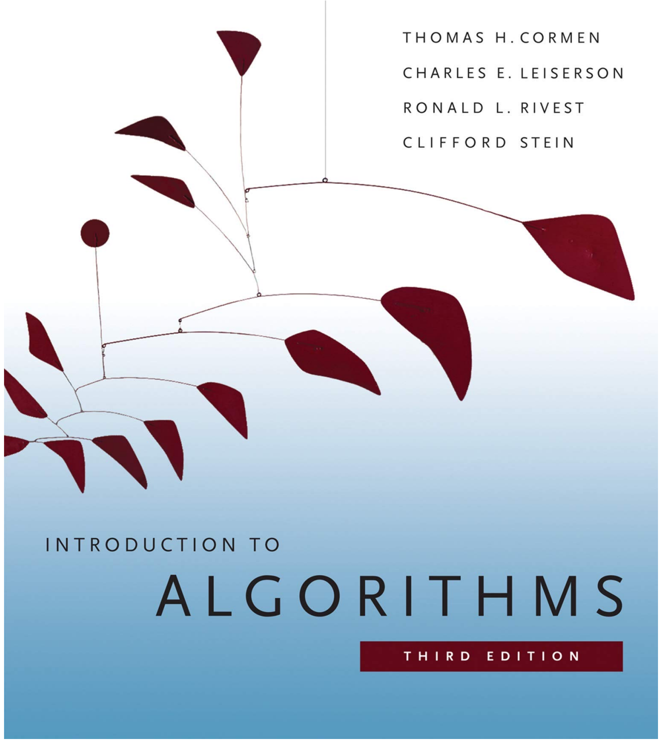
Algorithm



Chap 2: Asymptotic Notations

Outline

- Review of last lecture
- Sum of series
- Analyzing recursive algorithms

L' Hopital's rule

$$\lim_{n \to \infty} f(n) / g(n) = \lim_{n \to \infty} f(n)' / g(n)'$$

Condition:

If both lim f(n) and lim g(n) = ∞ or 0

Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

or

$$n! \approx \text{(constant)}\; n^{n+1/2} e^{-n}$$

Properties of asymptotic notations

- Textbook page 51
- Transitivity

  $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$

  $\Rightarrow f(n) = \Theta(h(n))$

  (holds true for o, O, ω, and Ω as well).
- Symmetry

  $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry

  $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

  $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$

logarithms

- $\lg n = \log_2 n$
- $\ln n = \log_e n$, $e \approx 2.718$
- $\lg^k n = (\lg n)^k$
- $\lg \lg n = \lg (\lg n) = \lg^{(2)} n$
- $\lg(^k) n = \lg \lg \lg \ldots \lg n$
- $\lg^2 4 = ?$
- $\lg^{(2)} 4 = ?$
- Compare $\lg^k n$ vs $\lg^{(k)} n$?

Useful rules for logarithms

- For all $a > 0$, $b > 0$, $c > 0$, the following rules hold
- $\log_b a = \log_c a / \log_c b = \lg a / \lg b$
- $\log_b a^n = n \log_b a$
- $b^{\log_b a} = a$
- $\log (ab) = \log a + \log b$
  - $\lg (2n) = ?$
- $\log (a/b) = \log (a) – \log(b)$
  - $\lg (n/2) = ?$
  - $\lg (1/n) = ?$
- $\log_b a = 1 / \log_a b$

Useful rules for exponentials

- For all $a > 0$, $b > 0$, $c > 0$, the following rules hold
- $a^0 = 1 \quad (0^0 = ?)$
- $a^1 = a$
- $a^{-1} = 1/a$
- $(a^m)^n = a^{mn}$
- $(a^m)^n = (a^n)^m$
- $a^m a^n = a^{m+n}$

More advanced dominance ranking

$$n^n >> n! >> 3^n >> 2^n >> n^3 >> n^2 >> n^{1+\varepsilon} >> n \log n \sim \log n!$$

$$>> n >> n / \log n >> \sqrt{n} >> n^\varepsilon >> \log^3 n >> \log^2 n >> \log n$$

$$>> \log n / \log \log n >> \log \log n >> \log^{(3)} n >> \alpha(n) >> 1$$

Find the order of growth for sums

- $T(n) = \sum_{i=1..n} i = \Theta(n^2)$
- $T(n) = \sum_{i=1..n} \log(i) = ?$
- $T(n) = \sum_{i=1..n} n / 2^i = ?$
- $T(n) = \sum_{i=1..n} 2^i = ?$
- …

- How to find out the actual order of growth?
  – Math…
  – Textbook Appendix A.1 (page 1058-60)

Arithmetic series

- An arithmetic series is a sequence of numbers such that the difference of any two successive members of the sequence is a constant.

  e.g.: 1, 2, 3, 4, 5

  or 10, 12, 14, 16, 18, 20
- In general:

$$a_j = a_{j-1} + d$$ ← Recursive definition

Or: $$a_j = a_1 + (j-1)d$$ ← Closed form, or explicit formula

Sum of arithmetic series

If $a_1, a_2, \ldots, a_n$ is an arithmetic series, then

$$\sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2}$$

e.g. 1 + 3 + 5 + 7 + … + 99 = ?

Geometric series

- A geometric series is a sequence of numbers such that the ratio between any two successive members of the sequence is a constant.

  e.g.: 1, 2, 4, 8, 16, 32

  or 10, 20, 40, 80, 160

  or 1, ½, ¼, 1/8, 1/16

- In general:

$$a_j = r a_{j-1}$$ ← Recursive definition

Or: $$a_j = r^{j-1} a_0$$ ← Closed form, or explicit formula

Sum of geometric series

$$\sum_{i=0}^{n} r^i = \begin{cases} (1-r^{n+1})/(1-r) & \text{if r < 1} \\ (r^{n+1}-1)/(r-1) & \text{if r > 1} \\ n+1 & \text{if r = 1} \end{cases}$$

$$\sum_{i=0}^{n} 2^i = ?$$

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{2^i} = ?$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = ?$$

Sum of geometric series

$$\sum_{i=0}^{n} r^i = \begin{cases} (1-r^{n+1})/(1-r) & \text{if r < 1} \\ (r^{n+1}-1)/(r-1) & \text{if r > 1} \\ n+1 & \text{if r = 1} \end{cases}$$

$$\sum_{i=0}^{n} 2^i = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1 \approx 2^{n+1}$$

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{2^i} = \lim_{n\to\infty} \sum_{i=0}^{n} (1/2)^i = \frac{1}{1-1/2} = 2$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = \lim_{n\to\infty} \sum_{i=0}^{n} \left(1/2\right)^0 - \left(1/2\right)^0 = 2 - 1 = 1$$

Important formulas

$$\sum_{i=1}^{n} 1 = n \in \Theta(n)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \in \Theta(n^2)$$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} \in \begin{cases} \Theta(1) & (r<1) \\ \Theta(r^n) & (r>1) \end{cases}$$

$$\sum_{i=1}^{n} i^2 \approx \frac{n^3}{3} \in \Theta(n^3)$$

$$\sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1} \in \Theta(n^{k+1})$$

$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2 \in \Theta(n2^n)$$

$$\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\lg n)$$

$$\sum_{i=1}^{n} \lg i \in \Theta(n \lg n)$$

Sum manipulation rules

$$\sum_i (a_i + b_i) = \sum_i a_i + \sum_i b_i$$

$$\sum_i c a_i = c \sum_i a_i$$

$$\sum_{i=m}^{n} a_i = \sum_{i=m}^{x} a_i + \sum_{i=x+1}^{n} a_i$$

Example:

$$\sum_{i=1}^{n} (4i + 2^i) = ?$$

$$\sum_{i=1}^{n} \frac{n}{2^i} = ?$$

Sum manipulation rules

$$\sum_i (a_i + b_i) = \sum_i a_i + \sum_i b_i$$

$$\sum_i c a_i = c \sum_i a_i$$

$$\sum_{i=m}^{n} a_i = \sum_{i=m}^{x} a_i + \sum_{i=x+1}^{n} a_i$$

Example:

$$\sum_{i=1}^{n} (4i + 2^i) = 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2^i = 2n(n+1) + 2^{n+1} - 2$$

$$\sum_{i=1}^{n} \frac{n}{2^i} = n \sum_{i=1}^{n} \frac{1}{2^i} \approx n$$

Examples

- $\sum_{i=1..n} n / 2^i = n * \sum_{i=1..n} (½)^i = ?$

- using the formula for geometric series:

  $\sum_{i=0..n} (½)^i = 1 + ½ + ¼ + \ldots (½)^n = 2$

- Application: algorithm for allocating dynamic memories

Examples

- $\sum_{i=1..n} \log(i) = \log 1 + \log 2 + \ldots + \log n$

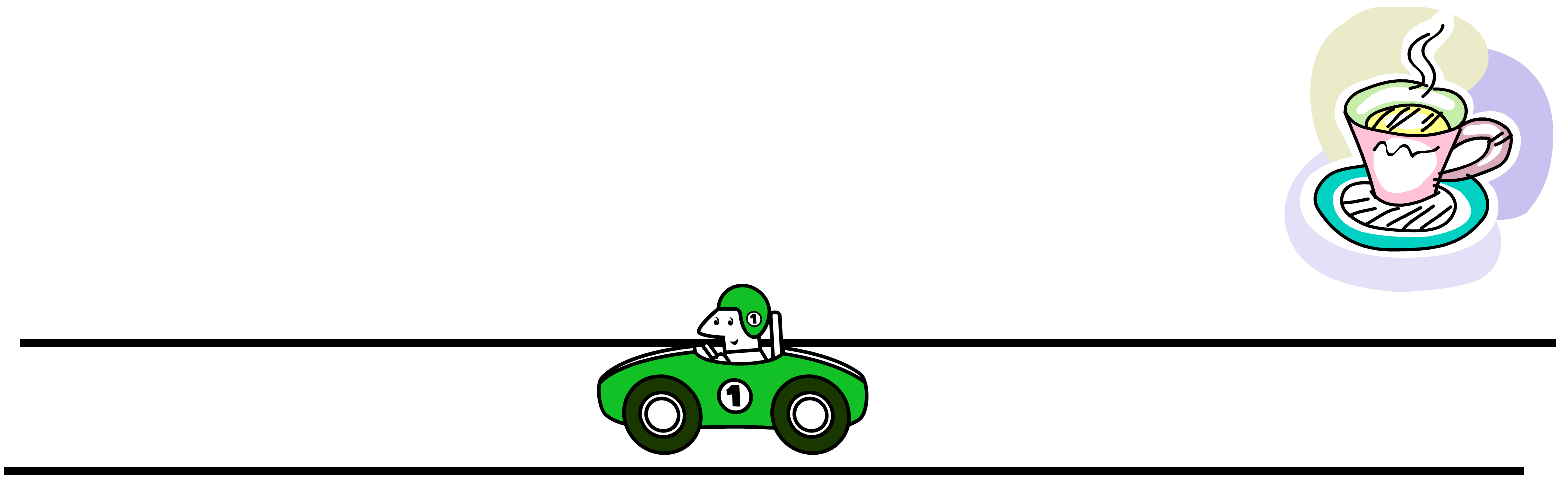
  $= \log 1 \times 2 \times 3 \times \ldots \times n$

  $= \log n!$

  $= n \log n$

- Application: algorithm for selection sort using priority queue

Problem of the day



How do you find a coffee shop if you don’t know on which direction it might be?

Recursive definition of sum of series

- $T(n) = \sum_{i=0..n} i$ is equivalent to:

$$\begin{cases} T(n) = T(n-1) + n & \longleftarrow \text{Recurrence relation} \\ T(0) = 0 & \longleftarrow \text{Boundary condition} \end{cases}$$

- $T(n) = \sum_{i=0..n} a^i$ is equivalent to:

$$\begin{cases} T(n) = T(n-1) + a^n \\ T(0) = 1 \end{cases}$$

Recursive definition is often intuitive and easy to obtain. It is very useful in analyzing recursive algorithms, and some non-recursive algorithms too.

Analyzing recursive algorithms

Recursive algorithms

- General idea:
  - Divide a large problem into smaller ones
    - By a constant ratio
    - By a constant or some variable
  - Solve each smaller one *recursively* or *explicitly*
  - Combine the solutions of smaller ones to form a solution for the original problem

Divide and Conquer

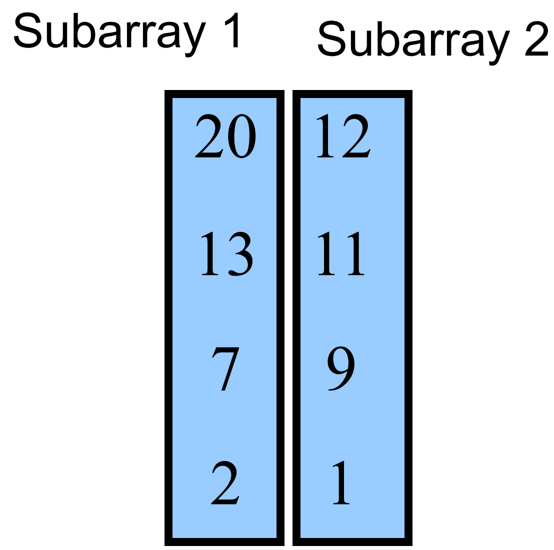
Merge sort

MERGE-SORT $A[1 \,.\,.\, n]$

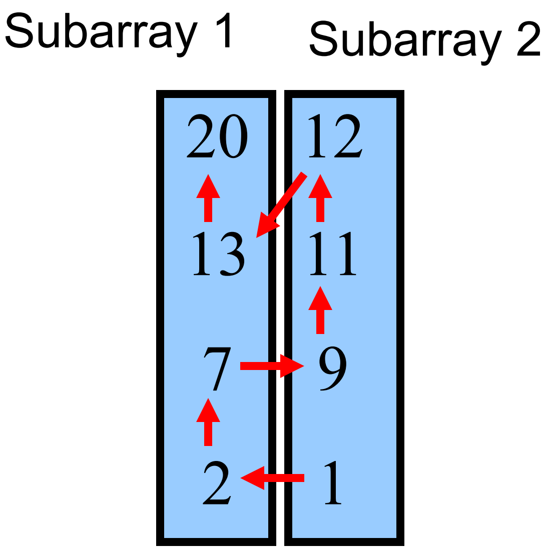
1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**

Merging two sorted arrays



Merging two sorted arrays



Merging two sorted arrays

20	12
13	11
7	9
2	1

Merging two sorted arrays

20	12
13	11
7	9
2	1

Merging two sorted arrays

20	12
13	11
7	9
2	1

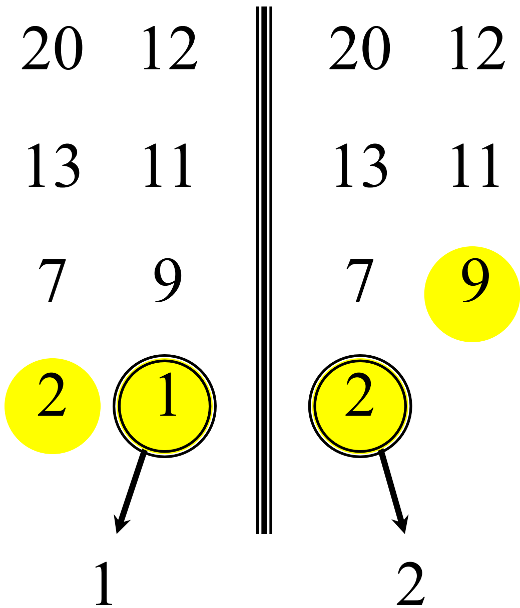
1

Merging two sorted arrays

20	12	20	12
13	11	13	11
7	9	7	9
2	1	2	

1

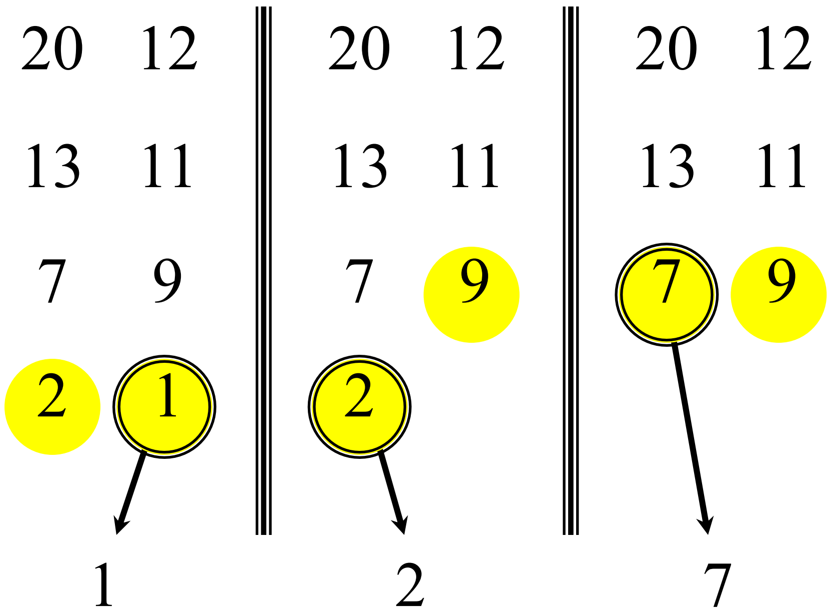
Merging two sorted arrays



Merging two sorted arrays

20	12		20	12		20	12
13	11		13	11		13	11
7	9		7	9		7	9
2	1		2				
1			2				

Merging two sorted arrays



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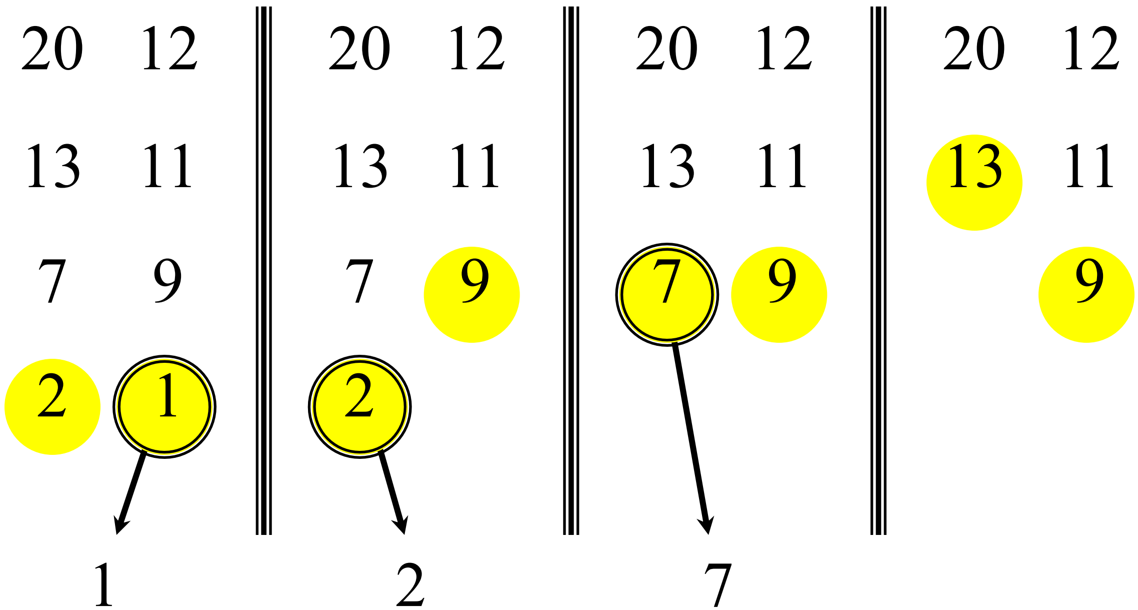

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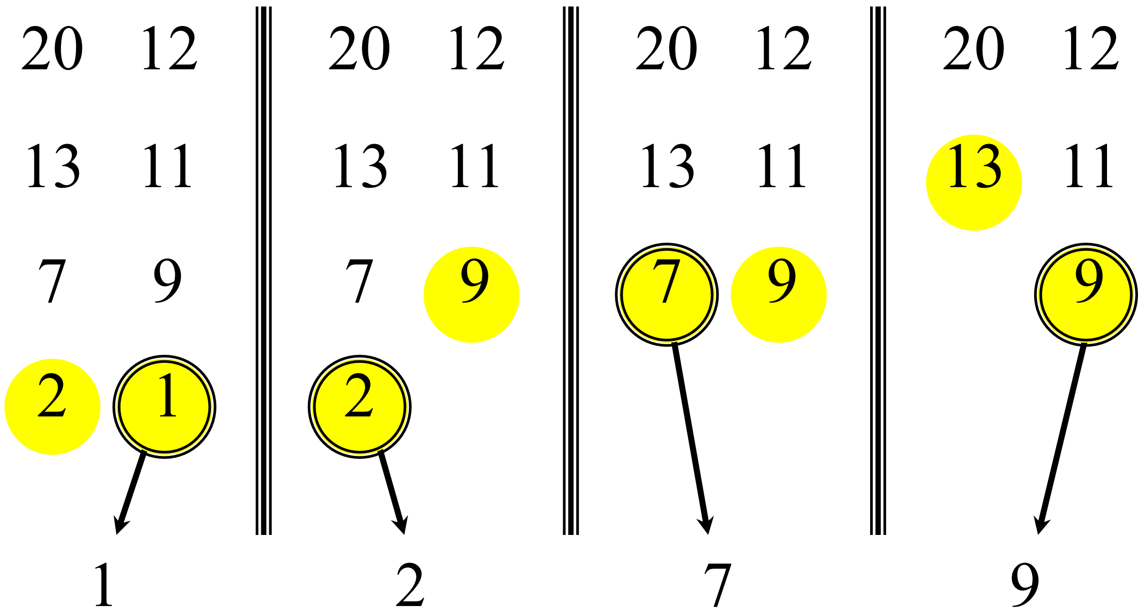

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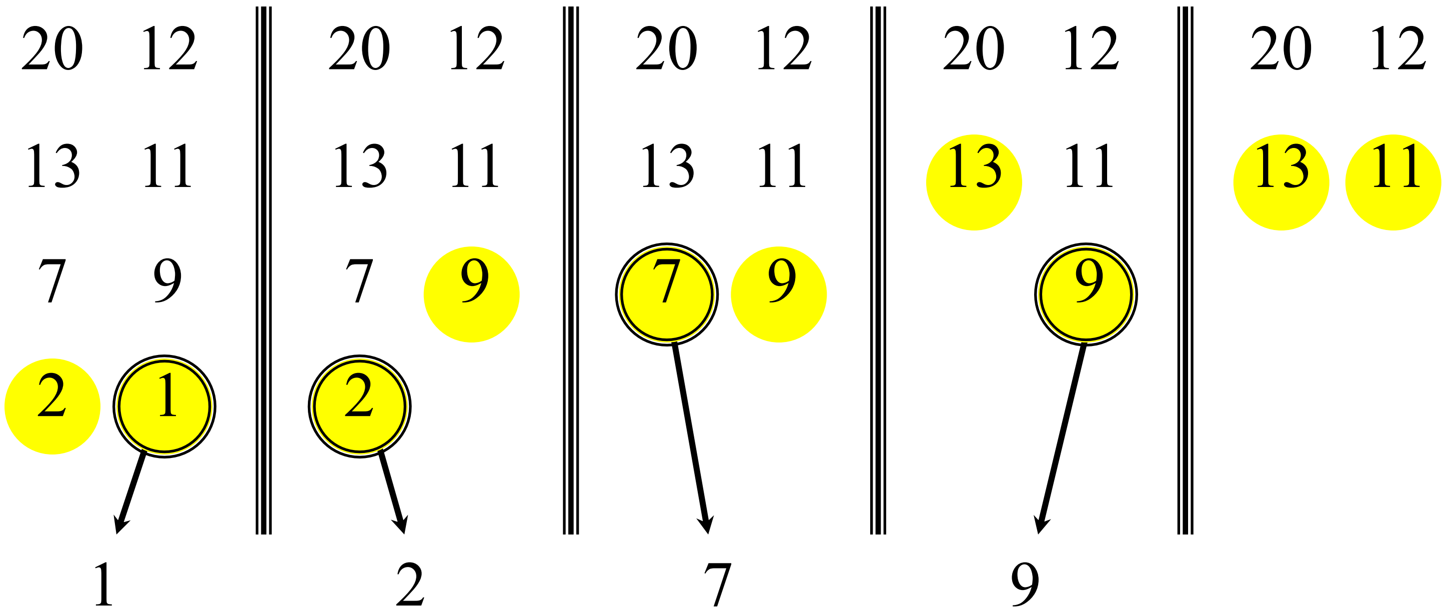

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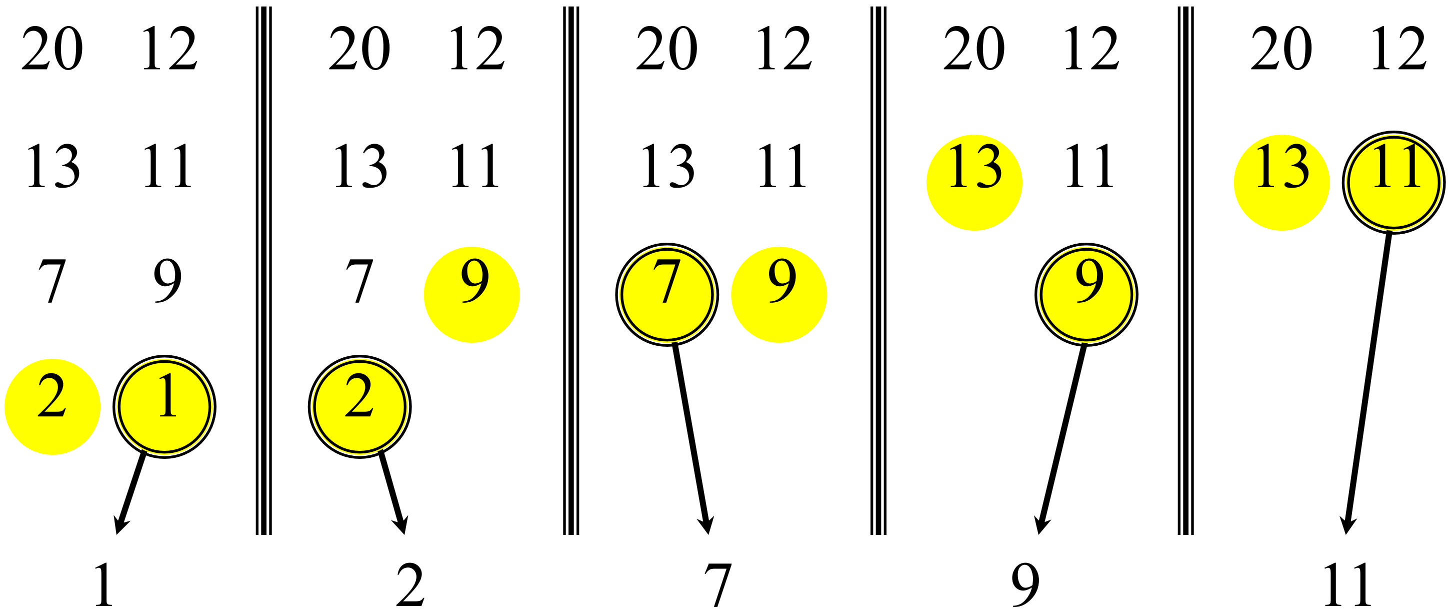

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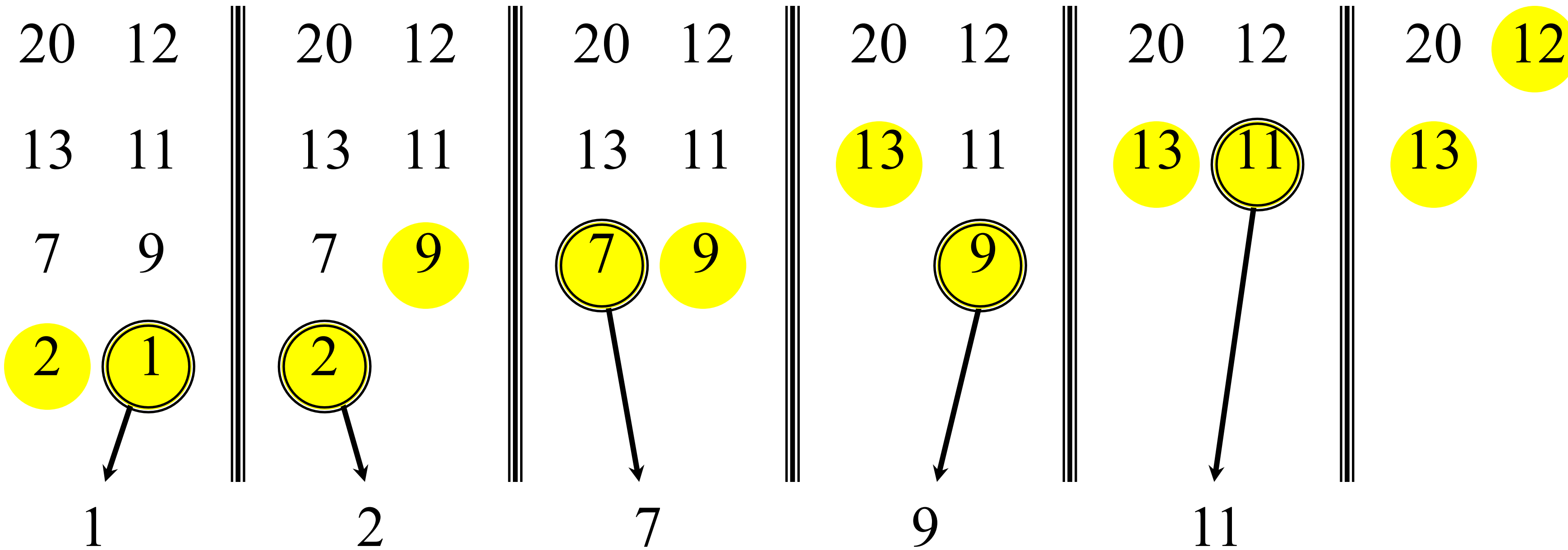

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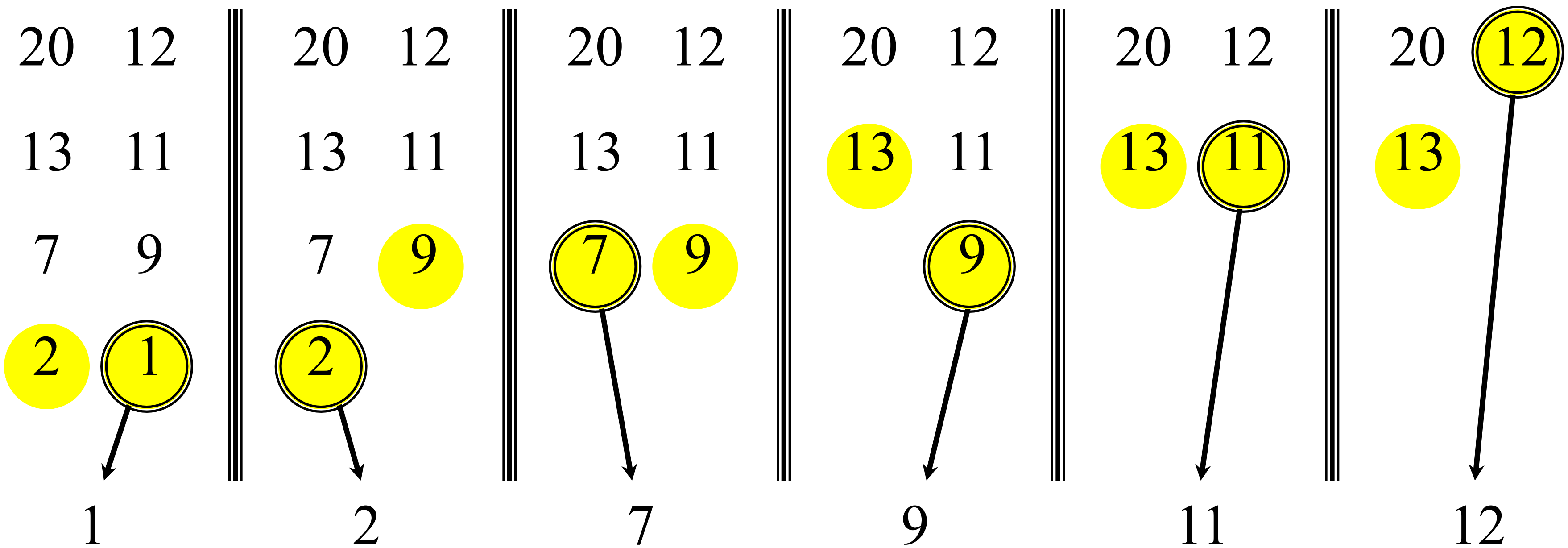


How to show the correctness of a recursive algorithm?

- By induction:
  - Base case: prove it works for small examples
  - Inductive hypothesis: assume the solution is correct for all sub-problems
  - Step: show that, if the inductive hypothesis is correct, then the algorithm is correct for the original problem.

Correctness of merge sort

MERGE-SORT $A[1 \mathrel{.\,.} n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \mathrel{.\,.} \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \mathrel{.\,.} n\,]$.
3. "***Merge***" the 2 sorted lists.

Proof:

1. Base case: if n = 1, the algorithm will return the correct answer because A[1..1] is already sorted.
2. Inductive hypothesis: assume that the algorithm correctly sorts $A[1..\lceil n/2 \rceil]$ and $A[\lceil n/2 \rceil+1..n]$.
3. Step: if $A[1..\lceil n/2 \rceil]$ and $A[\lceil n/2 \rceil+1..n]$ are both correctly sorted, the whole array $A[1..\lceil n/2 \rceil]$ and $A[\lceil n/2 \rceil+1..n]$ is sorted after merging.

How to analyze the time-efficiency of a recursive algorithm?

- Express the running time on input of size n as a function of the running time on smaller problems

Analyzing merge sort

$T(n)$	**MERGE-SORT** $A[1 \,.\,.\, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
$f(n)$	3. ***"Merge"*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.

Analyzing merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Merge two sorted subarrays

$$T(n) = 2\,T(n/2) + f(n) + \Theta(1)$$

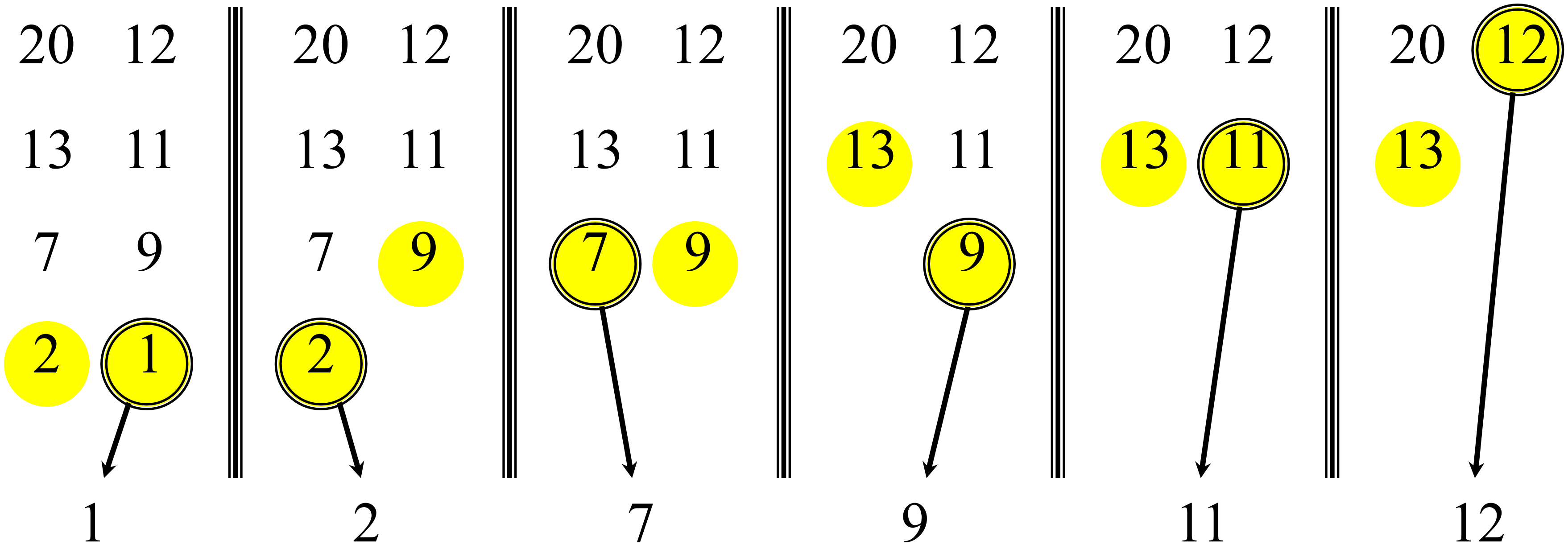
 # subproblems

subproblem size

Dividing and Combining

1. What is the time for the base case? Constant
2. What is $f(n)$?
3. What is the growth order of $T(n)$?

Merging two sorted arrays



$\Theta(n)$ time to merge a total of n elements (linear time).

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- Later we shall often omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

- But what does $T(n)$ solve to? I.e., is it $O(n)$ or $O(n^2)$ or $O(n^3)$ or …?

Binary Search

To find an element in a sorted array, we

1. Check the middle element
2. If ==, we've found it
3. else if less than wanted, search right half
4. else search left half

Example: Find 9

3	5	7	8	9	12	15

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4. else search left half

Example: Find 9

3 5 7 8 9 12 15

Binary Search

```
BinarySearch (A[1..N], value) {
    if (N == 0)
        return -1;                  // not found
    mid = (1+N)/2;
    if (A[mid] == value)
        return mid;                 // found
    else if (A[mid] < value)
        return BinarySearch (A[mid+1, N], value)
    else
        return BinarySearch (A[1..mid-1], value);
}
```

What’s the recurrence relation for its running time?

Recurrence for binary search

$$T(n) = T\left(\frac{n}{2}\right) + \Theta(1)$$

$$T(1) = \Theta(1)$$

Recursive Insertion Sort

RecursiveInsertionSort(A[1..n])

1. if (n == 1) do nothing;
2. ***RecursiveInsertionSort***(A[1..n-1]);
3. Find index *i* in A such that A[i] <= A[n] < A[i+1];
4. Insert A[n] after A[i];

Recurrence for insertion sort

$$T(n) = T(n-1) + \Theta(n)$$

$$T(1) = \Theta(1)$$

Compute factorial

Factorial (n)

```
if (n == 1) return 1;
return n * Factorial (n-1);
```

- Note: here we use n as the size of the input. However, usually for such algorithms we would use log(n), i.e., the bits needed to represent n, as the input size.

Recurrence for computing factorial

$$T(n) = T(n-1) + \Theta(1)$$

$$T(1) = \Theta(1)$$

- Note: here we use n as the size of the input. However, usually for such algorithms we would use log(n), i.e., the bits needed to represent n, as the input size.

What do these mean?

$$T(n) = T(n-1) + 1$$

$$T(n) = T(n-1) + n$$

$$T(n) = T(n/2) + 1$$

$$T(n) = 2T(n/2) + 1$$

Challenge: how to solve the recurrence to get a closed form, e.g. $T(n) = \Theta(n^2)$ or $T(n) = \Theta(n \lg n)$, or at least some bound such as $T(n) = O(n^2)$?

Solving recurrence

- Running time of many algorithms can be expressed in one of the following two recursive forms

$$T(n) = aT(n-b) + f(n)$$

or

$$T(n) = aT(n/b) + f(n)$$

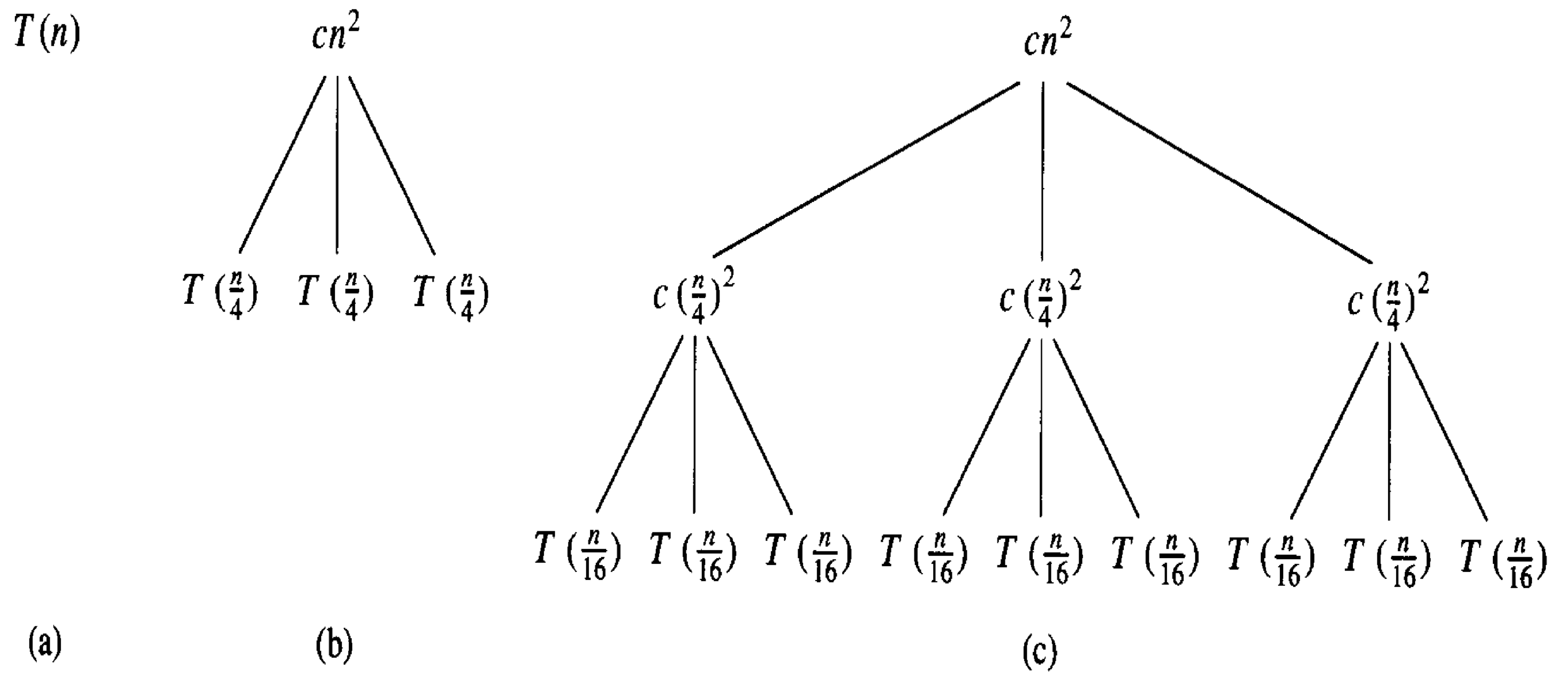
Both can be very hard to solve. We focus on relatively easy ones, which you will encounter frequently in many real algorithms (and exams…)

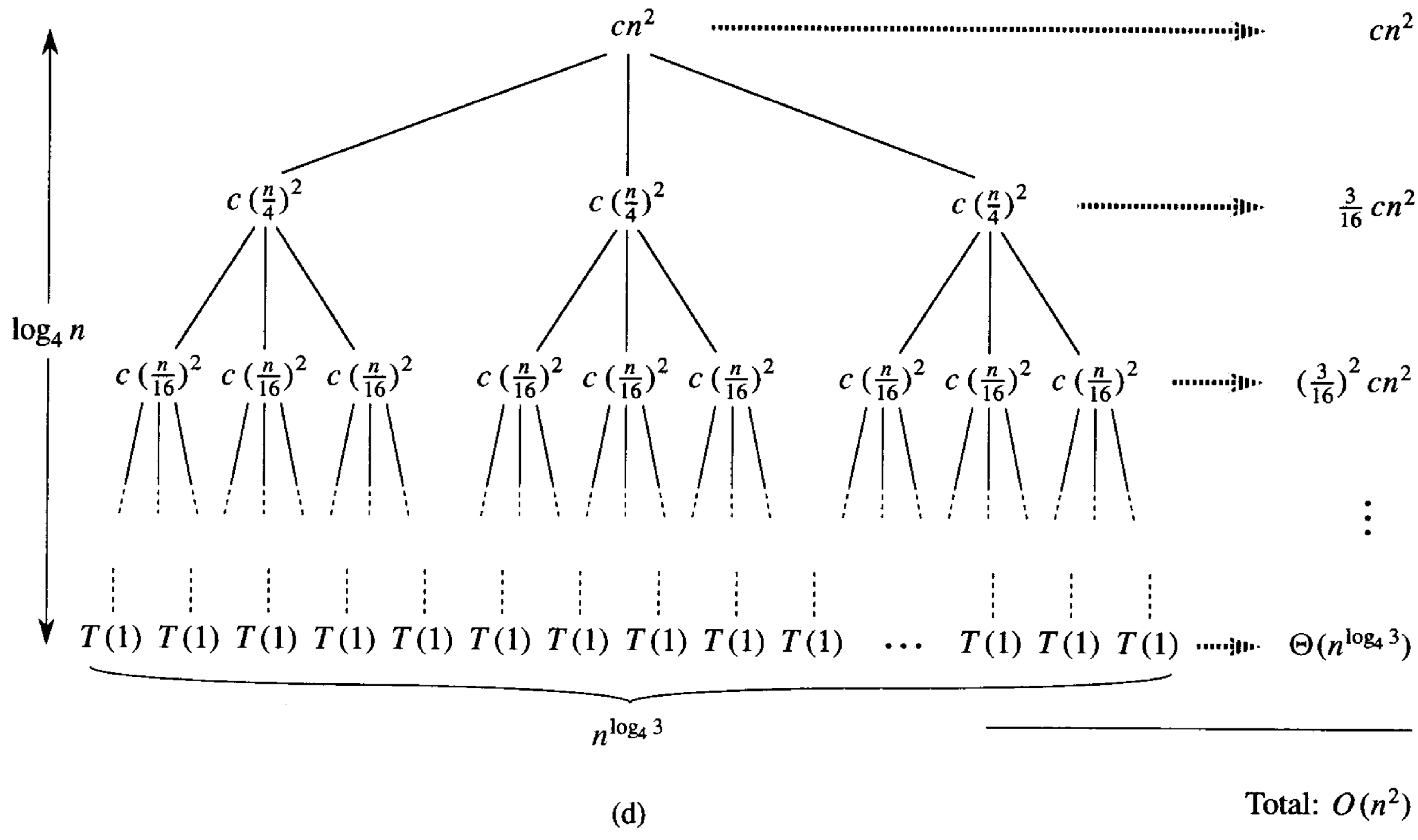
Solving recurrence

1. Recursion tree / iteration method
2. Master method
3. Math…

Recursion-tree method

$$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$$




cn^2
cn^2
$c\left(\frac{n}{4}\right)^2$
$c\left(\frac{n}{4}\right)^2$
$c\left(\frac{n}{4}\right)^2$
$\frac{3}{16}cn^2$
$\log_4 n$
$c\left(\frac{n}{16}\right)^2$
$\left(\frac{3}{16}\right)^2 cn^2$
$T(1)$
$\Theta(n^{\log_4 3})$
$n^{\log_4 3}$
Total: $O(n^2)$

(d)

The cost of the entire tree

$$T(n) = cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \ldots + \left(\frac{3}{16}\right)^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3})$$

$$= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right)$$

$$= \frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3}).$$

$$T(n) = \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right)$$

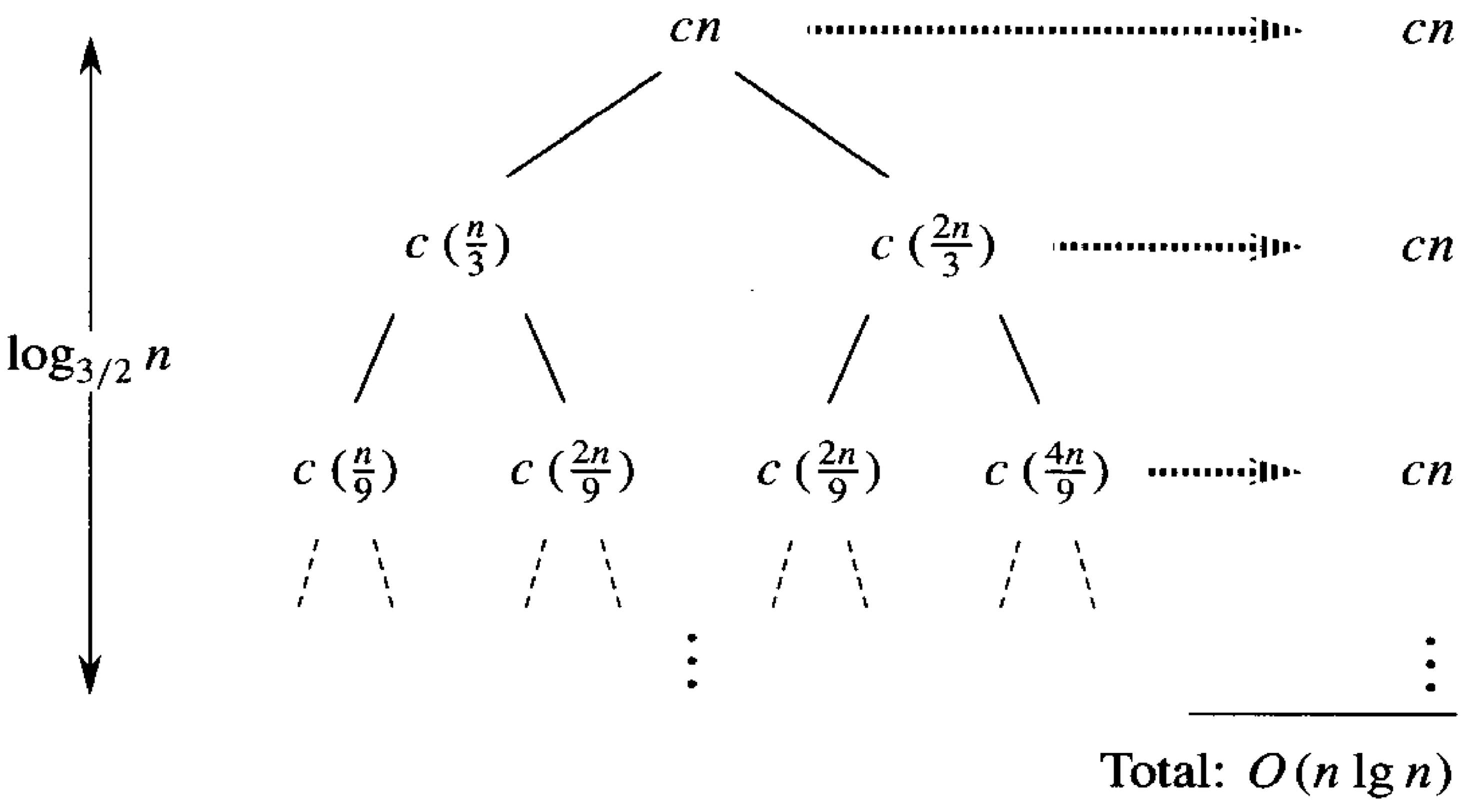
$$< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right)$$

$$= \frac{1}{1-(3/16)} cn^2 + \Theta\left(n^{\log_4 3}\right)$$

$$= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3})$$

$$= O(n^2)$$

$$T(n) = T(n/3) + T(2n/3) + cn$$



Solving recurrence

1. Recursion tree / iteration method
   - Good for guessing an answer
2. Substitution method
   - Generic method, rigid, but may be hard
3. Master method
   - Easy to learn, useful in limited cases only
   - Some tricks may help in other cases

The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

1. ***Divide*** the problem into a subproblems, **each** of size n/b
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions

   Divide + combine takes $f(n)$ time.

Master theorem

$$T(n) = a\,T(n/b) + f(n)$$

Key: compare $f(n)$ with $n^{\log_b a}$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $\underline{a\,f(n/b) \le c\,f(n)}$

Regularity Condition

$\Rightarrow T(n) = \Theta(f(n))$.

Case 1

$f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.

Alternatively: $n^{\log_b a} / f(n) = \Omega(n^{\varepsilon})$

Intuition: $f(n)$ grows polynomially slower than $n^{\log_b a}$

Or: $n^{\log_b a}$ dominates $f(n)$ by an n^{ε} factor for some $\varepsilon > 0$

Solution: $T(n) = \Theta(n^{\log_b a})$

$T(n) = 4T(n/2) + n$
$b = 2, a = 4, f(n) = n$
$\log_2 4 = 2$
$f(n) = n = O(n^{2-\varepsilon})$, or
$n^2 / n = n^1 = \Omega(n^{\varepsilon})$, for $\varepsilon = 1$
$\therefore T(n) = \Theta(n^2)$

$T(n) = 2T(n/2) + n/\log n$
$b = 2, a = 2, f(n) = n / \log n$
$\log_2 2 = 1$
$f(n) = n/\log n \notin O(n^{1-\varepsilon})$, or
$n^1 / f(n) = \log n \notin \Omega(n^{\varepsilon})$, for any $\varepsilon > 0$
$\therefore$ CASE 1 does not apply

Case 2

$f(n) = \Theta\ (n^{\log_b a})$.

Intuition: $f(n)$ and $n^{\log_b a}$ have the same asymptotic order.

Solution: $T(n) = \Theta(n^{\log_b a} \log n)$

e.g. $T(n) = T(n/2) + 1$ $\qquad \log_b a = 0$

$T(n) = 2\ T(n/2) + n$ $\qquad \log_b a = 1$

$T(n) = 4T(n/2) + n^2$ $\qquad \log_b a = 2$

$T(n) = 8T(n/2) + n^3$ $\qquad \log_b a = 3$

Case 3

$f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.

Alternatively: $f(n) / n^{\log_b a} = \Omega(n^{\varepsilon})$

Intuition: $f(n)$ grows polynomially faster than $n^{\log_b a}$

Or: $f(n)$ dominates $n^{\log_b a}$ by an n^{ε} factor for some $\varepsilon > 0$

Solution: $T(n) = \Theta(f(n))$

$T(n) = T(n/2) + n$

$b = 2,\ a = 1,\ f(n) = n$

$n^{\log_2 1} = n^0 = 1$

$f(n) = n = \Omega(n^{0+\varepsilon})$, or

$n / 1 = n = \Omega(n^{\varepsilon})$

$\therefore\ T(n) = \Theta(n)$

$T(n) = T(n/2) + \log n$

$b = 2,\ a = 1,\ f(n) = \log n$

$n^{\log_2 1} = n^0 = 1$

$f(n) = \log n \notin \Omega(n^{0+\varepsilon})$, or

$f(n) / n^{\log_2 1} / = \log n \notin \Omega(n^{\varepsilon})$

$\therefore$ *CASE 3 does not apply*

Regularity condition

- $a f(n/b) \leq c f(n)$ for some $c < 1$ and all sufficiently large n
- This is needed for the master method to be mathematically correct.
  - to deal with some non-converging functions such as sine or cosine functions
- For most *f(n)* you'll see (e.g., polynomial, logarithm, exponential), you can safely ignore this condition, because it is implied by the first condition $f(n) = \Omega(n^{\log_b a + \varepsilon})$

Examples

$T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$

$T(n) = 4T(n/2) + n^2$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

Case 2: $f(n) = \Theta(n^2).$

$\therefore\ T(n) = \Theta(n^2 \log n).$

Examples

$T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

Case 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.

$T(n) = 4T(n/2) + n^2/\lg n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.

Examples

$T(n) = 4T(n/2) + n^{2.5}$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^{2.5}.$

Case 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 0.5$

and $4(n/2)^{2.5} \le cn^{2.5}$ (reg. cond.) for $c = 0.75$.

$\therefore\ T(n) = \Theta(n^{2.5})$.

$T(n) = 4T(n/2) + n^2 \log n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2 \log n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\log n)$.

How do I know which case to use? Do I need to try all three cases one by one?

- Compare $f(n)$ with $n^{\log_b a}$

check if $n^{\log_b a} / f(n) \in \Omega(n^{\varepsilon})$

- $f(n) \in \begin{cases} o(n^{\log_b a}) & \text{Possible CASE 1} \\ \Theta(n^{\log_b a}) & \text{CASE 2} \\ \omega(n^{\log_b a}) & \text{Possible CASE 3} \end{cases}$

check if $f(n) / n^{\log_b a} \in \Omega(n^{\varepsilon})$

Examples

a. $T(n) = 4T(n/2) + n;$ $\log_b a$ = 2. n = $o(n^2)$ => Check case 1

b. $T(n) = 9T(n/3) + n^2;$ $\log_b a$ = 2. $n^2 = o(n^2)$ => case 2

c. $T(n) = 6T(n/4) + n;$ $\log_b a$ = 1.3. n = $o(n^{1.3})$ => Check case 1

d. $T(n) = 2T(n/4) + n;$ $\log_b a$ = 0.5. n = $\omega(n^{0.5})$ => Check case 3

e. $T(n) = T(n/2) + n \log n;$ $\log_b a$ = 0. nlogn = $\omega(n^0)$ => Check case 3

f. $T(n) = 4T(n/4) + n \log n.$ $\log_b a$ = 1. nlogn = $\omega(n)$ => Check case 3

More examples

$$T(n) = nT(n/2) + n$$

$$T(n) = 0.5T(n/2) + n\log n$$

$$T(n) = 3T(n/3) - n^2 + n$$

$$T(n) = T(n/2) + n(2 - \cos n)$$

Some tricks

- Changing variables

- Obtaining upper and lower bounds
  - Make a guess based on the bounds
  - Prove using the substitution method

Changing variables

$$T(n) = 2T(n-1) + 1$$

- Let $n = \log m$, i.e., $m = 2^n$

=> $T(\log m) = 2\, T(\log (m/2)) + 1$

- Let $S(m) = T(\log m) = T(n)$

=> $S(m) = 2S(m/2) + 1$

=> $S(m) = \Theta(m)$

=> $T(n) = S(m) = \Theta(m) = \Theta(2^n)$

Changing variables

$$T(n) = T(\sqrt{n}) + 1$$

- Let n $=2^m$

=> sqrt(n) = $2^{m/2}$

- We then have $T(2^m) = T(2^{m/2}) + 1$
- Let $T(n) = T(2^m) = S(m)$

=> $S(m) = S(m/2) + 1$

$\Rightarrow S(m) = \Theta(\log m) = \Theta(\log \log n)$

$\Rightarrow T(n) = \Theta(\log \log n)$

Changing variables

- $T(n) = 2T(n-2) + 1$
- Let $n = \log m$, i.e., $m = 2^n$

$\Rightarrow T(\log m) = 2\, T(\log m/4) + 1$

- Let $S(m) = T(\log m) = T(n)$

$\Rightarrow S(m) = 2S(m/4) + 1$

$\Rightarrow S(m) = m^{1/2}$

$\Rightarrow T(n) = S(m) = (2^n)^{1/2} = (\mathrm{sqrt}(2))^n \approx 1.4^n$

Obtaining bounds

Solve the Fibonacci sequence:

$$T(n) = T(n-1) + T(n-2) + 1$$

- $T(n) >= 2T(n-2) + 1$ [1]
- $T(n) <= 2T(n-1) + 1$ [2]

- Solving [1], we obtain $T(n) >= 1.4^n$
- Solving [2], we obtain $T(n) <= 2^n$
- Actually, $T(n) \approx 1.62^n$

Obtaining bounds

- $T(n) = T(n/2) + \log n$
- $T(n) \in \Omega(\log n)$
- $T(n) \in O(T(n/2) + n^{\varepsilon})$
- Solving $T(n) = T(n/2) + n^{\varepsilon}$,

  we obtain $T(n) = O(n^{\varepsilon})$, for any $\varepsilon > 0$
- So: $T(n) \in O(n^{\varepsilon})$ for any $\varepsilon > 0$
  - $T(n)$ is unlikely polynomial
  - Actually, $T(n) = \Theta(\log^2 n)$ by extended case 2

Extended Case 2

CASE 2: $f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$.

Extended CASE 2: (k >= 0)

$f(n) = \Theta(n^{\log_b a} \, log^k n) \Rightarrow T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.