MATH 321 1.4 LAY- PROPERTIES OF SET OPERATIONS; CROSS PRODUCTS

1. Last time

To prove $A \subseteq B$:

(1) Let $x \in A$ be arbitrary. (Assume nothing about x except that $x \in A$.)
(2) Show logically that $x \in B$ as well.

Theorem 1. *Let A be a set. Then $\emptyset \subseteq A$.*

Proof. To prove that $\emptyset \subseteq A$, we must establish that the implication

$$\text{if } x \in \emptyset, \text{ then } x \in A$$

is true. Since $\emptyset$ has no elements, it's true that every element of the empty set is also an element of A.

This is kind of like saying "if unicorns exist, then unicorns are pink". Since unicorns don't exist (sorry), we can say whatever we want about them following the clause "if unicorns exist", and it would be (vacuously) true of all unicorns. □

Exercise 2. [slide] Practice 1.6, 1.11

[start 9-10-18] Now that we've discussed the need for rigor in math and the lack thereof in the 19th century, what is the foundation for mathematical knowledge? It's not algebraic manipulation (you can't always assume that you can plug in numbers into an infinite series); it's not geometry (since intuition can sometimes lie to us). Mathematicians chose **sets!**

2. Properties of unions, intersections, and complements

Theorem 3. *(1.13) Let $A, B, C \subseteq U$. Then*

(1) $A \cup (U \setminus A) = U$
(2) $A \cap (U \setminus A) = \emptyset$
(3) $U \setminus (U \setminus A) = A$
(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(6) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(7) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
(8) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Example 4. Draw Venn diagram for (4). (shade intersections in different colors if possible)

Exercise 5. In your groups, draw Venn diagrams illustrating the truth of each of these statements. (You can split them up.)

Note 6. Your Venn diagrams **aren't proofs**! You might be missing a case; your diagram might not be general enough.

- Proving these statements is great practice of set proofs! Remember the strategy: to show $A = B$ as sets, show that
$$x \in A \implies x \in B \text{ AND } x \in B \implies x \in A.$$
  - In other words, we're showing there are no elements of A that are not also elements of B and vice versa.
  - The main goal of doing these types of proofs is to get you practice with proof strategies for set inclusion.
  - These statements are also valuable for proving statements in all fields of mathematics, since set theory is a sort of "common language" for mathematics.

- Steps to prove $A = B$:
  (1) Suppose $x \in A$.
  (2) Translate the definition of A to make some statement about sets that x is in or not in.
  (3) Rephrase those statements in order to show that $x \in B$.
  (4) Repeat to show that $y \in B \implies y \in A$.

Exercise 7. [slide] (Reading Question: Lay Practice 1.14-1.15) Practice 1.14, 1.15

Proof. 1.13(g). $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
(1) Suppose $x \in A \setminus (B \cap C)$.
(2) Then $x \in A$ and $x \notin B \cap C$. This means that $x \in A$ and x is NOT in BOTH B and C.
(3) This means that $x \notin B$ or $x \notin C$ (or both, though we can leave this part out because in set theory we use "inclusive or").
(4) Since x is also an element of A, this means that $x \in A \setminus B$ or $x \in A \setminus C$ (or both).
(5) Thus, $x \in (A \setminus B) \cup (A \setminus C)$.

Conversely,
(1) Suppose $y \in (A \setminus B) \cup (A \setminus C)$.
(2) Then $y \in A \setminus B$ or $y \in A \setminus C$ (or both).
(3) This means that $y \notin B$ or $y \notin C$.
(4) Hence, $y \notin B \cap C$.
(5) Since also $y \in A$, we have that $y \in A \setminus (B \cap C)$.

Note: we could also phrase both sides of this proof as a single string of "if and only if" statements, since at each step we're relying on **definitions** of complements, intersections, and unions, and definitions are always iff statements:

$$\begin{aligned} x \in A \setminus (B \cap C) &\iff x \in A \text{ and } x \notin B \cap C \\ &\iff x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\iff (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\iff x \in A \setminus B \text{ or } x \in A \setminus C \\ &\iff x \in (A \setminus B) \cup (A \setminus C). \end{aligned}$$

□

2.1. **Unions and intersections of indexed families.**

Definition 8. If for each element j of a nonempty set J there corresponds a set A_j, then

$$\mathcal{A} = \{A_j | j \in J\}$$

is called an **indexed family** of sets with J as the **index set**. Then1.18

$$\bigcup_{j \in J} A_j = \{x | x \in A_j \text{ for some } j \in J\}$$

$$\bigcap_{j \in J} A_j = \{x | x \in A_j \text{ for all } j \in J\}.$$

If $J = \{1, 2, \ldots, n\}$, we may write

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{j=1}^{n} A_j \text{ or } \cup_{j=1}^{n} A_j.$$

Exercise 9. [slide] TPS Practice 1.18

Proposition 10. *For each $k \in \mathbb{N}$, let $A_k = [0, 2 - 1/k]$. Then $\bigcup_{k=1}^{\infty} A_k = [0, 2)$.*

Exercise 11. Prove the Proposition. [Hint: the harder direction is the reverse inclusion. Suppose $y \in [0, 2)$. Then $2 - y > 0$. Choose $k \in \mathbb{N}$ such that $k > \frac{1}{2-y}$; finish the argument.]

- Rest of the reverse inclusion argument: then $2 - y > \frac{1}{k} > 0$. Thus $y < 2 - \frac{1}{k}$ for this particular k. So $y \in \bigcup_{k=1}^{\infty} A_k$.]

- Forward direction: if $x \in \bigcup_{k=1}^{\infty} A_k$, then $x \in A_k$ for **some** $k \in \mathbb{N}$. This means $0 \leq x \leq 2 - 1/k$ for some $k \in \mathbb{N}$. Then certainly $0 \leq x < 2$.

3. (if time) Cartesian products

Definition 12. The **ordered pair** (a, b) is the set whose members are $\{a\}$ and $\{a, b\}$. In symbols,

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Exercise 13. Why do we need this level of complexity to deal with the idea of caring about the order of elements? What other definitions could we use? Do those definitions do what we want?

Definition 14. If A, B are sets, then the **Cartesian product** (or **cross product**) of A and B, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In symbols,

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

- If A and B are intervals, then $A \times B$ is a rectangle in the usual coordinate system with A on the x-axis and B on the y-axis.

Exercise 15. (if time) [slide] TPS Lay Practice 2.3, 2.6