MATH 321 WEEK 5 CLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 1.5.9

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2}+\sqrt{3}$ are algebraic.

(i) $\sqrt{2}$ is a root of x^2-2.

(ii) $\sqrt[3]{2}$ is a root of x^3-2.

(iii) Let $x=\sqrt{2}+\sqrt{3}$. Then $(x-\sqrt{2})^2=(\sqrt{3})^2=3$, and hence

$$x^2-2\sqrt{2}x+2=3$$

so that $x^2-1=2\sqrt{2}x$. Squaring both sides gives

$$x^4-2x^2+1=4(2)x^2$$
$$x^4-10x^2+1=0.$$

Hence, $\sqrt{2}+\sqrt{3}$ is a root of x^4-10x^2+1.

(b) Show that $A_n=\{\text{roots of degree-}n\text{ polynomials with integer coefficients}\}$ is countable.
One way of doing this is as follows. Note that every polynomial has a finite number of roots. So if we could count degree-n polynomials with integer coefficients, we could count their roots–a countable union of finite sets is countable. (This fact is easily proved by slightly modifying the proof of Theorem 1.5.8(i).) Let $P(n,\mathbb{Z})$ be the set of degree-n polynomials with integer coefficients, and let $\mathbb{Z}^n_{\neq 0}=\{(a_1,a_2,\ldots,a_n):a_i\in\mathbb{Z}\forall i, a_1\neq 0\}$. Define $f:P(n,\mathbb{Z})\to\mathbb{Z}^n_{\neq 0}$ by

$$f(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0)=(a_n,a_{n-1},a_{n-2},\ldots,a_0).$$

We claim f is 1-1 and onto.

(i) 1-1: let $P_1=a_nx^n+\cdots+a_1x+a_0$ and $P_2=b_nx^n+b_{n-1}x^{n-1}+\cdots+b_1x+b_0$, and assume $f(P_1)=f(P_2)$. Then

$$(a_n,\ldots,a_1,a_0)=f(P_1)=f(P_2)=(b_n,\ldots,b_1,b_0)$$

implying that $a_n=b_n$, $a_{n-1}=b_{n-1}$, $\ldots$, $a_1=b_1$, and $a_0=b_0$. Thus, $P_1=P_2$, showing that f is 1-1.

(ii) Onto: let $a=(a_1,a_2,\ldots,a_n)\in\mathbb{Z}^n_{\neq 0}$. Then

$$a=f(a_1x^n+a_2x^{n-1}+\cdots+a_{n-1}x+a_n).$$

(c) Let A be the set of all algebraic numbers. Since countable unions of countable sets are countable by Theorem 1.5.8, and since A_n is countable for all $n\in\mathbb{N}$,

$$A=\bigcup_{n=1}^{\infty}A_n \text{ is countable.}$$

Let T be the set of transcendental numbers. Then $\mathbb{R}=A\sqcup T$, where $\sqcup$ denotes a disjoint union. Since A is countable, we must have that T is uncountable, for otherwise, $\mathbb{R}$ would be the union of two countable sets and hence countable itself.

(2) Prove Theorem 1.6.1:

Theorem 1. *The open interval* $(0,1)$ *is uncountable.*

(a) Since $b_1\neq a_{11}$ by definition ($b_1=2$ if $a_{11}\neq 2$ and $b_1=3$ if $a_{11}=2$), it must be that $x\neq f(1)$.

(b) Since $b_2\neq a_{22}$, and in general $b_n\neq a_{nn}$, by definition, it follows that $x\neq f(2)$ and, in general, $x\neq f(n)$ for any $n\in\mathbb{N}$.

(c) This contradicts the claim that we have enumerated all elements of $(0,1)$ in the list $\{f(n):n\in\mathbb{N}\}$, since $x\in(0,1)$ but $x\notin\{f(n):n\in\mathbb{N}\}$.

(3) 1.6.8
   (a) Suppose that $a' \in B$. This implies that $a' \in f(a') = B$. But B is defined as the set of all elements that aren't in their images, so that that $a' \notin B$ by definition. Contradiction.
   (b) Suppose that $a' \notin B$. This implies that $a' \notin f(a')$. But by the definition of B, this means that $a' \in B$, giving a contradiction.

(4) 1.6.9

Exercise 1.6.9. It is unlikely that there is a reasonably simple way to explicitly define a 1–1, onto mapping from $P(\mathbf{N})$ to $\mathbf{R}$. A more fruitful strategy is to make use of the ideas in Exercise 1.5.5 and 1.5.11. In particular, we have seen earlier in this section that $\mathbf{R} \sim (0,1)$. It is also true that $P(\mathbf{N}) \sim S$ where S is the set of all sequences consisting of 0s and 1s from Exercise 1.6.4. To see why, let $A \in P(\mathbf{N})$ be an arbitrary subset of $\mathbf{N}$. Corresponding to this set A is the sequence (a_n) where $a_n = 1$ if $n \in A$ and $a_n = 0$ otherwise. It is straightforward to show that this correspondence is both 1–1 and onto, and thus $P(\mathbf{N}) \sim S$.

With a nod to Exercise 1.5.5, we can conclude that $P(\mathbf{N}) \sim \mathbf{R}$ if we can demonstrate that $S \sim (0,1)$. Proving this latter fact is easier, but it is still not easy by any means. One way to avoid some technical details, is to use the Schröder–Bernstein Theorem (Exercise 1.5.11). Rather than finding a 1–1, onto function, the punchline of this result is that we will be done if we can find two 1–1 functions, one mapping $(0,1)$ into S, and the other mapping S into $(0,1)$. There are a number of creative ways to produce each of these functions.

Let's focus first on mapping $(0,1)$ into S. A fairly natural idea is to think in terms of binary representations. Given $x \in (0,1)$ let's inductively define a sequence (x_n) in the following way. First, bisect $(0,1)$ into the two parts $(0,1/2)$ and $[1/2,1)$. Then set $x_1 = 0$ if x is in the left half, and $x_1 = 1$ if x is in the right half. Now let I be whichever of these two intervals contains x, and bisect it using the same convention of including the midpoint in the right half. As before we set $x_2 = 0$ if x is in the left half of I and $x_2 = 1$ if x is in the right half. Continuing this process inductively, we get a sequence $(x_n) \in S$ that is uniquely determined by the given $x \in (0,1)$, and thus the mapping is 1–1.

It may seem like this mapping is onto S but it falls just short. Because of our convention about including the midpoint in the right half of each interval, we never get a sequence that is eventually all 1s, nor do we get the sequence of all 0s. This is fixable. The collection of all sequences in S that are NOT in the range of this mapping form a countable set, and it is not too hard to show that the cardinality of S with a countable set removed is the same as the cardinality of S. The other option is to use the Schröder–Bernstein Theorem mentioned previously. Having found a 1–1 function from $(0,1)$ into S, we just need to produce a 1–1 function that goes the other direction. An example of such a function would be the one that takes $(x_n) \in S$ and maps it to the real number with decimal expansion $.x_1x_2x_3x_4\ldots$. Because the only decimal expansions that aren't unique involve 9s, we can be confident that this mapping is 1–1.

The Schröder–Bernstein Theorem now implies $S \sim (0, 1)$, and it follows that $P(\mathbf{N}) \sim \mathbf{R}$.