MATH 321 FINAL EXAM F18 RUBRIC

KENAN INCE

In the following questions, $\mathbb{N}$ refers to the set $\{1, 2, 3, \dots\}$ of natural numbers and $\mathbb{I}$ refers to the set of irrational numbers.

(1) **(58 points)** True or false? Prove if true, and disprove or give a counterexample if false.

(a) **(8 points)** If (a_n) and (b_n) are sequences and both (a_n/b_n) and (b_n) converge, then (a_n) converges.

(i) True; define $c_n := a_n/b_n$ for all n. By assumption, $\lim c_n$ exists; call it c. Also by assumption, $\lim b_n$ exists; call it b. Now consider the sequence $(c_n b_n)$. We have that

$$c_n b_n = \left(\frac{a_n}{b_n}\right) b_n = a_n$$

for all n. Now, by part (iii) of the Algebraic Limit Theorem, since $(c_n) \to c$ and $(b_n) \to b$, it must be that

$$\lim a_n = \lim(c_n b_n) = cb$$

exists.

(b) **(8 points)** If $(x_n) \to x$, then $(|x_n|) \to |x|$.

(i) True; let $\epsilon > 0$ be arbitrary and assume that $(x_n) \to x$. Hence there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$. But then, by the reverse triangle inequality,

$$||x_n| - |x|| \leq |x - x_n| < \epsilon$$

whenever $n \geq N$, hence $(|x_n|) \to |x|$. (The converse, however, is false.)

(c) **(9 points)** If $K_1, K_2, \dots, K_n$ are compact sets, then $\bigcup_{i=1}^n K_i$ is compact.

(i) True. Boundedness is preserved by the finite union operation; just take the max of the bounds.

(ii) For closedness, you'll have to show that a limit point of $\bigcup_{i=1}^n K_i$ is a limit point of at least one of the K_i, hence is contained in the union. Assume x is a limit point of the union and let $\epsilon > 0$ be arbitrary. By the definition of limit point, there exists $y \in \bigcup_{i=1}^n K_i$ such that $y \in V_\epsilon(x)$. But then $y \in K_i$ for some i, which means x is a limit point of K_i since ϵ was arbitrary.

(d) **(9 points)** If $C_1, C_2, \dots, C_n$ are connected sets, then $\bigcap_{i=1}^n C_i$ is connected. [**Hint:** is $\emptyset$ connected?]

(i) Yes, $\emptyset$ is connected because it's explicitly excluded in the definition of disconnected.

(ii) True; for contradiction assume $C := \bigcap_{i=1}^n C_i$ is disconnected. Then there exist $A, B \subseteq \mathbb{R}$ such that $A \cup B = C$ and

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

If any of the C_i is empty or if $C = \emptyset$, then $C = \emptyset$ is connected. So assume none of the C_i, nor C, is empty and let $i \in \{1, \dots, n\}$ be arbitrary. Since $A \cap B \subseteq \overline{A} \cap B = \emptyset$, $A \cap B = \emptyset$, hence any $C_i \subseteq C$ is contained in either A or B, but not both. Define $E := A \cap C_i$ and $F := B \cap C_i$. Then certainly

$$E \cup F = (A \cap C_i) \cup (B \cap C_i) = (A \cup B) \cap C_i = C \cap C_i = C_i.$$

Moreover, $\overline{E} \cap F \subseteq \overline{A} \cap B = \emptyset$ and $E \cap \overline{F} \subseteq A \cap \overline{B} = \emptyset$, hence E and F are a separation of C_i, contradicting that C_i is connected. Hence C is connected.

(e) **(8 points)** The set

$$S := \{\sqrt{n} : n \in \mathbb{N}\} \subset \mathbb{I}$$

is uncountable.

   (i) False. The function $f : \mathbb{N} \to S$ given by $f(n) = \sqrt{n}$ is a 1-to-1 correspondence between $\mathbb{N}$ and S because $\sqrt{n} = \sqrt{m} \iff n = m$ and every element of S is the square root of a natural number. Since $\mathbb{N}$ is countable, it must be that S is countable as well.

 (f) **(8 points)** The set S from part (e) is closed.
   (i) False. Let $\epsilon > 0$ be arbitrary. You proved in your homework that, if $(x_n) \to x$, then $(\sqrt{x_n}) \to \sqrt{x}$. Thus, $(\sqrt{n}) \to \sqrt{0} = 0$, but $0 \notin S$.

 (g) **(8 points)** If E is a nonempty subset of $\mathbb{R}$ that is bounded below, then $\inf E + 1 \in E$.
   (i) False. Consider the set $E = \{\frac{1}{n} : n = 2, 3, 4, \dots\} \subseteq \mathbb{R}$. Certainly E is bounded below by 0, and in fact $\inf E = 0$. However, $\inf E + 1 = 1 \notin E$ by construction.

(2) **(25 points)** Let $a \in \mathbb{R}$ with $a > -1$. Prove that $(1 + a)^n \geq 1 + na$ for all $n \in \mathbb{N}$.
 (a) Base case: $n = 1$

$$(1 + a)^1 = 1 + a = 1 + (1)a.$$

 (b) Inductive step: assume that $(1 + a)^n \geq 1 + na$ and consider $(1 + a)^{n+1}$. Then

$$\begin{aligned}
(1 + a)^{n+1} &= (1 + a)(1 + a)^n \\
&\geq (1 + a)(1 + na) \\
&= 1 + na + a + na^2 \\
&= 1 + a(n + 1) + na^2 \\
&\geq 1 + (n + 1)a \text{ since } a > -1 \implies 0 \leq a^2 < 1 \text{ and } n > 0.
\end{aligned}$$

(3) **(17 points)** Compute

$$\lim_{n\to\infty} \Big(-\frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \frac{3}{16} \mp \cdots + (-1)^n \frac{3}{2^n} \Big).$$

 (a) Note that the term inside the limit is the nth partial sum of the series $\sum_{n=1}^{\infty} (-1)^n \frac{3}{2^n}$.
 (b) Now, note that

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^n \frac{3}{2^n} &= \sum_{n=0}^{\infty} (-1)^n \frac{3}{2^n} - \frac{3}{2^0} \\
&= 3 \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} - 3 \\
&= 3 \sum_{n=0}^{\infty} \Big(\frac{-1}{2}\Big)^n - 3
\end{aligned}$$

 (c) Hence, applying the geometric series formula yields

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^n \frac{3}{2^n} &= 3\Big(\frac{1}{1 - (-1/2)}\Big) - 3 \\
&= 3\Big(\frac{2}{3}\Big) - 3 \\
&= 2 - 3 = -1.
\end{aligned}$$

(4) **(Bonus)** Let T be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 3 and 8. Answer the following questions regarding T and explain why or why not.
 (a) **(3 points)** Is T countable?
   (i) No, T is uncountable. To see this, we mimic our proof that $[0, 1]$ is uncountable. Assume for contradiction that T is countable; then

$$T = \{x_1, x_2, \dots\}$$

   and each $x_i \in T$ has a decimal expansion of the form $x_i = .a_{i1}a_{i2}a_{i3}\dots$ where $a_{ij} \in \{3, 8\}$ for all $i, j \in \mathbb{N}$. Then define x via the decimal representation $x = .a_1a_2a_3\dots$, where

$$a_i = \begin{cases} 3 & \text{if } a_{ii} = 8 \\ 8 & \text{if } a_{ii} = 3. \end{cases}$$

Then $x \in T$ because its decimal expansion consists entirely of 3s and 8s, but $x \neq x_i$ for any i because they disagree in the ith decimal place. This contradiction shows that T is uncountable.

(b) **(5 points)** Is T compact?

(i) Clearly T is bounded by the interval $[-1, 1]$.

(ii) It remains to show T is closed. So let x be an arbitrary limit point of T. Then $x = \lim x_n$, where (x_n) is a sequence of points of T, hence each x_n has a decimal expansion $x_n = .a_{n1}a_{n2}a_{n3}\ldots$ with $a_{ij} \in \{3, 8\}$ for all $i, j \in \mathbb{N}$. Since $(x_n) \to x$, for all $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \frac{1}{10^k} \text{ whenever } n \geq N.$$

In other words, x_n and x must be equal up to the kth decimal place whenever $n \geq N$. Therefore, given any k, the decimal representation of $x$must consist entirely of 3s and 8s up to the kth decimal place. This implies that the decimal representation of x must consist entirely of (a perhaps infinite string of) 3s and 8s. Hence, $x \in T$, and since x was arbitrary, T is closed.

(iii) Since T is closed and bounded, T is compact.

(c) **(5 points)** Is T perfect?

(i) Since T is closed, it must be that T contains its set of limit points.

(ii) However, points of T with terminating decimal expansions, such as $\frac{3}{10} = 0.3$, are not limit points of T, hence T is not perfect. This is because we may consider the distance from $\frac{3}{10}$ to the next nearest point of T. The closest point of T which is less than 0.3 is 0.08, while the closest point of T which is greater than 0.3 is 0.33, a distance of 0.03. Hence, if we let $\epsilon < 0.03$, the neighborhood

$$V_\epsilon(0.3) \subseteq (0.27, 0.33)$$

does not contain any points of T other than 0.3. Hence 0.3 is an isolated point of T.