MATH 321 DAY 7 - FUNCTIONS & INDUCTION

1. Cartesian products

Definition 1. The **ordered pair** (a, b) is the set whose members are $\{a\}$ and $\{a, b\}$. In symbols,

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Exercise 2. Why do we need this level of complexity to deal with the idea of caring about the order of elements? What other definitions could we use? Do those definitions do what we want?

Definition 3. If A, B are sets, then the **Cartesian product** (or **cross product**) of A and B, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In symbols,

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

- If A and B are intervals, then $A \times B$ is a rectangle in the usual coordinate system with A on the x-axis and B on the y-axis.

Exercise 4. [slide] TPS Lay Practice 2.3, 2.6

2. Functions

Definition 5. Given two sets A and B, a **function** from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $f : A \to B$. Given an element $x \in A$, the expression $f(x)$ is used to represent the element of B associated with x by f. The set A is called the **domain** of f. The ***range*** of f is not necessarily equal to B, but refers to the subset of B given by $\{y \in B | y = f(x) \text{ for some } x \in A\}$.

Exercise 6. Why not use a simpler definition that avoids all that stuff about "associated with x", "rule", and "mapping"? Why not just define a function as a formula or equation?

- There are functions that don't (easily, or at all) admit formulas. For instance, how would you write a formula for the function "count the number of positive integers less than an integer n that share no prime factors with n"?
  - This is a very important function in number theory called **Euler's totient function** $\phi(n)$**.**
  - This function, together with Fermat's Little Theorem (a special case of Euler's Theorem), makes RSA encryption work and secures your credit card info online.
- There are functions that act on shapes rather than numbers. It's really hard to describe the function "take a knotted-up strand of DNA and pass one loop through another" in an equation.

Example 7. In 1829, Dirichlet proposed this unruly function, also known as the **characteristic function of** $\mathbb{Q}$**:**

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(CHi for CHaracteristic) The domain of g is all of $\mathbb{R}$, and the range is the set $\{0, 1\}$. There is no single formula for g in the usual sense, and it is quite difficult to graph, but it's still a function!

Example 8. The **absolute value function** is defined

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

With respect to multiplication and division, we have

(1) $|ab| = |a||b|$ and

(2) $|a+b| \leq |a| + |b|$. ("Proto-triangle inequality")

Exercise 9. [slide] TPS on Exercise 1.2.4

3. The Triangle Inequality

- We'll use the triangle inequality a ridiculous amount in this course in the following way. Say we want to find an upper bound on $|a-b|$, the distance from a to b on the number line. We have

$$|a-b| = |(a-c)+(c-b)| \leq |a-c| + |c-b|.$$

- If we pretend a, b, c are points in the plane instead of on the real line, this inequality says "the straight-line distance from a to b is less than the distance if we make a pit stop at c first" [draw].

Proposition 10. *(Triangle inequality) If $a, b, c \in \mathbb{R}$, then $|a-b| \leq |a-c| + |c-b|$.*

Exercise 11. (Reading Question) [slide] TPS on Exercises 1.2.6 and 1.2.7

4. Induction

- I like what your book says: "a proof is an essay of sorts. It is a set of carefully crafted directions which, when followed, should leave the reader absolutely convinced of the truth of the proposition in question. To achieve this, the steps in a proof must follow logically from previous steps or be justified by some other agreed-upon set of facts."
- We've seen proofs by contradiction and proofs of set inclusion and equality so far.
  - Proof by contradiction is an *indirect proof technique*: it starts by assuming what we want to show is false.
  - Proofs of set inclusion are *direct proofs*: they begin from the assumptions and arrive at the desired conclusion.
- Sometimes we use **both** direct and indirect proof techniques.
- Here's an example of a proof of an **if and only if** statement. "A if and only if B" is the same as saying BOTH $A \implies B$ and $B \implies A$.

Theorem 12. *Two real numbers a and b are equal if and only if, for every real number $\epsilon > 0$, it follows that $|a-b| < \epsilon$.*

Exercise 13. Prove this statement by proving:

(1) "If $a = b$, then for all $\epsilon > 0$, $|a-b| < \epsilon$" directly
(2) "If for all $\epsilon > 0$, $|a-b| < \epsilon$, then $a = b$" by contradiction. [Hint: take $\epsilon = |a-b|$ to get a contradiction.]

Proof. There are two key phrases in the statement of this proposition that warrant special attention. One is "for every," which will be addressed in a moment. The other is "if and only if." To say "if and only if" in mathematics is an economical way of stating that the proposition is true in two directions. In the forward direction, we must prove the statement:

($\Rightarrow$) *If $a = b$, then for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

We must also prove the converse statement:

($\Leftarrow$) *If for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$, then we must have $a = b$.*

For the proof of the first statement, there is really not much to say. If $a = b$, then $|a - b| = 0$, and so certainly $|a - b| < \epsilon$ no matter what $\epsilon > 0$ is chosen.

For the second statement, we give a proof by contradiction. The conclusion of the proposition in this direction states that $a = b$, so we assume that $a \neq b$. Heading off in search of a contradiction brings us to a consideration of the phrase "for every $\epsilon > 0$." Some equivalent ways to state the hypothesis would be to say that "for all possible choices of $\epsilon > 0$" or "no matter how $\epsilon > 0$ is selected, it is always the case that $|a - b| < \epsilon$." But assuming $a \neq b$ (as we are doing at the moment), the choice of

$$\epsilon_0 = |a - b| > 0$$

poses a serious problem. We are assuming that $|a - b| < \epsilon$ is true for *every* $\epsilon > 0$, so this must certainly be true of the particular ϵ_0 just defined. However, the statements

$$|a - b| < \epsilon_0 \quad \text{and} \quad |a - b| = \epsilon_0$$

Proof.

cannot both be true. This contradiction means that our initial assumption that $a \neq b$ is unacceptable. Therefore, $a = b$, and the indirect proof is complete. □

□

- Another style of direct proof: proof by induction.
- [slide] have you ever seen anyone set up a ton of dominoes and, simply by pushing the first domino over, knock them all down? That's what mathematical induction is.
- [slide] here's the idea: say we want to prove something for **all** $k \in \mathbb{N}$ (or $\mathbb{N} \cup \{0\}$). If we can prove that
  (1) the statement is true for $k = 1$ (or $k = 0$, whatever the first case you want is), and
  (2) if the statement is true for k, it's also true for $k + 1$,
- then we will have proven the statement for **all** $k \in \mathbb{N}$**. Statements are like dominoes**; set them up so k knocks down $k + 1$ for all $k \in \mathbb{N}$, knock the $k = 1$st domino down, and you've got them all!
- A great way of learning induction is by example and practicing.

Example 14. Let $x_1 = 1$, and for each $n \in \mathbb{N}$ define $x_{n+1} = \frac{x_n}{2} + 1$. Using this rule, we can compute $x_2 = (1/2)(1) + 1 = 3/2$, $x_3 = 7/4$, and so on.

This seems to be increasing; let's use induction to prove it!

Proposition 15. *For all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.*

Proof. We proceed by induction.

- Base case: $n = 1$. We want to show $x_1 \leq x_2$. But we already did this above since $1 < \frac{3}{2}$.
- Inductive step: we want to show that

$$\text{if } x_n \leq x_{n+1}, \text{ then it follows that } x_{n+1} \leq x_{n+2}.$$

Assume that $x_n \leq x_{n+1}$. We want to show that $x_{n+1} \leq x_{n+2}$. But this is the same as showing that $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$. This is certainly true; we can multiply the assumption $x_n \leq x_{n+1}$ by $\frac{1}{2}$ and add 1 to both sides to get that

$$x_{n+1} = \frac{1}{2}x_n + 1 \leq^{\text{assumption}} \frac{1}{2}x_{n+1} + 1 = x_{n+2}.$$

This completes the proof.

$\square$

Note 16. Consider the set S of natural numbers n for which we just showed that $x_n \leq x_{n+1}$. The base case showed that $1 \in S$. The inductive step says that whenever $n \in S$, then $n+1 \in S$. It must then be that $S = \mathbb{N}$; we proved this fact for all $n \in \mathbb{N}$ as desired!

Exercise 17. Use induction to prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}.$$

Exercise 18. Prove by induction that $7^n - 4^n$ is a multiple of 3, for all $n \in \mathbb{N}$.