MATH 321 MIDTERM FALL 2018

This exam will be due on Friday, October 19, at 4 pm. You may use your textbook, your class notes, and your homework for MATH 321, Fall 2018. You may not use other books, notes, or the Internet for this exam. You may not discuss any part of this exam with any other person besides myself until the exam has been handed in.

Please sign below to signify that you have abided by the above rules:

Signature:____

(1) Negate the following: given $\epsilon > 0$ there exists $y \in \mathbb{N}$ such that, for all $x \in \mathbb{R}$ such that $xy \le \epsilon$, $x \in A$ or $y \in B$.

(2) For each of the following statements, say whether it is true or false. If the statement is true, prove it. If it's false, give a counterexample.
   (a) A decreasing sequence is always bounded.
   (b) If A and B are sets so that $P(A) \subseteq P(B)$, then $A \subseteq B$. (Here $P(S)$ denotes the power set of S for any set S.)
   (c) If A, B, C are sets, then $(A \setminus B) \cup C = (A \cup C) \setminus (B \cup C)$.
   (d) If A and B are sets, $A = B \iff A \setminus B = \emptyset$.
   (e) If (a_n) is a sequence, $(a_n) \to a$, and there exists $N \in \mathbb{N}$ such that $a_n \ge 0$ for all $n \ge N$, then $a \ge 0$. (In other words, part (a) of the Order Limit Theorem holds if (a_n) is "eventually nonnegative".)
   (f) The set of natural numbers $\mathbb{N}$ has the same cardinality as the set $P = \{p \in \mathbb{N} : p \text{ is prime}\}$; in other words, $\mathbb{N} \sim P$. [**Hint:** among other things, you have to prove that there are infinitely many prime numbers. Assume for contradiction that there are finitely many and hence that $P = \{p_1, p_2, \dots, p_n\}$. Now consider the natural number $p_1 p_2 \cdots p_n + 1$.]

(3) Find the supremum and infimum of each of the following sets. No proofs are necessary.
   (a) $A = \{x \in \mathbb{I} : x^2 < 2\}$
   (b) $B = \{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$ (note that 1 is not considered a prime number for our purposes)

(4) Prove that a function $f : A \to B$ is onto if and only if, for all $y \in B$, the preimage $f^{-1}(\{y\})$ contains at least one point.

(5) Using the definition of convergence of a sequence, prove that
$$\lim \left(\frac{n-1}{n+1}\right) = 1.$$

(6) **(Bonus: 15 points)**: Define a function $f : \mathbb{N} \to \mathbb{Q}$ by
$$f(n) = \begin{cases} 2^n & \text{if } n \text{ is prime} \\ \frac{1}{2^n} & \text{if } n \text{ is not prime.} \end{cases}$$
Then define a sequence (a_n) by $a_n = f(n)$ for all $n \in \mathbb{N}$.
   (a) **(10 points)** Is (a_n) bounded? Prove your answer.
   (b) **(5 points)** Does (a_n) converge? Prove your answer.