MATH 321 WEEK 9 CLAIMED PROBLEM SOLUTIONS

(1) 2.4.10

(a) We want to find

$$\prod_{n=1}^{\infty}\Big(1+\frac{1}{n}\Big).$$

We know that the sequence of partial products

$$p_m = 2(\frac{3}{2})(\frac{4}{3})\cdots(\frac{m+1}{m}) = m+1 \to \infty$$

so that the sequence diverges. In the case of $\prod(1+\frac{1}{n^2})$, the first few terms are

$$2(\frac{5}{4})(\frac{10}{9})(\frac{17}{16})\cdots\Big(\frac{m^2+1}{m^2}\Big).$$

We conjecture this converges since $(\frac{1}{m^2}) \to 0$ so much faster than $(\frac{1}{n})$.

(b) We want to show that $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

($\implies$) Let $s_n = \sum_{i=1}^{n} a_i$. We assume $\prod_{n=1}^{\infty}(1+a_n)$ converges, so that (p_n) converges. By FOILing, we get that

$$\begin{aligned} p_n &= (1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n) \\ &\geq 1+a_1+a_2+a_3+\cdots+a_n \\ &= 1+\sum_{k=1}^{n} a_k = 1+s_n. \end{aligned}$$

Since (p_n) converges, it's bounded, so $1+s_n \leq M$ for some M, therefore $s_n \leq M-1$ for all n.

($\impliedby$) Assume that $\sum a_n = A$. Note that $(1+a_n) \leq 3^{a_n}$ for all n, so that

$$p_n = \prod_{k=1}^{n}(1+a_n) \leq 3^{a_1}3^{a_2}\cdots 3^{a_n} = 3^{\sum_{n-1}^{\infty} a_n} = 3^A.$$

In addition, p_n is monotone increasing, so the Monotone Convergence Theorem tells us (p_n) converges.

(2) 2.5.3: Prove that if an infinite series $\sum a_n$ converges to a limit L, then $\sum a_n$ satisfies the associative property. In other words, prove that any rearrangement

$$(a_1+\cdots+a_{n_1})+(a_{n_1+1}+\cdots+a_{n_2})+(a_{n_2+1}+\cdots+a_{n_3})+\ldots$$

leads to a series that also converges to L.

Exercise 2.5.3. (a) Letting $s_n = a_1+a_2+\cdots+s_n$, we are given that $\lim s_n = L$. For the regrouped series, let's write

$$\begin{aligned} b_1 &= a_1 + a_2 + \cdots + a_{n_1}, \\ b_2 &= a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}, \\ &\vdots \\ b_m &= a_{n_{m-1}+1} + \cdots + a_{n_m}, \end{aligned}$$

and the claim is that the series $\sum_{m=1}^{\infty} b_m$ converges to L as well.

To prove this, just observe that if (t_m) is the sequence of partial sums for the regrouped series, then

$$\begin{aligned} t_m &= b_1 + b_2 + \cdots + b_m \\ &= (a_1 + \cdots + a_{n_1}) + \cdots + (a_{n_{m-1}+1} + \cdots + a_{n_m}) = s_{n_m}. \end{aligned}$$

which means that (t_m) is a subsequence of (s_n) and therefore converges to L by Theorem 2.5.2.

(b) Our proof here does not apply to the example at the end of Section 2.1 because in that case the original series does not converge. The result proved here says that if the series converges, then the associative property holds, but it says nothing about what happens when the original series does not converge.

(3) 2.5.5: Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that $(a_n) \to a$.

Proof. Suppose for contradiction that $(a_n) \not\to a$. Then, there exists $\epsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $n_k \geq N$ with the property that $|a_{n_k} - a| \geq \epsilon$. As we let N range from 1 upward, deleting all n_k which are duplicates or earlier in the sequence than the previous n_k, we obtain a subsequence (a_{n_k}) with the property that $|a_{n_k} - a| \geq \epsilon$ for all $k \in \mathbb{N}$. Hence, $(a_{n_k}) \not\to a$. However, the problem is that (a_{n_k}) may diverge without giving a contradiction. So we pass again to a subsequence.

Since (a_n) is bounded, so is (a_{n_k}), which means that (a_{n_k}) contains a convergent subsequence $(a_{n_{k_l}})_{l=1}^{\infty}$. By assumption, $(a_{n_{k_l}}) \to a$. However, this contradicts that, for all $N \in \mathbb{N}$, there exists $n_{k_l} \geq N$ such that $|a_{n_{k_l}} - a| \geq \epsilon$—in other words, $(a_{n_{k_l}})$ cannot possibly converge to a. This contradiction completes the proof. $\square$

(4) 2.5.8: Given a sequence (x_n), a particular term x_m is a **peak term** if no later term in the sequence exceeds it; that is, if $x_m \geq x_n$ for all $n \geq m$.

(a) *Find exaamples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.* The sequence $(a_n) = n$ has zero peak terms, $(1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots)$ has one, $(2, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4})$ has two, and $(0, 1, 0, 1, 0, 1, \dots)$ has infinitely many peak terms (all the 1s).

(b) *Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano-Weierstrass Theorem.* Let (a_n) be an arbitrary sequence. If (a_n) has zero peak terms, then (a_n) is already monotone nonincreasing. If (a_n) has finitely many peak terms, the subsequence created by removing the peak terms is monotone nonincreasing. If (a_n) has infinitely many peak terms, the sequence of peak terms is monotone nondecreasing. In any case, (a_n) has a monotone subsequence. We can then prove the Bolzano-Weierstrass Theorem as follows: assuming (a_n) is bounded, any subsequence of (a_n) is also bounded. Moreover, we just showed that (a_n) has a monotone subsequence, so that sequence is both monotone and bounded, hence converges by the Monotone Convergence Theorem.