MATH 321 WEEK 5 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 1.5.3

Exercise 1.5.3. (a) Because A_1 is countable, there exists a 1–1 and onto function $f : \mathbf{N} \to A_1$.

If $B_2 = \emptyset$, then $A_1 \cup A_2 = A_1$ which we already know to be countable.

If $B_2 = \{b_1, b_2, \ldots, b_m\}$ has m elements then define $h : A_1 \cup B_2$ via

$$h(n) = \begin{cases} b_n & \text{if } n \leq m \\ f(n-m) & \text{if } n > m. \end{cases}$$

The fact that h is a 1–1 and onto follows immediately from the same properties of f.

If B_2 is infinite, then by Theorem 1.5.7 it is countable, and so there exists a 1–1 onto function $g : \mathbf{N} \to B_2$. In this case we define $h : A_1 \cup B_2$ by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that h is 1–1 and onto is derived directly from the fact that f and g are both bijections. Graphically, the correspondence takes the form

$$\begin{array}{cccccccc} \mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2: & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

To prove the more general statement in Theorem 1.5.8, we may use induction. We have just seen that the result holds for two countable sets. Now let's assume that the union of m countable sets is countable, and show that the union of $m+1$ countable sets is countable.

Given $m+1$ countable sets $A_1, A_2, \ldots, A_{m+1}$, we can write

$$A_1 \cup A_2 \cup \cdots \cup A_{m+1} = (A_1 \cup A_2 \cup \cdots \cup A_m) \cup A_{m+1}.$$

Then $C_m = A_1 \cup \cdots \cup A_m$ is countable by the induction hypothesis, and $C_m \cup A_{n+1}$ is just the union of two countable sets which we know to be countable. This completes the proof.

(b) Induction cannot be used when we have an infinite number of sets. It can only be used to prove facts that hold true for each value of $n \in \mathbf{N}$. See the discussion in Exercise 1.2.13 for more on this.

(c) Let's first consider the case where the sets $\{A_n\}$ are disjoint. In order to achieve 1-1 correspondence between the set $\mathbf{N}$ and $\bigcup_{n=1}^{\infty} A_n$, we first label the elements in each countable set A_n as

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}.$$

Now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in an array similar to the one for $\mathbf{N}$ given in the exercise:

$$\begin{array}{llllll}
A_1 = a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
A_2 = a_{21} & a_{22} & a_{23} & a_{24} & \cdots & \\
A_3 = a_{31} & a_{32} & a_{33} & \cdots & & \\
A_4 = a_{41} & a_{42} & \cdots & & & \\
A_5 = a_{51} & \cdots & & & & \\
\quad\vdots & & & & &
\end{array}$$

This establishes a 1–1 and onto mapping $g : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ where $g(n)$ corresponds to the element a_{jk} where (j, k) is the row and column location of n in the array for $\mathbf{N}$ given in the exercise.

If the sets $\{A_n\}$ are not disjoint then our mapping may not be 1–1. In this case we could again replace A_n with $B_n = A_n \backslash \{A_1 \cup \cdots \cup A_{n-1}\}$. Another approach is to use the previous argument to establish a 1–1 correspondence between $\bigcup_{n=1}^{\infty} A_n$ and an infinite *subset* of $\mathbf{N}$, and then appeal to Theorem 1.5.7.

(2) 1.5.7

Exercise 1.5.7. (a). The function $f(x) = (x, \frac{1}{3})$ is 1–1 from $(0, 1)$ to S.

(b) Given $(x, y) \in S$, let's write x and y in their decimal expansions

$$x = .x_1x_2x_3\dots \quad \text{and} \quad y = .y_1y_2y_3\dots$$

where we make the convention that we always use the terminating form (or repeated 0s) over the repeating 9s form when the situation arises.

Now define $f : S \to (0, 1)$ by

$$f(x, y) = .x_1y_1x_2y_2x_3y_3\dots$$

In order to show f is 1–1, assume we have two distinct points $(x, y) \neq (w, z)$ from S. Then it must be that either $x \neq w$ or $y \neq z$, and this implies that in at least one decimal place we have $x_i \neq w_i$ or $y_i \neq z_i$. But this is enough to conclude $f(x, y) \neq f(w, z)$.

The function f is not onto. For instance the point $t = .555959595\dots$ is not in the range of f because the ordered pair (x, y) with $x = .555\dots$ and $y = .5999\dots$ would not be allowed due to our convention of using terminating decimals instead of repeated 9s.

(3) 1.6.3

Exercise 1.6.3. (a) If we imitate the proof to try and show that $\mathbf{Q}$ is uncountable, we can construct a real number x in the same way. This x will again fail to be in the range of our function f, *but there is no reason to expect x to be rational.* The decimal expansions for rational numbers either terminate or repeat, and this will not be true of the constructed x.

(b) By using the digits 2 and 3 in our definition of b_n we eliminate the possibility that the point $x = .b_1b_2b_3\dots$ has some other possible decimal representation (and thus it cannot exist somewhere in the range of f in a different form.)