Name:____

MATH 321 MIDTERM SOLUTIONS

This exam will be due on Thursday, October 12, at 4 pm. You may use your textbook, your class notes, and your homework for MATH 321, Fall 2017. You may not use other books, notes, or the Internet for this exam. You may not discuss any part of this exam with any other person besides myself until the exam has been handed in.

Please sign below to signify that you have abided by the above rules:

Signature:____

(1) **(5 points)** Negate the following: for all $a, b \in \mathbb{R}$ satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + \frac{1}{n} < b$.

(1 point) There exist $a, b \in \mathbb{R}$ **(1 point)** satisfying $a < b$ such that, **(2 points)** for all $n \in \mathbb{N}$, **(1 point)** $a + \frac{1}{n} \geq b$.

(2) **(20 points)** Prove formally: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(a) **(10 points, 2 per step)** We first prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

(i) Let $x \in A \cap (B \cup C)$.
(ii) Then $x \in A$ and ($x \in B$ or $x \in C$).
(iii) Then ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$).
(iv) Thus $x \in A \cap B$ or $x \in A \cap C$.
(v) Therefore, $x \in (A \cap B) \cup (A \cap C)$.

(b) **(10 points, 2 per step)** The other direction just does these steps backward.

(i) Let $x \in (A \cap B) \cup (A \cap C)$.
(ii) Then $x \in A \cap B$ or $x \in A \cap C$.
(iii) Then ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$).
(iv) Then $x \in A$ and ($x \in B$ or $x \in C$).
(v) Therefore, $x \in A \cap (B \cup C)$.

(3) **(18 points)** Prove that a function $f : A \to B$ is 1-to-1 if and only if, for all $y \in B$, $f^{-1}(\{y\})$ contains at most one point.

(a) **(9 points)** ($\Longrightarrow$) Assume that $f : A \to B$ is 1-to-1. Suppose for contradiction that there exists $y \in B$ so that $f^{-1}(\{y\})$ contains more than one point. Let $a_1, a_2 \in f^{-1}(\{y\})$ such that $a_1 \neq a_2$. By definition of $f^{-1}(\{y\})$, $f(a_1) = y = f(a_2)$, though $a_1 \neq a_2$. This contradicts the assumption that f is 1-to-1.

(b) **(9 points)** ($\Longleftarrow$) Suppose that $a_1, a_2 \in A$ are such that $f(a_1) = f(a_2)$. Let $y = f(a_1) = f(a_2)$. Then, by assumption, $f^{-1}(\{y\})$ contains at most one point. But since $f(a_1) = y = f(a_2)$, we have that $a_1, a_2 \in f^{-1}(\{y\})$. Therefore, it must be that $a_1 = a_2$, and hence f is 1-to-1.

(4) **(17 points)** Let

$$A = \left\{ n + \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}.$$

Compute $\sup A$ and $\inf A$, if they exist, and prove your answers.

(a) **(8 points)** We show that $\sup A$ does not exist. In fact, A is not bounded above. Let $x \in \mathbb{R}$. Let $N = \lceil x \rceil$, the smallest natural number greater than or equal to x, and consider

$$y = (N+1) + \frac{(-1)^{N+1}}{N+1} \in A.$$

We must have that $\frac{(-1)^{N+1}}{N+1} > -1$ since $(-1)^{N+1} > -(N+1)$ for all $N \in \mathbb{N}$. Therefore, $y > (N+1) - 1 = N > x$, and thus x is not an upper bound for A. Therefore, A is not bounded above and $\sup A$ does not exist.

(b) **(9 points)** We show that $\inf A = 0$.

(i) We show that 0 is a lower bound for A. Note that

$$0 = 1 - 1 = 1 + \frac{(-1)^1}{1} \in A.$$

Suppose that $n > 1$. Then $\frac{(-1)^n}{n} > -1$ since $(-1)^n > -n$ for all $n > 1$. Therefore,

$$n + \frac{(-1)^n}{n} > n - 1 > 1 - 1 = 0$$

and hence 0 is a lower bound for A.

(ii) Let α be a lower bound for A. Since $0 \in A$, we must have that $\alpha \leq 0$ by definition. Hence, 0 is the greatest lower bound for A and $\inf A = 0$.

(5) **(20 points)** Given a sequence (a_n) and a natural number $k \in \mathbb{N}$, let (b_n) be the sequence defined by $b_n = a_{n+k}$. That is, the terms in (b_n) are the same as the terms in (a_n) once the first k terms have been skipped. Prove that (a_n) converges if and only if (b_n) converges, and if they converge, show that $\lim a_n = \lim b_n$. Thus the convergence of a sequence is not affected by omitting (or changing) a finite number of terms.

(a) **(10 points)** ($\Longrightarrow$)

(i) Suppose that (a_n) converges to a limit a. Let $\epsilon > 0$ be arbitrary.

(ii) [scratch work] $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \epsilon$. If $n \geq N$, then $b_n = a_{n+k}$, so that we're further out in the sequence (a_n) than N as well.

(iii) Choose $N \in \mathbb{N}$ such that, for all $n \geq N$, $|a_n - a| < \epsilon$.

(iv) Assume $n \geq N$.

(v) Then $n + k \geq N$, so $|b_n - a| = |a_{n+k} - a| < \epsilon$.

(b) **(10 points)** ($\Longleftarrow$)

(i) Suppose that $(b_n) = (a_{n+k})$ converges to a limit a. Let $\epsilon > 0$ be arbitrary.

(ii) [scratch work] $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|b_n - a| < \epsilon$. We want to show that $|a_n - a| < \epsilon$ as well. But $a_n = b_{n-k}$, so

$$|a_n - a| = |b_{n-k} - a| < \epsilon \text{ if } n - k \geq N.$$

In other words, we need $n \geq N + k$ in order for $|a_n - a| < \epsilon$.

(iii) Choose $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|b_n - a| < \epsilon$. Then let $N = N_1 + k$.

(iv) Assume $n \geq N$.

(v) Then $n \geq N_1 + k \implies n - k \geq N_1$. Therefore,

$$|a_n - a| = |b_{n-k} - a| < \epsilon,$$

as desired.

(6) **(20 points)** In what follows, if S and T are any two sets, we write $S \sim T$ if S and T have the same cardinality and let $P(S)$ denote the power set of S. Prove that if A and B are sets such that $A \sim B$, then $P(A) \sim P(B)$. **Hint**: Assuming that $f : A \to B$ is a 1-to-1 correspondence, construct a 1-to-1 correspondence $g : P(A) \to P(B)$.

- Let $S \subseteq A$. Then define $g : P(A) \to P(B)$ via
$$g(S) = \{f(x) : x \in S\} \subseteq B.$$
- **(10 points)** We show that g is 1-to-1. Let $S_1, S_2 \in P(A)$ be such that $g(S_1) = g(S_2)$. Then
$$\{f(x) : x \in S_1\} = \{f(x) : x \in S_2\}.$$
  - Let $x \in S_1$. Then $f(x) \in \{f(x) : x \in S_2)$ by assumption, so that in particular we have $f(x) = f(y)$ for some $y \in S_2$.
  - Since f is 1-to-1 by assumption, we have that $x = y \in S_2$, thus that $x \in S_2$. Therefore, $S_1 \subseteq S_2$.
  - Similarly, let $y \in S_2$. Then, by the same argument, $y \in S_1$, hence $S_1 = S_2$.
- **(10 points)** We show that g is onto. Let $S' \in P(B)$. Then define
$$S = \{f^{-1}(y) : y \in S'\} \subseteq P(A).$$
  - Then,
$$\begin{aligned} g(S) &= \{f(f^{-1}(y)) : y \in S'\} \\ &= \{y : y \in S'\} = S' \end{aligned}$$

  and g is onto.
- Therefore, $P(A) \sim P(B)$.

(7) **Bonus (15 points).** Prove that the set S of subsequences of a sequence has the same cardinality as the real numbers.

We show that $S \sim P(\mathbb{N})$. You've already showed in a claimed problem that $P(\mathbb{N}) \sim \mathbb{R}$, so this finishes the proof.

(a) **(5 points)** Let (x_n) be any sequence and let (x_{n_i}) be any subsequence. Then (x_{n_i}) is defined by choosing a subset of the terms x_n of the original sequence. Then $T \subseteq \mathbb{N}$. Define $f : S \to P(\mathbb{N})$ by
$$f(x_{n_i}) = \{n_i : x_{n_i} \text{ is a term in the subsequence } (x_{n_i})\}.$$

(b) **(5 points)** We show f is 1-to-1. Let $S_1, S_2 \in S$. Then $S_1 = (x_{n_{\alpha(i)}})$ and $S_2 = (x_{n_{\beta(i)}})$, say. Suppose that $f(S_1) = f(S_2)$. Then
$$\{n_{\alpha(i)}\} = \{n_{\beta(i)}\},$$
so that in fact $S_1 = S_2$.

(c) **(5 points)** We show f is onto. Let $S' \subseteq \mathbb{N}$. Define a subsequence of (x_n) by $T = (x_{n_i} : n_i \in S')$. Then
$$f(T) = \{n_i : n_i \in S'\} = S'$$
as desired.