MATH 321 WEEK 7 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.3.6: Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $1/n \to 0$ as given and using both the ALT and Exercise 2.3.1, show that $\lim b_n$ exists and find its value.

(a) Because we're told we need to use $1/n \to 0$, it makes sense to try to rewrite b_n in a way that introduces a denominator. Let's try multiplying by the conjugate $n + \sqrt{n^2 - 2n}$ on the top and bottom:

$$\begin{aligned} b_n &= \frac{(n - \sqrt{n^2 + 2n})(n + \sqrt{n^2 + 2n})}{n + \sqrt{n^2 + 2n}} \\ &= \frac{n^2 - (n^2 + 2n)}{n + \sqrt{n^2 + 2n}} \\ &= \frac{-2n}{n + \sqrt{n^2 + 2n}}. \end{aligned}$$

Dividing the numerator and denominator by n yields

$$b_n = \frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}.$$

Since the denominator is nonzero, we can use the Algebraic Limit Theorem and Exercise 2.3.1 to conclude that

$$\begin{aligned} \lim b_n &= \frac{\lim(-2)}{\lim(1) + \lim \sqrt{1 + \frac{2}{n}}} \\ &= \frac{-2}{1 + \sqrt{\lim(1 + \frac{2}{n})}} \\ &= \frac{-2}{1 + \sqrt{\lim(1) + \lim(\frac{2}{n})}} = \frac{-2}{2} = -1. \end{aligned}$$

(2) 2.4.2:

(a) Consider the recursively defined sequence $y_1 = 1$, $y_{n+1} = 3 - y_n$, and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$. What is wrong with this argument?

- The problem is that $\lim y_n$ does not exist, hence any reasoning about its value is invalid. To see this, note that the first few terms of the sequence are

$$1, 2, 1, 2, 1, 2, \ldots$$

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

- Yes; we use the Monotone Convergence Theorem. First, we show inductively that y_n is monotone increasing.
  - Base case: $y_1 = 1$, $y_2 = 3 - \frac{1}{1} = 2$. Hence $y_1 \leq y_2$.
  - Inductive step: suppose that $y_{n-1} \leq y_n$. Then $\frac{1}{y_n} \leq \frac{1}{y_{n-1}}$, and hence

$$y_{n+1} = 3 - \frac{1}{y_n} \geq 3 - \frac{1}{y_{n-1}} = y_n$$

as desired.

- We now show inductively that $y_n \geq 1$ for all n. In the next step, this will be used to show that $y_n \leq 3$ for all n as well.
  - Base case: $y_1 = 1 \geq 1$.
  - Inductive step: suppose that $y_n \geq 1$. Then
    $$y_{n+1} = 3 - \frac{1}{y_n} \geq 3 - \frac{1}{1} = 2 \geq 1$$
    as desired.
- Finally, we show that $y_n \leq 3$ for all n.
  - Base case: $y_1 = 1 \leq 3$.
  - Inductive step: suppose $y_n \leq 3$. Then
    $$y_{n+1} = 3 - \frac{1}{y_n} \leq 3 - \frac{1}{3} \leq 3$$
    as desired.
- Now, applying the Monotone Convergence Theorem to y_n shows that $\lim y_n$ exists; call it y. By the reasoning above, $\lim y_{n+1} = y$ as well. Thus, taking the limit of both sides of the recursive definition of y_{n+1} yields
  $$y = 3 - \frac{1}{y} \implies y^2 - 3y - 1 = 0$$
  $$y = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3 \pm \sqrt{13}}{2}.$$
  But $\sqrt{13} > \sqrt{9} = 3$, hence $(3+\sqrt{13})/2 > (3+3)/2 = 3$. And the Order Limit Theorem says that, since $y_n \leq 3$ for all n, $\lim y_n = y \leq 3$ as well. Therefore,
  $$y = \lim y_n = \frac{3 - \sqrt{13}}{2}.$$

(3) 2.4.3

(a) Show that $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ converges and find the limit.

- First, note that this sequence can be defined recursively as $y_1 = \sqrt{2}$, $y_{n+1} = \sqrt{2+y_n}$ for all n. We prove inductively that this sequence is monotone increasing and bounded above by 2. Then the MCT would imply that $y = \lim y_n$ exists.
  - Base case: $y_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = y_2$ because adding a positive number under the square root can only increase the value. Moreover, $y_1 = \sqrt{2} \leq 2$.
  - Inductive step: suppose that $y_n > y_{n-1}$. Then
    $$y_{n+1} = \sqrt{2+y_n} > \sqrt{2+y_{n-1}} = y_n$$
    as desired. Moreover, assume that $y_n \leq 2$. Then
    $$y_{n+1} = \sqrt{2+y_n} \leq \sqrt{2+2} = 2$$
    as desired.
- Taking the limit of both sides of our recursive definition of y_{n+1}, using the Algebraic Limit Theorem to bring the limit inside the square root, we see that
  $$\begin{aligned} y &= \sqrt{2+y} \\ y^2 &= 2+y \\ y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \\ y &= 2 \text{ or } -1. \end{aligned}$$
  Since all of the terms of the sequence are positive, the OLT implies that the limit y cannot be negative. Therefore, $\lim y_n = 2$.