MATH 321 WEEK 10 UNCLAIMED PROBLEM SOLUTIONS

(1) 2.5.6

- Let $b \geq 0$. Then the sequence $(b^{1/n})$ is monotone decreasing and bounded below by 0; therefore, $\lim(b^{1/n})$ exists, and we may call it L.
- To compute L, notice that $(b^{1/2n})$ is a subsequence, so $(b^{1/2n}) \to L$ by Theorem 2.5.2. But $(b^{1/2n})(b^{1/2n}) = b^{1/n}$, so by the Algebraic Limit Theorem,

$$L = \lim(b^{1/n}) = \lim(b^{1/2n})\lim(b^{1/2n}) = L \cdot L.$$

Because limits are unique, $L = L^2$, and thus $L = 0$.

(2) 2.6.2

(a) $(\frac{(-1)^n}{n})$ converges, hence is Cauchy, but is not monotone.

(b) Impossible. If (a_n) is Cauchy, then it converges, and therefore (a_n) must be bounded. Hence, it's impossible for a Cauchy sequence to have an unbounded subsequence.

(c) Impossible. If (a_{n_k}) is a Cauchy subsequence, then (a_{n_k}) converges to some limit a, hence is bounded: there exists $M \in \mathbb{R}$ so that $|a_{n_k}| \leq M$ for all k. If (a_n) were also bounded, it would converge by the Monotone Convergence Theorem; therefore, (a_n) must be unbounded. Suppose that (a_n) is monotone increasing; the decreasing case follows similarly. Then there exists N so that $a_N > M$, and in fact $a_n \geq a_N > M$ for all $n \geq N$ by monotonicity. In that case, we can choose k so that $n_k > N$ (otherwise we'd contradict the infinite-ness of (a_{n_k})). By monotonicity, $a_{n_k} \geq a_N > M$ as well, a contradiction since we assumed (a_{n_k}) was bounded by M. Therefore, no divergent, monotone sequence can have a Cauchy subsequence.

(d) The sequence $(0, 1, 0, 2, 0, 3, 0, 4, \dots)$ is certainly unbounded, but the subsequence $(0, 0, 0, 0, \dots)$ converges to 0, and hence is Cauchy.

(3) 2.6.4

(a) This sequence is Cauchy. Since (a_n) and (b_n) are Cauchy, there exist $N_1, N_2 \in \mathbb{N}$ so that when $n \geq N = \max\{N_1, N_2\}$, $|a_{n+1} - a_n|$ and $|b_{n+1} - b_n|$ are both less than $\frac{\epsilon}{2}$. Moreover, whenever $n \geq N$, the reverse triangle inequality tells us that

$$\begin{aligned}
\Big||a_{n+1} - b_{n+1}| - |a_n - b_n|\Big| &\leq \Big|(a_{n+1} - b_{n+1}) - (a_n - b_n)\Big| \\
&= |(a_{n+1} - a_n) + (b_n - b_{n+1})| \\
&\leq |a_{n+1} - a_n| + |b_{n+1} - b_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

(b) This sequence is not guaranteed to be Cauchy. Consider the sequence $s_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$ of partial sums of the alternating harmonic series. Then (s_n) converges because the series does, so (s_n) is Cauchy. Now, let's define $t_n = (-1)^n s_n$ and consider the differences between successive terms of (t_n). The difference between t_n and t_{n+1}, in absolute value, is

$$\begin{aligned}
|t_{n+1} - t_n| &= \Big|(-1)^{n+1}\sum_{k=1}^{n+1}\frac{(-1)^k}{k} - (-1)^n\sum_{k=1}^{n}\frac{(-1)^k}{k}\Big| \\
&= |(-1)^{n+1}|\Big|\sum_{k=1}^{n+1}\frac{(-1)^k}{k} + \sum_{k=1}^{n}\frac{(-1)^k}{k}\Big| \\
&= \Big|\sum_{k=1}^{n+1}\frac{(-1)^k}{k} + \sum_{k=1}^{n}\frac{(-1)^k}{k}\Big| = |s_{n+1} + s_n|,
\end{aligned}$$

which we can't hope to make smaller than any $\epsilon > 0$ given.

(c) Consider the sequence (a_n), where $a_n = \frac{(-1)^n}{n}$ for all n. Then

$$\lfloor a_n \rfloor = \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{if } n = 2, 4, 6, 8, \dots \\ -1 & if n = 3, 5, 7, 9, \dots \end{cases}$$

so that $(\lfloor a_n \rfloor) = (-1, 0, -1, 0, -1, 0, -1, 0, \dots)$ does not converge, hence is not Cauchy. However, $(a_n) \to 0$, so (a_n) is Cauchy.