MATH 321 WEEK 12 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.7.5

Exercise 2.7.5. By the Cauchy Condensation Test (Theorem 2.4.6) $\sum 1/n^p$ converges if and only if $\sum 2^n (1/2^n)^p$ converges. But notice that

$$\sum 2^n \left(\frac{1}{2^n}\right)^p = \sum \left(\frac{1}{2^n}\right)^{p-1} = \sum \left(\frac{1}{2^{p-1}}\right)^n.$$

By the Geometric Series Test (Example 2.7.5), this series converges if and only if $\left|\frac{1}{2^{p-1}}\right| < 1$. Solving for p we find that p must satisfy $p > 1$.

(2) 2.7.6

(a) False. The sequence $(1, 1, 1, \dots)$ is bounded, but its sequence of partial sums $(s_n) = (n)$, which has no convergent subsequence.

(b) True. If $\sum a_n$ converges, then its sequence of partial sums (s_n) is its own subsequence and (s_n) converges, hence $\sum a_n$ subverges.

(c) True. Assume that $\sum |a_n|$ subverges and that (s_{n_k}) is a convergent subsequence of (s_n), the sequence of partial sums of $\sum |a_n|$. Let (t_n) be the sequence of partial sums of $\sum a_n$. Let $\epsilon > 0$ be arbitrary. We will show that (t_{n_k}), indexed by the same k as (s_{n_k}), converges. Note that (s_{n_k}) is itself the sequence of partial sums of some other series $\sum b_k$, (t_{n_k}) is itself the sequence of partial sums of some other series $\sum c_k$, and $b_k = |c_k|$ for all k. Now, since $\sum b_k = \sum |c_k|$ converges and absolute convergence implies conditional convergence, it must be that $\sum c_k$, and hence (t_{n_k}), converges. Hence $\sum a_n$ subverges. (Whew!)

(d) False. Consider the sequence $(a_n) = (1, 1, -1, 2, -2, 3, -3, 4, -4, \dots)$. The corresponding sequence of partial sums is then $(s_n) = (1, 2, 1, 3, 1, 4, 1, 5, \dots)$, which has the convergent subsequence $(s_{n_k}) = (1, 1, 1, \dots)$, hence $\sum a_n$ subverges. However, any subsequence (a_{n_k}) of (a_n) is infinite, and thus, given any $M \in \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $a_{n_k} > M$. Hence (a_{n_k}) is unbounded, thus it diverges.

(3) 2.7.8

(a) True. Since $\sum |a_n|$ converges, it must be that $(|a_n|) \to 0$ by the contrapositive of the Divergence Test. Hence, there exists $N \in \mathbb{N}$ such that $|a_n| < 1$ whenever $n \geq N$. Multiplying both sides of this inequality by $|a_n|$, we see that whenever $n \geq N$, $a_n^2 = |a_n|^2 < |a_n|$. Now note that

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n^2| &= \sum_{n=1}^{\infty} a_n^2 \\ &= \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n^2 \\ &< \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} |a_n|.\end{aligned}$$

Since $\sum |a_n|$ converges and the first term is finite, it must be that $\sum a_n^2$ converges absolutely, as desired.

(b) False. Consider the sequences defined by $a_n = \frac{(-1)^n}{\sqrt{n}}$ and $b_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. Then $\sum a_n$ converges by the Alternating Series test (applied to the sequence $\frac{1}{\sqrt{n}}$) and $(b_n) \to 0$. However,

$$\sum a_n b_n = \sum \frac{1}{n}$$

diverges.

(c) True. We prove a stronger version of the contrapositive: if $\sum n^2 a_n$ converges, then $\sum a_n$ converges absolutely. (This implies that, if $\sum a_n$ either diverges or converges conditionally, then $\sum n^2 a_n$ diverges.) So let (a_n) be an arbitrary sequence such that $\sum n^2 a_n$ converges. By the Cauchy Criterion for Series, it must be that there exists $N \in \mathbb{N}$ such that, for all $n > m \geq N$,

$$n^2|a_{m+1} + a_{m+2} + \cdots + a_n| = |n^2 a_{m+1} + n^2 a_{m+2} + \cdots + n^2 a_n| < 1$$

and hence that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \frac{1}{n^2}.$$

If we take $n = m + 1$, this inequality becomes

$$|a_n| < \frac{1}{n^2}.$$

Now, we note that

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \\ &< \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Since the first sum is finite and the second is the tail of a convergent sequence, it must be that $\sum_{n=1}^{\infty} |a_n|$ converges, hence $\sum a_n$ converges absolutely.