MATH 321 3.3 - COMPACT SETS

- We saw that closed sets were somehow useful for solving optimization problems: $f(x) = x^2$ has a max and min on $[0,1]$ but not on $(0,1)$, for instance.
- However, being closed isn't enough to guarantee a max/min:
  - Note that the interval $[0,\infty)$, by virtue of being the complement of the open set $(-\infty,0)$, is closed.
  - However, we can't always solve optimization problems in $[0,\infty)$. For instance, say we want to minimize $f(x) = \frac{1}{x^2}$ on $[0,\infty)$. The global minimum doesn't exist!
- We need another criterion for maxes and mins to exist: compactness.

Definition 1. A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

- Note: these sets satisfy a sort of "generalized Bolzano-Weierstrass".
- We need one more definition before your reading question:

Definition 2. A set $A \subseteq \mathbb{R}$ is **bounded** if there exists $M > 0$ so that $|a| \leq M$ for all $a \in A$.

Exercise 3. (Reading question) Which of the following sets satisfy this generalized Bolzano-Weierstrass? Prove necessary conditions for a set K to satisfy generalized B-W.

(1) $[0,1]$: yes. If (a_n) is any sequence in $[0,1]$, then $|a_n| \leq 1$ for all n, so (a_n) is bounded. By (regular) Bolzano-Weierstrass, this means that (a_n) has a convergent subsequence (a_{n_k}). By the Algebraic Limit Theorem, since $0 \leq a_{n_k} \leq 1$ for all k, it must be that $0 \leq \lim a_{n_k} \leq 1$ as well. Hence, $\lim a_{n_k} \in [0,1]$.
(2) $(0,1)$: no; all subsequences of $(\frac{1}{n})$ converge to $0 \notin (0,1)$.
(3) $[0,\infty)$: no; no subsequence of (n) converges.
(4) $\mathbb{R}$: no; the sequence (n) has no convergent subsequence in $\mathbb{R}$.

- This shows that closed, bounded sets are compact. Are there any other compact sets, sets satisfying generalized B-W, besides the closed, bounded sets?

Theorem 4. ***(Characterization of Compactness in $\mathbb{R}$)*** *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. ($\Longleftarrow$) Exercise 3.3.3, unclaimed HW.

($\Longrightarrow$) Let K be compact and assume for contradiction that K is unbounded. We'll produce a sequence in K that marches off to ∞ in such a way that it can't have a convergent subsequence. To do this, notice that because K is unbounded, there exists an element $x_1 \in K$ satisfying $|x_1| > 1$. Likewise, there must exist $x_2 \in K$ with $|x_2| > 2$, and in general, given $n \in \mathbb{N}$, we can produce $x_n \in K$ so that $|x_n| > n$.

Now, because K is assumed to be compact, (x_n) should have a convergent subsequence (x_{n_k}). But the elements of the subsequence must satisfy $|x_{n_k}| > n_k$, and consequently (x_{n_k}) is unbounded. Because convergent sequences are bounded, we have a contradiction. Thus, K must be a bounded set.

Now, we'll show that K is also closed. To see that K contains its limit points, we let $x = \lim x_n$, where (x_n) is contained in K, and argue that $x \in K$ as well. Since K is bounded, so is (x_n), and by B-W, (x_n) must have a subsequence (x_{n_k}) converging to some limit x. Since K is compact, $x \in K$. Since subsequences of convergent sequences converge to the same limit, it must be that $(x_n) \to x \in K$ as well. Hence K is closed. $\square$

Exercise 5. What are some examples of compact sets that aren't closed intervals?

- The Cantor set
- The set $\{0,1\}$

- Really, compact sets are a sort of generalization of closed intervals: whenever a fact about closed intervals is true, it often remains true if we replace "closed interval" with "compact set".

- As an example, let's generalize the Nested Interval Property.

Theorem 6. ***(Nested Compact Set Property)*** *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ *is not empty.*

Proof. We know that each K_n is compact, so any sequence in any of the K_n, say K_1, has a subsequence converging to a point in K_1. For each $n \in \mathbb{N}$, we can construct a sequence by picking a point $x_n \in K_n$. □

Exercise 7. Finish the proof.

- Because each of these x_n is also in K_1, (x_n) has a convergent subsequence (x_{n_k}) whose limit $x = \lim x_n$ is an element of K_1.
- We want to argue that x is an element of every K_n; the reasoning is essentially the same. Given a particular $n_0 \in \mathbb{N}$, the terms in the sequence (x_n) are contained in K_{n_0} as long as $n \geq n_0$. Ignoring the finite number of terms for which $n_k < n_0$, the same subsequence (x_{n_k}) is thus also contained in K_{n_0}. The conclusion is that $x = \lim x_{n_k}$ is an element of K_{n_0}. Because n_0 was arbitrary, $x \in \bigcap_{n=1}^{\infty} K_n$.