Computer Graphics - CS402

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Homework # 4

1. Derive the matrix of the parallel projection onto the plane $x + y + z = 1$ in the direction of projector $\mathbf{v} = (-1, -1, -1)$.

Note that the projector $\mathbf{v} = -\mathbf{n}(1, 1, 1)$ is also normal vector to the plane $x + y + z = 1$. Hence our parallel projection is an axonometric projection. We have already derived the general axonometric projection matrix

$$\begin{pmatrix} 1-a^2 & -ab & -ac & ad_0 \\ -ab & 1-b^2 & -bc & bd_0 \\ -ac & -bc & 1-c^2 & cd_0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where (a, b, c) is a unit normal vector and $d_0 = ax_0 + by_0 + cz_0$ with $R(x_0, y_0, z_0)$ a reference point of the plane. $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is a unit normal and it is easy to find a reference point of the plane, for instance $R(1, 0, 0)$. Hence $d_0 = \frac{1}{\sqrt{3}}$. Hence the matrix is

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Derive the matrix of general perspective projection onto the plane with normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and reference point $R(x_0, y_0, z_0)$ and using $C(c_x, c_y, c_z)$ as the center of projection.

We have already derived the matrix of the perspective projection on a plane with normal vector $\mathbf{n}(a, b, c)$ and reference point $R(x_0, y_0, z_0)$ and with center of projection located at the origin, which is as follows

$$\begin{pmatrix} d_0 & 0 & 0 & 0 \\ 0 & d_0 & 0 & 0 \\ 0 & 0 & d_0 & 0 \\ a & b & c & 0 \end{pmatrix},$$

where $d_0 = ax_0 + by_0 + cz_0$. The easiest way to proceed is to use a translation to move the center of projection to the origin, do the perspective projection, and then translate back. Note that when the center of projection $C(c_x, c_y, c_z)$ is translated to the origin by vector $(-c_x, -c_y, -c_z)$, the projection plane is also translated by the same vector. While the normal vector to the translated plane is still the same, the reference point is translated to a new reference point $R'(x_0 - c_x, y_0 - c_y, z_0 - c_z)$. The perspective projection matrix on the translated plane with the origin as the center of projection is given by

$$\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ a & b & c & 0 \end{pmatrix},$$

with $d = a(x_0 - c_x) + b(y_0 - c_y) + c(z_0 - c_z) = ax_0 + by_0 + cz_0 - (ac_x + bc_y + cc_z) = d_0 - d_1$. It follows that the perspective projection matrix on the plane with normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and reference point $R(x_0, y_0, z_0)$ and using $C(c_x, c_y, c_z)$ as the center of projection is given by

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 & 0 & c_x \\ 0 & 1 & 0 & c_y \\ 0 & 0 & 1 & c_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ a & b & c & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} d + ac_x & bc_x & cc_x & -d_0c_x \\ ac_y & d + bc_y & cc_y & -d_0c_y \\ ac_z & bc_z & d + cc_z & -d_0c_z \\ a & b & c & -d_1 \end{pmatrix} \end{aligned}$$

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3. (a) Using the origin as the center of projection, derive the perspective transformation onto the plane passing through $(1, 0, 0)$ and having the normal vector $\mathbf{n} = (1, 1, 1)$.

   (b) What are the principal vanishing points for this perspective transformation.

(a) We have $d_0 = 1$ and therefore the matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

(b) In general, in order to find the vanishing point for a perspective projection centered at the origin on the plane with normal vector $\mathbf{n}(a, b, c)$ in the direction given by a vector $\mathbf{v}(v_1, v_2, v_3)$, we consider the family of parallel lines in the direction of vector $\mathbf{v}$ that can be written in parametric form as

$$\begin{cases} x &= v_1t + \alpha \\ y &= v_2t + \beta \\ z &= v_3t + \gamma, \end{cases}$$

where α, β, and γ are parameters. Applying the projection matrix on the family of parallel lines gives

$$\begin{pmatrix} d_0 & 0 & 0 & 0 \\ 0 & d_0 & 0 & 0 \\ 0 & 0 & d_0 & 0 \\ a & b & c & 0 \end{pmatrix} \begin{pmatrix} v_1t + \alpha \\ v_2t + \beta \\ v_3t + \gamma \\ 1 \end{pmatrix} = \begin{pmatrix} d_0(v_1t + \alpha) \\ d_0(v_2t + \beta) \\ d_0(v_3t + \gamma) \\ a(v_1t + \alpha) + b(v_2t + \beta) + c(v_3t + \gamma) \end{pmatrix}$$

It follows that the 3D coordinates of the projections are

$$\begin{cases} x' &= \frac{d_0(v_1t+\alpha)}{a(v_1t+\alpha)+b(v_2t+\beta)+c(v_3t+\gamma)} \\ \\ y' &= \frac{d_0(v_2t+\beta)}{a(v_1t+\alpha)+b(v_2t+\beta)+c(v_3t+\gamma)} \\ \\ z' &= \frac{d_0(v_3t+\gamma)}{a(v_1t+\alpha)+b(v_2t+\beta)+c(v_3t+\gamma)} \end{cases}$$

Taking the limit when $t \to +\infty$ and assuming the denominator is not 0, we get

$$\begin{cases} x^* & = & \frac{d_0 v_1}{av_1+bv_2+cv_3} \\ y^* & = & \frac{d_0 v_2}{av_1+bv_2+cv_3} \\ z^* & = & \frac{d_0 v_3}{av_1+bv_2+cv_3} \end{cases}$$

which are "eventually" the coordinates of the vanishing point in direction $\mathbf{v}$ when the limit makes sense. It is clear from the formulas that when direction $\mathbf{v}$ is parallel to the projection plane, hence orthogonal to $\mathbf{n}$, then the dot product $av_1 + bv_2 + cv_3 = 0$ and therefore there are no vanishing points in the direction $\mathbf{v}$. In the particular case given here, the principal vanishing points are those that correspond to the directions $\mathbf{i}(1,0,0)$, $\mathbf{j}(0,1,0)$, and $\mathbf{k}(0,0,1)$. Since $\mathbf{n} = (1,1,1)$ and $d_0 = 1$, then we get three principal vanishing points:

$$V_x = (1,0,0), \quad V_y = (0,1,0), \quad \text{and} \quad V_z = (0,0,1).$$

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4. Assume given the canonical perspective projection where the view plane is parallel to the xy-plane and located at distance $d > 0$ on the positive z-axis, and the center of projection located at the origin.

   (a) What is the projected image of a point $P(x, y, 0)$ located in the xy-plane.

   (b) What is the projected image of the line segment joining the points $P_1(-1, 1, -d)$ and $P_2(2, -2, d)$. transformation.

(a) The plane xy (or $z = 0$) is parallel to the projection plane $z = d$ and located at the center of projection $C(0, 0, 0)$. The point $P(x, y, 0)$ is a point in xy-plane and the line of projection $\overrightarrow{CP}$ lies entirely in the xy-plane and does not intersect the projection plane $z = d$. We then say that P is projected to infinity (∞).

(b) The line segment $\overrightarrow{P_1P_2}$ passes through the xy-plane. Writing the equation of the line segment, we have

$$\begin{cases} x & = & -1 + 3t \\ y & = & 1 - 3t \\ z & = & -d + (2d)t, \quad t \in [0, 1]. \end{cases}$$

Note that at $t = \frac{1}{2}$, we have $x = \frac{1}{2}$, $y = -\frac{1}{2}$, and $z = 0$, which are the coordinates of the intersection point I of the line segment $\overrightarrow{P_1P_2}$ and the xy-plane. Applying the perspective projection matrix to the equation of the line segment, we find

$$\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 + 3t \\ 1 - 3t \\ -d + (2d)t \\ 1 \end{pmatrix} = \begin{pmatrix} -d + 3dt \\ d - 3dt \\ -d^2 + (2d^2)t \\ -d + (2d)t \end{pmatrix}$$

Changing from homogeneous to three dimensional coordinates, the equations of the projected line segment are

$$x = \frac{-d + 3dt}{-d + (2d)t} = \frac{-1 + 3t}{-1 + 2t}, \quad y = \frac{d - 3dt}{-d + (2d)t} = \frac{1 - 3t}{-1 + 2t}, \quad z = \frac{-d^2 + (2d^2)t}{-d + (2d)t} = \frac{-d + (2d)t}{-1 + 2t} = d.$$

Note that $y = -x$ and $z = d$, hence it is the equation of a line. When $t = 0$, we are at point $(1, -1, d)$ which are the coordinates of the projection of P_1. When $t = 1$, we are at point $(2, -2, d)$ which are the coordinates of the projection of P_2. However, when $t = \frac{1}{2}$, the denominator is 0, thus the projection of the line segment is not a line segment but it passes through the point at infinity. Hence, it is consisting of two two half lines.