MATH 321 PRACTICE FINAL EXAM SOLUTIONS

KENAN INCE

(1) Let
$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \dots\right\} \text{ and } B = \{x \in \mathbb{Q} : 0 < x < 1\}.$$
Answer the following questions for each set:
   (a) What are the limit points?
      (i) for A: ± 1
      (ii) for B: 0 and 1
   (b) Is the set open? Closed?
      (i) A is not open because the point $1 = (-1)^1 + \frac{2}{1} \in A$ does not have an ϵ-neighborhood contained entirely within A. A is not closed because the limit point -1 is not contained in the set.
      (ii) B is open because every $x \in B$ has an ϵ-neighborhood of radius $\min\{1-x, x\}$ contained in the set. B is not closed because it does not contain either of its limit points.
   (c) Does the set contain any isolated points?
      (i) A contains isolated points; for example, consider $2 = (-1)^2 + \frac{2}{2} \in A$. The $\frac{1}{100}$-neighborhood about 2, for instance, contains no other points of A.
      (ii) B contains no isolated points. To see this, let $x \in B$ and let $\epsilon > 0$ be arbitrary. By the Density of $\mathbb{Q}$ in $\mathbb{R}$, $V_\epsilon(x)$ contains some point of $\mathbb{Q}$, hence x is a limit point of B.
   (d) Find the closure of the set.
      (i) The limit points of A are -1 and 1, and $1 = (-1)^1 + \frac{2}{1} \in A$. However, $-1 \notin A$. For assume that it is; then
$$-1 = (-1)^n + \frac{2}{n} \text{ for some } n \in \mathbb{N}.$$
Subtracting $(-1)^n$ from both sides yields
$$\begin{aligned} -1 - (-1)^n &= \frac{2}{n} \\ \implies \frac{2}{n} &= -2 \text{ or } 0 \end{aligned}$$
which is not true for any $n \in \mathbb{N}$. Therefore, $\overline{A} = A \cup \{-1\}$.
      (ii) The limit points of B, 0 and 1, are not contained in the set. Therefore, $\bar{B} = B \cup \{0, 1\}$.

(2) Let (a_n) be an increasing sequence of positive numbers. Prove, using the definition of convergence, that if (a_n) converges, so does $(7a_n^2)$.
   (a) Let $\epsilon > 0$ be arbitrary.
   (b) [scratch work] We guess that, if $(a_n) \to a$, then $(7a_n^2) \to 7a^2$. We want to make $|7a_n^2 - 7a^2| < \epsilon$. But
$$|7a_n^2 - 7a^2| = 7|a_n^2 - a^2| = 7|a_n - a||a_n + a|.$$
We can make $|a_n - a|$ as small as we want, so let's focus on $|a_n + a|$. We may insist that $|a - a_n| < 1$, so that $-1 < a_n - a < 1$, hence $a - 1 < a_n < a + 1$. This means that
$$2a - 1 < a_n + a < 2a + 1$$
and hence $|a_n + a| < M = \max\{|2a-1|, |2a+1|\}$.
   (c) Choose $N \in \mathbb{N}$ so that $|a_n - a| < \frac{\epsilon}{7M}$ whenever $n \geq N$.
   (d) Assume $n \geq N$.

(e) Then
$$\begin{aligned}|7a_n^2 - 7a^2| = 7|a_n^2 - a^2| = 7|a_n - a||a_n + a| \\ < 7\Big(\frac{\epsilon}{7M}\Big)M \\ = \epsilon.\end{aligned}$$

(3) Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$
Both. Since $K \cap F$ is the intersection of two closed sets, it's closed. Also, since K is bounded and $K \cap F \subseteq K$, $K \cap F$ is bounded as well. Thus, $K \cap F$ is compact.

(b) $\overline{F^c \cup K^c}$
(i) Definitely closed since it's a closure.
(ii) Not necessarily compact. For example, let $F = [0,1] = K$. Then $F^c = K^c = (-\infty, 0) \cup (1, \infty)$, and
$$\overline{F^c \cup K^c} = \overline{(-\infty, 0) \cup (1, \infty)} = (-\infty, 0] \cup [1, \infty)$$
which is not bounded, hence not compact.

(c) $K \setminus F$
Neither. Let $K = [0,1]$ and $F = \{0, 1\}$. Then $K \setminus F = (0,1)$, which is neither closed nor compact.

(d) $\overline{K \cap F^c}$
Both. It's definitely closed since it's a closure. It's definitely compact since $\overline{K \cap F^c} \subseteq \overline{K} = K$, which is bounded by virtue of being compact.

(4) Which of the following statements about 1-to-1 and onto functions are true? Prove if true, and give a counterexample if false.

(a) If f and g are onto, then $g \circ f$ is onto.
True; assume $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$. Let $c \in C$. Then, since g is onto, there exists $b \in B$ with $g(b) = c$. Moreover, since f is onto, there exists $a \in A$ so that $f(a) = b$. Therefore, $g(f(a)) = g(b) = c$.

(b) If f is onto and g is 1-to-1, then $g \circ f$ is onto.
False. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 0$ for all $x \in \mathbb{R}$ and let f be the identity function $f(x) = x$. Then $g(f(x)) = g(x) = 0$ for all $x \in \mathbb{R}$, hence $g \circ f$ is not onto.

(c) If f is onto and g is 1-to-1, then $g \circ f$ is 1-to-1.
False. Consider $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$ and $g(x) = x$. Then
$$(g \circ f)(-1) = g(1) = 1 = g(1) = (g \circ f)(1).$$

(d) If f and g are 1-to-1, then $g \circ f$ is 1-to-1.
True. Suppose that $g(f(x_1)) = g(f(x_2))$. Then, since g is 1-to-1, it must be that $f(x_1) = f(x_2)$. And since f is 1-to-1, it must be that $x_1 = x_2$.

(e) If f and g are 1-to-1 correspondences, then $g \circ f$ is a 1-to-1 correspondence.
True; this follows from (a) and (d) above.

(5) **(Bonus)** Let (a_n) be a bounded sequence and let $S = \{\lim(a_{n_k}) : (a_{n_k}) \text{ is any subsequence of } (a_n)\}$. Prove that S is closed.

- You are welcome to ignore this problem; I've made it too difficult. If you're interested in the proof, consider the explanation offered here, replacing each $d(x, y)$ (meaning the distance between x and y) with $|x - y|$.