MATH 321 WEEK 14 CLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 3.2.13: Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set.

Exercise 3.2.13. For contradiction, assume that there exists a nonempty set A, different from $\mathbf{R}$, that is both open and closed. Because $A \neq \mathbf{R}$, $B = A^c$ is also nonempty, and B is open and closed as well. Pick a point $a_1 \in A$ and $b_1 \in B$. We can assume, without loss of generality, that $a_1 < b_1$. Bisect the interval $[a_1, b_1]$ at $c = (b_1 - a_1)/2$. Now $c \in A$ or $c \in B$. If $c \in A$, let $a_2 = c$ and let $b_2 = b_1$. If $c \in B$, let $b_2 = c$ and let $a_2 = a_1$. Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$, where $a_n \in A$ and $b_n \in B$. By the Nested Interval Property, there exists an $x \in \bigcap_{n=1}^{\infty} I_n$. Because the lengths $(b_n - a_n) \to 0$, we can show $\lim a_n = x$ which implies that $x \in A$ because A is closed. However, it is also true that $\lim b_n = x$ and thus $x \in B$ because B is closed. Thus we have shown $x \in A$ and $x \in A^c$. This contradiction implies that no such A exists, and we conclude that $\mathbf{R}$ and $\emptyset$ are the only two sets that are both open and closed. (This argument is closely related to the discussion of connected sets in the next section.)

(2) 3.2.15: show that a closed interval $[a, b]$ is a G_δ set, the half-open interval $(a, b]$ is both a G_δ and F_σ set, $\mathbb{Q}$ is F_σ, and $\mathbb{I}$ is a G_δ set.

Exercise 3.2.15. (a) $[a, b] = \bigcap_{n=1}^{\infty}(a - 1/n, b + 1/n)$.

(b) $(a, b] = \bigcap_{n=1}^{\infty}(a, b + 1/n)$; $(a, b] = \bigcup_{n=1}^{\infty}[a + 1/n, b]$

(c) Because $\mathbf{Q}$ is countable, we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$. Note that each singleton set $\{r_n\}$ is closed and the complement $\{r_n\}^c$ is open. Then $\mathbf{Q} = \bigcup_{n=1}^{\infty}\{r_n\}$ shows that $\mathbf{Q}$ is an F_σ set, and $\mathbf{I} = \mathbf{Q}^c = \bigcap_{n=1}^{\infty}\{r_n\}^c$ shows that $\mathbf{I}$ is a G_δ set.

(3) 3.3.5 (T/F) (a) the arbitrary intersection of compact sets is compact. (b) the arbitrary union of compact sets is compact. (c) the intersection of an arbitrary set and compact set is compact. (d) the "Nested Closed Set Theorem".

Exercise 3.3.5. (a) True. By Theorem 3.2.14, an arbitrary intersection of closed sets is closed. Boundedness is also preserved by intersections; therefore, the arbitrary intersection of compact sets will be compact.

(b) False. The union of the compact sets $[1/n, 2]$ over all $n \in \mathbf{N}$ is equal to $(0, 2]$ which is not closed and thus not compact. The union of the compact sets $[n, n+1]$ over all $n \in \mathbf{N}$ is the unbounded set $[1, \infty)$.

(c) False. Take $A = (0, 1)$ and $K = [0, 1]$. Then $A \cap K = (0, 1)$ is not compact.

(d) False. Let $F_n = [n, \infty)$. Then F_n is closed for all n, but the intersection of these sets is empty.

(4) 3.4.4: Is the middle-fourths Cantor set compact? Perfect? What is its length? Dimension?

(a) The middle-fourths Cantor set C is closed and bounded, hence compact. The set C is closed because it is constructed by removing a union of open intervals $\cup U_n$, which is open, and $C = [0,1] \cap \left(\cup U_n\right)^c$, the intersection of two closed sets, which is closed. The set C is bounded because it is contained in the interval $[0, 1]$. Moreover, C is perfect; we can borrow the argument for showing the standard middle-thirds Cantor set is perfect. Let $x \in C$. Then, for each $k \in \mathbb{N}$, we denote by x_k an endpoint of the closed interval of C_i that contains x, which endpoint is not equal to x. Since the length of the interval of C_k containing x is $\left(\left(\frac{3}{8}\right)^k\right) \to 0$, we have that $(x_n) \to x$. Hence x is a limit point of C, and since x was arbitrary, C is perfect.

(b) We saw above that the length of each interval left at stage C_k is $(\frac{3}{8})^k$. There are 2^k intervals of C_k, so the total length of the intervals of C_k is

$$2^k \left(\frac{3}{8}\right)^k = \left(\frac{3}{4}\right)^k.$$

As $k \to \infty$, the total length of the intervals of C_k approaches $\lim \left(\frac{3}{4}\right)^k = 0$. Now, let's consider the dimension of C. (This is pretty tricky; be generous with points.) Consider the effect of scaling C by $\frac{8}{3}$. For C_1, this produces the interval $[0, \frac{8}{3}]$. Removing the middle fourth leaves us with the intervals $[0, 1] \cup [\frac{5}{3}, \frac{8}{3}]$. Each of these intervals is a copy of $[0, 1]$, so continuing this process will produce 2 copies of C! Therefore, whatever the dimension d of C is, it should satisfy

$$\left(\frac{8}{3}\right)^d = 2$$

or

$$d = \log_{(8/3)} 2 = \frac{\log 2}{\log \frac{8}{3}} \approx 0.706695.$$