MATH 321 DAY 10 - CONSEQUENCES OF COMPLETENESS, PART 2

1. Claimed homework presentations

(1) Abbott Exercise 1.2.3

Exercise 1.2.5. (a) If $x \in (A \cap B)^c$ then $x \notin (A \cap B)$. But this implies $x \notin A$ or $x \notin B$. From this we know $x \in A^c$ or $x \in B^c$. Thus, $x \in A^c \cup B^c$ by the definition of union.

(b) To show $A^c \cup B^c \subseteq (A \cap B)^c$, let $x \in A^c \cup B^c$ and show $x \in (A \cap B)^c$. So, if $x \in A^c \cup B^c$ then $x \in A^c$ or $x \in B^c$. From this, we know that $x \notin A$ or $x \notin B$, which implies $x \notin (A \cap B)$. This means $x \in (A \cap B)^c$, which is precisely what we wanted to show.

(c) In order to prove $(A \cup B)^c = A^c \cap B^c$ we have to show,

$$(A \cup B)^c \subseteq A^c \cap B^c \text{ and,} \tag{1}$$

$$A^c \cap B^c \subseteq (A \cup B)^c. \tag{2}$$

To demonstrate part (1) take $x \in (A \cup B)^c$ and show that $x \in (A^c \cap B^c)$. So, if $x \in (A \cup B)^c$ then $x \notin (A \cup B)$. From this, we know that $x \notin$A and $x \notin B$ which implies $x \in A^c$ and $x \in B^c$. This means $x \in (A^c \cap B^c)$.

Similarly, part (2) can be shown by taking $x \in (A^c \cap B^c)$ and showing that $x \in (A \cup B)^c$. So, if $x \in (A^c \cap B^c)$ then $x \in A^c$ and $x \in B^c$. From this, we know that $x \notin A$ and $x \notin B$ which implies $x \notin (A \cup B)$. This means $x \in (A \cup B)^c$. Since we have shown inclusion both ways, we conclude that $(A \cup B)^c = A^c \cap B^c$.

(2) Abbott Exercise 1.2.5
   (a) $|a-b| = |a+(-b)| \leq |a| + |-b| = |a| + |b|$
   (b) With the triangle inequality: suppose WLOG that $a \geq b$. Then note that

$$|a| = |(a-b)+b| \leq^{\Delta} |a-b| + |b|$$

so that

$$|a-b| \geq |a| - |b| = \Big||a| - |b|\Big|$$

   (c) (b) without the triangle inequality: $||a|-|b|| = |a-b|$ if $a, b \geq 0$. If $a \geq 0$ and $b < 0$, then

$$||a|-|b|| = |a-(-b)| = |a+b| \leq |a-b|$$

by the fact that b is negative. The same argument works if $a < 0, b \geq 0$ since $||a|-|b|| = ||b|-|a||$. If $a < 0, b < 0$, then

$$||a|-|b|| = |(-a)-(-b)| = |b-a| = |a-b|.$$

(3) Abbott Exercise 1.2.12
   Base case: De Morgan's Law

Inductive step: Suppose that, whenever $A_1, \ldots, A_{k-1}$ are sets, $(A_1 \cup \cdots \cup A_{k-1})^c = A_1^c \cap \cdots \cap A_{k-1}^c$. Then

$$\begin{aligned}(A_1 \cup \cdots \cup A_{k-1} \cup A_k)^c &= (A_1 \cup \cdots \cup (A_{k-1} \cup A_k))^c \\ &= A_1^c \cap \cdots \cap (A_{k-1} \cup A_k)^c \text{ by the induction assumption} \\ &= A_1^c \cap \cdots \cap A_{k-1}^c \cap A_k^c \text{ by De Morgan's Laws.}\end{aligned}$$

(4) Abbott Exercise 1.3.8

Suppose that $\sup A < \sup B$. Suppose for contradiction that no $b \in B$ is an upper bound for A. Then, for all $b \in B$, there exists an element $a \in A$ such that $a > b$. Since $\sup A > a$ for all $a \in A$, this tells us that, for all $b \in B$, $\sup A > b$ (and thus $\sup A$ is an upper bound for B). Therefore, by the definition of least upper bound, we must have that $\sup B \leq \sup A$, a contradiction.

(5) Abbott Exercise 1.3.9

Exercise 1.3.9. (a) Set $\epsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists an element $b \in B$ satisfying $\sup B - \epsilon < b$, which implies $\sup A < b$. Because $\sup A$ is an upper bound for A, then b is as well.

(b) Take $A = [0, 1]$ and $B = (0, 1)$.

(a) True
(b) False; take $A = (0, 1)$. Then $\sup A = 1$ although $a < 1$ for all $a \in (0, 1)$.
(c) Take $A = (0, 1)$ and $B = \{1\}$. Then $a < 1$ for all $a \in A$, but $\sup A = 1 = \inf B$.
(d) True. Since $\sup A \geq a$ for all $a \in A$ and $\sup B \geq b$ for all $b \in B$, it must be true that $\sup A + \sup B \geq a + b$ for all $a \in A$ and $b \in B$. So $\sup A + \sup B$ is an upper bound for $A + B$. Now, let $\epsilon > 0$. By Lemma 1.3.7, there exist elements $a \in A, b \in B$ satisfying $\sup A - \frac{\epsilon}{2} < a$ and $\sup B - \frac{\epsilon}{2} < b$. Then $(\sup A + \sup B) - \epsilon < a + b$, and by Lemma 1.3.7 $\sup A + \sup B$ is the lub for $A + B$.
(e) True. Set $\epsilon = \sup B - \sup A \geq 0$. By Lemma 1.3.7, there exists an element $b \in B$ satifying $\sup B - \epsilon < b$, which implies $\sup A < b$. Because $\sup A$ is an upper bound for A, then b is as well.

2. Consequences of Completeness, Part 2

2.1. The existence of square roots.

- We saw that $\sqrt{2} \notin \mathbb{Q}$. How do we know that $\sqrt{2} \in \mathbb{R}$?

Conjecture 1. *There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.*

Exercise 2. [slides] (Dis?)prove this.

Proof. Consider the set

$$T = \{t \in \mathbb{R} | t^2 < 2\}$$

and set $\alpha = \sup T$. We prove that $\alpha^2 = 2$ by ruling out the possibilities $\alpha^2 < 2$ and $\alpha^2 > 2$.

- We use both of the parts of the definition of $\sup T$.
- The strategy is to demonstrate that $\alpha^2 < 2$ violates the fact that α is an upper bound for T,
- and $\alpha^2 > 2$ violates that it's the least upper bound.

Assume for contradiction that $\alpha^2 < 2$. This should mean that we could find an element of T larger than α, giving us a contradiction. Let's try to show that $(\alpha + \frac{1}{n}) \in T$ for some n. This would contradict α being an upper bound for T.

We only have information about α^2, not α, so let's try squaring $\alpha + \frac{1}{n}$:

$$\begin{aligned}(\alpha + \frac{1}{n})^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &< \alpha^2 + \frac{2\alpha + 1}{n}\end{aligned}$$

Exercise 3. We want to show that $(\alpha + \frac{1}{n})^2 < 2$ and thus that $\alpha + \frac{1}{n} \in T$. Remember we have freedom in our choice of n. How big do we need to make n so that $\alpha^2 + \frac{2\alpha+1}{n} < 2$?

Proof. If we choose $n_0 \in \mathbb{N}$ large enough that

$$\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1},$$

then $(2\alpha+1)/n_0 < 2-\alpha^2$, and consequently that

$$\begin{aligned}\left(\alpha + \frac{1}{n_0}\right)^2 &= \alpha^2 + \frac{2\alpha}{n_0} + \frac{1}{n_0^2} \\ &< \alpha^2 + \frac{2\alpha+1}{n_0} \\ &< \alpha^2 + (2-\alpha^2) = 2\end{aligned}$$

and hence $\alpha + \frac{1}{n_0} \in T$, contradicting that $\alpha = \sup T$. Hence, $\alpha^2 < 2$. □

Now suppose that $\alpha^2 > 2$. Then we try to show that $\alpha - \frac{1}{n}$ is an upper bound for T for some n. We want to show that

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > 2$$

for an appropriate choice of n. The rest is left as an exercise. □

- A small modification of this proof shows that $\sqrt{x} \in \mathbb{R}$ for any $x \geq 0$.
- Using the binomial formula for expanding $(\alpha + 1/n)^m$, we can show that $\sqrt[m]{x} \in \mathbb{R}$ for any $m \in \mathbb{N}$.