MATH 321 2.3 - THE LIMIT OF A SEQUENCE

1. Introduction to sequences & series

Recall. In your Primary Source Project "Why Be So Critical?", you read Heinrik Abel's letter to Holmboe in which he stated, "Divergent series are on the whole devilish, and it is a shame that one dares to base any demonstration [proof] on them...can you imagine anything more appalling than to say

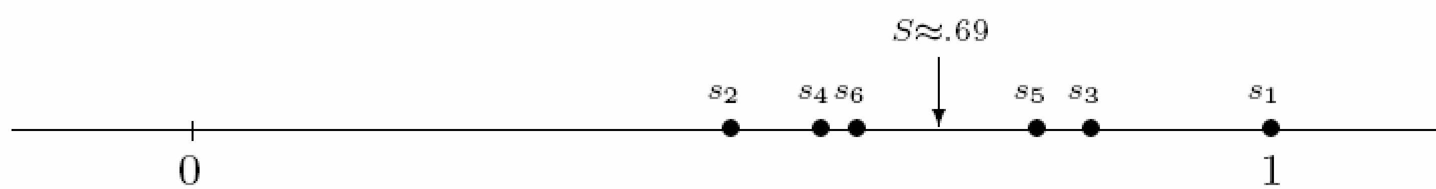
$$0 = 1 - 2^n + 3^n - 4^n + \text{etc} = \sum_{k=1}^{\infty} (-1)^{k+1} k^n$$

where n is a positive integer? *Risum teneatis amici*!

Example 1. Here's another example of a "devilish" infinite series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots.$$

- If we consider the partial sums from the right-hand side, i.e. taking s_n to be the sum of the first n terms of the series,
$$s_1 = 1, s_2 = 1/2, s_3 = 5/6, s_4 = 7/12, \dots$$
Note that the odd sums decrease ($s_1 > s_3 > s_5 > \dots$) while the even sums increase ($s_2 < s_4 < s_6 < \dots$).
- It seems reasonable that the sequence (s_n) eventually hones in on a value S where the odd and even partial sums "meet". Summing a few hundred terms reveals that $S \approx .69$.



- We're now tempted to write

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \tag{1.1}$$

However, it's not clear whether properties of addition and equality that are well understood for finite sums remain valid when applied to infinite sums.

Exercise 2. Write out the first 8 terms of the sum S and of $\frac{1}{2}S$. What are the first eight terms of the sequence that results when you add S and $\frac{1}{2}S$ together? How does the result compare to the original sequence S?

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

$$+\, S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \cdots$$

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} \cdots$$

- but the bottom equation contains precisely the same terms as the original sequence in a different order! How can $S = \frac{3}{2}S$?
- The problem is that addition, in the infinite setting, is not always commutative!
- You can't rearrange terms in an infinite series without potentially changing the sum.

- It is this "devilish" nature of infinite series that makes necessary a careful, deliberate examination of infinite objects such as sequences and series.
- Proof will be a check on our intuition to make sure we don't run into the traps of our own biases!

2. The limit of a sequence

- Most concepts in analysis can be reduced to statements about sequences, so we'll spend a lot of time focusing on those first.

Definition 3. A **sequence** is a function whose domain is $\mathbb{N}$. In other words, a sequence is an ordered, infinite list of real numbers. If the nth term of a sequence is a_n, we write the sequence as (a_n) or $(a_n)_{n=1}^{\infty}$.

Exercise 4. Give three examples of sequences.

- This is one of the most important definitions in this class:

Definition 5. (Convergence of a Sequence) A sequence (a_n) **converges** to a real number a if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, $|a_n - a| < \epsilon$. [draw a picture of a sequence on a number line bouncing closer and closer to a]

To indicate that (a_n) converges to a, we write either $\lim a_n = a$ or $(a_n) \to a$.

Exercise 6. (Reading question) What happens if we reverse the order of the quantifiers in the definition of convergence?

Definition 7. A sequence (x_n) **verconges** to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

(1) Give an example of a vercongent sequence.

The sequence $\{0, 1, 0, 1, 0, 1, \dots\}$ verconges to 0.5 because for $\epsilon = 1$, every element x_n of the sequence satisfies $|x_n - 0.5| = 0.5 < 1$.

(2) Is there an example of a vercongent sequence that is divergent?

The sequence above is divergent.

(3) Can a sequence vercongе to two different values?

Sure can! The sequence above also verconges to 1, 0, 37π, and $3,000,000 + e$. (Just choose a really big ϵ!)

(4) What exactly is being described in this strange definition?

This is really the definition of a bounded sequence.

Definition 8. A sequence (a_n) is **bounded** if there exists $K \in \mathbb{R}$ such that $|x_n| \leq K$ for all $n \in \mathbb{N}$.

Theorem 9. *A sequence is vercongent if and only if it is bounded.*

Proof. ($\Longleftarrow$) Suppose that (x_n) is bounded. In particular, there exists $K \in \mathbb{R}, N \in \mathbb{N}$ such that $x_n \leq K$ for all $n \geq N$. Then (x_n) verconges to x for all $x \in \mathbb{R}$, since for $n \geq N$ and $\epsilon = |K - x|$,

$$|x_n - x| \leq |K - x| = \epsilon.$$

($\Longrightarrow$) We show the contrapositive: an unbounded sequence cannot be vercongent for any x. Suppose (x_n) is unbounded. Then, negative the definition above, we have that for all $K \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists $n \geq N$ such that $x_n > K$. Then, for any $\epsilon > 0$ and any potential "milim" x to which (x_n) is vercongent, we may choose $K = |x - \epsilon|$. Then, by definition of unboundedness, there exists $n \in \mathbb{N}$ so that $x_n > |x - \epsilon| > 0$. Therefore, (x_n) is not vercongent to x. □

[start here 10-3-18]

- The order of quantifiers is important!
- In order to decipher the definition of "convergent", let's first focus on the ending phrase "$|a_n - a| < \epsilon$", and think about the points that satisfy an inequality like this.

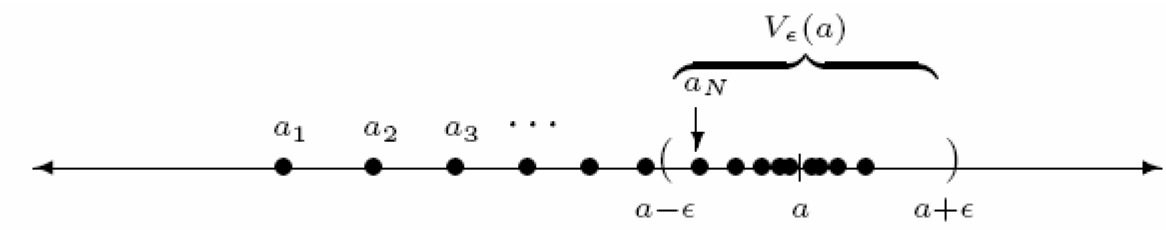
Definition 10. Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the **ϵ-neighborhood of a.**

- Note that $V_\epsilon(a)$ consists of all of those points whose distance from a is less than ϵ: $V_\epsilon(a) = (a-\epsilon, a+\epsilon)$ [draw the interval]
- Recasting the definition in terms of ϵ-neighborhoods:

Definition 11. (Convergence of a Sequence: Topological Version) A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood of a contains all but a finite number of the terms of (a_n).



- The natural number N in the original version is the point where (a_n) enters $V_\epsilon(a)$, never to leave.

Note 12. The value of N depends on the choice of ϵ. The smaller the ϵ-neighborhood, the larger N may have to be.

2.1. **Divergence (if time).**

- Consider the sequence
$$(a_n) = \left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots\right)$$
How can we argue the sequence does **not** converge to zero?

Exercise 13. (reading question) We need to talk about negating quantifiers such as "for all" and "there exists". Describe what we would have to demonstrate in order to disprove each of the following statements.

(1) At every college in the United States, there is a student who is at least seven feet tall.
  - There exists a college in the US at which every student is (strictly) less than seven feet tall.

(2) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
  - There exists a college in the US at which every professor gives at least one student a grade strictly lower than B.

(3) There exists a college in the United States where every student is at least six feet tall.
  - At every college in the US, there exists a student who is less than six feet tall.

- As we mentioned two weeks ago, negating a statement means **flipping** $\forall \leftrightarrow \exists$, reversing all inequalities, and flipping $= \leftrightarrow \neq$.
- How would we negate the definition of convergence?
  - We must produce a single value of ϵ for which **no** $N \in \mathbb{N}$ works.

Definition 14. [write definition and negation on top of each other] A sequence (x_n) is said to **diverge** if it does not converge. More specifically, (a_n) diverges if the following is true for all potential limits $x \in \mathbb{R}$:
there exists $\epsilon > 0$ so that, for all $N \in \mathbb{N}$, there exists $n \geq N$ so that $|x_n - x| \geq \epsilon$.

Example 15. The sequence (a_n) above does not converge to 0 because there's no N such that $n \geq N \implies a_n \in V_{1/10}(0)$. In other words, for **all** $N \in \mathbb{N}$, there exists $n \geq N$ so that $|a_n| \geq \epsilon = \frac{1}{10}$.

- Proving divergence using the definition is often harder than proving convergence because you have to show a sequence doesn't diverge to *anything,* though in each proof you only have to find a *single* $\epsilon > 0$ for which the sequence doesn't stay within that error tolerance.
- We'll postpone it until we have a more economical divergence criterion (section 2.5).

2.2. **Proving convergence.**

Question 16. *How do we know*

$$\lim\left(\frac{1}{\sqrt{n}}\right) = 0?$$

This translates to the statement

$$\forall \epsilon > 0 \exists N > 0 \text{ such that } \forall n \geq N, \left|\frac{1}{\sqrt{n}} - 0\right| < \epsilon.$$

Exercise 17. For each of the following ϵ-values, give the "target neighborhood" $V_\epsilon(0)$ that we're aiming for, and give the smallest value of N so that $\frac{1}{\sqrt{n}}$ falls into that neighborhood whenever $n \geq N$.

(1) $\epsilon = 1/10$
 $N > 100$
(2) $\epsilon = 1/50$
 $N > 50^2 = 2500$
(3) $\epsilon = 1/k$ for general k
 $N > k^2$

- Your work here says that, to pick the right N for a given ϵ, we have to take the reciprocal of that ϵ and then square it.

Example 18. Let's use your work to give a proof that $\lim(\frac{1}{\sqrt{n}}) = 0$.

Proof. Let $\epsilon > 0$ be an arbitrary positive number. Choose a natural number N satisfying

$$N > \frac{1}{\epsilon^2}.$$

We now verify that this choice of N has the desired property. Let $n \geq N$. Then

$$n > \frac{1}{\epsilon^2} \implies \frac{1}{\sqrt{n}} < \epsilon \implies |a_n - 0| < \epsilon.$$

□

- This is always how proving that a specific sequence (x_n) converges to $x$goes. Here's a template for a convergence proof:
  (1) "Let $\epsilon > 0$ be arbitrary."
  (2) [off to the side, as scratch work] Starting from the expression $|x_n - x| < \epsilon$, do algebra and rearrange until you get an inequality of the form $f(\epsilon) < n$, where $f(\epsilon)$ is some expression/function in terms of ϵ.
  (3) [back on your proof sheet] "Choose $N > f(\epsilon)$."
  (4) "Assume $n \geq N$."
  (5) Perform your scratch work backwards to show that $N > f(\epsilon) \implies |x_n - x| < \epsilon$.

Exercise 19. Show that

$$\lim\left(\frac{n+1}{n}\right) = 1.$$

Proof.

(1) Let $\epsilon > 0$ be arbitrary.
(2) [scratch work]

$$\left|\frac{n+1}{n} - 1\right| < \epsilon$$
$$\left|1 + \frac{1}{n} - 1\right| < \epsilon$$
$$\left|\frac{1}{n}\right| < \epsilon$$
$$\frac{1}{n} < \epsilon$$
$$\frac{1}{\epsilon} < n$$

(3) Choose $N > \frac{1}{\epsilon}$.
(4) Assume $n \geq N$.
(5) Then $n \geq N > \frac{1}{\epsilon}$, hence $\frac{1}{\epsilon} < n$. Therefore, $\frac{1}{n} < \epsilon$, and hence

$$\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \epsilon$$

as desired.

□

Theorem 20. ***(Uniqueness of Limits)*** *The limit of a sequence, when it exists, must be unique.*

Proof. Unclaimed Homework □

Exercise 21. (if time) Determine the limits of the following sequences, if they exist. For those limits that exist, prove it.

(1) (a_n) where $a_n = \frac{1}{n^2}$

Proof. We show that $\lim a_n = 0$.
(a) Let $\epsilon > 0$ be arbitrary.
(b) [scratch work] $\frac{1}{n^2} < \epsilon \implies \frac{1}{\epsilon} < n^2 \implies \sqrt{\frac{1}{\epsilon}} < n$ (where we don't consider the negative square root because n is positive)
(c) Choose $N > \sqrt{\frac{1}{\epsilon}}$.
(d) Assume $n \geq N$.
(e) Then $n > \sqrt{\frac{1}{\epsilon}}$, therefore $n^2 > \frac{1}{\epsilon}$, hence

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \epsilon$$

as desired.

□

(2) (b_n) where $b_n = n$
(3) (c_n) where $c_n = \frac{1}{\log n}$.
(a) Guess that this sequence converges to 0.
(b) Let $\epsilon > 0$ be arbitrary.
(c) [scratch work, off to the side]:

$$\begin{aligned} \left|\frac{1}{\log n}\right| < \epsilon &\implies \frac{1}{\log n} < \epsilon \\ &\implies \log n > \frac{1}{\epsilon} \\ &\implies n > e^{1/\epsilon}. \end{aligned}$$

(d) Choose $N > e^{1/\epsilon}$. Assume $n \geq N$. Then

$$\begin{aligned} \left|\frac{1}{\log n}\right| &= \frac{1}{\log n} \\ &< \frac{1}{\log(e^{1/\epsilon})} \\ &= \frac{1}{1/\epsilon} = \epsilon \end{aligned}$$

as desired.