MATH 321 3.2 - OPEN AND CLOSED SETS

- The Cantor set extends the horizons of our intuition about the nature of subsets of the real line.
- Last time, we found that the Cantor set was both big (in the sense of cardinality) and small (in the sense of length). It doesn't contain any intervals!
  - Read section 3.1 of your book to find out that the Cantor set is also "medium-sized" in the sense of **dimension**–it's about $\log(2)/\log(3) \approx .631$-dimensional!
- [slides] The notion of a fractional dimension is the impetus behind the term "fractal", coined in 1975 by Benoit Mandlebrot to describe a class of sets whose intricate structures have much in common with the Cantor set.

Exercise 1. (reading question)

(1) Consider the set given by starting with $D_0 = [0, 1]$ and removing middle fourths at each stage (so that $D_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$, D_2 is given by removing a segment of size $\frac{3}{32}$ from each of the intervals of D_1, and so on. What is the length of D? What is the cardinality of D? Does D contain any intervals?

(2) Now consider the set F given by varying the proportion of the intervals removed at each step in the following way. As usual, let $F_0 = [0, 1]$. Start by removing the middle fourth again, so that $F_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$. For the next step, only remove an interval of length $\frac{1}{16}$ from each of the intervals of F_1. At step 3, we remove an interval of length $\frac{1}{64}$ from each of the four intervals remaining. We continue this pattern to get the set F, called a Smith-Volterra-Cantor set or a fat Cantor set. What is the length of F? The cardinality? Does F contain any intervals?

(1)
  (a) As before, we first consider the lengths of the removed intervals.
    (i) At stage 1, we remove one interval of length $\frac{1}{4}$.
    (ii) At stage 2, we begin with the intervals $[0, \frac{3}{8}]$ and $[\frac{5}{8}, 1]$, each of length $\frac{3}{8}$. We then remove the middle fourth of each of these two intervals, which has length $\frac{1}{4}(\frac{3}{8}) = \frac{3}{32}$. The total length removed at stage 2 is thus $2(\frac{1}{4})(\frac{3}{32}) = \frac{3}{16}$.
    (iii) At stage 3, we begin with four intervals each of length $\frac{3}{8} - \frac{3}{32} = \frac{12}{32} - \frac{3}{32} = \frac{9}{32}$. We then remove the middle fourth of each interval, which has length $\frac{1}{4}(\frac{9}{32}) = \frac{9}{128}$. The total length removed at stage 3 is thus $4(\frac{1}{4})(\frac{9}{32}) = \frac{9}{32}$.
    (iv) At stage n, the pattern continues: we have 2^{n-1} intervals and remove a total length of $\frac{1}{4}(\frac{3}{4})^{n-1}$.
    (v) The sum of the lengths of the removed intervals is thus
$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{4}\Big(\frac{3}{4}\Big)^{n-1} &= \frac{1}{4}\sum_{n=1}^{\infty}\Big(\frac{3}{4}\Big)^{n-1} \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\Big(\frac{3}{4}\Big)^{n} \\ &= \frac{1}{4}\Big(\frac{1}{1-3/4}\Big) \\ &= \frac{1}{4}\Big(\frac{1}{1/4}\Big) = 1.\end{aligned}$$
    Thus, the length of the middle-fourths Cantor set is still 0.
  (b) By a similar argument to the regular Cantor set, we can construct a 1-to-1 correspondence $f : D \to S$, where S is the set of sequences of 0s and 1s, by giving the set of "directions" for locating any given point of D in each of the D_i.
  (c) D does not contain any intervals since its length is 0. Every time we had an interval, we made sure to remove part of it.

(2)
  (a) We consider the lengths of the removed intervals of F.
    (i) At stage 1, we remove a length of $2^0 \times \frac{1}{4} = \frac{1}{4}$.
    (ii) At stage 2, we remove two intervals, each of length $\frac{1}{16}$, for a total removed length of $\frac{1}{8}$.
    (iii) At stage 3, we remove four intervals, each of length $\frac{1}{64}$, for a total removed length of $\frac{1}{16}$.
    (iv) At stage n, we remove a total length of $\frac{1}{4}(\frac{1}{2})^{n-1}$.
    (v) The total length removed is thus

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{4}\Big(\frac{1}{2}\Big)^{n-1} &= \frac{1}{4}\sum_{n=1}^{\infty}\Big(\frac{1}{2}\Big)^{n-1} \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^{n} \\ &= \frac{1}{4}\Big(\frac{1}{1-1/2}\Big) \\ &= \frac{1}{4}\Big(\frac{1}{1/2}\Big) = \frac{1}{2}.\end{aligned}$$

    (vi) The length of the Smith-Volterra-Cantor set is thus $1 - \frac{1}{2} = \frac{1}{2}$!
  (b) Because the Smith-Volterra-Cantor set has nonzero length, it must have cardinality equal to that of $\mathbb{R}$.
  (c) Even though the S-V-C set has nonzero length, it contains no intervals! Given $a < b$ inside F, there exists $x \in (a, b)$ such that x was removed at some stage of the construction of F. Therefore, $(a, b) \notin F$.

- Today, we'll further explore and attempt to classify the *topology* of subsets of the real line.
  - In this exploration, it won't matter "how big" subsets are so much as what their endpoints look like. For example, in topology, $(0,\infty)$ and $(0,1)$ are pretty much the same, while $(0,1)$ and $[0,1]$ are different.
  - If you're interested in taking a broader view of topology, take my class next semester!
- The Cantor Set will be a testing ground as we develop more theory about the often-elusive nature of subsets of the real line.
- Last week, we learned that every Cauchy sequence (a_n) with terms in $\mathbb{R}$ converges to some limit $a \in \mathbb{R}$.
- What happens when we replace $\mathbb{R}$ with some subset of $\mathbb{R}$?

Exercise 2. (reading question)

(1) Last week, we learned that every Cauchy sequence (a_n), where $a_n \in \mathbb{R}$ for all n, converges to some limit $a \in \mathbb{R}$. What happens if we replace $\mathbb{R}$ with some subset of $\mathbb{R}$? In other words, if $a_n \in X$ for all n, where $X \subseteq \mathbb{R}$, can we say that (a_n) converges to some limit $a \in X$?
(2) For which of the following sets X is the statement above true (every Cauchy sequence with all terms in X converges to a limit in X)? If you think the statement is true, you don't need to prove it, just jot down some justification for why you think it's true. If you think the statement is false for that set X, give an example of a Cauchy sequence with every term in X whose limit does not lie in X. Then think about what the sets that contain the limits of their Cauchy sequences have in common.

$$\begin{aligned} X &= (0,1) \\ X &= [0,1] \\ X &= (-\infty,0) \\ X &= (-\infty,0] \end{aligned}$$

- It turns out that every Cauchy sequence in $[0,1]$ or $(-\infty,0]$ converges. Intuitively, this is because these sets "contain their endpoints".

- Contrast this with the situation in $(0,1)$, where the sequence $(\frac{1}{n}) \to 0 \notin (0,1)$, or $(-\infty, 0)$, where the sequence $(-\frac{1}{n}) \to 0 \notin (-\infty, 0)$. Here the problem is that the sequences "reach the endpoint", but the endpoint isn't in the set.
- We call sets that "contain all their endpoints" (for some definition of "endpoints") **closed sets.**
- It turns out that a set is closed if and only if every Cauchy sequence converges inside that set.

- In addition to being the sets where Cauchy sequences converge, if we want to solve optimization problems, we need our constraint (interval) to be closed!

Example 3. Cutting out $h \times h$ squares from a 4×6 sheet of cardboard, then folding up the squares, gives us a box. What's the maximum volume we can store in the box if its height must satisfy $0 < h < 0.75$? $0 \leq h \leq 0.75$?

- [slide from MATH 201 3.3] The maximum in $(0, 0.75)$ doesn't exist.
- In $[0, 0.75]$, the max is $(0.75, 8.44)$.

- So closed sets are very important! It's no exaggeration to say that analysts (people who do analysis) love closed sets most of all.
- Topologists like myself are bigger fans of open sets :)
- Open sets are more intuitively characterized than closed sets, so let's start by characterizing open sets.

1. Open sets

- The prototypical open set is the ϵ-neighborhood you've already encountered:

Definition 4. Given $a \in \mathbb{R}$ and $\epsilon > 0$, the **ϵ-neighborhood of** a is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon).$$

A set $O \subseteq \mathbb{R}$ is **open** if for all points $a \in O$, there exists an ϵ-neighborhood $V_\epsilon(a)$ that's also contained in O.

Example 5. Consider the open interval

$$(c, d) = \{x \in \mathbb{R} : c < x < d\}.$$

Let $x \in (c, d)$ be arbitrary. [draw] If we take $\epsilon = \min\{x - c, d - x\}$ [the distance to the closest endpoints; draw], then it follows that $V_\epsilon(x) = (x - \epsilon, x + \epsilon)$ is contained in (c, d). This is because $x - \epsilon \geq x - (x - c) = c$ and $x + \epsilon \leq x + (d - x) = d$.

Exercise 6. [slide] Crannell worksheet Qs

Question 7. *Why don't we define open sets to be the same as open intervals (a, b) where $a < b$ and $a, b \in \mathbb{R}$? Who can give me an example of an open set that's not a (finite) interval?*

- $\mathbb{R} = (-\infty, \infty)$.
- $\emptyset$*, trivially: there are no points in* $\emptyset$*!*

Exercise 8. Does this argument work if the interval contains one of its endpoints? What if we replace (c, d) with $(a, b) \cup (c, d)$, the union of two open intervals?

- The union of open intervals is also open. This observation leads to the next result:

Theorem 9. *(i) The union of an arbitrary (potentially uncountable) collection of open sets is open.*
(ii) The intersection of a finite collection of open sets is open.

Proof. (i) How do we label the elements of a potentially-uncountable set O? Well, no matter how many elements there are, we can put the elements of the set into one-to-one correspondence with *some* (perhaps uncountable) set, call it Λ. This effectively "labels" each element of O with a corresponding $\lambda \in \Lambda$, so we can call each element of O O_λ for some λ. Then we can write $\{O_\lambda : \lambda \in \Lambda\}$ to represent our collection of open sets, no matter how many there are. This is standard notation; mathematicians often index uncountable sets as $\{U_\lambda\}_{\lambda \in \Lambda}$, where the U's are the elements of our uncountable set and the λ's are labelings.

Now, let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Let $a \in O$. We want to show that there exists an ϵ so that $(a - \epsilon, a + \epsilon) \subseteq O$.

Since $a \in O$, we must have that $a \in O_\lambda$ for some $\lambda \in \Lambda$.

Because we assume O_λ to be open, there exists a neighborhood $V_\epsilon(a) \subseteq O_\lambda$. But since $O_\lambda \subseteq O$, we must have $V_\epsilon(a) \subseteq O$ as well, completing the proof.

(ii) Let $\{O_1, O_2, \dots, O_N\}$ be a finite collection of open sets. Now, if $a \in \bigcap_{k=1}^N O_k$, then a is an element of each of the open sets. By the definition of an open set, we know that, for each $1 \leq k \leq N$, there exists $V_{\epsilon_k}(a) \subseteq O_k$.

We are in search of a *single* ϵ-neighborhood that is contained in every O_k, so the trick is to take the smallest one. Let

$$\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}.$$

It follows that $V_\epsilon(a) \subseteq V_{\epsilon_k}(a) \subseteq O_k$ for all k, and thus $V_\epsilon(a) \subseteq \bigcap_{k=1}^N O_k$ as well. □

Exercise 10. Why doesn't the proof of (ii) generalize to an arbitrary intersection of open sets? We'd need to replace min with inf in the proof, but why doesn't the proof work in this case? Can you give a counterexample to show that, in fact, it's possible for the arbitrary intersection of open sets to not be open?

- The problem is that $\inf\{e_\lambda : \lambda \in \Lambda\}$ might be 0. For example, consider the collection $O_k = (-\frac{1}{k}, \frac{1}{k})$ for $k = 1, 2, 3, \dots$. Then, for the point $0 \in O_k$, the ϵ-neighborhood $V_{1/k}(0) \subseteq O_k$ for all k (in other words, $\epsilon_k = \frac{1}{k}$ for all k. However, $\inf\{\frac{1}{k} : k \in \mathbb{N}\} = 0$, and we have to have $\epsilon > 0$ in this proof.
  - It makes sense that the proof doesn't work in this case, because $\bigcap_{k=1}^\infty (-\frac{1}{k}, \frac{1}{k}) = \{0\}$, and $\{0\}$ is not actually an open set! (No ϵ-neighborhood of 0 is contained in $\{0\}$.)

2. Closed sets

- Let's try to characterize closed sets now.

Definition 11. A point x is a **limit point** of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

Example 12. 0 is a limit point of $(0, 1)$ because every ϵ-neighborhood of 0 looks like $(-\epsilon, \epsilon)$. Since $\epsilon > 0$, it must be that $\epsilon \in V_\epsilon(0) \cap (0, 1)$.

- We claimed earlier that closed-ness had something to do with limits of sequences; let's try and put that link together.

Theorem 13. *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Exercise 14. (reading question) Prove the backwards direction; that is, prove that if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$, then x is a limit point of A.

Proof. ($\Longleftarrow$) Assume $\lim a_n = x$ where $a_n \in A$ but $a_n \neq x$, and let $V_\epsilon(x)$ be an arbitrary ϵ-neighborhood. The (topological) definition of convergence assures us that there is a term a_N in the sequence satisfying $a_N \in V_\epsilon(x)$, and the proof is complete.

($\Longrightarrow$) Assume x is a limit point of A. In order to produce a sequence (a_n) converging to x, we'll consider the ϵ-neighborhoods obtained using $\epsilon = 1/n$. Since x is a limit point of A, every neighborhood $V_{1/n}(x)$ intersects A in some point other than x. Thus, for each $n \in \mathbb{N}$, we can choose a point

$$a_n \in V_{1/n}(x) \cap A$$

with the stipulation that $a_n \neq x$. Then it seems like $\lim a_n = x$. To justify this, let $\epsilon > 0$ be arbitrary, and choose N by the Archimedean Property so that $1/N < \epsilon$. It follows that $|a_n - x| < \epsilon$ for all $n \geq N$. □

Exercise 15. Why do we need to stipulate that $a_n \neq x$ in the theorem?

- We don't want to allow, for example, 0 to be a limit point of the set $\{0\}$. In fact, it shouldn't be by definition, since $V_\epsilon(0) \cap \{0\} = \{0\}$ for any $\epsilon > 0$, and the definition of limit point says $V_\epsilon(0)$ must intersect $\{0\}$ in some point other than 0 to be a limit point.
- However, if we don't stipulate that $a_n \neq x$ in the theorem, then we'd have the sequence $(0, 0, 0, 0, \dots)$, every element contained in $\{0\}$, converging to 0, and that would imply 0 was a limit point.

- In this case, 0 is called an *isolated point* of $\{0\}$:

Definition 16. A point $a \in A$ is an **isolated point** of A if it is not a limit point of A.

Example 17. Since 0 is not a limit point of $\{0\}$ by the argument above, 0 is an isolated point of $\{0\}$.

Question 18. *Do sets always contain their limit points?*

- *No; the point* 0 *is a limit point of the set* $(0,1)$*, as shown earlier.*

Definition 19. A set $F \subseteq \mathbb{R}$ is **closed** if it contains all of its limit points.

- Usually, mathematicians use "closed" to mean that some operation on the elements of a given set does not take us out of the set.
- In this case, the operation in question is the taking a limit.

Theorem 20. *A set* $F \subseteq \mathbb{R}$ *is closed if and only if every Cauchy sequence contained in* F *has a limit that is also an element of* F*.*

Proof. Homework □

Exercise 21. [slide: Crannell worksheet 3.2] Which of the following sets are closed? If a set is closed, prove it. If a set is not closed, give an example of a limit point of the set that is not contained in the set.

(1) $\emptyset$
Closed; the set of limit points of $\emptyset$ is $\emptyset$, and $\emptyset \subseteq \emptyset$.

(2) $\mathbb{R}$
Closed; let x be a limit point of $\mathbb{R}$. Then x is the limit of a sequence (a_n) of real numbers. But sequences of real numbers converge to real numbers, so $x \in \mathbb{R}$ as well.

(3) $\mathbb{R} \setminus \{5\}$
Not closed; the sequence $(4, 4.9, 4.99, 4.999, \dots)$ has limit 5, but $5 \notin \mathbb{R} \setminus \{5\}$.

(4) $\{5\}$
Closed; the only sequence contained in $\{5\}$ is the constant sequence $(5, 5, 5, \dots)$, which has limit 5. And $5 \in \{5\}$!

(5) $\mathbb{Q}$
Not closed; the sequence $(3, 3.1, 3.14, 3.141, 3.1415, \dots)$ has limit π, hence $\pi$is a limit point of $\mathbb{Q}$, but $\pi \notin \mathbb{Q}$.

(6) $(2,4)$

(7) $[c,d] = \{x \in \mathbb{R} : c \leq x \leq d\}$
Closed; let's prove it. If x is a limit point of $[c,d]$, then there exists $(x_n) \subseteq [c,d]$ with $(x_n) \to x$. We need to prove that $x \in [c,d]$. But by the Order Limit Theorem, since $c \leq x_n \leq d$, it must be that $c \leq x \leq d$ as well. Thus, $x \in [c,d]$, and $[c,d]$ is closed.

(8) $[2,4)$
Not closed; the sequence $(4 - \frac{1}{n})$ has all its terms contained in $[2,4)$, but its limit $4 \notin [2,4)$.

(9) Name a nonempty, bounded, closed set that is not of form $[a,b]$.
  - $\{5\}$ above gives an example!
  - Also $[0,1] \cup [2,3]$!

Note 22. Mathematicians don't use "open" and "closed" in the same way that laypeople do. In particular, notice that $\mathbb{R}$ and $\emptyset$ is **both** open and closed!

Question 23. *Where does the argument that* $[c,d]$ *is closed break down if we try to use it to show that* $(0,1)$ *is closed?*

- *The problem is that the Order Limit Theorem only works for non-strict inequalities. In other words, if* $0 < x_n < 1$ *for all* n*, that doesn't mean that* $0 < \lim(x_n) < 1$ *as well, as we see with the sequence* $(\frac{1}{n})$*.*

2.1. Interlude: when closed is not enough.

- We saw that closed sets were somehow useful for solving optimization problems: $f(x) = x^2$ has a max and min on $[0,1]$ but not on $(0,1)$, for instance.
- However, being closed isn't enough to guarantee a max/min:
  - We saw that $\mathbb{R}$ was closed, but the function x^3 has neither a global max nor a global min on $\mathbb{R}$.
  - For a less strange example, the interval $(-\infty, 2]$ is closed because its limit points are all contained in $(-\infty, 2]$, but the function x^3 has no global min on this interval!
- We need another criterion for maxes and mins to exist: compactness.

3. Closure

- Given the nice behavior of closed sets, it's nice to have a way of turning any set into a closed set.

Definition 24. Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The **closure** of A is defined to be $A \cup L$ and is denoted $\overline{A}$ (LaTeX: \overline{A}).

Example 25. What is the closure of $(0, 1)$?

- The limit points of $(0, 1)$ are 0 and 1. Any other point is not a limit point because if (x_n) is a sequence in $(0, 1)$, $0 < x_n < 1$ for all x, and by the OLT, this means that $0 \leq \lim(x_n) \leq 1$ as well.

Exercise 26.

(1) What is the closure of $(3, 5]$?
(2) What is the closure of $\mathbb{R}$?
(3) What is the closure of $\mathbb{Q}$?
  - It's actually $\mathbb{R}$!
  - Let $y \in \mathbb{R}$ be arbitrary, and consider any neighborhood $V_\epsilon(y) = (y - \epsilon, y + \epsilon)$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $r \neq y$ in this neighborhood (in fact, in the neighborhood $(y - \epsilon, y)$ alone there are infinitely many). Thus, y is a limit point of $\mathbb{Q}$.

Theorem 27. *For any $A \subseteq \mathbb{R}$, the closure $\bar{A}$ is a closed set and is the smallest closed set containing A.*

4. Complements

Theorem 28. *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

- We prove the first sentence; the second will follow.

Exercise 29. (Reading Question) Prove $(\implies)$.

Proof. $(\implies)$ Let $O \subseteq \mathbb{R}$ be an open set. We want to show that O^c is closed. Let x be a limit point of O^c. Then every neighborhood of x contains some point of O^c. But this means that x cannot be in the open set O, because $x \in O$ implies there exists a neighborhood $V_\epsilon(x) \subseteq O$. Thus, $x \in O^c$, as desired. Since x was arbitrary, it must be that O^c contains all its limit points, hence O^c is closed.

$(\impliedby)$ Assume O^c is closed. Let $x \in O$ be arbitrary; we want to produce an ϵ-neighborhood $V_\epsilon(x) \subseteq O$. Because O^c is closed, we can be sure that x is *not* a limit point of O^c. Hence, by definition, there must be some neighborhood $V_\epsilon(x)$ of x that does not intersect the set O^c. But this means $V_\epsilon(x) \subseteq O$, as desired. Since x was arbitrary, O is open.

The second statement follows quickly from the first using the observation that $(E^c)^c = E$ for any set $E \subseteq \mathbb{R}$. □

- Finally, we want to make some statement about when unions and intersections of closed sets are closed, similar to our statement about open sets.
- Topologists take these statements as defining properties of open and closed sets.
  - To a topologist, a list of subsets of a given set are given and decreed to be "open". These sets must have the property that countable unions and finite intersections of open sets are open.
  - Then, by definition, the closed sets are all the complements of the open sets.

Question 30. *Based on the fact that closed sets are the complements of open sets, give some examples of closed sets.*

Example 31. $(-\infty, 0] \cup [1, \infty)$ is closed because it's the complement of $(0, 1)$, an open set.

- $\mathbb{R}$ is closed because it's the complement of the open set $\emptyset$.
- $\emptyset$ is closed because it's the complement of the open set $\mathbb{R}$.

Theorem 32. *(i) The union of a finite collection of closed sets is closed.*
(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof. Note that these are the "opposite" properties of those satisfied by open sets. De Morgan's Laws state that for any collection of sets $\{E_\lambda : \lambda \in \Lambda\}$ it is true that

$$\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \text{ and } \Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c.$$

The details are left as Exercise 3.2.9. □

5. Preview of Compactness: when closed is not enough

Definition 33. A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

- Note: these sets satisfy a sort of "generalized Bolzano-Weierstrass".
- We need one more definition before your reading question:

Definition 34. A set $A \subseteq \mathbb{R}$ is **bounded** if there exists $M > 0$ so that $|a| \leq M$ for all $a \in A$.

Exercise 35. (Reading question) Which of the following sets satisfy this generalized Bolzano-Weierstrass? Prove necessary conditions for a set K to satisfy generalized B-W.

(1) $[0, 1]$: yes.
(2) $(0, 1)$: no; all subsequences of $(\frac{1}{n})$ converge to $0 \notin (0, 1)$.
(3) $[0, \infty)$: no; no subsequence of (n) converges.
(4) $\mathbb{R}$: no; the sequence (n) has no convergent subsequence in $\mathbb{R}$.

- This shows that closed, bounded sets are compact. Are there any other compact sets, sets satisfying generalized B-W, besides the closed, bounded sets?