MATH 321 WEEK 13 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 3.2.3

Exercise 3.2.3. (a) Neither. Given any point in $\mathbf{Q}$, there is no ϵ-neighborhood contained in $\mathbf{Q}$ because the irrational numbers are dense. The set of limit points not contained in $\mathbf{Q}$ is $\mathbf{I}$.

(b) Closed. Given any point in $\mathbf{N}$, there is no ϵ-neighborhood of that point contained in the set.

(c) Open. The limit point 0 is not contained in the set $\{x \in \mathbf{R} : x \neq 0\}$.

(d) Neither. None of the points in this set have ϵ-neighborhoods contained in the set. The set is not closed because this particular sequence converges to some finite limit (i.e., $\sum 1/n^2$ converges) and this limit, whatever it might be, is not an element of the set.

(e) Closed. There is no ϵ-neighborhood of any point in the set contained in $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbf{N}\}$. Because the series $\sum 1/n$ diverges, this particular sequence of points does not have a limit point in $\mathbf{R}$, and is thus a closed set.

(2) 3.2.8

(a) It's a closure, so the set is definitely closed. It's not definitely open; for example, take $A = (0, 1)$, $B = [0, 1]$. Then $\overline{A \cup B} = [0, 1]$.

(b) Definitely open. Since B is closed, its complement is open, and $A \backslash B = A \cap B^c$ is the intersection of two open sets, so must be open.

(c) Definitely open. $A^c \cup B$ is the union of two closed sets, so is closed, hence its complement is open.

(d) Definitely closed, since

$$(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = \mathbb{R} \cap B = B.$$

(e) Definitely open. If $x \in \bar{A}^c$, then $x \notin \bar{A}$ and definitely $x \notin A$, thus $x \in A^c \subseteq \overline{A^c}$. Therefore, $\bar{A}^c \subseteq \overline{A^c}$, and the intersection is just $\bar{A}^c$. Since this is the complement of a closed set, it must be open.

(3) 3.2.10

(a) Can't exist; let$A$ be a countable set contained in $[0, 1]$ and consider any sequence of points $(a_n) \subseteq A$. Then, by Bolzano-Weierstrass, (a_n) has a convergent subsequence $(a_{n_k}) \to a \in \mathbb{R}$. But certainly $(a_{n_k}) \subseteq A$, and hence a is a limit point of A.

(b) This can exist; one example is $R = C \cap \mathbb{Q}$, where C is the Cantor set. If $x \in R$, we can define a sequence (a_n) converging to x as follows. Choose $a_n \in R \cap C_n$ to be an endpoint of the interval of C_n containing x that is not equal to x. Then $|x - a_n| \leq \frac{1}{3^n}$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be arbitrary. Since $(\frac{1}{3^n}) \to 0$, we can choose N so that whenever $n \geq N$, $\frac{1}{3^n} < \epsilon$. Now, if $n \geq N$,

$$|x - a_n| \leq \frac{1}{3^n} < \epsilon$$

and hence x is a limit point of R.

(c) Can't exist. Let X be a set that contains only isolated points. This means that, for every $x \in X$, there exists $\epsilon(x) > 0$ so that $V_{\epsilon(x)}(x)$ contains no point of X. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose a rational number $r(x) \in V_{\epsilon(x)}(x)$ for every $x \in X$. Thus, X is in 1-1 correspondence with some (not necessarily proper) subset of $\mathbb{Q}$, and hence X can't be uncountable.