MATH 321 CLAIMED HW #7 SOLUTIONS

(1) 2.3.3

Exercise 2.3.3. Let $\epsilon > 0$ be arbitrary. We must show that there exists an N such that $n \geq N$ implies $|y_n - l| < \epsilon$. In terms of ϵ-neighborhoods (which are a bit easier to use in this case), we must equivalently show $y_n \in (l - \epsilon, l + \epsilon)$ for all $n \geq N$.

Because $(x_n) \to l$, we can pick an N_1 such that $x_n \in (l - \epsilon, l + \epsilon)$ for all $n \geq N_1$. Similarly, because $(z_n) \to l$ we can pick an N_2 such that $z_n \in (l-\epsilon, l+\epsilon)$ whenever $n \geq N_2$. Now, because $x_n \leq y_n \leq z_n$, if we let $N = \max\{N_1, N_2\}$, then it follows that $y_n \in (l - \epsilon, l + \epsilon)$, for all $n \geq N$. This completes the proof.

(2) 2.3.10

(a) True; by the Algebraic Limit Theorem, if $\lim(a_n) = a$ and $\lim(b_n) = b$, then

$$0 = \lim(a_n - b_n) = a - b,$$

and hence $\lim a_n = a = b = \lim b_n$.

(b) False; consider $b_n = \sum_{i=1}^{n} \frac{(-1)^{n+1}}{n}$. Then $\lim b_n$ exists by the Alternating Series Test, but

$$|b_n| = \sum_{i=1}^{n} \frac{1}{n},$$

and we know that the harmonic series does not converge.

(c) True; by the Algebraic Limit Theorem, assuming $(a_n) \to 0$,

$$0 = \lim(b_n - a_n) = \lim b_n - \lim a_n = \lim b_n - 0$$

so that $\lim b_n = 0$ as well.

(d) True. Let $\epsilon > 0$ be arbitrary. By the definition of $(a_n) \to 0$, we can choose $N \in \mathbb{N}$ so that $|a_n| = |a_n - 0| < \epsilon$ whenever $n \geq N$. Thus, whenever $n \geq N$,

$$|b_n - b| \leq a_n < \epsilon,$$

hence $(b_n) \to b$ by definition.

(3) 2.3.11

Exercise 2.3.11. Let $\epsilon > 0$ be arbitrary. Then we need to find an N such that $n \geq N$ implies $|y_n - L| < \epsilon$. Because $(x_n) \to L$, we know that there exists $M > 0$ such that $|x_n - L| < M$ for all n. Also, there exists an $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $|x_n - L| < \epsilon/2$. Now for $n \geq N_1$ we can write

$$\begin{aligned}
|y_n - L| &= \left| \frac{x_1 + x_2 + \cdots + x_{N_1} + \cdots + x_n}{n} - \frac{nL}{n} \right| \\
&= \left| \frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n} + \frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n} \right| \\
&\leq \left| \frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n} \right| + \left| \frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n} \right| \\
&\leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon(n - N_1)}{2n}.
\end{aligned}$$

Because N_1 and M are fixed constants at this point, we may choose N_2 so that $\frac{(N_1-1)M}{n} < \epsilon/2$ for all $n \geq N_2$. Finally, let $N = \max\{N_1, N_2\}$ be the desired N. To see that this works, keep in mind that $\frac{n-N_1}{n} < 1$ and observe

$$|y_n - L| \leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon(n - N_1)}{2n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$. This completes the proof.

The sequence $(x_n) = (1, -1, 1, -1, \cdots)$ does not converge, but the averages satisfy $(y_n) \to 0$.

(4) 2.3.12
   (a) True. (no proof required) Assume for contradiction that a is not an upper bound for B; then there exists $b \in B$ so that $a < b$. Since $(a_n) \to a$, if we take $\epsilon = b - a > 0$, then there is an $n \in \mathbb{N}$ so that

$$|a_n - a| < b - a$$

   whenever $n \geq N$. But this implies in particular that $|a_N - a| < b - a$. Since a_N is an upper bound for B, $a_N > b > a$, so we can rewrite this inequality as

$$\begin{aligned} a_N - a &< b - a \\ a_N &< b \end{aligned}$$

   contradicting that a_N is an upper bound for B.
   (b) True (no proof required).
   (c) False; let a_n be the first n digits of π (or e or $\sqrt{2}$ or any other irrational number).