## MATH 321 LAY 2.1 - INTRODUCTION TO SETS

- Tip: start putting definitions and theorems in a master list (on a big sheet of paper or in mind-mapping software) and connect them to show how they relate; I'll check on your progress at the midterm and final.
  - These are great study aids!
- The existence of irrational numbers shook up the Greek idea that all line segments had rational lengths. [draw a unit square and its diagonal]
  - The idea that numbers were rational was built into the Greek definition of "number'.
  - The Pythagoreans dealt with this by saying that some lengths were not expressible by numbers.
  - We'll deal with it by expanding our notion of "number" past the rationals!

**Definition 1.** We start with the set of **natural numbers** $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

- Leopold Kronecker: "The natural numbers are the work of God. All of the rest is the work of mankind."

The set of **integers** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

The set of **rational numbers** $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$

- The colons in these descriptions are read as "such that". So $\mathbb{Q}$ is the set of fractions $\frac{a}{b}$ *such that* both $a$ and $b$ are integers **and** $b \neq 0$.
- $\mathbb{Q}$ is particularly nice because it's what's known as a *field*: a set in which we can do addition, subtraction, multiplication, and division and still stay inside the set.
- Problem: we can **approximate** $\sqrt{2}$ within $\mathbb{Q}$ by decimals such as 1.4, 1.41, 1.414, ..., but there's a "hole" in $\mathbb{Q}$ (seen as a number line) where $\sqrt{2}$ should be.

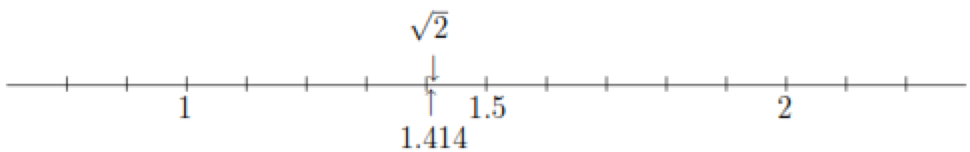


Figure 1.2: Approximating $\sqrt{2}$ with rational numbers.

- We'll find out later how to construct the real numbers from $\mathbb{Q}$, but it's not too far off to think of it as "filling in the holes" by defining an **irrational number** wherever there's a hole in the number line and placing it there.

**Exercise 2.** What are some questions we could ask about the rational and irrational numbers? See if you have any intuition or thoughts about how you might begin seeking an answer to your question.

**Question 3.** *What properties does the set of irrational numbers have?*

*How do the sets of rationals and irrationals fit together?*

*Is there a kind of symmetry between the rationals and irrationals, or is there some sense in which we can argue that one is more common?*

*Can all irrational numbers be expressed as algebraic combinations of roots and rational numbers, or are there others?*

- The unifying foundation to all branches of math is sets.
  - "Math= logic+ set theory"

**Goals**:

- discuss the set theory that we need for analysis
- practice proof techniques and logic

0.1. **Introductory problem(s) (in groups).**

(1) [slide] Practice 1.2, 1.6, 1.11, and 1.12 from Lay

0.2. **Basic set operations.**

**Definition 4.** A **set** is a collection of objects characterized by some defining property that allows us to think of the objects as a whole. The objects in a set are called **elements** or **members** of the set.

- We use capital letters to denote sets and the symbol $\in$ to denote membership in a set. The symbol $\notin$ means "not in" a set.

**Exercise 5.** How can we define a set without referring to other undefined terms?

- Although initially *naive set theory*, which defines a set merely as any well-defined collection, was well accepted, it soon ran into several obstacles. It was found that this definition spawned several paradoxes, most notably:
  - Russell's paradox—It shows that the "set of all sets that do not contain themselves," i.e. the "set" $\{x : x \text{ is a set and } x \notin x\}$ does not exist.
  - Cantor's paradox—It shows that "the set of all sets" cannot exist.
- The reason is that the phrase *well-defined* is not very well-defined. It was important to free set theory of these paradoxes because nearly all of mathematics was being redefined in terms of set theory. In an attempt to avoid these paradoxes, set theory was axiomatized based on first-order logic, and thus *axiomatic set theory* was born.
- For most purposes, however, naive set theory is still useful.

**Example 6.** Examples of sets.

- $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}$, $\mathbb{Q}$
- $\mathbb{R}$, the set of real numbers
- the empty set, denoted $\emptyset$.
- the *singleton* set consisting of the number 1 alone: $\{1\}$.
- the singleton set whose only element is the singleton set consisting of the number one: $\{\{1\}\}$.
- Many of these sets are represented using blackboard bold or bold typeface. Special sets of numbers include
- $\mathbb{P}$, denoting the set of all primes: $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, ...\}$

**Exercise 7.** Which sets above have every one of their elements also contained in another set above?

**Definition 8.** Let $A$ and $B$ be sets. We say that $A$ is a **subset** of $B$ if every element of $A$ is an element of $B$, and we denote this by writing $A \subseteq B$ (or occasionally $B \supseteq A$).

If $A$ is a subset of $B$ and there exists an element of $B$ that is not in $A$, then $A$ is called a **proper subset** of $B$, write $A \subset B$.

**Proof technique.** To show that $A \subseteq B$, we must show that the statement

$$\text{if } x \in A, \text{ then } x \in B$$

is true.

**Definition 9.** The **empty set**, denoted $\emptyset$, is the set with no elements.

**Theorem 10.** *Let $A$ be a set. Then $\emptyset \subseteq A$.*

*Proof.* To prove that $\emptyset \subseteq A$, we must establish that the implication

$$\text{if } x \in \emptyset, \text{ then } x \in A$$

is true. Since $\emptyset$ has no elements, it's true that every element of the empty set is also an element of $A$.

This is kind of like saying "if unicorns exist, then unicorns are pink". Since unicorns don't exist (sorry), we can say whatever we want about them following the clause "if unicorns exist", and it would be (vacuously) true of all unicorns. □

**Definition 11.** Let $A$ and $B$ be sets. We say that $A$ is **equal** to $B$, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

- Combined with the definition of subset, we see that proving $A = B$ is equivalent to proving
$$x \in A \implies x \in B \text{ and } x \in B \implies x \in A.$$

**Example 12.** Let
$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{x | x = 2k \text{ for some } k \in \mathbb{N}\}$$
$$C = \{x \in \mathbb{N} | x < 6\}.$$
Then $\{4, 3, 2\} \subseteq A$, $3 \notin B$, and $C \subseteq A$.

**Exercise 13.** Which of the following are also true of these sets? Prove if true (using the Proof Technique above) or give a counterexample if false.

(1) $A \subseteq C$
(2) $C \subseteq B$
(3) $\{2\} \in A$
(4) $\{2, 4, 6, 8\} \subseteq B$
(5) $A = C$

- There are three basic ways to form new sets from old ones.
  - *Union*: gluing together two sets to get a third. Denoted by the symbol $\cup$, think of this as meaning "or". For example, $x \in A \cup B$ if and only if $x \in A$ OR $x \in B$.
  - *Intersection*: using one set as a "cookie cutter" on another set; the "cookie" is the intersection. Denoted by the symbol $\cap$, meaning "and". For example, $x \in A \cap B$ if and only if $x \in A$ AND $x \in B$.
  - *Complement*: what remains after throwing out a subset $B$ of a larger set $A$. Denoted $B^c$ if it's obvious what $A$ is (e.g. $A = \mathbb{R}$) or $A \setminus B$ to make explicit the larger set.
- More formally,

**Definition 14.** Let $A$ and $B$ be sets. Then
$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$
$$A \setminus B = \{x \in A | x \notin B\}.$$
If $A \cap B = \emptyset$, $A$ and $B$ are said to be **disjoint**.

**Exercise 15.** Which of the following sets contain $\sqrt{2}$?

(1) $A = \{x \in \mathbb{Q} | x^2 < 3\}$
(2) $B = \{x \in \mathbb{R} | x^2 < 3\}$
(3) $C = A \cup B$
(4) $D = A \cap B$
(5) $E = A^c$

**3.** Let $A = \{2,4,6,8\}$, $B = \{6,7,8,9\}$, and $C = \{2,8\}$. Which of the following statements are true? ☆

(a) $\{8,7\} \subseteq B$ (b) $\{7\} \subseteq B \cap C$
(c) $(A \setminus B) \cap C = \{2\}$ (d) $C \setminus A = \varnothing$
(e) $\varnothing \in B$ (f) $A \cap B \cap C = 8$
(g) $B \setminus A = \{2,4\}$ (h) $(B \cup C) \setminus A = \{7,9\}$

**4.** Let $A = \{2,4,6,8\}$, $B = \{6,8,10\}$, and $C = \{5,6,7,8\}$. Find the following sets.

(a) $A \cap B$ (b) $A \cup B$
(c) $A \setminus B$ (d) $B \cap C$
(e) $B \setminus C$ (f) $(B \cup C) \setminus A$
(g) $(A \cap B) \setminus C$ (h) $C \setminus (A \cup B)$

**Exercise 16.** [slide]