MATH 321 DAY 10 - CONSEQUENCES OF COMPLETENESS, PART 1

1. Warmup

Recall the Axiom of Completeness. every nonempty set of real numbers that is bounded above has a least upper bound.

Exercise 1. (Warmup) (T/F) Prove if true, counterexample if false.

(1) The rational numbers satisfy the Axiom of Completeness.
   (a) False; consider $S = \{r \in \mathbb{Q} : r^2 \leq 2\}$
(2) The integers satisfy the Axiom of Completeness.
   (a) True; let S be a nonempty set of integers which is bounded above by some number N and consider the set

$$T := \begin{cases} S \cap [0, N], & N > 0 \\ S \cap [N, 0], & N < 0 \\ S \cap [s, 0], \text{ where } s \in S \text{ is arbitrary}, & N = 0. \end{cases}$$

   Now T must be a finite set, hence $\max T$ exists. We claim $\sup S = \max T$.
(3) The natural numbers satisfy the Axiom of Completeness.
   (a) True by the reasoning above.
(4) Every set $A \subseteq \mathbb{R}$ has at most one least upper bound.
   (a) True; suppose s and t are least upper bounds for A. Then, in particular, both s and t are upper bounds for A. Since each of s, t are least upper bounds for S, it must be that $s \leq t$ (because t is an upper bound for S) and $t \leq s$ (because s is an upper bound for s). Hence $s = t$.
(5) Every set $A \subseteq \mathbb{R}$ has at least one least upper bound.
(6) Every set $A \subseteq \mathbb{R}$ which is bounded above has at least one least upper bound.
   (a) False; consider $A = \emptyset$
(7) Every set $A \subseteq \mathbb{R}$ has at most one greatest lower bound.
(8) Every set $A \subseteq \mathbb{R}$ has at least one greatest lower bound.
(9) Every set $A \subseteq \mathbb{R}$ which is bounded below has at least one greatest lower bound.
   (a) False; consider $A = \emptyset$.

2. Consequences of Completeness, Part 1

2.1. The Nested Interval Property.

Theorem 2. ***(The Nested Interval Property)*** *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} | a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

- This is one way of stating that "the real line has no holes".
  - If the real line had holes, we could center our nested intervals on a hole and violate the Nested Interval Property [draw picture]
- [slide] Now, consider the following exercises:

1. We begin with an example of nested open intervals. For every $n \in \mathbf{N}$, let $J_n = (0, \frac{1}{n}$
of the statements below are true?
   (a) $J_1 \subset J_2 \subset J_3 \subset J_4 \subset \dots$
   (b) $J_1 \supset J_2 \supset J_3 \supset J_4 \supset \dots$
   (c) $\cup_{n=1}^{m} J_n$ is nonempty for every $m \in \mathbf{N}$.
   (d) $\cap_{n=1}^{m} J_n$ is nonempty for every $m \in \mathbf{N}$.
   (e) $\cup_{n=1}^{\infty} J_n$ is nonempty.
   (f) $\cap_{n=1}^{\infty} J_n$ is nonempty.
2. In the exercise above, put a star next to the statement that explains why these are called "nested". ("Open" means—sort of—that the endpoints are not includ difference between "open" , "closed" , and neither open nor closed will be so import we'll spend more time on this later in the course.)

(1)
   (a) Since $\frac{1}{n}$ decreases as n increases, J_1 is the biggest interval, and
   $$J_1 \supset J_2 \supset J_3 \supset \dots$$
   so that (a) is false and (b) is true.
   (b) Since $J_1 \neq \emptyset$, (c) and (e) are true.
   (c) Fix $m \in \mathbb{N}$ and let $b_m = \frac{1}{m}$. (By the Archimedean Property, to be proven later today), there exists an integer k such that $0 < 1/k < b_m$. Moreover, since $1/k < b_n$, $1/k < b_j$ for all $j \leq m$. Therefore, $1/n \in \cap_{n=1}^{m} J_n$ and, in particular, $\cap_{n=1}^{m} J_n \neq \emptyset$ for all $m \in \mathbb{N}$.
   (d) However, infinite intersections behave strangely. Let $x \in \mathbb{R}$. Then, by the Archimedean Property, there is an integer n so that $0 < \frac{1}{n} < x$. Then the interval $J_n = (0, \frac{1}{n})$ does not contain x, so that $x \notin \cap_{n=1}^{\infty} J_n$. Therefore, $\cap_{n=1}^{\infty} J_n = \emptyset$.

(2) Statement (b) is why we call these intervals "nested".

- Now let's prove the NIP using the AoC. How can we construct a bounded set in such a way that its sup will help us?

3. With $I_n = [a_n, b_n]$ as in the NIP statement above, let $A = \{a_n\} = \{a_1, a_2, a_3, \dots\}$.
   (a) Does A satisfy the hypotheses of the AoC (Axiom of Completeness)?
   (b) Therefore, A has a what? (We will call that number that the AoC gives us x).
   (c) Is $x \in I_1$? Why or why not? Is $x \in I_n$ for any/all $n \in \mathbf{N}$? Why or why not?
   (d) Explain how, therefore, we know that $\cap_{n=1}^{\infty} I_n$ is nonempty.

Proof. We will use the AoC to show that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. The AoC is a statement about bounded sets, and the one we want to consider is the set $A = \{a_n : n \in \mathbb{N}\}$ of left-handed endpoints of the I_n.

(1) Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we can write
$$x = \sup A$$
by the AoC.
(2) Therefore, A has a least upper bound x.
(3) Because x is an upper bound for A, we have $a_1 \leq x$ and $a_n \leq x$ for all n. Because every b_n is an upper bound for A and x is the least upper bound, $x \leq b_n$ for all n.
(4) Therefore, $a_n \leq x \leq b_n$ for all n, and hence $x \in I_n$ for all n. Therefore, $x \in \cap_{n=1}^{\infty} I_n$.

□

2.2. **The Density of $\mathbb{Q}$ in $\mathbb{R}$.**

- $\mathbb{Q}$ is an extension of $\mathbb{N}$, and $\mathbb{R}$ is an extension of $\mathbb{Q}$. The following results indicate how $\mathbb{N}$ and $\mathbb{Q}$ sit inside of $\mathbb{R}$.

Exercise 3. [RQ] 1.4.1

(1) If $a, b \in \mathbb{Q}$, then $a = \frac{p}{q}$ and $b = \frac{r}{s}$ for some $p, r \in \mathbb{Z}$ and nonzero $q, s \in \mathbb{Z}$. Then $ab = \frac{pr}{qs}$ where $qs \neq 0$ (because both $q, s \neq 0$) and pr are both integers. Hence $ab \in \mathbb{Q}$. Similarly,
$$a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs} \in \mathbb{Q}.$$
(2) Let $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ be arbitrary, and suppose for contradiction that $a + t$ is a rational number; call it r. Then
$$a + t = r \implies t = r - a$$
is the sum of the rational numbers r and $-a$, so by (a) $t \in \mathbb{Q}$, a contradiction. Similarly, if $at := s \in \mathbb{Q}$, then t is the product of the rational numbers s and $\frac{1}{a}$, hence by (a) $t \in \mathbb{Q}$, a contradiction.
(3) $\mathbb{I}$ is closed neither under addition nor multiplication: $\sqrt{2} + (-\sqrt{2}), \sqrt{2} \cdot \sqrt{2} \notin \mathbb{I}$.

---[start here 9-19-18]

Exercise 4. How, if at all, did reading the chapter "Pure or Applied?" from *Letters to a Young Mathematician* change your view of mathematics as a discipline? Of this course in particular? Do you think this course is pure, applied, both, or neither?

Theorem 5 (**Archimedean Property**). (1) *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$. (In other words, $\mathbb{N}$ is not bounded above.)*
(2) *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

Proof. Why should we prove statements that sound obvious? In the case of (i), we prove it because we can, and because we'll use it a lot in the future and it's good to have a solid logical foundation for the statement. There are statements that sound obvious but are false, such as: (i) sets with infinitely many elements have the same number of elements; (ii) there are more fractions than whole numbers; (iii) you can't integrate a function unless it's continuous; (iv) all integrable functions have antiderivatives (counterexample: e^{x^2}).

(1) Suppose, for contradiction, that $\mathbb{N}$ is bounded above. Then, by the AoC, it has a least upper bound $\alpha = \sup \mathbb{N}$. Since $\alpha - 1 < \alpha$, $\alpha - 1$ is not an upper bound for $\mathbb{N}$, and hence there exists $n \in \mathbb{N}$ satisfying $\alpha - 1 < n$. But this is equivalent to $\alpha < n + 1$. Because $n + 1 \in \mathbb{N}$, we have a contradiction to the fact that α is an upper bound for $\mathbb{N}$.
(2) This follows from (i) by letting $x = 1/y$.

□

Exercise 6. **(RQ)** 1.4.3: Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. This demosntrates that the intervals in the NIP must be closed for the conclusion of the theorem to hold.

Note that, if $x \leq 0$, certainly $x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$. Now let $x > 0$ be arbitrary. Thus, by the Archimedean Property, there exists $N \in \mathbb{N}$ so that $\frac{1}{N} < x$. Therefore, $x \notin (0, 1/N)$, hence $x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$ and $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

- This familiar property of $\mathbb{N}$ is key to an extremely important fact about how $\mathbb{Q}$ fits inside of $\mathbb{R}$.

Theorem 7 (**Density of $\mathbb{Q}$ in $\mathbb{R}$**). *For every two real numbers a, b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

- We want to find integers m, n so that

$$a < \frac{m}{n} < b. \tag{2.1}$$

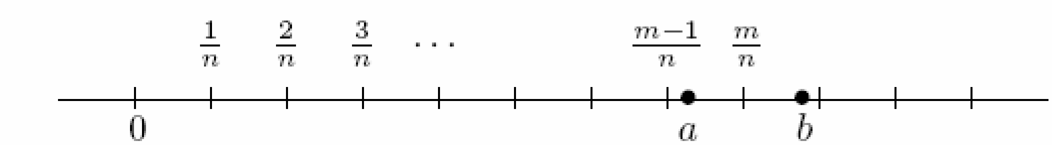
Exercise 8. Suppose $a = 0.1$ and $b = 0.2$.

(1) How would we have to choose $n \in \mathbb{N}$ so that it would be possible for $0.1 < \frac{m}{n} < 0.2$?
(2) How would we then choose m?
(3) Can you use this example to begin a proof of the density of $\mathbb{Q}$ in $\mathbb{R}$?

Proof.

(1) First, we choose the denominator n large enough so that consecutive increments of size $1/n$ are too close together to "step over" the interval (a, b), or in other words,

$$\frac{1}{n} < b - a :$$



(2) Now we want to pick m. Multiplying (2.1) by n, we want to pick m so that $na < m < nb$. Since we want to "squeeze" $\frac{m}{n}$ into (a, b), pick m to be the smallest natural number greater than na. In other words, pick $m \in \mathbb{N}$ such that

$$m - 1 \leq^{(3)} na <^{(4)} m. (*)$$

$\square$

Exercise 9. Finish the proof to show that $a < \frac{m}{n} < b$.

(1) From (4), note that $na < m \implies a < \frac{m}{n}$.
(2) Adding 1 to $(*)$ gives

$$m \leq na + 1 < m + 1.$$

Dividing through by $n$gives that

$$\frac{m}{n} \leq a + \frac{1}{n}$$

(we ignore the rest of the inequality for now). By our choice of n such that $\frac{1}{n} < b - a$, we have that

$$\frac{m}{n} \leq a + \frac{1}{n} < a + (b - a) = b$$

as desired.

Exercise 10. Which of these statements are true for all $a, b \in \mathbb{R}$ such that $0 < a < b$? If true, prove the statement. If false, give a counterexample.

(1) $\exists q \in \mathbb{Z}$ s.t. $a < \frac{5}{q} < b$.

FALSE. From the proof above, since we had to carefully pick m in terms of n, it seems that restricting $m = 5$ might "break" things. Looking at the step where we picked $m - 1 \leq na < m$, we guess that choosing a so that $4 \leq qa \leq 5$ is false for all $q \in \mathbb{Z}$ might give us a counterexample. So pick $a = 6$ and, say, $b = 7$. Then

$$\frac{5}{q} < 6 = a$$

for all $q \in \mathbb{Z}$ since $6q > 5$ for all $q \in \mathbb{Z}$.

(2) $\exists p \in \mathbb{Z}$ s.t. $a < \frac{p}{5} < b$.

FALSE. In the proof above, we had to pick the denominator n so that $\frac{1}{n} < b - a$. If $\frac{1}{5} \geq b - a$, it may happen that our increments of size $1/n$ are too far apart and "step over" the interval (a, b). For example, take $a = 0, b = \frac{1}{10}$. Then, for all $p \in \mathbb{Z}$, $\frac{p}{5} > \frac{1}{10} = b - a$, so that $\frac{p}{5}$ is not between a and b [draw number line picture].

(3) $\exists p, q \in \mathbb{Z}$ s.t. $a < \frac{p}{q} < b$.

This is true and is the same statement of the Density theorem.

2.3.