Computer Graphics - CS402

Madjid Allili ~ Johnson 125 ~ ext. 2740 ~ mallili@ubishops.ca

Homework # 2

1. Find the equation of the plane determined by the three points $P_0(1,5,-7)$, $P_1(2,6,1)$ and $P_2(0,1,2)$.

(a) First, we compute a normal vector to the plane using the cross product of two (independent) vectors of the plane, for instance: $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$.

$$\overrightarrow{\mathbf{n}} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & 8 \\ -1 & -4 & 9 \end{vmatrix} = 41\overrightarrow{i} - 17\overrightarrow{j} - 3\overrightarrow{k}$$

(b) Let $M(x,y,z)$ be a generic point of the plane. We must have $\overrightarrow{\mathbf{n}} \cdot \overrightarrow{P_0M} = 0$.

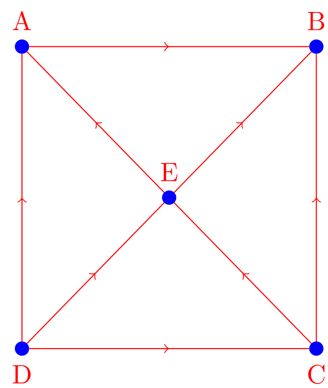
$$\begin{aligned} \overrightarrow{\mathbf{n}} \cdot \overrightarrow{P_0M} &= 41(x-1) + (-17)(y-5) + (-3)(z+7) \\ &= 41x - 17y - 3z + 23 \\ &= 0, \end{aligned}$$

which is the cartesian equation of the plane.

---

2. Use vectors to show that, if the two diagonals of a rectangle are perpendicular to each other, the rectangle is a square.

Consider the rectangle with vertices A, B, C, and D given in this order in the clockwise orientation. We assume that the diagonals of the rectangle are perpendicular. Hence, the vectors $\overrightarrow{AC}$ and $\overrightarrow{DB}$ are normal to each other.



We recall that two vectors are normal to each other if only if their dot product is zero. The dot product of tow vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as

$$\overrightarrow{u} \cdot \overrightarrow{v} = ||\overrightarrow{u}||||\overrightarrow{v}|| \cos(\overrightarrow{u}, \overrightarrow{v}).$$

Vectorially, we can write

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} \quad \text{and} \quad \overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB}.$$

It follows that

$$\begin{aligned}
\overrightarrow{AC} \cdot \overrightarrow{DB} &= (\overrightarrow{AD} + \overrightarrow{DC}) \cdot (\overrightarrow{DC} + \overrightarrow{CB}) \\
&= \overrightarrow{AD} \cdot \overrightarrow{DC} + \overrightarrow{AD} \cdot \overrightarrow{CB} + \overrightarrow{DC} \cdot \overrightarrow{DC} + \overrightarrow{DC} \cdot \overrightarrow{CB} \\
&= 0.
\end{aligned}$$

Since $A\,B\,C\,D$ is a rectangle, $\overrightarrow{AD}$ and $\overrightarrow{DC}$ are perpendicular, hence $\overrightarrow{AD} \cdot \overrightarrow{DC} = 0$. The same observation can be made about $\overrightarrow{DC}$ and $\overrightarrow{CB}$. Moreover, $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are parallel and have the same magnitude and therefore they represent the same vector, i.e. $\overrightarrow{AD} = \overrightarrow{BC}$. Taking into account all these observations, we have

$$\overrightarrow{AC} \cdot \overrightarrow{DB} = -||\overrightarrow{AD}||^2 + ||\overrightarrow{DC}||^2 = 0.$$

It follows that $||\overrightarrow{AD}|| = ||\overrightarrow{DC}||$ and therefore $A\,B\,C\,D$ is a square.

---

3. Find the equation of the line passing through $P_0(1, -5, 2)$ and $P_1(6, 7, -3)$.

The parametric equation of the line passing through P_0 and P_1 is given in vectorial form by

$$P_0 + t\overrightarrow{P_0P_1} = (1, -5, 2) + t(5, 12, -5),\ t \in \mathbb{R}.$$

This translates into three parametric equations in coordinates form:

$$\begin{cases}
x &= 1 + 5t, \\
y &= -5 + 12t, \\
z &= 2 - 5t,\ t \in \mathbb{R}.
\end{cases}$$

---

4. Let a plane be determined by the normal $\overrightarrow{n} = \overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$ and the point $P_0(2, 3, -1)$. Find the distance from point $P(5, 2, 7)$ to the plane.

Let Q be the orthogonal projection of P on the plane. The distance from P to the plane is defined as $d = ||\overrightarrow{PQ}||$. We can write $\overrightarrow{P_0P} = \overrightarrow{P_0Q} + \overrightarrow{QP}$ and

$$\begin{aligned}
|\overrightarrow{P_0P} \cdot \overrightarrow{\mathbf{n}}| &= |\overrightarrow{P_0Q} \cdot \overrightarrow{\mathbf{n}} + \overrightarrow{QP} \cdot \overrightarrow{\mathbf{n}}| &\quad (1) \\
&= 0 + ||\overrightarrow{QP}||\,||\overrightarrow{\mathbf{n}}|| &\quad (2)
\end{aligned}$$

Note that equation (1) is true because the dot product is distributive. Equation (2) is true because $\overrightarrow{P_0Q}$ and $\overrightarrow{\mathbf{n}}$ are perpendicular and $\overrightarrow{QP}$ and $\overrightarrow{\mathbf{n}}$ are in the same direction (parallel). It follows that

$$\begin{aligned}
d = ||\overrightarrow{QP}|| &= \frac{|\overrightarrow{P_0P} \cdot \overrightarrow{\mathbf{n}}}{||\overrightarrow{\mathbf{n}}||} \\
&= \frac{|(5-2) \times 1 + (2-3) \times (-1) + (7+1) \times 1|}{\sqrt{1^2 + (-1)^2 + 1^2}} \\
&= \frac{12}{\sqrt{3}}.
\end{aligned}$$

---

5. Given the plane $5x - 3y + 6z - 7 = 0$:

   (a) Find a normal vector to the plane.

   (b) Determine whether $P_1(1, 5, 2)$ and $P_2(-3, -1, 2)$ are on the same side of the plane.

   (a) From the cartesian equation of the plane, a normal vector to the plane is $\overrightarrow{\mathbf{n}} = (5, -3, 6)$.

   (b) We can easily see that $P_0(2, 1, 0)$ lies on the plane as it satisfies the equation of the plane. Given a point $P(x, y, z)$, we have the following:

      i. P lies on the plane if and only if $\overrightarrow{\mathbf{n}} \cdot \overrightarrow{P_0P} = 0$ (equation of the plane).

      ii. If the angle between $\overrightarrow{\mathbf{n}}$ and $\overrightarrow{P_0P}$ is smaller than 90^0, then $\overrightarrow{\mathbf{n}} \cdot \overrightarrow{P_0P} > 0$.

      iii. If the angle between $\overrightarrow{\mathbf{n}}$ and $\overrightarrow{P_0P}$ is greater than 90^0, then $\overrightarrow{\mathbf{n}} \cdot \overrightarrow{P_0P} < 0$.

   If follows that two points P_1, P_2 are on the same side of the plane if the dot products of $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ with $\overrightarrow{\mathbf{n}}$ are of the same sign. In this specific case, we have $\overrightarrow{\mathbf{n}} \cdot \overrightarrow{P_0P_1} = (5 - 15 + 12 - 7) = -5$ and $\overrightarrow{\mathbf{n}} \cdot \overrightarrow{P_0P_2} = (-15 + 3 + 12 - 7) = -7$. Hence, P_1, P_2 are on the same side of the plane.

---

6. Three vertices (in 3D) determine a triangle if they do not lie in the same line. Devise a test for collinearity of three vertices.

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ be three vertices in 3D. If they are collinear, one vertex is a linear combination of the other two and the determinant of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

will be zero.

---

7. We can use vertices in 3D to define objects such as 3D polygons. Given a set of vertices, find a test to determine whether the polygon that they determine is planar.

One test is to use the first three (non-collinear) vertices to find the equation of the plane $ax + by + cz + d = 0$ they determine. Although there are four coefficients in the equation only three are independent so we can select one arbitrarily. Then we can successively evaluate $ax + bc + cz + d$ for the other vertices. A vertex will be on the plane if the equation evaluates to zero for its coordinates. Assuming (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are not collinear, an equivalent test is to form the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_i \\ y_1 & y_2 & y_3 & y_i \\ z_1 & z_2 & z_3 & z_i \end{bmatrix}$$

for each $i \geq 4$. The i-th vertex is in the plane determined by the first three if and only if the determinant of this matrix is zero.

---

8. Devise a test for the convexity of a two-dimensional polygon.

I will describe two methods for testing convexity.

(a) By definition, an angle on the inside of a polygon formed by each pair of adjacent sides is called an **interior angle** of the polygon. There are many (equivalent) definitions of a convex polygon in 2D. We will rely here on the following. A polygon is convex if and only if all its interior angles are less than 180^0. We can assign a direction for each edge of the polygon by traversing the polygon in a counter-clockwise order. The interior angle formed by two adjacent edges is less than 180^0 iff the cross product of any two adjacent edges (used as vectors with the third coordinate set to 0) in the order defined is a normal vector to the plane of the polygon in the direction given by the right-hand orientation. This is equivalent to saying that the determinant of the 2×2 matrix whose first and second rows are the two adjacent edges in the same order is positive.

(b) Consider the lines defined by the sides of the polygon. We can assign a direction for each of these lines by traversing the vertices in a counter-clockwise order. One very simple test is obtained by noting that any point inside the object is on the left of each of these lines. Thus, if we substitute the point into the equation for each of the lines $(ax + by + c)$, we should always get the same sign.

---

9. (a) What three dimensional line is determined by the homogeneous coordinate point $(1, 5, -1)$?

   (b) Do the homogeneous coordinates $(1, 5, -1)$ and $(-2, -10, -3)$ represent the same projective point?

(a) The three dimensional line determined by the homogeneous coordinate point $(1, 5, -1)$ is

$$L : (1, 5, -1)t, \quad t \in \mathbb{R}.$$

(b) The homogeneous coordinates $(1, 5, -1)$ and $(-2, -10, -3)$ represent the same projective point if there exists $t_0 \in \mathbb{R}$ such that $(-2, -10, -3) = t_0(1, 5, -1)$. The first two equations require that $t_0 = -2$, but the third equation requires that $t_0 = 3$. Hence, the two homogeneous coordinates do not represent the same projective point.
