MATH 321 WEEK 9 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.5.1

Exercise 2.5.1. (a) Impossible. Theorem 2.5.5 guarantees that all bounded sequences have convergent subsequences. If a subsequence is bounded, then that subsequence has a convergent subsequence which would necessarily be a convergent subsequence of the original sequence.

(b) $(1/2, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 4/5, \cdots, 1/n, (n-1)/n, \cdots)$

(c) The sequence

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 1, \cdots\right)$$

has this property. Notice that there is also a subsequence converging to 0. We shall see that this is unavoidable.

(d) Impossible. This is a slightly technical argument but the idea is straightforward. If there are subsequences converging to every point in the set

$$\{1, 1/2, 1/3, 1/4, \ldots\},$$

then there necessarily has to be a subsequence that converges to 0 as well. To construct it, pick $a_{n_1} \in V_1(1)$. Then choose $n_2 > n_1$ such that $a_{n_2} \in V_{\frac{1}{2}}(1/2)$. The reason such an a_{n_2} must exist is because we know there is a subsequence converging to $1/2$. Continue this process. Having selected $a_{n_k} \in V_{\frac{1}{k}}(1/k)$, select $n_{k+1} > n_k$ so that $a_{n_{k+1}} \in V_{\frac{1}{k+1}}(1/(k+1))$. The resulting subsequence satisfies $0 < a_{n_k} < 2/k$, which is enough to conclude that $(a_{n_k}) \to 0$.

(2) 2.5.2

(a) True. If every proper subsequence of (x_n) converges, then, in particular, the subsequence (x_{n+1}) given by removing the first term of (x_n) converges to some limit L. By an exam problem, this means that (x_n) must converge to L as well.

(b) True. If (x_n) converged, then every subsequence of (x_n) must also converge to the same limit. Therefore, if some subsequence (x_{n_k}) diverges, then (x_n) must diverge as well.

(c) True. Since (x_n) is bounded, it must contain a convergent subsequence $(x_{n_k}) \to L$ by Bolzano-Weierstrass. Since (x_n) does not converge to L, by definition, there must exist $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n \geq N$ with $|x_n - L| \geq \epsilon$. Define a new subsequence of (x_n), call it (y_N), by letting y_N be the term x_n of (x_n) with $n \geq N$ and $|x_n - L| \geq \epsilon$. The sequence (y_N) is bounded and therefore has a convergent subsequence (y_{N_k}). But it must be that (y_{N_k}) does not converge to L, since for all $k \in \mathbb{N}$, $|y_{N_k} - L| \geq \epsilon$.

(d) False. Consider the sequence (s_n) of partial sums of $\sum \frac{1}{k}$. It's certainly monotone increasing. We know that $\sum \frac{1}{k}$ diverges, thus (s_n) diverges as well. But the subsequence $(s_{n^2}) = (1 + \frac{1}{4} +$

$\frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2}$) must converge because $\sum \frac{1}{k^2}$ converges. So (s_n) is a monotone sequence with a convergent subsequence that does not converge.

(3) 2.5.4

Proof. Suppose the NIP is true, and let S be a nonempty set of real numbers that is bounded above. We want to show that S has a least upper bound.

Since S is bounded above, there is some number M_1 so that $s \leq M_1$ for all $s \in S$. Let $s_1 \in S$ be an arbitrary element of S and consider the interval $I_1 = [s_1, M_1]$. Bisect the interval $I_1 = [s, M]$ and choose a closed half-interval $I_2 \subseteq I_1$ by letting I_2 be the right half-interval $[\frac{s_1+M_1}{2}, M_1]$ if there is an element of $S \in [\frac{s+M}{2}, M_1]$ and letting I_2 be the left half-interval otherwise. (If neither is possible, then S is empty, a contradiction.) Repeat this process, at each stage choosing the closed half-interval $I_k \subseteq I_{k-1}$ to be the right half-interval $[\frac{s_{k-1}+M_{k-1}}{2}, M_{k-1}]$ if the right half-interval contains an element of S and the left half-interval $[s_{k-1}, \frac{s_{k-1}+M_{k-1}}{2}]$ otherwise. At each stage, one of the halves must contain an element of S, otherwise there were no elements of S at the previous stage. Then

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

is a nested sequence of closed, bounded intervals, so there exists an element $x \in \bigcap_{k=1}^{\infty} I_k$. We claim that $x = \sup S$.

To prove this, note that the length of I_k is half the length of I_{k-1} for all k, and the length of I_1 was $M - s$, so the length of I_k, call it $|I_k|$, is

$$|I_k| = \frac{M-s}{2^{k-1}}.$$

Since we are assuming that $(1/2^n) \to 0$, the Algebraic Limit Theorem tells us that $|I_k| = (M - s)(\frac{1}{2^{k-1}}) \to 0$ as well. Thus, $\bigcap_{k=1}^{\infty} I_k = \{x\}$.

Now, suppose for contradiction that there exists $s \in S$ with $x < s$. Then it must be that s is an upper bound for I_k for some $k \in \mathbb{N}$, hence $s > M_k$, which is impossible because we chose M_k to be an upper bound for S. Therefore, x is an upper bound for S.

Let α be another upper bound for S. Suppose for contradiction that $\alpha < x$. Then there exists k so that α is a lower bound for I_k, hence $\alpha < s_k$ for some k, which is impossible since α is an upper bound for S. Therefore, x is the least upper bound of S, and the Axiom of Completeness is proven. $\square$