Name:______________________________

MATH 321 TAKE-HOME FINAL

This exam will be due on Friday, December 15 at noon, slipped under my office door. You may use your textbook, your class notes, and your homework for MATH 321, Fall 2017. You may not use other books, notes, or the Internet for this exam. You may not discuss any part of this exam with any other person besides myself until the exam has been handed in.

Please sign below to signify that you have abided by the above rules:

Signature:______________________________

20

(1) Let S and T be nonempty sets. Prove that there exists a function $f : S \to T$ that is 1-to-1 if and only if there exists an function $g : T \to S$ that is onto.

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3 — g is well-def'd

(a) ($\Longrightarrow$) Suppose that there exists a function $f : S \to T$ that is 1-to-1. We want to define an onto function $f : T \to S$. Since f is 1-to-1, we know that if $t = f(s_1) = f(s_2)$, then $s_1 = s_2$. In other words, we know that $f^{-1}\{t\}$ contains at most one element. Since S is nonempty, we know there exists some point $r \in S$. Define a function $g : T \to S$ by

4

$$g(t) = \begin{cases} \text{the unique element } s \text{ of } f^{-1}\{t\} & \text{if } f^{-1}\{t\} \neq \emptyset \\ r & \text{if } f^{-1}\{t\} = \emptyset. \end{cases}$$

3

We want to show that this function is onto. Let $x \in S$. Then $f(x)$ is equal to some point $t \in T$, so by definition $x \in f^{-1}\{t\}$. Since f is 1-to-1, x is the unique element of $f^{-1}\{t\}$, hence $x = g(t)$.

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5: def f

5: show 1-1

(b) ($\Longleftarrow$) Assume $g : T \to S$ is onto. Then, for each $s \in S$, we may choose an arbitrary element $t_s \in g^{-1}\{s\}$. We then define $f : S \to T$ by setting $f(s) = t_s$ for all $s \in S$. This function is well-defined because each t_s is in the preimage of a different $s \in S$, so by definition $f(s)$ only contains one element. It remains to show that f is 1-to-1. Suppose that $f(s_1) = f(s_2)$ for some $s_1, s_2 \in S$. This means that $t_{s_1} = t_{s_2}$. Applying g to both sides of this equation gives that $s_1 = g(t_{s_1}) = g(t_{s_2}) = s_2$. Thus, f is 1-to-1.

32

(2) Are the following true or false? Prove or give a counterexample. (2 hr 15 min)

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(a) The sequence defined by $s_1 = 1$ and $s_{n+1} = \sqrt{1 + s_n}$ for all $n \geq 1$ converges.

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(i) True. (s_n) is bounded: we argue by induction that $s_n \leq 2$ for all n.
  (A) **Base case**: $s_1 = 1 < 2$.
  (B) **Inductive step**: suppose that $s_n < 2$. Then

$$s_{n+1} = \sqrt{1 + s_n} < \sqrt{1 + 2} = \sqrt{3} < 2.$$

  Therefore, (s_n) is bounded above by 2.

3

(ii) (s_n) is increasing: we also use induction.
  (A) **Base case:** $s_1 = 1 < \sqrt{2} = s_2$.
  (B) **Inductive step:** assume that $s_n < s_{n+1}$. Then

$$s_{n+1} = \sqrt{1 + s_n} < \sqrt{1 + s_{n+1}} = s_{n+2}$$

  and hence (s_n) is increasing.

2

(iii) We have shown that (s_n) is an increasing sequence that is bounded above. Thus it converges by MCT.

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(b) If $f : \mathbb{R} \to \mathbb{R}$ is any function, $(a_n) \to c$, and $a_n \neq c$ for all n, then $f(a_n) \to f(c)$. (20 min)

(i) False; consider the function

4: def counterex

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0. \end{cases}$$

4: prove it's a counterex

Then the sequence $(\frac{1}{n}) \to 0$ satisfies $f(\frac{1}{n}) = 1$ for all n, so $(f(\frac{1}{n})) \to 1$, but $f(\lim \frac{1}{n}) = f(0) = 0 \neq 1$.

(c) Every finite set is compact. (30 min)

3 bdd

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(i) True. Every finite set $X = \{x_1, x_2, \dots, x_n\}$ is bounded, so any sequence (a_n) with values in a finite set has a convergent subsequence $(a_{n_k}) \to a \in \mathbb{R}$. But the only way for (a_{n_k}) to converge is for all of its terms to eventually be equal to x_i for some i. This is because, if a_{n_k} converges and each $a_{n_k} = x_{n_j}$ for some $0 \le n_j \le n$, then for there exists $N \in \mathbb{N}$ so that whenever $k \ge N$,

5 clsd

$$|a_{n_k} - a| < \epsilon = \min\{|a_{n_k} - x_j| : x_j \neq a_{n_k} \in X\}.$$

Therefore, it must be that whenever $k \ge N$, $a = a_{n_k} \in X$. Thus, $a = \lim(a_{n_k}) \in X$ and X is compact.

(ii) Alternatively, X must be closed because it is the finite union of the closed sets $\{x_1\}, \{x_2\}, \dots, \{x_n\}$.

(d) The union of two perfect sets is perfect. (30 min)

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3 (i) True. Let P and Q be two perfect sets. The finite union of closed sets is closed, so $U = P \cup Q$ is closed. It remains to show that U contains no isolated points. Let $x \in U$.

5 Then $x \in P$ or $x \in Q$. In either case (or in the case that both are true), x is a limit point of P or x is a limit point of Q. Since $P, Q \subseteq U$, x is a limit point of U as well. Hence U is perfect.

28

(3) Given the series $\sum a_n$ and $\sum b_n$, suppose there exists a natural number N such that $a_n = b_n$ for all $n \ge N$. Prove that $\sum a_n$ is convergent if and only if $\sum b_n$ is convergent. Thus, the convergence of a series is not affected by changing a finite number of terms. (Of course, the value of the sum may change.)

(a) $(\Longrightarrow)$ Suppose that $\sum a_n$ converges and let (s_n) be the sequence of partial sums for $\sum a_n$ and (t_n) the sequence of partial sums for $\sum b_n$.

(i) Let $\epsilon > 0$ be arbitrary.

4 (ii) Since $\sum a_n$ converges to some $A \in \mathbb{R}$, there exists $N_1 \in \mathbb{N}$ so that, whenever $n \ge N_1$,

$$|s_n - A| < \epsilon.$$

(iii) Let $N' = \max\{N, N_1\}$.

14

4 (iv) Then, if $n \ge N'$,

$$t_n - s_n = (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_{N-1} - a_{N-1})$$

6 and we may define $B = A + (t_n - s_n)$. Then

$$\begin{aligned} |t_n - B| &= |t_n - s_n + s_n - B| \\ &= |s_n - (B - (t_n - s_n))| \\ &= |s_n - A| < \epsilon \end{aligned}$$

and hence $\sum b_n = B$.

(b) $(\Longleftarrow)$ Suppose that $\sum b_n$ converges and that there exists $N \in \mathbb{N}$ so that $a_n = b_n$ for all $n \ge N$. Let (s_n) be the sequence of partial sums of $\sum a_n$ and (t_n) the sequence of partial sums for $\sum b_n$.

(i) Let $\epsilon > 0$ be arbitrary.

4 (ii) Since $\sum b_n$ converges to some $B \in \mathbb{R}$, there exists $N_1 \in \mathbb{N}$ so that, whenever $n \ge N_1$,

$$|t_n - B| < \epsilon.$$

14

(iii) Let $N' = \max\{N, N_1\}$.

4 (iv) Then, if $n \ge N'$,

$$t_n - s_n = (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_{N-1} - a_{N-1})$$

6 and we may define $A = B - (t_n - s_n)$. Then

$$\begin{aligned} |s_n - A| &= |s_n - t_n + t_n - A| \\ &= |t_n - (A + t_n - s_n)| \\ &= |t_n - B| < \epsilon \end{aligned}$$

and hence $\sum a_n = A$.

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(4) A set E is **totally disconnected** if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$. Show that $\mathbb{Q}$ is totally disconnected.

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8 (a) Let $x, y \in \mathbb{Q}$ be distinct rational numbers. Assume without loss of generality that $x < y$. Then, by the density of $\mathbb{I}$ in $\mathbb{R}$, there exists an irrational number $t \in (x, y)$.

8 (b) Consider the sets $A = \mathbb{Q} \cap (-\infty, t)$ and $B = \mathbb{Q} \cap (t, \infty)$. Since $t \notin \mathbb{Q}$, we have that $A \cup B = \mathbb{Q}$. Moreover, $\bar{A} = \mathbb{Q} \cap (-\infty, t]$ and $\bar{B} = \mathbb{Q} \cap [t, \infty)$. Therefore,

$$\bar{A} \cap B = \emptyset = A \cap \bar{B},$$

hence A and B are separated.

4 (c) Since $x < t < y$, we have that $x \in A$ and $y \in B$, as desired. Therefore, $\mathbb{Q}$ is totally disconnected.

(5) **(Bonus)** Note that the definition of $\lim_{x \to c} f(x) = L$ is that, for all $\epsilon > 0$, there exists $\delta > 0$ so that whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. Prove the Squeeze Theorem for functions. That is, let f, g, and h be functions from $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a limit point of A. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in D$ with $x \neq c$, and suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$. Prove that $\lim_{x \to c} g(x) = L$ as well. You may assume the Algebraic Limit Theorem for functions.

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(a) Assume that $f, g, h : A \to \mathbb{R}$, c is a limit point of A, $f(x) \leq g(x) \leq h(x)$ for all $x \in A$ with $x \neq c$, and $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$.

+1 (b) Let $\epsilon > 0$ be arbitary.

(c) ~~[scratch work]~~ we want to find $\delta > 0$ so that whenever $0 < |x - c| < \delta$, we have $|g(x) - L| < \epsilon$. We have that we can make $|f(x) - L|$ and $|h(x) - L|$ arbitrarily small. Note that, since

+3 $g(x) \leq h(x)$ for all x, the ALT for functions implies that $\lim_{x \to c} g(x) \leq \lim_{x \to c} h(x) = L$. Therefore, $|g(x) - L| = L - g(x)$. Since there exists $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$, we have that $L - \epsilon < f(x) \leq g(x)$. Therefore,

$$|g(x) - L| = L - g(x) < \epsilon$$

and hence $\lim_{x \to c} g(x) = L$, as desired.

$|f(x) - L| < \varepsilon$

+2 $\Rightarrow L - f(x) < \varepsilon$

+2 $\Rightarrow L - \varepsilon < f(x) \leq g(x)$

+2 $\Rightarrow |g(x) - L| = L - g(x) < \varepsilon$