MATH 321 DAY 2 - PYTHAGOREAN INTERLUDE; INTRODUCTION TO SETS

0.1. Mathematical notation.

- In this course, we will use the following mathematical notation on the board. However, when you write out solutions to problems (or anything formal, such as a mathematical paper), it is considered good form not to use any of these notational abbreviations, and instead to write out your statements in words.
  - $\forall$: for all, for every
  - $\exists$: there exists at least one
  - $\in$: in, element of
  - s.t.: such that
  - $\implies$: implies
  - $\iff$, iff: if and only if

Exercise 1. Translate each statement into words, then attempt to prove it (if it's true) or give a counterexample if it's false. Assume that $x, f_n(x)$, and $f(x) \in \mathbb{R}$ and $M, N \in \mathbb{N}$.

(1) $\exists x$ s.t. $x < 8$
(2) $\forall x$, $x > 1 \implies x^2 > 1$.
(3) $\forall M > 0$, $\exists N > 0$ s.t. $N < 1/M$.

- Tip: start putting definitions and theorems in a master list (on a big sheet of paper or in mind-mapping software) and connect them to show how they relate; I'll check on your progress at the midterm and final.
  - These are great study aids!

1. Pythagorean interlude: $\sqrt{2}$ and the irrational numbers

- Toward the end of his career, renowned British mathematician G. H. Hardy laid out a justification of a life spent studying math in his *Mathematician's Apology*.
- At the center of his defense (and mine) is that math is an aesthetic discipline. "Real mathematics," as Hardy referred to it, "must be justified as art if it can be justified at all."
- To help make his point, Hardy laid out two theorems from classical Greek mathematics which, in his opinion, possess an aesthetic beauty that's easy to recognize.

Theorem 2. *(School of Pythagoras, 500 B.C.) $\sqrt{2}$ cannot be expressed as a ratio of integers.*

- The Pythagoreans were a philosophic guild that took the great Greek mathematician Pythagoras's teachings as the basis for a philosophical system.
- Legend has it that Hippasus, a follower of Pythagoras, discovered a proof that $\sqrt{2}$ was not a rational number.
- They took him out to sea, and he didn't come back.
- The Pythagoreans suppressed this information.
  - They had religious objections to numbers that couldn't be written as ratios of whole numbers.
- The proof uses what's known as **proof by contradiction**. [Does anyone know how proof by contradiction works?] It goes like this:
  - Assume statement A is true.
  - If statement A is true, then we come to a contradiction of our assumption that A is true.
  - Therefore, A couldn't have been true to start with. Ready?
- Who's seen this proof before? [put one person who's seen it before in each group and have students try to prove it]

Proof. Assume for contradiction that $\sqrt{2}$ can be expressed as a ratio of integers $\frac{a}{b}$. And let's say this fraction is in lowest terms, meaning a and b share no common factors that we could cancel, in other words $\gcd(a,b) = 1$.

(1) Well, let's play around with the resulting equation $\sqrt{2} = \frac{a}{b}$. We don't yet know much about $\sqrt{2}$, so let's square both sides:
$$2 = \frac{a^2}{b^2}.$$
(2) Equations involving fractions are not as nice as equations without them, so let's multiply both sides by b^2:
$$2b^2 = a^2.$$
(3) All this tells us is that a^2 is even. What does this mean? This means that a must be even. [Why? We'll fill in this gap in a second.]
(4) If a is even, what else do we know? Well, remember we're trying to contradict our assumption that $\sqrt{2} = \frac{a}{b}$ **in lowest terms**. We don't seem to have a way to contradict that $\sqrt{2} = \frac{a}{b}$ directly, but what about the lowest terms part?
(5) If a is even, say $a = 2k$, then $a^2 = 4k^2$. So a^2 is actually divisible by 4!
(6) But if $a^2 = 4k^2$ and $\frac{a^2}{b^2} = 2$, then $b^2 = 2k^2$. So b^2 is even as well!
(7) If b^2 is even, then b must be even as well.
(8) But if a and b are both even, then we can cancel a 2 from top and bottom of $\frac{a}{b}$, contradicting that $\frac{a}{b}$ is in lowest terms!
(9) Therefore, our initial assumption was false, and $\sqrt{2}$ is not a rational number.

□

- You're lucky you live in a time where this proof won't get you thrown off a ship!
- Incidentally, standard printer paper is designed to have a common ratio between the two edges even when we fold the paper in half (for scaling). It turns out that the only way to do this is to make at least one of the sides of the paper have length an irrational number.

Exercise 3. [in groups] Prove by contradiction that, if p^2 is even, p must be even as well.

Proof. Assume for contradiction that p is not even. Then it's odd, so $p = 2n+1$, where n is a whole number. But then
$$p^2 = (2n+1)^2 = 4n^2 + 4n + 1.$$
Since $4n^2 + 4n$ is clearly even, $4n^2 + 4n + 1$ must be odd. This is a contradiction of our original assumption that p^2 is even! Since the only thing we assumed for contradiction was that p was not even, that original assumption must be wrong. So p is even! □

Exercise 4. [if time] Start with a rectangular piece of paper with long side a and short side b. Show that, if we want the ratio of long side to short side to be the same after folding the long side of the paper in half, then at least one of a and b must be irrational. [Hint: show that $\frac{a}{b} = \sqrt{2}$.]

Solution. The original ratio of sides is a/b. The new ratio is $\frac{b}{\frac{1}{2}a}$. Solve for $\frac{a}{b}$.

Exercise 5. Can we slightly modify the proof that $\sqrt{2} \notin \mathbb{Q}$ to show that $\sqrt{3}$ is not a rational number? That $\sqrt{9}$ is an irrational number? That $\sqrt{25}$ is an irrational number? Why or why not?

(1) It works for $\sqrt{3}$:
   (a) $3 = \frac{a^2}{b^2}$
   (b) $3b^2 = a^2$, so a^2 is divisible by 3.
   (c) This means that $\sqrt{3}b = a$. Since a is an integer, b must be a multiple of $\sqrt{3}$.
   (d) Therefore $a = 3k$ for some integer k.
   (e) Thus $a^2 = 9k^2$ and $3 = \frac{9k^2}{b^2}$.
   (f) Hence b^2 must be divisible by 3.
   (g) By the same reasoning as above, b must be divisible by 3.
   (h) Contradiction!
(2) It doesn't work for $\sqrt{25}$–and it better not since 5 is a rational number!

(a) $25 = \frac{a^2}{b^2}$
(b) $25b^2 = a^2$, so a^2 is divisible by 25.
(c) This **doesn't** mean that a is divisible by 25, though it must be divisible by 5!
(d) This doesn't tell us anything about what b^2 is divisible by!

- The existence of irrational numbers shook up the Greek idea that all line segments had rational lengths. [draw a unit square and its diagonal]
- The idea that numbers were rational was built into the Greek definition of "number'.
- The Pythagoreans dealt with this by saying that some lengths were not expressible by numbers.
- We'll deal with it by expanding our notion of "number" past the rationals!
- We start with the set of **natural numbers** $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.
  - Leopold Kronecker: "The natural numbers are the work of God. All of the rest is the work of mankind."
  - The set of **integers** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
  - The set of **rational numbers** $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}\}$
    * The vertical lines in these descriptions are read as "such that". So $\mathbb{Q}$ is the set of fractions $\frac{a}{b}$ *such that* $\frac{a}{b}$ is in lowest terms ($\gcd(a, b) = 1$) **and** both a and b are integers.
    * $\mathbb{Q}$ is particularly nice because it's what's known as a *field*: a set in which we can do addition, subtraction, multiplication, and division and still stay inside the set.

Definition 6. A **field** is any set where addition and multiplication are well-defined operations that stay inside the set and satisfy the following properties:

- Addition and multiplication are commutative: $a + b = b + a$, $ab = ba$
- Addition and multiplication are associative: $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$
- Addition and multiplication obey the distributive property: $a(b + c) = ab + ac$
- There is an additive identity, and every element has an additive inverse.
- There is a multiplicative identity, and every element has a multiplicative inverse.

Exercise 7. Convince yourself that this definition is the same as saying we can add, subtract, multiply, and divide in a field. Are $\mathbb{N}$ and $\mathbb{Z}$ fields? Why or why not? If not, which of the properties in the definition fail to be satisfied?

- Problem: we can **approximate** $\sqrt{2}$ within $\mathbb{Q}$ by decimals such as 1.4, 1.41, 1.414, ..., but there's a "hole" in $\mathbb{Q}$ (seen as a number line) where $\sqrt{2}$ should be.

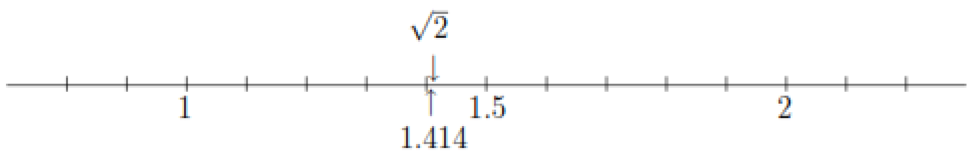

Figure 1.2: APPROXIMATING $\sqrt{2}$ WITH RATIONAL NUMBERS.

- We'll find out later how to construct the real numbers from $\mathbb{Q}$, but it's not too far off to think of it as "filling in the holes" by defining an **irrational number** wherever there's a hole in the number line and placing it there.

Question 8. *What properties does the set of irrational numbers have?*

How do the sets of rationals and irrationals fit together?

Is there a kind of symmetry between the rationals and irrationals, or is there some sense in which we can argue that one is more common?

Can all irrational numbers be expressed as algebraic combinations of roots and rational numbers, or are there others?

Exercise 9. [in groups] Do there exist irrational numbers a and b such that a^b is rational? If not, prove there's not. If yes, give an example [Hint: $(a^b)^c = a^{bc}$.]

- Proof: if $(\sqrt{2})^{\sqrt{2}}$ is rational, we're done. If not, $[(\sqrt{2})^{\sqrt{2}}]^{\sqrt{2}} = \sqrt{2}^2$ certainly is.
- This proof uses **cases**: either A is true or B is true, but in either case, the assertion holds.

- This can also be phrased as a **proof by contradiction**: suppose not. Then $(\sqrt{2})^{\sqrt{2}}$ is irrational. But then
$$((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 \in \mathbb{Q}.$$

Homework: