MATH 321 1.5 - CARDINALITY

1. Warmup

- Both the irrational and rational numbers are dense in $\mathbb{R}$; we can think of both sets as "dotting" the real line in such a way that there are infinitely many "dots" between any two points on the line.
- A priori, it would seem that the sets $\mathbb{Q}$ and $\mathbb{I}$ would thus be the same size. This is false! In a sense, there are "as many" irrationals as reals, and fewer rationals than irrationals: $|\mathbb{Q}| < |\mathbb{I}| = |\mathbb{R}|$!

Definition 1. A function $f : A \to B$ is **one-to-one** if, whenever $f(a_1) = f(a_2)$ in B, $a_1 = a_2$ in A. [The **contrapositive** of your book's definition.] The function f is **onto** if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$. If f is both 1-to-1 and onto, we say that f is a **1-to-1 correspondence** between A and B.

- If functions throw darts from points in A to points in B,
  - one-to-one means that no two darts hit the same point in B, and
  - onto means that every point in B is hit by a dart.

Definition 2. Two sets A and B have the same **cardinality** if there exists a function $f : A \to B$ that is one-to-one and onto. Write $A \sim B$.

- This makes sense because 1-1 and onto means no two darts hit the same spot and every spot is hit. This means there are the same number of points of B as darts.
- Counting a set A is the same as covering A in darts labeled with natural numbers.

Definition 3. We say A is **countable** if $A \sim \mathbb{N}$. If A is neither finite nor countable, then we say A is **uncountable**.

Note 4. Under this terminology, finite sets aren't countable. Weird!

[start here 9-24-18]

Exercise 5. [slide] **Reading question**: finish the following proof that if A is an infinite subset of B and B is countable, then A is countable:

Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, whic
and onto. Let $A \subseteq B$ be an infinite subset of B. We must show th
countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of g :
set $g(1) = f(n_1)$. Show how to inductively continue this process to pr
1–1 function g from $\mathbf{N}$ onto A.

Proof. Next let $n_2 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1\}\}$ and set $g(2) = f(n_2)$. In other words, we set $g(1)$ to be the smallest natural number that maps into A, $g(2)$ to be the second smallest, etc.

We now use induction to define the function $g : \mathbb{N} \to A$ for all $n \in \mathbb{N}$. This is a common use of induction; the statement we'd like to prove is $P(n) =$"$g(n)$ is defined" for all $n \in \mathbb{N}$. So we have to do our base case by defining $g(1)$ as above, then do an inductive step.

In general, assume we have defined $g(m-1)$, and let $g(m) = f(n_m)$ where

$$n_m = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1), \ldots, f(n_{m-1})\}\}.$$

We show that $g : \mathbb{N} \to A$ is 1-to-1 and onto:

(1) **1-1:** We first note that our definition of n_m above actually defines a function $n_m : \mathbb{N} \to \mathbb{N}$. We first show that this function $n_m : \mathbb{N} \to \mathbb{N}$ is 1-1, then use this fact to show that g is 1-1.
  (a) Suppose that $n_m = n_{m'}$. Suppose for contradiction that $m < m'$ (if $m > m'$, relabel m as m' and vice versa). Then, by definition, $f(n_{m'}) \in A \setminus \{f(n_1), \ldots, , f(n_m), \ldots, f(n_{m'-1})\}$. In particular, $f(n_{m'}) \neq f(n_m)$. But this contradicts that $n_m = n_{m'}$.
  (b) Now, it follows that if $g(m) = g(m')$, then

$$f(n_m) = g(m) = g(m') = f(n_{m'})$$

  by definition, and since f is 1-to-1, this means that $n_m = n_{m'}$. Since n_m is 1-to-1, this means that $m = m'$.

(2) **Onto**: suppose that $a \in A$. Then $a \in B$, and since f is onto, $a = f(k)$ for some $k \in \mathbb{N}$. It must be that $k \in \{n : f(n) \in A\}$, and as we inductively remove the minimal element, k must eventually be the minimum by at least the $k-1$st step.

□

Note 6. The fact that, if $A \subseteq B$ and B is countable, then A is either finite or countable isn't too surprising. If a set can be arranged into a single list, then deleting some elements from the list results in another (shorter, and possibly terminating) list.

2. Proving that $A \sim B$

Definition 7. For any set A, the **power set** of A, $P(A)$, is the set of all possible subsets of A.

Proof technique. To show that $A \sim B$, or equivalently that there exists a 1-to-1 correspondence $f : A \to B$, it suffices to find a way of "labeling" each element of A with a distinct element of B. If $A = \mathbb{N}$, this is called a "counting strategy" for A. This doesn't always lead to a function describable via a formula, but as long as it's clear that your counting strategy defines a 1-1 correspondence, you don't need a formula.

Exercise 8.

1. If $B = \{e, \pi, \sqrt{2}\}$, then list all elements of $P(B)$. (Hint: there are 8 of them).
2. [T/F] If $A = \{1, 2, 3\}$ and $B = \{e, \pi, \sqrt{2}\}$ then $A \sim B$.
3. [T/F] If $A = \{1, 2, 3\}$ and $C = \{x \in \mathbf{R} : (x^2 - 1)(x^2 - 4) = 0\}$ then $A \sim C$.
4. [T/F] The even integers $2\mathbf{Z}$ have the same cardinality as the integers; that is, $2\mathbf{Z} \sim \mathbf{Z}$.

Proof. We label $2k \in 2\mathbb{Z}$ with $k \in \mathbb{Z}$. In other words, let $f : \mathbb{Z} \to 2\mathbb{Z}$ be the function given by $f(k) = 2k$. Then

(1) f **is 1-1**: if $f(k_1) = f(k_2)$, then $2k_1 = 2k_2 \implies k_1 = k_2$.
(2) f **is onto**: if $x \in 2\mathbb{Z}$, then $x = 2k = f(k)$ for some $k \in \mathbb{Z}$.

□

Exercise 9. [slide] (Note that 6-8 show that "having the same cardinality" is an *equivalence relation.*)

6. [T/F] $A \sim A$ for every set A.
7. [T/F] If $A \sim B$, then $B \sim A$.
8. [T/F] If $A \sim B$ and $B \sim C$, then $A \sim C$.
9. (a) Make a table of all positive rational numbers so that each fraction $\frac{p}{q}$ appears in t
   column and the qth row. (Okay, just go out as far as $p, q = 5$.)
   (b) Cross out duplicates that are not in lowest terms.
   (c) Turn your table 45° clockwise, so that $\frac{1}{1}$ is in the top "row". There should b
   numbers in the next row, and more numbers as you move further down. Readi
   twisted table, list in order the first dozen numbers you encounter.
   (d) [T/F] $\mathbf{Q}$ is countable.

(1) True; the identity function $id : A \to A$ is a 1-to-1 correspondence between A and itself.
(2) True; we are given that $A \sim B$, hence there exists a 1-to-1 correspondence $f : A \to B$. Since f is 1-to-1 and onto, it must be invertible, hence f^{-1} exists. But $f^{-1} : B \to A$ is also 1-to-1 and onto, hence $B \sim A$.
(3) True; we are given that 1-to-1 correspondences $f : A \to B$ and $g : B \to C$ exist. Now the function $g \circ f : A \to C$ is 1-to-1 and onto by a quick argument.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**...**
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	$\cancel{\frac{2}{6}}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\cancel{\frac{6}{2}}$	$\cancel{\frac{6}{3}}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
⋮	⋮								

(4) [slide]
   (a) This table shows that $\mathbb{Q}$ is countable. Label $\frac{1}{1}$ with the natural number 1, $\frac{2}{1}$ with 2, and continue inductively to define a 1-to-1 correspondence $f : \mathbb{N} \to \mathbb{Q}$.

3. Is $\mathbb{R}$ countable?

Exercise 10. (if time; otherwise do as a class) [slide]

10. Let $\{x_1, x_2, x_3, \dots\}$ be a sequence of real numbers.
   (a) [T/F] It is possible to construct a non-empty closed interval $I_1 \subset \mathbf{R}$ so that x_1
   (b) [T/F] It is possible to construct a non-empty closed interval $I_2 \subset I_1$ so that x_2
   (c) [T/F] For every $n \in \mathbf{N}$, it is possible to construct a non-empty closed interval I_n
   that does not contain x_{n+1}.
   (d) $\cap_{n=1}^{\infty} I_n$ contains x_k for some $k \in \mathbf{N}$.
   (e) $\cap_{n=1}^{\infty} I_n$ is empty/non-empty. (Choose one and prove it).
   (f) $\mathbf{R}$ is countable/uncountable. (Choose one and prove it).

(1) True; let $I_1 = [x_1 + 1, x_1 + 2]$
(2) True; consider the intervals $[x_1 + 1, x_1 + \frac{4}{3}]$ and $[x_1 + \frac{5}{3}, x_1 + 2]$. Then x_2 can be in at most one of these intervals; choose one of these intervals that does not contain x_2 to be I_2.
(3) True; we may repeat this process inductively. Suppose we have constructed I_n so that $x_1, \dots, x_n \notin I_n$. We may then divide I_n into thirds; consider the first third and last third. Then x_{n+1} is in at most one of these thirds; choose one of the two subintervals that does not contain x_{n+1} and call it I_{n+1}.
(4) False; for all $k \in \mathbb{N}$, $x_k \notin I_k$, and thus $x_k \notin \bigcap_{n=1}^{\infty} I_n$.
(5) By the Nested Interval Property, $\cap_{n=1}^{\infty} I_n \neq \emptyset$. Choose an element x of $\cap_{n=1}^{\infty} I_n$; then $x \in \mathbb{R}$, but $x \notin \{x_1, x_2, x_3, \dots\}$, since if it were, x could not be in $\cap_{n=1}^{\infty} I_n$.

Theorem 11. $\mathbb{R}$ *is uncountable.*

Proof. Assume for contradiction that $\mathbb{R}$ is countable. Then we may enumerate $\mathbb{R} = \{x_1, x_2, x_3, \dots, \}$. We construct $\{I_n\}_{n=1}^{\infty}$ as above so that $x_k \notin I_k$ for all $x_k \in \mathbb{R}$. But by the Nested Interval Prop, $\cap_{n=1}^{\infty} I_n$ is nonempty, and hence contains a real number $x \in \mathbb{R}$. Thus $x \neq x_k$ for any k, for otherwise $x = x_k \notin I_k$ and thus $x \notin \cap_{n=1}^{\infty} I_n$. This contradicts our enumeration of $\mathbb{R}$. Therefore, $\mathbb{R}$ is uncountable. □

- The force of the theorem is that the cardinality of $\mathbb{R}$ is a "larger type of infinity" than countably infinite, i.e. than $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$.
- This means the countable sets are the smallest type of infinite set. $\mathbb{R}$ is bigger.

Exercise 12. (reading question) Explain the flaw in the following proof that $\mathbb{Q}$ is uncountable: suppose for contradiction that $\mathbb{Q} = \{r_1, r_2, \dots\}$. We construct $\{I_n\}_{n=1}^{\infty}$ so that $r_k \notin I_k$ for all $r_k \in \mathbb{Q}$. But by the NIP, $\cap_{n=1}^{\infty} I_n$ is nonempty, and hence contains a rational number $r \in \mathbb{Q}$. Thus $r \neq r_k$ for any k, otherwise $r = r_k \notin I_k$ and thus $x \notin \cap_{n=1}^{\infty} I_n$. This contradicts our enumeration of $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is uncountable.

- Show $\mathbb{Q}$ does not satisfy the NIP. In other words, give an example of a sequence $\{I_n \cap \mathbb{Q}\}_{n=1}^{\infty}$ of "closed bounded intervals of $\mathbb{Q}$" such that $\bigcap_{n=1}^{\infty} (I_n \cap \mathbb{Q}) = \emptyset$.

[start here 9-26-18]

4. Cantor's diagonalization method

- Even if we union together countably many copies of $\mathbb{Q}$, then the result is still countable. So no amount of (countably) unioning together $\mathbb{Q}$ can give us $\mathbb{R}$ or even fill in the "holes" in $\mathbb{Q}$ left by $\mathbb{I}$.
- In fact, $\mathbb{R}$ is bigger than any countable union of countable sets:

Theorem 13.

(1) If $A_1, A_2, \dots, A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \dots \cup A_m$ is countable.
(2) If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is countable. ("A countable union of countable sets is countable.")

Proof. Exercise 1.5.3. □

- We have mathematician Georg Cantor to thank for our knowledge that $\mathbb{R}$ is uncountable. In fact, Cantor proved much more.
- Cantor's proof that $\mathbb{R}$ is uncountable is very similar to the proof above, but it was initially resisted.
- His work eventually produced a paradigm shift in the way mathematicians understand the infinite.
- Cantor also proved the following:

Theorem 14. *The open interval* $(0, 1)$ *is uncountable.*

Exercise 15. Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable. This shows that what follows is an alternate proof that $\mathbb{R}$ is uncountable.

- We show that there is a 1-1 correspondence betweem $(0, 1)$ and $\mathbb{R}$.
- We'd like a function $f : (0, 1) \to \mathbb{R}$ that passes the horizontal line test and stretches "all the way" from $-\infty$ to ∞.
- Notice that the tangent function almost does it–$g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $g(x) = \tan(x)$ is a 1-1 correspondence!
- What we need to do is scale the tangent function so that it hits its full period in $(0, 1)$: we want $x = 0$ to be input into the tan function as $-\frac{\pi}{2}$ and 1 to be input as $\frac{\pi}{2}$. Try

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right).$$

- Then $f : (0, 1) \to \mathbb{R}$ is 1-1 and onto.

Proof. **(Claimed Problem Presentation)** We proceed by contradiction and assume that there does exist a function $f : \mathbb{N} \to (0, 1)$ that is 1-1 and onto.

- For each $m \in \mathbb{N}$, $f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\ldots$$

- Here, for each $m, n \in \mathbb{N}$, a_{mn} is the digit from the set $\{0, 1, 2, \ldots, 9\}$ that represents the nth digit in the decimal expansion of $f(m)$.
- The 1-1 correspondence between $\mathbb{N}$ and $(0, 1)$ can be summarized in the doubly indexed array

$$\begin{array}{ccccccccccc}
\mathbf{N} & & (0,1) & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\boldsymbol{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \boldsymbol{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \boldsymbol{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \boldsymbol{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \boldsymbol{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \boldsymbol{a_{66}} & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

- Here, **every** real number in $(0, 1)$ is assumed to appear somewhere on the list.

- Now for the pearl of the argument–define a real number $x \in (0,1)$ with the decimal expansion $x = .b_1b_2b_3b_4\dots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

  - To compute the digit b_1, we look at the digit a_{11} in the upper left-hand corner of the array. If $a_{11} = 2$, we choose $b_1 = 3$; otherwise, we set $b_1 = 2$.

Exercise 16.

(1) Explain why the real number $x = .b_1b_2b_3b_4\dots$ cannot be $f(1)$.
   (a) This is because $b_1 \neq a_{11}$ by definition.
(2) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.
   (a) This is because, for all n, $b_n \neq a_{nn}$ by definition.
(3) Point out the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.
   (a) We have constructed a real number $x = .b_1b_2b_3\dots \in (0,1)$ that is not in the image of f. This contradicts f being onto. Hence $(0,1)$ is uncountable.

□

Exercise 17. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(1) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is countable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of the theorem must be flawed.
   - The step where we define x doesn't work because all rationals have terminating decimal expansions.
(2) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or $.4999\dots$. Doesn't this cause some problems?
   - Since b_n has no 9s in it, much less repeating 9s, this won't be an issue.

Exercise 18. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \dots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1,0,1,0,\dots) \in S$, as is $(1,1,1,1,\dots)$. Give a rigorous argument showing that S is uncountable.

Exercise 19. (Claimed Problem Presentation) Theorem 1.6.2 (Cantor's Theorem)

Theorem 20. *Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.*

Proof. It's easier to assume such a function exists and get a contradiction than to try a direct proof. Thus, assume for contradiction that $f : A \to P(A)$ is onto. For each element $a \in A$, $f(a)$ is a **subset** of A.

Since f is onto, every subset of A appears as $f(a)$ for some $a \in A$.

Construct a subset $B \in P(A)$ as follows. For each element $a \in A$, consider the subset $f(a)$. This subset may contain a or may not, depending on f. If $f(a)$ does not contain a, then we include a in our set B. More precisely, let

$$B = \{a \in A : a \notin f(a)\}.$$

Because we have assumed f is onto, it must be that $B = f(a')$ for some $a' \in A$.

Case 1. Suppose that $a' \in B$. This implies that $a' \in f(a') = B$. But B is defined as the set of all elements that aren't in their images, so that $a' \notin B$ by definition. Contradiction.

Case 2. Suppose that $a' \notin B$. This implies that $a' \notin f(a')$. But by the definition of B, this means that $a' \in B$, giving a contradiction.

$\square$