MATH 321 4.2 - FUNCTIONAL LIMITS

- Last time, we noticed that the Dirichlet function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

  seemed to have "different limits" at $x = 1/2$ depending on whether we approached $1/2$ via rational numbers ("$\lim g(x) = 1$") or irrational numbers ("$\lim g(x) = 0$").
- This necessitates a definition of functional limit that doesn't depend on "what way" you approach $1/2$.
- If c is a limit point of the domain of f, then, intuitively, the statement

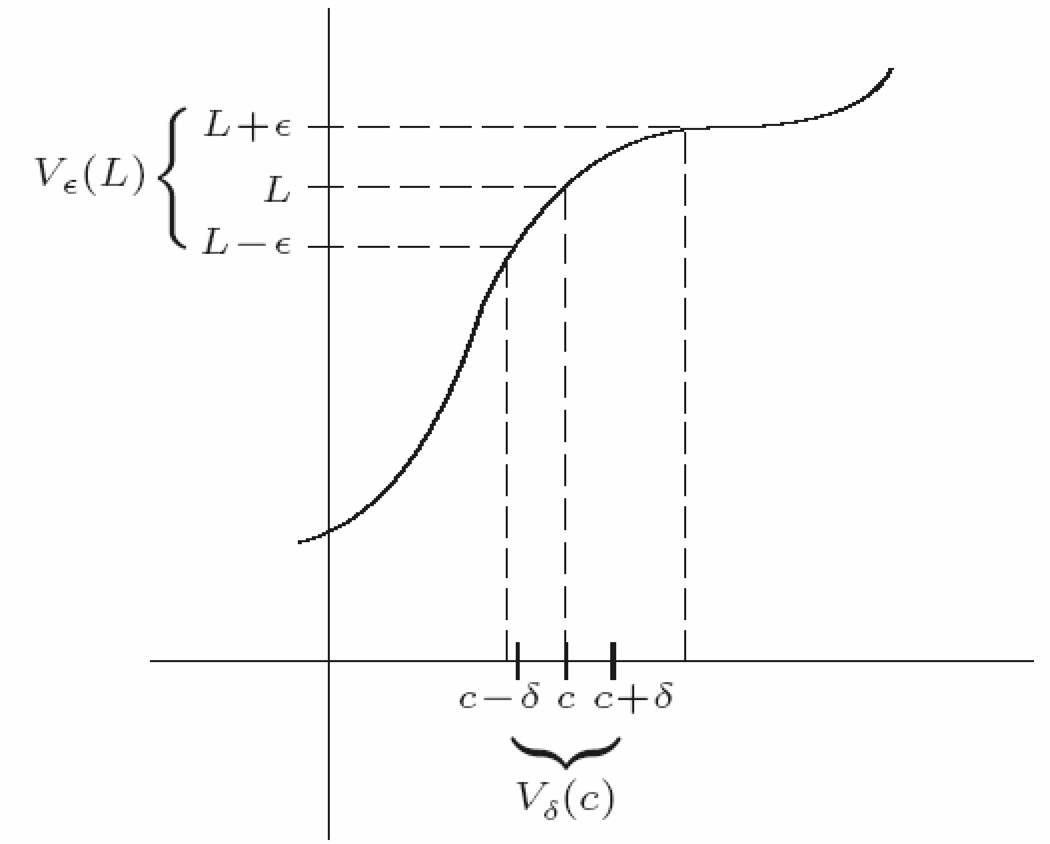
$$\lim_{x \to c} f(x) = L$$

  is intended to convey that values of $f(x)$ get arbitrarily close to L as x is chosen closer and closer to c.
  - Limits don't care about the value of f **at** c, or even whether f is defined at c.
- We'll take inspiration from the "challenge-response" pattern established in the definition for the limit of a sequence:
  - Let (a_n) be a sequence of real numbers. Then $\lim a_n = L$ if for any given error tolerance $\epsilon > 0$, I can make a_n within that tolerance of L by taking n large enough.
  - Each ϵ is a particular challenge, and each N is the respective response.
- For functional limit statements such as $\lim_{x \to c} f(x) = L$, the challenges are still made in the form of an arbitrary ϵ-neighborhood centered at L, but the response this time is a δ-neighborhood centered at c.

Definition 1. (Functional Limit) Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x \to c} f(x) = L$ provided that, for all $\epsilon > 0$ (challenge), there exists a $\delta > 0$ (response) such that whenever $0 < |x - c| < \delta$ (and $x \in A$), it follows that $|f(x) - L| < \epsilon$.

Exercise 2. (Reading question)

(1) How is this definition equivalent to a "challenge-response" definition?



(2) Prove that, if $f(x) = 3x + 1$, then $\lim_{x\to 2} f(x) = 7$.
   (a) Let $\epsilon > 0$ be arbitrary.
   (b) [scratch work] Notice that

$$|f(x) - 7| = |(3x + 1) - 7| = |3x - 6| = 3|x - 2|.$$

   (c) Choose $\delta = \epsilon/3$.
   (d) Assume $0 < |x - 2| < \delta$.
   (e) Then

$$|f(x) - 7| = 3|x - 2| < 3(\epsilon/3) = \epsilon,$$

   as desired.

- Notice that the steps to prove $\lim_{x\to c} f(x) = L$ are the same as those for proving convergence, with N replaced by δ and "$n \geq N$" replaced by "$0 < |x - c| < \delta$":
  (1) "let $\epsilon > 0$ be arbitrary"
  (2) [scratch work] rewrite $|f(x) - L|$ in terms of $|x - c|$
  (3) Choose δ in terms of ϵ.
  (4) "Assume $0 < |x - c| < \delta$."
  (5) Then our scratch work implies that $|f(x) - L| < \epsilon$.
- The above definition of functional limit is often called the "ϵ-δ version" of the definition for functional limits.

Question 3. *How can we convert the ϵ-δ version of functional limit to a topological version?*

Definition 4. (Functional Limit: Topological Version) Let c be a limit point of the domain of $f : A \to \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every ϵ-neighborhood $V_\epsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$), it follows that $f(x) \in V_\epsilon(L)$.

- We'll often leave off the reminder ($x \in A$), since $f(x)$ doesn't make any sense if x is not in the domain of f.

Example 5. Show that $\lim_{x\to 2} g(x) = 4$, where $g(x) = x^2$.

(1) Let $\epsilon > 0$ be arbitrary.
(2) Our goal is to make $|g(x) - 4| < \epsilon$ by restricting $|x - 2|$ to be smaller than some δ. A little algebra reveals
$$|g(x) - 4| = |x^2 - 4| = |x + 2||x - 2|.$$
We need an upper bound on $|x + 2|$ in order to choose δ. Note that x is approaching 2, so we can agree that our δ-neighborhood around $c = 2$ must have radius no bigger than $\delta = 1$ so that $1 \le x \le 3$; thus we get the upper bound $|x + 2| \le |3 + 2| = 5$ for all $x \in V_\delta(c)$. Thus, we also need to make $|x - 2| < \epsilon/5$ in order to make
$$|x + 2||x - 2| \le 5(\epsilon/5) = \epsilon.$$
Note that it's possible for $\epsilon/5$ to be bigger than 1 and thus invalidate our reasoning about $|x + 2|$, so we should choose δ to be a minimum.
(3) Choose $\delta = \min\{1, \epsilon/5\}$.
(4) Assume that $0 < |x - 2| < \delta$.
(5) Then, since $\delta \le 1$, $|x + 2| \le |3 + 2| = 5$. In addition, since $\delta \le \epsilon/5$,
$$|g(x) - 4| = |x^2 - 4| = |x + 2||x - 2| < 5(\epsilon/5) = \epsilon.$$

0.1. **Sequential criterion for functional limits.**

- Results like the Algebraic and Order Limit Theorems significantly helped us in evaluating the limits of sequences. It'd be nice to have similar statements for functional limits!
- In order to do so, it's useful to re-characterize functional limits in terms of sequential limits, as we discussed at the beginning of this chapter.
- This will let us use the Algebraic Limit Theorem for sequences to quickly prove a version of the ALT for functions.

Theorem 6. ***(Sequential Criterion for Functional Limits)*** *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

(1) $\lim\limits_{x\to c} f(x) = L$.
(2) *For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.*

Exercise 7. Prove the forward direction.

Proof. ($\implies$): Assume that $\lim\limits_{x\to c} f(x) = L$. Consider an arbitrary (x_n) converging to c and satisfying $x_n \neq c$. Our goal is to show that $(f(x_n)) \to L$. It's easiest to use the topological formulation of the definition.

Let $\epsilon > 0$ be arbitrary. The topological definition of functional limit assures us that there exists $V_\delta(c)$ with the property that all $x \in V_\delta(c)$ different from c satisfy $f(x) \in V_\epsilon(L)$. All we need to do then is argue that our particular sequence (x_n) is eventually in $V_\delta(c)$. Since $(x_n) \to c$, there exists a point x_N after which $x_n \in V_\delta(c)$. Hence, $n \ge N$ implies $f(x_n) \in V_\epsilon(L)$, as desired.

($\impliedby$): We prove the contrapositive. That is, we assume that $\lim\limits_{x\to c} f(x) \neq L$ and produce a sequence $(x_n) \subseteq A$ with $x_n \neq c$ and $(x_n) \to c$ so that $f(x_n) \not\to L$. This statement is equivalent to $(ii) \implies (i)$.

The idea is that, if the limit is not L, there's a sequence that stays a certain distance $\epsilon_0 > 0$ from L, hence can't converge to L. [draw]

If $\lim\limits_{x\to c} f(x) \neq L$, there exists at least one particular $\epsilon_0 > 0$ for which no δ is a suitable response. In other words, no matter what $\delta > 0$ we try, there will always be at least one point
$$x \in V_\delta(c) \text{ with } x \neq c \text{ for which } f(x) \notin V_{\epsilon_0}(L).$$
Now consider $\delta_n = 1/n$. Then for each $n \in \mathbb{N}$, we may pick $x_n \in V_{\delta_n}(c)$ with $x_n \neq c$ and $f(x_n) \notin V_{\epsilon_0}(L)$. But now notice that the result of this is a sequence $(x_n) \to c$ with $x_n \neq c$, where the image sequence $f(x_n)$ certainly does *not* converge to L. $\square$

- Having this theorem under our belts gives us an economical proof of the Algebraic Limit Theorem for functions, as well as a quick way to show a limit doesn't exist by finding two sequences converging to the same point whose functional limits converge to different points.

Corollary 8 (Algebraic Limit Theorem for Functional Limits)**.** *Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then*

(1) $\lim_{x\to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$,
(2) $\lim_{x\to c}[f(x) + g(x)] = L + M$,
(3) $\lim_{x\to c}[f(x)g(x)] = LM$, *and*
(4) $\lim_{x\to c} f(x)/g(x) = L/M$, *provided* $M \neq 0$.

Proof. These follow from the Sequential Criterion and the Algebraic Limit Theorem for sequences. □

Exercise 9. (reading question) Prove (i) and (ii).

(1) Assume $\lim_{x\to c} f(x) = L$ and let (x_n) be an arbitrary sequence satisfying $x_n \neq c$ converging to c. Then by the Sequential Criterion, $(f(x_n)) \to L$. Moreover, the sequence $(kf(x_n)) \to kL$ by the Algebraic Limit Theorem for sequences. Since (x_n) was arbitrary, for any $(x_n) \to c$ with $x_n \neq c$, we have that $(kf(x_n)) \to kL$. By the Sequential Criterion, this means that $\lim_{x\to c} kf(x) = kL$, as desired.
(2) We follow the same logic as in (i). Let (x_n) be an arbitrary sequence satisfying $x_n \neq c$ converging to c. Then, by the Sequential Criterion, $(f(x_n)) \to L$ and $(g(x_n)) \to M$. By the Algebraic Limit Theorem for sequences, this implies that $(f(x_n)+g(x_n)) \to L+M$, which implies that $\lim_{x\to c}[f(x)+g(x)] = L+M$.

- The proofs of the other parts are similar.

Corollary 10 (Divergence Criterion for Functional Limits)**.** *Let f be a function defined on A, and let c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and*

$$\lim x_n = \lim y_n = c \text{ but } \lim f(x_n) \neq \lim f(y_n),$$

then we can conclude that the functional limit $\lim_{x\to c} f(x)$ does not exist.

Proof. This just negates the Sequential Criterion. □

Example 11. [slide] Assuming the familiar properties of $\sin(x)$, we can prove that $\lim_{x\to 0} \sin(1/x)$ does not exist.

Exercise 12. Let $x_n = 1/2n\pi$ and $y_n = 1/(2n\pi + \pi/2)$. Compute $\lim(x_n)$, $\lim(y_n)$, $\lim \sin(1/x_n)$, and $\lim \sin(1/y_n)$, then use these limits to prove that $\lim_{x\to 0} \sin(1/x)$ does not exist.

- Since this course is called "advanced calculus", let's close by using our new definition of functional limits to define the derivative.

Definition 13. Let $f : A \to \mathbb{R}$ be a function. The **derivative** of f **at a point** $x \in A$ is the limit

$$f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

if it exists.