MATH 321 WEEK 6 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.2.4
  (a) $(1, -1, 1, -1, 1, -1, \dots)$
  (b) This is impossible. Suppose that $(a_n) \to a$ with $a \neq 1$. Then, taking $\epsilon = |a-1|$ in the definition of convergence, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, $|a_n - a| < |a-1|$. If $a_n = 1$ for some $n \geq N$, then $|1-a| < |a-1| = |1-a|$, a contradiction. Therefore, $a_n \neq 1$ for all $n \geq N$, and thus, (a_n) contains at most $N-1$ ones.
  (c) $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, \dots)$

(2) 2.2.5
  (a) We show $\lim a_n = 0$.
    (i) Let $\epsilon > 0$ be arbitrary.
    (ii) [scratch work] $|[[5/n]]| = [[5/n]] = 0$ whenever $n > 5$. For when $n > 5$, $0 < 5/n < 1$.
    (iii) Choose $N = 6$.
    (iv) Assume $n \geq N$.
    (v) Then

$$|[[5/n]]| = [[5/n]] = 0 < \epsilon.$$

  (b) b) Here the limit of a_n is 1. Let $\epsilon > 0$ be arbitrary. By picking $N = 7$ we have that for $n \geq N$,

$$\left| \left[\left[\frac{12+4n}{3n} \right]\right] - 1 \right| = |1-1| < \epsilon,$$

because $[[(12+4n)/3n]] = 1$ for all $n \geq 7$.

In these exercises, the choice of N does not depend on ϵ in the usual way. In exercise (b) for instance, setting $N = 7$ is a suitable response for every choice of $\epsilon > 0$. Thus, this is a rare example where a smaller $\epsilon > 0$ does not require a larger N in response.

(3) 2.2.6

Assume $(a_n) \to a$ and $(a_n) \to b$. We show that $a = b$ by appealing to the following theorem (Theorem 1.2.6 in your book):

Theorem 1. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a-b| < \epsilon$.*

  (a) Let $\epsilon > 0$ be arbitrary.
  (b) [scratch work] We want to show that $|a-b| < \epsilon$. But we know that we can make $|a_n - a|$ and $|a_n - b|$ as small as we want by the definition of convergence. Then we notice that

$$\begin{aligned} |a-b| &= |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| \\ &= |a_n - a| + |a_n - b| \end{aligned}$$

where the inequality is due to the Triangle Inequality. Thus, if $|a_n - a|$ and $|a_n - b|$ were both less than $\epsilon/2$, we'd be done.
  (c) Choose $N_1 \in \mathbb{N}$ so that, whenever $n \geq N_1$, $|a_n - a| < \epsilon/2$. Similarly, choose $N_2 \in \mathbb{N}$ so that whenever $n \geq N_2$, $|a_n - b| < \epsilon/2$. Then let $N = \max\{N_1, N_2\}$.
  (d) Assume $n \geq N$.

(e) Then

$$\begin{aligned} |a-b| &= |(a-a_n)+(a_n-b)| \\ &\leq^{\Delta} |a_n-a|+|a_n-b| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon. \end{aligned}$$

Thus, by Theorem 1.2.6, we must have that $a=b$, as desired.

(4) 2.3.1

Exercise 2.3.1. (a) Let $\epsilon > 0$ be arbitrary. We must find an N such that $n \geq N$ implies $|\sqrt{x_n}-0| < \epsilon$. Because $(x_n) \to 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - 0| = x_n < \epsilon^2$. Using this N, we have $\sqrt{(x_n)^2} < \epsilon^2$, which gives $|\sqrt{x_n}-0| < \epsilon$ for all $n \geq N$, as desired.

(b) Part (a) handles the case $x=0$, so we may assume $x > 0$. Let $\epsilon > 0$. This time we must find an N such that $n \geq N$ implies $|\sqrt{x_n}-\sqrt{x}| < \epsilon$, for all $n \geq N$. Well,

$$\begin{aligned} |\sqrt{x_n}-\sqrt{x}| &= |\sqrt{x_n}-\sqrt{x}|\left(\frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n}+\sqrt{x}}\right) \\ &= \frac{|x_n-x|}{\sqrt{x_n}+\sqrt{x}} \\ &\leq \frac{|x_n-x|}{\sqrt{x}} \end{aligned}$$

Now because $(x_n) \to x$ and $x > 0$, we can choose N such that $|x_n - x| < \epsilon\sqrt{x}$ whenever $n \geq N$. And this implies that for all $n \geq N$,

$$|\sqrt{x_n}-\sqrt{x}| < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$$

as desired.

(5) 2.3.3

Exercise 2.3.3. Let $\epsilon > 0$ be arbitrary. We must show that there exists an N such that $n \geq N$ implies $|y_n - l| < \epsilon$. In terms of ϵ-neighborhoods (which are a bit easier to use in this case), we must equivalently show $y_n \in (l-\epsilon, l+\epsilon)$ for all $n \geq N$.

Because $(x_n) \to l$, we can pick an N_1 such that $x_n \in (l-\epsilon, l+\epsilon)$ for all $n \geq N_1$. Similarly, because $(z_n) \to l$ we can pick an N_2 such that $z_n \in (l-\epsilon, l+\epsilon)$ whenever $n \geq N_2$. Now, because $x_n \leq y_n \leq z_n$, if we let $N = \max\{N_1, N_2\}$, then it follows that $y_n \in (l-\epsilon, l+\epsilon)$, for all $n \geq N$. This completes the proof.