MATH 321 WEEK 12 CLAIMED PROBLEM SOLUTIONS

(1) 2.7.7

Exercise 2.7.7. (a) The idea here is that eventually the terms a_n "look like" a non-zero constant times $1/n$, and we know that any series of this form diverges. To make this precise, let $\epsilon_0 = l/2 > 0$. Because $(na_n) \to l$, there exists $N \in \mathbf{N}$ such that $na_n \in V_{\epsilon_0}(l)$ for all $n \geq N$. A little algebra shows that this implies we must have $na_n > l/2$, or

$$a_n > (l/2)\,(1/n) \qquad \text{for all } n \geq N.$$

Because this inequality is true for all but some finite number of terms, we may still appeal to the Comparison Test to assert that $\sum a_n$ diverges.

(b) Assume that $\lim(n^2 a_n) \to L \geq 0$. The definition of convergence (with $\epsilon_0 = 1$) tells us that there exists an N such that $n^2 a_n < L + 1$ for all $n \geq N$. This means that eventually $a_n < (L+1)/n^2$. We know that the series $\sum 1/n^2$ converges, and by the Algebraic Limit Theorem for series (Theorem 2.7.1), $\sum (L+1)/n^2$ converges as well. Thus, by the Comparison Test $\sum a_n$ must converge.

(2) 2.7.9

Exercise 2.7.9. (a) First, pick an ϵ-neighborhood around r of size $\epsilon_0 = |r - r'|$. Because $\lim \left|\frac{a_{n+1}}{a_n}\right| = r$, there exists an N such that $n \geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right| \in V_{\epsilon_0}(r)$. It follows that $\left|\frac{a_{n+1}}{a_n}\right| \leq r'$ for all $n \geq N$, and this implies the statement in (a)

(b) Having chosen N, $|a_N|$ is now a fixed number. Also, $\sum (r')^n$ is a geometric series with $|r'| < 1$, so it converges. Therefore, by the Algebraic Limit Theorem $|a_N| \sum (r')^n$ converges.

(c) From (a) we know that there exists an N such that $|a_{N+1}| \leq |a_N| r'$. Extending this we find $|a_{N+2}| \leq |a_{N+1}| r' \leq |a_N| (r')^2$, and using induction we can say that

$$|a_k| \leq |a_N| (r')^{k-N} \qquad \text{for all } k \geq N.$$

Thus, $\sum_{k=N}^{\infty} |a_k|$ converges by the Comparison Test and part (b). Because

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k|$$

and $\sum_{k=1}^{N-1} |a_k|$ is just a finite sum, the series $\sum_{k=1}^{\infty} |a_k|$ converges.

(3) 2.7.12

We use the observation that $x_j = s_j - s_{j-1}$ to rewrite

$$\begin{aligned}
\sum_{j=m}^{n} s_j (y_j - y_{j+1}) &= s_m y_m - s_m y_{m+1} + s_{m+1} y_{m+1} - s_{m+1} y_{m+2} \pm \cdots + s_n y_n - s_n y_{n+1} \\
&= s_m y_m + y_{m+1}(s_{m+1} - s_m) + y_{m+2}(s_{m+2} - s_{m+1}) \pm \cdots + y_n (s_n - s_{n-1}) - s_n y_{n+1} \\
&= s_m y_m + x_{m+1} y_{m+1} + x_{m+2} y_{m+2} + \cdots + x_n y_n - s_n y_{n+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{j=m}^{n} x_j y_j - \sum_{j=m}^{n} s_j(y_j - y_{j+1}) &= x_m y_m - s_m y_m + s_n y_{n+1} \\ &= y_m(x_m - s_m) + s_n y_{n+1}.\end{aligned}$$

Now observe that $x_m - s_m = -(s_m - x_m) = -s_{m-1}$, and hence

$$\sum_{j=m}^{n} x_j y_j - \sum_{j=m}^{n} s_j(y_j - y_{j+1}) = -s_{m-1} y_m + s_n y_{n+1}.$$

Moving the $\sum s_j(y_j - y_{j+1})$ to the right side of this equation finishes the proof.