MATH 321 DAY 14 - POWER SETS AND CANTOR'S THEOREM

Definition 1. Given a set A, the **power set** $P(A)$ is the collection of all subsets of A. $P(A)$ is a set of subsets of A.

- We care about power sets because, given a set A, $P(A)$ is "much bigger" than A (and thus

Exercise 2.

(1) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)
  - $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$

(2) If A is finite with n elements, show that $P(A)$ has 2^n elements.
  - Let $A = \{x_1, \ldots, x_n\}$ and $S \subseteq A$. Then, for all $a \in A$, either $a \in S$ or $a \notin S$. This means that we can represent S by a string of numbers
$$a_1, a_2, \ldots, a_n$$
where for all i,
$$a_i = \begin{cases} 0 & \text{if } x_i \notin S \\ 1 & \text{if } x_i \in S. \end{cases}$$
Each $\{a_i\}_{i=1}^n$ represents a unique subset of A, since a subset is uniquely determined by which elements are in it. The number of strings $\{a_i\}_{i=1}^n$ is 2^n. Hence $P(A)$ has 2^n elements.

Exercise 3.

(1) Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A into $P(A)$.
  (a) $a \mapsto \{a\}$ for all $a \in A$
  (b) $a \mapsto A \setminus \{a\}$

(2) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1-1 map $g : C \to P(C)$.
  (a) $n \mapsto \{n\}$ for all $n \in C$.

(3) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.
  (a) Since $A, C, P(A), P(C)$ are finite sets and $|A| < |P(A)|$, $|C| < |P(C)|$, once we choose where each of the elements of A or C go, there still remain $2^3 - 3$ elements of $P(A)$ and $2^4 - 4$ elements of $P(C)$ which aren't mapped to.

- Cantor's Theorem says this is impossible even for infinite sets:

Theorem 4. *Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.*

Proof. Assume for contradiction that $f : A \to P(A)$ is onto. For each $a \in A$, $f(a) \subseteq A$. The assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$. To arrive at a contradiction, we will produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.

For each element $a \in A$, consider the subset $f(a)$. If $a \notin f(a)$, we include a in our set B. More precisely, let
$$B = \{a \in A : a \notin f(a)\}.$$

Exercise 5. Return to the particular functions constructed in the previous exercise and construct the subset B that results using the previous rule. In each case, note that B is not in the range of the function used.

Because we have assumed that $f : A \to P(A)$ is onto, it must be that $B = f(a')$ for some $a' \in A$. The contradiction arises when we consider whether $a' \in B$.

Exercise 6.

(1) First, show that the case $a' \in B$ leads to a contradiction.

(2) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

$\square$

- Cantor's Theorem implies that there's no function from $\mathbb{N}$ to $P(\mathbb{N})$; in other words, $P(\mathbb{N})$ is uncountable!

Question 7. *How does the cardinality of the uncountable set $P(\mathbb{N})$ compare to that of the uncountable set $\mathbb{R}$?*

- In fact, one can show that $P(\mathbb{N}) \sim S \sim (0,1) \sim \mathbb{R}$, where S is the set of sequences of 0s and 1s. Hence, $P(\mathbb{N}) \sim \mathbb{R}$.

Exercise 8. [take-home challenge!] Prove that $S \sim (0,1)$ by constructing 1-1 functions $f : S \to (0,1)$ and $g : (0,1) \to S$. It's a fact that, if we can construct such functions, then the two sets they map between are in 1-1 correspondence.

Exercise 9. Answer each of the following by establishing a 1-1 correspondence with a set of known cardinality.

(1) Is the set of all functions from $\{0,1\}$ to $\mathbb{N}$ countable or uncountable?
(2) Is the set of all functions from $\mathbb{N}$ to $\{0,1\}$ countable or uncountable?
(3) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbb{N})$ contain an uncountable antichain?