MATH 321 WEEK 4 CLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 1.3.9

Exercise 1.3.9. (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Exercise 1.3.9. (a) Set $\epsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists an element $b \in B$ satisfying $\sup B - \epsilon < b$, which implies $\sup A < b$. Because $\sup A$ is an upper bound for A, then b is as well.

(b) Take $A = [0, 1]$ and $B = (0, 1)$.

(2) 1.3.11

Exercise 1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Exercise 1.3.11. (a) True. Observe that all elements of B are contained in A and hence $\sup A \geq b$ for all $b \in B$. By Definition 1.3.2 part (ii), $\sup B$ is less than or equal to any other upper bounds of B. Because $\sup A$ is an upper bound for B, it follows that $\sup B \leq \sup A$.

(b) True. Let $c = (\sup A + \inf B)/2$ from which it follows that

$$a \leq \sup A < c < \inf B \leq b.$$

(c) False. Consider, the open sets $A = (d, c)$ and $B = (c, f)$. Then $a < c < b$ for every $a \in A$ and $b \in B$, but $\sup A = c = \inf B$.

(3) 1.4.5

Exercise 1.4.5. We have to show the existence of an irrational number between any two real numbers a and b. By applying Theorem 1.4.3 on the real numbers $a-\sqrt{2}$ and $b-\sqrt{2}$ we can find a rational number r satisfying $a-\sqrt{2} < r < b-\sqrt{2}$. This implies $a < r + \sqrt{2} < b$. From Exercise 1.4.1(b) we know $r + \sqrt{2}$ is an irrational number between a and b.

(4) 1.4.6. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.
 (a) The set of all rational numbers p/q with $q \leq 10$: no such number exists between $\frac{1}{11}$ and $\frac{2}{11}$, for example.
 (b) The dyadic rationals are dense in $\mathbb{R}$. The idea is to first expand the interval $[a, b]$ so be at least 1 unit of length long and then collapse the interval back down. We know that $\exists n \in \mathbb{N}$ such that
$$(b-a)2^n > 1$$
thus we know there is an $m \in \mathbb{Z}$ such that $m \in (2^n a, 2^n b)$ (since the interval is of length greater than 1). Now we have
$$2^n a < m < 2^n b \implies a < \frac{m}{2^n} < b,$$
as desired.
 (c) The set of all rational numbers p/q with $10|p| \geq q$.
This inequality implies $|p|/q \geq \frac{1}{10}$. So the interval $(\frac{1}{20}, \frac{1}{10})$, for example, contains no rational numbers of this form.

(5) 1.4.8
 (a) Let $A = (0,1) \cap \mathbb{Q}$, $B = (0,1) \cap \mathbb{I}$. Then $A \cap B = \emptyset$, $\sup A = 1 = \sup B$ by the density of $\mathbb{Q}$ and $\mathbb{I}$, $1 \notin A$, and $1 \notin B$, as desired.
 (b) Let $J_n = (-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $\cap_{n=1}^{\infty} J_n = \{0\}$.
 (c) Let $L_n = [n, \infty)$ for all $n \in \mathbb{N}$. Then if $x \in \mathbb{R}$, by the Archimedean Principle there exists $n' \in \mathbb{N}$ with $n' > x$. The interval $L_{n'} = [n', \infty)$ then does not contain x. Therefore, $\cap_{n=1}^{\infty} L_n = \emptyset$.
 (d) The answer here depends on whether we consider "unbounded closed intervals" to be closed. The definition of "closed set" is "a set whose complement is open", so we should consider those intervals to be closed. But things get interesting if we don't:
  (i) If we consider unbounded closed intervals to be closed, then the example $L_n = [n, \infty)$ above works, and is even "nested"!
  (ii) If we don't consider unbounded closed intervals to be closed, this is impossible. Let $\{I_n\}$ be a collection of closed intervals such that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$. Then define
$$J_n = I_n \cap I_{n-1} \text{ for all } 1 < n \in \mathbb{N}.$$
Then $J_n \neq \emptyset$ for all n and the J_n are nested. But J_n might just be a point, not an interval! What do we do about this? Well, according to the definition of "closed interval" in your book, it's possible to have a closed interval $[a, b]$ with $a = b$. So in any case, the proof of the Nested Interval Property works even if one of the I_n has $a_n = b_n$. Now, $\{J_n\}$ satisfies the hypotheses of the Nested Interval Property, and
$$\bigcap_{n=2}^{\infty} J_n \neq \emptyset.$$
But
$$\begin{aligned}\bigcap_{n=2}^{\infty} J_n &= \bigcap_{n=2}^{\infty} (I_n \cap I_{n-1}) = (I_2 \cap I_1) \cap (I_3 \cap I_2) \cap \ldots \\ &= I_1 \cap I_2 \cap I_3 \cap \ldots \\ &= \bigcap_{n=1}^{\infty} I_n,\end{aligned}$$
so that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.