MATH 321 WEEK 11 CLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.7.1

Exercise 2.7.1. (a) Here we show that the sequence of partial sums (s_n) converges by showing that it is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. We need to find an N such that $n > m \geq N$ implies $|s_n - s_m| < \epsilon$. First recall,

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|.$$

Because (a_n) is decreasing and the terms are positive, an induction argument shows that for all $n > m$ we have

$$|a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n| \leq a_{m+1}.$$

So, by virtue of the fact that $(a_n) \to 0$, we can choose N so that $m \geq N$ implies $a_m < \epsilon$. But this implies

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + \cdots \pm a_n| \leq a_{m+1} < \epsilon$$

whenever $n > m \geq N$, as desired.

(b) Let I_1 be the closed interval $[0, s_1]$. Then let I_2 be the closed interval $[s_2, s_1]$, which must be contained in I_1 as (a_n) is decreasing. Continuing in this fashion, we can construct a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots .$$

By the Nested Interval Property there exists at least one point S satisfying $S \in I_n$ for every $n \in \mathbf{N}$. We now have a candidate for the limit, and it remains to show that $(s_n) \to S$.

Let $\epsilon > 0$ be arbitrary. We need to demonstrate that there exists an N such that $|s_n - S| < \epsilon$ whenever $n \geq N$. By construction, the length of I_n is $|s_n - s_{n-1}| = a_n$. Because $(a_n) \to 0$ we can choose N such that $a_n < \epsilon$ whenever $n \geq N$. Thus,

$$|s_n - S| \leq a_n < \epsilon$$

because both $s_n, S \in I_n$.

(c) The subsequence (s_{2n}) is increasing and bounded above (by a_1 for instance.) The Monotone Convergence Theorem allows us to assert that there exists an $S \in \mathbf{R}$ satisfying $S = \lim(s_{2n})$. One way to prove that the other subsequence (s_{2n+1}) converges to the same value is to use the Algebraic Limit Theorem and the fact that $(a_n) \to 0$ to write

$$\lim(s_{2n+1}) = \lim(s_{2n} + a_{2n+1}) = S + \lim(a_{2n+1}) = S + 0 = S.$$

The fact that both (s_{2n}) and (s_{2n+1}) converge to S implies that $(s_n) \to S$ as well. (See Exercise 2.3.5.)

(2) 2.7.7

Exercise 2.7.7. (a) The idea here is that eventually the terms a_n "look like" a non-zero constant times $1/n$, and we know that any series of this form diverges. To make this precise, let $\epsilon_0 = l/2 > 0$. Because $(na_n) \to l$, there exists $N \in \mathbf{N}$

such that $na_n \in V_{\epsilon_0}(l)$ for all $n \geq N$. A little algebra shows that this implies we must have $na_n > l/2$, or

$$a_n > (l/2)\,(1/n) \quad \text{for all } n \geq N.$$

Because this inequality is true for all but some finite number of terms, we may still appeal to the Comparison Test to assert that $\sum a_n$ diverges.

(b) Assume that $\lim(n^2 a_n) \to L \geq 0$. The definition of convergence (with $\epsilon_0 = 1$) tells us that there exists an N such that $n^2 a_n < L + 1$ for all $n \geq N$. This means that eventually $a_n < (L+1)/n^2$. We know that the series $\sum 1/n^2$ converges, and by the Algebraic Limit Theorem for series (Theorem 2.7.1), $\sum (L+1)/n^2$ converges as well. Thus, by the Comparison Test $\sum a_n$ must converge.

(3) 2.7.9

Exercise 2.7.9. (a) First, pick an ϵ-neighborhood around r of size $\epsilon_0 = |r - r'|$. Because $\lim \left|\frac{a_{n+1}}{a_n}\right| = r$, there exists an N such that $n \geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right| \in V_{\epsilon_0}(r)$. It follows that $\left|\frac{a_{n+1}}{a_n}\right| \leq r'$ for all $n \geq N$, and this implies the statement in (a)

(b) Having chosen N, $|a_N|$ is now a fixed number. Also, $\sum (r')^n$ is a geometric series with $|r'| < 1$, so it converges. Therefore, by the Algebraic Limit Theorem $|a_N| \sum (r')^n$ converges.

(c) From (a) we know that there exists an N such that $|a_{N+1}| \leq |a_N| r'$. Extending this we find $|a_{N+2}| \leq |a_{N+1}| r' \leq |a_N| (r')^2$, and using induction we can say that

$$|a_k| \leq |a_N| (r')^{k-N} \quad \text{for all } k \geq N.$$

Thus, $\sum_{k=N}^{\infty} |a_k|$ converges by the Comparison Test and part (b). Because

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k|$$

and $\sum_{k=1}^{N-1} |a_k|$ is just a finite sum, the series $\sum_{k=1}^{\infty} |a_k|$ converges.

(4) 2.7.13

Exercise 2.7.13. (a) Let $s_n = \sum_{k=1}^{n} x_k$. By hypothesis, (s_n) converges to a limit L. Among other things, this implies that there exists $M > 0$ satisfying $|s_n| \leq M$ for all $n \in \mathbf{N}$.

For each $n \in \mathbf{N}$, Exercise 2.7.12 implies

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}). \tag{1}$$

(b) We would like to take the limit across equation (1) as $n \to \infty$. We know (s_n) and (y_{n+1}) both converge, but what about the sum? Well, using a telescoping argument we can show that it converges absolutely. More precisely, observe that

$$\begin{aligned} \sum_{k=1}^{n} |s_k (y_k - y_{k+1})| &\leq \sum_{k=1}^{n} M (y_k - y_{k+1}) \\ &= M (y_1 - y_{n+1}), \end{aligned}$$

and (y_{n+1}) converges as $n \to \infty$. This proves $\sum_{k=1}^{n} s_k (y_k - y_{k+1})$ converges absolutely. Applying the Algebraic Limit Theorem to equation (1) gives the result.