MATH 321 WEEK 4 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 1.3.4

(a) We conjecture that $\sup(A_1 \cup A_2) = \max\{\sup A_1, \sup A_2\}$. In fact, we conjecture in general that

$$\sup\Big(\bigcup_{k=1}^{n} A_k\Big) = \max\{\sup A_1, \sup A_2, \dots, \sup A_n\}.$$

To prove this statement, we induct on n. Let $s_i = \sup A_i$ for all i. We may suppose without loss of generality that

$$s_1 \le s_2 \le \cdots \le s_n$$

by renumbering if necessary.

(i) Base case: suppose WLOG that $s_2 \ge s_1$. Then if $x \in A_1$, $x \le s_1 \le s_2$, and if $x \in A_2$, $x \le s_2$ by definition of s_2, so either way s_2 is an upper bound for $A_1 \cup A_2$. Let α be another upper bound for $A_1 \cup A_2$. Then, in particular, α is an upper bound for A_2, so that $s_1 \le s_2 \le \alpha$. Thus, $\sup(A_1 \cup A_2) = \max\{\sup A_1, \sup A_2\}$.

(ii) Inductive step: suppose that, for any collection of n sets $A_1, \dots, A_n$,

$$\sup\Big(\bigcup_{k=1}^{n} A_k\Big) = \max\{\sup A_1, \dots, \sup A_n\} = \sup A_n.$$

Suppose that $s_{n+1} = \sup A_{n+1} \ge s_n$. Define $B_1 = A_1 \cup A_2$, $B_2 = A_3$, $B_3 = A_4$, and so on. We may write

$$\bigcup_{k=1}^{n+1} A_k = \bigcup_{k=1}^{n} B_k.$$

Applying the induction assumption to $\bigcup_{k=1}^{n} B_k$, we find that

$$\begin{aligned}
\sup\Big(\bigcup_{k=1}^{n+1} A_k\Big) = \sup\Big(\bigcup_{k=1}^{n} B_k\Big) &= \max\{\sup(A_1 \cup A_2), s_3, s_4, \dots, s_n, s_{n+1}\}\\
&= \max\{s_2, s_3, \dots, s_n, s_{n+1}\}\\
&= s_{n+1},
\end{aligned}$$

as desired.

- Alternatively, fix $n \in \mathbb{N}$ and let $A = \cup_{k=1}^{n} A_k$. Let $S = \max\{\sup A_1, \sup A_2, \dots, \sup A_n\}$. We prove directly that $S = \sup A$.

  (i) S is an upper bound for A. For if $a \in A$, then $a \in A_k$ for some $1 \le k \le n$. Thus, $a \le \sup A_k$. Since S is defined as a maximum of a set of sups including $\sup A_k$, we have $a \le \sup A_k \le S$.

  (ii) Let α be another upper bound for A. Then α is an upper bound for each of $A_1, \dots, A_n$ by definition. But this means that $\sup A_k \le \alpha$ for all $1 \le k \le n$. Therefore, by definition of the maximum, $S \le \alpha$ as well.

(b) Yes; the formula becomes

$$\sup\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sup\{s_1, s_2, s_3, \dots\}.$$

Let $s = \sup\Big(\bigcup_{k=1}^{\infty} A_k\Big)$. We show that $s = \sup\{s_1, s_2, s_3, \dots\}$.

(i) We first show that s is an upper bound for $\{s_1, s_2, s_3, \dots\}$. By definition of sup, $s \geq x \forall x \in \bigcup_{k=1}^{\infty} A_k$. Thus, s is an upper bound for $A_1, A_2, \dots$, meaning that $s_k \leq s$ for all $k \in \mathbb{N}$ (since $s_k = \sup A_k$) and hence s is an upper bound for $\{s_1, s_2, \dots\}$.

(ii) We now show that s is the least upper bound for $\{s_1, s_2, s_3, \dots\}$. If α is any upper bound for $\{s_1, s_2, \dots\}$, then $\alpha \geq s_k$ for all k, and hence α is an upper bound for $\bigcup_{k=1}^{\infty} A_k$. Thus, since $s = \sup \left(\bigcup_{k=1}^{\infty} A_k \right)$, $s \leq \alpha$. So, by definition, $s = \sup\{s_1, s_2, \dots\}$.

(2) 1.3.5

Exercise 1.3.5. (a) In the case $c = 0$, $cA = \{0\}$ and without too much difficulty we can argue that $\sup(cA) = 0 = c \sup A$. So let's focus on the case where $c > 0$. Observe that $c \sup A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c \sup A \leq d$. We know $ca \leq d$ for all $a \in A$, and thus $a \leq d/c$ for all $a \in A$. This means d/c is an upper bound for A, and by Definition 1.3.2 $\sup A \leq d/c$. But this implies $c \sup A \leq c(d/c) = d$, which is precisely what we wanted to show.

(b) Assuming the set A is bounded below, we claim $\sup(cA) = c \inf A$ for the case $c < 0$. In order to prove our claim we first show $c \inf A$ is an upper bound for cA. Since $\inf A \leq a$ for all $a \in A$, we multiply both sides of the equation to get $c \inf A \geq ca$ for all $a \in A$. This shows that $c \inf A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c \inf A \leq d$. We know $ca \leq d$ for all $a \in A$, and thus $d/c \leq a$ for all $a \in A$. This means d/c is a lower bound for A and from Exercise 1.3.1, $d/c \leq \inf A$. But this implies $c \inf A \leq c(d/c) \leq d$, which is precisely what we wanted to show.

(3) 1.3.7

Exercise 1.3.7. Since a is an upper bound for A, we just need to verify the second part of the definition of supremum and show that if d is any upper bound then $a \leq d$. By the definition of upper bound $a \leq d$ because a is an element of A. Hence, by Definition 1.3.2, a is the supremum of A.

(4) 1.3.10

(a) Use the Axiom of Completeness to prove the Cut Property.
Note that, since $a < b$ for all $a \in A, b \in B$ and $A \sqcup B = \mathbb{R}$, there must exist some n such that, whenever $m \geq n$, $m \notin A$. Hence, A is bounded above by n, so we can define $c = \sup A$. Then, by definition, c is an upper bound for A, hence $x \leq c$ whenever $x \in A$. It remains to show that $x \geq c$ whenever $x \in B$. Note that, if $x \in B$, then x is an upper bound for a because $a < x$ for all $a \in A$. Since c is the least upper bound, we have $c \leq x$, as desired.

(b) Use the Cut Property to prove the Axiom of Completeness.
Suppose that $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Let B be the set of all upper bounds for E, and define $A = \mathbb{R} \setminus B$. Then, by the Cut Property, there exists $c \in \mathbb{R}$ so that $x \leq c$ whenever $x \notin B$ and $x \geq c$ whenever $x \in B$. We claim $c = \sup E$.

(i) We want to show that c is an upper bound for E. To that end, let $x \in E$. If $x \in B$, then $x \leq c$ by definition of upper bound. If $x \notin B$, then $x \leq c$ by the Cut Property. In either case, $x \leq c$, and hence c is an upper bound for E.

(ii) Let b be an upper bound for E. Then $b \in B$, in which case $c \leq b$. Hence, $c = \sup E$, as desired.

(5) 1.4.2

(a) We show by contradiction that s is an upper bound for A. Suppose not; then there exists an element $a \in A$ so that $s < a$. By the Archimedean Property, we can choose $n \in \mathbb{N}$ such that

$\frac{1}{n} < a - s$. Then $s + \frac{1}{n} < s + (a - s) = a$, contradicting that $s + \frac{1}{n}$ is an upper bound for A for all n.

(b) We show by contradiction that, if α is any other upper bound for A, then $s \leq \alpha$. Suppose that α is an upper bound for A and $\alpha < s$. By the Archimedean Property, choose $n \in \mathbb{N}$ so that $\frac{1}{n} < s - \alpha$. Then $s - \frac{1}{n} > s - (s - \alpha) = \alpha$, and by assumption $s - \frac{1}{n}$ is not an upper bound for A. Thus, α can't be an upper bound for A either, giving a contradiction.

(6) 1.4.4
  (a) We show that b is an upper bound for T. Let $r \in T$; then $r \leq b$ by definition.
  (b) Let α be an upper bound for T. Suppose $\alpha < b$. By the density of the rational numbers, there exists $r \in \mathbb{Q}$ such that $\alpha < r < b$. Moreover, $r \in T$, contradicting that α is an upper bound for T.