MATH 321 2.5 - SUBSEQUENCES AND THE BOLZANO-WEIERSTRASS THEOREM

- **Last time**: we showed that the sequence of partial sums (s_m) of the harmonic series does not converge by focusing on a particular subsequence (s_{2^k}) of the original sequence:

Theorem 1. ***(Cauchy Condensation Test)*** *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \dots$$

converges.

Corollary 2. *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

- A rigorous argument for this corollary requires a few basic facts about geometric series, which we'll deal with in Section 2.7.
- **Today**: we'll put infinite series aside and focus on the important concept of subsequences.

Definition 3. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < \dots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \dots)$$

is called a **subsequence** of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Example 4. The order of the terms in a subsequence must be the same as in the original sequence, and repetitions are not allowed. Thus if $(a_n) = (\frac{1}{n})$, then $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots)$ and $(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$ are legitimate subsequences, while

$$\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \dots\right) \text{ and } \left(1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \dots\right)$$

are not.

- We'll see in a second that if $0 < b < 1$, then $\lim(b^n) = 0$. Use this fact to prove:

Exercise 5. (reading question) Show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Proof. ($\Longrightarrow$) Suppose that $\lim(b^n) = 0$, and suppose that $b > 1$. (The case $b < -1$ will follow.) We show that b^n becomes arbitrarily large, and thus that b^n diverges.

Proof. Let $x \in \mathbb{R}$ be a real number. Then, if we take $n = \log_b x + 1$, we have that

$$b^n = b^{\log_b x + 1} = b^{\log_b x} \cdot b = bx > x,$$

hence $(b^n) \to \infty$. **[Why doesn't this proof work for $-1 < b < 1$?]** □

($\Longleftarrow$) Suppose that $-1 < b < 1$. If $0 < b < 1$, Example 2.5.3 shows that $\lim(b^n) = 0$. Thus, we concern ourselves with the case $-1 < b < 0$. We show using the definition that $\lim(b^n) = 0$ in this case.

(1) Let $\epsilon > 0$ be arbitrary.
(2) [scratch work] we want to show that $|b^n| < \epsilon$. Since $-1 < b < 0$, we have that $b = -a$ for some $1 > a > 0$. Thus
$$|b^n| = |(-a)^n| = |(-1)^n a^n| = a^n.$$
But we know that (a^n) converges for any $0 < a < 1$.
(3) Since (a^n) converges if $0 < a < 1$, we may choose $N \in \mathbb{N}$ such that $a^n < \epsilon$ whenever $n \geq N$.
(4) Assume $n \geq N$.
(5) Then
$$|b^n| = a^n < \epsilon$$
and hence $\lim(b^n) = 0$, as desired.

□

Exercise 6. [slide] [Crannell Worksheet 2.5]

Theorem 7. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Exercise 8. (Reading question) Prove the Theorem.

Proof. Assume $(a_n) \to a$ and let (a_{n_k}) be a subsequence. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Because $n_k \geq k$ for all k, the same N will suffice for the subsequence; that is, $|a_{n_k} - a| < \epsilon$ whenever $k \geq N$. □

Example 9. (Back to geometric sequences) Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > \cdots > 0,$$

the sequence (b^n) is decreasing and bounded below. The MCT lets us conclude that (b^n) converges to some limit l satisfying $b > l \geq 0$.

To compute l, note that (b^{2n}) is a subsequence, so $(b^{2n}) \to l$ by the Theorem. Because limits are unique, $l^2 = l$, and thus $l = 0$.

- We can use this result in some clever ways to understand when infinite sums are associative (Claimed HW).
- We'll also use it to prove divergence:

Example 10. We now have the tools to prove divergence. Show that

$$\Big(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\Big)$$

diverges.

Proof. Note that

$$\Big(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\Big)$$

is a subsequence that converges to $1/5$. Also,

$$\Big(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots\Big)$$

is a subsequence that converges to $-1/5$. Because we have two subsequences converging to two different limits, we can rigorously conclude that the original sequence diverges. □

Exercise 11. Show that the sequence $((-1)^n)$ diverges.

- The subsequences (1) and (-1) converge to different limits.

(**T**/F) If every subsequence of (x_n) converges to the same limit, then (x_n) converges.

- True; (x_n) is a subsequence of (x_n), so (x_n) converges.

(T/**F**) Every sequence of real numbers contains a convergent subsequence.

- False; consider (n).

(T/**F**) Every increasing sequence of real numbers contains a convergent subsequence.

- False; consider (n).

- In our divergence example, we spotted convergent subsequences hiding in the original sequence.
- For bounded sequences, it turns out that we can always find at least one convergent subsequence.

Theorem 12. ***(Bolzano-Weierstrass Theorem)*** *Every bounded sequence contains a convergent subsequence.*

- Importance of B-W:
  - next time, lets us show that Cauchy sequences converge
  - equivalent to Axiom of Completeness as characterizing property of $\mathbb{R}$
  - completely characterizes "compact sets", an important topological concept, as closed & bounded sets in $\mathbb{R}$

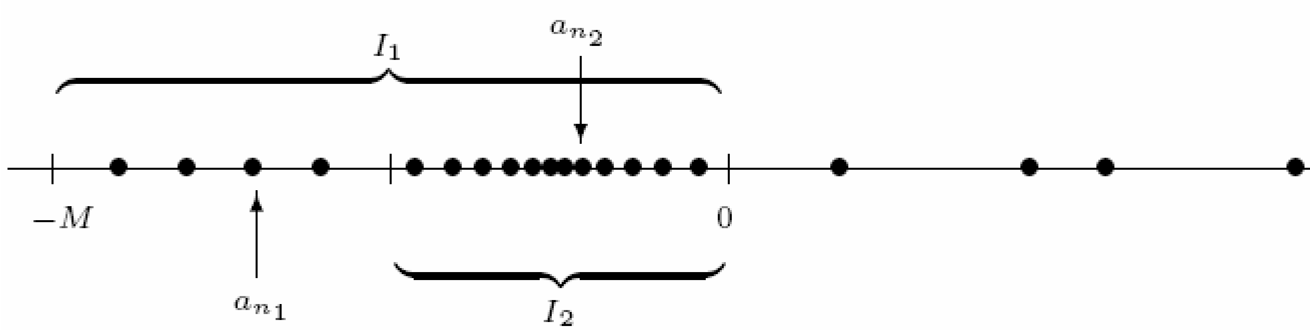
- lets us solve the optimization problem $\max\{f(x) : x \in X \subseteq \mathbb{R}\}$ where $X \subseteq \mathbb{R}$ is closed and bounded and f is continuous.
    * used in economics to argue that it's possible for consumers to maximize utility subject to a budget constraint
    * used in economics to argue that it's possible to efficiently allocate resources among members of a society
    * used to argue that the competitive equilibrium price (where demand = supply in all markets) depends on the existence of demand

Exercise 13. Prove the Bolzano-Weierstrass Theorem by doing the following:

(1) Let (a_n) be a bounded sequence. Recall the definition of "bounded" for sequences and translate this into an inequality that each term a_n must satisfy.
(2) Translate your statement from (1) into interval notation and bisect the interval. Argue that one of the two resulting intervals (call it I_1) contains infinitely many terms of (a_n). Define a_{n_1} to be one of the infinitely many terms of (a_n) in one of the two intervals.
(3) Bisect I_1 again and repeat the argument in (2).
(4) Write a general argument for how to choose the interval I_n and the term a_{n_k} of (a_n).
(5) Use the Nested Interval Property to get a candidate, x, for $\lim a_{n_k}$.
(6) Find a formula for the length of I_k. Use this formula to prove that $(a_{n_k}) \to x$.

Proof. The idea isn't too different from one way of defining the 1-1 correspondence $f : (0, 1) \to S$, where S is the set of sequences of 0s and 1s. (remember this involved bisecting the interval $(0, 1)$ over and over.)

(1) Let (a_n) be a bounded sequence. Then there exists $M > 0$ satisfying $|a_n| \le M$ for all $n \in \mathbb{N}$. [draw]
(2) Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$.
(3) It must be that one of these closed intervals contains infinitely many terms of (a_n). (Else (a_n) contains only finitely many terms, but sequences are defined to be infinite.)
(4) Select one interval that contains infinitely many terms of (a_n) and label the interval I_1.
(5) Let a_{n_1} be some term in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



(6) Now we bisect I_1 again into closed intervals of equal length and let I_2 be a half that again contains an infinite number of terms of the original sequence. Because there are an infinite number of terms of (a_n) to choose from, we can select an a_{n_2} from the original sequence with $n_2 > n_1$ and $a_{n_2} \in I_2$.
(7) In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of terms of (a_n) and then select

$$n_k > n_{k-1} > \dots > n_2 > n_1$$

so that $a_{n_k} \in I_k$.

We want to argue that (a_{n_k}) is a convergent subsequence, but we need a candidate for the limit. Whatever the limit is, it seems likely that it's in all of the intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

but wait! These form a nested sequence of closed intervals, and by the Nested Interval Property there's at least one $x \in \mathbb{R}$ contained in every I_k! Let's guess that x is the limit. It just remains to show that $(a_{n_k}) \to x$.

Let $\epsilon > 0$. How close can we make x to the terms of (a_n)? It has to do with the size of the intervals. By construction, the length of I_k is $M(1/2)^{k-1}$, which converges to 0. (This follows from your "geometric sequence" reading question and the Algebraic Limit Theorem.)

Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Because x and a_{n_k} are both in I_k, it follows that $|a_{n_k} - x| < \epsilon$. $\square$