Power system state estimation: a survey

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Recent developments in the solution methods for state estimation are reviewed. Concepts of decoupling, ill-conditioning and robustness in state estimation are discussed. Derivations of decoupled estimators, stable estimators and robust estimators are reviewed. Future directions for research are suggested.

Keywords: state estimation, weighted least square estimation, security monitoring

Prologue

'Shall we go that way?'

'But there is no trail over there.'

'I don't know. We don't have to follow the trail.'

Fred Schweppe grinned and started to walk on, loudly crackling the autumn leaves whenever he stepped. Margaret and I had been invited to spend the weekend in his cabin in Maine. We finally reached a place where we had a wonderful view of the whole mountain. One day, I remember we got on a historical log train. And he was like a kid showing off, proudly explaining to us the intricate working of the pipes and the valves.

Fred appreciated and respected tradition. And yet, he dared to take a different approach to expand the horizons. He was the man who led as to state estimates and spot pricing—totally new planes of power system engineering. We shall forever be grateful.

I. Introduction

Fred Schweppe introduced state estimation to power systems in 1968[1-3]. He defined the state estimator as 'a data processing algorithm for converting redundant meter readings and other available information into an estimate of the state of an electric power system'. Today, state estimation is an essential part in almost every energy management system throughout the world[4]. It is a very basic tool in ensuring secure operation of a power system[5].

Real-time measurements are collected in power systems through the SCADA (supervisory control and data acquisition) system[6]. Typical data include real and reactive power line flows, real and reactive power injections at the buses, and bus voltages. These telemetered data contain errors. The errors arise from (i) inaccurate transducer calibration, (ii) the effect of analogue-to-digital conversion, (iii) noise in communication channels, (iv) unbalanced phases, etc. State estimation is a process to clean up the erroneous data. The reason why this can be done is that the physical quantities (line flows, voltages, etc.) and hence the measurements in a connected network are related. If there is *redundancy* in the measurement set (more measurements than necessary to determine the condition of the network), a systematic cross-checking should be able to correct the errors. *State estimation* is indeed a systematic procedure—a mathematical procedure—to process the set of real-time measurements to come up with the best estimate of the current state of the system. The result of state estimation provides the real-time database for other applications, such as security assessment and control, economic dispatch, etc.

Several excellent review papers on state estimation have been published in the past few years[7-9]. Some practical experience has been reported[10,11]. The objective of this paper is twofold: first, to present a tutorial on state estimation and its solution methods, and secondly, to survey recent developments in state estimation methodologies. The focus of this paper is on the formulation and derivation of methods. The real-time implementation of state estimators and practical experiences are reviewed in a separate paper by Dy Liacco in this issue[12].

The paper is organized as follows. Section II presents the original weighted least square formulation of state estimation. Recent advances in real and reactive power decoupling of state estimation are discussed in Section III. The problem of ill-conditioning in state estimation that surfaces in large systems is discussed in Section IV. Several methods for alleviating the problem are surveyed in Section V. The effect of bad data on state estimation is discussed in Section VI. Several robust estimators which are less sensitive to bad data are surveyed in Section VII. A short note on historical developments is included in Section VIII. Suggestions for future research in state estimation are given in Section IX.

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II. Basic WLS estimator

In layman's terms, the *state* of a power system refers to its operating condition relative to overload, overvoltage, etc., i.e. the amount of power flowing through the lines, transformers, substations, etc. and their voltage readings. Mathematically, all these quantities can be computed once the set of bus voltage magnitudes and phase angles is known. Therefore, technically, the *state* of a power system is defined as the set of bus voltage magnitudes and phase angles.

The mathematical model of state estimation is based on the mathematical relations between the measurements and the state variables. Let **z** denote the set of telemetered measurements, **x** the vector of state variables (bus voltage magnitudes and phase angles), **h** the equations relating the measurement to the state variable and υ the measurement error vector; we have

$$\mathbf{z}=\mathbf{h}(\mathbf{x})+\upsilon \tag{1}$$

The errors $\{\upsilon_1, \upsilon_2, \dots, \upsilon_m\}$ are assumed to be independent random variables with Gaussian distribution whose mean is zero. The variance σ_i^2 of the measurement error υ_i, provides an indication of the certainty about that particular measurement. A large variance indicates that the corresponding measurement is not very accurate. Ways to assign values to σ_i^2 in practice are reported in Reference 11. Let us denote the measurement error covariance matrix by **R**, i.e.

$$\mathbf{R}:=E\{\upsilon\upsilon^T\}=\begin{bmatrix}\sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_m^2\end{bmatrix} \tag{2}$$

In the measurement–state relationship (1), **x** is the unknown true state—a deterministic quantity. Since the errors v are random variables, the measurements **z** are random variables as well. Moreover, **z** has a Gaussian distribution with mean $\mathbf{h}(\mathbf{x})$ and covariance **R**. The probability density function of **z** can be written as

$$f(\mathbf{z})=\frac{1}{(2\pi)^{m/2}(\det\mathbf{R})^{1/2}} \times \exp\{-\tfrac{1}{2}[\mathbf{z}-\mathbf{h}(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x})]\} \tag{3}$$

For the state estimation problem, a set of measurements **z** has been observed based on the fact that we want to estimate the state **x**. It is reasonable to select an **x** which makes the observed **z** most likely to have been observed, i.e. the **x** which maximizes the probability density function (3). An estimate $\hat{\mathbf{x}}$ thus obtained is called the *maximum likelihood estimate*.

It is clear from the property of the exponential function that maximizing $f(\mathbf{z})$ in (3) is equivalent to minimizing the quadratic term in the exponent:

$$\min_{\mathbf{x}} J(\mathbf{x})=\frac{1}{2}[\mathbf{z}-\mathbf{h}(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x})] \tag{4}$$

$$=\sum\frac{1}{2}\frac{\{\mathbf{z}_i-\mathbf{h}_i(\mathbf{x})\}^2}{\sigma_i^2} \tag{5}$$

Since the maximum likelihood estimate in this case minimizes the error squared weighted by the measurement accuracy, it is commonly called the *weighed least square* (WLS) estimate.

The solution of the WLS problem (4) gives the estimated state $\hat{\mathbf{x}}$, which must satisfy the following optimality condition:

$$\frac{\partial J}{\partial \mathbf{x}}=0 \Rightarrow \mathbf{g}(\hat{\mathbf{x}})=\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\hat{\mathbf{x}})]=\mathbf{0} \tag{6}$$

where

$$\mathbf{H}(\mathbf{x})=\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \tag{7}$$

is the Jacobian matrix of the measurement function

The solution $\hat{\mathbf{x}}$ of the non-linear equation (6) may be obtained by an iterative method in which a linear equation is solved at each iteration to compute the correction $\Delta\mathbf{x}^k=\mathbf{x}^{k+1}-\mathbf{x}^k$:

$$[\mathbf{A}(\mathbf{x}^k)]\Delta\mathbf{x}^k=-\mathbf{H}^T(\mathbf{x}^k)\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x}^k)] \tag{8}$$

where $[\mathbf{A}(\mathbf{x}^k)]$ is a non-singular matrix which depends on the method used. It should be pointed out that as long as the sequence of points $\{\mathbf{x}^k\}$ generated by the iterative method converges, it will converge to the solution of (6) provided $[\mathbf{A}(\mathbf{x})]$ is non-singular[13]. One method which guarangees local convergence is Newton's method, for which the $[\mathbf{A}(\mathbf{x})]$ matrix is given by

$$[\mathbf{A}(\mathbf{x})]=\frac{\partial \mathbf{g}}{\partial \mathbf{x}} \tag{9}$$

The ijth element of $\partial\mathbf{g}/\partial\mathbf{x}$ is[7]:

$$\frac{\partial g_i}{\partial x_j}=\left[\frac{\partial^2\mathbf{h}(\mathbf{x})}{\partial x_i\,\partial x_j}\right]^T\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x})]-\left(\frac{\partial \mathbf{h}}{\partial x_i}\right)^T\mathbf{R}^{-1}\left(\frac{\partial \mathbf{h}}{\partial x_j}\right) \tag{10}$$

The basic WLS method ignores the second-derivative terms and chooses instead

$$[\mathbf{A}(\mathbf{x})]=\mathbf{H}^T(\mathbf{x})\mathbf{R}^{-1}\mathbf{H}(\mathbf{x}) \tag{11}$$

Hence equation (8) can be rewritten as

$$\mathbf{G}(\mathbf{x}^k)\Delta\mathbf{x}^k=\mathbf{H}^T(\mathbf{x}^k)\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x}^k)] \tag{12}$$

where

$$\mathbf{G}(\mathbf{x}^k):=\mathbf{H}^T(\mathbf{x}^k)\mathbf{R}^{-1}\mathbf{H}(\mathbf{x}^k) \tag{13}$$

is called the *gain matrix*. Equation (12) is called the *normal equation* of the WLS problem. It should be noted that if the set of measurements is sufficient and well distributed, then the measurement Jacobian matrix $\mathbf{H}(\mathbf{x})$ will have full rank and hence $\mathbf{G}(\mathbf{x})$ is non-singular. In such a case the network is said to be observable. Observability is reviewed in the paper by Clements in this issue[14]. Recall that if $\mathbf{G}(\mathbf{x})$ is non-singular and the iterative method generates a convergent sequence, then what it converges to is the solution of (6).

The rows of the measurement Jacobian matrix correspond to measurements. It can be shown that they have mostly zeros. Let $\mathbf{h}_k(\mathbf{x})$ be the kth row of $\mathbf{H}(\mathbf{x})$. The gain matrix can be expressed as

$$\mathbf{G}(\mathbf{x}) = \sum_{k=1}^{m} \frac{\mathbf{h}_k^T(\mathbf{x})\mathbf{h}_k(\mathbf{x})}{\sigma_k^2} \tag{14}$$

Therefore, if $\mathbf{h}_k(\mathbf{x})$ has, say, three non-zero elements, then $\mathbf{h}_k^T(\mathbf{x})\mathbf{h}_k(\mathbf{x})/\sigma_k^2$ introduce 9 non-zero elements in $\mathbf{G}(\mathbf{x})$. Indeed, the gain matrix can be interpreted as the bus admittance matrix of an augmented graph[15]. The gain matrix is a symmetric, positive definite matrix which is also relatively sparse. Sparse matrix triangular factorization with optimal ordering[16], i.e.

$$\mathbf{G} = \mathbf{U}^T\mathbf{U} \tag{15}$$

is used to solve equation (12).

Another way of interpreting the normal equation (12) is given below. First notice that equation (12) is the solution of the *linear* WLS problem (the superscript k is dropped for convenience)

$$\min_{\Delta\mathbf{x}} J(\Delta\mathbf{x}) := [\Delta\mathbf{z} - \mathbf{H}(\mathbf{x})]^T\mathbf{T}\mathbf{R}^{-1}[\Delta\mathbf{z} - \mathbf{H}(\mathbf{x})] \tag{16}$$

where $\Delta\mathbf{x} := \mathbf{z} - \mathbf{h}(\mathbf{x})$. Equivalently, it is the solution of the least square problem obtained by absorbing the weights into the measurements and the Jacobian:

$$\min_{\Delta\mathbf{x}} J(\Delta\mathbf{x}) := [\Delta\bar{\mathbf{z}} - \bar{\mathbf{H}}\Delta\mathbf{x}]^T\mathbf{R}^{-1}[\Delta\bar{\mathbf{z}} - \bar{\mathbf{H}}\Delta\mathbf{x}] \tag{17}$$

where $\bar{\mathbf{H}} := \mathbf{R}^{-12}\mathbf{H}$ and $\Delta\bar{\mathbf{z}} := \mathbf{R}^{-1/2}\Delta\mathbf{z}$. The least square problem is a desired (minimum norm) solution to the overdetermined system of equations:

$$\bar{\mathbf{H}}\Delta\mathbf{x} \cong \Delta\bar{\mathbf{z}} \tag{18}$$

where $\cong$ indicates that there are more equations than unknowns. Equation (12) may be expressed as the solution of equation (18) using the notion of the pseudo-inverse $\bar{\mathbf{H}}^+ := (\bar{\mathbf{H}}^T\bar{\mathbf{H}})^{-1}\bar{\mathbf{H}}^T$ of $\bar{\mathbf{H}}$:

$$\Delta\mathbf{x} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\Delta\bar{\mathbf{z}} := (\mathbf{H})^+\Delta\bar{\mathbf{z}} \tag{19}$$

III. Decoupled estimators

The state variables consist of two sets: (i) bus voltage magnitudes and (ii) bus voltage phase angles. The measurements can also be grouped into two categories: (i) real power flows and injections and (ii) reactive power flows and injections, as well as voltages. It is well known that the real powers predominantly affect the phase angles while the reactive powers predominantly affect the voltage magnitudes. Several decoupled versions of state estimations have been proposed in the literature[17-21] based on this general principle and experimentation. We shall present a more analytical derivation of decoupled state estimation recently introduced in Reference 22.

Let $\boldsymbol{\theta}$ denote the set of bus voltage angles, $\mathbf{V}$ the set of bus voltage magnitudes, subscript P associate the quantity with real power measurements, subscript Q with reactive power measurements, subscript $\boldsymbol{\theta}$ with angle variables and subscript V with magnitude variables. Hence the solution $\Delta\mathbf{x} = (\Delta\boldsymbol{\theta}, \Delta\mathbf{V})$ at each iteration is the solution of the overdetermined system of equations (18), which may be rewritten as

$$\begin{bmatrix} \bar{\mathbf{H}}_{P\theta} & \bar{\mathbf{H}}_{PV} \\ \bar{\mathbf{H}}_{Q\theta} & \bar{\mathbf{H}}_{QV} \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\mathbf{V} \end{bmatrix} \cong \begin{bmatrix} \Delta\bar{\mathbf{z}}_P \\ \Delta\bar{\mathbf{z}}_Q \end{bmatrix} \tag{20}$$

By treating coupling terms as measurement terms, equation (20) can be transformed into

$$\begin{aligned} \tilde{\mathbf{H}}_{P\theta}\Delta\boldsymbol{\theta} &\cong \Delta\tilde{\mathbf{z}}_P \\ \tilde{\mathbf{H}}_{QV}\Delta\mathbf{V} &\cong \Delta\tilde{\mathbf{z}}_Q \end{aligned} \tag{21}$$

where

$$\tilde{\mathbf{H}}_{QV} := \bar{\mathbf{H}}_{QV} - \bar{\mathbf{H}}_{Q\theta}\bar{\mathbf{H}}_{PQ}^{+}\bar{\mathbf{H}}_{PV} \tag{22}$$

$$\Delta\tilde{\mathbf{z}}_Q := \Delta\bar{\mathbf{z}}_Q - \bar{\mathbf{H}}_{Q\theta}\bar{\mathbf{H}}_{P\theta}^{+}\Delta\bar{\mathbf{z}}_P \tag{23}$$

$$\Delta\tilde{\mathbf{z}}_P := \Delta\bar{\mathbf{z}}_P - \bar{\mathbf{H}}_{PV}\bar{\mathbf{H}}_{QV}^{+}\Delta\bar{\mathbf{z}}_Q \tag{24}$$

Equation (21) is equivalent to equation (20). It can be shown[22] that the following decoupled algorithm solves equation (21) exactly (within the first approximation of the non-linear equation $\mathbf{h}$).

(i) Perform $P\theta$ iteration:

$$\Delta\theta^k = \bar{\mathbf{H}}_{P\theta}^{+}\Delta\tilde{\mathbf{z}}_P(\boldsymbol{\theta}^k, \mathbf{V}^k)$$

$$\boldsymbol{\theta}^k \leftarrow \boldsymbol{\theta}^k + \Delta\boldsymbol{\theta}^k$$

(ii) Perform QV iteration:

$$\Delta\mathbf{V}^k = \tilde{\mathbf{H}}_{QV}^{+}\Delta\bar{\mathbf{z}}_Q(\boldsymbol{\theta}^k, \mathbf{V}^k)$$

$$\mathbf{V}^k \leftarrow \mathbf{V}^k + \Delta\mathbf{V}^k$$

Remarks

1. The elements of the Jacobian matrix $\bar{\mathbf{H}}$ are functions of $(\boldsymbol{\theta}, \mathbf{V})$, hence they vary from iteration to iteration. However, it turns out that if the real and reactive power measurements are normalized by the corresponding voltage, i.e. replacing the flow and injection measurements P_{km}, Q_{km}, P_k and Q_k by P_{km}/V_k, Q_{km}/V_k, P_k/V_k and Q_k/V_k, then $\bar{\mathbf{H}}$ can be approximated very well by a constant matrix which is $\bar{\mathbf{H}}$ evaluated at a flat voltage profile ($V = 1.0, \theta = 0$). This is indeed how it is commonly done in the decoupled state estimator. Consequently, two constant matrices are used throughout the iterations, thus reducing the total amount of computation significantly.

2. It is shown[22] to be true in two cases and is conjectured to be a good approximation in general that the matrix $\tilde{\mathbf{H}}_{QV}$ in the QV iteration can be obtained directly from the network topology and element impedances; in the same way as obtaining the Jacobian submatrix $\bar{\mathbf{H}}_{QV}$ except that branch susceptances b_{km} are replaced by $-1/x_{km}$.

3. There is a dual version of the decoupled state estimator in which a modified $\tilde{\mathbf{H}}_{P\theta}$ is used in the $P\theta$ iteration (obtained in the same way as $\bar{\mathbf{H}}_{P\theta}$ except that branch susceptances b_{km} are replaced by $-1/x_{km}$) and

the original $\tilde{\mathbf{H}}_{QV}$ is used in the PV iteration. This version has in fact been an established approach for the decoupled state estimator for some time[20-21]. However, it is shown in Reference 22 that the validity of the approximation regarding $\tilde{\mathbf{H}}_{P\theta}$ is more stringent and experimental results favour the primal version presented here.

IV. Ill-conditioning

In addition to the telemetered measurements, there are two other types of 'measurements' that may be included in $\mathbf{z}$. *Pseudo-measurements* are manufactured data, such as generator output or substation load demand, that are based on historical data or the dispatcher's objective guesses. Pseudo-measurements are introduced in order to make an unobservable network observable. *Virtual measurements* are the kind of information that does not require metering, e.g. *zero injection* at a switching station. One way of incorporating pseudo-measurements and virtual measurements is to treat the former as less accurate measurements and the latter as more accurate measurements. The assignment of very large and very small weighting factors has caused convergence problems in the implementation of state estimation in large systems. This may be explained by the notion of numerical ill-conditioning.

The state estimation solution method generates a sequence of points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_3, \ldots$. At each iteration a subproblem is solved, i.e. the next point $\mathbf{x}_{k+1}$ is generated by using the current point $\mathbf{x}_k$ and the parameter value $\mathbf{p}$ (impedances, etc.). This procedure can be represented by a function $\mathbf{x}_1 = \varphi(\mathbf{x}_0, \mathbf{p})$, $\mathbf{x}_2 = \varphi(\mathbf{x}_1, \mathbf{p}), \ldots$. The iterative process converges if $\mathbf{x}_n$ approaches the solution $\mathbf{x}$. Because of finite precision representation a number $\mathbf{x}_1$ is actually stored as an approximation $\mathbf{x}_1^*$, the difference being the round-off error. The effect of round-off error is that $\mathbf{x}_2^* = \varphi(\mathbf{x}_1^*, \mathbf{p})$ or $\mathbf{x}_2^{**} = \varphi(\mathbf{x}_1^*, \mathbf{p}^*)$ is computed rather than $\mathbf{x}_2 = \varphi(\mathbf{x}_1, \mathbf{p})$. An algorithm is ill-conditioned if for a given $(\mathbf{x}_1, \mathbf{p})$ the difference between $\varphi(\mathbf{x}_1, \mathbf{p})$ and $\varphi(\mathbf{x}_1^*, \mathbf{p})$ or between $\varphi(\mathbf{x}_1, \mathbf{p})$ and $\varphi(\mathbf{x}_1, \mathbf{p}^*)$ is large for $\mathbf{x}_1$ and $\mathbf{x}_1^*$ very close and $\mathbf{p}$ and $\mathbf{p}^*$ very close. Therefore, for a normally fast convergent solution method, owing to ill-conditioning, the effect of round-off error may lead it to slow convergence or failure to converge at all. One way of measuring the degree of ill-conditioning in solving the linear equations $\mathbf{Ax} = \mathbf{b}$ is by the *condition number* of the coefficient matrix $\mathbf{A}$. It is well known that (Reference 23, pp 184–199) the relative error in $\mathbf{A}$ or $\mathbf{b}$ may be magnified by as much as the condition number of $\mathbf{A}$ in passing to the solution. The condition number of the gain matrix $\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}$ in equation (12) is the square of the condition number of the Jacobian matrix $\mathbf{H}$ (Reference 23, pp 233–224). Therefore the basic WLS formulation is numerically not very well conditioned.

In addition to the disparity in weighting factors, other potential sources of ill-conditioning in the WLS state estimator have been identified. One of them is the presence of a large number of injection measurements in the system[24] and the other is the existence of the connection of a long transmission line (large impedance) with a short line (small impedence)[25]. Several methods have been suggested to alleviate the ill-conditioning problem. They will be reviewed in the next section.

V. Stable estimators

There are two basic ideas in various proposed methods for alleviating the numerical ill-conditioning problem in state estimation. The first is to avoid the formation of the gain matrix $\mathbf{G} = \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}$ and the second is to treat virtual measurements as equality constraints. Four methods will be reviewed here, namely (1) the orthogonal transformation method[26-29], (2) the hybrid method[24], (3) normal equations with equality constraints[30] and (4) Hachtel's method[31,32].

V.1 *Orthogonal transformation method*

Recall that at each iteration the state estimation problem is the solution of the least square problem (17):

$$\min_{\Delta\mathbf{x}} J(\Delta\mathbf{x}) = [\Delta\bar{\mathbf{z}} - \bar{\mathbf{H}}\Delta\mathbf{z}]^T[\Delta\bar{\mathbf{z}} - \bar{\mathbf{H}}\Delta\mathbf{x}] \tag{25}$$

Let $\mathbf{Q}$ be an orthogonal matrix ($\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$). Then we have

$$\begin{aligned} J(\Delta\mathbf{x}) &= [\Delta\bar{\mathbf{z}} - \bar{\mathbf{H}}\Delta\mathbf{x}]^T\mathbf{Q}^T\mathbf{Q}[\Delta\bar{\mathbf{z}} - \bar{\mathbf{H}}\Delta\mathbf{x}] \\ &= \|\mathbf{Q}\Delta\bar{\mathbf{z}} - \mathbf{Q}\bar{\mathbf{H}}\Delta\mathbf{x}\|^2 \end{aligned} \tag{26}$$

Now if $\mathbf{Q}$ is chosen such that

$$\boxed{\mathbf{Q}}\,\boxed{\bar{\mathbf{H}}} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \boxed{\mathbf{Q}}\,\boxed{\Delta\bar{\mathbf{z}}} = \begin{bmatrix} \Delta\mathbf{y}_1 \\ \Delta\mathbf{y}_2 \end{bmatrix} \tag{27}$$

where $\mathbf{R}_1$ is an upper triangular matrix, then

$$J(\Delta\mathbf{x}) = \|\Delta\mathbf{y}_1 - \mathbf{R}_1\Delta\mathbf{x}\|^2 + \|\Delta\mathbf{y}_2\|^2 \tag{28}$$

The minimum of $J(\Delta\mathbf{x})$ occurs at

$$\mathbf{R}_1\Delta\mathbf{x} = \Delta\mathbf{y}_1 \tag{29}$$

The orthogonal transformation method starts with the orthogonal transformation (27) of $\bar{\mathbf{H}}$ and $\Delta\bar{\mathbf{z}}$ and then solves equation (29) by backward substitutions. Fast Givens is an efficient method for orthogonal transformation[33].

V.2 *Hybrid method*

The hybrid method is based on the fact that

$$\mathbf{G} = \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} = \bar{\mathbf{H}}^T\bar{\mathbf{H}} = (\mathbf{Q}\bar{\mathbf{H}})^T(\mathbf{Q}\mathbf{H}) = \mathbf{R}_1^T\mathbf{R}_1.$$

Therefore the normal equations may be solved by substitutions using $\mathbf{H}_U$:

$$\mathbf{R}_1^T\mathbf{R}_1\Delta\mathbf{x} = \mathbf{H}^T\mathbf{R}^{-1}\Delta\mathbf{z} \tag{30}$$

Two major steps are involved in the hybrid method. The first step is to perform the orthogonal transformation $\mathbf{Q}$ on $\bar{\mathbf{H}}$ as in (27) and the second step is to solve the normal equations by forward and backward substitutions (30).

V.3 *Normal equations with equality constraints*

The virtual measurements (zero injections) may be separated from the telemetered measurements and treated as equality constraints. Now the measurements $\mathbf{z}$ include only telemetered (and pseudo, if any) measurements. The problem is to find an estimate of the state vector **x** which minimizes the weighted least square $J(\mathbf{x})=[\mathbf{z}-\mathbf{h}(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x})]$ while the equality constraints $\mathbf{c}(\mathbf{x})=\mathbf{0}$ are satisfied.

The method of Lagrange multipliers may be applied to solve the constrained minimization problem. The Lagrangian of the problem is defined as

$$\mathbf{L}(\mathbf{x},\boldsymbol{\lambda})=\tfrac{1}{2}[\mathbf{z}-\mathbf{h}(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x})]-\boldsymbol{\lambda}^T\mathbf{c}(\mathbf{x}) \tag{31}$$

At the optimal solution the derivatives of the Lagrangian with respect to **x** and $\boldsymbol{\lambda}$ must vanish, i.e.

$$\mathbf{H}^T(\mathbf{x})\mathbf{R}^{-1}[\mathbf{z}-\mathbf{h}(\mathbf{x})]+\mathbf{C}^T(\mathbf{x})\boldsymbol{\lambda}=0 \tag{32}$$

$$\mathbf{c}(\mathbf{x})=0$$

The optimal solution may be obtained by an iterative solution method for the non-linear equations (32). At each iteration the following linearized equation is solved.

$$\begin{bmatrix}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}(\mathbf{x}) & \mathbf{C}^T(\mathbf{x})\\ \mathbf{C}(\mathbf{x}) & 0\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}\\ \boldsymbol{\lambda}\end{bmatrix}=\begin{bmatrix}\mathbf{H}^T(\mathbf{x})\mathbf{R}^{-1}\Delta\mathbf{z}\\ \Delta\mathbf{c}\end{bmatrix} \tag{33}$$

where $\Delta\mathbf{z}:=\mathbf{z}-\mathbf{h}(\mathbf{x})$ and $\Delta\mathbf{c}=-\mathbf{C}(\mathbf{x})$

The coefficient matrix in equation (33) is no longer positive definite. Care must be exercised in the triangular factorization of the matrix[34].

V.4 *Hachtel's augmented matrix method*

Hachtel's augmented matrix method separates the (incremental) residuals as independent variables and they are solved simultaneously with $\Delta\mathbf{x}$ (and $\boldsymbol{\lambda}$). We shall present the derivation for the case of state estimation with equality constraints. The case without equality constraints is similar and easier.

Let us define

$$\Delta\mathbf{r}:=\Delta\mathbf{z}-\mathbf{H}(\mathbf{x})\Delta\mathbf{x} \tag{34}$$

The first set of equations in (33) becomes

$$\mathbf{H}^T(\mathbf{x})\Delta\mathbf{r}+\boldsymbol{\lambda}^T\mathbf{C}(\mathbf{x})=0 \tag{35}$$

Combining (34), (35) and the second set of equations in (33), we have

$$\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{C}(\mathbf{x})\\ \mathbf{0} & \mathbf{R}^{-1} & \mathbf{H}(\mathbf{x})\\ \mathbf{C}^T(\mathbf{x}) & \mathbf{H}^T(\mathbf{x}) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\lambda}\\ \mathbf{R}^{-1}\Delta\mathbf{r}\\ \Delta\mathbf{x}\end{bmatrix}=\begin{bmatrix}\Delta\mathbf{c}\\ \Delta\mathbf{z}\\ \mathbf{0}\end{bmatrix} \tag{36}$$

Hachtel's method solves equation (36) at each iteration. Note that the gain matrix is never formed. The coefficient matrix in this case is again non-definite.

V.5 *Comparison of methods*

A comparison of the state estimation methods presented above has been conducted[35]. Both theoretical comparison and extensive numerical testing on various size systems are carried out. The comparison is made in terms of:

- numerical stability,
- computational efficiency,
- implementation complexity.

The results indicate that both Hachtel's method and the hybrid method are a good compromise between numerical stability and computational efficiency, with reasonable complexity in implementation.

VI. Bad data

The mathematical model of WLS state estimation, as shown in Section II, assumes *a priori* knowledge of the probability distributions of measurement errors (Gaussian with zero mean and known variance). Occasionally, large errors or bad data occur owing to meter failure, wrong polarity, incorrect calibration, etc. The presence of bad data invalidates the Gaussian assumption. There are two schools of thought on handling bad data in state estimation. The first approach uses the result of WLS state estimation to detect, identify and remove bad data. The second approach is to replace the WLS estimator by a more robust estimator which is less sensitive to small departures from the assumptions.

Intuitively, if bad data are present, the *residual* $r_i=z_i-h_i(\hat{\mathbf{x}})$, which is the difference between the measured value and the calculated value using the estimated state, will be large, as will the WLS error $J(\hat{\mathbf{x}})$. This suggests a way of detecting bad data. It can be shown using linear approximation that under the hypothesis that there are no bad data the residual vector **r** has a Gaussian distribution with zero mean and covariance $\mathbf{R}-\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T$ while the WLS error $J(\hat{\mathbf{x}})$ has a chi-squared distribution[36]. A statistical method of hypothesis testing can be applied to determine whether the value of **r** or $J(\mathbf{x})$ indicates that the hypothesis is true. Experience has shown that the use of the normalized residual is fairly reliable for bad data detection.

It has been shown that for single or multiple non-interacting bad data the largest normalized residual corresponds to a bad measurement. This fact has been successfully used as a basis to identify bad data by successively removing the measurement with the largest normalized residual when bad data are detected. For multiple interacting bad data, three approaches have been proposed to identify and remove them. These are based on:

- combinatorial optimization[37],
- hypothesis testing identification[38],
- geometric approach[39].

A comparative study of bad data identification has been conducted[40]. More detailed discussion on bad data processing can be found in the article by Koglin *et al.* in this issue[41].

The approach by robust estimators will be reviewed in the next section.

VII. Robust estimators

The WLS estimator minimizes the weighted sum of the *square* of the residuals. Therefore the solution is sensitive to bad data (or outliers with large residual), which violates the Gaussian assumption of measurement errors. An estimator, such as WLS, which is vulnerable to outliers is not robust. Robust estimators have been proposed for power system state estimation. A more comprehensive review of robust estimation theory and its application to state estimation is presented in Reference 42. We categorize robust estimators into three groups.

VII.1 *Non-quadratic estimators*

For this class of estimators the estimated state $\hat{\mathbf{x}}$ is found by minimizing the objective function $J(\mathbf{x})$ which is given by

$$J(\mathbf{x})=\sum \rho\left(\frac{z_i-h_i(\mathbf{x})}{\sigma_i}\right)=\sum \rho(r_{w_i}) \tag{37}$$

where r_{w_i} is the ith component of the weighted residual. The function ρ is quadratic when the weighted residual is small and changes to a less dramatic function, such as linear or square-root, when the weighted residual is large. It is expected that such a selection of ρ will suppress the influence of bad data. Several functions have been proposed:

- quadratic straight,
- multiple-segment,
- quadratic square-root,
- quadratic constant.

An analysis of non-quadratic estimators can be found in References 36 and 42.

VII.2 *Weighted least absolute value (WLAV) estimator*

The WLAV estimator minimizes

$$J(\mathbf{x})=\sum_1^m \frac{1}{\sigma_i}|z_i-h_i(\mathbf{x})| \tag{38}$$

The solution is obtained by interpolating only n (the number of state variables) among all m measurements. Bad data are thus rejected in the process of estimating the state of the system. Linear programming is used in the solution process. The main problem with this approach is the computation time and memory storage requirements. Several methods to improve the solution efficiency have been proposed[43–46].

VII.3 *Least median of squares (LMS) estimator*

The LMS estimator minimizes the median of the squared residuals. The procedure has been applied to power systems. It selects measurements of size n for which the network is observable to solve for the state variables. The LMS estimator selects the state variables which give rise to the smallest value of the median of the square residuals. The LMS estimator fits only the majority of the data. It can withstand a fraction of outliers. Many advantageous statistical properties of the LMS estimator are discussed in Reference 42. The computational burden associated with the combinatorial nature of the problem is evident. A re-sampling technique which would guarantee a high probability of including interesting samples within a small sample is proposed in Reference 47. More development work seems to be needed to make the method practical.

VIII. Historical note

Schweppe introduced state estimation to power systems for security monitoring of transmission networks in 1968[1–3]. The concept was immediately embraced by the industry, although his proposed solution method (basic WLS) was not adopted. Two alternative methods were developed. Dopazo *et al.*[48,49] used only the line flow measurements and developed a computationally simpler method after going through a transformation. There were questionable approximations in the derivation of the method. Nevertheless, the method was implemented in the AEP system. Larson *et al.*[50,51] suggested processing the redundant measurements sequentially and applied the Kalman filter technique for the estimation. After a few years, both methods were abandoned[21] and a modified version of Schweppe's original method was generally accepted.

Among the major factors that made WLS estimators attractive were (i) the use of sparse matrix techniques on the gain matrix and (ii) the computational savings with the fast decoupled version. After several earlier attempts[17–19], Garcia *et al.*[20] and Allemong *et al.*[21] independently conducted extensive testing to pronounce a preferred version of fast decoupled state estimation. An analytical justification of fast decoupled state estimation appeared only recently. In that same paper, Monticelli and Garcia[22] also showed that a slightly modified version would do better. Similar fast decoupled load flow had been proposed and tested earlier.

Among those who pointed out the numerical ill-conditioning problem in state estimation, Aschmoneit *et al.*[30] were the first to propose the inclusion of zero injections as equality constraints in the state estimation formulation. Gjelsvik *et al.*[31] proposed the use of Hachtel's augmented matrix method. In the meantime, Clements and co-workers[24] and Quintana and co-workers[26–28] proposed the use of other, more stable numerical solution methods. These methods were adopted in various energy management systems.

Merrill and Schweppe back in 1971 suggested the use of non-quadratic estimators to suppress bad data[52]. Handschin *et al.*[36] conducted an extensive study on various non-quadratic estimators, as well as methods for bad data detection and identification. Irving *et al.*[43] were among the first proponents of the least absolute value estimators using linear programming. Recently, Mili *et al.*[42] rekindled the interest in robust estimators.

IX. Concluding remarks

In 1968, Fred Schweppe had just moved from Lincoln Laboratory to the Electrical Engineering Department at MIT and switched his research to power systems. State estimation was his first contribution to power engineering. Twenty years later, state estimation has become one of the standard tools in power system operation and a part of the regular power engineering curriculum in universities.

Many individuals after Schweppe have contributed to the success of state estimation development and implementation. In this paper we have reviewed only some major developments in the last few years in the solution methods for state estimation. Most of the researchers referenced have made a name for their contributions. Many others whose names are not referenced here, either because they do not appear in publications or whose publications were inadvertently missed, have made equally important contributions.

One of the assumptions in the state estimation models is that the network topology and line impedances are accurately given. From the very begining it was recognized that the assumptions may not be valid, especially on network parameters[53]. Errors in topology and parameters seriously degrade the accuracy of state estimation results. Some progress has been made recently in the detection and identification of topology errors[54-56] as well as network parameters[53,57]. More efficient parameter estimation methods are needed and further research is warranted.

State estimation has been applied to larger and larger portions of the network, down from 765 kV networks to 69 kV networks in the last few years. Interest in automation has penetrated into distribution systems[58]. State estimation for distribution systems may soon be justified. Distributed processing is necessary to accommodate the ever-increasing size of systems. Some pioneering work in hierarchical state estimation has been conducted by Van Cutsem and Pavella[59]. A very fruitful research direction in state estimation would be along the similar path leading to distributed algorithms[60].

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