MATH 321 WEEK 11 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.7.5

Exercise 2.7.5. By the Cauchy Condensation Test (Theorem 2.4.6) $\sum 1/n^p$ converges if and only if $\sum 2^n (1/2^n)^p$ converges. But notice that

$$\sum 2^n \left(\frac{1}{2^n}\right)^p = \sum \left(\frac{1}{2^n}\right)^{p-1} = \sum \left(\frac{1}{2^{p-1}}\right)^n.$$

By the Geometric Series Test (Example 2.7.5), this series converges if and only if $|\frac{1}{2^{p-1}}| < 1$. Solving for p we find that p must satisfy $p > 1$.

(2) 2.7.6

(a) False. Consider $a_n = \frac{1}{n}$ for all n. In this case, $|a_n| \leq 1$ for all n, but $\sum a_n$ is an unbounded, divergent series, hence its sequence of partial sums (s_n) is unbounded, monotone increasing, and divergent. Therefore, for instance by Exercise 2.6.2(c), any subsequence of (s_n) must also diverge.

(b) True. If $\sum a_n$ converges, then its sequence of partial sums (s_n) converges. But (s_n) is a subsequence of itself, thus $\sum a_n$ subverges as well.

(c) True. Suppose that $\sum |a_n|$ subverges; then there's a convergent subsequence of the sequence (s_n) of partial sums. That convegent subsequence represents the partial sums of some other convergent series $\sum |b_n|$, where all terms are nonnegative. Since $\sum |b_n|$ converges, we must have that $\sum b_n$ converges as well. But the sequence of partial sums of $\sum b_n$ is a subsequence of the sequence of partial sums of $\sum a_n$, so that $\sum a_n$ subverges.

(d) False. Consider the sequence

$$(a_n) = (0, 0, 1, -1, 2, -2, 3, -3, 4, -4, \dots, n, -n, \dots).$$

Let (s_n) denote the sequence of partial sums of $\sum a_n$. Then

$$(s_{2n}) = (0 + 0, 1 - 1, 2 - 2, 3 - 3, \dots) = (0, 0, 0, \dots)$$

converges to 0, so $\sum a_n$ subverges. However, any subsequence of (a_n) must be unbounded, and hence no subsequence of (a_n) converges.

(3) 2.7.8

(a) True. Suppose that $\sum |a_n|$ converges. This means that the sequence of partial sums (s_n) for this series must converge to some number A. By the Algebraic Limit Theorem (iii), we must then have that $(s_n^2) \to A^2$. However, note that for all $n \in \mathbb{N}$,

$$s_n^2 = \Big(|a_1| + |a_2| + \cdots + |a_n|\Big)^2 \geq \Big(|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2\Big)$$

(FOIL the left-hand side and note that all OI terms are positive). But the right-hand side is the nth partial sum (t_n) of $\sum |a_n|^2$. Since (s_n^2) is monotone increasing, it must be that A^2 is an upper bound for (s_n^2), hence for (t_n). Moreover, (t_n) is also monotone increasing. Thus, by the Monotone Convergence Theorem, (t_n) must converge, and hence so must $\sum |a_n|^2$.

(b) False. Consider $a_n = \frac{(-1)^n}{n}$ and $b_n = \frac{(-1)^n}{\log(n)}$ for $n \geq 2$ (otherwise $b_1 = \frac{-1}{\log(1)} = -\frac{1}{0}$). Then we know $\sum a_n$ and (b_n) both converge, but

$$\sum_{n=2}^{\infty} a_n b_n = \sum_{n=2}^{\infty} \frac{1}{n\log(n)}.$$

We want to show $\sum a_n b_n$ diverges. To do so, we use the Cauchy Condensation Test:

$$\sum_{n=2}^{\infty} \frac{1}{n\log(n)} \text{ converges } \iff \sum_{n=2}^{\infty} \frac{2^n}{2^n \log(2^n)} \text{ converges.}$$

But

$$\sum \frac{2^n}{2^n\log(2^n)} = \sum \frac{1}{\log(2^n)} = \sum \frac{1}{n\log(2)} = \frac{1}{\log(2)} \sum \frac{1}{n} \text{ diverges,}$$

hence so does $\sum a_n b_n$.

(c) True. We prove the contrapositive: if $\sum n^2 a_n$ converges, then so does $\sum |a_n|$. Assume that $\sum n^2 a_n$ converges. Then, by the Divergence Test, $n^2 a_n \to 0$, so there exists $N \in \mathbb{N}$ so that, whenever $n \geq N$, $|n^2 a_n| < 1$. This implies that $|a_n| < \frac{1}{n^2}$ for all $n \geq N$. But by the Comparison Test, since $\sum \frac{1}{n^2}$ converges, it must be that $\sum |a_n|$ also converges.