MATH 321 4.1 - DISCUSSION: EXAMPLES OF DIRICHLET AND THOMAE

- Mathematicians didn't really consider (dis)continuous functions until the 19th century, when they were developing power series and Fourier series.
- Recall that a **power series for** f **about** a lets us write a function that is infinitely differentiable as the limit of polynomials:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$
$$= \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$$

  where $c_n \in \mathbb{R}$ for all n.
- Of course, polynomials are continuous functions. Does this necessarily mean that any function f which can be written as a power series is continuous?
- In general, if $f(x) = \lim f_n(x)$ is the limit of a sequence of functions (f_n), each of which is continuous, does that mean that f is continuous? It turns out this isn't true in general!
- Any significant progress on this question requires us to be able to define continuity in a rigorous way, not just as "a function having no holes or gaps".

Definition 1. We say that f is continuous at c if

$$\lim_{x\to c} f(x) = f(c).$$

- The problem is, at present, that we only have a definition for the limit of a sequence, and it's not entirely clear what is meant by $\lim_{x\to c} f(x)$.
- Consider the following family of examples, based on an idea of the German mathematician Peter Lejeune Dirichlet:

Example 2. Let

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

[slide] It's technically impossible to draw a graph of this function, but it would look something like this:

Figure 4.1: Dirichlet's Function, $g(x)$.

Exercise 3. (reading question) Does it make any sense to attach a value to the expression $\lim_{x\to 1/2} g(x)$? One idea is to consider a sequence $(x_n) \to 1/2$ and define $\lim_{x\to 1/2} g(x)$ as the limit of the sequence $g(x_n)$:

$$\lim_{x\to a} g(x) = \lim_{n\to\infty} g(x_n) \text{ where } (x_n) \to x.$$

(1) If $x_n = \frac{1}{2} - \frac{1}{n}$ for all $n \in \mathbb{N}$, what is $\lim x_n$? What is $\lim_{n\to\infty} g(x_n)$?
(2) If $y_n = \frac{1}{2} - \frac{\sqrt{2}}{n}$ for all $n \in \mathbb{N}$, what is $\lim x_n$? What is $\lim_{n\to\infty} g(x_n)$?
(3) What do you think is the actual value of $\lim_{x\to 1/2} g(x)$?

Here's the problem: this value depends on how the sequence (x_n) is chosen! If each x_n is rational, then

$$\lim_{n\to\infty} g(x_n) = 1.$$

If x_n is irrational for each n, then

$$\lim_{n\to\infty} g(x_n) = 0!$$

- Whatever definition of functional limit we agree on, it should lead to the conclusion that $\lim_{x\to 1/2} g(x)$ does not exist.
- In any case, Dirichlet's function can't be continuous at $c = 1/2$, because its limit doesn't exist there!
- There's nothing unique about $c = 1/2$: because both $\mathbb{Q}$ and $\mathbb{I}$ are dense in the real line, for any $z \in \mathbb{R}$ we can find sequences $(x_n) \subseteq \mathbb{Q}$ and $(y_n) \subseteq \mathbb{I}$ so that

$$\lim x_n = \lim y_n = z.$$

- Because $\lim g(x_n) \neq \lim g(y_n)$, the same reasoning as above shows that g is not continuous at z. Dirichlet's function is *nowhere-continuous*.

Exercise 4. Can you adjust the definition of g to define a new function h on $\mathbb{R}$ that is discontinuous at every point **except** 0?

[slide] Let's adjust the definition of $g(x)$ to define a new function h on $\mathbb{R}$ by

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then if we take $c \neq 0$, as before we can construct sequences $(x_n) \to c$ of rationals and $(y_n) \to c$ of irrationals so that

$$\lim h(x_n) = c \text{ and } \lim h(y_n) = 0.$$

Thus, h is not continuous at every point $c \neq 0$.

However, if $c = 0$, both these limits are equal to $h(0) = 0$. In fact, it appears that no matter how we construct a sequence $(z_n) \to 0$, we'll always have $\lim h(z_n) = 0$.

- This is really what we want the definition of functional limits to be:

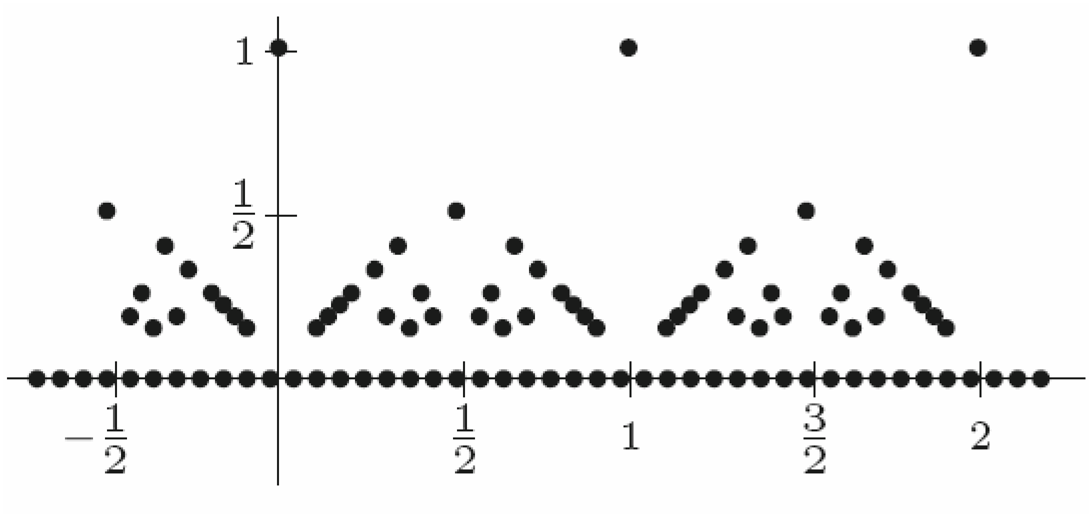
$$\lim_{x\to c} h(x) = L \text{ if } \lim_{n\to\infty} h(z_n) = L \text{ for all sequences } (z_n) \to c.$$

- For reasons we'll see later, we'll fashion the definition of functional limits in terms of neighborhoods constructed around c and L, but it'll be equivalent to this definition.

Exercise 5. [**RQ;** slide] Consider **Thomae's function** $t : \mathbb{R} \to \mathbb{R}$ defined by

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

At what points is t continuous? Discontinuous?



- If $c \in \mathbb{Q}$, then $t(c) > 0$. Because the set $\mathbb{I}$ is dense in $\mathbb{R}$, we can find a sequence (y_n) of irrationals converging to c. The result is that $\lim t(y_n) = 0 \neq t(c)$, and Thomae's function is discontinuous on $\mathbb{Q}$.
- If $c \in \mathbb{I}$, say $c = \sqrt{2} \approx 1.414213\ldots$, this argument breaks down. Consider the sequence of rational approximations for $\sqrt{2}$
$$\left(1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \ldots\right).$$
Then the sequence $t(x_n)$ begins
$$\left(1, \frac{1}{5}, \frac{1}{100}, \frac{1}{500}, \frac{1}{5000}, \frac{1}{100000}, \ldots\right) \to 0 = t(\sqrt{2})$$
and t is continuous at $\sqrt{2}$. This always happens: the closer a rational number is to a fixed irrational number, the larger its denominator (its number of decimal places) must be.
- Thus, t is continuous at every irrational and discontinuous at every rational!

Question 6. *What questions about continuity are brought up by these examples?*

- *Is there a function defined on $\mathbb{R}$ which is discontinuous precisely on $\mathbb{I}$?*
- *Can the set of discontinuities of a particular function be arbitrary?*
- *If we are given some set $A \subseteq \mathbb{R}$, is it always possible to find a function that is continuous only on the set A^c?*
- *What conclusions can we draw about the discontinuities of functions that don't have such erratic oscillations (e.g. monotone functions)?*
- *We'll answer each of these questions in this chapter.*