MATH 321 DAY 12 - CARDINALITY

1. Warmup

- Both the irrational and rational numbers are dense in $\mathbb{R}$; we can think of both sets as "dotting" the real line in such a way that there are infinitely many "dots" between any two points on the line.
- A priori, it would seem that the sets $\mathbb{Q}$ and $\mathbb{I}$ would thus be the same size. This is false! In a sense, there are "as many" irrationals as reals, and fewer rationals than irrationals.

Definition 1. A function $f : A \to B$ is **one-to-one** if, whenever $f(a_1) = f(a_2)$ in B, $a_1 = a_2$ in A. [The **contrapositive** of your book's definition.] The function f is **onto** if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$. If f is both 1-to-1 and onto, we say that f is a **1-to-1 correspondence** between A and B.

- If functions throw darts from points in A to points in B,
  - one-to-one means that no two darts hit the same point in B, and
  - onto means that every point in B is hit by a dart.

Definition 2. Two sets A and B have the same **cardinality** if there exists a function $f : A \to B$ that is one-to-one and onto. Write $A \sim B$.

- This makes sense because 1-1 and onto means no two darts hit the same spot and every spot is hit. This means there are the same number of points of B as darts.
- Counting a set A is the same as covering A in darts labeled with natural numbers.

Definition 3. We say A is **countable** if $A \sim \mathbb{N}$. If A is neither finite nor countable, then we say A is **uncountable**.

Note 4. Under this terminology, finite sets aren't countable. Weird!

Exercise 5. [slide] **Reading question**: finish the following proof that if $A \subseteq B$ and B is countable, then A is either countable or finite:

Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, whic
and onto. Let $A \subseteq B$ be an infinite subset of B. We must show th
countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of g :
set $g(1) = f(n_1)$. Show how to inductively continue this process to pr
1–1 function g from $\mathbf{N}$ onto A.

Proof. Next let $n_2 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1)\}\}$ and set $g(2) = f(n_2)$. In general, assume we have defined $g(k)$ for $k < m$, and let $g(m) = f(n_m)$ where

$$n_m = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(n_1), \ldots, f(n_{m-1})\}\}.$$

We show that $g : \mathbb{N} \to A$ is 1-to-1 and onto:

(1) **1-1:** We first show that $n_m : \mathbb{N} \to \mathbb{N}$ is 1-1, then use this fact to show that g is 1-1.
   (a) Suppose that $n_m = n_{m'}$. Suppose WLOG that $m \le m'$. Then $f(m') \in A \backslash \{f(n_1), \ldots, , f(n_{m'-1})\}$ by definition. If $m < m'$, then $n_{m'} = f(m') \neq f(m) = n_m$, a contradiction. Therefore, $n_m = n_{m'} \implies m = m'$.

(b) Now, it follows that if $g(m) = g(m')$, then

$$f(n_m) = g(m) = g(m') = f(n_{m'})$$

by definition, and since f is 1-to-1, this means that $n_m = n_{m'}$. By (i), this means that $m = m'$.

(2) **Onto**: suppose that $a \in A$. Then $a \in B$, and since f is onto, $a = f(n_0)$ for some $n \in \mathbb{N}$. It must be that $n_0 \in \{n : f(n) \in A\}$, and as we inductively remove the minimal element, n_0 must eventually be the minimum by at least the $n_0 - 1$st step.

□

Definition 6. For any set A, the **power set** of A, $P(A)$, is the set of all possible subsets of A.

Proof technique. To show that A is countable, you must find a way of "labeling" each element of A with a distinct natural number. This is called a "counting strategy" for A. This doesn't always lead to a function describable via a formula, but as long as it's clear that yoru function is 1-1 and onto, you don't need a formula.

Exercise 7.

1. If $B = \{e, \pi, \sqrt{2}\}$, then list all elements of $P(B)$. (Hint: there are 8 of them).
2. [T/F] If $A = \{1, 2, 3\}$ and $B = \{e, \pi, \sqrt{2}\}$ then $A \sim B$.
3. [T/F] If $A = \{1, 2, 3\}$ and $C = \{x \in \mathbf{R} : (x^2 - 1)(x^2 - 4) = 0\}$ then $A \sim C$.
4. [T/F] The even integers $2\mathbf{Z}$ have the same cardinality as the integers; that is, $2\mathbf{Z} \sim \mathbf{Z}$.

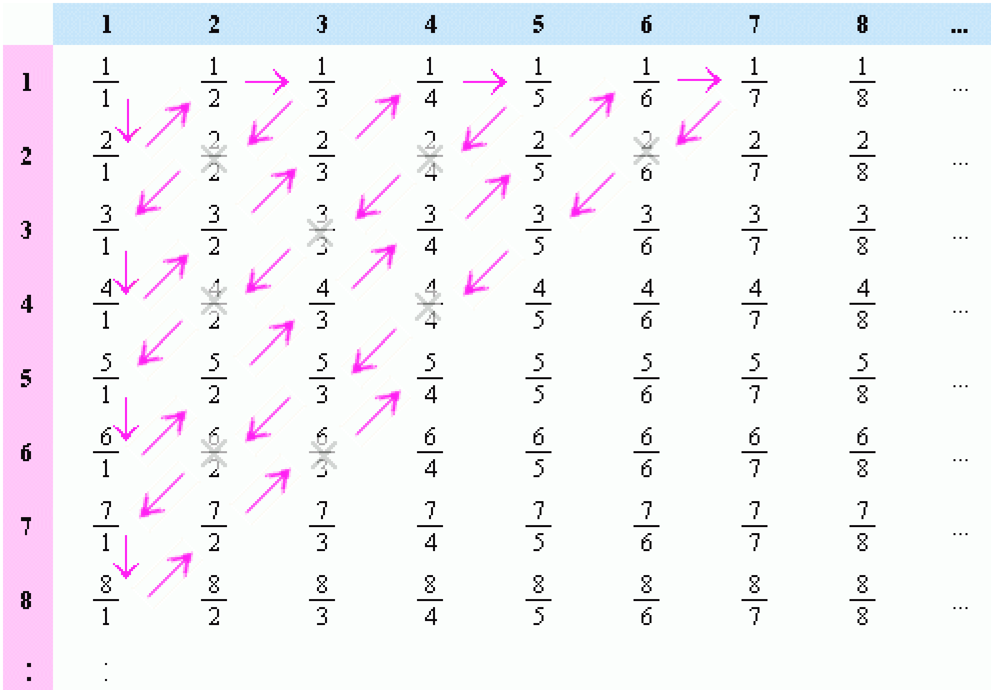
Proof. We label $2k \in 2\mathbb{Z}$ with $k \in \mathbb{Z}$. In other words, let $f : \mathbb{Z} \to 2\mathbb{Z}$ be the function given by $f(k) = 2k$. Then

(1) f **is 1-1**: if $f(k_1) = f(k_2)$, then $2k_1 = 2k_2 \implies k_1 = k_2$.
(2) f **is onto**: if $x \in 2\mathbb{Z}$, then $x = 2k = f(k)$ for some $k \in \mathbb{Z}$.

□

Exercise 8.

6. [T/F] $A \sim A$ for every set A.
7. [T/F] If $A \sim B$, then $B \sim A$.
8. [T/F] If $A \sim B$ and $B \sim C$, then $A \sim C$.
9. (a) Make a table of all positive rational numbers so that each fraction $\frac{p}{q}$ appears in t column and the qth row. (Okay, just go out as far as $p, q = 5$.)
   (b) Cross out duplicates that are not in lowest terms.
   (c) Turn your table 45° clockwise, so that $\frac{1}{1}$ is in the top "row". There should l numbers in the next row, and more numbers as you move further down. Readin twisted table, list in order the first dozen numbers you encounter.
   (d) [T/F] $\mathbf{Q}$ is countable.


1 2 3 4 5 6 7 8 ...
1 1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 ...
2 2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 ...
3 3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 ...
4 4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 ...
5 5/1 5/2 5/3 5/4 5/5 5/6 5/7 5/8 ...
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Proof. □