MATH 321 DAY 12 - CARDINALITY CONTINUED

Exercise 1.

10. Let $\{x_1, x_2, x_3, \dots\}$ be a sequence of real numbers.

(a) [T/F] It is possible to construct a non-empty closed interval $I_1 \subset \mathbf{R}$ so that $x_1 \notin$

(b) [T/F] It is possible to construct a non-empty closed interval $I_2 \subset I_1$ so that $x_2 \notin$

(c) [T/F] For every $n \in \mathbf{N}$, it is possible to construct a non-empty closed interval I_n that does not contain x_{n+1}.

(d) $\cap_{n=1}^{\infty} I_n$ contains x_k for some $k \in \mathbf{N}$.

(e) $\cap_{n=1}^{\infty} I_n$ is empty/non-empty. (Choose one and prove it).

(f) $\mathbf{R}$ is countable/uncountable. (Choose one and prove it).

(1) True; let $I_1 = [x_1 + 1, x_1 + 2]$
(2) True; consider the intervals $[x_1 + 1, x_1 + \frac{4}{3}]$ and $[x_1 + \frac{5}{3}, x_1 + 2]$. Then x_2 can be in at most one of these intervals; choose one of these intervals that does not contain x_2 to be I_2.
(3) True; we may repeat this process inductively. Suppose we have constructed I_n so that $x_1, \dots, x_n \notin I_n$. We may then divide I_n into thirds; consider the first third and last third. Then x_{n+1} is in at most one of these thirds; choose one of the two subintervals that does not contain x_{n+1} and call it I_{n+1}.
(4) False; for all $k \in \mathbb{N}$, $x_k \notin I_k$, and thus $x_k \notin \bigcap_{n=1}^{\infty} I_n$.
(5) By the Nested Interval Property, $\cap_{n=1}^{\infty} I_n \neq \emptyset$. Choose an element x of $\cap_{n=1}^{\infty} I_n$; then $x \in \mathbb{R}$, but $x \notin \{x_1, x_2, x_3, \dots\}$, since if it were, x could not be in $\cap_{n=1}^{\infty} I_n$.

Theorem 2. *$\mathbb{R}$ is uncountable.*

Proof. Assume for contradiction that $\mathbb{R}$ is countable. Then we may enumerate $\mathbb{R} = \{x_1, x_2, x_3, \dots, \}$. We construct $\{I_n\}_{n=1}^{\infty}$ as above so that $x_k \notin I_k$ for all $x_k \in \mathbb{R}$. But by the Nested Interval Prop, $\cap_{n=1}^{\infty} I_n$ is nonempty, and hence contains a real number $x \in \mathbb{R}$. Thus $x \neq x_k$ for any k, for otherwise $x = x_k \notin I_k$ and thus $x \notin \cap_{n=1}^{\infty} I_n$. This contradicts our enumeration of $\mathbb{R}$. Therefore, $\mathbb{R}$ is uncountable. □

- The force of the theorem is that the cardinality of $\mathbb{R}$ is a "larger type of infinity" than countably infinite, i.e. than $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$.
- It's an important exercise to show that any subset of a countable set must be either countable or finite [left for homework].
  - This isn't too surprising. If a set can be arranged into a single list, then deleting some elements from the list results in another (shorter, and possibly terminating) list.
- This means the countable sets are the smallest type of infinite set. $\mathbb{R}$ is bigger.
- In fact, $\mathbb{R}$ is bigger than any countable union of countable sets:

Exercise 3. (reading question) Explain the flaw in the following proof that $\mathbb{Q}$ is uncountable: suppose for contradiction that $\mathbb{Q} = \{r_1, r_2, \dots\}$. We construct $\{I_n\}_{n=1}^{\infty}$ so that $r_k \notin I_k$ for all $r_k \in \mathbb{Q}$. But by the NIP, $\cap_{n=1}^{\infty} I_n$ is nonempty, and hence contains a rational number $r \in \mathbb{Q}$. Thus $r \neq r_k$ for any k, otherwise $r = r_k \notin I_k$ and thus $x \notin \cap_{n=1}^{\infty} I_n$. This contradicts our enumeration of $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is uncountable.

- Show $\mathbb{Q}$ does not satisfy the NIP. In other words, give an example of a sequence $\{I_n \cap \mathbb{Q}\}_{n=1}^{\infty}$ of "closed bounded intervals of $\mathbb{Q}$" such that $\bigcap_{n=1}^{\infty} (I_n \cap \mathbb{Q}) = \emptyset$.

- In fact, if we union together countably many copies of $\mathbb{Q}$, then the result is still countable. So no amount of (countably) unioning together $\mathbb{Q}$ can give us $\mathbb{R}$ or even fill in the "holes" in $\mathbb{Q}$ left by $\mathbb{I}$.

Theorem 4.

(1) *If $A_1, A_2, \ldots, A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.*
(2) *If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is countable. ("A countable union of countable sets is countable.")*

Proof. Exercise 1.5.3. □

- We have mathematician Georg Cantor to thank for our knowledge that $\mathbb{R}$ is uncountable. In fact, Cantor proved much more.
- Cantor's proof that $\mathbb{R}$ is uncountable is very similar to the proof above, but it was initially resisted.
- His work eventually produced a paradigm shift in the way mathematicians understand the infinite.

0.1. **Cantor's diagonalization method.**

- Cantor also proved the following:

Theorem 5. *The open interval $(0, 1)$ is uncountable.*

Exercise 6. Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable. This shows that what follows is an alternate proof that $\mathbb{R}$ is uncountable.

- We show that there is a 1-1 correspondence betweem $(0, 1)$ and $\mathbb{R}$.
- We'd like a function $f : (0, 1) \to \mathbb{R}$ that passes the horizontal line test and stretches "all the way" from $-\infty$ to ∞.
- Notice that the tangent function almost does it–$g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $g(x) = \tan(x)$ is a 1-1 correspondence!
- What we need to do is scale the tangent function so that it hits its full period in $(0, 1)$: we want $x = 0$ to be input into the tan function as $-\frac{\pi}{2}$ and 1 to be input as $\frac{\pi}{2}$. Try

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right).$$

- Then $f : (0, 1) \to \mathbb{R}$ is 1-1 and onto.

Proof. We proceed by contradiction and assume that there does exist a function $f : \mathbb{N} \to (0, 1)$ that is 1-1 and onto.

- For each $m \in \mathbb{N}$, $f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\ldots$$

- Here, for each $m, n \in \mathbb{N}$, a_{mn} is the digit from the set $\{0, 1, 2, \ldots, 9\}$ that represents the nth digit in the decimal expansion of $f(m)$.
- The 1-1 correspondence between $\mathbb{N}$ and $(0, 1)$ can be summarized in the doubly indexed array

N		$(0,1)$								
1	$\longleftrightarrow$	$f(1)$	$=$	$.\boldsymbol{a_{11}}$	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
2	$\longleftrightarrow$	$f(2)$	$=$	$.a_{21}$	$\boldsymbol{a_{22}}$	a_{23}	a_{24}	a_{25}	a_{26}	$\cdots$
3	$\longleftrightarrow$	$f(3)$	$=$	$.a_{31}$	a_{32}	$\boldsymbol{a_{33}}$	a_{34}	a_{35}	a_{36}	$\cdots$
4	$\longleftrightarrow$	$f(4)$	$=$	$.a_{41}$	a_{42}	a_{43}	$\boldsymbol{a_{44}}$	a_{45}	a_{46}	$\cdots$
5	$\longleftrightarrow$	$f(5)$	$=$	$.a_{51}$	a_{52}	a_{53}	a_{54}	$\boldsymbol{a_{55}}$	a_{56}	$\cdots$
6	$\longleftrightarrow$	$f(6)$	$=$	$.a_{61}$	a_{62}	a_{63}	a_{64}	a_{65}	$\boldsymbol{a_{66}}$	$\cdots$
$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- Here, **every** real number in $(0,1)$ is assumed to appear somewhere on the list.
- Now for the pearl of the argument–define a real number $x \in (0,1)$ with the decimal expansion $x = .b_1b_2b_3b_4\dots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2. \end{cases}$$

  – To compute the digit b_1, we look at the digit a_{11} in the upper left-hand corner of the array. If $a_{11} = 2$, we choose $b_1 = 3$; otherwise, we set $b_1 = 2$.

□

Exercise 7.

(1) Explain why the real number $x = .b_1b_2b_3b_4\dots$ cannot be $f(1)$.
(2) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbb{N}$.
(3) Point out the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.

Exercise 8. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(1) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is countable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of the theorem must be flawed.
    - The step where we define x doesn't work because all rationals have terminating decimal expansions.
(2) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or $.4999\dots$. Doesn't this cause some problems?
    - Since b_n has no 9s in it, much less repeating 9s, this won't be an issue.
    - Some versions of this proof define $x = .b_1b_2b_3b_4\dots$ using the rule $b_n = a_{nn}$; would this version still work?

Exercise 9. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \dots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, \dots) \in S$, as is $(1, 1, 1, 1, \dots)$. Give a rigorous argument showing that S is uncountable.