MATH 321 2.3 - THE ALGEBRAIC AND ORDER LIMIT THEOREMS

1. Reading Question

Exercise 1. [Reading question] Using only the definition of convergence of a sequence, prove that if $(x_n) \to 2$, then

(1) $(\frac{2x_n-1}{3}) \to 1$
  (a) "Let $\epsilon > 0$ be arbitrary."
  (b) (scratch work) starting from the expression we want to show, rearrange until we get back to our assumption.
    (i) We want to show $|\frac{2x_n-1}{3} - 1| < \epsilon$.
    (ii) We are given that $(x_n) \to 2$, so we know we can make $|x_n - 2|$ as small as we like.
    (iii) Rearranging what we want to show:

$$\begin{aligned}\left|\frac{2x_n-1}{3} - 1\right| &= \left|\frac{2x_n-4}{3}\right| \\ &= \frac{|2x_n-4|}{3} < \epsilon \\ |2x_n - 4| &< 3\epsilon \\ 2|x_n - 2| &< 3\epsilon \\ |x_n - 2| &< \frac{3\epsilon}{2}\end{aligned}$$

  (c) Choose N. Since $(x_n) \to 2$, we can find an $N \in \mathbb{N}$ such that, whenever $n \geq N$, $|x_n - 2| < 3\epsilon/2$.
  (d) "Assume $n \geq N$."
  (e) Perform the scratch work backwards: if $n \geq N$, then

$$\begin{aligned}\left|\frac{2x_n-1}{3} - 1\right| &= \left|\frac{2x_n-4}{3}\right| \\ &= \frac{|2x_n-4|}{3} \\ &= \frac{2|x_n-2|}{3} < \frac{2}{3}\left(\frac{3\epsilon}{2}\right) = \epsilon.\end{aligned}$$

  - Note that the strategy here, as it will be in the future, is to bound the quantity we want to be less than ϵ with some algebraic combination of quantities over which we have control.

(2) $(1/x_n) \to 1/2$.
  (a) Let $\epsilon > 0$ be arbitrary.
  (b) [scratch work]

$$\begin{aligned}\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2-x_n}{2x_n}\right| &= \frac{|x_n-2|}{2|x_n|} < \epsilon \\ |x_n - 2| &< 2|x_n|\epsilon.\end{aligned}$$

  But $|x_n|$ changes with n, so we can't choose N so that $\forall n \geq N$, $|x_n - 2| < 2|x_n|\epsilon$. We need to get a lower bound L on $|x_n|$ first, then choose N so that, for all $n \geq N$,

$$|x_n - 2| < 2L\epsilon \leq 2|x_n|\epsilon.$$

  (c) Note that, since $(x_n) \to 2$ by assumption, there exists $N_1 \in \mathbb{N}$ so that, for all $n \geq N_1$, $|x_n| > 1$. Moreover, there exists $N_2 \in \mathbb{N}$ so that, for all $n \geq N_2$,

$$|x_n - 2| < 2\epsilon.$$

  Choose $N = \max\{N_1, N_2\}$.
  (d) Suppose that $n \geq N$.

(e) Then

$$\begin{aligned}\left|\frac{1}{x_n} - \frac{1}{2}\right| &= \frac{|x_n - 2|}{2|x_n|} \\ &< \frac{|x_n - 2|}{2} \\ &< \frac{2\epsilon}{2} = \epsilon\end{aligned}$$

as desired.

- In calculus, you used the fact that the limit of a sequence behaves nicely with respect to addition, subtraction, multiplication, and division. Why does it?
  - This "nice behavior" is called by your book the Algebraic Limit Theorem.
  - Today we'll justify this "nice behavior" mathematically.
  - I promised you'd get to spam the triangle inequality and I haven't delivered so far. Today is your day to use it to prove something!

2. Goals

(1) Prove that the limiting process is well-behaved with respect to algebraic operations such as $+, -, \times, \div$:

Theorem 2 (**Algebraic Limit Theorem**). *Let* $\lim a_n = a$ *and* $\lim b_n = b$. *Then*
(a) $\lim(ca_n) = ca$ *for all* $c \in \mathbb{R}$*;*
(b) $\lim(a_n + b_n) = a + b$*;*
(c) $\lim(a_n b_n) = ab$*;*
(d) $\lim(a_n/b_n) = a/b$*, provided* $b \neq 0$*.*

(2) Prove that the limiting process is also well-behaved with respect to the "order operation", in other words with respect to $\leq$ and $\geq$:

Theorem 3 (**Order Limit Theorem**). *Let* $\lim a_n = a$ *and* $\lim b_n = b$.
(a) *If* $a_n \geq 0$ *for all* $n \in \mathbb{N}$*, then* $a \geq 0$.
(b) *If* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $a \leq b$.
(c) *If there exists* $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $c \leq b$*. Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$*, then* $a \leq c$.

3. The Algebraic Limit Theorem

- In general, the real purpose for creating a rigorous definition of convergence is to *confidently prove statements about convergent sequences in general.*
- Before we get to proving the Algebraic Limit Theorem, a statement about sums/products of convergent sequences, we need one tool:

Definition 4. A sequence (x_n) is **bounded** if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

- Geometrically, this means we can find an interval $[-M, M]$ so that every $x_n \in [-M, M]$.

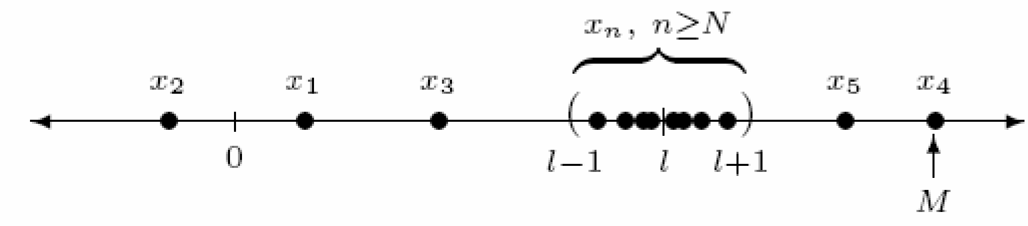
Theorem 5. *Every convergent sequence is bounded.*

Proof. Assume $(x_n) \to L$. We'll show that the sequence must be bounded because it's eventually within a small distance of its limit.

(1) This means that for any ϵ, say $\epsilon = 1$, we know there exists $n \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - l| < 1$.
(2) This means that, for all $n \geq N$, $x_n \in (l-1, l+1)$.
(3) Not knowing whether l is positive or negative, we can certainly conclude that

$$|x_n| < |l| + 1$$

for all $n \geq N$.



(4) We still need to worry about the terms in the sequence that come before the Nth term. Because there are only finitely many of these, we can let

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l| + 1\}.$$

It follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$, as desired.

□

- Now we can try to tackle the Algebraic Limit Theorem.

Theorem 6. ***(Algebraic Limit Theorem)*** *Let* $\lim a_n = a$ *and* $\lim b_n = b$. *Then*

(1) $\lim(ca_n) = ca$ *for all* $c \in \mathbb{R}$;
(2) $\lim(a_n + b_n) = a + b$;
(3) $\lim(a_n b_n) = ab$;
(4) $\lim(a_n/b_n) = a/b$, *provided* $b \neq 0$.

Proof. (i) Let's consider your proof that, if $(x_n) \to 2$, then $(\frac{2x_n - 1}{3}) \to 1$. In that proof, you used the fact that $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ $|a_n - 2| < \frac{3\epsilon}{2}$. This is because you wanted to be able to "cancel" the constant $\frac{2}{3}$ in the expression

$$\Big|\frac{2}{3}(a_n - 2)\Big| = \frac{2}{3}|a_n - 2| < \frac{2}{3}\Big(\frac{3\epsilon}{2}\Big) = \epsilon.$$

This proof will go similarly. Consider the case where $c \neq 0$. We follow the proof template from last time:

(1) "Let $\epsilon > 0$ be arbitrary."
(2) (scratch work) starting from the expression we want to show, rearrange until we get back to our assumption.
   (a) We want to show $|ca_n - ca| < \epsilon$. Factoring out a c, we get $|c||a_n - a| < \epsilon$, or $|a_n - a| < \frac{\epsilon}{|c|}$.
   (b) We are given that $(a_n) \to a$, so we know we can make $|a_n - a|$ as small as we like. That means we can make $|a_n - a| < \frac{\epsilon}{|c|}$ by taking terms further out than some $N \in \mathbb{N}$.
(3) Choose N. Since $(a_n) \to a$, we can find an $N \in \mathbb{N}$ such that, whenever $n \geq N$, $|a_n - a| < \frac{\epsilon}{|c|}$.
(4) "Assume $n \geq N$."
(5) Perform the scratch work backwards: if $n \geq N$, then

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon.$$

Note that the strategy here, as it will be in the future, is to bound the quantity we want to be less than ϵ with some algebraic combination of quantities over which we have control. □

Exercise 7. [ALT workshop; split students into three groups and have each present one part, with you providing hints] Prove (ii)-(iv) by following the four steps below:

Complete the proof of (ii) by doing the following. These four steps are useful in general for proving convergence.

(1) Write our assumptions (what quantities do we know we can make arbitrarily small?) and our goal (what quantity do we want to show we can make arbitrarily small?).
   - We are given that we can make $|a_n - a|$ and $|b_n - b|$ arbitrarily small by taking $n \geq N_1$ for some choice of N_1 (in the case of $|a_n - a|$) or by taking $n \geq N_2$ for some choice of N_2 (in the case of $|b_n - b|$).
   - We need to argue that

$$|(a_n + b_n) - (a + b)|$$

can be made less than ϵ using the assumptions that $|a_n - a|$ and $|b_n - b|$ can be made arbitrarily small.

(2) Rewrite the quantity you want to make arbitrarily small in terms of the quantities you know you can make arbitrarily small.

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)|$$

(3) Find a bound for the quantity you want to make arbitrarily small in terms of the quantities you know can be made arbitrarily small. [Hint: apply the triangle inequality.]

$$|(a_n - a) - (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

(4) Choose N in which a way that, whenever $n \geq N$, the right side of your inequality is less than ϵ.
- We know we want to make $|a_n - a| + |b_n - b| < \epsilon$. Perhaps we could make each term less than $\epsilon/2$!
- There exists $N_1 \in \mathbb{N}$ s.t. whenever $n \geq N_1, |a_n - a| < \epsilon/2$. Same for b_n and N_2. What should we make N?
- Certainly if N is bigger than **both** N_1 and N_2, then both $|a_n - a|$ and $|b_n - b|$ will be less than $\epsilon/2$. So take $N = \max\{N_1, N_2\}$.

(5) Write your proof from the initial "let $\epsilon > 0$ be arbitrary", using all of the previous parts.

Proof. (iii) To show that $(a_n b_n) \to ab$, we need to rewrite $a_n b_n - ab$ in such a way that we can use our assumptions that $a_n \to a$ and $b_n \to b$. We begin by observing that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b|. \end{aligned}$$

Essentially, we have broken up the distance from $a_n b_n$ to ab with a midway point ab_n and are using the sum of the two distances to overestimate the original distance. This clever trick will be spammed over and over again. □

Exercise 8. Complete the proof of (iii) by doing the following:

(1) Argue that we can make $|a||b_n - b| < \frac{\epsilon}{2}$ by choosing n large enough.
(2) Using the fact that convergent sequences are bounded, we know that $|b_n| \leq M$ for some M. Thus, argue that we can make $|b_n||a_n - a| < \frac{\epsilon}{2}$ by choosing n large enough.
(3) Choose N so that **both** terms on the right are less than $\epsilon/2$.
(4) Write a complete proof of (iii) by combining the previous parts.

Proof. (iv) This final statement will follow from (iii) if we can prove that

$$(b_n) \to b \implies \left(\frac{1}{b_n}\right) \to \frac{1}{b}$$

whenever $b \neq 0$, because then

$$\left(\frac{a_n}{b_n}\right) = (a_n)\left(\frac{1}{b_n}\right) \to a\left(\frac{1}{b}\right) = \frac{a}{b}.$$

This is the goal of the next exercise. □

Exercise 9. Begin the proof of (iv) by doing the following:

(1) Write the statement we want to show $((\frac{1}{b_n}) \to \frac{1}{b})$ in terms of the definition of convergence.

We want to show that, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\frac{1}{b_n} - \frac{1}{b}| < \epsilon$ whenever $n \geq N$.

(2) Algebraically manipulate the quantity we want to bound to write it in terms of quantities we know we can make small, like $|b - b_n|$ and $|b_n|$.

We get a common denominator on the left side of the inequality we want to show:

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{bb_n}\right| = \frac{|b - b_n|}{|b||b_n|} < \epsilon$$

(3) Argue that we can make the numerator of the resulting quantity as small as we like by choosing n large.

Let $\epsilon > 0$ be arbitrary. Because $b_n \to b$, we know that there exists N_1 such that, whenever $n \geq N_1$, $|b - b_n| = |b_n - b| < \epsilon$.

Proof. (continued) What's left is the denominator $|b||b_n|$. Because the b_n terms are in the denominator, we no longer want an upper bound but rather a *lower bound* on $|b_n|$ so we know that making the numerator small will make the whole fraction small. This will lead to a bound on the size of $1/(|b||b_n|)$.

It'd be nice to lower-bound $|b_n|$ by some expression in terms of b. The trick is to look far enough out in the sequence (b_n) so that the terms are closer to b than they are to 0.

Consider the particular value $\epsilon_0 = |b|/2$. Because $(b_n) \to b$, there exists an $N_2 \in \mathbb{N}$ such that $|b_n - b| < \epsilon_0$ for all $n \geq N_2$. This implies that the distance from b_n to b is less than the distance from b to $b/2$, $|b - \frac{b}{2}| = |b|/2$ [draw]. Thus, $|b_n| > |b|/2$. □

Exercise 10. Finish the proof.

If we let $N = \max\{N_1, N_2\}$, where $|b - b_n| < \epsilon/2$ whenever $n \geq N_2$, then $n \geq N$ implies

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = |b - b_n| \frac{1}{|b||b_n|} < \frac{\epsilon |b|^2}{2} \frac{1}{|b||\frac{b}{2}|} = \epsilon.$$

4. The Order Limit Theorem

- Earlier, we saw that limits behaved really nicely with respect to algebraic combinations of sequences. For instance, if $\lim a_n = a$, and $\lim b_n = b$, we know that
$$\lim \left(\frac{14\pi a_n - 43 b_n}{4}\right) = \frac{14\pi a - 43b}{4}.$$
- It turns out the limiting process is also well-behaved with respect to the "order operation", in other words with respect to $\leq$ and $\geq$:

Theorem 11 (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(1) *If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.*
(2) *If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.*
(3) *If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.*

Exercise 12. (reading question) Which parts of the Order Limit Theorem still hold if we replace every inequality with a strict inequality? (For example, the first part would read "If $a_n > 0$ for all $n \in \mathbb{N}$, then $a > 0$".) For those parts that still hold, sketch a proof. For those parts that don't still hold, give a counterexample.

(1) If $a_n > 0$ for all $n \in \mathbb{N}$, then $a \geq 0$: false; consider $\{a_n\}$ where $a_n = \frac{1}{n}$. Then $a_n > 0$ for all $n \in \mathbb{N}$, but $\lim a_n = 0$.
(2) Consider $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $a_n < 0 < b_n$ for all n, but $\lim a_n = 0 = \lim b_n$.
(3) $\frac{1}{n} > 0$ for all $n$but $\lim(\frac{1}{n}) = 0 \not> 0$. Similarly, $-\frac{1}{n} < 0$ for all n but $\lim(-\frac{1}{n}) = 0$.

Exercise 13. [Workshop] Each group proves one part of OLT (assuming the previous parts).

Proof.

(1) Assume for contradiction that $a < 0$. The idea is to produce a term in the sequence (a_n) which is also less than 0.
  - We'll have to use the definition of convergence somehow. In particular, if we can show for some n that a_n is closer to a than a is to 0 [draw picture], that is enough to show that $a_n < 0$.
  - This is the same as trying to find n so that $|a_n - a| \leq |a|$.
  - If we set $\epsilon = |a|$, the definition of convergence says we can find $N \in \mathbb{N}$ so that $|a_n - a| < \epsilon = |a|$ for all $n \geq N$.
  - In particular, $|a_N - a| < |a|$.
  - This means that $a_N < 0$, a contradiction.
  - Hence $a \geq 0$.

Exercise 14. How could we use (1) to prove (2) and (3)?

(2) This is the same as trying to show that if $b_n - a_n \geq 0$ for all n, then $b - a \geq 0$. We can apply (i):
   - The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b - a$.
   - Because $b_n - a_n \geq 0$ by assumption, (i) implies that $b - a \geq 0$, hence that $a \leq b$.

(3) Let $a_n = c$ for all $n \in \mathbb{N}$. Then certainly $\lim a_n = c$, and by (ii), if $a_n = c \leq b_n$ for all n, then $c \leq b$. Similarly, let $b_n = c$ for all $n \in \mathbb{N}$; then by (ii), if $a_n \leq c = b_n$ for all n, $a \leq c$.

□

Exercise 15. T/F? Prove if true, counterexample if false.

(1) If $(a_n) \to a$ and there exists N_1 such that $a_n \geq 0$ for all $n \geq N_1$, then $a \geq 0$.
   (a) True

(2) If $(a_n) \to a$ and $a \geq 0$, then $a_n \geq 0$ for all n.
   (a) False; consider $(-\frac{1}{n})$.

(3) If $(a_n) \to a$ and $a \geq 0$, then there exists $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$.
   (a) False; consider $(-\frac{1}{n})$.

(4) If $(a_n) \to a$ and $(b_n) \to b$, and there exists N_1 such that $a_n \geq b_n$ for all $n \geq N_1$, then $a \geq b$.
   (a) True

(5) If $(a_n) \to a$ and $(b_n) \to b$ with $a \geq b$, then $a_n \geq b_n$ for all n.
   (a) False; let $a_n = -\frac{1}{n}$ and $b_n = 0$ for all n.

(6) If $(a_n) \to a$ and $(b_n) \to b$ with $a \geq b$, then there exists $N \in \mathbb{N}$ such that $a_n \geq b_n$ for all $n \geq N$.
   (a) False; let $a_n = -\frac{1}{n}$ and $b_n = 0$ for all n.

Note 16. Loosely speaking, limits and their properties don't depend at all what happens at the beginning of the sequence, but are instead determined by what happens when n gets large–their "tails".

We could thus rephrase part (i) of the OLT to only assume that there exists some point N_1 where $a_n \geq 0$ for all $n \geq N_1$.

The theorem remains true, and in fact the same proof is valid with the provision that when N is chosen it be at least as large as N_1.

Many upcoming results will have this property as well.