MATH 321 WEEK 3 CLAIMED PROBLEM SOLUTIONS

(1) 1.2.7

Exercise 1.2.7. (a) $f(A) = [0, 4]$ and $f(B) = [1, 16]$. In this case, $f(A \cap B) = f(A) \cap f(B) = [1, 4]$ and $f(A \cup B) = f(A) \cup f(B) = [0, 16]$.

(b) Take $A = [0, 2]$ and $B = [-2, 0]$ and note that $f(A \cap B) = \{0\}$ but $f(A) \cap f(B) = [0, 4]$.

(c) We have to show $y \in g(A \cap B)$ implies $y \in g(A) \cap g(B)$. If $y \in g(A \cap B)$ then there exists an $x \in A \cap B$ with $g(x) = y$. But this means $x \in A$ and $x \in B$ and hence $g(x) \in g(A)$ and $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cap g(B)$.

(d) Our claim is $g(A \cup B) = g(A) \cup g(B)$. In order to prove it, we have to show,

$$g(A \cup B) \subseteq g(A) \cup g(B) \text{ and,} \tag{1}$$

$$g(A) \cup g(B) \subseteq g(A \cup B). \tag{2}$$

To demonstrate part (1), we let $y \in g(A \cup B)$ and show $y \in g(A) \cup g(B)$. If $y \in g(A \cup B)$ then there exists $x \in A \cup B$ with $g(x) = y$. But this means

$x \in A$ or $x \in B$, and hence $g(x) \in g(A)$ or $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cup g(B)$.

To demonstrate the reverse inclusion, we let $y \in g(A) \cup g(B)$ and show $y \in g(A \cup B)$. If $y \in g(A) \cup g(B)$ then $y \in g(A)$ or $y \in g(B)$. This means we have an $x \in A$ or $x \in B$ such that $g(x) = y$. This implies, $x \in A \cup B$, and hence $g(x) \in g(A \cup B)$. Since we have shown parts (1) and (2), we can conclude $g(A \cup B) = g(A) \cup g(B)$.

(2) 1.2.8

Exercise 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1–1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1–1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1–1.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1–1 and onto.

(a) $f(n) = n^2$ is one example.

(b) This one's a bit tricky, but we can define

$$f(n) = \begin{cases} n-1, & n \geq 1 \\ 0, & n = 0 \end{cases}.$$

Then $f(0) = 0 = f(1)$, so this function is not onto, but $k = f(k+1)$ for all $k \neq 0$, making the function onto.

(c) One such function is $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(x) = \begin{cases} x/2, & x \text{ even} \\ -(x+1)/2, & x \text{ odd} \end{cases}.$$

Since 1-1 correspondence is an equivalence relation, students may equivalently define $g : \mathbb{Z} \to \mathbb{N}$ that is 1-1 and onto. Define

$$g(n) = \begin{cases} 2|n| - 1 & \text{if } n < 0 \\ 2n & \text{if } n \geq 0 \end{cases}.$$

Then check it's 1-1 and onto.

(3) 1.2.9

Exercise 1.2.9. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Exercise 1.2.9. (a) $f^{-1}(A) = [-2,2]$ and $f^{-1}(B) = [-1,1]$. In this case, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) = [-1,1]$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = [-2,2]$.

(b) In order to prove $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show,

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cap B)$ and show $x \in g^{-1}(A) \cap g^{-1}(B)$. So, if $x \in g^{-1}(A \cap B)$ then $g(x) \in (A \cap B)$. But this means $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This implies, $x \in g^{-1}(A) \cap g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cap g^{-1}(B)$ and show $x \in g^{-1}(A \cap B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. This implies $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This means, $x \in g^{-1}(A \cap B)$.

Similarly, in order to prove $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$, we have to show,

$$g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cup B)$ and show $x \in g^{-1}(A) \cup g^{-1}(B)$. So, if $x \in g^{-1}(A \cup B)$ then $g(x) \in (A \cup B)$. But this means $g(x) \in A$ or $g(x) \in B$, which implies $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. From this we know $x \in g^{-1}(A) \cup g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cup g^{-1}(B)$ and show $x \in g^{-1}(A \cup B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. This implies $g(x) \in A$ or $g(x) \in B$, and hence $g(x) \in A \cup B$. This means, $x \in g^{-1}(A \cup B)$.

(4) 1.2.10: Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:
   (a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.
      (i) False. $1 < 1 + \epsilon$ for every $\epsilon > 0$, yet $1 \not< 1$.
   (b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.
      (i) False. $1 < 1 + \epsilon$ for every $\epsilon > 0$, yet $1 \not< 1$.
   (c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.
      (i) We prove the contrapositive: two real numbers satisfy $a > b$ if and only if there exists $\epsilon > 0$ such that $a \geq b + \epsilon$.
      (ii) ($\Longrightarrow$) Suppose that $a > b$. Then define $\epsilon = a - b > 0$. Now,

$$b + \epsilon = b + (a - b) = a \leq a.$$

      (iii) ($\Longleftarrow$) Suppose that there exists $\epsilon > 0$ so that $a \geq b + \epsilon$. Then $a - b \geq \epsilon > 0$, hence $a > b$.

(5) 1.2.13

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated valid statement. Provide a proof that does not use induction.

Exercise 1.2.13. (a) From Exercise 1.2.5 we know $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ which proves the base case. Now we want to show that

if we have $(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$, then it follows that

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = A_1^c \cap A_2^c \cap \cdots \cap A_{n+1}^c.$$

Since the union of sets obey the associative law,

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = ((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1})^c$$

which is equal to

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c.$$

Now from our induction hypothesis we know that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

which implies that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1}^c.$$

By induction, the claim is proved for all $n \in \mathbf{N}$.

(b) Example 1.2.2 illustrates this phenomenon.

(c) In order to prove $(\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c$ we have to show,

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq \bigcap_{n=1}^{\infty} A_n^c \text{ and,} \tag{1}$$

$$\bigcap_{n=1}^{\infty} A_n^c \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right)^c. \tag{2}$$

To demonstrate part (1), we let $x \in (\bigcup_{n=1}^{\infty} A_n)^c$ and show $x \in \bigcap_{n=1}^{\infty} A_n^c$. So, if $x \in (\bigcup_{n=1}^{\infty} A_n)^c$ then $x \notin A_n$ for all $n \in \mathbf{N}$. This implies x is in the complement of each A_n and by the definition of intersection $x \in \bigcap_{n=1}^{\infty} A_n^c$.

To demonstrate the reverse inclusion, we let $x \in \bigcap_{n=1}^{\infty} A_n^c$ and show $x \in (\bigcup_{n=1}^{\infty} A_n)^c$. So, if $x \in \bigcap_{n=1}^{\infty} A_n^c$ then $x \in A_n^c$ for all $n \in \mathbf{N}$ which means $x \notin A_n$ for all $n \in \mathbf{N}$. This implies $x \notin (\bigcup_{n=1}^{\infty} A_n)$ and we can now conclude $x \in (\bigcup_{n=1}^{\infty} A_n)^c$.