Name:____

MATH 321 TAKE-HOME FINAL EXAM

This exam will be due on Friday, December 15 at noon, slipped under my office door. You may use your textbook, your class notes, and your homework for MATH 321, Fall 2017. You may not use other books, notes, or the Internet for this exam. You may not discuss any part of this exam with any other person besides myself until the exam has been handed in.

Please sign below to signify that you have abided by the above rules:

Signature:____

(1) **(20 points)** Let S and T be nonempty sets. Prove that there exists a function $f : S \to T$ that is 1-to-1 if and only if there exists an function $g : T \to S$ that is onto.

(2) **(32 points)** Are the following true or false? Prove if true, or give a counterexample if false.
  (a) The sequence defined by $s_1 = 1$ and $s_{n+1} = \sqrt{1 + s_n}$ for all $n \geq 1$ converges.
  (b) If $f : \mathbb{R} \to \mathbb{R}$ is any function, $(a_n) \to c$, and $a_n \neq c$ for all n, then $f(a_n) \to f(c)$.
  (c) Every finite set is compact.
  (d) The union of two perfect sets is perfect.

(3) **(28 points)** Given the series $\sum a_n$ and $\sum b_n$, suppose there exists a natural number N such that $a_n = b_n$ for all $n \geq N$. Prove that $\sum a_n$ is convergent if and only if $\sum b_n$ is convergent. Thus, the convergence of a series is not affected by changing a finite number of terms. (Of course, the value of the sum may change.)

(4) **(20 points)** A set E is **totally disconnected** if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$. (Here, **separated** means that $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.) Show that $\mathbb{Q}$ is totally disconnected.

(5) **(Bonus: 10 points)** We say that the **limit of** $f(x)$ **equals** L, written $\lim_{x \to c} f(x) = L$, if, for all $\epsilon > 0$, there exists $\delta > 0$ so that whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. Prove the Squeeze Theorem for functions. That is, let f, g, and h be functions from $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a limit point of A. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in D$ with $x \neq c$, and suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$. Prove that $\lim_{x \to c} g(x) = L$ as well. You may assume the Algebraic Limit Theorem for functions.