WEEK 8 CLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 2.4.4

(a) **Show that MCT implies the Archimedean Property:** Let $x \in \mathbb{R}$ be arbitrary. We want to show that there exists $n \in \mathbb{N}$ so that $x \leq n$. Suppose for contradiction that $n < x$ for all $n \in \mathbb{N}$. In that case, the sequence $(n)_{n \in \mathbb{N}}$ of natural numbers is increasing and bounded above by x, so by MCT, it converges to some number m. Moreover, by the Algebraic Limit Theorem, the sequence $(n+1)$ converges to $m+1$. However, by your exam problem, "lopping off" the first term of a sequence doesn't change the limit, so $m = m+1$, a contradiction.

(b) **Show that MCT implies the Nested Interval Property:** For all $n \in \mathbb{N}$, let I_n be a closed interval such that $I_n = [a_n, b_n]$. Assume that, for all n, $I_{n+1} \subseteq I_n$. Consider the sequence (a_n) of left endpoints of these intervals. Since the intervals are nested, (a_n) is monotone increasing, and it is bounded above by all the b_n (and in particular by b_1). Therefore, by MCT, (a_n) converges to some number, call it a. We claim $a \in \bigcap I_n$. To prove this, we must show that $a_n \leq a \leq b_n$ for all n. But $a \leq b_n$ follows from the Order Limit Theorem since $a_m \leq b_n$ for all m. Suppose for contradiction that $a < a_k$ for some k. But by the definition of convergence, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$,

$$|a - a_n| < a_k - a.$$

Let $M = \max\{N, k\}$. Then $|a - a_M| < a_k - a$ (in particular, $a_M - a < |a - a_M| < a_k - a$), implying that $a_M < a_k$. However, $a_M \geq a_k$ by virtue of being "more nested". This contradiction implies that $a \geq a_n$ for all n, and hence $a \in \cap I_n$.

(2) 2.4.6 **(arithmetic-geometric mean)**

(a) **Explain why $\sqrt{xy} \leq (x+y)/2$ for all $x, y \in \mathbb{R}$:** Note that $(x+y)^2 = x^2 + 2xy + y^2$. Therefore, $xy = \frac{(x+y)^2 - x^2 - y^2}{2} \leq \frac{(x+y)^2}{2}$. Take the square root of both sides to obtain the desired inequality.

(b) **Let $0 \leq x_1 \leq y_1$ and define**

$$x_{n+1} = \sqrt{x_n y_n} \textbf{ and } y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n = \lim y_n$ **exist**: We must show that (x_n) and (y_n) are both monotone and bounded. First, we must show that $x_n \leq y_n$ for all n. Note that $x_1 \leq y_1$ and assume for induction that $x_n \leq y_n$. Then

$$x_{n+1} = \sqrt{x_n y_n} \leq \frac{x_n + y_n}{2} = y_{n+1}$$

where the inequality is by part (a). Thus, $x_n \leq y_n$ for all n. Now note that, for all n,

$$x_{n+1} = \sqrt{x_n y_n} \geq \frac{x_n + y_n}{2} \geq \frac{x_n + x_n}{2} = x_n.$$

Moreover,

$$y_{n+1} = \frac{x_n + y_n}{2} \leq \frac{y_n + y_n}{2} = y_n$$

Therefore,

$$x_1 \leq x_n = \sqrt{x_{n-1} y_{n-1}} \leq \frac{x_{n-1} + y_{n-1}}{2} = y_n \leq y_1.$$

Hence, (x_n) is bounded above by y_1, and (y_n) is bounded below by x_1. Combining these facts with the above reasoning, we have that $x = \lim(x_n)$ and $y = \lim(y_n)$ exist by the MCT. Now, taking the limit of both sides of the equation $y_{n+1} = \frac{x_n + y_n}{2}$, we get

$$y = \frac{x+y}{2} \implies \frac{y}{2} = \frac{x}{2} \implies x = y,$$

as desired.

(3) 2.4.7 **(Limit Superior) Let (a_n) be a bounded sequence.**
   (a) **Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.**
   (b) **The *limit superior* of (a_n), or $\limsup a_n$, is defined by**

$$\limsup a_n = \lim y_n,$$

   where y_n is the sequence from (a). Provide a reasonable definition of $\liminf a_n$ and briefly explain why it exists for any bounded sequence.
   (c) **Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.**
   (d) **Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.**

Exercise 2.4.7. (a) For each $n \in \mathbf{N}$, set $A_n = \{a_k : k \geq n\}$ so that $y_n = \sup A_n$. Because $A_{n+1} \subseteq A_n$ it follows that $y_{n+1} \leq y_n$ and so (y_n) is decreasing. If L is a lower bound for (a_n), then for all $n \in \mathbf{N}$ it must be that $y_n \geq a_n \geq L$. Thus (y_n) is both decreasing and bounded, and it follows from the Monotone Convergence Theorem that (y_n) converges.

(b)Define the *limit inferior* of (a_n) as

$$\liminf a_n = \lim z_n,$$

where $z_n = \inf\{a_k : k \geq n\}$. The sequence (z_n) is increasing (because we are taking the greatest lower bound of a smaller set each time) and bounded above (because (a_n) is bounded.) Thus (z_n) converges by MCT.

(c) For each $n \in \mathbf{N}$ we have $y_n \geq z_n$, so by the Order Limit Theorem (Theorem 2.3.4) $\lim y_n \geq \lim z_n$. This shows $\limsup a_n \geq \liminf a_n$ for every bounded sequence.

The sequence $(a_n) = (1, 0, 1, 0, 1, 0, \cdots)$ has $\limsup a_n = 1$ and $\liminf a_n = 0$. Notice that this sequence is not convergent.

(d) First let's prove that if $\lim y_n = \lim z_n = l$, then $\lim a_n = l$ as well. Let $\epsilon > 0$. There exists an $N \in \mathbf{N}$ such that $y_n \in V_\epsilon(l)$ and $z_n \in V_\epsilon(l)$ for all $n \geq N$. Because $z_n \leq a_n \leq y_n$, it must also be the case that $a_n \in V_\epsilon(l)$ for all $n \geq N$. Therefore $\lim a_n$ exists and is equal to l.

Next, let's show that if $\lim a_n = l$, then $\lim y_n = l$. (The proof that $\lim z_n = l$ is similar.) Let $\epsilon > 0$ be arbitrary. Because $\lim a_n = l$, there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies $a_n \in V_\epsilon(l)$. This means that $l - \epsilon$ and $l + \epsilon$ are lower and upper bounds for the set $\{a_n, a_{n+1}, a_{n+2}, \cdots\}$. It follows that $l - \epsilon \leq y_n \leq l + \epsilon$ for all $n \geq N$. Keeping in mind that we already know $y = \lim y_n$ exists, we can use the Order Limit Theorem to assert that $l - \epsilon \leq y \leq l + \epsilon$, and because ϵ is arbitrary we must have $y = l$. (Theorem 1.2.6 could be referenced in this last step.)

(4) 2.4.9

Exercise 2.4.9. We will show that if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges by again exploiting a relationship between the partial sums

$$s_m = b_1 + b_2 + \cdots + b_m, \quad \text{and} \quad t_k = b_1 + 2b_2 + \cdots + 2^k b_{2^k}.$$

Because $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, its monotone sequence of partial sums (t_k) must be unbounded. To show that (s_m) is unbounded it is enough to show that for all $k \in \mathbf{N}$, there is a term s_m satisfying $s_m \geq t_k/2$. This argument is similar to the one for the forward direction, only to get the inequality to go the other way we group the terms in s_m so that the *last* (and hence smallest) term in each group is of the form b_{2^k}.

Given an arbitrary k, we focus our attention on s_{2^k} and observe that

$$\begin{aligned}
s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\
&\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\
&= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k} \\
&= \frac{1}{2}\left(2b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^k b_{2^k}\right) \\
&= b_1/2 + t_k/2.
\end{aligned}$$

Because (t_k) is unbounded, the sequence (s_m) must also be unbounded and cannot converge. Therefore, $\sum_{n=1}^{\infty} b_n$ diverges.