MATH 321 DAY 9 - THE AXIOM OF COMPLETENESS

Exercise 1. Present claimed Week 2 HW. Class: participate by offering suggestions when mistakes are made or if you have a different idea of how to do the proof.

(1) Abbott Exercise 1.2.3
(2) Abbott Exercise 1.2.5
(3) Abbott Exercise 1.2.12
(4) Abbott Exercise 1.3.8

Set	sup	inf
$\{m/n : m, n \in \mathbb{N}, m < n\}$	1	0
$\{(-1)^m/n : m, n \in \mathbb{N}\}$	1	-1
$\{n/(3n+1) : n \in \mathbb{N}\}$	1/3	1/4
$\{m/(m+n) : m, n \in \mathbb{N}\}$	1	0

1. Completeness

- Two weeks ago, we discussed the need to rigorously address questions about the real numbers in the "Why Be So Critical?" module.
- Before we can ask such questions, we need a definition of the real numbers to start from.
- If we assume we know what rational numbers are, can we define the real numbers without resorting to unproven axioms?

Question 2. *What is a real number?*

- *An extension of the rational numbers $\mathbb{Q}$ in which there are no holes or gaps*
- *But what kind of extension?*
- *How do we define $\mathbb{R}$ as a set (if we assume $\mathbb{Q}$ is already defined)?*

Definition 3. $\mathbb{R}$ is **an ordered field** which satisfies the **Axiom of Completeness** and contains $\mathbb{Q}$ as a **subfield.**

(1) $\mathbb{R}$ is a set containing $\mathbb{Q}$.
(2) $+, \times$ extend from $\mathbb{Q}$ in such a way that every element of $\mathbb{R}$ has an additive inverse and every nonzero element of $\mathbb{R}$ has a multiplicative inverse.
(3) $\mathbb{R}$ is a **field**: $+, \times$ are commutative, associative, and distributive.
(4) The ordering $<$ on $\mathbb{Q}$ extends to $\mathbb{R}$ in such a way that the familiar properties of the ordering still hold.
(5) **Axiom of Completeness:** every nonempty set of real numbers that is bounded above has a least upper bound.

[start here 9/12]

(1) An **axiom** in mathematics is a statement that's accepted, to be used without proof.
(2) Why should we accept the Axiom of Completeness?
    (a) It's actually impossible to do calculus without accepting an axiom like this.
    (b) It's impossible to prove this statement with the existing axioms; we have to assume it.

Definition: For a set $A \subset \mathbf{R}$, we say

- A is *bounded above* if there exists a number $b \in \mathbf{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A.
- $a_0 = \max(A)$ if $a_0 \in A$ and $a_0 \geq a$ for all $a \in A$ We might also say a_0 is the *maximum* of A.
- A real number s is the *least upper bound* (or sometimes, we use the Latin word *supremum*) for a set $A \subset \mathbf{R}$ if s is an upper bound for A and if for every upper bound b of A, we have $s \leq b$. If s exists, we use the notation $s = \sup(A)$.

Exercise 4. Let
$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}.$$
The set A is bounded above and below.

(1) Show that $\sup A = 1$ by showing:
   (a) 1 is an upper bound for A.
      - $n \geq 1 \forall n \in \mathbb{N} \implies \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$.
   (b) If b is an upper bound for A, we have $b \geq 1$.
      - Since $1 \in A$ and b is an upper bound for A, $b \geq 1$.

(2) Does the Axiom of Completeness hold for $\mathbb{Q}$?
   - The set $S = \{r \in \mathbb{Q} : r^2 < 2\}$. Since $\sqrt{2}^2 = 2$, $\sqrt{2} = \sup S \in \mathbb{R}$. Since $\sqrt{2} \notin \mathbb{Q}$, the Axiom of Completeness fails for $\mathbb{Q}$.

- How does the Axiom of Completeness mean that $\mathbb{Q}$ has "holes" which are filled in in $\mathbb{R}$?

Exercise 5. (**Reading Question**) TPS Exercises 1.3.1, 1.3.2

Lemma 1.3.8. *Assume $s \in \mathbf{R}$ is an upper bound for a set $A \subseteq \mathbf{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

Exercise 1.3.1. (a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Exercise 1.3.2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

- 1.3.1(b): Consider $-A = \{-a : a \in A\}$. Then, by Lemma 1.3.8, $-s = \sup(-A)$ if and only if, for every $\epsilon > 0$, there exists an $a \in A$ satisfying $-s - \epsilon < -a$. But $-s = \sup(-A) \iff s = \inf A$ $(*)$. Thus,
$$\begin{aligned} s = \inf A &\iff -s = \sup(-A) \iff \forall \epsilon > 0 \exists -a \in -A \text{ s.t. } -s - \epsilon < -a \\ &\iff \forall \epsilon > 0 \exists a \in A \text{ s.t. } a - \epsilon < s. \end{aligned}$$
  Now, prove $(*)$.
  - Suppose $-s = \sup(-A)$. Then $-s \geq -a$ for all $-a \in -A$ and, if α is another upper bound for $-A$, then $-s \leq \alpha$. This means that $s \leq a$ for all $a \in A$, so that s is a lower bound for A. Suppose that L is another lower bound for A; then $-L$ is an upper bound for $-A$, meaning that $-s \leq -L$, implying that $s \geq L$. Thus, $s = \inf A$.

- 1.3.2
  (1) $B = \{0\}$
  (2) Impossible; all finite sets contain their suprema. For if $A = \{x_1, \ldots, x_n\}$ is finite and $s = \sup(A)$, then $M := \max\{x_1, \ldots, x_n\}$ is an upper bound for A, so $s \leq M$. But since s is an upper bound for A and $M \in A$, $s \geq M$. Hence $s = M \in A$.
  (3) $C = \{\frac{1}{n} : n \in \mathbb{N}\}$

Exercise 6. [slide] If false, give a counterexample. If true, prove it.

For some of the problems below (*e.g.* the ones involving ϵ), it may help to draw a number line.

1. If $A = \{\frac{n}{n+1} \mid n \in \mathbf{N}\}$, then which if any of these numbers are an upper bound for A: $\frac{1}{2}$, 1, 5?
2. When does $\max(A) \neq \sup(A)$?
3. [T/F] An upper bound for $A \subset \mathbf{R}$ is necessarily an element of A.
4. [T/F] A least upper bound for $A \subset \mathbf{R}$ is necessarily an element of A.
5. [T/F] A set $A \subset \mathbf{R}$ has at least one maximum.
6. [T/F] A set $A \subset \mathbf{R}$ has at most one maximum.
7. [T/F] A set $A \subset \mathbf{R}$ has at least one upper bound.
8. [T/F] A set $A \subset \mathbf{R}$ has at most one upper bound.
9. [T/F] A set $A \subset \mathbf{R}$ has at least one least upper bound.
10. [T/F] A set $A \subset \mathbf{R}$ has at most one least upper bound.

In the problems below, assume s is an upper bound for A and that $A \neq \emptyset$.

11. [T/F] Then $s = \sup(A)$ if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.
12. [T/F] Then $s = \sup(A)$ only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.
13. [T/F] Then $s = \sup(A)$ if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon > a$.
14. [T/F] Then $s = \sup(A)$ only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon > a$.
15. [T/F] Then $s = \sup(A)$ if for every $\epsilon > 0$, there exists $a \in A$ such that $s + \epsilon > a$.
16. [T/F] Then $s = \sup(A)$ only if for every $\epsilon > 0$, there exists $a \in A$ such that $s + \epsilon > a$.

(1) $1, 5$
(2) When A is an infinite set which is bounded above, $\max(A)$ may not exist, (e.g. $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ has no maximum) while $\sup(A)$ always does by the Axiom of Completeness. When A is a finite set which is bounded above, $\max(A)$ exists and equals $\sup(A)$ by the above reasoning.
(3) False; consider the set A from #1 and the upper bound 5.
(4) False; consider the set A from #1 and its least upper bound 1.
(5) False; let $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ as above.
(6) True; suppose that M and N are both maxima of A. Then $M \in A$ and $M \geq a$ for all $a \in A$, and similarly for N. But since $N \in A$, $M \geq N$, and since $M \in A$, $N \geq M$. Hence $N = M$.
(7) False; let $A = \mathbb{N}$.
(8) False; the set A from #1 has infinitely many upper bounds.
(9) False; let $A = \mathbb{N}$.
(10) True; suppose that s and t are suprema of A and suppose for contradiction (WLOG) that $s > t$. Define $\epsilon := s - t > 0$. Then by Lemma 1.3.8 there exists an element $a \in A$ satisfying $s - \epsilon < a$. But

this implies that

$$t = s - (s - t) = s - \epsilon < a$$

contradicting that t is an upper bound for A.

(11) True; this is the forward direction of Lemma 1.3.8.

(12) True; this is the backward direction of Lemma 1.3.8.

(13)