MATH 321 DAY 4 - SUBSETS AND PROVING SET INCLUSION; SET OPERATIONS

1. Notes

- ADD/DROP DEADLINE FRIDAY

2. Introduction to Sets

2.1. Subsets and proving set inclusion.

Definition 1. Let A and B be sets. We say that A is a **subset** of B if every element of A is an element of B, and we denote this by writing $A \subseteq B$ (or occasionally $B \supseteq A$).

If A is a subset of B and there exists an element of B that is not in A, then A is called a **proper subset** of B, write $A \subset B$.

Proof technique. To show that $A \subseteq B$, we must show that the statement

$$\text{if } x \in A, \text{ then } x \in B$$

is true.

Definition 2. Let A and B be sets. We say that A is **equal** to B, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

- Combined with the definition of subset, we see that proving $A = B$ is equivalent to proving

$$x \in A \implies x \in B \text{ and } x \in B \implies x \in A.$$

Example 3. Let

$$\begin{aligned} A &= \{1, 2, 3, 4, 5\} \\ B &= \{x | x = 2k \text{ for some } k \in \mathbb{N}\} \\ C &= \{x \in \mathbb{N} | x < 6\}. \end{aligned}$$

Then $\{4, 3, 2\} \subseteq A$, $3 \notin B$, and $C \subseteq A$.

Exercise 4. Which of the following are also true of these sets? Prove if true (using the Proof Technique above) or give a counterexample if false.

(1) $A \subseteq C$
(2) $C \subseteq B$
(3) $\{2\} \in A$
(4) $\{2, 4, 6, 8\} \subseteq B$
(5) $A = C$

Definition 5. The **empty set**, denoted $\emptyset$, is the set with no elements.

Theorem 6. *Let A be a set. Then $\emptyset \subseteq A$.*

Proof. To prove that $\emptyset \subseteq A$, we must establish that the implication

$$\text{if } x \in \emptyset, \text{ then } x \in A$$

is true. Since $\emptyset$ has no elements, it's true that every element of the empty set is also an element of A.

This is kind of like saying "if unicorns exist, then unicorns are pink". Since unicorns don't exist (sorry), we can say whatever we want about them following the clause "if unicorns exist", and it would be (vacuously) true of all unicorns. □

2.2. Basic set operations.

- There are three basic ways to form new sets from old ones.
  - *Union*: gluing together two sets to get a third. Denoted by the symbol $\cup$, think of this as meaning "or". For example, $x \in A \cup B$ if and only if $x \in A$ OR $x \in B$.
  - *Intersection*: using one set as a "cookie cutter" on another set; the "cookie" is the intersection. Denoted by the symbol $\cap$, meaning "and". For example, $x \in A \cap B$ if and only if $x \in A$ AND $x \in B$.
  - *Complement*: what remains after throwing out a subset B of a larger set A. Denoted B^c if it's obvious what A is (e.g. $A = \mathbb{R}$) or $A \setminus B$ to make explicit the larger set.
- More formally,

Definition 7. Let A and B be sets. Then

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$
$$A \setminus B = \{x \in A | x \notin B\} = B^c.$$

If $A \cap B = \emptyset$, A and B are said to be **disjoint**.

Exercise 8. Which of the following sets contain $\sqrt{2}$?

(1) $A = \{x \in \mathbb{Q} | x^2 < 3\}$
(2) $B = \{x \in \mathbb{R} | x^2 < 3\}$
(3) $C = A \cup B$
(4) $D = A \cap B$
(5) $E = A^c$

Exercise 9. [slide] Practice 1.6, 1.11