MATH 321 WEEK 13 CLAIMED PROBLEM SOLUTIONS

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(1) 3.2.5: Prove that a set F is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Exercise 3.2.5. ($\Rightarrow$) Assume that the set $F \subseteq \mathbf{R}$ is closed. Then F contains its limit points. We will show that that every Cauchy sequence (a_n) contained in F has its limit in F by showing that the limit of (a_n) is either a limit point

or possibly an isolated point of F. Because (a_n) is Cauchy, we know $x = \lim a_n$ exists. If $a_n \neq x$ for all n, then it follows from Theorem 3.2.5 that x is a limit point of F. Now consider a Cauchy sequence a_n where $a_n = x$ for some n. Because $(a_n) \subseteq F$ it follows that $x \in F$ as well. (Note that if a_n is eventually equal to x, then it may not be true that x is a limit point of F.)

($\Leftarrow$) Assume that every Cauchy sequence contained in F has a limit that is also an element of F. To show that F is closed we want to show that it contains its limit points. Let x be a limit point of F. By Theorem 3.2.5, $x = \lim a_n$ for some sequence (a_n). Because (a_n) converges, it must be a Cauchy sequence. So x is contained in F, and therefore F is closed.

(2) 3.2.6: Decide whether the following statements are true or false.

(a) **An open set that contains every rational number must necessarily be all of $\mathbb{R}$.** False. The set $\mathbb{R} \setminus \{\sqrt{2}\}$ is open because its complement, $\{\sqrt{2}\}$, is a closed set (it has no limit points, hence it contains all of its limit points). In addition, since $\sqrt{2} \notin \mathbb{Q}$, $\mathbb{R} \setminus \{\sqrt{2}\} \supseteq \mathbb{Q}$.

(b) **The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set".** False. Consider the nested closed sets $F_1 = [0,1]$, $F_2 = \{0\}$, $\emptyset = F_3 = F_4 = \dots$. We know that $\emptyset$ is closed. Moreover, the empty set is a subset of every F_n, since for all $x \in \emptyset$ (there are none), $x \in F_n$. But $\cap F_n = \emptyset$.

(c) **Every nonempty open set contains a rational number.** True. Let U be a nonempty open set. Then there exists $x \in U$, and since U is open, there exists $\epsilon > 0$ so that $V_\epsilon(x) \subseteq U$ as well. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ so that $|r - x| < \epsilon$, and therefore $r \in V_\epsilon(x) \subseteq U$.

(d) **Every bounded infinite closed set contains a rational number.** False; consider the set $\{\sqrt{2}\} \cup \{\sqrt{2} + \frac{1}{n} : n \in \mathbb{N}\}$. This set is closed because its only limit point, $\sqrt{2}$, is in the set. But $\sqrt{2}, \sqrt{2} + \frac{1}{n} \in \mathbb{I}$ for all n because the sum of an irrational number and a rational number is always irrational.

(e) **The Cantor set is closed.** True. The Cantor set is constructed by removing a (countable) collection of open intervals $\{U_i\}$ from $[0,1]$. Therefore, the complement of the Cantor set is $(-\infty, 0) \cup (1, \infty) \bigcup_{i \in \mathbb{N}} U_i$, a (countable) union of open intervals. Since unions of open intervals are open, the complement of C is open, so C is closed.

(3) 3.2.7. Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if A is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12 (the closure is a closed set and the smallest closed set containing A).

Exercise 3.2.7. (a) Let L be the set of limit points of A, and suppose that x is a limit point of L. We want to show that x is an element of L; in other words, that x is a limit point of A. Let $V_\epsilon(x)$ be arbitrary. By the definition of a limit point, $V_\epsilon(x)$ intersects L at a point $l \in L$, where $l \neq x$. Now choose $\epsilon' > 0$ small enough so that $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$ and $x \notin V_{\epsilon'}(l)$. Since $l \in L$, l is a limit point of A and so $V_{\epsilon'}(l)$ intersects A. This implies $V_\epsilon(x)$ intersects A at a point different than x, and therefore x is a limit point of A and thus an element of L.

(b) Assume x is a limit point of $A \cup L$ and consider the ϵ-neighborhood $V_\epsilon(x)$ for an arbitrary $\epsilon > 0$. We know $V_\epsilon(x)$ must intersect $A \cup L$ and we would like to argue that it in fact intersects A. If $V_\epsilon(x)$ intersects A at a point different than x we are done, so let's assume that there exists an $l \in L$ with $l \in V_\epsilon(x)$. Using the same argument employed in (a), we take $\epsilon' > 0$ small enough so that $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$, and $x \notin V_{\epsilon'}(l)$. Because l is a limit point of A we have that there exists an $a \in V_{\epsilon'}(l) \subseteq V_\epsilon(x)$ and thus $V_\epsilon(x)$ intersects A at some point other than x, as desired.

Because any limit point of $A \cup L$ is a limit point of A (and thus an element of L), it follows that $A \cup L$ contains its limit points; i.e., $\overline{A} = A \cup L$ is a closed set. This proves Theorem 3.2.12.

...well, almost, but we still have to argue the closure is the *smallest* closed set containing A. Suppose not; then there exists a set $B \subsetneq C$ containing A which is closed. Since $C = A \cup L$ and $A \subseteq B$, it must be that there exists $x \in L$ such that $x \notin B$. This means that some limit point of A is missing from B, contradicting that B is closed.

(4) 3.2.9 (De Morgan's Laws)
   (a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \text{ and } \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c.$$

Exercise 3.2.9. (a) Let $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. Then x is not an element of E_λ for all λ. Hence $x \in E_\lambda^c$ for all λ. So $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. We have just shown that $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Now we will show that $\bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. Let $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Then for all λ, $x \notin E_\lambda$. So $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$, and hence $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$. Therefore

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c.$$

Secondly, we want to show that

$$\left(\bigcap_{\lambda\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c.$$

Let $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$. Then there exists a $\lambda' \in \Lambda$ for which x is not an element of $E_{\lambda'}$. Therefore $x \in E_{\lambda'}^c$. So $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$, and we have $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Now assume $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c$. Then there exists a $\lambda' \in \Lambda$ such that $x \notin E_{\lambda'}$. Therefore $x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$, so $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$. So it is also true that $\bigcup_{\lambda\in\Lambda} E_\lambda^c \subseteq \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$ and we have reached our desired conclusion.

(b) (i) Suppose that E_λ is a finite collection of closed sets. Then their complements, E_λ^c are a finite collection of open sets. We know by Theorem 3.2.3 that the intersection of a finite collection of open sets is open. In symbols,

$$\bigcap_{\lambda\in\Lambda} E_\lambda^c = \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$$

is an open set. Therefore the union of a finite collection of closed sets, $\bigcup_{\lambda\in\lambda} E_\lambda$ is closed.

(ii) Now suppose that E_λ is an arbitrary collection of closed sets. Then $\bigcup_{\lambda\in\Lambda} E_\lambda^c$ is open by Theorem 3.2.3. By De Morgan's Laws,

$$\bigcup_{\lambda\in\Lambda} E_\lambda^c = \left(\bigcap_{\lambda\in\lambda} E_\lambda\right)^c.$$

It then follows from Theorem 3.2.13 that the intersection of an arbitrary collection of closed sets is closed.