MATH 321 DAY 5 - WHY BE SO CRITICAL?

[See Why Be So Critical? Instructor Resources]

IMPLEMENTATION METHOD I. Students are assigned to read the entire PSP and respond (in writing) to the questions therein prior to class discussion. Typically, the author assigns this reading one week prior to a class discussion of it; other instructors have confirmed that sufficient time for careful advance reading is important for high quality in-class discussions. Students are encouraged to discuss the readings and PSP questions with each other or with the instructor (outside of class time) before the assigned due date (provided their written responses are their own). While there is no prohibition against using additional resources to complete the PSP (e.g., a calculus text), it is important to assure students that there is no need to do any historical research in order to complete it.

On the assignment due date, a whole class discussion (45 - 50 minutes) of the reading is conducted by the instructor, with student responses to various PSP questions elicited during that discussion. An instructor-prepared handout containing solutions to select questions (especially Question #2) can be helpful during this discussion. The completed written work is typically collected at the close of that class period; however, the discussion could also be conducted after the instructor has collected and read students' written PSP work. The author does evaluate students' individual written work for a grade. That evaluation and grade is based primarily on completeness, but also takes into account both presentation (e.g., use of complete sentences) and accuracy (particularly with regard to the mathematical details in Questions # 2, 5, 6).

A brief set of summary notes that could be used by an instructor during a whole class discussion of the PSP is offered below (pages 13–14). Although some type of summarizing discussion is highly recommended, that discussion need not adhere to the notes provided here.

1. Summary Discussion

- Caution that one of the difficulties with historical readings is that the meanings of words change over time; for example, 'geometer' referred to any mathematician (not just someone who worked with geometry)
- Overview of pre-nineteenth century calculus themes:

Time Period	Focus *What objects should we study?*	Primary justification of "correctness" *How do we know our mathematics is "true"?*
17th century	Calculus of CURVES (using algebra as a tool)	New methods produce results that matched "old" (known) results (obtained from geome
18th century	Calculus of FUNCTIONS (with physics as primary motivation) NEW QUESTION: What is a function really? Related historical controversies: Fourier Series Convergence Vibrating String Problem	Methods produce correct predictions (in physics) NEW CONCERN: Is it valid to borrow "truths" from one domain (e.g., geometry, physics) to justify truths in another (e.g., mathemat

- Overview of the situation at the end of 18th/start of 19th century (Four main points, I – IV)
  (1) Increasing mistrust of "geometric" intuition as valid proof method for "analytic" truths (and more general frustration that analytic "truths" are being verified by non-analytic 'proofs")
    (a) Ask for evidence of this in the assigned reading.
  (2) Concern that existing 'algebraic' proof methods lack adequate rigor
    (a) Ask for evidence of this in the assigned reading; two subthemes to elicit here:

   (i) Euclid had long been a model of rigor; nineteenth century mathematicians express desire to bring back something like an axiomatic approach as a foundation for certain knowledge –
   (ii) algebra allows too much generality (e.g., unrestricted) Makes it too easy to assume that properties (e.g., continuity, rationality) that hold at all "lower" values will also hold in the limit (elicit or mention Abel power series example here)

(3) Use of power series (in particular) lacks firm foundation, though some mathematicians at the time advocated for its centrality in calculus. Ask for evidence of this in the assigned reading; two mathematical points to elicit in particular:
   (a) Discuss current views about $\sum_{n=1}^{\infty} x^n$ (converges for $-1 < x < 1$ but diverges for $x = \pm 1$)
   (b) Discuss Abel's use of the phrase 'x less than 1' here (where today we would write '$|x| < 1$'). –
   (c) Abel mentions we could also have convergence for $|x| \leq 1$ with $\lim \phi(x) \neq \phi(1)$.
   (d) Ask students for their answers to Question 4 and 5 here.

(4) General concerns about foundations: If we don't base calculus on power series, what do we use instead?
   (a) Some possibilities (and early proponents of each):
      (i) Fluxions (Newton); Infinitesimals (Leibniz); **Limits** (d'Alembert) ← **The "winner"!** –
      (ii) Chosen option of 'limit' raises yet another new question: What is a limit really??
      (iii) Ultimate nineteenth century response to this set of concerns: Require FORMAL PROOFS as way to certify knowledge via RIGOROUS use of INEQUALITIES as way to talk about 'being close'
      (iv) Historical Aside: Another factor that influenced this direction were new teaching & research situations (École Polytechnique) that required thinking carefully about ideas in order to explain them to others. – This nineteenth century response, which forms the basis of the work we will do together throughout this course, is often described as 'the arithmetization of analysis'.
      (v) Primary justification of "correctness" How do we know our mathematics is "true"?
         (A) New methods produce results that matched "old" (known) results (obtained from geometry)
         (B) Methods produce correct predictions (in physics)
         (C) NEW CONCERN: Is it valid to borrow "truths" from one domain (e.g., geometry, physics) to justify truths in another (e.g., mathematics)?

Exercise 1. (Question 5)

(1) The power series $1 + x + x^2 + x^3 + \cdots = \sum_{r=1}^{\infty} x^r$ converges for $|x| < 1$ to $\phi(x) = \frac{1}{1-x}$ (Geometric Series formula)
(2) If $\lim_{x \to 1} \phi(x) \neq \phi(1)$, we say ϕ is **discontinuous** at $a = 1$.

Exercise 2. (Question 6)

(1) Instead of a polynomial, it's a sum of sines evaluated at multiples of x.
(2) For $x = \pi$, this formula says

$$\frac{\pi}{2} = \sin \pi - \frac{1}{2} \sin 2\pi + \frac{1}{3} \sin 3\pi - \dots$$

but the sine function is 0 at all multiples of π, hence the equation above becomes $\frac{\pi}{2} = 0$.

(3) Differentiating both sides of the series equation, where the series is differentiated term-by-term, yields

$$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \dots .$$

Now, again plugging in $x = \pi$, we obtain

$$\begin{aligned}\frac{1}{2} &= \cos\pi - \cos 2\pi + \cos 3\pi - \dots \\ &= 1 - (-1) + 1 - (-1) + \dots \\ &= \sum_{k=1}^{\infty} 1\end{aligned}$$

which is absurd.