MATH 321 WEEK 14 UNCLAIMED PROBLEM SOLUTIONS

KENAN INCE

(1) 3.2.14

(a)

(i) We first show that E is closed if and only if $\bar{E} = E$. ($\implies$): assume E is closed. Then E contains all its limit points, hence E is the smallest set containing E and all its limit points. But this is the defining property of $\bar{E}$; therefore, $E = \bar{E}$. ($\impliedby$): Assume that the smallest closed set containing E and its limit points is E. This must mean that E contains all its limit points, thus E is closed.

(ii) We now show that E is open if and only if $E^\circ = E$. ($\implies$): assume E is open. Then, if $x \in E$, there exists $V_\epsilon(x) \subseteq E$. Therefore, $E \subseteq E^\circ$. But by definition $E^\circ \subseteq E$, hence $E^\circ = E$. ($\impliedby$): Suppose that $E^\circ = E$. Then, if $x \in E$, $x \in E^\circ$. But this means that if $x \in E$, there exists $V_\epsilon(x) \subseteq E$. This is the definition of E being open.

(b) We first show that $\bar{E}^c = (E^c)^\circ$. Note that

$$
\begin{aligned}
x \in \bar{E}^c &\iff x \notin \bar{E} \\
&\iff x \notin E \text{ and } x \text{ is not a limit point of } E \\
&\iff x \in E^c \text{ and there exists } \epsilon > 0 \text{ so that } V_\epsilon(x) \text{ contains no points of } E \\
&\iff x \in E^c \text{ and there exists } \epsilon > 0 \text{ so that } V_\epsilon(x) \subseteq E^c \\
&\iff x \in (E^c)^\circ.
\end{aligned}
$$

Now, we show that $(E^\circ)^c = \overline{E^c}$. Note that

$$
\begin{aligned}
x \in (E^\circ)^c &\iff x \notin E^\circ \\
&\iff x \notin E \text{ or } \forall \epsilon > 0, V_\epsilon(x) \nsubseteq E \\
&\iff x \in E^c \text{ or } \forall \epsilon > 0, V_\epsilon(x) \text{ contains a point of } E^c \\
&\iff x \in E^c \text{ or } x \text{ is a limit point of } E^c \\
&\iff x \in E^c \cup L, \text{ where } L \text{ is the set of limit points of } E^c \\
&\iff x \in \overline{E^c}.
\end{aligned}
$$

(2) 3.3.3

Exercise 3.3.3. Let $K \subseteq \mathbf{R}$ be closed and bounded. Since K is bounded, the Bolzano-Weierstrass Theorem guarantees that for any sequence (a_n) contained in K, we can find a convergent subsequence (a_{n_k}). Because the set is closed, the limit of this subsequence is also in K. Hence K is compact.

(3) 3.3.7

Exercise 3.3.7. (a) Fix $s \in [0,2]$. We want to find an $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$. We know that $C_1 = [0,1/3] \cup [2/3,1]$. Then we have that:

$$[0,1/3] + [0,1/3] = [0,2/3]$$

$$[0,1/3] + [2/3,1] = [2/3,4/3]$$

$$[2/3,1] + [2/3,1] = [4/3,1].$$

Hence $C_1 + C_1 = [0,2/3] \cup [2/3,4/3] \cup [4/3,2] = [0,2]$, so for any $s \in [0,2]$, we can find an $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$.

A convenient way to visualize this result in the (x,y)–plane is to shade in the four squares corresponding to the components of $C_1 \times C_1$ (see Figure 3.1) and observe that, for each $s \in [0,2]$, the line $x + y = s$ must intersect at least one of the squares. For each n we can draw a similar picture (with increasing numbers of smaller squares), and our job is to argue that the line $x + y = s$ continues to intersect at least one of the smaller squares

To argue by induction, suppose that we can find $x_n, y_n \in C_n$ such that $x_n + y_n = s$. To show that this must hold for $n+1$, let's focus attention on a square from the nth stage where $x_n + y_n = s$ holds (i.e., where $x + y = s$ intersects an nth stage square). Moving to the $n+1$th stage means removing the open middle third of this shaded region. But this results in a situation precisely like the one in Figure 3.1, implying that the line $x + y = s$ must intersect a $(n+1)$st stage square. This shows that there exist $x_{n+1}, y_{n+1} \in C_{n+1}$ where $x_{n+1} + y_{n+1} = s$.

(b) We have (x_n) and (y_n) with $x_n, y_n \in C_n$ and $x_n + y_n = s$ for all n. The sequence (x_n) doesn't converge, but (x_n) is bounded so by the Bolzano-Weierstrass Theorem there exists a convergent subsequence (x_{n_k}). Set $x = \lim x_{n_k}$. Now look at the corresponding subsequence $(y_{n_k}) = s - x_{n_k}$. Using the Algebraic Limit Theorem, we see that this subsequence converges to $y = \lim(x - x_{n_k}) = s - x$. This shows $x + y = s$. We now need to argue that $x, y \in C$.

One temptation is to say that because C is closed, $x = \lim(x_{n_k})$ must be in C. However, we don't know (and it probably isn't true) that (x_{n_k}) is in C. We *can* say that (x_{n_k}) is in C_1, and because C_1 is closed we may conclude $x \in C_1$. In fact, given any fixed n_0, we can argue that $x \in C_{n_0}$ because x_{n_k} is (with the exception of some finite number of terms) contained in C_{n_0}. This implies $x \in \bigcap_{n=1}^{\infty} C_n = C$ as desired, and a similar argument works for y.