followed by a reflection through the line $y = -x$ is a pure rotation.

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The reflection about $x$-axis leaves $x$ coordinates unchanged and negates the $y$ coordinates which results in the following matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The reflection about $y = -x$ exchanges $x$ coordinate with $-y$ and $y$ coordinate with $-x$ which results in the following matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

It follows that the combination of the two transformations has a matrix given by

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is clearly an **orthogonal** matrix whose determinant is 1. Hence, it is a pure rotation (it is the counter-clockwise rotation around the origin with angle $\frac{3\pi}{2}$).

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2. Find the vertices of the rotated triangle obtained by performing a $45^0$ rotation of triangle $A(0,0), B(1,1), C(5,2)$.

   (a) About the origin.

   (b) About $P(-1,-1)$.

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We represent the triangle by a matrix formed from the homogeneous coordinates of the vertices

$$\begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) The matrix of rotation is

$$R_{45^0} = \begin{pmatrix} \cos 45^0 & -\sin 45^0 & 0 \\ \sin 45^0 & \cos 45^0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

So the coordinates of the vertices $A'$, $B'$, $C'$ of the rotated triangle $ABC$ are

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{7\sqrt{2}}{2} \\ 1 & 1 & 1 \end{pmatrix}$$

Thus $A' = (0,0)$, $B' = (0, \sqrt{2})$, and $C' = (\frac{3}{2}\sqrt{2}, \frac{7}{2}\sqrt{2})$.

(b) The $45^0$ rotation matrix about $P(-1,-1)$ is a combination of a translation by vector $(1,1)$, followed by a $45^0$ rotation about the origin, followed by a translation by vector $(-1,-1)$:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2}-1 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that the coordinates of the vertices $A'$, $B'$, $C'$ of the rotated triangle $ABC$ are

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2}-1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & \frac{3}{2}\sqrt{2}-1 \\ \sqrt{2}-1 & 2\sqrt{2}-1 & \frac{9}{2}\sqrt{2}-1 \\ 1 & 1 & 1 \end{pmatrix}$$

Thus $A' = (-1, \sqrt{2}-1)$, $B' = (-1, 2\sqrt{2}-1)$, and $C' = (\frac{3}{2}\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1)$.

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3. Write the transformation matrix that magnifies the triangle with vertices $A(0,0)$, $B(1,1)$, and $C(5,2)$ to twice its size while keeping $C(5,2)$ fixed.

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The scaling matrix with respect to $C(5,2)$ is a combination of a translation by vector $(-5,-2)$, followed by a scaling about the origin, followed by a translation by vector $(5,2)$:

$$\begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that the coordinates of the vertices $A'$, $B'$, $C'$ of the magnified triangle $ABC$ are

$$\begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -3 & 5 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Thus $A' = (-5,-2)$, $B' = (-3,0)$, and $C' = C = (5,2)$.

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4. Let $L$ be the line that passes through the origin in the direction of $u = (1,1,1)$. Find the matrix for the rotation about $L$ with angle $45^0$.

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We can use directly the matrix derived in the slides as follows

$$R_{(u,\theta)} = \begin{pmatrix} u_x^2 + \cos\theta(1-u_x^2) & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y \sin\theta & 0 \\ u_x u_y(1-\cos\theta) + u_z \sin\theta & u_y^2 + \cos\theta(1-u_y^2) & u_y u_z(1-\cos\theta) - u_x \sin\theta & 0 \\ u_x u_z(1-\cos\theta) - u_y \sin\theta & u_y u_z(1-\cos\theta) + u_x \sin\theta & u_z^2 + \cos\theta(1-u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(u_x, u_y, u_z)$ is the unit vector giving the direction of the axis of rotation and $\theta$ is the rotation angle. Thus, in our case $(u_x, u_y, u_z) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $\cos\theta = \sin\theta = \frac{\sqrt{2}}{2}$. Hence

$$R_{(u,\theta)} = \begin{pmatrix} \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}-\sqrt{6}}{6} & \frac{2-\sqrt{2}+\sqrt{6}}{6} & 0 \\ \frac{2-\sqrt{2}+\sqrt{6}}{6} & \frac{1+\sqrt{2}}{3} & \frac{2-\sqrt{2}-\sqrt{6}}{6} & 0 \\ \frac{2-\sqrt{2}-\sqrt{6}}{6} & \frac{2-\sqrt{2}+\sqrt{6}}{6} & \frac{1+\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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5. Let $L$ be the line that passes through $(1,1,2)$ in the direction of $u=(1,1,1)$. Find the matrix for the rotation about $L$ with angle $60^0$.

First, we follow the same steps as in the previous exercise to derive the $60^0$ rotation matrix about the line $L'$ in direction $u=(1,1,1)$ through the origin. Taking $(u_x,u_y,u_z)=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ and $\cos\theta=\frac{1}{2}$, $\sin\theta=\frac{\sqrt{3}}{2}$

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix for the rotation about $L$ with angle $60^0$ is a combination of the matrix of the translation by vector $(-1,-1,-2)$, followed by the the $60^0$ rotation matrix about the line $L'$, followed by the matrix of the translation by vector $(1,1,2)$.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Find the matrix of the transformation that aligns the vector $\mathbf{u}=(1,1,1)$ with the vector $\mathbf{v}=(2,1,1)$.

We recall that a matrix $A_{\mathbf{v},\mathbf{k}}$ aligning a unit vector $\mathbf{v}=(v_x,v_y,v_z)$ with vector $\mathbf{k}=(0,0,1)$ is given by

$$A_{\mathbf{v},\mathbf{k}}=\begin{pmatrix} d & 0 & -v_x \\ 0 & 1 & 0 \\ v_x & 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{v_z}{d} & -\frac{v_y}{d} \\ 0 & \frac{v_y}{d} & \frac{v_z}{d} \end{pmatrix}$$

where $d=\sqrt{v_y^2+v_z^2}$. It follows that the matrix that does the reverse, i.e. that aligns $\mathbf{k}=(0,0,1)$ with $\mathbf{v}=(v_x,v_y,v_z)$ is given by

$$A_{\mathbf{k},\mathbf{v}}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{v_z}{d} & \frac{v_y}{d} \\ 0 & -\frac{v_y}{d} & \frac{v_z}{d} \end{pmatrix}\begin{pmatrix} d & 0 & v_x \\ 0 & 1 & 0 \\ -v_x & 0 & d \end{pmatrix}$$

in virtue of the algebraic relation $(AB)^{-1}=B^{-1}A^{-1}$. Hence, in order to align $\mathbf{u}$ with $\mathbf{v}$, we can proceed by reusing the known matrices that is, align $\mathbf{u}$ with $\mathbf{k}$, then align $\mathbf{k}$ with $\mathbf{v}$. Consequently, after normalizing the vectors $\mathbf{u}$ and $\mathbf{v}$, we obtain

$$\begin{aligned} A_{\mathbf{u},\mathbf{v}} &= A_{\mathbf{k},\mathbf{v}}A_{\mathbf{u},\mathbf{k}} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 1 & 0 \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} & \frac{2\sqrt{2}+3}{6} & \frac{2\sqrt{2}-3}{6} \\ -\frac{1}{3\sqrt{2}} & \frac{2\sqrt{2}-3}{6} & \frac{2\sqrt{2}+3}{6} \end{pmatrix} \end{aligned}$$