WEEK 8 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.3.6

Rationalizing, we obtain

$$\begin{aligned} b_n &= n - \sqrt{n^2+2n} \cdot \frac{n+\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}} \\ &= \frac{n^2-n^2-2n}{n+\sqrt{n^2+2n}} = \frac{-2n}{n(1+\sqrt{1+\frac{2}{n}})} \\ &= \frac{-2}{1+\sqrt{1+\frac{2}{n}}}. \end{aligned}$$

Since $\lim(1/n) = 0$, the Algebraic Limit Theorem implies that $\lim(2/n) = 0$ as well, and applying the ALT to the definition of b_n yields

$$\lim(b_n) = \frac{-2}{1+\sqrt{1+0}} = \frac{-2}{2} = -1.$$

(2) 2.4.2

(a) The problem is that the limit of this sequence doesn't exist. In fact, notice that the terms of the sequence are

$$1, 2, 1, 2, 1, 2, \ldots$$

which is a divergent sequence.

(b) Note that the first few terms of this sequence are

$$1, 2, 2\frac{1}{2}, 2\frac{3}{5}, \ldots.$$

Based on these terms, we conjecture that (y_n) is increasing and bounded above by 3. To show (y_n) is increasing, note that $y_1 < y_2$ and assume for induction that $y_n < y_{n+1}$. Then

$$y_{n+1} = 3 - \frac{1}{y_n} < 3 - \frac{1}{y_{n+1}} = y_{n+2},$$

proving that (y_n) is increasing. Finally, note that $y_1 < 3$ and assume that $y_n < 3$. Then

$$y_{n+1} = 3 - \frac{1}{y_n} < 3 - \frac{1}{3} = \frac{8}{3} < 3$$

as desired. Therefore, by the Monotone Convergence Theorem, (y_n) converges to some limit y. Taking the limit of both sides of the recursive definition of (y_n) yields

$$y = 3 - \frac{1}{y} \implies y^2 - 3y + 1 = 0 \implies y = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

(3) 2.4.3

Exercise 2.4.3. (a) One way to describe the sequence in this exercise is to set $a_1 = \sqrt{2}$ and let $a_{n+1} = \sqrt{2 + a_n}$.

To prove that (a_n) is increasing we use induction. Clearly $a_1 < a_2$. Now assume $a_n < a_{n+1}$. Adding 2 to both sides and then taking the square root preserves the inequality, and so we have

$$\sqrt{2 + a_n} < \sqrt{2 + a_{n+1}},$$

which is equivalent to asserting $a_{n+1} < a_{n+2}$. By induction, (a_n) is increasing. (Notice that this proof takes as given that the square root function is increasing.) We can also use induction to prove (a_n) is bounded above by 2. It's clear that $a_1 < 2$. Assuming $a_n < 2$, it follows that $a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2$.

Because (a_n) is bounded and increasing, it converges to a limit L by MCT. Now, as with the previous exercises, we are justified in taking the limit across the recursive equation to get that L satisfies $L = \sqrt{2 + L}$. A little algebra yields the equation $L^2 - L - 2 = 0$, from which we conclude that $L = 2$.

We should note that the last steps in this problem involved taking the limit inside a square root sign, and this is not a manipulation that is justified by the Algebraic Limit Theorem. Instead we should reference Exercise 2.3.1 to support this part of the argument.

(b) This is remarkably similar to part (a), and in fact has the same answer. First, rewrite the sequence in a recursive way: $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$.

Let's prove that the sequence is increasing by induction. For the base case we observe that

$$x_1 = 2 < \sqrt{2\sqrt{2}} = x_2,$$

so we just need to prove that $x_n < x_{n+1}$ implies $x_{n+1} < x_{n+2}$. But if $x_n < x_{n+1}$ then $\sqrt{x_n} < \sqrt{x_{n+1}}$, and multiplying by $\sqrt{2}$ gives $\sqrt{2x_n} < \sqrt{2x_{n+1}}$. Thus we have $x_{n+1} < x_{n+2}$ and the sequence is increasing.

To show the sequence is bounded above by 2 we first observe that $x_1 < 2$. Now if $x_n < 2$, then $x_{n+1} = \sqrt{2x_n} < \sqrt{2 \cdot 2} = 2$ as well, and (x_n) is bounded.

Therefore this sequence converges by the Monotone Convergence Theorem and we can assert that both (x_n) and (x_{n+1}) converge to some real number l. Taking limits across the recursive equation $x_{n+1} = \sqrt{2x_n}$ yields $l = \sqrt{2l}$, which implies $l = 2$.