MATH 321 MIDTERM RUBRIC FALL 2018

This exam will be due on Friday, October 19, at 4 pm. You may use your textbook, your class notes, and your homework for MATH 321, Fall 2018. You may not use other books, notes, or the Internet for this exam. You may not discuss any part of this exam with any other person besides myself until the exam has been handed in.

Please sign below to signify that you have abided by the above rules:

Signature:____

(1) [**5 points**] Negate the following: given $\epsilon > 0$ there exists $y \in \mathbb{N}$ such that, for all $x \in \mathbb{R}$ such that $xy \leq \epsilon$, $x \in A$ or $y \in B$.
   - [**1 point**] There exists $\epsilon > 0$ such that, [**2 points**] for all $y \in \mathbb{N}$, [**1 point**] there exists $x \in \mathbb{R}$ such that [**1 point**] $xy \leq \epsilon$ but [**1 point**] both $x \notin A$ and $y \notin B$.

(2) [**45 points**] For each of the following statements, say whether it is true or false. If the statement is true, prove it. If it's false, give a counterexample.

(a) [**5 points**] A decreasing sequence is always bounded.
   - False; $(a_n) = (-n)$ is not bounded below.

(b) [**7 points**] If A and B are sets so that $P(A) \subseteq P(B)$, then $A \subseteq B$. (Here $P(S)$ denotes the power set of S for any set S.)
   - True. Suppose that $P(A) \subseteq P(B)$. This means that, if $S \in P(A)$, then $S \in P(B)$. Consider the particular example $S = A$. Now $A \in P(A)$ because $A \subseteq A$. By assumption, $A \in P(B)$, which means that $A \subseteq B$.

(c) [**6 points**] If A, B, C are sets, then $(A \setminus B) \cup C = (A \cup C) \setminus (B \cup C)$.
   - False. Consider $A = \mathbb{R}$, $B = \mathbb{I}$, and $C = \mathbb{Q}$. Then $A \setminus B = \mathbb{Q}$, $(A \setminus B) \cup C = \mathbb{Q} \cup \mathbb{Q} = \mathbb{Q}$, $A \cup C = \mathbb{R}$, and $B \cup C = \mathbb{R}$, hence $(A \cup C) \setminus (B \cup C) = \mathbb{R} \setminus \mathbb{R} = \emptyset$. Since $\mathbb{Q} \neq \emptyset$, the statement is false.0

(d) [**7 points**] If A and B are sets, $A = B \iff A \setminus B = \emptyset$.
   - False. Consider $A = \mathbb{Q}$ and $B = \mathbb{R}$. Then

$$A \setminus B = \mathbb{Q} \setminus \mathbb{R} = \{x \in \mathbb{Q} : x \notin \mathbb{R}\} = \emptyset.$$

   However, $A = \mathbb{Q} \neq \mathbb{R} = B$.

(e) [**10 points**] If (a_n) is a sequence, $(a_n) \to a$, and there exists $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$, then $a \geq 0$. (In other words, part (a) of the Order Limit Theorem holds if (a_n) is "eventually nonnegative".)
   - True. Assume $(a_n) \to a$ and that there exists $N_1 \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$ and assume for contradiction that $a < 0$. Since $(a_n) \to a$, for all $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that, for all $n \geq N$, $|a_n - a| < |a|/2$. This means that, if $n \geq N_2$,

$$-\frac{|a|}{2} < a - a_n < \frac{|a|}{2}$$

   - Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, both $a_n \geq 0$ and $\frac{a_n}{2} < a < \frac{3a_n}{2}$. Therefore,

$$0 \leq \frac{a_n}{2} < a$$

   as desired.

(f) [**10 points**] The set of natural numbers $\mathbb{N}$ has the same cardinality as the set $P = \{p \in \mathbb{N} : p \text{ is prime}\}$; in other words, $\mathbb{N} \sim P$. [**Hint:** among other things, you have to prove that there are infinitely many prime numbers. Assume for contradiction that there are finitely many and hence that $P = \{p_1, p_2, \ldots, p_n\}$. Now consider the natural number $p_1 p_2 \cdots p_n + 1$.]

Lemma 1. *There are infinitely many prime numbers.*

Proof. **(Lemma)** Suppose for contradiction that there are finitely many prime numbers, so that $P = \{p_1, p_2, \ldots, p_n\}$. Consider the natural number $p := p_1 p_2 \cdots p_n + 1$. Since p has remainder 1 upon division by any of $p_1, \ldots, p_n$, p is not divisible by any prime number. But this means that p itself is prime. Since $p > p_i$ for $i = 1, \ldots, n$, $p \in P$, contradicting that our list was complete. Therefore, there are infinitely many prime numbers. $\square$

Proof. ($\mathbb{N} \sim P$) Note that $P \subseteq \mathbb{N}$ by definition. Since $\mathbb{N}$ is countable, P must be either countable (meaning $P \sim \mathbb{N}$) or finite. Since we've shown that P is not finite, it must be that $P \sim \mathbb{N}$. $\square$

(3) [**10 points**] Find the supremum and infimum of each of the following sets. No proofs are necessary.
  (a) [**5 points**] $A = \{x \in \mathbb{I} : x^2 < 2\}$
    - $\sup A = \sqrt{2}$
    - $\inf A = -\sqrt{2}$
  (b) [**5 points**] $B = \{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$
    - $\sup B = \frac{1}{2}$
    - $\inf B = 0$

(4) [**20 points**] Prove that a function $f : A \to B$ is onto if and only if, for all $y \in B$, the preimage $f^{-1}(\{y\})$ contains at least one point.

Proof. ($\implies$) Suppose that $f : A \to B$ is onto and let $y \in B$ be arbitrary. Since f is onto, there exists $x \in A$ such that $f(x) = y$. But this means that $x \in f^{-1}(\{y\})$, so that $f^{-1}(\{y\})$ contains at least one point.

($\impliedby$) Suppose that, for all $y \in B$, $f^{-1}(\{y\})$ contains at least one point. Let $b \in B$ be arbitrary. Then, by assumption, $f^{-1}(\{b\})$ contains at least one point. Choose an element of $f^{-1}(\{b\})$ and call it a. Since $f^{-1}(\{b\}) := \{a \in A : f(a) = b\}$, it must be that $f(a) = b$. Since b was arbitrary, f is onto. $\square$

(5) [**20 points**] Using the definition of convergence of a sequence, prove that

$$\lim \left(\frac{n-1}{n+1}\right) = 1.$$

  (a) [**3 points**] Let $\epsilon > 0$ be arbitrary.
  (b) [scratch work]

$$\begin{aligned} \left|\frac{n-1}{n+1} - 1\right| &< \epsilon \\ \left|\frac{n-1-(n+1)}{n+1}\right| &< \epsilon \\ \left|\frac{-2}{n+1}\right| &< \epsilon \\ \frac{2}{n+1} &< \epsilon \\ \frac{2}{\epsilon} &< n+1 \\ \frac{2}{\epsilon} - 1 &< n \end{aligned}$$

  (c) [**4 points**] Choose $N > \frac{2}{\epsilon} - 1$.
  (d) [**3 points**] Assume $n \geq N$.
  (e) [**10 points**] Then $n > \frac{2}{\epsilon} - 1$, and hence

$$\begin{aligned} \frac{2}{\epsilon} < n+1 &\implies \frac{2}{n+1} < \epsilon \\ &\implies \left|\frac{-2}{n+1}\right| < \epsilon \\ &\implies \left|\frac{n-1}{n+1} - 1\right| < \epsilon \end{aligned}$$

as desired.

(6) **(Bonus: 15 points)**: Define a function $f : \mathbb{N} \to \mathbb{Q}$ by

$$f(n) = \begin{cases} 2^n & \text{if } n \text{ is prime} \\ \frac{1}{2^n} & \text{if } n \text{ is not prime.} \end{cases}$$

Then define a sequence (a_n) by $a_n = f(n)$ for all $n \in \mathbb{N}$.

(a) **(10 points)** Is (a_n) bounded? Prove your answer.

- (a_n) is not bounded.

Proof. Assume for contradiction that (a_n) is bounded; then there exists $M \in \mathbb{R}$ so that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence, in particular, $2^p \leq M$ for all prime numbers p. But since we proved above that there are infinitely many prime numbers, there is in fact a prime number p greater than M. Hence $a_p = 2^p > 2^M > M$, contradicting that (a_n) is bounded. $\square$

(b) **(5 points)** Does (a_n) converge? Prove your answer.

Proof. (a_n) does not converge, for if (a_n) converged, it would be bounded. $\square$