MATH 321 2.4 - THE MONOTONE CONVERGENCE THEOREM & INFINITE SERIES

- Last time, we showed that convergent sequences are bounded (hence convergence implies vercongence).
- However, this is certainly not true the other way around.
- Who can give me an example of a bounded sequence that is not convergent?
- On the other hand, if a bounded sequence is *monotone*, then it is convergent.

Definition 1. A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotone** if it is either increasing or decreasing.

Exercise 2. [slides] [Annalisa Crannell Worksheets 2.3-2.4 ALT/OLT/MCT & 2.4: Series]

[10/29/18: Crannell 4-7; state theorems below, then go on to worksheet 2.5]

Theorem 3. ***(Monotone Convergence Theorem)*** *If a sequence is monotone and bounded, then it converges.*

- non-constructive
- but can be used to compute limits!

Exercise 4. Reading questions

(1) Prove that the sequence defined by $x_1 = 3$ and
$$x_{n+1} = \frac{1}{4 - x_n}$$
converges.

The first few terms are
$$\left(3, 1, \frac{1}{3}, \frac{3}{11}, \dots\right)$$
We want to show that the sequence satisfies the requirements of the Monotone Convergence Theorem. We need to show the following:

(a) $\{x_n\}$ is monotone decreasing. We use induction:

(i) Base case: $x_2 < x_1$. We compute
$$x_2 = \frac{1}{4 - x_1} = \frac{1}{4 - 3} = 1;$$
hence $x_2 = 1 < 3 = x_1$.

(ii) Inductive step: suppose $x_n < x_{n-1}$. Then
$$x_{n+1} = \frac{1}{4 - x_n} < \frac{1}{4 - x_{n-1}} = x_n$$
as desired.

(b) $\{x_n\}$ is bounded. This follows since

(i) $\{x_n\}$ is decreasing and hence bounded above by $x_1 = 3$; and

(ii) $\{x_n\}$ is bounded below by 0. We show this by induction: certainly $x_1 = 3 \geq 0$. Suppose $x_n \geq 0$. We know as well that $x_n \leq 3$, and hence that
$$x_{n+1} = \frac{1}{4 - x_n} \geq 0,$$
as desired.

(2) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

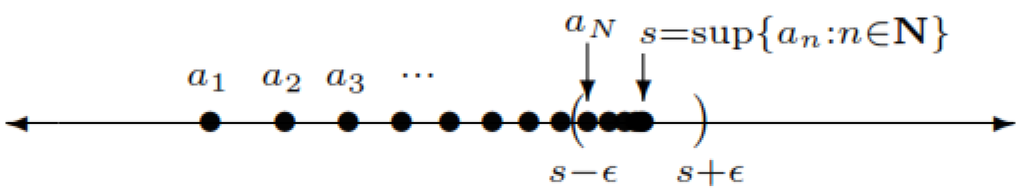
- Fix $\epsilon > 0$. Then, since $\{x_n\}$ converges to some limit x, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $|x_n - x| < \epsilon$.

- If we let $N = M - 1$, then whenever $n \geq N = M - 1$, we have that $n + 1 \geq M$, and hence
$$|x_{n+1} - x| < \epsilon,$$
showing that $\{x_{n+1}\} \to x$ as well.

(3) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.
- Let $x = \lim x_n$. Then by (b), we may take the limit of both sides to obtain
$$x = \lim x_{n+1} = \frac{1}{4 - \lim x_n} = \frac{1}{4 - x}.$$
- Multiplying both sides by $4 - x$ yields
$$4x - x^2 = x(4 - x) = 1$$
so that
$$x^2 - 4x + 1 = 0$$
and hence
$$x = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2}.$$
- Since $\{x_n\}$ is monotone decreasing and $x_1 = 3 < 4 + \sqrt{12}$, we conclude that
$$x = \lim x_n = \frac{4 - \sqrt{12}}{2}.$$

Proof. **(Monotone Convergence Theorem)** Let (a_n) be monotone and bounded.

- To prove (a_n) converges using the definition of convergence, we're going to need a candidate for the limit. Can anyone think of one using just what we know about the real numbers? (Operations, Axiom of Completeness, Algebraic & Order Limit Theorems)
- Let's assume the sequence is increasing (the decreasing case is handled similarly), and consider the *set* of points $\{a_n : n \in \mathbb{N}\}$.
- By assumption, this set is bounded, so we can let
$$s = \sup\{a_n : n \in \mathbb{N}\}.$$
- It seems reasonable to guess that $\lim a_n = s$.



- To prove this, let $\epsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ so that, whenever $n \geq N$, $|a_n - s| < \epsilon$.
  - Since $s = \sup\{a_n : n \in \mathbb{N}\}$, $s \geq a_n$ for all n, so that $|a_n - s| = s - a_n$.
  - Therefore, what we want to prove is equivalent to $s - a_n < \epsilon \implies s - \epsilon < a_n$ for all $n \geq N$.
- Because s is the least upper bound for $\{a_n : n \in \mathbb{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N so that $s - \epsilon < a_N$.
- Now, the fact that (a_n) is increasing implies that if $n \geq N$, then $a_N \leq a_n$, and hence for all $n \geq N$,
$$s - \epsilon < a_N < a_n \implies |a_n - s| = s - a_n < \epsilon,$$
as desired.

$\square$

Exercise 5. How would you prove the Monotone Convergence Theorem for a decreasing sequence?

- The Monotone Convergence Theorem is useful to show that series converge as well as sequences.

Definition 6. Let (x_n) be a sequence. An **infinite series** (or just **series**) is a formal expression of the form

$$\sum x_n = \sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \dots$$

Normally, when we think of a sum like "$3+5$", we think of it as the same as "8" or "$4+4$" or other equivalent expressions. This becomes trickier when we think of infinite series; for these we will consider these as what mathematicians describe as formal sums, meaning $3+5$ is just those three symbols, in that order.

The corresponding **sequence of partial sums** (s_m) is given by $s_m = x_1 + x_2 + \dots + x_m$, and we say the series $\sum_{n=1}^{\infty} x_n$ converges to S if and only if the sequence (s_m) converges to S. In this case, we write

$$\sum_{n=1}^{\infty} x_n = S.$$

- We have to say "formal expression" for reasons we discussed earlier–infinite sums don't always obey the rules that finite sums do. Order matters.

Exercise 7. (T/F) If true, prove it. If false, give a counterexample.

(1) If (x_n) is a sequence of positive real numbers, then the partial sums for the series $\sum x_n$ form a bounded sequence.

Example 8. False. Consider the sequence $(x_n) = (n)$. Then the partial sum $s_m = \sum_{k=1}^{m} k = \frac{m(m+1)}{2} \to \infty$, hence is unbounded.

(2) If (x_n) is a sequence of positive real numbers, then the partial sums for the series $\sum x_n$ form a monotone sequence.

Proof. True. Let (x_n) be a sequence of positive real nmbers. Then

$$s_{m+1} = \sum_{k=1}^{m+1} x_k = \sum_{k=1}^{m} x_k + x_{m+1} > \sum_{k=1}^{m} x_k = s_m$$

since in particular $x_{m+1} > 0$. Therefore, (s_m) is a monotone sequence. □

(3) The telescoping series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{n+1}\Big)$$

converges. (You can assume that infinite addition remains associative.)

Proof. The mth partial sum of the telescoping series is

$$\begin{aligned} s_m = \sum_{n=1}^{m} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) &= \Big(\frac{1}{1} - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \dots + \Big(\frac{1}{m} - \frac{1}{m+1}\Big) \\ &= 1 + \Big(-\frac{1}{2} + \frac{1}{2}\Big) + \Big(-\frac{1}{3} + \frac{1}{3}\Big) + \dots + \Big(-\frac{1}{m} + \frac{1}{m}\Big) - \frac{1}{m+1} \\ &= 1 - \frac{1}{m+1}. \end{aligned}$$

Hence, $\lim s_m = \lim(1 - \frac{1}{m+1}\Big) = 1 - 0 = 1$ by the Algebraic Limit Theorem and the fact that $\lim(\frac{1}{m}) = \lim(\frac{1}{m+1}) = 0$. □

(4) The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (Hint: consider the above sequence and the Monotone Convergence Theorem.)

Proof. Here we consider the sequence of partial sums (s_m) where

$$\begin{aligned} s_m &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{m^2} \\ &< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \cdots + \frac{1}{m(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= 1 + 1 - \frac{1}{m} \\ &< 2. \end{aligned}$$

Thus, 2 is an upper bound for the sequence of partial sums, so by the Monotone Convergence Theorem, $\sum_{n=1}^{\infty} 1/n^2$ converges to some (for the moment) unknown limit less than 2. (Note: the actual sum is $\pi^2/6$; the mathematician Leonhard Euler (of Euler's Theorem and Euler's Formula) gained fame when he determined the actual sum in 1735.] □

Example 9. Now consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

- Note that the harmonic series has an increasing sequence of partial sums,

$$s_m = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m},$$

that upon naive inspection seems like it may be bounded. However, 2 is no longer an upper bound because

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2.$$

- A similar calculation shows that $s_8 > 2\frac{1}{2}$, and we can see that in general

$$\begin{aligned} s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + 2^{k-1}\left(\frac{1}{2^k}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + k\left(\frac{1}{2}\right), \end{aligned}$$

which is unbounded. Thus, despite the incredibly slow pace, the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$ eventually surpasses every number on the positive real line.
- Because convergent sequences are bounded, the harmonic series diverges.
- This technique works in general:

Theorem 10. ***(Cauchy Condensation Test)*** *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \ldots$$

converges.

Corollary 11. *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

- A rigorous argument for this corollary requires a few basic facts about geometric series, which we'll deal with in Section 2.7.

Exercise 12. (T/F; prove or give a counterexample)

(1) If $\sum_{n=1}^{\infty} x_n$ converges, then $(x_n) \to 0$.
(2) If $(x_n) \to 0$, then $\sum_{n=1}^{\infty} x_n$ converges.