MATH 321 3.4 - PERFECT SETS AND CONNECTED SETS

- One of the underlying goals of topology is to strip away all extraneous information from our intuitive picture of $\mathbb{R}$ and isolate just those properties that are responsible for the phenomenon we are studying.
- For example, the compactness (boundedness and closed-ness) of a closed interval is more important than its interval-ness.
- We'll show that the property of $[0, 1]$ and the Cantor set that makes them uncountable is that they're nonempty closed sets that do not contain isolated points.

0.1. **Perfect Sets.**

Definition 1. A set $P \subseteq \mathbb{R}$ is **perfect** if it is closed and does not contain isolated points.

Exercise 2. (Reading questions) Recall that (one definition of) a limit point of a set X is a point $x \in X$ for which there exists a sequence (a_n), with each term a_n contained in X but not equal to x, so that $(a_n) \to x$. A point that is not a limit point of X is called an isolated point of X. We call a set that is both closed and contains no isolated points a perfect set. Which of the following closed sets are perfect? For those that are, prove it. For those that aren't, give an example of a point in the set that's isolated.

(1) $[0, 1]$
(2) $\{0\}$
(3) $\mathbb{Z}$
(4) the Cantor set C
   (a) Hint 1: let $x \in C$. Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| < \frac{1}{3}$.
   (b) Hint 2: then show that, for each $n \in \mathbb{N}$, there exists $x_n \in C \cap C_n$, different from x, satisfying $|x - x_n| < \frac{1}{3^n}$.

Proof. (the Cantor set is perfect) We defined the Cantor set to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n,$$

where each C_n is a finite union of closed intervals. By Theorem 3.2.14, each C_n is closed, and by the same theorem, C is closed as well. It remains to show that no point in C is isolated.

Let $x \in C$ be arbitrary. To convince ourselves that x is not isolated, we must construct a sequence (x_n) of points in C, different from x, that converges to x. From our earlier discussion, we know that C at least contains the endpoints of the intervals that make up each C_n.

Since $x \in C_1$, we know that one of the endpoints x_1 of C_1 is not equal to x and satisfies $|x - x_1| < \frac{1}{3}$ by definition of C_1. Similarly, for each $n \in \mathbb{N}$, we can take one of the endpoints x_n of the interval of C_n that contains x, with $x_n \neq x$. Thus, by definition, $|x - x_n| < \frac{1}{3^n}$.

Therefore, the sequence (x_n) has none of its terms equal to x, but given arbitrary $\epsilon > 0$, we can choose N large enough so that $\frac{1}{3^n} < \epsilon$. Therefore, if $n \geq N$,

$$|x - x_n| < \frac{1}{3^n} < \epsilon$$

and hence $(x_n) \to x$, as desired. $\square$

- This gives us an alternate, and perhaps more satisfying, proof that C is uncountable. You'll construct it:

Exercise 3. Suppose that $P \subseteq \mathbb{R}$ is a perfect set. Prove that P is uncountable by proving the following in order:

(1) P must be infinite.
(2) [draw] If $x_0 \in P \cap (a_0, b_0)$ for some $a_0, b_0 \in \mathbb{R}$, then there exists some $x_1 \in P \cap (a_0, b_0)$ with $x_1 \neq x_0$.

   (a) else x_0 would be isolated.
(3) [draw] If $x_0, x_1 \in P \cap (a_0, b_0)$, then there exist $a_1, b_1 \in \mathbb{R}$ so that $x_1 \in (a_1, b_1) \subset (a_0, b_0)$ but $x_0 \notin [a_1, b_1]$.
   (a) True; assume WLOG that $x_0 < x_1$. Then we just need to choose a_1, b_1 so that $a_0 < x_0 < a_1 < x_1 < b_1 < b_0$.
   (b) Why do such a_1, b_1 have to exist? [we know $x_0 < x_1$, so we must be able to choose a point between them]
(4) There exist sequences (a_n), (b_n), and (x_n) so that for every $n \in \mathbb{N}$, $x_n \in P \cap (a_n, b_n)$ but $x_n \notin [a_{n+1}, b_{n+1}]$.
(5) A non-empty perfect set is uncountable.

Proof. If P is perfect and nonempty, it must be infinite because otherwise it would consist only of isolated points. Assume for contradiction that P is countable, so that

$$P = \{x_1, x_2, x_3, \dots\},$$

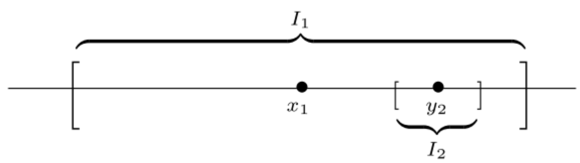
where every element of P appears on the list. The idea is to construct a sequence of nested compact sets K_n, all contained in P, with the property that $x_1 \notin K_1$, $x_2 \notin K_2$, $x_3, \notin K_4, \dots$.

Some care must be taken to show that each K_n is nonempty, for then we can use the Nested Compact Set Property to produce

$$x \in \bigcap_{n=1}^{\infty} K_n \subseteq P$$

that cannot be on the list $\{x_1, x_2, x_3, \dots\}$.

Let I_1 be a closed interval that contains x_1 in its interior (i.e. x_1 is not an endpoint of I_1). Now, x_1 is not isolated, so there exists some other point $y_2 \in P$ that is also in the interior of I_1 (and y_2 must be in our list of elements of P). Construct a closed interval I_2, centered on y_2, so that $I_2 \subseteq I_1$ but $x_1 \notin I_2$.



This process can be continued. Because $y_2 \in P$ is not isolated, there must exist another point $y_3 \in P$ in the interior of I_2, and we may insist that $y_3 \neq x_2$. Now, construct I_3 centered on y_3 and small enough so that $x_2 \notin I_3$ and $I_3 \subseteq I_2$. Observe that $I_3 \cap P \neq \emptyset$ because this intersection contains at least y_3.

If we carry out this construction inductively, the result is a sequence of closed intervals I_n satisfying:

(1) $I_{n+1} \subseteq I_n$,
(2) $x_n \notin I_{n+1}$, and
(3) $I_n \cap P \neq \emptyset$.

To finish the proof, we let $K_n = I_n \cap P$. For each $n \in \mathbb{N}$, we have that K_n is closed because it is the intersection of closed sets, and bounded because it is contained in the bounded set I_n. Hence, K_n is compact. By construction, K_n is not empty and $K_{n+1} \subseteq K_n$. Thus, by the Nested Compact Set Property, the intersection

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

But each K_n is a subset of P, and the fact that $x_n \notin I_{n+1}$ leads to the conclusion that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, which is the sought-after contradiction. ☐

0.2. **Connected sets.**

- Although the two open intervals $(1, 2)$ and $(2, 5)$ have the limit point $x = 2$ in common, they're still "further apart" than $[1, 2]$ and $(2, 5)$, where the shared limit point is actually contained in one of the sets.
- We want to make that difference explicit:

Definition 4. Two nonempty sets $A, B \subseteq \mathbb{R}$ are **separated** if $\overline{A} \cap B$ and $A \cap \bar{B}$ are both empty. A set $E \subseteq \mathbb{R}$ is **disconnected** if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.

A set that is not disconnected is called a **connected** set.

Example 5. (i) If we let $A = (1, 2)$ and $B = (2, 5)$, then it is not difficult to verify that $E = (1, 2) \cup (2, 5)$ is disconnected. Note that the sets $C = (1, 2]$ and $D = (2, 5)$ are not separated because $C \cap \bar{D} = \{2\}$ is not empty. This is comforting because $C \cup D = (1, 5)$ better be disconnected!

Exercise 6. (reading question) Classify the following subsets of $\mathbb{R}$ according to the characteristics closed, bounded, compact, perfect, and disconnected.

$$\mathbf{Q}$$

$$\mathbf{R}$$

$$\{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$

$$\{0\} \cup \{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$$

(1) $[-4, -2] \cup [2, 4]$
  (a) closed because it's a finite union of closed sets
  (b) bounded by $[-4, 4]$
  (c) compact because it's closed and bounded
  (d) perfect: the set of limit points of $[-4, -2]$ is $[-4, -2]$ and the set of limit points of $[2, 4]$ is $[2, 4]$. Since the intervals in the union are disjoint, no sequenc can manage to converge to another point. (In fact, it's true in general that, for any sets $A_1, A_2 \subseteq \mathbb{R}$, the set of limit points of $A_1 \cup A_2$ is $L_1 \cup L_2$, the union of their sets of limit points. The proof is left as an exercise.)
  (e) Disconnected: the set in question is the union of $A = [-4, -2]$ and $B = [2, 4]$, which satisfy $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

(2) $\mathbb{Q}$
  (a) not closed since its complement $\mathbb{I}$ is not open. To see this, consider the point $\sqrt{2} \in \mathbb{I}$, say. Then any ϵ-neighborhood of $\sqrt{2}$ contains a rational number by the Density of $\mathbb{Q}$ in $\mathbb{R}$, so there is no ϵ for which $V_\epsilon(\sqrt{2}) \subseteq \mathbb{I}$.
  (b) Unbounded (can use e.g. Archimedean Principle)
  (c) Not compact
  (d) Not perfect since it isn't closed
  (e) is disconnected. If we let

$$A = \mathbb{Q} \cap (-\infty, \sqrt{2}) \text{ and } B = \mathbb{Q} \cap (\sqrt{2}, \infty),$$

  then $\mathbb{Q} = A \cup B$, and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

(3) $\mathbb{R}$
  (a) closed
  (b) unbounded
  (c) not compact
  (d) for any $x \in \mathbb{R}$, the sequence $(x + \frac{1}{n})$ converges to x, so $\mathbb{R}$ is perfect.
  (e) connected

(4) $A = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$
  (a) not closed because the limit point 0 is not in the set
  (b) bounded
  (c) not compact
  (d) not perfect
  (e) disconnected: we can write

$$A = C \cup D \text{ where } C = A \cap (0, \frac{5}{12}), D = A \cap (\frac{5}{12}, 1).$$

(5) $A \cup \{0\}$
  (a) closed because the only limit point, 0, is contained in the set

(b) bounded
(c) compact
(d) not perfect; $\frac{1}{2}$ is an isolated point since $V_{\frac{1}{12}}(\frac{1}{2})$ does not intersect the set at any point other than $\frac{1}{2}$
(e) still disconnected; the same example as above works

- Our current definition of connected is "not disconnected". It'd be nice to have a positive characterization of connected sets as well.

Theorem 7. *A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.*

Proof. Unclaimed HW □

- Connectedness is more interesting when working with subsets of the plane or higher-dimensional spaces, because connected sets in $\mathbb{R}$ coincide perfectly with (potentially infinite) intervals.

Theorem 8. *A set $E \subseteq \mathbb{R}$ is connected if and only if, whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well. (In other words, E is an interval.)*

Proof. ($\implies$) Assume E is connected, and let $a, b \in E$ and $a < c < b$. Set

$$A = (-\infty, c) \cap E \text{ and } B = (c, \infty) \cap E.$$

□

Exercise 9. Finish the forward direction.

We want to show that $c \in E$. If $c \notin E$, then $E = A \cup B$ and

$$\overline{A} \cap B = (-\infty, c] \cap (c, \infty) \cap E = (-\infty, c) \cap (c, \infty) \cap E = \emptyset$$

and similarly, $A \cap \overline{B} = \emptyset$. Thus E is disconnected, a contradiction. This contradiction shows that $A \cup B$ is missing some point of E, and the only possibility is c. Thus, $c \in E$.

Proof. ($\impliedby$) Suppose that E is an interval in the sense that whenever $a, b \in E$ satisfy $a < c < b$ for some c, then $c \in E$. Our intent is to use the characterization of connectedness in Theorem 7, so let $E = A \cup B$, where A and B are nonempty and disjoint. We need to show that one of these sets contains a limit point of the other.

The idea is to use the Nested Interval Property to construct a sequence of intervals $([a_n, b_n])$ so that the point

$$x \in \bigcap_{n=0}^{\infty} [a_n, b_n]$$

guaranteed by the NIP is both $\lim(a_n)$ and $\lim(b_n)$. Then if $x \in A$, it's a limit point of B via the sequence (b_n), and if $x \in B$, it's a limit point of A via the sequence (a_n).

[draw] Pick $a_0 \in A$ and $b_0 \in B$, and for the sake of the argument, assume $a_0 < b_0$. Because E is itself an interval, the interval $I = [a_0, b_0]$ is contained in E. □

Exercise 10. Finish the proof by constructing (a_n) and (b_n), both converging to the x guaranteed by NIP, as desired.

Proof. Now, bisect I_0 into two equal halves. The midpoint of I_0 must be in A or B, and so choose $I_1 = [a_1, b_1]$ to be the half that allows us to have $a_1 \in A$ and $b_1 \in B$. Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$, where $a_n \in A$, $b_n \in B$, and the length $(b_n - a_n) \to 0$. By the NIP, there exists

$$x \in \bigcap_{n=0}^{\infty} I_n,$$

and it's straightforward to show that the sequences of endpoints each satisfy $\lim a_n = x$ and $\lim b_n = x$. But now $x \in E$ must belong to either A or B, thus making it a limit point of the other. □