MATH 321 2.7 - PROPERTIES OF INFINITE SERIES

- **Theme**: what features of finite addition hold when we're adding up infinitely many numbers?

1. Properties of Infinite Series

- Given an infinite series, make sure to keep a clear distinction between
  - the sequence of **terms** $(a_1, a_2, a_3, \dots)$ and
  - the sequence of **partial sums** $(s_1, s_2, s_3, \dots)$, where $s_n = a_1 + a_2 + \cdots + a_n$.
- The statement
$$\sum_{k=1}^{\infty} a_k = A \text{ means that } \lim(s_n) = A.$$
- Because of this, we can immediately translate many results from the study of sequences into statements about infinite series.

Theorem 1. ***(Algebraic Limit Theorem for Series)*** *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*
(1) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*
(2) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof. This is an exercise in translating between statements about series convergence and statements about the convergence of the sequence of partial sums.

(1) We want to show that the sequence of partial sums
$$t_m = ca_1 + ca_2 + ca_3 + \cdots + ca_m$$
converges to cA. But we are given that $\sum_{k=1}^{\infty} a_k$ converges to A, meaning that the partial sums
$$s_m = a_1 + a_2 + a_3 + \cdots + a_m$$
converge to A. Because $t_m = cs_m$, applying the Algebraic Limit Theorem for sequences yields $(t_m) \to cA$, as desired.

□

Exercise 2. **(Reading question)** Prove (b).

- One way to summarize the ALTS(i) is that infinite addition still satisfies the distributive property so long as the series converges.
- ALTS(ii) means that series can be added in the usual way so long as they converge.
- However, we don't yet know anything about the product of two infinite series.
  - This relies on when infinite addition is commutative and will either be postponed or omitted, depending on time constraints. If you're interested, I'm glad to discuss it further outside of class.

Theorem 3. ***(Cauchy Criterion for Series)*** *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$, it follows that*
$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Exercise 4. Prove it.

- Observe that
$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$
and apply the Cauchy Criterion for Sequences.

- The Cauchy Criterion leads to economical proofs of several basic facts about series.

Theorem 5. ***(Alternating Series Test)*** *Let (a_n) be a sequence satisfying*
(1) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \dots$ *and*

(2) $(a_n) \to 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof. Unclaimed HW $\square$

Exercise 6. True or false? Prove if true, counterexample if false. In what follows, (a_k) and (b_k) are sequences.

(1) If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.
   (a) this is the "Divergence Test" for series divergence: consider the special case $n = m+1$ in the Cauchy Criterion for Series.

(2) If $(a_k) \to 0$, then the series $\sum_{k=1}^{\infty} a_k$ converges.
   (a) False; $\sum \frac{1}{n}$

---

(3) If $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} b_k$ converges $\implies$ $\sum_{k=1}^{\infty} a_k$ converges.
   (a) **Note:** if we don't assume $0 \le a_k$, this is false! Consider $(a_k) = (-1, 0, -1, 0, \dots)$ and $(b_k) = (0)$.
   (b) True; follows from Cauchy Criterion and the observation that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \le |b_{m+1} + b_{m+2} + \cdots + b_n|.$$

(4) If $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} a_k$ diverges $\implies$ $\sum_{k=1}^{\infty} b_k$ diverges.
   (a) True; follows from Cauchy Criterion and the observation that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \le |b_{m+1} + b_{m+2} + \cdots + b_n|.$$

(5) If $r \in \mathbb{R}$, then

$$(1-r)(1 + r + r^2 + r^3 + \cdots + r^{m-1}) = 1 - r^m.$$

   (a) True; multiply.

(6) The partial sums s_m of the series $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$ satisfy

$$s_m = \frac{a(1-r^m)}{1-r}.$$

   (a) True; divide both sides of (5) by $1-r$ and multiply both sides by a.

(7) The series $\sum_{k=0}^{\infty} ar^k$ converges if and only if $|r| < 1$, in which case

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

   (a) True. From the above, if $|r| < 1$, the sequence of partial sums converges to

$$\lim_{m\to\infty} \left(\frac{a(1-r^m)}{1-r} \right) = \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$

   (b) If $r = 1$, then

$$\sum_{k=0}^{\infty} ar^k = \sum_{k=0}^{\infty} a \to \pm\infty$$

   depending on the sign of a.
   (c) If $r = -1$, then

$$\sum_{k=0}^{\infty} ar^k = \sum_{k=0}^{\infty} a(-1)^k = a - a + a - a + a - a + \dots$$

   diverges.
   (d) If $|r| > 1$, then

$$\sum_{k=0}^{\infty} ar^k = \lim_{m\to\infty} \frac{a(1-r^m)}{1-r} \to +\infty$$

   because if $r < -1$, then the numerator and denominator are both positive, and if $r > 1$, they're both negative.

(8) If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

(a) True. Because $\sum_{n=1}^{\infty} |a_n|$ converges, we know by the Cauchy Criterion that, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all $n > m \geq N$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|,$$

so the sufficiency of the Cauchy Criterion guarantees that $\sum_{n=1}^{\infty} a_n$ also converges.

(9) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ converges as well.

(a) False. Consider the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots.$$

Taking absolute value of the terms gives us the harmonic series $\sum_{n=1}^{\infty} 1/n$, which we have seen diverges. However, the alternating harmonic series converges because of the Alternating Series Test below.

Definition 7. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ **converges absolutely.** If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

- In terms of this new definition, we've shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally, whereas

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \sum_{n=1}^{\infty} \frac{1}{2^n}, \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

converge absolutely. In particular, any convergent series with all but finitely many positive terms must converge absolutely.

2. (if time) Rearrangements

Question 8. *How should we define a **rearrangement** of a series?*

- *A rearrangement of a series is obtained by permuting the terms in the sum into some other order.*
- *We have to eventually use all the terms in the original series, and no term can get repeated.*

Example 9. Earlier, we considered the rearrangement of the alternating harmonic series produced by taking two positive terms for every negative term:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots.$$

There are certainly an infinite number of rearrangements of any sum; however, neither

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

nor

$$1 + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{11} - \frac{1}{12} + \ldots$$

is considered a rearrangement of the original alternating harmonic series.

Definition 10. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a **rearrangement** of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

- We can now answer the question from the beginning of our discussion about series, about when rearrangements of convergent series converge to the same limit as the original series.

Theorem 11. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let's use

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

for the partial sums of the original series and use

$$t_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + \cdots + b_m$$

for the partial sums of the rearranged series. Thus we want to show that $(t_m) \to A$.

Let $\epsilon > 0$. By hypothesis, $(s_n) \to A$, so choose N_1 such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Because the convergence is absolute, we can choose N_2 so that

$$\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{\infty} |a_k| < \frac{\epsilon}{2}$$

for all $n > m \geq N_2$. Now, take $N = \max\{N_1, N_2\}$. We know that the finite set of terms $\{a_1, a_2, a_3, \ldots, a_N\}$ must all appear in the rearranged series, and we want to move far enough out in the series $\sum_{n=1}^{\infty} b_n$ so that we have included all of these terms. Thus, choose

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

It should now be evident that if $m \geq M$, then $(t_m - s_n)$ consists of a finite set of terms, the absolute values of which appear in the tail $\sum_{k=N+1}^{\infty} |a_k|$. Our choice of N_2 earlier then guarantees $|t_m - s_N| < \epsilon/2$, and so

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $m \geq M$. □

Exercise 12. How does this proof break down if $\sum a_k$ converges conditionally?

- In that case, we can't make $|t_m - s_n|$ as small as we'd like. In that case, all we know is that $(t_m - s_n)$ consists of a finite set of terms which lie in the tail *up to sign*.
- But now we care about the signs of those terms, and with their signs they might not lie in the tail and might thus be impossible to make arbitrarily small.

3. Epilogue

- The Theorem makes it clear that absolute convergence is an extremely valuable quality to have when manipulating series. The hypothesis of absolute convergence essentially allows us to treat infinite sums as though they were finite sums.
  - Experimenting with rearrangements of $\sum(-1/2)^n$ like
  $$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + \frac{1}{256} + \frac{1}{1024} - \frac{1}{32} \ldots$$
  always will yield a sum of $\frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$ because $\sum(-1/2)^n$ converges absolutely!
- On the other hand, conditionally convergent series are, in your book's words, "delightfully pathological"—that is, we can't be sure that rearrangements converge to the same limit or even converge at all!
  - Recall that if $S = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, then
  $$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots \text{''} = \text{''} S.$$
  We now know what went wrong in that argument: it's not that $S = 0$, but rather that the alternating harmonic series doesn't converge absolutely.