MATH 321 WEEK 10 CLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.6.5

Exercise 2.6.5. Note that the Pseudo-Cauchy definition only requires that the difference between consecutive terms in the sequence become arbitrarily small, whereas the real Cauchy property requires that any two terms beyond a certain point in the sequence differ by an arbitrarily small amount.

(i) Pseudo-Cauchy sequences are not necessarily bounded. A counterexample would be the sequence of partial sums of the harmonic series:

$$(1), (1 + 1/2), (1 + 1/2 + 1/3), (1 + 1/2 + 1/3 + 1/4), \ldots$$

Because $s_{n+1} - s_n = 1(n+1)$, it follows that the sequence is Pseudo-Cauchy, and we have seen in a previous example that this sequence is unbounded.

(ii) This is true and can be proved with a straightforward triangle inequality proof.

Let $\epsilon > 0$ be arbitrary. We need to find an N so that $n \geq N$ implies $|(x_n + y_n) - (x_{n+1} + y_{n+1})| < \epsilon$. Because (x_n) and (y_n) are Pseudo-Cauchy we can pick N so that when $n, m \geq N$ it follows that $|x_n - x_{n+1}| < \epsilon/2$ and $|y_n - y_{n+1}| < \epsilon/2$. But in this case we have

$$|(x_n + y_n) - (x_{n+1} + y_{n+1})| \leq |x_n - x_{n+1}| + |y_n - y_{n+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

(2) 2.6.6

(a) Any convergent sequence is quasi-increasing, so any convergent sequence that isn't monotone or eventually monotone will do as an example. For instance, $(a_n) = ((-1)^n/n)$ is quasi-increasing but not monotone or eventually monotone.

Lemma 1. *Every convergent sequence is quasi-increasing.*

Proof. Assume that (a_n) is a convergent sequence, thus (a_n) is Cauchy. Therefore, given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ so that whenever $n > m \geq N$,

$$|a_n - a_m| < \epsilon.$$

But this implies that

$$a_n > a_m - \epsilon$$

as desired. □

(b) Consider the sequence (b_n) defined by

$$b_n = \begin{cases} n, & n \neq 2 \\ 47, & n = 2 \end{cases}.$$

Then (b_n) is not monotone and (b_n) diverges. To show (b_n) is quasi-increasing, let $\epsilon > 0$ be arbitrary. Note that, if $n > m \geq 3$,

$$|b_n - b_m| = n - m.$$

Thus, if $n > m \geq 3$,

$$b_n = n > m > m - \epsilon = b_m - \epsilon.$$

Therefore, (b_n) is quasi-increasing.

(c) It is true that, if (a_n) is bounded and quasi-increasing, then (a_n) converges. To see this, we can mimic the proof of the MCT. Let's consider the set of points $\{a_n : n \in \mathbb{N}\}$. By assumption, this set is bounded, so we can let

$$s = \sup\{a_n : n \in \mathbb{N}\}.$$

We claim that $\lim a_n = s$. To prove this, let $\epsilon > 0$. Because s is the least upper bound for $\{a_n : n \in \mathbb{N}\}$, $s - \frac{\epsilon}{2}$ is not an upper bound, so there exists a point in the sequence a_{N_1} so that $s - \frac{\epsilon}{2} < a_{N_1}$. Now, the fact that (a_n) is quasi-increasing implies that there exists an $N_2 \in \mathbb{N}$ such that, if $n > m \geq N_2$, then $a_n > a_m - \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and assume $n > m \geq N$. Then

$$s - \epsilon = \left(s - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} < a_{N_1} - \frac{\epsilon}{2} < a_n \leq s < s + \epsilon,$$

which implies $|a_n - s| < \epsilon$, as desired.

- Note that we've shown that a sequence (a_n) is convergent if and only if it is quasi-increasing and bounded.

(3) 2.6.7

Exercise 2.6.7. (a) Let (a_n) be a bounded increasing sequence. We want to argue that (a_n) converges. Because (a_n) is bounded, we can appeal to the Bolzano-Weierstrass to assert that (a_n) has a convergent subsequence (a_{n_k}). Set $L = \lim a_{n_k}$. The goal is to show that the original sequence converges to this same limit.

Step one is to argue that L is an upper bound for all the terms in (a_n). Assume, for contradiction, that there exists $a_m > L$ and set $\epsilon_0 = a_m - L$. The fact that (a_n) is increasing implies

$$a_n - L \geq a_m - L = \epsilon_0 > 0$$

for all $n \geq m$, which is impossible if $L = \lim a_{n_k}$.

Having established that $a_n \leq L$ for all n, we can show $\lim a_n = L$. Given $\epsilon > 0$, we know there exists a term in the subsequence, call it a_{n_K}, satisfying $L - a_{n_K} < \epsilon$. If $n \geq n_K$ then $L - a_n \leq L - a_{n_K} < \epsilon$, and the result follows.

(b) Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all n. Our goal is to use the Cauchy Criterion to produce a convergent subsequence.

First construct the sequence of closed intervals and the subsequence with $a_{n_k} \in I_k$ according to the method described in the proof of the Bolzano-Weierstrass Theorem in the text. Rather than using NIP to produce a candidate for the limit of this subsequence, we can argue that (a_{n_k}) is convergent by appealing to the Cauchy Criterion.

Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. (Note that this is the place where the Archimedean Property is required. In particular, we need some way to know that $(1/2)^k \to 0$ that doesn't make implicit use of BW or something equivalent to it.) Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. So for any $s, t \geq N$, because a_{n_s} and a_{n_t} are in I_k, it follows that $|a_{n_s} - a_{n_t}| < \epsilon$. Having shown (a_{n_k}) is a Cauchy sequence, we know it converges.

(c) The rational numbers are an ordered field where the Archimedean Property holds. Since AoC is most definitely not true in $\mathbf{Q}$, it follows that there is no way to prove AoC using only properties possessed by $\mathbf{Q}$.