MATH 321 WEEK 6 CLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.2.7

Exercise 2.2.7. (a) The sequence $(-1)^n$ is *frequently* in the set 1.

(b) Definition (i) is stronger. "Frequently" does not imply "eventually", but "eventually" implies "frequently".

(c) A sequence (a_n) converges to a real number a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is *eventually* in $V_\epsilon(a)$.

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2, then (x_n) is *frequently* in the interval (1.9, 2.1). However, (x_n) is not necessarily *eventually* in the interval (1.9, 2.1). Consider the sequence $(2, 0, 2, 0, 2, \cdots)$, for instance.

(2) 2.2.8

(a) Yes; the definition means there's a "window size" for which, arbitrarily far out in the sequence, there are zeroes in all windows of that size. In this case, take $M = 2$.

(b) Yes; suppose not. Then there exists N_1 so that, for all $n \geq N$, $x_n \neq 0$. But then, for any $M \in \mathbb{N}$ and this N, all n satisfying $N \leq n \leq N + M$ have $x_n \neq 0$.

(c) No. Consider the sequence $(x_n) = (1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots)$. Then, no matter the "window size" M you choose, there's a window of that size that doesn't always "show" a zero as you go arbitrarily far out. In other words, there exists $N \in \mathbb{N}$ so that, for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

(d) A sequence is not zero-heavy if, for all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ so that for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

(3) 2.3.7

(a) $(x_n) = (-1)$, $(y_n) = (1)$

(b) Impossible; Algebraic Limit Theorem implies that if $(x_n + y_n)$ converges and (x_n) converges, then $y_n = (x_n + y_n) - (x_n)$ converges as well.

(c) Consider the sequence $(b_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$. Then $(1/b_n) = (1, 2, 3, 4, \ldots)$ which is unbounded and diverges. It may be tempting to say something like $(1/b_n)$ "converges to infinity." Although it is possible to give a rigorous meaning to the phrase *converges to infinity*, we can avoid the whole issue by adding alternating negative signs to the original sequence.

(c)

(d) A convergent sequence is bounded, so since (b_n) is convergent, $|b_n| \leq M$ for all n. Since $(a_n - b_n)$ is bounded, $|a_n - b_n| \leq N$ for all n. Thus,

$$|a_n| = |a_n - b_n + b_n| \leq |a_n - b_n| + |b_n| \leq M + N$$

must be bounded as well.

(e) Let $a_n = 0$ for all n and $b_n = (-1)^n$. Then (a_n) and $(a_n b_n) \to 0$, but b_n diverges.

(4) 2.3.13

Exercise 2.3.13. (a) First fix $n \in \mathbf{N}$ and let

$$b_n = \lim_{m\to\infty} a_{mn} = \lim_{m\to\infty} \frac{1}{1+n/m} = \frac{1}{1+0} = 1.$$

Thus

$$\lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} = \lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 = 1.$$

In the other order, we first fix m and compute the limit along each row of the a_{mn} array to get

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{m\to\infty} \left(\lim_{n\to\infty} \frac{m/n}{(m/n)+1} \right) = \lim_{m\to\infty} \frac{0}{0+1} = 0.$$

From this example we see that it is possible for

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} \neq \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n},$$

and so defining doubly indexed limits in this fashion would be problematic to say the least. Instead, we use the definition given in the exercise. The question to ask is how the definition of the limit relates to the iterated values. Does the existence of one of these imply the existence of the others? Can they all be different? When are they equal?

(b) For the case $a_{mn} = \frac{1}{m+n}$, it straightforward to verify that $\lim_{m,n\to\infty} = 0$. Given $\epsilon > 0$, we choose $N > 1/2\epsilon$. If both n and m are larger than N, the sum $n + m \geq 2N > 1/\epsilon$, and it follows that $1/(m+n) < \epsilon$ as desired.

It is easy to check that both iterated limits from part (a) also yield a value of 0 in this case.

The case $a_{mn} = mn/(m^2 + n^2)$ is more subtle. As in the previous example, both iterated limits yield 0. To see this, we can write

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{mn}{m^2+n^2} = \lim_{m\to\infty} \left(\lim_{n\to\infty} \frac{m/n}{(m^2/n^2)+1} \right) = \lim_{m\to\infty} \frac{0}{0+1} = 0.$$

The calculation of the other iterated limit is analogous.

In this case, however, $\lim_{m,n\to\infty} a_{mn}$ does not exist. Perhaps the best way to see this is to note that when $m = n$ we have $a_{mn} = 1/2$. But when $m = 2n$ we get $a_{mn} = 2/5$. Thus, no matter what we choose for our limiting value a, it is going to be impossible to force $|a_{mn} - a|$ to be arbitrarily small for all large values of m and n.

(c) The array $a_{mn} = \frac{(-1)^n}{m} + \frac{(-1)^m}{n}$ has this property. The iterated limits can't be computed but the overall limit is most definitely zero.

(d) As in all convergence proofs, it is important in this argument to be very thoughtful (and explicit) about the order in which we assert the existence of the various variables involved.

Let $\epsilon > 0$. Our job is to produce an $N \in \mathbf{N}$ such that $|b_m - a| < \epsilon$ whenever $m \geq N$. Now for *any* values of m and n the triangle inequality gives us

$$|b_m - a| \leq |b_m - a_{mn}| + |a_{mn} - a|,$$

and the hypothesis of the problem gives us control over the quantities on the right hand side of this inequality, provided we are careful.

Because $\lim_{m,n\to\infty} a_{mn} = a$, there exists an $N \in \mathbf{N}$ such that $|a_{mn} - a| < \epsilon/2$ for all $m, n \geq N$. We now argue that this same N works for our purposes. To see this, consider an arbitrary $m \geq N$. For this particular m, we are given that $b_m = \lim_{n\to\infty} a_{mn}$. Among other things, this implies that there exists an n_0 such that $|b_m - a_{mn_0}| < \epsilon/2$, and we are on solid ground insisting that $n_0 \geq N$ as well. It follows that

$$\begin{aligned} |b_m - a| &\leq |b_m - a_{mn_0}| + |a_{mn_0} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and the result is proved.

(e) This follows directly from part (d).