THINGS TO DISCUSS IN MATH 321 12-4-17: CLASSIFYING SUBSETS TOPOLOGICALLY

Definition 1. A set $P \subseteq \mathbb{R}$ is **perfect** if P is closed and every point of P is a limit point of P.

Two nonempty sets $A, B \in \mathbb{R}$ are **separated** if $\bar{A} \cup B$ and $A \cup \bar{B}$ are both empty.

A set $E \subseteq \mathbb{R}$ is **disconnected** if it can be written as $E = A \cup B$, where A, B are both nonempty, separated sets.

Exercise 2. (Reading question) For each of the following subsets of $\mathbb{R}$, state which of the following properties applies: closed, bounded, compact, perfect, disconnected. Explain your answers.

(1) $\{x \in \mathbb{R} : 2 \leq |x| \leq 4\}$
(2) $\mathbb{Q}$
(3) $\mathbb{R}$
(4) $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$
(5) $\{0\} \cup \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$

(1) $\{x \in \mathbb{R} : 2 \leq |x| \leq 4\}$
  (a) this is the union of two closed intervals $[-4, -2] \cup [2, 4]$, hence is closed. It's not open because 2 has no ϵ-neighborhood contained in the set.
  (b) bounded (in $[-4, 4]$)
  (c) compact
  (d) perfect
  (e) disconnected: let $A = (-5, -1)$ and $B = (1, 5)$
(2) $\mathbb{Q}$
  (a) not closed since its complement $\mathbb{I}$ is not open. To see this, consider the point $\sqrt{2} \in \mathbb{I}$, say. Then any ϵ-neighborhood of $\sqrt{2}$ contains a rational number by the Density of $\mathbb{Q}$ in $\mathbb{R}$, so there is no ϵ for which $V_\epsilon(\sqrt{2}) \subseteq \mathbb{I}$.
  (b) Unbounded (can use e.g. Archimedean Principle)
  (c) Not compact
  (d) Not perfect since it isn't closed
  (e) is disconnected. If we let

$$A = \mathbb{Q} \cap (-\infty, \sqrt{2}) \text{ and } B = \mathbb{Q} \cap (\sqrt{2}, \infty),$$

  then $\mathbb{Q} = A \cup B$, and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.
(3) $\mathbb{R}$
  (a) closed
  (b) unbounded
  (c) not compact
  (d) for any $x \in \mathbb{R}$, the sequence $(x + \frac{1}{n})$ converges to x, so $\mathbb{R}$ is perfect.
  (e) connected
(4) $A = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$
  (a) not closed because the limit point 0 is not in the set
  (b) bounded
  (c) not compact
  (d) not perfect
  (e) disconnected: we can write

$$A = C \cup D \text{ where } C = A \cap (0, \frac{5}{12}), D = A \cap (\frac{5}{12}, 1).$$

(5) $A \cup \{0\}$
  (a) closed because the only limit point, 0, is contained in the set

(b) bounded
(c) compact
(d) not perfect; $\frac{1}{2}$ is an isolated point since $V_{\frac{t}{12}}(\frac{1}{2})$ does not intersect the set at any point other than $\frac{1}{2}$
(e) still disconnected; the same example as above works

- Though we've made a lot of statements about closed, compact, and perfect sets, it remains to prove a handful of them.
- Remember, we have to prove everything rigorously to make sure we aren't getting fooled by our intuition.
- We have some powerful characterizations of exactly the sets in $\mathbb{R}$ that are compact, and it's very useful to know that closed sets are the complements of open sets and vice versa.
- **Today**: "cleaning up" by:
  (1) proving characterizations of open/closed sets in terms of their complements
  (2) proving a characterization of compact sets as closed and bounded
  (3) proving that the NIP still holds if you replace "closed interval" with "compact set"

1. Open and Closed Sets Are Complements

Theorem 3. *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

- We prove the first sentence; the second will follow.

Exercise 4. (Reading Question) Prove ($\Longrightarrow$).

Proof. ($\Longrightarrow$) Let $O \subseteq \mathbb{R}$ be an open set. We want to show that O^c is closed. Let x be a limit point of O^c. Then every neighborhood of x contains some point of O^c. But this means that x cannot be in the open set O, because $x \in O$ implies there exists a neighborhood $V_\epsilon(x) \subseteq O$. Thus, $x \in O^c$, as desired.

($\Longleftarrow$) Assume O^c is closed. Let $x \in O$ be arbitrary; we want to produce an ϵ-neighborhood $V_\epsilon(x) \subseteq O$. Because O^c is closed, we can be sure that x is *not* a limit point of O^c. Hence, by definition, there must be some neighborhood $V_\epsilon(x)$ of x that does not intersect the set O^c. But this means $V_\epsilon(x) \subseteq O$, as desired.

The second statement follows quickly from the first using the observation that $(E^c)^c = E$ for any set $E \subseteq \mathbb{R}$. □

- Finally, we want to make some statement about when unions and intersections of closed sets are closed, similar to our statement about open sets.
- Topologists take these statements as defining properties of open and closed sets.
  - To a topologist, a list of subsets of a given set are given and decreed to be "open". These sets must have the property that countable unions and finite intersections of open sets are open.
  - Then, by definition, the closed sets are all the complements of the open sets.

2. Compact Sets

- All of the compact sets we know are closed and bounded. How do we know there aren't any others?

Definition 5. A set $A \subseteq \mathbb{R}$ is **bounded** if there exists $M > 0$ so that $|a| \leq M$ for all $a \in A$.

A set $K \subseteq \mathbb{R}$ is **compact** if it satisfies "generalized B-W": every sequence (a_n) with $a_n \in K$ for all n has a subsequence (a_{n_k}) converging to some $a \in K$.

Theorem 6. ***(Characterization of Compactness in*** $\mathbb{R}$***)*** *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. ($\Longleftarrow$) Exercise 3.3.3, unclaimed HW.

($\Longrightarrow$) Let K be compact and assume for contradiction that K is unbounded. We'll produce a sequence in K that marches off to ∞ in such a way that it can't have a convergent subsequence. To do this, notice that because K is unbounded, there exists an element $x_1 \in K$ satisfying $|x_1| > 1$. Likewise, there must exist $x_2 \in K$ with $|x_2| > 2$, and in general, given $n \in \mathbb{N}$, we can produce $x_n \in K$ so that $|x_n| > n$.

Now, because K is assumed to be compact, (x_n) should have a convergent subsequence (x_{n_k}). But the elements of the subsequence must satisfy $|x_{n_k}| > n_k$, and consequently (x_{n_k}) is unbounded. Because convergent sequences are bounded, we have a contradiction. Thus, K must be a bounded set.

Now, we'll show that K is also closed. To see that K contains its limit points, we let $x = \lim x_n$, where (x_n) is contained in K, and argue that $x \in K$ as well. Since K is bounded, so is (x_n), and by B-W, (x_n) must have a subsequence (x_{n_k}) converging to some limit x. Since K is compact, $x \in K$. Since subsequences of convergent sequences converge to the same limit, it must be that $(x_n) \to x \in K$ as well. Hence K is closed. □

Exercise 7. What are some examples of compact sets that aren't closed intervals?

- The Cantor set
- The set $\{0, 1\}$

- Really, compact sets are a sort of generalization of closed intervals: whenever a fact about closed intervals is true, it often remains true if we replace "closed interval" with "compact set".
- As an example, let's generalize the Nested Interval Property.

Theorem 8. ***(Nested Compact Set Property)*** *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ *is not empty.*

Proof. We know that each K_n is compact, so any sequence in any of the K_n, say K_1, has a subsequence converging to a point in K_1. For each $n \in \mathbb{N}$, we can construct a sequence by picking a point $x_n \in K_n$. □

Exercise 9. Finish the proof.

- Because each of these x_n is also in K_1, (x_n) has a convergent subsequence (x_{n_k}) whose limit $x = \lim x_n$ is an element of K_1.
- We want to argue that x is an element of every K_n; the reasoning is essentially the same. Given a particular $n_0 \in \mathbb{N}$, the terms in the sequence (x_n) are contained in K_{n_0} as long as $n \geq n_0$. Ignoring the finite number of terms for which $n_k < n_0$, the same subsequence (x_{n_k}) is thus also contained in K_{n_0}. The conclusion is that $x = \lim x_{n_k}$ is an element of K_{n_0}. Because n_0 was arbitrary, $x \in \bigcap_{n=1}^{\infty} K_n$.