MATH 321 2.6 - THE CAUCHY CRITERION

- **Last time**: we proved the Bolzano-Weierstrass Theorem (every bounded sequence has a convergent subsequence) by first conjecturing a candidate for the limit of such a sequence using the NIP, then proving the sequence converged to that limit.
- **Question:** how do we prove a sequence converges when we don't have any guesses about what it converges to?
  - What if we prove the terms of the sequence get really close together eventually?

Definition 1. A sequence (a_n) is a **Cauchy sequence** if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, whenever $m, n \geq N$, it follows that $|a_n - a_m| < \epsilon$.

- Think: the terms are getting Cauchy if the terms are getting cozy (with each other)
- Compare to the definition of convergence:

Definition 2. A sequence (a_n) **converges** to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that $|a_n - a| < \epsilon$.

- A sequence converges if, whenever we go far enough out in the sequence, the terms get really close to *some number a*.
- A sequence is Cauchy if, whenever we go far enough out in the sequence, the terms get really close to *each other.*
- We'll argue today that these definitions are equivalent: Convergent sequences are Cauchy, and Cauchy sequences converge.
  - If the terms of a sequence eventually cluster really close together, there must be a limit in the cluster somewhere.

Theorem 3. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (x_n) converges to x. To prove that (x_n) is Cauchy, we must find a point in the sequence after which we have $|x_n - x_m| < \epsilon$. We can finish the proof using the Triangle Inequality. □

Exercise 4. (Reading Question) Complete the proof of Theorem 2.6.2. (Try mimicking a convergence proof but replacing step 4 with "assume $m, n \geq N$".)

Proof.

(1) Assume (x_n) converges to x and let $\epsilon > 0$ be arbitrary.
(2) We do some scratch work:

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq^{\Delta} |x_n - x| + |x_m - x| \\ &< \epsilon/2 + \epsilon/2. \end{aligned}$$

(3) So we need to choose $N \in \mathbb{N}$ so that, whenever $n \geq N$, $|x_n - x| < \epsilon/2$. This can be done since we assumed that $(x_n) \to x$.
(4) Assume $m, n \geq N$.
(5) Then

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq^{\Delta} |x_n - x| + |x_m - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

as desired.

□

- The converse is a bit difficult to prove mainly because, in order to prove that a sequence converges, we must have a proposed limit for the sequence in question.

Question 5. *How could we find a potential limit for a given Cauchy sequence?*

- Our strategy here is to use the Bolzano-Weierstrass Theorem to get a subsequence having a limit, then show the whole sequence converges to that limit.
- What assumptions do we need in order to use B-W?
- First, we have to show:

Lemma 6. *Cauchy sequences are bounded.*

Proof. [The idea is that the terms of the sequence get really close together eventually, so they can't go to $\pm\infty$.] Choose $\epsilon = 1$ in the definition of Cauchy: there exists N so that $|x_m - x_n| < 1$ whenever $m, n \geq N$. Thus, we must have (take $m = N$)

$$|x_N - x_n| < 1$$
$$|x_n| < |x_N| + 1$$

for all $n \geq N$. (The distance from x_N to x_n is less than 1, so the distance from x_n to zero is at most the distance from x_N to 0 plus 1.) It follows that

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence (x_n). □

Theorem 7. ***(Cauchy Criterion)*** *A sequence converges if and only if it is a Cauchy sequence.*

Proof. ($\Longrightarrow$): reading question

($\Longleftarrow$): Let (x_n) be a Cauchy sequence. Then the lemma shows (x_n) is bounded, so by Bolzano-Weierstrass it must have a convergent subsequence (x_{n_k}). Set

$$x = \lim x_{n_k}.$$

The idea is to show that $(x_n) \to x$ as well. Once again, we use a triangle inequality argument.

[The idea is that subsequences are infinite, so no matter how far out in (x_n) we go, we can't be too far from a term of (x_{n_k}). By Cauchy-ness, these terms must be close together, and by convergence of the subsequence, the terms of (x_{n_k}) must be close to x.]

Since the terms of the subsequence get close to x, and the terms in the tail of the sequence are close to each other, we can make each of these differences less than $\epsilon/2$.

Let $\epsilon > 0$. Becausre (x_n) is Cauchy, there exists N so that

$$|x_n - x_m| < \frac{\epsilon}{2}$$

whenever $m, n \geq N$. Now, since $(x_{n_k}) \to x$, we can choose a term in the subsequence, call it x_{n_K}, with $n_K \geq N$ and

$$|x_{n_K} - x| < \frac{\epsilon}{2}.$$

To see that N has the desired property for (x_n), observe that if $n \geq N$,

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_K} + x_{n_K} - x| \\ &\leq |x_n - x_{n_K}| + |x_{n_K} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Note 8. Suppose we assume the Archimedean Property is true (take it as an axiom). Then the following axioms are equivalent to the Axiom of Completeness and could be used in the place of the Axiom of Completeness as the defining property of $\mathbb{R}$. Then we could prove the AoC, as well as the other theorems in the list, as a consequence of our new axiom.

(1) Monotone Convergence Theorem (Exercise 2.4.4)
(2) Nested Interval Property (Exercise 2.5.4)

(3) Bolzano-Weierstrass Theorem (Exercise 2.6.7)
(4) Cauchy Criterion (Exercise 2.6.7)