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Introduction

The goal of this book is to teach you how to score a perfect 800 on the SAT math section as consistently as possible. With that in mind, this book was written with three requirements:

- It must leave no stone unturned. All tested topics and concepts must be covered no matter how obscure or rarely tested they are.
- It must place a greater focus on the most difficult questions. Those last few tough questions are what separate a 700 from an 800.
- It must have a lot of practice questions. The more you do, the better you score. Ideally, every question type becomes a simple reflex, something you know how to handle instinctively because you've seen it so many times before.

Who this book is for

The subtitle of this book is *The Advanced Guide and Workbook*. If you generally do well in math class or take advanced math classes, then this book is for you. If you're ambitious, studious, and have your eyes on an 800, then this book is for you. If you get poor grades in math class or score below a 560 on the SAT Math section, then I recommend starting with more beginner-friendly resources (e.g. Khanacademy).

One other prerequisite is that you must know how to factor quadratic expressions. We assume, for example, that you know how to factor $x^2 - 8x - 33$ into $(x - 11)(x + 3)$. If you're not comfortable with factoring, there are plenty of online tutorials you can check out (e.g. Khanacademy).

The Digital SAT Math section

The SAT Math section is divided into two modules, taken one after the other. Each module is 35 minutes long and consists of 22 questions.

The first module is the same for everyone, but the questions may be in a different order for each student. Despite any differences, the questions in both modules are always arranged such that easier questions show up first and harder questions show up last.

There are two possible second modules, one easier and one harder. If you do well enough on the first module, you'll be given the harder second module. If you don't do well enough, you'll be given the easier second module, and your highest possible score will be capped, likely in the 540-600 range. This means that the only way to get a perfect score (or any score higher than a 600) is to get to the harder second module. The number of correct answers needed on the first module to qualify for the harder second module is not set in stone. For the first 4 official Bluebook practice tests, 15 is the approximate threshold.

Your performance on the first module affects your score more than your performance on the second module, so it's important that you do your best on the first module.

There is no penalty for guessing, so you should answer all questions.

Finally, two of the questions in each module are experimental, which means they're for data-gathering purposes only and don't count towards your score. There's no way to tell which questions are experimental and which are not, so you should treat every question as though it is being scored.

How to use this book

All chapters come with exercises. Do them. You won't master the material until you think through the questions yourself. If you get stuck on a question, give yourself 4-5 minutes to figure it out. If you're still stuck, then look at the solution and take the time to fully understand it. Then circle the question number or make a note of it somewhere so that you can redo the question later. Revisiting questions you've missed is the best way to improve your score.

With the exception of 4 chapters, all chapters have two exercises with SAT practice questions. The second exercise will always have a greater number of difficult questions than the first. This isn't to reflect the difficulty difference between module 1 and module 2, but to provide a smoother progression from easy to hard for learning purposes. This progression, by the way, does not mean that there are no hard questions on the first exercise, just fewer of them, just as the second exercise will have some easy questions, but fewer of them.

With that in mind, here are a few ways of going through this book:

- Straight Through—The most thorough way but the most time-consuming.
- Double Pass—Go through the book once, skipping the hardest exercise of each chapter (the last one). Then go back and do the exercises that were skipped.
- Last Exercise—If you're already a high scorer and only want to tackle the toughest questions to save time, then just do the last exercise of every chapter. The last few questions of the exercise prior may also be worth doing.
- Jump Around—Each chapter was written to be independent of the others as much as possible. If you're already proficient in some topics yet weak in others, feel free to focus on the chapters that are most relevant to your improvement.

Are the questions in this book harder than the ones on the actual test?

This is one of the most commonly asked questions from students. The answer is yes and no. The exercises in this book may *feel* harder because there is a disproportionate number of hard questions, but the hardest questions in this book are no more difficult than the hardest questions you'll get on the SAT. Every practice question in this book is "true to the test," but the difficulty distribution isn't. For example, you might get only one hard triangles question on the real SAT, but in the triangles chapter of this book, you'll get more than 10.

A note on the Bluebook tests

When significant changes to the SAT were made in 2015-2016, the College Board published 4 practice tests to give students a sense of what the new exam would be like. As time went by, however, it became clearer and clearer that the questions on those initial 4 practice tests did not accurately reflect the ones on the real exams that were being released. In 2021, the College Board removed 2 of those 4 practice tests, claiming that they were no longer representative of the SAT in its current form.

With the release of the new digital SAT in 2024, I expect history to repeat itself. The College Board has again published 4 initial "Bluebook" practice tests, but over time, these tests will be less and less representative of the exam. The reason is simple—designing a new standardized test is impossible to get right the first time, especially without enough real student data, so test questions must be fine-tuned at a later stage. After all, the test can't be too easy, it can't be too hard, and the questions have to be worded in just the right way. As more data points are collected on how students perform on certain questions, subtle changes to the test will be made, and these changes will never be announced. For this reason, it's important that we view the first four Bluebook practice tests as a resource rather than a completely accurate depiction of the test and our potential performance on it.

At the time of this writing, the first digital SAT has been administered (March 2024) and students are already grumbling about the significant increase in the difficulty of the math section compared to the Bluebook practice tests.

As the SAT evolves, this book will evolve with it. Minor updates will be made on a semiregular basis to the current edition. Any major updates will be published in new editions.

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Desmos

On the digital SAT, you will have access to Desmos, a graphing calculator built into the test-taking app. Desmos is very powerful—some questions that would otherwise be difficult or time-consuming to solve by hand can be solved effortlessly using Desmos. Familiarizing yourself with this tool will not only improve your speed and accuracy but also arm you with new ways of getting the answer without "doing the math."

When students first find out how useful Desmos can be, many are shocked that such a tool would be allowed. *It's like a cheat code. Couldn't you get a high score just by abusing it?* Unfortunately, the answer is no. The makers of the SAT are not that naïve. As powerful as Desmos is, you will not be able to "Desmos" your way to an 800. Here are some of the ways the SAT limits its usefulness:

- Questions without answer choices—sorry, no backsolving or answer checking.
- Time limit—spend all your time brute-forcing answers on Desmos and you won't get to every question.
- Extra variables—you can graph $y = 3x + 5$, but you can't graph $y = mx + b$.
- Non-numerical answers—a calculator can't interpret what a certain number means or determine whether a certain statement is true.
- Questions designed to "dodge" the calculator—sure, you can graph the equation of a circle and tell me what its radius is, but can you tell me what its equation is if I increase its radius by 2? And if I also move the circle 5 units to the right?
- Topics that are more concept-based than calculation-based—exponential models, triangle similarity theorems, extraneous solutions to equations, probability, statistics, the list goes on.

In short, there are going to be a lot of questions for which Desmos is useless.

Don't be fooled by high-scoring students who claim that nearly every question can be easily solved using Desmos. These students are often so proficient in math that they would do well even without a graphing calculator. They possess the skill set to use Desmos to its fullest potential, but they often miss the fact that it's their ability to use the tool, and not so much the tool itself, that makes the tool valuable. They may feel like Desmos is solving most of the questions for them, but in reality, they're only remembering the questions for which Desmos helped them avoid a bunch of work they already knew how to do. A tool is only as effective as the person wielding it.

Don't get me wrong—Desmos is fantastic. It's very handy and it can certainly save you on more than a few questions, but it's not the shortcut to success that some students make it out to be.

For all these reasons, Desmos is not a major focus of this book. This book is about the actual math that's tested on the SAT—the concepts and problem-solving strategies you absolutely need to know to get a high score, no matter your ability with a graphing calculator. This section will teach you everything you need to know about Desmos to reap 90% of its benefits, but in the rest of this book, Desmos is only mentioned when there's a particularly noteworthy calculator solution. Note that Desmos is allowed on all the exercises in this book (unless otherwise stated), and we encourage you to familiarize yourself with it as much as possible.

With that out of the way, let's go over the most important things you need to know about Desmos:

1. Although Desmos allows you to enter inputs with a mouse or trackpad (by clicking a virtual keyboard), you should default to using your computer's keyboard. Typing things in will save so much time.

   Exponents can be typed with the caret symbol, ^ (shift + 6). Multiplication can be typed with the asterisk symbol, * (shift + 8). Division can be typed with a forward slash, /. Your keyboard already has + and – keys. Trigonometric expressions such as sin x can be typed in. The square root sign can be entered by typing "sqrt". The inequality signs for "less than or equal to" and "greater than or equal to" can be entered by typing <= and >=, respectively. Use the left and right arrow keys to move the cursor as necessary when typing in inputs.

   The following expression, for example, can be typed in without using a mouse or trackpad.

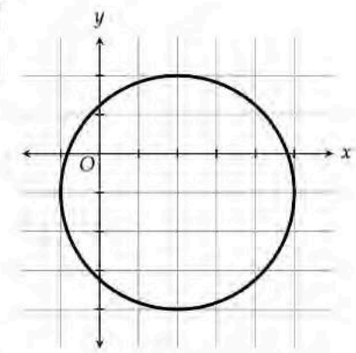
   $$\frac{x^2}{3} \cdot \cos^2(x) + \sqrt{(10x)}$$

2. Desmos can convert decimal answers into fractions and vice versa. Just press the "$\frac{\square}{\square}$" button that appears to the left when the answer is a fraction or decimal.

   $$\frac{1}{2} + \frac{1}{4}$$
   $$= 0.75$$

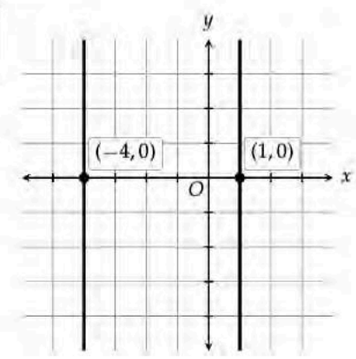
3. You do not have to isolate y before you graph an equation. Just type in any equation as it is and Desmos will graph it. This feature allows you to graph more complicated equations, such as those of circles, without any upfront work.

   $$x^2 - 4x + y^2 + 2y = 4$$



4. Desmos can solve equations. Just enter the equation and click on the x-intercepts of the vertical lines. The x-coordinates are the solutions to the equation. In the example below, the solutions to the equation $x^2 + 3x - 4 = 0$ are $x = -4$ and $x = 1$, as indicated by the two vertical lines.

   $$x^2 + 3x - 4 = 0$$



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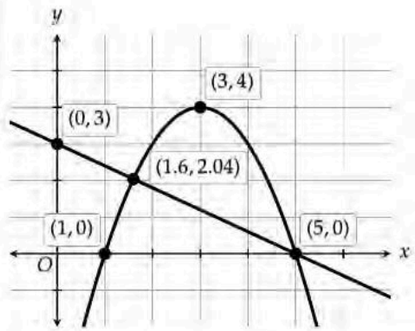
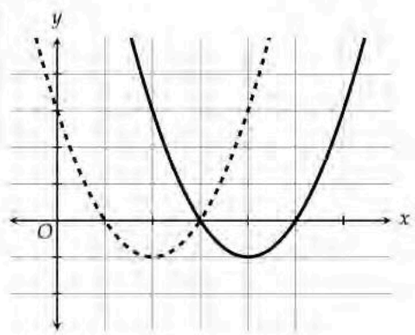
5. Desmos can find all the important points on a graph—intersection points, y-intercepts, x-intercepts, maximums, and minimums. Just look for the gray points and click on them to get their coordinates. You may have to zoom in and out a little (the '+' and '–' buttons) or click close to the desired point for the clickable gray point to appear.

	$y = -0.6x + 3$
	$y = -x^2 + 6x - 5$



6. Desmos can perform function transformations. Just enter your functions using function notation (i.e. $f(x)$ and $g(x)$). In the example below, the graph of $g(x)$ (shown as solid) is produced by translating the graph of $f(x)$ (shown as dashed) 2 units to the right. We cover function transformations in chapter 15.

	$f(x) = x^2 - 4x + 3$
	$g(x) = f(x - 2)$



7. When doing trigonometry calculations, make sure the appropriate unit of angle measure is selected—radians or degrees. You can make this selection by clicking on the "wrench" button in the upper right ("Graph Settings").

There's a lot more that you can do with Desmos, but these are the tips that every student should know. I recommend watching a few online tutorials if you're interested in learning more.

Finally, I strongly recommend that you bring your own pen/pencil and graphing calculator (e.g. TI-84) on test day. There are many questions that you'll have to do on spare paper, and one of the best ways to avoid silly mistakes is to write everything out. A physical graphing calculator can be more convenient to use when crunching through calculations on paper because 1) you don't have to look up and down between your paper and your computer, 2) you may find two-thumb typing to be quicker, and 3) you're able to, on almost all graphing calculators, copy and paste answers from previous calculations and use them in your next calculation, which is something Desmos doesn't allow you to do. With Desmos, you have to type out previously outputted answers you want to use.

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Failure is making 100 mistakes *during* the test.
Success is making 1000 mistakes *before* the test.

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1

Exponents & Radicals

Given the expression x^n, x is called the **base** and n is called the **exponent** or **power**. Here are the laws of exponents you should know:

Law	Example
$x^1 = x$	$3^1 = 3$
$x^0 = 1$	$3^0 = 1$
$x^m \cdot x^n = x^{m+n}$	$3^4 \cdot 3^5 = 3^9$
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{3^7}{3^3} = 3^4$
$(x^m)^n = x^{mn}$	$(3^2)^4 = 3^8$
$(xy)^m = x^m y^m$	$(2 \cdot 3)^3 = 2^3 \cdot 3^3$
$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$	$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$
$x^{-m} = \frac{1}{x^m}$	$3^{-4} = \frac{1}{3^4}$

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The difference between

$$(-3)^2 \text{ and } -3^2$$

comes down to order of operations (PEMDAS), which dictates that parentheses are prioritized first. So,

$$(-3)^2 = (-3)\cdot(-3) = 9$$

Without parentheses, exponents take priority:

$$-3^2 = -3\cdot 3 = -9$$

Notice that the negative is not applied until the exponent operation is carried through.
Sometimes, the result turns out to be the same, as in:

$$(-2)^3 \text{ and } -2^3$$

Both yield -8.

EXAMPLE 1: Which of the following is equivalent to $-x^2(-x)^5$?

A) $-x^{10}$ B) $-x^7$ C) x^7 D) x^{10}

The $-x^2$ term cannot be simplified any further because the negative is not inside any parentheses. The $(-x)^5$ term has an odd exponent. An odd exponent yields an odd number of negative signs, which in turn yields a negative result. So, $(-x)^5$ simplifies to $-x^5$. The entire expression then boils down to

$$-x^2(-x)^5 = -x^2\cdot -x^5 = x^{2+5} = x^7$$

Answer $\boxed{(C)}$.

EXAMPLE 2: Which expression is equivalent to $4^{x+1}\cdot 3^{2x}$, where $x > 0$?

A) $4(12)^x$ B) $4(24)^x$ C) $4(36)^x$ D) $8(12)^x$

We can use our laws of exponents to simplify the expression. First, the law $x^{m+n} = x^m\cdot x^n$ allows us to simplify 4^{x+1} to $4^x\cdot 4^1$. Then the law $x^{mn} = (x^m)^n$ allows us to turn 3^{2x} into $(3^2)^x = 9^x$. After applying these laws, we get

$$4^{x+1}\cdot 3^{2x} = 4^x\cdot 4^1\cdot 9^x = 4\cdot(4^x\cdot 9^x) = 4(36)^x$$

Answer $\boxed{(C)}$. In the last step, we used the law $x^m y^m = (xy)^m$ to get $4^x\cdot 9^x = 36^x$.

Roots are the opposites of exponents. Whereas 5^2 will give you 25, the square root of 25, denoted by $\sqrt{25}$, will give you back the 5. Similarly, taking the cube root of 6^3 gives $\sqrt[3]{6^3} = 6$ and taking the fourth root of b^4 gives $\sqrt[4]{b^4} = b$. Notice how the operations cancel each other out.

Instead of using the radical sign, $\sqrt[n]{}$, we can also denote the nth-root using the equivalent fractional exponent $\frac{1}{n}$. For example,

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

What about $x^{\frac{2}{3}}$? The 2 on top means to square x. The 3 on the bottom means to cube root it:

$$\sqrt[3]{x^2}$$

We can see this more clearly if we break it down using our laws of exponents:

$$x^{\frac{2}{3}} = \left(x^2\right)^{\frac{1}{3}} = \sqrt[3]{x^2}$$

The order in which we do the squaring and the cube-rooting doesn't matter. We could've broken it down this way as well:

$$x^{\frac{2}{3}} = \left(x^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{x}\right)^2$$

You'll see $\sqrt[3]{x^2}$ more often than $\left(\sqrt[3]{x}\right)^2$ because the outside cube root avoids the need for parentheses.

EXAMPLE 3: Which of the following is equivalent to $\sqrt[4]{x^5}$?

A) x B) $x^5 - x^4$ C) $x^{\frac{5}{4}}$ D) $x^{\frac{4}{5}}$

The fourth root equates to a fractional exponent of $\frac{1}{4}$, so

$$\sqrt[4]{x^5} = \left(x^5\right)^{\frac{1}{4}} = x^{\frac{5}{4}}$$

Answer $\boxed{(C)}$.

EXAMPLE 4: The expression $\dfrac{25^{2x}}{\left(\sqrt{5}\right)^x}$ is equivalent to 5^{kx}, where k is a constant. What is the value of k ?

Because most of the laws of exponents involve terms with the same base, an often-used tactic is to convert each term to have the same base. Here, we're asked to simplify to 5^{kx}, so we should convert everything to have the same base of 5. As is common in this type of question, we see that we have a perfect square, 25, which can be written as 5^2. In the denominator, we have $\sqrt{5}$, which can be written as $5^{\frac{1}{2}}$.

$$\frac{25^{2x}}{\left(\sqrt{5}\right)^x} = \frac{(5^2)^{2x}}{(5^{\frac{1}{2}})^x} = \frac{5^{4x}}{5^{\frac{1}{2}x}} = 5^{4x-\frac{1}{2}x} = 5^{\frac{7}{2}x}$$

Therefore, $k = \boxed{\frac{7}{2}}$.

EXAMPLE 5: If the square of a is equal to the cube of b, where $a \geq 0$ and $b \geq 0$, for what value of x is $\sqrt{a^x}$ equal to b ?

Essentially, we need to solve

$$\sqrt{a^x} = b$$

for x given that $a^2 = b^3$. Let's first cube root both sides of $a^2 = b^3$ to isolate b:

$$\sqrt[3]{a^2} = \sqrt[3]{b^3}$$

$$a^{\frac{2}{3}} = b$$

Substituting $a^{\frac{2}{3}}$ for b in the equation above, we get

$$\sqrt{a^x} = a^{\frac{2}{3}}$$

$$a^{\frac{1}{2}x} = a^{\frac{2}{3}}$$

Since both sides have the same base of a, we can just equate the exponents and solve for x:

$$\frac{1}{2}x = \frac{2}{3}$$

$$x = \boxed{\frac{4}{3}}$$

The SAT will also test you on simplifying square roots (also called "surds"). To simplify a square root, factor the number inside the square root and take out any pairs:

$$\sqrt{48} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = \sqrt{\boxed{2 \cdot 2} \cdot \boxed{2 \cdot 2} \cdot 3} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$$

In the example above, we take a 2 out for the first $\boxed{2 \cdot 2}$. Then we take another 2 out for the second pair $\boxed{2 \cdot 2}$. Finally, we multiply the two 2's outside the square root to get 4 and leave the 3 inside. Of course, a quicker route would have been to break down 48 into bigger factor pairs:

$$\sqrt{48} = \sqrt{\boxed{4 \cdot 4} \cdot 3} = 4\sqrt{3}$$

Here's another example:

$$\sqrt{72} = \sqrt{\boxed{2 \cdot 2} \cdot \boxed{3 \cdot 3} \cdot 2} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}$$

To go backwards, take the number outside and put it back under the square root as a pair:

$$6\sqrt{2} = \sqrt{6 \cdot 6 \cdot 2} = \sqrt{72}$$

To simplify a cube root such as $\sqrt[3]{16}$, take out any triplets:

$$\sqrt[3]{16} = \sqrt[3]{\boxed{2 \cdot 2 \cdot 2} \cdot 2} = 2\sqrt[3]{2}$$

EXAMPLE 6: Which of the following is equivalent to $\left(x^2\right)^{\frac{3}{4}}$, where $x > 0$?

A) $\sqrt{x}$ B) $x\sqrt{x}$ C) $\sqrt[3]{x^2}$ D) $\sqrt[4]{x}$

Solution 1:

$$\left(x^2\right)^{\frac{3}{4}} = x^{(2 \cdot \frac{3}{4})} = x^{\frac{3}{2}} = \sqrt{x^3} = \sqrt{\boxed{x \cdot x} \cdot x} = x\sqrt{x}$$

Answer $\boxed{(B)}$.

Solution 2: Since $\left(x^2\right)^{\frac{3}{4}} = x^{(2 \cdot \frac{3}{4})} = x^{\frac{3}{2}}$, we can compare this exponent of $\frac{3}{2}$ to the exponent of x in each of the answer choices.

Choice A: $\sqrt{x} = x^{\frac{1}{2}}$

Choice B: $x\sqrt{x} = x^1 \cdot x^{\frac{1}{2}} = x^{(1+\frac{1}{2})} = x^{\frac{3}{2}}$

Choice C: $\sqrt[3]{x^2} = x^{\frac{2}{3}}$

Choice D: $\sqrt[4]{x} = x^{\frac{1}{4}}$

These results confirm that the answer is $\boxed{(B)}$.

EXERCISE 1: Evaluate WITHOUT a calculator. Answers for this exercise start on page 311.

1. $(-1)^{10}$
2. $(-1)^{15}$
3. $(-1)^{8}$
4. -1^{8}
5. $-(-1)^{8}$
6. $(-3)^{3}$
7. -3^{3}
8. $-(-6)^{2}$
9. $-(-4)^{3}$
10. $2^{3} \times 3^{2} \times (-1)^{5}$
11. 4^{-1}
12. 5^{0}
13. 3^{-2}
14. $\left(\frac{1}{2}\right)^{-1}$
15. $\left(\frac{4}{3}\right)^{-2}$

EXERCISE 2: Simplify so that your answer contains only positive exponents. The first one has been done for you. Answers for this exercise start on page 312.

1. $2k^{-4} \cdot 4k^{2} = \frac{2 \cdot 4k^{2}}{k^{4}} = \frac{8}{k^{2}}$
2. $3x^{2} \cdot 2x^{3}$
3. $5x^{4} \cdot 3x^{-2}$
4. $7m^{3} \cdot -3m^{-3}$
5. $(2x^{2})^{-3}$
6. $-3a^{2}b^{-3} \cdot 3a^{-5}b^{8}$
7. $\frac{3x^{4}}{(x^{-2})^{2}}$
8. $\frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$
9. $(2m)^{2} \cdot (3m^{3})^{2}$
10. $(a^{-1} \cdot a^{-2})^{2}$
11. $(b^{-2})^{-3} \cdot (b^{3})^{2}$
12. $\frac{(m^{2}n)^{3}}{(mn^{2})^{2}}$
13. $\frac{mn}{m^{2}n^{3}}$
14. $\frac{k^{-2}}{k^{-3}}$
15. $\frac{x^{2}y^{3}z^{4}}{x^{-3}y^{-4}z^{-5}}$

EXERCISE 3: Simplify the radicals or solve for x. Answers for this exercise start on page 312.

1. $\sqrt{12}$
2. $\sqrt{96}$
3. $\sqrt{45}$
4. $\sqrt{18}$
5. $3\sqrt{75}$
6. $\sqrt{32}$
7. $5\sqrt{2} = \sqrt{x}$
8. $3\sqrt{x} = \sqrt{45}$
9. $2\sqrt{2} = \sqrt{4x}$
10. $4\sqrt{6} = 2\sqrt{3x}$
11. $3\sqrt{8} = x\sqrt{2}$
12. $x\sqrt{x} = \sqrt{216}$

EXERCISE 4: Answers for this exercise start on page 313.

1

Which expression is equivalent to $(x^{2}y)(x^{4}y^{-3})$, where x, y, and z are positive numbers?

A) $x^{6}y^{-3}$
B) $x^{6}y^{-2}$
C) $x^{8}y^{-3}$
D) $x^{8}y^{-2}$

2

Which of the following is equivalent to $\sqrt{\frac{x}{64}}$ for all $x > 0$?

A) $\frac{x^{2}}{8}$
B) $\frac{x^{2}}{32}$
C) $\frac{x^{\frac{1}{2}}}{8}$
D) $\frac{x^{\frac{1}{2}}}{32}$

3

Which expression is equivalent to $\frac{x^{\frac{5}{6}}}{\sqrt[3]{x}}$, where $x \neq 0$?

A) $x^{\frac{1}{2}}$
B) $x^{\frac{5}{2}}$
C) $x^{\frac{2}{3}}$
D) $x^{\frac{5}{3}}$

4

Which expression is equivalent to $b^{\frac{9}{7}}$, where $b > 0$?

A) $\sqrt[15]{b^{105}}$
B) $\sqrt[15]{b^{135}}$
C) $\sqrt[105]{b^{135}}$
D) $\sqrt[135]{b^{105}}$

5

Which expression represents the product of $(b^{6}c^{-2}d^{-5})$ and $(b^{8}c^{-3} + c^{4}d^{5})$, where b, c, and d are positive?

A) $b^{14}c^{-5} + c^{2}d^{-10}$
B) $b^{14}c^{-5} + c^{2}$
C) $b^{14}c^{-5}d^{-5} + b^{6}c^{2}$
D) $b^{14}c^{-5}d^{-5} + c^{2}$

6

If $x \neq 0$ and $y \neq 0$, which of the following is equivalent to $\frac{8x^{2}}{\sqrt{4x^{6}y^{4}}}$?

A) $2xy^{-2}$
B) $4x^{-2}y^{2}$
C) $4x^{-1}y^{-2}$
D) $4xy^{2}$

7

If r and s are positive, which of the following expressions is equivalent to $r^{\frac{6}{7}}s^{\frac{3}{7}}$?

A) $\frac{1}{\sqrt[6]{r^7s^{14}}}$

B) $\sqrt[6]{r^7s^{14}}$

C) $\frac{1}{\sqrt[7]{r^6s^3}}$

D) $\sqrt[7]{r^6s^3}$

8

$$\sqrt[b]{a} = \sqrt[d]{c}$$

The given equation relates the distinct positive numbers a, b, c, and d. Which equation correctly expresses c in terms of a, b, and d ?

A) $c = \frac{d}{a^{\frac{1}{b}}}$

B) $c = a^{d-b}$

C) $c = a^{\frac{b}{d}}$

D) $c = a^{\frac{d}{b}}$

9

Which expression is equivalent to $\sqrt{16a^{\frac{2}{3}}}$, where $a > 0$?

A) $4a^{\frac{1}{3}}$

B) $4a^{\frac{4}{3}}$

C) $8a^{\frac{1}{3}}$

D) $8a^{\frac{4}{3}}$

10

Which expression is equivalent to $\sqrt{rs}\,(\sqrt{r}+\sqrt{s})$, where $r \geq 0$ and $s \geq 0$?

A) $\sqrt{r^2s+rs^2}$

B) $r\sqrt{s}+s\sqrt{r}$

C) $rs\sqrt{r+s}$

D) $\sqrt{rs+r+s}$

11

The expression $\left(\sqrt[3]{x^2}\right)^n$, where n is a constant, is equivalent to x^8. What is the value of n ?

12

Which expression is equivalent to $2^{-2k} \cdot 3^k$?

A) $(3\sqrt{2})^k$

B) $\left(\frac{1}{36}\right)^k$

C) $\left(\frac{3}{4}\right)^k$

D) $\left(\frac{9}{4}\right)^k$

13

Which expression is equivalent to $a^{\frac{2}{9}}(a^{\frac{2}{3}})^{\frac{2}{3}}$, where a is positive?

A) $\sqrt[3]{a^2}$

B) $\sqrt[3]{a^{16}}$

C) $\sqrt[9]{a^8}$

D) $\sqrt[9]{a^{14}}$

14

Which expression is equivalent to $\frac{4}{9}\left(\frac{2}{3}\right)^a$?

A) $\left(\frac{2}{3}\right)^{\frac{a}{2}}$

B) $\left(\frac{2}{3}\right)^{a-2}$

C) $\left(\frac{2}{3}\right)^{a+2}$

D) $\left(\frac{2}{3}\right)^{2a}$

15

$$\sqrt[4]{x^3} \cdot \sqrt[12]{x^5}$$

The given expression is equivalent to $\sqrt[m]{x^n}$, where m and n are positive constants each less than 10 and $x > 1$. What is the value of $\frac{m}{n}$?

16

Two numbers, p and q, are each greater than zero, and the square of p is equal to the cube root of q. For what value of x is p^{1-x} equal to the square root of q ?

EXERCISE 5: Answers for this exercise start on page 314.

1

Which expression is equivalent to $\sqrt{100x^{36}}$?

A) $10x^6$
B) $50x^6$
C) $10x^{18}$
D) $50x^{18}$

2

If $(5^3)^{4k} = (5^{\frac{1}{3}})^{24}$, what is the value of k ?

A) -6
B) $\frac{2}{3}$
C) $\frac{3}{4}$
D) 2

3

Which of the following is equivalent to $3y^{\frac{1}{2}}$ for all $y > 0$?

A) $\sqrt{3y}$
B) $\sqrt{9y}$
C) $\sqrt{\frac{3}{y}}$
D) $\sqrt{\frac{9}{y}}$

4

Which expression is equivalent to 4^{m+2} ?

A) 16^m
B) $16 + 4^m$
C) $8(4^m)$
D) $16(4^m)$

5

Which expression is equivalent to $\sqrt[4]{x^2y^4}$, where $x > 0$ and $y > 0$?

A) $\sqrt{xy}$
B) $y\sqrt{x}$
C) $\frac{1}{x^2}$
D) x^2y

6

Which expression is equivalent to $p^{\frac{1}{3}} \cdot (\sqrt[3]{p})^2$, where p is greater than 0 ?

A) $p^{\frac{2}{3}}$
B) $p^{\frac{2}{9}}$
C) $p^{\frac{7}{3}}$
D) p

7

Which expression is equivalent to $x^{\frac{2a}{b}}$ for all positive values of x, where a and b are positive integers?

A) $\sqrt[b]{ax^2}$
B) $\sqrt[b]{x^{2a}}$
C) $\sqrt[b]{x^{a+2}}$
D) $\sqrt[2a]{x^b}$

8

If $\frac{9^7}{\sqrt[4]{9^{10}}} = 9^{6t}$, what is the value of t ?

9

Which expression is equivalent to $a^{\frac{3}{4}}b^{\frac{1}{2}}$, where $a \geq 0$ and $b \geq 0$?

A) $\sqrt[4]{a^3b}$
B) $\sqrt[4]{a^3b^2}$
C) $\sqrt{a^3b}$
D) $\sqrt{a^4b^2}$

10

Which expression is equivalent to $h^{\frac{5}{12}}(h^{\frac{-1}{4}})^{\frac{7}{3}}$, where $h > 0$?

A) $\frac{1}{h^6}$
B) $\sqrt{h^5}$
C) $\frac{1}{\sqrt[6]{h}}$
D) $\frac{1}{\sqrt[5]{h^2}}$

11

$$\sqrt[8]{41k}\left(\sqrt[9]{41k}\right)^{12}$$

For what value of x is the given expression equivalent to $(41k)^{25x}$, where $k > 1$?

12

The expression 4^{18x} is equivalent to k^{3x}, where k is a constant. What is the value of k ?

13

Which expression is equivalent to $9^{\frac{1}{n}}(4^{\frac{1}{2n}})$, where n is a positive integer?

A) $24^{\frac{1}{n}}$
B) $12^{\frac{1}{n}}$
C) $\sqrt[n]{18}$
D) $\sqrt[n]{6}$

14

$$\frac{10\sqrt[6]{9^3x^{48}}}{\sqrt[4]{6^4x}}$$

The given expression is equivalent to ax^b, where $a > 0$, $b > 0$, and $x > 1$. What is the value of ab ?

15

If m and n are both positive numbers, and $4m$ is equal to the cube root of the square of n, for what value of x is m^x equal to n when $m = 2$?

16

If $2^{x+3} - 2^x = k(2^x)$, what is the value of k ?

A) 3
B) 5
C) 7
D) 8

2 Expressions

Algebraic expressions are just combinations of numbers and variables. Both $x^2 + y$ and $\frac{3m-k}{2}$ are examples of expressions. In this chapter, we'll cover some fundamental techniques that will allow you to deal with questions involving expressions quickly and effectively.

1. Combining Like Terms

When adding or subtracting, be careful of terms that look like they can go together but can't. For example, you cannot combine $b^2 + b$ to make b^3, nor can you combine $a + ab$ to make $2ab$. For an addition or subtraction to be valid, the variables in each term have to exactly match.

EXAMPLE 1: Which expression is equivalent to $2(2a^2 - 3a^2b^2 - 4b^2) - (a^2 + 5a^2b^2 - 10b^2)$?

A) $-6a^2b^2$ B) $3a^2 - 11a^2b^2 - 18b^2$ C) $3a^2 - 11a^2b^2 + 2b^2$ D) $5a^2 + 2a^2b^2 + 2b^2$

$$\begin{aligned} 2(2a^2 - 3a^2b^2 - 4b^2) - (a^2 + 5a^2b^2 - 10b^2) &= 4a^2 - 6a^2b^2 - 8b^2 - a^2 - 5a^2b^2 + 10b^2 \\ &= 3a^2 - 11a^2b^2 + 2b^2 \end{aligned}$$

Answer $\boxed{(C)}$.

2. Simplifying Fractional Expressions

Fractional expressions can be simplified when 1) the numerator is either a product of expressions or a common factor itself, 2) the denominator is either a product of expressions or a common factor itself, and 3) there is a factor common to both the numerator and denominator.

Some examples:

- $\frac{4xy(x+y)(x-y)}{6x(x-y)}$ can be simplified because both the numerator and denominator are products that share common factors of 2, x, and $x - y$: $\frac{{}^{2}\cancel{4x}y(x+y)\cancel{(x-y)}}{{}^{3}\cancel{6x}\cancel{(x-y)}} = \frac{2y(x+y)}{3}$.
- $\frac{4x}{4y+1}$ cannot be simplified because the denominator is neither a product of expressions nor a common factor itself. Doing the following would be incorrect: $\frac{\cancel{4}x}{\cancel{4}y+1}$. Similarly, $\frac{4y+1}{4x}$ cannot be simplified either.
- $\frac{4(x+1)}{x+1}$ can be simplified because the denominator itself is a common factor: $\frac{4\cancel{(x+1)}}{\cancel{x+1}} = 4$.
- $\frac{x+8}{x+4}$ cannot be simplified because neither the numerator nor the denominator is a product of expressions or a common factor. Doing the following would be incorrect: $\frac{x+\cancel{8}^{2}}{x+\cancel{4}^{1}} = \frac{x+2}{x+1}$.
- $\frac{8x}{4xy+4x}$ seems like it cannot be simplified because the denominator is neither a product nor a common factor itself. However, the denominator can be factored into a product: $\frac{8x}{4x(y+1)}$. And now we can simplify: $\frac{{}^{2}\cancel{8x}}{\cancel{4x}(y+1)} = \frac{2}{y+1}$.

3. Expansion and Factoring

EXAMPLE 2: Which expression is equivalent to $2(x-4)(2x+3)$?

A) $4x^2 - 10x - 24$ B) $4x^2 + 10x - 24$ C) $4x^2 + 10x + 24$ D) $8x^2 - 20x - 24$

Binomials are expressions with only two terms, such as the $x - 4$ and $2x + 3$ in this example. To get started, we can distribute the "2" to either the first binomial or the second, but not both. Let's distribute it to the first binomial:

$$2(x-4)(2x+3) = (2x-8)(2x+3)$$

Now we can use a method called FOIL (first, outer, inner, last) to expand this product of two binomials. If you haven't heard of FOIL, it's the same as distributing each term.

$$(2x-8)(2x+3)$$

First: $2x \cdot 2x = \boxed{4x^2}$, Outer: $2x \cdot 3 = \boxed{6x}$, Inner: $-8 \cdot 2x = \boxed{-16x}$, Last: $-8 \cdot 3 = \boxed{-24}$

Putting these results together, we get

$$(2x-8)(2x+3) = 4x^2 + 6x - 16x - 24 = 4x^2 - 10x - 24$$

Answer $\boxed{(A)}$.

EXAMPLE 3: Which expression is equivalent to $9x^3y + 6x^2y^2$?

A) $3x^2y(3x+2y)$ B) $3x^2y(3x+2y^2)$ C) $3x^2y^2(3x+2y)$ D) $3x^2y^2(3x+2y^2)$

We can factor this expression using the greatest common factor of both terms, $3x^2y$:

$$9x^3y + 6x^2y^2 = 3x^2y(3x+2y)$$

Answer $\boxed{(A)}$.

EXAMPLE 4: Which expression is equivalent to $\dfrac{2x(x+4)-3(x+4)}{4x-6}$, where $x > \dfrac{3}{2}$?

A) $\dfrac{x+4}{2}$ B) $\dfrac{2x-3}{4}$ C) $\dfrac{2x^2-3x-12}{4x-6}$ D) $\dfrac{2x^2+5x-8}{4x-6}$

Don't be too quick to expand the numerator. Notice that both terms have a $(x+4)$ in common. We can factor out this $(x+4)$ from the terms in the numerator and a 2 from the terms in the denominator:

$$\frac{2x(x+4)-3(x+4)}{4x-6} = \frac{(x+4)(2x-3)}{4x-6} = \frac{(x+4)(2x-3)}{2(2x-3)} = \frac{(x+4)\cancel{(2x-3)}}{2\cancel{(2x-3)}} = \frac{x+4}{2}$$

Answer $\boxed{(A)}$.

If you're confused at all about how we factored the numerator, another way to look at it is to let $A = (x+4)$. Then the numerator becomes $2xA - 3A = A(2x-3) = (x+4)(2x-3)$.

In addition to the techniques shown in the previous examples, there are three essential patterns you must know, summarized by the formulas below:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $a^2 - b^2 = (a+b)(a-b)$

Memorize these forwards and backwards. They show up very often.

EXAMPLE 5: Which expression is equivalent to $16x^4 - 81$?

A) $(4x^2+9)^2$ B) $(2x+3)^2(2x-3)^2$ C) $(4x^2+27)(4x^2-3)$ D) $(4x^2+9)(2x+3)(2x-3)$

Here, we have an expression of the form $a^2 - b^2$, a difference of two squares, where $a = \sqrt{16x^4} = 4x^2$ and $b = \sqrt{81} = 9$. Since $a^2 - b^2 = (a+b)(a-b)$,

$$16x^4 - 81 = (4x^2+9)(4x^2-9)$$

But wait! We're not done. In this result is another difference of two squares, $4x^2 - 9$. We have to factor this as well.

$$(4x^2+9)(4x^2-9) = (4x^2+9)(2x+3)(2x-3)$$

Answer $\boxed{(D)}$.

As you can see, memorizing a formula is not enough. You have to know when and how to apply it. The more you do these types of problems, the better you'll get at recognizing these patterns and their variations.

EXAMPLE 6: Which expression is equivalent to $(3x-5)^{\frac{2}{3}}$?

A) $(9x^2+25)^3$ B) $\sqrt[3]{9x^2+25}$ C) $\sqrt[3]{9x^2-15x+25}$ D) $\sqrt[3]{9x^2-30x+25}$

As we learned in the chapter on exponents, the numerator of a fractional exponent is the power and the denominator is the root. Here's the step-by-step simplification:

$$(3x-5)^{\frac{2}{3}} = [(3x-5)^2]^{\frac{1}{3}} = \sqrt[3]{(3x-5)^2}$$

We now have an expression of the form $(a-b)^2$ inside the root, where $a = 3x$ and $b = 5$. Using the formula $(a-b)^2 = a^2 - 2ab + b^2$ to expand it, we get

$$\sqrt[3]{(3x-5)^2} = \sqrt[3]{(3x)^2 - 2(3x)(5) + (5)^2} = \sqrt[3]{9x^2 - 30x + 25}$$

Answer $\boxed{(D)}$.

EXAMPLE 7: Which expression is equivalent to $16x^4 - 8x^2y^2 + y^4$?

A) $(4x^2+y^2)^2$ B) $(2x-y)^4$ C) $(2x+y)^2(2x-y)^2$ D) $(4x+y)^2(x-y)^2$

The given expression is of the form $a^2 - 2ab + b^2$, where $a = 4x^2$ and $b = y^2$. Using the formula $a^2 - 2ab + b^2 = (a-b)^2$ to factor it, we get

$$16x^4 - 8x^2y^2 + y^4 = (4x^2 - y^2)^2$$

But this result is not in the answer choices. We have to take it one step further—the expression inside the parentheses is a difference of two squares, which we know how to factor:

$$(4x^2 - y^2)^2 = [(2x+y)(2x-y)]^2 = (2x+y)^2(2x-y)^2$$

Answer $\boxed{(C)}$.

4. Combining Fractions

When you're adding simple fractions,

$$\frac{1}{3} + \frac{1}{4}$$

the first step is to find the least common multiple of the denominators. We do this so that we can get a common denominator. In a lot of cases, it's just the product of the denominators, as it is here, $3 \times 4 = 12$.

$$\frac{1}{3} + \frac{1}{4} = \frac{1}{3} \cdot \frac{4}{4} + \frac{1}{4} \cdot \frac{3}{3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

Now when we're adding fractions with expressions in the denominator, the idea is the same.

EXAMPLE 8: Which expression is equivalent to $\frac{1}{x+2}+\frac{2}{x-2}$, where $x \neq -2$ and $x \neq 2$?

A) $\frac{3x-2}{(x+2)(x-2)}$ B) $\frac{3x+2}{(x+2)(x-2)}$ C) $\frac{3}{(x+2)(x-2)}$ D) $\frac{2}{(x+2)(x-2)}$

The common denominator is just the product of the two denominators: $(x+2)(x-2)$. With this in mind, we can multiply the top and bottom of each fraction by the factor they don't have:

$$\frac{1}{x+2}+\frac{2}{x-2}=\frac{1}{x+2}\cdot\frac{x-2}{x-2}+\frac{2}{x-2}\cdot\frac{x+2}{x+2}=\frac{x-2}{(x+2)(x-2)}+\frac{2(x+2)}{(x+2)(x-2)}=\frac{(x-2)+2(x+2)}{(x+2)(x-2)}$$
$$=\frac{3x+2}{(x+2)(x-2)}$$

Answer $\boxed{(B)}$.

EXAMPLE 9: Which expression is equivalent to $\frac{6}{x+3}+\frac{18}{x^2+3x}$, where $x \neq -3$ and $x \neq 0$?

A) $\frac{6}{x}$ B) $\frac{24}{x(x+3)}$ C) $\frac{30x+72}{x(x+3)}$ D) $\frac{24}{x^2+4x+3}$

First, factor out an x from the denominator of the second fraction to get $\frac{6}{x+3}+\frac{18}{x(x+3)}$. Now we need a common denominator to add these two fractions. Both denominators already have a factor of $(x+3)$, so multiplying them to get a common denominator would be overdoing it. All we need for the first denominator to match the second is a factor of x, so let's just multiply the top and bottom of the first fraction by x:

$$\frac{6}{x+3}+\frac{18}{x(x+3)}=\frac{6}{x+3}\cdot\frac{x}{x}+\frac{18}{x(x+3)}=\frac{6x}{x(x+3)}+\frac{18}{x(x+3)}=\frac{6x+18}{x(x+3)}=\frac{6\cancel{(x+3)}}{x\cancel{(x+3)}}=\frac{6}{x}$$

Answer $\boxed{(A)}$.

5. Flipping (Dividing) Fractions

What's the difference between $\frac{\frac{1}{2}}{3}$ and $\frac{1}{\frac{2}{3}}$?

The difference is where the longer fraction line is. The first is $\frac{1}{2}$ divided by 3. The second is 1 divided by $\frac{2}{3}$. They're not the same.

$$\frac{\frac{1}{2}}{3}=\frac{1}{2}\div 3=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$$
$$\frac{1}{\frac{2}{3}}=1\div\frac{2}{3}=1\times\frac{3}{2}=\frac{3}{2}$$

To summarize, when there is a fraction $\frac{a}{b}$ in the denominator, we flip a and b: $\frac{c}{\frac{a}{b}}=\frac{bc}{a}$.

When there is a fraction $\frac{a}{b}$ in the numerator, a and b remain where they are: $\frac{\frac{a}{b}}{c}=\frac{a}{bc}$.

EXAMPLE 10:

$$\frac{6}{\left(\frac{24}{x}\right)}$$

Which of the following is equivalent to the given expression?

A) $4x$ B) $\frac{x}{4}$ C) $\frac{1}{4x}$ D) $\frac{4}{x}$

$$\frac{6}{\left(\frac{24}{x}\right)}=\frac{6x}{24}=\frac{x}{4}$$

Answer $\boxed{(B)}$.

6. Splitting Fractions

EXAMPLE 11: Which expression is equivalent to $\frac{30+c}{6}$?

A) $\frac{5+c}{6}$ B) $\frac{10+c}{2}$ C) $5+c$ D) $5+\frac{c}{6}$

We can split the fraction into two:

$$\frac{30+c}{6}=\frac{30}{6}+\frac{c}{6}=5+\frac{c}{6}$$

Answer $\boxed{(D)}$. As you can see, this process reverses fraction addition.

Note that while you can split up the numerators of fractions, you cannot do so with denominators. So,

$$\frac{3}{x+y}\neq\frac{3}{x}+\frac{3}{y}$$

In fact, you cannot break up a fraction like $\frac{3}{x+y}$ any further.

EXERCISE 1: Answers for this exercise start on page 316.

1

Which expression is equivalent to $6x^2y + 6xy^2$?

A) $6xy(x+y)$
B) $12xy(x+y)$
C) $6x^2y^2(y+x)$
D) $12x^3y^3$

2

Which expression is equivalent to $(3x^3 + 8x^2 - 4x) + (7x^2 - 11x - 7)$?

A) $3x^3 + x^2 - 15x - 7$
B) $3x^3 + 15x^2 - 15x - 7$
C) $10x^5 - 7x - 7$
D) $15x^4 + 3x^3 - 15x^2 - 7$

3

Which expression is equivalent to $\frac{1}{a} + \frac{3}{4}$, where $a \neq 0$?

A) $\frac{3+4a}{4a}$
B) $\frac{4+3a}{4a}$
C) $\frac{7}{4a}$
D) $\frac{4}{a+4}$

4

The expression $(15x + 34)(-8x + 3)$ is equivalent to the expression $ax^2 + bx + c$, where a, b, and c are constants. What is the value of b ?

5

Which expression is equivalent to $\frac{4+8x}{12x}$, where $x \neq 0$?

A) $\frac{1+8x}{3x}$
B) $\frac{4+2x}{3x}$
C) $\frac{1+2x}{3x}$
D) 1

6

Which expression is equivalent to $484x^2 - 324$?

A) $(18x-22)(18x+22)$
B) $(18x-22)(18x-22)$
C) $(22x-18)(22x+18)$
D) $(22x-18)(22x-18)$

7

Which expression is equivalent to $(x^2 + y)(y + z)$?

A) $x^2z + y^2 + yz$
B) $x^2y + x^2z + y^2 + yz$
C) $x^2y + y^2 + x^2z$
D) $x^2 + x^2z + y^2 + yz$

8

The expression $\frac{15x}{36x^2 + 18x}$ is equivalent to $\frac{5}{k(2x+1)}$, where k is a constant and $x > 0$. What is the value of k ?

A) 3
B) 6
C) 12
D) 18

9

Which expression is equivalent to $3x^4 - 3$?

A) $3(x^2+1)^2$
B) $3(x^2-1)^2$
C) $3(x^3-1)(x+1)$
D) $3(x^2+1)(x+1)(x-1)$

10

Which expression is equivalent to $\frac{\left(\frac{3}{x-1}\right)}{6x-6}$, where $x \neq 1$?

A) $\frac{1}{2}$
B) 18
C) $\frac{1}{2x-2}$
D) $\frac{1}{2(x-1)^2}$

11

Which expression is equivalent to $2(3x^2-5)^2 - (3x^2-5)^2$?

A) $9x^4 - 15x^2 + 75$
B) $9x^4 - 30x^2 + 25$
C) $9x^4 - 60x^2 + 75$
D) $18x^4 - 90x^2 + 25$

12

Which expression is equivalent to $\frac{xy - x^2}{xy - y^2}$, where $y \neq 0$ and $x \neq y$?

A) $-\frac{y}{x}$
B) $\frac{y}{x}$
C) $\frac{x}{y}$
D) $-\frac{x}{y}$

13

Which expression is equivalent to $\frac{x^2(2x-3) - (2x-3)}{x-1}$, where $x > 1$?

A) $2x^2 - 3x$
B) $(2x-3)(x+1)$
C) $\frac{2x^2-3}{x-1}$
D) $\frac{2x^3 - 5x^2 + 3}{x-1}$

14

Which expression is equivalent to $\frac{3}{3x-14} - \frac{1}{x-3}$?

A) $\frac{2}{2x-11}$
B) $-\frac{23}{(3x-14)(x-3)}$
C) $\frac{5}{(3x-14)(x-3)}$
D) $-\frac{3x-10}{(3x-14)(x-3)}$

15

Which expression is equivalent to $x^2(x+2)(x-2) + 4$?

A) $(x^2-2)^2$
B) $(x^2+2)^2$
C) $(x-1)^2(x+2)^2$
D) $(x+1)^2(x-2)^2$

EXERCISE 2: Answers for this exercise start on page 317.

1

Which expression is equivalent to $3xy^6 + 3x^3y^3$?

A) $3xy^3(y^2 + x^2)$
B) $3xy^3(y^2 + x^3)$
C) $3xy^3(y^3 + x^2)$
D) $3xy^3(y^3 + x^3)$

2

Which expression is equivalent to $(5a + 3\sqrt{a}) - (2a + 5\sqrt{a})$?

A) $-2a\sqrt{a}$
B) $a\sqrt{a}$
C) $3a - 2\sqrt{a}$
D) $3a + 8\sqrt{a}$

3

Which expression is equivalent to $\frac{2}{x} - \frac{1}{3x}$, where $x > 0$?

A) $-\frac{1}{2x}$
B) $\frac{2}{x}$
C) $\frac{2}{3x}$
D) $\frac{5}{3x}$

4

Which expression is equivalent to $(x - 8)(5 - 3x^2)$?

A) $-3x^3 - 6x^2 + 24$
B) $-3x^3 + 11x^2 + 5x - 40$
C) $-3x^3 + 24x^2 + 5x - 40$
D) $-3x^3 + 29x^2 + 24$

5

Which expression is equivalent to $\frac{x^2 - 3}{x + \sqrt{3}}$, where $x \neq -\sqrt{3}$?

A) $\frac{x\sqrt{3}}{3}$
B) $x - 1$
C) $x - \sqrt{3}$
D) $x + \sqrt{3}$

6

$$(4x^2 + 4x - 1)(x^2 - 3x)$$

If the given expression is rewritten in the form $ax^4 + bx^3 + cx^2 + dx$, where a, b, c, and d are constants, what is the value of $b + c$?

7

Which expression is equivalent to $5b - \frac{b + k}{b}$, where $b > 0$?

A) $\frac{4b - k}{b}$
B) $\frac{4b + k}{b}$
C) $\frac{5b^2 - b - k}{b}$
D) $\frac{5b^2 - b + k}{b}$

8

The expression $\frac{(6x + 8)(3x + 6)}{12x + 24}$ is equivalent to $\frac{ax + b}{2}$, where a and b are constants and $x > 0$. What is the value of $a + b$?

9

Which expression is equivalent to $(3\sqrt{a} - \sqrt{b})^{\frac{2}{7}}$, where $a > b$ and $b > 0$?

A) $\sqrt[7]{9a - b}$
B) $\sqrt[7]{9a - 6ab + b}$
C) $\sqrt[7]{9a - 3\sqrt{ab} + b}$
D) $\sqrt[7]{9a - 6\sqrt{ab} + b}$

10

Which expression is equivalent to $x^4 - 8x^2 + 16$?

A) $(x - 2)^4$
B) $(x - 2)^3(x + 2)$
C) $(x - 2)^2(x + 2)^2$
D) $(x^2 + 4)(x + 2)(x - 2)$

11

Which expression is equivalent to $\frac{y - 3xy}{2x^2 - 2xy} + \frac{1 + 3y}{x - y}$?

A) $\frac{2x - 3xy + 4y}{2x^2 - 2xy}$
B) $\frac{2x + 3xy + y}{2x^2 - 2xy}$
C) $\frac{2x + 3xy + y}{2x^3 - 4x^2y + 2xy^2}$
D) $\frac{4y - 3xy + 1}{2x^3 - 4x^2y + 2xy^2}$

12

The expression $8x^2 - \frac{1}{2}y^2$ is equivalent to $8(x - cy)(x + cy)$, where c is a positive constant. What is the value of c ?

A) $\frac{1}{16}$
B) $\frac{1}{8}$
C) $\frac{1}{4}$
D) $\frac{\sqrt{2}}{4}$

13

Which expression is equivalent to $(x + 1)^2 + 2(x + 1)(y + 1) + (y + 1)^2$?

A) $(x + y + 1)^2$
B) $(x + y + 2)^2$
C) $(x + y)^2 + 2$
D) $(x + y)^2 - x - y$

14

Which expression is equivalent to $\frac{x}{x - 2} + \frac{x}{2(2 - x)}$, where $x \neq 2$?

A) $-\frac{x}{x - 2}$
B) $-\frac{x}{2(x - 2)}$
C) $\frac{x}{2(x - 2)}$
D) $\frac{3x}{2(x - 2)}$

3

Manipulating & Solving Equations

On the SAT, there is a huge emphasis on equations. To get these types of questions right, you must learn how to isolate the variables and expressions on one side of the equation. Let's dive into some tips and techniques.

1. Don't forget to combine like terms

You should be ruthless in finding like terms and combining them. Doing so will simplify things and allow you to figure out the next step.

EXAMPLE 1: If $2(a+b+2c+3d+1) = 3a+2b+4c+6d$, what is the value of a ?

The same four variables are on both sides of the equation. So before we say to ourselves that this question is impossible because there are 4 variables and one equation, we should first get all the terms on one side and combine the ones that are alike. Sounds simple but you won't believe how many students judge a problem before they've taken the first step.

$$2(a+b+2c+3d+1) = 3a+2b+4c+6d$$
$$2a+\cancel{2b}+\cancel{4c}+\cancel{6d}+2 = 3a+\cancel{2b}+\cancel{4c}+\cancel{6d}$$
$$\boxed{2} = a$$

As you can see, the b, c, and d terms canceled quite nicely.

2. Square equations correctly

When squaring equations to remove a square root, the most important thing to remember is that you're not squaring individual elements—you're squaring the entire side.

EXAMPLE 2:

$$\sqrt{ab} = a - b$$

If $a > 0$ and $b > 0$, the given equation is equivalent to which of the following?

A) $ab = a^2 - b^2$ B) $ab = a^2 + b^2$ C) $2ab = a^2 - b^2$ D) $3ab = a^2 + b^2$

The square root in the problem should scream to you that the equation should be squared. Most students know they have to get rid of the square root, but all too often, their approach is to square each individual element:

$$ab = a^2 - b^2$$

This result, of course, is incorrect. When modifying equations, you must apply any given operation to the entire SIDE, like so:

$$(\sqrt{ab})^2 = (a-b)^2$$

If it helps, wrap each side in parentheses before applying the operation. By the way, the same holds true for all other operations, including multiplication and division. When you multiply or divide both sides of an equation, what you're actually doing is wrapping each side in parentheses, but because of the distributive property, it just so happens that multiplying or dividing each individual element gets you the same result. For example, if we had the equation

$$x + 2 = y$$

and we wanted to multiply both sides by 3, what we're actually doing is

$$3(x+2) = 3(y)$$

which turns out to be the same as

$$3x + 6 = 3y$$

Anyway, back to the problem:

$$(\sqrt{ab})^2 = (a-b)^2$$
$$ab = a^2 - 2ab + b^2$$
$$3ab = a^2 + b^2$$

The answer is $\boxed{(D)}$.

EXAMPLE 3:

$$\sqrt{x+3}+y=z$$

The given equation relates the positive real numbers x, y, and z. Which equation correctly expresses x in terms of y and z ?

A) $x=z^2-y^2-3$ B) $x=z^2+y^2-3$ C) $x=(z-y)^2-3$ D) $x=(z+y)^2-3$

Another common mistake is squaring each side before the equation is "ready" to be squared. For this example, here's what we'd get if we squared each side right away:

$$(\sqrt{x+3}+y)^2=z^2$$

Imagine expanding this out and trying to isolate x. We'd be in a world of pain. The right approach is to isolate the square root on the left by subtracting y from both sides:

$$\sqrt{x+3}=z-y$$

Now we can square each side:

$$\begin{aligned} x+3&=(z-y)^2 \\ x&=(z-y)^2-3 \end{aligned}$$

Answer $\boxed{(C)}$.

3. Square root equations correctly

Now, when it comes to taking the square root of both sides, most students forget the plus or minus ($\pm$). Always remember that an equation such as $x^2=25$ has two solutions:

$$\begin{aligned} \sqrt{x^2}&=\sqrt{25} \\ x&=\pm 5 \end{aligned}$$

However, the plus or minus only applies when you're taking the square root to solve an equation. Otherwise, square roots, by definition, always refer to the positive root. That's why $\sqrt{9}=3$, not ± 3, and $\sqrt{x}=-3$ has no real solution. It's only in situations where the square root is used to solve an equation that the plus or minus gets tacked on.

EXAMPLE 4: If $(x+3)^2=121$, what is the sum of the two possible values of x ?

$$\begin{aligned} (x+3)^2&=121 \\ \sqrt{(x+3)^2}&=\pm\sqrt{121} \\ x+3&=\pm 11 \\ x&=-3\pm 11 \end{aligned}$$

So x could be either 8 or -14. The sum of those two possibilities is $\boxed{-6}$.

4. Cross-multiply when fractions are set equal to each other

Whenever a fraction is equal to another fraction,

$$\frac{a}{b}=\frac{c}{d}$$

you can cross-multiply: $ad=bc$.

EXAMPLE 5: If $\frac{4}{5}x=\frac{10}{3}$, what is the value of x ?

$$\begin{aligned} \frac{4}{5}x&=\frac{10}{3} \\ 12x&=50 \\ x&=\boxed{\frac{25}{6}} \end{aligned}$$

EXAMPLE 6:

$$\frac{2}{b}-\frac{5c}{a+bc}=0$$

The given equation relates the positive numbers a, b, and c. Which equation correctly expresses a in terms of b and c ?

A) $2bc$ B) $\frac{3}{2}bc$ C) $\frac{2}{5bc}-bc$ D) $\frac{2-5c}{b-bc}$

$$\begin{aligned} \frac{2}{b}-\frac{5c}{a+bc}&=0 \\ \frac{2}{b}&=\frac{5c}{a+bc} \\ 2(a+bc)&=5bc \\ 2a+2bc&=5bc \\ 2a&=3bc \\ a&=\frac{3}{2}bc \end{aligned}$$

Answer $\boxed{(B)}$.

5. Factor to isolate variables that are common to multiple terms

Some equations have variables that are tougher to isolate. For a lot of these equations, you will have to do some shifting around to factor out the variable you want.

EXAMPLE 7:

$$b = \frac{a}{3a+c}$$

Which of the following expresses a in terms of b and c ?

A) $\frac{bc}{1-3b}$ B) $\frac{bc}{3b+1}$ C) $\frac{1-3b}{bc}$ D) $\frac{3b+1}{bc}$

$$\begin{aligned} b &= \frac{a}{3a+c} \\ b(3a+c) &= a \\ 3ab + bc &= a \\ bc &= a - 3ab \\ bc &= a(1-3b) \\ \frac{bc}{1-3b} &= a \end{aligned}$$

See what we did? We expanded everything out and put every term containing a on the right side. Then we were able to factor out a and isolate it. The answer is $\boxed{(A)}$.

6. Factor to get a product of expressions

EXAMPLE 8:

$$x^2(x-3) - (x-3) = 0$$

What is the smallest solution to the given equation?

We can factor out $(x-3)$ to get

$$(x-3)(x^2-1) = 0$$

If you didn't quite catch that, here's another way to see it. If we let $A = (x-3)$, the equation becomes $x^2A - A = 0$, and we can factor out A to get $A(x^2-1) = 0$. Substituting $(x-3)$ back in for A, we get $(x-3)(x^2-1) = 0$. The takeaway here is that any shared term or expression can be factored out.

Looking at the result, we see that we have a difference of two squares, $x^2 - 1$, which can be factored further:

$$\begin{aligned} (x-3)(x^2-1) &= 0 \\ (x-3)(x+1)(x-1) &= 0 \end{aligned}$$

With this product of expressions, it's easy to see that we have 3 possible cases (solutions): $x-3$ is equal to 0, $x+1$ is equal to 0, or $x-1$ is equal to 0. So the solutions are $x = 3, -1,$ or 1. Of these, $\boxed{-1}$ is the smallest.

7. Be comfortable solving for expressions, rather than any one variable

EXAMPLE 9: If $3x + 9y = 9$, what is the value of $x + 3y$?

Get in the habit of looking for what you want **before** you solve for anything specific. Is there any way to get the answer without solving for x and y?

Yes! Dividing both sides of the given equation by 3 gives $x + 3y = \boxed{3}$.

EXAMPLE 10: If $4\left(\frac{3-x}{2}\right) - 5 = 23$, what is the value of $\frac{3-x}{2}$?

The expression $\frac{3-x}{2}$ shows up in the equation itself, so we can solve for it directly without worrying about the value of x.

$$\begin{aligned} 4\left(\frac{3-x}{2}\right) - 5 &= 23 \\ 4\left(\frac{3-x}{2}\right) &= 28 \\ \left(\frac{3-x}{2}\right) &= \boxed{7} \end{aligned}$$

EXAMPLE 11: If $\frac{x}{y} = 3$, what is the value of $\frac{y}{2x}$?

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) $\frac{3}{2}$

Here, we have no choice but to solve for the expression. We're given x over y but we want y over x. We can flip the given equation to get

$$\frac{y}{x} = \frac{1}{3}$$

Then we can divide both sides by 2 to obtain the $\frac{y}{2x}$ we're looking for: $\frac{y}{2x} = \frac{1}{2 \cdot 3} = \frac{1}{6}$. Answer $\boxed{(A)}$.

8. Watch out for extraneous solutions and missing solutions

Extraneous solutions are values we may think are solutions but actually aren't. They are also called *false* solutions.

EXAMPLE 12:

$$\frac{(x-5)(x-3)}{x-5} = 0$$

What is a possible solution to the given equation?

A good first step to solving this equation might be to multiply both sides by $x-5$ to get rid of the fraction. We would end up with

$$(x-5)(x-3) = 0$$

At this point, it's easy to see that either $x - 5 = 0$ or $x - 3 = 0$, so the possible solutions are $x = 5$ and $x = 3$.

However, plugging in $x = 5$ into the original equation results in $\frac{0}{0}$, which is undefined because of division by zero. Therefore, $x = 5$ is not a valid solution to this equation. It's an extraneous solution. The only possible solution here is $\boxed{3}$.

Extraneous solutions can also occur when working with square roots. As a simple example, the equation

$$\sqrt{x} = -4$$

has no solution because the square root of something can never be negative. However, if we square both sides, we get $x = (-4)^2$, which implies that $x = 16$ is a solution even though it clearly does not satisfy the original equation.

Why do these extraneous solutions occur? Well, manipulating equations in certain ways can bring about solutions that satisfy the transformed equation but not the original. Squaring both sides of an equation, for instance, has the effect of turning negative values into positive ones, and once this distinction is lost, false solutions can arise.

Let's take a look at an example of a missing solution.

EXAMPLE 13:

$$(x + 4)(x - 9) = (x + 4)$$

What is the sum of the solutions to the given equation?

Because $(x + 4)$ appears on both sides of the equation, let's simplify by dividing both sides by $(x + 4)$:

$$\frac{\cancel{(x+4)}(x-9)}{\cancel{(x+4)}} = \frac{\cancel{(x+4)}}{\cancel{(x+4)}}$$

$$(x - 9) = 1$$

$$x = 10$$

Plugging in $x = 10$ yields 14 on both sides of the original equation, so $x = 10$ is definitely a solution. But is 10 the only solution? There doesn't seem to be any more even though the question suggests that there's more than one.

The problem is that in dividing both sides by $(x + 4)$ to find one solution, we made another (more obvious) solution disappear: $x = -4$. Plugging in $x = -4$ into the original equation yields 0 on both sides, so it's clearly a solution. How come it was so easy to miss?

Remember that in math, division by zero is not allowed. So when we divided both sides by $(x + 4)$, we made the implicit assumption that x is not -4 (otherwise, $x + 4$ would be equal to zero). That division and all the steps we took afterwards were based on that assumption. However, in making that assumption, we neglected to consider that $x = -4$ itself might also be a solution. That's why it was missed. Since it is a solution, the answer here is $10 + (-4) = \boxed{6}$.

So what are the takeaways from this section?

- Dividing by an expression can cause you to miss obvious solutions.
- Multiplying by an expression can give rise to extraneous solutions.
- Equations with square roots and/or fractional expressions are more prone to have extraneous solutions. For these equations, get in the habit of plugging your results back into the original equation to see whether they are valid.

EXERCISE 1: Answers for this exercise start on page 319.

1

$$48 = \frac{8}{-x}$$

What is the solution to the given equation?

2

If $18 + 9x = 54$, what is the value of $2 + x$?

3

$$9 = -24 + 3x$$

Which equation has the same solution as the given equation?

A) $-3x = -15$
B) $3x = -15$
C) $-3x = -33$
D) $3x = -33$

4

If $3x - 8 = -23$, what is the value of $6x - 7$?

A) -5
B) -21
C) -30
D) -37

5

$$3(5 - 2x)(3 + x) = 0$$

What positive value of x is a solution to the given equation?

6

$$-3(r - 9) + 5(r - 9) = -4$$

What value of r is the solution to the given equation?

7

If $x^2 + 3x + 16 = 0$, what is the value of $5x^2 + 15x$?

8

If $\frac{12 - x}{9} = \frac{12 - x}{4}$, the value of $12 - x$ is between which of the following pairs of values?

A) -14 and -9
B) -8 and -4
C) -3 and 4
D) 5 and 13

9

$$\frac{64x^2}{36} = 64$$

What is the negative solution to the given equation?

10

If $\frac{4}{9} = \frac{8}{3}m$, what is the value of m ?

A) $\frac{1}{6}$

B) $\frac{2}{3}$

C) $\frac{5}{6}$

D) 6

11

$$y + 2kx = kx^2 + 5$$

In the given equation, k is a constant. If $y = 23$ when $x = 3$, what is the value of k ?

A) -6

B) 3

C) 6

D) 9

12

$$f = p\left(\frac{(1+j)^n - 1}{j}\right)$$

The given equation relates the positive numbers f, j, n, and p. Which of the following correctly expresses p in terms of f, j, and n ?

A) $p = \frac{fj}{(1+j)^n - 1}$

B) $p = \frac{(1+j)^n - 1}{fj}$

C) $p = \frac{f - j}{(1+j)^n - 1}$

D) $p = fj + 1 - (1+j)^n$

13

If $\frac{m}{2n} = 2$, what is the value of $\frac{n}{2m}$?

A) $\frac{1}{8}$

B) $\frac{1}{4}$

C) $\frac{1}{2}$

D) 1

14

$$-4d = 3e - h$$

The given equation relates the numbers d, e, and h. Which equation correctly expresses d in terms of e and h ?

A) $d = \frac{-3e - h}{4}$

B) $d = -\frac{3e}{4} - h$

C) $d = 3e + \frac{h}{4}$

D) $d = \frac{h - 3e}{4}$

15

If $\frac{x}{6} = \frac{x+12}{42}$, what is the value of $\frac{6}{x}$?

A) $\frac{1}{3}$

B) 2

C) 3

D) 6

16

$$(n-4)^2 = (n+4)^2$$

What value of n satisfies the given equation?

17

$$\sqrt{x^2 - 25} = n$$

In the given equation, n is a positive constant. Which of the following is one of the solutions to the given equation?

A) $n + 5$

B) $\sqrt{n + 25}$

C) $\sqrt{n^2 + 5}$

D) $-\sqrt{n^2 + 25}$

18

If $3(x - 2y) - 3z = 0$, which of the following equations correctly expresses x in terms of y and z ?

A) $\frac{2y + 3z}{3}$

B) $2y + z$

C) $y + 2z$

D) $6y + 3z$

19

$$d = a\left(\frac{c+1}{24}\right)$$

Doctors use Cowling's rule, defined by the equation shown, to determine the correct dosage d, in milligrams, of medication for a child based on the adult dosage a, in milligrams, and the child's age c, in years. Ben is a patient who is in need of a certain medication. If a doctor uses Cowling's rule to prescribe Ben a dosage that is half the adult dosage, what is Ben's age, in years?

A) 7

B) 9

C) 11

D) 13

20

$$\frac{x^2 - 9}{x - 3} = 0$$

Which value is a solution to the given equation?

A) 3

B) $\sqrt{3}$

C) $-\sqrt{3}$

D) -3

21

$$T = 2P\sqrt{\frac{L}{G}}$$

The given equation relates the positive real numbers G, L, P, and T. Which equation correctly expresses G in terms of L, P, and T ?

A) $\frac{2PL}{T^2}$

B) $\frac{T^2}{4LP^2}$

C) $\frac{LT^2}{4P^2}$

D) $\frac{4LP^2}{T^2}$

22

$$x(x-7) + 9(x-7) = 0$$

What is the negative solution to the given equation?

23

$$x(y-2) = -y$$

The given equation relates the positive numbers x and y. Which equation correctly expresses y in terms of x ?

A) $y = \frac{x}{x-2}$

B) $y = \frac{x}{x+2}$

C) $y = \frac{2x}{x-1}$

D) $y = \frac{2x}{x+1}$

24

$$V = P(1-r)^t$$

The value V of a car depreciates over t years according to the given equation, where P is the original price and r is the annual rate of depreciation. Which of the following correctly expresses r in terms of V, P, and t ?

A) $r = 1 - \sqrt[t]{\frac{V}{P}}$

B) $r = 1 + \sqrt[t]{\frac{V}{P}}$

C) $r = \sqrt[t]{\frac{V}{P}} - 1$

D) $r = 1 - \frac{\sqrt[t]{V}}{P}$

25

$$\frac{4}{x^2 - 6x + 9} = 9$$

If x is a solution to the given equation, which of the following could be the value of $x - 3$?

A) $\frac{2}{3}$

B) $\frac{3}{2}$

C) $\frac{7}{3}$

D) $\frac{9}{2}$

26

$$\sqrt{2x+8} - 3\sqrt{x-3} = 0$$

What value of x is the solution to the given equation?

27

$$x - c = x^2 - c^2$$

Which of the following are solutions to the given equation, where c is a positive constant?

I. c
II. $-c$
III. $1 - c$

A) I only
B) II only
C) I and III only
D) I, II, and III

28

$$20 - \sqrt{x} = \frac{2}{3}\sqrt{x} + 10$$

If $x > 0$, what value of x satisfies the given equation?

EXERCISE 2: Answers for this exercise start on page 321.

1

$$\sqrt{x-1} - 2 = 11$$

What value of x is the solution to the given equation?

2

If $\frac{5x}{6} - 4 = 12 - \frac{7x}{6}$, what is the value of $4x$?

3

$$B + 8A = 5(4C + B)$$

The given equation can be rewritten as $B = mA - nC$, where m and n are constants. What is the value of $m + n$?

4

$$(2x+7)^2 = 25$$

What is the sum of the solutions to the given equation?

5

$$\frac{2}{3}t + m = 40$$

A tennis player uses the given equation to estimate the tension t, in pounds, in his racquet strings after m tennis matches. Which of the following equations correctly expresses t in terms of m ?

A) $t = -\frac{3}{2}(40 - m)$

B) $t = -\frac{3}{2}(40) - m$

C) $t = \frac{3}{2}(40 - m)$

D) $t = \frac{3}{2}(40) - m$

6

$$\frac{3}{5}(x+11) - \frac{2}{3}(x+11) = -8$$

What value of x is the solution to the given equation?

7

If $\frac{4}{k+2} = \frac{x}{3}$, where $k \neq -2$, which equation correctly expresses k in terms of x ?

A) $k = \frac{12-2x}{x}$

B) $k = \frac{12+2x}{x}$

C) $k = \frac{x}{12+2x}$

D) $k = 12x - 2$

8

$$\sqrt{x^2+p^2}=q$$

In the given equation, p and q are positive constants. Which of the following is one of the solutions to the given equation?

A) $-\sqrt{q^2-p^2}$

B) $\sqrt{p^2+q^2}$

C) $\sqrt{q-p^2}$

D) $-p^2-q^2$

9

$$\sqrt{(x+6)(x-6)}=8$$

What is the positive solution to the given equation?

10

$$2u=\frac{18}{v}-6$$

The given equation relates the numbers u and v, where $u>0$ and $v>0$. Which equation correctly expresses v in terms of u ?

A) $v=\frac{9}{u+3}$

B) $v=\frac{u}{9}-3$

C) $v=\frac{u}{9}+3$

D) $v=\frac{u+3}{9}$

11

$$\frac{k}{3x+3y}=k+1$$

The given equation relates the distinct positive numbers x, y, and k. Which equation correctly expresses $x+y$ in terms of k ?

A) $x+y=\frac{k}{3k+1}$

B) $x+y=\frac{k}{3k+3}$

C) $x+y=\frac{3k}{k+1}$

D) $x+y=\frac{3k+3}{k}$

12

$$\left(\frac{1}{8}+\frac{1}{x}\right)t=1$$

The given equation represents the number of days t it will take Jim and Andy to paint a 300-foot fence working together given that it would take Jim x days to paint the fence working alone. If it takes 6 days for Jim and Andy to paint the fence working together, how many days would it have taken Jim to paint the fence working alone?

13

$$\frac{a}{b}=\frac{a+1}{2c}$$

The given equation relates the distinct positive numbers a, b, and c. Which equation correctly expresses a in terms of b and c ?

A) $a=\frac{b}{b-2c}$

B) $a=\frac{b}{2c-b}$

C) $a=\frac{b+c}{2c}$

D) $a=\frac{c-b}{2c}$

14

$$\frac{3r}{t}=6\sqrt{2s}$$

The given equation relates the positive numbers r, s, and t. Which equation correctly expresses s in terms of r and t ?

A) $s=\frac{r^2}{2t^2}$

B) $s=\frac{r^2}{4t^2}$

C) $s=\frac{r^2}{8t^2}$

D) $s=\frac{3r^2}{4t^2}$

15

$$ax+7a+x+7=b$$

In the given equation, a and b are positive constants. Which equation correctly expresses $x+7$ in terms of a and b ?

A) $x+7=\frac{b}{a}-7$

B) $x+7=\frac{b}{2a}$

C) $x+7=\frac{b}{a+1}$

D) $x+7=\frac{b-1}{a}$

16

If $(x+1)(x-2)=7x-18$, what is the value of $7x-18$?

17

$$xy-1=z-xz$$

The given equation relates the positive numbers x, y, and z. Which equation correctly expresses x in terms of y and z ?

A) $\frac{z}{y}+1$

B) $\frac{z}{y+z}+1$

C) $\frac{z+1}{yz}$

D) $\frac{z+1}{y+z}$

18

If $\frac{2\sqrt{x+4}}{3}=6$ and $x>0$, what is the value of x ?

19

If $\frac{2x+y}{x-2y}=3$, what is the value of $\frac{x+y}{x-y}$?

20

$$\frac{x-n}{3} = x(x-n)$$

Which of the following are solutions to the given equation, where n is a positive constant greater than 1 ?

I. $\frac{1}{3}$

II. $\frac{n}{3}$

III. n

A) I and II only

B) I and III only

C) II and III only

D) I, II, and III

21

If $x + y = \sqrt{x^2 + y^2 + 16}$, what is the value of xy ?

22

$$\sqrt{x+9} = \frac{x}{2} - 3$$

What is the solution to the given equation?

23

$$\sqrt{2(x-k)} = x - k$$

In the given equation, k is a constant. Which of the following is the sum of the solutions to the given equation?

A) $2k$

B) $k + 1$

C) $2k - 2$

D) $2k + 2$

24

$$\frac{x}{\sqrt{x+b}} = 7 - \frac{b}{\sqrt{x+b}}$$

In the given equation, b is a positive constant. Which of the following is the solution to the given equation?

A) $7 - b$

B) $49 - b$

C) $b - 49$

D) $b^2 + 49$

25

If $\sqrt{x+2} = 2 - \sqrt{x}$, what is the value of $2 - \sqrt{x}$?

A) $\frac{1}{2}$

B) $\frac{3}{2}$

C) $\sqrt{\frac{5}{2}}$

D) 2

4

Other Types of Equation Questions

In this chapter, we'll cover more question types involving equations.

1. Matching terms

EXAMPLE 1: The expressions $(x+10)(x+b)$ and $x^2+3bx+c$, where b and c are constants, are equivalent. What is the value of c ?

We essentially have two variables and one equation. Let's expand $(x+10)(x+b)$ first and see if that takes us anywhere.

$$(x+10)(x+b) = x^2+bx+10x+10b = x^2+(b+10)x+10b$$

Notice that we grouped the x terms, bx and $10x$. You'll see why in a moment. So now we have

$$x^2+(b+10)x+10b = x^2+3bx+c$$

For both sides to be equal to each other, the corresponding terms on each side must be equal. For example, the coefficient of the x term on the left side must equal the coefficient of the x term on the right side. Let's underline the corresponding terms:

$$x^2+\underline{(b+10)x}+\underline{\underline{10b}} = x^2+\underline{3bx}+\underline{c}$$

Equating the corresponding coefficients and constants gives us

$$b+10 = 3b$$
$$10b = c$$

Solving the first equation, we get $b=5$. Plugging this result into the second equation, we get $c = 10(5) = \boxed{50}$.

2. Infinitely many solutions

Another way that the SAT tests this "matching terms" concept is to frame the question in the context of **infinitely many solutions** for a *single* equation (we'll talk about infinitely many solutions for a *system* of equations in another chapter).

A single equation has infinitely many solutions when both sides of the equation are equivalent.

For instance,

$$3x + 6 = 3x + 6$$

has infinitely many solutions because no matter what the value of x is, the equation is always true ($x = 1$ is a solution, $x = 2$ is a solution, $x = 3$ is a solution, ...) Notice that the equation boils down to $0 = 0$, which, again, is always true.

Equations like $3x + 6 = 3x + 6$ are a bit weird because what's the point of dealing with them in the first place? Of course $3x + 6$ is equal to $3x + 6$! But keep in mind that these equations are not meant to be solved; they're meant to demonstrate the concept of infinitely many solutions. Let's take a look at how this concept might appear in an SAT question.

EXAMPLE 2:

$$a(x^2 - 2b) = 4x^2 - 12$$

In the given equation, a and b are constants. The equation has infinitely many solutions. What is the value of b ?

As in Example 1, we can expand the left side and match the terms so that both sides are equivalent. Only when both sides are equivalent does the equation have infinitely many solutions.

$$a(x^2 - 2b) = 4x^2 - 12$$
$$ax^2 - 2ab = 4x^2 - 12$$

Comparing corresponding terms, we can see that $a = 4$ and $-2ab = -12$. Now we can solve for b.

$$-2ab = -12$$
$$-2(4)b = -12$$
$$b = \frac{-12}{-8} = \boxed{1.5}$$

EXAMPLE 3:

$$kx + 3(5 - 2x) = 15$$

In the given equation, k is a constant. If the equation is true for all values of x, what is the value of k ?

This question is just another way of testing the infinitely many solutions concept. Expanding the left side of the equation, we get

$$kx + 15 - 6x = 15$$

Since the right side is just a constant of 15, we need to cancel out the x terms on the left side in order for both sides to be equivalent. It's easy to see that $k = \boxed{6}$ does the job. If it helps, you can think of the right side as having a $0x$ term. The end result is that no matter what the value of x is, 15 equals 15. Yes, I know these equations are weird, but that's how you get infinitely many solutions.

3. No solutions

The opposite of infinitely many solutions is **no solutions**. When an equation has no solutions, there are no values of x that satisfy it. To illustrate, the equation

$$3x + 6 = 3x + 10$$

has no solutions because there is no value of x that can ever make $3x + 6$ equal to $3x + 10$. The equation itself is a contradiction. This is even more obvious if we subtract $3x$ from both sides: we're left with $6 = 10$, which is fundamentally false.

Now let's go back to the original equation, $3x + 6 = 3x + 10$, for a moment. Notice that the x terms on each side have the same coefficient of 3, but the constants of 6 and 10 are different. These are the signs of an equation with no solution.

A single equation has no solutions when both sides of the equation are equivalent EXCEPT for the constants. The constants must be different.

EXAMPLE 4:

$$3cx - 4(x + 1) = 2(x - 1)$$

In the given equation, c is a constant. The equation has no solution. What is the value of c ?

Expanding each side, we get

$$3cx - 4(x + 1) = 2(x - 1)$$
$$3cx - 4x - 4 = 2x - 2$$
$$3cx - 4 = 6x - 2$$

The constants are already different, so we just need to get the coefficients of x to match. Very simply, $3c = 6$, which gives $c = \boxed{2}$.

EXAMPLE 5: How many solutions does the equation $6(x + 2) = 6x + 8$ have?

A) Zero B) Exactly one C) Exactly two D) Infinitely many

Expanding the left side, we get

$$6x + 12 = 6x + 8$$

Both sides are equivalent except for the constants, 12 and 8, so this equation must have no solution. Answer $\boxed{(A)}$. Note that subtracting $6x$ from both sides yields $12 = 8$, a contradiction that further confirms our answer.

4. Absolute Value

The absolute value of x, denoted by $|x|$, is the distance x is from 0. In other words, absolute value makes everything positive. If the value is positive, it stays positive. If it's negative, it becomes positive.

EXAMPLE 6:

$$|x+3| = 8$$

What is one possible solution to the given equation?

Because of the absolute value, the expression inside can take on a positive or negative value, so we have two possibilities:

- $x+3=8$
- $x+3=-8$

Solving these equations, we get $x = 8-3 = \boxed{5}$ and $x = -8-3 = \boxed{-11}$ as possible solutions.

5. Finding the number of solutions to various types of equations

EXAMPLE 7:

$$(2x-3)^2 = -1$$

How many distinct real solutions does the given equation have?

A) Zero B) Exactly one C) Exactly two D) Infinitely many

The square of anything can never be negative, so this equation has zero real solutions. Answer $\boxed{(A)}$. We'll talk more extensively about the number of solutions to quadratic equations in a later chapter.

EXAMPLE 8:

$$|x-10| = 0$$

How many solutions does the given equation have?

A) Zero B) One C) Two D) Three

As we saw earlier, absolute value equations typically have two solutions, one for the positive case and one for the negative case. The equation in this example, however, has a constant of zero, and because zero does not have positive and negative counterparts, the only possibility is $x-10=0$, which has just one solution, $x=10$. Because there is only one solution, the answer is $\boxed{(B)}$.

EXERCISE 1: Answers for this exercise start on page 325.

1

$$\frac{1}{4}x - 6 = px + 5$$

In the given equation, p is a constant. The equation has no solution. What is the value of p?

2

$$|x+20| = 8$$

What is one possible solution to the given equation?

3

$$3x + 6 = \frac{12x + r}{4}$$

In the given equation, r is a constant. The equation has infinitely many solutions. What is the value of r?

A) 6
B) 8
C) 12
D) 24

4

$$|x-15| = 32$$

What is the sum of the solutions to the given equation?

5

How many solutions does the equation $3(x-2) = 6(x+1) - 3x$ have?

A) Zero
B) Exactly one
C) Exactly two
D) Infinitely many

6

$$\frac{1}{5}x = k(x-k)$$

In the given equation, k is a constant. The equation has no solution. What is the value of k?

7

If $|3x+9| = 156$, what is the positive value of $x+3$?

8

$$6x = x - 3x(2n - 1)$$

In the given equation, n is a constant. The equation has infinitely many solutions. What is the value of n ?

A) $-\frac{2}{3}$

B) $-\frac{1}{3}$

C) $\frac{4}{3}$

D) $\frac{5}{3}$

9

How many solutions does the equation $-|x + 6| = 21$ have?

A) Zero

B) Exactly one

C) Exactly two

D) More than two

10

$$2(x + k) = 2x + 2$$

In the given equation, k is a constant greater than 1. How many solutions does the equation have?

A) Zero

B) Exactly one

C) Exactly two

D) Infinitely many

11

$$12x + 16nx = 20$$

In the given equation, n is a constant. The equation has no solution. What is the value of n ?

12

How many solutions does the equation $-14(x - 5) = 5(14 - x)$ have?

A) Zero

B) Exactly one

C) Exactly two

D) Infinitely many

13

$$(-x)^2 = 6$$

How many distinct real solutions does the given equation have?

A) Zero

B) Exactly one

C) Exactly two

D) Infinitely many

14

$$3|5 - x| = 42 - 4|5 - x|$$

What is the positive solution to the given equation?

15

$$(x - 4)^2 - a = x^2 + bx + b$$

In the given equation, a and b are constants. If the equation is true for all values of x, what is the value of a ?

16

$$|x| = x$$

How many solutions does the given equation have?

A) Zero

B) Exactly one

C) Exactly two

D) Infinitely many

17

The expressions $bx^2 + 11x$ and $x(4x + a) - 3x$, where a and b are constants, are equivalent. What is the value of $a + b$?

A) 12

B) 15

C) 18

D) 21

18

The expression $(ax + 5)(bx - 5)$, where a and b are constants, can be rewritten as $cx^2 + 15x - 25$, where c is a constant. What is the value of $b - a$?

19

The expression $12x^2 - mx - 20$, where m is a constant, is equivalent to $(kx - 5)(nx + 4)$, where k and n are integer constants. Which of the following must be an integer?

A) $\frac{12}{m}$

B) $\frac{12}{n}$

C) $\frac{m}{k}$

D) $\frac{m}{n}$

EXERCISE 2: Answers for this exercise start on page 326.

1

$$ax - b = 3(2x + 1)$$

In the given equation, a and b are constants. The equation has no solution. Which of the following could be the values of a and b ?

A) $a = 2$ and $b = -3$
B) $a = 2$ and $b = 3$
C) $a = 6$ and $b = -3$
D) $a = 6$ and $b = 3$

2

$$|-8x - 12| = 34$$

Exactly how many solutions does the given equation have?

A) Zero
B) One
C) Two
D) Four

3

$$9 - 6x = -\frac{1}{11}(66x - 99)$$

How many solutions does the given equation have?

A) Zero
B) Exactly one
C) Exactly two
D) More than two

4

$$\frac{3}{2}(x + m) = mx$$

In the given equation, m is a constant. If the equation has exactly one solution, which of the following could NOT be the value of m ?

A) $-\frac{3}{2}$
B) 0
C) $\frac{1}{2}$
D) $\frac{3}{2}$

5

The expression $1.56x^2 - 0.84x - 1.08$ can be rewritten as $n(26x^2 - 14x - 18)$, where n is a constant. What is the value of n ?

6

$$x - \frac{1}{2}(3x + 8) = 2\left(2 - \frac{1}{4}x\right)$$

How many solutions are there to the given equation?

A) Zero
B) Infinitely many
C) Exactly one
D) Exactly two

7

$$\frac{2}{3}ax + 3 = \frac{8}{3}x + 9b$$

In the given equation, a and b are constants. The equation has infinitely many solutions. What is the value of $\frac{a}{b}$?

A) $\frac{3}{4}$
B) $\frac{4}{3}$
C) 6
D) 12

8

$$3x + a(3 - 2x) = 12 - 7x$$

In the given equation, a is a constant. The equation has no solution. What is the value of a ?

A) -2
B) 2
C) 4
D) 5

9

$$2|10 - 4x| + 9 = 77$$

What is the sum of the solutions to the given equation?

10

$$\sqrt{-x} = x$$

How many distinct real solutions does the given equation have?

A) Zero
B) Exactly one
C) Exactly two
D) Infinitely many

11

$$|x - 13| = |x + 5|$$

What is the solution to the given equation?

A) -8
B) -4
C) 4
D) 8

12

$$-10x + 16jx = 0$$

In the given equation, j is a constant. The equation has infinitely many solutions. What is the value of j ?

13

$$g(x) = \left|\frac{7}{9}x - 40\right|$$

The function g is defined by the given equation. For which of the following values of a does $g(a) = a$?

A) -180
B) $\frac{27}{2}$
C) $\frac{45}{2}$
D) $\frac{360}{7}$

14

How many solutions does the equation $(x + 1)(x - 1) = x(x - 1)$ have?

A) Zero
B) One
C) Two
D) Infinitely many

15

$$34x^2 = -68x^2$$

How many solutions does the given equation have?

A) Zero
B) Exactly one
C) Exactly two
D) Infinitely many

16

$$\frac{1}{3}(bx + c) = \frac{14}{27}x + \frac{4}{9}$$

In the given equation, b and c are constants and $c > 3$. The equation has no solution. What is the value of b ?

17

$$(2x + 3)(ax - 5) = 12x^2 + bx - 15$$

In the given equation, a and b are constants. If the equation is true for all values of x, what is the value of b ?

A) 6
B) 8
C) 10
D) 12

18

The expression $5x^5 + 6x^4 - 8x^3$ can be rewritten as $(x^3 - hx^2)(5x^2 + 10x)$, where h is a constant. What is the value of h ?

19

$$18x^2 - 8 = 2(ax + b)(ax - b)$$

In the given equation, a and b are constants. If the equation is true for all values of x, which of the following could be the value of ab ?

A) 6
B) 9
C) 12
D) 36

20

The expression $(x + a)(x + 6)$, where a is a constant, is equivalent to $x^2 + bx + c$, where b and c are positive integer constants. Which of the following must be an integer?

A) $\frac{c}{6}$
B) $\frac{b}{a}$
C) $\frac{c}{b}$
D) $\frac{6}{a}$

21

$$4x^2 + mx + 9 = (2x + n)^2$$

In the given equation, m and n are constants and $n < 0$. If the equation is true for all values of x, what is the value of $m + n$?

A) -15
B) -9
C) -3
D) 12

5 Functions

A function is a machine that takes an input, transforms it, and spits out an output. In math, functions are typically denoted by $f(x)$, with x being the input. So for the function

$$f(x) = x^2 + 1$$

every input is squared and then added to one to get the output. It's important to understand that x is a completely arbitrary label—it's just a placeholder for the input. Anything can be an input, including other functions and expressions defined in terms of x:

$$f(2x) = (2x)^2 + 1$$
$$f(a) = a^2 + 1$$
$$f(b + 1) = (b + 1)^2 + 1$$
$$f(g(x)) = (g(x))^2 + 1$$
$$f(\text{Panda}) = (\text{Panda})^2 + 1$$

Note the careful use of parentheses. In the first equation, for example, writing $2x^2$ instead of $(2x)^2$ would be incorrect. Wrap the input in parentheses and you'll never go wrong.

EXAMPLE 1: The function f is defined by $f(x) = (x + 1)^x$. What is the value of $f(0) + f(1) + f(2) + f(3)$?

Just plug in the given inputs.

$$\begin{aligned} f(0) + f(1) + f(2) + f(3) &= (0 + 1)^0 + (1 + 1)^1 + (2 + 1)^2 + (3 + 1)^3 \\ &= 1^0 + 2^1 + 3^2 + 4^3 \\ &= 1 + 2 + 9 + 64 \\ &= \boxed{76} \end{aligned}$$

EXAMPLE 2: The function g is defined by $g(x) = 7x + 8$. For what value of x is $g(x) = -13$?

Here, we're asked to find the input that results in an output of -13. To find this input, we can set up an equation:

$$\begin{aligned} g(x) &= -13 \\ 7x + 8 &= -13 \\ 7x &= -21 \\ x &= \boxed{-3} \end{aligned}$$

EXAMPLE 3: The functions f and g are defined by $f(x) = 3x - 2$ and $g(x) = 2x - 18$. If the function h is defined by $h(x) = f(x) + g(x)$, what is the value of $h(9)$?

Functions can be added, subtracted, multiplied, and divided to create new functions. Here, function h is created by adding functions f and g:

$$h(x) = f(x) + g(x) = (3x - 2) + (2x - 18) = 5x - 20$$

Therefore, $h(9) = 5(9) - 20 = \boxed{25}$.

EXAMPLE 4: Functions f and g are defined by $f(x - 1) = 6x$ and $g(x) = x + 3$. What is the value of $f(g(2))$?

Whenever you see composite functions (functions that are written inside of another function), start from the inside and work your way out. First,

$$g(2) = 2 + 3 = 5$$

Now we have to figure out the value of $f(5)$.

Well, we can plug in $x = 6$ into $f(x - 1)$ to get $f(5) = 6(6) = \boxed{36}$. Don't be thrown off by the $x - 1$ in $f(x - 1)$. It simply means that $x - 1$ is the input rather than x. Remember that the input can be any expression.

EXAMPLE 5: Functions f and g are defined by $f(x) = x + 1$ and $g(x) = \frac{x}{2}$. If $f(g(k)) = 10$, what is the value of k ?

Again, we start from the inside and work our way out. Since $g(k) = \frac{k}{2}$,

$$f(g(k)) = f\left(\frac{k}{2}\right) = \frac{k}{2} + 1$$

Setting this expression equal to 10, we get

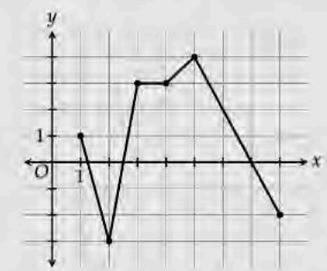
$$\begin{aligned} \frac{k}{2} + 1 &= 10 \\ \frac{k}{2} &= 9 \\ k &= \boxed{18} \end{aligned}$$

As we've mentioned, a function takes an input and returns an output. Well, these input and output pairs allow us to graph any function as a set of points in the xy-plane, with the input as x and the output as y. Questions on the SAT will often use phrases like "the graph of $y = f(x)$" or "the graph of $y = g(x)$" to make it clear that the output of a function is being represented as the y-coordinate. In other words, a function like $f(x) = x^2 + 1$ can be viewed as $y = x^2 + 1$.

Essentially, in a graphing context, $f(x)$ and y are interchangeable. So, a statement that $f(x) > 0$, for example, means that all the points on the graph have positive y-values (i.e. the graph is always above the x-axis).

Learning to think of points on a graph as the inputs and outputs of a function will help you solve many SAT questions.

EXAMPLE 6:



The graph of $y = f(x)$ is shown in the xy-plane. For what value of x does $f(x)$ reach its maximum?

Again, when it comes to graphs, think of $f(x)$ as the y. So we're looking for the point on the graph with the highest y-value, the "peak" of the graph. That point is $(5, 4)$, so the value of x we're looking for is $\boxed{5}$.

EXAMPLE 7: The function f is defined by $f(x) = ax^2 + 3$, where a is a constant. The graph of $y = f(x)$ in the xy-plane passes through the point $(1, 2)$. What is the value of a ?

Remember—a point is just an input and an output, an x and a y. Because $(1, 2)$ is a point on the graph of the function, we can plug in 1 for x and 2 for y. Doing so gives us an equation to solve for a:

$$\begin{aligned} 2 &= a(1)^2 + 3 \\ 2 &= a + 3 \\ a &= \boxed{-1} \end{aligned}$$

EXAMPLE 8: The function f is defined by $f(x) = 3(6)^x - 8$. What is the y-intercept of the graph of $y = f(x)$ in the xy-plane?

A) $(0, -8)$ B) $(0, -5)$ C) $(0, 6)$ D) $(0, 10)$

The y-intercept is the point at which the graph crosses the y-axis. The y-intercept always has an x-coordinate of 0, so to find the y-coordinate, we can just plug $x = 0$ into the function:

$$f(0) = 3(6)^0 - 8 = 3(1) - 8 = -5$$

Therefore, the y-intercept of the graph is $(0, -5)$. Answer $\boxed{(B)}$.

EXAMPLE 9: The function g is defined by $g(x) = 8x + 4$. What is the x-intercept of the graph of $y = g(x)$ in the xy-plane?

A) $(-8, 0)$ B) $(-2, 0)$ C) $\left(-\frac{1}{2}, 0\right)$ D) $(4, 0)$

An x-intercept is a point at which the graph crosses the x-axis. By definition, x-intercepts have a y-coordinate of 0, so to find the x-coordinate, we can set $g(x) = 0$ and solve for x:

$$\begin{aligned} 8x + 4 &= 0 \\ 8x &= -4 \\ x &= -\frac{1}{2} \end{aligned}$$

In this case, we just have one x-intercept at $\left(-\frac{1}{2}, 0\right)$. Answer $\boxed{(C)}$.

EXAMPLE 10:

$$q(x) = x(x + 5)(x - 6)$$

If the given function q is graphed in the xy-plane, where $y = q(x)$, what is an x-intercept of the graph?

A) $(-6, 0)$ B) $(-5, 0)$ C) $(-1, 0)$ D) $(5, 0)$

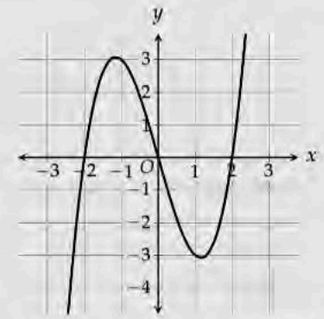
To find the x-intercepts, we set $q(x) = 0$ and solve for x:

$$x(x + 5)(x - 6) = 0$$

The product on the left side of the equation is equal to 0 only when $x = 0$, $x + 5 = 0$, or $x - 6 = 0$. The solutions are then $x = 0, -5$, and 6, which means the x-intercepts are at $(0, 0)$, $(-5, 0)$, and $(6, 0)$. Of these, only $(-5, 0)$ is in the answer choices. Answer $\boxed{(B)}$.

Note that x, $x + 5$, and $x - 6$ are **factors** of $q(x)$. These factors make it easy to identify the x-intercepts.

EXAMPLE 11:



The graph of $y = h(x)$ is shown, where the function h is defined by $h(x) = ax^3 + bx$, where a and b are constants. For how many values of x does $h(x) = 0$?

We can reword this question in several ways: How many different inputs result in an output of 0? How many points on the graph have a y-value of 0?

Essentially, this question is just asking for the number of x-intercepts. Since the given graph crosses the x-axis 3 distinct times, there are $\boxed{3}$ values of x that result in $h(x) = 0$.

Note that the x-intercepts of the graph are $(-2, 0)$, $(0, 0)$, and $(2, 0)$. The x-values of $-2, 0$, and 2 indicate that $(x + 2)$, x, and $(x - 2)$, respectively, must be factors of $h(x)$. With these factors, the equation of $h(x)$ should look something like

$$h(x) = kx(x - 2)(x + 2)$$

where k is a constant.

The **zeros, roots, and x-intercepts** of a function are all just different terms for the same thing—the values of x that make $f(x) = 0$ (or $y = 0$). Graphically, they refer to the values of x where the function crosses the x-axis.

If the graph of a function has an x-intercept at $(n, 0)$, then $(x - n)$ must be a factor of that function.

EXAMPLE 12:

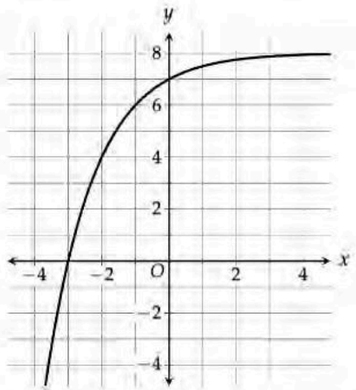
$$f(x) = \frac{4}{(x - 5)^2}$$

The function f is given. For what value of x is $f(x)$ undefined?

A function is undefined for any input that results in an operation that is "not allowed" or gives an undefined answer. Division by zero is the most common case. Since $f(x)$ has a denominator of zero when $x = 5$, $f(x)$ is undefined when $x = \boxed{5}$.

EXERCISE 1: Answers for this exercise start on page 329.

1



What is the y-intercept of the graph shown?

A) $(-3, 0)$
B) $(7, 0)$
C) $(0, -3)$
D) $(0, 7)$

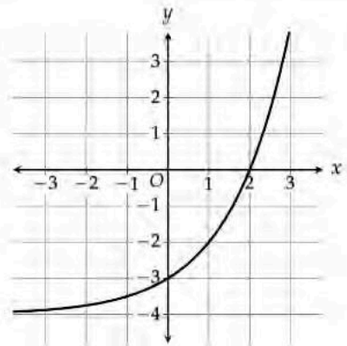
2

x	$f(x)$
0	20
1	21
3	29

For the function f, the table shows three values of x and their corresponding values of $f(x)$. Which equation could define f ?

A) $f(x) = 20x$
B) $f(x) = x + 20$
C) $f(x) = x - 20$
D) $f(x) = x^2 + 20$

3



The graph of the exponential function f is shown in the xy-plane. For what value of x does $f(x) = 0$?

A) -4
B) -3
C) 0
D) 2

4

What is the y-coordinate of the y-intercept of the graph of $y = -4\left(\frac{1}{2}\right)^x + 9$?

5

The function f is defined by $f(x) = \frac{16 + x^2}{2x}$. What is the value of $f(-4)$?

A) -8
B) -4
C) 4
D) 8

6

A ball falls from a platform that is 150 feet above the ground. The function $h(t) = 150 - 16t^2$ gives the ball's height, in feet, above the ground t seconds after it falls. Which of the following is the best interpretation of $h(3) = 6$ in this context?

A) The ball is 3 feet above the ground 6 seconds after it falls.
B) The ball is 3 feet below the platform 6 seconds after it falls.
C) The ball is 6 feet above the ground 3 seconds after it falls.
D) The ball is 6 feet below the platform 3 seconds after it falls.

7

If $f(x) = x^2$, for which of the following values of c is $f(c) < c$?

A) $\frac{1}{2}$
B) 1
C) $\frac{3}{2}$
D) 2

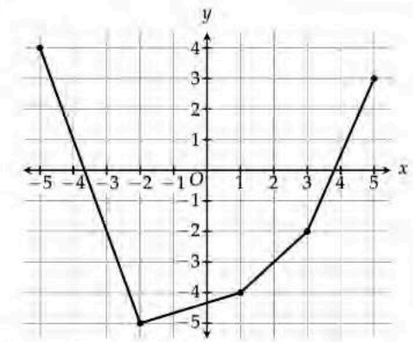
8

$$f(x) = 4x - 3$$
$$g(x) = 3x + 5$$

The functions f and g are defined by the given equations. Which of the following is equal to $f(8)$?

A) $g(1)$
B) $g(3)$
C) $g(5)$
D) $g(8)$

9



The graph of $y = f(x)$ is shown in the xy-plane. If $f(a) = f(3)$, which of the following could be the value of a ?

A) -4
B) -3
C) -2
D) 1

10

$$f(x) = x^2 + 1$$
$$g(x) = x^2 - 1$$

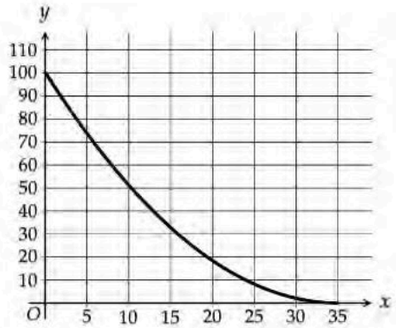
The functions f and g are defined by the given equations. What is the value of $f(g(2))$?

A) 3
B) 5
C) 10
D) 17

11

The function f is defined by $f(x) = 3x + 2$ and the function g is defined by $g(x) = f(2x) - 1$. What is the value of $g(10)$?

12



The graph gives the atmospheric pressure y, in kilopascals, that corresponds to an elevation of x kilometers above sea level. Which statement is the best interpretation of the y-intercept in this context?

A) The atmospheric pressure at sea level is about 35 kilopascals.
B) The atmospheric pressure at sea level is about 100 kilopascals.
C) The atmospheric pressure at 35 kilometers above sea level is about 0 kilopascals.
D) The atmospheric pressure at 100 kilometers above sea level is about 0 kilopascals.

13

The function f is defined by $f(x) = -3x + 5$, and $\frac{1}{2}f(a) = 10$, where a is a constant. What is the value of a ?

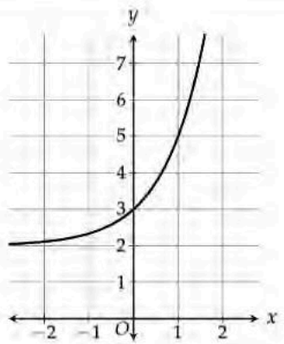
A) -8
B) -5
C) 5
D) 8

14

The graph of $y = f(x)$ in the xy-plane has exactly two x-intercepts at $(r, 0)$ and $(s, 0)$, where r and s are positive constants. Which of the following equations could define f ?

A) $f(x) = (x + r)(x + s)$
B) $f(x) = x(x + r)(x + s)$
C) $f(x) = (x - r)(x - s)$
D) $f(x) = x(x - r)(x - s)$

15



The graph of $y = 3^x + b$ is shown, where b is a constant. What is the value of b ?

A) 1
B) 2
C) 3
D) 4

16

Function f is defined by $f(x) = x(x + 2)(x + 6)$. Which table of values represents $y = f(x - 2)$?

A)

x	y
-8	0
-4	0
-2	0

B)

x	y
-4	0
0	0
2	0

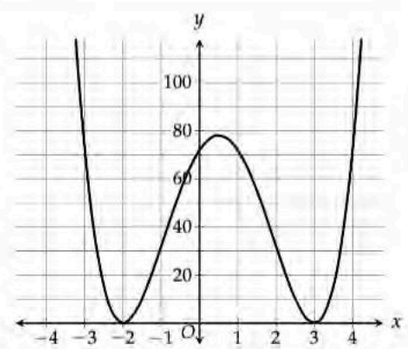
C)

x	y
-2	0
0	0
4	0

D)

x	y
2	0
4	0
8	0

17



Which of the following could be the equation of the graph shown?

A) $y = (x + 2)^2(x - 3)^2$
B) $y = (x - 2)^2(x + 3)^2$
C) $y = 2(x + 2)^2(x - 3)^2$
D) $y = 2(x - 2)^2(x + 3)^2$

18

Function f is defined by $f(x) = (x + 2)(x - 2)^2$. The graph of $y = f(x)$ in the xy-plane passes through the point (p, q), where $-2 \le p \le 2$. Which of the following is NOT a possible value of q ?

A) 12
B) 8
C) 4
D) 0

19

If $f(x + 1) = \frac{1}{4}f(x)$, which of the following equations could define the function f ?

A) $f(x) = \frac{1}{4}\left(\frac{1}{8}\right)^x$
B) $f(x) = 8\left(\frac{1}{4}\right)^x$
C) $f(x) = 8(4)^x$
D) $f(x) = \frac{1}{4}(8)^x$

20

x	0	1	2
$f(x)$	-2	3	18

Function f is defined by $f(x) = ax^2 + b$, where a and b are constants. The table gives three values of x and their corresponding values of $f(x)$. What is the value of $f(3)$?

A) 23
B) 39
C) 43
D) 56

EXERCISE 2: Answers for this exercise start on page 330.

1

The function $g(x) = 0.002x^2 + 5x$ models the revenue, in dollars, generated by an online advertisement that is clicked half as many times as it is viewed, where x is the number of times it is viewed. Which of the following is the best interpretation of $g(2{,}000) = 18{,}000$ in this context?

A) If the advertisement is clicked 2,000 times, then it will generate 18,000 dollars in revenue.
B) If the advertisement is viewed 2,000 times, then it will generate 18,000 dollars in revenue.
C) If the advertisement is viewed 18,000 times, then it will generate 2,000 dollars in revenue.
D) If the advertisement is clicked 18,000 times, then it will generate 2,000 dollars in revenue.

2

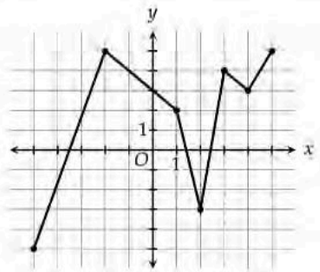
The function f is defined by $f(x) = \frac{x+18}{x-6}$. For what value of x does $f(x) = -1$?

3

Function g is defined by $g(x) = \sqrt{3x}$. If $g(a) = 6$, what is the value of a ?

A) 3
B) 6
C) 9
D) 12

4



The graph of $y = f(x)$ is shown in the xy-plane. For how many values of x does $f(x) = 3$?

A) Two
B) Three
C) Four
D) Five

5

The function f is defined by $f(x) = \frac{a^{-x}}{b}$, where a and b are positive constants. What is the y-intercept of the graph of $y = f(x)$ in the xy-plane?

A) $\left(0, \frac{1}{b}\right)$
B) $\left(0, \frac{a}{b}\right)$
C) $\left(0, \frac{1}{ab}\right)$
D) $(0, 1)$

6

x	$f(x)$	$g(x)$
−2	3	4
−1	5	2
0	−2	−3
1	3	5
2	6	7
3	7	1

For the functions f and g, the table gives six values of x and their corresponding values of $f(x)$ and $g(x)$. If $g(c) = 5$, where c is a constant, what is the value of $f(c)$?

A) −2
B) 3
C) 5
D) 6

7

$$f(x) = -x(x-a)(x+a)(x+b)(x+b)$$

In the given function f, a and b are positive constants and $a \neq b$. For how many distinct values of x does $f(x) = 0$?

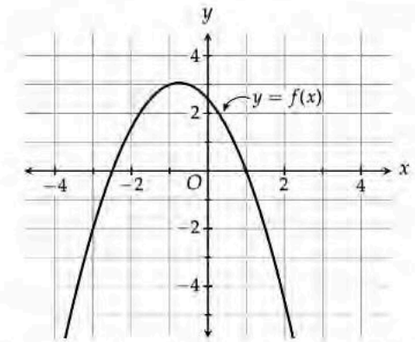
A) Two
B) Three
C) Four
D) Five

8

In the xy-plane, the graph of $y = f(x)$ reaches its maximum value at the point $(3, f(3))$. The function g is defined by $g(x) = f(x) + 7$. At which of the following points in the xy-plane does the graph of $y = g(x)$ reach its maximum value?

A) $(10, f(10) + 7)$
B) $(f(3), f(3) + 7)$
C) $(3, f(10))$
D) $(3, f(3) + 7)$

9



The graph of $y = f(x)$ is shown in the xy-plane. If the function g is defined by $g(x) = (x+3)(x-1)$, for which of the following values of x is $f(x) > g(x)$?

A) −3
B) −2
C) 1
D) 2

10

x	$f(x)$
−4	3
−2	5
0	2
2	16
3	4
4	8

For the function f, the table shows several values of x and their corresponding values of $f(x)$. Function g is defined by $g(x) = 2f(x)$. What is the value of k if $g(k) = 8$?

A) 2
B) 3
C) 4
D) 8

11

The function f is defined by $f(x) = \frac{(x+3)(12-x)}{x}$. The value of $f(a-9)$ is 0, where a is a constant. What is the greatest possible value of a ?

12

Function g is defined by $g(x) = \frac{x-8}{2}$. The graph of $y = g(x)$ in the xy-plane passes through the points $(20, c)$ and (c, d), where c and d are constants. What is the value of d ?

A) -5
B) -1
C) 3
D) 6

13

The graph of $y = f(x)$ has a y-intercept at $(0, 12)$ and x-intercepts at $(-3, 0)$ and $(2, 0)$. Which of the following equations could define f ?

A) $f(x) = (x+3)^2(x-2)$
B) $f(x) = (x+3)(x-2)^2$
C) $f(x) = (x-3)^2(x+2)$
D) $f(x) = (x-3)(x+2)^2$

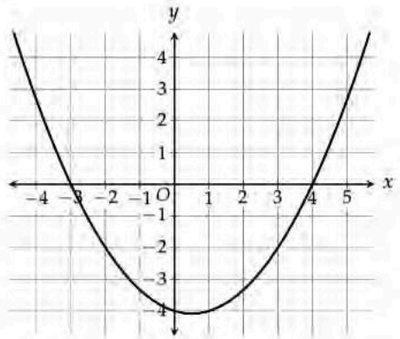
14

$$f(x) = \sqrt{x-2}$$

The function f is defined by the given equation. Which of the following is equal to $f(18) - f(11)$?

A) $f(3)$
B) $f(5)$
C) $f(6)$
D) $f(7)$

15



An equation for the graph shown is $y = a(x+3)(x+b)$, where a and b are constants. What is the value of ab ?

A) -2
B) $-\frac{4}{3}$
C) $\frac{1}{2}$
D) 8

16

The function f is defined by $f(x) = \frac{11}{3}x + 34$, and the function g is defined by $g(x) = \frac{5}{3}x - 8$. If the function h is defined by $h(x) = f(x) - g(x)$, what is the x-coordinate of the x-intercept of the graph of $y = h(x)$ in the xy-plane?

17

The function f is defined by $f(x) = 125 \cdot a^x - b$, where a and b are constants. In the xy-plane, the graph of $y = f(x)$ has an x-intercept at $(3, 0)$ and a y-intercept at $(0, b-3)$. What is the value of b ?

18

The function f is defined by $f(x) = \frac{x+32}{x+b}$, where b is a constant. If $f(-8) = 4$, what is the value of $f(10)$?

19

The function f is defined by $f(x) = 7(4)^{-x} + 13$. Which of the following statements must be true?

I. $f(x) > 7$
II. $f(x) \leq 20$

A) I only
B) II only
C) Neither I nor II
D) I and II

20

The function f is defined by $f(x) = b \cdot a^x$, where a and b are positive constants, and $g(x) = f(x+2)$. The graph of $y = f(x)$ in the xy-plane passes through the points $(0, 36)$ and $\left(-3, \frac{9}{16}\right)$. Which equation defines the function g ?

A) $g(x) = 216 \cdot 3^x$
B) $g(x) = 324 \cdot 3^x$
C) $g(x) = 288 \cdot 4^x$
D) $g(x) = 576 \cdot 4^x$

21

Function f is defined by $f(x) = a\sqrt{x} + b$, where a and b are constants. In the xy-plane, the graph of $y = f(x)$ has an x-intercept at $(16, 0)$, and the graph of $y = f(x) - 10$ has an x-intercept at $(9, 0)$. What is the value of b ?

22

$$f(x) = b\sqrt{a-x} + 7\sqrt{a-x}$$

For the function f, a and b are constants and $f(16) = 0$. If $f(9) > 0$, which of the following could be the value of $a + b$?

A) -3
B) 7
C) 9
D) 15

6

Lines & Linear Relationships

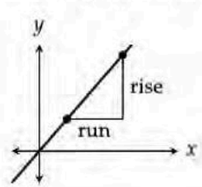
Lines in the xy-plane represent functions of the form $f(x) = mx + b$, which are often referred to as linear functions. Let's dive into some properties you'll need to know.

Given any two points (x_1, y_1) and (x_2, y_2) on a line, the

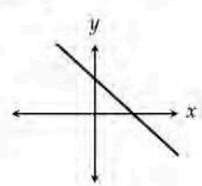
$$\text{slope of the line} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope is a measure of the steepness of a line—the larger the slope, the more steep the line is. The rise is the distance between the y coordinates and the run is the distance between the x coordinates. A slope of 2 means that the line goes 2 units up for every 1 unit to the right, or 2 units down for every 1 unit to the left. A slope of $-\frac{2}{3}$ means that the line goes 2 units down for every 3 units to the right, or 2 units up for every 3 units to the left.

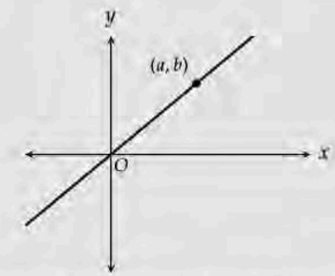
Lines with positive slope always go up and to the right as shown below.



Lines with negative slope go down and to the right:



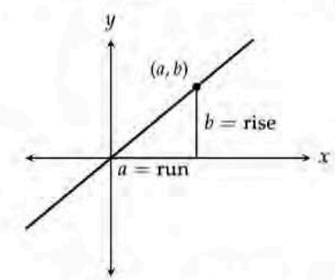
EXAMPLE 1:



The line shown in the xy-plane passes through the point $(0, 0)$ and the point (a, b), where $a > b$. Which of the following could be the slope of this line?

A) $-\frac{1}{2}$ B) $\frac{3}{4}$ C) 1 D) $\frac{3}{2}$

First, notice that the slope is positive. The slope, $\frac{\text{rise}}{\text{run}}$, is also equal to $\frac{b}{a}$.



Since $a > b$, $\frac{b}{a}$ is always less than 1. For example, if $a = 5$ and $b = 3$, the slope would be $\frac{3}{5}$. The only choice that's both positive and less than 1 is answer $\boxed{(B)}$.

EXAMPLE 2: Line m passes through the points $(k, 7)$ and $(3, k-4)$ in the xy-plane. If the slope of line m is 3, what is the value of k ?

$$\text{Slope} = \frac{(k-4)-7}{3-k} = 3$$

$$k - 11 = 3(3 - k)$$

$$k - 11 = 9 - 3k$$

$$4k = 20$$

$$k = \boxed{5}$$

In addition to slope, you also need to know what x and y intercepts are. The x-intercept is where the graph crosses the x-axis. Likewise, the y-intercept is where the graph crosses the y-axis.

Let's say we have the line defined by

$$2x + 3y = 12$$

To find the x-intercept, set y equal to 0.

$$\begin{aligned} 2x + 3(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

The x-intercept is $(6, 0)$.

To find the y-intercept, set x equal to 0.

$$\begin{aligned} 2(0) + 3y &= 12 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

The y-intercept is $(0, 4)$.

EXAMPLE 3: The function f is defined by $f(x) = 3x + 12$. What is the x-intercept of the graph of $y = f(x)$ in the xy-plane?

A) $(12, 0)$ B) $(4, 0)$ C) $(-4, 0)$ D) $(-12, 0)$

The graph of the function in the xy-plane is defined by the equation $y = 3x + 12$. To find the x-intercept, set y equal to 0:

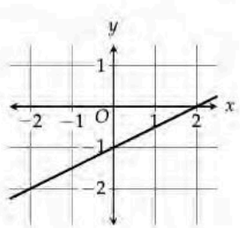
$$\begin{aligned} 0 &= 3x + 12 \\ -12 &= 3x \\ -4 &= x \end{aligned}$$

Therefore, the x-intercept of the graph is $(-4, 0)$, answer $\boxed{\text{(C)}}$.

All lines can be expressed in **slope-intercept form**:

$$y = mx + b$$

where m is the slope and b is the y-intercept. So for the line $y = \frac{1}{2}x - 1$, the slope is $\frac{1}{2}$ and the y-intercept is -1:



While all lines can be expressed in slope-intercept form, sometimes it'll take some work to get there. If you're given a slope and a y-intercept, then of course it's most convenient to express the line in slope-intercept form. But what if we're handed a slope and a point instead of a slope and a y-intercept? Then it'll be more convenient to use **point-slope form**:

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is the given point.

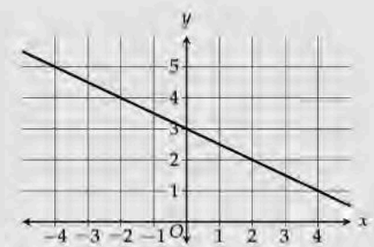
Let's say we want to find the equation of a line that has a slope of 3 and passes through the point $(1, -2)$. The equation of the line in point-slope form is then

$$y - (-2) = 3(x - 1)$$

Once it's in this form, we can manipulate the equation to get the slope-intercept form of the line if needed:

$$\begin{aligned} y - (-2) &= 3(x - 1) \\ y + 2 &= 3x - 3 \\ y &= 3x - 5 \end{aligned}$$

EXAMPLE 4:



What is an equation of the graph shown?

A) $y = -2x + 3$ B) $y = \frac{1}{2}x + 3$ C) $y = -\frac{1}{2}x + 3$ D) $y = 2x - 3$

All the answer choices are in slope-intercept form. To get the equation of the line in slope-intercept form, $y = mx + b$, we need to find the slope m and the y-intercept b. The line crosses the y-axis at 3, so $b = 3$. The line goes down 1 unit for every 2 units to the right, so the slope m is $-\frac{1}{2}$. Therefore, the equation of the line is $y = -\frac{1}{2}x + 3$. Answer $\boxed{\text{(C)}}$.

EXAMPLE 5: Line l passes through the points $(-2,3)$ and $(3,13)$ in the xy-plane. What is the y-coordinate of the y-intercept of line l ?

The slope of line l is $\frac{y_2-y_1}{x_2-x_1}=\frac{13-3}{3-(-2)}=2$. With the given point $(3,13)$, we can use point-slope form to find the equation of the line:

$$y-13=2(x-3)$$

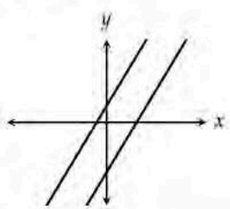
Note that we could've used the other point $(-2,3)$; the result will be the same. The next step is to simplify this equation to get the slope-intercept form of the line:

$$\begin{aligned} y-13&=2(x-3)\\ y&=2x-6+13\\ y&=2x+7 \end{aligned}$$

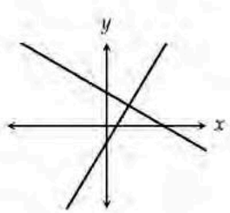
Now we can easily see that the y-coordinate of the y-intercept is $\boxed{7}$.

There are a few more things you need to know about lines.

Two lines are parallel if they have the same slope.



Two lines are perpendicular if the product of their slopes is -1. In other words, if one slope is the negative reciprocal of the other (e.g. 2 and $-\frac{1}{2}$).



EXAMPLE 6: Line m has a slope of $\frac{2}{3}$ and passes through the point $(4,3)$ in the xy-plane. Line n is perpendicular to line m and passes through the same point $(4,3)$. Which equation defines line n ?

A) $y=-\frac{2}{3}x+9$ B) $y=-\frac{3}{2}x-3$ C) $y=-\frac{3}{2}x+6$ D) $y=-\frac{3}{2}x+9$

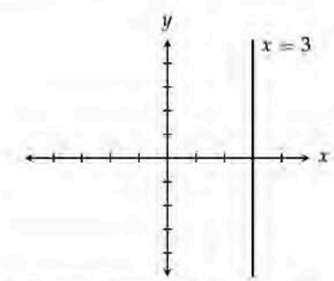
Because it's perpendicular to line m, line n must have a slope of $-\frac{3}{2}$ (the negative reciprocal). Now we can use point-slope form and simplify:

$$\begin{aligned} y-3&=-\frac{3}{2}(x-4)\\ y&=-\frac{3}{2}x+6+3\\ y&=-\frac{3}{2}x+9 \end{aligned}$$

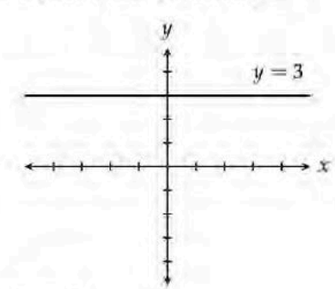
Answer $\boxed{(D)}$.

Finally, let's take a look at the equations of horizontal and vertical lines.

The equation of the vertical line that passes through $(3,0)$ is $x=3$. Every point on the line has an x-coordinate of 3.



The equation of the horizontal line that passes through $(0,3)$ is $y=3$.



Every point on the line has a y-coordinate of 3.

EXERCISE 1: Answers for this exercise start on page 333.

1

What is the slope of a line that is perpendicular to the graph of $y = -\frac{1}{4}x + 5$ in the xy-plane?

2

What is the x-intercept of the graph of $8x + 12y = 48$ in the xy-plane?

A) $(4, 0)$
B) $(6, 0)$
C) $(8, 0)$
D) $(12, 0)$

3

In the xy-plane, the y-intercept of the graph of $5x - 8y = 40$ is $(0, c)$. What is the value of c ?

4

In the xy-plane, line w has a slope of -8 and passes through the point $\left(\frac{1}{2}, -1\right)$. Which equation defines line w ?

A) $y = \frac{1}{2}x - 1$
B) $y = -8x - 5$
C) $y = -8x + 3$
D) $y = 8x - 5$

5

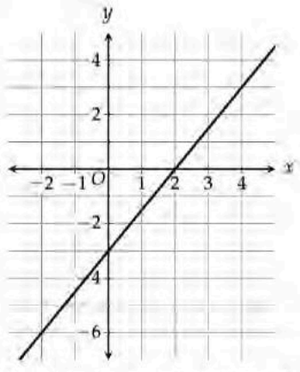
What is the slope of the graph of $y = -11x - \frac{2}{5}(18 - 6x)$ in the xy-plane?

6

In the xy-plane, a line has an x-intercept at $(-2, 0)$ and a y-intercept at $(0, -4)$. What is the slope of the line?

A) -2
B) $-\frac{1}{2}$
C) $\frac{1}{2}$
D) 2

7



The graph of line l is shown in the xy-plane. Which of the following is an equation of a line that is parallel to line l ?

A) $y = -\frac{2}{3}x + 2$
B) $y = \frac{2}{3}x + 10$
C) $y = \frac{3}{2}x - 4$
D) $y = 3x - 1$

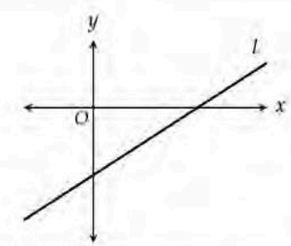
8

Day	Average speed, s (miles per hour)	Number of calories burned, c
Monday	7.2	616
Thursday	6.8	584
Friday	7.9	672
Saturday	8.5	720

On certain days of the week, Elaine runs for an hour on a treadmill. For each day that she ran on the treadmill last week, the table shows the average speed s at which she ran, in miles per hour, and the number of calories c she burned during the run. If the relationship between c and s can be modeled by a linear function, which of the following functions models this relationship?

A) $c(s) = 30s + 400$
B) $c(s) = 60s + 210$
C) $c(s) = 80s + 40$
D) $c(s) = 90s - 30$

9



The graph of line l is shown in the xy-plane. If line l can be represented by the equation $y = mx + b$, where m and b are constants, which of the following must be true?

A) $mb > 0$
B) $mb < 0$
C) $mb = 0$
D) $mb = 1$

10

Line k passes through the point $(1, 5)$ in the xy-plane. If line k is perpendicular to the line with equation $y = \frac{1}{2}x + 3$, which equation defines line k ?

A) $y = -2x + 3$
B) $y = -2x + 7$
C) $y = -\frac{1}{2}x + \frac{11}{2}$
D) $y = \frac{1}{2}x + \frac{9}{2}$

11

In the xy-plane, a line with a slope of $\frac{1}{3}$ passes through the points $(-1, 1)$ and $(5, n)$. What is the value of n ?

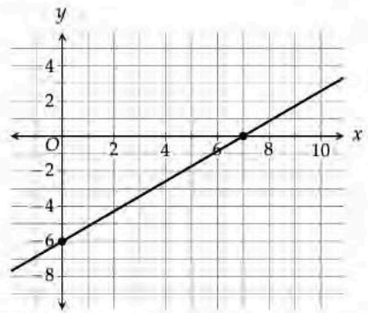
A) 9
B) 4
C) 3
D) $\frac{7}{3}$

12

x	$f(x)$
-3	10
3	14
9	18

The table shows several values of x and their corresponding values of $f(x)$. There is a linear relationship between x and $f(x)$ that is defined by the equation $f(x) = ax + b$, where a and b are constants. What is the value of ab ?

13



Line p is shown in the xy-plane, and the point with coordinates $(5, r)$ lies on line p. What is the value of r ?

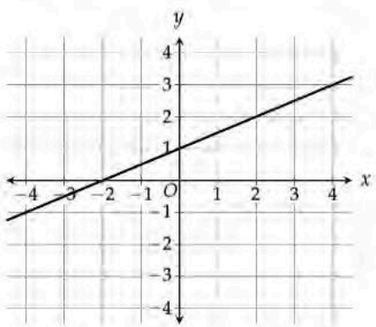
A) $-\frac{5}{3}$

B) $-\frac{11}{6}$

C) $-\frac{11}{7}$

D) $-\frac{12}{7}$

14



Line r is shown in the xy-plane. Line s (not shown) is parallel to line r and passes through the point $(1, -1)$. Which of the following points lies on line s ?

A) $(-3, -3)$

B) $(-2, -2)$

C) $(0, -2)$

D) $(4, 0)$

15

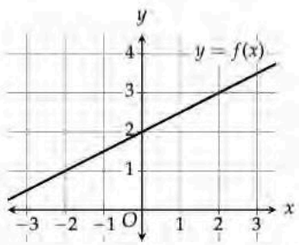
$$6x + y = 17$$
$$hx - 9y = -8$$

In the given pair of equations, h is a constant. The graphs of the equations in the xy-plane are perpendicular lines. What is the value of h ?

16

A line in the xy-plane passes through the point $(6, -3)$ and crosses the x-axis at the point $(15, 0)$. If the line also passes through the point $(-9, t)$, what is the value of t ?

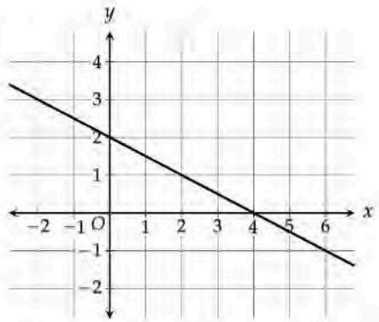
17



In the xy-plane, the graph of the linear function f is perpendicular to the graph of the linear function g (not shown). If the graphs of f and g intersect at the point $\left(1, \frac{5}{2}\right)$, what is the value of $g(-1)$?

EXERCISE 2: Answers for this exercise start on page 335.

1



The graph of the linear function f is shown in the xy-plane. What is the slope of the graph of f ?

A) -2

B) $-\frac{1}{2}$

C) $\frac{1}{4}$

D) $\frac{1}{2}$

2

Line j is defined by $4x + 20y = 5$. Line k is parallel to line j in the xy-plane. What is the slope of line k ?

A) -5

B) $-\frac{1}{5}$

C) $\frac{1}{5}$

D) 5

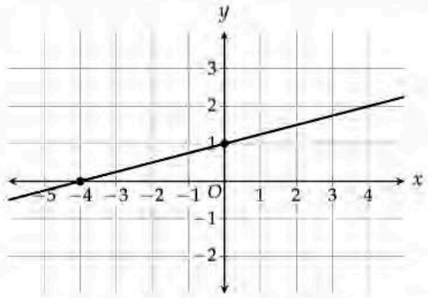
3

The linear function f is defined by $f(x) = ax + b$, where a and b are constants. If the graph of $y = f(x)$ in the xy-plane has a slope of 6 and passes through the point $(-5, -14)$, what is the value of b ?

4

What is the y-coordinate of the y-intercept of the graph of $7x = 6\left(15 + \frac{9x}{4} - \frac{4y}{7}\right)$ in the xy-plane?

5



The graph of the equation $ax + by = -4$, where a and b are constants, is shown in the xy-plane. What is the value of ab ?

6

A line in the xy-plane is defined by the equation $y = ax + b$, where a and b are constants. The line passes through the points $(0, -79)$ and $(5, 91)$. What is the value of $a + b$?

7

In the xy-plane, the line with equation $ax - \frac{1}{3}y = 8$, where a is a constant, passes through the point $(2, 6)$. What is the x-coordinate of the x-intercept of the line?

8

x	y
-6	$-114 - p$
-2	$-57 - p$
2	$-p$

The table shows three values of x and their corresponding values of y, where p is a constant. There is a linear relationship between x and y. What is the slope of the line that represents this relationship in the xy-plane?

A) $-\frac{57}{4}$

B) $\frac{57}{4}$

C) $\frac{p-57}{-4}$

D) $\frac{2p+57}{4}$

9

The graph of the equation $5x - 8y = k$, where k is a constant, is a line in the xy-plane. The line has an x-intercept at $(m, 0)$ and a y-intercept at $(0, n)$, where m and n are constants. What is the value of $\frac{m}{n}$?

A) $-\frac{8}{5}$

B) $-\frac{5}{8}$

C) $\frac{5}{8}$

D) $\frac{8}{5}$

10

In the xy-plane, the graph of the linear function f has a negative slope and a y-intercept at $(0, k)$, where k is a positive constant. Which of the following must be true?

I. If $a > b$, then $f(a) < f(b)$.
II. If $a < 0$, then $f(a) > k$.
III. If $a > 0$, then $f(a) > 0$.

A) I and II only

B) I and III only

C) II and III only

D) I, II, and III

11

In the xy-plane, the points $(-3, 5)$ and $(6, 8)$ lie on line l. Which of the following points also lies on line l ?

A) $(27, 15)$

B) $(33, 19)$

C) $(42, 23)$

D) $(54, 27)$

12

$$y = \frac{a}{b}x + c$$

$$y = \frac{d}{e}x + c$$

In the given pair of equations, a, b, c, d, and e are constants. The pair of equations represent two perpendicular lines in the xy-plane. If $0 < \frac{a}{b} < 1$, which of the following must be true?

A) $\frac{d}{e} < -1$

B) $-1 < \frac{d}{e} < 0$

C) $0 < \frac{d}{e} < 1$

D) $\frac{d}{e} > 1$

13

In the xy-plane, line p passes through the point $(1, n)$, where n is a constant. Line q intersects line p at the point $(0, 0)$ and is perpendicular to line p. Which of the following points must lie on line q ?

A) $(-n, -1)$

B) $(n, -1)$

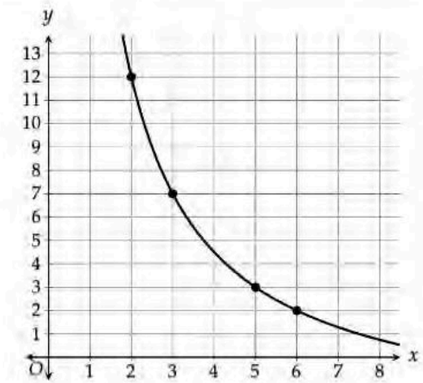
C) $(n, 1)$

D) $(1, -n)$

14

In the xy-plane, line m has an x-intercept at $(a, b + 4)$, where a and b are constants. If line m passes through the points $(b, -11)$ and $(b + 6, 7)$, what is the value of a ?

15



The function g is defined by $g(x) = \frac{f(x) + 6}{x}$, where f is a linear function. The partial graph of $y = g(x)$ is shown. What is the y-intercept of the graph of $y = f(x)$ in the xy-plane?

A) $(0, 8)$

B) $(0, 24)$

C) $(0, 30)$

D) $(0, 48)$

16

In the xy-plane, the line with equation $ax - by = 5$, where a and b are constants, is perpendicular to the line with equation $11x - 4y = 5$. Which of the following equations represents a line that is perpendicular to the line with equation $11x + 12y = 5$?

A) $ax - 3by = 5$

B) $ax + 3by = 5$

C) $3ax - by = 5$

D) $3ax + by = 5$

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7

Interpreting Linear Equations & Models

On the SAT, you will encounter linear model questions that are a direct extension of our previous chapters on lines and word problems. You'll be asked to figure out the meaning of certain variables and numbers in these models within a real world context.

EXAMPLE 1:

$$V(t) = 240{,}000 - 5{,}000t$$

The given function V models the value, in dollars, of a home from 2006 to 2015, where t is the number of years after 2006. Which of the following is the best interpretation of 240,000 in this context?

A) The value, in dollars, of the home in 2006
B) The value, in dollars, of the home in 2015
C) The average value, in dollars, of the home from 2006 to 2015
D) The increase, in dollars, in the value of the home from 2006 to 2015

We're given a linear function in $y = mx + b$ form. The y-intercept b will typically designate an initial value, the value when $x = 0$. In this case, the y-intercept is 240,000 and it describes the value of the home when $T = 0$, zero years after 2006, which, of course, is 2006. Answer $\boxed{(A)}$.

EXAMPLE 2:

$$V(t) = 240{,}000 - 5{,}000t$$

The given function V models the value, in dollars, of a home from 2006 to 2015, where t is the number of years after 2006. Which of the following is the best interpretation of 5,000 in this context?

A) The number of homes sold each year
B) The yearly decrease, in dollars, in the value of the home
C) The difference, in dollars, between the value of the home in 2006 and in 2015
D) The yearly decrease, in dollars, in the value of the home per square foot

This time, we're asked to interpret 5,000. Again, we should look at the given function as a linear model expressed in $y = mx + b$ form. In this case, the slope m is $-5{,}000$. Since the slope, by definition, is a rate of change, the increase or decrease in y (value of the home) for each unit increase in x (each year), the number 5,000 can best be interpreted as the amount, in dollars, the value of the home decreases by each year. Answer $\boxed{(B)}$.

It's important that you don't get tricked into choosing a rate that looks right but ultimately doesn't fit the context set by the variable definitions—in this case, the definitions of t and V. Answer A is wrong because we're not dealing with the number of homes sold; we're dealing with the value of a home. Answer D is wrong because the numbers in the equation aren't on a per-square-foot basis. Always be aware of how the variables in the problem are defined.

EXAMPLE 3: Evan spent a total of \$275 on x cups of regular coffee and y cups of specialty coffee. The equation $3x + 5y = 275$ represents this situation. Which of the following is the best interpretation of 5 in this context?

A) The number of cups of regular coffee Evan bought
B) The number of cups of specialty coffee Evan bought
C) The price, in dollars, of one cup of regular coffee
D) The price, in dollars, of one cup of specialty coffee

Although this is an easy question, we're going to use this example to illustrate a technique that's helpful in dealing with these types of questions. The technique involves looking at the units of each term in the equation.

Here, we're given a linear equation of the form $ax + by = c$. The 275 on the right side of the equation represents the 275 dollars that were spent in total. Since the left side of the equation sums to this 275 dollars, the terms $3x$ and $5y$ must each be in dollars for the equation to make sense:

$$3x \text{ dollars} + 5y \text{ dollars} = 275 \text{ dollars}$$

Therefore, $5y$ represents a dollar amount. Since y is defined as the number of cups of specialty coffee, the only way the units work out is if the coefficient 5 represents a dollar amount per cup of specialty coffee.

$$5\frac{\text{dollars}}{\text{cup of specialty coffee}} \times y \text{ cups of specialty coffee} = 5y \text{ dollars}$$

Therefore, the only reasonable interpretation of 5 is that it's the price, in dollars, per cup of specialty coffee. Answer $\boxed{(D)}$.

Of course, it's not hard to get to this answer intuitively. It's easy to see that $5y$ must be the amount Evan spent on specialty coffee, so 5 must be the price per cup. There's no need to write out all the units if this is an easy question for you, but this technique may serve you on more difficult questions.

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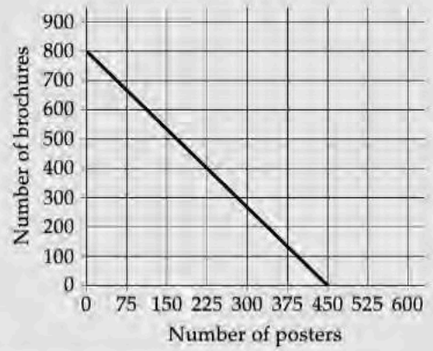
EXAMPLE 4:



A marketing agency is given a $7,200 budget to print posters and brochures for a client. The line shown models the possible combinations of posters and brochures that the agency could print. According to the model, what is the cost of printing each poster?

A) $9
B) $12
C) $16
D) $25

When dealing with a linear model that represents possible combinations, the key is to look at either the x-intercept or y-intercept. Since we want to find the cost of printing each poster, we should look at the x-intercept—450 posters can be printed when no brochures are printed. Therefore, the cost of printing each poster is $\frac{7{,}200}{450} = \$16$. Answer $\boxed{\text{(C)}}$.

EXAMPLE 5:

$$h(x) = \frac{4x+6}{5}$$

The function h models the maximum height, in inches, of a plant that was grown using x grams of fertilizer. Based on this model, using 1 additional gram of fertilizer in growing the plant would increase its maximum height by how many inches?

What we're essentially looking for is the change in h (maximum height) for every 1 unit increase in x (fertilizer)—in other words, the slope. To find the slope, we can put the equation into $y = mx + b$ form by splitting up the fraction: $h(x) = \frac{4}{5}x + \frac{6}{5}$. Now we can easily see that the slope is $\frac{4}{5}$ or $\boxed{.8}$. To summarize, the slope of 0.8 indicates that each additional gram of fertilizer increases the maximum height of the plant by 0.8 inches.

EXAMPLE 6:

$$h(x) = \frac{4x+6}{5}$$

The function h models the maximum height, in inches, of a plant that was grown using x grams of fertilizer. Based on this model, how many more grams of fertilizer should be used in growing the plant to raise the maximum height of the plant by exactly 1 inch?

Because this question is asking for the change in x for every 1 unit increase in h (the reverse of the previous example), we need to rearrange the equation so that we have x in terms of h.

$$\frac{4x+6}{5} = h$$
$$4x + 6 = 5h$$
$$4x = 5h - 6$$
$$x = \frac{5}{4}h - \frac{3}{2}$$

The slope of this resulting equation indicates that x must increase by $\frac{5}{4}$, or $\boxed{1.25}$, when we want h to increase by 1.

A shortcut for this type of question is to take the reciprocal of the slope of the original equation: the reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$, or 1.25.

EXAMPLE 7: A can of soda is put into a freezer. The function $T(m) = 65 - 6m$ models the temperature of the soda, in degrees Fahrenheit, m minutes after the can has been put into the freezer. According to the model, what is the decrease in the temperature of the soda, in degrees Fahrenheit, for every 5 minutes the can is left in the freezer?

The slope of -6 represents the change in the temperature of the soda for every 1 minute the can is left in the freezer. So for every 5 minutes left in the freezer, the temperature of the soda decreases by $5 \times 6 = \boxed{30}$ degrees Fahrenheit.

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EXERCISE 1: Answers for this exercise start on page 338.

1

The function $w(x) = 100 - 3x$ gives the water level, in feet, in a large aquarium x days after the aquarium was last refilled. Which statement is the best interpretation of the slope of the graph of $y = w(x)$ in the xy-plane in this context?

A) The water level decreases by 3 feet each day.
B) The water level increases by 3 feet each day.
C) The water level decreases by 100 feet each day.
D) The water level increases by 100 feet each day.

2

Annie collects educational puzzle sets for her classroom. She has B beginner-level puzzle sets and E intermediate-level puzzle sets. There are a total of 1,400 puzzle pieces in all her puzzle sets. The equation $36B + 64E = 1{,}400$ represents this situation. An intermediate-level puzzle set has how many more puzzle pieces than a beginner-level puzzle set?

3

$$f(x) = 24 + 6x$$

The given function f models the weight, in ounces, of a serving tray containing x empty glasses. According to the model, how many ounces does each empty glass weigh?

A) 4
B) 6
C) 12
D) 24

4

A total of 175 ounces of flour was used to make bagels and muffins. The equation $4.8x + 1.5y = 175$ represents this situation, where x is the number of bagels made and y is the number of muffins made. According to the equation, how many ounces of flour are used to make each bagel?

5

A local football team consists of 11 starting players and additional backup players. At the start of the football season, a total of 159 jerseys were purchased for the team, and each player received 3 jerseys. The equation $3(x + 11) = 159$ represents this situation. Which of the following is the best interpretation of the expression $x + 11$ in this context?

A) The number of backup players on the team
B) The number of starting players on the team
C) The number of players on the team
D) The number of jerseys purchased for the starting players on the team

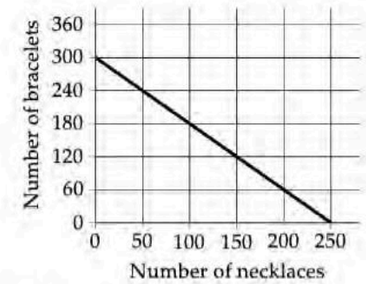
6

$$y = 177 + 42x$$

The equation gives the number of juniper trees y in a certain forest x years after 2010. What is the best interpretation of $(x, y) = (5, 387)$ in this context?

A) The number of juniper trees in the forest increased by 387 in 2015.
B) The number of juniper trees in the forest increased by 387 from 2010 to 2015.
C) There were 387 juniper trees in the forest in 2005.
D) There were 387 juniper trees in the forest in 2015.

7



The line shown models the possible combinations of the number of bracelets and necklaces a certain company with 25 jewelers can handcraft in a month, based on the constant rate at which the jewelers work. Based on this model, what is the maximum number of bracelets that one jeweler at this company could handcraft in a month?

A) 10
B) 12
C) 21
D) 30

8

Ivy bought an 80-ounce block of cheese. She then cut the block into x slices of cheese and y cubes of cheese. The equation $0.6x + 0.25y = 80$ represents this situation. Which is the best interpretation of $0.6x$ in this context?

A) The amount of cheese, in ounces, in each cube
B) The amount of cheese, in ounces, in each slice
C) The total amount of cheese, in ounces, Ivy cut into cubes
D) The total amount of cheese, in ounces, Ivy cut into slices

9

During a road trip, Carl used 12 gallons of fuel driving a minivan and 17 gallons of fuel driving a campervan. He traveled a total distance of 590 miles driving both vehicles. The equation $12x + 17y = 590$ represents this situation. Which is the best interpretation of y in this context?

A) The total number of gallons of fuel Carl used driving the campervan
B) The total number of miles Carl drove the campervan for
C) The fuel efficiency, in miles per gallon, of the campervan
D) The average fuel efficiency, in miles per gallon, of the minivan and the campervan

10

Of the 585 photos in a photo album, 315 are color photos and the rest are black-and-white. The equation $6p + 315 = 585$ describes this situation, where p represents the number of pages that contain black-and-white photos. What is the best interpretation of $6p$ in this context?

A) There are $6p$ black-and-white photos.
B) There are $6p$ photos on each page.
C) There are $6p$ pages in total.
D) There are $6p$ photos in total.

11

Stacy is playing a game in which she needs to collect a certain number of points to win. The function p defined by $p(x) = 52 - 4x$ models the number of additional points Stacy needs to win the game, where x is the number of rounds played. Which of the following is the best interpretation of $p(8) = 20$?

A) Stacy collected a total of 20 points in the first 8 rounds.
B) It will take Stacy 8 rounds to collect the 20 additional points she needs to win the game.
C) After the first 8 rounds, Stacy needs to collect 20 more points to win the game.
D) In the 8th round, Stacy collected 20 more points than she did the previous round.

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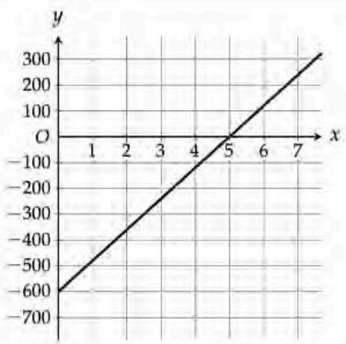
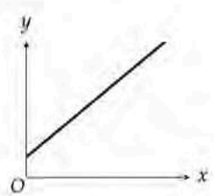
12



The graph models the daily profit y, in dollars, of a bakery after x cakes are sold. Which of the following is the best interpretation of the y-intercept of the graph in this context?

A) The price of each cake
B) The cost of making each cake
C) The daily cost of running the bakery
D) The daily cost of making the cakes that weren't able to be sold

13



A train began its route a certain distance away from a train depot and traveled in a straight path directly away from it. The line shown represents the train's distance from the depot, in miles, x hours after it began its route. What is the best interpretation of the slope of the line?

A) The speed of the train, in miles per hour, during its route
B) The distance, in miles, between the train depot and the train at the start of its route
C) The total distance, in miles, the train has traveled x hours after it began its route
D) The number of hours it took for the train to finish its route

14

$$T = 56 + 5h$$

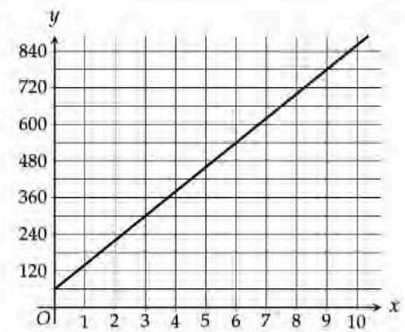
The given equation models the temperature T of a room, in degrees Fahrenheit, h hours after a heater is turned on. According to the model, the temperature of the heated room increases by how many degrees Fahrenheit every 30 minutes?

15

A recipe suggests sweetening honey tea with sugar according to the equation $s = 10 - 2h$, where s is the amount of sugar, in teaspoons, that should be added to a tea beverage containing h teaspoons of honey. What is the best interpretation of 2 in this context?

A) For every teaspoon of honey in the beverage, two more teaspoons of sugar should be added.
B) For every teaspoon of honey in the beverage, two fewer teaspoons of sugar should be added.
C) For every two teaspoons of honey in the beverage, one more teaspoon of sugar should be added.
D) For every two teaspoons of honey in the beverage, one fewer teaspoon of sugar should be added.

16



The graph represents the total time y, in minutes, a lawyer spent reviewing a case and providing x legal consultations. What is the best interpretation of the slope of the graph in this context?

A) The lawyer spent 60 minutes reviewing the case.
B) Each legal consultation lasted 60 minutes.
C) The lawyer spent 80 minutes reviewing the case.
D) Each legal consultation lasted 80 minutes.

17

$$T(x) = \frac{2x + 9}{5}$$

The function T models the time, in seconds, it takes to load a web page with x images. According to the model, by how many seconds does each image increase the load time of a web page?

18

Jamil reached the end of a 26-mile track by walking at a speed of 5 miles per hour and running at a faster speed than he walked. The equation $5t + 9(4 - t) = 26$ represents this situation, where t is the total amount of time, in hours, Jamil spent walking. Which of the following is the best interpretation of 4 in this context?

A) The total distance, in miles, that Jamil ran
B) The speed at which Jamil ran, in miles per hour
C) The total number of hours Jamil spent running
D) The total number of hours Jamil spent walking and running

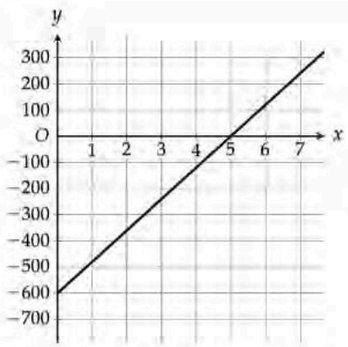
19

$$A = \frac{7}{2}(9 + x)$$

The function A gives the area, in square inches, of a trapezoid with a height of 7 inches, a shorter base of 9 inches, and a longer base of x inches. If the length of the longer base is increased by 5 inches, by how many square inches does the area of the trapezoid increase?

EXERCISE 2: Answers for this exercise start on page 339.

1



The graph models the daily profit y, in dollars, of a bakery after x cakes are sold. Which of the following is the best interpretation of the slope of the graph in this context?

A) The price of each cake
B) The profit generated from each cake sold
C) The daily profit generated from all the cakes that were sold
D) The number of cakes that need to be sold to make a daily profit of 100 dollars

2

Potassium sorbate is mixed into yogurt to increase its shelf life. The function $f(x) = 4 + 3x$ models the shelf life, in days, of 120 ounces of yogurt with x grams of potassium sorbate mixed in. Which of the following is the best interpretation of $f(4) = 16$ in this context?

A) The shelf life of 120 ounces of yogurt with 4 grams of potassium sorbate mixed in is about 16 days.
B) The shelf life of 120 ounces of yogurt with 16 grams of potassium sorbate mixed in is about 4 days.
C) Mixing in 4 grams of potassium sorbate increases the shelf life of 120 ounces of yogurt by about 16 days.
D) Mixing in 16 grams of potassium sorbate increases the shelf life of 120 ounces of yogurt by about 4 days.

3

A tank initially contained 204 gallons of water. After a hole was punched in the tank, the amount of water in the tank decreased at a constant rate each minute until 36 gallons of water remained in the tank. The equation $204 - 4.2x = 36$ represents this situation, where x is the number of minutes after the hole was punched. Which is the best interpretation of the number 4.2 in this context?

A) The percentage decrease in the amount of water in the tank
B) The rate at which water left the tank, in gallons per minute
C) The number of minutes it took for every 1-gallon decrease in the amount of water in the tank
D) The total decrease, in gallons, in the amount of water in the tank x minutes after the hole was punched

4

A seafood vendor earned \$725 for selling a pounds of fish and b pounds of shrimp. The equation $12a + 18b = 725$ represents this situation. What is the best interpretation of $18b$ in this context?

A) The total number of shrimp the vendor sold
B) The total weight, in pounds, of all the fish and shrimp the vendor sold
C) The total amount, in dollars, the vendor earned for b pounds of shrimp sold
D) The total amount, in dollars, the vendor earned for each pound of shrimp sold

5

A total of 200 dollars was spent to purchase notebooks and binders. The equation $2.50x + 3.50y = 200$ represents this situation, where x is the number of notebooks purchased and y is the number of binders purchased. What is the best interpretation of $(x, y) = (24, 40)$ in this context?

A) If 24 notebooks were purchased, then 40 binders were purchased.
B) If 24 notebooks were purchased, then 40 dollars was spent to purchase them.
C) If 40 notebooks were purchased, then 24 binders were purchased.
D) If 40 notebooks were purchased, then 24 dollars was spent to purchase them.

6

A website provides video tutorials on computer programming to its subscribers. The function $s(x) = 500 + 200x$ gives the number of subscribers when there were x videos available on the site. What is the best interpretation of 500 in this context?

A) The site gained 500 subscribers for every additional video it made available.
B) The site initially had 500 videos available.
C) The site had 500 subscribers when there were no videos available.
D) The site gained 500 subscribers for every 200 additional videos it made available.

7

An office supply store sells desks at a discount to customers who purchase at least 5 desks. For a purchase of x desks, the price per desk y, in dollars, is given by the equation $y = 500 - 5x$, where $5 \leq x \leq 30$. Which statement is the best interpretation of $(x, y) = (6, 470)$ in this context?

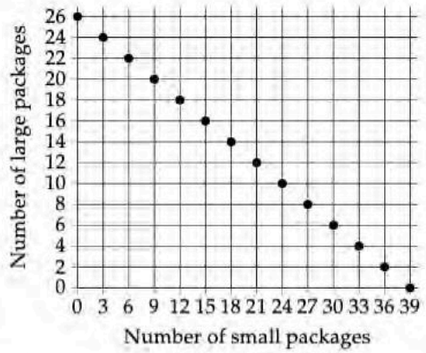
A) A customer purchasing 6 desks will pay a total of 470 dollars.
B) A customer purchasing 6 desks will receive a discount of 470 dollars.
C) A customer purchasing 6 desks will pay 470 dollars for each desk.
D) A customer purchasing 470 desks will receive a discount of 6 dollars on each desk.

8

A total of 400 patients were treated at a health clinic run by doctors and nurse practitioners. Each patient was treated by either a doctor or a nurse practitioner. Of the patients treated, 127 were treated by doctors. The equation $21n + 127 = 400$ represents this situation, where n is the number of patients treated by each nurse practitioner at the clinic. Which of the following is the best interpretation of $21n$ in this context?

A) The number of nurse practitioners at the clinic
B) The number of patients treated by nurse practitioners at the clinic
C) The number of doctors at the clinic
D) The difference between the number of patients treated by nurse practitioners at the clinic and the number of patients treated by doctors at the clinic

9



Olivia sells figurines and ships them in small and large packages. It takes her 10 minutes to prepare each small package and 15 minutes to prepare each large package. The graph represents the possible combinations of the number of small packages and the number of large packages Olivia can prepare in exactly k minutes. What is the value of k ?

A) 270
B) 330
C) 390
D) 450

10

A walking trail consists of a level path that is 7 miles long and a sloped path that is 4 miles long. It took Oscar a total of 5 hours to walk the entire trail. The equation $7x + 4y = 5$ represents this situation. Which of the following is the best interpretation of y in this context?

A) The total number of hours it took Oscar to walk the sloped path
B) Oscar's average speed, in miles per hour, while walking the sloped path
C) The average number of hours it took Oscar to walk one mile of the sloped path
D) Oscar's average speed, in miles per hour, while walking the level path

11

A party store sells premade banners and custom banners. Yesterday, the store sold a total of 80 banners, earning x dollars in revenue from premade banners and y dollars in revenue from custom banners. The equation $\frac{x}{14} + \frac{y}{25} = 80$ represents this situation. Which of the following is the best interpretation of 14 in this context?

A) The store sold 14 premade banners yesterday.
B) The store earns 14 dollars in revenue for each premade banner sold.
C) The store sold 14 custom banners yesterday.
D) The store earns 14 dollars in revenue for each custom banner sold.

12

$$h(t) = 100 - 4t$$

The given function h models the number of hours until a gallon of milk held at a temperature of t, in degrees Celsius, goes sour. Which of the following is the best interpretation of the number 4 in this context?

A) An increase of 1°C will make a gallon of milk go sour 4 hours faster.
B) An increase of 1°C will make 4 gallons of milk go sour 1 hour faster.
C) An increase of 4°C will make a gallon of milk go sour 1 hour faster.
D) An increase of 4°C will make a gallon of milk go sour 4 hours faster.

13

$$h(t) = 3.5\left(\frac{t}{50} + 2\right)$$

The given function h models the height of a tree, in inches, t days after it is planted. According to the model, about how many inches does it grow each day?

A) 0.02
B) 0.07
C) 1.75
D) 3.5

14

An antique lamp was sold at an auction. The price p of the lamp, in dollars, during the auction can be modeled by the equation $p = 900 - 10t$, where t is the number of seconds left in the auction. What is the best interpretation of 900 in this context?

A) The starting auction price of the lamp
B) The final auction price of the lamp
C) The increase in the price of the lamp per second
D) The time it took to auction off the lamp, in seconds

15

A teacher printed sticker sheets with 7 stickers on each sheet. At a school party, she gave out 2 stickers to each student who attended. The relationship between the number of students who attended the party, x, and the number of sticker sheets that the teacher printed, y, is represented by the equation $7y - 32 = 2x$. What is the best interpretation of 32 in this context?

A) The teacher gave out a total of 32 stickers at the party.
B) The teacher had 32 sticker sheets remaining after the party.
C) The teacher printed 32 more stickers than she gave out at the party.
D) The teacher printed a total of 32 sticker sheets.

16

$$y = 1.30x - 1.50$$

The given equation is used by a bank to convert x U.S. dollars into y euros. The bank performs the conversion using an exchange rate and charges a fee for the conversion. Which of the following is the best interpretation of 1.50 in this context?

A) The bank charges 1.50 euros to do the currency exchange.
B) The bank charges 1.50 U.S. dollars to do the currency exchange.
C) One U.S. dollar is worth 1.50 euros.
D) One euro is worth 1.50 U.S. dollars.

17

$$C(x) = 0.07(x - 180) + 62.6$$

The function C gives the monthly cost, in dollars, of using an international phone plan for x minutes each month. If the phone plan is used for 75 more minutes each month, by how many dollars does the monthly cost of the plan increase?

A) 1.75
B) 5.25
C) 55.25
D) 67.85

18

$$C(x) = 1.5 + 2.5x$$

The given function C models the cost, in dollars, of mailing a shipment weighing x pounds. According to the model, an increase of 10 dollars in the mailing cost corresponds to an increase of how many pounds in the weight of the shipment?

A) 2
B) 2.5
C) 4
D) 5

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8

Percent

EXAMPLE 1: Among 10th graders at a school, 40% of the students are Red Sox fans. Among those Red Sox fans, 20% are also Celtics fans. What percent of the 10th graders at the school are both Red Sox fans and Celtics fans?

A) 8% B) 12% C) 20% D) 30%

In percent questions, the word "of" should be translated as multiplication. If we let the number of 10th graders be x, then the number of Red Sox fans is $40\% \times x = 0.40x$, and the number of Celtics fans is $0.20(0.40x) = 0.08x$, which means 8% of the 10th graders at the school are fans of both teams. Answer $\boxed{(A)}$.

If you're not as comfortable working with variables, a common strategy in percent questions is to make up a number to represent the total, typically 100. Let's suppose that the number of 10th graders at the school is 100 and see if we get the same answer.

$$\text{Red Sox fans} = 40\% \text{ of } 100 = 40$$
$$\text{Celtics \& Red Sox fans} = 20\% \text{ of } 40 = 8$$

Finally, $\dfrac{8}{\text{Number of 10th graders}} = \dfrac{8}{100} = 8\%$, which is indeed the same answer.

EXAMPLE 2: The number y is 18% greater than the number x, where $x > 0$. Which expression represents y in terms of x ?

A) $118x$ B) $18x$ C) $1.18x$ D) $0.18x$

The number y is the number x increased by 18%:

$$y = x + 0.18x = (1 + 0.18)x = 1.18x$$

Answer $\boxed{(C)}$. This example can be generalized. To perform any **percent increase** (or "percent greater than"), we multiply the starting value by one plus the percentage (converted to a fraction/decimal). Why the "one plus"? Because a percent increase is the starting value plus 18% on top. So to increase a value by 20%, we multiply that value by 1.20. To increase a value by 44%, we multiply that value by 1.44, and so on.

EXAMPLE 3: The number y is 35% less than the number of x, where $x > 0$. Which expression represents y in terms of x ?

A) $0.35x$ B) $0.65x$ C) $\dfrac{x}{1.35}$ D) $\dfrac{x}{1.65}$

This is a percent decrease problem. The number y is the number x decreased by 35%:

$$y = x - 0.35x = (1 - 0.35)x = 0.65x$$

Answer $\boxed{(B)}$. To perform any **percent decrease** (or "percent less than"), we multiply the starting value by one minus the percentage (converted to a fraction/decimal). Why the "one minus"? Because it gives us the remaining portion. So to decrease a value by 15%, we multiply by $1 - 0.15 = 0.85$ because 0.85 of that value remains. To decrease a value by 60%, we multiply by $1 - 0.60 = 0.40$ because 0.40 of that value remains, and so on.

EXAMPLE 4: A clothing store sells a certain jacket for 20% more than its suggested retail price. Using a rewards card, a customer is able to purchase the jacket for 25% less than what the store sells it for. The customer's purchase price is what percent of the jacket's suggested retail price?

A) 80% B) 84% C) 90% D) 95%

To summarize the question, we essentially have a price increase of 20% and then a decrease of 25%. We'll let the original retail price of the jacket be p. The store's price is then $1.20p$ and the customer's purchase price is

$$1.20p \times 0.75 = 1.20 \times 0.75 \times p = 0.90p$$

This $0.90p$ result means that the final purchase price is 90% of the original price (suggested retail price). Answer $\boxed{(C)}$. This example shows that percent changes can be "chained" and calculated all at once to get the final result:

$$20\% \text{ increase, then } 25\% \text{ decrease} \Rightarrow 1.20 \times 0.75p$$

And let's imagine that the customer had to pay a sales tax of 5% on the final price. We could "chain" that on as well

$$1.20 \times 0.75 \times 1.05p$$

and get the final price without calculating any intermediate values.

The mistake students make is going one step at a time. So if the jacket costs \$100, they'll take 20% of \$100 to get \$20 and add that back to the \$100 to get \$120. Then they'll take 25% of \$120 to get \$30 and subtract it from the \$120. This process is tedious and error-prone.

For percent questions, don't calculate what the actual changes or intermediate values are at each step. Get directly to the end result by taking the product of the corresponding multiplication factors.

EXAMPLE 5: On Monday, Julie received 240 emails. On Tuesday, the number of emails Julie received was p% less than the number she received on Monday. Which expression represents the number of emails Julie received on Tuesday?

A) $240(1 - p)$ B) $240(p - 1)$ C) $240\left(1 - \dfrac{p}{100}\right)$ D) $240\left(\dfrac{p}{100} - 1\right)$

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As we've already stated, to find a value that is $p\%$ less than some starting value, we multiply that starting value by one minus $p\%$ (converted to a fraction/decimal).

So to find the value that is $p\%$ less than 240, we first represent $p\%$ as $\frac{p}{100}$ and subtract it from one: $1 - \frac{p}{100}$. Then we multiply 240 by this factor: $240\left(1 - \frac{p}{100}\right)$. Answer $\boxed{(C)}$.

Don't be fooled by trap answers such as $240(1 - p)$. If p were 25, this expression would result in an unexpected negative value. Always remember that by definition, $p\% = \frac{p}{100}$ and that this fraction/decimal is often what you want to be working with, not the whole number value of p itself.

Let's summarize what we've learned so far.

	Translates to ...	Example
$p\%$ of x	$\frac{p}{100} \times x$	15% of $x \Rightarrow \frac{15}{100} \times x = 0.15x$
$p\%$ greater than x	$\left(1 + \frac{p}{100}\right)x$	15% greater than $x \Rightarrow \left(1 + \frac{15}{100}\right)x = 1.15x$
$p\%$ less than x	$\left(1 - \frac{p}{100}\right)x$	15% less than $x \Rightarrow \left(1 - \frac{15}{100}\right)x = 0.85x$

You should also be able to translate in reverse as well. So when you see $1.15x$, that's equivalent to 15% greater than x, and when you see $0.85x$, that's equivalent to 15% less than x.

EXAMPLE 6: The value of x is $\frac{27}{20}$ times the value of y, where y is a positive number. The value of x is what percent greater than the value of y ?

A) 26% B) 35% C) 65% D) 74%

Since $x = \frac{27}{20}y = 1.35y$, x is 35% greater than y. Answer $\boxed{(B)}$.

EXAMPLE 7: If the number m is 80% less than the number n, and the number n is 250% greater than 84, what is the value of m ?

$$n = 84\left(1 + \frac{250}{100}\right) = 84(1 + 2.5) = 294$$

$$m = 294(1 - 0.80) = 294(0.20) = \boxed{58.8}$$

The SAT will often give you percentages you don't see very often, such as "80% less than" or "250% greater than," to trick you into multiplying by 0.80 or 2.5. Don't fall for it! A percent increase requires that we add one to account for the original number. A percent decrease requires that we subtract from one to get the remaining portion.

EXAMPLE 8: The chickens on a farm laid 30% more eggs this year than they did last year. If they laid 5,200 eggs this year, how many did they lay last year?

To summarize, this year's number is 30% more than last year's number:

$$\text{Last Year (?)} \xrightarrow{\text{30\% increase}} \text{This Year (5,200)}$$

Since we're given the number of eggs for this year, the temptation is to decrease this year's number by 30% to get last year's number (to "reverse" the percent change, in other words):

$$\text{Last Year (?)} \xleftarrow{\text{30\% decrease}} \text{This Year (5,200)}$$

Doing so would yield $5{,}200(1 - 0.30) = 5{,}200(0.70) = 3{,}640$. However, this answer is incorrect and illustrates a common mistake that students make. We cannot simply "reverse" a percent change this way because percent change depends on the starting value. A 30% increase from last year is not equal in measure to a 30% decrease from this year because the 30% increase is based off of last year's value while the 30% decrease is based off of this year's value.

As a quick example, if we start with 100 and increase it by 50%, we get 150. But if we take this 150 and decrease it by the same percentage, 50%, we get 75, not the original 100 we started with.

So to solve this problem, we need to set up an equation. A 30% increase from last year means that we can multiply last year's number by 1.30 to get this year's number:

$$\begin{aligned} \text{This Year} &= (1.30)(\text{Last Year}) \\ 5{,}200 &= (1.30)(\text{Last Year}) \\ \frac{5{,}200}{1.30} &= \text{Last Year} \\ \boxed{4{,}000} &= \text{Last Year} \end{aligned}$$

Remember: If x is 50% greater than y, that does not mean y is 50% less than x. When in doubt, set up an equation!

EXAMPLE 9: The number of students at a school decreased 20% from 2010 to 2011. If the number of students enrolled in 2011 was k, which of the following expresses the number of students enrolled in 2010 in terms of k ?

A) $0.75k$ B) $1.20k$ C) $1.25k$ D) $1.5k$

This example illustrates the same concept as Example 8 but with variables instead of numbers. The answer is NOT $1.20k$. Again, we need to set up an equation. If x is the number of students in 2010, then

$$\begin{aligned} 0.80x &= k \\ x &= \frac{1}{0.80}k \\ x &= 1.25k \end{aligned}$$

Therefore, there were 25% more students in 2010 than in 2011. Answer $\boxed{(C)}$.

So far, we've dealt with problems where the percent change is given to us. In problems where we have to calculate the percent change, we use the following formula:

$$\text{Percent change} = \frac{\text{new value} - \text{original value}}{\text{original value}} \times 100\%$$

For example, if the price of a dress increases from 80 dollars to 90 dollars, the percent change is:

$$\frac{90 - 80}{80} \times 100\% = 12.5\%$$

If percent change is positive, it's a percent increase. Negative? Percent decrease. From the formula, we can see that percent change is always based off of the original value, which serves as the denominator.

EXAMPLE 10: In a particular store, the number of TVs sold during the first week of November was 685. The number of TVs sold during the second week was 500. The number of TVs sold during the second week was what percent less than the number of TVs sold during the first week (rounded to the nearest percent)?

A) 17% B) 27% C) 37% D) 47%

For percent change questions worded as "percent greater than" or "percent less than," the quantity that comes after the word "than" should be considered the "original value" in the context of the percent change formula because it's the quantity being compared against. In this case, the number of TVs sold during the first week is the basis for comparison and should be used as the "original value":

$$\text{Percent change} = \frac{\text{new value} - \text{original value}}{\text{original value}} \times 100\% = \frac{500 - 685}{685} \times 100\% \approx -27\%$$

Answer $\boxed{(B)}$.

EXAMPLE 11: 200 is x% greater than 50. What is the value of x ?

It's tempting to answer 400% since 200 is 4 times 50, but this answer is wrong. It's true that 200 is 400% *of* 50, but 200 is not 400% *greater than* 50 (400% greater than 50 is actually 50(1 + 4) = 250).

It's best to use the formula:

$$\frac{\text{new value} - \text{original value}}{\text{original value}} \times 100\% = \frac{200 - 50}{50} \times 100\% = 3 \times 100\% = 300\%$$

Therefore, $x = \boxed{300}$.

EXERCISE 1: Answers for this exercise start on page 341.

1

280 is 32% of what number?

2

If n% of 40 is 9.6, what is n% of 250 ?

A) 45
B) 60
C) 75
D) 80

3

Approximately 15% of a town's residents live within the town's downtown area. If the town has a total of r residents, which expression best represents the number of residents who live outside the downtown area?

A) $0.15r$
B) $0.85r$
C) $1.15r$
D) $6.67r$

4

The result of decreasing a positive number x by p% can be represented by the expression $0.12x$. What is the value of p ?

A) 0.12
B) 0.88
C) 12
D) 88

5

A factory's electricity consumption increased by 2% from 2022 to 2023. If the factory's electricity consumption in 2023 is n times its electricity consumption in 2022, what is the value of n ?

A) 0.02
B) 0.98
C) 1.02
D) 1.2

6

What is the result of increasing 150 by 300%?

7

In 2019, the population of Lisbon, Portugal was about 2.3 million. The population of Lisbon in 2019 was about 28% of the total population of Portugal. Which of the following is the best estimate of the total population of Portugal in 2019?

A) 1.8 million
B) 3.2 million
C) 8.2 million
D) 12.2 million

8

Joanne used a coupon to receive a 20% discount on the purchase of a book. She paid $29.00 after the coupon was applied. What was the price, in dollars, of the book before the coupon was applied?

9

90 is u% less than 450. What is the value of u ?

10

The number of dishes served by a restaurant during dinner was 17.5% greater than the number of dishes served during lunch. If the restaurant served 940 dishes during dinner, how many more dishes did the restaurant serve during dinner than during lunch?

11

Sims spent 76% more on groceries in 2016 than in 2015, and she spent 145% more on groceries in 2017 than in 2016. If Sims spent p times as much on groceries in 2017 as in 2015, what is the value of p ?

A) 1.102
B) 1.862
C) 2.552
D) 4.312

12

The number p is 60% of the positive number q, and p is 390% of the number r. If q is x% of r, what is the value of x ?

13

Of the clothing items in a store, 24% are vests. Of the vests, 36% are blue. Of the blue vests, 78% are fleece. Which of the following is closest to the percentage of the clothing items in the store that are not blue fleece vests?

A) 10.70%
B) 67.39%
C) 89.30%
D) 93.26%

14

The volume of air in a balloon at the start of an experiment is 258% of the volume of air in the balloon at the end of the experiment. If the volume of air in the balloon at the end of the experiment is n% of the volume of air in the balloon at the start of the experiment, what is the value of n, to the nearest whole number?

15

In 2016, County A and County B collected the same amount of taxes. In 2017, the amount of taxes collected by County A decreased by 25% and the amount of taxes collected by County B increased by 20%. If County A collected 60 million dollars of taxes in 2017, what was the amount of taxes, in millions of dollars, County B collected in 2017?

A) 54
B) 78
C) 90
D) 96

EXERCISE 2: Answers for this exercise start on page 342.

1

In March, a city zoo attracted 32,000 visitors to its polar bear exhibit. In April, the number of visitors to the exhibit increased by 15%. How many visitors did the zoo attract to its polar bear exhibit in April?

A) 32,150
B) 32,480
C) 35,200
D) 36,800

2

A product's price is now 1.84 times its original price. By what percentage did the product's price increase from its original price?

A) 0.84%
B) 1.84%
C) 84%
D) 184%

3

A company has a marketing budget of $22,500. The marketing budget is set to decrease by k% to $18,750. What is the value of k ?

4

The result of increasing the quantity y by 2,400% is 10,200. What is the value of y ?

5

Biologists have found that 60% of a large population of mice are male. Of the males in the population, 0.05% carry a rare genetic mutation. What percentage of the population of mice are male and carry the rare genetic mutation?

A) 0.0003%
B) 0.003%
C) 0.03%
D) 0.3%

6

John was charged a total of $67.50 for a mirror after an 8% sales tax was added. What was the cost, in dollars, of the mirror before the sales tax was added?

7

The wholesale price of a travel bag is 10% less than the retail price. James managed to purchase the travel bag at 80% off the wholesale price at a promotional event. What percentage of the retail price did James pay?

A) 8%
B) 18%
C) 72%
D) 82%

8

If 220% of c is 880, and c is p% greater than 50, what is the value of p ?

9

To paint a shed, Keaton used 8% of the paint in a container. If there were 57.5 liters of paint left in the container after Keaton painted the shed, how many liters of paint were in the container before he painted the shed?

10

At an auction, the starting price of a gemstone is set at 60% less than its assessed value of $30,000. Eli makes an offer that is 370% of the starting price. What is the value of Eli's offer, in dollars?

11

The number r is 70% less than the number s. The number r is 60% greater than the number t. If $s = 180$, what is the value of t ?

12

The mass of an iceberg decreased by 14% from 2005 to 2010 and then increased by 188% from 2010 to 2015. What was the net percentage increase in the mass of the iceberg from 2005 to 2015?

A) 40.32%
B) 147.68%
C) 161.68%
D) 174.00%

13

The number x is 75% less than the positive number y. The number z is 84% greater than x. The number z is how many times y ?

14

The value of a is $\frac{25}{16}$ times the value of b, where $b > 0$. By what percent is the value of b less than the value of a ?

A) 36.00%
B) 43.75%
C) 56.25%
D) 64.00%

15

The sum of positive numbers a and b is 4.76 times the positive number c, and a is 124% of b. What percent of c is a ?

A) 216.85%
B) 238.00%
C) 263.50%
D) 295.12%

16

In 2010, the number of houses built in Town A was n percent greater than the number of houses built in Town B. If 700 houses were built in Town A in 2010, which expression represents the number of houses built in Town B in 2010?

A) $700\left(\frac{100+n}{100}\right)$
B) $700\left(\frac{100}{100+n}\right)$
C) $700\left(\frac{100-n}{100}\right)$
D) $700\left(\frac{100}{100-n}\right)$

9

Exponential vs. Linear Growth

The concepts in this chapter build on what we learned earlier in the percents chapter. Let's see how.

EXAMPLE 1: Jonas has a savings account that earns 3 percent interest compounded annually. His initial deposit was $800. Which of the following expressions gives the value of the account, in dollars, after 10 years?

A) $800(1.30)^{10}$ B) $800 + 30(10)$ C) $800(1.03)(10)$ D) $800(1.03)^{10}$

A 3 percent interest rate compounded annually means Jonas earns 3 percent on the account once a year. This isn't just 3% on the original $800. This is 3% on whatever's in his account at the time, including any interest he's already earned in previous years. So if we're in year 5, Jonas would earn 3% interest on the original $800 and 3% interest on the total interest accumulated in years one through four. This interest that's earned on top of interest is what defines **compound interest**.

If we try to calculate the interest earned in the account after each and every year, this problem would take forever. Instead, let's take what we learned from the percents chapter and apply it here. As we know, a 3% increase is equivalent to a multiplication factor of 1.03. Therefore, the total value of the account after each year is just 1.03 times whatever the value was last year:

End of Year 1 total: $800(1.03) = 800(1.03)^1$
End of Year 2 total: $800(1.03)(1.03) = 800(1.03)^2$
End of Year 3 total: $800(1.03)(1.03)(1.03) = 800(1.03)^3$
End of Year 4 total: $800(1.03)(1.03)(1.03)(1.03) = 800(1.03)^4$

See the pattern in the exponents? The value of the account after t years is just $800(1.03)^t$. Therefore, the value of the account after 10 years is $800(1.03)^{10}$, answer $\boxed{(D)}$. No calculations needed.

EXAMPLE 2: The population of an ant colony doubles every month. If there were 500 ants in the colony on May 1st, which function represents the population of the ant colony, $A(t)$, t months after May 1st?

A) $A(t) = 500(2t)$ B) $A(t) = 2(500)^t$ C) $A(t) = 500 + 2t$ D) $A(t) = 500(2)^t$

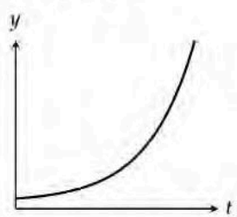
Since the population doubles every month, the population after the first month is 500(2). The population after the second month is double the population after the first month: $500(2)(2) = 500(2)^2$. The population after the third month is double the population after the second month: $500(2)(2)(2) = 500(2)^3$. See where this is going? The pattern is exactly the same as in Example 1 except the initial value here is 500 and the multiplication factor is 2. Therefore, the ant population after t months is $500(2)^t$, answer $\boxed{(D)}$.

A bank account earning 3 percent every year and an ant population doubling every month are examples of **exponential growth**, which occurs when a value grows periodically by a factor greater than 1. As we've seen, exponential growth can be modeled as an equation that looks like

$$y = ab^t$$

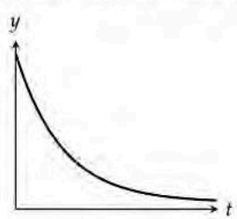
where y is the value after t specified units of time (e.g. years), a is the initial value, and b is a growth factor greater than one. The initial value a is sometimes referred to as the **coefficient** and the growth factor b is sometimes referred to as the **base**. In our bank account example, 800 was the initial value and 1.03 was the growth factor. In our ant population example, 500 was the initial value and 2 was the growth factor.

Graphs of exponential growth have the following shape:



Notice how the graph creeps up slowly at first but then shoots up faster and faster over time. That's exponential growth.

Exponential decay, however, is the opposite. Imagine a radioactive substance that loses mass over time. It loses a lot of its mass in the beginning and then loses it more and more slowly as time goes by.



It's worth memorizing the shapes of these graphs of exponential growth and decay. The SAT may test you explicitly on them.

The equation for exponential decay is the same as the equation for exponential growth:

$$y = ab^t$$

The only difference is that the growth factor, b, is less than 1. So an equation that models exponential decay might look like

$$y = 400(0.6)^t$$

where y is the mass, in grams, of a radioactive substance t years from now. The 400 indicates that the substance currently has a mass of 400 grams, and the 0.6 indicates that at the end of each year, the substance is left with 60% of the mass it started the year with. In other words, the substance loses 40% of its mass each year.

So far, the examples we've discussed have been relatively simple. To model more complicated cases of exponential growth and decay, such as a bank account growing by 3% every 2 years or a radioactive substance losing half its mass every 9 months, we'll need to use a more generalized equation, one that you absolutely need to memorize:

$$y = ab^{\frac{t}{k}}$$

where y is the value after t specified units of time, a is the initial value, and k is the time required for the value of y to change by a factor of b. Note that t and k must be in the same units. So if t is in years, k should also be in years.

Let's go over some examples so that you fully understand what k means.

EXAMPLE 3:

$$M(t) = 400(1.05)^{\frac{t}{3}}$$

The function M models the mass, in nanograms, of a particle after t years. Which of the following is the best interpretation of $(1.05)^{\frac{t}{3}}$ in this context?

A) The mass of the particle increases by 5% every 4 months.
B) The mass of the particle increases by 5% every 3 years.
C) The mass of the particle increases by 5 nanograms every 4 months.
D) The mass of the particle increases by 5 nanograms every 3 years.

We have an exponential equation with $k = 3$ and $b = 1.05$, which means it takes 3 years for the mass of the particle to change by a factor of 1.05. In other words, the particle's mass increases by 5% every 3 years. And indeed, if we plug in $t = 3$, we get $400(1.05)^1 = 400(1.05)$, which represents one 5% increase after 3 years. Answer $\boxed{(B)}$.

Let's see why the generalized equation $y = ab^{\frac{t}{k}}$ works by taking a deeper look at this example. The mass of the particle changes by a factor of 1.05 for every increase of 1 in the value of the exponent, $\frac{t}{3}$. If we start the value of the exponent at 0, then $\frac{t}{3} = 0$. Multiplying both sides by 3 yields $t = 0$ years. Now if we set the value of the exponent to 1, then $\frac{t}{3} = 1$. Multiplying both sides by 3 yields $t = 3$ years. Since the value of the exponent increases by 1 from $t = 0$ years to $t = 3$ years, 3 years must be the time required for the particle's mass to change by a factor of 1.05. Repeat these steps with k instead of 3, and you'll have proven why k represents the time required for the value of y to change by a factor of b in the generalized equation.

EXAMPLE 4:

$$M(t) = 400(0.6)^{3t}$$

The function M models the mass, in nanograms, of a particle after t years. Which of the following is the best interpretation of $(0.6)^{3t}$ in this context?

A) The mass of the particle decreases by 60% every 4 months.
B) The mass of the particle decreases by 60% every 3 years.
C) The mass of the particle decreases by 40% every 4 months.
D) The mass of the particle decreases by 40% every 3 years.

With $y = ab^{\frac{t}{k}}$ as a reference, we see that the exponent of the given equation is not in the form of $\frac{t}{k}$, so how are we supposed to find the value of k ? We have to use a little arithmetic trick:

$$M(t) = 400(0.6)^{3t} = 400(0.6)^{t/(1/3)}$$

Multiplying by 3 is the same as dividing by $\frac{1}{3}$. Now we can see that $k = \frac{1}{3}$, which means that the particle's mass changes by a factor of 0.6 every $\frac{1}{3}$ year. That's equivalent to a $1 - 0.6 = 40\%$ decrease every 4 months. Answer $\boxed{(C)}$. Note that this is an example of exponential decay.

EXAMPLE 5: A non-profit organization currently has 50 volunteers. If the organization is able to double the number of volunteers every 8 months, which of the following equations best models the number of volunteers, v, the organization will have t months from now?

A) $v = 50(2)^{\frac{t}{8}}$ B) $v = 50(2)^{\frac{t}{4}}$ C) $v = 50(2)^{t}$ D) $v = 50(2)^{8t}$

Again, we'll use $y = ab^{\frac{t}{k}}$ as a reference. Based on the given information, $a = 50$ and $b = 2$. Since the number of volunteers increases by a factor of 2 every 8 months, and t is in months, $k = 8$. Therefore, the equation that best models the number of volunteers is $v = 50(2)^{\frac{t}{8}}$. Answer $\boxed{(A)}$.

EXAMPLE 6: A non-profit organization currently has 50 volunteers. If the organization is able to double the number of volunteers every 8 months, which of the following equations best models the number of volunteers, v, the organization will have t years from now?

A) $v = 50(2)^{\frac{t}{8}}$ B) $v = 50(2)^{\frac{2t}{3}}$ C) $v = 50(2)^{\frac{3t}{2}}$ D) $v = 50(2)^{8t}$

This question is the same as the one in Example 3, except we now have inconsistent units of time. We again have a growth factor of 2, and since k (from our reference equation) is the amount of time it takes for a single change by the growth factor, we might assume that $k = 8$. However, this answer would be wrong because k is in months when the t in our equation needs to be in years.

To form the correct equation, t and k must be in the same units, so we have to convert 8 months into years: $k = \frac{8}{12} = \frac{2}{3}$ years. The equation is then $v = 50(2)^{t/(2/3)} = 50(2)^{\frac{3t}{2}}$. Answer $\boxed{(C)}$.

EXAMPLE 7:

$$f(m) = 25.5(1.72)^{m}$$

The given function f models the temperature of a substance m minutes after it is exposed to sulfur dioxide, where $0 \le m \le 10$. Which of the following functions best models the temperature of the substance s seconds after its exposure to sulfur dioxide, where $0 \le s \le 600$?

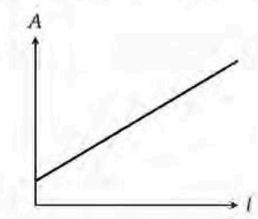
A) $g(s) = 25.5\left(\frac{1.72}{60}\right)^{s}$ B) $g(s) = 25.5\left(\frac{1.72}{60}\right)^{\frac{s}{60}}$ C) $g(s) = 25.5(1.72)^{\frac{s}{60}}$ D) $g(s) = 25.5(1.72)^{60s}$

The given function f tells us that the temperature of the substance increases by a factor of 1.72 (72% increase) every minute. That's the same as a 1.72 factor increase every 60 seconds. This conversion to seconds is important—because s is in seconds, the "k-value" has to be converted to seconds. So which of the answer choices has a growth factor of 1.72 and a k-value of 60? Only answer $\boxed{(C)}$.

Now let's compare exponential growth and decay with linear growth and decay. As you may already know, **linear growth** can be modeled by a line with a positive slope. For example, if Ann has a piggybank with 100 dollars already in it, and she adds 5 dollars every month, the total amount in the piggybank can be modeled by

$$A = 5t + 100$$

where A is the total amount, t is the number of months, and 100 (the y-intercept) is the initial amount.

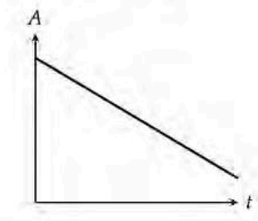


Unlike exponential growth, **linear growth** is constant and consistent. There is no slowing down or speeding up. The total goes up by the same amount each time.

Now imagine that Ann takes 5 dollars out of her piggybank every month instead of adding to it. The total balance would decrease by a constant amount each month, resulting in **linear decay**. The total amount A in the piggybank could then be modeled by

$$A = 100 - 5t$$

The graph of such an equation is a line with a negative slope:



EXERCISE 1: Answers for this exercise start on page 344.

1

Albert has a large book collection. He decides to trade in two of his used books for one new book each month at a local bookstore. Which of the following best describes the relationship between time (in months) and the total number of books in Albert's collection?

A) Increasing linear
B) Decreasing linear
C) Exponential growth
D) Exponential decay

2

A scientist counts 80 cells in a petri dish and finds that each one splits into two new cells every hour. He uses the function $A(t) = cr^t$ to calculate the total number of cells in the petri dish after t hours. Which of the following assigns the correct values to c and r ?

A) $c = 40, r = 2$
B) $c = 80, r = 0.5$
C) $c = 80, r = 1.5$
D) $c = 80, r = 2$

3

The number of deer in a population is expected to decrease by 11% each year. If the number of deer in the population is currently 12,000, which of the following is closest to the expected number of deer in the population 3 years from now?

A) 8,040
B) 8,460
C) 8,770
D) 9,320

4

Two variables, x and y, are related such that the value of y decreases by 25% for every increase in the value of x by 1. If $y = 4$ when $x = 0$, which of the following equations represents this relationship?

A) $y = 0.75(4)^x$
B) $y = 4(0.75)^x$
C) $y = 4(1.25)^x$
D) $y = 16(0.25)^{x+1}$

5

The employees at a new bookstore must stock a certain number of shelves so that the store is ready for its opening in two weeks. The employees stock shelves at a constant rate throughout the two weeks. If $p(t)$ is the number of shelves left to be stocked after t days, which of the following statements best describes the function p ?

A) The function p is an increasing exponential function.
B) The function p is a decreasing exponential function.
C) The function p is an increasing linear function.
D) The function p is a decreasing linear function.

6

The initial value of a car is \$10,000. The function v estimates that the value of the car at the end of each year is 65% of its value at the end of the previous year. Which equation defines v, where $v(k)$ is the estimated value, in dollars, of the car at the end of the kth year?

A) $v(k) = 10{,}000(0.35)^k$
B) $v(k) = 10{,}000(0.65)^k$
C) $v(k) = 10{,}000(1.35)^k$
D) $v(k) = 10{,}000(1.65)^k$

7

The function $p(t) = 13(1.1)^t$ models the population, in thousands, of a certain town from 2008 to 2020, where t represents the number of years after 2008. Which of the following is the best interpretation of "the value of $p(4)$ is approximately 19" in this context?

A) The population of the town grew by approximately 19 percent each year from 2008 to 2012.
B) The population of the town grew by approximately 19,000 from 2008 to 2012.
C) The population of the town was approximately 19,000 in 2012.
D) The population of the town grew by approximately 19,000 every 4 years between 2008 and 2020.

8

$$f(t) = 20\left(1 + \frac{15}{100}\right)^t$$

The function f models the temperature, in degrees Celsius, of a metal alloy used in an experiment, where t is the number of seconds after the experiment began. Which of the following is the best interpretation of the number 15 in this context?

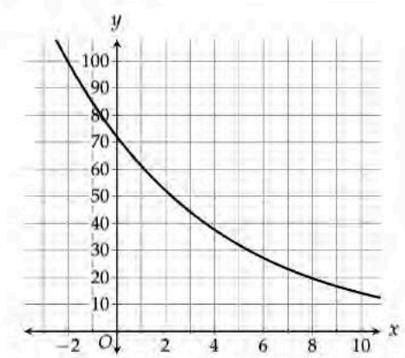
A) The temperature, in degrees Celsius, of the metal alloy at the beginning of the experiment
B) The increase in the temperature, in degrees Celsius, of the metal alloy every 100 seconds during the experiment
C) The percent by which the temperature, in degrees Celsius, of the metal alloy decreased from each second to the next during the experiment
D) The percent by which the temperature, in degrees Celsius, of the metal alloy increased from each second to the next during the experiment

9

The equation $m(t) = 31.6(1.278)^t$ gives the estimated number of members, in thousands, of a wholesale club for $0 \le t \le 30$, where t is the number of years after 1976. Which of the following is the best interpretation of the number 31.6 in this context?

A) The estimated number of members, in thousands, in 1976
B) The estimated number of members, in thousands, t years after 1976
C) The estimated increase in the number of members, in thousands, each year after 1976
D) The estimated percent by which the number of members increased from each year to the next after 1976

10



The graph of $y = f(x)$ is shown, where the function f is defined by $f(x) = a(b)^x$, and a and b are positive constants. Which of the following is closest to the value of $\frac{a}{b}$?

A) 63
B) 85
C) 102
D) 480

11

Omar currently holds a government bond that has a market value of \$900. Each year, the market value of the bond is expected to be 20% greater than its market value the year before. If the expression $900(1+p)$, where p is a constant, represents the expected market value, in dollars, of the bond after 3 years, what is the value of p ?

12

Which of the following equivalent forms of the exponential function f displays the value of $f(4)$ as the base or the coefficient?

A) $f(x) = 2\left(\frac{1}{2}\right)^x$

B) $f(x) = 2\left(\frac{1}{4}\right)^{\frac{x}{2}}$

C) $f(x) = 2\left(\frac{1}{8}\right)^{\frac{x}{3}}$

D) $f(x) = 2\left(\frac{1}{16}\right)^{\frac{x}{4}}$

13

The population of trees in a forest has been decreasing by 6 percent every 4 years. The population at the beginning of 2015 was estimated to be 14,000. If P represents the population of trees in the forest t years after 2015, which of the following equations best defines P ?

A) $P = 14{,}000(0.06)^{\frac{t}{4}}$

B) $P = 14{,}000 + 0.94(4t)$

C) $P = 14{,}000(0.94)^{4t}$

D) $P = 14{,}000(0.94)^{\frac{t}{4}}$

14

$$v = 40(1.6)^t$$

The given equation models the resale value of a collectible toy t years after its first release. Which of the following equations best models the resale value, v, of the toy m months after its first release?

A) $v = 40(1.6)^m$

B) $v = 40(1.6)^{\frac{m}{12}}$

C) $v = 40(1.6)^{12m}$

D) $v = 40(1.6)^{\frac{12}{m}}$

15

$$N(h) = 1{,}000(0.97)^{4h}$$

The function N models the number of bacteria in a petri dish after h hours. According to the model, the number of bacteria is predicted to decrease by 3% every k minutes. What is the value of k ?

A) $\frac{1}{4}$

B) 4

C) 15

D) 240

16

The amount of data, in gigabytes, stored in a database increases by 2% every 15 hours. If 16 gigabytes worth of data are currently stored in the database, which of the following functions g gives the amount of data, in gigabytes, that will be stored in the database t days from now?

A) $g(t) = 16(2)^{15t}$

B) $g(t) = 16(1.02)^{\frac{t}{15}}$

C) $g(t) = 16(1.02)^{\frac{5t}{8}}$

D) $g(t) = 16(1.02)^{\frac{8t}{5}}$

EXERCISE 2: Answers for this exercise start on page 345.

1

In 2004, a streaming service had 10,000 subscribers. The number of subscribers increased by 160% each year. Which of the following equations best models the number of subscribers, s, the streaming service had t years after 2004?

A) $s = 10{,}000 + 16{,}000t$

B) $s = 10{,}000(1.16)^t$

C) $s = 10{,}000(1.6)^t$

D) $s = 10{,}000(2.6)^t$

2

$$r(t) = 11(0.92)^t$$

The function r models the roundworm population, in thousands, in a pond after t years. Which of the following is the best interpretation of the number 0.92 in this context?

A) The roundworm population decreases by 0.92 thousand each year.

B) It will take 0.92 years for the roundworm population to halve.

C) The proportion of the roundworm population that remains after each year is 0.92.

D) The roundworm population decreases by 0.92 percent each year.

3

Each week, the weight of a certain crocodile increases by 0.75% of its initial birth weight. Which of the following functions best models the weight of this crocodile over time?

A) Decreasing exponential

B) Decreasing linear

C) Increasing exponential

D) Increasing linear

4

The function $f(x) = 2.9(1.07)^x$ gives the estimated value, in millions of dollars, of a missing piece of artwork, where x is the number of years since the artwork disappeared, and $0 \le x \le 6$. If $y = f(x)$ is graphed in the xy-plane, which of the following is the best interpretation of the y-intercept of the graph in this context?

A) The average estimated value of the artwork was 2.9 million dollars during the first 6 years of its disappearance.

B) The average estimated value of the artwork was 3.1 million dollars during the first 6 years of its disappearance.

C) The estimated value of the artwork was 2.9 million dollars at the time it disappeared.

D) The estimated value of the artwork was 3.1 million dollars at the time it disappeared.

5

$$C(t) = 100(2)^{\frac{t}{5}}$$

To examine how a certain virus spreads, scientists introduced the virus to cells in a test tube and found that the number of infected cells in the test tube grew exponentially over time. The given function C models the number of infected cells in the test tube t days after the virus was introduced. Which is the best interpretation of $(2)^{\frac{t}{5}}$ in this context?

A) The predicted number of infected cells in the test tube doubled every 5 days.

B) The predicted number of infected cells in the test tube grew by a factor of 5 every two days.

C) The predicted number of infected cells in the test tube doubled every day.

D) The predicted number of infected cells in the test tube grew by a factor of 5 every day.

6

Laura purchases a stock valued at \$326.50. According to her model, the stock is predicted to have a value of \$413.60 after 8 years. If Laura's model can be represented by the exponential function V, where $V(t)$ is the value of the stock, in dollars, t years after its purchase, which of the following equations could define V ?

A) $V(t) = 87.10(1.03)^t$
B) $V(t) = 87.10(1.3)^t$
C) $V(t) = 326.50(1.03)^t$
D) $V(t) = 326.50(1.3)^t$

7

For the function f, the value of $f(x)$ is equal to half the value of x. Which of the following is the best description of this function?

A) Decreasing linear
B) Decreasing exponential
C) Increasing linear
D) Increasing exponential

8

Ben paid off a loan by following a repayment plan. On the first day of the repayment plan, Ben paid \$30. On each day after the first, Ben paid 1.5 times the amount he had paid the previous day. If r represents the amount, in dollars, Ben paid on the nth day of the repayment plan, which equation gives r in terms of n ?

A) $r = 30(1.5)^n$
B) $r = 30(1.5)^{n-1}$
C) $r = 30(2.5)^n$
D) $r = 30(2.5)^{n-1}$

9

$$h(t) = 10{,}000(3)^{bt}$$

The given function h models the number of honeybees in a certain beehive, where b is a constant and t is the number of days after the beehive was first found. According to the model, there were 30,000 honeybees in the beehive 10 days after the beehive was first found. What is the value of b ?

10

$$W(t) = 830(1.09)^{\frac{6}{n}t}$$

The function W gives the value, in dollars, of a certain investment after t years. If the value of the investment increases by 9% every n months, what is the value of n ?

11

$$g(x) = 256(0.61)^{\frac{x}{30}}$$

The exponential function g is defined by the given equation. The equation can be rewritten as

$g(x) = 256\left(1 - \frac{k}{100}\right)^{\frac{x}{10}}$, where k is a constant.

What is the value of k, to the nearest whole number?

A) 15
B) 20
C) 27
D) 77

12

Since the beginning of 1990, the value of a certain diamond pendant has decreased by 4% from the beginning of each year to the beginning of the following year. At the beginning of 1991, the value of the pendant was \$840. Which of the following equations best models the value, in dollars, of the pendant t years after the beginning of 1990?

A) $v = 840(0.96)^t$
B) $v = 840(1.04)(0.96)^t$
C) $v = 875(0.96)^t$
D) $v = 875 - 875(0.04)^t$

13

The table below shows the price P, in dollars, of a barrel of crude oil t days after the beginning of an oil shortage.

Number of days, t	Price, P (dollars)
0	50.00
10	60.00
20	72.00
30	86.40
40	103.68

If the equation $P = 50(r)^{\frac{t}{n}}$ is used to model the relationship between t and P, which of the following could be the values of r and n ?

A) $r = 1.2$ and $n = 5$
B) $r = 1.44$ and $n = 5$
C) $r = 1.2$ and $n = 20$
D) $r = 1.44$ and $n = 20$

14

The function f is defined for all $x \geq 0$. Which of the following equivalent forms of the function f displays, as a constant or coefficient, the minimum value of f, where $x \geq 0$?

A) $f(x) = 50(2.5)(1.25)^{x-2}$
B) $f(x) = 125(0.8)(1.25)^{x-1}$
C) $f(x) = 64(1.25)^{x+1}$
D) $f(x) = 80(0.64)(1.25)^{x+2}$

15

$$C = 4(1.002)^{2t}$$

The given equation can be used to model the number of cars, in millions, registered in a certain state t years after 2009. According to the model, the number of cars registered in the state increases by n% every 6 months. What is the value of n ?

A) 0.002
B) 0.04
C) 0.2
D) 2

16

A music festival increases the number of tickets offered for sale each year by about 5% of the number offered for sale the previous year. In its fourth year, the music festival offered 40,000 tickets for sale. Which of the following equations best models the number of tickets offered for sale in the nth year of the music festival?

A) $T = 40{,}000(0.05)^{n-4}$
B) $T = 40{,}000(1.05)^n$
C) $T = 40{,}000(1.05)^{n-4}$
D) $T = 40{,}000(1.05)^{n+4}$

17

The function $p(t) = 870(1.14)^t$ models the number of thistle plants in a field from 2000 to 2020, where t is the number of years after 2000. Which of the following functions best models the number of thistle plants in the field from 2000 to 2020, where n is the number of years after 2003 ?

A) $q(n) = 870(1.14)^{n-3}$
B) $q(n) = 870(1.14)^{3n}$
C) $q(n) = 870(1.14)^3(1.14)^{\frac{n}{3}}$
D) $q(n) = 870(1.14)^3(1.14)^n$

10

Rates, Conversions, & Ratios

The best way to approach most questions involving rates is to use conversion factors (if you've gone through high school chemistry, you should know what I'm talking about). Conversion factors are multipliers that help you go from one set of units to another. For example, $\frac{12 \text{ months}}{1 \text{ year}}$ is a conversion factor that converts years into months. Not only can they represent standard unit conversions (e.g. $\frac{60 \text{ minutes}}{1 \text{ hour}}$), they can also represent information provided by the question (e.g. $\frac{1 \text{ ticket}}{50 \text{ dollars}}$). When using conversion factors to solve a problem, you must set them up in the right sequence and with the appropriate numerators and denominators.

EXAMPLE 1: A bicycle manufacturer can produce 20 bicycles per hour. How many hours would it take the manufacturer to produce 320 bicycles?

Intuitive Approach: We divide the total by the rate to get $320 \div 20 = \boxed{16}$ hours. Easy.

Conversion Factor Approach: The given rate of 20 bicycles per hour can be used as a conversion factor: either $\frac{20 \text{ bicycles}}{1 \text{ hour}}$ or its reciprocal, $\frac{1 \text{ hour}}{20 \text{ bicycles}}$. In this case, we want to use the conversion factor $\frac{1 \text{ hour}}{20 \text{ bicycles}}$ so that when we multiply by the given 320 bicycles, the "bicycles" unit cancels out and the "hours" unit is what remains:

$$320 \cancel{\text{bicycles}} \times \frac{1 \text{ hour}}{20 \cancel{\text{bicycles}}} = \frac{320 \text{ hours}}{20} = \boxed{16} \text{ hours}$$

Having the correct unit remaining at the end confirms that we used the correct conversion factor. If the question had asked us to convert a given number of hours into the number of bicycles that can be made, we would've used the other conversion factor, $\frac{20 \text{ bicycles}}{1 \text{ hour}}$.

Although the conversion factor approach seems tedious, I recommend using this approach by default to keep your work organized and to avoid silly mistakes, especially for the harder problems. Ultimately, you are using conversion factors in some form or fashion whether you reason out the conversions in your head or write them out explicitly.

EXAMPLE 2: A box at the supermarket can hold 6 oranges each. Each orange costs 20 cents. Given that the supermarket has a budget of \$540 to stock oranges, how many boxes will the supermarket be able to fill?

Intuitive Approach: If each orange is 20 cents, then a dollar would be enough for 5 oranges. Five hundred forty dollars would then be enough for $540 \times 5 = 2700$ oranges, which would fill $2700 \div 6 = \boxed{450}$ boxes.

Conversion Factor Approach: Typically, our setup will begin with a starting constraint given by the question (in this case, 540 dollars) or a target that needs to be reached (e.g. 320 bicycles in Example 1).

$$540 \cancel{\text{dollars}} \times \frac{100 \cancel{\text{cents}}}{1 \cancel{\text{dollar}}} \times \frac{1 \cancel{\text{orange}}}{20 \cancel{\text{cents}}} \times \frac{1 \text{ box}}{6 \cancel{\text{oranges}}} = \frac{540 \times 100 \text{ boxes}}{20 \times 6} = \boxed{450} \text{ boxes}$$

Notice how the units cancel as we go along. If the units are canceling, chances are we're doing things right. And since the answer should be in boxes, the "box(es)" unit that remains confirms that we've indeed done things correctly.

EXAMPLE 3: A car can travel 1 mile in 1 minute and 15 seconds. At this rate, how many miles can the car travel in 1 hour?

$$1 \cancel{\text{hour}} \times \frac{60 \cancel{\text{minutes}}}{1 \cancel{\text{hour}}} \times \frac{60 \cancel{\text{seconds}}}{1 \cancel{\text{minute}}} \times \frac{1 \text{ mile}}{75 \cancel{\text{seconds}}} = \frac{60 \times 60 \text{ miles}}{75} = \boxed{48} \text{ miles}$$

EXAMPLE 4: Tom drives at an average speed of 50 miles per hour. Which expression represents the number of hours it will take Tom to drive $5x$ miles at this speed?

A) $\frac{x}{10}$ B) $\frac{10}{x}$ C) $10x$ D) $250x$

$$5x \cancel{\text{miles}} \times \frac{1 \text{ hour}}{50 \cancel{\text{miles}}} = \frac{5x \text{ hours}}{50} = \frac{x}{10} \text{ hours}$$

Answer $\boxed{(A)}$.

EXAMPLE 5: To prepare for class, Mr. Chu has to print a number of booklets with p pages per booklet. If every 5 pages cost c cents to print and he spent a total of d dollars, how many booklets did Mr. Chu print in terms of p, c, and d?

A) $\frac{cp}{500d}$ B) $\frac{100d}{cp}$ C) $\frac{500d}{cp}$ D) $\frac{5d}{cp}$

$$d \cancel{\text{dollars}} \times \frac{100 \cancel{\text{cents}}}{1 \cancel{\text{dollar}}} \times \frac{5 \cancel{\text{pages}}}{c \cancel{\text{cents}}} \times \frac{1 \text{ booklet}}{p \cancel{\text{pages}}} = \frac{d(100)(5)}{cp} \text{ booklets} = \frac{500d}{cp} \text{ booklets}$$

Answer $\boxed{(C)}$.

EXAMPLE 6: A certain beach has an area of 87,412 square feet. What is the area, in square meters, of this beach? (Use 1 meter = 3.28 feet.)

A) 2,477.13 B) 8,125 C) 26,650 D) 286,711.36

To convert from one square unit to another square unit using a conversion factor that isn't in square units, we have to use the same conversion factor twice to ensure the old units are adequately cancelled out and the new unit is correctly squared.

$$87{,}412\ \text{feet}^2 \times \frac{1\ \text{meter}}{3.28\ \text{feet}} \times \frac{1\ \text{meter}}{3.28\ \text{feet}} = \frac{87{,}412\ \text{meters}^2}{(3.28)(3.28)} = 8{,}125\ \text{meters}^2$$

Answer $\boxed{(B)}$.

EXAMPLE 7: The speed of an object is 210 inches per second. What is the speed, in feet per minute? (1 foot = 12 inches)

A) 17.5 B) 42 C) 1,050 D) 12,600

When converting speeds, we have to watch out for two units: a unit of length and a unit of time. In this example, our conversion factors need to be set up in such a way that we get $\frac{\text{feet}}{\text{minute}}$ at the end.

$$\frac{210\ \text{inches}}{1\ \text{second}} \times \frac{1\ \text{foot}}{12\ \text{inches}} \times \frac{60\ \text{seconds}}{1\ \text{minute}} = \frac{210 \times 60}{12}\ \frac{\text{feet}}{\text{minute}} = 1{,}050\ \frac{\text{feet}}{\text{minute}}$$

Answer $\boxed{(C)}$.

So far, the examples in this chapter have all dealt with conversion factors. Let's do a simple example where we have to find a rate rather than use one to perform a conversion. To be more specific, we'll sometimes be asked to find the **average rate of change**, which, in most situations, is the change in the value of a quantity divided by the elapsed time.

EXAMPLE 8: A rocket has 360 gallons of fuel left after 2 hours of flight, and only 100 gallons after 6 hours of flight. On average, it burns n gallons of fuel for every hour of flight, where n is a constant. What is the value of n ?

In $6 - 2 = 4$ hours of flight, the rocket burned $360 - 100 = 260$ gallons of fuel. Therefore, the rocket, on average, burns $\frac{260}{4} = \boxed{65}$ gallons of fuel every hour.

Ratios

Let's say that the ratio of cars to trucks in a parking lot is 5:2 (5 to 2). Because ratios can be written as fractions, this ratio is equivalent to $\frac{5}{2}$. A ratio of 5:2 is also equivalent to a ratio of 10:4, since the latter reduces to the former.

In this context, a ratio of 5:2 means that for every 5 cars, there are 2 trucks. And assuming that there are only cars and trucks parked in the lot, this ratio also means that there are 5 cars for every 7 vehicles. By the same token, there are 2 trucks for every 7 vehicles.

EXAMPLE 9: Minyoung bought croissants and bagels for a breakfast event. The ratio of the number of croissants she bought to the number of bagels she bought was 3 to 4. If Minyoung bought 72 bagels, how many croissants did she buy?

According to the given ratio, Minyoung bought 3 croissants for every 4 bagels. Since she bought 72 bagels, she must have bought

$$\frac{3\ \text{croissants}}{4\ \text{bagels}} \times 72\ \text{bagels} = \boxed{54}\ \text{croissants}$$

EXAMPLE 10: Arfand is following a recipe for a seasoning blend that requires sea salt, black pepper, and paprika. According to the recipe, the ratio of grams of sea salt to grams of black pepper should be 1:2, and the ratio of grams of black pepper to grams of paprika should be 4:3. How many grams of paprika should Arfand use to make 108 grams of the seasoning blend?

Since black pepper is involved in both the given ratios, we can use it to establish a "common denominator" for comparison. First, multiply the sea salt to black pepper ratio by 2 to get 2:4. Why multiply by 2? Because now the "4" in this sea salt to black pepper ratio lines up with the "4" in the black pepper to paprika ratio. Now that we have this common number in both ratios, it's easy to see that the ratio between the three ingredients is 2:4:3 (sea salt to black pepper to paprika).

According to this ratio, 3 grams of paprika should be used for every $2 + 4 + 3 = 9$ grams of the blend.

Therefore, Arfand should use $\frac{3}{9} \times 108 = \boxed{36}$ grams of paprika.

EXAMPLE 11: If the ratio of $y : 2.4$ is equivalent to $2.7 : 3.6$, what is the value of y ?

A) $\frac{3}{2}$ B) $\frac{4}{3}$ C) $\frac{7}{3}$ D) $\frac{9}{5}$

Because ratios are just fractions, we can set up the following equation:

$$\frac{y}{2.4} = \frac{2.7}{3.6}$$
$$y = \frac{2.7}{3.6}(2.4) = 1.8 = \frac{9}{5}$$

Answer $\boxed{(D)}$.

EXERCISE 1: Answers for this exercise start on page 348.

1

Idina can type 90 words in 2.5 minutes. How many words can she type in 12 minutes?

2

The ratio of markers to pencils in a drawer is 8 to 3. If there are 48 pencils in the drawer, how many markers are there?

3

Emma is given a total of \$270 to buy blankets and pillows. She buys 5 blankets for \$12 each. If each pillow costs \$15 each, how many pillows can she buy?

4

The ratio a to b is equivalent to the ratio 36 to z. When $a = 3$, what is the value of b in terms of z ?

A) $\frac{z}{3}$

B) $12z$

C) $\frac{z}{12}$

D) $108z$

5

A distance of 9 furlongs is equivalent to how many inches? (1 furlong = 220 yards and 1 yard = 36 inches)

6

The ratio of blueberries to grapes in a fruit salad is 5 to 12. If there are x blueberries in the fruit salad, which expression represents the number of grapes in the fruit salad?

A) $\frac{5}{12}x$

B) $\frac{12}{5}x$

C) $\frac{12}{17}x$

D) $60x$

7

An empty pool can be filled in 5 hours if water is pumped in at 300 gallons an hour. How many hours would it take to fill the pool if water is pumped in at 500 gallons an hour?

8

Which of the following speeds is equivalent to 990 miles per hour? (1 mile = 1,760 yards)

A) 33.75 yards per second

B) 484 yards per second

C) 2,025 yards per second

D) 29,040 yards per second

9

If a apples cost d dollars, which of the following expressions gives the cost of 20 apples, in dollars?

A) $\frac{20a}{d}$

B) $\frac{20d}{a}$

C) $\frac{a}{20d}$

D) $\frac{20}{ad}$

10

At a food stand, the price of nachos is d dollars. The price of french fries is $0.3d$ dollars. What is the ratio of the price of nachos to the price of french fries?

A) 1 to 3

B) 3 to 1

C) 7 to 3

D) 10 to 3

11

Henry drives 150 miles at 30 miles per hour and then another 200 miles at 50 miles per hour. What was his average speed, in miles per hour, for the entire journey, to the nearest hundredth?

A) 38.89

B) 40.00

C) 42.33

D) 43.58

12

The ratio of milk to flour in a mixture is 16 to 20. If 10 cups of milk are added to the mixture, how many cups of flour must be added to the mixture to maintain this ratio?

13

The area of Wompatuck State Park in Massachusetts is 5.5 square miles. Which of the following is closest to the area of this park, in square kilometers? (Use 1 kilometer = 0.62 mile.)

A) 2.1

B) 3.4

C) 8.9

D) 14.3

14

There are a total of 270 coupons in an advertising brochure. The ratio of expired coupons to unexpired coupons is 5 to 1. How many unexpired coupons are there in the advertising brochure?

15

At the start of dinner service, there were 30 diners seated inside a restaurant. At 45 minutes into dinner service, there were 120 diners seated inside. If the number of diners seated inside the restaurant increased at a constant rate after the start of dinner service, how many diners were seated inside at 24 minutes into dinner service?

A) 42
B) 48
C) 72
D) 78

16

A flask contains an acidic solution with a concentration of 7.1×10^{15} hydrogen ions per milliliter. If 4.8×10^{23} hydrogen ions have a total mass of 0.8 grams, which of the following is closest to the concentration, in grams per liter, of the acidic solution?

A) 1.2×10^{-5}
B) 1.2×10^{-8}
C) 1.5×10^{-5}
D) 1.5×10^{-8}

17

An island in the Galápagos has an estimated population of 740 sea lions and an area of 4,200 hectares. Which of the following is closest to the estimated population density, in sea lions per square mile, of the island? (Use 1 square mile = 259 hectares.)

A) 0.0007
B) 0.02
C) 45.6
D) 12,000

18

Jared and Robert are accountants who are tasked with reviewing financial reports. It takes Jared 15 hours, working at a constant rate, to review a report containing 240 pages of financial statements. If Robert works at twice Jared's rate, how many minutes would it take Robert to review a report containing 120 pages of financial statements?

A) 225
B) 345
C) 450
D) 900

19

During a race on a circular racetrack, a racecar burns fuel at a constant rate. After lap 4, the racecar has 22 gallons left in its tank. After lap 7, the racecar has 18 gallons left in its tank. Assuming the racecar does not refuel, after which lap will the racecar have 6 gallons left in its tank?

A) Lap 13
B) Lap 15
C) Lap 16
D) Lap 19

EXERCISE 2: Answers for this exercise start on page 349.

1

A bookstore ordered an initial shipment of 10 paperback copies and 4 hardcover copies of a newly published book. The store must order a second shipment with the same ratio of paperback copies to hardcover copies as in the initial shipment. If the store orders 50 hardcover copies of the book for the second shipment, how many paperback copies should the store order?

2

A model of a building has a scale where a width of 5 inches in the model represents an actual width of 16 feet. A room in the building has an actual width of $10w$ feet. Which expression represents the width, in inches, of the room in the model?

A) $\frac{5}{8}w$
B) $\frac{25}{8}w$
C) $8w$
D) $32w$

3

As a submarine descends into the deep ocean, the pressure it must withstand increases. At an altitude of -700 meters, the pressure is 50 atm (atmospheres), and at an altitude of -900 meters, the pressure is 70 atm. For every 10 meters the submarine descends, the pressure it faces increases by n, where n is a constant. What is the value of n?

A) 0.1
B) 1
C) 2
D) 10

4

A train covers 32 kilometers in 14.5 minutes. If it continues to travel at the same rate, which of the following is closest to the distance it will travel in 2 hours?

A) 54 kilometers
B) 265 kilometers
C) 364 kilometers
D) 928 kilometers

5

A recipe for soap calls for $1\frac{1}{2}$ cups of lye for every $\frac{2}{5}$ cup of castor oil. How many cups of lye are needed for a batch of soap that uses 3 cups of castor oil?

A) $1\frac{4}{5}$
B) 5
C) $9\frac{4}{5}$
D) $11\frac{1}{4}$

6

The ratio $2\frac{1}{4}$ to $1\frac{1}{2}$ is equivalent to n to 2. What is the value of n?

7

9 pikol = 2 large bahar
400 kulack = 29 pikol

The given formulas represent the relationships between some units of weight that were once used in Indonesia. How many large bahar are equivalent to 1,000 kulack?

8

A plastic supplier sells polyethylene at a price of \$5.75 per kilogram. The density of polyethylene is 0.92 kilogram per cubic meter. What is the price of polyethylene from this supplier, in dollars per cubic meter? (Disregard the \$ sign when entering your answer. For example, if your answer is \$1.44, enter 1.44)

9

An 8 inch by 10 inch piece of cardboard costs \$2.00. If the cost of a piece of cardboard is proportional to its area, what is the cost of a piece of cardboard that is 16 inches by 20 inches?

A) \$4.00
B) \$8.00
C) \$12.00
D) \$16.00

10

For the values x and y, the ratio of x to y is 15 to 19. If x is increased by 50%, what must y be multiplied by in order to maintain the same ratio?

11

The student body at an after-school program consists only of 6th graders, 7th graders, and 8th graders. The ratio of 6th graders to 8th graders is 17 to 28. If a total of 110 students attend the program, n of whom are 7th graders, what is the least possible value of n ?

12

A pressure washer pumps water at a constant rate of 24,960 fluid ounces per minute. At what rate, in gallons per second, does the pressure washer pump water?
(1 gallon = 128 fluid ounces)

A) 3.25
B) 195
C) 11,700
D) 53,248

13

A pass at an arcade venue allows for 8 rounds of a certain game to be played. Each round lasts 14 minutes. If the pass costs \$40, what is the approximate cost per minute of play, in cents?

A) 2.8
B) 22.9
C) 35.7
D) 70.0

14

In a trail mix, the ratio of pistachios to walnuts is 5 to 6, and the ratio of walnuts to cashews is 4 to 15. If the ratio of pistachios to cashews in the trail mix is 2 to k, what is the value of k ?

15

The price of Product X is 25% greater than the price of Product Y. The price of Product Z is 25% less than the price of product Y. What is the ratio of the price of Product X to the price of Product Z?

A) 3 to 2
B) 4 to 3
C) 5 to 2
D) 5 to 3

16

An object is expanding so that its surface area is increasing at a constant rate of 3 square inches per hour. There are 144 square inches in 1 square foot. Of the following, which is closest to the rate at which the surface area of the object is increasing, in square meters per hour? (Use 1 foot = 0.3048 meter.)

A) 0.0019
B) 0.0064
C) 0.0684
D) 0.2242

17

Boxed are unloaded from a delivery truck at a constant rate. By 1:00 PM, a total of 40 boxes had been unloaded from the truck. By 3:30 PM, a total of 65 boxes had been unloaded from the truck. What is the total number of boxes that will have been unloaded from the truck by 7:00 PM?

18

Brett currently spends \$160 each month on gas. His current car is able to travel 30 miles per gallon of gas. He decides to switch his current car for a new car that is able to travel 40 miles per gallon of gas. Assuming the price of gas stays the same, how much will he spend on gas each month with the new car?

A) \$100
B) \$120
C) \$130
D) \$140

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11

Word Problems

This chapter will focus on questions that require you to construct an equation or expression from a given situation. You will develop an instinct for translating words into math, setting the right variables, and finally solving for the answer (if necessary).

EXAMPLE 1: A retail store has monthly fixed costs of \$3,000 and monthly salary costs of \$2,500 for each employee. If the store maintains a total of x employees during a certain year, which of the following equations represents the total cost c, in dollars, of running the store for that year?

A) $c = 3{,}000 + 2{,}500x$
B) $c = 12(3{,}000 + 2{,}500x)$
C) $c = 12(3{,}000) + 2{,}500x$
D) $c = 3{,}000 + 12(2{,}500x)$

The monthly cost of running the store is $3{,}000 + 2{,}500x$. To get the cost for an entire year, we multiply by 12 months: $c = 12(3{,}000 + 2{,}500x)$. Answer $\boxed{(B)}$.

EXAMPLE 2: Karter traveled a total of 20 miles by jogging at a speed of 4 miles per hour and biking at a speed of b miles per hour. He spent one more hour jogging than he did biking. Which equation represents this situation, where x represents the number of hours Karter spent jogging?

A) $4x + b(x+1) = 20$ B) $4x + b(x-1) = 20$ C) $\frac{x}{4} + \frac{x+1}{b} = 20$ D) $\frac{x}{4} + \frac{x-1}{b} = 20$

Since Karter spent x hours jogging, he must have spent $x - 1$ hours biking. By jogging at 4 miles per hour for x hours, he traveled $4x$ miles, and by biking at b miles per hour for $x - 1$ hours, he traveled $b(x-1)$ miles. Since he traveled a total of 20 miles, $4x + b(x-1) = 20$. Answer $\boxed{(B)}$.

EXAMPLE 3: The sum of three consecutive integers is 72. What is the largest of these three integers?

The most important technique in solving word problems is to let a variable be one of the things you don't know. In this problem, we don't know any of the three integers, so we let the smallest one be x. It doesn't matter which number we set as x, as long as we're consistent throughout the problem.

So if x is the smallest, then our consecutive integers are

$$x, x+1, x+2$$

Because they sum to 72, we can set up an equation:

$$\begin{aligned} x + (x+1) + (x+2) &= 72 \\ 3x + 3 &= 72 \\ 3x &= 69 \\ x &= 23 \end{aligned}$$

Because x is the smallest, our three consecutive integers must then be 23, 24, and $\boxed{25}$ (the largest).

But what would the solution have looked like if we had let x be the largest integer? Our three integers would've been

$$x-2, x-1, x$$

And our equation would've been

$$\begin{aligned} (x-2) + (x-1) + x &= 72 \\ 3x - 3 &= 72 \\ 3x &= 75 \\ x &= 25 \end{aligned}$$

And because x was set to be the largest of the three integers in this scenario, we're already at the answer!

The lesson here is that you should think about which unknown you want to set as the variable. Often times, that unknown will be what the question is asking for. Other times, it will be an unknown you specifically choose to make the problem easier to set up and solve. And sometimes, as was the case in this example, it doesn't matter which unknown you pick; you'll end up with the same answer with the same amount of effort.

EXAMPLE 4: One number is 3 times another number. If they sum to 44, what is the larger of the two numbers?

In this problem, we want to set x to be the smaller of the two numbers. That way, the two numbers can be expressed as x and $3x$. If we let x be the larger of the two, we would have to work with x and $\frac{x}{3}$, and fractions are yucky. Setting up our equation, we get

$$\begin{aligned} x + 3x &= 44 \\ 4x &= 44 \\ x &= 11 \end{aligned}$$

Be careful—we're not done yet! The question asks for the larger of the two, so we have to multiply x by 3 to get $\boxed{33}$.

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EXAMPLE 5: Of the children enrolled at a daycare program, 70% attend full-time and the rest attend part-time. If the number of children who attend full-time is 150 more than the number of children who attend part-time, what is the total number of children enrolled at the daycare program?

Since 70% attend full-time, 30% must attend part-time. Now let x be the total number of children enrolled at the program. Then $0.70x$ attend full-time and $0.30x$ attend part-time. Setting up an equation, we get

$$\begin{aligned} 0.70x - 0.30x &= 150 \\ 0.40x &= 150 \\ x &= \frac{150}{0.40} = \boxed{375} \end{aligned}$$

EXAMPLE 6: Albert is 7 years older than Henry. In 5 years, Albert will be twice as old as Henry. How many years old is Albert now?

Let x be Albert's age now. We could've assigned x to be Henry's age, but as we mentioned earlier, assigning the variable to be what the question is asking for is typically the fastest route. Now at this point, some of you might be thinking of assigning another variable to Henry's age. While that would certainly work, it would only add more steps to the solution. Try to stick to one variable unless the question clearly calls for more.

If Albert is x years old now, then Henry must be $x - 7$ years old.
Five years from now, Albert will be $x + 5$ and Henry will be $x - 7 + 5 = x - 2$ years old.

$$\begin{aligned} x + 5 &= 2(x - 2) \\ x + 5 &= 2x - 4 \\ x &= \boxed{9} \end{aligned}$$

EXAMPLE 7: After each of her workouts, Iris consumes 8 ounces of a particular drink that is sold in 16-ounce bottles. Each bottle contains 20 grams of protein. Iris would like to increase her protein intake by 3 grams after each workout. Which equation can Iris use to determine how many additional ounces, x, of the drink she should consume after each workout?

A) $\frac{16}{20}x = 3$ B) $\frac{16}{20}x = 13$ C) $\frac{20}{16}x = 3$ D) $\frac{20}{16}x = 13$

There are 20 grams of protein in every 16 ounces. Therefore, in x additional ounces of the drink, there are $\frac{20}{16}x$ grams of protein. This expression needs to equal 3 since that's the amount of protein Iris would like to increase her intake by. Answer $\boxed{\text{(C)}}$.

We can also approach this question starting from the 3 additional grams of protein. There are again 20 grams of protein in every 16 ounces of the drink, so to get those 3 additional grams of protein, Iris would need to consume $\frac{16}{20}(3)$ additional ounces. So, $x = \frac{16}{20}(3)$, which is equivalent to the equation $\frac{20}{16}x = 3$.

EXAMPLE 8: Jake can run 60 yards per minute. Amy can run 120 yards per minute for the first 10 minutes but then slows down to 20 yards per minute thereafter. If they start running at the same time, after how many minutes t will both Jake and Amy have run the same distance, assuming $t > 10$?

The problem already gives us a variable t to work with. We want to equate Jake's distance run with Amy's.

Jake's distance: $60t$
Amy's distance: $120(10) + 20(t - 10)$

$$\begin{aligned} 60t &= 120(10) + 20(t - 10) \\ 60t &= 1{,}200 + 20t - 200 \\ 40t &= 1{,}000 \\ t &= 25 \end{aligned}$$

After $\boxed{25}$ minutes, they will have run the same distance.

EXAMPLE 9: A tank contains 30 liters of a 10% acetone solution. How many liters of a 50% acetone solution must be added to the tank to obtain a 20% acetone solution?

Wrong Solution: The tank currently contains $30(0.10) = 3$ liters of acetone. To obtain a 20% acetone solution, there would need to be $30(0.20) = 6$ liters of acetone. Therefore, $6 - 3 = 3$ liters of acetone need to be added. Given that we have a 50% acetone solution to work with, we would need to add $3(2) = \boxed{6}$ liters of this solution to contribute the necessary 3 liters of acetone.

This solution is wrong and illustrates a common mistake: forgetting that the total amount of solution in the tank changes once we add to it. When we add 6 liters of a 50% solution to the tank, it's not just the amount of acetone that increases; the total amount of solution in the tank also increases. So we can't just take 20% of the original 30 liters and set that as the target because the target shifts once we add more solution.

Correct Solution: The tank currently contains $30(0.10) = 3$ liters of acetone. Let x be the amount of 50% solution that must be added. Adding x liters of the 50% acetone solution will contribute an additional $0.50x$ liters of acetone and increase the total amount of solution in the tank to $30 + x$ liters. Since we want to obtain a 20% solution, we can set up the following equation:

$$\begin{aligned} \frac{\text{Liters of acetone}}{\text{Liters of solution}} &= 0.20 \\ \frac{3 + 0.50x}{30 + x} &= 0.20 \\ 3 + 0.50x &= 0.20(30 + x) \\ 3 + 0.50x &= 6 + 0.20x \\ 0.30x &= 3 \\ x &= \boxed{10} \end{aligned}$$

So to obtain a 20% solution, 10 liters of the 50% solution must be added. This type of word problem with percentages is very common in chemistry and is typically known as a "mixture" problem.

EXAMPLE 10: When Alex and Barry work separately from each other, Alex can paint a house in 6 days, and Barry can paint a house in 12 days. Assuming that they each work at a constant rate, how many days will it take Alex and Barry to paint a house if they work together?

This is the typical "work-rate" problem that involves two individuals who work at different rates. The general approach is to use the formula $W = rt$, where W is the amount of work done, r is the overall rate at which work is being done, and t is the time spent. The key thing to note is that the overall rate, r, can be found by summing up the individual rates.

Since Alex can paint a house in 6 days, his rate is $\frac{1}{6}$ of a house per day. Since Barry can paint a house in 12 days, his rate is $\frac{1}{12}$ of a house per day.

Working together, they can paint $\frac{1}{6} + \frac{1}{12} = \frac{2}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$ of a house per day. Now we can use $W = rt$, where $W = 1$ (i.e. 1 house) and $r = \frac{1}{4}$, to find the time it will take them to paint one house.

$$W = rt$$
$$1 = \frac{1}{4}t$$
$$4 = t$$

Therefore, it will take Alex and Barry $\boxed{4}$ days to paint one house. This answer makes sense because if Alex can finish a house in 6 days by himself, then it should take less than 6 days if Barry is working alongside him.

EXERCISE 1: Answers for this exercise start on page 351.

1

According to a model, the temperature of an object was 18 degrees Celsius when it was placed in an oven and increased by 2.4 degrees Celsius per minute while it was in the oven. The model is defined by the equation $y = bx + c$, where y is the temperature, in degrees Celsius, of the object x minutes after it was placed in the oven, and b and c are constants. What is the value of c ?

2

If 3 is subtracted from 3 times the number x, the result is 21. What is the result when 8 is added to half of x ?

A) 1
B) 5
C) 8
D) 12

3

A cheese vendor currently has 175 pounds of mozzarella available for sale. If each pound of mozzarella sells for \$8.75, which of the following functions gives the amount of mozzarella M, in pounds, still available for sale after d dollars worth has been sold?

A) $M(d) = 175 - \frac{d}{8.75}$
B) $M(d) = 175 - 8.75d$
C) $M(d) = 175 - \frac{8.75}{d}$
D) $M(d) = 175(8.75) - d$

4

John and David work as home inspectors. For each inspection, John charges \$0.30 per square foot plus a base fee of \$500. David charges \$0.45 per square foot plus a fee of \$25 per bedroom. If they each charge the same amount to inspect a certain 5-bedroom house, what is the size, in square feet, of the house?

5

Marcia baked 240 cookies and distributed a third of them evenly among c classmates. Each classmate received 4 cookies. Which equation represents this situation?

A) $\frac{3c}{240} = 4$
B) $\frac{240c}{3} = 4$
C) $\frac{240}{3c} = 4$
D) $\frac{240(3)}{c} = 4$

6

Point B lies on line segment AC such that the length of AB is 7 more than twice the length of BC. If the length of AC is 55, what is the length of AB ?

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7

A garden has $\frac{3}{5}$ as many shrubs as flowers. If the garden has 180 more flowers than shrubs, how many flowers are in the garden?

8

To commute from his house to his office in the morning, Carl takes a 20-mile route and drives at an average speed of a miles per hour. To commute from his office back to his house in the evening, Carl takes a 24-mile route and drives at an average speed of b miles per hour. Carl's morning commute takes the same amount of time as his evening commute. Which of the following equations represents the information given?

A) $20a + 24b = 44$

B) $\frac{a}{24} = \frac{b}{20}$

C) $\frac{20}{a} = \frac{24}{b}$

D) $\frac{a}{20} + \frac{b}{24} = 1$

9

Each player in a golf league is assigned to one of 38 groups. The number of groups with 4 players is 3 times the number of groups g with 5 players. There are 2 groups with 6 players. Which equation must be true for the value of g ?

A) $3g + 2 = 38$

B) $4g + 5(3g) + 2(6) = 38$

C) $4g + 2 = 38$

D) $4(3g) + 5g + 2(6) = 38$

10

At a store, the price of a tie is k dollars less than three times the price of a shirt. If a shirt costs \$40 and a tie costs \$30, what is the value of k ?

11

A taxicab charges \$4.75 for the first mile traveled and \$3.50 for each additional mile. A particular passenger travels x miles in the taxicab and is charged a total of \$24. Which equation represents this situation?

A) $3.50x - 4.75 = 24$

B) $8.25x - 3.50 = 24$

C) $3.50x + 1.25 = 24$

D) $3.50x + 4.75 = 24$

12

Joey and Matt were tasked with bringing popcorn to a certain event. Joey brought x pounds of popcorn and Matt brought 4 fewer pounds than Joey did. During the event, 60% of the popcorn they had brought was consumed. If Joey and Matt divided the leftover popcorn evenly between them, which of the following represents the number of pounds of leftover popcorn each of them took?

A) $0.2x - 0.8$

B) $0.4x - 0.8$

C) $0.6x - 1.2$

D) $0.8x - 1.6$

13

Mitch works 42 hours each week clearing land. It takes him 14 hours to clear 1 acre of land. This week, Mitch would like to work additional hours to earn \$200 more than he usually does. If Mitch works at a constant rate and earns \$875 for each acre of land cleared, which of the following equations can he use to solve for h, the additional number of hours he should work this week?

A) $\frac{14}{875}h = 200$

B) $\frac{14}{875}h = 2{,}825$

C) $\frac{875}{14}h = 200$

D) $\frac{875}{14}h = 2{,}825$

14

Joseph can harvest a field of wheat in 12 days. Patrick can harvest the same field twice as fast as Joseph can. If Joseph and Patrick work together at their respective rates to harvest the entire field in t days, which of the following equations represents this situation?

A) $\frac{12}{t} + \frac{6}{t} = 2$

B) $\frac{1}{12}t + \frac{1}{6}t = 1$

C) $\frac{12}{t} + \frac{24}{t} = 2$

D) $\frac{1}{12}t + \frac{1}{24}t = 1$

15

Mark and Kevin own $\frac{1}{4}$ and $\frac{1}{3}$ of the books on a shelf, respectively. Lori owns the remainder of the books. If Kevin owns 9 more books than Mark, how many books does Lori own?

16

A water tank is connected to two pipes, Pipe A and Pipe B. It takes 4 hours to fill the tank when only Pipe A is in use, and it takes 6 hours to fill the tank when only Pipe B is in use. If it takes m minutes to fill the tank when both Pipe A and Pipe B are in use, what is the value of m ?

17

A flask contains 24 milliliters of a 30% acid solution. How many milliliters of an 8% acid solution must be added to the flask to produce a 16% acid solution?

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EXERCISE 2: Answers for this exercise start on page 353.

1

Which of the following represents the square of the sum of x and y, decreased by the product of x and y ?

A) $x^2 + y^2 - xy$

B) $x^2y^2 - xy$

C) $(x+y)^2 - (x+y)$

D) $(x+y)^2 - xy$

2

An internet service provider charges a one time setup fee of \$100 and \$50 each month for service. If c customers join at the same time and are on the service for m months, which of the following expressions represents the total amount, in dollars, the provider has charged these customers?

A) $100c + 50m$

B) $100c + 50cm$

C) $150cm$

D) $100m + 50cm$

3

To prepare for landing, a plane descends so that its altitude decreases at a constant rate from 24,500 feet to 17,900 feet in 12 minutes. Which of the following equations gives the altitude A, in feet, of the plane t minutes after its descent began, for $0 \le t \le 12$?

A) $A = 17{,}900 - 550t$

B) $A = 17{,}900 + 550t$

C) $A = 24{,}500 - 550t$

D) $A = 24{,}500 + 550t$

4

A customer at a grocery store bought 4 times as many bags of rice as bags of pasta. Each bag of rice cost \$2.86 and each bag of pasta cost \$3.41. If the customer spent a total of \$103.95 on rice and pasta, how many bags of rice did the customer buy?

5

The Pirates won exactly 4 of their first 15 games. They then played N remaining games and won all of them. If they won exactly half of all the games they played, what is the value of N ?

6

When ordering a certain book in bulk, a school pays \$1,720 for the first 80 copies and a discounted price for each additional copy. The total cost for 200 copies of the book is \$3,700. Which function f gives the total cost, in dollars, for x copies of the book, where $x \ge 80$?

A) $f(x) = 18.50x + 240$

B) $f(x) = 18.50x + 1{,}720$

C) $f(x) = 16.50x + 400$

D) $f(x) = 16.50x + 1{,}720$

7

It will take 5 hours for a group of n volunteers to plant a total of 180 trees. Each tree takes 15 minutes to plant. Which equation represents this situation?

A) $\left(\frac{180}{15}\right)n = (5)(60)$

B) $\left(\frac{60}{15}\right)n = (180)(5)$

C) $(5)(60)n = (180)(15)$

D) $(5)(15)n = (180)(60)$

8

A tub can be filled by a hose and emptied by a drain. When the drain is closed, the hose can fill $\frac{1}{3}$ of the tub in 1 minute. When the hose is turned off, the drain can empty $\frac{2}{11}$ of the tub in 1 minute. If it takes t minutes for the hose to fill the tub while the drain is open, which of the following equations represents this situation?

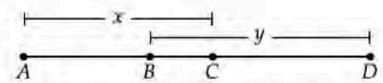
A) $\frac{1}{3}t - \frac{2}{11} = 1$

B) $\frac{1}{3} - \frac{2}{11} = \frac{1}{t}$

C) $\frac{1}{3}t + \frac{2}{11} = 1$

D) $\frac{1}{3} + \frac{2}{11} = \frac{1}{t}$

9



Note: Figure not drawn to scale.

For line segment $\overline{AD}$ shown, $AD = 50$ and $BC = 10$. Which equation shows the relationship between the lengths of AC, BD, and AD ?

A) $x + y + 10 = 50$

B) $x + y - 5 = 50$

C) $x + y - 10 = 50$

D) $x + y - 20 = 50$

10

An accounting company currently has three times as many junior accountants as senior accountants. The company plans to hire 140 more senior accountants so that the number of junior accountants is twice the number of senior accountants. How many junior accountants does the accounting company currently have?

11

Alice and Trinity pack boxes of specialized equipment for delivery. When they are each working alone, Alice can pack 10 boxes every 3 hours and Trinity can pack 15 boxes every x hours. Working together at their respective rates, Alice and Trinity can pack 115 boxes in 12 hours. Which of the following equations represents this situation?

A) $115\left(\frac{3}{10} + \frac{x}{15}\right) = 12$

B) $12\left(\frac{3}{10} + \frac{x}{15}\right) = 115$

C) $12\left(\frac{10}{3} + \frac{15}{x}\right) = 1$

D) $12\left(\frac{10}{3} + \frac{15}{x}\right) = 115$

12

A bakery gave out coupons to celebrate its grand opening. Each coupon was worth either \$1, \$3, or \$5. Twice as many \$1 coupons were given out as \$3 coupons, and 3 times as many \$3 coupons were given out as \$5 coupons. The total value of all the coupons given out was \$360. How many \$3 coupons were given out?

A) 40

B) 45

C) 48

D) 54

13

Yoona runs at a constant rate of 1 yard per second. Jessica runs 4 times as fast as Yoona. If Jessica gives Yoona a head start of 30 yards in a race, how many yards must Jessica run to catch up to Yoona?

14

Of the 200 jellybeans in a jar, 70% are green and the rest are red. How many green jellybeans must be removed so that 60% of the remaining jellybeans are green?

15

To move into a new studio space, the m members of an art club agreed to split the first month's rent of r dollars equally among themselves. If k of the members fail to pay their share, which of the following represents the additional amount, in dollars, that each of the remaining members must pay to cover the first month's rent?

A) $\frac{r}{m-k}$

B) $\frac{kr}{m-k}$

C) $\frac{kr(m-k)}{m}$

D) $\frac{kr}{m(m-k)}$

16

Terry is hired to pave a parking lot and finishes $\frac{1}{3}$ of the parking lot before Andy is hired to work alongside him. They each work at a constant rate, but Terry works faster than Andy does. The equation $9\left(\frac{1}{x}+\frac{1}{x+4}\right)=\frac{2}{3}$ can be used to find the total number of days x it would have taken Terry to pave the entire parking lot by himself. Which of the following is the best interpretation of the number 9 in the equation?

A) The number of days it would have taken Terry and Andy to pave the entire parking lot if they had worked together from the start.

B) The number of days it will take Terry and Andy to pave the remainder of the parking lot working together.

C) The number of days it would take Andy to pave the remainder of the parking lot if he were working alone.

D) The number of days it would take Terry to pave the remainder of the parking lot if he were working alone.

17

A chemist wants to make 800 milliliters of a 40% alcohol solution by mixing a 25% alcohol solution with a 50% alcohol solution. How many milliliters of the 25% alcohol solution should the chemist use?

12 Quadratic Equations

Just as lines were one group of functions that have their own properties, quadratics are another. A quadratic is a function of the form

$$f(x) = ax^2 + bx + c$$

in which the highest power of x is 2. A quadratic equation is an equation that can be written as

$$ax^2 + bx + c = 0$$

In this chapter, we'll review how to solve quadratic equations. In a future chapter, we'll look at certain properties of quadratic functions and their graphs.

EXAMPLE 1:

$$x^2 + 5x - 14 = 0$$

What is one solution to the given equation?

When solving quadratic equations, factoring is the standard first approach. In this example, we can factor the left-hand side of the equation to get

$$(x+7)(x-2) = 0$$

For the factored expression to equal 0, either $x + 7 = 0$ or $x - 2 = 0$, so the possible solutions are $x = \boxed{-7}$ or $x = \boxed{2}$. A lesson on factoring quadratics is outside the scope of this book, so if it's something you struggle with, feel free to find an algebra textbook or an online tutorial to brush up.

In cases where a quadratic expression cannot be factored or factoring is too difficult, the quadratic formula can be used to find the solutions.

The Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Applying the quadratic formula to Example 1 yields the same answers as factoring:

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(-14)}}{2(1)} = \frac{-5 \pm \sqrt{81}}{2} = \frac{-5 \pm 9}{2} = -7 \text{ or } 2$$

EXAMPLE 2:

$$2x^2 - 10x + 3 = 0$$

One solution to the given equation can be written as $\frac{5-\sqrt{n}}{2}$, where n is a constant. What is the value of n?

The given quadratic expression cannot be factored, so let's use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(2)(3)}}{2(2)} = \frac{10 \pm \sqrt{76}}{4} = \frac{10 \pm 2\sqrt{19}}{4} = \frac{5 \pm \sqrt{19}}{2}$$

Therefore, one of the solutions is $\frac{5-\sqrt{19}}{2}$, and the value of n must be $\boxed{19}$.

The next few examples involve the following formulas, which you should memorize.

If $ax^2 + bx + c = 0$, then **the sum of the solutions** $= -\frac{b}{a}$ and **the product of the solutions** $= \frac{c}{a}$.

EXAMPLE 3:

$$x^2 + 5x - 14 = 0$$

What is the sum of the solutions to the given equation?

We can find the sum of the solutions without finding the solutions themselves.

$$\text{Sum of the solutions} = -\frac{b}{a} = -\frac{5}{1} = \boxed{-5}$$

Note that we already found the solutions to this equation in Example 1: -7 and 2. The sum of these solutions is indeed -5.

EXAMPLE 4:

$$2x^2 - 5x - 8 = 0$$

What is the product of the solutions to the given equation?

$$\text{Product of the solutions} = \frac{c}{a} = \frac{-8}{2} = \boxed{-4}$$

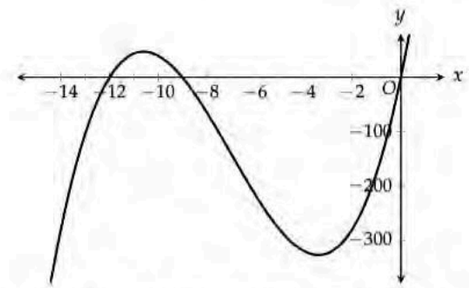
EXAMPLE 5: One of the factors of $2x^3 + 42x^2 + 216x$ is $x + c$, where c is a positive constant. What is the smallest possible value of c?

Solution 1: Factor out the greatest common factor, $2x$, from the expression and then factor the quadratic:

$$2x^3 + 42x^2 + 216x = 2x(x^2 + 21x + 108) = 2x(x+9)(x+12)$$

As you can see, $2x$, $x+9$, and $x+12$ are all factors of the expression. Since the question specifies that c is positive, c must be either 9 or 12. The smallest possible value of c is therefore $\boxed{9}$.

Solution 2: If you have trouble with factoring, an alternative approach is to graph the given expression as a function and derive the factors from the x-intercepts of the graph. Here's the graph of $y = 2x^3 + 42x^2 + 216x$:



In the functions chapter, we learned that if the graph of a function has an x-intercept at $(n, 0)$, then $(x - n)$ must be a factor of that function. Because the graph above has x-intercepts at -12, -9, and 0, $(x+12)$, $(x+9)$, and x must be factors of the original expression. Therefore, the smallest possible (positive) value of c is indeed $\boxed{9}$.

Side Note: If the graph of a function has an x-intercept at 1, then $(x - 1)$ must be a factor of that function. But what if the x-intercept is at $\frac{1}{2}$ or $\frac{2}{3}$? While we could write the corresponding factors as $\left(x - \frac{1}{2}\right)$ or $\left(x - \frac{2}{3}\right)$, we typically want to avoid any fractions in these expressions. The factor $\left(x - \frac{1}{2}\right)$ is more often written as $(2x - 1)$, and $\left(x - \frac{2}{3}\right)$ is more often written as $(3x - 2)$. An easy way to tell whether two factors are equivalent is if they both equal zero for the same value of x. The factors $\left(x - \frac{1}{2}\right)$ and $(2x - 1)$ are equivalent because they both equal zero when $x = \frac{1}{2}$.

EXAMPLE 6:

$$\sqrt{x+5} + 1 = x$$

What is the solution to the given equation?

When a square root is involved, one common mistake students make is squaring each side before the square root is isolated on one side. In this example, we can't square each side right away to get rid of the square root. We first have to move the "1" from the left-hand side to the right-hand side:

$$\sqrt{x+5} = x - 1$$

Now we can square both sides.

$$\begin{aligned}(\sqrt{x+5})^2 &= (x-1)^2\\ x + 5 &= x^2 - 2x + 1\\ 0 &= x^2 - 3x - 4\\ 0 &= (x-4)(x+1)\\ x &= -1, 4\end{aligned}$$

So, the solutions are -1 and 4, but hold on! We're not actually done yet. When there are square roots in the original equation, we have to check for false solutions by testing each of our values in the original equation. When $x = -1$, the left-hand side is $\sqrt{-1+5}+1 = 3$ and the right-hand side is -1. These values don't match, so -1 is not actually a solution. When $x = 4$, the left-hand side is $\sqrt{4+5}+1 = 4$ and the right-hand side is also 4. The values from both sides match so $\boxed{4}$ is the only solution.

Why do false solutions occur? Because squaring both sides has the effect of turning negative values into positive ones, which can cause a positive-negative mismatch. If we plug $x = -1$ into $\sqrt{x+5} = x-1$, we get 2 on the left but -2 on the right. One side is positive and the other is negative. Once we square both sides, however, this distinction gets lost since both sides become 4.

In summary, when you're dealing with a square root in an equation, isolate the square root before squaring each side. Once you have a set of possible solutions, check for false solutions by plugging your results into the original equation.

EXAMPLE 7:

$$(x+1)^2 + 5(x+1) - 24 = 0$$

What is the positive solution to the given equation?

To simplify the equation and make it easier to factor, we can treat $(x+1)$ as a "single entity" by substituting an arbitrary variable such as A for $x+1$. In other words, let $A = x+1$. Once we've factored the equation, we can substitute $x+1$ back in for A.

$$\begin{aligned}(x+1)^2 + 5(x+1) - 24 &= 0\\ A^2 + 5A - 24 &= 0\\ (A+8)(A-3) &= 0\\ (x+1+8)(x+1-3) &= 0\\ (x+9)(x-2) &= 0\\ x &= -9 \text{ or } 2\end{aligned}$$

Because the question asks for the positive solution, the answer is $\boxed{2}$.

EXAMPLE 8: In the xy-plane, the graph of $y = x^2 + 2x - 12$ passes through the point $(m, 6m)$, where m is a positive constant. What is the value of m?

We're given a point on the graph, so let's plug it in.

$$\begin{aligned}y &= x^2 + 2x - 12\\ 6m &= m^2 + 2m - 12\\ 0 &= m^2 - 4m - 12\\ 0 &= (m-6)(m+2)\\ m &= 6, -2\end{aligned}$$

The question states that m is positive, so $m = \boxed{6}$.

EXERCISE 1: Answers for this exercise start on page 356.

1

$$x^2 - 5x - 24 = 0$$

What is one of the solutions to the given equation?

2

$$x^2 - 3x - 12 = x - 7$$

What is the positive solution to the given equation?

3

$$x^2 + 7x + 19 = 28 - 5x$$

What is the sum of the solutions to the given equation?

A) -12
B) -9
C) -7
D) 12

4

$$x^2 - 10x + 7 = 0$$

Which of the following is a solution to the given equation?

A) $5 + 6\sqrt{2}$
B) $5 - 3\sqrt{2}$
C) $10 + 8\sqrt{3}$
D) $10 - 6\sqrt{3}$

5

$$-x(x+3) = x + 4$$

What is the solution to the given equation?

6

$$x^2 + 4x + 2 = 0$$

What are the solutions to the given equation?

A) $x = -2 \pm \sqrt{2}$
B) $x = 2 \pm 2\sqrt{2}$
C) $x = -2 \pm 2\sqrt{2}$
D) $x = -4 \pm 2\sqrt{2}$

7

$$\frac{84}{x} = x - 8$$

What is the positive solution to the given equation?

8

Which of the following is a factor of $2x^2 + x - 15$?

A) $x - 3$
B) $x + 5$
C) $2x - 5$
D) $2x + 3$

9

$$(2x-3)^2 = 4x+5$$

What is the sum of the solutions to the given equation?

10

One of the factors of $3x^3 + 69x^2 + 336x$ is $x+k$, where k is a positive constant. What is the largest possible value of k ?

11

$$x^2 - 12x + 6 = 0$$

One solution to the given equation can be written as $6+\sqrt{n}$, where n is a constant. What is the value of n ?

12

$$(x+3)(x-3) = 7x-3$$

What is the product of the solutions to the given equation?

A) -12
B) -9
C) -6
D) 7

13

One of the factors of $5x^3 - 50x^2 + 120x$ is $x-m$, where m is a positive constant. What is one possible value of m ?

14

$$4x^2 - (2k+1)x + k + 1 = 0$$

In the given equation, k is a positive constant. The product of the solutions to the given equation is $w(k+1)$, where w is a constant. What is the value of w ?

A) 4
B) 2
C) $\frac{1}{2}$
D) $\frac{1}{4}$

15

$$\sqrt{(x+6)^2} = \sqrt{6-5x}$$

What is the smallest solution to the given equation?

16

$$(8x)^2 - 5(8x) = 6$$

What is the positive solution to the given equation?

EXERCISE 2: Answers for this exercise start on page 358.

1

$$4x^2 - 9x - 1 = 0$$

What are the solutions to the given equation?

A) $x = \frac{-9 \pm \sqrt{97}}{8}$
B) $x = \frac{-9 \pm \sqrt{65}}{8}$
C) $x = \frac{9 \pm \sqrt{97}}{8}$
D) $x = \frac{9 \pm \sqrt{65}}{8}$

2

$$-\frac{1}{2}x^2 + x = -24$$

What is the positive solution to the given equation?

3

$$x^2 + x - 5 = 0$$

One solution to the given equation can be written as $\frac{\sqrt{k}-1}{2}$, where k is a constant. What is the value of k ?

4

$$2x(-x+17) = 54 - x(5-3x)$$

What is the sum of the solutions to the given equation?

5

$$-3x^2 + 6x - 1 = 0$$

What is one solution to the given equation?

A) $-1 + \frac{\sqrt{2}}{3}$
B) $-1 + \frac{\sqrt{6}}{3}$
C) $1 - \frac{\sqrt{2}}{3}$
D) $1 - \frac{\sqrt{6}}{3}$

6

$$x(x-a)(x+4) = 0$$

In the given equation, a is a positive constant. The sum of the solutions to the given equation is 6. What is the value of a ?

7

$$x^2 - 8x = -9$$

One solution to the given equation can be written as $a - \sqrt{b}$, where a and b are constants. What is the value of $a + b$?

A) 8
B) 11
C) 18
D) 36

8

$$x^2 - 8kx - (k - 4) = 0$$

In the given equation, k is a positive constant. What is the product of the solutions to the given equation?

A) $4 - k$
B) $k - 4$
C) $-8k$
D) $8k$

9

In the xy-plane, the graph of $y = ax + 12$, where a is a positive constant, passes through the point $(-a, a)$. What is the value of a ?

10

$$2x^2 - 7x + 3 = 0$$

What is the smallest solution to the given equation?

11

Which of the following expressions is (are) a factor of $2x^2 + 11x + 12$?

I. $x + 2$
II. $2x + 1$

A) Neither I nor II
B) I only
C) II only
D) I and II

12

$$3x^2 + (108k - 51)x + 93k + 21 = 0$$

In the given equation, k is a positive constant. The sum of the solutions to the given equation is $ak + b$, where a and b are constants. What is the value of $a + b$?

13

$$72x^2 - (18p + 9q)x + 64p = -32q$$

In the given equation, p and q are positive constants. The product of the solutions to the given equation is $n(2p + q)$, where n is a constant. What is the value of n ?

14

If $4x + k$ is a factor of $8x^3 - 26x^2 - 24x$, where k is a positive constant, what is the value of k ?

15

$$2\sqrt{4x - 19} = x - 3$$

What is the largest solution to the given equation?

16

$$(x + 11)^2 - 21(x + 11) + 54 = 0$$

What is the positive solution to the given equation?

17

$$3x^2 + 10x = 8$$

The two solutions to the given equation are a and b, and $a > b$. What is the value of b^2 ?

A) $\frac{4}{9}$
B) $\frac{2}{3}$
C) 4
D) 16

18

$$x^2 + px + 18 = 0$$

In the given equation, p is a positive constant. The solutions to the given equation are integers. Which of the following is (are) a possible value of p ?

I. 9
II. 17
III. 19

A) I only
B) II only
C) I and III only
D) I, II, and III

13

Systems of Equations

A system of equations refers to 2 or more equations that deal with the same set of variables.

$$-5x + y = -7$$
$$-3x - 2y = -12$$

There are two main ways of solving systems of 2 equations: substitution and elimination.

Substitution is all about isolating one variable, either x or y, in the fastest way possible.

Taking the example above, we can see that it's easiest to isolate y in the first equation because it has no coefficient. Adding $5x$ to both sides, we get

$$y = 5x - 7$$

We can now substitute the y in the second equation with $5x - 7$ and solve from there.

$$\begin{aligned} -3x - 2(5x - 7) &= -12 \\ -3x - 10x + 14 &= -12 \\ -13x &= -26 \\ x &= 2 \end{aligned}$$

Plugging $x = 2$ back into $y = 5x - 7$, we get $y = 5(2) - 7 = 3$.

The solution is then $x = 2, y = 3$, which can be denoted as $(2, 3)$.

Elimination is about getting the same coefficient for one variable across the two equations so that adding or subtracting the equations eliminates that variable.

Using the same example as before, we can multiply the first equation by 2 so that the y's have the same coefficient (we don't have to worry about the sign because we can add or subtract the equations).

$$\begin{aligned} -10x + 2y &= -14 \\ -3x - 2y &= -12 \end{aligned}$$

To eliminate y, we add the equations.

$$\begin{aligned} -10x + 2y &= -14 \\ -3x - 2y &= -12 \\ \hline -13x &= -26 \end{aligned}$$

Now, we can see that $x = 2$. This result can be used in either of the original equations to solve for y. Let's pick the first equation.

$$\begin{aligned} -10(2) + 2y &= -14 \\ -20 + 2y &= -14 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

And finally, we get the same solution as we got using substitution: $x = 2, y = 3$.

When solving systems of equations, you can use either method, but one of them will typically be faster. If you see a variable with no coefficient, like in $-5x + y = -7$ above, substitution is likely the best route. If you see matching coefficients, or you see that it's easy to get matching coefficients, elimination is likely the best route. The example above was simple enough for both methods to work well (though substitution was slightly faster). In these cases, it comes down to your personal preference.

Infinitely many solutions

In the *Other Types of Equation Questions* chapter, we learned that a single equation has infinitely many solutions when both sides of the equation are equivalent.

Similarly, a system of equations has infinitely many solutions when both equations are equivalent. Here, we have to clarify what we mean by "equivalent." Both equations in the system

$$3x + 2y = 5$$
$$3x + 2y = 5$$

have exactly the same coefficients and constants, and therefore the system has infinitely many solutions ($(1, 1), (3, -2), (5, -5)$, to name just a few). However, the equations in the system

$$6x + 4y = 10$$
$$3x + 2y = 5$$

do not have the same coefficients and constants, yet the system still has infinitely many solutions. Why is that? Because the two equations are *essentially* the same: the first equation can be divided by 2 to obtain the second equation. In other words, the two equations are equivalent.

> Two equations are equivalent when one equation can be manipulated to form the other. A system has infinitely many solutions when the equations in the system are equivalent.

EXAMPLE 1:

$$3x - 5y = 8$$
$$mx - ny = 32$$

In the given system of equations, m and n are constants. The system has infinitely many solutions. What is the value of $m + n$?

For there to be infinitely many solutions, both equations need to be equivalent. We can multiply the first equation by 4 to get the constants to match:

$$12x - 20y = 32$$
$$mx - ny = 32$$

Now we can clearly see that $m = 12$ and $n = 20$. Therefore, $m + n = \boxed{32}$.

No solutions

In the *Other Types of Equation Questions* chapter, we saw that a single equation has no solutions when both sides of the equation are equivalent except for the constants.

Similarly, a system of equations has no solutions when both equations are equivalent except for their constants. The constants must be different. For example, the system

$$3x + 2y = 5$$
$$3x + 2y = -4$$

has no solutions because the different constants (5 vs. -4) result in equations that contradict each other. There isn't an x and a y that can possibly satisfy both equations at the same time. Note that the system

$$3x + 2y = 5$$
$$6x + 4y = -8$$

also has no solution. Why? Because the second equation can be divided by 2 to get the same system as before.

A system has no solution when the equations are equivalent except for their constants—in other words, when one equation can be manipulated to have the same coefficients as the other. The constants need not match and must not match.

EXAMPLE 2:

$$-ax - 12y = 15$$
$$4x + 3y = -2$$

In the given system of equations, a is a constant. The system has no solution. What is the value of a ?

We must get the coefficients of y to match so that we can compare the two equations. To do that, we can multiply the second equation by -4:

$$-ax - 12y = 15$$
$$-16x - 12y = 8$$

Now that the coefficients of y are both -12, it's easy to see that when $a = \boxed{16}$, the equations are equivalent except for the constants, which must be the case for a system with no solution.

Word problems

You will most definitely run into a question that asks you to translate a situation into a system of equations. Here's a classic example:

EXAMPLE 3: A group of 30 students order lunch from a restaurant. Each student gets either a burger or a salad. The price of a burger is \$5 and the price of a salad is \$6. If the group spent a total of \$162, how many students ordered burgers?

Let x be the number of students who ordered burgers and y be the number who ordered salads. Based on the given information, we can construct a system of two equations:

$$x + y = 30$$
$$5x + 6y = 162$$

Make sure you understand how we came up with these equations. To solve this system, we'll use elimination. Multiply the first equation by 6 and subtract:

$$\begin{aligned} 6x + 6y &= 180 \\ 5x + 6y &= 162 \\ \hline x &= \boxed{18} \end{aligned}$$

18 students got burgers.

Other Systems of Equations Questions

EXAMPLE 4:

$$3a + 6b = 13$$
$$-2a - 7b = 7$$

The solution to the given system of equations is (a, b). What is the value of $a - b$?

In this example, the question does not ask for the value of a or b, but the value of the expression $a - b$. In these cases, you can often get directly to the answer by adding or subtracting the equations, rather than solving for each variable individually. Here, we can add the two equations to get $a - b = \boxed{20}$.

EXAMPLE 5:

$$0.05x + 0.8 = 4$$
$$3(0.05x + 0.8) + 5(3y - 0.78) = 20$$

The solution to the given system of equations is (x, y). What is the value of $3y - 0.78$?

We can solve for $3y - 0.78$ as if it were a single variable or unit. Also, because $0.05x + 0.8$ shows up in both equations, we can substitute its value given by the first equation into the second:

$$\begin{aligned} 3(0.05x + 0.8) + 5(3y - 0.78) &= 20 \\ 3(4) + 5(3y - 0.78) &= 20 \\ 5(3y - 0.78) &= 8 \\ 3y - 0.78 &= \boxed{1.6} \end{aligned}$$

You might encounter systems of equations that are a bit more complicated than the standard ones we've already shown. For these systems, substitution and some equation manipulation will typically do the trick.

EXAMPLE 6:

$$y + 3x = 0$$
$$x^2 + 2y^2 = 76$$

The solution to the given system of equations is (x, y). What is a possible value of y ?

In the first equation, we isolate y to get $y = -3x$. Plugging this into the second equation,

$$x^2 + 2(-3x)^2 = 76$$
$$x^2 + 2(9x^2) = 76$$
$$x^2 + 18x^2 = 76$$
$$19x^2 = 76$$
$$x^2 = 4$$
$$x = \pm 2$$

When $x = 2$, $y = -3(2) = -6$. When $x = -2$, $y = -3(-2) = 6$. A possible value of y is then either $\boxed{-6}$ or $\boxed{6}$.

Graphs

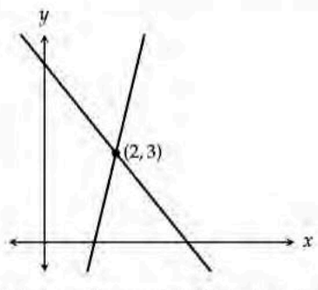
Graphing systems of equations can inform our understanding of them.

More specifically, **when the equations in a system are graphed, the intersection points are the solutions to that system.** Therefore, the number of solutions to a system of equations is equal to the number of intersection points.

Take, for example, the system of equations at the beginning of this chapter:

$$-5x + y = -7$$
$$-3x - 2y = -12$$

We can transform both equations into $y = mx + b$ form (we won't show that here) and graph them to get the following lines.



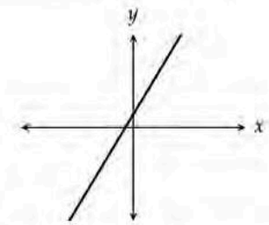
The solution to the system, $(2, 3)$, is the intersection point. There is only one intersection point, so there is only one solution.

What about the graphs of systems that have infinitely many solutions or no solution?

Graphing the following system,

$$2y - 4x = 2$$
$$y - 2x = 1$$

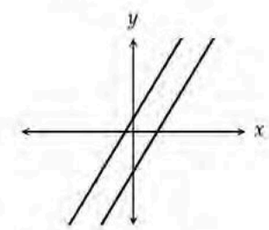
which has infinitely many solutions because its equations are equivalent, we get



It's just one line! Well, actually it's two lines, but because they're the same line, they overlap and intersect at an infinite number of points. Hence, an infinite number of solutions.

Now let's graph a system with no solution:

$$y - 2x = 1$$
$$y - 2x = -3$$



What do you notice about the lines? They're parallel! They have no intersection points. Hence, no solution.

EXAMPLE 7: In the xy-plane, the graphs of the equations $y = 3x - 5$ and $y = -2x + 10$ intersect at the point (h, k). What is the value of k ?

As mentioned earlier, the solutions to a system of equations are the intersection points of the graphs of those equations, and vice versa. **So to find the point(s) where two graphs intersect, solve the system consisting of their equations.** In this problem, that system is

$$y = 3x - 5$$
$$y = -2x + 10$$

Substituting the first equation into the second, we get

$$3x - 5 = -2x + 10$$
$$5x = 15$$
$$x = 3$$

When $x = 3$, $y = 3(3) - 5 = 4$. So the two lines intersect at $(3, 4)$ and $k = \boxed{4}$.

EXAMPLE 8:

$$y = x^2 - 5x + 6$$
$$y = x + 1$$

The graphs of the equations in the given system of equations intersect at the point (x, y) in the xy-plane. What is a possible value of y ?

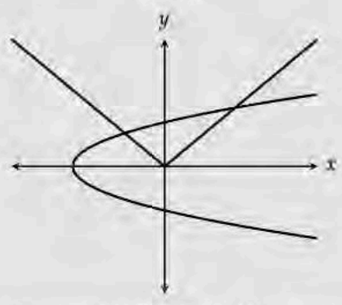
The solutions to the system are the intersection points, so let's solve the system. Substituting the first equation into the second, we get

$$x^2 - 5x + 6 = x + 1$$
$$x^2 - 6x + 5 = 0$$
$$(x - 1)(x - 5) = 0$$
$$x = 1 \text{ or } 5$$

When $x = 1$, $y = 1 + 1 = 2$. When $x = 5$, $y = 5 + 1 = 6$. So the graphs of the two equations intersect at $(1, 2)$ and $(5, 6)$, which means the possible values of y are $\boxed{2}$ and $\boxed{6}$.

EXAMPLE 9:

$$y^2 = x + 3$$
$$y = |x|$$



The given system of equations is represented in the graph. How many solutions does the system have?

A) Zero B) Exactly one C) Exactly two D) Infinitely many

Simple. The graphs intersect at two points, so the system has two solutions. Answer $\boxed{(C)}$.

EXERCISE 1: Answers for this exercise start on page 360.

1

$$y = -5x + 65$$
$$y = 25$$

Which ordered pair (x, y) is a solution to the given system of equations?

A) $(-8, 25)$
B) $(8, 25)$
C) $(25, -8)$
D) $(25, 8)$

2

$$3x - 5y = -11$$
$$x = 1 - 3y$$

The solution to the given system of equations is (x, y). What is the value of y ?

A) -1
B) 0
C) 1
D) 2

3

$$y + 2x = 20$$
$$6x - 5y = 12$$

What is the solution (x, y) to the given system of equations?

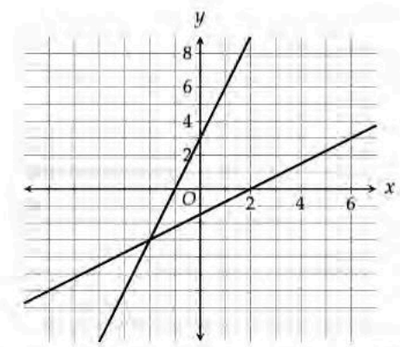
A) $(-7, 6)$
B) $(-6, 7)$
C) $(6, 7)$
D) $(7, 6)$

4

$$y = x^2 + 10x + 23$$
$$x - 9 = -15$$

The solution to the given system of equations is (x, y). What is the value of y ?

5



The graph of a system of two linear equations is shown. What is the solution (x, y) to the system?

A) $(-4, -6)$
B) $(-3, -2)$
C) $(-2, -3)$
D) $(-1, 2)$

6

$$2x + 5y = 24$$
$$x + 4y = 15$$

The solution to the given system of equations is (x, y). What is the value of $x + y$?

A) 7
B) 8
C) 9
D) 10

7

$$2x - 4y = 8$$
$$x + 2y = 4$$

How many solutions (x, y) does the given system of equations have?

A) Zero
B) One
C) Two
D) More than two

8

An ice cream vendor sells small and large cups of ice cream. Each small cup of ice cream is sold for \$5, and each large cup is sold for \$7. During an event, the vendor sold a total of 200 cups of ice cream and collected \$160 more from sales of small cups than from sales of large cups. If x is the number of small cups sold at the event and y is the number of large cups sold at the event, which system of equations represents this situation?

A) $x - y = 160$
$5x + 7y = 200$

B) $x - y = 160$
$7x + 5y = 200$

C) $x + y = 200$
$5x = 7y + 160$

D) $x + y = 200$
$7y = 5x + 160$

9

If $\frac{a}{b} = 4.5$ and $\frac{a}{kb} = 12.5$, what is the value of k ?

10

$$6x - 3y = 9$$

One of the two linear equations in a system is given. The system has no solution. Which equation could be the second equation in the system?

A) $3x - y = 2$
B) $2x - y = 3$
C) $\frac{1}{3}x - \frac{1}{6}y = 2$
D) $\frac{1}{3}x - \frac{1}{6}y = \frac{1}{2}$

11

$$x^2 - y^2 = 48$$
$$x - y = 6$$

If (x, y) is the solution to the given system of equations, what is the value of $x + y$?

A) 8
B) 12
C) 16
D) 24

12

$$2x - 5y = a$$
$$bx + 10y = -8$$

In the given system of equations, a and b are constants. If the system has infinitely many solutions, what is the value of a ?

A) -4
B) $\frac{1}{4}$
C) 4
D) 16

13

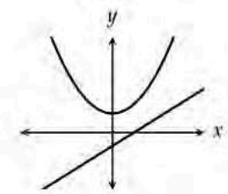
$$y = ax + b$$
$$y = -bx$$

In the given system of equations, a and b are constants. The graphs of the equations in the xy-plane intersect at the point $(2, 8)$. What is the value of a ?

A) 2
B) 4
C) 6
D) 8

14

$$y = x^2 + 1$$
$$y = x - 1$$



The given system of equations is represented in the graph. How many solutions does the system have?

A) Zero
B) Exactly one
C) Exactly two
D) Exactly three

15

$$3x - \frac{5}{16} = 4$$
$$y = 2\left(3x - \frac{5}{16}\right)^2$$

The solution to the given system of equations is the ordered pair (x, y). What is the value of y ?

A) 8
B) 16
C) 32
D) 64

16

$$ax + 2y = 5$$
$$3x - 6y = 20$$

In the given system of equations, a is a constant. The system has exactly one solution. Which of the following could NOT be the value of a ?

A) -1
B) $\frac{3}{4}$
C) 1
D) 3

17

$$3x = 16$$
$$80 = 4y^2 + 3x$$

The graphs of the equations in the given system of equations intersect at the point (x, y) in the xy-plane. What is a possible value of y ?

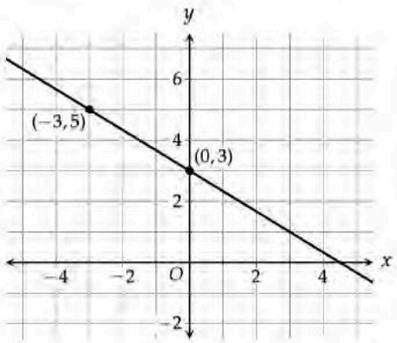
A) $-\frac{16}{3}$
B) -4
C) $\frac{4}{3}$
D) 8

18

$$-5x = 8 - ky$$
$$18y + 5x = -10x + 21$$

In the given system of equations, k is a constant. If the system has no solution, what is the value of k ?

19



The graph of one of the two equations in a linear system is shown in the xy-plane. The system has no solution. Which equation could be the second equation in this system?

A) $y = -\frac{3}{2}x - 1$

B) $y = \frac{3}{2}x + 3$

C) $y = -\frac{2}{3}x - 1$

D) $y = -\frac{2}{3}x + 3$

20

In the xy-plane, the graph of $y = x^2 - 7x + 7$ intersects the graph of $y = 2x - 1$ at the points $(1, 1)$ and (p, q). What is the value of p ?

21

Shannon drove 340 miles from Melrose to Norwich, passing Hanover along the way. From Melrose to Hanover, her average speed was 50 miles per hour, and from Hanover to Norwich, her average speed was 30 miles per hour. The drive took 9 hours. How many miles did Shannon drive from Melrose to Hanover?

22

$$\frac{4}{3} + \frac{5}{4}x = 28y + \frac{5}{8}x$$
$$my = \frac{1}{2}(5x - 8)$$

In the given system of equations, m is a constant. If the system has no solution, what is the value of m ?

EXERCISE 2: Answers for this exercise start on page 363.

1

$$4x - \frac{1}{3}y = -8$$
$$y = 4x + 16$$

Which ordered pair (x, y) is the solution to the given system of equations?

A) $(-2, 8)$

B) $(-1, 12)$

C) $(1, 20)$

D) $(3, 28)$

2

$$9x = -7y + 22$$
$$5x = 7y - 76$$

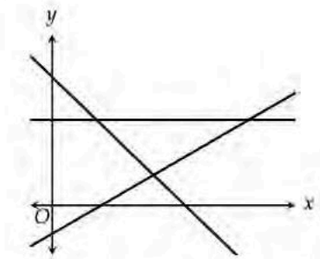
The solution to the given system of equations is (x, y). What is the value of $7x$?

3

$$y = 0.5x + 14$$
$$x - y = -18$$

The solution to the given system of equations is (x, y). What is the value of y ?

4



A system of three linear equations is represented in the graph. How many solutions does the system have?

A) Zero

B) Exactly one

C) Exactly two

D) Exactly three

5

Which of the following systems of linear equations has no solution?

A) $y = 8$
$x = 8$

B) $y = 8$
$y = 8x + 8$

C) $y = 8x$
$y = 8x + 8$

D) $y = -8x + 8$
$y = 8x + 8$

6

$$y = 2x - 14$$
$$y = (x+5)(x-7)$$

The solution to the given system of equations is (x, y). What is a possible value of x ?

A) -7
B) -3
C) 4
D) 9

7

$$3x - 6y = 15$$
$$-2x + 4y = -10$$

At how many points do the graphs of the given equations intersect in the xy-plane?

A) Zero
B) One
C) Two
D) More than two

8

$$y = -10x$$
$$x^2 + y = 96$$

A solution to the given system of equations is (x, y), where $y < 0$. What is the value of y ?

9

If $\frac{x+y}{x} = 9$ and $\frac{ay}{3x} = 32$, where a is a constant, what is the value of a ?

10

$$9(2x+1) + 27(y-4) = 810$$
$$(2x+1) + (y-4) = -393$$

The solution to the given system of equations is (x, y). What is the value of $8(y-4)$?

11

A restaurant has two types of tables, rectangular ones that can each seat 4 people and circular tables that can each seat 8 people. If 144 people are enough to fill all 30 tables at the restaurant, how many rectangular tables does the restaurant have?

A) 12
B) 16
C) 20
D) 24

12

$$mx - 6y = 10$$
$$2x - ny = 5$$

In the given system of equations, m and n are constants. The system has infinitely many solutions. What is the value of $\frac{m}{n}$?

A) $\frac{1}{12}$
B) $\frac{1}{3}$
C) $\frac{4}{3}$
D) 3

13

One of the equations in a system of two linear equations is $x - 2y = 5$. The system has exactly one solution. Which equation could be the other equation in the system?

A) $-x + 2y = 0$
B) $x - 2y = 0$
C) $-x + 2y = -5$
D) $x + 2y = -5$

14

$$x^2 - y^2 = \frac{1}{12}$$
$$x - 2y = 0$$

If (x, y) satisfies the given system of equations, what is a possible value of y ?

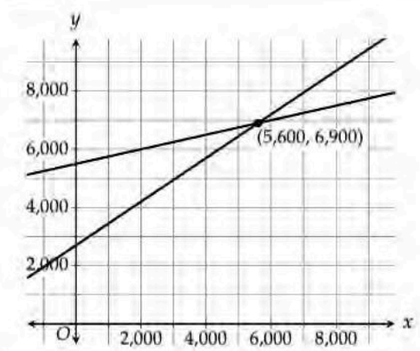
A) $-\frac{1}{6}$
B) $-\frac{1}{\sqrt{12}}$
C) $\frac{1}{4}$
D) $\frac{1}{2}$

15

$$\frac{3}{10}y + \frac{b}{3}x = \frac{3}{4} - \frac{1}{5}y$$
$$\frac{3}{8}x + \frac{3}{5} = -\frac{1}{4}y + \frac{8}{5}$$

In the given system of equations, b is a constant. If the system has no solution, what is the value of b ?

16



What system of linear equations is represented by the lines shown?

A) $-\frac{x}{0.01} + \frac{y}{0.04} = 90$
$-40x + 30y = 220$

B) $-\frac{x}{0.01} + \frac{y}{0.04} = 220$
$-40x + 30y = 90$

C) $-0.01x + 0.04y = 90$
$-\frac{x}{40} + \frac{y}{30} = 220$

D) $-0.01x + 0.04y = 220$
$-\frac{x}{40} + \frac{y}{30} = 90$

17

The function f is defined by $f(x) = ax^3 + bx$, where a and b are constants. In the xy-plane, the graph of $y = f(x)$ passes through the points $(-1, -10)$ and $(2, 38)$. What is the value of ab ?

A) 9
B) 21
C) 24
D) 39

18

$$x^2 + 2x = y - 1$$
$$-4x = y + 8$$

The solution to the given system of equations is (x, y). What is the value of x ?

19

When Kenji uses his push mower, he mows 57.75 square yards of grass per minute, and when he uses his self-propelled mower, he mows 80.25 square yards of grass per minute. If Kenji spent a total of 2 hours using his two mowers in a week and mowed a total of 9,000 square yards of grass, how many minutes did he spend using his push mower?

20

$$y = \sqrt{x} + 3$$
$$\sqrt{4x} - y = 3$$

The solution to the given system of equations is (x, y). What is the value of y ?

21

$$-15x + 3y = 12$$
$$y + 132 = x^2 - 4x$$

The graphs of the given equations in the xy-plane intersect at the point (x, y). What is a possible value of x ?

22

$$5x - 3y = 8$$
$$-20x + 12y = -32$$

Which of the following ordered pairs (x, y) is a solution to the given system of equations for all real values of n ?

A) $\left(n, \frac{5}{3}n - 8\right)$
B) $\left(-\frac{3}{5}n + \frac{8}{5}, n\right)$
C) $\left(n - 1, \frac{5}{3}n - 1\right)$
D) $\left(\frac{3}{5}n + 1, n - 1\right)$

14 Inequalities

Just as we had equations and systems of equations, we can have inequalities and systems of inequalities.

The only difference is that you must reverse the sign every time you either multiply or divide both sides by a negative number.

For example, let's say we're trying to isolate x in the following inequality:

$$2x + 3 < 9$$

Do we have to reverse the sign at any point? Well, the first step is to subtract both sides by 3 to get $2x < 6$. Now we can divide both sides by 2 to get $x < 3$. At no point during this process did we multiply or divide by a negative number. Therefore, the sign stays the same.

Let's look at another example:

$$3x + 5 < 4x + 4$$

The first step is to combine like terms. We can subtract both sides by $4x$ to get the x's on the left-hand side. Then we can subtract both sides by 5 to get the constants on the right-hand side:

$$3x - 4x < 4 - 5$$
$$-x < -1$$

Notice that the sign hasn't changed yet. Now, to get rid of the negative in front of the x, we need to **multiply** both sides by -1. Doing so requires that we reverse the sign:

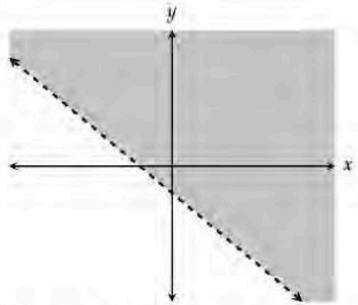
$$x > 1$$

Negative numbers in inequalities are the source of many mistakes, so let's be clear. Just working with negative numbers does NOT mean you need to change the sign. Some students see that they're dividing a negative number and impulsively reverse the sign. Don't do that. Only reverse the sign when you multiply or divide both sides **by** a negative number.

Graphs of inequalities

Most of the inequality questions on the SAT will not test your ability to solve inequalities. Rather, they'll test your understanding of how they're represented in the xy-plane.

Any inequality expressed in terms of x and y can be graphed in the xy-plane. For example, the inequality $y > -x - 1$ has the following graph:

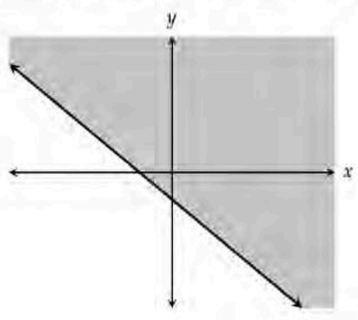


As you can see, the graph consists of the dashed line $y = -x - 1$ and a shaded region. Since the inequality we graphed is $y > -x - 1$, it makes sense that the shaded region consists of all the points above the line $y = -x - 1$. The shaded region also represents all the solutions to the inequality. In other words, any point in the shaded region (e.g. $(0, 0)$) satisfies $y > -x - 1$.

If the sign in the inequality were reversed, the shaded region would lie below the line $y = -x - 1$, and any point within that shaded region would be a solution to $y < -x - 1$.

If you ever have a hard time determining whether a region is above or below a line, use the y-axis as a reference or draw a vertical line through the graph yourself if the y-axis isn't shown. For any line that cuts across the y-axis (or your vertical line), the top part of the y-axis is always in the "above" region, and the bottom part of the y-axis is always in the "below" region.

Lastly, we mentioned that the line is dashed. The dashed line indicates that the points on the line $y = -x - 1$ itself do not satisfy the inequality. If the inequality were $y \geq -x - 1$ instead of $y > -x - 1$, then the line in the graph would be solid, indicating that the points on the line do satisfy the inequality:



EXAMPLE 1:

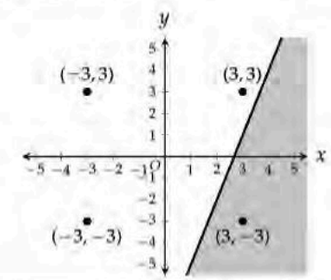
$$y \leq 3x - 8$$

Which point (x, y) is a solution to the given inequality in the xy-plane?

A) $(-3, -3)$ B) $(-3, 3)$ C) $(3, -3)$ D) $(3, 3)$

Test each point and see which one satisfies the inequality. Plugging in $(-3, -3)$ yields $-3 \leq 3(-3) - 8 \Rightarrow -3 \leq -17$, a false statement. Plugging in $(-3, 3)$ yields $3 \leq 3(-3) - 8 \Rightarrow 3 \leq -17$, another false statement. Plugging in $(3, -3)$ yields $-3 \leq 3(3) - 8 \Rightarrow -3 \leq 1$, the only true statement. Answer $\boxed{(C)}$.

Graphing the inequality from this example, we get



Notice that the point $(3, -3)$ is the only answer choice that's inside the shaded solution region.

Compound inequalities are two or more inequalities that are combined to place more than one condition on the same single variable. For example, the compound inequality $2 \leq 5x \leq 3$ consists of $2 \leq 5x$ and $5x \leq 3$. When solving a compound inequality with three parts, any mathematical operations we perform must be applied to all three parts of the inequality.

EXAMPLE 2: If $-7 \leq -2x + 3 \leq 15$, which of the following must be true?

A) $5 \leq x \leq 6$ B) $-6 \leq x \leq -5$ C) $-6 \leq x \leq 5$ D) $-5 \leq x \leq 6$

To isolate x, we first subtract 3 from all three parts:

$$-7 - 3 \leq -2x + 3 - 3 \leq 15 - 3$$
$$-10 \leq -2x \leq 12$$

Now we divide everything by -2, reversing the signs:

$$\frac{-10}{-2} \geq \frac{-2x}{-2} \geq \frac{12}{-2}$$
$$5 \geq x \geq -6$$

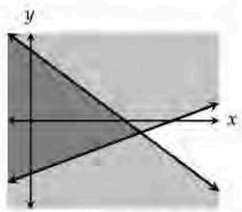
Since $5 \geq x \geq -6$ is equivalent to $-6 \leq x \leq 5$, the answer is $\boxed{(C)}$.

Systems of inequalities

To graph a system of inequalities, graph each inequality in the system one by one, shading in their individual solution regions. The solution region of the system is where all the individual shaded regions overlap. Let's look at the following example:

$$y \leq -x + 4$$
$$y \geq \frac{1}{2}x - 3$$

We need to shade the region below $y = -x + 4$ and the region above $y = \frac{1}{2}x - 3$.



The darkened overlapping region on the left contains all the points that are solutions to the system. These are the points that satisfy both inequalities in the system, not just one.

EXAMPLE 3:

$$x \leq 1$$
$$y \geq -2x + 1$$

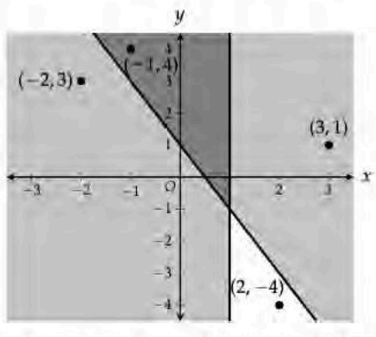
Which point (x, y) is a solution to the given system of inequalities in the xy-plane?

A) $(-2, 3)$ B) $(-1, 4)$ C) $(2, -4)$ D) $(3, 1)$

Let's plug in each point to see whether it satisfies both inequalities in the system.

The point $(-2, 3)$ yields $-2 \leq 1$, a true statement, from the first inequality and $3 \geq -2(-2) + 1 \Rightarrow 3 \geq 5$, a false statement, from the second inequality.

The point $(-1, 4)$ yields $-1 \leq 1$, a true statement, from the first inequality and $4 \geq -2(-1) + 1 \Rightarrow 4 \geq 3$, a true statement, from the second inequality. Since both statements are true, $(-1, 4)$ satisfies the system, and we can stop here. Answer $\boxed{(B)}$. To check our answer, let's graph the system of inequalities from this example:



As you can see, the only point from the answer choices that lies inside the overlapping solution region is $(-1, 4)$.

EXAMPLE 4:

$$y \leq 17$$
$$3x - 2y \leq -10$$

In the xy-plane, the point $(5, y)$ is a solution to the given system of inequalities. Which of the following could be the value of y ?

A) 6 B) 11 C) 14 D) 18

According to the first inequality, y can be no greater than 17. Since the second inequality contains an x term, we can plug the given point into the inequality to obtain more information about y:

$$3(5) - 2y \leq -10$$
$$-2y \leq -25$$
$$y \geq 12.5$$

So y must be at least 12.5 but no greater than 17. The only answer choice within this range is 14. Answer $\boxed{(C)}$.

Inequality word problems

EXAMPLE 5: James used no more than 400 grams of sugar to make chocolate chip cookies and shortbread cookies. He used 5 grams of sugar for each chocolate chip cookie and 7 grams of sugar for each shortbread cookie. Which of the following inequalities represents the possible numbers of chocolate chip cookies c and shortbread cookies s James could have made?

A) $\frac{5}{c} + \frac{7}{s} < 400$ B) $\frac{5}{c} + \frac{7}{s} \leq 400$ C) $5c + 7s < 400$ D) $5c + 7s \leq 400$

James used a total of $5c$ grams of sugar for the chocolate chip cookies and a total of $7s$ grams of sugar for the shortbread cookies. Altogether, he used $5c + 7s$ grams of sugar, an amount that must be less than or equal to 400 grams. Therefore, $5c + 7s \leq 400$. Answer $\boxed{(D)}$.

EXAMPLE 6: Hudson can work no more than 30 hours each week as a security guard. He wants to earn at least \$1,800 each week working x regular hours and y overtime hours. If he earns \$50 per regular hour and \$80 per overtime hour, which of the following inequalities represents this situation?

A) $x + y \leq 30$, $50x + 80y \geq 1{,}800$
B) $x + y \leq 30$, $80x + 50y \geq 1{,}800$
C) $x + y \geq 30$, $50x + 80y \geq 1{,}800$
D) $x + y \geq 30$, $80x + 50y \geq 1{,}800$

Since Hudson can't work more than 30 hours each week, $x + y \leq 30$. The amount he earns from his regular hours is $50x$ and the amount he earns from his overtime hours is $80y$. Since Hudson wants his total earnings to be at least \$1,800, $50x + 80y \geq 1{,}800$. Answer $\boxed{(A)}$.

Minimum & Maximum word problems

EXAMPLE 7: Corinne is a graphic designer who earns \$275 for every logo she designs. What is the minimum number of logos she would have to design to earn at least \$4,000 ?

Intuitive Approach: To earn at least \$4,000, Corinne would have to design at least $\frac{4{,}000}{275} \approx 14.5$ logos. That's 14 logos and half a logo. We have to assume, however, that a fraction of a logo cannot be designed and sold, so we must round up to $\boxed{15}$ logos.

Inequality Approach: If we let x be the number of logos Corinne designs, we can construct and solve the following inequality:

$$275x \geq 4{,}000$$
$$x \geq \frac{4{,}000}{275}$$
$$x \geq 14.5$$

The smallest whole number that satisfies this inequality is $\boxed{15}$.

When a whole number answer is implied, finding the minimum requires that we round up from a decimal answer. Finding the maximum requires that we round down from a decimal answer.

EXAMPLE 8: If one tray of flatbread can be made from 8 cups of flour and 6 cups of yogurt, what is the maximum number of whole trays of flatbread that can be made from 150 cups of flour and 100 cups of yogurt?

In these types of questions, one of the resources (e.g. either flour or yogurt) will be used up before the other, and that limiting resource will determine the amount of output that can be produced. To determine which resource is the limiting one, we have to consider each resource separately.

If we consider only the flour requirement, $\frac{150}{8} = 18.75$ trays of flatbread can be made. If we consider only the yogurt requirement, $\frac{100}{6} \approx 16.7$ trays of flatbread can be made. Since the amount that can be produced from the yogurt is less than the amount that can be produced from the flour, the yogurt is the **limiting factor**. There isn't enough of it to use up the flour, so we're limited to 16.7 trays of flatbread. As a result, the maximum whole number of trays that can be made is $\boxed{16}$. Remember that we round down from a decimal answer when finding the maximum.

EXAMPLE 9: An art teacher needs to buy a total of 36 paintbrushes for a painting class. Each paintbrush must be either an acrylic brush, which costs \$5, or a watercolor brush, which costs \$3. If no more than \$150 can be spent on the paintbrushes, what is the minimum number of watercolor brushes the art teacher can buy?

Let a be the number of acrylic brushes and w be the number of watercolor brushes. Now we can set up a system consisting of an equation and an inequality:

$$a + w = 36$$
$$5a + 3w \leq 150$$

The goal now is to get a single inequality that we can solve for w. The first step is to isolate a in the first equation to get $a = 36 - w$. Now we can substitute $36 - w$ for a in the inequality:

$$5(36 - w) + 3w \leq 150$$
$$180 - 5w + 3w \leq 150$$
$$180 - 2w \leq 150$$
$$-2w \leq -30$$
$$w \geq 15$$

Based on this result, the minimum number of watercolor brushes that can be bought is $\boxed{15}$.

EXAMPLE 10: Ava is decorating two-tier and three-tier wedding cakes. It takes her 20 minutes to decorate each two-tier wedding cake and 35 minutes to decorate each three-tier wedding cake. If Ava needs to decorate at least 14 wedding cakes today, and she can spend no more than 6 hours doing so, what is the maximum number of three-tier wedding cakes she could decorate today?

Let x be the number of two-tier cakes and y be the number of three-tier cakes Ava decorates. Now we can set up the following system:

$$20x + 35y \leq 360$$
$$x + y \geq 14$$

Note that we converted 6 hours to 360 minutes to set up the first inequality, which we can reduce to get the following system:

$$4x + 7y \leq 72$$
$$x + y \geq 14$$

Since we have a system of two inequalities, we can't just do what we did in Example 8, which gave us a system with one equation and one inequality. Instead, we have to use one of the following approaches:

Approach 1: The maximum possible value of y (i.e. maximum number of three-tier wedding cakes) can be found by minimizing x (number of two-tier wedding cakes). Since the second inequality in the system gives $x \geq 14 - y$, the minimum possible value of x is equivalent to $14 - y$. Substituting this result into the first inequality, we get

$$4(14 - y) + 7y \leq 72$$
$$56 - 4y + 7y \leq 72$$
$$3y \leq 16$$
$$y \leq 5.33\ldots$$

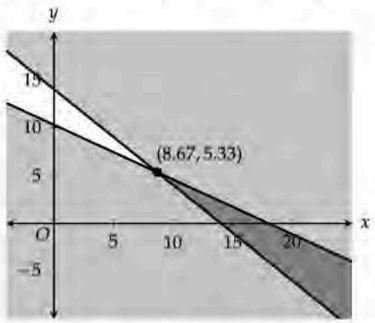
The maximum possible value of y is then $\boxed{5}$ (since it's implied that Ava decorates cakes in whole numbers).

Approach 2: Inequalities can be added together if their signs point in the same direction. Note that inequalities should never be subtracted from one another; think only in terms of addition. In this case, we can multiply the second inequality by -4 to align both the signs and the coefficients of x:

$$4x + 7y \leq 72$$
$$-4x - 4y \leq -56$$

Remember that multiplying by a negative number reverses the sign. Now we can add these two inequalities together to get $3y \leq 16$, which simplifies to $y \leq 5.33$. Based on this result, the maximum possible (whole number) value of y is $\boxed{5}$.

Approach 3 (best): We can graph the system of inequalities on Desmos and look for the point with the maximum value of y in the overlapping solution region (the highest point in the region).



The highest point in the overlapping solution region turns out to be $(8.67, 5.33)$, which indicates that $y \leq 5.33$. The maximum possible (whole number) value of y is then $\boxed{5}$.

EXERCISE 1: Answers for this exercise start on page 366.

1

$$y > 7x - 19$$

Which of the following points in the xy-plane satisfies the given inequality?

A) $(-1, -32)$
B) $(0, -20)$
C) $(2, -3)$
D) $(4, -14)$

2

Katherine has 28 classroom calculators that each require a set of 4 batteries. If her school supplies her with batteries in packs of 6, what is the least number of packs needed to provide every classroom calculator with a complete set of batteries?

3

$$y < -4x - 1$$
$$y > x + 2$$

Which point (x, y) is a solution to the given system of inequalities in the xy-plane?

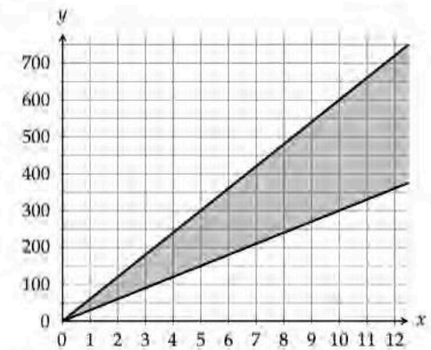
A) $(1, 4)$
B) $(-1, 1)$
C) $(-2, 2)$
D) $(-3, -3)$

4

A hot dog stand sells more than \$800 worth of hot dogs each day. The price of a hot dog is \$3 during lunch and \$5 during dinner. Let l represent the number of hot dogs sold during lunch yesterday, and let d represent the number of hot dogs sold during dinner yesterday. Which of the following inequalities represents all possible values of l and d ?

A) $3l + 5d > 800$
B) $3l + 5d \geq 800$
C) $5l + 3d > 800$
D) $5l + 3d \geq 800$

5



A train travels at a speed of at least 30 miles per hour and no more than 60 miles per hour. The graph shows the possible distances y, in miles, the train has traveled after x hours. Which ordered pair (x, y) represents a possible distance y, in miles, the train has traveled after x hours?

A) $(4, 375)$
B) $(6, 450)$
C) $(9, 250)$
D) $(11, 625)$

6

In the xy-plane, the solution region of an inequality contains the points $(2, 2)$, $(-1, -1)$, and $(-2, -4)$. Which of the following could be the inequality?

A) $y \leq \frac{3}{2}x - 1$

B) $y \geq \frac{3}{2}x - 1$

C) $y \leq -3x + 2$

D) $y \geq -3x + 2$

7

During a week-long fishing trip, Ashleigh caught nine less than three times the number of fish Naomi caught. If they caught at least 45 fish combined, what is the minimum number of fish that Naomi could have caught?

8

To restock supplies, a nail salon purchases toolkits that each include 80 nail files and 150 nail buffers. If the nail salon needs to restock at least 1,800 nail files and at least 4,000 nail buffers, what is the minimum number of toolkits the salon must purchase?

9

$$x + y < 25$$
$$2x - 3y < 77$$

In the xy-plane, the point $(18, y)$ is a solution to the given system of inequalities. Which of the following could be the value of y ?

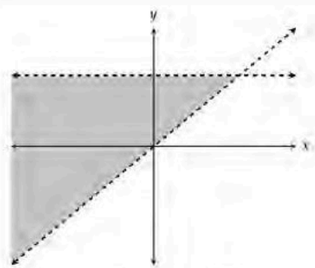
A) -14

B) -6

C) 7

D) 11

10



The shaded region shown could represent the solutions to which of the following systems of inequalities?

A) $y > 3$
$y > x$

B) $y < 3$
$y < x$

C) $y < 3$
$y > x$

D) $y > 3$
$y < x$

11

To get to work each day, Harry must travel 8 miles by bus and 16 miles by train. The bus travels at an average speed of x miles per hour and the train travels at an average speed of y miles per hour. If Harry's daily commute never takes more than 1 hour, which of the following inequalities represents the possible average speeds of the bus and train during the commute?

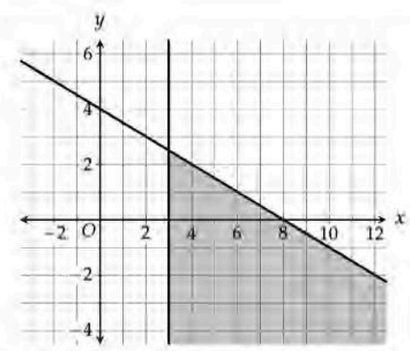
A) $\frac{8}{x} + \frac{16}{y} \leq 1$

B) $\frac{16}{x} + \frac{8}{y} \leq 1$

C) $\frac{x}{8} + \frac{y}{16} \leq 1$

D) $8x + 16y \leq 1$

12



The shaded region of the graph represents the solution to which of the following system of inequalities?

A) $y \geq 3$
$y \leq -2x + 4$

B) $y \geq 3$
$y \geq -2x + 4$

C) $x \geq 3$
$2y \leq -x + 8$

D) $x \geq 3$
$2y \geq -x + 8$

13

Each month, Eric spends 40% of his monthly salary and contributes the rest to his investment account. His goal is to contribute at least $50,000 to his investment account each year. Which of the following is the closest to the minimum possible monthly salary Eric needs to earn to achieve his goal each year?

A) $5,788

B) $6,945

C) $8,334

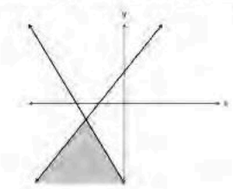
D) $10,417

14

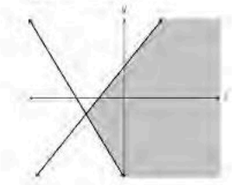
$$y \geq \frac{3}{2}x + 2$$
$$y \leq -2x - 5$$

Which of the following graphs in the xy-plane could represent the given system of inequalities?

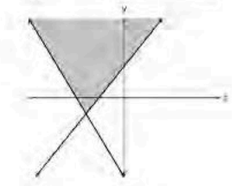
A)



B)



C)



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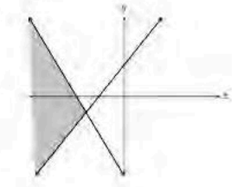


15

Tina works no more than 30 hours at a nail salon each week. She can do a manicure in 20 minutes and a pedicure in 30 minutes. Each manicure earns her \$25 and each pedicure earns her \$40, and she must earn at least \$900 to cover her expenses. If m represents the number of manicures Tina does in a week and p represents the number of pedicures she does in a week, which system of inequalities describes all possible values of m and p for which Tina covers her expenses?

A) $3m + 2p \leq 30$
$25m + 40p \geq 900$

B) $2m + 3p \leq 30$
$25m + 40p \geq 900$

C) $\frac{m}{3} + \frac{p}{2} \leq 30$
$25m + 40p \geq 900$

D) $\frac{m}{3} + \frac{p}{2} \geq 900$
$25m + 40p \leq 30$

16

During hibernation, adult male black bears lose 6 to 12 pounds in weight each week. An adult male black bear weighed 580 pounds after 5 weeks of hibernation. Which of the following inequalities gives all possible values for the black bear's weight w, in pounds, after 15 weeks of hibernation?

A) $400 \leq w \leq 490$
B) $460 \leq w \leq 520$
C) $460 \leq w \leq 580$
D) $610 \leq w \leq 640$

17

Manasa uses an automated system that pays her taxes on the day they are due. The system will pay 25% of her taxes from her checking account and the remainder of her taxes from her savings account. If there is \$3,300 in Manasa's checking account and \$4,500 in her savings account on the day her taxes are due, which inequality gives the amount in taxes t, in dollars, for which her checking account will have a higher balance than her savings account after the taxes are paid?

A) $t < 2{,}400$
B) $t > 2{,}400$
C) $t < 2{,}700$
D) $t > 2{,}700$

18

A manufacturer ships only 3-pound and 5-pound rugs to its stores. Each shipment must contain at least 85 rugs, and the total weight of the rugs can be no more than 400 pounds. What is the greatest possible number of 5-pound rugs each shipment can contain?

19

Katy needs to cover an area of at least 800 square feet using ceramic and porcelain tiles. Each ceramic tile costs \$4.70 and covers an area of 4 square feet. Each porcelain tile costs \$15.20 and covers an area of 7 square feet. If Katy has a budget of \$1,600 to spend on these tiles, what is the minimum number of ceramic tiles she can purchase?

EXERCISE 2: Answers for this exercise start on page 368.

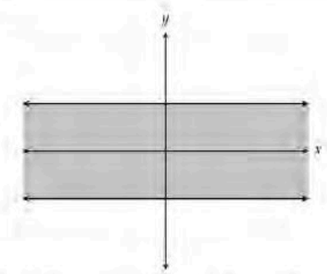
1

$$y > 3x - 20$$
$$y < \frac{1}{2}x + 5$$

Which point (x, y) is a solution to the given system of inequalities in the xy-plane?

A) $(-10, 0)$
B) $(10, 0)$
C) $(10, 10)$
D) $(-10, -10)$

2



The shaded region of the graph could represent the solutions to which of the following systems of inequalities?

A) $y \geq 3$
$y \leq -3$

B) $y \leq 3$
$y \geq -3$

C) $x \geq 3$
$x \leq -3$

D) $x \leq 3$
$x \geq -3$

3

A gift shop held a weekend sale with the goal of selling at least \$8,000 worth of greeting cards and gift boxes. Each greeting card was sold for \$5, and each gift box was sold for \$7. If 400 gift boxes were sold during the sale, what is the minimum number of greeting cards the shop could have sold to meet its goal?

A) 1,040
B) 1,160
C) 1,280
D) 1,400

4

The point $(5, 0)$ in the xy-plane is a solution to which of the following systems of inequalities?

A) $y > x$
$y > -x$

B) $y > x$
$y < -x$

C) $y < x$
$y > -x$

D) $y < x$
$y < -x$

5

For an inequality graphed in the xy-plane, the points $(1, 4)$ and $(3, 0)$ are contained in the solution region, and the point $(1, -2)$ is not contained in the solution region. Which of the following could be the inequality?

A) $y > -2x + 1$
B) $y \geq x - 4$
C) $x < 2$
D) $y \leq -3x + 7$

6

An ice cream distributor contracts out to two different companies to manufacture cartons of ice cream. Company A can produce 80 cartons per hour and Company B can produce 140 cartons per hour. To fulfill an order of over 1,100 cartons in 10 total hours of contract time, the distributor contracts out x hours to Company A and the remaining hours to Company B. Which of the following inequalities gives all possible values of x in the context of this problem?

A) $\frac{80}{x} + \frac{140}{10 - x} > 1{,}100$
B) $140x + 80(10 - x) > 1{,}100$
C) $80x + 140(10 - x) > 1{,}100$
D) $80x + 140(x - 10) > 1{,}100$

7

Melanie burns 120 to 135 calories for each mile that she jogs. Which inequality represents the total number of calories, y, Melanie could burn from jogging 14 miles?

A) $120 \le y + 14 \le 135$
B) $120 \le y - 14 \le 135$
C) $120 \le 14y \le 135$
D) $120 \le \frac{y}{14} \le 135$

8

A pharmacy produces a certain medication in a daytime variety and a nighttime variety. A bottle of the daytime variety contains 2 ounces of the active ingredient and 6 ounces of flavored syrup. A bottle of the nighttime variety contains 3 ounces of the active ingredient and 5 ounces of flavored syrup. The pharmacy currently has no more than 385 ounces of the active ingredient and no more than 850 ounces of flavored syrup available. If at least 65 bottles of the daytime variety must be filled, what is the maximum number of bottles of the nighttime variety that can be filled?

A) 78
B) 85
C) 92
D) 106

9

Denise is given a \$20 bill to buy 45-cent stamps and 60-cent stamps at the post office. She is told that she must return with at least \$5 in change. Which of the following inequalities represents this situation, where x is the number of 45-cent stamps Denise can buy, and y is the number of 60-cent stamps Denise can buy? (Assume there is no sales tax.)

A) $0.45x + 0.60y \ge 5$
B) $0.45x - 0.60y \le 5$
C) $0.45x + 0.60y \ge 15$
D) $0.45x + 0.60y \le 15$

10

The number k is 7 more than half the maximum value of y. Which inequality shows the possible values of y?

A) $y \le 2k - 7$
B) $y \ge 2k - 7$
C) $y \le 2k - 14$
D) $y \ge 2k - 14$

11

Sandra is given a budget of \$250 to buy pencils and enough pencil cases to hold all of them. Each pencil case costs \$4.50 and holds up to 12 pencils. Each pencil costs \$1.50. Which of the following systems of inequalities represents this situation in terms of the number of pencils Sandra can buy, x, and the number of pencil cases she can buy, y?

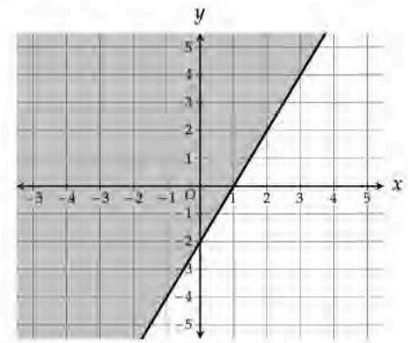
A) $1.50x + 4.50y \le 250$
$y \ge 12x$

B) $1.50x + 4.50y \ge 250$
$y \le 12x$

C) $1.50x + 4.50y \le 250$
$y \ge \frac{x}{12}$

D) $1.50x + 4.50y \ge 250$
$y \ge \frac{x}{12}$

12



The solutions to which of the following inequalities are represented by the shaded region of the graph?

A) $x - 2y \le 4$
B) $x - 2y \ge 4$
C) $2x - y \le 2$
D) $2x - y \ge 2$

13

A tool manufacturer wants to produce a wrench set that consists of a toolbox and 7 wrenches. The toolbox costs t dollars to produce and each wrench costs w dollars to produce. The total cost of producing the wrench set is at least 4 times but no more than 6 times the cost of producing the toolbox. Which inequality represents all possible values of w?

A) $\frac{3}{7}t \le w \le \frac{5}{7}t$
B) $3t \le w \le 5t$
C) $\frac{4}{7}t \le w \le \frac{6}{7}t$
D) $4t \le w \le 6t$

14

Paul wants to purchase and install 30 new home windows. The installation fee on each window is \$50 plus 10% of the price of each window. The most Paul can spend on the windows, including the installation fees, is \$25,000. Which of the following is closest to the maximum possible price per window, before the installation fee, Paul could pay?

A) \$712.12
B) \$756.06
C) \$783.33
D) \$828.90

15

Jerry estimates that there are m marbles in a jar. Harry, who knows the actual number of marbles in the jar, notes that the actual number, n, is within 10 marbles (inclusive) of Jerry's estimate. Which of the following inequalities represents the relationship between Jerry's estimate and the actual number of marbles in the jar?

A) $n + 10 \le m \le n - 10$
B) $m - 10 \le n \le m + 10$
C) $n \le m \le 10n$
D) $\frac{m}{10} \le n \le 10m$

16

$$y > 15x + a$$
$$y < 5x + b$$

In the given system of inequalities, a and b are constants. If the point $(1, 20)$ is a solution to the system in the xy-plane, which of the following could be the value of $b - a$?

A) 6
B) 8
C) 10
D) 12

17

John makes cedar and oak birdhouses to sell in his shop. To fill his shop's showroom, he must make at least 75 birdhouses. Each cedar birdhouse takes 165 minutes to make and each oak birdhouse takes 75 minutes to make. If John can spend at most 160 hours making birdhouses for his showroom, what is the maximum number of cedar birdhouses he could make?

18

A school play had 240 audience members consisting of adults and children. There was at least one adult for every 2 children. Which of the following inequalities gives the possible values of a, the number of adults in the audience?

A) $a > 80$
B) $a \geq 80$
C) $a > 120$
D) $a \geq 120$

19

A barista wants to use Arabica and Robusta beans to make a cup of coffee containing less than 280 milligrams of caffeine. Each ounce of Arabica beans contains 40 milligrams of caffeine and each ounce of Robusta beans contains 60 milligrams of caffeine. If the barista is required to use twice as many Arabica beans as Robusta beans, which inequality gives all possible values for the amount a, in ounces, of Arabica beans the barista could use?

A) $0 < a < 1\frac{3}{4}$
B) $0 < a < 2$
C) $0 < a < 3\frac{1}{2}$
D) $0 < a < 4$

20

During each of her workouts, Mina burns at least 700 calories from walking and running, and she runs for at least 20 minutes longer than she walks. Mina burns 5.8 calories per minute while walking and 10.2 calories per minute while running. What is the minimum amount of time, in minutes, she could spend running during each workout?

21

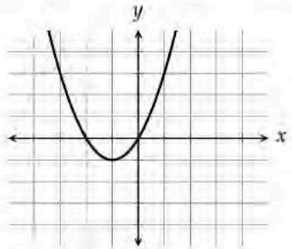
Lianne wants to make a seasoning that consists of 75% sea salt and 25% black pepper. If sea salt costs \$2 per pound and black pepper costs \$8 per pound, and Lianne can spend no more than \$210 on these ingredients, what is the maximum number of pounds of seasoning that she will be able to make?

A) 42
B) 50
C) 56
D) 60

15 Function Transformations

Function transformations are changes we can make to the equation of a function to "transform" its graph in specific ways. The transformations you might encounter on the SAT are vertical shift (a.k.a. vertical translation) and horizontal shift (a.k.a. horizontal translation).

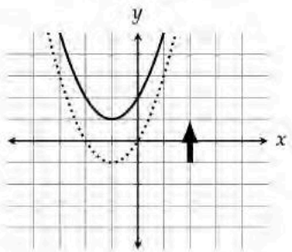
Let's start with the example function $f(x) = x^2 + 2x$, whose graph looks like



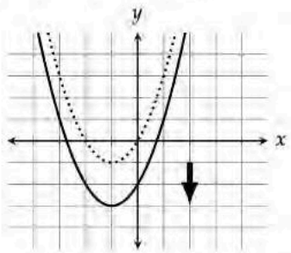
To shift the graph of $f(x)$ up, add a constant to $f(x)$. For example, $y = f(x) + 2 = x^2 + 2x + 2$ produces a graph that is 2 units above the graph of $y = f(x)$.

To shift the graph of $f(x)$ down, subtract a constant from $f(x)$. For example, $y = f(x) - 2 = x^2 + 2x - 2$ produces a graph that is 2 units below the graph of $y = f(x)$.

$y = f(x) + 2$



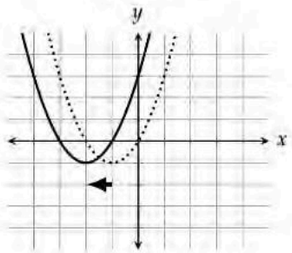
$y = f(x) - 2$



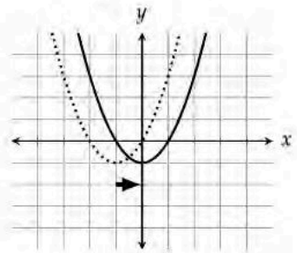
To shift the graph of $f(x)$ to the left by a units, substitute $x + a$ in for x. For example, $y = f(x+1) = (x+1)^2 + 2(x+1)$ produces a graph that is 1 unit to the left of the graph of $y = f(x)$.

To shift the graph of $f(x)$ to the right by a units, substitute $x - a$ in for x. For example, $y = f(x-1) = (x-1)^2 + 2(x-1)$ produces a graph that is 1 unit to the right of the graph of $y = f(x)$.

$y = f(x+1)$



$y = f(x-1)$



Here's a trick I like to use for interpreting horizontal shifts:

For horizontal shifts, the value of x that makes the substituted expression equal to 0 will tell you what the horizontal shift is. For instance, when we have $f(x-1)$, what value of x makes $x - 1$ equal to 0? $x = 1$. So $f(x-1)$ is 1 unit to the right of $f(x)$. What about $f(x+4)$? Well, $x = -4$ makes $x + 4$ equal to 0, so in relation to $f(x)$, $f(x+4)$ is shifted 4 units to the left (negative value = shift to the left). And what about $f(3x-2)$? A value of $x = \frac{2}{3}$ makes $3x - 2$ equal to 0, so $f(3x-2)$ is a shift of $\frac{2}{3}$ of a unit to the right of $f(x)$.

Note that horizontal and vertical shifts are commonly referred to as **translations**.

EXAMPLE 1: In the xy-plane, the graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ down 3 units and to the left by 5 units. If the function f is defined by $f(x) = (x-3)^2 + 1$, which equation defines the function g?

A) $g(x) = (x-8)^2 - 2$ B) $g(x) = (x+2)^2 - 2$ C) $g(x) = (x-8)^2 + 4$ D) $g(x) = (x+2)^2 + 4$

The order in which we apply horizontal and vertical translations does not matter. Applying a shift of 3 units down, we get $f(x) - 3$. Shifting this result to the left by 5 units gives $f(x+5) - 3$ (we substitute $x + 5$ for x). Therefore,

$$g(x) = f(x+5) - 3$$
$$= ((x+5)-3)^2 + 1 - 3 = (x+2)^2 - 2$$

Answer $\boxed{(B)}$. If you graph this answer, you will see that it is 3 units down from and 5 units to the left of the graph of $f(x)$.

To summarize, for a function $f(x)$,

- $f(x) + a$ results in an upward shift of a units; $f(x) - a$ results in a downward shift of a units
- $f(x+a)$ results in a horizontal shift of a units to the left; $f(x-a)$ results in a horizontal shift of a units to the right

EXERCISE: Answers for this exercise start on page 371.

1

The function f is an exponential function. The y-intercept of the graph of $y = f(x)$ in the xy-plane is $(0, 5)$. What is the y-intercept of the graph of $y = f(x) - 8$?

A) $(-8, 5)$
B) $(8, 5)$
C) $(0, -3)$
D) $(0, 13)$

2

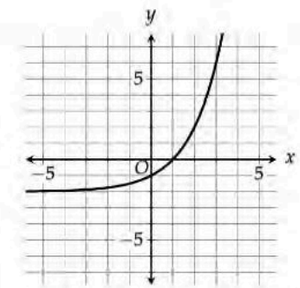
In the xy-plane, which of the following translations of the graph of $y = 2x^2 - 2$ results in the graph of $y = 2x^2 + 4$?

A) A translation 2 units downward
B) A translation 6 units upward
C) A translation 2 units to the left
D) A translation 6 units to the right

3

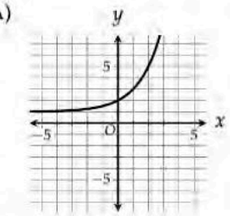
The function f is defined by $f(x) = -(x-5)^2$. In the xy-plane, the graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ up 8 units. What is the y-coordinate of the y-intercept of the graph of $y = g(x)$?

4

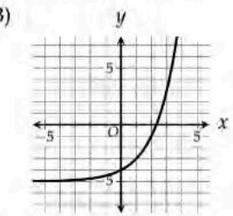


The graph of $y = g(x)$ is shown. What is the graph of $y = g(x) + 3$?

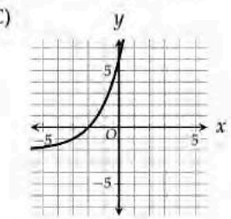
A)



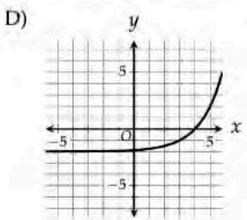
B)



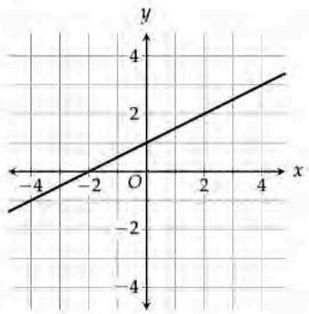
C)



D)



5



The graph of $y = h(x) + 9$ is shown. Which equation defines function h ?

A) $h(x) = 2x - 8$

B) $h(x) = \frac{1}{2}x - 8$

C) $h(x) = 2x + 10$

D) $h(x) = \frac{1}{2}x + 10$

6

The function f is defined by $f(x) = \frac{x+1}{2}$. In the xy-plane, the graph of $y = g(x)$ is the result of shifting the graph of $y = f(x)$ up 7 units. Which equation defines function g ?

A) $g(x) = \frac{x-6}{2}$

B) $g(x) = \frac{x+8}{2}$

C) $g(x) = \frac{x+15}{2}$

D) $g(x) = \frac{7x+7}{2}$

7

The graph of $y = -x + 3$ is translated 20 units to the right in the xy-plane. Which equation represents the resulting graph?

A) $y = -x - 17$

B) $y = -x + 17$

C) $y = -x + 23$

D) $y = -20x + 3$

8

The function f is defined by $f(x) = ax(x - 7)$, where a is a constant. The graph of $y = f(x)$ in the xy-plane is translated 11 units to the left to produce the graph of $y = g(x)$. Which equation defines function g ?

A) $g(x) = a(x + 11)(x - 7)$

B) $g(x) = a(x + 11)(x + 4)$

C) $g(x) = a(x - 11)(x - 7)$

D) $g(x) = a(x - 11)(x - 18)$

9

The function f is defined by $f(x) = \frac{2}{3}x - 6$. The graph of $y = f(x)$ in the xy-plane is translated 3 units to the left. What is the y-intercept of the resulting graph?

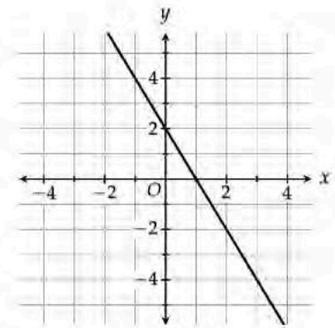
A) $(0, -3)$

B) $(0, -4)$

C) $(0, -8)$

D) $(0, -9)$

10



The graph of $y = f(x)$ is shown in the xy-plane. The function g is defined by $g(x) = f(x) + 4$. What is the x-intercept of the graph of $y = g(x)$?

A) $(-1, 0)$

B) $(2, 0)$

C) $(3, 0)$

D) $(5, 0)$

11

In the xy-plane, the graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ to the right by $1\frac{1}{2}$ units. Which of the following equations defines $g(x)$?

A) $g(x) = f(3x - 2)$

B) $g(x) = f(3x + 2)$

C) $g(x) = f(2x - 3)$

D) $g(x) = f(2x + 3)$

12

In the xy-plane, the graph of $y = f(x)$ is a line that passes through the points $(-1, 82)$ and $(3, 46)$. The graph of $y = g(x)$ is the result of translating the graph of $y = f(x)$ down 14 units. For what value of x does $g(x) = 0$?

13

$$g(x) = 3^x - 5$$

The exponential function g is defined by the given equation. The graph of $y = g(x)$ in the xy-plane is translated 2 units down and 3 units to the left to produce the graph of $y = h(x)$. What is the value of $h(0)$?

14

$$f(x) = -2 + \frac{k+6}{x}$$

In the given function, k is a constant. The graph of $y = f(x)$ in the xy-plane is translated 4 units up and 5 units to the right to produce the graph of $y = g(x)$. Which equation defines function g ?

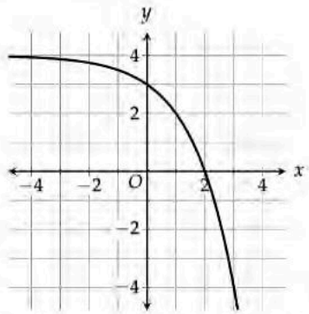
A) $g(x) = -2 + \frac{k+10}{x-5}$

B) $g(x) = -2 + \frac{k+10}{x+5}$

C) $g(x) = 2 + \frac{k+6}{x-5}$

D) $g(x) = 2 + \frac{k+6}{x+5}$

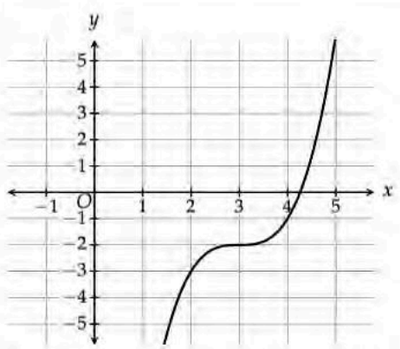
15



The graph of $y = f(x) - 2$ is shown. Which equation defines the function f ?

A) $f(x) = -2^x + 4$
B) $f(x) = -2^x + 5$
C) $f(x) = -2^x + 6$
D) $f(x) = -3^x + 6$

16



The graph of $y = g(x)$ is shown, where the function g is defined by $g(x) = f(x + a) + b$ and a and b are constants. If the function f is defined by $f(x) = x^3$, what is the value of $a + b$?

A) -5
B) -1
C) 1
D) 5

17

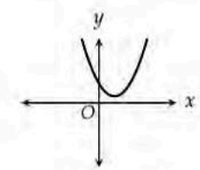
The function f is defined by $f(x) = (x - 3)^2$, and the function g is defined by $g(x) = x^2 + 4x + 4$. In the xy-plane, the graph of $y = g(x)$ is the result of shifting the graph of $y = f(x)$ to the left by k units. What is the value of k ?

18

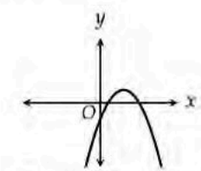
Line p is defined by $12x + 5y = 56$. Line q is the result of translating line p down 6 units in the xy-plane. What is the x-coordinate of the x-intercept of line q ?

16 Quadratic Functions

In a previous chapter, we learned how to solve quadratic equations. In this chapter, we'll discuss quadratic functions and their graphs, called **parabolas**. A parabola is a "U" shape that opens either upward or downward.



upward-opening



downward-opening

We can determine whether a parabola opens upward or downward without graphing it just by looking at the standard form of its equation.

The equation of a quadratic function in **standard form** is

$$y = ax^2 + bx + c$$

where a, b, and c are constants. The coefficient of x^2, a, determines whether the graph (parabola) opens upward or downward.

- When a is positive, the parabola opens upward.
- When a is negative, the parabola opens downward.

The constant c is the y-intercept of the graph. In other words, the graph passes through $(0, c)$.

Let's walk through an example quadratic function to demonstrate.

EXAMPLE QUADRATIC 1:

$$y = x^2 - 4x - 21$$

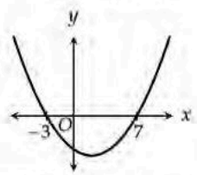
This example equation is written in standard form, where $a = 1$, $b = -4$, and $c = -21$. Because a is positive, the graph of this equation opens upward. And because $c = -21$, the graph intercepts the y-axis at $(0, -21)$.

Let's keep going with this example.

We already know the y-intercept, so let's find the x-intercepts by using what we learned in the quadratic equations chapter. The x-intercepts are the points at which a graph crosses the x-axis. To find them, set y equal to 0 and solve for x. In this case, setting y equal to 0 gives us a plain old quadratic equation we already know how to solve, by factoring or by applying the quadratic formula. Here's the factoring approach:

$$y = x^2 - 4x - 21$$
$$0 = (x+3)(x-7)$$
$$x = -3, 7$$

Therefore, the graph of Example Quadratic 1 has x-intercepts at $(-3, 0)$ and $(7, 0)$, as shown below.



The x-coordinates of the x-intercepts are sometimes referred to as the **roots** of a quadratic, the values of x for which $y = 0$. In this example, the roots are -3 and 7.

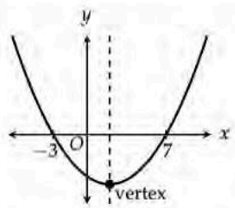
The x-intercepts of a quadratic function are obvious when its equation is written in **intercept form**:

$$y = a(x-p)(x-q)$$

where p and q are the x-coordinates of the x-intercepts (the roots), and a is a constant equal to the value of a from the standard form equation of the quadratic ($y = ax^2 + bx + c$).

Note that all the formulas we learned in the quadratic equations chapter apply to this chapter as well. The sum of the roots, $-3 + 7 = 4$, is equivalent to $-\frac{b}{a}$ ($-\frac{-4}{1} = 4$), and the product of the roots, $-3 \times 7 = -21$, is equivalent to $\frac{c}{a}$ ($\frac{-21}{1} = -21$). These formulas allow us to obtain the sum or product of the roots without having to solve for the roots themselves.

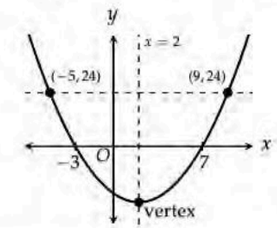
Now let's take the graph above and draw a line down the middle of the parabola such that each side is a mirror of the other.



This line is called the **axis of symmetry** and always passes through a parabola's **vertex**. You can think of the vertex as the "midpoint" of a parabola. For parabolas that open upward, the vertex is the point where they reach their minimum (their lowest point). For parabolas that open downward, the vertex is the point where they reach their maximum (their highest point).

Now how do we find the coordinates of the vertex? Let's start by finding the equation of the axis of symmetry.

If we draw a horizontal line across our example parabola, the line will intersect the parabola at two points.



For the line we've drawn above, the intersection points are $(-5, 24)$ and $(9, 24)$.

Notice that the axis of symmetry runs through the middle of these two intersection points. This would be true for any horizontal line we could have drawn.

Because the axis of symmetry lies in the middle, we can find its location by computing the midpoint of the x-coordinates of the two intersection points. The midpoint of any two values is just their average, so the midpoint of -5 and 9 is $\frac{-5+9}{2} = 2$. The equation of the axis of symmetry must then be $x = 2$.

Since the vertex lies on the axis of symmetry, it must have an x-coordinate of 2. To find the y-coordinate of the vertex, we plug $x = 2$ into the equation of the parabola (Example Quadratic 1):

$$y = (2)^2 - 4(2) - 21 = -25$$

Therefore, the vertex is at $(2, -25)$, and this parabola reaches a minimum value of -25.

Now you might be thinking that we were only able to find the vertex this way because we plucked those intersection points out of nowhere. Indeed, finding the vertex would be very tedious if, for every parabola, we had to draw a horizontal line and solve for the intersection points ourselves.

Thankfully, there's an easier way. Look back at the graph above, and notice that the x-axis itself is a horizontal line—one for which we already knew the intersection points: $(-3, 0)$ and $(7, 0)$, the x-intercepts! If we take the average of their x-coordinates, -3 and 7, we get $x = \frac{-3+7}{2} = 2$, the same value we got earlier.

The conclusion here is that for any given parabola, the x-coordinate of its vertex is simply the midpoint, or average, of its roots.

- Given any two distinct points on a parabola that have the same y-value, say (x_1, y) and (x_2, y), the x-coordinate of the vertex is the average of the two x-coordinates, $\frac{x_1 + x_2}{2}$.
- By extension, the x-coordinate of the vertex is equal to the average of the roots.
- For a parabola whose equation is in standard form, $y = ax^2 + bx + c$, the sum of the roots is $-\frac{b}{a}$. Since the average of the roots is the sum of the roots divided by 2, the x-coordinate of the vertex is equal to $-\frac{b}{2a}$.

Let's confirm that the formula $-\frac{b}{2a}$ works for locating the vertex. The parabola above, which has a vertex at $(2, -25)$, is defined by $y = x^2 - 4x - 21$ (Example Quadratic 1). For this equation, the value of $-\frac{b}{2a}$ is $-\frac{-4}{2(1)} = 2$, which is indeed the x-coordinate of the vertex.

Now that we know how to find the vertex of a parabola, let's talk about vertex form.

The equation of a quadratic function in **vertex form** is

$$y = a(x-h)^2 + k$$

where (h, k) is the vertex, and a is a constant equal to the value of a from the standard form equation of the quadratic ($y = ax^2 + bx + c$).

Let's express Example Quadratic 1 in vertex form. Its equation in standard form is $y = x^2 - 4x - 21$ and its vertex, which we already found, is $(2, -25)$. The equation of this quadratic in vertex form is then

$$y = (x-2)^2 - 25$$

Because the coefficient of x^2 in standard form is 1, the value of a in vertex form is also 1. Note that if we expand this result, we get back the original standard form:

$$y = (x-2)^2 - 25 = (x^2 - 4x + 4) - 25 = x^2 - 4x - 21$$

Vertex form is called vertex form for a reason—we're able to find the vertex just by looking at the numbers in the equation. Here are some quick examples:

- The graph of $y = -4(x-3)^2 - 16$ has a vertex at $(3, -16)$.
- The graph of $y = 3(x-4)^2 + 10$ has a vertex at $(4, 10)$.
- The graph of $y = -2(x+5)^2 + 8$ has a vertex at $(-5, 8)$.

One of the most common mistakes students make is to look at an equation like $y = -2(x+5)^2 + 8$ and think the vertex is at $(5, 8)$ instead of $(-5, 8)$. The routine I use to avoid this mistake is to ask, *What value of x would make the expression inside the parentheses equal to zero? Well, $x = -5$ would make $x + 5$ equal to 0. Therefore, the vertex is at $x = -5$.*

The Discriminant

Let's say that we have a quadratic function expressed in standard form, $y = ax^2 + bx + c$, and we have to find the x-intercepts of its graph. To do so, we set y equal to 0 and solve the resulting quadratic equation:

$$0 = ax^2 + bx + c$$

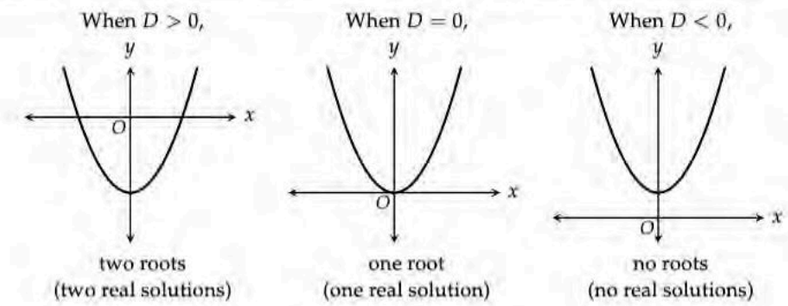
A generalized solution is given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice the expression tucked underneath the square root. This expression, $b^2 - 4ac$, is called the discriminant.

- When $b^2 - 4ac > 0$, the plus or minus in the quadratic formula yields two distinct real solutions to the equation. Thus, the graph of the quadratic function intersects the x-axis twice (two x-intercepts).
- When $b^2 - 4ac = 0$, a zero ends up underneath the square root, and the square root of zero is zero. Since there is no difference between plus zero and minus zero, the quadratic formula yields only one distinct real solution to the equation. Thus, the graph of the quadratic function intersects the x-axis only once (one x-intercept).
- When $b^2 - 4ac < 0$, a negative number ends up underneath the square root. Since the square root of a negative number cannot be found, the quadratic formula does not yield any real solutions to the equation. Thus, the graph of the quadratic function does not intersect the x-axis at all (zero x-intercepts).

The value of the discriminant does not matter. What matters is whether it's positive, negative, or zero. Here's another summary of the three possibilities, with D representing the value of the discriminant:



Let's calculate the discriminant for Example Quadratic 1, $y = x^2 - 4x - 21$,

$$\text{Discriminant} = b^2 - 4ac = (-4)^2 - 4(1)(-21) = 100$$

Because the discriminant is positive, the graph must have two x-intercepts. And indeed it does: $(-3, 0)$ and $(7, 0)$, the two x-intercepts we found earlier.

Now that we've illustrated all the properties of quadratics you need to know, let's reinforce the concepts by going through one more example quadratic.

EXAMPLE QUADRATIC 2:

$$y = -2x^2 - 12x - 25$$

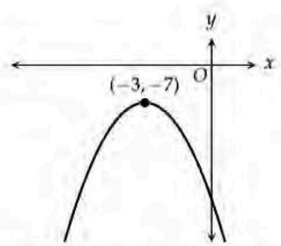
- Again, we're given an equation in standard form. Because the coefficient of x^2, -2, is negative, the graph of this quadratic opens downward.
- The constant term, -25, indicates that the graph has a y-intercept at $(0, -25)$.
- The discriminant is equal to $b^2 - 4ac = (-12)^2 - 4(-2)(-25) = -56$. Because the discriminant is negative, the graph does not intersect the x-axis (there are no x-intercepts or roots).
- The sum of the roots is $-\frac{b}{a} = -\frac{-12}{-2} = -6$, and the product of the roots is $\frac{c}{a} = \frac{-25}{-2} = 12.5$. But wait! We just determined that there were no roots, so how can there be a sum and a product of roots that don't exist? The answer has to do with imaginary numbers. This quadratic doesn't have any *real* roots, but it does have *imaginary* roots, and their sum and product are equal to the values we just computed. On the SAT, you won't be tested on imaginary numbers—it's good to be aware of them, but ultimately they're not something you need to worry about.

- The vertex of the graph is located at $x = -\frac{b}{2a} = -\frac{-12}{2(-2)} = -3$. Note that this formula works for any quadratic function, even ones that don't have any real roots. Now we can plug $x = -3$ into the equation to find the vertex's y-coordinate: $y = -2(-3)^2 - 12(-3) - 25 = -7$. Therefore, the vertex is at $(-3, -7)$.
- Now that we've found the vertex, we can express Example Quadratic 2 in vertex form:

$$y = -2(x+3)^2 - 7$$

  The -2 comes from the coefficient of x^2 given by the standard form equation. Note that a process called *completing the square* can also be used to convert from standard form to vertex form. Knowing how to complete the square is not necessary for the quadratics questions you'll encounter on the SAT, but it is necessary for other question types. We'll be covering how to complete the square in a later chapter.
- Because the graph opens downward, it reaches a maximum at its vertex. Thus, the graph reaches a maximum value of -7 at $x = -3$.

Finally, a graph of Example Quadratic 2:



Wow! We just covered pretty much everything you need to know about quadratics. We're not quite done yet though as there are several unique question types you need to be exposed to.

EXAMPLE 1:

$$f(x) = -2(x-1)(x-13)$$

The function f is defined by the given equation. For what value of x does $f(x)$ reach its maximum?

Solution 1: Since the graph of f is a parabola that opens downward, $f(x)$ reaches its maximum at its vertex. Now how do we locate the vertex? Well, f has two roots, $x = 1$ and $x = 13$. The x-coordinate of the vertex is equal to the average of these two roots: $\frac{1+13}{2} = 7$. Therefore, $f(x)$ reaches its maximum at $x = \boxed{7}$.

Solution 2: We can expand $f(x)$ like so,

$$f(x) = -2(x^2 - 14x + 13) = -2x^2 + 28x - 26$$

and use the following formula to find the x-coordinate of the vertex:

$$x = -\frac{b}{2a} = -\frac{28}{2(-2)} = \boxed{7}$$

EXAMPLE 2:

$$-3x^2 + 10x - 7 = 0$$

How many distinct real solutions does the given equation have?

A) Exactly one B) Exactly two C) Zero D) Infinitely many

The discriminant, $b^2 - 4ac$, tells us how many solutions a quadratic equation has. In this case,

$$b^2 - 4ac = (10)^2 - 4(-3)(-7) = 100 - 84 = 16$$

The discriminant is positive, so the given equation must have two distinct real solutions. Answer $\boxed{(B)}$.

EXAMPLE 3:

$$2x^2 + 12x + c = 0$$

In the given equation, c is an integer constant. If the equation has no real solution, what is the least possible value of c?

For a quadratic equation to have no real solution, the discriminant must be less than zero:

$$\begin{aligned} b^2 - 4ac &< 0 \\ (12)^2 - 4(2)(c) &< 0 \\ 144 - 8c &< 0 \\ -8c &< -144 \\ c &> 18 \end{aligned}$$

Given that c is an integer constant, the least possible value of c is $\boxed{19}$.

EXAMPLE 4:

$$y = x + 5$$
$$y = x^2 + 11x + c$$

In the given system of equations, c is a constant. The system has exactly one distinct real solution. What is the value of c?

Remember how we solve a system of equations? We use substitution or elimination to get a single equation. Then we solve the single equation for x. The solutions we get from the single equation are the solutions to the system.

So for this question, let's start off by substituting the first equation into the second,

$$x + 5 = x^2 + 11x + c$$

and moving all the terms to one side:

$$0 = x^2 + 10x + c - 5$$

We now have a single quadratic equation. Remember—the solutions to this equation are the solutions to the system. **Therefore, for the system to have only one solution, this equation must have only one solution.**

And what determines how many solutions a quadratic equation has? That's right—the discriminant. For a quadratic equation to have only one real solution, its discriminant must be equal to zero:

$$\begin{aligned} b^2 - 4ac &= 0 \\ (10)^2 - 4(1)(c-5) &= 0 \\ 100 - 4(c-5) &= 0 \\ 100 - 4c + 20 &= 0 \\ -4c &= -120 \\ c &= \boxed{30} \end{aligned}$$

EXAMPLE 5: In the xy-plane, a parabola with equation $y = x^2 + 5x + 6$ intersects a line at exactly one point. If the line has equation $y = ax - 10$, where a is a positive constant, what is the value of a?

To find the intersection point(s) of two graphs, we solve the system consisting of their equations. The solutions to the system are the intersection points.

Therefore, for two graphs to have only one intersection point, the system consisting of their equations must have only one solution.

This question is essentially just a variation of Example 4. In this case, the system is

$$y = ax - 10$$
$$y = x^2 + 5x + 6$$

Substituting the first equation into the second, we get

$$ax - 10 = x^2 + 5x + 6$$

Moving all the terms to the right-hand side gives the following quadratic equation:

$$0 = x^2 + (5-a)x + 16$$

Notice how we grouped the terms. Now, for the system to have only one solution, this equation must have only one solution. And for that to be true, the discriminant must be equal to zero (like in Example 4):

$$\begin{aligned} b^2 - 4ac &= 0 \\ (5-a)^2 - 4(1)(16) &= 0 \\ (5-a)^2 &= 64 \\ 5 - a &= \pm 8 \\ -a &= -5 \pm 8 \\ -a &= -13 \text{ or } 3 \\ a &= 13 \text{ or } -3 \end{aligned}$$

Since the question specifies that a is a positive constant, $a = \boxed{13}$.

Review:

1. The equation of a quadratic function in standard form is
$$y = ax^2 + bx + c$$
where a, b, and c are constants. The graph of a quadratic function in the xy-plane is a parabola.
2. If $a > 0$, the parabola opens upward and has a minimum.
If $a < 0$, the parabola opens downward and has a maximum.
3. The parabola has a y-intercept at $(0, c)$.
4. The equation of a quadratic function in intercept form is
$$y = a(x-p)(x-q)$$
where p and q are the x-coordinates of the x-intercepts (the roots), and a is a constant equal to the value of a from the standard form equation of the quadratic.
5. When the equation of a quadratic is not in intercept form, the x-intercepts can be found in the following ways:
   - Factoring (i.e. converting the equation into intercept form)
   - The quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   - Graphing on Desmos
6. The sum of the roots $= -\dfrac{b}{a}$
7. The product of the roots $= \dfrac{c}{a}$
8. The discriminant, $D = b^2 - 4ac$, can be used to determine how many real solutions a quadratic equation $ax^2 + bx + c = 0$ has.
   - When $D > 0$, the equation has two real solutions (the parabola crosses the x-axis twice).
   - When $D = 0$, the equation has one real solution (the parabola crosses the x-axis once).
   - When $D < 0$, the equation has no real solutions (the parabola does not cross the x-axis at all).
9. The x-coordinate of the vertex can be found by
   - Averaging the x-coordinates of two points on the parabola that have the same y-coordinate (e.g. the x-intercepts/roots).
   - Using the formula $x = -\dfrac{b}{2a}$.

   The x-coordinate can then be plugged into the equation to find the y-coordinate of the vertex.
10. The equation of a quadratic function in vertex form is
$$y = a(x-h)^2 + k$$
where (h, k) is the vertex, and a is a constant equal to the value of a from the standard form equation of the quadratic.
11. The minimum or maximum of a quadratic function is always reached at its vertex.

EXERCISE 1: Answers for this exercise start on page 373.

1

In the xy-plane, the graph of the equation $y = x^2 - 3x - 10$ has an x-intercept at $(5, 0)$. What is the other x-intercept of the graph?

A) $(-5, 0)$
B) $(-2, 0)$
C) $(-1, 0)$
D) $(2, 0)$

2

$$E(x) = 50x^2 - 800x + 10{,}000$$

The given equation models the relationship between the total monthly expenses E, in dollars, of a restaurant and the number of tables x in its dining area, where $0 \le x \le 25$. Which of the following is the best interpretation of 10,000 in this context?

A) The maximum number of tables that can be placed in the dining area
B) The average monthly expenses, in dollars, for each table in the dining room
C) The total monthly expenses, in dollars, when there are zero tables in the dining area
D) The total monthly expenses, in dollars, when the number of tables in the dining area is maximized

3

$$v(t) = -17(t - 26)^2 + 14{,}750$$

The given function v models the value, in dollars, of an artifact t years after its excavation. According to this model, how many years after its excavation was the value of the artifact the greatest?

4

$$g(x) = -5x^2 - 16x + 4$$

The given equation defines the function g. For what value of x does $g(x)$ reach its maximum?

5

$$-12x^2 - 6x + d = 0$$

In the given equation, d is a constant. If the equation has exactly one solution, what is the value of d?

A) $-\frac{1}{4}$
B) $-\frac{2}{3}$
C) $-\frac{3}{4}$
D) $-\frac{3}{2}$

6

The function $f(x) = -\frac{1}{2}(x - 5)^2 + 8$ gives the profit, in hundreds of thousands of dollars, a company makes when x thousand units of its product are produced and sold, where $1 \le x \le 9$. Which of the following is the best interpretation of the vertex of the graph of $y = f(x)$ in the xy-plane?

A) The company can make a maximum profit of 500,000 dollars.
B) The company can make a maximum profit of 800,000 dollars.
C) The company makes a profit of 0 dollars when 5,000 units of its product are produced and sold.
D) The company makes a profit of 0 dollars when 8,000 units of its product are produced and sold.

7

In the xy-plane, a parabola has vertex $\left(\frac{7}{3}, -4\right)$ and intersects the x-axis at $(10, 0)$ and $(a, 0)$, where a is a constant. What is the value of a?

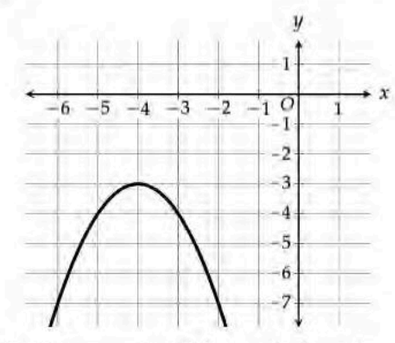
A) -10
B) $-\frac{23}{3}$
C) $-\frac{19}{3}$
D) $-\frac{16}{3}$

8

$$ax^2 + 30x - 9 = 0$$

In the given equation, a is a constant. The equation has no real solutions if $a < k$. What is the greatest possible value of k?

9



What is an equation of the graph shown?

A) $y = -(x - 4)^2 - 3$
B) $y = -(x + 4)^2 - 3$
C) $y = (x - 4)^2 - 3$
D) $y = (x + 4)^2 - 3$

10

$$9x^2 + 2cx + 36 = 0$$

In the given equation, c is a constant. For which of the following values of c will the equation have no real solution?

A) 26
B) 18
C) -11
D) -19

11

In the xy-plane, the graph of the equation $y = a(x - b)^2 - c$, where a, b, and c are positive constants, could have a vertex at which of the following points?

A) $(7, 10)$
B) $(7, -10)$
C) $(-7, 10)$
D) $(-7, -10)$

12

$$f(x) = -4x^2 + 22x$$

The function f gives the download speed, in megabytes per second, of a file x minutes after a download of the file was started. The graph of $y = f(x)$ in the xy-plane has x-intercepts at $(0, 0)$ and $(b, 0)$, where b is a constant. Which of the following is the best interpretation of b in this context?

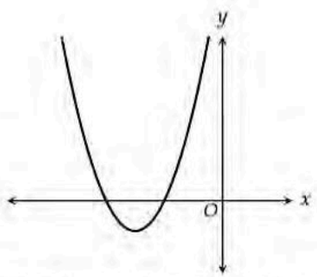
A) At the start of the download, the download speed was b megabytes per second.
B) During the download, the download speed reached a maximum of b megabytes per second.
C) The download speed was at its highest b minutes after the download was started.
D) The download was completed b minutes after it was started.

13

$$-x^2 + bx - 961 = 0$$

In the given equation, b is a positive integer. The equation has exactly two distinct real solutions. What is the least possible value of b ?

14



The graph of $y = f(x)$ is shown in the xy-plane. If a, b, and c are positive constants, which of the following equations could define the function f ?

A) $f(x) = -a(x+b)(x+c)$
B) $f(x) = -a(x-b)(x-c)$
C) $f(x) = a(x+b)(x+c)$
D) $f(x) = a(x-b)(x-c)$

15

In the xy-plane, a line with equation $y = x + 3$ intersects a parabola at exactly one point. If the parabola has equation $y = kx^2 - 5x + 7$, where k is a constant, what is the value of k ?

16

$$y = 3$$
$$y = x^2 + 6x + r$$

In the given system of equations, r is a constant. For which of the following values of r does the given system have no real solution?

A) -50
B) 1
C) 12
D) 38

17

$$3y = c$$
$$y = -4x^2 + 10x$$

In the given system of equations, c is a constant. The graphs of the equations in the given system intersect at exactly one point in the xy-plane. What is the value of c ?

18

$$f(x) = 2x^2 + bx + c$$

The function f is defined by the given equation, where b and c are constants. The graph of $y = f(x)$ in the xy-plane is a parabola that has a vertex at (h, k), where h and k are both negative constants. Which of the following must be true?

I. $b > 0$
II. $c \geq 0$

A) I only
B) II only
C) I and II
D) Neither I nor II

EXERCISE 2: Answers for this exercise start on page 375.

1

$$2x^2 - 13x + 21 = 0$$

How many distinct real solutions does the given equation have?

A) Zero
B) Exactly one
C) Exactly two
D) Infinitely many

2

$$P(x) = -0.03(x-20)^2 + 12$$

The given function P models the total profit, in thousands of dollars, that a charity group expects to earn from selling boxes of cookies for x dollars each, where $0 \leq x \leq 40$. Based on the model, which of the following is the best interpretation of $P(10) = 9$ in this context?

A) If each box of cookies is sold for \$9, the total profit will likely be \$10,000.
B) If each box of cookies is sold for \$9, the total profit will likely be \$21,000.
C) If each box of cookies is sold for \$10, the total profit will likely be \$9,000.
D) If each box of cookies is sold for \$10, the total profit will likely be \$22,000.

3

$$f(x) = x^2 - 24x + 180$$

For a manufacturer of x-ray machines, the cost per unit, in thousands of dollars, can be modeled by the given function f, where x is the weekly number of units produced, and $0 \leq x \leq 30$. How many units should the manufacturer produce each week to minimize the cost per unit?

4

$$y = 190x - x(12 + x)$$

The given equation relates the variables x and y. What is the maximum value of y ?

5

$$f(x) = 2(x+15)(x-4)$$

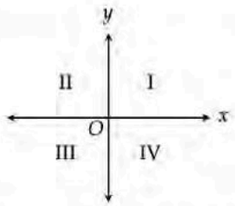
The function f is defined by the given equation. For what value of x does $f(x)$ reach its minimum?

A) -11
B) $-\frac{11}{2}$
C) $\frac{11}{2}$
D) 11

6

A quadratic function models the height, in feet, of a projectile above the ground in terms of the time, in seconds, after it is launched. The model indicates that the projectile reaches a height of 34 feet above the ground 6 seconds after being launched and then reaches its maximum height of 49 feet above the ground 10 seconds after being launched. Based on the model, how many seconds after reaching its maximum height does the projectile return to a height of 34 feet above the ground?

7



In the xy-plane, the quadrants are labeled as shown. The graph of the equation $y = 3(x + a)^2 + b$, where a and b are positive constants, has a vertex in which quadrant?

A) Quadrant I
B) Quadrant II
C) Quadrant III
D) Quadrant IV

8

$$-6(3x^2 - 8x) = q$$

In the given equation, q is an integer constant. If the equation has exactly two distinct real solutions, what is the greatest possible value of q ?

9

$$f(x) = m\left[(x - m)^2 - 1\right]$$

In the given quadratic function, m is a positive constant. The graph of $y = f(x)$ in the xy-plane is a parabola. Which of the following statements about the parabola is true?

A) Its minimum is reached at $(m, -1)$.
B) Its minimum is reached at $(m, -m)$.
C) Its maximum is reached at $(m, -1)$.
D) Its maximum is reached at $(m, -m)$.

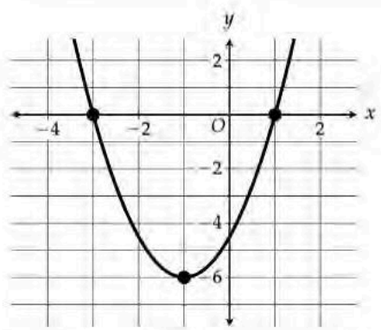
10

$$g(x) = -3x^2 + 18x$$

The function g gives the download speed, in megabytes per second, of a file x minutes after a download of the file was started. The graph of $y = g(x)$ in the xy-plane has x-intercepts at $(0, 0)$ and $(c, 0)$, where c is a constant. Which of the following is the best interpretation of the value of $\frac{c}{2}$ in this context?

A) At the start of the download, the download speed was $\frac{c}{2}$ megabytes per second.
B) During the download, the download speed reached a maximum of $\frac{c}{2}$ megabytes per second.
C) The download speed was at its highest $\frac{c}{2}$ minutes after the download was started.
D) The download was completed $\frac{c}{2}$ minutes after it was started.

11



The graph of $y = ax^2 + 3x + c$ is shown, where a and c are constants. What is the value of $a + c$?

12

The function f is defined by $f(x) = -6x^2 + 48x - 95$, and the function g is defined by $g(x) = f(x - 3)$. For what value of x does $g(x)$ reach its maximum?

A) -2
B) 1
C) 4
D) 7

13

In the xy-plane, the line $y = 5.5$ intersects the graph of the equation $y = -x^2 + 16x - a$ at exactly one point. What is the value of a ?

A) $-\frac{139}{2}$
B) $\frac{128}{11}$
C) $\frac{27}{2}$
D) $\frac{117}{2}$

14

$$y = 4x$$
$$y = 3x^2 + bx + 3$$

In the given system of equations, b is a positive constant. The system has exactly one distinct real solution. What is the value of b ?

15

$$f(x) = a(x - 3)(x - k)$$

In the given quadratic function, a and k are constants. The graph of $y = f(x)$ in the xy-plane is a parabola with vertex $(5, -32)$. What is the value of a ?

A) 2
B) 5
C) 6
D) 8

16

$$y = 2x^2 + 15x + 57$$
$$y = -9x + p$$

In the given system of equations, p is a constant. The system has exactly one distinct real solution, (x, y). What is the value of y ?

17

The function g is defined by $g(x) = ax^2 + bx + c$, where a, b, and c are constants. The graph of $y = g(x)$ in the xy-plane passes through the points $(-4, 0)$ and $(8, 0)$. If the sum of b and c is equal to 14, what is the value of a ?

18

A golf ball was struck from a certain height above the ground. The ball reached its maximum height of 72 feet above the ground 3 seconds after it was struck, and the ball hit the ground 7 seconds after it was struck. A quadratic function can be used to model the height, in feet, of the ball above the ground in terms of the time, in seconds, after the ball was struck. Based on the model, what was the height, in feet, of the ball 5 seconds after it was struck?

19

x	y
6	28
10	−20
14	28

The table gives three values of x and their corresponding values of y, where $y = f(x) - 2$ and f is a quadratic function. What is the value of $f(20)$?

20

$$g(x) = ax^2 + 8x + c$$

The quadratic function g is defined by the given equation, where a and c are constants. The graph of $y = g(x)$ in the xy-plane has x-intercepts at $(0, 0)$ and $(n, 0)$, where n is a nonzero constant. If the maximum value of $g(x)$ is 25, which of the following must be true?

I. $a < 0$
II. $c = 0$
III. $n > 0$

A) I only
B) II only
C) I and II only
D) I, II, and III

21

In the xy-plane, a parabola with equation $y = ax^2 + bx + c$, where a, b, and c are constants, has a vertex at the point $(-5, -100)$. If $c > 0$, which of the following is a possible value of b ?

A) −60
B) −40
C) 40
D) 60

22

$$f(x) = 3x^2 + bx + c$$

In the given quadratic function, b and c are constants. The graph of $y = f(x)$ in the xy-plane does not intersect the x-axis. If $f(-7) = f(9)$, which of the following could be the value of $b + c$?

A) −1
B) −3
C) −5
D) −7

18

A golf ball was struck from a certain height above the ground. The ball reached its maximum height of 72 feet above the ground 3 seconds after it was struck, and the ball hit the ground 7 seconds after it was struck. A quadratic function can be used to model the height, in feet, of the ball above the ground in terms of the time, in seconds, after the ball was struck. Based on the model, what was the height, in feet, of the ball 5 seconds after it was struck?

19

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14	28

The table gives three values of x and their corresponding values of y, where $y = f(x) - 2$ and f is a quadratic function. What is the value of $f(20)$?

20

$$g(x) = ax^2 + 8x + c$$

The quadratic function g is defined by the given equation, where a and c are constants. The graph of $y = g(x)$ in the xy-plane has x-intercepts at $(0, 0)$ and $(n, 0)$, where n is a nonzero constant. If the maximum value of $g(x)$ is 25, which of the following must be true?

I. $a < 0$
II. $c = 0$
III. $n > 0$

A) I only
B) II only
C) I and II only
D) I, II, and III

21

In the xy-plane, a parabola with equation $y = ax^2 + bx + c$, where a, b, and c are constants, has a vertex at the point $(-5, -100)$. If $c > 0$, which of the following is a possible value of b?

A) −60
B) −40
C) 40
D) 60

22

$$f(x) = 3x^2 + bx + c$$

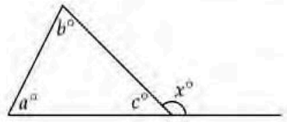
In the given quadratic function, b and c are constants. The graph of $y = f(x)$ in the xy-plane does not intersect the x-axis. If $f(-7) = f(9)$, which of the following could be the value of $b + c$?

A) −1
B) −3
C) −5
D) −7

17 Angles

Exterior Angle Theorem

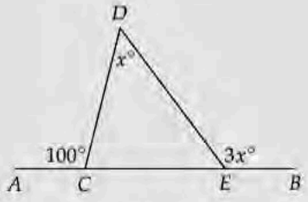
An exterior angle is formed when any side of a triangle is extended. In the triangle below, $x°$ designates an exterior angle.



An exterior angle is always equal to the sum of the two angles in the triangle furthest from it:

$$x = a + b$$

EXAMPLE 1:

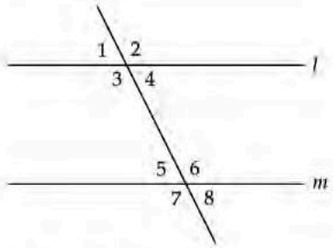


In the figure, points C and E lie on $\overline{AB}$. What is the value of x?

Since $\angle ACD$ and $\angle DCE$ lie on the same side of a straight line, they are supplementary (sum to 180°). The measure of $\angle DCE$ is then $180 - 100 = 80°$. Now, $\angle DEB$ is an exterior angle to $\angle DCE$ and $\angle CDE$. Using the exterior angle theorem, we get

$$80 + x = 3x$$
$$80 = 2x$$
$$x = \boxed{40}$$

Parallel Lines



When a line intersects two parallel lines, the following are true:

- Vertical angles are congruent (e.g. $\angle 1 \cong \angle 4$ and $\angle 6 \cong \angle 7$).
- Alternate interior angles are congruent (e.g. $\angle 3 \cong \angle 6$ and $\angle 4 \cong \angle 5$).
- Alternate exterior angles are congruent (e.g. $\angle 1 \cong \angle 8$ and $\angle 2 \cong \angle 7$).
- Corresponding angles are congruent (e.g. $\angle 1 \cong \angle 5$ and $\angle 4 \cong \angle 8$).
- Same side interior angles are supplementary (e.g. $\angle 3 + \angle 5 = 180°$ and $\angle 4 + \angle 6 = 180°$).

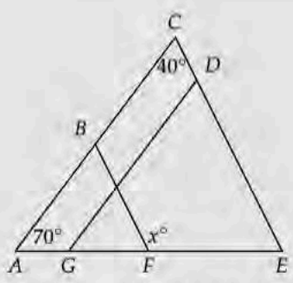
Essentially, when a line crosses two parallel lines, there are two sets of congruent angles:

$$\angle 1 \cong \angle 4 \cong \angle 5 \cong \angle 8$$
$$\angle 2 \cong \angle 3 \cong \angle 6 \cong \angle 7$$

Given two lines intersected by a third line, the two lines must be parallel if any one of the following congruencies exist:

- Congruent alternate interior angles (e.g. $\angle 4 \cong \angle 5$)
- Congruent alternate exterior angles (e.g. $\angle 2 \cong \angle 7$)
- Congruent corresponding angles (e.g. $\angle 2 \cong \angle 6$)

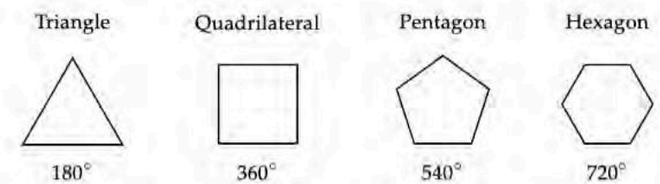
EXAMPLE 2:



In the figure, $\overline{AC}$ is parallel to $\overline{GD}$, and $\overline{BF}$ is parallel to $\overline{CE}$. What is the value of x ?

Here is the fastest approach: $\angle ACE = \angle ABF = 40°$ because they are corresponding angles ($\overline{AC}$ cuts parallel lines $\overline{BF}$ and $\overline{CE}$). Since $\angle BFE$ is an exterior angle to $\triangle ABF$, $x = 70 + 40 = \boxed{110}$ by the exterior angle theorem.

Polygons



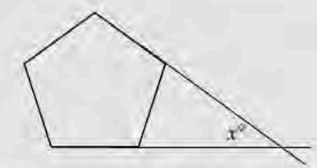
As you can see, each additional side in a polygon increases the sum of the interior angles by 180°. For any polygon, the sum of the interior angles is given by

$$180(n-2) \text{ where } n \text{ is the number of sides}$$

A **regular polygon** is a polygon in which all sides and angles are equal. The polygons shown above are regular. To find the measure of each interior angle in a regular polygon, divide the sum of the interior angles by the number of interior angles. A regular hexagon, for example, has interior angles that each measure $720 \div 6 = 120°$.

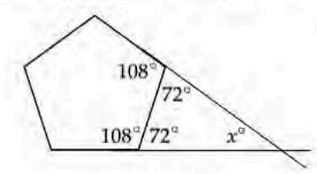
On the SAT, you are mainly tested on triangles and quadrilaterals, but it's good to know the basics for any polygon.

EXAMPLE 3:



In the figure, two sides of a regular pentagon are extended to form a triangle. What is the value of x ?

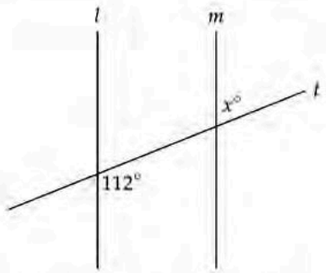
The total number of degrees in a pentagon is $180(5-2) = 540°$, so each interior angle measures $540° \div 5 = 108°$. The supplementary angles are then $180 - 108 = 72°$.



Since the angles in a triangle sum to 180°, $x = 180 - 72 - 72 = \boxed{36}$.

EXERCISE: Answers for this exercise start on page 379.

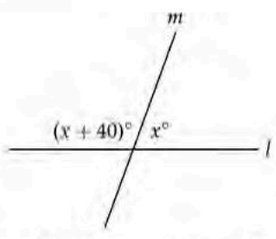
1



Note: Figure not drawn to scale.

In the figure, line l is parallel to line m. What is the value of x ?

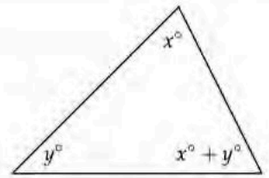
2



In the figure, line l intersects line m. What is the value of x ?

A) 60
B) 70
C) 75
D) 80

3

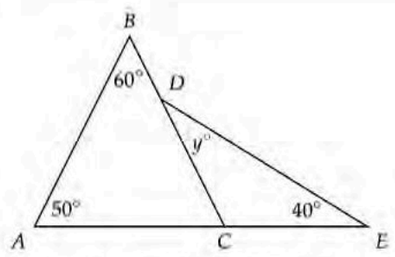


Note: Figure not drawn to scale.

In the figure, $x = 40$. What is the value of y ?

A) 40
B) 50
C) 80
D) 90

4

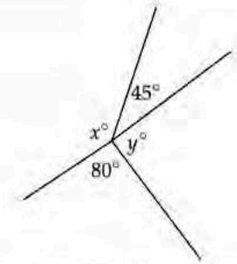


Note: Figure not drawn to scale.

In the figure, point C lies on $\overline{AE}$, and point D lies on $\overline{BC}$. What is the value of y ?

A) 30
B) 40
C) 50
D) 70

5

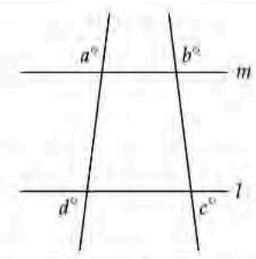


Note: Figure not drawn to scale.

In the figure, four line segments intersect at a point. What is the value of $x + y$?

A) 125
B) 180
C) 235
D) 280

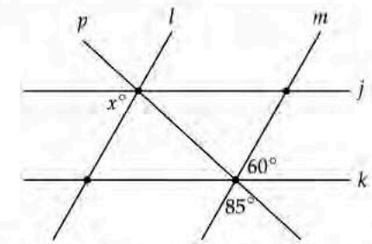
6



In the figure, lines l and m are parallel. What is the value of $a + b + c + d$?

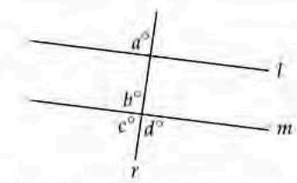
A) 270
B) 360
C) 720
D) It cannot be determined from the information given.

7



In the figure, lines j and k are parallel, and lines l and m are parallel. What is the value of x ?

8

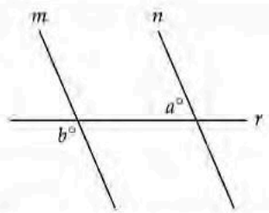


Note: Figure not drawn to scale.

In the figure, line r intersects lines l and m. Which of the following is sufficient to prove that lines l and m are parallel?

A) $a = c$
B) $a = d$
C) $b = c$
D) $b = d$

9

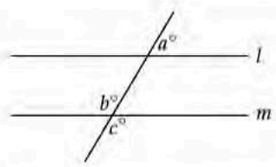


Note: Figure not drawn to scale.

In the figure shown, lines m and n are parallel, and line r intersects both lines. If $a > 74$, which of the following must be true?

A) $b < 106$
B) $b > 106$
C) $b < 180 - a$
D) $b > 180 - a$

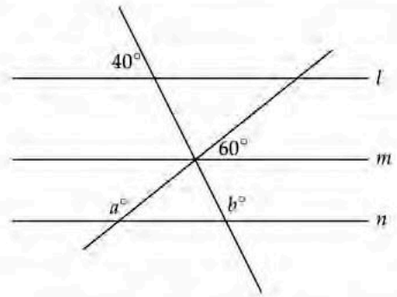
10



Note: Figure not drawn to scale.

In the figure, lines l and m are parallel. If $b = 9k + 15$ and $c = 16k - 76$, what is the value of a ?

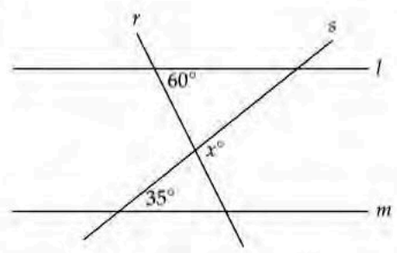
11



Note: Figure not drawn to scale.

In the figure, lines l, m, and n are parallel. What is the value of $a + b$?

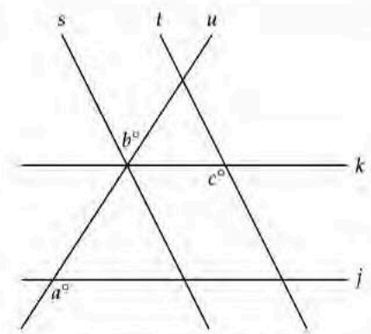
12



Note: Figure not drawn to scale.

In the figure, lines l and m are parallel. What is the value of x ?

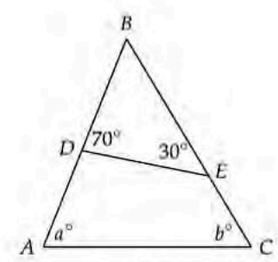
13



Note: Figure not drawn to scale.

In the figure, line u intersects parallel lines k and j and parallel lines s and t. If $a = 118$ and $b = 52$, what is the value of c ?

14

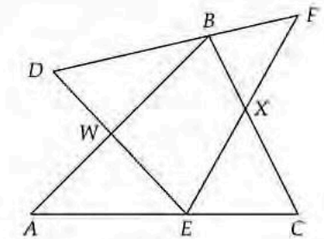


Note: Figure not drawn to scale.

In the figure, point D lies on $\overline{AB}$, and point E lies on $\overline{BC}$. What is the value of $a + b$?

A) 80
B) 100
C) 110
D) 120

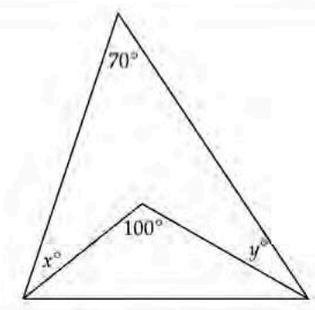
15



Note: Figure not drawn to scale.

In the figure shown, point B lies on line segment DF, point E lies on line segment AC, line segments AB and DE intersect at point W, and line segments BC and EF intersect at point X. The measure of $\angle A$ is $40°$, the measure of $\angle C$ is $60°$, the measure of $\angle D$ is $68°$, and the measure of $\angle FBC$ is $72°$. What is the measure, in degrees, of $\angle AED$?

16



Note: Figure not drawn to scale.

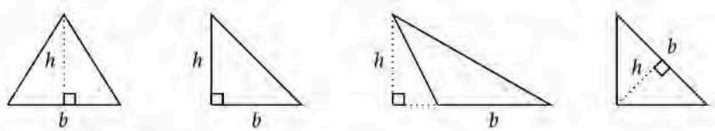
In the figure, two triangles share a common side. What is the value of $x + y$?

A) 10
B) 20
C) 30
D) 50

18 Triangles

Area of a Triangle

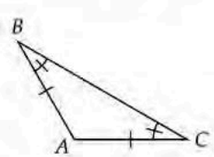
The area of a triangle is given by $\frac{1}{2}bh$, where b is the length of its base and h is its height. Any side of the triangle can be chosen as the base. Once the base is chosen, the height is defined as the perpendicular line segment from the opposite vertex (corner) of the triangle to the line containing the base. Here are some examples:



The altitude of a triangle is just another term for its height.

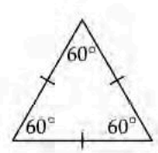
Isosceles & Equilateral Triangles

An isosceles triangle is one that has two sides of equal length. The angles opposite those sides are equal.



Because $AB = AC$, the measures of angles B and C are equal.

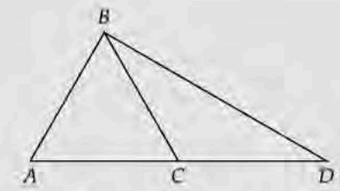
In an equilateral triangle, all sides have the same length. Because equal sides imply equal angles, each angle measures $180 \div 3 = 60°$.



EXAMPLE 1: In an isosceles triangle, the measure of one of the angles is $50°$, and the measure of another angle is $x°$. What is the greatest possible value of x ?

An isosceles triangle has not only two equal sides but also two equal angles. There are two possibilities for an isosceles triangle with an angle of $50°$: one of the unknown angles could be $50°$, forming a 50–50–80 triangle, or the two unknown angles themselves could be equal, forming a 50–65–65 triangle. Given these two possibilities, $\boxed{80}$ is the greatest possible value of x.

EXAMPLE 2:

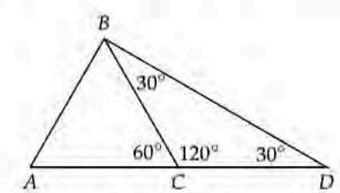


Note: Figure not drawn to scale.

In the figure, triangle ABC is equilateral and $BC = CD$. What is the measure, in degrees, of angle BDC ?

A) 15° B) 30° C) 45° D) 60°

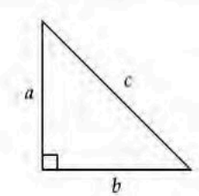
Each angle of an equilateral triangle measures $60°$. Since angles ACB and BCD are supplementary (form a straight line), angle BCD measures $180 - 60 = 120°$. Since $BC = CD$, angles CBD and BDC must have the same measure: $(180 - 120) \div 2 = 30°$.



Answer $\boxed{(B)}$.

Right Triangles

Right triangles are made up of two legs and a hypotenuse (the side opposite the right angle). The hypotenuse is always the longest side.



The side lengths of any right triangle adhere to the **pythagorean theorem**: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse.

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EXAMPLE 3: A right triangle has legs with lengths of 21 feet and 22 feet. What is the length, in feet, of the hypotenuse of this triangle?

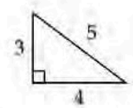
A) $5\sqrt{37}$ B) $15\sqrt{5}$ C) 43 D) 925

Let h be the length of the hypotenuse. Applying the pythagorean theorem, we get

$$21^2 + 22^2 = h^2$$
$$925 = h^2$$
$$h = \sqrt{925} = \sqrt{25 \cdot 37} = 5\sqrt{37}$$

Answer $\boxed{(A)}$.

When working with right triangles, certain side lengths come up repeatedly. For example, the 3–4–5 triangle:



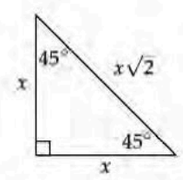
A set of three whole numbers that satisfy the pythagorean theorem is called a **pythagorean triple**. Though not necessary, it'll save you some time and effort if you learn to recognize the most common triples that show up:

3–4–5 · 6–8–10 · 5–12–13 · 7–24–25 · 8–15–17

Note that the 6–8–10 right triangle is just a multiple of the 3–4–5 right triangle.

Special Right Triangles

There are two special right triangles you have to know: the 45°–45°–90° triangle and the 30°–60°–90° triangle. They're "special" because of the way their sides relate to one another. Let's look at the 45–45–90 triangle first:



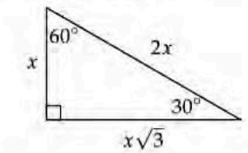
The 45–45–90 triangle is a right isosceles triangle, which means its two legs are equal. If the length of each leg is x, then the length of the hypotenuse is $x\sqrt{2}$ (square root 2 times longer than x).

Letting h be the length of the hypotenuse, we can prove this relationship using the pythagorean theorem:

$$x^2 + x^2 = h^2$$
$$2x^2 = h^2$$
$$\sqrt{2x^2} = \sqrt{h^2}$$
$$x\sqrt{2} = h$$

These proofs are shown not because they will be tested on the SAT, but because they illustrate important problem-solving concepts that you may have to use on certain SAT questions.

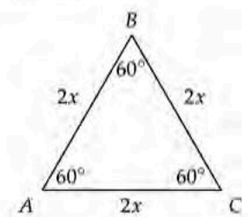
The second special right triangle is the 30°–60°–90° triangle:



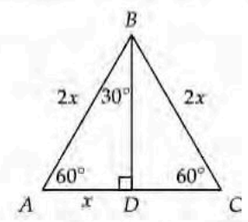
Because 30° is the smallest angle, the side opposite to it is the shortest. If we let the length of the shortest side be x, then the hypotenuse, the largest side, turns out to be twice as long as x, and the side opposite 60° turns out to be $\sqrt{3}$ times longer than x.

One common mistake students make is to think that because 60° is twice 30°, the side opposite 60° must be twice as long as the side opposite 30°. This line of thinking is flawed. You cannot extrapolate the ratio of the sides from the ratio of the angles. It's true that the side opposite 60° is longer than the side opposite 30°, but it isn't twice as long.

We can prove the 30–60–90 relationship by using an equilateral triangle. Let each side be $2x$ (we could use x but you'll see why $2x$ makes things easier in a bit):



Drawing a perpendicular line from B to $\overline{AC}$ creates two 30–60–90 triangles:



Because an equilateral triangle is symmetrical, AD is half of $2x$, or just x. That's why we used $2x$—to avoid any fractions. We can now use the pythagorean theorem to find BD:

$$AD^2 + BD^2 = AB^2$$
$$x^2 + BD^2 = (2x)^2$$
$$BD^2 = 4x^2 - x^2$$
$$BD^2 = 3x^2$$
$$BD = \sqrt{3x^2} = x\sqrt{3}$$

The x, $x\sqrt{3}$, and $2x$ side lengths of triangle ABD are proof of the 30–60–90 triangle relationship.

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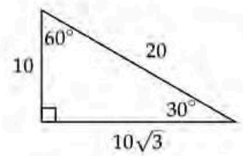
EXAMPLE 4: In a right triangle, the measure of one of the acute angles is 30°, and the length of the hypotenuse is 20 inches. What is the area, in square inches, of this triangle?

A) $50\sqrt{3}$ B) 100 C) $100\sqrt{3}$ D) 200

Since the triangle has a right angle and a 30° angle, the remaining angle must be 60°. We can then use the 30–60–90 triangle relationship to get the following lengths:

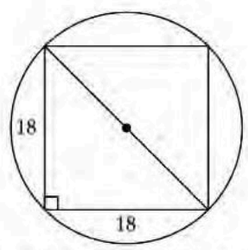


The area of this triangle is then $\frac{1}{2}bh = \frac{1}{2}(10\sqrt{3})(10) = 50\sqrt{3}$ square inches. Answer $\boxed{(A)}$.

EXAMPLE 5: A square with a side length of 18 centimeters is inscribed in a circle. What is the radius, in centimeters, of the circle?

A) 9 B) $9\sqrt{2}$ C) $12\sqrt{3}$ D) $18\sqrt{2}$

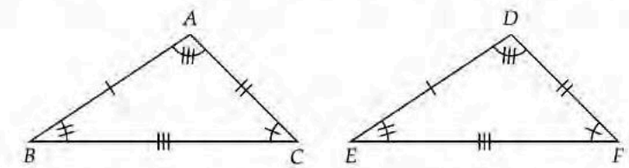
For any regular polygon inscribed in a circle, the center of the polygon is also the center of the circle. In this example, the center of the square is also the center of the circle. Because the centers are the same, a diagonal of the square is a diameter of the circle:



The diagonal splits the square into two 45–45–90 triangles (isosceles right triangles). According to the 45–45–90 triangle relationship, the length of the diagonal must be $18\sqrt{2}$ centimeters. The radius of the circle is then $18\sqrt{2} \div 2 = 9\sqrt{2}$ centimeters. Answer $\boxed{(B)}$.

Congruent Triangles

Two triangles are congruent if they have congruent corresponding angles and side lengths. For example, triangles ABC and DEF are congruent since $AB = DE$, $AC = DF$, $BC = EF$, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.



Congruence implies that each vertex, angle, and side in one triangle has a matching counterpart in the other triangle. In math, we call these matching pairs *corresponding vertices*, *corresponding sides*, and *corresponding angles*. For instance, vertex A *corresponds to* vertex D, $\angle A$ *corresponds to* $\angle D$, side $\overline{AB}$ *corresponds to* side $\overline{DE}$, and so on.

Congruent triangles are named so that corresponding vertices/angles are in the same order. Take a look at how the congruent triangles ABC and DEF are named. The first letters are the corresponding vertices A and D. The second letters are the corresponding vertices B and E. Finally, the last letters are the corresponding vertices C and F. Mathematically, $\triangle ABC \cong \triangle DEF$. It would not be correct to write $\triangle ABC \cong \triangle EDF$ since A and E do not correspond and B and D do not correspond.

Because congruent triangles are named this way, we can determine which sides correspond to which just by looking at the names. For triangles ABC and DEF, the first two letters are AB and DE, respectively, so $\overline{AB}$ corresponds to $\overline{DE}$. Similarly, the last two letters are BC and EF, respectively, so $\overline{BC}$ corresponds to $\overline{EF}$. Finally, the first and last letters are AC and DF, respectively, so $\overline{AC}$ corresponds to $\overline{DF}$.

EXAMPLE 6: Triangles ABC and DEF are congruent, where A corresponds to D, and angles B and E each have measure 50°. The measure of angle C is 25°. What is the measure of angle D ?

A) 25° B) 65° C) 75° D) 105°

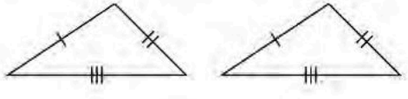
Since congruent triangles have congruent corresponding angles and angle C (25°) corresponds to angle F, the measure of angle F must also be 25°. The measure of angle D is then $180^\circ - \angle E - \angle F = 180 - 50 - 25 = 105^\circ$. Answer $\boxed{(D)}$.

Triangle Congruence Theorems

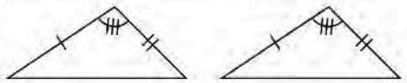
Now how do we prove that two triangles are congruent? The process would be very tedious if we had to check every side and every angle. Fortunately, there are ways of proving congruence without knowing all the sides and angles. Once two triangles are proven to be congruent, then all their corresponding parts are congruent.

Triangle congruence theorems express the minimum amount of information needed to prove that two triangles are congruent:

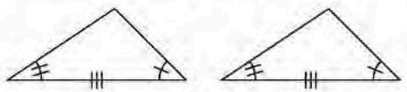
- SSS (side-side-side): If all pairs of corresponding sides are congruent, then the triangles are congruent.



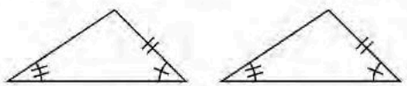
- SAS (side-angle-side): If two pairs of corresponding sides are congruent and the corresponding angles between those sides are congruent, then the triangles are congruent.



- ASA (angle-side-angle): If two pairs of corresponding angles are congruent and the corresponding sides between those angles are congruent, then the triangles are congruent.



- AAS (angle-angle-side): If two pairs of corresponding angles are congruent and the corresponding sides opposite to one of those pairs of angles are congruent, then the triangles are congruent.



ASA and AAS are essentially the same because if you know two angles of a triangle, you know the third. Put simply, two angles and a side are enough to prove congruence.

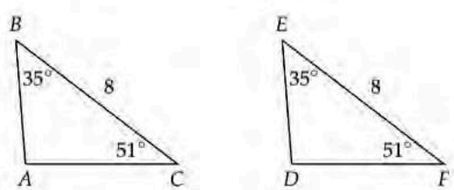
Note that SSA (side-side-angle) is NOT a triangle congruence theorem because it only proves congruence when the angle is at least 90°. When the angle is acute, more than one triangle is possible, so congruence isn't guaranteed. For example, the following two triangles satisfy SSA but are obviously not congruent.

You will not be tested on the nuances of SSA. Just know that it is not a triangle congruence theorem.

EXAMPLE 7: In triangles ABC and DEF, angles C and F each have measure 51°, $BC = 8$, and $EF = 8$. Which additional piece of information is sufficient to prove that triangle ABC is congruent to triangle DEF?

A) $AB = 9$ and $DE = 9$
B) $AC = 14$ and $DE = 14$
C) Angles B and E each have measure 35°.
D) Angles B and D each have measure 43°.

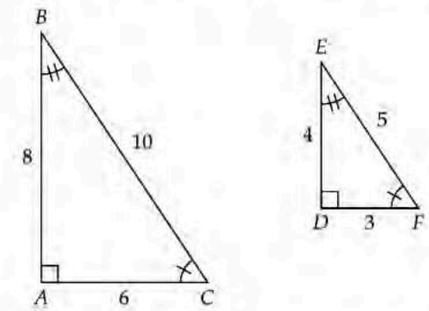
Angles B and E are corresponding angles. If they had the same measure, then the two triangles would be congruent by ASA (angle-side-angle).



Answer $\boxed{(C)}$. Answer A is wrong because SSA is not a valid triangle congruence theorem. Answer B is wrong because $\overline{AC}$ and $\overline{DE}$ are not corresponding sides—their congruence doesn't help satisfy any of the triangle congruence theorems. Answer D is wrong because the given measures do not result in congruent corresponding angles between the two triangles.

Similar Triangles

Two triangles are similar when their corresponding angles are congruent and their corresponding sides are proportional in the same ratio. Put simply, similar triangles are different size versions of one another.



In similar triangles ABC and DEF above, $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, $AB = 2DE$, $AC = 2DF$, and $BC = 2EF$. As a result, triangle ABC is a larger version of triangle DEF—the triangles have the same shape but not the same size.

With similar triangles, we use the same naming scheme as we do with congruent triangles. If $\triangle ABC \sim \triangle DEF$ (the $\sim$ symbol means "similar"), then A and D, the first respective letters, are corresponding vertices, B and E, the second respective letters, are corresponding vertices, and C and F, the last respective letters, are corresponding vertices. The same logic applies when determining which sides correspond to which. The first two respective letters of each name, AB and DE, indicate that side $\overline{AB}$ corresponds to side $\overline{DE}$, and so on.

It would not be correct to write $\triangle ABC \sim \triangle EDF$ since A and E do not correspond and B and D do not correspond.

In summary, the two most important aspects of similar triangles is that 1) their corresponding angles are congruent and 2) their corresponding sides are proportional in the same ratio. In our example above,

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = 2$$

This equation also implies that

$$\frac{AB}{AC} = \frac{DE}{DF} \quad \text{and} \quad \frac{AB}{BC} = \frac{DE}{EF} \quad \text{and} \quad \frac{AC}{BC} = \frac{DF}{EF}$$

In other words, not only are ratios between corresponding sides equivalent but "corresponding ratios" between sides are also equivalent.

Setting up these types of equations will be your goal in many of the harder similar triangles questions.

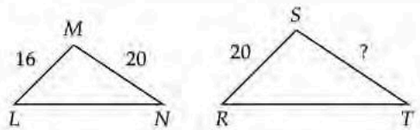
EXAMPLE 8: Isosceles triangles FGH and JKL are similar, where F and G correspond to J and K, respectively. Angle G has a measure of 68°. What is the measure of angle K?

A) 44° B) 56° C) 68° D) 112°

Since similar triangles have congruent corresponding angles and angle G (68°) corresponds to angle K, the measure of angle K must also be 68°. Answer $\boxed{(C)}$. The fact that the triangles are isosceles is irrelevant.

EXAMPLE 9: Triangle LMN is similar to RST such that L, M, and N correspond to R, S, and T, respectively. The length of $\overline{LM}$ is 16 and the lengths of $\overline{MN}$ and $\overline{RS}$ are each 20. What is the length of $\overline{ST}$?

A) 16 B) 20 C) 24 D) 25



Similar triangles have corresponding sides in the same proportion. Since $\overline{RS}$ and $\overline{LM}$ are corresponding sides, triangle RST must have sides that are $\frac{20}{16} = \frac{5}{4}$ times longer than the corresponding sides of triangle LMN. Therefore, the length of $\overline{ST}$ is $\frac{5}{4}$ times longer than corresponding side $\overline{MN}$: $ST = \frac{5}{4}MN = \frac{5}{4}(20) = 25$.

Answer $\boxed{(D)}$. Here is the solution expressed as an equation that can be solved for ST:

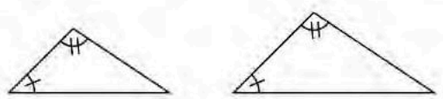
$$\frac{RS}{LM} = \frac{ST}{MN} \Rightarrow \frac{20}{16} = \frac{ST}{20} \Rightarrow ST = \frac{20}{16}(20) = 25$$

Triangle Similarity Theorems

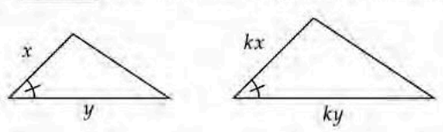
Now how do we determine whether two triangles are similar in the first place? Just as we have triangle congruence theorems, we have triangle similarity theorems to help us prove similarity without needing to check every side and angle. Once two triangles are proven to be similar, then all corresponding angles are congruent and all corresponding sides are proportional according to the same ratio.

Triangle similarity theorems express the minimum amount of information needed to prove that two triangles are similar:

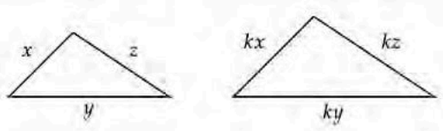
- AA(A) (angle-angle-(angle)): If two pairs of corresponding angles are congruent, then the triangles are similar. Note that AA is the same as AAA because if two pairs of angles are congruent, then the third pair must also be congruent.



- SAS (side-angle-side): If two pairs of corresponding sides are proportional according to the same ratio and the corresponding angles between those sides are congruent, then the triangles are similar.



- SSS (side-side-side): If all pairs of corresponding sides are proportional according to the same ratio, then the triangles are similar.



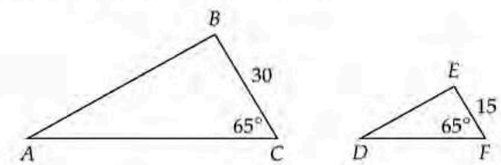
Among the three similarity theorems, AA (angle-angle) is by far the most commonly tested.

EXAMPLE 10: In triangle ABC, the measure of angle C is 65° and the length of side BC is 30. In triangle DEF, the measure of angle F is 65° and the length of side EF is 15. Which additional piece of information is sufficient to prove that triangle ABC is similar to triangle DEF ?

I. The length of side AB is twice the length of side DE.
II. The length of side AC is twice the length of side DF.

A) I is sufficient, but II is not.
B) II is sufficient, but I is not.
C) Either I or II is sufficient.
D) Neither I nor II is sufficient.

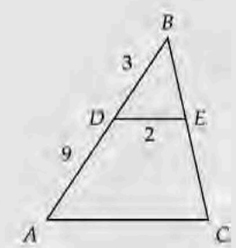
Let's start with a picture, which isn't necessarily drawn to scale.



If statement I ($AB = 2DE$) were true, then we would have two pairs of corresponding sides in the same proportion. However, SSA (side-side-angle) is not a valid triangle similarity theorem, so I is not sufficient.

Now if statement II ($AC = 2DF$) were true, then we would again have two pairs of corresponding sides in the same proportion. This time, however, the congruent angles would lie between the sides in each triangle. Since SAS (side-angle-side) is a valid triangle similarity theorem, II is sufficient. Answer $\boxed{(B)}$.

EXAMPLE 11:



Note: Figure not drawn to scale.

In triangle ABC, $\overline{DE}$ is parallel to $\overline{AC}$, $AD = 9$, $DB = 3$, and $DE = 2$. What is the length of $\overline{AC}$?

Because $\overline{DE}$ and $\overline{AC}$ are parallel, $\angle BDE$ is congruent to $\angle BAC$ and $\angle BED$ is congruent to $\angle BCA$. Therefore, triangle BDE and triangle BAC are similar by AA (angle-angle).

Now the answer is NOT 6. A common mistake that students make whenever they see similar triangles is to assume that certain segments correspond to each other when they do not. Of course, some questions are designed so that this mistake is easy to make. In this particular case, it's easy to assume that because AD is 3 times BD, AC must be 3 times DE, but this would be incorrect. When figuring out the sides of similar triangles, we have to look at, well, the actual *sides* of the triangles, and AD is not a side of any triangle, just a portion of one.

One correct approach is to set up an equation that relates the relevant corresponding sides:

$$\frac{BD}{BA} = \frac{DE}{AC}$$
$$\frac{3}{3+9} = \frac{2}{AC}$$

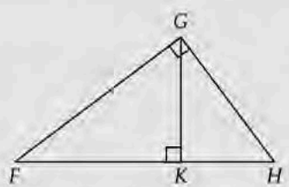
Cross multiplying, we get

$$3AC = 24$$
$$AC = \boxed{8}$$

In this example, the similarity between the two triangles was easy to recognize, and it was obvious which sides corresponded to each other. In tougher questions, similar triangles and their corresponding sides are more difficult to spot and keep track of.

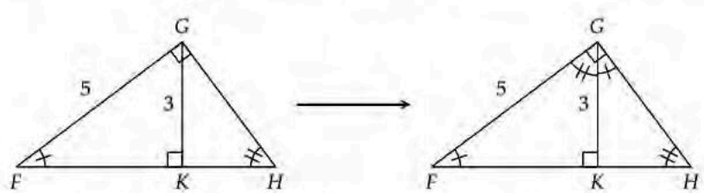
For these tougher questions, I recommend labeling congruent angles with tick marks. The tick marks will help you determine which vertices correspond and which sides correspond. Here's an example to illustrate:

EXAMPLE 12:



In the figure, $FG = 5$ and $GK = 3$. What is the length of $\overline{KH}$?

At first glance, this does not look like a similar triangles question. But if we label the angles of the outside triangle FGH, something interesting happens. Let's first label $\angle F$ with one tick mark and $\angle H$ with two tick marks.



Now whenever two angles of one triangle are equal in measure to two angles of another, the remaining angles must also be equal to each other. Because outside triangle FGH and triangle FGK on the left both have right angles and share $\angle F$, $\angle FGK$ must have the same measure as $\angle H$ (two tick marks). Likewise, outside triangle FGH and triangle GKH on the right both have right angles and share $\angle H$, so $\angle KGH$ must have the same measure as $\angle F$ (one tick mark).

The result is that outside triangle FGH, triangle FKG on the left, and triangle GKH on the right all have congruent angles, which means they're all similar to one another by AA(A) (angle-angle-(angle))!

Note that the tick marks helped us name these triangles by making it easy to identify the corresponding vertices. If the outer triangle is named FGH (1-tick, right angle, 2-tick), then the triangle on the left should be named FKG (1-tick, right angle, 2-tick).

Since triangle FKG is a 3–4–5 triangle with $FK = 4$, an equation that relates the corresponding sides of triangles FKG and GKH would allow us to solve for KH.

To identify the corresponding sides, we can use the order of the letters in the triangles' names, as discussed earlier, or we can use our tick marks: sides opposite from angles with the same number of tick marks correspond to each other.

$$\frac{KH \text{ (opposite 1-tick angle in } \triangle GKH)}{KG \text{ (opposite 1-tick angle in } \triangle FKG)} = \frac{GK \text{ (opposite 2-tick angle in } \triangle GKH)}{FK \text{ (opposite 2-tick angle in } \triangle FKG)}$$
$$\frac{KH}{3} = \frac{3}{4}$$
$$KH = \boxed{\frac{9}{4}}$$

Similarity in Other Shapes

Triangles aren't the only shapes that can be similar to one another. Since similarity exists when there are different size versions of the same shape, other polygons can exhibit similarity as well.

When dealing with similar polygons, just remember these fundamental properties of similarity:

- Congruent corresponding angles
- Corresponding sides that are proportional in the same ratio

EXAMPLE 13: Quadrilateral $ABCD$ is similar to quadrilateral $WXYZ$, where A, B, C and D correspond to W, X, Y, and Z, respectively. The measure of angle A is $50°$, the measure of angle B is $100°$, and the measure of angle Y is $140°$. What is the measure of angle Z ?

A) $50°$ B) $70°$ C) $90°$ D) $120°$

Since corresponding angles are congruent and angles A ($50°$) and B ($100°$) correspond to angles W and X, respectively, the measure of angle W must be $50°$ and the measure of angle X must be $100°$. The measure of angle Z is then $360° - \angle W - \angle X - \angle Y = 360 - 50 - 100 - 140 = 70°$. Answer $\boxed{(B)}$.

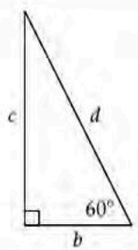
EXERCISE 1: Spare paper recommended. Answers for this exercise start on page 381.

1

A square has a side length of 6 inches. What is the length, in inches, of the square's diagonal?

A) $3\sqrt{2}$
B) 6
C) $6\sqrt{2}$
D) $6\sqrt{3}$

2



Note: Figure not drawn to scale.

For the right triangle shown, $d = 10$. What is the value of b ?

A) 5
B) $5\sqrt{2}$
C) $5\sqrt{3}$
D) $10\sqrt{3}$

3

Triangles LMN and PQR are congruent, where L corresponds to P and M corresponds to Q. The measures of angles L and R are 94° and 22°, respectively. What is the measure of angle M ?

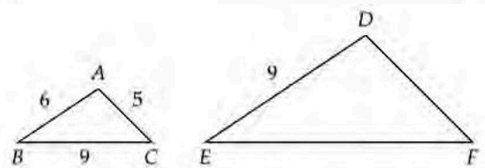
A) 22°
B) 64°
C) 94°
D) 116°

4

Triangles ABC and KLM are similar, where A and B correspond to K and L, respectively. The measure of angle A is 15°, $BC = 24$, and $KM = 3AC$. What is the length of $\overline{LM}$?

A) 8
B) 45
C) 72
D) 216

5



In the figure shown, triangle ABC is similar to triangle DEF. What is the perimeter of triangle DEF ?

A) 20
B) 26.8
C) 30
D) 36.2

6

Triangles ABC and JKL are similar, where A corresponds to J, and B and K are right angles. The measure of angle A is 34°, and $KL = \frac{1}{2}BC$. What is the measure of angle L ?

A) 17
B) 28
C) 34
D) 56

7

A right triangle has a hypotenuse of length 31 inches. One of the legs of this triangle has a length of 25 inches and the other leg has a length of $4\sqrt{n}$ inches, where n is an integer constant. What is the value of n ?

8

A right triangle has a hypotenuse of length 42 centimeters and an angle measuring 30°. What is the perimeter, in centimeters, of this triangle?

A) $21 + 21\sqrt{3}$
B) $56 + 14\sqrt{3}$
C) $63 + 21\sqrt{3}$
D) $63 + 42\sqrt{3}$

9

In triangles LMN and RST, angles L and R each have measure 127°. Which additional piece of information is sufficient to determine whether triangle LMN is similar to triangle RST ?

A) The measures of angles N and T
B) The lengths of sides LM and RS
C) The measures of angles S and T
D) The lengths of sides MN and ST

10

When adjusted to a height of 35 feet, a pole casts a shadow that is 14 feet long. By how many feet must the height of the pole be increased so that it casts a shadow that is 18 feet long?

11

In triangle XYZ, the measure of angle Y is 70°. Which of the following pieces of information is sufficient to prove that triangle XYZ is an isosceles triangle?

I. The measure of angle X is 40°.
II. The measure of angle Z is 55°.

A) I is sufficient, but II is not.
B) II is sufficient, but I is not.
C) I is sufficient and II is sufficient.
D) Neither I nor II is sufficient.

12

The hypotenuse of an isosceles right triangle has a length of 12 units. What is the area of the triangle, in square units?

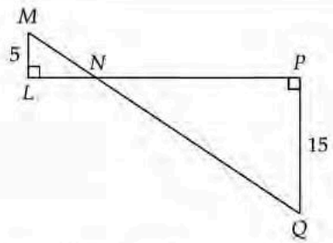
A) $18\sqrt{2}$
B) 36
C) 48
D) $36\sqrt{2}$

13

In triangles ABC and DEF, angles A and D each have measure 72° and angles B and E each have measure 36°. Which additional piece of information is sufficient to prove that triangle ABC is congruent to triangle DEF ?

A) The measure of angle A is equal to the measure of angle C.
B) The length of side AC is equal to the length of side DF.
C) The measure of angle C is equal to the measure of angle F.
D) The length of side DE is equal to the length of side EF.

14



In the figure, $\overline{MQ}$ and $\overline{LP}$ intersect at point N, and $MQ = 38$. What is the length of $\overline{NQ}$?

15

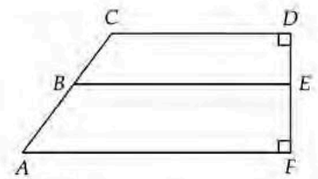
In triangles CDE and FGH, angles C and F each have measure $75°$, $CD = 20$, and $FG = 5$. Which additional piece of information is sufficient to prove that triangle CDE is similar to triangle FGH ?

A) The measures of angles D and H are $64°$ and $16°$, respectively.
B) The measures of angles E and G are $70°$ and $35°$, respectively.
C) $CE = 8$ and $FH = 4$
D) $DE = 16$ and $GH = 4$

16

An equilateral triangle has a perimeter of $42\sqrt{3}$ inches. What is the height, in inches, of this triangle?

17



Note: Figure not drawn to scale.

In the figure, trapezoid $BCDE$ is similar to trapezoid $ABEF$. The length of $\overline{AB}$ is 4, the length of $\overline{DE}$ is 2, and the length of $\overline{BC}$ is equal to the length of $\overline{EF}$. What is the length of $\overline{EF}$?

A) $\sqrt{6}$
B) $2\sqrt{2}$
C) $2\sqrt{3}$
D) 3

18

A right triangle has an area of 450 square inches. The triangle has sides of length $a\sqrt{3}$, $a\sqrt{12}$, and $3a$ inches, where a is a constant. To the nearest tenth of an inch, what is the length of the shortest side of this triangle?

A) 16.1
B) 19.4
C) 21.2
D) 22.8

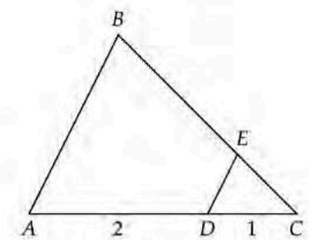
19

In triangle DEF, the measure of angle D is $90°$ and $\overline{DG}$ is an altitude of the triangle. The length of $\overline{DE}$ is 5, and the length of $\overline{EG}$ is 4. Which of the following ratios has a value of $\frac{3}{4}$?

A) $\frac{DF}{EF}$
B) $\frac{DG}{DF}$
C) $\frac{DG}{FG}$
D) $\frac{FG}{DG}$

EXERCISE 2: Spare paper recommended. Answers for this exercise start on page 383.

1

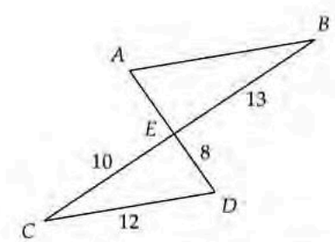


Note: Figure not drawn to scale.

In the figure shown, triangle ABC is similar to triangle DEC such that A corresponds to D and B corresponds to E. The measure of $\angle ABC$ is $120°$. What is the measure of $\angle DEC$?

A) $40°$
B) $60°$
C) $80°$
D) $120°$

2



Note: Figure not drawn to scale.

In the figure shown, $\overline{AD}$ and $\overline{BC}$ intersect at point E, and $\overline{AB}$ is parallel to $\overline{CD}$. What is the length of $\overline{AB}$?

A) $\frac{130}{8}$
B) $\frac{130}{12}$
C) $\frac{156}{8}$
D) $\frac{156}{10}$

3

Triangles ABC and DEF are similar, where A corresponds to D and B corresponds to E. The measure of angle A is $20°$, $EF = 6$, and $DF = 6$. What is the measure of angle C ?

A) $20°$
B) $80°$
C) $140°$
D) $160°$

4

In triangles RST and WXY, angles R, T, W, and Y each have measure $29°$. Which additional piece of information is sufficient to prove that triangle RST is congruent to triangle WXY ?

A) The measures of angle S and angle X are equal.
B) The lengths of sides RT and WX are equal.
C) The lengths of sides RT and WY are equal.
D) No additional information is necessary.

5

A right triangle has legs with lengths of $16\sqrt{23}$ inches and $30\sqrt{23}$ inches. What is the length of the hypotenuse of this triangle, in inches?

A) $34\sqrt{23}$
B) $36\sqrt{23}$
C) $38\sqrt{23}$
D) $42\sqrt{23}$

6

Right triangles ABC and DEF are similar, where A and D are right angles, and B corresponds to E. If $AB = 12$, $BC = 60\sqrt{2}$, and $DF = 105$, what is the length of side EF ?

A) $48\sqrt{2}$
B) $75\sqrt{2}$
C) $100\sqrt{2}$
D) $525\sqrt{2}$

7

Isosceles triangles ABC and DEF are similar, where A corresponds to D and B corresponds to E. The measure of angle A is equal to the measure of angle B, $AB = 10$, $AC = 5$, and $EF = 40$. What is the length of side DE ?

A) 20
B) 40
C) 60
D) 80

8

$$A(x) = \frac{1}{2}x(7x)$$

The function A gives the area, in square feet, of a certain right triangle, where x is the length, in feet, of one of the legs of the triangle. If the length of this triangle's hypotenuse, in feet, can be written in the form $ax\sqrt{b}$, where a and b are integers, what is the value of $a + b$?

9

In triangle ABC, the measure of angle A is $30°$ and $AB = AC$. In triangle DEF, the measure of angle E is $75°$. Which additional piece of information is sufficient to determine whether triangle ABC is similar to triangle DEF ?

A) The lengths of $\overline{AB}$ and $\overline{DE}$
B) The lengths of $\overline{BC}$ and $\overline{EF}$
C) The measure of angle B
D) The measure of angle F

10

According to the triangle inequality theorem, the length of any side of a triangle must be smaller than the sum of the lengths of the other two sides. If a triangle has side lengths of 9 inches, 15 inches, and x inches, which inequality represents all possible values of x ?

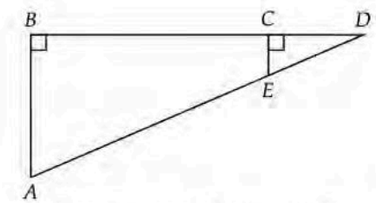
A) $0 < x < 24$
B) $6 < x < 24$
C) $12 < x < 24$
D) $x > 24$

11

In triangles LMN and PQR, angles M and Q each have measure $54°$, $MN = 16$, and $QR = 16$. Which additional piece of information is sufficient to prove that triangle LMN is congruent to triangle PQR ?

A) Angles L and R each have measure $44°$.
B) Angles N and P each have measure $44°$.
C) $LM = 11$ and $PQ = 11$
D) $LN = 11$ and $PR = 11$

12



Note: Figure not drawn to scale.

In the figure shown, $CE = 9$ and $BC = 6CD$. What is the length of line segment AB ?

13

A circle is inscribed in an equilateral triangle. The radius of the circle is 9 inches. What is the perimeter, in inches, of the triangle?

A) $18\sqrt{3}$
B) $27\sqrt{3}$
C) $54\sqrt{3}$
D) 108

14

In triangle ABC, angle B measures $90°$, $AC = 106$, and $BC = 90$. Point D on side AC is connected by a line segment with point E on side BC such that line segment DE is parallel to side AB, and $DE = 14$. What is the length of line segment BE ?

15

In triangle FGH, $FG = 21$ and $GH = 27$. In triangle XYZ, $XY = 28$ and $YZ = 36$. Which additional piece of information is sufficient to prove that triangles FGH and XYZ are similar?

I. The measure of angle G is equal to the measure of angle Y.
II. $FH = 15$ and $XZ = 20$

A) I is sufficient, but II is not.
B) II is sufficient, but I is not.
C) I is sufficient and II is sufficient.
D) Neither I nor II is sufficient.

16

An equilateral triangle is inscribed in a circle. The triangle has a side length of $22\sqrt{3}$ feet. What is the radius, in feet, of the circle?

A) $11\sqrt{3}$
B) 22
C) 33
D) 44

17

In triangle ABC, angle A is a right angle, point D lies on $\overline{AC}$, point E lies on $\overline{BC}$, and $\overline{DE}$ is parallel to $\overline{AB}$. The length of $\overline{DE}$ is 36 inches, the length of $\overline{AB}$ is 54 inches, and the length of $\overline{AD}$ is 15 inches. What is the area, in square inches, of triangle CDE ?

18

In triangle ABC, angle B is a right angle, point D lies on $\overline{AB}$, point E lies on $\overline{AC}$, and $\overline{DE}$ is perpendicular to $\overline{AC}$. If the length of $\overline{AB}$ is 12 units, the length of $\overline{AC}$ is 13 units, and the length of $\overline{DE}$ is 3 units, what is the length of $\overline{AE}$, in units?

19

In triangle ABC, $AB = 6$, $BC = 6$, and the measure of angle ABC is $120°$. What is the area of triangle ABC ?

A) $2\sqrt{3}$
B) $4\sqrt{3}$
C) $6\sqrt{3}$
D) $9\sqrt{3}$

20

Square $ABCD$ has a side length of 12. Point F lies on $\overline{BC}$, and $\overline{AF}$ and $\overline{BD}$ intersect at point E. The length of $\overline{BF}$ is 4. What is the length of $\overline{BE}$?

A) 3
B) $2\sqrt{2}$
C) $3\sqrt{2}$
D) $4\sqrt{2}$

21

A right triangle has a perimeter of $78 + 78\sqrt{3}$ centimeters and an angle measuring $30°$. What is the length, in centimeters, of the shortest side of this triangle?

A) 26
B) $26\sqrt{3}$
C) 39
D) $39\sqrt{3}$

22

In triangle QRS, angle Q is a right angle and $\overline{QT}$ is an altitude of the triangle. The length of $\overline{QS}$ is 15 units and the length of $\overline{RS}$ is 17 units. What is the length of $\overline{TS}$ to the nearest tenth of a unit?

A) 12.6
B) 12.8
C) 13.2
D) 13.4

23

In triangle WXY, point Z lies on $\overline{WY}$, and the measures of $\angle W$ and $\angle YXZ$ are each $36°$. Which of the following ratios is equal to $\frac{WY}{XY}$?

A) $\frac{WX}{XZ}$
B) $\frac{XY}{WX}$
C) $\frac{XY}{YZ}$
D) $\frac{YZ}{XZ}$

19 Circles

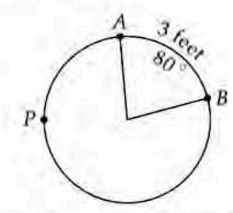
First, the basics. Given a circle with a radius of length r,

- the diameter of the circle is $2r$
- the area of the circle is πr^2
- the circumference of the circle is $2\pi r$

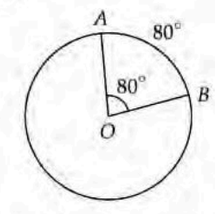
Arc Length vs. Arc Measure

An arc is a portion of a circle. Every arc has an arc length and an arc measure. Arc length is the actual distance along the circle from one point to another (e.g. 3 feet). Arc measure is the number of degrees of rotation along the circle from one point to another (e.g. $80°$). A full rotation is $360°$.

Minor arcs, such as arc AB below, have a measure of less than $180°$ and are named using two letters. Major arcs, such as arc APB below, are named using three letters.



The measure of an arc is equal to the measure of the central angle that intercepts it. In the circle below, arc AB has the same measure as central angle AOB.



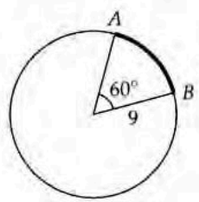
By definition, a central angle of a circle is an angle whose vertex is at the center of the circle and whose legs are radii. For any given arc, the central angle that intercepts it is sometimes referred to as the arc's **associated angle**. In the example above, angle AOB is arc AB's associated angle.

The length of an arc can be found by using the formula

$$\frac{\theta}{360} \times 2\pi r$$

where θ is the degree measure of the central angle that intercepts the arc and r is the radius. The expression $\frac{\theta}{360}$ represents the fraction of the circle's circumference constituted by the arc, and $2\pi r$ is the circle's circumference.

In the circle below, arc AB constitutes $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the circumference. Therefore, its length is $\frac{1}{6} \times 2\pi(9) = 3\pi$ units.



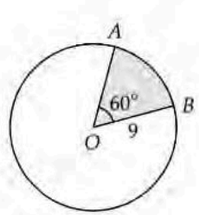
Area of a Sector

A sector is a "slice" of a circle. The area of a sector is given by

$$\frac{\theta}{360} \times \pi r^2$$

where θ is the degree measure of the central angle that encloses the sector and r is the radius. The expression $\frac{\theta}{360}$ represents the fraction of the circle's area constituted by the sector, and πr^2 is the circle's area.

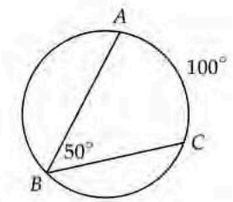
In the circle below, sector OAB constitutes $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the area. Therefore, its area is $\frac{1}{6} \times \pi(9)^2 = \frac{27}{2}\pi$ square units.



Angles Inscribed in a Circle

An inscribed angle of a circle is an angle whose vertex lies on the circle and whose legs are chords (line segments whose endpoints both lie on the circle).

The measure of an inscribed angle is half the measure of the arc it intercepts. In the circle below, chords $\overline{AB}$ and $\overline{BC}$ make up inscribed angle ABC, which has half the measure of arc AC.

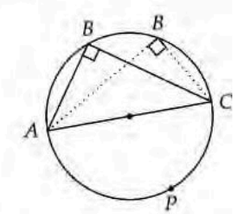


Angles Inscribed in a Semicircle

A semicircle is a half of a circle. We obtain an angle inscribed in a semicircle by designating a point on the circle as the vertex and drawing line segments to the endpoints of a diameter.

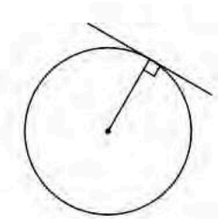
Because angles inscribed in a semicircle always intercept half a circle (180°), they always measure 90° (half of 180°).

In the circle below, $\overline{AC}$ is a diameter and angle ABC is inscribed in semicircle ABC. Because angle ABC intercepts arc APC (180°), angle ABC must be a right angle. Point B could be moved to any distinct location on the circle and angle ABC would still be 90°.



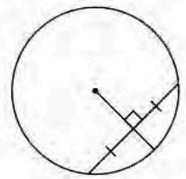
Radius Drawn to a Tangent Line

A line is tangent to a circle if it touches the circle at exactly one point. A radius drawn to that point of tangency is perpendicular to the line:



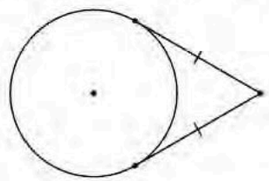
Radius that Bisects a Chord

A radius that splits a chord into two equal parts (bisects a chord) is perpendicular to that chord. The converse is also true: a radius that is perpendicular to a chord bisects that chord.

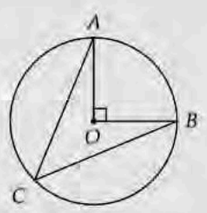


Tangent Segments to a Circle

Tangent segments to a circle drawn from the same external point are congruent.



EXAMPLE 1:



The circle shown has center O, and points A, B, and C lie on the circle. Angle AOB is a right angle. What is the measure, in degrees, of angle ACB ?

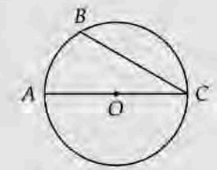
A) 15 B) 30 C) 45 D) 60

Since central angle AOB (90°) intercepts arc AB, arc AB also measures 90°. Since angle ACB is an inscribed angle that intercepts arc AB, its measure must be half the measure of arc AB: $90^\circ \div 2 = 45^\circ$. Answer $\boxed{(C)}$.

EXAMPLE 2: A circle has radius of length 9 inches. The length of arc AB on this circle is 4π inches. What is the measure, in degrees, of arc AB ?

The circumference of the circle is $2\pi r = 2\pi(9) = 18\pi$ inches, which means arc AB constitutes $\frac{4\pi}{18\pi} = \frac{2}{9}$ of the circle. Given that a full circle measures 360°, arc AB must have a measure of $\frac{2}{9} \times 360^\circ = \boxed{80}$ degrees.

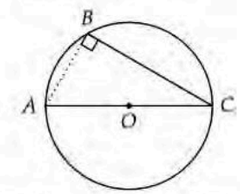
EXAMPLE 3:



In the circle shown, point O is the center, and diameter $\overline{AC}$ has a length of 10 inches. The measure of arc AB is 60°. What is the length, in inches, of $\overline{BC}$?

A) 6 B) $4\sqrt{3}$ C) 8 D) $5\sqrt{3}$

First, the line that's begging to be drawn is $\overline{AB}$. Since $\overline{AC}$ is a diameter, arc ABC is a semicircle in which angle ABC is inscribed. Angle ABC must therefore be a right angle.



Since arc AB measures 60°, inscribed angle ACB must measure $60^\circ \div 2 = 30^\circ$. Thus, triangle ABC is a 30–60–90 triangle with hypotenuse $\overline{AC}$ of length 10 inches.

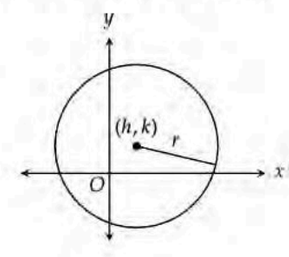
In a 30–60–90 triangle, the shorter leg (e.g. $\overline{AB}$) is half as long as the hypotenuse and the longer leg (e.g. $\overline{BC}$) is $\sqrt{3}$ times longer than the shorter leg. Therefore, the length of side $\overline{AB}$ is $10 \div 2 = 5$ inches, and the length of side $\overline{BC}$ is $5\sqrt{3}$ inches. Answer $\boxed{(D)}$.

Equation of a Circle in the xy-plane

The standard form equation of a circle in the xy-plane is

$$(x-h)^2 + (y-k)^2 = r^2$$

where (h, k) is the center of the circle and r is the length of its radius.



Let's illustrate with a quick example. Let's say we have the following equation of a circle:

$$(x+6)^2+(y-5)^2=64$$

This equation is in standard form with $h=-6$, $k=5$, and $r=\sqrt{64}=8$. Therefore, the circle is centered at $(-6,5)$ and has a radius of 8.

When finding the center of a circle, the foolproof method I like to use is to ask myself what values make the expressions inside the parentheses equal to zero. *What value of x makes $x+6$ equal to zero? And what value of y makes $y-5$ equal to zero?* Well, $x=-6$ and $y=5$. Therefore, the center is at $(-6,5)$.

EXAMPLE 4: The equation $x^2+y^2=9$ represents a circle in the xy-plane. A new circle is drawn whose center is 8 units down from the center of this circle and whose radius is 7 units longer than the radius of this circle. Which of the following is an equation of this new circle?

A) $x^2+(y-8)^2=16$
B) $x^2+(y+8)^2=16$
C) $x^2+(y-8)^2=100$
D) $x^2+(y+8)^2=100$

The given circle has center $(0,0)$ and a radius of $\sqrt{9}=3$. The new center is 8 units down, so its coordinates must be $(0,0-8)=(0,-8)$. The new radius is 7 units longer, so its length must be $3+7=10$. The equation of the new circle is then

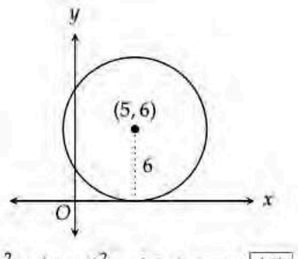
$$(x-0)^2+(y-(-8))^2=10^2$$
$$x^2+(y+8)^2=100$$

Answer $\boxed{(D)}$.

EXAMPLE 5: A circle in the xy-plane has center $(5,6)$ and intersects the x-axis at exactly one point. Which of the following equations represents this circle?

A) $(x-5)^2+(y-6)^2=25$
B) $(x+5)^2+(y+6)^2=25$
C) $(x-5)^2+(y-6)^2=36$
D) $(x+5)^2+(y+6)^2=36$

The circle's center is 5 units to the right of the y-axis and 6 units above the x-axis. For the circle to intersect the x-axis at exactly one point (i.e. be tangent to the x-axis), it must have a radius that is equal to the center's distance from the x-axis. Therefore, the circle must have a radius of 6.



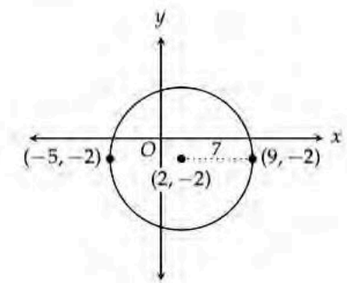
The equation of the circle is then $(x-5)^2+(y-6)^2=36$. Answer $\boxed{(C)}$.

EXAMPLE 6: A circle in the xy-plane has a diameter with endpoints $(-5,-2)$ and $(9,-2)$. Which of the following equations represents this circle?

A) $(x-2)^2+(y+2)^2=49$
B) $(x-2)^2+(y+2)^2=196$
C) $(x-4)^2+(y+2)^2=49$
D) $(x-4)^2+(y+2)^2=196$

The center of the circle is at the midpoint of the diameter's endpoints. The midpoint between two points can be found by averaging the x-coordinates and averaging the y-coordinates. Thus, the center is at $\left(\frac{-5+9}{2},\frac{-2+(-2)}{2}\right)=(2,-2)$.

Since the endpoints of the given diameter have the same y-coordinates, the length of the diameter is just the distance between the x-coordinates: $9-(-5)=14$. The radius is then $14\div 2=7$, and the equation of the circle is $(x-2)^2+(y+2)^2=49$.



Answer $\boxed{(A)}$.

Completing the Square

So far, the equations of circles we've dealt with have all been in standard form. But we can also get equations of circles that look like $x^2-10x+y^2-y=-5$. To convert these raw equations into standard form, we need to know how to complete the square. **Completing the square** is a technique used to convert quadratic expressions such as x^2+6x+2 into the form $(x+a)^2+b$ (i.e. vertex form, remember?).

How to Complete the Square

Let's say we need to complete the square for the following quadratic expression:

$$x^2+6x+2$$

Step 1) We know what the end result should look like. Create a template for it with a blank space:

$$(x\quad)^2+2$$

Step 2) Divide the coefficient of x by 2. Put the resulting number inside the parentheses and remember it for the next step. In this case, the coefficient of x is 6, and 6 divided by 2 is 3, so we put positive 3 inside the parentheses and remember it for the next step:

$$(x+3)^2+2$$

Step 3) Square the number you remembered from the previous step. Subtract the result outside the parentheses. In this case, $3^2 = 9$, so we subtract 9 outside the parentheses:

$$(x+3)^2 + 2 - 9$$

Step 4) Combine like terms:

$$(x+3)^2 - 7$$

Step 5) And we're done! Let's check our work. Expanding $(x+3)^2 - 7$ gives $x^2 + 6x + 9 - 7 = x^2 + 6x + 2$, which is the expression we started with.

Now let's convert the equation of the circle given earlier into standard form by completing the square twice, once for the x's and once for the y's. The equation was

$$x^2 - 10x + y^2 - y = -5$$

Let's start with x and ignore the y's.

$$\underline{(x \quad)^2} + y^2 - y = -5$$
$$\underline{(x-5)^2 - 25} + y^2 - y = -5$$

Since the coefficient of x is -10 and half of -10 is -5, we put -5 inside the parentheses and subtract $(-5)^2 = 25$ outside the parentheses. Now we complete the square a second time for y:

$$(x-5)^2 - 25 + \underline{y^2 - y} = -5$$
$$(x-5)^2 - 25 + \underline{(y \quad)^2} = -5$$
$$(x-5)^2 - 25 + \underline{\left(y - \frac{1}{2}\right)^2 - \frac{1}{4}} = -5$$

Since the coefficient of y is -1 and half of -1 is $-\frac{1}{2}$, we put $-\frac{1}{2}$ inside the parentheses and subtract $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ outside the parentheses. Combining like terms, we get

$$(x-5)^2 - 25 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} = -5$$
$$(x-5)^2 + \left(y - \frac{1}{2}\right)^2 = 25 + \frac{1}{4} - 5$$
$$(x-5)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{81}{4}$$

And voilà, the equation is now in standard form. The circle is centered at $\left(5, \frac{1}{2}\right)$ and has a radius of $\sqrt{\frac{81}{4}} = \frac{9}{2}$.

Completing the square may seem tedious, but it's actually not. You don't need to rewrite the entire equation for each step (we only do so for illustrative purposes). Normally, you would

$$\underbrace{(x-5)^2 - 25}_{\text{write this part first}} + \underbrace{\left(y - \frac{1}{2}\right)^2 - \frac{1}{4}}_{\text{and then write this part second}} = -5$$

all on one line. Then combine like terms.

EXAMPLE 7:

$$9x^2 + 9y^2 + 18x - 36y = 4$$

The graph of the given equation in the xy-plane is a circle. What is the length of the circle's radius?

Before we can convert the given equation into standard form, we need to make the equation easier to work with. Let's first divide both sides by 9 so that the coefficients of x^2 and y^2 are 1:

$$x^2 + y^2 + 2x - 4y = \frac{4}{9}$$

Now let's rearrange the terms so that the x's are together and the y's are together:

$$x^2 + 2x + y^2 - 4y = \frac{4}{9}$$

Now we're ready to complete the square, first for the x's and then for the y's:

$$\underline{(x \quad)^2} + y^2 - 4y = \frac{4}{9}$$
$$\underline{(x+1)^2 - 1} + y^2 - 4y = \frac{4}{9}$$
$$(x+1)^2 - 1 + \underline{(y \quad)^2} = \frac{4}{9}$$
$$(x+1)^2 - 1 + \underline{(y-2)^2 - 4} = \frac{4}{9}$$
$$(x+1)^2 + (y-2)^2 = 1 + 4 + \frac{4}{9}$$
$$(x+1)^2 + (y-2)^2 = \frac{49}{9}$$

Now that the equation is in standard form, we can see that the radius of the circle is $\sqrt{\frac{49}{9}} = \boxed{\frac{7}{3}}$.

Desmos Note: If completing the square is a struggle for you, don't worry too much about it. It's not tested very often on the SAT, and when it is, you can usually just solve the problem by using Desmos, which can graph the equations of circles. For example, let's say you're given the equation of a circle and asked to find its radius. Instead of completing the square, you could graph the equation and calculate the radius from the coordinates of a few select points on the graph.

EXERCISE 1: Answers for this exercise start on page 388.

1

What is the circumference of the circle in the xy-plane with equation $(x+7)^2+(y-5)^2=100$?

A) 10π
B) 20π
C) 100π
D) 200π

2

Point C lies on a circle with center O and diameter $\overline{AB}$. The measure of angle BAC is $47°$. What is the measure, in degrees, of arc BC ?

3

Which of the following equations represents a circle in the xy-plane with a radius of length 4 and center at $(0,1)$?

A) $x^2+(y-1)^2=4$
B) $(x-1)^2+y^2=4$
C) $x^2+(y-1)^2=16$
D) $(x-1)^2+y^2=16$

4

A circle in the xy-plane has the equation $(x-2)^2+(y-9)^2=169$. Which point (x,y) lies on this circle?

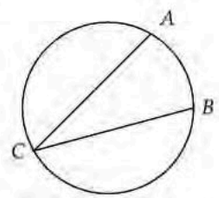
A) $(7,-3)$
B) $(7,-1)$
C) $(7,1)$
D) $(7,3)$

5

Circle A has an area of 32π and circle B has an area of 384π. The radius of circle B is how many times longer than the radius of circle A?

A) $2\sqrt{3}$
B) $3\sqrt{2}$
C) 6
D) 12

6



Note: Figure not drawn to scale.

Points A, B, and C lie on the circle shown. The measure of angle ACB is $30°$. The length of arc AB is what fraction of the circumference of the circle?

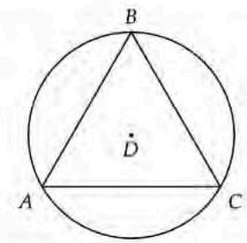
A) $\frac{1}{3}$
B) $\frac{1}{4}$
C) $\frac{1}{6}$
D) $\frac{1}{12}$

7

A circle in the xy-plane has the equation $x^2+y^2=9$. The line $y=0$ intersects this circle at two points. Which of the following is one of the points of intersection?

A) $(-3,0)$
B) $(0,-3)$
C) $(0,0)$
D) $(0,3)$

8



The circle shown has center D and area 36π. Equilateral triangle ABC is inscribed in this circle. What is the length of arc AB ?

A) 2π
B) 3π
C) 4π
D) 6π

9

Point O is the center of a circle with a radius of 10 feet. The measure of arc CD on this circle is $72°$. What is the length, in feet, of arc CD ?

A) 2π
B) 4π
C) 5π
D) 20π

10

Circle A in the xy-plane has the equation $(x-1)^2+(y+2)^2=64$. Circle B is obtained by shifting circle A up 7 units. Which of the following equations represents circle B?

A) $(x-1)^2+(y-9)^2=64$
B) $(x-1)^2+(y+9)^2=64$
C) $(x-1)^2+(y-5)^2=64$
D) $(x-1)^2+(y+5)^2=64$

11

Point P is the center of a circle. The measure of arc QR on this circle is $144°$. What is the measure, in degrees, of angle PQR ?

12

A circle in the xy-plane has the equation $(x+4)^2+(y+6)^2=k$, where k is a constant. This circle intersects the x-axis at exactly one point. What is the value of k ?

13

A circle in the xy-plane has a diameter with endpoints $(-8,-5)$ and $(-8,5)$. Which equation represents this circle?

A) $(x+8)^2+(y-5)^2=100$
B) $(x+8)^2+y^2=100$
C) $(x+8)^2+(y-5)^2=25$
D) $(x+8)^2+y^2=25$

14

$$x^2+2x+y^2-8y=64$$

The graph of the given equation in the xy-plane is a circle. What is the length of the circle's radius?

EXERCISE 2: Answers for this exercise start on page 389.

1

The point $(-2, 0)$ is the center of a circle in the xy-plane. The circle has an area of 49π. Which of the following equations represents this circle?

A) $(x-2)^2 + y^2 = 7$

B) $(x+2)^2 + y^2 = 7$

C) $(x-2)^2 + y^2 = 49$

D) $(x+2)^2 + y^2 = 49$

2

The equation $(x+3)^2 + (y-6)^2 = 9$ represents circle A. Circle B is obtained by shifting circle A to the right by 5 units in the xy-plane. What are the coordinates of the center of circle B?

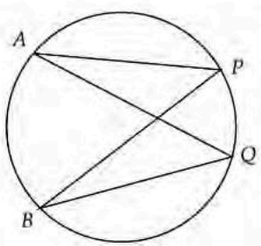
A) $(2, 6)$

B) $(2, -6)$

C) $(8, 6)$

D) $(8, -6)$

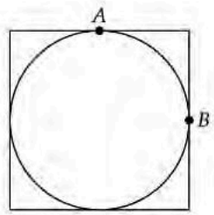
3



Note: Figure not drawn to scale.

Angles APB and AQB are inscribed in the circle shown. The measure of angle APB is $40°$. What is the measure, in degrees, of angle AQB ?

4



In the figure, a circle is inscribed in a square with a side length of 14 inches. What is the length, in inches, of arc AB ?

A) $\frac{7}{4}\pi$

B) $\frac{7}{2}\pi$

C) 7π

D) $\frac{49}{4}\pi$

5

The center of a circle in the xy-plane is at $(1, -1)$ and the point $(-5, 7)$ lies on the circle. Which of the following equations represents this circle?

A) $(x+1)^2 + (y-1)^2 = 10$

B) $(x-1)^2 + (y+1)^2 = 10$

C) $(x+1)^2 + (y-1)^2 = 100$

D) $(x-1)^2 + (y+1)^2 = 100$

6

A circle has center O, and points M and N lie on the circle. The measure of arc MN is $24°$ and the length of arc MN is 8π centimeters. What is the length, in centimeters, of the radius of the circle?

A) 15

B) 45

C) 60

D) 120

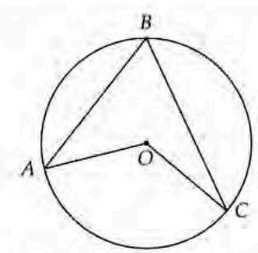
7

Triangle ABC is inscribed in a circle. The measure of angle A is equal to the measure of angle C. The measure of arc AC is $48°$. What is the measure, in degrees, of arc AB ?

8

In the xy-plane, circle A is defined by the equation $(x-3)^2 + (y-4)^2 = 16$, and circle B is defined by the equation $(x-3)^2 + (y-4)^2 = n$, where n is a constant. The diameter of circle B is 48 units longer than the diameter of circle A. What is the value of n ?

9



The circle shown has center O, and points A, B, and C lie on the circle. The measure of angle BAO is $38°$ and the measure of angle BCO is $25°$. What is the measure, in degrees, of arc AC ?

10

Which of the following equations represents a circle in the xy-plane that intersects the line $x = 5$ at exactly one point?

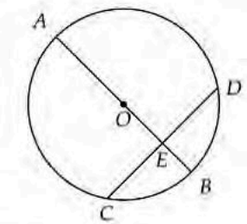
A) $(x-1)^2 + (y-7)^2 = 4$

B) $(x-5)^2 + (y-1)^2 = 4$

C) $(x-7)^2 + (y-5)^2 = 4$

D) $(x-13)^2 + (y-13)^2 = 4$

11



Note: Figure not drawn to scale.

The circle shown has center O and a radius of length 16. Diameter $\overline{AB}$ bisects $\overline{CD}$ at point E, and $CD = 28$. If the length of $\overline{OE}$ can be written in the form $2\sqrt{k}$, where k is an integer, what is the value of k ?

12

In the xy-plane, a circle with a radius of 1 has its center at $(9, -12)$. This circle can be represented by the equation $x^2 + y^2 + ax + by + c = 0$, where a, b, and c are constants. What is the value of $a + b + c$?

13

A circle in the xy-plane is defined by the equation $(x-h)^2+(y-k)^2=144$, where h and k are constants. The point $(8, 27)$ lies on this circle. Which of the following is a possible value for k ?

A) -8
B) 12
C) 39
D) 43

14

In the xy-plane, the points $(-4, 3)$ and $(8, 3)$ are endpoints of a diameter of a circle. An equation of this circle is $(x-a)^2+(y-b)^2=c^2$, where a, b, and c are positive constants. What is the value of $a+b+c$?

15

A circle has center A, and points B and C lie on the circle. Line segment BD is tangent to the circle at point B, line segment CD is tangent to the circle at point C, and the length of line segment BD is 63 units. If the circle has a radius of 16 units, what is the area, in square units, of quadrilateral $ABCD$?

16

$$x^2+y^2+7x+y=-\frac{7}{2}$$

The given equation represents a circle in the xy-plane. What is the length of the radius of the circle?

17

In the xy-plane, a circle has center D with coordinates $(m, 16)$, where m is a constant. Points E and F lie on the circle. Point E lies on the circle and has coordinates $(m+4\sqrt{10}, 34)$, and point F has coordinates $(n, 16)$, where n is a constant greater than m. What is the area, in square units, of triangle DEF ?

A) 198
B) $72\sqrt{10}$
C) 242
D) $99\sqrt{10}$

18

A circle in the xy-plane has the equation $(x-2)^2+(y-2)^2=25$. Line m is tangent to this circle at the point $(5, 6)$. What is the y-coordinate of the y-intercept of line m ?

20 Radians

A radian is simply another unit of angle measure. Just as we have feet and meters for distances and pounds and kilograms for weights, we have both degrees and radians for angles.

$$\pi \text{ radians} = 180^\circ$$

If you've never used radians before, don't be put off by the π. After all, it's just a number. We could've written

$$3.14 \text{ radians} \approx 180^\circ$$

instead, but 3.14 is only an approximation. Values are typically expressed in terms of π when radians are being used.

EXAMPLE 1: An angle has a measure of 45 degrees. What is the measure of this angle, in radians?

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$

To convert from degrees to radians, we use the conversion factor $\frac{\pi \text{ radians}}{180^\circ}$.

$$45^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4} \text{ radians}$$

Answer $\boxed{(B)}$. Notice that the degree units (represented by the circle symbols) cancel out just as they should in any conversion problem.

EXAMPLE 2: An angle has a measure of $\frac{3\pi}{2}$ radians. What is the measure of the angle, in degrees?

This time, we're going from radians to degrees, so we should flip the conversion factor we used in Example 1:

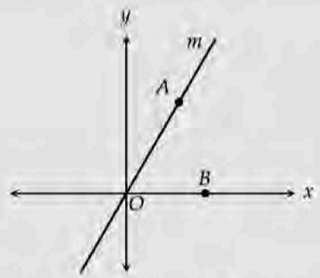
$$\frac{3\pi}{2} \text{ radians} \times \frac{180^\circ}{\pi \text{ radians}} = \boxed{270}^\circ$$

You might be wondering why we even need radians. Why not just stick with degrees? Is this another difference between the U.S. and the rest of the world, like it is with feet and meters? Nope. Some calculations are much simpler when the angles are in radians. For example, the length of an arc or the area of a sector can be found very easily when the central angle is in radians:

$$\text{Arc Length} = r\theta \qquad \text{Area of a Sector} = \frac{1}{2}r^2\theta$$

In these formulas, r is the length of the circle's radius and θ is the radian measure of the central angle. These formulas are nice to know, but you don't need to have them memorized for the SAT.

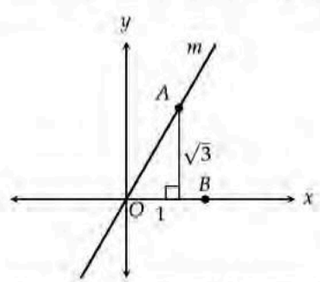
EXAMPLE 3:



In the xy-plane, point A lies on line m and point B lies on the x-axis as shown. If line m passes through the point $(0, 0)$ and has a slope of $\sqrt{3}$, what is the measure, in radians, of angle AOB?

A) $\frac{\pi}{6}$ B) $\frac{\pi}{5}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{3}$

We can draw a line segment from point A down to the x-axis to make a right triangle. Because the slope of line m is $\sqrt{3}$, the ratio of the height of this triangle to its base is always $\sqrt{3}$ to 1 (rise over run).



In other words, this right triangle has legs whose lengths are in a ratio of $\sqrt{3}$ to 1. It must therefore be a 30–60–90 triangle, in which the angle opposite the $\sqrt{3}$-side is $60°$. The measure of angle AOB is then

$$60° \times \frac{\pi \text{ radians}}{180°} = \frac{\pi}{3} \text{ radians}$$

Answer $\boxed{(D)}$.

EXERCISE: Answers for this exercise start on page 392.

1

An angle has a measure of $\frac{17\pi}{12}$ radians. What is the measure of the angle, in degrees?

2

An angle has a measure of 225 degrees. What is the measure of this angle, in radians?

A) $\frac{3\pi}{4}$
B) $\frac{7\pi}{6}$
C) $\frac{5\pi}{4}$
D) $\frac{3\pi}{2}$

3

The measure of angle A is $\frac{5\pi}{9}$ radians. The measure of angle B is $\frac{7\pi}{36}$ radians less than the measure of angle A. What is the measure of angle B, in degrees?

A) 35
B) 65
C) 100
D) 135

4

The measure of angle P is 162 degrees greater than the measure of angle R. How much greater is the measure of angle P than the measure of angle R, in radians?

A) $\frac{9}{20}\pi$
B) $\frac{9}{10}\pi$
C) $\frac{10}{9}\pi$
D) $\frac{20}{9}\pi$

5

An arc of a circle has a measure of 3.4 radians. What is the measure, in degrees, of this arc, to the nearest degree? (Disregard the degree sign when entering your answer. For example, if your answer is 36°, enter 36)

6

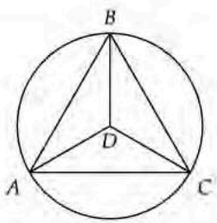
A circle has center O, and points A and B lie on the circle. The length of arc AB is $\frac{1}{5}$ of the circumference of the circle. What is the measure, in radians, of angle AOB?

A) $\frac{\pi}{10}$
B) $\frac{\pi}{5}$
C) $\frac{2\pi}{5}$
D) $\frac{5\pi}{2}$

7

An angle has a measure of 900 degrees. The number of radians in this angle can be written as $k\pi$, where k is a constant. What is the value of k ?

8



In the figure shown, equilateral triangle ABC is inscribed in a circle with center D. What is the measure, in radians, of angle ADB ?

A) $\frac{2\pi}{3}$

B) $\frac{3\pi}{4}$

C) $\frac{4\pi}{5}$

D) $\frac{5\pi}{6}$

9

Point O is the center of a circle. The measure of arc MN on this circle is $\frac{4\pi}{3}$ radians. The length of arc MN is what fraction of the circumference of the circle?

10

A circle has center O, and points X and Y lie on the circle. The measure of angle OXY is $\frac{\pi}{9}$ radians. What is the measure, in radians, of angle XOY ?

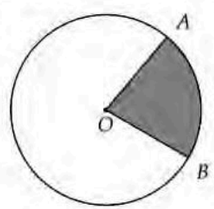
A) $\frac{5\pi}{18}$

B) $\frac{5\pi}{9}$

C) $\frac{2\pi}{3}$

D) $\frac{7\pi}{9}$

11



In the figure shown, a circle has center O and a radius of 6 inches. The area of the shaded sector is 10π square inches. What is the measure, in radians, of angle AOB ?

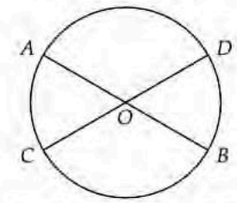
A) $\frac{2\pi}{5}$

B) $\frac{4\pi}{9}$

C) $\frac{5\pi}{9}$

D) $\frac{5\pi}{8}$

12



Note: Figure not drawn to scale.

The circle shown has center O and diameters $\overline{AB}$ and $\overline{CD}$. The length of arc AD is three times the length of arc BD. What is the measure of angle AOD, in radians?

A) $\frac{\pi}{4}$

B) $\frac{\pi}{3}$

C) $\frac{2\pi}{3}$

D) $\frac{3\pi}{4}$

13

In the xy-plane, segment OA has endpoints $(0, 0)$ and $(\sqrt{3}, \sqrt{3})$. What is the measure, in radians, of the acute angle formed by segment OA and the line $y = 0$?

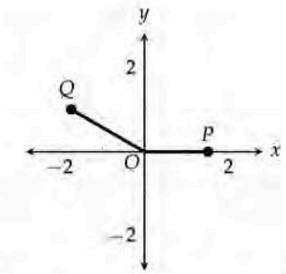
A) $\frac{\pi}{6}$

B) $\frac{\pi}{4}$

C) $\frac{\pi}{3}$

D) $\frac{\pi}{2}$

14



In the figure shown, the coordinates of point Q are $(-\sqrt{3}, 1)$. What is the measure, in radians, of angle POQ ?

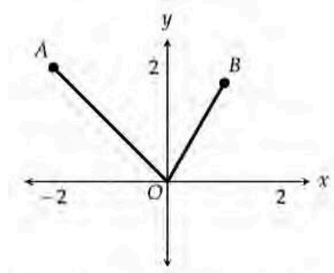
A) $\frac{2\pi}{3}$

B) $\frac{3\pi}{4}$

C) $\frac{5\pi}{6}$

D) $\frac{11\pi}{12}$

15



In the figure shown, points A and B have coordinates $(-2, 2)$ and $(1, \sqrt{3})$, respectively. What is the measure, in radians, of angle AOB ?

A) $\frac{3}{8}\pi$

B) $\frac{5}{12}\pi$

C) $\frac{4}{9}\pi$

D) $\frac{7}{12}\pi$

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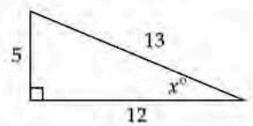
21 Trigonometry

Let's use a 5–12–13 right triangle to illustrate the functions sine, cosine, and tangent—the three trigonometric functions you'll need to know:

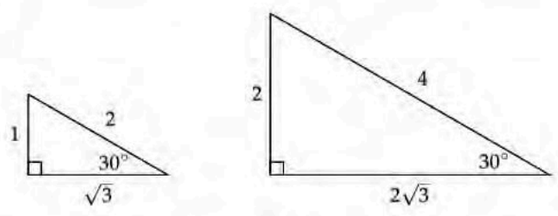


For the x° angle above,

$$\sin x^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} \qquad \cos x^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} \qquad \tan x^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$$

where "opposite" refers to the length of the side opposite the x° angle and "adjacent" refers to the length of the side next to the x° angle (the one that's not the hypotenuse).

The outputs of trigonometric functions are just ordinary numbers, so don't let the strange names confuse you. They're just ratios. For instance, the sine of 30° ($\sin 30^\circ$) is always equal to $\frac{1}{2}$. Why? Let's take a look at the 30–60–90 triangles below.



As you can see, in the triangle on the left, $\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$, and in the triangle on the right, $\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{2}{4} = \frac{1}{2}$. Because all right triangles with a 30° angle are similar, the ratios between side lengths are always the same. Therefore, $\sin 30^\circ$ is always equal to $\frac{1}{2}$.

In trigonometry, angles are often expressed in radians. For example, $\sin \frac{\pi}{6} = \frac{1}{2}$, where $\frac{\pi}{6}$ radians is 30°. On the SAT, you should assume that any trigonometric expression involving π is in radians, not degrees.

Many students over-complicate trigonometry because they view $\sin x$, $\cos x$, and $\tan x$ as different from regular numbers. Perhaps because of the notation, students sometimes make mistakes like the following:

$$\frac{\sin 2x}{x} = \sin 2$$

The operation above is not possible because $\sin 2x$ is one "entity." You cannot separate sin and $2x$ and treat them independently just like you can't mathematically separate $f(x)$ into f and x.

The definitions of sine, cosine, and tangent are best memorized through the acronym **SOH-CAH-TOA**, S for sine (opposite over hypotenuse), C for cosine (adjacent over hypotenuse), and T for tangent (opposite over adjacent).

Aside from the definitions, you should also memorize the following relationship between the sine and cosine of complementary angles (angles whose measures sum to 90°):

The sine of an angle is equal to the cosine of the complementary angle.

$$\sin x^\circ = \cos(90^\circ - x^\circ)$$

The reverse is true. The cosine of an angle is equal to the sine of the complementary angle.

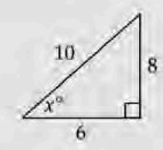
$$\cos x^\circ = \sin(90^\circ - x^\circ)$$

Expressed in radians,

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \text{and} \qquad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

where θ is the measure of the angle in radians.

EXAMPLE 1:



In the triangle shown, what is the value of $\cos x^\circ$?

A) $\frac{3}{5}$ B) $\frac{3}{4}$ C) $\frac{4}{5}$ D) $\frac{4}{3}$

$$\cos x^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$

Answer $\boxed{(A)}$.

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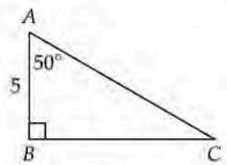
EXAMPLE 2: In triangle ABC, angle B is a right angle, angle A has a measure of $50°$, and $AB = 5$. Which of the following represents the length of $\overline{BC}$?

A) $5\cos 50°$ B) $\dfrac{5}{\cos 50°}$ C) $5\tan 50°$ D) $\dfrac{5}{\tan 50°}$

This is the classic trigonometry problem that illustrates the basic purpose of sine, cosine, and tangent: to help us find certain side lengths of a right triangle.



We're given the measure of angle A. The side we're given, $\overline{AB}$, is adjacent to angle A, and the side we're looking for, $\overline{BC}$, is opposite angle A. Now which of the three trig functions relates the opposite and adjacent sides? Tangent. So let's construct an equation using $\tan(A)$:

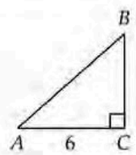
$$\tan(A) = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AB}$$

$$\tan 50° = \frac{BC}{5}$$

$$5\tan 50° = BC$$

Answer $\boxed{(C)}$. Using a calculator to get a better sense of the actual length, we get $BC = 5\tan 50° = 5.96$.

EXAMPLE 3: In triangle ABC, angle C is a right angle, $AC = 6$, and $\cos(A) = \frac{3}{4}$. The length of $\overline{BC}$ can be written as $2\sqrt{k}$, where k is a constant. What is the value of k ?



Since $\cos(A) = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{AC}{AB}$,

$$\frac{AC}{AB} = \frac{3}{4}$$

Cross-multiplying yields

$$3AB = 4AC$$

$$AB = \frac{4}{3}AC = \frac{4}{3}(6) = 8$$

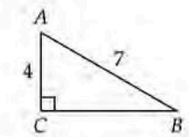
By the pythagorean theorem, $BC = \sqrt{AB^2 - AC^2} = \sqrt{8^2 - 6^2} = \sqrt{28} = 2\sqrt{7}$. Therefore, $k = \boxed{7}$.

EXAMPLE 4: In triangle ABC, $\sin(B) = \frac{4}{7}$ and angle C is a right angle. What is the value of $\cos(A)$?

A) $\dfrac{\sqrt{33}}{7}$ B) $\dfrac{4}{7}$ C) $\dfrac{33}{49}$ D) $\dfrac{16}{49}$

Solution 1: Since C is the right angle in the triangle, the measures of angles A and B must sum to $90°$. Since the cosine of an angle is equal to the sine of the complementary angle, $\cos(A) = \sin(90 - A) = \sin(B) = \frac{4}{7}$. Essentially, $\cos(A)$ is equal to $\sin(B)$, so the value of $\cos(A)$ is also $\frac{4}{7}$. Answer $\boxed{(B)}$.

Solution 2: Since $\sin(B) = \frac{4}{7}$, we can let AC, the side opposite angle B, have a length of 4 and AB, the hypotenuse, have a length of 7.



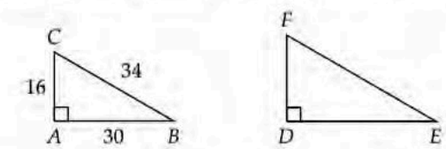
Note that these may not be the actual side lengths of triangle ABC—we're just using convenient numbers to help us derive the value of $\cos(A)$, and we're able to do so because the actual side lengths, whatever they are, must be in the same ratio. With these lengths in place, we get $\cos(A) = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{AC}{AB} = \dfrac{4}{7}$. Answer $\boxed{(B)}$. Note that $\tan(A)$ can also be found by first using the pythagorean theorem to find BC.

EXAMPLE 5:

$$AB = 30$$
$$BC = 34$$
$$AC = 16$$

The side lengths of right triangle ABC are given. Triangle ABC is similar to triangle DEF, where A corresponds to D and B corresponds to E. What is the value of $\sin(E)$?

Since BC is the longest side, it must be the hypotenuse. Angle A is then the right angle.



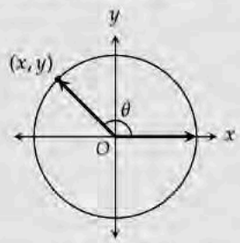
Since angle E corresponds to angle B, they must have equal measures. Thus,

$$\sin(E) = \sin(B) = \frac{AC}{BC} = \frac{16}{34} = \boxed{\frac{8}{17}}$$

As we learned at the beginning of this chapter, the sine of an angle in one right triangle is equal to the sine of the corresponding angle in a similar triangle because the ratios between side lengths are the same across similar triangles. This, of course, is true not just for sine but cosine and tangent as well.

So far, we've only looked at trig functions in the context of right triangles, which don't have any angles greater than 90 degrees. But what about angles that are greater than or equal to 90°? If sine is opposite over hypotenuse, is $\sin 135^\circ$ even possible? Yes, but only because the definitions of sine, cosine, and tangent we've been using so far are too narrow. These definitions help introduce students to trigonometry, but they are incomplete. To find the value of a trigonometric function for an obtuse angle, we have to use the following definitions:

Given an angle in standard position (its vertex is at $(0,0)$ and its initial side is along the positive x-axis) drawn on the unit circle (center at $(0,0)$ and a radius of 1),

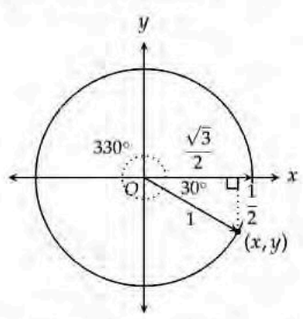


- The sine of the angle is equal to the y-coordinate of the point where the angle terminates on the unit circle.
- The cosine of the angle is equal to the x-coordinate of the point where the angle terminates on the unit circle.
- The tangent of the angle is equal to the sine of the angle divided by the cosine of the angle (the y-coordinate divided by the x-coordinate).

Let's do a couple of examples.

1. What are the values of $\sin 330^\circ$, $\cos 330^\circ$, and $\tan 330^\circ$?

   First, draw the unit circle and figure out which quadrant the angle terminates in. In this case, 330° is almost a full circle and terminates in the fourth quadrant.



The terminal side of the angle forms a $360^\circ - 330^\circ = 30^\circ$ angle with the x-axis. This acute angle is sometimes called the **reference angle**. To find the coordinates of the desired point on the unit circle, draw a line from the point to the x-axis to form a right triangle. Since the hypotenuse of this right triangle is a radius of the unit circle, its length is 1. We therefore have a 30–60–90 triangle with a hypotenuse of 1. As we've already learned, the side opposite 30° (the shorter leg) is half as long as the hypotenuse, and the side opposite 60° (the longer leg) is $\sqrt{3}$ times longer than the shorter leg. Thus, the y-coordinate is $-\frac{1}{2}$ (negative because the point is in the fourth quadrant) and the x-coordinate is $\frac{\sqrt{3}}{2}$.

The values of the three trig functions are then $\sin 330^\circ = -\frac{1}{2}$, $\cos 330^\circ = \frac{\sqrt{3}}{2}$, and

$\tan 330^\circ = \frac{\sin 330^\circ}{\cos 330^\circ} = -\frac{1}{2} \div \frac{\sqrt{3}}{2} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$. Rationalizing the denominator, we get

$\tan 330^\circ = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

2. What are the values of $\sin \frac{35\pi}{4}$, $\cos \frac{35\pi}{4}$, and $\tan \frac{35\pi}{4}$?

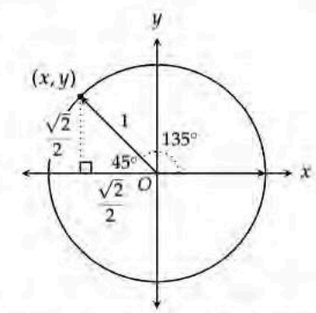
   Since the angle is in terms of π, we should assume radian measure. To make this example easier to follow, we'll convert the angle to degrees:

$$\frac{35\pi}{4} \text{ radians} \times \frac{180^\circ}{\pi \text{ radians}} = 1{,}575^\circ$$

Since angles that are 360° (2π radians) apart terminate at the same point, we can simplify this problem by continually subtracting 360° (2π radians) from the angle until we get a resulting angle less than 360° (2π radians). In this case, $1{,}575 \div 360 \approx 4$ multiples of 360° go into 1,575°, so for the purposes of this question,

$$1{,}575^\circ = 1{,}575^\circ - 4 \times 360^\circ = \underline{135^\circ}$$

which terminates in the 2nd quadrant.



Since the reference angle is $180^\circ - 135^\circ = 45^\circ$, we have a 45–45–90 triangle with a hypotenuse of 1. In any 45–45–90 triangle, the hypotenuse is $\sqrt{2}$ times longer than the legs. Therefore, the length of each leg in the triangle above is $\frac{1}{\sqrt{2}}$, or $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, when the denominator is rationalized.

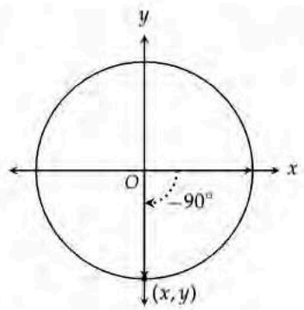
The y-coordinate of the desired point is then $\frac{\sqrt{2}}{2}$ and the x-coordinate is $-\frac{\sqrt{2}}{2}$ (negative because the point is in the second quadrant). Given these coordinates, the values of the three trig functions are

$\sin \frac{35\pi}{4} = \sin 135^\circ = \frac{\sqrt{2}}{2}$, $\cos \frac{35\pi}{4} = \cos 135^\circ = -\frac{\sqrt{2}}{2}$, and

$\tan \frac{35\pi}{4} = \tan 135^\circ = \frac{\sin 135^\circ}{\cos 135^\circ} = \frac{\sqrt{2}}{2} \div -\frac{\sqrt{2}}{2} = -1$.

3. What are the values of $\sin\left(-\frac{\pi}{2}\right)$, $\cos\left(-\frac{\pi}{2}\right)$, and $\tan\left(-\frac{\pi}{2}\right)$?

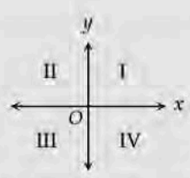
$-\frac{\pi}{2}$ radians is equivalent to $-90°$, or $-90° + 360° = 270°$. Let's look at the unit circle.



In this case, no special right triangle is needed. Since the unit circle has a radius of 1, it's obvious that the coordinates of the intercepted point are $(0, -1)$. Therefore, $\sin\left(-\frac{\pi}{2}\right) = -1$, $\cos\left(-\frac{\pi}{2}\right) = 0$, and $\tan\left(-\frac{\pi}{2}\right) = \frac{-1}{0} =$ undefined.

Desmos Note: The process we've outlined for finding trig values by hand only works when the angle terminates at a point whose coordinates are obvious or can be found using special right triangle relationships. For all other angles, such as 88°, you would have to use a calculator. And practically speaking, you should always use Desmos to find trig values on the SAT, not the unit circle. Even when the answer choices are exact values such as $\frac{\sqrt{2}}{2}$, it's still easier to compare your decimal answer from Desmos with the decimal value of each answer choice. Why did we cover the unit circle in this chapter then? Because it improves your understanding of trigonometry and illustrates concepts that could be helpful for certain SAT questions.

Lastly, when determining the value of a trig function, it's good to know beforehand what the sign of the result will be (positive or negative):



- Sine, cosine, and tangent are all positive in the first quadrant ($0° < \theta < 90°$).
- Only sine is positive in the second quadrant ($90° < \theta < 180°$).
- Only tangent is positive in the third quadrant ($180° < \theta < 270°$).
- Only cosine is positive in the fourth quadrant ($270° < \theta < 360°$).

This information is best memorized using the acronym **ASTC** (All Students Take Calculus). All the functions are positive in the first quadrant, only sine is positive in the second, and so on.

EXERCISE 1: Answers for this exercise start on page 394.

1

In triangle ABC, angle C is a right angle, $AC = 3$, and $BC = 4$. What is the value of $\tan(A)$?

A) $\frac{3}{4}$
B) $\frac{4}{5}$
C) $\frac{4}{3}$
D) $\frac{5}{3}$

2

In triangle ABC, the measure of angle C is 90° and the length of $\overline{AB}$ is 30. If $\cos(A) = \frac{5}{6}$, what is the length of $\overline{AC}$?

3

In triangle ABC, the measure of angle C is 90° and $\sin(A) = \frac{12}{13}$. What is the value of $\cos(B)$?

A) $\frac{5}{13}$
B) $\frac{12}{13}$
C) $\frac{13}{12}$
D) $\frac{13}{5}$

4

In triangle DEF, the measure of angle F is 64°, angle E is a right angle, and the length of $\overline{DF}$ is 15. Which of the following represents the length of $\overline{EF}$?

A) $15\cos 26°$
B) $15\cos 64°$
C) $\frac{15}{\cos 26°}$
D) $\frac{15}{\cos 64°}$

5

Triangle ABC is similar to triangle DEF, where angle A corresponds to angle D and angles C and F are right angles. If $\sin(B) = 0.352$, what is the value of $\sin(E)$?

6

$AB = 198$
$BC = 390$
$AC = 336$

The side lengths of right triangle ABC are given. Triangle ABC is similar to triangle PQR, where A corresponds to P and B corresponds to Q. What is the value of $\cos(Q)$?

A) $\frac{33}{65}$
B) $\frac{33}{56}$
C) $\frac{56}{65}$
D) $\frac{56}{33}$

7

In right triangle ABC, the measures of angle A and angle C sum to 90 degrees. The value of $\cos(A) = \frac{24}{25}$. What is the value of $\sin(C)$?

8

In triangle DEF, $\sin(E) = \frac{24}{74}$ and angle D is a right angle. What is the value of $\sin(F)$?

9

Triangle ABC is similar to triangle DEF, where angle A corresponds to angle D and angles C and F are right angles. If $\tan(B) = 4$, what is the value of $\tan(D)$?

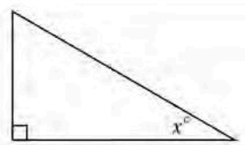
10

Triangle JKL is similar to triangle UVW, where angle J corresponds to angle U and angles K and V are right angles. If $\cos J = \frac{65}{97}$ and $\cos L = \frac{72}{97}$, what is the value of $\tan W$?

11

Triangle ABC is similar to triangle RST, where angle A corresponds to angle R and angles B and S are right angles. The length of each side of triangle ABC is 2 times the length of the corresponding side of triangle RST. If $\tan(C) = \frac{28}{45}$, what is the value of $\sin(T)$?

12



Note: Figure not drawn to scale.

In the triangle shown, $\cos(90° - x°) = \frac{8}{17}$. What is the value of $\cos x°$?

A) $\frac{8}{15}$
B) $\frac{17}{15}$
C) $\frac{8}{17}$
D) $\frac{15}{17}$

13

Triangle FGH is similar to triangle LMN, where F corresponds to L and H corresponds to N. Angles H and N are right angles. If $\cos(F) = \frac{2}{3}$ and $LM = 18$, what is the length of $\overline{MN}$?

A) 12
B) $6\sqrt{5}$
C) 15
D) $9\sqrt{5}$

EXERCISE 2: Answers for this exercise start on page 396.

1

In triangle ABC, the measure of angle A is 25°, angle C is a right angle, and the length of $\overline{BC}$ is 16. Which of the following represents the length of $\overline{AC}$?

A) $16 \sin 25°$
B) $16 \tan 25°$
C) $\frac{16}{\sin 25°}$
D) $\frac{16}{\tan 25°}$

2

In triangle ABC, the measure of angle A is 40° and B is a right angle. Which of the following additional pieces of information is sufficient to determine the length of $\overline{AB}$?

I. The length of $\overline{AC}$
II. The length of $\overline{BC}$

A) I is sufficient, but II is not.
B) II is sufficient, but I is not.
C) I is sufficient and II is sufficient.
D) Neither I nor II is sufficient.

3

In a right triangle, the sine of one of the two acute angles is $\frac{\sqrt{3}}{2}$. What is the sine of the other acute angle?

A) $\frac{1}{2}$
B) $\frac{\sqrt{3}}{2}$
C) $\frac{1}{\sqrt{3}}$
D) $\frac{\sqrt{2}}{2}$

4

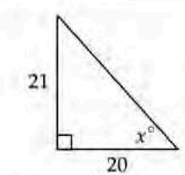
What is the value of $\sin \frac{76\pi}{3}$?

A) $\frac{\sqrt{3}}{2}$
B) $\frac{1}{2}$
C) $-\frac{1}{2}$
D) $-\frac{\sqrt{3}}{2}$

5

In right triangle RST, $\cos T = \sin 52°$. What is the measure, in degrees, of angle T ?

6



Note: Figure not drawn to scale.

In the triangle shown, what is the value of $\tan(90° - x°)$?

A) $\frac{20}{29}$
B) $\frac{20}{21}$
C) $\frac{21}{29}$
D) $\frac{21}{20}$

7

The value of $\tan \frac{49\pi}{6}$ can be written as $k\sqrt{3}$, where k is a constant. What is the value of k ?

A) $\frac{1}{3}$

B) $\frac{1}{\sqrt{3}}$

C) 1

D) 3

8

Triangle LMN is similar to triangle QRS, where L corresponds to Q and angles M and R are right angles. If $\tan(N) = \frac{5}{9}$ and $QR = 4$, what is the length of $\overline{RS}$?

9

Triangle JKL is similar to triangle RST, where angle J corresponds to angle R and angles K and S are right angles. If $\cos(L) = \frac{119}{169}$, what is the value of $\tan(T)$?

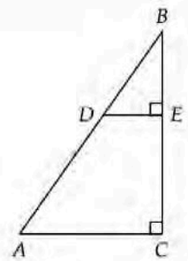
A) $\frac{120}{169}$

B) $\frac{119}{120}$

C) $\frac{120}{119}$

D) $\frac{169}{119}$

10



Note: Figure not drawn to scale.

In right triangle ABC shown, $DE = 10$ and $\tan(A) = \frac{5}{4}$. What is the length of $\overline{BE}$?

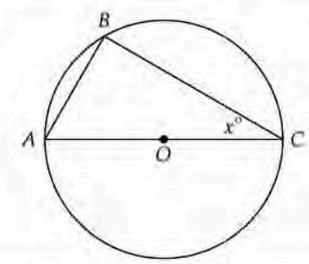
11

Triangle ABC is similar to triangle DEF, where angle A corresponds to angle D and angles B and E are right angles. If $AB = 2$, $DF = 12\sqrt{2}$, and $\tan(A) = \sqrt{7}$, what is the length of $\overline{DE}$?

12

In triangle ABC, the measure of angle A is equal to the measure of angle C. If $\cos A = \frac{3}{8}$ and $AB = 20$, what is the length of $\overline{AC}$?

13



The circle shown has center O and diameter $\overline{AC}$. Point B lies on the circle. If $AC = 1$, which of the following represents the area of triangle ABC ?

A) $\frac{\tan x}{2}$

B) $2 \sin x$

C) $\frac{\sin x \cos x}{2}$

D) $\frac{\sin x \tan x}{2}$

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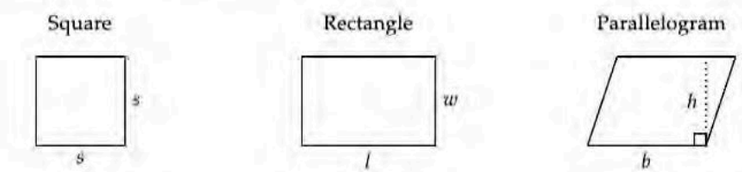
22

Area, Perimeter, & Volume

Since we've already discussed polygons, triangles, and circles in previous chapters, the focus of this chapter will be on geometrical shapes and concepts we haven't covered yet.

Squares, rectangles, and parallelograms



Square: Area $= s^2$, Perimeter $= 4s$

Rectangle: Area $= lw$, Perimeter $= 2l + 2w$

Parallelogram: Area $= bh$

Parallelograms are quadrilaterals in which opposite sides are parallel and equal in length. Rectangles are parallelograms that have four right angles. Squares are rectangles with four sides of equal length.

EXAMPLE 1: A square has a perimeter of 76 inches. What is the area of this square, in square inches?

A) 19 B) 361 C) 1,444 D) 5,776

The square must have a side length of $76 \div 4 = 19$ inches. Therefore, its area is $s^2 = 19^2 = 361$ square inches. Answer $\boxed{(B)}$.

EXAMPLE 2: A rectangle has a width that is 3 inches less than its length. If the area of the rectangle is 108 square inches, what is the perimeter, in inches, of the rectangle?

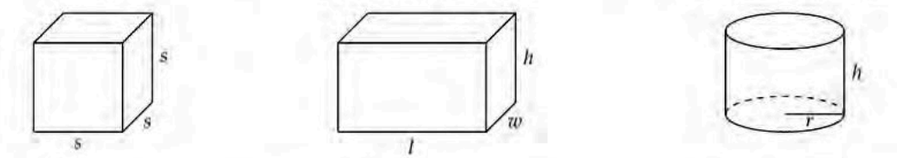
Let the length of the rectangle be l. Its width w is then $l - 3$. Since the area of the rectangle is 108 in^2,

$$lw = 108$$
$$l(l-3) = 108$$
$$l^2 - 3l - 108 = 0$$
$$(l-12)(l+9) = 0$$

Since the length of a rectangle has to be positive, $l = 12$ and $w = l - 3 = 12 - 3 = 9$. Finally, the perimeter is $2l + 2w = 2(12) + 2(9) = \boxed{42}$ inches.

Right prisms and cylinders

In geometry, a **solid** is a 3-dimensional shape. A **prism** is a solid with flat sides and two identical ends (often called bases). A **right prism** is a prism in which the sides are perpendicular to the bases. In other words, a right prism "stands up straight." The prisms you encounter on the SAT will be right prisms:



Cube: Volume $= s^3$, Surface area $= 6s^2$

Right Rectangular Prism: Volume $= lwh$, Surface area $= 2lw + 2lh + 2wh$

Right Circular Cylinder: Volume $= \pi r^2 h$

The volume of any right prism can be found using the following formula:

$$\text{Volume} = \text{Area of base} \times \text{Height}$$

This formula explains why the volume of a cube is s^3 (base area of s^2 and height of s), the volume of a right rectangular prism is lwh (base area of lw and height of h), and the volume of a right circular cylinder is $\pi r^2 h$ (base area of πr^2 and height of h). Note that a cylinder is technically not a prism by definition but the concept still applies.

The **surface area** of a solid is the total area that the surface of the solid occupies. Since a cube has 6 faces and each face has an area of s^2, the surface area of a cube is $6s^2$. A right rectangular prism, on the other hand, has 2 faces each with an area of lw, 2 faces each with an area of lh, and 2 faces each with an area of wh, which together yield a surface area of $2lw + 2lh + 2wh$. The surface area of a right circular cylinder is derived in Example 4.

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EXAMPLE 3: A right rectangular prism has a length of 15 inches and a width of 18 inches. If the volume of the prism is 7,020 cubic inches, what is the surface area, in square inches, of this prism?

First, we need to solve for the height h. Since the volume of the prism is 7,020 cubic inches,

$$lwh = 7{,}020$$
$$(15)(18)h = 7{,}020$$
$$h = 7{,}020 \div (15 \cdot 18) = 26 \text{ inches}$$

The surface area of the prism is then $2lw + 2lh + 2wh = 2(15)(18) + 2(15)(26) + 2(18)(26) = \boxed{2256}\ \text{in}^2$.

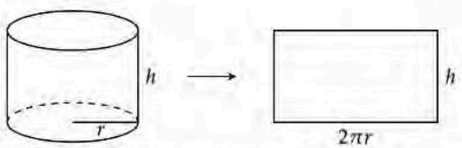
EXAMPLE 4: A right circular cylinder has a radius of 4 centimeters (cm) and a height of 5 cm. What is the surface area, in cm^2, of this cylinder?

A) 40π B) 60π C) 72π D) 81π

For the sake of learning, let's first derive the formula for the surface area of a cylinder with radius r and height h. Then we'll apply the formula to this question.

The surface area of a cylinder consists of the areas of its two bases and the area of the curved surface that goes around the cylinder. Since the two bases are circles and each circle has an area of πr^2, the two bases contribute a total surface area of $2\pi r^2$. That's the easy part.

To find the area of the curved surface that goes around the cylinder, we have to imagine that the curved surface is unraveled like wrapping paper:



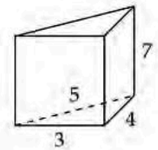
Unraveling the curved surface results in a rectangle with a length equal to the cylinder's base circumference, $2\pi r$, and a width equal to the height of the cylinder, h. The area of this rectangle is the area of the curved surface: $lw = 2\pi rh$.

Summing up the areas of the bases and the area of the curved surface, we get the following formula:

$$\text{Surface area of a cylinder} = 2\pi r^2 + 2\pi rh$$

Applying this formula to the cylinder given in the question, we get a surface area of $2\pi(4)^2 + 2\pi(4)(5) = 32\pi + 40\pi = 72\pi$. Answer $\boxed{(C)}$.

EXAMPLE 5: A triangular prism has a height of 7 inches. The base of the prism is a right triangle with sides of length 3, 4, and 5 inches. What is the volume, in cubic inches, of the prism?

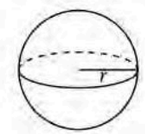


Since the base of the prism is a 3–4–5 right triangle (the 5-inch side being the hypotenuse), the area of the base is $\frac{1}{2}(\text{leg}_1)(\text{leg}_2) = \frac{1}{2}(3)(4) = 6$ square inches. The volume of the prism is then the area of the base multiplied by the height of the prism: $6 \times 7 = \boxed{42}$ cubic inches.

Spheres, Cones, and Pyramids

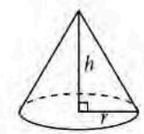
The following solids also appear on the SAT, but not as often as cubes, rectangular prisms, and cylinders. Even though the SAT gives you these formulas at the beginning of each math section, all the volume formulas presented in this chapter are worth memorizing.

Sphere



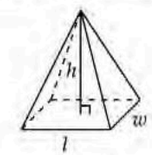
Volume $= \frac{4}{3}\pi r^3$

Cone



Volume $= \frac{1}{3}\pi r^2 h$

Square Pyramid



Volume $= \frac{1}{3}lwh$

Changes that affect area and volume

How does doubling both the length and the width of a rectangle affect its area? How does increasing the radius of a sphere by 50% affect its volume? The following examples will show you how to tackle these types of questions.

EXAMPLE 6: Circle A has a radius that is 30% longer than the radius of circle B. The area of circle A is k percent greater than the area of circle B. What is the value of k?

A) 9 B) 30 C) 36 D) 69

If we let r_a be the radius of circle A and r_b be the radius of circle B, then $r_a = 1.30r_b$ (if you don't understand why, see the Percent chapter). The area of circle A can then be expressed as

$$\pi r_a^2 = \pi(1.30r_b)^2 = \pi \cdot (1.30)^2 \cdot r_b^2 = (1.30)^2 \pi r_b^2 = 1.69\pi r_b^2$$

Note the use of parentheses when we substitute $1.30r_b$ for r_a in the first step. These parentheses are very important. In the following steps, we just rearrange the terms until there is a single coefficient, 1.69, in front of πr_b^2, the area of circle B.

This coefficient indicates that the area of circle A is 1.69 times the area of circle B. The area of circle A is therefore 69 percent greater than the area of circle B. Answer $\boxed{(D)}$. Here's another way to interpret this result—increasing the radius of a circle by 30% will increase its area by 69%.

EXAMPLE 7: Sphere P has a radius that is 5 times the radius of sphere Q. The volume of sphere P is how many times the volume of sphere Q?

If we let r_p be the radius of sphere P and r_q be the radius of sphere Q, then $r_p = 5r_q$. The volume of sphere P is then

$$\frac{4}{3}\pi r_p^3 = \frac{4}{3}\pi(5r_q)^3 = \frac{4}{3}\pi \cdot (5)^3 \cdot r_q^3 = (5)^3\left(\frac{4}{3}\pi r_q^3\right) = 125\left(\frac{4}{3}\pi r_q^3\right) = \boxed{125} \times \text{the volume of sphere Q}$$

Notice that 125 is just 5^3. The factor of 5 gets cubed because of the r^3 in the volume formula. Using this "shortcut," we can easily determine what happens in other scenarios. For example, increasing the radius of a sphere by a factor of 6 would result in a volume that is $6^3 = 216$ times larger. By reverse logic, a sphere whose volume is 216 times the volume of a smaller sphere must have a radius that is $\sqrt[3]{216} = 6$ times longer than that of the smaller sphere.

EXAMPLE 8: Cones A and B are right circular cones. Cone A has a radius that is $\frac{1}{2}$ the radius of cone B and a height that is 3 times the height of cone B. Cone B has a volume of 720 cubic inches. What is the volume of cone A, in cubic inches?

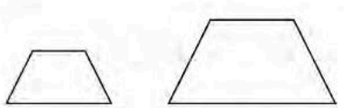
Let r_a and h_a be the radius and height of cone A, and let r_b and h_b be the radius and height of cone B. Based on the information given, $r_a = \frac{1}{2}r_b$ and $h_a = 3h_b$. The volume of cone A can then be expressed as

$$\frac{1}{3}\pi r_a^2 h_a = \frac{1}{3}\pi\left(\frac{1}{2}r_b\right)^2(3h_b) = \frac{1}{3}\pi \cdot \left(\frac{1}{2}\right)^2 \cdot r_b^2 \cdot 3 \cdot h_b = \left(\frac{1}{2}\right)^2 \cdot 3 \cdot \left(\frac{1}{3}\pi r_b^2 h_b\right) = \frac{3}{4}\left(\frac{1}{3}\pi r_b^2 h_b\right)$$

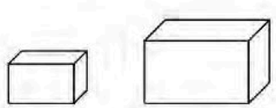
Therefore, the volume of cone A is $\frac{3}{4}$ the volume of cone B: $720 \times \frac{3}{4} = \boxed{540}$ in^3.

Similar polygons and solids

Two polygons are similar if their corresponding angles are congruent and their corresponding sides are in the same proportion. Two solids are similar if they are of the same type and their corresponding dimensions are in the same proportion. Similar polygons and similar solids are essentially different size versions of the same shape/solid.



Similar Trapezoids



Similar Rectangular Prisms

If two polygons are similar, then the ratio of their areas is equal to the **square** of the ratio of their corresponding side lengths. If two solids are similar, then the ratio of their volumes is equal to the **cube** of the ratio of their corresponding dimensions.

So let's say that triangles A and B are similar. If triangle A has side lengths that are 3 times the corresponding side lengths of triangle B (ratio of 3:1), then the area of triangle A is 9 times the area of triangle B (ratio of $3^2:1^2 = 9:1$).

EXAMPLE 9: Polygon A is similar to polygon B. Each side of polygon A has a length of 15 units, and each side of polygon B has a length of 3 units. The area of polygon A is how many times the area of polygon B?

Polygon A has sides that are $15 \div 3 = 5$ times as long as the corresponding sides of polygon B. Therefore, the area of polygon A is $5^2 = \boxed{25}$ times the area of polygon B.

EXAMPLE 10: Rectangular prisms A and B are similar. Each edge length of prism B is 12 times the corresponding edge length of prism A. The volume of prism A is 42 cubic inches. What is the volume, in cubic inches, of prism B?

Since the edges of prism B are 12 times as long as the corresponding edges of prism A, the volume of prism B must be $12^3 = 1{,}728$ times the volume of prism A. The volume of prism B is therefore $1{,}728 \times 42 = \boxed{72576}$ cubic inches.

Density is a substance's mass per unit of volume. Memorize the formula:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

Denser objects are heavier relative to their size.

EXAMPLE 11:

The volume of a sample of plastic is 1.5 cubic feet, and its mass is 126 pounds. What is the density, in pounds per cubic foot, of this sample?

A) 42 B) 84 C) 124.5 D) 189

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{126 \text{ pounds}}{1.5 \text{ cubic feet}} = 84 \text{ pounds per cubic foot}$$

Answer $\boxed{(B)}$.

EXERCISE 1: Answers for this exercise start on page 398.

1

The mass of a foam block is 8 kilograms. The volume of the block is 20 cubic meters. What is the density, in kilograms per cubic meter, of the block?

A) 0.4
B) 2.5
C) 12
D) 28

2

A cube has a volume of 42,875 cubic inches. What is the surface area, in square inches, of the cube?

3

A right square pyramid has a height of 33 centimeters (cm) and a square base with a side length of 14 cm. What is the volume, in cm^3, of this pyramid?

4

A cylinder has a volume of 980π cubic feet and a height of 5 feet. What is the base diameter, in feet, of the cylinder?

5

Square A has an area of 64 square centimeters. The perimeter of square B is 4 times the perimeter of square A. What is the side length, in centimeters, of square B?

A) 16
B) 32
C) 64
D) 128

6

Quadrilateral X is similar to quadrilateral Y, where the longest side of quadrilateral X corresponds to the longest side of quadrilateral Y. The longest side of quadrilateral X has a length of 3 inches, and the longest side of quadrilateral Y has a length of 9 inches. The area of quadrilateral Y is how many times the area of quadrilateral X?

A) 3
B) 9
C) 27
D) 36

7

A right rectangular prism has a length of 16 inches, a width of 18 inches, and a height of 20 inches. What is the surface area, in square inches, of this prism?

8

The perimeter of square A is 144 inches greater than the perimeter of square B. The side length of square A is how many inches greater than the side length of square B?

A) 12
B) 36
C) 72
D) 144

9

A rectangle has a width of $(x - 8)$ units and a length of $(x + 5)$ units. If the rectangle has an area of 68 square units, what is the value of x ?

A) 9
B) 12
C) 15
D) 17

10

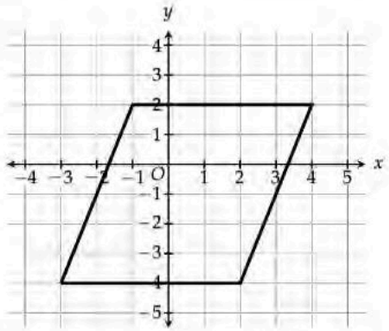
Parallelogram $ABCD$ is similar to parallelogram $PQRS$ such that A, B, C, and D correspond to P, Q, R, and S, respectively. The ratio of the length of side AB to the length of side PQ is 3 to 2. The area of $PQRS$ is 72 square units. What is the area, in square units, of $ABCD$?

A) 32
B) 108
C) 162
D) 324

11

The length of rectangle A is 54 times the length of rectangle B and the width of rectangle A is 45 times the width of rectangle B. The area of rectangle A is n times the area of rectangle B. What is the value of n ?

12



What is the area, in square units, of the parallelogram shown?

13

An object in the shape of a cube has a density of 390 grams per cubic centimeter. The object has a mass of 225 grams. To the nearest hundredth of a centimeter, what is the length of one edge of the object?

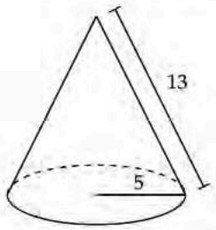
A) 0.58
B) 0.76
C) 0.83
D) 1.20

14

The volume of a sphere is equal to the volume of a cube with an edge length of 15 inches. To the nearest tenth of an inch, what is the length of the radius of the sphere?

A) 6.4
B) 9.3
C) 11.3
D) 13.6

15



The right circular cone shown has a radius of 5 inches and a slant height of 13 inches. The volume of this cone is $k\pi$ cubic inches. What is the value of k ?

16

Cylinders A and B are right circular cylinders. The radius of cylinder A is 14 times the radius of cylinder B, and the height of cylinder A is half the height of cylinder B. The volume of cylinder A is how many times the volume of cylinder B?

EXERCISE 2: Answers for this exercise start on page 399.

1

The area of square A is 576 times the area of square B. The perimeter of square A is how many times the perimeter of square B?

A) 6
B) 12
C) 24
D) 96

2

The surface area, in square inches, of a cube can be represented by the expression $24a^2$, where a is a positive constant. Which expression represents the volume, in cubic inches, of the cube?

A) $8a^3$
B) $16a^3$
C) $64a^6$
D) $216a^6$

3

A right circular cylinder is carved from a stone slab that has a density of 2.4 grams per cubic centimeter. The height of the cylinder is 4 centimeters, and the base of the cylinder has an area of 28 square centimeters. To the nearest whole number, what is the mass, in grams, of this cylinder?

A) 3
B) 17
C) 47
D) 269

4

Rectangular prism X is similar to rectangular prism Y. Each edge length of rectangular prism Y is 3 times the corresponding edge length of rectangular prism X. The volume of rectangular prism Y is 525 cubic meters. What is the volume, in cubic meters, of rectangular prism X?

5

$$f(x) = \pi x^3 + \pi x^2$$

The given function f represents the volume, in cubic feet, of a right circular cylinder with a radius of x feet. Which expression represents the height, in feet, of the cylinder?

A) $x+1$
B) πx^3
C) x
D) x^3+1

6

Triangle ABC is similar to triangle XYZ such that A, B, and C correspond to X, Y, and Z, respectively. The area of triangle ABC is 16 times the area of triangle XYZ. If the length of side AB is 68 inches, what is the length, in inches, of side XY ?

7

$$2xy + 100x + 100y$$

The given expression represents the surface area, in square centimeters, of a right rectangular prism with a length of x centimeters and a width of y centimeters. What is the height, in centimeters, of the rectangular prism?

A) 2
B) 25
C) 50
D) 100

8

$$A = x\left(\frac{75 - 2x}{2}\right)$$

The given equation represents the area A, in square inches, of a rectangle with a length of x inches. Which of the following is the best interpretation of 75 in this context?

A) The length of the rectangle is 75 inches greater than its width.
B) The sum of the length and the width of the rectangle is 75 inches.
C) The area of the rectangle cannot be greater than 75 square inches.
D) The perimeter of the rectangle is 75 inches.

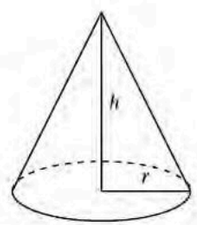
9

A rectangular image on a screen has an area of 464 square centimeters. The image is edited so that its length is 25% greater than its original length and its width is 10% less than its original width. What is the area, in square centimeters, of the edited image?

10

A rectangle has a width that is 6 inches less than 5 times its length. The area of the rectangle is 287 square inches. What is the length of the rectangle, in inches?

11



Cone M is a right circular cone with a radius of r and height of h, as shown. The volume of cone N (not shown) is 144 times the volume of cone M. Which of the following could represent the radius of cone N in terms of r and the height of cone N in terms of h?

A) radius $= 4r$, height $= 6h$
B) radius $= 4r$, height $= 36h$
C) radius $= 6r$, height $= 4h$
D) radius $= 6r$, height $= 12h$

12

A rectangular prism has a width that is 6 inches greater than its length and a height that is 12 inches less than its width. The equation $y = x(x - 6)(x + 6)$ represents this situation, where y is the volume, in cubic inches, of the rectangular prism and $y > 0$. Which of the following is the best interpretation of $x - 6$ in this context?

A) The length of the rectangular prism, in inches
B) The width of the rectangular prism, in inches
C) The height of the rectangular prism, in inches
D) The difference between the width and the length of the rectangular prism, in inches

13

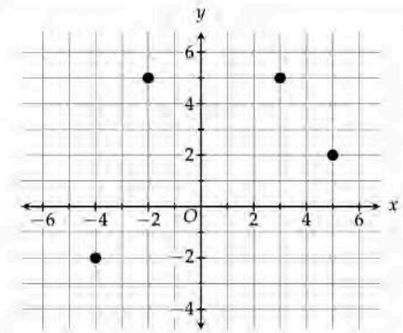
An artifact in the shape of a right circular cylinder has a volume of 112 cubic inches. A replica of the artifact is made in which the radius and height of the original artifact are each increased by 50%. What is the volume, in cubic inches, of the replica?

14

A sphere has a diameter of 10 centimeters. A solid right circular cylinder is inside the sphere, such that the two bases of the cylinder are distinct parallel cross-sections of the sphere. Each base has a diameter of 6 centimeters. To the nearest cubic centimeter, what is the volume of the space in the sphere not taken up by the cylinder?

A) 226
B) 241
C) 297
D) 354

15



A quadrilateral is formed by connecting the four points shown. What is the area, in square units, of this quadrilateral?

16

Identical rectangles X and Y each have an area of $\frac{49}{5}$ square meters and a perimeter of P meters. If one of the shorter sides of rectangle X is glued to one of the shorter sides of rectangle Y, the resulting rectangle has a perimeter of $\frac{14}{9}P$ meters. What is the value of P?

23 Reading Data

The SAT will test your ability to read and interpret bar charts, line graphs, and scatterplots. These questions tend to be easier because the graphs themselves are self-explanatory once you've seen a few examples and any math involved is generally straightforward. In fact, there's no math in this chapter that we haven't already covered, so we'll just look at one example and then it's off to the exercise.

EXAMPLE 1:

The line graph shows the value of a watch, in dollars, appraised every 3 years after it was manufactured. What is the greatest average rate of change in the value of the watch, in dollars per year, between two successive appraisals shown in the graph?

Average rate of change was discussed briefly in the "Rates" chapter. It's the change in a quantity divided by the elapsed time. In other words, it's the slope. The greater the average rate of change between two points, the greater the slope between them.

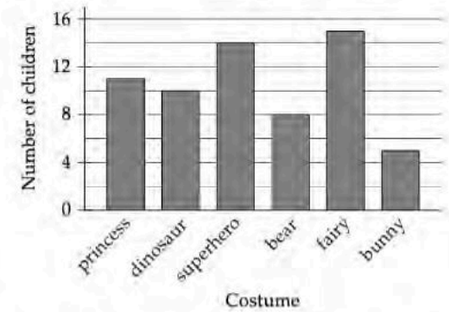
In this line graph, the segment from 9 to 12 years has the greatest slope and therefore the greatest average rate of change:

$$\text{Avg. rate of change}_{(9 \text{ to } 12)} = \frac{\text{Change in price}}{\text{Elapsed time}} = \frac{\$21{,}300 - \$20{,}100}{12 \text{ years} - 9 \text{ years}} = \frac{\$1{,}200}{3 \text{ years}} = \boxed{400} \text{ dollars per year}$$

EXERCISE: Answers for this exercise start on page 402.

1

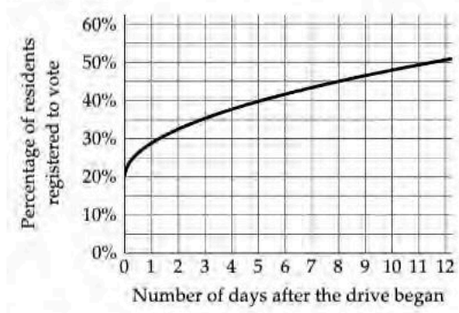
The bar graph shows the distribution of costumes worn by children at a school party.



Based on the graph, how many children did not wear a "dinosaur" costume or a "fairy" costume?

2

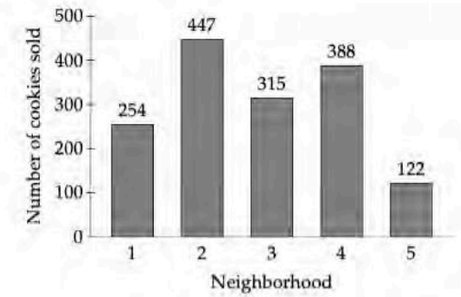
The graph shows the percentage of residents in a town registered to vote after a voter registration drive began.



Approximately how many days did it take after the registration drive began for 45% of the residents to be registered to vote?

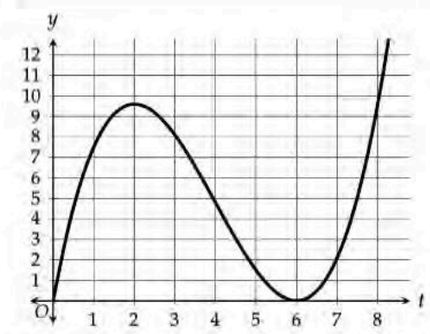
A) 3
B) 5
C) 8
D) 10

3



A local youth organization sold cookies in five different neighborhoods. The bar graph shows the number of cookies sold in each of the five neighborhoods. How many more cookies were sold in neighborhoods 2 and 4 than were sold in neighborhoods 1, 3, and 5?

4

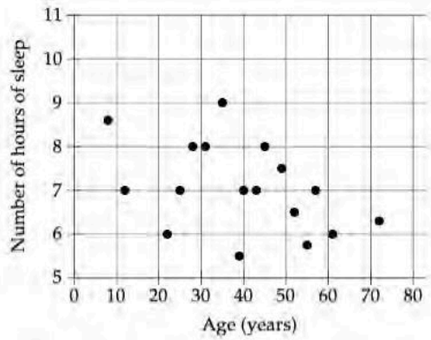


The graph shown models the kinetic energy, y, in joules, of a ball t seconds after an experiment began. For which of the following intervals of time was the kinetic energy of the ball strictly decreasing?

A) From $t = 0$ to $t = 2$
B) From $t = 1$ to $t = 4$
C) From $t = 3$ to $t = 5$
D) From $t = 4$ to $t = 7$

5

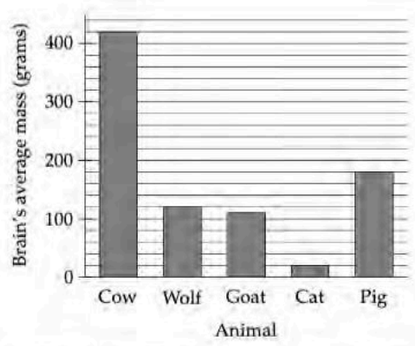
Researchers surveyed a group of 17 candidates applying for the same job. The scatterplot shows their ages and the number of hours they slept the night before the job interview.



Which of the following is the closest to the age, in years, of the candidate who slept the least number of hours the night before the job interview?

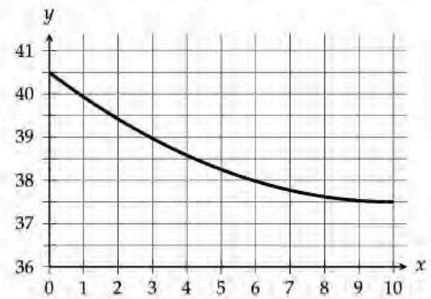
A) 35
B) 40
C) 55
D) 60

6



According to the bar graph shown, the average mass of a wolf's brain is what fraction of the average mass of a pig's brain?

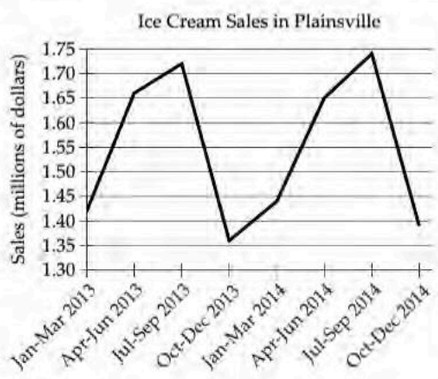
7



The graph models the body temperature y, in degrees Celsius, of a pronghorn antelope x minutes after the end of a 3-mile run, where $0 \le x \le 10$. Based on the model, which of the following is closest to the predicted body temperature, in degrees Celsius, of the pronghorn antelope at the end of its 3-mile run?

A) 36
B) 37.5
C) 39
D) 40.5

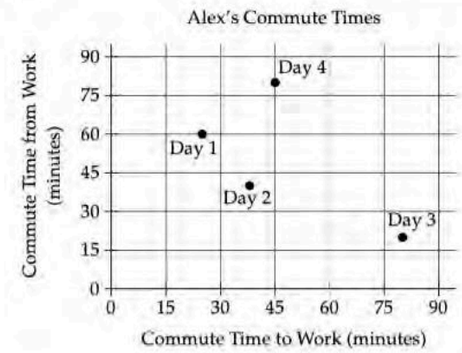
8



According to the line graph shown, ice cream sales in Plainsville were highest both in 2013 and in 2014 during which three-month period?

A) January to March
B) April to June
C) July to September
D) October to December

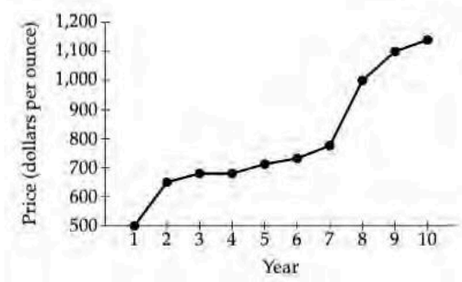
9



The scatterplot shows Alex's commute time *to* work and his commute time *from* work for four days. For which of the four days was his total commute time to and from work the greatest?

A) Day 1
B) Day 2
C) Day 3
D) Day 4

10

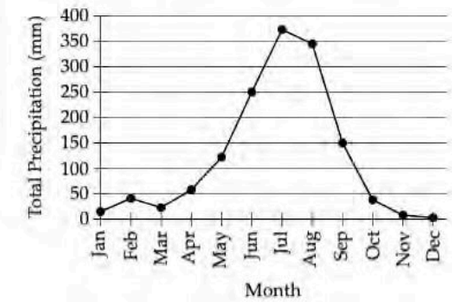


The graph shows the price of platinum, in dollars per ounce, over a 10-year period. Between which two years was the average rate of change, in dollars per ounce per year, of the price of platinum the greatest?

A) Year 1 and year 2
B) Year 3 and year 4
C) Year 7 and year 8
D) Year 8 and year 9

11

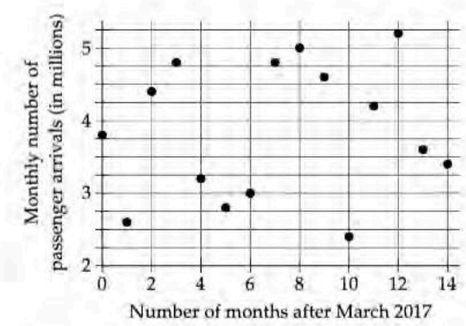
The line graph shows the monthly total precipitation in Kathmandu from January to December last year.



Based on the line graph, the total precipitation in September was what percentage of the total precipitation in June?

A) 40%
B) 50%
C) 60%
D) 75%

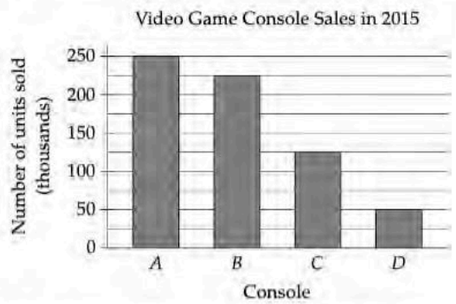
12



The graph shown models the monthly number of passenger arrivals, in millions, at a certain airport from March 2017 to May 2018. According to the model, which of the following is closest to this airport's monthly number of passenger arrivals, in millions, for March 2017?

A) 2.4
B) 3.3
C) 3.8
D) 5.2

13

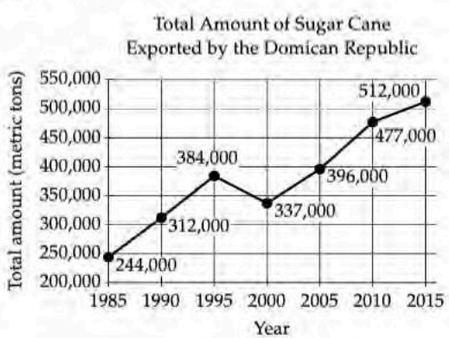


The bar graph shows the number of units sold in 2015 for four different video game consoles. The prices of consoles A, B, C, and D in 2015 were \$100, \$150, \$250, and \$300, respectively. Which of the four consoles generated the highest total revenue in 2015?

A) Console A
B) Console B
C) Console C
D) Console D

14

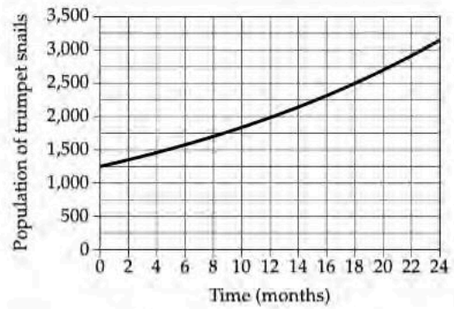
The line graph shows the total amount, in metric tons, of sugar cane exported by the Dominican Republic from 1985 to 2015.



What is the average rate of change, in metric tons per year, of the total amount of sugar cane exported by the Dominican Republic from 2005 to 2010?

15

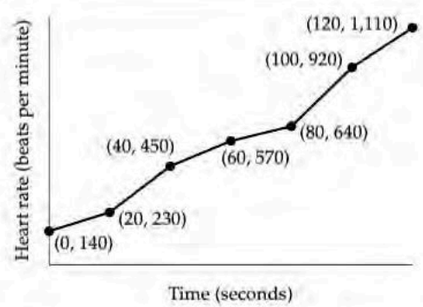
The graph shown models the population of trumpet snails in a certain area over time, in months, after an initial population was introduced.



According to the graph, how many months does it take for the population of trumpet snails in the area to double?

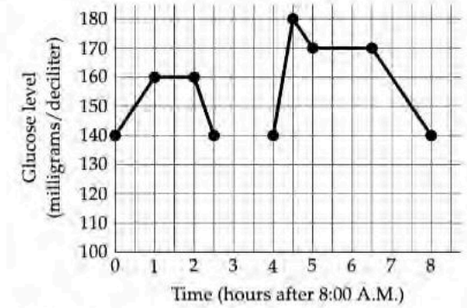
A) 5
B) 9
C) 12
D) 18

16



The line graph shows the heart rate, in beats per minute, of a hummingbird measured every 20 seconds. What is the smallest average rate of change in the hummingbird's heart rate, in beats per minute per second, between two measurements shown in the graph?

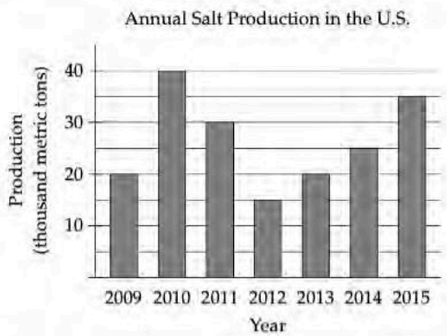
17



On the day of a medical evaluation, Greg ate breakfast at 8:00 A.M. and lunch at 12:00 P.M. During each meal, doctors recorded his glucose levels in the graph shown until they were able to calculate his *glucose recovery time*, the time it took for his body's glucose level to return to its initial value at the start of the meal. According to the graph, by how many hours was Greg's glucose recovery time after lunch greater than his glucose recovery time after breakfast?

A) 1.5
B) 2
C) 3
D) 5.5

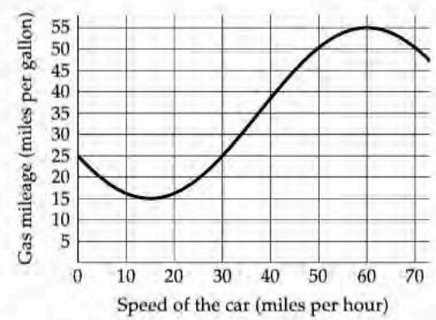
18



Based on the bar graph shown, for which of the following two consecutive years was the percent increase in U.S. annual salt production equal in absolute value to the percent decrease from 2010 to 2011?

A) 2009 to 2010
B) 2012 to 2013
C) 2013 to 2014
D) 2014 to 2015

19



The graph models the gas mileage of a car, in miles per gallon, as a function of its speed, in miles per hour. Based on the model, how many gallons of gas are needed to drive this car for 5 hours at a constant speed of 30 miles per hour?

24 Probability

Generally speaking, probability can be defined as

$$\frac{\text{Number of target outcomes}}{\text{Number of total possible outcomes}}$$

Most of the probability questions on the SAT will involve tables of data. For these table questions, probability can be more narrowly defined as

$$\frac{\text{Number in target group within group being selected from}}{\text{Number in group being selected from}}$$

EXAMPLE 1:

	Beef	Chicken
First Class	18	27
Coach	62	138

The table summarizes the distribution of dinner preferences for passengers on a particular flight. If a first class passenger is selected at random from this flight, what is the probability that the selected passenger prefers a beef dinner?

A) $\frac{9}{40}$ B) $\frac{2}{5}$ C) $\frac{3}{5}$ D) $\frac{2}{3}$

The group we're selecting from is the group of first class passengers, of which there are $18 + 27 = 45$. The target group, which must be a subset of the group we're selecting from, consists of first class passengers who prefer a beef dinner, of which there are 18. The desired probability is then

$$\frac{\text{Number of first class passengers who prefer a beef dinner}}{\text{Number of first class passengers}} = \frac{18}{45} = \frac{2}{5}$$

Answer $\boxed{(B)}$.

EXAMPLE 2: The table gives the distribution of vehicles on a lot by type and year of manufacture.

	Cars	Trucks	Total
2012	173	126	299
2013	182	143	325
2014	165	109	274
Total	520	378	898

If one of these vehicles is selected at random, which of the following is closest to the probability that the vehicle is either a car manufactured in 2012 or a truck manufactured in 2014?

A) 0.193 B) 0.314 C) 0.352 D) 0.421

In this question, we're selecting from all vehicles on the lot, a total of 898 vehicles. The target group consists of cars manufactured in 2012 and trucks manufactured in 2014, a total of $173 + 109 = 282$ vehicles. The desired probability is then

$$\frac{\text{Number of 2012 cars on the lot + Number of 2014 trucks on the lot}}{\text{Total number of vehicles on the lot}} = \frac{282}{898} \approx 0.314$$

Answer $\boxed{(B)}$.

EXAMPLE 3: The table shows the distribution of occupation and years of experience for workers in a trade union.

	Years of Experience					
	1	2	3	4	5+	Total
Electrician	15	45	20	30	50	160
Welder	25	35	55	10	40	165
Plumber	60	5	65	70	55	255
Carpenter	15	25	50	35	60	185
Total	115	110	190	145	205	765

One of these workers will be selected at random. What is the probability of selecting a plumber, given that the worker has at least 4 years of experience? (Express your answer as a decimal or fraction, not as a percent.)

In probability questions, the phrase *"given that,"* which is just another way of saying *"assuming that,"* always leads to information that narrows down the group being selected from.

In this case, it's assumed that the selected worker will be one of the ones who have at least 4 years of experience, so we're not selecting from the entire group of workers but from those who have at least 4 years of experience. The desired probability is then

$$\frac{\text{Number of plumbers who have 4+ years of experience}}{\text{Number of workers who have 4+ years of experience}} = \frac{70 + 55}{145 + 205} = \frac{125}{350} = \boxed{\frac{5}{14}}$$

EXERCISE 1: Answers for this exercise start on page 404.

1

A box contains a total of 80 cookies. A cookie is to be selected at random from the box. If the probability that a chocolate chip cookie will be selected is 0.45, how many chocolate chip cookies are in the box?

2

The table summarizes the distribution of color and size for 450 shirts.

	Small	Medium	Large	Total
Gray	55	80	30	165
Navy	45	70	20	135
Black	50	65	35	150
Total	150	215	85	450

If one of these shirts is selected at random, what is the probability of selecting a gray shirt? (Express your answer as a decimal or fraction, not as a percent.)

3

In a display case, there are 11 gold, 30 silver, 17 platinum, and 6 titanium rings. If one of these rings is selected at random, what is the probability of selecting a ring that is neither silver nor titanium?

A) $\frac{17}{64}$

B) $\frac{7}{16}$

C) $\frac{9}{16}$

D) $\frac{7}{9}$

4

The table shows the distribution of sunglass lenses in a box by type and shape.

	Photochromic	Polarized
Rectangular	9	7
Round	5	6

If one of these lenses is selected at random, what is the probability of selecting a round photochromic lens?

A) $\frac{5}{9}$

B) $\frac{5}{11}$

C) $\frac{5}{14}$

D) $\frac{5}{27}$

5

Color	Red	Blue	Black	White	Silver
Percent	20%	33%	10%	14%	

A car manufacturer produces cars in red, blue, black, white, and silver. The incomplete table shows the percentage of cars it produces in each color. If a car produced by the manufacturer is selected at random, what is the probability of selecting a red or silver car?

A) 23%

B) 33%

C) 37%

D) 43%

6

A fair die has 12 sides each labeled with a number from 1 through 12, with a different number on each side. Two students each roll this die once. The first student rolls a 3. What is the probability that the second student also rolls a 3?

A) $\frac{1}{4}$

B) $\frac{3}{4}$

C) $\frac{1}{12}$

D) $\frac{1}{144}$

7

The table shows the distribution by type and ink color of pens stored in a bin.

Type	Ink color	Number of pens
Ballpoint	Blue	67
	Black	112
	Red	41
	Green	93
Felt-tip	Blue	24
	Black	136
	Red	58

Based on the table, what fraction of the ballpoint pens stored in the bin have red ink?

A) $\frac{41}{531}$

B) $\frac{41}{313}$

C) $\frac{41}{272}$

D) $\frac{58}{313}$

8

A whole number greater than 10 but less than 20 will be selected at random. What is the probability that the selected number will be even?

A) $\frac{4}{8}$

B) $\frac{4}{9}$

C) $\frac{6}{9}$

D) $\frac{4}{10}$

9

	Violation Type			
	Speeding	Stop sign	Parking	Total
Truck	68	39	17	124
Car	83	51	26	160
Total	151	90	43	284

The table shows the distribution of driving violations recorded by a local police department by type and vehicle. Based on the table, what fraction of the recorded stop sign violations were committed by truck drivers?

A) $\frac{39}{284}$

B) $\frac{39}{124}$

C) $\frac{39}{90}$

D) $\frac{51}{90}$

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10

	Won	Lost	Total
Underdog	10	35	45
Favorite	25	5	30
Total	35	40	75

The table shows the results of 75 baseball games played by a certain team, categorized by whether the team was considered the favorite (expected to win) in the game or the underdog (expected to lose). What fraction of the games in which the team was considered the underdog did the team win?

A) $\frac{2}{5}$

B) $\frac{2}{7}$

C) $\frac{2}{9}$

D) $\frac{2}{15}$

11

A tank contains a total of n fish, of which g are guppies. If a fish is selected at random from the tank, what is the probability of selecting a fish that is not a guppy?

A) $\frac{n-g}{g+n}$

B) $\frac{n}{g+n}$

C) $1-\frac{g}{n-g}$

D) $1-\frac{g}{n}$

12

	Defective	Not defective	Total
Assembly line A	300	5,700	6,000
Assembly line B	500	3,500	4,000
Total	800	9,200	10,000

A manufacturer uses two assembly lines to produce refrigerator units. The table summarizes the results of a quality control test used to check for defective units. If a refrigerator unit that was checked during this test is selected at random, what is the probability that the unit was produced by assembly line A, given that the unit is defective?

A) $\frac{1}{20}$

B) $\frac{3}{8}$

C) $\frac{3}{5}$

D) $\frac{5}{8}$

13

Data set A: 3, 5, 13, 15, 17, 17

Data set B: 5, 9, 13, 17, 21, 25

The lists give the values in data sets A and B. If a number is selected at random from data set A, what is the probability that the selected number is not a number in data set B?

A) $\frac{1}{6}$

B) $\frac{1}{3}$

C) $\frac{1}{2}$

D) $\frac{2}{3}$

14

	$11–20	$21–30	$31+	Total
Cashmere	9	14	28	51
Wool	18	13	44	75
Cotton	40	15	9	64
Total	67	42	81	190

The table summarizes 190 scarves sold at a store by price and material. If a wool scarf is selected at random from this store, what is the probability that the scarf has a price of at least 21 dollars? (Express your answer as a decimal or fraction, not as a percent.)

15

The given table summarizes the distribution of 180 beverage items by type and sugar content, in grams.

	Less than 20 g	20–30 g	Greater than 30 g	Total
Soda	17	35	43	95
Juice	22	56	7	85
Total	39	91	50	180

One of these beverage items will be selected at random. What is the probability of selecting a juice beverage, given that the beverage's sugar content is no more than 30 grams?

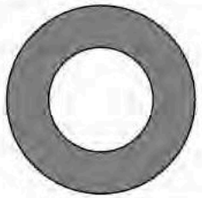
A) $\frac{7}{50}$

B) $\frac{13}{30}$

C) $\frac{3}{5}$

D) $\frac{78}{85}$

16



Note: Figure not drawn to scale.

In the figure shown, a circle with a radius of 4 inches has the same center as a circle with a radius of 6 inches. If a point within the figure is selected at random, what is the probability that the point is within the shaded region? (Express your answer as a decimal or fraction, not as a percent.)

17

Favorite flavor	Percent
Vanilla	28%
Chocolate	14%
Strawberry	24%
Cookies and cream	18%
Coffee	12%
Butter pecan	4%

The table shows the results of a survey in which respondents were asked to specify their favorite ice cream flavor. If a respondent is selected at random, what is the probability that the respondent's favorite flavor is cookies and cream, given that the respondent's favorite flavor is neither vanilla nor coffee?

A) $\frac{9}{50}$

B) $\frac{3}{10}$

C) $\frac{3}{7}$

D) $\frac{9}{20}$

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EXERCISE 2: Answers for this exercise start on page 405.

1

The table gives the distribution of residential buildings in a neighborhood by type and year of construction.

	2005	2010	2015	Total
Single-family home	24	49	41	114
Duplex	37	28	63	128
Condominium	61	53	8	122
Total	122	130	112	364

One of these buildings will be selected at random. What is the probability of selecting a building that was constructed in 2015?

A) $\frac{41}{364}$

B) $\frac{4}{13}$

C) $\frac{61}{182}$

D) $\frac{4}{9}$

2

	In stock	Out of stock
Brand A	12	21
Brand B	18	14
Brand C	11	24

The given table summarizes the distribution of 100 items at a store by brand and availability. If one brand A item is selected at random, what is the probability that the selected item is in stock?

A) $\frac{3}{25}$

B) $\frac{12}{41}$

C) $\frac{4}{11}$

D) $\frac{4}{7}$

3

In a group of 420 nurses, some specialize in pediatrics, some specialize in cardiac care, and the rest specialize in oncology. If one of these nurses is chosen at random, the probability of choosing a nurse who specializes in cardiac care is $\frac{2}{7}$ and the probability of choosing a nurse who specializes in oncology is $\frac{15}{28}$. How many nurses in the group specialize in pediatrics?

4

There are h jars of honey, m jars of mayonnaise, and p jars of peanut butter in a cupboard. If one of these jars is selected at random, which expression represents the probability of selecting either a jar of honey or a jar of peanut butter?

A) $\frac{h}{h+p}$

B) $\frac{h+p}{h+p-m}$

C) $\frac{h+p}{h+m+p}$

D) $\frac{h+p-m}{h+m+p}$

5

	Number of soil samples	Percent of samples contaminated with glyphosate
Area 1	450	8%
Area 2	550	6%

The table summarizes the results of a research study in which soil samples collected from two areas were tested for glyphosate contamination. If one of these soil samples is selected at random, what is the probability of selecting a soil sample that is contaminated with glyphosate?

A) 0.067

B) 0.069

C) 0.070

D) 0.072

6

The table shows the results of a survey in which students at three colleges were asked whether they live on campus or off campus.

	Off campus	On campus	Total
College A	335	689	1,024
College B	124	482	606
College C	596	274	870
Total	1,055	1,445	2,500

If one of the students surveyed is selected at random, which of the following probabilities has the greatest value?

A) The probability that the student lives off campus, given that the student attends College C.

B) The probability that the student lives on campus, given that the student attends College C.

C) The probability that the student attends college C, given that the student lives off campus.

D) The probability that the student attends college C, given that the student lives on campus.

7

Scientists at Companies A and B

	Has spoken at conferences before	Has not spoken at conferences before
Company A	40	80
Company B	30	120

A national conference invites scientists from certain research companies to speak on advancements in their field. The table summarizes the prior speaking experiences of scientists at two research companies that will be represented at the conference. A scientist from company A is randomly selected to be its representative at the conference, and a scientist from company B is randomly selected to be its representative. The probability that company A's representative has spoken at conferences before is how much greater than the probability that company B's representative has spoken at conferences before?

A) $\frac{2}{15}$

B) $\frac{1}{4}$

C) $\frac{3}{10}$

D) $\frac{1}{3}$

8

On Friday, a bakery produced 700 bagels per hour for the first 3 hours of operation and 300 bagels per hour for the remaining 5 hours of operation. If a bagel produced by the bakery on Friday is selected at random, what is the probability the bagel was produced during the first 4 hours of operation?

9

A new medical test is designed to give a positive indicator to a patient who has a certain virus and a negative indicator to a patient who does not. The table summarizes the results for 1,000 patients who have taken this test.

	Negative indicator	Positive indicator	Total
Has virus	30	370	400
Does not have virus	550	50	600
Total	580	420	1,000

Based on the table, what is the probability that the test gives the wrong indicator?

A) 5%
B) 8%
C) 10%
D) 12%

10

The table summarizes the distribution of age and favorite sport for 150 members of a country club.

	Squash	Tennis	Golf	Total
30–44 years	18	10	22	50
45–59 years	11	14	25	50
60+ years	21	26	3	50
Total	50	50	50	150

One of these members will be selected at random. What is the probability of selecting a member whose favorite sport is tennis, given that the member is at least 45 years of age? (Express your answer as a decimal or fraction, not as a percent.)

11

There are a total of 500 three-digit numbers from 101 to 600. If one of these numbers is chosen at random, what is the probability that the number is less than 301? (Express your answer as a decimal or fraction, not as a percent.)

12

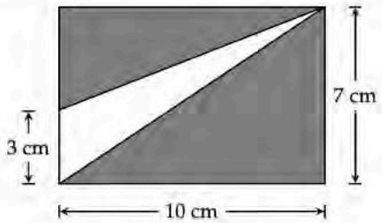
At a certain school, each student was asked whether the school's budget surplus should be spent on gym equipment or computers. The number of students who answered "computers" is 160. If a student at the school is selected at random, the probability of selecting a student who answered "gym equipment" is $\frac{1}{3}$. What is the total number of students at this school?

13

A collection of books is organized into 7 stacks of paperback books and 9 stacks of hardcover books. Each stack of paperback books consists of 4 books published in 2010 and 6 books published in 2011. Each stack of hardcover books consists of 8 books published in 2010 and 8 books published in 2011. If a book from one of these stacks is selected at random, what is the probability of selecting a paperback book, given that the book was published in 2011?

A) $\frac{21}{107}$
B) $\frac{7}{19}$
C) $\frac{3}{7}$
D) $\frac{3}{5}$

14



Note: Figure not drawn to scale.

The figure shows a rectangle and a triangle. If a point within the figure is to be selected at random, how much greater is the probability of selecting a point in the shaded region than the probability of selecting a point in the unshaded region?

A) $\frac{4}{7}$
B) $\frac{3}{5}$
C) $\frac{24}{35}$
D) $\frac{11}{14}$

15

A positive integer less than or equal to 125 is to be selected at random. What is the probability of selecting a two-digit number that is a multiple of 3? (Express your answer as a decimal or fraction, not as a percent.)

16

The table summarizes the distribution of participants in a health study by weight group.

Weight group	Percent
Less than 110 pounds	10%
110–149 pounds	25%
150–175 pounds	47%
Greater than 175 pounds	18%

If a participant in this study is selected at random, what is the probability of selecting a participant who weighs at least 150 pounds, given that the participant weighs at least 110 pounds? (Express your answer as a decimal or fraction, not as a percent.)

17

A field inhabited by plants has a side with rich soil and a side with poor soil. The table summarizes the distribution of plant species, in percent, for each side.

	Clover	Holly	Laurel	Privet	Total
Poor soil	45%	25%	20%	10%	100%
Rich soil	20%	35%	15%	30%	100%

Twice as many plants inhabit the side with rich soil as the side with poor soil. If a laurel plant is selected at random from the field, what is the probability that the laurel plant inhabits the side with rich soil?

25 Statistics I

A data set is simply a collection of values. Let data set A consist of the following 6 values:

$$5, 6, 2, 2, 2, 7$$

The **mean** of data set A is the average of its values:

$$\frac{5+6+2+2+2+7}{6} = \boxed{4}$$

The **median** of a data set is the number in the middle when the values in the data set are ordered from least to greatest. For example, the median of $\{1, 2, 3, 4, 5\}$ is 3. For data set A, the values from least to greatest are

$$2, 2, 2, 5, 6, 7$$

However, there isn't a single number in the middle that we can consider the median. That's because there's an even number of numbers in the data set. When that's the case, the median is the average of the two middle numbers, 2 and 5:

$$\text{Median of data set A} = \frac{2+5}{2} = \boxed{3.5}$$

Now what if there were 100 numbers in data set A? How would we determine the median then? Well, if we're looking for the middle number, it would make sense to take half of 100 to get 50, which designates the 50th number. But is the 50th number really the median? Probably not since there's an even number of numbers. Maybe it's the average of the 50th and 51st numbers? Or is it the average of the 49th and 50th numbers? With large data sets, it's hard to tell.

Here's the foolproof technique I use to find the median of any data set: **regardless of whether the data set has an odd or even number of numbers, we always "go up."** So for a data set of 100 numbers, we divide by 2 to get 50 and "go up" to 51. Since we have two whole numbers, 50 and 51, the median is the average the 50th and 51st numbers. For a list of 101 numbers, we divide by 2 to get 50.5 and "go up" to 51. Since we only have one whole number, 51, the median is the 51st number.

This technique may seem strange but it works. Just for reassurance, let's test it out on a data set of 3 numbers, for which the 2nd number is obviously the median, and a data set of 4 numbers, for which the average of the 2nd and 3rd numbers is the median. For a data set of 3 numbers, we take half of 3 to get 1.5, and then "go up" to 2, which confirms that the 2nd number is the median. For a data set of 4 numbers, we take half of 4 to get 2, and then "go up" to 3. The whole numbers 2 and 3 confirm that the average of the 2nd and 3rd numbers is the median.

The **mode** of a data set is the value that appears most often in the data set. For data set A, it's $\boxed{2}$.

The **range** of a data set is the difference between the largest value in the data set and the smallest value:

$$\text{Range of data set A} = 7 - 2 = \boxed{5}$$

The **standard deviation** of a data set is a measure of how spread out the values in the data set are—in other words, the extent to which they "deviate" from the mean. The standard deviation is smaller when more numbers are closer to the mean. The standard deviation is greater when more numbers are spread out further away from the mean. For example, data set A,

$$2, 2, 2, 5, 6, 7$$

would have a higher standard deviation than the data set

$$5, 5, 5, 5, 6, 7$$

because this second data set has values that are closer together overall. It turns out that the standard deviation of data set A is 2.28 and the standard deviation of this second data set is 0.83. Don't worry about how we got these values—you'll never be asked to calculate standard deviations on the SAT. You'll only be asked to compare them, as we just did.

When assessing the standard deviation of a data set, it's important to note that the actual values in the data set don't matter, only their spacing in relation to one another. For example, consider the following two data sets:

Data set 1: 100, 101, 102, 103 Data set 2: 5, 10, 15, 20

Data set 1 has a smaller standard deviation than data set 2 even though its values are much larger. That's because the values in data set 1 are closer together (spaced one unit apart) whereas the values in data set 2 are more spread out (spaced five units apart).

EXAMPLE 1:

Data set A: 26, 32, 38, 44, 50, 56

Data set B: 26, 26, 32, 38, 44, 50, 56

The values in data sets A and B are shown. Which statement correctly compares the means of data sets A and B?

A) The mean of data set B is greater than the mean of data set A.
B) The mean of data set B is less than the mean of data set A.
C) The mean of data set B is equal to the mean of data set A.
D) There is not enough information to compare the means of the two data sets

When a new value is added to a data set, it brings the mean up if it is greater than the existing mean and pulls the mean down if it is less than the existing mean.

Here, data sets A and B are identical except for the additional 26 in data set B. Because 26 is the minimum value in the data set (definitely lower than the mean), adding another 26 only serves to pull the mean down.

Therefore, the mean of data set B must be less than the mean of data set A. Answer $\boxed{(B)}$. As proof, the mean of data set A is $\frac{26+32+38+44+50+56}{6} = 41$, whereas the mean of data set B is $\frac{26+26+32+38+44+50+56}{7} \approx 38.86$.

EXAMPLE 2: The mean weight of a group of pandas is 200 pounds. A new panda weighing 230 pounds is added to the group, increasing the mean weight of the entire group to 205 pounds. How many pandas were in the original group?

When dealing with word problems involving means or averages, think in terms of sums or totals. You can always find the sum of a data set by multiplying its mean by the number of values.

First, let the number of pandas in the original group be x. These x pandas weigh a total of $200x$ pounds. After the new panda is added, the $x+1$ pandas in the expanded group weigh a new total of $205(x+1)$ pounds.

Since the new panda weighs 230 pounds,

$$\begin{aligned} 200x + 230 &= 205(x+1) \\ 200x + 230 &= 205x + 205 \\ -5x &= -25 \\ x &= 5 \end{aligned}$$

Therefore, there were $\boxed{5}$ pandas in the original group.

EXAMPLE 3:

Value	Frequency
5	6
8	5
9	2
12	4

The frequency table summarizes the values in a data set. Which of the following statements correctly compares the mean and the median of this data set?

A) The mean is 2 greater than the median.
B) The mean is 1 greater than the median.
C) The mean is equal to the median.
D) The mean is 1 less than the median.

The **frequency** of a value is the number of times that value appears in a data set. The frequency table in this example indicates that there are six 5's, five 8's, two 9's, and four 12's in the data set:

$$5, 5, 5, 5, 5, 5, 8, 8, 8, 8, 8, 9, 9, 12, 12, 12, 12$$

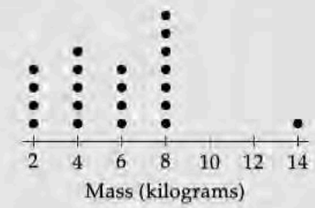
We can calculate the sum of all these values by multiplying each value by its frequency and summing the results:

$$\text{Sum of the data set} = (5 \times 6) + (8 \times 5) + (9 \times 2) + (12 \times 4) = 136$$

Since there are $6 + 5 + 2 + 4 = 17$ values in the data set, the mean is $\frac{136}{17} = 8$.

Now let's use the technique discussed earlier to find the median. The median of this data set is the $17 \div 2 = 8.5 \rightarrow$ 9th value going from smallest to largest. Counting up to 9th value in the frequency table, we count past the six 5's and land within the five 8's, so the 9th value is 8. Thus, the median is 8 and is equal to the mean. Answer $\boxed{(C)}$.

EXAMPLE 4:



The masses, in kilograms, of 21 rock samples were recorded and summarized in the dot plot shown. It was found that the sample with the greatest recorded mass actually has a mass that is 4 kilograms more than what was recorded. If the record is corrected, which of the following measures of the data will increase?

I. The mean
II. The median
III. The range

A) I only B) I and III only C) II and III only D) I, II, and III

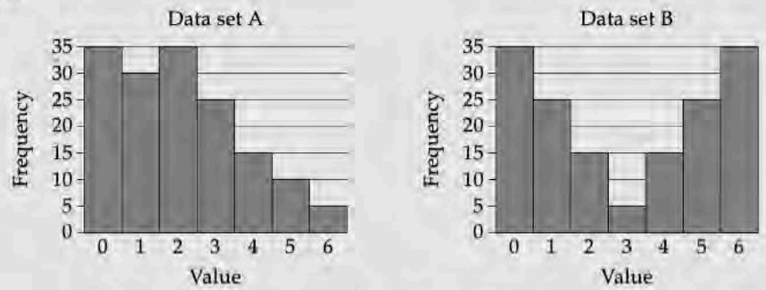
In a dot plot, the number of dots above a value represents the number of times that value appears in the data set (i.e. its frequency). A dot plot conveys the same information as a frequency table but allows us to more easily visualize how the values are spread out. In the given dot plot, the greatest recorded mass is 14 kilograms. In statistics, this data point is called an **outlier**, an extreme value that is further away from the rest of the data.

Because every value is taken into account when calculating the mean, the mean is easily affected by changes to the data. In fact, increasing any of the values in a data set increases the mean because the sum of the values in the data set increases in lockstep. Therefore, replacing the outlier value of 14 with an even greater value of 18 will undoubtedly increase the mean.

The median is less affected by changes to the data, especially changes to outliers. In this example, there are 21 values, so the median is the $21 \div 2 = 10.5 \rightarrow$ 11th value, which is 6. Changing the value of the outlier does not change the location or value of the median. Therefore, the median will remain the same.

The range is the difference between a data set's maximum and minimum value. Because the value of 14 is also the maximum of the data set, increasing this value to 18 will increase the maximum, thereby increasing the range. In the end, the mean and the range will increase, but the median will remain the same. Answer $\boxed{(B)}$.

EXAMPLE 5:



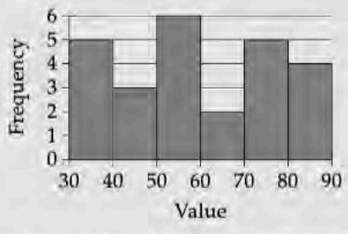
Data sets A and B are summarized in the bar graphs shown. Each data set consists of 155 values. Which of the following statements correctly compares the standard deviation of data set A and the standard deviation of data set B?

A) The standard deviation of data set A is greater than the standard deviation of data set B.
B) The standard deviation of data set A is less than the standard deviation of data set B.
C) The standard deviation of data set A is equal to the standard deviation of data set B.
D) There is not enough information to compare the standard deviations.

The bar graph for data set B is symmetrical. Because the lower values and the higher values are weighted equally, the mean of data set B is equal to its median of 3. We can estimate the mean of data set A to be roughly 2 since there are many more values from 0 to 2 than from 3 to 6.

Now most of the values in data set B are either at the low end or the high end, spread out away from its mean of 3. The values in data set A, on the other hand, are clustered towards the low end, where the mean also resides. Because the values in data set A are, on the whole, closer together and more tightly packed around their mean than those in data set B, the standard deviation of data set A is less than the standard deviation of data set B. Answer $\boxed{(B)}$.

EXAMPLE 6:



The histogram summarizes a data set consisting of 25 values. The first bar represents the frequency of values greater than or equal to 30, but less than 40. The second bar represents the frequency of values greater than or equal to 40, but less than 50, and so on. How many values in the data set are greater than or equal to 50 but less than 80?

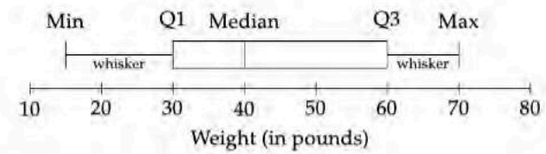
A **histogram** is a bar chart that groups data values into intervals. When it's relevant, histogram questions on the SAT will clarify whether each interval includes its starting or ending value. In this example, there are 6 values in the data set that are greater than or equal to 50 but less than 60, 2 values greater than or equal to 60 but less than 70, and 5 values greater than or equal to 70 but less than 80. Therefore, there must be

$6 + 2 + 5 = \boxed{13}$ values greater than or equal to 50 but less than 80.

Box plots

Like dot plots and histograms, box plots are just another way of visualizing numerical data. Five statistical measures of a data set are needed to construct a box plot: the minimum, the first quartile, the median, the third quartile, and the maximum.

Here's an example of a box plot that summarizes the weights of 30 tortoises:



The left and right sides of the "box" represent the first and third quartiles (Q1 and Q3), respectively, and the line segment inside the box designates the median. Line segments are drawn from the left and right sides of the box to the minimum and the maximum. These segments are sometimes called **whiskers**.

From the box plot, we can see that the minimum weight among the 30 tortoises is 15 pounds, the first quartile is 30 pounds, the median is 40 pounds, the third quartile is 60 pounds, and the maximum is 70 pounds.

The **first quartile**, also known as the lower quartile or 25th percentile, is the value for which 25% of the data set is less than. The **third quartile**, also known as the upper quartile or 75th percentile, is the value for which 75% of the data set is less than. So in this example, a first quartile of 30 pounds means that 25% of the tortoises represented in the data set weigh less than 30 pounds. A third quartile of 60 pounds means that 75% of the tortoises represented in the data set weigh less than 60 pounds.

To use another example, if your score on an exam is equal to the first quartile, then you scored better than 25% of the people who took the exam. If your score on an exam is equal to the third quartile, then you scored better than 75% of the people who took the exam.

Note that the **median** can be defined as the 50th percentile, the value for which 50% of the data set is less than.

Box plot questions on the SAT mainly test your ability to read a box plot. You will not need to know how to calculate the first and third quartiles from a raw data set.

EXERCISE 1: Answers for this exercise start on page 407.

1

Data set P: 41, 43, 45, 45, 47, 47, 51, 51, 55

Data set Q: 33, 33, 36, 38, 38, 40, 40, 40

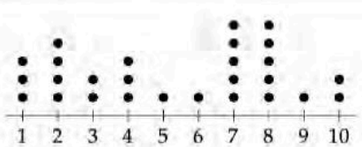
The data values for data sets P and Q are shown. The median of data set P is how much greater than the median of data set Q?

2

Value	Frequency
17	15
19	21
23	27
28	33

What is the range of the data set represented in the frequency table?

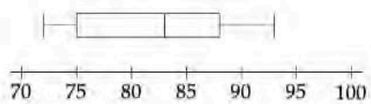
3



A data set consisting of 27 values is represented in the dot plot shown. What is the median of this data set?

4

Final Exam Scores
(out of 100 points)



The box plot summarizes the final exam scores of 26 students in a math class. Based on the box plot, which of the following best estimates the number of points by which the median score exceeds the lowest individual score?

A) 3
B) 5
C) 8
D) 11

5

The mean height of the 14 students in a particular class is 63 inches. The mean height of the 21 students in another class is 68 inches. If these two classes are combined, what is the mean height, in inches, of the students in the combined class?

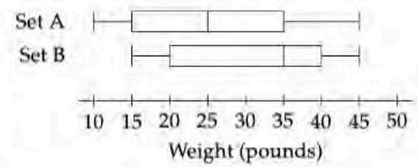
A) 64.5
B) 65
C) 66
D) 66.5

6

A food company hires an independent research agency to determine its product's shelf life, the length of time it may be stored before it expires. Using a random sample of 40 units of the product, the research agency finds that the product's shelf life has a range of 3 days. Which of the following must be true about the units in the sample?

A) All the units expired within 3 days.
B) The unit with the longest shelf life took 3 days longer to expire than the unit with the shortest shelf life.
C) The mean shelf life of the units is 3 more than the median.
D) The median shelf life of the units is 3 more than the mean.

7



The box plots summarize the weights, in pounds, of two sets of kettlebells. Which of the following statements must be true?

A) The number of kettlebells in set A is equal to the number of kettlebells in set B.
B) The median kettlebell weight in set A is equal to the median kettlebell weight in set B.
C) The range of kettlebell weights in set A is equal to the range of kettlebell weights in set B.
D) The maximum kettlebell weight in set A is equal to the maximum kettlebell weight in set B.

8

Data set A: 30, 24, 61, 37

Data set B: 77, 63, 114, x

Data set A and data set B each consist of 4 integers. If the mean of data set B is twice the mean of data set A, what is the value of x ?

9

Low temperature (°F)	Frequency
60	3
61	4
63	4
67	10
70	7

The table gives the distribution of daily low temperatures for a city over 28 days. What is the median low temperature, in degrees Fahrenheit (°F), of the city for these 28 days?

10

For which of the following data sets is the mean different than the median?

A) 3, 3, 3, 13, 13, 13, 23, 23, 23
B) 4, 4, 4, 4, 8, 16, 16, 16, 16
C) 7, 7, 7, 7, 7, 7, 7, 7, 7
D) 8, 16, 24, 32, 40, 48, 56, 64, 72

11

Data set A

Value	Frequency
30	11
50	11
70	11

The frequency table shown represents data set A. Which of the following frequency tables represents a data set that has a mean greater than the mean of data set A?

A)

Value	Frequency
30	12
50	12
70	12

B)

Value	Frequency
30	9
50	10
70	11

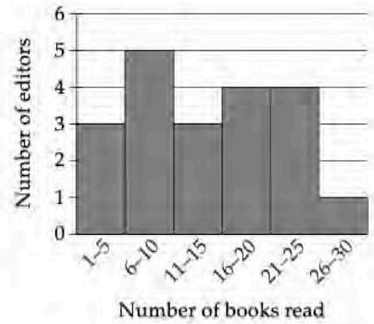
C)

Value	Frequency
30	12
50	11
70	10

D)

Value	Frequency
30	13
50	11
70	13

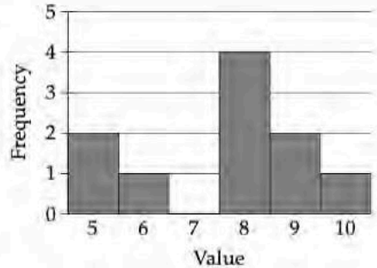
12



The histogram summarizes the number of books read last year by 20 editors at a publishing company. Which of the following could be the median number of books read by the 20 editors?

A) 9
B) 12
C) 18
D) 24

13



The bar graph shows the frequency of each value in a certain data set. Which of the following correctly gives the mean and the range of this data set?

A) Mean = 7.6, Range = 4
B) Mean = 7.6, Range = 5
C) Mean = 8.2, Range = 4
D) Mean = 8.2, Range = 5

14

91, 96, 88, 91, 84

The list shown gives the values in a data set. If a value of 100 is added to the data set, which of the following correctly describes how the mean and the median of the data set will change?

A) The mean and the median will increase.
B) The mean and the median will decrease.
C) The mean will increase, and the median will remain the same.
D) The mean will decrease, and the median will remain the same.

15

Data set A

Value	Frequency
10	2
11	3
12	7
13	3
14	9
15	5

Data set B

Value	Frequency
13	2
14	3
15	7
16	3
17	9
18	5

The tables show the frequencies of the values for data sets A and B. Which of the following statements best compares the means and the standard deviations of data sets A and B?

A) The mean of data set B is equal to the mean of data set A, and the standard deviation of data set B is equal to the standard deviation of data set A.
B) The mean of data set B is equal to the mean of data set A, and the standard deviation of data set B is greater than the standard deviation of data set A.
C) The mean of data set B is greater than the mean of data set A, and the standard deviation of data set B is equal to the standard deviation of data set A.
D) The mean of data set B is greater than the mean of data set A, and the standard deviation of data set B is greater than the standard deviation of data set A.

16

Value	Frequency
24	4
27	6
30	2
33	4
36	2

Which of the following statements correctly compares the mean and the median of the data set summarized in the frequency table?

A) The mean is 2 greater than the median.
B) The mean is 1.5 greater than the median.
C) The mean is equal to the median.
D) The mean is 1 less than the median.

17

Quiz	1	2	3	4	5	6	7
Score	87	75	90	83	98	87	91

The table shows Jay's scores on seven math quizzes. Which of the following are true about his scores?

I. The mode is greater than the median.
II. The median is greater than the mean.
III. The range is greater than 20.

A) II only
B) III only
C) II and III
D) I, II, and III

EXERCISE 2: Answers for this exercise start on page 408.

1

A plant has 40 leaves. The median length of the leaves is 5 inches. Which of the following statements must be true?

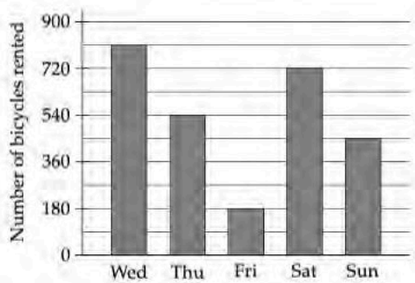
A) The sum of the lengths of the leaves is 200 inches.

B) The average of the length of the shortest leaf and the length of the longest leaf is 5 inches.

C) The difference between the length of the shortest leaf and the length of the longest leaf is 5 inches.

D) At least half of the leaves have lengths that are greater than or equal to 5 inches.

2



The bar graph shows the number of bicycles rented on each day of a five-day festival. Which statement best compares the mean and the median number of bicycles rented per day for the five days?

A) The mean is less than the median.

B) The mean is greater than the median.

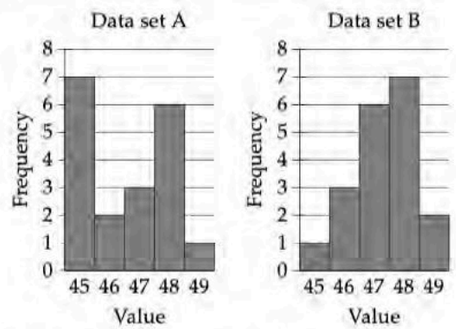
C) The mean is equal to the median.

D) There is not enough information to compare the mean and the median.

3

The mean value of the 19 pins in a collection is 98 dollars. If a chosen pin is removed from the collection, the mean value of the remaining pins is 101 dollars. What is the value, in dollars, of the chosen pin?

4



Two data sets of 19 values each are summarized in the bar graphs shown. Which of the following statements best compares the median of data set A and the median of data set B?

A) The median of data set A is less than the median of data set B.

B) The median of data set A is greater than the median of data set B.

C) The median of data set A is equal to the median of data set B.

D) There is not enough information to compare the medians of the two data sets.

5

Data set A consists of 6 data values and has a mean of 92. Data set B consists of 4 data values. If data sets A and B are combined into one data set, the mean of the combined data set is 94. What is the mean of data set B?

A) 95

B) 96

C) 97

D) 98

6

To evaluate the water quality of lakes A and B, researchers took 81 samples of water from each lake. The mean pH of the samples from lake A is 7.3, and the mean pH of the samples from lake B is 6.8. Which of the following must be true?

I. The median pH of the samples from lake A is greater than the median pH of the samples from lake B.

II. The highest pH measured among the samples from lake A is greater than the highest pH measured among the samples from lake B.

A) I only

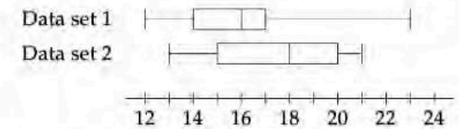
B) II only

C) I and II

D) Neither I nor II

7

Two data sets are summarized by the box plots shown.



Based on the box plots, which of the following statements must be true?

I. The mean of data set 2 is greater than the mean of data set 1.

II. The range of data set 2 is greater than the range of data set 1.

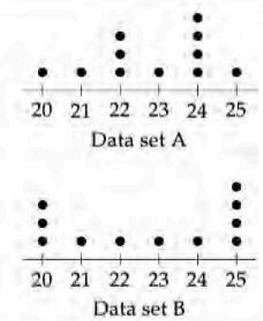
A) Neither I nor II

B) I only

C) II only

D) I and II

8



Data sets A and B are represented in the dot plots shown. Which of the following statements must be true?

I. The mean of data set B is equal to the mean of data set A.

II. The standard deviation of data set B is equal to the standard deviation of data set A.

A) I only

B) II only

C) Neither I nor II

D) I and II

9

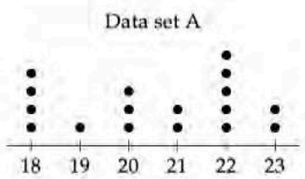
The frequency table summarizes the prices of 30 items in a shopping cart.

Price (dollars)	Frequency
25	8
40	2
45	4
55	10
70	6

If an item with a price of 25 dollars is removed from the shopping cart, what will be the effect on the mean and median price of the items in the shopping cart?

A) The mean and median will both decrease.
B) The mean and median will both increase.
C) The mean will decrease, and the median will remain the same.
D) The mean will increase, and the median will remain the same.

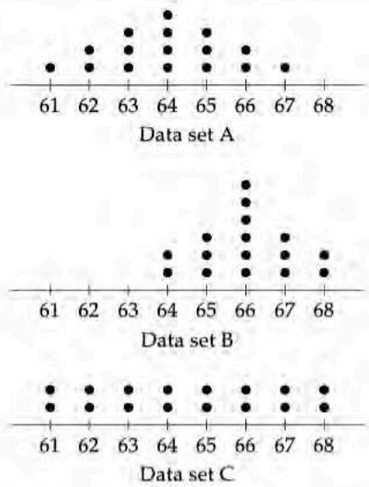
10



The dot plot shows the distribution of values in data set A. Data set B is created by multiplying each value in data set A by 2. Which of the following statements about the medians and the ranges of data sets A and B must be true?

A) The median of data set B is equal to the median of data set A, and the range of data set B is equal to the range of data set A.
B) The median of data set B is twice the median of data set A, and the range of data set B is equal to the range of data set A.
C) The median of data set B is equal to the median of data set A, and the range of data set B is twice the range of data set A.
D) The median of data set B is twice the median of data set A, and the range of data set B is twice the range of data set A.

11



The dot plots represent the distributions of values in data sets A, B, and C. The standard deviations of data sets A, B, and C are a, b, and c, respectively. Which of the following correctly compares a, b, and c?

A) $a < b < c$
B) $b < a < c$
C) $c < a < b$
D) $c < b < a$

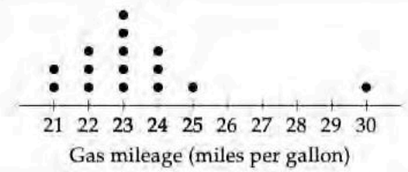
12

Data set A consists of 29 integers between 370 and 440. Data set B consists of all the integers in data set A as well as two integers between 210 and 250. Which of the following must be greater for data set A than for data set B?

I. The mean
II. The median
III. The range

A) I only
B) I and II only
C) II and III only
D) I, II, and III

13



The dot plot gives the gas mileages, in miles per gallon, of 15 different cars. If the dot representing the car with the greatest gas mileage is removed from the dot plot, which of the following measures of the data set will decrease?

I. The mean
II. The median
III. The standard deviation

A) I only
B) I and III only
C) II and III only
D) I, II, and III

14

72, 73, 76, 79, 80, 81, 85

The list gives 7 of the 8 positive integers in data set Q. The mean of these 7 integers is 78. The range of data set Q is greater than 20 but less than 30. If the mean of data set Q is an integer that is less than 78, what is the value of the smallest integer in data set Q?

15

The tables give the distribution of daily travel times between two towns for bus A and bus B over a 40-day period.

Bus A

Travel time (minutes)	Frequency
44	5
45	10
47	15
48	10

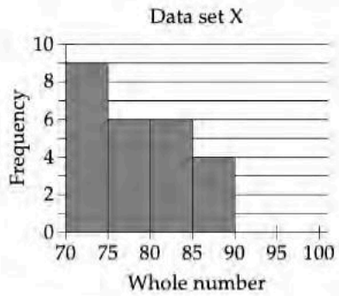
Bus B

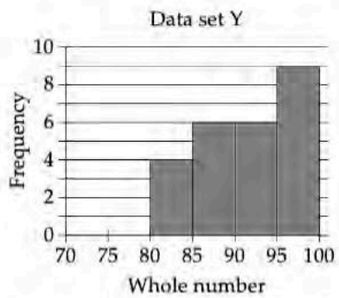
Travel time (minutes)	Frequency
25	5
30	10
35	15
40	10

Which statement correctly compares the standard deviations of travel times for bus A and bus B?

A) The standard deviation of travel times for bus A is less than the standard deviation of travel times for bus B.
B) The standard deviation of travel times for bus B is less than the standard deviation of travel times for bus A.
C) The standard deviation of travel times for bus A is equal to the standard deviation of travel times for bus B.
D) There is not enough information to compare these standard deviations.

16





The histograms shown summarize two data sets, X and Y. Each data set consists of 25 whole numbers. In each histogram, the first interval represents the frequency of whole numbers greater than or equal to 70, but less than 75. The second interval represents the frequency of whole numbers greater than or equal to 75, but less than 80, and so on. What is the largest possible difference between the median of data set Y and the mean of data set X?

17

A digital marketing agency has three groups of clients. The first group consists of 8 clients who are charged a monthly fee of at least \$170 but no more than \$190. The second group consists of 11 clients who are charged a monthly fee of at least \$200 but no more than \$230. The third group consists of 21 clients who are charged a monthly fee of at least \$240 but no more than \$300. If a new client who is charged a monthly fee of \$250 is added to the third group, what will be the effect on the mean monthly fee charged by the agency?

A) The mean will increase.

B) The mean will decrease.

C) The mean will remain the same.

D) There is not enough information to determine how the mean will change.

26 Statistics II

The goal of statistics is to be able to make predictions and estimates based on limited information. For example, we might want to find the mean weight of all raccoons in the United States. The problem is that it's impossible to survey the entire raccoon population. In fact, by the time that could be accomplished, not only would the data be out of date but there would be new raccoons in the population. The best we can do is to estimate the mean by selecting a certain number of raccoons at random from the population and weighing them. In statistics, selecting a group of members from a population is called **sampling**, and the resulting group of selected members is called a **sample**. A group of members selected at random is called a **random sample**.

In statistics, randomness is very important because 1) it ensures that each member of the population has an equal chance of being included, and 2) it minimizes bias and the risk of favoring certain characteristics or groups, which could distort the results. Using a random sample to estimate something about the entire population is a core tenet of statistics. Here are some examples of how random samples can be used:

- The mean weight of a random sample of raccoons is used to estimate the mean weight of the entire raccoon population.
- A journalist conducts a poll of a random sample of a city's residents to estimate the total number of residents who support a new law.
- A random sample of college students is surveyed to estimate the percentage of all college students who live with roommates.

On the SAT, statistical conclusions are only valid when random samples are used.

EXAMPLE 1: A gym chain has a total of 18,000 members. A manager selected 1,000 members at random and asked how many workout partners each of them had. The results of the survey are shown in the table.

Number of workout partners	0	1	2	3 or more
Number of members	350	300	200	150

Based on the table, what is the best estimate of the total number of members who have 2 workout partners?

Using the sample data, we can estimate the total number of members who have 2 workout partners to be $\frac{200}{1{,}000} \times 18{,}000 = \boxed{3{,}600}$.

EXAMPLE 2: In a study, a random sample of turtles was selected from a certain population. Based on the sample, it is estimated that the proportion of the population that has spotted shells is 0.35, with an associated margin of error of 0.06. Based on this estimate and margin of error, which of the following is the most appropriate conclusion about the proportion of the population that has spotted shells?

A) The proportion is exactly 0.35.
B) It is plausible that the proportion is less than 0.29.
C) It is not possible that the proportion is greater than 0.41.
D) It is plausible that the proportion is between 0.29 and 0.41.

As we've learned, samples of a population can be used to estimate characteristics of the whole population. Because an estimate obtained from a sample is not guaranteed to be the actual value for the population, an **associated margin of error** is often given to express how far off the sample estimate might be (i.e. the room for error). Given a margin of error, we can establish a range of values within which the actual value is likely to fall. Note that on the SAT, you will never have to calculate the margin of error. It will always be given.

In this example, the sample proportion is 0.35, with an associated margin of error of 0.06. Given this margin of error, it is plausible that the population proportion is between $0.35 - 0.06 = 0.29$ and $0.35 + 0.06 = 0.41$.

Answer $\boxed{(D)}$. Note that the word *plausible* means *likely* or *reasonable*.

Answer A is incorrect because an estimate obtained from a sample is just an estimate. It's not guaranteed to be the true value for the whole population. Answer B is incorrect because the plausible range is from 0.29 to 0.41. A proportion less than 0.29 is not plausible. Answer C is incorrect because although a proportion greater than 0.41 is not plausible, it's still *possible*. After all, luck plays a role in statistics and the chosen sample may not be representative of the whole population.

EXAMPLE 3: Based on a random sample of a population of dolphins, the mean dorsal fin height of the population is estimated to be 14.6 inches, with an associated margin of error of 1.5 inches. Based on this estimate and margin of error, which of the following is the most appropriate conclusion?

A) The majority of dolphins in the population have a dorsal fin height that is between 13.1 and 16.1 inches.
B) All dolphins in the population have a dorsal fin height of 14.6 inches.
C) It is plausible that the mean dorsal fin height of the population is between 13.1 and 16.1 inches.
D) No dolphins in the population have a dorsal fin height that is less than 13.1 inches or greater than 16.1 inches.

This example question is essentially the same as the previous one, but we have to watch out for answers that try to assign a population characteristic to the members within a population. Let me explain.

Imagine that we estimated the mean weight of your family members to be 145 pounds, with an associated margin of error of 5 pounds. The key word here is *mean*. Our estimate of the mean is not enough for us to conclude that your grandmother weighs 145 pounds or that your father weighs 145 pounds, nor is it enough to conclude that most of your family members weigh between 140 pounds and 150 pounds. After all, if half your family members weighed 50 pounds each and the other half weighed 240 pounds each, the mean weight would still be 145 pounds. The only conclusion we would be able to make is that the *mean* weight of your family members is likely between 140 pounds and 150 pounds. An estimate of the mean applies only to the mean.

In this question, answers A, B, and D may seem valid but are all incorrect because they are conclusions about the actual dorsal fin heights of the dolphins within the population, rather than the *mean* dorsal fin height.

Only answer $\boxed{(C)}$ is appropriate. Based on the given estimate and margin of error, it is plausible that the population mean is between $14.6 - 1.5 = 13.1$ inches and $14.6 + 1.5 = 16.1$ inches.

Before we move onto the next example, let's talk about **generalizability**. In statistics, the results of any survey or study can only be applied to the population from which the random sample was drawn. Here are some examples:

- In a study, a random sample of elephants in Thailand was used. The results of the study can only be generalized to all elephants in Thailand.
- For a survey, 100 pilots over the age of 45 were selected at random. The results of the survey can only be generalized to pilots over the age of 45.
- For a poll, 50 members of an alumni association were selected at random. The results of the poll can only be applied to the members of that particular alumni association.

As you've probably noticed, samples must be randomly selected. If a sample is not randomly selected, then any results based on that sample cannot be generalized to the population.

EXAMPLE 4: Malden is a town in the state of Massachusetts. In a study, a real estate agent randomly selected 50 houses in Malden and estimated that the mean house value in Malden is $750,000, with an associated margin of error of $80,000. The agent wants to repeat the study to obtain another estimate of the mean house value in Malden, but one that has a smaller margin of error. Which of the following samples would be most appropriate for the agent to use when the study is repeated?

A) 30 randomly selected houses in Malden
B) 80 randomly selected houses in Massachusetts
C) 80 randomly selected houses in Malden
D) 100 randomly selected houses in one of the neighborhoods of Malden

To obtain an estimate specific to Malden, no houses outside of Malden can be included in the sample, so we can eliminate answer B. The estimate also needs to be generalizable to all houses in Malden, not just one of its neighborhoods, so we can eliminate answer D. Only a sample of houses randomly selected from the entire town of Malden can be used.

Now when it comes to margin of error, its value is influenced by two main factors:

- Variability/Spread in the population data set (often measured by standard deviation)
- Sample size

Let's briefly talk about the first factor, even though it's typically not something we can control. Variability refers to how spread out a data set is. The higher the variability in a population data set, the higher the margin of error for any estimate based on a sample of that data set. If all the houses in Malden had a value between $700,000 and $710,000, a sample estimate of the mean value would likely have a smaller margin of error than if all the houses in Malden had a value between $100,000 and $800,000. When a population has a wide array of values, obtaining a sample that is representative of the whole population is much more difficult. Therefore, most samples result in estimates that are less accurate.

The second factor is sample size. A larger sample size reduces the margin of error. This makes sense because the more data points we include in a sample, the more representative it is of the whole population. Thus, estimates based on larger samples tend to be more accurate.

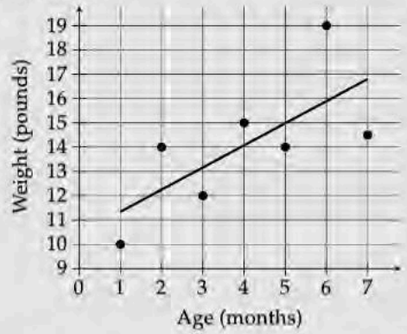
Given this discussion of sample size, it's clear that the answer to the question at hand is $\boxed{(C)}$. The real estate agent initially used a random sample of 50 houses. To achieve a smaller margin of error, a larger sample size is required. Between answers A and C, only answer C offers a sample size larger than 50.

Why don't all studies use large sample sizes? Because gathering more sample data can be too costly and time-consuming.

Scatterplots

So far, we've only dealt with univariate data (one variable). In situations where we want to estimate or predict a value based on another value, we need to work with bivariate data (two variables), which can be visualized using a scatterplot.

EXAMPLE 5: The scatterplot shows the relationship between age, in months, and weight, in pounds, for 7 infants. A line of best fit is also shown.



Based on the line of best fit, which of the following is closest to the predicted weight, in pounds, of a 5-month-old infant?

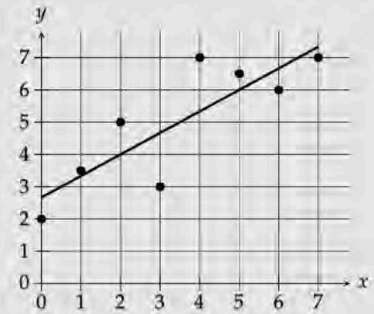
A) 12 B) 14 C) 15 D) 16

You interpret the points in a scatterplot in the same way you would interpret points in the xy-coordinate plane. For example, the data point at $(1, 10)$ represents an infant who is 1 month old and weighs 10 pounds.

Predictions from scatterplot data can be made using a linear model called the line of best fit, which is the line that best approximates all the points in the data. The line of best fit is found through a process called linear regression, which is not something the SAT will test you on. On the SAT, the line of best fit will either be shown or easily approximated by hand.

For this question, the line of best fit predicts a weight of 15 pounds ($y = 15$) for an infant who is 5 months old ($x = 5$). Answer $\boxed{(C)}$. Note that the answer is not 14 pounds—we want the *predicted* weight given by the line of best fit, not the actual weight of a 5-month-old from the data set.

EXAMPLE 6: The scatterplot shows the relationship between two variables, x and y. A line of best fit is also shown.



Which of the following equations best represents the line of best fit shown?

A) $y = 0.67x - 2.7$ B) $y = 0.67x + 2.7$ C) $y = 1.5x - 2.7$ D) $y = 1.5x + 2.7$

The y-intercept of the line of best fit is approximately 2.7. The slope can be estimated from any two points on the line. Using the points $(2, 4)$ and $(5, 6)$, we can calculate the slope to be $\frac{y_2 - y_1}{x_2 - x_1} = \frac{6-4}{5-2} = \frac{2}{3} \approx 0.67$. Given these values, the equation of the line of best fit is best represented by $y = 0.67x + 2.7$. Answer $\boxed{(B)}$.

Experimental design

EXAMPLE 7: Researchers conducted an experiment to determine whether exercise improves student exam scores. They randomly selected 200 students and tracked how much they exercised over the course of a year. At the end of the year, the researchers found that the students who did more exercise performed significantly better on the same exams than the students who did less exercise. Based on the design and results of this study, which of the following is an appropriate conclusion?

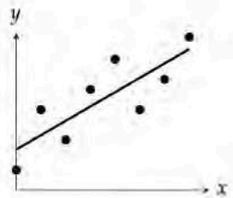
A) Exercise likely improves exam scores.
B) Students who do not exercise at all cannot improve their exam scores.
C) Any student who starts exercising more will improve his or her exam scores.
D) There is a positive association between exercise and exam scores.

This question deals with a classic case of **association** (also called correlation) vs. **causation** (cause and effect relationship). Just because the students who exercised more got better exam scores doesn't mean that exercise *causes* an improvement in exam scores. All we can conclude is that more exercise is *associated* with better exam scores. Perhaps students who exercise more tend to have more discipline. Perhaps they're more likely to have a consistent routine that allows for dedicated study time. We don't know. Due to the way the experiment was designed, we can't tell if exercise is the cause or if some other underlying factor is the cause.

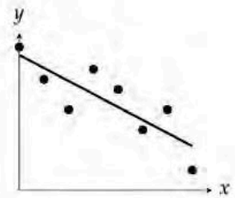
Therefore, answer A is wrong because it implies causation. Answer B is wrong because it makes an illogical generalization—the experiment wasn't about whether the students were capable of improving their exam scores. Answer C is wrong because it implies causation and makes a strong claim that is not guaranteed to be true. Even if exercise did improve exam scores, not every single student who starts exercising would improve their scores. After all, exercise is not the sole determinant of exam scores.

Thus, the answer is $\boxed{(D)}$. There is a **positive association** between exercise and exam scores.

Note that a **positive association** refers to a relationship between two variables in which an increase in one variable is associated with an increase in the other variable. A **negative association**, on the other hand, refers to a relationship between two variables in which an increase in one variable is associated with a decrease in the other variable. A positive association is often modeled by a line with a positive slope (as x increases, y also increases), and a negative association is often modeled by a line with a negative slope (as x increases, y decreases).



positive association



negative association

In the experiment, the researchers wanted to determine whether or not exercise *causes* an improvement in exam scores. What should they have done differently to prove causation rather than association? The answer is **random assignment**. Instead of tracking how much each student exercised, they should have randomly assigned each student to one of two groups: the "more exercise" group or the "less exercise" group. Everyone assigned to the "more exercise" group exercises frequently each week, and everyone assigned to the "less exercise" group exercises infrequently each week or not at all. If the "more exercise" group performs significantly better on the same exams than the "less exercise" group, then we can conclude that exercise causes an improvement in exam scores. Of course, conducting this type of experiment can be extremely difficult, which is why proving causation is usually a monumental task.

Why is random assignment needed to prove causation? Imagine that the researchers assigned all the students over the age of 15 to the "more exercise" group and all the students under the age of 15 to the "less exercise" group, and the "more exercise" group still performed significantly better on the same exams. Did they perform better because they exercised more or because they were older? Because age is now a **confounding variable** that could explain the difference in exam scores, we can't conclude that there is a causal link between exercise and exam scores. Random assignment controls for these potential confounding variables by ensuring that participants and their characteristics are equally distributed across experimental and control groups.

EXERCISE 1: Answers for this exercise start on page 411.

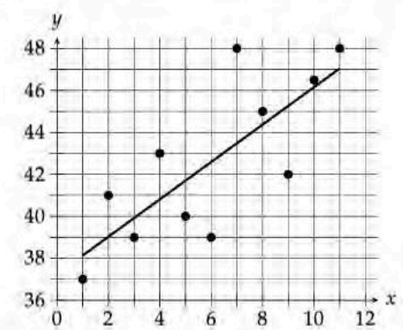
1

An online service currently has 8,400 users. A random sample of 350 users was selected and asked whether they will renew their subscription to the service. Of those surveyed, 12% stated that they will not renew their subscription. Based on this survey, which of the following is the best estimate of the total number of users who will not renew their subscription?

A) 42
B) 200
C) 1,008
D) 7,392

2

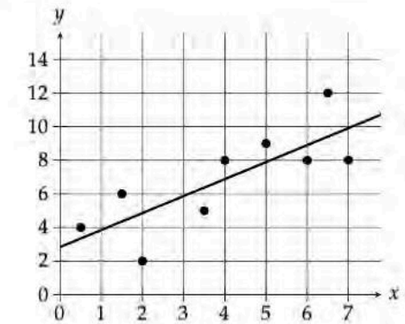
The scatterplot shows the relationship between two variables, x and y. A line of best fit is also shown.



Which of the following is closest to the predicted value of y at $x = 8.5$?

A) 42.1
B) 43.9
C) 44.8
D) 45.3

3



The scatterplot shows the relationship between two variables, x and y. A line of best fit is also shown. For how many of the data points does the line of best fit predict a y-value that is greater than the actual y-value?

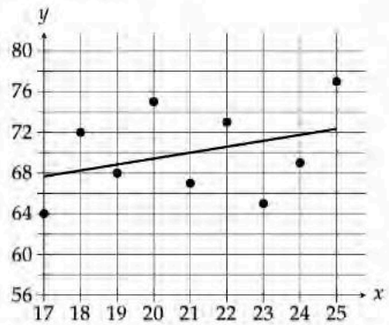
4

A random sample of 400 people from a town were interviewed about their attitudes toward community wellness programs. Based on this sample, it is estimated that 58% of the town's population supports additional funding for community wellness programs, with an associated margin of error of 6%. Which of the following is the most appropriate conclusion from these data?

A) Exactly 58% of the town's population supports additional funding for community wellness programs.
B) It is plausible that between 52% and 64% of the town's population supports additional funding for community wellness programs.
C) It is plausible that less than 52% of the town's population supports additional funding for community wellness programs.
D) It is plausible that more than 64% of the town's population supports additional funding for community wellness programs.

5

The scatterplot shows the relationship between two variables, x and y. A line of best fit for the data is also shown.



At $x = 21$, which of the following is closest to the amount by which the y-value predicted by the line of best fit exceeds the actual y-value?

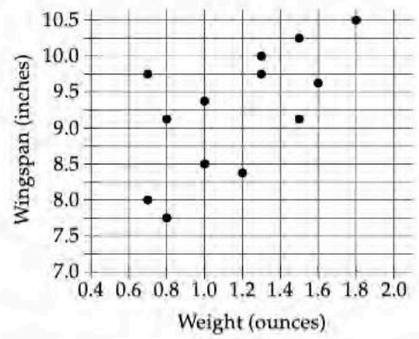
A) 1.5
B) 3
C) 4
D) 6

6

For a poll, an organizer randomly selected 21 professors from a certain liberal arts program at a university. These professors were asked whether they were interested in using a new textbook for their classes, and 14 of them responded "Yes." What is the largest group to which the results of the poll can be generalized?

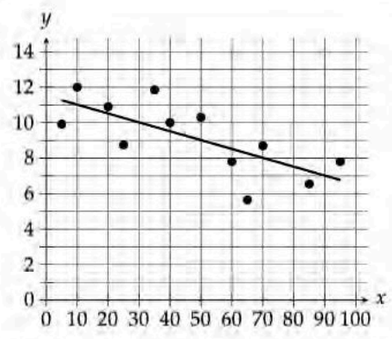
A) The 14 professors who responded "Yes"
B) All liberal arts professors at the university
C) All professors in the program
D) All professors at the university

7



The scatterplot shows the weight, in ounces, and the wingspan, in inches, for 13 sparrows. How many of these sparrows weigh less than 1.4 ounces and have a wingspan greater than 9.25 inches?

8

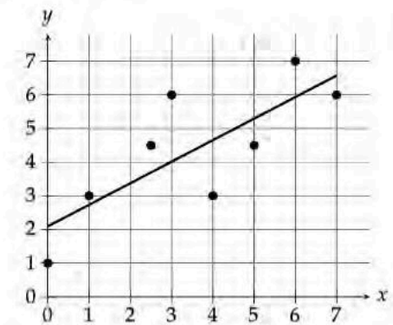


In the scatterplot, a line of best fit for the data is shown. Which of the following equations best represents this line of best fit?

A) $y = 11.5 - 0.05x$
B) $y = 11.5 - 0.5x$
C) $y = -11.5 + 0.05x$
D) $y = -11.5 + 0.5x$

9

In the given scatterplot, a line of best fit for the data is shown.



Which of the following is closest to the slope of the line of best fit?

A) -1.56
B) -0.64
C) 0.64
D) 1.56

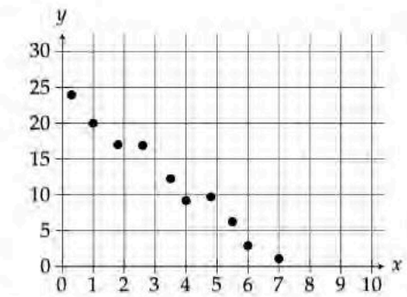
10

An economist conducted research to determine whether there is a relationship between the price of food and population density. He collected data from a random sample of 100 U.S. cities and found significant evidence that the price of food is lower in places with a high population density. Which of the following conclusions is best supported by these results?

A) In U.S. cities, there is a positive association between the price of food and population density.
B) In U.S. cities, there is a negative association between the price of food and population density.
C) In U.S. cities, a decrease in the price of food is caused by an increase in the population density.
D) In U.S. cities, an increase in the population density is caused by a decrease in the price of food.

11

The scatterplot shows the relationship between x and y.



Which of the following equations is the most appropriate linear model for the data shown?

A) $y = -0.7x - 24.2$
B) $y = -0.7x + 24.2$
C) $y = -3.4x - 24.2$
D) $y = -3.4x + 24.2$

12

In a poll, 500 people were chosen at random from a population of 36,000 and asked how many times they used public transportation in the last month. The table shows the results of the poll.

0 times	150
1 time	125
2 times	50
3 times	75
4 or more times	100

Based on these results, which of the following is the best estimate of the total number of people in the population who used public transportation at least two times in the last month?

A) 3,600
B) 9,000
C) 16,200
D) 19,800

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13

A restaurant manager asked a random sample of 100 customers how long they waited for their order. Based on the responses, the manager estimated that the mean wait time for an order is 7 minutes, with an associated margin of error of 2 minutes. Which of the following is the most appropriate conclusion from this information?

A) It is likely that most customers wait 7 minutes for their order.
B) It is likely that all customers wait between 5 and 9 minutes for their order.
C) It is plausible that the mean wait time for an order is between 5 and 9 minutes.
D) It is not possible that the mean wait time for an order is less than 5 minutes or greater than 9 minutes.

14

A researcher conducted a study to determine if certain memorization strategies could improve the performance of chess players. For the study, the researcher selected a random sample of adults who are competing in a chess tournament. At the end of the study, 80% of those sampled reported that they did not find the memorization strategies to be useful during their chess games. Which of the following is the largest population to which the results of this study can be applied?

A) All chess players who compete in tournaments
B) All adults who compete in chess tournaments
C) All chess players competing in the tournament
D) All adults competing in the tournament

15

For a quality control test, a food manufacturing company selected a random sample of 100 granola bars from a large batch it produced. Based on the results of this test, it was estimated that the mean amount of fiber per bar is 5 grams, with an associated margin of 1.4 grams. The company then repeated the test with a random sample of 200 granola bars from the same batch. If the mean and margin of error were calculated the same way for both tests, which of the following statements is most likely true?

A) The mean from the second test is smaller than the mean from the first test.
B) The mean from the second test is larger than the mean from the first test.
C) The margin of error from the second test is smaller than the margin of error from the first test.
D) The margin of error from the second test is larger than the margin of error from the first test.

16

Researchers must conduct an experiment to see whether a new vaccine is effective in relieving certain allergies. They have selected a random sample of 100 patients suffering from allergies. Some of the patients are assigned to the new vaccine while the rest are assigned to the traditional treatment. Which of the following methods of assigning each patient's treatment is most likely to lead to a reliable conclusion about the effectiveness of the new vaccine?

A) Those who have more than one allergy are assigned to the new vaccine.
B) The median weight of the patients is found. Those who weigh less than the median weight are assigned to the new vaccine.
C) The patients divide themselves evenly into two groups. A coin is tossed to decide which group is assigned to the new vaccine.
D) Each patient is given a random number. Those with an even number are assigned to the new vaccine.

EXERCISE 2: Answers for this exercise start on page 412.

1

A city has a total of 75,300 voters. A journalist polled a random sample of 900 voters and found that 660 of those polled will vote for the current mayor in the next election. Based on this poll, what is the expected total number of city voters who will vote for the current mayor in the next election?

A) 20,080
B) 48,640
C) 55,220
D) 62,360

2

A group of school administrators wanted to gather opinions on new teacher certification requirements that were introduced in a certain state. The group surveyed a random sample of math teachers from one of the school districts in the state. The survey showed that the majority of the teachers in the sample support the new requirements. Which of the following is the largest population to which the results of the survey can be generalized?

A) All math teachers in the school district
B) All teachers in the school district
C) All math teachers in the state
D) The teachers in the sample

3

A nutritionist wants to conduct a survey to estimate the mean number of servings of vegetables eaten by a certain group of people each day. The nutritionist has not yet decided which group of people to survey. Surveying a random sample from which of the following groups would most likely result in the smallest margin of error?

A) Residents of the same city
B) Customers of a certain restaurant
C) Viewers of the same television show
D) Students who are following the same daily diet plan

4

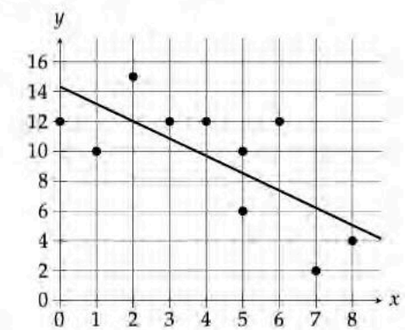
In a study, a biologist selected a random sample of 125 crabs from a local population and found that the mean carapace width of the crabs in the sample was 5.4 inches. Based on this information, which of the following statements must be true?

I. The mean carapace width of all the crabs in the local population is 5.4 inches.
II. If another random sample of 125 crabs were selected from the local population, the mean carapace width of the crabs in the sample would be 5.4 inches.

A) I only
B) II only
C) Neither I nor II
D) Both I and II

5

The scatterplot shows the relationship between two variables, x and y. A line of best fit is also shown.



For how many of the data points is the actual y-value at least 2 greater than the y-value predicted by the line of best fit?

A) 1
B) 2
C) 3
D) 4

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6

In a psychological study, 600 college gymnasts were randomly selected and put in high-stakes competitions to determine whether they performed better under pressure. Based on the results of this study, it is estimated that 65% of all college gymnasts perform better under pressure, with an associated margin of error of 4%. Which of the following inferences can appropriately be drawn based on the given margin of error?

A) The percentage of all college gymnasts who perform better under pressure is between 65% and 69%.

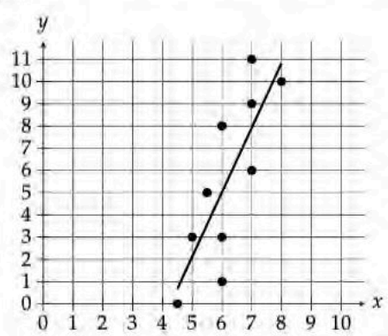
B) Approximately 4% of the college gymnasts in the study who performed better under pressure do not actually perform better under pressure.

C) It is unlikely that the percentage of all college gymnasts who perform better under pressure is 60%.

D) It is not possible that the percentage of all college gymnasts who perform better under pressure is 70%.

7

The scatterplot shows the relationship between two variables, x and y. A line of best fit is also shown.



Which equation best represents the line of best fit shown?

A) $y = -2.9x + 12.4$

B) $y = 2.9x + 12.4$

C) $y = -2.9x - 12.4$

D) $y = 2.9x - 12.4$

8

Two candidates are running for governor of a state. A recent poll reports that out of a random sample of 250 voters, 110 support candidate A and 140 support candidate B. An estimated 500,000 state residents are expected to vote on election day. According to the poll, candidate B is expected to receive how many more votes than candidate A?

A) 60,000

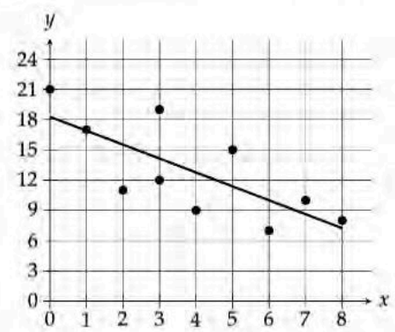
B) 130,000

C) 220,000

D) 280,000

9

The scatterplot shows the relationship between two variables, x and y. A line of best fit is also shown.



Which of the following is closest to the slope of the line of best fit shown?

A) −2.16

B) −1.38

C) −0.72

D) −0.46

10

All the residents of a town were mailed a survey about proposed renovations to the town's high school. Approximately 10% of the residents completed the survey and mailed it back. Based on the responses, it was estimated that 72% of the residents support the proposed renovations, with an associated margin of error of 12%. Which of the following best explains why these survey results are unlikely to represent the opinions of all the town's residents?

A) The survey included residents who do not attend the high school.

B) Not all residents were aware of the proposed renovations before the survey.

C) Responses to the survey were not obtained from a random sample of the town's residents.

D) The margin of error is greater than the percentage of residents who responded to the survey.

11

A company with 900 employees selected a random sample of 300 employees and asked each of them, "Are you satisfied with the company's training materials?" Based on the sample, it is estimated that 82% of the company's employees are satisfied with the company's training materials, with an associated margin of error of 6%. Which of the following is the most plausible conclusion?

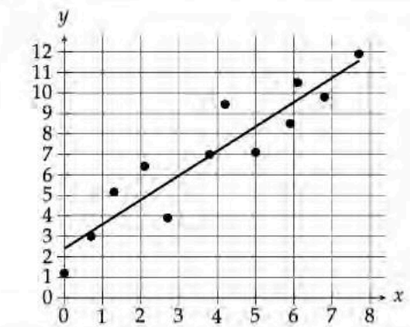
A) Between 228 and 264 employees at the company are satisfied with the company's training materials.

B) Between 684 and 792 employees at the company are satisfied with the company's training materials.

C) Because the sample included employees who are not satisfied with the company's training materials, no conclusion can be made.

D) Because the sample did not include all the employees who are satisfied with the company's training materials, no conclusion can be made.

12



The scatterplot shows the relationship between two variables, x and y, for data set A. A line of best fit is shown. Data set B is created by adding 1.6 to the y-coordinate of each data point in data set A. Which of the following equations best represents a line of best fit for data set B?

A) $y = 2.4 + 1.2x$

B) $y = 2.4 + 2.8x$

C) $y = 4 + 1.2x$

D) $y = 4 + 2.8x$

13

A researcher wanted to determine whether studying at a library is more effective than studying at home. He selected a random sample of 200 students and gave them three days to study for the same test. The 100 students who lived closest to a library were asked to study at a library, and the remaining 100 students were asked to study at home. After the students took the test, the scores of those who studied at a library were compared to the scores of those who studied at home. How should this experiment be changed so that the researcher can conclude whether studying at a library is more effective than studying at home?

A) The students assigned to study at a library should study at the same library.

B) The students should be randomly assigned to study either at a library or at home.

C) The students assigned to study at a library should take a more difficult test than the students assigned to study at home.

D) No changes to the experiment are necessary.

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14

A company consists of full-time and part-time employees. A sample of 60 employees was selected at random from the company and given a survey to determine how many social events the company should have each year. There are 59 part-time employees and 1 full-time employee in the sample. Which of the following is the largest population to which the results of the survey can be generalized?

A) All part-time employees at the company
B) All employees at the company
C) The 59 part-time employees in the sample
D) The 60 employees in the sample

15

For a study, a pet supply store selected two random samples of its customers. Based on data gathered from each sample, the percentage of customers who own a cat was estimated. The results are shown in the table.

Sample	Percentage of customers who own a cat	Margin of error
A	27%	4%
B	30%	6%

If the margin of error was calculated the same way for both samples, which of the following is the most likely reason that the margin of error for sample B is greater than the margin of error for sample A?

A) Sample B included a greater number of customers who own a cat than sample A.
B) Sample B included a greater percentage of customers who own a cat than sample A.
C) Sample B included more customers than sample A.
D) Sample B included fewer customers than sample A.

16

A residential building has 15 floors. To estimate the mean age of all the residents who live in the building, Rami randomly selected 3 floors and surveyed the 102 residents who live on those floors. The mean age of the residents surveyed is 31 years. Which of the following statements must be true?

A) The mean age of all the residents who live in the building is 31 years.
B) It is plausible that the mean age of all the residents who live in the building is 31 years.
C) The mean age of all the residents who live in the building should not be estimated because the sample size is too small.
D) The mean age of all the residents who live in the building should not be estimated because the sampling method is flawed.

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Answers to the Exercises

Chapter 1: Exponents & Radicals

EXERCISE 1:

1. 1
2. -1
3. 1
4. -1
5. -1
6. -27
7. -27
8. -36
9. 64
10. -72
11. $\frac{1}{4}$
12. 1
13. $\frac{1}{9}$
14. 2
15. $\frac{9}{16}$

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EXERCISE 2:

1. $\frac{2\cdot 4k^2}{k^4} = \boxed{\frac{8}{k^2}}$

2. $6x^{2+3} = \boxed{6x^5}$

3. $15x^{4-2} = \boxed{15x^2}$

4. $-21m^{3-3} = -21m^0 = \boxed{-21}$

5. $\frac{1}{2^3(x^2)^3} = \boxed{\frac{1}{8x^6}}$

6. $-9a^{2-5}b^{-3+8} = -9a^{-3}b^5 = \boxed{-\frac{9b^5}{a^3}}$

7. $\frac{3x^4}{x^{-4}} = 3x^4\cdot x^4 = 3x^{4+4} = \boxed{3x^8}$

8. $x^{\frac{3}{2}-\frac{1}{2}} = x^1 = \boxed{x}$

9. $2^2m^2\cdot 3^2m^{3\cdot 2} = 36m^{2+6} = \boxed{36m^8}$

10. $(a^{-1-2})^2 = (a^{-3})^2 = a^{-3\cdot 2} = a^{-6} = \boxed{\frac{1}{a^6}}$

11. $b^{-2\cdot -3}\cdot b^{3\cdot 2} = b^6\cdot b^6 = b^{6+6} = \boxed{b^{12}}$

12. $\frac{m^{3\cdot 2}n^3}{m^2n^{2\cdot 2}} = \frac{m^6n^3}{m^2n^4} = \frac{m^{6-2}}{n} = \boxed{\frac{m^4}{n}}$

13. The m and n in the numerator cancel out one m and one n in the denominator: $\boxed{\frac{1}{mn^2}}$

14. $\frac{k^3}{k^2} = k^{3-2} = \boxed{k}$

15. $x^{2+3}y^{3+4}z^{4+5} = \boxed{x^5y^7z^9}$

EXERCISE 3:

1. $\boxed{2\sqrt{3}}$

2. $\boxed{4\sqrt{6}}$

3. $\boxed{3\sqrt{5}}$

4. $\boxed{3\sqrt{2}}$

5. $\boxed{15\sqrt{3}}$

6. $\boxed{4\sqrt{2}}$

7. $\boxed{x=50}$

$$\sqrt{5\cdot 5\cdot 2} = \sqrt{x}$$
$$\sqrt{50} = \sqrt{x}$$
$$x = 50$$

8. $\boxed{x=5}$

$$\sqrt{3\cdot 3\cdot x} = \sqrt{45}$$
$$9x = 45$$
$$x = 5$$

9. $\boxed{x=2}$

$$\sqrt{2\cdot 2\cdot 2} = \sqrt{4x}$$
$$8 = 4x$$
$$x = 2$$

10. $\boxed{x=8}$

$$\sqrt{4\cdot 4\cdot 6} = \sqrt{2\cdot 2\cdot 3x}$$
$$96 = 12x$$
$$x = 8$$

11. $\boxed{x=6}$

$$\sqrt{3\cdot 3\cdot 8} = \sqrt{2\cdot x\cdot x}$$
$$72 = 2x^2$$
$$x^2 = 36$$
$$x = 6$$

12. $\boxed{x=6}$

$$\sqrt{x\cdot x\cdot x} = \sqrt{216}$$
$$x^3 = 216$$
$$x = \sqrt[3]{216} = 6$$

EXERCISE 4:

1. $\boxed{B}$ $x^{2+4}y^{1-3} = x^6y^{-2}$

2. $\boxed{C}$ $\sqrt{\frac{x}{64}} = \frac{\sqrt{x}}{\sqrt{64}} = \frac{x^{\frac{1}{2}}}{8}$

3. $\boxed{A}$ $\frac{x^{\frac{5}{6}}}{x^{\frac{1}{3}}} = x^{\frac{5}{6}-\frac{1}{3}} = x^{\frac{3}{6}} = x^{\frac{1}{2}}$

4. $\boxed{C}$ Only answer C boils down to $b^{\frac{9}{7}}$:

$$\sqrt[105]{b^{135}} = b^{\frac{135}{105}} = b^{\frac{27}{21}} = b^{\frac{9}{7}}$$

5. $\boxed{C}$ Distribute and add the exponents:

$$\left(b^6c^{-2}d^{-5}\right)\left(b^8c^{-3}+c^4d^5\right) =$$
$$b^{6+8}c^{-2-3}d^{-5} + b^6c^{-2+4}d^{-5+5} =$$
$$b^{14}c^{-5}d^{-5} + b^6c^2d^0 =$$
$$b^{14}c^{-5}d^{-5} + b^6c^2$$

6. $\boxed{C}$ $\frac{8x^2}{\left(4x^6y^4\right)^{\frac{1}{2}}} = \frac{8x^2}{2x^3y^2} = 4x^{-1}y^{-2}$

7. $\boxed{D}$ $r^{\frac{6}{7}}s^{\frac{3}{7}} = \left(r^6s^3\right)^{\frac{1}{7}} = \sqrt[7]{r^6s^3}$

8. $\boxed{D}$ To isolate c, raise each side to the dth power.

$$\left(\sqrt[b]{a}\right)^d = \left(\sqrt[d]{c}\right)^d$$
$$\left(a^{\frac{1}{b}}\right)^d = \left(c^{\frac{1}{d}}\right)^d$$
$$a^{\frac{d}{b}} = c^1$$

9. $\boxed{A}$ $\sqrt{16a^{\frac{2}{3}}} = \sqrt{16}\cdot\left(a^{\frac{2}{3}}\right)^{\frac{1}{2}} = 4a^{\frac{2}{3}\cdot\frac{1}{2}} = 4a^{\frac{1}{3}}$

10. $\boxed{B}$

$$\sqrt{rs}\left(\sqrt{r}+\sqrt{s}\right) = \sqrt{r^2s}+\sqrt{rs^2} = r\sqrt{s}+s\sqrt{r}$$

11. $\boxed{12}$ Simplify and equate the exponents:

$$\left(\sqrt[3]{x^2}\right)^n = x^8$$
$$\left(x^{\frac{2}{3}}\right)^n = x^8$$
$$\frac{2}{3}n = 8$$
$$n = 8\cdot\frac{3}{2} = 12$$

12. $\boxed{C}$ $2^{-2k}\cdot 3^k = \left(2^{-2}\right)^k\cdot 3^k = \left(\frac{1}{2^2}\right)^k\cdot 3^k = \left(\frac{1}{4}\right)^k\cdot 3^k = \left(\frac{3}{4}\right)^k$

13. $\boxed{A}$ $a^{\frac{2}{9}}\left(a^{\frac{2}{3}}\right)^{\frac{2}{3}} = a^{\frac{2}{9}}\left(a^{\frac{2}{3}\cdot\frac{2}{3}}\right) = a^{\frac{2}{9}}\left(a^{\frac{4}{9}}\right) = a^{\frac{2}{9}+\frac{4}{9}} = a^{\frac{6}{9}} = a^{\frac{2}{3}} = \sqrt[3]{a^2}$

14. $\boxed{C}$ $\frac{4}{9}\left(\frac{2}{3}\right)^a = \left(\frac{2}{3}\right)^2\left(\frac{2}{3}\right)^a = \left(\frac{2}{3}\right)^{a+2}$

15. $\boxed{\frac{6}{7}\text{ or }.8571}$ $\sqrt[4]{x^3}\cdot\sqrt[12]{x^5} = x^{\frac{3}{4}}\cdot x^{\frac{5}{12}} = x^{\frac{3}{4}+\frac{5}{12}} = x^{\frac{9}{12}+\frac{5}{12}} = x^{\frac{14}{12}} = x^{\frac{7}{6}} = \sqrt[6]{x^7}$. Therefore, $m=6$ and $n=7$.

16. $\boxed{-2}$ We need to solve

$$p^{1-x} = \sqrt{q}$$

for x. Given that $p^2 = \sqrt[3]{q}$, we can cube both sides to get

$$\left(p^2\right)^3 = \left(\sqrt[3]{q}\right)^3$$
$$p^6 = q$$

With this result, we can substitute for q in the equation we need to solve.

$$p^{1-x} = \sqrt{p^6}$$
$$p^{1-x} = p^3$$

Equating the exponents on both sides, we get $1-x=3$, which gives $x=-2$.

EXERCISE 5:

1. $\boxed{C}$ $\sqrt{100x^{36}} = \sqrt{100} \cdot \sqrt{x^{36}} = 10\left(x^{36}\right)^{\frac{1}{2}} = 10x^{18}$

2. $\boxed{B}$ Multiply the exponents:

$$(5^3)^{4k} = (5^{\frac{1}{3}})^{24}$$
$$5^{12k} = 5^8$$

Since the bases are the same, we can equate the exponents: $12k = 8$ and so $k = \frac{8}{12} = \frac{2}{3}$.

3. $\boxed{B}$ $3y^{\frac{1}{2}} = 3\sqrt{y} = \sqrt{3 \cdot 3 \cdot y} = \sqrt{9y}$

4. $\boxed{D}$ $4^{m+2} = 4^m \cdot 4^2 = 4^m \cdot 16 = 16(4^m)$

5. $\boxed{B}$ $\sqrt[4]{x^2y^4} = \left(x^2y^4\right)^{\frac{1}{4}} = x^{(2\cdot\frac{1}{4})}y^{(4\cdot\frac{1}{4})} = x^{\frac{1}{2}}y^1 = y\sqrt{x}$

6. $\boxed{D}$ $p^{\frac{1}{3}} \cdot (\sqrt[3]{p})^2 = p^{\frac{1}{3}} \cdot (p^{\frac{1}{3}})^2 = p^{\frac{1}{3}} \cdot p^{\frac{2}{3}} = p^{\frac{1}{3}+\frac{2}{3}} = p^1 = p$

7. $\boxed{B}$ The $2a$ means raised to the $2a$ power and the b on the bottom means the bth root.

8. $\boxed{\frac{3}{4} \text{ or } .75}$ Simplify the left-hand side: $\frac{9^7}{\sqrt[4]{9^{10}}} = 9^{7-\frac{10}{4}} = 9^{\frac{14}{2}-\frac{5}{2}} = 9^{\frac{9}{2}}$. Thus, the equation becomes $9^{\frac{9}{2}} = 9^{6t}$. Since both sides have the same base, we can equate the exponents: $6t = \frac{9}{2}$, which yields $t = \frac{9}{12} = \frac{3}{4}$.

9. $\boxed{B}$ $a^{\frac{3}{4}}b^{\frac{1}{2}} = a^{\frac{3}{4}}b^{\frac{2}{4}} = \left(a^3b^2\right)^{\frac{1}{4}} = \sqrt[4]{a^3b^2}$

10. $\boxed{C}$ $h^{\frac{5}{12}}(h^{-\frac{1}{4}})^{\frac{7}{3}} = h^{\frac{5}{12}}(h^{-\frac{1}{4}\cdot\frac{7}{3}}) = h^{\frac{5}{12}}(h^{-\frac{7}{12}}) = h^{\frac{5}{12}-\frac{7}{12}} = h^{-\frac{2}{12}} = h^{-\frac{1}{6}} = \frac{1}{\sqrt[6]{h}}$

11. $\boxed{\frac{7}{120} \text{ or } .0583}$ $\sqrt[8]{41k}\left(\sqrt[9]{41k}\right)^{12} = (41k)^{\frac{1}{8}} \cdot (41k)^{\frac{12}{9}} = (41k)^{\frac{1}{8}} \cdot (41k)^{\frac{4}{3}} = (41k)^{\frac{1}{8}+\frac{4}{3}} = (41k)^{\frac{3}{24}+\frac{32}{24}} = (41k)^{\frac{35}{24}}$.

Since this result is equivalent to $(41k)^{25x}$, we can equate the exponents: $25x = \frac{35}{24}$. Dividing both sides by 25 yields $x = \frac{35}{24} \times \frac{1}{25} = \frac{7}{120}$.

12. $\boxed{4096}$ Since $4^{18x} = k^{3x}$ can be written as $(4^{18})^x = (k^3)^x$, we can see that $k^3 = 4^{18}$. Taking the cube root of both sides, we get $k = \sqrt[3]{4^{18}} = (4^{18})^{\frac{1}{3}} = 4^6 = 4096$.

13. $\boxed{C}$

Solution 1: $9^{\frac{1}{n}}(4^{\frac{1}{2n}}) = 9^{\frac{1}{n}}(4^{\frac{1}{2}})^{\frac{1}{n}} = 9^{\frac{1}{n}} \cdot 2^{\frac{1}{n}} = (9 \cdot 2)^{\frac{1}{n}} = 18^{\frac{1}{n}} = \sqrt[n]{18}$

Solution 2: $9^{\frac{1}{n}}(4^{\frac{1}{2n}}) = 9^{\frac{1}{n}}\left(2^2\right)^{\frac{1}{2n}} = 9^{\frac{1}{n}}2^{(2\cdot\frac{1}{2n})} = 9^{\frac{1}{n}} \cdot 2^{\frac{1}{n}} = (9 \cdot 2)^{\frac{1}{n}} = 18^{\frac{1}{n}} = \sqrt[n]{18}$

14. $\boxed{\frac{155}{4} \text{ or } 38.75}$ $\frac{10\sqrt[6]{9^3x^{48}}}{\sqrt[4]{6^4x}} = \frac{10\left(9^3x^{48}\right)^{\frac{1}{6}}}{\left(6^4x\right)^{\frac{1}{4}}} = \frac{10 \cdot 9^{\frac{3}{6}} \cdot x^{\frac{48}{6}}}{6^{\frac{4}{4}} \cdot x^{\frac{1}{4}}} = \frac{10 \cdot 9^{\frac{1}{2}} \cdot x^8}{6^1 \cdot x^{\frac{1}{4}}} = \frac{30x^8}{6x^{\frac{1}{4}}} = 5x^{8-\frac{1}{4}} = 5x^{\frac{31}{4}}$.

Therefore, $a = 5$, $b = \frac{31}{4} = 7.75$, and $ab = 38.75$.

15. $\boxed{\frac{9}{2} \text{ or } 4.5}$ We're being asked to solve

$$m^x = n$$

for x. Given that $4m = \sqrt[3]{n^2}$ and $m = 2$, we can isolate n by first cubing each side and then raising each side to the $\frac{1}{2}$ power:

$$(4 \cdot 2)^3 = \left(\sqrt[3]{n^2}\right)^3$$
$$8^3 = n^2$$
$$\left(8^3\right)^{\frac{1}{2}} = \left(n^2\right)^{\frac{1}{2}}$$
$$8^{\frac{3}{2}} = n$$

Substituting this result for n in the equation we're trying to solve, we get

$$m^x = n$$
$$2^x = 8^{\frac{3}{2}}$$
$$2^x = \left(2^3\right)^{\frac{3}{2}}$$
$$2^x = 2^{(3\cdot\frac{3}{2})}$$
$$2^x = 2^{\frac{9}{2}}$$
$$x = \frac{9}{2} = 4.5$$

Notice that we substituted 2^3 for 8 in order to get the same base on both sides.

16. $\boxed{C}$

$$2^{x+3} - 2^x = k(2^x)$$
$$(2^x)(2^3) - 2^x = k(2^x)$$

Now we factor out 2^x from the two terms on the left side:

$$2^x(2^3 - 1) = k(2^x)$$
$$2^x(7) = k(2^x)$$
$$7 = k$$

Chapter 2: Expressions

EXERCISE 1:

1. $\boxed{A}$ We factor out $6xy$ from both terms to get $6xy(x+y)$.

2. $\boxed{B}$ Combining like terms, we get $3x^3 + (8x^2 + 7x^2) + (-4x - 11x) - 7 = 3x^3 + 15x^2 - 15x - 7$.

3. $\boxed{B}$ The least common denominator is $4a$. So, $\frac{1}{a} + \frac{3}{4} = \frac{4}{4a} + \frac{3a}{4a} = \frac{4+3a}{4a}$

4. $\boxed{-227}$ Expand: $(15x+34)(-8x+3) = -120x^2 + 45x - 272x + 102 = -120x^2 - 227x + 102$. Therefore, $b = -227$.

5. $\boxed{C}$ Factor out 4 from the numerator and simplify: $\frac{4(1+2x)}{12x} = \frac{1+2x}{3x}$. Another way to get the same answer is to split the fractions and simplify.

6. $\boxed{C}$ Using the difference of two squares formula, $a^2 - b^2 = (a+b)(a-b)$, we get $484x^2 - 324 = (22x+18)(22x-18)$, which is equivalent to answer C.

7. $\boxed{B}$ Expand: $(x^2+y)(y+z) = x^2y + x^2z + y^2 + yz$

8. $\boxed{B}$ Factor out $18x$ from the denominator. The x's cancel out and $\frac{15}{18}$ reduces to $\frac{5}{6}$: $\frac{15x}{36x^2+18x} = \frac{15x}{18x(2x+1)} = \frac{5}{6(2x+1)}$. Therefore, the value of k is 6.

9. $\boxed{D}$ $3x^4 - 3 = 3(x^4-1) = 3(x^2+1)(x^2-1) = 3(x^2+1)(x+1)(x-1)$

10. $\boxed{D}$ $\frac{\left(\frac{3}{x-1}\right)}{6x-6} = \frac{3}{x-1} \div (6x-6) = \frac{3}{x-1} \times \frac{1}{6x-6} = \frac{3}{x-1} \times \frac{1}{6(x-1)} = \frac{1}{2(x-1)^2}$

11. $\boxed{B}$ $2(3x^2-5)^2 - (3x^2-5)^2 = 2(3x^2-5)^2 - 1(3x^2-5)^2 = (3x^2-5)^2 = (3x^2)^2 - 2(3x^2)(5) + 5^2 = 9x^4 - 30x^2 + 25$. To expand $(3x^2-5)^2$ above, we took advantage of the formula $(a-b)^2 = a^2 - 2ab + b^2$.

12. $\boxed{D}$ $\frac{xy - x^2}{xy - y^2} = \frac{x(y-x)}{y(x-y)} = \frac{-x(x-y)}{y(x-y)} = -\frac{x}{y}$

13. $\boxed{B}$ Factor out $(2x-3)$ from both terms in the numerator. Doing so yields the expression (x^2-1), which factors into $(x+1)(x-1)$.

$$\frac{x^2(2x-3) - (2x-3)}{x-1} = \frac{(2x-3)(x^2-1)}{x-1} = \frac{(2x-3)(x+1)\cancel{(x-1)}}{\cancel{x-1}} = (2x-3)(x+1)$$

14. $\boxed{C}$ Multiply the top and bottom of the first fraction by $(x-3)$ and the top and bottom of the second fraction by $(3x-14)$ to get common denominators of $(3x-14)(x-3)$:

$$\frac{3}{3x-14} - \frac{1}{x-3} = \frac{3(x-3)}{(3x-14)(x-3)} - \frac{3x-14}{(3x-14)(x-3)} = \frac{3(x-3) - (3x-14)}{(3x-14)(x-3)}$$
$$= \frac{3x - 9 - 3x + 14}{(3x-14)(x-3)} = \frac{5}{(3x-14)(x-3)}$$

15. $\boxed{A}$ First, expand: $x^2(x+2)(x-2) + 4 = x^2(x^2-4) + 4 = x^4 - 4x^2 + 4$. Now we can apply the formula $a^2 - 2ab + b^2 = (a-b)^2$, where $a = x^2$ and $b = 2$, to get $x^4 - 4x^2 + 4 = (x^2-2)^2$.

EXERCISE 2:

1. $\boxed{C}$ When we factor out $3xy^3$, we end up with $3xy^3(y^3 + x^2)$. Remember that we add the exponents when multiplying expressions with the same base, so $y^3 \cdot y^3 = y^6$ and $x \cdot x^2 = x^3$.

2. $\boxed{C}$ Combining like terms, we get $5a - 2a = 3a$ and $3\sqrt{a} - 5\sqrt{a} = -2\sqrt{a}$.

3. $\boxed{D}$ Multiply the top and bottom of the first fraction by 3 to get a common denominator of $3x$:

$$\frac{2}{x} - \frac{1}{3x} = \frac{2}{x} \cdot \frac{3}{3} - \frac{1}{3x} = \frac{6}{3x} - \frac{1}{3x} = \frac{5}{3x}$$

4. $\boxed{C}$ Expand: $(x-8)(5-3x^2) = 5x - 3x^3 - 40 + 24x^2 = -3x^3 + 24x^2 + 5x - 40$

5. $\boxed{C}$ Use the difference of two squares formula, $a^2 - b^2 = (a+b)(a-b)$:

$$\frac{x^2-3}{x+\sqrt{3}} = \frac{(x+\sqrt{3})(x-\sqrt{3})}{x+\sqrt{3}} = x - \sqrt{3}$$

6. $\boxed{-21}$ Expand: $(4x^2+4x-1)(x^2-3x) = 4x^2(x^2-3x) + 4x(x^2-3x) - 1(x^2-3x) = 4x^4 - 12x^3 + 4x^3 - 12x^2 - x^2 + 3x = 4x^4 - 8x^3 - 13x^2 + 3x$. Therefore, $b = -8$, $c = -13$, and $b + c = -8 - 13 = -21$.

7. $\boxed{C}$ $5b - \frac{b+k}{b} = \frac{5b^2}{b} - \frac{b+k}{b} = \frac{5b^2 - (b+k)}{b} = \frac{5b^2 - b - k}{b}$

8. $\boxed{7}$ In the numerator, we can factor out a 2 from $(6x+8)$ and a 3 from $(3x+6)$. In the denominator, we can factor out 12.

$$\frac{(6x+8)(3x+6)}{12x+24} = \frac{2 \cdot 3(3x+4)(x+2)}{12(x+2)} = \frac{6(3x+4)\cancel{(x+2)}}{12\cancel{(x+2)}} = \frac{3x+4}{2}$$

Therefore, $a + b = 3 + 4 = 7$.

9. $\boxed{D}$ First, we can manipulate the exponent: $(3\sqrt{a} - \sqrt{b})^{\frac{2}{7}} = [(3\sqrt{a} - \sqrt{b})^2]^{\frac{1}{7}} = \sqrt[7]{(3\sqrt{a} - \sqrt{b})^2}$. Now we can expand the inside expression using the formula $(x-y)^2 = x^2 - 2xy + y^2$:

$$\sqrt[7]{(3\sqrt{a} - \sqrt{b})^2} = \sqrt[7]{(3\sqrt{a})^2 - 2(3\sqrt{a})(\sqrt{b}) + (\sqrt{b})^2} = \sqrt[7]{9a - 6\sqrt{ab} + b}$$

10. $\boxed{C}$ We can apply the formula $a^2 - 2ab + b^2 = (a-b)^2$ to the given expression, where $a = x^2$ and $b = 4$: $x^4 - 8x^2 + 16 = (x^2-4)^2$. Now we have a difference of two squares, $x^2 - 4$, which can be factored into $(x-2)(x+2)$:

$$(x^2-4)^2 = [(x-2)(x+2)]^2 = (x-2)^2(x+2)^2$$

11. $\boxed{B}$ Factor $2x$ out from the denominator of the first fraction and it's easy to see that $2x(x-y)$ should be the common denominator:

$$\frac{y-3xy}{2x^2-2xy}+\frac{1+3y}{x-y}=\frac{y-3xy}{2x(x-y)}+\frac{1+3y}{x-y}=\frac{y-3xy}{2x(x-y)}+\frac{2x(1+3y)}{2x(x-y)}=\frac{y-3xy+2x(1+3y)}{2x(x-y)}$$
$$=\frac{y-3xy+2x+6xy}{2x(x-y)}=\frac{2x+3xy+y}{2x(x-y)}=\frac{2x+3xy+y}{2x^2-2xy}$$

12. $\boxed{C}$ First factor out an 8 from both terms. Then use the formula $a^2-b^2=(a-b)(a+b)$.

$$8x^2-\frac{1}{2}y^2=8\left(x^2-\frac{1}{16}y^2\right)=8\left(x-\frac{1}{4}y\right)\left(x+\frac{1}{4}y\right)$$

Therefore, $c=\frac{1}{4}$.

13. $\boxed{B}$ The expression follows the $(a+b)^2=a^2+2ab+b^2$ pattern, where $a=x+1$ and $b=y+1$. Therefore, the expression is equivalent to $((x+1)+(y+1))^2=(x+y+2)^2$.

14. $\boxed{C}$ To get a common denominator of $2(x-2)$ for both fractions, we first factor out a negative from the second fraction and then we multiply the top and bottom of the first fraction by 2:

$$\frac{x}{x-2}+\frac{x}{2(2-x)}=\frac{x}{x-2}+\frac{x}{-2(x-2)}=\frac{x}{x-2}-\frac{x}{2(x-2)}=\frac{2x}{2(x-2)}-\frac{x}{2(x-2)}=\frac{x}{2(x-2)}$$

Chapter 3: Manipulating & Solving Equations

EXERCISE 1:

1. $\boxed{-\frac{1}{6} \text{ or } -.1666 \text{ or } -.1667}$ Multiply both sides by $-x$: $-48x=8$. Now we can divide both sides by -48 to get $x=\frac{8}{-48}=-\frac{1}{6}$.

2. $\boxed{6}$ Divide both sides by 9 to get $2+x=6$.

3. $\boxed{C}$ Subtract $3x$ from both sides and then subtract 9 from both sides. The resulting equation is $-3x=-33$.

4. $\boxed{D}$ If $3x-8=-23$, then $3x=-15$. Multiplying both sides by 2, we get $6x=-30$ and $6x-7=-37$.

5. $\boxed{\frac{5}{2} \text{ or } 2.5}$ Either $5-2x=0$ or $3+x=0$. $5-2x=0$ gives the positive solution 2.5.

6. $\boxed{7}$ First, combine the $(r-9)$ terms by summing their coefficients:

$$\begin{aligned}-3(r-9)+5(r-9)&=-4\\2(r-9)&=-4\\r-9&=-2\\r&=7\end{aligned}$$

7. $\boxed{-80}$ Subtracting both sides by 16 gives $x^2+3x=-16$. Then multiplying both sides by 5 gives us the answer: $5x^2+15x=-80$.

8. $\boxed{C}$ Cross multiplying gives

$$\begin{aligned}4(12-x)&=9(12-x)\\0&=5(12-x)\\0&=12-x\end{aligned}$$

Therefore, the value of $12-x$ is 0, which is between -3 and 4.

9. $\boxed{-6}$ Multiply both sides by 36. The 64's cancel out.

$$\begin{aligned}64x^2&=64(36)\\x^2&=36\\x&=\pm\sqrt{36}=-6 \text{ or } 6\end{aligned}$$

The negative solution is therefore -6.

10. $\boxed{A}$ Cross multiply.

$$\begin{aligned}\frac{4}{9}&=\frac{8}{3}m\\12&=72m\\\frac{1}{6}&=m\end{aligned}$$

11. $\boxed{C}$ Plug in the given x and y values.

$$\begin{aligned}y+2kx&=kx^2+5\\23+2(k)(3)&=k(3)^2+5\\23+6k&=9k+5\\-3k&=-18\\k&=6\end{aligned}$$

12. $\boxed{A}$

$$\begin{aligned}f&=p\left(\frac{(1+j)^n-1}{j}\right)\\fj&=p((1+j)^n-1)\\\frac{fj}{(1+j)^n-1}&=p\end{aligned}$$

13. $\boxed{A}$ Multiply both sides by 2 to get $\frac{m}{n}=4$, which means $\frac{n}{m}=\frac{1}{4}$. Then, $\frac{n}{2m}=\frac{1}{2}\cdot\frac{n}{m}=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$

14. $\boxed{D}$ Divide both sides by -4. Notice that the negative eventually gets distributed to both terms in the numerator.

$$d = \frac{3e-h}{-4} = -\frac{3e-h}{4} = \frac{-(3e-h)}{4} = \frac{h-3e}{4}$$

15. $\boxed{C}$ Cross multiply.

$$\begin{aligned} \frac{x}{6} &= \frac{x+12}{42} \\ 42x &= 6x+72 \\ 36x &= 72 \\ x &= 2 \end{aligned}$$

Now, $\frac{6}{x} = \frac{6}{2} = 3$.

16. $\boxed{0}$ The answer should be obvious just by looking at it. Testing $n = 0$ gives us:

$$\begin{aligned} (0-4)^2 &= (0+4)^2 \\ (-4)^2 &= (4)^2 \\ 16 &= 16 \end{aligned}$$

We could also expand and solve like so:

$$\begin{aligned} (n-4)(n-4) &= (n+4)(n+4) \\ n^2-8n+16 &= n^2+8n+16 \\ -16n &= 0 \\ n &= 0 \end{aligned}$$

17. $\boxed{D}$ First, square both sides to remove the square root.

$$\begin{aligned} \sqrt{x^2-25} &= n \\ x^2-25 &= n^2 \\ x^2 &= n^2+25 \\ x &= \pm\sqrt{n^2+25} \end{aligned}$$

Don't forget about the $\pm$ when you square root both sides of an equation. In this case, only the negative solution appears in the answer choices.

18. $\boxed{B}$

$$\begin{aligned} 3(x-2y)-3z &= 0 \\ 3x-6y-3z &= 0 \\ 3x &= 6y+3z \\ x &= 2y+z \end{aligned}$$

19. $\boxed{C}$

$$\begin{aligned} d &= a\left(\frac{c+1}{24}\right) \\ \frac{a}{2} &= a\left(\frac{c+1}{24}\right) \\ \frac{1}{2} &= \frac{c+1}{24} \\ 12 &= c+1 \\ 11 &= c \end{aligned}$$

20. $\boxed{D}$ Test each answer choice. Only -3 satisfies the equation. Note that 3 is NOT a solution because it results in division by zero.

21. $\boxed{D}$ Divide both sides by $2P$ and then square both sides.

$$\begin{aligned} T &= 2P\sqrt{\frac{L}{G}} \\ \frac{T}{2P} &= \sqrt{\frac{L}{G}} \\ \frac{T^2}{4P^2} &= \frac{L}{G} \end{aligned}$$

Now, cross-multiply.

$$\begin{aligned} GT^2 &= 4LP^2 \\ G &= \frac{4LP^2}{T^2} \end{aligned}$$

22. $\boxed{-9}$ Factor $(x-7)$ out from both products on the left-hand side:

$$\begin{aligned} x(x-7)+9(x-7) &= 0 \\ (x-7)(x+9) &= 0 \\ x &= 7, -9 \end{aligned}$$

The negative solution is -9.

23. $\boxed{D}$ Distribute and then move all the terms with y to one side.

$$\begin{aligned} x(y-2) &= -y \\ xy-2x &= -y \\ xy+y &= 2x \\ y(x+1) &= 2x \\ y &= \frac{2x}{x+1} \end{aligned}$$

24. $\boxed{A}$ Divide both sides by P and take the tth root of both sides.

$$\begin{aligned} V &= P(1-r)^t \\ \frac{V}{P} &= (1-r)^t \\ \sqrt[t]{\frac{V}{P}} &= 1-r \\ r &= 1-\sqrt[t]{\frac{V}{P}} \end{aligned}$$

25. $\boxed{A}$

$$\begin{aligned} \frac{4}{x^2-6x+9} &= 9 \\ \frac{4}{(x-3)^2} &= 9 \\ \frac{4}{9} &= (x-3)^2 \\ \pm\sqrt{\frac{4}{9}} &= \sqrt{(x-3)^2} \\ \pm\frac{2}{3} &= x-3 \end{aligned}$$

26. $\boxed{5}$

$$\begin{aligned} \sqrt{2x+8}-3\sqrt{x-3} &= 0 \\ \sqrt{2x+8} &= 3\sqrt{x-3} \\ 2x+8 &= 9(x-3) \\ 2x+8 &= 9x-27 \\ 35 &= 7x \\ 5 &= x \end{aligned}$$

27. $\boxed{C}$ The right-hand side is a difference of two squares, so we can factor like so:

$$x-c = (x+c)(x-c)$$

Now we can divide both sides by $x-c$ to get

$$1 = x+c$$

which gives the solution $x = 1-c$. Because we divided both sides by an expression to solve this equation, we have to watch out for missing solutions. And indeed, $x-c$, the expression we divided by, is itself the source of a solution, $x = c$. Plugging $x = c$ into the original equation yields zero on both sides. Therefore, the solutions are $x = c$ and $x = 1-c$.

28. $\boxed{36}$

$$\begin{aligned} 20-\sqrt{x} &= \frac{2}{3}\sqrt{x}+10 \\ 10 &= \frac{5}{3}\sqrt{x} \\ 6 &= \sqrt{x} \\ 36 &= x \end{aligned}$$

EXERCISE 2:

1. $\boxed{170}$ Isolate the square root and then square both sides.

$$\begin{aligned} \sqrt{x-1}-2 &= 11 \\ \sqrt{x-1} &= 13 \\ x-1 &= 169 \\ x &= 170 \end{aligned}$$

2. $\boxed{32}$

$$\begin{aligned} \frac{5x}{6}-4 &= 12-\frac{7x}{6} \\ \frac{5x}{6}+\frac{7x}{6} &= 4+12 \\ \frac{12x}{6} &= 16 \\ 2x &= 16 \\ x &= 8 \end{aligned}$$

3. $\boxed{7}$

$$B + 8A = 5(4C + B)$$
$$B + 8A = 20C + 5B$$
$$8A - 20C = 4B$$
$$2A - 5c = B$$

Therefore, $m = 2$, $n = 5$, and $m + n = 2 + 5 = 7$.

4. $\boxed{-7}$ Square root both sides. Don't forget the plus or minus sign.

$$(2x + 7)^2 = 25$$
$$2x + 7 = \pm 5$$
$$x = \frac{-7 \pm 5}{2}$$
$$x = -6, -1$$

Therefore, the sum of the solutions is $-6 + (-1) = -7$.

5. $\boxed{C}$ Move m to the right-hand side and multiply both sides by $\frac{3}{2}$. Note the parentheses.

$$\frac{2}{3}t + m = 40$$
$$\frac{2}{3}t = 40 - m$$
$$t = \frac{3}{2}(40 - m)$$

6. $\boxed{109}$ Multiply both sides by 15 to get rid of the fractions. Then combine the $(x + 11)$ terms.

$$\frac{3}{5}(x + 11) - \frac{2}{3}(x + 11) = -8$$
$$9(x + 11) - 10(x + 11) = -120$$
$$-(x + 11) = -120$$
$$x + 11 = 120$$
$$x = 109$$

7. $\boxed{A}$ Cross multiply to get $12 = kx + 2x$. Then, $k = \frac{12 - 2x}{x}$.

8. $\boxed{A}$ Start by squaring both sides.

$$\sqrt{x^2 + p^2} = q$$
$$x^2 + p^2 = q^2$$
$$x^2 = q^2 - p^2$$
$$x = \pm\sqrt{q^2 - p^2}$$

Don't forget about the $\pm$ when you square root both sides of an equation. In this case, only the negative solution appears in the answer choices.

9. $\boxed{10}$ Square both sides. Then expand and solve.

$$\sqrt{(x + 6)(x - 6)} = 8$$
$$x^2 - 36 = 64$$
$$x^2 = 100$$
$$x = \pm\sqrt{100} = \pm 10$$

Since the question asks for the positive solution, the answer is 10.

10. $\boxed{A}$ Multiply both sides by v. Then move all the terms with v to one side.

$$2u = \frac{18}{v} - 6$$
$$2uv = 18 - 6v$$
$$2uv + 6v = 18$$
$$uv + 3v = 9$$
$$v(u + 3) = 9$$
$$v = \frac{9}{u + 3}$$

11. $\boxed{B}$

$$\frac{k}{3x + 3y} = k + 1$$
$$\frac{k}{3(x + y)} = k + 1$$
$$\frac{k}{3} = (k + 1)(x + y)$$
$$\frac{k}{3(k + 1)} = x + y$$
$$\frac{k}{3k + 3} = x + y$$

12. $\boxed{24}$ Plug in $t = 6$ and solve for x.

$$\left(\frac{1}{8} + \frac{1}{x}\right)(6) = 1$$
$$\frac{3}{4} + \frac{6}{x} = 1$$
$$\frac{6}{x} = \frac{1}{4}$$
$$x = 24$$

We cross-multiplied in the last step.

13. $\boxed{B}$ Cross-multiply.

$$\frac{a}{b} = \frac{a + 1}{2c}$$
$$2ac = ab + b$$
$$2ac - ab = b$$
$$a(2c - b) = b$$
$$a = \frac{b}{2c - b}$$

14. $\boxed{C}$

$$\frac{3r}{t} = 6\sqrt{2s}$$
$$\frac{3r}{6t} = \sqrt{2s}$$
$$\frac{r}{2t} = \sqrt{2s}$$
$$\frac{r^2}{4t^2} = 2s$$
$$\frac{r^2}{8t^2} = s$$

15. $\boxed{C}$

$$ax + 7a + x + 7 = b$$
$$a(x + 7) + (x + 7) = b$$
$$(x + 7)(a + 1) = b$$
$$x + 7 = \frac{b}{a + 1}$$

16. $\boxed{10}$ First, expand the left-hand side. Then combine like terms and factor.

$$(x + 1)(x - 2) = 7x - 18$$
$$x^2 - x - 2 = 7x - 18$$
$$x^2 - 8x + 16 = 0$$
$$(x - 4)^2 = 0$$

Therefore, $x = 4$ and $7x - 18 = 7(4) - 18 = 10$.

17. $\boxed{D}$

$$xy - 1 = z - xz$$
$$xy + xz = z + 1$$
$$x(y + z) = z + 1$$
$$x = \frac{z + 1}{y + z}$$

18. $\boxed{77}$

$$\frac{2\sqrt{x + 4}}{3} = 6$$
$$2\sqrt{x + 4} = 18$$
$$\sqrt{x + 4} = 9$$
$$x + 4 = 81$$
$$x = 77$$

19. $\boxed{\frac{4}{3} \text{ or } 1.333}$

$$\frac{2x + y}{x - 2y} = 3$$
$$2x + y = 3(x - 2y)$$
$$2x + y = 3x - 6y$$
$$7y = x$$

We can substitute this result into the expression we're trying to evaluate:

$$\frac{x + y}{x - y} = \frac{7y + y}{7y - y} = \frac{8y}{6y} = \frac{4}{3}$$

20. $\boxed{B}$ Dividing both sides by $(x-n)$ gives the solution $\frac{1}{3}=x$. Because we divided both sides by an expression to solve this equation, we have to watch out for missing solutions. And indeed, $x-n$, the expression we divided by, is itself the source of a solution, $x=n$. Plugging $x=n$ into the original equation yields zero on both sides. Therefore, the solutions are $x=\frac{1}{3}$ and $x=n$.

21. $\boxed{8}$ Square both sides of the equation.

$$(x+y)^2=\left(\sqrt{x^2+y^2+16}\right)^2$$
$$x^2+2xy+y^2=x^2+y^2+16$$
$$2xy=16$$
$$xy=8$$

22. $\boxed{16}$ Multiply both sides by 2 to get rid of the fraction and then square both sides.

$$\sqrt{x+9}=\frac{x}{2}-3$$
$$2\sqrt{x+9}=x-6$$
$$4(x+9)=(x-6)^2$$
$$4x+36=x^2-12x+36$$
$$0=x^2-16x$$
$$0=x(x-16)$$
$$x=0,\ 16$$

Because $x=0$ does not satisfy the original equation, it is extraneous. Therefore, $x=16$ is the only solution.

23. $\boxed{D}$ First, square both sides.

$$\sqrt{2(x-k)}=x-k$$
$$2(x-k)=(x-k)^2$$

Now divide both sides by $x-k$.

$$2=x-k$$
$$k+2=x$$

So $x=k+2$ is a solution. We should check that it is indeed a solution by plugging it into the original equation. Doing so yields 2 on both sides, so we can be confident this solution is valid. Now because we divided both sides by an expression, $x-k$, to solve this equation, we have to watch out for missing solutions. As it turns out, the expression $x-k$ is the source of a solution, $x=k$, which, when plugged in, yields zero on both sides of the original equation. Therefore, the solutions are $x=k+2$ and $x=k$. The sum of the solutions is then $k+2+k=2k+2$.

24. $\boxed{B}$ In one of the steps below, we'll use the following law of exponents: $\frac{x^m}{x^n}=x^{m-n}$.

$$\frac{x}{\sqrt{x+b}}=7-\frac{b}{\sqrt{x+b}}$$
$$\frac{x}{\sqrt{x+b}}+\frac{b}{\sqrt{x+b}}=7$$
$$\frac{x+b}{\sqrt{x+b}}=7$$
$$\frac{(x+b)^1}{(x+b)^{\frac{1}{2}}}=7$$
$$\sqrt{x+b}=7$$
$$x+b=49$$
$$x=49-b$$

25. $\boxed{B}$ Square both sides.

$$\sqrt{x+2}=2-\sqrt{x}$$
$$x+2=(2-\sqrt{x})^2$$
$$x+2=2^2-2(2)(\sqrt{x})+(\sqrt{x})^2$$
$$x+2=4-4\sqrt{x}+x$$
$$4\sqrt{x}=2$$
$$\sqrt{x}=\frac{1}{2}$$

The value of $2-\sqrt{x}$ is then $2-\frac{1}{2}=\frac{3}{2}$.

Chapter 4: Other Types of Equation Questions

EXERCISE 1:

1. $\boxed{\frac{1}{4} \text{ or } .25}$ For an equation to have no solution, both sides must be equivalent except for the constants. Therefore, p must equal $\frac{1}{4}$ to match the coefficient of x on the left side of the equation.

2. $\boxed{-28 \text{ or } -12}$ Either $x+20=8$ or $x+20=-8$. Solving these equations, we get $x=-12$ or $x=-28$.

3. $\boxed{D}$ For an equation to have infinitely many solutions, both sides must be equivalent. Multiplying both sides by 4 gives

$$12x+24=12x+r$$

Comparing both sides, we can see that r must equal 24.

4. $\boxed{30}$ Either $x-15=32$ or $x-15=-32$. Solving these equations, we get $x=47$ or $x=-17$. The sum of these solutions is then $47+(-17)=30$.

5. $\boxed{A}$ Expand each side of the equation:

$$3x-6=6x+6-3x$$
$$3x-6=3x+6$$

Since both sides of the equation are equivalent except for the constants, -6 and 6, the equation has no solution.

6. $\boxed{\frac{1}{5} \text{ or } .2}$ For an equation to have no solution, both sides must be equivalent except for the constants. Therefore, k must equal $\frac{1}{5}$ so the coefficient of x on the right side of the equation is equal to the coefficient of x on the left side of the equation. It's important to note that $k=\frac{1}{5}$ does not equate the constants on either side (0 on the left vs. $-\frac{1}{25}$ on the right).

7. $\boxed{52}$ The positive case for this absolute value equation is $3x+9=156$. We can divide both sides by 3 to get $x+3=52$.

8. $\boxed{B}$ For an equation to have infinitely many solutions, both sides must be equivalent. First, let's expand the right side of the equation:

$$6x=x-6nx+3x$$
$$6x=4x-6nx$$
$$2x=-6nx$$

Now when we compare the coefficients on both sides, we get $2=-6n$, which gives $n=\frac{2}{-6}=-\frac{1}{3}$.

9. $\boxed{A}$ Multiply both sides of the equation by -1, and we get $|x+6|=-21$. The absolute value of something (in this case, $|x+6|$) can never be negative. Therefore, the equation has zero solutions.

10. $\boxed{A}$ The left side of the equation expands to $2x+2k$. Since $k>1$, the constant $2k$ can never equal 2, which means the constants on either side of the equation will always be different despite the x terms being the same. Therefore, the equation has no solutions.

11. $\boxed{-\frac{3}{4} \text{ or } -.75}$ For an equation to have no solution, both sides must be equivalent except for the constants. Since there is no x term on the right side of the equation (i.e. $0x$), we need to find a value of n such that the x terms on the left side of the equation cancel out. In other words, the coefficients of x on the left should sum to zero:

$$12+16n=0$$
$$16n=-12$$
$$n=-\frac{12}{16}=-\frac{3}{4}$$

12. $\boxed{B}$ The equation simplifies to $-14x + 70 = 70 - 5x$. Since the coefficients of x do not match, this equation can be solved for exactly one solution. It turns out $x = 0$.

13. $\boxed{C}$ $(-x)^2$ evaluates to x^2, so the equation boils down to $x^2 = 6$, which has two solutions, $x = \sqrt{6}$ and $x = -\sqrt{6}$.

14. $\boxed{11}$

$$3|5 - x| = 42 - 4|5 - x|$$
$$7|5 - x| = 42$$
$$|5 - x| = 6$$

So now we have two cases: either $5 - x = 6$ or $5 - x = -6$. Solving these two equations, we get $x = -1$ and $x = 11$, which is the positive solution.

15. $\boxed{24}$ First, expand the left side.

$$(x - 4)^2 - a = x^2 + bx + b$$
$$x^2 - 8x + 16 - a = x^2 + bx + b$$

Comparing the coefficients and constants on both sides of the equation, we can see that $-8 = b$ and $16 - a = b$. Substituting the first result into the second, we get $16 - a = -8$, which gives $a = 24$.

16. $\boxed{D}$ Any value of x greater than or equal to 0 satisfies the equation. Therefore, the equation has infinitely many solutions.

17. $\boxed{C}$

$$bx^2 + 11x = x(4x + a) - 3x$$
$$bx^2 + 11x = 4x^2 + ax - 3x$$
$$bx^2 + 11x = 4x^2 + (a - 3)x$$

Comparing the coefficients on either side, we can see that $b = 4$ and $11 = a - 3$, which gives $a = 14$. Therefore, $a + b = 14 + 4 = 18$.

18. $\boxed{3}$

$$(ax + 5)(bx - 5) = cx^2 + 15x - 25$$
$$abx^2 + 5bx - 5ax - 25 = cx^2 + 15x - 25$$
$$abx^2 + (5b - 5a)x - 25 = cx^2 + 15x - 25$$

Comparing the coefficients of x on either side, we can see that $5b - 5a = 15$. Dividing both sides by 5, we get $b - a = 3$.

19. $\boxed{B}$

$$(kx - 5)(nx + 4) = 12x^2 - mx - 20$$
$$knx^2 + 4kx - 5nx - 20 = 12x^2 - mx - 20$$
$$knx^2 + (4k - 5n)x - 20 = 12x^2 - mx - 20$$

Comparing the coefficients, we can see that $kn = 12$. Divide both sides by n, and we get $k = \frac{12}{n}$. Since k is an integer constant (as specified in the problem), $\frac{12}{n}$ must also be an integer constant.

EXERCISE 2:

1. $\boxed{D}$ For an equation to have no solutions, the coefficients of the x terms must be the same on either side but the constants must be different. If we expand the right side, we get

$$ax - b = 6x + 3$$

Therefore, $a = 6$ and $b \neq -3$. Only answer choice D satisfies these conditions. Choice C would result in infinitely many solutions since both sides would be equivalent.

2. $\boxed{C}$ There are two cases: $-8x - 12 = 34$ and $-8x - 12 = -34$. We don't have to solve these equations, but it's clear that they're solvable and each have one solution. Therefore, the equation has two solutions.

3. $\boxed{D}$ Expanding the right side of the equation, we get $9 - 6x = -6x + 9$. Because both sides of the equation are equivalent, the equation has infinitely many solutions.

4. $\boxed{D}$ Since m is the coefficient of x on the right side of the equation, the value of m cannot equal the coefficient of x on the left side of the equation, $\frac{3}{2}$. Otherwise, the equation would have no solution or infinitely many solutions. In this case, the equation would have no solution if $m = \frac{3}{2}$.

5. $\boxed{\frac{3}{50} \text{ or } .06}$ Equating corresponding coefficients gives $1.56 = 26n$, $0.84 = 14n$, and $1.08 = 18n$. Solving any one of these equations gives $n = 0.06$.

6. $\boxed{A}$ Expand.

$$x - \frac{3}{2}x - 4 = 4 - \frac{1}{2}x$$
$$-\frac{1}{2}x - 4 = 4 - \frac{1}{2}x$$

The coefficients of x on either side are the same but the constants are different (-4 vs. 4). Therefore, the equation has no solutions.

7. $\boxed{D}$ An equation has infinitely many solutions when both sides are equivalent. Comparing the terms on both sides, we can see that $\frac{2}{3}a = \frac{8}{3}$ and $3 = 9b$. Solving these equations, we get $a = 4$ and $b = \frac{1}{3}$. Thus, $\frac{a}{b} = \frac{4}{\frac{1}{3}} = 12$.

8. $\boxed{D}$ For an equation to have no solution, the coefficients of the x terms must be the same on either side but the constants must be different. First, let's expand the left side.

$$3x + 3a - 2ax = 12 - 7x$$
$$3a - 2ax = 12 - 10x$$

Comparing the coefficients of the x terms, we get $-2a = -10$, which yields $a = 5$.

9. $\boxed{5}$

$$2|10 - 4x| + 9 = 77$$
$$2|10 - 4x| = 68$$
$$|10 - 4x| = 34$$

So now we have two cases: $10 - 4x = 34$ and $10 - 4x = -34$. Solving these two equations, we get $x = -6$ and $x = 11$, which sum to 5.

10. $\boxed{B}$ x cannot be positive because $\sqrt{-x}$ would be taking the square root of a negative number, which is not possible with real numbers. x cannot be negative either: because the square root of something can never be negative, it's not possible for both sides to have the same value. As an example, if $x = -3$, then $\sqrt{-(-3)} \neq -3$. The only value of x that can satisfy the equation is 0. Therefore, the equation has only one distinct real solution.

11. $\boxed{C}$ The best approach is to test the answer choices. Only $x = 4$ satisfies the equation.

Here's the mathematical approach. If $x - 13$ and $x + 5$ are both positive, then there is no solution since $x - 13 = x + 5$ leads to $-13 = 5$. This means that if there is a solution, $x - 13$ must be the negative of $x + 5$ (or vice versa). Therefore, $x - 13 = -(x + 5)$, which, when solved, yields $x = 4$.

12. $\boxed{\frac{5}{8} \text{ or } .625}$ For an equation to have infinitely many solutions, both sides must be equivalent. Since there is no x term on the right side of the equation (i.e. $0x$), we need to find a value of j such that the x terms on the left side of the equation cancel out. In other words, the coefficients of x on the left should sum to zero: $-10 + 16j = 0$. Solving this equation, we get $j = \frac{10}{16} = \frac{5}{8}$.

13. $\boxed{C}$ If $g(a) = a$, then $\left|\frac{7}{9}a - 40\right| = a$. Now the best approach is to test each answer choice. Answer C yields $\left|\frac{7}{9}\left(\frac{45}{2}\right) - 40\right| = \left|-\frac{45}{2}\right| = \frac{45}{2}$. Therefore, the answer is C. For a more mathematical approach, we can set up two equations: $\frac{7}{9}a - 40 = a$ for the positive case and $\frac{7}{9}a - 40 = -a$ for the negative case. The positive case yields $a = -180$, which is an extraneous solution because it doesn't work when plugged back into the original equation (with the absolute value). The negative case yields $a = \frac{45}{2}$, which is not extraneous.

14. $\boxed{B}$ Divide both sides by $x-1$ to get $(x+1)=x$. Since $x+1$ can never equal x, this equation has no solution. However, because we divided both sides by the expression $x-1$, we have to watch out for missing solutions. And indeed, $x-1$ is the source of a solution, $x=1$, which satisfies the original equation. Therefore, the given equation has one solution.

15. $\boxed{B}$ Add $68x^2$ to both sides to get $102x^2=0$. Now divide both sides by 102 to get $x^2=0$, which yields $x=0$, the only solution. Note that dividing both sides of the original equation by x^2 results in a missing solution, so "Zero solutions" is an incorrect answer.

16. $\boxed{\frac{14}{9} \text{ or } 1.555 \text{ or } 1.556}$ For an equation to have no solution, both sides must be equivalent except for the constants. So, the coefficients of x on either side must match: $\frac{1}{3}b=\frac{14}{27}$. Solving this equation, we get $b=\frac{14}{27}\cdot 3=\frac{14}{9}$. Note that since $c>3$, the constant on the left has to be greater than 1 and can never equal the constant of $\frac{4}{9}$ on the right.

17. $\boxed{B}$ Expand $(2x+3)(ax-5)$, and we get

$$2ax^2+3ax-10x-15=12x^2+bx-15$$
$$2ax^2+(3a-10)x-15=12x^2+bx-15$$

Comparing both sides, we get $2a=12$ and $b=3a-10$, which yields $a=6$ and $b=3a-10=3(6)-10=8$.

18. $\boxed{\frac{4}{5} \text{ or } .8}$ After expanding, we have

$$5x^5+10x^4-5hx^4-10hx^3=5x^5+6x^4-8x^3$$

Comparing the coefficients of x^3, we can see that $-10h=-8$, which yields $h=\frac{-8}{-10}=\frac{4}{5}$.

19. $\boxed{A}$ **Solution 1:** Remember, $p^2-q^2=(p+q)(p-q)$. We can apply this formula here once we take out a 2: $18x^2-8=2(9x^2-4)=2(3x+2)(3x-2)$, which equals $2(ax+b)(ax-b)$. Comparing the coefficients, we can see that $a=3$ and $b=2$. Therefore, $ab=(3)(2)=6$. Note that this approach assumes that the constants are positive, but that's okay since all the answer choices are positive. It's possible that $a=-3$ and $b=2$ or $a=3$ and $b=-2$, for instance, but these are cases we didn't need to worry about.

Solution 2: Expand the right side of the equation: $18x^2-8=2a^2x^2-2b^2$. Comparing the coefficients on either side, we get $18=2a^2$ and $8=2b^2$. Solving these equations gives $a=\pm 3$ and $b=\pm 2$. Of the answer choices, only 6 is a possible value of ab (when $a=3$ and $b=2$ or when $a=-3$ and $b=-2$).

20. $\boxed{A}$ Let's set up an equation and expand $(x+a)(x+6)$:

$$x^2+ax+6x+6a=x^2+bx+c$$
$$x^2+(a+6)x+6a=x^2+bx+c$$

Now let's compare the coefficients. Since b is an integer, $a+6$ must be an integer. And if $a+6$ is an integer, a itself must be an integer. After all, subtracting one integer from another integer always results in an integer, and the same holds true for addition. So $a, b,$ and c are all integers. Comparing the constants on either side, we get $6a=c$. Dividing both sides by 6 gives $a=\frac{c}{6}$. Since a is an integer (as we've shown), $\frac{c}{6}$ must also be an integer. Note that the other answer choices are integers in some cases but not in others.

21. $\boxed{A}$ Expanding the right side, we get

$$4x^2+mx+9=4x^2+4nx+n^2$$

Comparing both sides, we can see that $9=n^2$ and $m=4n$. Given that $n<0$, $n=-3$ and $m=4(-3)=-12$. Finally, $m+n=-12+(-3)=-15$.

Chapter 5: Functions

EXERCISE 1:

1. $\boxed{D}$ The graph crosses the y-axis at $(0,7)$. This is the y-intercept.

2. $\boxed{D}$ Check each answer choice to see whether $f(0)=20$, $f(1)=21$, and $f(3)=29$. The only function that satisfies all three is answer D.

3. $\boxed{D}$ The graph crosses the x-axis at $(2,0)$. Therefore, $f(x)=0$ when $x=2$.

4. $\boxed{5}$ When $x=0$,

$$y=-4\left(\frac{1}{2}\right)^0+9=-4(1)+9=5.$$

5. $\boxed{B}$ $f(-4)=\frac{16+(-4)^2}{2(-4)}=\frac{32}{-8}=-4.$

6. $\boxed{C}$ Since $h(t)$ is defined as the ball's height, in feet, above the ground t seconds after it falls, $h(3)=6$ means that the ball's height is 6 feet above the ground 3 seconds after it falls.

7. $\boxed{A}$ Plug in the answer choices and check. $f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$, which is less than $\frac{1}{2}$. The answer is A.

8. $\boxed{D}$ $f(8)=4(8)-3=29$. Testing each answer choice to see which one yields 29, we see that $g(8)=3(8)+5=29$.

9. $\boxed{B}$ $f(3)=-2$. Now where else does f output -2? At $x=-3$. So a must be -3.

10. $\boxed{C}$ $g(2)=2^2-1=3$. So, $f(g(2))=f(3)=3^2+1=10$.

11. $\boxed{61}$ First, $g(10)=f(20)-1$. Since $f(20)=3(20)+2=62$, $g(10)=62-1=61$.

12. $\boxed{B}$ The y-intercept of the graph is at $(0,100)$, where $x=0$ and $y=100$. Given how x and y are defined by the question, these values indicate that the atmospheric pressure is 100 kilopascals at an elevation of 0 kilometers above sea level (in other words, at sea level).

13. $\boxed{B}$ From the second equation, $f(a)=20$. So,

$$f(a)=-3a+5$$
$$20=-3a+5$$
$$15=-3a$$
$$-5=a$$

14. $\boxed{C}$ The given x-intercepts indicate that $x-r$ and $x-s$ must be factors of $f(x)$. These factors narrow down the answer choices to C and D. Since the question states that there are exactly two x-intercepts, the answer must be C since D has a factor x that yields a third x-intercept at $x=0$.

15. $\boxed{B}$ The graph indicates that when $x=0$, $y=3$. We can plug this y-intercept into the equation to solve for b:

$$3=3^0+b$$
$$3=1+b$$
$$2=b$$

16. $\boxed{B}$ To find $f(x-2)$, replace x with $x-2$: $f(x-2)=(x-2)(x-2+2)(x-2+6)=(x-2)(x)(x+4)$. With this equation, it's easy to see that the x-intercepts of $f(x-2)$ are at $x=2, 0$, and -4. These values line up with the ones shown in answer B.

17. $\boxed{C}$ The graph has x-intercepts at $x=-2$ and $x=3$, so the equation must have the factors $(x+2)$ and $(x-3)$. This narrows it down to answers A and C. To decide between them, we'll use the y-intercept as a point to test against. When $x=0$, the graph gives a y-value of about 72. Answer A gives $y=(0+2)^2(0-3)^2=36$ when $x=0$, and answer C gives $y=2(0+2)^2(0-3)^2=72$ when $x=0$. Therefore, the answer is C.

18. $\boxed{A}$ Graph the function on Desmos. For values of x between -2 and 2, the graph gives y-values that range between 0 and 9.481 (inclusive). Since the answer choice of 12 is outside this range, q cannot be 12.

19. $\boxed{B}$ This question requires you to know your laws of exponents, specifically $x^{m+n} = x^m \cdot x^n$. The overall strategy is to test each answer choice. Answer A yields $f(x+1) = \frac{1}{4}\left(\frac{1}{8}\right)^{x+1} = \frac{1}{4}\left(\frac{1}{8}\right)^x \cdot \left(\frac{1}{8}\right)^1 = \frac{1}{8}f(x)$. Answer B yields $f(x+1) = 8\left(\frac{1}{4}\right)^{x+1} = 8\left(\frac{1}{4}\right)^x \cdot \left(\frac{1}{4}\right)^1 = \frac{1}{4}f(x)$. Therefore, the answer is B.

20. $\boxed{C}$ We plug in values to solve for a and b. Plugging in $(0, -2)$, we get $-2 = a(0)^2 + b = b$. So, $b = -2$. Plugging in $(1, 3)$, we get

$$\begin{aligned} 3 &= a(1)^2 + b \\ 3 &= a - 2 \\ 5 &= a \end{aligned}$$

So, $a = 5$ and $f(x) = 5x^2 - 2$. Finally, $f(3) = 5(3)^2 - 2 = 43$.

EXERCISE 2:

1. $\boxed{B}$ The function g models the revenue generated by an advertisement, and the variable in the model, x, is defined as the number of times the advertisement is viewed. Therefore, $g(2{,}000) = 18{,}000$ means that the advertisement will generate 18,000 dollars in revenue if it is viewed 2,000 times. The information about clicks serves to provide a detail about the advertisement and an underlying assumption of the function g, but it is otherwise irrelevant.

2. $\boxed{-6}$ Setting up an equation, we get

$$\begin{aligned} \frac{x+18}{x-6} &= -1 \\ x + 18 &= 6 - x \\ 2x &= -12 \\ x &= -6 \end{aligned}$$

3. $\boxed{D}$ Setting up an equation, we get

$$\begin{aligned} g(a) &= 6 \\ \sqrt{3a} &= 6 \\ 3a &= 36 \\ a &= 12 \end{aligned}$$

4. $\boxed{C}$ Draw a horizontal line at $y = 3$. This line intersects $f(x)$ four times, so there are four solutions (four values of x for which $f(x) = 3$).

5. $\boxed{A}$ $f(0) = \frac{a^{-0}}{b} = \frac{1}{b}$. Therefore, the y-intercept of the graph is $\left(0, \frac{1}{b}\right)$.

6. $\boxed{B}$ If $g(c) = 5$, then $c = 1$ since 1 is the only input to $g(x)$ that gives an output of 5. Then, $f(c) = f(1) = 3$.

7. $\boxed{C}$ Because $f(x)$ has the factors $-x, (x-a), (x+a)$, and $(x+b)$, the distinct values $0, a, -a$, and $-b$, respectively, yield $f(x) = 0$. That's 4 distinct values.

8. $\boxed{D}$ Since $g(x)$ just adds a constant of 7 to every value of $f(x)$, the maximum of $g(x)$ must occur at the same x-value as the maximum of $f(x)$, namely $x = 3$. So, the maximum of $g(x)$ is reached at the point $(3, g(3))$, and since $g(x) = f(x) + 7$, this point is equivalent to $(3, f(3) + 7)$.

9. $\boxed{B}$ Test each of the answer choices. When $x = -3$, $f(x) = -2$ according to its graph, and $g(x) = (-3+3)(-3-1) = 0$. In this case, $f(x)$ is not greater than $g(x)$. When $x = -2$, $f(x) \approx 1.5$ and $g(x) = (-2+3)(-2-1) = -3$. In this case, $f(x) > g(x)$ so we have our answer.

10. $\boxed{B}$ With the given information, we get the following:

$$\begin{aligned} g(k) &= 8 \\ 2f(k) &= 8 \\ f(k) &= 4 \end{aligned}$$

Looking at the chart, $f(x) = 4$ only when $x = 3$. So $k = 3$.

11. $\boxed{21}$ First, $f(a-9) = \frac{((a-9)+3)(12-(a-9))}{(a-9)} = \frac{(a-6)(21-a)}{a-9}$. Now this expression is equal to 0 only when the numerator is equal to 0. Therefore, $a = 6$ or $a = 21$, which is the greatest possible value.

12. $\boxed{B}$ First find c by plugging in 20: $c = g(20) = \frac{20-8}{2} = 6$. Now we can find d: $d = g(c) = g(6) = \frac{6-8}{2} = -1$.

13. $\boxed{B}$ The x-intercepts of -3 and 2 indicate that the equation of $f(x)$ must have the factors $(x+3)$ and $(x-2)$. That eliminates answer choices C and D. A y-intercept of 12 means that when we plug in $x = 0$, $f(x) = 12$. Only answer B meets all these conditions.

14. $\boxed{A}$ Since $f(18) = \sqrt{18-2} = \sqrt{16} = 4$ and $f(11) = \sqrt{11-2} = \sqrt{9} = 3$, $f(18) - f(11) = 4 - 3 = 1$. Testing each answer choice, we see that $f(3)$ is the only one that also yields 1.

15. $\boxed{B}$ The graph has x-intercepts at $(4, 0)$ and $(-3, 0)$, so the equation must have factors $(x-4)$ and $(x+3)$. Since the given equation already has the factor $(x+3)$, b must be equal to -4. To solve for a, we can simply plug in another point on the graph. For convenience, we'll use the y-intercept, $(0, -4)$.

$$\begin{aligned} y &= a(x+3)(x-4) \\ -4 &= a(0+3)(0-4) \\ -4 &= -12a \\ \frac{1}{3} &= a \end{aligned}$$

Therefore, $ab = \frac{1}{3} \cdot -4 = -\frac{4}{3}$.

16. $\boxed{-21}$ First, $h(x) = \left(\frac{11}{3}x + 34\right) - \left(\frac{5}{3}x - 8\right) = 2x + 42$. Now we can solve for the x-intercept by setting $h(x)$ equal to 0: $2x + 42 = 0$ yields $x = -21$, which is the x-coordinate of the x-intercept.

17. $\boxed{64}$ Plugging in the point $(0, b-3)$, we get

$$\begin{aligned} b - 3 &= 125 \cdot a^0 - b \\ b - 3 &= 125 \cdot 1 - b \\ 2b &= 128 \\ b &= 64 \end{aligned}$$

18. $\boxed{\frac{7}{4} \text{ or } 1.75}$ Given that $f(-8) = 4$,

$$\begin{aligned} \frac{-8+32}{-8+b} &= 4 \\ 24 &= 4(-8+b) \\ 6 &= -8+b \\ 14 &= b \end{aligned}$$

Now, $f(10) = \frac{10+32}{10+14} = \frac{42}{24} = 1.75$.

19. $\boxed{A}$ The expression $7(4)^{-x}$ is always positive, no matter the value of x. Therefore, $7(4)^{-x} + 13$ is always greater than 13. Since the output of f is always greater than 13 ($f(x) > 13$), it must also be true that $f(x) > 7$. Thus, statement I is true. Statement II, however, is not true. Plugging in $x = -2$, for example, yields a value that is greater than 20: $f(-2) = 7(4)^2 + 13 = 125$. An alternative solution is to graph the function on Desmos. The graph of $y = f(x)$ reaches a minimum value of approximately 13 (the graph approaches but never touches 13, even though Desmos will round to 13). And since the graph keeps going up and up, we can consider the maximum value of f to be infinity (certainly not 20).

20. $\boxed{D}$ To solve for a and b, we have to plug in the given points. Plugging in $(0, 36)$ gives $36 = b \cdot a^0 \Rightarrow 36 = b$ since $a^0 = 1$. Plugging in $\left(-3, \frac{9}{16}\right)$ gives

$$\frac{9}{16} = 36 \cdot a^{-3}$$
$$\frac{1}{64} = a^{-3}$$
$$\frac{1}{64} = \frac{1}{a^3}$$
$$64 = a^3$$
$$a = \sqrt[3]{64} = 4$$

Therefore, $f(x) = 36 \cdot 4^x$, and $g(x) = f(x+2) = 36 \cdot 4^{x+2} = 36 \cdot 4^x \cdot 4^2 = 576 \cdot 4^x$.

21. $\boxed{40}$ Plugging the point $(16, 0)$ into $y = f(x)$ gives $0 = a\sqrt{16} + b$, and plugging the point $(9, 0)$ into $y = f(x) - 10$ gives $0 = a\sqrt{9} + b - 10$. Let's evaluate the square roots and write these two equations as a system we can then solve:

$$0 = 4a + b$$
$$0 = 3a + b - 10$$

Subtracting the second equation from the first, we get

$$0 = a + 10$$
$$a = -10$$

Plugging this value back into the first equation gives us $0 = 4(-10) + b \Rightarrow 40 = b$.

22. $\boxed{D}$ Factoring $\sqrt{a-x}$ out from each term yields $f(x) = (b+7) \cdot \sqrt{a-x}$. Since $f(16) = 0$,

$$(b+7) \cdot \sqrt{a-16} = 0$$

Therefore, either $b = -7$ or $a = 16$ (these are our two cases). Since $f(9) > 0$,

$$(b+7) \cdot \sqrt{a-9} > 0$$

Note that a square root can never produce a negative. So for $(b+7) \cdot \sqrt{a-9}$ to yield a positive number, $(b+7)$ must be positive and $\sqrt{a-9}$ must be positive. For that to be the case,

$$b > -7 \quad \text{and} \quad a > 9$$

Given our two cases and these conditions, we can conclude that b cannot be -7, so a has to be 16 (note that 16 is greater than 9). Since $a = 16$ and $b > -7$, the value of $a + b$ must be greater than $16 - 7 = 9$. The only answer choice greater than 9 is 15.

Chapter 6: Lines & Linear Relationships

EXERCISE 1:

1. $\boxed{4}$ Perpendicular lines have slopes that are the negative reciprocal of each other. Since the given line has a slope of $-\frac{1}{4}$, a line that is perpendicular must have a slope of 4.

2. $\boxed{B}$ To find the x-intercept, set y equal to 0: $8x + 12(0) = 48$, which yields $x = 6$. Therefore, the x-intercept of the graph is $(6, 0)$.

3. $\boxed{-5}$ To find the y-intercept, set x equal to 0: $5(0) - 8y = 40$, which yields $y = -5$. Therefore, the y-intercept of the graph is $(0, -5)$, and $c = -5$.

4. $\boxed{C}$ Since we're given a slope and a point, we can use point-slope form to find the equation:

$$y - (-1) = -8\left(x - \frac{1}{2}\right)$$
$$y + 1 = -8x + 4$$
$$y = -8x + 3$$

Alternatively, we could've narrowed down the answer choices to B and C because those are the only answer choices with a slope of -8. Then we could've decided between them by testing each equation with the given point.

5. $\boxed{-\frac{43}{5} \text{ or } -8.6}$

$$y = -11x - \frac{2}{5}(18 - 6x)$$
$$= -11x - \frac{2}{5}(18) - \frac{2}{5}(-6x)$$
$$= -11x - 7.2 + 2.4x = -8.6x - 7.2$$

The slope of the line is then -8.6.

6. $\boxed{A}$ Slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 0}{0 - (-2)} = \frac{-4}{2} = -2$

7. $\boxed{C}$ The graph of line l goes up three units for every two units to the right, which means its slope is $\frac{3}{2}$. A parallel line must have the same slope—only answer C gives an equation of a line with the same slope.

8. $\boxed{C}$ We can use the values from any two days to find the line that best models the data. Let's use the values from Monday and Thursday to calculate the slope: $\frac{616 - 584}{7.2 - 6.8} = 80$. At this point, we can tell the answer is probably going to be choice C, but let's find the y-intercept just to be sure. Currently, we have $c = 80s + b$. Plugging in $s = 7.2$ and $c = 616$ from Monday, we get $616 = 80(7.2) + b$, which gives $b = 616 - 80(7.2) = 40$. Therefore, $c(s) = 80s + 40$.

An alternative solution is to test each of the answer choices by plugging in the values from the table. Answer A works for Monday ($c(7.2) = 30(7.2) + 400 = 616$) but not for any of the other days. Answer B works for Saturday ($c(8.5) = 60(8.5) + 210 = 720$) but not for any of the other days. Answer D does not give the correct value of c for any of the given values of s. Only answer C gives the correct value of c for each of the given values of s. These types of questions require you to be thorough. Don't just choose the first thing that "works." You have to evaluate all the answer choices.

9. $\boxed{B}$ The graph shows a positive slope m and a negative y-intercept b. Thus, $mb < 0$.

10. $\boxed{B}$ Line k must have a slope that is the negative reciprocal of $\frac{1}{2}$, which is -2. The equation of line k is then of the form $y = -2x + b$. Plugging in the point $(1, 5)$, we get $5 = -2(1) + b$, which gives $7 = b$. Now that we have the value of b, the equation of line k is $y = -2x + 7$.

11. $\boxed{C}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{3}$$
$$\frac{n - 1}{5 - (-1)} = \frac{1}{3}$$
$$\frac{n - 1}{6} = \frac{1}{3}$$
$$n - 1 = 2$$
$$n = 3$$

20. $\boxed{D}$ To solve for a and b, we have to plug in the given points. Plugging in $(0, 36)$ gives $36 = b \cdot a^0 \Rightarrow 36 = b$ since $a^0 = 1$. Plugging in $\left(-3, \frac{9}{16}\right)$ gives

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Note that a square root can never produce a negative. So for $(b+7) \cdot \sqrt{a-9}$ to yield a positive number, $(b+7)$ must be positive and $\sqrt{a-9}$ must be positive. For that to be the case,

$$b > -7 \quad \text{and} \quad a > 9$$

Given our two cases and these conditions, we can conclude that b cannot be -7, so a has to be 16 (note that 16 is greater than 9). Since $a = 16$ and $b > -7$, the value of $a + b$ must be greater than $16 - 7 = 9$. The only answer choice greater than 9 is 15.

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The slope of the line is then -8.6.

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10. $\boxed{B}$ Line k must have a slope that is the negative reciprocal of $\frac{1}{2}$, which is -2. The equation of line k is then of the form $y = -2x + b$. Plugging in the point $(1, 5)$, we get $5 = -2(1) + b$, which gives $7 = b$. Now that we have the value of b, the equation of line k is $y = -2x + 7$.

11. $\boxed{C}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{3}$$
$$\frac{n - 1}{5 - (-1)} = \frac{1}{3}$$
$$\frac{n - 1}{6} = \frac{1}{3}$$
$$n - 1 = 2$$
$$n = 3$$

12. $\boxed{8}$ Using the first two rows (points), we can calculate the slope: $\frac{14-10}{3-(-3)} = \frac{4}{6} = \frac{2}{3}$. Now we can find the equation of the line using point-slope form:

$$y - 10 = \frac{2}{3}(x - (-3))$$
$$y - 10 = \frac{2}{3}x + 2$$
$$y = \frac{2}{3}x + 12$$

Therefore, $a = \frac{2}{3}$, $b = 12$, and $ab = \frac{2}{3} \cdot 12 = 8$.

13. $\boxed{D}$ Looking at the intercepts, we can see that the slope of the line is $\frac{6}{7}$ (6 units up, 7 units to the right). And since the y-intercept is $(0, -6)$, the equation of the line is $y = \frac{6}{7}x - 6$. The value of r is then $\frac{6}{7}(5) - 6 = \frac{30}{7} - \frac{42}{7} = -\frac{12}{7}$.

14. $\boxed{A}$ From the graph, line r has a slope of $\frac{1}{2}$ (1 unit up for every 2 units to the right). Since line s is parallel, line s must also have a slope of $\frac{1}{2}$. We can now find the equation of line s using point-slope form:

$$y - (-1) = \frac{1}{2}(x - 1)$$
$$y + 1 = \frac{1}{2}x - \frac{1}{2}$$
$$y = \frac{1}{2}x - \frac{3}{2}$$

Testing each answer choice, we see that only answer A, $(-3, -3)$, satisfies this equation. A faster approach is to "draw" line s on the graph starting at $(1, -1)$ and then see which point the line crosses.

15. $\boxed{\frac{3}{2} \text{ or } 1.5}$ The first line has a slope of -6 and the second line has a slope of $\frac{-h}{-9} = \frac{h}{9}$. Since the slopes of perpendicular lines are negative reciprocals of each other, $\frac{h}{9} = \frac{1}{6}$, which gives $h = \frac{9}{6} = \frac{3}{2}$.

16. $\boxed{-8}$ The slope of the line is $\frac{0-(-3)}{15-6} = \frac{3}{9} = \frac{1}{3}$. We can now find the equation of the line using point-slope form and the given point $(15, 0)$: $y - 0 = \frac{1}{3}(x - 15)$, which simplifies to $y = \frac{1}{3}x - 5$. The value of t is then $\frac{1}{3}(-9) - 5 = -8$.

17. $\boxed{\frac{13}{2} \text{ or } 6.5}$ From the graph, we can see that f goes up 1 unit for every 2 units to the right, which means its slope is $\frac{1}{2}$. Since g is perpendicular to f, the slope of g must be -2 (the negative reciprocal). Since g passes through the point $\left(1, \frac{5}{2}\right)$, we can use point-slope form to find the equation of g:

$$y - \frac{5}{2} = -2(x - 1)$$
$$y = -2x + 2 + \frac{5}{2}$$
$$y = -2x + \frac{9}{2}$$

Finally, $g(-1) = -2(-1) + \frac{9}{2} = 2 + 4\frac{1}{2} = 6\frac{1}{2}$. A quicker way would've been to work backwards from the point $\left(1, \frac{5}{2}\right)$, knowing that the slope is -2. So on the graph of g, 1 unit to the left brings us to $\left(0, \frac{5}{2} + 2\right) = \left(0, 4\frac{1}{2}\right)$, and 1 more unit to the left brings us to $\left(-1, 4\frac{1}{2} + 2\right) = \left(-1, 6\frac{1}{2}\right)$. Therefore, $g(-1) = 6.5$.

EXERCISE 2:

1. $\boxed{B}$ Look at the x-intercept and the y-intercept. The line goes down 2 units for every 4 units to the right. Therefore, the slope of the line is $\frac{-2}{4} = -\frac{1}{2}$.

2. $\boxed{B}$ From the equation, line j has a slope of $\frac{-4}{20} = -\frac{1}{5}$. Since parallel lines have the same slope, line k must also have a slope of $-\frac{1}{5}$.

3. $\boxed{16}$ We can use point-slope form to find the equation:

$$y - (-14) = 6(x - (-5))$$
$$y + 14 = 6x + 30$$
$$y = 6x + 16$$

Therefore, the value of b is 16.

4. $\boxed{\frac{105}{4} \text{ or } 26.25}$ To find the y-intercept, plug in $x = 0$ and solve for y:

$$7(0) = 6\left(15 + \frac{9(0)}{4} - \frac{4y}{7}\right)$$
$$0 = 6\left(15 - \frac{4y}{7}\right)$$
$$0 = 15 - \frac{4y}{7}$$
$$-15 = -\frac{4y}{7}$$
$$\frac{105}{4} = y$$

Of course, the easiest solution is to just use Desmos.

5. $\boxed{-4}$ The line shown has a slope of $\frac{1}{4}$ and a y-intercept at $(0, 1)$. The equation of this line in slope-intercept form is therefore $y = \frac{1}{4}x + 1$. To get a and b, we need to convert this equation into standard form. First, multiply both sides by -4 to get $-4y = -x - 4$. Now add x to both sides to get $x - 4y = -4$. Thus, $a = 1$, $b = -4$, and $ab = -4$. An alternative approach involves plugging in points such as $(0, 1)$ and $(-4, 0)$ to solve for a and b.

6. $\boxed{-45}$ Plugging in $(0, -79)$, we get $-79 = a(0) + b$, which yields $-79 = b$, the y-intercept. Plugging in $(5, 91)$, we get $91 = a(5) - 79$, which simplifies to $170 = 5a$ and finally $34 = a$ (the slope). Given these values, $a + b = 34 + (-79) = -45$.

7. $\boxed{\frac{8}{5} \text{ or } 1.6}$ First, plug the point $(2, 6)$ into the equation of the line so that we can solve for a:

$$a(2) - \frac{1}{3}(6) = 8$$
$$2a - 2 = 8$$
$$2a = 10$$
$$a = 5$$

So the equation of the line is $5x - \frac{1}{3}y = 8$. The x-intercept always has a y-coordinate of 0, so if we set y equal to 0, we get $5x = 8$, which gives $x = \frac{8}{5} = 1.6$.

8. $\boxed{B}$ We can calculate the slope from any two of the points. To demonstrate, we'll use the 2nd and 3rd points: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-p - (-57 - p)}{2 - (-2)} = \frac{57}{4}$.

9. $\boxed{A}$ When $x = 0$, $5(0) - 8y = k \Rightarrow y = -\frac{k}{8}$. This is the y-intercept, n. When $y = 0$, $5x - 8(0) = k \Rightarrow x = \frac{k}{5}$. This is the x-intercept, m. Therefore, $\frac{m}{n} = \frac{\frac{k}{5}}{-\frac{k}{8}} = \frac{k}{5} \cdot -\frac{8}{k} = -\frac{8}{5}$.

12. $\boxed{8}$ Using the first two rows (points), we can calculate the slope: $\frac{14-10}{3-(-3)} = \frac{4}{6} = \frac{2}{3}$. Now we can find the equation of the line using point-slope form:

$$y - 10 = \frac{2}{3}(x - (-3))$$
$$y - 10 = \frac{2}{3}x + 2$$
$$y = \frac{2}{3}x + 12$$

Therefore, $a = \frac{2}{3}$, $b = 12$, and $ab = \frac{2}{3} \cdot 12 = 8$.

13. $\boxed{D}$ Looking at the intercepts, we can see that the slope of the line is $\frac{6}{7}$ (6 units up, 7 units to the right). And since the y-intercept is $(0, -6)$, the equation of the line is $y = \frac{6}{7}x - 6$. The value of r is then $\frac{6}{7}(5) - 6 = \frac{30}{7} - \frac{42}{7} = -\frac{12}{7}$.

14. $\boxed{A}$ From the graph, line r has a slope of $\frac{1}{2}$ (1 unit up for every 2 units to the right). Since line s is parallel, line s must also have a slope of $\frac{1}{2}$. We can now find the equation of line s using point-slope form:

$$y - (-1) = \frac{1}{2}(x - 1)$$
$$y + 1 = \frac{1}{2}x - \frac{1}{2}$$
$$y = \frac{1}{2}x - \frac{3}{2}$$

Testing each answer choice, we see that only answer A, $(-3, -3)$, satisfies this equation. A faster approach is to "draw" line s on the graph starting at $(1, -1)$ and then see which point the line crosses.

15. $\boxed{\frac{3}{2} \text{ or } 1.5}$ The first line has a slope of -6 and the second line has a slope of $\frac{-h}{-9} = \frac{h}{9}$. Since the slopes of perpendicular lines are negative reciprocals of each other, $\frac{h}{9} = \frac{1}{6}$, which gives $h = \frac{9}{6} = \frac{3}{2}$.

16. $\boxed{-8}$ The slope of the line is $\frac{0-(-3)}{15-6} = \frac{3}{9} = \frac{1}{3}$. We can now find the equation of the line using point-slope form and the given point $(15, 0)$: $y - 0 = \frac{1}{3}(x - 15)$, which simplifies to $y = \frac{1}{3}x - 5$. The value of t is then $\frac{1}{3}(-9) - 5 = -8$.

17. $\boxed{\frac{13}{2} \text{ or } 6.5}$ From the graph, we can see that f goes up 1 unit for every 2 units to the right, which means its slope is $\frac{1}{2}$. Since g is perpendicular to f, the slope of g must be -2 (the negative reciprocal). Since g passes through the point $\left(1, \frac{5}{2}\right)$, we can use point-slope form to find the equation of g:

$$y - \frac{5}{2} = -2(x - 1)$$
$$y = -2x + 2 + \frac{5}{2}$$
$$y = -2x + \frac{9}{2}$$

Finally, $g(-1) = -2(-1) + \frac{9}{2} = 2 + 4\frac{1}{2} = 6\frac{1}{2}$. A quicker way would've been to work backwards from the point $\left(1, \frac{5}{2}\right)$, knowing that the slope is -2. So on the graph of g, 1 unit to the left brings us to $\left(0, \frac{5}{2} + 2\right) = \left(0, 4\frac{1}{2}\right)$, and 1 more unit to the left brings us to $\left(-1, 4\frac{1}{2} + 2\right) = \left(-1, 6\frac{1}{2}\right)$. Therefore, $g(-1) = 6.5$.

EXERCISE 2:

1. $\boxed{B}$ Look at the x-intercept and the y-intercept. The line goes down 2 units for every 4 units to the right. Therefore, the slope of the line is $\frac{-2}{4} = -\frac{1}{2}$.

2. $\boxed{B}$ From the equation, line j has a slope of $\frac{-4}{20} = -\frac{1}{5}$. Since parallel lines have the same slope, line k must also have a slope of $-\frac{1}{5}$.

3. $\boxed{16}$ We can use point-slope form to find the equation:

$$y - (-14) = 6(x - (-5))$$
$$y + 14 = 6x + 30$$
$$y = 6x + 16$$

Therefore, the value of b is 16.

4. $\boxed{\frac{105}{4} \text{ or } 26.25}$ To find the y-intercept, plug in $x = 0$ and solve for y:

$$7(0) = 6\left(15 + \frac{9(0)}{4} - \frac{4y}{7}\right)$$
$$0 = 6\left(15 - \frac{4y}{7}\right)$$
$$0 = 15 - \frac{4y}{7}$$
$$-15 = -\frac{4y}{7}$$
$$\frac{105}{4} = y$$

Of course, the easiest solution is to just use Desmos.

5. $\boxed{-4}$ The line shown has a slope of $\frac{1}{4}$ and a y-intercept at $(0, 1)$. The equation of this line in slope-intercept form is therefore $y = \frac{1}{4}x + 1$. To get a and b, we need to convert this equation into standard form. First, multiply both sides by -4 to get $-4y = -x - 4$. Now add x to both sides to get $x - 4y = -4$. Thus, $a = 1$, $b = -4$, and $ab = -4$. An alternative approach involves plugging in points such as $(0, 1)$ and $(-4, 0)$ to solve for a and b.

6. $\boxed{-45}$ Plugging in $(0, -79)$, we get $-79 = a(0) + b$, which yields $-79 = b$, the y-intercept. Plugging in $(5, 91)$, we get $91 = a(5) - 79$, which simplifies to $170 = 5a$ and finally $34 = a$ (the slope). Given these values, $a + b = 34 + (-79) = -45$.

7. $\boxed{\frac{8}{5} \text{ or } 1.6}$ First, plug the point $(2, 6)$ into the equation of the line so that we can solve for a:

$$a(2) - \frac{1}{3}(6) = 8$$
$$2a - 2 = 8$$
$$2a = 10$$
$$a = 5$$

So the equation of the line is $5x - \frac{1}{3}y = 8$. The x-intercept always has a y-coordinate of 0, so if we set y equal to 0, we get $5x = 8$, which gives $x = \frac{8}{5} = 1.6$.

8. $\boxed{B}$ We can calculate the slope from any two of the points. To demonstrate, we'll use the 2nd and 3rd points: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-p - (-57 - p)}{2 - (-2)} = \frac{57}{4}$.

9. $\boxed{A}$ When $x = 0$, $5(0) - 8y = k \Rightarrow y = -\frac{k}{8}$. This is the y-intercept, n. When $y = 0$, $5x - 8(0) = k \Rightarrow x = \frac{k}{5}$. This is the x-intercept, m. Therefore, $\frac{m}{n} = \frac{\frac{k}{5}}{-\frac{k}{8}} = \frac{k}{5} \cdot -\frac{8}{k} = -\frac{8}{5}$.

12. $\boxed{8}$ Using the first two rows (points), we can calculate the slope: $\frac{14-10}{3-(-3)} = \frac{4}{6} = \frac{2}{3}$. Now we can find the equation of the line using point-slope form:

$$y - 10 = \frac{2}{3}(x - (-3))$$
$$y - 10 = \frac{2}{3}x + 2$$
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Therefore, $a = \frac{2}{3}$, $b = 12$, and $ab = \frac{2}{3} \cdot 12 = 8$.

13. $\boxed{D}$ Looking at the intercepts, we can see that the slope of the line is $\frac{6}{7}$ (6 units up, 7 units to the right). And since the y-intercept is $(0, -6)$, the equation of the line is $y = \frac{6}{7}x - 6$. The value of r is then $\frac{6}{7}(5) - 6 = \frac{30}{7} - \frac{42}{7} = -\frac{12}{7}$.

14. $\boxed{A}$ From the graph, line r has a slope of $\frac{1}{2}$ (1 unit up for every 2 units to the right). Since line s is parallel, line s must also have a slope of $\frac{1}{2}$. We can now find the equation of line s using point-slope form:

$$y - (-1) = \frac{1}{2}(x - 1)$$
$$y + 1 = \frac{1}{2}x - \frac{1}{2}$$
$$y = \frac{1}{2}x - \frac{3}{2}$$

Testing each answer choice, we see that only answer A, $(-3, -3)$, satisfies this equation. A faster approach is to "draw" line s on the graph starting at $(1, -1)$ and then see which point the line crosses.

15. $\boxed{\frac{3}{2} \text{ or } 1.5}$ The first line has a slope of -6 and the second line has a slope of $\frac{-h}{-9} = \frac{h}{9}$. Since the slopes of perpendicular lines are negative reciprocals of each other, $\frac{h}{9} = \frac{1}{6}$, which gives $h = \frac{9}{6} = \frac{3}{2}$.

16. $\boxed{-8}$ The slope of the line is $\frac{0-(-3)}{15-6} = \frac{3}{9} = \frac{1}{3}$. We can now find the equation of the line using point-slope form and the given point $(15, 0)$: $y - 0 = \frac{1}{3}(x - 15)$, which simplifies to $y = \frac{1}{3}x - 5$. The value of t is then $\frac{1}{3}(-9) - 5 = -8$.

17. $\boxed{\frac{13}{2} \text{ or } 6.5}$ From the graph, we can see that f goes up 1 unit for every 2 units to the right, which means its slope is $\frac{1}{2}$. Since g is perpendicular to f, the slope of g must be -2 (the negative reciprocal). Since g passes through the point $\left(1, \frac{5}{2}\right)$, we can use point-slope form to find the equation of g:

$$y - \frac{5}{2} = -2(x - 1)$$
$$y = -2x + 2 + \frac{5}{2}$$
$$y = -2x + \frac{9}{2}$$

Finally, $g(-1) = -2(-1) + \frac{9}{2} = 2 + 4\frac{1}{2} = 6\frac{1}{2}$. A quicker way would've been to work backwards from the point $\left(1, \frac{5}{2}\right)$, knowing that the slope is -2. So on the graph of g, 1 unit to the left brings us to $\left(0, \frac{5}{2} + 2\right) = \left(0, 4\frac{1}{2}\right)$, and 1 more unit to the left brings us to $\left(-1, 4\frac{1}{2} + 2\right) = \left(-1, 6\frac{1}{2}\right)$. Therefore, $g(-1) = 6.5$.

EXERCISE 2:

1. $\boxed{B}$ Look at the x-intercept and the y-intercept. The line goes down 2 units for every 4 units to the right. Therefore, the slope of the line is $\frac{-2}{4} = -\frac{1}{2}$.

2. $\boxed{B}$ From the equation, line j has a slope of $\frac{-4}{20} = -\frac{1}{5}$. Since parallel lines have the same slope, line k must also have a slope of $-\frac{1}{5}$.

3. $\boxed{16}$ We can use point-slope form to find the equation:

$$y - (-14) = 6(x - (-5))$$
$$y + 14 = 6x + 30$$
$$y = 6x + 16$$

Therefore, the value of b is 16.

4. $\boxed{\frac{105}{4} \text{ or } 26.25}$ To find the y-intercept, plug in $x = 0$ and solve for y:

$$7(0) = 6\left(15 + \frac{9(0)}{4} - \frac{4y}{7}\right)$$
$$0 = 6\left(15 - \frac{4y}{7}\right)$$
$$0 = 15 - \frac{4y}{7}$$
$$-15 = -\frac{4y}{7}$$
$$\frac{105}{4} = y$$

Of course, the easiest solution is to just use Desmos.

5. $\boxed{-4}$ The line shown has a slope of $\frac{1}{4}$ and a y-intercept at $(0, 1)$. The equation of this line in slope-intercept form is therefore $y = \frac{1}{4}x + 1$. To get a and b, we need to convert this equation into standard form. First, multiply both sides by -4 to get $-4y = -x - 4$. Now add x to both sides to get $x - 4y = -4$. Thus, $a = 1$, $b = -4$, and $ab = -4$. An alternative approach involves plugging in points such as $(0, 1)$ and $(-4, 0)$ to solve for a and b.

6. $\boxed{-45}$ Plugging in $(0, -79)$, we get $-79 = a(0) + b$, which yields $-79 = b$, the y-intercept. Plugging in $(5, 91)$, we get $91 = a(5) - 79$, which simplifies to $170 = 5a$ and finally $34 = a$ (the slope). Given these values, $a + b = 34 + (-79) = -45$.

7. $\boxed{\frac{8}{5} \text{ or } 1.6}$ First, plug the point $(2, 6)$ into the equation of the line so that we can solve for a:

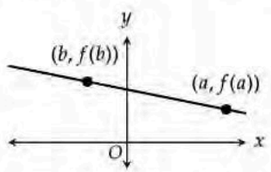
$$a(2) - \frac{1}{3}(6) = 8$$
$$2a - 2 = 8$$
$$2a = 10$$
$$a = 5$$

So the equation of the line is $5x - \frac{1}{3}y = 8$. The x-intercept always has a y-coordinate of 0, so if we set y equal to 0, we get $5x = 8$, which gives $x = \frac{8}{5} = 1.6$.

8. $\boxed{B}$ We can calculate the slope from any two of the points. To demonstrate, we'll use the 2nd and 3rd points: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-p - (-57 - p)}{2 - (-2)} = \frac{57}{4}$.

9. $\boxed{A}$ When $x = 0$, $5(0) - 8y = k \Rightarrow y = -\frac{k}{8}$. This is the y-intercept, n. When $y = 0$, $5x - 8(0) = k \Rightarrow x = \frac{k}{5}$. This is the x-intercept, m. Therefore, $\frac{m}{n} = \frac{\frac{k}{5}}{-\frac{k}{8}} = \frac{k}{5} \cdot -\frac{8}{k} = -\frac{8}{5}$.

10. $\boxed{A}$ I must be true because the line has a negative slope: the larger the x, the smaller the y. See the graph below.



II must be true because for a line with a negative slope, $f(a)$ is always greater than the line's y-intercept when $a < 0$. III is not necessarily true—a negative slope guarantees that $f(a)$ will be negative for large enough values of a.

11. $\boxed{A}$ The slope of line l is $\frac{8-5}{6-(-3)} = \frac{3}{9} = \frac{1}{3}$. The equation of the line in point-slope form is then $y - 8 = \frac{1}{3}(x - 6)$, which simplifies to $y = \frac{1}{3}x + 6$. Now we can test each answer choice by plugging in the x-coordinate and verifying the y-coordinate. Only answer A, $(27, 15)$, satisfies the equation of the line. A second approach is to test each answer choice by calculating the slope with one of the given points.

12. $\boxed{A}$ One easy way to approach this problem is to make up numbers for a and b. Let $a = 1$ and $b = 2$ so that $\frac{a}{b} = \frac{1}{2}$. Since the second line is perpendicular to the first, $\frac{d}{e} = -2$, which satisfies the condition in answer choice A.

13. $\boxed{B}$ Since both lines intersect at $(0, 0)$, line p must pass through $(0, 0)$. Therefore, the slope of line p is $\frac{n-0}{1-0} = n$. The slope of line q is then $-\frac{1}{n}$ (the negative reciprocal), and its equation is $y = -\frac{1}{n}x$ (given that its y-intercept is 0). Now we can check each of the answer choices. Plugging in $x = n$ into our equation, we get $y = -\frac{1}{n}(n) = -1$. Therefore, the point $(n, -1)$ must lie on line q.

14. $\boxed{-\frac{1}{3} \text{ or } -.3333}$ Since x-intercepts, by definition, have a y-coordinate of 0, $b + 4 = 0$, which means $b = -4$. The two given points then become $(-4, -11)$ and $(2, 7)$. The slope of line m is then $\frac{-11-7}{-4-2} = \frac{-18}{-6} = 3$, and the equation of the line in point-slope form is $y - 7 = 3(x - 2)$, which simplifies to $y = 3x + 1$. To solve for the x-intercept, a, we set y equal to 0:

$$0 = 3x + 1$$
$$-1 = 3x$$
$$-\frac{1}{3} = x$$

Thus, $a = -\frac{1}{3}$.

15. $\boxed{B}$ Since f is a linear function, let $f(x) = mx + b$. Then $g(x) = \frac{mx + b + 6}{x}$. Since the graph of g passes through the point $(3, 7)$, we can plug it in to get $7 = \frac{3m + b + 6}{3}$, which yields $21 = 3m + b + 6$. Since the graph also passes through $(5, 3)$, $3 = \frac{5m + b + 6}{5}$, which yields $15 = 5m + b + 6$. So now we have a system of equations:

$$21 = 3m + b + 6$$
$$15 = 5m + b + 6$$

Subtracting the second equation from the first yields $6 = -2m$, so $m = -3$. Plugging this result back into the first equation gives $21 = 3(-3) + b + 6$. Solving this equation yields $b = 24$, which is the y-coordinate of the y-intercept of the graph of f.

16. $\boxed{D}$ The equation of the first given line can be rewritten as $y = \frac{a}{b}x - \frac{5}{b}$, from which we can tell the slope is $\frac{a}{b}$. The equation of the second given line can be rewritten as $y = \frac{11}{4}x - \frac{5}{4}$, from which we can tell the slope is $\frac{11}{4}$. Since the slopes of perpendicular lines are negative reciprocals,

$$\frac{a}{b} = -\frac{4}{11}$$

Now let's get to the question. The slope of $11x + 12y = 5$ is $-\frac{11}{12}$. A line perpendicular to this one must have a slope of $\frac{12}{11}$ (the negative reciprocal). This is the slope we need to look for in the answer choices.

- The slope of the equation in answer A is $\frac{a}{3b} = \frac{1}{3}\left(\frac{a}{b}\right) = \frac{1}{3} \cdot -\frac{4}{11} = -\frac{4}{33}$.
- The slope of the equation in answer B is $-\frac{a}{3b} = -\frac{1}{3}\left(\frac{a}{b}\right) = -\frac{1}{3} \cdot -\frac{4}{11} = \frac{4}{33}$.
- The slope of the equation in answer C is $\frac{3a}{b} = 3\left(\frac{a}{b}\right) = 3 \cdot -\frac{4}{11} = -\frac{12}{11}$.
- The slope of the equation in answer D is $-\frac{3a}{b} = -3\left(\frac{a}{b}\right) = -3 \cdot -\frac{4}{11} = \frac{12}{11}$.

Therefore, the answer is D.

Chapter 7: Interpreting Linear Equations & Models

EXERCISE 1:

1. $\boxed{A}$ The slope is -3, which means the water level decreases by 3 feet each day.

2. $\boxed{28}$ Since the left side of the equation sums to 1,400 puzzle pieces, $36B$ and $64E$ must represent quantities of puzzle pieces. And since B and E are the numbers of beginner-level and intermediate-level puzzle sets, respectively, 36 must be the number of pieces in each beginner-level set and 64 must be the number of pieces in each intermediate-level set. Therefore, an intermediate-level set has $64 - 36 = 28$ more pieces than a beginner-level set.

3. $\boxed{B}$ The coefficient of x is 6. Therefore, each empty glass weighs 6 ounces.

4. $\boxed{4.8}$ Since the 175 on the right-hand side of the equation is in ounces of flour and x is defined as the number of bagels made, $4.8x$ must be the total amount of flour, in ounces, used to make x bagels. Thus, 4.8 must be the number of ounces of flour used to make each bagel.

5. $\boxed{C}$ The expression $x + 11$ represents the 11 starting players and x additional backup players—in other words, the total number of players on the team. In the equation, $x + 11$ is multiplied by 3 since each player receives 3 jerseys.

6. $\boxed{D}$ $(x, y) = (5, 387)$ is just another way of stating that $y = 387$ when $x = 5$. Since y is defined as the number of juniper trees in the forest x years after 2010, 387 must be the number of juniper trees in the forest 5 years after 2010 (in 2015). Note that the answer is not A because there is a difference between the number of juniper trees and an increase in the number of juniper trees.

7. $\boxed{B}$ If all 25 jewelers made bracelets only, they would be able to make 300 bracelets (the y-intercept). This implies that each jeweler works at a rate of $\frac{300}{25} = 12$ bracelets per month (assuming no necklaces are made). Therefore, 12 is the maximum.

8. $\boxed{D}$ Since the left side of the equation sums to 80 ounces of cheese, $0.6x$ and $0.25y$ must each represent ounces of cheese. And since x represents the number of slices, the coefficient 0.6 must then be the number of ounces of cheese in each slice. Therefore, $0.6x$ is the total amount of cheese, in ounces, cut into slices.

9. $\boxed{C}$ We have 590 miles on the right side of the equation and 17 gallons of fuel as the coefficient of y. Look at the units. To "turn" gallons of fuel into miles, y must be in miles per gallon. Therefore, the best interpretation of y is the fuel efficiency of the campervan, in miles per gallon.

10. $\boxed{A}$ Since 315 is the number of color photos and 585 is the total number of photos, $6p$ must be the number of black-and-white photos.

11. $\boxed{C}$ Since $p(x)$ is defined as the number of additional points Stacy needs to win the game, where x is the number of rounds played, $p(8)$ represents the number of additional points Stacy needs to win the game after 8 rounds have been played. Therefore, $p(8) = 20$ means that after the first 8 rounds, Stacy needs 20 additional points to win the game.

12. $\boxed{C}$ The y-intercept is negative—it's the bakery's profit when no cakes are sold. Therefore, anything that varies with the number of cakes sold is incorrect. For example, answer D is wrong because the cost of the cakes that didn't sell depends on how many the bakery did sell. It's not a fixed number like the y-intercept is. The best interpretation of the y-intercept is the cost of running the bakery (rent, labor, machinery, etc.), which is likely a fixed number.

13. $\boxed{A}$ The slope of the line is the rise (change in y) over the run (change in x). In this case, it's the number of miles traveled over the time it took, which is another way of saying the speed of the train.

14. $\boxed{\frac{5}{2} \text{ or } 2.5}$ The slope of the equation is 5, which means the temperature goes up by 5 degrees every hour. So every half hour (30 minutes), the temperature goes up by $0.5 \times 5 = 2.5$ degrees.

15. $\boxed{B}$ The number 2 refers to the slope of -2, which indicates that two fewer teaspoons of sugar should be added for every teaspoon of honey already in the beverage. Don't be fooled by answers (C) and (D), which "reverse" the x and the y (h and s, in this case). The slope is always the change in y for each unit increase in x, not the other way around.

16. $\boxed{D}$ The line crosses the points $(0, 60)$ and $(6, 540)$. The slope of the line is then $\frac{540 - 60}{6 - 0} = 80$. Because the slope is the change in y (total time) over the change in x (number of legal consultations), it represents the time taken per legal consultation. Essentially, the slope of 80 indicates that each legal consultation lasted 80 minutes.

17. $\boxed{\frac{2}{5} \text{ or } .4}$ To see the answer more clearly, we can put the function into $y = mx + b$ form: $T(x) = \frac{2}{5}x + \frac{9}{5}$. The slope is $\frac{2}{5}$, or 0.4, which means the load time increases by 0.4 seconds for each image on the page.

18. $\boxed{D}$ Given the equation, it's obvious that Jamil spent t hours walking at 5 miles per hour and $4 - t$ hours running at 9 miles per hour. Since t, the number of hours spent walking, is being subtracted from 4 to get the number of hours spent running, 4 must be the total number of hours spent walking and running.

19. $\boxed{\frac{35}{2} \text{ or } 17.5}$ If we look at the equation as if it were a line, the slope would be $\frac{7}{2}$, or 3.5. This means that every 1-inch increase in the length of the longer base yields a 3.5-square-inch increase in the area of the trapezoid. So, increasing the length of the longer base by 5 inches would yield a $3.5 \times 5 = 17.5$-square-inch increase in the area of the trapezoid.

EXERCISE 2:

1. $\boxed{B}$ The slope is the change in y (daily profit) for each unit increase in x (cakes sold). Therefore, the slope is the change in daily profit (i.e. profit generated) for each additional cake sold.

2. $\boxed{A}$ Since $f(x)$ is defined as the shelf life, in days, of 120 ounces of yogurt with x grams of potassium sorbate, $f(4)$ represents the shelf life, in days, of 120 ounces of yogurt with 4 grams of potassium sorbate. Therefore, $f(4) = 16$ means that the shelf life of 120 ounces of yogurt with 4 grams of potassium sorbate is 16 days. Note that the answer is not C because there is a difference between shelf life and an increase in shelf life.

3. $\boxed{B}$ The tank started with 204 gallons and ended with 36 gallons. The term $4.2x$ must then represent the total amount of water that left the tank. Since x is the number of minutes after the hole was punched, the coefficient 4.2 must be the rate at which water left the tank, in gallons per minute.

4. $\boxed{C}$ Since the left side of the equation sums to 725 dollars, $12a$ and $18b$ must each be dollar amounts. Since b is the number of pounds of shrimp sold, $18b$ must be the total dollar amount the vendor earned for b pounds of shrimp sold, with 18 being the dollar price per pound.

5. $\boxed{A}$ $(x, y) = (24, 40)$ is just another way of stating that $y = 40$ when $x = 24$. Since y is defined as the number of binders purchased and x is defined as the number of notebooks purchased, 40 must be the number of binders purchased when the number of notebooks purchased is 24.

6. $\boxed{C}$ The y-intercept of 500 means that when $x = 0$ (when no videos were available), the site had 500 subscribers. Note that the answer is not B because 500 is the y-intercept of $s(x)$, which gives the number of subscribers, not the number of videos.

7. $\boxed{C}$ $(x, y) = (6, 470)$ is just another way of stating that $y = 470$ when $x = 6$. Since y is defined as the price per desk for a purchase of x desks, 470 must be the price per desk, in dollars, for a purchase of 6 desks.

8. $\boxed{B}$ Since the left side of the equation sums to 400 patients, $21n$ must represent a certain number of patients. Since n is the number of patients treated by each nurse practitioner at the clinic, $21n$ must then be the total number of patients treated by nurse practitioners, with 21 being the number of nurse practitioners at the clinic.

9. $\boxed{C}$ Pick any point. Let's say 0 small packages and 26 large packages. This combination will take her $0 \times 10 + 26 \times 15 = 390$ minutes. Therefore, $k = 390$. As specified in the question, any point (i.e. combination) given by the graph will yield the same answer.

10. $\boxed{C}$ Since the left side of the equation sums to 5 hours, the terms $7x$ and $4y$ must each represent a certain number of hours. Since the coefficient of y is 4 (4 miles being the length of the sloped path), the units of y must be hours per mile to ensure that $4y$ is in hours. Given these units and the coefficient of 4, y must be the average number of hours it took Oscar to walk each mile of the sloped path.

11. $\boxed{B}$ Since the left side of the equation sums to 80 banners, the terms $\frac{x}{14}$ and $\frac{y}{25}$ must each represent a certain number of banners. Since x is the dollar amount earned from premade banners, 14 must be the dollar amount earned for each premade banner sold. This way, the term $\frac{x}{14}$ gives the number of premade banners sold.

12. $\boxed{A}$ The number 4 refers to the slope of -4, which indicates that an increase of 1° C decreases the number of hours until a gallon of milk goes sour by 4. In other words, the milk goes sour 4 hours faster.

13. $\boxed{B}$ Expanding $h(t)$, we get $h(t) = 0.07t + 7$. This is a linear model with a slope of 0.07, which indicates that the height of the tree grows by 0.07 inches for each unit increase in t (i.e. each day).

14. $\boxed{B}$ When $t = 0$, there is no time left in the auction. The auction has finished. Therefore, 900 is the final auction price of the lamp.

15. $\boxed{C}$ $7y$ represents the total number of stickers the teacher printed and $2x$ represents the total number of stickers the teacher gave out at the party. According to the equation, $2x$ is 32 less than $7y$, which means the total number of stickers the teacher gave out is 32 less than the total number she printed. In other words, she printed 32 more stickers than she gave out. This interpretation might be easier to see if we add 32 to both sides of the equation to get $7y = 2x + 32$.

16. $\boxed{A}$ Because it's the slope, the 1.30 can be thought of as the exchange rate, converting U.S. dollars into euros. But after the conversion, 1.50 is subtracted away, which means you get 1.50 euros less than you should. Thus, the 1.50 is best interpreted as a 1.50 euro fee the bank charges to do the conversion.

17. $\boxed{B}$ The function $C(x)$ is a linear model with a slope of 0.07, which indicates that the monthly cost of the plan increases by 0.07 dollars for each minute increase in the usage per month. Therefore, increasing the usage per month by 75 minutes would increase the monthly cost by $75 \times 0.07 = 5.25$ dollars.

18. $\boxed{C}$ Because this question is asking for the change in "x" per change in "y" (the reverse of slope), we need to rearrange the given equation $C = 1.5 + 2.5x$ to get x in terms of C. Dividing each term in the equation by 2.5, we get $0.4C = 0.6 + x$, which yields $x = 0.4C - 0.6$. The slope here is 0.4, which means the weight of a shipment increases by 0.4 pounds per dollar increase in the mailing cost. So a 10 dollar increase in the mailing cost is equivalent to a weight increase of $10 \times 0.4 = 4$ pounds.

Chapter 8: Percent

EXERCISE 1:

1. $\boxed{875}$ Let the number be x. Then $280 = 0.32x$, which yields $x = \frac{280}{0.32} = 875$.

2. $\boxed{B}$

$$\frac{n}{100} \times 40 = 9.6$$
$$40n = 960$$
$$n = 24$$

Then 24% of $250 = (0.24)(250) = 60$

3. $\boxed{B}$ If 15% of the town's residents live within the downtown area, then 100% - 15% = 85% of the residents must live outside the downtown area. Since the town has r residents in total, $0.85r$ residents live outside the downtown area.

4. $\boxed{D}$ A 0.12 multiplication factor corresponds to a $1 - 0.12 = 0.88 = 88\%$ decrease. Put another way, $0.12x$ is 12% of x, and 12% of x is equivalent to x decreased by 88%.

5. $\boxed{C}$ A 2% increase corresponds to a 1.02 multiplication factor. Here's the proof. First, let x be the electricity consumption in 2022. Then the electricity consumption in 2023 is $x + 0.02x = 1.02x$. Therefore, $n = 1.02$.

6. $\boxed{600}$ $150\left(1 + \frac{300}{100}\right) = 150(1 + 3) = 600$

7. $\boxed{C}$ Let the total population of Portugal be x million. Then $0.28x = 2.3$, and $x = \frac{2.3}{0.28} \approx 8.2$ million.

8. $\boxed{\frac{145}{4} \text{ or } 36.25}$ If the pre-coupon price of the book is x, then the discounted price is $0.80x$. Since $0.80x = 29$, $x = \frac{29}{0.80} = 36.25$.

9. $\boxed{80}$ This is a percent decrease problem, where 90 is the "new" value and 450 is the "original" value:

$$\frac{90 - 450}{450} \times 100\% = -80\%$$

Therefore, $u = 80$. Note that the negative sign should be ignored since the wording of the question already implies u is a percent decrease.

10. $\boxed{140}$ Let x be the number of dishes served during lunch. Then

$$1.175x = 940$$
$$x = 800$$

Therefore, $940 - 800 = 140$ more dishes were served during dinner.

11. $\boxed{D}$ If x is the amount Sims spent in 2015, then $1.76x$ is the amount she spent in 2016. In 2017, she spent 145% more than what she spent in 2016:
$(1 + 1.45)(1.76x) = (2.45)(1.76x) = 4.312x$.
Therefore, $p = 4.312$.

12. $\boxed{650}$ Since p is equal to both $0.60q$ and $3.90r$,

$$0.60q = 3.90r$$
$$q = \frac{3.90}{0.60}r = 6.5r$$

Since q is 6.5 times r, q is $6.5 \times 100\% = 650\%$ of r. Therefore, $x = 650$.

13. $\boxed{D}$ If the number of clothing items in the store is x, then the number of vests is $0.24x$. Since 36% of the vests are blue, the number of blue vests is $(0.36)(0.24x)$. Since 78% of the blue vests are fleece, the number of blue fleece vests is $(0.78)(0.36)(0.24x) = 0.067392x$. Therefore, 6.7392% of the clothing items in the store are blue fleece vests, which means that $100\% - 6.7392\% \approx 93.26\%$ of the clothing items are not blue fleece vests.

14. $\boxed{39}$ Based on the information, $V_{\text{start}} = 2.58V_{\text{end}}$. Dividing both sides of this equation by 2.58 yields $\frac{1}{2.58}V_{\text{start}} = V_{\text{end}}$. Therefore, V_{end} is $\frac{1}{2.58} \approx 0.39 = 39\%$ of V_{start}, which means $n = 39$.

15. $\boxed{D}$ Let x be the amount of taxes, in millions of dollars, collected by County A in 2016. Since the taxes decreased by 25% from 2016 to 2017,

$$(1 - 0.25)x = 60$$
$$0.75x = 60$$
$$x = 80$$

Because County B collected the same amount as County A in 2016, County B also collected 80 million dollars of taxes in 2016. In 2017, County B collected 20% more than in 2016, so County B must have collected $80(1.20) = 96$ million dollars in 2017.

EXERCISE 2:

1. $\boxed{D}$ $32{,}000(1.15) = 36{,}800$

2. $\boxed{C}$ A multiplication factor of 1.84 corresponds to an 84% increase. Here's the proof. If we let x be the original price, then the current price is $1.84x$. Now we can use the percent change formula:
$\frac{1.84x - x}{x} \times 100\% = \frac{0.84x}{x} \times 100\% = 84\%$.

3. $\boxed{\frac{50}{3} \text{ or } 16.66 \text{ or } 16.67}$ Use the percent change formula, where 22,500 is the original value and 18,750 is the new value:

$$\frac{18{,}750 - 22{,}500}{22{,}500} \times 100\% = -16.67\%$$

Therefore, $k = 16.67$. Note that the negative sign should be ignored since the question already specifies k as a percent decrease. Also note that 16.66 and $\frac{50}{3}$ are acceptable answers because we can truncate decimals or use improper fractions on the SAT.

4. $\boxed{408}$ Setting up an equation, we get $(1 + \frac{2{,}400}{100})y = 10{,}200$, which simplifies to $25y = 10{,}200$, so $y = \frac{10{,}200}{25} = 408$.

5. $\boxed{C}$ Let x be the total number of mice. Then $0.60x$ are male, and $0.0005(0.60x) = 0.0003x$ are male and carry the rare genetic mutation. This result is equivalent to $0.0003 = 0.03\%$ of the mice population.

6. $\boxed{\frac{125}{2} \text{ or } 62.5}$ If the cost before tax is x, then the cost after tax is $1.08x$. Since $1.08x = 67.50$, $x = \frac{67.50}{1.08} = 62.5$ dollars. Note that trailing zeros are not necessary when entering your answer on the SAT.

7. $\boxed{B}$ Let the retail price of the travel bag be $\$x$. Then the wholesale price is $(1 - 0.10)x = 0.9x$. James bought the bag at a price of $(1 - 0.80)(0.9x) = (0.2)(0.9x) = 0.18x$, which is equivalent to 18% of the retail price.

8. $\boxed{700}$ $2.20c = 880$, so $c = \frac{880}{2.20} = 400$. The number 400 is $\frac{400 - 50}{50} \times 100\% = 700\%$ greater than 50 (by the percent change formula).

9. $\boxed{\frac{125}{2} \text{ or } 62.5}$ Let x be the number of liters of paint in the container at the start. To paint the shed, Keaton used 8% of the paint, so 92% remains. The remaining number of liters is then $0.92x$. The question states that 57.5 liters remain, so $0.92x = 57.5$, and $x = \frac{57.5}{0.92} = 62.5$.

10. $\boxed{44400}$ First, we find the starting price: $30{,}000(1 - 0.60) = 30{,}000(0.4) = 12{,}000$ dollars. Eli's offer must then be $3.70(12{,}000) = 44{,}400$ dollars.

11. $\boxed{\frac{135}{4} \text{ or } 33.75}$ Based on the information, $r = (1 - 0.70)s = 0.30s$ and $r = 1.60t$. Since $s = 180$, $r = 0.30(180) = 54$ and $t = \frac{r}{1.60} = \frac{54}{1.60} = 33.75$.

12. $\boxed{B}$ Let the mass of the iceberg be x in 2005. Its mass in 2010 is then $(1 - 0.14)x = 0.86x$, and its mass in 2015 is $(1 + 1.88)(0.86x) = (2.88)(0.86x) = 2.4768x$. By the percent change formula, the net percentage increase from 2005 to 2015 is $\frac{2.4768x - x}{x} \times 100\% = 147.68\%$.

13. $\boxed{\frac{23}{50} \text{ or } .46}$ Based on the given information, $x = (1 - 0.75)y = 0.25y$ and $z = 1.84x$. Using the first equation to substitute for x in the second equation, we get $z = 1.84(0.25y) = 0.46y$. Therefore, z is 0.46 times y.

14. $\boxed{A}$ Since $a = \frac{25}{16}b$, we can multiply both sides by 16 and divide both sides by 25 to get $b = \frac{16}{25}a = 0.64a$. A multiplication factor of 0.64 corresponds to a percent decrease of $1 - 0.64 = 0.36 = 36\%$. Therefore, b is 36% less than a. Note that the answer is NOT 56.25%. It's true that a is 56.25% greater than b, but that does not mean b is less than a by 56.25%.

15. $\boxed{C}$ Setting up an equation, we get

$$a + b = 4.76c$$

We ultimately want a in terms of c so we need to substitute for b. Since $a = 1.24b$, $b = \frac{a}{1.24}$. Substituting this result for b in the equation, we get

$$a + \frac{a}{1.24} = 4.76c$$
$$1.24a + a = (4.76)(1.24)c$$
$$2.24a = (4.76)(1.24)c$$
$$a = \frac{(4.76)(1.24)c}{2.24} = 2.635c = 263.50\% \text{ of } c$$

Note that in the second step, we multiplied both sides by 1.24 to get rid of the fraction.

16. $\boxed{B}$

$$\text{Town A} = \left(1 + \frac{n}{100}\right)(\text{Town B})$$
$$700 = \left(\frac{100}{100} + \frac{n}{100}\right)(\text{Town B})$$
$$700 = \left(\frac{100 + n}{100}\right)(\text{Town B})$$

Now multiply both sides by 100 and divide both sides by $100 + n$ to get

$$700\left(\frac{100}{100 + n}\right) = \text{Town B}$$

Chapter 9: Exponential vs. Linear Growth

EXERCISE 1:

1. $\boxed{B}$ Each month, Albert loses a book. Because this is a constant decrease, the relationship is linear decay (decreasing linear).

2. $\boxed{D}$ The cell count doubles every hour so the growth factor, r, is 2. The initial count is 80 so $c = 80$.

3. $\boxed{B}$ $12{,}000(0.89)^3 \approx 8{,}460$.

4. $\boxed{B}$ We're given that $y = 4$ when $x = 0$ so 4 is our initial value. Now each decrease in y by 25% means that 75% (or 0.75) of y remains. So when x is increased to 1, $y = 4(0.75)$. When $x = 2$, $y = 4(0.75)(0.75) = 4(0.75)^2$. When $x = 3$, $y = 4(0.75)(0.75)(0.75) = 4(0.75)^3$. To generalize, $y = 4(0.75)^x$.

5. $\boxed{D}$ Since the employees stock shelves at a constant rate, the number of shelves left to be stocked decreases at a constant rate over time. Therefore, the function p is a decreasing linear function.

6. $\boxed{B}$ At the end of the 1st year, the value of the car is $10{,}000(0.65)$. At the end of the 2nd year, the value of the car is 65% of what it was at the end of the 1st year: $10{,}000(0.65)(0.65) = 10{,}000(0.65)^2$. Generalizing this pattern yields $v(k) = 10{,}000(0.65)^k$.

7. $\boxed{C}$ Since $p(t)$ models the town population t years after 2008, the value of $p(4)$ must be the population of the town 4 years after 2008 (the year 2012). Answers A, B, and D are wrong because the output of $p(t)$ is a population size, not a change in population.

8. $\boxed{D}$ The given definition of f is in the form of an exponential equation, where 20 is the metal alloy's temperature at the beginning of the experiment and 15 is the percent by which it increased each second. We know the temperature increased each second because the growth factor, $1 + \frac{15}{100} = 1.15$, is greater than 1. For answer C to be correct, the growth factor would have to be $1 - \frac{15}{100} = 0.85$.

9. $\boxed{A}$ The number 31.6 is the initial value of $m(t)$ when $t = 0$. Note that $m(0) = 31.6$. Since t represents the number of years after 1976, $t = 0$ represents zero years after 1976, or just 1976. Therefore, the number 31.6 is the number of members, in thousands, the wholesale club had in 1976.

10. $\boxed{B}$ The graph gives a value of 72 when $x = 0$. Therefore, $f(0) = a(b)^0 = a = 72$. Now that we know the value of a, we can easily solve for b by using another point on the graph. Since the graph passes through $(1, 61)$, roughly speaking, $61 = a(b)^1$, which yields $61 = 72b$. Thus, $b \approx \frac{61}{72} \approx 0.85$. Finally, $\frac{a}{b} = \frac{72}{0.85} \approx 85$.

11. $\boxed{.728}$ After 3 years, the market value of the bond is $900(1.2)^3 = 900(1.728) = 900(1 + 0.728)$. Therefore, $p = 0.728$.

12. $\boxed{C}$ Since all the answer choices are equivalent forms of the same function $f(x)$, we can use any one of them to calculate $f(4)$. Let's use answer A: $f(4) = 2\left(\frac{1}{2}\right)^4 = 2\left(\frac{1}{16}\right) = \frac{1}{8}$. The only answer choice that has $\frac{1}{8}$ as the base or the coefficient is answer C, which has $\frac{1}{8}$ as the base.

13. $\boxed{D}$ Since we have a case of exponential decay, we can use the equation $y = ab^{\frac{t}{k}}$ as a reference. The population of trees starts at 14,000 and decays exponentially by a factor of $1 - 0.06 = 0.94$ every 4 years. Therefore, $a = 14000$, $b = 0.94$, $k = 4$, and the equation is $P = 14{,}000(0.94)^{\frac{t}{4}}$.

14. $\boxed{B}$ The given equation tells us that the resale value increases by a factor of 1.6 every year. That's the same as a 1.6 factor increase every 12 months. This conversion to months is important: because m is in months, the k-value also needs to be in months. So which answer choice has a growth factor of 1.6 and a k-value of 12? Only answer choice B.

15. $\boxed{C}$ Based on the model, which we can express as $N(h) = 1{,}000(0.97)^{4h} = 1{,}000(0.97)^{h/(1/4)}$, 97% of the bacteria remain after every $\frac{1}{4}$ hour. That's a 3% decrease every $\frac{1}{4} \times 60 = 15$ minutes.

16. $\boxed{D}$ Let's use the standard exponential equation $y = ab^{\frac{t}{k}}$ as a reference. From the given information, $a = 16$ (the initial amount) and $b = 1.02$ (the growth factor). Since the standard exponential equation requires t and k to be in the same units, we have to convert 15 hours into days, which gives $k = \frac{15}{24} = \frac{5}{8}$ days. Putting everything together, we get $g(t) = 16(1.02)^{t/(5/8)} = 16(1.02)^{\frac{8t}{5}}$.

EXERCISE 2:

1. $\boxed{D}$ Given the percent increase each year, s is best represented by an exponential function of the form ar^t, where a is the initial value and r is the growth factor. Based on the information given, the initial value is 10,000. The growth factor is 1 plus the percent change (expressed as a decimal): $1 + \frac{160}{100} = 2.6$. Therefore, $s = 10{,}000(2.6)^t$.

2. $\boxed{C}$ Notice that $r(0) = 11$, $r(1) = 11(0.92)$, $r(2) = 11(0.92)(0.92)$, and so on. For each year, the value of $r(t)$ is 0.92 times the value from the year before. In other words, each year's population is 92% of the population from the year before. Therefore, 0.92 is best interpreted as the proportion of the population that remains after each year. It's worth noting that this is equivalent to an 8% decrease in the population each year.

3. $\boxed{D}$ 0.75% percent of the crocodile's initial birth weight is a constant value that doesn't change. A weekly increase of this fixed amount constitutes linear growth.

4. $\boxed{C}$ The y-intercept is the point at which $x = 0$: $y = f(0) = 2.9(1.07)^0 = 2.9$. Since x is defined as the number of years since the artwork disappeared, $x = 0$ can be interpreted as "0 years since the artwork disappeared"—in other words, the exact time it disappeared. And since y is defined as the estimated value of the artwork, in millions of dollars, the y-intercept of 2.9 is best interpreted as the estimated value of the artwork, in millions of dollars, at the time it disappeared.

5. $\boxed{A}$ Since the growth factor is 2 and $\frac{t}{5}$ is the exponent, the predicted number of infected cells doubled every 5 days. So after 5 days, $C(5) = 80(2)^1 = 160$. After 10 days, $C(10) = 80(2)^2 = 320$. And so on.

6. $\boxed{C}$ The general form of an exponential function is ar^t, where r is the growth rate, t is the time that has passed, and a is the initial value (when $t = 0$). Here, the initial value is \$326.50, so we can eliminate answer choices A and B. We can decide between C and D by checking whether they give $V(t) = 413.60$ when $t = 8$. Only choice C outputs a value of 413.60 when $t = 8$.

7. $\boxed{C}$ The function f can be defined by $f(x) = \frac{1}{2}x$. This is a line with a positive slope of $\frac{1}{2}$, which means for each unit increase in x, $f(x)$ increases by $\frac{1}{2}$. Therefore, f is an increasing linear function. Don't be fooled by the word "half." What determines the increasing/decreasing behavior of a function is how the output changes as the input changes, not how big or small the output is relative to the input.

8. $\boxed{B}$ Test the answer choices. Only B and D give the correct value of \$30 when $n = 1$ (the first day's payment). Between B and D, only B is an exponential function with the correct growth factor of 1.5. Note that the nuance to this question is that n starts at 1 rather than 0.

9. $\boxed{\frac{1}{10} \text{ or } .1}$ **Solution 1:** Notice that the growth factor given by the equation is 3, and 30,000 and 10,000 are one factor of 3 apart. Therefore, it takes 10 days for the number of bees to increase by the given growth factor, which means the k-value is 10 and the function h can be written as $h(t) = 10{,}000(3)^{\frac{t}{10}}$. The value of b is then $\frac{1}{10}$.

Solution 2: We're given that $h(t) = 30{,}000$ when $t = 10$, so we can form the following equation:

$$\begin{aligned} 30{,}000 &= 10{,}000(3)^{10b} \\ 3^1 &= 3^{10b} \\ 1 &= 10b \\ \frac{1}{10} &= b \end{aligned}$$

Note that we were able to equate the exponents once both sides had the same base of 3.

10. $\boxed{16}$ Let's find the k-value by manipulating the exponent:

$$830(1.09)^{\frac{6}{8}t} = 830(1.09)^{t/\left(\frac{8}{6}\right)}$$

Now we can easily see that the k-value is $\frac{8}{6}$, or $\frac{4}{3}$. Given that t is in years, this k-value must also be in years. In context, this k-value means that the value of the investment increases by 9% every $\frac{4}{3}$ years. Since $\frac{4}{3}$ years is equivalent to $\frac{4}{3} \times 12 = 16$ months, the answer is 16.

11. $\boxed{A}$ $g(x) = 256(0.61)^{\frac{x}{30}} = 256(0.61)^{\frac{x}{3 \cdot 10}} = 256\left((0.61)^{\frac{1}{3}}\right)^{\frac{x}{10}} = 256(\sqrt[3]{0.61})^{\frac{x}{10}} \approx 256(0.85)^{\frac{x}{10}}$. Since 0.85 is 0.15 less than 1, we can write $0.85 = 1 - \frac{15}{100}$. Therefore, $k = 15$.

12. $\boxed{C}$ **Solution 1:** The twist to this question is that it gives us the value of the pendant in 1991 but wants us to model its value after 1990. To do so, let's find the value of the pendant in 1990. If we let x be the value of the pendant in 1990, then its value in 1991 is $0.96x$ (a decrease of 4%). Setting up an equation, we get $0.96x = 840$, which gives $x = \frac{840}{0.96} = 875$. The exponential model is then $v = 875(0.96)^t$.

Solution 2: Since 1991 is one year after 1990, we can plug $t = 1$ into each of the answer choices to see which of them produces a value of 840. Only answers C and D. Between these two answers, only answer C is in the form of an exponential decay model. Another way to decide between these two answers is to find the value of the pendant in 1992 (simply 840(0.96)) and then compare the values for $t = 2$.

13. $\boxed{D}$ From the numbers in the table, we can see that the price increases by a factor of 1.2 every 10 days (e.g. $\frac{60}{50} = 1.2$ from Day 0 to Day 10). Since r represents the growth factor and n represents the time period needed for a single change by the growth factor, a possible solution is $r = 1.2$ and $n = 10$. Unfortunately, this is not one of the answer choices. However, an increase by a factor of 1.2 every 10 days is equivalent to two increases by a factor of 1.2 every 20 days. Therefore, another possible solution is $r = (1.2)(1.2) = 1.44$ and $n = 20$, which is an answer choice.

14. $\boxed{D}$ From the answer choices, it's clear that the function f is an exponential function with a growth factor of 1.25. Since the growth factor is greater than 1, the minimum of f occurs when the exponent of the growth factor is as small as possible. In all the answer choices, the exponent is minimized when $x = 0$. So, we can plug in $x = 0$ into any of the answer choices to find the minimum value of f. Using answer choice A, we get $f(0) = 50(2.5)(1.25)^{-2} = 80$. The minimum value of f is therefore 80. Only answer choice D displays the number 80 as a constant or coefficient.

15. $\boxed{C}$ Since the growth factor in the equation is 1.002, the number of cars registered increases by $1.002 - 1 = 0.002 = 0.2\%$ every $\frac{1}{2}$ year (6 months). Therefore, $n = 0.2$.

16. $\boxed{C}$ **Solution 1:** Process of elimination. Because of the 5% increase each year, the answer must have a growth factor of 1.05. This requirement eliminates answer choice A. Given that $T = 40{,}000$ when $n = 4$, we can plug $n = 4$ into the remaining answer choices to see which of them produces 40,000. Only answer choice C.

Solution 2: We can work backwards to find the initial number of tickets offered for sale in the festival's theoretical "0th year." Although the "0th year" doesn't quite make sense in the context of this problem, we can just think of it as an initial value for the purposes of this model. If we let x be the number of tickets in the 0th year, then $x(1.05)^4$ represents the number of tickets in the festival's 4th year. Setting up an equation, we get $x(1.05)^4 = 40{,}000$, which gives $x = \frac{40{,}000}{(1.05)^4}$. The final model is then

$$T = \frac{40{,}000}{(1.05)^4}(1.05)^n = 40{,}000 \cdot \frac{(1.05)^n}{(1.05)^4} = 40{,}000(1.05)^{n-4}$$

If we just think about it intuitively, we need an exponent of $n - 4$ to account for the fact that we're starting from year 4 rather than year 0. This solution is the underlying proof.

17. $\boxed{D}$ Since n is the number of years after 2003, we want the "initial value" of our new model $q(n)$ to be the number of thistle plants in 2003. To calculate this initial value, we can use the given function $p(t)$: $p(3) = 870(1.14)^3$. Note that our new model will have the same growth factor of 1.14. Putting everything together, we get

$$q(n) = (\text{initial value})(\text{growth factor})^n = 870(1.14)^3(1.14)^n$$

Chapter 10: Rates, Conversions, & Ratios

EXERCISE 1:

1. $\boxed{432}$ $12\ \cancel{\text{minutes}} \times \dfrac{90\text{ words}}{2.5\ \cancel{\text{minutes}}} = 432$ words

2. $\boxed{128}$ There are 8 markers for every 3 pencils, so there must be $\dfrac{8}{3} \times 48 = 128$ markers.

3. $\boxed{14}$ After spending a total of $5 \times 12 = \$60$ on blankets, Emma has $270 - 60 = \$210$ left to spend on pillows. Since each pillow costs \$15, she can buy $210 \div 15 = 14$ pillows.

4. $\boxed{C}$ When $a = 3$, $\dfrac{3}{b} = \dfrac{36}{z}$. Cross multiplying yields $36b = 3z$, which results in $b = \dfrac{3z}{36} = \dfrac{z}{12}$.

5. $\boxed{71280}$ $9\ \cancel{\text{furlongs}} \times \dfrac{220\ \cancel{\text{yards}}}{1\ \cancel{\text{furlong}}} \times \dfrac{36\text{ inches}}{1\ \cancel{\text{yard}}} = 71{,}280$ inches

6. $\boxed{B}$ There are 12 grapes for every 5 blueberries. The fruit salad must then have $\dfrac{12\text{ grapes}}{5\text{ blueberries}} \times x\text{ blueberries} = \dfrac{12}{5}x$ grapes.

7. $\boxed{3}$ The pool has a capacity of $5 \times 300 = 1{,}500$ gallons. At an increased rate of 500 gallons per hour, it would only take $1{,}500 \div 500 = 3$ hours to fill the pool.

8. $\boxed{B}$ $\dfrac{990\ \cancel{\text{miles}}}{1\ \cancel{\text{hour}}} \times \dfrac{1{,}760\text{ yards}}{1\ \cancel{\text{mile}}} \times \dfrac{1\ \cancel{\text{hour}}}{60\ \cancel{\text{minutes}}} \times \dfrac{1\ \cancel{\text{minute}}}{60\text{ seconds}} = \dfrac{484\text{ yards}}{1\text{ second}}$

9. $\boxed{B}$ $20\ \cancel{\text{apples}} \times \dfrac{d\text{ dollars}}{a\ \cancel{\text{apples}}} = \dfrac{20d}{a}$ dollars

10. $\boxed{D}$ $\dfrac{d}{0.3d} = \dfrac{1}{0.3} = \dfrac{1}{3/10} = \dfrac{10}{3}$

11. $\boxed{A}$ The first 150 miles took $150 \div 30 = 5$ hours. The next 200 miles took $200 \div 50 = 4$ hours. His average speed, total distance over total time, was $(150 + 200)/(5 + 4) \approx 38.89$ miles per hour.

12. $\boxed{\dfrac{25}{2}\text{ or } 12.5}$ To maintain the same ratio, things must be added or removed in the same ratio. Since a ratio of 16:20 simplifies to 4:5, there are 4 cups of milk for every 5 cups of flour in the mixture. So, adding more milk means that there must be 5 additional cups of flour for every 4 additional cups of milk. For 10 additional cups of milk, we need $\dfrac{5}{4} \times 10 = 12.5$ additional cups of flour.

13. $\boxed{D}$ $5.5\ \cancel{\text{miles}}^2 \times \dfrac{1\text{ kilometer}}{0.62\ \cancel{\text{mile}}} \times \dfrac{1\text{ kilometer}}{0.62\ \cancel{\text{mile}}} = \dfrac{5.5}{(0.62)(0.62)}\text{ kilometers}^2 \approx 14.3\text{ kilometers}^2$

14. $\boxed{45}$ The 5 to 1 ratio means that 1 out of every 6 coupons is unexpired: $\dfrac{1}{6} \times 270 = 45$ unexpired coupons.

15. $\boxed{D}$ The number of diners increased at a rate of $\dfrac{120 - 30}{45} = 2$ per minute, which means that the number of diners at 24 minutes was $24 \times 2 = 48$ greater than at the start. Since there were 30 diners at the start, there must have been $30 + 48 = 78$ diners at 24 minutes.

16. $\boxed{A}$ $\dfrac{7.1 \times 10^{15}\ \cancel{\text{hydrogen ions}}}{1\ \cancel{\text{mL}}} \times \dfrac{0.8\text{ grams}}{4.8 \times 10^{23}\ \cancel{\text{hydrogen ions}}} \times \dfrac{1000\ \cancel{\text{mL}}}{1\text{ L}} \approx 1.2 \times 10^{-5}\ \dfrac{\text{grams}}{\text{L}}$

17. $\boxed{C}$ $\dfrac{740\text{ sea lions}}{4{,}200\ \cancel{\text{hectares}}} \times \dfrac{259\ \cancel{\text{hectares}}}{1\text{ mile}^2} = \dfrac{740 \times 259\text{ sea lions}}{4{,}200\ \text{mile}^2} \approx 45.6\ \dfrac{\text{sea lions}}{\text{mile}^2}$

18. $\boxed{A}$ Jared's rate is $\dfrac{240}{15} = 16$ pages per hour. Robert's rate must then be $16 \times 2 = 32$ pages per hour. It would take Robert $\dfrac{120}{32} = 3.75$ hours to review the 120-page report. That's $3.75 \times 60 = 225$ minutes.

19. $\boxed{C}$ The racecar burned $22 - 18 = 4$ gallons of fuel in $7 - 4 = 3$ laps. To get to 6 gallons left, the racecar will have to consume $18 - 6 = 12$ more gallons. That's $12\ \cancel{\text{gallons}} \times \dfrac{3\text{ laps}}{4\ \cancel{\text{gallons}}} = 9$ more laps, which is Lap $7 + 9 = 16$.

EXERCISE 2:

1. $\boxed{125}$ $\dfrac{10\text{ paperback}}{4\text{ hardcover}} \times 50\text{ hardcover} = 125$ paperback copies

2. $\boxed{B}$ $10w\ \cancel{\text{feet}} \times \dfrac{5\text{ inches}}{16\ \cancel{\text{feet}}} = \dfrac{50w}{16}$ inches $= \dfrac{25}{8}w$ inches

3. $\boxed{B}$ The pressure increases by $70 - 50 = 20$ atm while the submarine descends $-900 - (-700) = -200$ meters. That's $20 \div 200 = 0.1$ atm per meter, or 1 atm per 10 meters.

4. $\boxed{B}$ $2\ \cancel{\text{hours}} \times \dfrac{60\ \cancel{\text{minutes}}}{1\ \cancel{\text{hour}}} \times \dfrac{32\text{ kilometers}}{14.5\ \cancel{\text{minutes}}} \approx 265$ kilometers

5. $\boxed{D}$ $3\ \cancel{\text{cups of castor oil}} \times \dfrac{3/2\text{ cups of lye}}{2/5\ \cancel{\text{cup of castor oil}}} = 11.25$ cups of lye

6. $\boxed{3}$ Remember that ratios are essentially just fractions. So the given ratio is equivalent to $\dfrac{2.25}{1.5} = 1.5 = \dfrac{3}{2} = 3:2$. Therefore, $n = 3$.

7. $\boxed{16}$ $1{,}000\ \cancel{\text{kulack}} \times \dfrac{29\ \cancel{\text{pikol}}}{400\ \cancel{\text{kulack}}} \times \dfrac{2\text{ large bahar}}{9\ \cancel{\text{pikol}}} \approx 16$ large bahar

8. $\boxed{5.29}$ $\dfrac{\$5.75}{1\ \cancel{\text{kilogram}}} \times \dfrac{0.92\ \cancel{\text{kilogram}}}{1\text{ meter}^3} = \dfrac{\$5.29}{1\text{ meter}^3}$

9. $\boxed{B}$ An 8 inch by 10 inch piece of cardboard has an area of $8 \times 10 = 80$ square inches. A 16 inch by 20 inch piece of cardboard has an area of $16 \times 20 = 320$ square inches. The cost of the 16 by 20 inch piece is then $320\ \cancel{\text{square inches}} \times \dfrac{2\text{ dollars}}{80\ \cancel{\text{square inches}}} = 8$ dollars.

10. $\boxed{1.5\text{ or }\dfrac{3}{2}}$ If x is increased by 50% (i.e. multiplied by 1.5), then y must also be increased by 50% to maintain the same ratio: $\dfrac{1.5x}{1.5y} = \dfrac{x}{y}$. Therefore, y must be multiplied by 1.5.

11. $\boxed{20}$ The given ratio of 17 to 28 does not reduce or simplify any further, which means that sixth graders and eighth graders must come in "batches" of $17 + 28 = 45$. In other words, the total number of sixth and eighth graders must be a multiple of 45. In the case that this total is the first multiple of 45, 45 itself, the remaining $110 - 45 = 65$ students must be seventh graders. In the case that this total is $45 \times 2 = 90$, the remaining $110 - 90 = 20$ students must be seventh graders. Therefore, the possible values of n are 20 and 65, with 20 being the least possible value. Notice that we don't have to consider multiples of 45 higher than 90 since those multiples would exceed the total number of students.

12. $\boxed{A}$ $\frac{24{,}960 \text{ fluid ounces}}{1 \text{ minute}} \times \frac{1 \text{ gallon}}{128 \text{ fluid ounces}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 3.25 \frac{\text{gallons}}{\text{second}}$

13. $\boxed{C}$ A pass allows for a total of $8 \times 14 = 112$ minutes of play. Since a pass costs $40 \times 100 = 4{,}000$ cents, the cost per minute of play is then $\frac{4{,}000}{112} \approx 35.7$ cents.

14. $\boxed{9}$ Walnuts are the "common denominator" between the two ratios. Since the least common multiple of 6 and 4 is 12, we can multiply the ratio of pistachios to walnuts by 2 to get 10:12, and we can multiply the ratio of walnuts to cashews by 3 to get 12:45. Now that we have this common denominator of 12 in both ratios, it's easy to see that the ratio of pistachios to cashews is 10 to 45, which simplifies to 2 to 9. Therefore, $k = 9$.

15. $\boxed{D}$ Let y be the price of Product Y. Then the price of Product X is $1.25y$ and the price of Product Z is $0.75y$. Simplifying the ratio of their prices, we get $\frac{1.25y}{0.75y} = \frac{1.25}{0.75} = \frac{125}{75} = \frac{5}{3}$.

16. $\boxed{A}$ $\frac{3 \text{ inches}^2}{1 \text{ hour}} \times \frac{1 \text{ feet}^2}{144 \text{ inches}^2} \times \frac{0.3048 \text{ meter}}{1 \text{ foot}} \times \frac{0.3048 \text{ meter}}{1 \text{ foot}} = \frac{3(0.3048)^2}{144} \frac{\text{meters}^2}{\text{hour}} = 0.0019 \frac{\text{meters}^2}{\text{hour}}$

17. $\boxed{100}$ It took 2.5 hours for $65 - 40 = 25$ boxes to be unloaded. There are 3.5 hours from 3:30PM to 7:00PM. In 3.5 hours, $3.5 \text{ hours} \times \frac{25 \text{ boxes}}{2.5 \text{ hours}} = 35$ more boxes will be unloaded. That's a total of $65 + 35 = 100$ boxes.

18. $\boxed{B}$ **Solution 1:** I'll explain the most intuitive solution first. Imagine that the new car is able to travel 60 miles per gallon instead of 40. Since 60 miles per gallon is twice the current car's 30 miles per gallon, the new car would be able to travel twice the distance per gallon. Twice the distance means that Brett would only need to spend half as much on gas. It's a reciprocal relationship. If Brett could travel 3 times as far on each gallon, he'd only be spending $\frac{1}{3}$ as much. Now let's apply this logic to 40 miles per gallon. If the new car gets 40 miles per gallon, it's able to travel $\frac{4}{3}$ times as far as the current car (per gallon). Brett would then only need to spend $\frac{3}{4}$ as much, which would be $\frac{3}{4} \times \$160 = \$120$.

Solution 2: What's tricky is that we don't know the distance Brett travels each month or the number of gallons he uses each month. But we need to start somewhere, so let's say he needs 2 gallons of gas each month (you can make up any number you want). That means he travels $30 \times 2 = 60$ miles each month and each gallon costs $160 \div 2 = 80$ dollars (ridiculous, I know). Now if he switches to the new car, he'll only need $60 \div 40 = 1.5$ gallons of gas each month (distance of 60 miles divided by the 40 miles per gallon). Because the price of gas stays the same, that amount of gas will cost him $1.5 \times 80 = 120$ dollars each month. The answer ends up being 120 no matter what number we make up for the number of gallons of gas Brett uses each month.

Solution 3: If we let x be the price per gallon of gas, then Brett currently uses $\frac{160}{x}$ gallons of gas each month. Using that amount of gas means that he drives $\frac{160}{x}(30) = \frac{4{,}800}{x}$ miles each month. With the new car, he will need $\frac{4{,}800}{x} \div 40 = \frac{120}{x}$ gallons of gas each month to drive that distance. Since each gallon of gas costs x dollars, he will need to spend $\frac{120}{x} \times x = 120$ dollars on gas each month.

Chapter 11: Word Problems

EXERCISE 1:

1. $\boxed{18}$ After x minutes, the object's temperature increases by $2.4x$ degrees Celsius. Since its starting temperature was 18 degrees Celsius, its final temperature, y, after x minutes is $2.4x + 18$. Therefore, $c = 18$.

2. $\boxed{D}$ Set up an equation to find x:

$$\begin{aligned} 3x - 3 &= 21 \\ 3x &= 24 \\ x &= 8 \end{aligned}$$

Finally, 8 added to half of x is $8 + \frac{1}{2}(8) = 12$.

3. $\boxed{A}$ For d dollars worth of mozzarella to have been sold, $\frac{d}{8.75}$ pounds must have been sold. That leaves $175 - \frac{d}{8.75}$ pounds still available for sale.

4. $\boxed{2500}$ Let x be the square footage of the house. Now equate John's charge with David's charge:

$$\begin{aligned} 0.30x + 500 &= 0.45x + 5(25) \\ -0.15x &= -375 \\ x &= \frac{-375}{-0.15} = 2{,}500 \end{aligned}$$

5. $\boxed{C}$ Marcia distributed a total of $\frac{240}{3}$ cookies to her classmates. Since the cookies were divided evenly among c classmates, each classmate received $\frac{240}{3} \div c = \frac{240}{3} \times \frac{1}{c} = \frac{240}{3c}$ cookies, and this expression is equal to the given value of 4.

6. $\boxed{39}$ Let the length of BC be x. Then the length of AB is $7 + 2x$. Since the lengths of AB and BC sum to the length of AC,

$$\begin{aligned} 7 + 2x + x &= 55 \\ 7 + 3x &= 55 \\ 3x &= 48 \\ x &= 16 \end{aligned}$$

The length of AB is then $7 + 2(16) = 39$.

7. $\boxed{450}$ Let the number of flowers be x. The number of shrubs is then $\frac{3}{5}x$. Since there are 180 more flowers than shrubs,

$$\begin{aligned} x - \frac{3}{5}x &= 180 \\ \frac{2}{5}x &= 180 \\ x &= 180 \times \frac{5}{2} = 450 \end{aligned}$$

8. $\boxed{C}$ Carl's morning commute takes $\frac{20}{a}$ hours (distance divided by speed equals time taken). Carl's evening commute takes $\frac{24}{b}$ hours. Since Carl's morning commute takes the same amount of time as his evening commute, $\frac{20}{a} = \frac{24}{b}$.

9. $\boxed{C}$ Since the number of groups with 5 players is g, the number of groups with 4 players is $3g$. Then there are 2 groups with 6 players. Since the total number of groups is 38,

$$\begin{aligned} g + 3g + 2 &= 38 \\ 4g + 2 &= 38 \end{aligned}$$

10. $\boxed{90}$ Three times the price of a shirt is 120. Since a tie, which costs 30, is k less than that, k must be $120 - 30 = 90$. Expressed as an equation,

$$\begin{aligned} \text{tie} &= 3(\text{shirt}) - k \\ 30 &= 3(40) - k \\ 30 &= 120 - k \\ k &= 90 \end{aligned}$$

11. $\boxed{C}$ The passenger was charged \$4.75 for the first mile and \$3.50 for each of the remaining $(x-1)$ miles. That's a total charge of $4.75 + 3.50(x-1) = 4.75 + 3.50x - 3.50 = 3.50x + 1.25$ dollars. Since the total charge was \$24, $3.50x + 1.25 = 24$.

12. $\boxed{B}$ Matt brought $x - 4$ pounds of popcorn. Together, Joey and Matt brought $x + (x-4) = 2x - 4$ pounds of popcorn. Since 60% of the popcorn was consumed, there must have been $0.40(2x-4)$ pounds of popcorn leftover (i.e. 40%). Splitting the leftover popcorn evenly, they each took $\frac{0.40(2x-4)}{2} = 0.20(2x-4) = 0.4x - 0.8$ pounds of popcorn.

13. $\boxed{C}$ Since Mitch earns \$875 for each acre cleared and it takes him 14 hours to clear each acre, he earns \$875 every 14 hours. So for h additional hours of work, he will earn $\frac{875}{14}h$ dollars. We can set this expression equal to \$200 since that is the additional amount he wants to earn.

14. $\boxed{B}$ Patrick can harvest the field twice as fast as Joseph can, so Patrick can harvest the field in 6 days. Patrick's rate of harvesting is then $\frac{1}{6}$ of the field per day, and Joseph's rate is $\frac{1}{12}$ of the field per day. If they work together, they would be able to harvest at a combined rate of $\frac{1}{6} + \frac{1}{12}$ of the field per day. Using the work equation $rt = W$, where W is 1 field, r is their combined rate, and t is the time needed to finish the work, we get the equation $\left(\frac{1}{12} + \frac{1}{6}\right)t = 1$, which is equivalent to the equation in choice B.

15. $\boxed{45}$ Let x be the total number of books. Mark then has $\frac{1}{4}x$ books and Kevin has $\frac{1}{3}x$ books. Kevin owns 9 more than Mark, so

$$\frac{1}{3}x - \frac{1}{4}x = 9$$

Multiplying both sides by 12, we get

$$4x - 3x = 108$$

which yields $x = 108$, the total number of books. Mark owns $\frac{1}{4} \times 108 = 27$ books and Kevin owns $\frac{1}{3} \times 108 = 36$ books. Lori must then own $108 - 27 - 36 = 45$ books.

16. $\boxed{144}$ Pipe A by itself can fill $\frac{1}{4}$ of the tank each hour, and Pipe B by itself can fill $\frac{1}{6}$ of the tank each hour. When used together, they can fill $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ of the tank each hour. Now we can use the formula $W = rt$, where W is 1 tank and $r = \frac{5}{12}$ of the tank per hour, to find t: $1 = \frac{5}{12}t$, which yields $t = \frac{12}{5}$. Therefore, it takes $\frac{12}{5}$ hours for both pipes to fill the tank. Converting this result to minutes, we get $m = \frac{12}{5} \times 60 = 144$.

17. $\boxed{42}$ There are $0.30(24) = 7.2$ milliliters of acid currently in the flask. Let x be the number of milliliters of 8% acid solution that need to be added. Adding those x milliliters would contribute an additional $0.08x$ milliliters of acid and increase the total amount of solution in the flask to $x + 24$ milliliters. Now we can form the following equation:

$$\begin{aligned} \frac{7.2 + 0.08x}{x + 24} &= 0.16 \\ 7.2 + 0.08x &= 0.16(x + 24) \\ 7.2 + 0.08x &= 0.16x + 3.84 \\ -0.08x &= -3.36 \\ x &= \frac{-3.36}{-0.08} = 42 \end{aligned}$$

EXERCISE 2:

1. $\boxed{D}$ The square of the sum of x and y is $(x+y)^2$. The product of x and y is xy. The difference is then $(x+y)^2 - xy$.

2. $\boxed{B}$ The setup fees amount to $100c$, \$100 for each customer. The monthly cost for all the customers amounts to $50c$, \$50 for each customer. Over m months, the monthly charges add up to $50c \times m$, or $50cm$. The total charge is therefore $100c + 50cm$.

3. $\boxed{C}$ The plane descends at a rate of $\frac{24{,}500 - 17{,}900}{12} = \frac{6{,}600}{12} = 550$ feet per minute. Since the plane started its descent at an altitude of 24,500 feet, its altitude after t minutes can be represented by $A = 24{,}500 - 550t$.

4. $\boxed{28}$ If we let x be the number of bags of pasta bought, then $4x$ is the number of bags of rice bought. Since the total cost was \$103.95,

$$\begin{aligned} 2.86(4x) + 3.41x &= 103.95 \\ 11.44x + 3.41x &= 103.95 \\ 14.85x &= 103.95 \\ x &= \frac{103.95}{14.85} = 7 \end{aligned}$$

Therefore, the customer bought $4x = 4(7) = 28$ bags of rice.

5. $\boxed{7}$ All in all, they played in $15 + N$ games and won $4 + N$ of them. Setting up an equation, we get

$$\frac{4 + N}{15 + N} = \frac{1}{2}$$

Cross multiplying,

$$\begin{aligned} 2(4 + N) &= 15 + N \\ 8 + 2N &= 15 + N \\ N &= 7 \end{aligned}$$

6. $\boxed{C}$ After the first 80 copies, each additional copy costs $\frac{3{,}700 - 1{,}720}{200 - 80} = \$16.50$. The total cost for x copies of the book, where $x \geq 80$, is the cost of the first 80 copies plus \$16.50 for each of the remaining $x - 80$ copies: $f(x) = 1{,}720 + 16.50(x - 80) = 1{,}720 + 16.50x - 1{,}320 = 16.50x + 400$.

7. $\boxed{C}$ 180 trees will take a total of 180(15) minutes to plant. Each volunteer has 5(60) minutes. Therefore, the number of volunteers n must be $\frac{180(15)}{5(60)}$. Setting up an equation and rearranging, we get

$$\begin{aligned} n &= \frac{180(15)}{5(60)} \\ (5)(60)n &= (180)(15) \end{aligned}$$

8. $\boxed{B}$ After t minutes, the hose has output an amount equal to $\frac{1}{3}t$ of the tub and the drain has emptied an amount equal to $\frac{2}{11}t$ of the tub. The tub being filled after t minutes means that the hose has output 1 full tubs-worth more than the drain has emptied in those t minutes. In other words, the hose has "outpaced" the drain by an amount equal to 1 full tub. Therefore, $\frac{1}{3}t - \frac{2}{11}t = 1$, which is equivalent to the equation in choice B.

9. $\boxed{C}$ If we sum the lengths of AC and BD, we get the length of AD except we double-count BC. To account for that, we can just subtract the length of BC once, giving us the equation $x + y - 10 = 50$.

10. $\boxed{840}$ Let the number of seniors be x. The number of juniors is then $3x$. Once 140 more seniors are hired, the number of seniors will be $x + 140$. Setting up an equation with the information given, we get

$$\begin{aligned} 3x &= 2(x + 140) \\ 3x &= 2x + 280 \\ x &= 280 \end{aligned}$$

The number of junior accountants is then $3x = 3(280) = 840$.

11. $\boxed{D}$ Alice's rate is $\frac{10}{3}$ boxes per hour and Trinity's rate is $\frac{15}{x}$ boxes per hour. Their combined rate of packing is then $\frac{10}{3} + \frac{15}{x}$ boxes per hour. Therefore, in 12 hours, they can pack a total of $12\left(\frac{10}{3} + \frac{15}{x}\right)$ boxes, and this expression is equal to the given value of 115.

12. $\boxed{D}$ Let the number of \$5 coupons given out be x. Then the number of \$3 coupons given out is $3x$, and the number of \$1 coupons given out is $2(3x) = 6x$.

$$\begin{aligned} 5(x) + 3(3x) + 1(6x) &= 360 \\ 5x + 9x + 6x &= 360 \\ 20x &= 360 \\ x &= 18 \end{aligned}$$

The number of \$3 coupons given out is then $3x = 3(18) = 54$.

13. $\boxed{40}$ Jessica runs at a rate of 4 yards per second. Let t be the time it takes for Jessica to overtake Yoona. We can construct an equation with the left side being Yoona's distance and the right side being Jessica's distance.

$$\begin{aligned} 30 + t &= 4t \\ 30 &= 3t \\ 10 &= t \end{aligned}$$

It will take 10 seconds for Jessica to catch up to Yoona, which means Jessica must run $4(10) = 40$ yards to catch up.

14. $\boxed{50}$ The answer is NOT 20. You can't just take 10% of the green jellybeans away because as you do that, the total number of jellybeans also goes down. We first find that there are $\frac{7}{10} \times 200 = 140$ green jellybeans. We need to remove x of them so that 60% of what's left is green:

$$\frac{\text{green jellybeans left}}{\text{total jellybeans left}} = 60\%$$

$$\frac{140 - x}{200 - x} = \frac{6}{10}$$

Cross multiplying,

$$\begin{aligned} 10(140 - x) &= 6(200 - x) \\ 1{,}400 - 10x &= 1{,}200 - 6x \\ 200 &= 4x \\ x &= 50 \end{aligned}$$

15. $\boxed{D}$ Initially, the members are each responsible for paying $\frac{r}{m}$ dollars. But if k members fail to pay, then $m - k$ members remain and each one becomes responsible for $\frac{r}{m-k}$ dollars. This amount is greater than the original amount by

$$\frac{r}{m-k} - \frac{r}{m} = \frac{mr}{m(m-k)} - \frac{r(m-k)}{m(m-k)} = \frac{mr - mr + kr}{m(m-k)} = \frac{kr}{m(m-k)} \text{ dollars}$$

Here is an alternative solution. Since each member is initially responsible for $\frac{r}{m}$ dollars, the club loses out on $\frac{r}{m}(k)$ dollars when the k members fail to pay. To make up this lost amount, each of the remaining $m - k$ members must pay an additional

$$\frac{\frac{r}{m}(k)}{m-k} = \frac{\frac{kr}{m}}{m-k} = \frac{kr}{m(m-k)} \text{ dollars}$$

16. $\boxed{B}$ This problem follows the format of a standard $W = rt$ word problem, where W is the amount of work done, r is the overall rate at which work is being done, and t is the time spent. Given that x is the total number of days it takes Terry to pave the entire parking lot by himself, Terry, by definition, paves $\frac{1}{x}$ of the parking lot each day. For example, if it takes 5 days for Terry to pave the parking lot by himself, by definition he paves $\frac{1}{5}$ of the parking lot each day. Since the $\frac{1}{x}$ in the equation is Terry's rate, the $\frac{1}{x+4}$ in the equation must be Andy's rate, so Andy paves $\frac{1}{x+4}$ of the parking lot each day. The $x + 4$ means that Andy would take 4 days longer than Terry to pave the lot if they worked separately. Working together, they pave at a combined rate of $\frac{1}{x} + \frac{1}{x+4}$ of the parking lot each day. This is the overall rate r. The number $\frac{2}{3}$ is the remaining fraction of the parking lot that has yet to be paved. This is W. So that leaves the number 9, which must be t. Given what we have for W and r, t must be the number of days it will take Terry and Andy to pave the remainder of the parking lot working together.

17. $\boxed{320}$ Let x be the number of milliliters of 25% solution the chemist should use. If the chemist uses x milliliters of the 25% solution, then $800 - x$ milliliters of the 50% solution will have to be used. The x milliliters of 25% solution will contribute $0.25x$ milliliters of alcohol, and the $800 - x$ milliliters of 50% solution will contribute $0.50(800 - x)$ milliliters of alcohol. Setting up an equation, we get

$$\begin{aligned} \frac{0.25x + 0.50(800 - x)}{800} &= 0.40 \\ 0.25x + 0.50(800 - x) &= 0.40(800) \\ 0.25x + 400 - 0.50x &= 320 \\ -0.25x &= -80 \\ x &= \frac{-80}{-0.25} = 320 \end{aligned}$$

Since we set x to be what the question is asking for, the answer is 320.

Chapter 12: Quadratic Equations

EXERCISE 1:

1. $\boxed{-3 \text{ or } 8}$ Factoring the left-hand side, we get $(x+3)(x-8)=0$. The solutions are then $x=-3$ or $x=8$.

2. $\boxed{5}$

$$x^2-3x-12=x-7$$
$$x^2-4x-5=0$$
$$(x-5)(x+1)=0$$
$$x=5,-1$$

The positive solution is $x=5$.

3. $\boxed{A}$ Moving all terms to the left-hand side, we get $x^2+12x-9=0$. The sum of the solutions is then $-\frac{b}{a}=-\frac{12}{1}=-12$.

4. $\boxed{B}$ Using the quadratic formula, we get

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(7)}}{2(1)}=$$
$$\frac{10\pm\sqrt{72}}{2}=\frac{10\pm6\sqrt{2}}{2}=5\pm3\sqrt{2}.$$

5. $\boxed{-2}$

$$-x(x+3)=x+4$$
$$-x^2-3x=x+4$$
$$-x^2-4x-4=0$$
$$x^2+4x+4=0$$
$$(x+2)^2=0$$
$$x=-2$$

6. $\boxed{A}$ Using the quadratic formula, we get

$$x=\frac{-4\pm\sqrt{(4)^2-4(1)(2)}}{2(1)}=\frac{-4\pm\sqrt{8}}{2}$$
$$=\frac{-4\pm2\sqrt{2}}{2}$$
$$=-2\pm\sqrt{2}$$

7. $\boxed{14}$ Multiplying both sides by x gives

$$84=x^2-8x$$
$$0=x^2-8x-84$$
$$0=(x-14)(x+6)$$
$$x=14,-6$$

The positive solution is 14.

8. $\boxed{C}$ Factoring the expression, we get $(x+3)(2x-5)$. Therefore, $x+3$ and $2x-5$ are both factors of the expression. Alternatively, graph the expression as a function or use the quadratic formula. The x-intercepts at -3 and $\frac{5}{2}$ indicate that $x+3$ and $2x-5$ must be factors.

9. $\boxed{4}$ Expanding everything, we get

$$(2x-3)^2=4x+5$$
$$4x^2-12x+9=4x+5$$
$$4x^2-16x+4=0$$

The sum of the solutions is then $-\frac{b}{a}=-\frac{-16}{4}=4$.

10. $\boxed{16}$ Factoring the expression, we get $3x(x^2+23x+112)=3x(x+7)(x+16)$. Therefore, 16 is the largest possible value of k. Alternatively, graph the expression as a function. The x-intercepts at -16, -7, and 0 indicate that $x+16$, $x+7$, and x must be factors.

11. $\boxed{30}$ Using the quadratic formula, we get

$$x=\frac{-(-12)\pm\sqrt{(-12)^2-4(1)(6)}}{2(1)}=$$

$\frac{12\pm\sqrt{120}}{2}=\frac{12\pm2\sqrt{30}}{2}=6\pm\sqrt{30}$. The value of n must then be 30.

12. $\boxed{C}$ Expanding the left-hand side, we get

$$x^2-9=7x-3$$
$$x^2-7x-6=0$$

The product of the solutions is then $\frac{c}{a}=\frac{-6}{1}=-6$.

13. $\boxed{4 \text{ or } 6}$ Factoring the expression, we get $5x(x^2-10x+24)=5x(x-4)(x-6)$. Therefore, the possible values of m are 4 and 6. Alternatively, graph the expression as a function. The x-intercepts at 0, 4, and 6 indicate that x, $x-4$, and $x-6$ must be factors.

14. $\boxed{D}$ The product of the solutions is equal to $\frac{c}{a}=\frac{(k+1)}{4}=\frac{1}{4}(k+1)$. Therefore, $w=\frac{1}{4}$.

15. $\boxed{-15}$ First, square both sides to get rid of the square roots. Then we have

$$(x+6)^2=6-5x$$
$$x^2+12x+36=6-5x$$
$$x^2+17x+30=0$$
$$(x+2)(x+15)=0$$

The solutions are $x=-2$ and $x=-15$. The smallest solution is then $x=-15$.

16. $\boxed{\frac{3}{4} \text{ or } .75}$ To make the equation easier to factor, let $A=8x$. The equation then becomes

$$A^2-5A=6$$
$$A^2-5A-6=0$$
$$(A-6)(A+1)=0$$
$$A=6,-1$$

Now we can go back and solve for x. Since $A=8x$, we have $6=8x$, which yields $x=\frac{3}{4}$ (the positive solution we're looking for), and $-1=8x$, which yields $x=-\frac{1}{8}$.

EXERCISE 2:

1. $\boxed{C}$ Using the quadratic formula, we get $x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(-1)}}{2(4)} = \frac{9 \pm \sqrt{97}}{8}$.

2. $\boxed{8}$ To get rid of the fraction, we can multiply both sides by 2:

$$-\frac{1}{2}x^2 + x = -24$$
$$-x^2 + 2x = -48$$
$$0 = x^2 - 2x - 48$$
$$0 = (x-8)(x+6)$$
$$x = 8, -6$$

The positive solution is $x = 8$.

3. $\boxed{21}$ Using the quadratic formula, we get $x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-5)}}{2(1)} = \frac{-1 \pm \sqrt{21}}{2}$. Therefore, one of the solutions can be written as $x = \frac{\sqrt{21} - 1}{2}$. The value of k is then 21.

4. $\boxed{\frac{39}{5} \text{ or } 7.8}$

$$2x(-x+17) = 54 - x(5-3x)$$
$$-2x^2 + 34x = 54 - 5x + 3x^2$$
$$0 = 5x^2 - 39x + 54$$

The sum of the solutions is then $-\frac{b}{a} = -\frac{-39}{5} = \frac{39}{5}$.

5. $\boxed{D}$ Using the quadratic formula, we get $x = \frac{-6 \pm \sqrt{(6)^2 - 4(-3)(-1)}}{2(-3)} = \frac{-6 \pm \sqrt{24}}{-6} = \frac{-6 \pm 2\sqrt{6}}{-6} = \frac{-6}{-6} \pm \frac{2\sqrt{6}}{-6} = 1 \pm \frac{\sqrt{6}}{3}$.

6. $\boxed{10}$ Given that the left-hand side is already factored, it's easy to see that the solutions to the equation are $x = 0, a$, and -4. Since of the sum of these solutions is 6,

$$0 + a + (-4) = 6$$
$$a - 4 = 6$$
$$a = 10$$

7. $\boxed{B}$ Move everything to the left-hand side: $x^2 - 8x + 9 = 0$. Using the quadratic formula, we get $x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(9)}}{2(1)} = \frac{8 \pm \sqrt{28}}{2} = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$. Therefore, $a + b = 4 + 7 = 11$.

8. $\boxed{A}$ The product of the solutions is equal to $\frac{c}{a} = \frac{-(k-4)}{1} = -(k-4) = 4 - k$.

9. $\boxed{3}$ Plugging in $(-a, a)$, we get

$$a = a(-a) + 12$$
$$a = -a^2 + 12$$
$$a^2 + a - 12 = 0$$
$$(a+4)(a-3) = 0$$
$$a = -4, 3$$

Since a is positive, $a = 3$.

10. $\boxed{\frac{1}{2} \text{ or } .5}$

$$2x^2 - 7x + 3 = 0$$
$$(2x-1)(x-3) = 0$$

If you had trouble factoring this, remember that you can always use the quadratic formula. Now that the equation is factored, it's easy to see that $x = \frac{1}{2}$ or $x = 3$. Since we want the smallest solution, the answer is $\frac{1}{2}$.

11. $\boxed{A}$ Factoring the expression, we get $(x+4)(2x+3)$. Neither of the given options is a factor. Alternatively, graph the expression as a function or use the quadratic formula. The x-intercepts at -4 and $-\frac{3}{2}$ indicate that $x + 4$ and $2x + 3$ must be factors. Again, we can see that neither of the given options is a factor.

12. $\boxed{-19}$ The sum of the solutions is equal to $\frac{-b}{a} = \frac{-(108k - 51)}{3} = \frac{-108k + 51}{3} = -36k + 17$. Therefore, $a = -36$, $b = 17$, and $a + b = -36 + 17 = -19$.

13. $\boxed{\frac{4}{9} \text{ or } .4444}$ Add $32q$ to both sides to get a quadratic equation in standard $ax^2 + bx + c = 0$ form:

$$72x^2 - (18p + 9q)x + 64p + 32q = 0$$

The product of the solutions is then equal to $\frac{c}{a} = \frac{(64p + 32q)}{72} = \frac{32(2p+q)}{72} = \frac{4}{9}(2p+q)$. Therefore, $n = \frac{4}{9}$.

14. $\boxed{3}$ Factoring the given expression, we get $2x(4x^2 - 13x - 12) = 2x(x-4)(4x+3)$. The value of k must then be 3. Alternatively, graph the given expression as a function. The x-intercepts at $-\frac{3}{4}$, 0, and 4 indicate that $4x + 3$, x, and $x - 4$ must be factors.

15. $\boxed{17}$ Square both sides to get rid of the square root:

$$(2\sqrt{4x - 19})^2 = (x-3)^2$$
$$4(4x - 19) = x^2 - 6x + 9$$
$$16x - 76 = x^2 - 6x + 9$$
$$0 = x^2 - 22x + 85$$
$$0 = (x-5)(x-17)$$
$$x = 5, 17$$

The largest solution is 17.

16. $\boxed{7}$ To make the equation easier to factor, let $A = x + 11$. The equation then becomes

$$A^2 - 21A + 54 = 0$$
$$(A-3)(A-18) = 0$$
$$A = 3, 18$$

Now we can go back and solve for x. Since $A = x + 11$, we have $3 = x + 11$, which yields $x = -8$, and $18 = x + 11$, which yields $x = 7$. Thus, the positive solution is $x = 7$.

17. $\boxed{D}$ Move the 8 to the left-hand side to get $3x^2 + 10x - 8 = 0$. Now, we can either factor the equation or use the quadratic formula. Factoring the equation gives $(x+4)(3x-2) = 0$. So, $x = -4$ or $x = \frac{2}{3}$. Since $a > b$, b must be -4 and $b^2 = (-4)^2 = 16$.

18. $\boxed{C}$ If the equation is to have integer solutions, the value of p must be the sum of two factors of 18. For example, 6 and 3 multiply to 18 and sum to 9, so p could be 9, in which case the equation would factor to $(x+6)(x+3)$. Listing all the unique factor pairs of 18, we get 1 and 18 (a sum of 19), 2 and 9 (a sum of 11), and 3 and 6 (a sum of 9). Therefore, the possible values of p are 19, 11, and 9—I and III only. Note that we only considered positive factors of 18 because the question specifies that p is a positive constant.

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Chapter 13: Systems of Equations

EXERCISE 1:

1. $\boxed{B}$ Given the second equation, we can substitute 25 for y in the first equation:

$$\begin{aligned} 25 &= -5x + 65 \\ -40 &= -5x \\ 8 &= x \end{aligned}$$

Therefore, $x = 8$ and $y = 25$.

2. $\boxed{C}$ Substituting the second equation into the first, we get

$$\begin{aligned} 3(1-3y) - 5y &= -11 \\ 3 - 9y - 5y &= -11 \\ 3 - 14y &= -11 \\ -14y &= -14 \\ y &= 1 \end{aligned}$$

3. $\boxed{D}$ From the first equation, $y = 20 - 2x$. Plugging this into the second equation, we get

$$\begin{aligned} 6x - 5(20 - 2x) &= 12 \\ 6x - 100 + 10x &= 12 \\ 16x &= 112 \\ x &= 7 \end{aligned}$$

We already know the answer is (D) at this point, but just in case, we can verify that $y = 20 - 2(7) = 6$.

4. $\boxed{-1}$ Adding 9 to both sides of the second equation yields $x = -6$. Substituting -6 for x in the first equation yields

$y = (-6)^2 + 10(-6) + 23 = 36 - 60 + 23 = -1$.

5. $\boxed{C}$ The solution to the system is the intersection point of the two lines: $(-2, -3)$.

6. $\boxed{C}$ The fastest way to do this problem is to subtract the second equation from the first, which yields $x + y = 9$.

7. $\boxed{B}$ Divide the first equation by 2 to get $x - 2y = 4$. We can't get the coefficients to match (-2 vs. 2 for the y's). Therefore, the system has one solution. In fact, we can even solve this system by adding the two equations to get $2x = 8$, $x = 4$, which gives $y = 0$.

8. $\boxed{C}$ Since a total of 200 cups were sold, $x + y = 200$. The vendor collected $5x$ dollars from sales of small cups and $7y$ dollars from sales of large cups. Since the vendor collected \$160 more from sales of small cups than from sales of large cups, $5x$ exceeds $7y$ by 160: $5x - 7y = 160$, which is equivalent to $5x = 7y + 160$.

9. $\boxed{\frac{9}{25} \text{ or } .36}$ The first equation yields $a = 4.5b$. Substituting $4.5b$ for a in the second equation, we get

$$\begin{aligned} \frac{4.5b}{kb} &= 12.5 \\ \frac{4.5}{k} &= 12.5 \\ 4.5 &= 12.5k \\ k &= \frac{4.5}{12.5} = \frac{45}{125} = \frac{9}{25} \end{aligned}$$

10. $\boxed{C}$ A system has no solution when the equations are equivalent except for the constants. Dividing the given equation by 18 gives $\frac{1}{3}x - \frac{1}{6}y = \frac{1}{2}$, which is equivalent to answer C except for the constants. Hence, answer C would result in a system with no solution. Note that answer D would result in a system with infinitely many solutions. Answer B would do the same. Answer A would result in a system with one solution since the coefficients cannot be made to match.

11. $\boxed{A}$ The first equation factors to $(x + y)(x - y) = 48$. Substituting the value of $x - y$ given by the second equation into the first, we get $(x + y)(6) = 48$, which gives $x + y = 8$.

12. $\boxed{C}$ A system has infinitely many solutions when the equations are equivalent (both coefficients and constants). To get the same coefficients, multiply the first equation by -2 to get $-4x + 10y = -2a$. Now we can see that $-2a = -8$, $a = 4$.

13. $\boxed{C}$ If the two graphs intersect at the point $(2, 8)$, then $(2, 8)$ is a solution to the system. Plugging this point into the second equation allows us to solve for b:

$$\begin{aligned} y &= -bx \\ 8 &= -b(2) \\ -4 &= b \end{aligned}$$

With this result, we can now plug $(2, 8)$ into the first equation to solve for a:

$$\begin{aligned} y &= ax + b \\ 8 &= a(2) - 4 \\ 12 &= 2a \\ 6 &= a \end{aligned}$$

14. $\boxed{A}$ The two graphs do not intersect at all, so there are no solutions.

15. $\boxed{C}$ We don't have to solve for x. We can substitute the value of $3x - \frac{5}{16}$ given by the first equation into the second equation: $y = 2(4)^2 = 32$.

16. $\boxed{A}$ Multiply the first equation by -3 to get $-3ax - 6y = -15$. The constant a cannot be -1. Otherwise, the second equation's coefficients would then be equal to the first equation's coefficients, resulting in a system with no solution.

17. $\boxed{B}$ We can substitute the value of $3x$ given by the first equation into the second equation:

$$\begin{aligned} 80 &= 4y^2 + 16 \\ 64 &= 4y^2 \\ 16 &= y^2 \\ \pm 4 &= y \end{aligned}$$

18. $\boxed{-6}$ A system has no solution when the equations are equivalent except for the constants. Let's first combine like terms and line up the coefficients:

$$\begin{aligned} ky - 5x &= 8 \\ 18y + 15x &= 21 \end{aligned}$$

Now we can multiply the first equation by -3 to get the coefficients of x to match:

$$\begin{aligned} -3ky + 15x &= -24 \\ 18y + 15x &= 21 \end{aligned}$$

Looking at the coefficients of y, we can see that $-3k = 18$, $k = -6$.

19. $\boxed{C}$ A system of two linear equations has no solution when their graphs are parallel (have the same slope). The line shown has a slope of $-\frac{2}{3}$ (down 2 units, over 3 units) and a y-intercept of 3. Answer C has the same slope and a y-intercept of -1, making it a distinct line parallel to the one shown. Hence, answer C would result in a system with no solution. Note that answer D is incorrect because it's the same line as the one shown, which means it would result in a system with infinitely many solutions (infinitely many intersection points). Answers A and B are incorrect because they are not parallel to the line shown (different slopes).

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20. $\boxed{8}$ To find the point(s) where two graphs intersect, solve the system consisting of their equations. In this problem, that system is

$$y = x^2 - 7x + 7$$
$$y = 2x - 1$$

Substituting for y in the second equation yields

$$x^2 - 7x + 7 = 2x - 1$$
$$x^2 - 9x + 8 = 0$$
$$(x-1)(x-8) = 0$$
$$x = 1 \text{ or } 8$$

So the x-coordinates of the points of intersection are 1 and 8. Since the question already gave us the point $(1, 1)$, p must be equal to 8.

21. $\boxed{175}$ Let x be the number of hours Shannon spent driving from Melrose to Hanover, and let y be the number of hours she spent driving from Hanover to Norwich. Based on the given information, we can construct the following system of equations:

$$50x + 30y = 340$$
$$x + y = 9$$

To solve for x, we can multiply the second equation by 30 to get $30x + 30y = 270$. Subtracting this resulting equation from the first equation gives $20x = 70$, $x = 3.5$. Therefore, Shannon must have driven $50x = 50(3.5) = 175$ miles from Melrose to Hanover.

22. $\boxed{112}$ To get rid of the fractions, multiply the first equation by 24 and the second equation by 2:

$$32 + 30x = 672y + 15x$$
$$2my = 5x - 8$$

Now move things around so that the x's and y's are lined up:

$$15x - 672y = -32$$
$$-5x + 2my = -8$$

A system has no solution when the equations are equivalent except for the constants. To get the coefficients of x to match, we can multiply the second equation by -3:

$$15x - 672y = -32$$
$$15x - 6my = 24$$

Now we can match the coefficients of y: $6m = 672$, which yields $m = 112$. Note that the constants -32 and 24 are different.

EXERCISE 2:

1. $\boxed{B}$ First, multiply the first equation by 3 to get rid of the fraction: $12x - y = -24$. Next, substitute the second equation into the first:

$$12x - (4x + 16) = -24$$
$$12x - 4x - 16 = -24$$
$$8x = -8$$
$$x = -1$$

Finally, $y = 4(-1) + 16 = 12$.

2. $\boxed{-27}$ Add both equations to cancel the y terms and get $14x = -54$. Divide this resulting equation by 2 to get $7x = -27$.

3. $\boxed{10}$ We can isolate x in the second equation to get $x = y - 18$. Substituting this into the first equation, we get

$$y = 0.5(y - 18) + 14$$
$$y = 0.5y - 9 + 14$$
$$0.5y = 5$$
$$y = 10$$

4. $\boxed{A}$ All three lines never intersect at a single point, so the system has no solution.

5. $\boxed{C}$ Answer C has no solution because the two equations have the same coefficients of x but different constants (0 vs. 8). Furthermore, the graph of the system is two parallel lines (no intersection point = no solution).

6. $\boxed{B}$ Substitute the first equation into the second:

$$2x - 14 = (x + 5)(x - 7)$$
$$2(x - 7) = (x + 5)(x - 7)$$
$$2 = x + 5$$
$$-3 = x$$

Note that in the second step, we divided both sides by $x - 7$, which is the source of the missing solution $x = 7$. This solution is not in the answer choices, so we don't need to worry about it.

7. $\boxed{D}$ Asking for the number of intersection points is another way of asking for the number of solutions. Divide the first equation by 3 to get $x - 2y = 5$. Divide the second equation by -2 to get $x - 2y = 5$. The equations are the same, so there are an infinite number of solutions (intersection points).

8. $\boxed{-160}$ Substitute $-10x$ for y in the second equation:

$$x^2 - 10x = 96$$
$$x^2 - 10x - 96 = 0$$
$$(x + 6)(x - 16) = 0$$

Thus, we have a solution at $x = -6$ and another solution at $x = 16$. When $x = -6$, $y = -10(-6) = 60$. When $x = 16$, $y = -10(16) = -160$. Since the question specifies that $y < 0$, $y = -160$.

9. $\boxed{12}$

$$\frac{x + y}{x} = 9$$
$$x + y = 9x$$
$$y = 8x$$

Now we can substitute $8x$ for y in the second equation.

$$\frac{a(8x)}{3x} = 32$$
$$\frac{8}{3}a = 32$$
$$a = 32 \cdot \frac{3}{8} = 12$$

10. $\boxed{1932}$ Divide the first equation by 9 to get $(2x + 1) + 3(y - 4) = 90$. Now the second equation can be subtracted from this equation to get $2(y - 4) = 483$. Finally, we can arrive at the value of $8(y - 4)$ by multiplying this resulting equation by 4: $8(y - 4) = 483 \cdot 4 = 1932$.

11. $\boxed{D}$ Let r and c be the number of rectangular tables and circular tables, respectively, at the restaurant. Based on the information, we can construct the following two equations:

$$4r + 8c = 144$$
$$r + c = 30$$

To solve for r, multiply the second equation by 8 and subtract to get $-4r = -96$, $r = 24$.

12. $\boxed{C}$ For a system to have infinitely many solutions, the equations need to be equivalent. We can multiply the second equation by 2 to get the constants to match:

$$mx - 6y = 10$$
$$4x - 2ny = 10$$

Now it's easy to see that $m = 4$ and $2n = 6$, $n = 3$. Finally, $\frac{m}{n} = \frac{4}{3}$.

13. $\boxed{D}$ For a system of two linear equations, there is exactly one solution when the corresponding coefficients cannot be made to match. The answer is D because even though the coefficients of x match, the coefficients of y do not. Compare this to an equation like the one in answer A, where the coefficients of x and y can be made to match by multiplying the equation by -1. Answers A and B would result in no solution. Answer C would result in infinitely many solutions.

14. $\boxed{A}$ From the second equation, we have $x = 2y$. Plugging this into the first equation, we get

$$(2y)^2 - y^2 = \frac{1}{12}$$
$$4y^2 - y^2 = \frac{1}{12}$$
$$3y^2 = \frac{1}{12}$$
$$y^2 = \frac{1}{36}$$
$$y = \pm\frac{1}{6}$$

15. $\boxed{\frac{9}{4} \text{ or } 2.25}$ A system has no solution when the equations are equivalent except for the constants. First, let's simplify. For the first equation, we can add $\frac{1}{5}y$ to both sides, and for the second equation, we can subtract $\frac{3}{5}$ from both sides. The resulting system is

$$\frac{1}{2}y + \frac{b}{3}x = \frac{3}{4}$$
$$\frac{1}{4}y + \frac{3}{8}x = 1$$

One way to get the coefficients of y to match is to multiply the second equation by 2:

$$\frac{1}{2}y + \frac{b}{3}x = \frac{3}{4}$$
$$\frac{1}{2}y + \frac{6}{8}x = 2$$

Note that the constants are different. Equating the coefficients of x, we get $\frac{b}{3} = \frac{6}{8}$, which we can cross multiply to get $8b = 18$, $b = \frac{18}{8} = \frac{9}{4}$.

16. $\boxed{D}$ There are many ways to tackle this problem. The simplest is to graph each of the answer choices using Desmos. Another way is to see which answer choice is satisfied by (5,600, 6,900), the solution represented by the intersection point. Only the system of equations in answer D is satisfied by (5,600, 6,900).

17. $\boxed{B}$ The given points indicate that $f(-1) = -10$ and $f(2) = 38$. Since $f(-1) = a(-1)^3 + b(-1)$ and $f(2) = a(2)^3 + b(2)$, we can construct the following system of equations:

$$-a - b = -10$$
$$8a + 2b = 38$$

To eliminate b and solve for a, we can multiply the first equation by 2 to get $-2a - 2b = -20$ and then add this resulting equation to the second equation: $6a = 18$, $a = 3$. Plugging this result back into the first equation gives $-3 - b = -10$, $b = 7$. Therefore, $ab = 3 \cdot 7 = 21$.

18. $\boxed{-3}$ First, subtract 8 from both sides of the second equation to get $-4x - 8 = y$. Then substitute this result for y in the first equation:

$$x^2 + 2x = -4x - 8 - 1$$
$$x^2 + 2x = -4x - 9$$
$$x^2 + 6x + 9 = 0$$
$$(x+3)^2 = 0$$
$$x = -3$$

19. $\boxed{28}$ Let p be the number of minutes Kenji spent using his push mower and s be the number of minutes he spent using his self-propelled mower. Since Kenji spent 2 hours (120 minutes) mowing 9,000 square yards of grass, we can construct the following system of equations:

$$57.75p + 80.25s = 9{,}000$$
$$p + s = 120$$

Now we can multiply the second equation by 80.25 to get $80.25p + 80.25s = 9{,}630$. Subtracting this equation from the first equation gives $-22.5p = -630$, $p = \frac{630}{22.5} = 28$.

20. $\boxed{9}$ Plugging the first equation into the second, we get

$$\sqrt{4x} - (\sqrt{x} + 3) = 3$$
$$2\sqrt{x} - \sqrt{x} - 3 = 3$$
$$\sqrt{x} = 6$$
$$x = 36$$

Therefore, $y = \sqrt{36} + 3 = 9$.

21. $\boxed{-8 \text{ or } 17}$ Divide the first equation by 3 to get $-5x + y = 4$. Now add $5x$ to both sides to isolate y: $y = 5x + 4$. Substituting this result into the second equation, we get

$$5x + 4 + 132 = x^2 - 4x$$
$$0 = x^2 - 9x - 136$$
$$0 = (x+8)(x-17)$$
$$x = -8, 17$$

22. $\boxed{D}$ Dividing the second equation by -4 gives $5x - 3y = 8$, so both equations are the same. This means that any given solution only needs to satisfy $5x - 3y = 8$. From this equation, we can isolate x to get

$$x = \frac{3}{5}y + \frac{8}{5}$$

and we can isolate y to get

$$y = \frac{5}{3}x - \frac{8}{3}$$

Now it's easy to check each answer choice. According to our equations above, when $x = n$,

$$y = \frac{5}{3}n - \frac{8}{3},$$

so answer A is incorrect. When $y = n$,

$$x = \frac{3}{5}n + \frac{8}{5},$$

so answer B is incorrect. When $x = n - 1$,

$$y = \frac{5}{3}(n-1) - \frac{8}{3} = \frac{5}{3}n - \frac{5}{3} - \frac{8}{3} = \frac{5}{3}n - \frac{13}{3},$$

so answer C is incorrect. Finally, when $y = n - 1$,

$$x = \frac{3}{5}(n-1) + \frac{8}{5} = \frac{3}{5}n - \frac{3}{5} + \frac{8}{5} = \frac{3}{5}n + 1,$$

so answer D is correct.

Here's another approach. We can find one solution to $5x - 3y = 8$ and use that solution to check each answer choice. For example, $(1, -1)$ is a solution. Checking answer A with this solution would mean that $n = 1$ (so that the x-value is 1). However, plugging in $n = 1$ does not give $(1, -1)$, so answer A is incorrect. We can eliminate answers B and C in the same manner. Let's skip ahead to answer D. Checking answer D with the solution $(1, -1)$ would mean that $n = 0$ (so that the y-value is -1). Plugging in $n = 0$ gives the desired $(1, -1)$, so after eliminating the other answers, we can be certain answer D is correct.

Chapter 14: Inequalities

EXERCISE 1:

1. $\boxed{C}$ Test each point. Answer C gives $-3 > 7(2) - 19 \Rightarrow -3 > -5$, which is the only true resulting statement from the given answer choices.

2. $\boxed{19}$ Katherine will need a total of $28 \times 4 = 112$ batteries. Given that there are 6 batteries in a pack, she will need $112 \div 6 \approx 18.67$ packs. Since it's implied that packs only come in whole numbers, she needs to round up to 19 packs to ensure she has enough batteries.

3. $\boxed{C}$ Test each point. Plugging in $(-2, 2)$ from answer C results in $2 < -4(-2) - 1 \Rightarrow 2 < 7$ from the first inequality and $2 > -2 + 2 \Rightarrow 2 > 0$ from the second inequality. Because both resulting statements are true, $(-2, 2)$ is a solution to the system.

4. $\boxed{A}$ The hot dog stand sold $3l$ dollars worth during lunch and $5d$ dollars worth during dinner. In total, that's $3l + 5d$ dollars, which must be greater than 800. Therefore, $3l + 5d > 800$.

5. $\boxed{D}$ The shaded region represents the possible distances, y, the train has traveled after x hours (in other words, the solution set). The only point from the answer choices that's inside the shaded region is $(11, 625)$, answer D.

6. $\boxed{B}$ Test each inequality by checking whether it's satisfied by all three given points. Only the inequality in answer B is satisfied by all three given points. The point $(2, 2)$ results in the true statement $2 \geq \frac{3}{2}(2) - 1 \Rightarrow 2 \geq 2$. The point $(-1, -1)$ results in the true statement $-1 \geq \frac{3}{2}(-1) - 1 \Rightarrow -1 \geq -2.5$. And finally, the point $(-2, -4)$ results in the true statement $-4 \geq \frac{3}{2}(-2) - 1 \Rightarrow -4 \geq -4$.

7. $\boxed{14}$ Let n be the number of fish Naomi caught. Then Ashleigh must have caught $3n - 9$ fish. Since they caught at least 45 fish combined,

$$\begin{aligned} n + (3n - 9) &\geq 45 \\ 4n &\geq 54 \\ n &\geq 13.5 \end{aligned}$$

Since it's implied that fish are caught in whole numbers, the minimum possible value of n is 14.

8. $\boxed{27}$ To get the required number of nail files, the salon needs at least $\frac{1{,}800}{80} = 22.5 \to 23$ toolkits. To get the required number of nail buffers, the salon needs at least $\frac{4{,}000}{150} \approx 26.7 \to 27$ toolkits. Based on these numbers, a minimum of 27 toolkits must be purchased to meet both requirements.

9. $\boxed{B}$ Plugging in $x = 18$ into the system, we get $18 + y < 25 \Rightarrow y < 7$ from the first inequality and $2(18) - 3y < 77 \Rightarrow y > -13.67$ from the second inequality. The only answer that satisfies $y < 7$ and $y > -13.67$ is -6.

10. $\boxed{C}$ The shaded region falls below the horizontal line $y = 3$, so $y < 3$. The shaded region also stays above $y = x$, so $y > x$.

11. $\boxed{A}$ The time Harry spends on the bus is $\frac{8}{x}$ hours and the time he spends on the train is $\frac{16}{y}$ hours. Since the total number of hours is never greater than 1, $\frac{8}{x} + \frac{16}{y} \leq 1$.

12. $\boxed{C}$ The shaded region lies to the right of the vertical line $x = 3$, so $x \geq 3$. The shaded region also lies below the line with slope $-\frac{1}{2}$ (down 1 unit, over 2 units) and y-intercept 4: $y \leq -\frac{1}{2}x + 4$. Multiplying both sides by 2 gives $2y \leq -x + 8$. Therefore, the shaded region represents the solutions to the system $x \geq 3$ and $2y \leq -x + 8$.

13. $\boxed{B}$ Let m be Eric's monthly salary. Then $0.6m$ is his monthly investment account contribution since 60% is what remains after his spending. Since he wants to contribute at least \$50,000 each year,

$$\begin{aligned} 12(0.6m) &\geq 50{,}000 \\ 7.2m &\geq 50{,}000 \\ m &\geq 6{,}944.44 \end{aligned}$$

Answer B is closest to this minimum value of m.

14. $\boxed{D}$ The line going from the bottom-left to the top-right must be $y = \frac{3}{2}x + 2$ and the line going from the top-left to the bottom-right must be $y = -2x + 5$ (based on the slopes and y-intercepts). Answer D correctly shades in the region above $y = \frac{3}{2}x + 2$ and below $y = -2x + 5$.

15. $\boxed{C}$ One manicure takes 1/3 of an hour. One pedicure takes 1/2 an hour. The total number of hours Tina spends doing manicures and pedicures must be less than or equal to 30, so $\frac{1}{3}m + \frac{1}{2}p \leq 30$. Now Tina earns $25m$ for the manicures and $40p$ for the pedicures. Since these earnings combined must be at least \$900 to cover her expenses, $25m + 40p \geq 900$.

16. $\boxed{B}$ After 15 weeks, the bear has lost an additional 10 weeks worth of weight relative to the 5-week mark. That 10-week weight loss amounts to a minimum of $10 \times 6 = 60$ pounds and a maximum of $10 \times 12 = 120$ pounds. Since the bear weighed 580 pounds after 5 weeks of hibernation, the bear could weigh $580 - 120 = 460$ pounds to $580 - 60 = 520$ pounds after 15 weeks of hibernation: $460 \leq w \leq 520$.

17. $\boxed{B}$ Manasa pays $0.25t$ from her checking account and $0.75t$ from her savings account. In order for her checking account to have a higher balance than her savings account after taxes are paid,

$$\begin{aligned} 3{,}300 - 0.25t &> 4{,}500 - 0.75t \\ 0.50t &> 1{,}200 \\ t &> 2{,}400 \end{aligned}$$

18. $\boxed{72}$ Let x be the number of 3-pound rugs and y be the number of 5-pound rugs. Based on the information given, we can construct the following system:

$$\begin{aligned} x + y &\geq 85 \\ 3x + 5y &\leq 400 \end{aligned}$$

At this point, we can graph these inequalities on Desmos and look for the point with the maximum value of y in the shaded region (the highest point), which turns out to be $(12.5, 72.5)$. The maximum possible number of 5-pound rugs is then 72 (we round down for maximums).

For a mathematical approach, we can multiply the first equation by -3 to get $-3x - 3y \leq -255$. Then we can add this result to the second equation to get $2y \leq 145 \Rightarrow y \leq 72.5$.

19. $\boxed{35}$ Let x be the number of ceramic tiles and y be the number of porcelain tiles. Based on the given information, we can construct the following system:

$$\begin{aligned} 4x + 7y &\geq 800 \\ 4.70x + 15.20y &\leq 1{,}600 \end{aligned}$$

At this point, we can graph these inequalities on Desmos and look for the point with the minimum value of x in the shaded region (the leftmost point), which turns out to be $(34.4, 94.6)$. The minimum number of ceramic tiles Katy can purchase is then 35 (we round up for minimums).

For a mathematical approach, we can multiply the first equation by -15.2 and the second equation by 7 to get

$$\begin{aligned} -60.8x - 106.4y &\leq -12{,}160 \\ 32.9x + 106.4y &\leq 11{,}200 \end{aligned}$$

Now that the inequality signs and the coefficients of y match, we can add these two inequalities to get

$$\begin{aligned} -27.9x &\leq -960 \\ x &\geq 34.4 \end{aligned}$$

EXERCISE 2:

1. $\boxed{D}$ Test each point. Plugging in $(-10, -10)$ from answer D results in $-10 > 3(-10) - 20 \Rightarrow -10 > -50$ from the first inequality and $-10 < \frac{1}{2}(-10) + 5 \Rightarrow -10 < 0$ from the second inequality. Because both resulting statements are true, $(-10, -10)$ is a solution to the system.

2. $\boxed{B}$ The shaded region is below the horizontal line $y = 3$ but above the horizontal line $y = -3$. Therefore, $y \geq -3$ and $y \leq 3$.

3. $\boxed{A}$ Since each gift box was sold for \$7, the shop sold $400 \times \$7 = \$2{,}800$ worth of gift boxes, which means the shop needed to have sold at least $\$8{,}000 - \$2{,}800 = \$5{,}200$ worth of greeting cards to meet its goal. Since each greeting card was sold for \$5, the shop could have sold a minimum of $\$5{,}200 \div \$5 = 1{,}040$ greeting cards to meet its goal.

4. $\boxed{C}$ Test each system of inequalities by checking whether it's satisfied by $(5, 0)$. For the system given by answer C, the first inequality yields $0 < 5$ and the second inequality yields $0 > -5$ when we plug in $(5, 0)$. Both are true statements so $(5, 0)$ is a solution to the system given by answer C.

5. $\boxed{A}$ Test each inequality by checking whether it's satisfied by $(1, 4)$ and $(3, 0)$ and not by $(1, -2)$. Only the inequality in answer A meets all of these conditions. The point $(1, 4)$ results in the true statement $4 > -2(1) + 1 \Rightarrow 4 > -1$. The point $(3, 0)$ results in the true statement $0 > -2(3) + 1 \Rightarrow 0 > -5$. And finally, the point $(1, -2)$ results in the false statement $-2 > -2(1) + 1 \Rightarrow -2 > -1$.

6. $\boxed{C}$ If the distributor contracts out to Company A for x hours, then it contracts out to Company B for $10 - x$ hours. Company A then produces $80x$ cartons and Company B produces $140(10 - x)$ cartons. Setting up the inequality, we get

$$80x + 140(10 - x) > 1{,}100$$

7. $\boxed{D}$ From a 14-mile jog, Melanie could burn between 14(120) calories and 135(14) calories: $14(120) \leq y \leq 135(14)$. Divide all three parts of this compound inequality by 14 and we get $120 \leq \frac{y}{14} \leq 135$.

8. $\boxed{B}$ Because we're trying to maximize the number of nighttime bottles, we assume that only 65 daytime bottles are filled. The 65 daytime bottles will use up $65 \times 2 = 130$ ounces of the active ingredient and $65 \times 6 = 390$ ounces of flavored syrup, leaving behind $385 - 130 = 255$ ounces of the active ingredient and $850 - 390 = 460$ ounces of flavored syrup. The remaining amount of active ingredient is enough for $\frac{255}{3} = 85$ nighttime bottles, and the remaining amount of flavored syrup is enough for $\frac{460}{5} = 92$ nighttime bottles. Based on these numbers, we're limited by the remaining amount of active ingredient, so the maximum number of nighttime bottles that can be filled is 85.

9. $\boxed{D}$ Denise will spend $0.45x$ on 45-cent stamps and $0.60y$ on 60-cent stamps. To return with at least \$5 in change, Denise can spend no more than \$15 altogether. Expressing this condition as an inequality, we get $0.45x + 0.60y \leq 15$.

10. $\boxed{C}$ Let Y be the maximum value of y. Then

$$k = 7 + 0.5Y$$
$$k - 7 = 0.5Y$$
$$2k - 14 = Y$$

Since the maximum value of y is equal to $2k - 14$, it must be true that $y \leq 2k - 14$.

11. $\boxed{C}$ The amount that Sandra spends on pencils is $1.50x$ and the amount she spends on pencil cases is $4.50y$. Since she cannot spend more than \$250, $1.50x + 4.50y \leq 250$. To hold all of the pencils she buys, she will need at least $\frac{x}{12}$ pencil cases (by assuming that each pencil case will hold the maximum of 12 pencils, we get the minimum number of cases she will need to buy). Hence, $y \geq \frac{x}{12}$.

12. $\boxed{C}$ The shaded region lies above the line with slope 2 (up 2 units for every 1 unit over to the right) and y-intercept -2. Therefore, the shaded region represents the inequality $y \geq 2x - 2$. To find the answer that's equivalent, we can subtract $2x$ from both sides: $-2x + y \geq -2$. Then we can multiply both sides by -1 (remember to switch the sign): $2x - y \leq 2$.

13. $\boxed{A}$ Based on the given information,

$$4t \leq t + 7w \leq 6t$$

To isolate w, first subtract all three parts of this inequality by t:

$$3t \leq 7w \leq 5t$$

Then divide everything by 7:

$$\frac{3}{7}t \leq w \leq \frac{5}{7}t$$

14. $\boxed{A}$ If we let p be the price per window, then $30p$ is the amount Paul spends on the windows and $30(50 + 0.10p)$ is the total amount he spends installing them. We can now set up the following inequality:

$$30p + 30(50 + 0.10p) \leq 25{,}000$$
$$30p + 1{,}500 + 3p \leq 25{,}000$$
$$33p \leq 23{,}500$$
$$p \leq 712.1212\ldots$$

The maximum possible price per window is then $\approx \$712.12$.

15. $\boxed{B}$ Let's say Jerry's estimate, m, is 100 marbles. If the actual number of marbles is within 10 of that estimate, then the actual number must be at least 90 and at most 110. Using variables, $m - 10 \leq n \leq m + 10$.

16. $\boxed{D}$ Plug in $x = 1$, $y = 20$ into the first inequality to get $20 > 15 + a$, $5 > a$. Do the same for the second inequality to get $20 < 5 + b$, $15 < b$. So, a is less than 5 and b is greater than 15. The difference between the two must be more than $15 - 5 = 10$. Among the answer choices, 12 is the only one that is greater than 10.

17. $\boxed{44}$ Let x be the number of cedar birdhouses and y be the number of oak birdhouses. Based on the information given (after converting minutes to hours), we can construct the following system:

$$x + y \geq 75$$
$$2.75x + 1.25y \leq 160$$

At this point, we can graph these inequalities on Desmos and look for the point with the maximum value of x in the shaded region (the rightmost point), which turns out to be $(44.17, 30.83)$. The maximum possible number of cedar birdhouses John could make is then 44 (we round down for maximums).

For a mathematical approach, we can multiply the first equation by -1.25 to get $-1.25x - 1.25y \leq -93.75$. Then we can add this result to the second equation to get $1.5x \leq 66.25 \Rightarrow x \leq 44.17$.

18. $\boxed{B}$ Assume the case in which the number of adults is at a minimum: exactly one adult for every 2 children. In this case, if the number of adults is a, the number of children is $2a$. Since there a total of 240 audience members,

$$a + 2a = 240$$
$$3a = 240$$
$$a = 80$$

Since this is the minimum possible number of adults, $a \geq 80$.

19. $\boxed{D}$ If a ounces of Arabica beans are used, then $\frac{a}{2}$ ounces of Robusta beans must be used. The corresponding amounts of caffeine are then $40a$ milligrams and $60 \cdot \frac{a}{2} = 30a$ milligrams, respectively. Since the total amount of caffeine in the cup must be less than 280 milligrams,

$$\begin{aligned} 40a + 30a &< 280 \\ 70a &< 280 \\ a &< 4 \end{aligned}$$

And because a cup of coffee has more than zero beans (otherwise it wouldn't be coffee), $0 < a < 4$.

20. $\boxed{51}$ Let x be the number of minutes spent walking and y be the number of minutes spent running. Based on the information given, we can construct the following system:

$$\begin{aligned} 5.8x + 10.2y &\geq 700 \\ y &\geq x + 20 \end{aligned}$$

At this point, we can graph these inequalities on Desmos and look for the point with the minimum value of y in the shaded region (the lowest point), which turns out to be $(31, 51)$. The minimum number of minutes Mina could spend running is then 51.

For a mathematical approach, we can subtract x from both sides of the second equation and then multiply both sides by 5.8:

$$\begin{aligned} 5.8x + 10.2y &\geq 700 \\ -5.8x + 5.8y &\geq 116 \end{aligned}$$

Now we can add these two inequalities to get $16y \geq 816 \Rightarrow y \geq 51$.

21. $\boxed{D}$ For 1 pound of seasoning, Lianne will need 0.75 pounds of sea salt and 0.25 pounds of black pepper. The sea salt will cost $0.75 \times \$2 = \$1.50$ and the black pepper will cost $0.25 \times \$8 = \$2$. Altogether, that's $\$1.50 + \$2 = \$3.50$ for each pound of seasoning. Since Lianne can spend no more than \$210, she is limited to making $\frac{\$210}{\$3.50} = 60$ pounds of seasoning. Therefore, 60 is the maximum.

Chapter 15: Function Transformations

EXERCISE:

1. $\boxed{C}$ Because $f(x) - 8$ represents a translation of 8 units down from $f(x)$, the y-intercept of $y = f(x) - 8$ must be 8 units down from the y-intercept of $y = f(x)$: $(0, 5 - 8) = (0, -3)$.

2. $\boxed{B}$ The difference between $2x^2 - 2$ and $2x^2 + 4$ is a constant of 6. In other words, 6 needs to be added on to $y = 2x^2 - 2$ to get $y = 2x^2 + 4$. That entails a translation 6 units upward.

3. $\boxed{-17}$ Function g must be defined by $g(x) = f(x) + 8 = -(x-5)^2 + 8$. The y-coordinate of the y-intercept is then $g(0) = -(0-5)^2 + 8 = -(-5)^2 + 8 = -25 + 8 = -17$.

4. $\boxed{A}$ The graph of $y = g(x)$ has a y-intercept at $(0, -1)$. The graph of $y = g(x) + 3$, which is 3 units up from the graph of $y = g(x)$, must then have a y-intercept at $(0, -1 + 3) = (0, 2)$. The only graph that has a y-intercept at $(0, 2)$ is answer A.

5. $\boxed{B}$ The graph shown is a line with equation $y = \frac{1}{2}x + 1$ (slope of $\frac{1}{2}$ and y-intercept at 1). Since the graph shown is $h(x) + 9$, the graph of $h(x)$ must be 9 units down: $h(x) = \frac{1}{2}x + 1 - 9 = \frac{1}{2}x - 8$.

6. $\boxed{C}$ $g(x) = f(x) + 7 = \frac{x+1}{2} + 7 = \frac{x+1}{2} + \frac{14}{2} = \frac{x+15}{2}$

7. $\boxed{C}$ Before applying any translations to the graph of $y = -x + 3$, we first make sure y is isolated on one side of the equation, which it is. We then replace x with $x - 20$ to get the resulting equation after a translation of 20 units to the right (note the parentheses): $y = -(x - 20) + 3 = -x + 20 + 3 = -x + 23$.

8. $\boxed{B}$ To translate $f(x)$ to the left by 11 units, we replace x with $x + 11$: $g(x) = f(x + 11) = a(x + 11)(x + 11 - 7) = a(x + 11)(x + 4)$.

9. $\boxed{B}$ $f(x + 3)$ represents a translation of 3 units to the left from $f(x)$. Since $f(x + 3) = \frac{2}{3}(x + 3) - 6 = \frac{2}{3}x + \frac{2}{3}(3) - 6 = \frac{2}{3}x - 4$, the resulting graph has a y-intercept at $(0, -4)$.

10. $\boxed{C}$ The graph of g is the result of translating the graph of f up 4 units. The easiest approach is to just draw the resulting graph and see where the x-intercept is.

    Another approach is to actually solve for the x-intercept. Using slope-intercept form, we get $f(x) = -2x + 2$. Adding 4 to get the equation of g, $g(x) = -2x + 6$. Setting $g(x) = 0$ and solving for x to get the x-intercept, we get $x = 3$, which means the x-intercept is at $(3, 0)$.

    Yet another approach is an intuitive one: the graph of g is 4 units up from where f is, but because the slope of f is -2, the x- and y-intercepts of g will not change by the same number of units. They'll change in a ratio of 2:1. So when the y-intercept gets shifted up by 4, the x-intercept gets shifted to the right by 2. The new x-intercept is therefore at $(1 + 2, 0) = (3, 0)$.

11. $\boxed{C}$ For horizontal shifts, the trick is to find the value of x that makes the substituted expression equal to 0. This value is equal to the horizontal shift. For choice A, $x = \frac{2}{3}$ makes $3x - 2$ equal to 0, so the horizontal shift is $\frac{2}{3}$ units to the right. For choice B, that value is $-\frac{2}{3}$, so the shift is $\frac{2}{3}$ units to the left. For choice C, $x = \frac{3}{2}$ makes the expression $2x - 3$ equal to 0, so the shift is $\frac{3}{2}$ units to the right. This is the answer. For choice D, the shift is $\frac{3}{2}$ units to the left.

12. $\boxed{\frac{59}{9} \text{ or } 6.555 \text{ or } 6.556}$ The slope of $y = f(x)$ is $\frac{46-82}{3-(-1)} = \frac{-36}{4} = -9$. We can now use point-slope form to find the equation of the line defined by $f(x)$: $y - 82 = -9(x-(-1))$, which simplifies to $y = -9x + 73$. The equation of $g(x)$ is then $g(x) = -9x + 73 - 14 = -9x + 59$. Setting $g(x) = 0$, we get $-9x + 59 = 0$, which gives $x = \frac{-59}{-9} = \frac{59}{9}$.

13. $\boxed{20}$ To translate $g(x)$ down 2 units and to the left by 3 units, we subtract 2 from $g(x)$ and replace x with $x+3$: $h(x) = g(x+3) - 2 = 3^{x+3} - 5 - 2 = 3^{x+3} - 7$. Therefore, $h(0) = 3^{0+3} - 7 = 27 - 7 = 20$.

14. $\boxed{C}$ To translate $f(x)$ up 4 units and to the right by 5 units, we add 4 to $f(x)$ and replace x with $x-5$: $g(x) = f(x-5) + 4 = -2 + \frac{k+6}{x-5} + 4 = 2 + \frac{k+6}{x-5}$.

15. $\boxed{C}$ The graph shown passes through the points $(0,3)$ and $(2,0)$ (we can pick other points but these are the easiest to work with). Since the graph of $y = f(x)$ must be 2 units up from the given graph of $y = f(x) - 2$, the graph of f must pass through the points $(0, 3+2) = (0,5)$ and $(2, 0+2) = (2,2)$. Now we can test each of the answer choices using these points. When $x = 0$, the equations in answers C and D yield 5. To decide between these two possible answers, we plug in $x = 2$, and only answer C yields the target output of 2.

16. $\boxed{A}$ Graph the functions. The graph of $f(x) = x^3$ is "centered" at $(0,0)$. The graph of $g(x)$ is "centered" at $(3,-2)$. Comparing these points of reference, we can see that $g(x)$ is shifted 3 units to the right and 2 units downward from $f(x)$. Therefore, $g(x) = f(x-3) - 2$, which means $a = -3$ and $b = -2$. The sum $a + b$ is then $-3 + (-2) = -5$.

17. $\boxed{5}$ Factoring g, we get $g(x) = x^2 + 4x + 4 = (x+2)^2$. Since $g(x)$ is $f(x)$ shifted k units to the left, $g(x) = f(x+k)$, which means

$$\begin{aligned}(x+2)^2 &= (x+k-3)^2\\ x+2 &= x+k-3\\ 2 &= k-3\\ 5 &= k\end{aligned}$$

Alternatively, we could've solved this question by comparing $f(x)$ and $g(x)$ to $y = x^2$. The graph of $f(x) = (x-3)^2$ is 3 units to the right of $y = x^2$, and the graph of $g(x) = (x+2)^2$ is 2 units to the left of $y = x^2$. Therefore, $g(x)$ is 5 units to the left of $f(x)$.

18. $\boxed{\frac{13}{6} \text{ or } 2.166 \text{ or } 2.167}$ Given the equation of line p, let's isolate y: $12x + 5y = 56$ means that $5y = -12x + 56$, which yields $y = -\frac{12}{5}x + \frac{56}{5}$. The equation of line q is then $y = -\frac{12}{5}x + \frac{56}{5} - 6 = -\frac{12}{5}x + \frac{26}{5}$. Note that directly subtracting 6 from the 56 in the equation of line p to get the equation of line q would have been a mistake. The issue is the coefficient of y. You want the division by 5 to happen before the 6 is subtracted because otherwise the division compresses the translation by a factor of 5. Just remember that you must isolate y before performing any transformations. In questions that give you a function definition such as $f(x) = x^2$, you don't have to worry about this because "$f(x)$" is already isolated. Now to find the x-intercept of line q, we set $y = 0$ and solve for x:

$$0 = -\frac{12}{5}x + \frac{26}{5} \quad\Rightarrow\quad 0 = -12x + 26 \quad\Rightarrow\quad 12x = 26 \quad\Rightarrow\quad x = \frac{26}{12} = \frac{13}{6}$$

Chapter 16: Quadratic Functions

EXERCISE 1:

1. $\boxed{B}$ We can factor the equation to find the x-intercepts.

$$y = x^2 - 3x - 10 = (x-5)(x+2)$$

The x-intercepts must then be $(5,0)$ and $(-2,0)$. Alternatively, graph the equation and locate the x-intercepts.

2. $\boxed{C}$ The number 10,000 is the y-intercept, the total monthly expenses when x, the number of tables, is 0. These expenses are likely to be rent, equipment, worker salaries, etc.

3. $\boxed{26}$ The value of the artifact, modeled by $v(t)$, was at its maximum at the vertex of $v(t)$. Since $v(t)$ is written in vertex form, it's easy to see that the vertex is at $(26, 14750)$. Therefore, the value of the artifact was greatest 26 years after its excavation.

4. $\boxed{-\frac{8}{5} \text{ or } -1.6}$ $g(x)$ reaches its maximum at its vertex, which is at $x = -\frac{b}{2a} = -\frac{-16}{2(-5)} = \frac{16}{-10} = -1.6$.

5. $\boxed{C}$ The equation has exactly one solution when the discriminant is equal to 0:

$$\begin{aligned}b^2 - 4ac &= 0\\ (-6)^2 - 4(-12)(d) &= 0\\ 48d &= -36\\ d &= \frac{-36}{48} = -\frac{3}{4}\end{aligned}$$

6. $\boxed{B}$ Since f has a negative coefficient for the x^2 term, $f(x)$ reaches a maximum at its vertex (its graph is a parabola that opens downward). Given that f is written in vertex form, it's easy to see that the vertex is at $(5,8)$. This means that when 5 thousand units are produced and sold, the company makes the most it can make—a maximum profit of 8 hundred thousand dollars. Only answer B is accurate.

7. $\boxed{D}$ When a parabola has two x-intercepts, the x-coordinate of the vertex is equal to the midpoint (average) of those two x-intercepts:

$$\begin{aligned}\frac{a+10}{2} &= \frac{7}{3}\\ a + 10 &= \frac{14}{3}\\ a &= \frac{14}{3} - 10 = \frac{14}{3} - \frac{30}{3} = -\frac{16}{3}\end{aligned}$$

8. $\boxed{-25}$ For a quadratic equation to have no real solutions, the discriminant must be less than 0:

$$\begin{aligned}b^2 - 4ac &< 0\\ (30)^2 - 4(a)(-9) &< 0\\ 900 + 36a &< 0\\ 36a &< -900\\ a &< -25\end{aligned}$$

Therefore, $k = -25$. Note that k cannot be any greater because that would allow for invalid values of a.

9. $\boxed{B}$ The parabola opens downward so the coefficient of x^2 must be negative, which narrows down the answer choices to A and B. Since the vertex of the parabola is $(-4,-3)$, we can eliminate A. Only answer B gives an equation in vertex form that results in a vertex of $(-4,-3)$.

10. $\boxed{C}$ For a quadratic equation to have no real solution, the discriminant must be less than 0:

$$\begin{aligned}b^2 - 4ac &< 0\\ (2c)^2 - 4(9)(36) &< 0\\ 4c^2 &< 4(9)(36)\\ c^2 &< 324\end{aligned}$$

The only answer choice that gives a value less than 324 when squared is -11, answer C. Note that to solve the inequality above all the way to c, we have to consider the negative case: $-18 < c < 18$.

11. $\boxed{B}$ The equation is given in vertex form, so it's easy to see that the vertex is at $(b, -c)$. Since b and c are positive, the x-coordinate of the vertex must be positive and the y-coordinate must be negative. Only answer B satisfies these criteria.

12. $\boxed{D}$ Since $(0,0)$ and $(b,0)$ are x-intercepts, $f(x)$, the download speed, is 0 at $x = 0$ and $x = b$. First, why would the download speed be 0 at $x = 0$? Well, the download is just starting so no megabytes have been downloaded yet. Now why would the download speed be 0 at $x = b$? The best answer is that the file download has been completed, so there are no more megabytes left to download—just like a car's speed is 0 when it comes to a stop. Therefore, b represents the number of minutes it took for the download to complete.

13. $\boxed{63}$ For a quadratic equation to have two real solutions, the discriminant must be greater than 0:

$$b^2 - 4ac > 0$$
$$(b)^2 - 4(-1)(-961) > 0$$
$$b^2 > 3844$$
$$b > 62$$

Since b is a positive integer, we don't have to consider the negative case, and the least possible value is 63.

14. $\boxed{C}$ The given parabola opens upward and has two negative x-intercepts. Now for a parabola to open upward, the coefficient of x^2 must be positive. Since a is positive, only answers C and D meet this requirement. Between C and D, only answer C has two negative x-intercepts, $(-b, 0)$ and $(-c, 0)$, given that b and c are positive.

15. $\boxed{\frac{9}{4} \text{ or } 2.25}$ Since the graphs intersect at exactly one point, the system consisting of their two equations must have only one solution. Substituting the first equation into the second, we get

$$x + 3 = kx^2 - 5x + 7$$
$$0 = kx^2 - 6x + 4$$

For this equation to yield only one solution, its discriminant must be equal to 0:

$$b^2 - 4ac = 0$$
$$(-6)^2 - 4(k)(4) = 0$$
$$-16k = -36$$
$$k = \frac{-36}{-16} = \frac{9}{4}$$

16. $\boxed{D}$ Substituting the first equation into the second, we get

$$3 = x^2 + 6x + r$$
$$0 = x^2 + 6x + r - 3$$

For the system to have no real solution, this equation must have no real solution. And for that to be the case, its discriminant must be less than zero:

$$b^2 - 4ac < 0$$
$$(6)^2 - 4(1)(r - 3) < 0$$
$$36 - 4r + 12 < 0$$
$$-4r < -48$$
$$r > 12$$

The only answer choice greater than 12 is answer D.

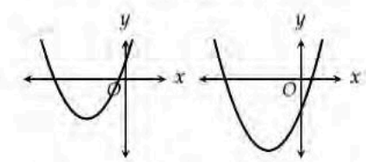
17. $\boxed{\frac{75}{4} \text{ or } 18.75}$ Since the graphs intersect at exactly one point, the given system must have only one solution. Substituting the first equation into the second, we get

$$\frac{c}{3} = -4x^2 + 10x$$
$$4x^2 - 10x + \frac{c}{3} = 0$$

For this equation (and therefore the system) to yield only one solution, the discriminant must be equal to 0:

$$b^2 - 4ac = 0$$
$$(-10)^2 - 4(4)\left(\frac{c}{3}\right) = 0$$
$$100 - \frac{16c}{3} = 0$$
$$-16c = -300$$
$$c = \frac{-300}{-16} = \frac{75}{4}$$

18. $\boxed{A}$ For a quadratic function of the form $f(x) = ax^2 + bx + c$, the x-coordinate of the vertex is given by $-\frac{b}{2a}$. In this case, the vertex of f is at (h, k), so $-\frac{b}{2(2)} = h$, which yields $b = -4h$. Since h is negative, b must be positive. Therefore, I is true. However, we don't know the value of b nor the values of h and k. Therefore, it's impossible to determine whether c, which represents the y-intercept, is positive or negative. Here are two upward-opening parabolas that each have a vertex in the third quadrant.



In the first case, the value of the y-intercept is positive. In the second case, it's negative. Therefore, II is not necessarily true.

EXERCISE 2:

1. $\boxed{C}$ The discriminant determines the number of solutions: $b^2 - 4ac = (-13)^2 - 4(2)(21) = 1$. Since the discriminant is positive, the equation has two distinct real solutions.

2. $\boxed{C}$ Since x is defined as the price each box is sold for and $P(x)$ represents the total profit, in thousands of dollars, $P(10)$ is the total profit, in thousands of dollars, if each box is sold for \$10. Therefore, $P(10) = 9$ means that the total profit is estimated to be \$9,000 if each box is sold for \$10.

3. $\boxed{12}$ We're looking for the value of x that results in the minimum value of $f(x)$. Since the graph of f is a parabola that opens upwards in a "U" shape, the minimum of f occurs at the vertex, which is located at $x = -\frac{b}{2a} = -\frac{-24}{2(1)} = 12$. Therefore, the manufacturer should produce 12 units each week to minimize the cost per unit.

4. $\boxed{7921}$ Expanding the given equation, we get $y = 190x - 12x - x^2 = 178x - x^2 = -x^2 + 178x$. The maximum occurs at the vertex, which is located at $x = -\frac{b}{2a} = -\frac{178}{2(-1)} = 89$. The maximum value of y is then $-(89)^2 + 178(89) = 7{,}921$.

5. $\boxed{B}$ The graph of f has x-intercepts at $x = -15$ and $x = 4$. The minimum of f is reached at the vertex, whose x-coordinate is the midpoint (average) of these two x-intercepts. The minimum of f is therefore reached at $x = \frac{-15 + 4}{2} = -\frac{11}{2}$.

6. $\boxed{4}$ A path that can be modeled by a quadratic function is symmetrical with respect to its maximum (or minimum). The projectile took $10 - 6 = 4$ seconds to go from a height of 34 feet to its maximum height of 49 feet. Therefore, it must have also taken 4 seconds for the projectile to fall back down to a height of 34 feet after reaching its maximum height.

7. $\boxed{B}$ Since we're given the vertex form of the equation, it's easy to see that the vertex is at $(-a, b)$. Since a and b are positive constants, the x-coordinate of the vertex must be negative and the y-coordinate must be positive. Thus, the vertex is in Quadrant II.

8. $\boxed{31}$ First, expand and rearrange the quadratic equation into standard form:

$$-18x^2 + 48x = q$$
$$-18x^2 + 48x - q = 0$$

For the equation to have two distinct real solutions, the discriminant must be greater than zero:

$$b^2 - 4ac > 0$$
$$(48)^2 - 4(-18)(-q) > 0$$
$$-72q > -(48)^2$$
$$q < \frac{-(48)^2}{-72} = 32$$

Since q is an integer, the greatest possible value of q is 31.

9. $\boxed{B}$ The minimum or maximum of a parabola always occurs at its vertex. Since m is positive, the parabola opens upwards in a "U" shape, which means this parabola has a minimum at its vertex. Expanding the equation gives us a vertex form of $f(x) = m\left[(x-m)^2 - 1\right] = m(x-m)^2 - m$. This resulting vertex form indicates that the vertex is at $(m, -m)$, which, as we stated earlier, is also the parabola's minimum.

10. $\boxed{C}$ Given that $g(x)$ is a parabola and $(0,0)$ and $(c,0)$ are its two x-intercepts, $x = \frac{c}{2}$, being the midpoint of these two x-intercepts, must be the x-coordinate of the vertex. In this case, the vertex of g represents a maximum, since its graph opens downwards in an upside-down "U" shape. Therefore, $g(x)$ reaches its maximum at $x = \frac{c}{2}$. In the context of this question, the download speed was at its highest (maximum) $\frac{c}{2}$ minutes after the download was started.

11. $\boxed{-3}$ There are several ways to approach this question. One way is to plug in two points and solve for a and c. Another way is to use intercept form. Since the graph passes through $(-3,0)$ and $(1,0)$, the equation of the graph can be written as $y = a(x+3)(x-1)$. Since the graph passes through vertex $(-1,-6)$, we can plug this in to solve for a: $-6 = a(-1+3)(-1-1)$, which yields $-6 = -4a$. Thus, $a = \frac{-6}{-4} = 1.5$. The equation is then $y = 1.5(x+3)(x-1) = 1.5(x^2+2x-3) = 1.5x^2 + 3x - 4.5$. Finally, $a + c = 1.5 + (-4.5) = -3$.

12. $\boxed{D}$ Since $f(x-3)$ represents a translation of 3 units to the right from $f(x)$, $g(x)$ reaches its maximum 3 units to the right of where $f(x)$ reaches its maximum. Since f is a quadratic function, $f(x)$ reaches its maximum at its vertex, where $x = -\frac{b}{2a} = -\frac{48}{2(-6)} = 4$. Therefore, $g(x)$ reaches its maximum at $x = 4 + 3 = 7$. If you didn't know that the vertex of a quadratic is found at $x = -\frac{b}{2a}$, an alternative approach would've been to graph the function on Desmos and click on the vertex for its coordinates. Yet another approach is to expand $f(x-3)$ first to find g and then use $x = -\frac{b}{2a}$ on g directly.

13. $\boxed{D}$ Since the graphs intersect at exactly one point, the system consisting of their two equations must have only one solution. Substituting the first equation into the second, we get

$$5.5 = -x^2 + 16x - a$$
$$x^2 - 16x + a + 5.5 = 0$$

For this equation to yield only one solution, its discriminant $(b^2 - 4ac)$ must be equal to 0:

$$(-16)^2 - 4(1)(a + 5.5) = 0$$
$$256 - 4a - 22 = 0$$
$$-4a = -234$$
$$a = \frac{-234}{-4} = \frac{117}{2}$$

14. $\boxed{10}$ Substitute the first equation into the second:

$$4x = 3x^2 + bx + 3$$
$$0 = 3x^2 + (b-4)x + 3$$

For the system to have exactly one solution, this resulting equation must have exactly one solution. And for that to be the case, its discriminant must be equal to zero:

$$b^2 - 4ac = 0$$
$$(b-4)^2 - 4(3)(3) = 0$$
$$(b-4)^2 = 36$$
$$b - 4 = \pm 6$$
$$b = 4 \pm 6 = -2 \text{ or } 10$$

Since b is a positive constant, $b = 10$.

15. $\boxed{D}$ One of the x-intercepts of the graph of f is at $x = 3$. Since the x-coordinate of the vertex (in this case, 5) is always the midpoint of a parabola's two x-intercepts, the other x-intercept must be at $x = 7$. Therefore, $k = 7$, giving us $f(x) = a(x-3)(x-7)$. We can now use the vertex as a point to plug in to solve for a:

$$-32 = a(5-3)(5-7)$$
$$-32 = a(2)(-2)$$
$$-32 = -4a$$
$$a = 8$$

16. $\boxed{39}$ Substitute the first equation into the second:

$$2x^2 + 15x + 57 = -9x + p$$
$$2x^2 + 24x + (57 - p) = 0$$

For the system to have exactly one solution, this resulting equation must have exactly one solution. And for that to be the case, its discriminant must be equal to zero:

$$b^2 - 4ac = 0$$
$$24^2 - 4(2)(57 - p) = 0$$
$$576 - 456 + 8p = 0$$
$$8p = -120$$
$$p = -15$$

Now that we have the value of p, we can graph both equations on Desmos and see that they intersect at $(-6, 39)$. Therefore, $y = 39$. Alternatively, we can solve our initial equation:

$$2x^2 + 24x + (57 - p) = 0$$
$$2x^2 + 24x + 72 = 0$$
$$x^2 + 12x + 36 = 0$$
$$(x+6)^2 = 0$$
$$x = -6$$

When $x = -6$,
$y = -9x + p = -9(-6) - 15 = 39$.

17. $\boxed{-\frac{7}{18} \text{ or } -.3888 \text{ or } -.3889}$ Since the graph of $y = g(x)$ has x-intercepts at $(-4,0)$ and $(8,0)$, the equation of g in intercept form is $g(x) = a(x+4)(x-8) = a(x^2 - 4x - 32) = ax^2 - 4ax - 32a$, where a is the same constant as in standard form. Since this equation must be equivalent to the standard form equation given by the question ($y = ax^2 + bx + c$), we can compare coefficients and see that $b = -4a$ and $c = -32a$. Since b and c sum to 14, $-4a + (-32a) = 14$, which yields $-36a = 14$. Thus, $a = \frac{14}{-36} = -\frac{7}{18}$. A faster solution is to use the sum and product of the roots formulas to express b and c in terms of a.

18. $\boxed{54}$ We can use a quadratic function in vertex form to model the height of the ball. Since the ball reached a maximum height of 72 feet after 3 seconds, we can consider the vertex to be $(3, 72)$. This gives us a vertex form of

$$h(t) = a(t-3)^2 + 72$$

where a is a constant. Since the ball hit the ground after 7 seconds, we can plug in the point $(7, 0)$ to find the value of a:

$$0 = a(7-3)^2 + 72$$
$$-16a = 72$$
$$a = -4.5$$

Now that we have the value of a,
$h(5) = -4.5(5-3)^2 + 72 = 54$.

19. $\boxed{282}$ Since the values of y are 2 less than the values of $f(x)$, we can find the corresponding three values of $f(x)$ by adding 2 to the three values of y. In other words, $f(x) = y + 2$, which gives the following x- and y-values for f: $(6, 30)$, $(10, -18)$, and $(14, 30)$. Note that f has the same value at $x = 6$ and $x = 14$. Since f is a quadratic, it is symmetrical with respect to its minimum. Therefore, the minimum of f occurs at the midpoint (average) of $x = 6$ and $x = 14$, which is $x = \frac{6+14}{2} = 10$. We know it's a minimum rather than a maximum because the y-value of -18 at $x = 10$ is smaller than the y-values at $x = 6$ and $x = 14$. These values give us a vertex form of $f(x) = a(x-10)^2 - 18$, where a is a constant. Plugging in the point $(6, 30)$, we get

$$\begin{aligned} 30 &= a(6-10)^2 - 18 \\ 30 &= 16a - 18 \\ 48 &= 16a \\ 3 &= a \end{aligned}$$

Therefore, $f(x) = 3(x-10)^2 - 18$, and $f(20) = 3(20-10)^2 - 18 = 300 - 18 = 282$.

20. $\boxed{D}$ Since g is a quadratic function that has a maximum value, its graph must be a parabola that opens downward. Downward-opening parabolas always have a negative coefficient of x^2, so $a < 0$, and I must be true.

Since the graph of g passes through $(0, 0)$, we can plug this point into the equation:

$$\begin{aligned} 0 &= a(0)^2 + 8(0) + c \\ 0 &= c \end{aligned}$$

Therefore, II must be true.

Since $c = 0$, the equation of g becomes $g(x) = ax^2 + 8x = x(ax + 8)$. From this factored equation, we can see that the x-intercepts of g are at $x = 0$ and $x = -\frac{8}{a}$. Since a is negative, the x-intercept at $-\frac{8}{a}$ must be positive, which means $n > 0$. Therefore, III must be true.

21. $\boxed{D}$ Given the vertex at $(-5, -100)$, we can write the equation of the parabola in vertex form and then expand:

$$\begin{aligned} y &= a(x+5)^2 - 100 \\ &= a(x^2 + 10x + 25) - 100 \\ &= ax^2 + 10ax + 25a - 100 \end{aligned}$$

Since this equation must be equivalent to the one given to us by the question $(y = ax^2 + bx + c)$, we can compare the coefficients and see that $b = 10a$ and $c = 25a - 100$. Since $c > 0$, $25a - 100 > 0$, which yields $a > 4$. Since $a > 4$, b must be greater than $10(4) = 40$. Only answer D gives a value greater than 40.

22. $\boxed{A}$ For any quadratic function f, if $f(a) = f(b)$, then the x-coordinate of the vertex is always equidistant from a and b. In other words, the x-coordinate of the vertex is the midpoint or average of a and b. Since $f(-7) = f(9)$, the x-coordinate of the vertex in this case must be the average of -7 and 9: $\frac{-7+9}{2} = 1$. Since the graph of f opens upward and does not intersect the x-axis, we can consider its vertex to lie at $(1, k)$, where k must be a positive constant. An equation of f in vertex form is then

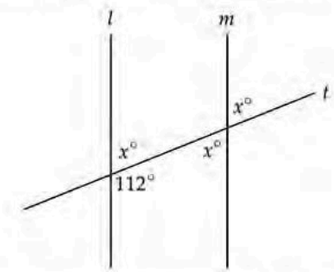
$$\begin{aligned} f(x) &= 3(x-1)^2 + k \\ &= 3(x^2 - 2x + 1) + k \\ &= 3x^2 - 6x + 3 + k \end{aligned}$$

Since this equation must be equivalent to the one given to us by the question $(f(x) = 3x^2 + bx + c)$, we can compare the coefficients and see that $b = -6$ and $c = 3 + k$. Now, $b + c = -6 + (3 + k) = -3 + k$. Since k must be positive, $b + c$ must be greater than -3. Only answer A gives a value greater than -3.

Chapter 17: Angles

EXERCISE:

1. $\boxed{68}$ Because vertical angles are equivalent and alternate interior angles are equivalent in the context of parallel lines, we can label several angles as being equivalent to $x°$:



Now, it's easy to see that $x°$ and $112°$ are supplementary, meaning they sum to $180°$. Therefore, $x = 180 - 112 = 68$.

2. $\boxed{B}$ The two angles form a line, so they sum to $180°$:

$$\begin{aligned} (x + 40) + x &= 180 \\ 2x + 40 &= 180 \\ 2x &= 140 \\ x &= 70 \end{aligned}$$

3. $\boxed{B}$

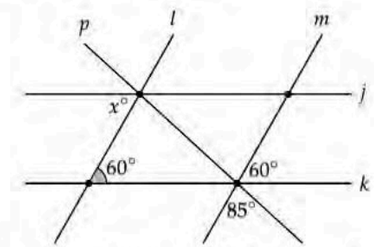
$$\begin{aligned} x + y + (x + y) &= 180 \\ 40 + y + (40 + y) &= 180 \\ 2y + 80 &= 180 \\ 2y &= 100 \\ y &= 50 \end{aligned}$$

4. $\boxed{A}$ The measure of $\angle ACB$ is $180° - 60° - 50° = 70°$. This angle is an exterior angle to the triangle on the right, $\triangle CDE$. So, given the exterior angle theorem, $70 = y + 40$, which yields $y = 30$.

5. $\boxed{C}$ The angles form a circle, so they sum to $360°$: $x + y = 360° - 45° - 80° = 235°$.

6. $\boxed{B}$ Since vertical angles are congruent, $a + b + c + d$ is equal to the sum of the interior angles of the quadrilateral. Since the interior angles of any quadrilateral sum to $360°$, $a + b + c + d = 360$.

7. $\boxed{60}$ Because line k intersects parallel lines l and m, the given $60°$ angle and the marked angle below are corresponding angles, which means they are equivalent.



Now line l intersects parallel lines j and k, so $x°$ and the marked angle are alternate interior angles, which are also always equivalent. Therefore, $x = 60$. Note that there are many other ways to do this problem.

8. $\boxed{B}$ Only one of the following is sufficient to prove that two lines are parallel: a pair of equal corresponding angles, a pair of equal alternate interior angles, or a pair of equal alternate exterior angles. Since $a°$ and $d°$ are alternate exterior angles, $a = d$ suffices.

9. $\boxed{A}$ The vertical angle to angle b also has a measure of $b°$. Since this vertical angle is also a same side interior angle to angle a, $a + b = 180$ (same side interior angles are supplementary), which yields $a = 180 - b$. Substituting $180 - b$ for a in the inequality $a > 74$, we get

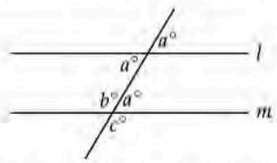
$$\begin{aligned} 180 - b &> 74 \\ -b &> -106 \\ b &< 106 \end{aligned}$$

Intuitively, the greater the value of a, the smaller the value of b to compensate (given that they must sum to 180). Since 74 is the lower limit of a, $180 - 74 = 106$ must be the upper limit of b. Note that answers C and D are wrong because they are equivalent to $a + b < 180$ and $a + b > 180$, respectively, both of which are not true.

10. $\boxed{48}$ Since $b°$ and $c°$ are vertical angles, they are congruent:

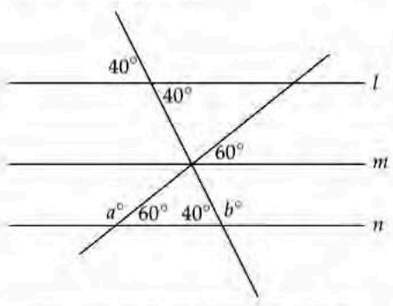
$$9k + 15 = 16k - 76$$
$$-7k = -91$$
$$k = 13$$

The value of b is then $9k + 15 = 9(13) + 15 = 132$. Since vertical angles are congruent and alternate interior angles are congruent, $a°$ and $b°$ are supplementary, as shown below.

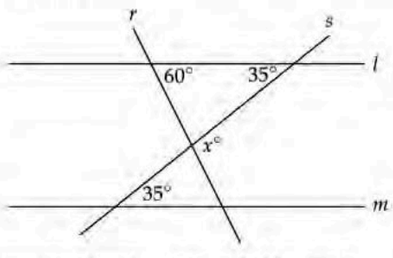


The value of a is therefore $180 - 132 = 48$.

11. $\boxed{260}$ Angle a is equal to $180 - 60 = 120$. Angle b is equal to $180 - 40 = 140$. Finally, $a + b = 120 + 140 = 260$.

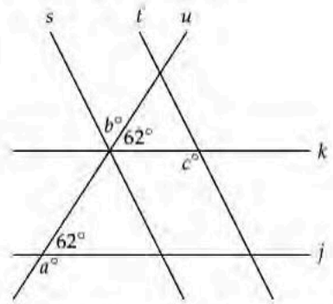


12. $\boxed{95}$ Because alternate interior angles are equal, one of the missing angles of the upper triangle is also 35°:



Since x is an exterior angle to the upper triangle, $x = 60 + 35 = 95$.

13. $\boxed{114}$ The fastest way is to first find the measure of the supplementary angle of a: $180 - 118 = 62°$. The corresponding angle to this angle is also 62° (see below). Since $b + 62$ and c are the measures of alternate interior angles with respect to lines s and t, $c = b + 62 = 52 + 62 = 114°$.



14. $\boxed{B}$ Since the angles of a triangle sum to 180°, the measure of $\angle B$ is $180 - 70 - 30 = 80°$. Now look at outer triangle ABC. Subtracting away $\angle B$ leaves us with $a + b$. Therefore, $a + b = 180 - 80 = 100°$.

15. $\boxed{56}$ In $\triangle ABC$, the measure of $\angle ABC$ is $180 - \angle A - \angle C = 180 - 40 - 60 = 80°$. The measure of $\angle DBW$ is then $180 - \angle ABC - \angle FBC = 180 - 80 - 72 = 28°$. Since $\angle BWE$ is an exterior angle to $\triangle BDW$, the measure of $\angle BWE$ is $\angle D + \angle DBW = 68 + 28 = 96°$. Since $\angle BWE$ is also an exterior angle to $\triangle AEW$, the measure of $\angle AED$ is $\angle BWE - \angle A = 96 - 40 = 56°$.

16. $\boxed{C}$ The two missing angles in the smaller inner triangle add up to 80°. The two bottom angles in the larger outer triangle add up to $180 - 70 = 110$. If we subtract the two missing angles of the smaller inner triangle from the two bottom angles of the larger outer triangle, we'll end up with $x + y$:

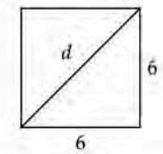
$$x + y = 110 - 80 = 30°$$

Alternatively, the angle above the 100° angle is $360 - 100 = 260°$. Since the upper shape is a quadrilateral, $x + y = 360 - 260 - 70 = 30°$.

Chapter 18: Triangles

EXERCISE 1:

1. $\boxed{C}$



A diagonal cuts the square into two 45–45–90 triangles, each with legs of length 6 inches. Since the diagonal is the hypotenuse of each triangle, its length is $6\sqrt{2}$ inches according to the 45–45–90 triangle relationship.

2. $\boxed{A}$ In a 30–60–90 triangle, the side opposite the 30° angle is half the hypotenuse. Therefore, $b = 10 \div 2 = 5$.

3. $\boxed{B}$ Since angles N and R are corresponding angles, angle N must also be 22°. The measure of angle M is then $180 - \angle L - \angle N = 180 - 94 - 22 = 64°$.

4. $\boxed{C}$ Since KM and AC are corresponding sides and $KM = 3AC$, the side lengths of triangle KLM must be 3 times the side lengths of triangle ABC. Therefore, $LM = 3BC = 3(24) = 72$.

5. $\boxed{C}$ The sides of triangle DEF are $9 \div 6 = 1.5$ times longer than the respective sides of triangle ABC. Therefore, $EF = 9 \times 1.5 = 13.5$ and $DF = 5 \times 1.5 = 7.5$. The perimeter of triangle DEF is then $9 + 13.5 + 7.5 = 30$.

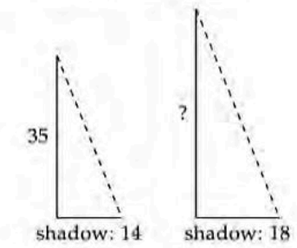
6. $\boxed{D}$ Since angles A and J are corresponding angles, angle J must also be 34°. The measure of angle L is then $180 - \angle J - \angle K = 180 - 34 - 90 = 56°$. Note that the given information on KL and BC is irrelevant.

7. $\boxed{21}$ By the pythagorean theorem, the length of the second leg is $\sqrt{31^2 - 25^2} = \sqrt{336} = \sqrt{16 \cdot 21} = 4\sqrt{21}$. Therefore, $n = 21$.

8. $\boxed{C}$ The triangle in question is a 30–60–90 triangle, in which the shorter leg is half as long as the hypotenuse and the longer leg is $\sqrt{3}$ times longer than the shorter leg. Given this relationship, the shorter leg is $42 \div 2 = 21$ and the longer leg is $21\sqrt{3}$. The perimeter is then $21 + 21\sqrt{3} + 42 = 63 + 21\sqrt{3}$.

9. $\boxed{A}$ Similarity depends on whether or not two triangles have congruent corresponding angles. Since we're given the measures of $\angle L$ and $\angle R$, all we would need to get the remaining angle measures is one additional angle measure from each triangle. Therefore, the measures of angles N and T would be sufficient. Note that the measures of angles S and T would be insufficient because they belong to the same triangle.

10. $\boxed{10}$ This question is a similar triangles problem that boils down to a proportions problem. Let's draw a diagram:



Since the triangles are similar, the sides are proportional. So for the shadow to be $\frac{18}{14} = \frac{9}{7}$ times longer, the pole must also be $\frac{9}{7}$ times longer. Therefore, the pole must be extended to $35 \times \frac{9}{7} = 45$ feet, which is an additional $45 - 35 = 10$ feet.

11. $\boxed{C}$ If the measure of angle X is 40°, then the measure of angle Z is $180 - 70 - 40 = 70°$. Since there are two angles of equal measure (70°), the triangle must be isosceles. If the measure of angle Z is 55°, then the measure of angle X is $180 - 70 - 55 = 55°$. With two angles of equal measure (55°), the triangle is again isosceles. I is sufficient and II is sufficient.

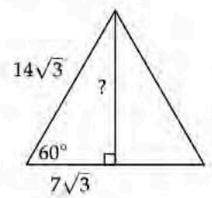
12. $\boxed{B}$ An isosceles right triangle is a 45–45–90 triangle, in which the hypotenuse is $\sqrt{2}$ times longer than each of the legs. If the hypotenuse is 12 units long, then each leg is $\frac{12}{\sqrt{2}}$ units long. The area is then $\frac{1}{2}bh = \frac{1}{2}\left(\frac{12}{\sqrt{2}}\right)\left(\frac{12}{\sqrt{2}}\right) = \frac{1}{2}\cdot\frac{144}{2} = 36$ square units.

13. $\boxed{B}$ Answer B is correct because it results in AAS (angle-angle-side) congruence between the two triangles. Answer A is incorrect because angles A and C belong to the same triangle and additional information about triangle DEF would be needed. Answer C is incorrect because the measures of angles C and F are enough to prove similarity but not congruence. Answer D is incorrect because sides DE and EF belong to the same triangle and additional information about triangle ABC would be needed.

14. $\boxed{\frac{57}{2} \text{ or } 28.5}$ Triangles LMN and PQN are similar by AA (angle-angle) because they both have right angles and the vertical angles at point N are congruent. Since $\overline{PQ}$ corresponds to $\overline{LM}$, the side lengths of triangle PQN are longer than the side lengths of triangle LMN by a factor of $\frac{15}{5} = 3$, which means that if we let the length of $\overline{MN}$ be x, then the length of $\overline{NQ}$ is $3x$. Setting up an equation, we get $x + 3x = 38$, which gives $x = \frac{38}{4} = 9.5$. The length of $\overline{NQ}$ is then $3x = 3(9.5) = 28.5$.

15. $\boxed{B}$ Take a look at answer B. If angles E and G are 70° and 35°, respectively, then angle D is $180 - 75 - 70 = 35°$ and angle H is $180 - 75 - 35 = 70°$. Now, we can see that all corresponding angles are congruent, so the triangles are similar. Answer A is incorrect because the given measures do not result in congruent corresponding angles. Answer C is incorrect because the given lengths are not in the same proportion as CD and FG. Answer D is incorrect because although the given lengths are in the same proportion as CD and FG, SSA (side-side-angle) is not a valid triangle similarity theorem.

16. $\boxed{21}$ Since the triangle is equilateral, each side has a length of $42\sqrt{3} \div 3 = 14\sqrt{3}$. Drawing the height, we get two congruent 30–60–90 triangles:



In a 30–60–90 triangle, the side opposite the 60° angle is $\sqrt{3}$ times longer than the side opposite the 30° angle. Therefore, the height is $7\sqrt{3}\cdot\sqrt{3} = 7\cdot 3 = 21$.

17. $\boxed{B}$ Since trapezoids $BCDE$ and $ABEF$ are similar, the ratio between corresponding sides $\overline{BC}$ and $\overline{AB}$ must be equal to the ratio between corresponding sides $\overline{DE}$ and $\overline{EF}$. Letting the lengths of $\overline{BC}$ and $\overline{EF}$ each be x, we can set up the following equation:

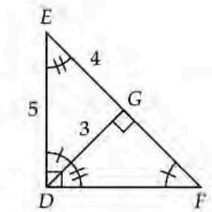
$$\frac{x}{4} = \frac{2}{x}$$
$$x^2 = 8$$
$$x = 2\sqrt{2}$$

18. $\boxed{D}$ For a right triangle, we can use its legs as the base and the height. The legs are always the two shortest sides (since the hypotenuse is always the longest side). Since $\sqrt{3} \approx 1.73$ and $\sqrt{12} \approx 3.46$, $a\sqrt{3}$ and $3a$ represent the two legs and $a\sqrt{12}$ represents the hypotenuse. Setting up an equation for the area, we get

$$\frac{1}{2}bh = 450$$
$$\frac{1}{2}(a\sqrt{3})(3a) = 450$$
$$a^2 = \frac{300}{\sqrt{3}}$$
$$a = \sqrt{\frac{300}{\sqrt{3}}} \approx 13.1607$$

Since $a\sqrt{3}$ is the shortest side, its length is $13.1607 \times \sqrt{3} \approx 22.8$.

19. $\boxed{D}$ Since $\overline{DG}$ is an altitude, it is perpendicular to $\overline{EF}$. Since triangle EDG is a right triangle, it must be a 3–4–5 triangle with $DG = 4$. At this point, it's obvious that $\frac{DG}{EG} = \frac{3}{4}$, but this ratio isn't in the answer choices, so we need to look at the similar triangles. Once angles E and F are labeled with tick marks, the other corresponding angles become apparent.



Looking at the tick marks, we can see that upper triangle DEG is similar to lower triangle FDG. Since $\overline{DG}$ and $\overline{EG}$ from the upper triangle correspond respectively to $\overline{FG}$ and $\overline{DG}$ from the lower triangle (sides opposite the 2-tick angles correspond to each other, and sides opposite the 1-tick angles correspond to each other), $\frac{DG}{EG} = \frac{FG}{DG} = \frac{3}{4}$.

EXERCISE 2:

1. $\boxed{D}$ Since the triangles are similar and $\angle ABC$ corresponds to $\angle DEC$, the two angles must be congruent. Thus, the measure of $\angle DEC$ is 120°.

2. $\boxed{D}$ Because $\overline{AB}$ is parallel to $\overline{CD}$, angles A and D have the same measure and angles B and C have the same measure. By AA (angle-angle), triangles ABE and DCE are similar. Therefore,

$$\frac{BE}{CE} = \frac{AB}{CD}$$
$$\frac{13}{10} = \frac{AB}{12}$$
$$AB = \frac{13}{10}(12) = \frac{156}{10}$$

3. $\boxed{C}$ Since triangles ABC and DEF are similar, corresponding angles are congruent and corresponding sides are proportional. Since angle A corresponds to angle D, angle D must measure 20°. Since $EF = DF$, triangle DEF is isosceles and the angles opposite EF and DF are equal. Thus, angle E is also 20°, and angle F is then $180 - 20 - 20 = 140°$. Finally, angle C corresponds to angle F, so angle C must be 140° as well.

4. $\boxed{C}$ If $RT = WY$, then the triangles would be congruent by ASA (angle-side-angle). Answer A is incorrect because congruent angles only help to prove similarity, not congruence. Answer B is incorrect because sides RT and WX do not correspond, and their lengths being equal does not help satisfy any of the triangle congruence theorems. Answer D is incorrect because side length information is necessary to prove congruence, and only angle measure information was given.

5. $\boxed{A}$ By the pythagorean theorem, the length of the hypotenuse is $\sqrt{(16\sqrt{23})^2 + (30\sqrt{23})^2} = \sqrt{26{,}588}$. At this point, we can check the decimal value of $\sqrt{26{,}588}$ against each of the answer choices. Or we can take a cue from the answer choices, all of which have a factor of $\sqrt{23}$, and assume that 23 goes into 26,588. Thus, $\sqrt{26{,}588} = \sqrt{1{,}156\cdot 23} = \sqrt{1{,}156}\cdot\sqrt{23} = 34\sqrt{23}$. Note that this triangle is a multiple of the 8–15–17 right triangle.

6. $\boxed{B}$ Using the pythagorean theorem to find AC, we get $AC = \sqrt{(60\sqrt{2})^2 - 12^2} = 84$. Since triangles ABC and DEF are similar, corresponding sides AC and DF indicate that the side lengths of triangle DEF are $\frac{105}{84} = \frac{5}{4}$ times the corresponding side lengths of triangle ABC. Therefore, $EF = \frac{5}{4}(BC) = \frac{5}{4}(60\sqrt{2}) = 75\sqrt{2}$.

7. $\boxed{D}$ Since the measures of angles A and B are equal, the sides opposite them have equal lengths. Thus, $BC = AC = 5$. Because triangles ABC and DEF are similar, corresponding sides BC and EF indicate that the side lengths of triangle DEF are $\frac{40}{5} = 8$ times the corresponding side lengths of triangle ABC. Therefore, $DE = 8AB = 8(10) = 80$.

8. $\boxed{7}$ The area of any triangle is given by the formula $A = \frac{1}{2}bh$, where b is the base and h is the height. For a right triangle, the legs can be used as the base and the height. Now let's look at the given function A. Since x is the length of one of the legs (i.e. the base), $7x$ must be the length of the other leg (i.e. the height). Using the pythagorean theorem to find the hypotenuse h, we get

$$(x)^2 + (7x)^2 = h^2$$
$$50x^2 = h^2$$
$$h = \sqrt{50x^2} = \sqrt{25x^2 \cdot 2} = 5x\sqrt{2}$$

Therefore, $a = 5$, $b = 2$, and $a + b = 7$.

9. $\boxed{D}$ Since $AB = AC$ in triangle ABC, the angles opposite sides AB and AC must be congruent. Thus, angles B and C must each have measure $(180 - 30) \div 2 = 75°$. Since triangles are similar if they have congruent angles, all we would need to prove that triangles ABC and DEF are similar is an angle measure from triangle DEF. Therefore, the measure of angle F suffices. Answers A and B are incorrect because one pair of side lengths is not enough to satisfy any of the triangle similarity theorems (i.e. SSS or SAS).

10. $\boxed{B}$ Based on the triangle inequality theorem, the following three conditions must all be met:

$$9 < 15 + x$$
$$15 < 9 + x$$
$$x < 9 + 15$$

These inequalities simplify down to

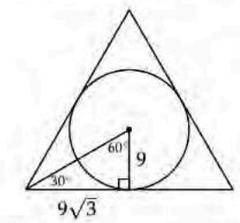
$$-6 < x$$
$$6 < x$$
$$x < 24$$

respectively. The first inequality, $-6 < x$, is always true if the second inequality is true, so we can discard the first inequality. Putting the remaining two inequalities together, we get $6 < x < 24$. The third side must have a length between 6 and 24 inches for there to be a valid triangle.

11. $\boxed{C}$ If $LM = PQ$, then the two triangles would be congruent by SAS (side-angle-side). It's important that the angle lies between the sides. Answer D is incorrect because SSA is not a valid triangle congruence theorem. Answers A and B are incorrect because they equate angles that do not correspond to each other.

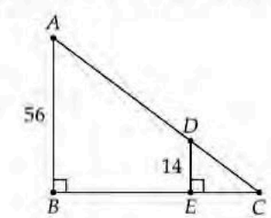
12. $\boxed{63}$ Since triangles ABD and ECD both have right angles and share angle D, they are similar by AA (angle-angle). If we let the length CD be x, then $BC = 6x$, and $BD = 6x + x = 7x$. Since side BD, which has length $7x$, corresponds to side CD, which has length x, the sides of triangle ABD are 7 times longer than the corresponding sides of triangle ECD. Therefore, $AB = 7(CE) = 7(9) = 63$.

13. $\boxed{C}$ The center of the inscribed circle is also the center of the equilateral triangle (called the centroid). Since each angle of an equilateral triangle measures 60°, a line drawn from any vertex to the center splits the vertex angle into two equal angles of 30° each. Next, a radius drawn perpendicular to a side connected to the above vertex completes a 30–60–90 triangle:



Using the 30–60–90 triangle relationship, we can determine that the bottom side is $9\sqrt{3}$. Each side of the equilateral triangle is twice as long: $2 \times 9\sqrt{3} = 18\sqrt{3}$. The perimeter is then $3 \times 18\sqrt{3} = 54\sqrt{3}$.

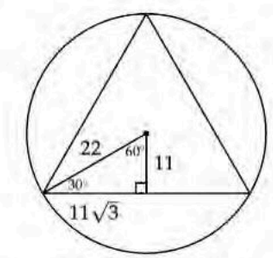
14. $\boxed{\frac{135}{2} \text{ or } 67.5}$ By the pythagorean theorem, $AB = \sqrt{AC^2 - BC^2} = \sqrt{106^2 - 90^2} = 56$.



Since triangles ABC and DEC both have right angles and share angle C, they are similar by AA (angle-angle). Because AB and DE are corresponding sides, the sides of triangle DEC are $\frac{14}{56} = \frac{1}{4}$ as long as the corresponding sides of triangle ABC. Thus, the length of side EC is $\frac{1}{4}$ the length of corresponding side BC: $EC = \frac{1}{4}(90) = 22.5$. Finally, $BE = BC - EC = 90 - 22.5 = 67.5$.

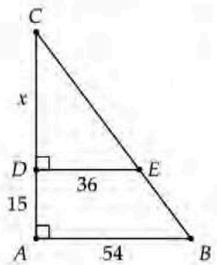
15. $\boxed{C}$ Note that $\frac{XY}{FG} = \frac{4}{3}$ and $\frac{YZ}{GH} = \frac{4}{3}$, so these pairs of corresponding sides are in the same proportion. If angles G and Y had the same measure, then the triangles would be similar by SAS (side-angle-side), so I is sufficient. Now if $FH = 15$ and $XZ = 20$, then $\frac{XZ}{FH} = \frac{4}{3}$ (the same proportion as the other corresponding pairs), and the triangles would be similar by SSS (side-side-side). Therefore, II is also sufficient.

16. $\boxed{B}$ The center of the inscribed equilateral triangle (called the centroid) is also the center of the circle. Since each angle of an equilateral triangle measures 60°, a radius of the circle drawn to a vertex of the triangle forms two equal angles of 30° each. Next, a perpendicular line drawn from the centroid to one of the sides completes a 30–60–90 triangle:



The length of the bottom side is half the side length of the equilateral triangle: $22\sqrt{3} \div 2 = 11\sqrt{3}$. By the 30–60–90 triangle relationship, the shortest side has a length of 11 and the hypotenuse has a length of 22. Since the hypotenuse is a radius of the circle, the length of the circle's radius is 22.

17. $\boxed{540}$ To find the area of triangle CDE, we need to find the length of $\overline{CD}$, which we'll denote as x.



Since triangles ABC and DEC both have right angles and share angle C, they are similar by AA (angle-angle). Therefore,

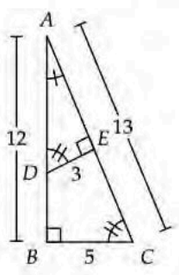
$$\frac{CD}{CA} = \frac{DE}{AB}$$
$$\frac{x}{x+15} = \frac{36}{54}$$

Cross-multiplying yields

$$54x = 36(x+15)$$
$$54x = 36x + 540$$
$$18x = 540$$
$$x = 30$$

Finally, the area of triangle CDE is $\frac{1}{2}(DE)(CD) = \frac{1}{2}(36)(30) = 540$.

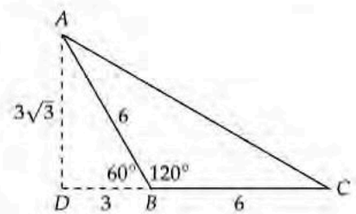
18. $\boxed{\frac{36}{5} \text{ or } 7.2}$ By the pythagorean theorem, triangle ABC is a 5–12–13 right triangle ($BC = 5$).



Triangles ABC and AED are similar by AA (angle-angle) since they both have right angles and share angle A. Since $\overline{AE}$ corresponds to $\overline{AB}$ and $\overline{DE}$ corresponds to $\overline{CB}$, here's one setup we could use to find AE:

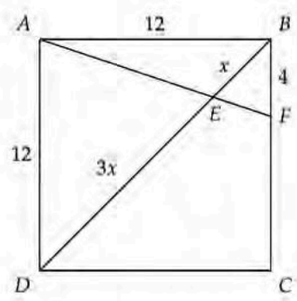
$$\frac{AE}{AB} = \frac{DE}{CB}$$
$$AE = \frac{DE}{CB}(AB) = \frac{3}{5}(12) = \frac{36}{5} = 7.2$$

19. $\boxed{D}$ Here's one approach. Draw the height from A as shown below. Triangle ADB turns out to be a 30–60–90 triangle, and its side lengths can be determined from its hypotenuse of 6.



The area is then $\frac{1}{2}bh = \frac{1}{2}(6)(3\sqrt{3}) = 9\sqrt{3}$.

20. $\boxed{C}$



Since sides AD and BC of the square are parallel, triangles ADE and FBE are similar by AA (angle-angle). The sides of triangle ADE are $12 \div 4 = 3$ times longer than the corresponding sides of triangle FBE. Because triangle ABD is a 45–45–90 triangle, the length of BD is $12\sqrt{2}$. If we let $BE = x$, then $DE = 3x$.

$$x + 3x = 12\sqrt{2}$$
$$4x = 12\sqrt{2}$$
$$x = 3\sqrt{2}$$

21. $\boxed{B}$ A right triangle with a 30° angle is a 30–60–90 triangle. If we let the length of the shortest side be x, then the remaining sides have lengths $x\sqrt{3}$ and $2x$ (by the 30–60–90 triangle relationship). Given the perimeter, we have

$$x + x\sqrt{3} + 2x = 78 + 78\sqrt{3}$$
$$3x + x\sqrt{3} = 78 + 78\sqrt{3}$$

At this point, we have several approaches. The first approach is to compare the terms on both sides. For example, $3x = 78$ and $x\sqrt{3} = 78\sqrt{3}$ seems natural. However, this doesn't work since the two equations result in different values of x. How about $3x = 78\sqrt{3}$ and $x\sqrt{3} = 78$? The first equation yields $x = 26\sqrt{3}$, which satisfies the second equation. Therefore, $x = 26\sqrt{3}$.

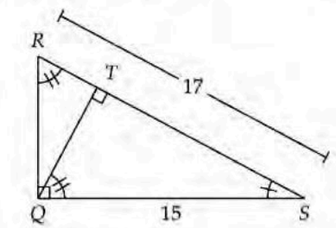
The second approach is to continue to solve the equation:

$$3x + x\sqrt{3} = 78 + 78\sqrt{3}$$
$$x(3+\sqrt{3}) = 78 + 78\sqrt{3}$$
$$x = \frac{78+78\sqrt{3}}{3+\sqrt{3}} = \frac{(78+78\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}$$
$$= \frac{234 - 78\sqrt{3} + 234\sqrt{3} - 234}{3^2 - (\sqrt{3})^2}$$
$$= \frac{156\sqrt{3}}{6} = 26\sqrt{3}$$

Notice how we got rid of the $\sqrt{3}$ in the denominator by multiplying the top and bottom of the fraction by $3 - \sqrt{3}$ (the *conjugate* of the denominator). This process is called *rationalizing the denominator*.

The third approach is to stop at $x = \frac{78+78\sqrt{3}}{3+\sqrt{3}}$ in the second approach above and use your calculator to compare the decimal value of x with the decimal values of the answer choices.

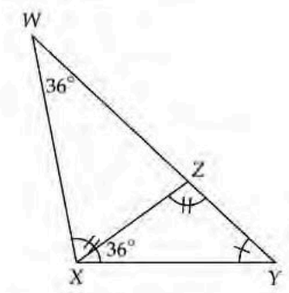
22. $\boxed{C}$ Notice that both outer triangle RQS and triangle QTS have right angles and share angle S. Because they have the same angle measures, they are similar.



Since $\overline{TS}$ and $\overline{QS}$ in triangle QTS correspond to $\overline{QS}$ and $\overline{RS}$, respectively, in outer triangle RQS, we can equate the following ratios:

$$\frac{TS}{QS} = \frac{QS}{RS}$$
$$\frac{TS}{15} = \frac{15}{17}$$
$$TS = \frac{15}{17}(15) \approx 13.2$$

23. $\boxed{C}$ Draw a picture.



Outer triangle WXY and inner triangle XZY both have 36° angles and share $\angle Y$. Therefore, they are similar by AA (angle-angle). Label the angles with tick marks to see which sides correspond to which. Sides $\overline{WY}$ and $\overline{XY}$ in triangle WXY correspond to sides $\overline{XY}$ and $\overline{YZ}$ in triangle XZY, respectively (sides opposite the 2-tick angles correspond to each other, and sides opposite the 36° angles correspond to each other). For similar triangles, corresponding ratios between sides are equivalent.

Therefore, $\frac{WY}{XY} = \frac{XY}{YZ}$.

Chapter 19: Circles

EXERCISE 1:

1. $\boxed{B}$ The standard form of a circle with center (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$. Therefore, the given circle has a radius of $\sqrt{100}=10$ and a circumference of $2\pi r = 2\pi(10) = 20\pi$.

2. $\boxed{94}$ Angle BAC is an inscribed angle that intercepts arc BC. Therefore, the measure of arc BC is twice the measure of angle BAC: $47^\circ \times 2 = 94^\circ$.

3. $\boxed{C}$ The standard form of a circle with center (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$. Therefore, a circle with center $(0,1)$ and radius 4 can be represented by the equation

$$(x-0)^2+(y-1)^2=4^2$$
$$x^2+(y-1)^2=16$$

4. $\boxed{A}$ The best approach is to test each point and see if it satisfies the equation. A second approach is to realize that all the answer choices have an x-coordinate of 7. We can plug this into the equation to find the y-coordinate:

$$(7-2)^2+(y-9)^2=169$$
$$25+(y-9)^2=169$$
$$(y-9)^2=144$$
$$y-9=\pm\sqrt{144}$$
$$y=9\pm 12=-3 \text{ or } 21$$

Therefore, the points $(7,-3)$ and $(7,21)$ lie on the circle. Of these, only $(7,-3)$ is found in the answer choices.

5. $\boxed{A}$ Since the area of a circle is given by πr^2, circle A must have a radius of $\sqrt{32}$ and circle B must have a radius of $\sqrt{384}$. The radius of circle B is longer than the radius of circle A by a factor of
$\frac{\sqrt{384}}{\sqrt{32}} = \sqrt{\frac{384}{32}} = \sqrt{12} = \sqrt{2\cdot 2\cdot 3} = 2\sqrt{3}$.

6. $\boxed{C}$ The measure of arc AB must be twice the measure of inscribed angle ACB. Therefore, arc AB measures $30^\circ \times 2 = 60^\circ$, which means arc AB makes up $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the circumference.

7. $\boxed{A}$ To find the points of intersection between any two graphs, treat their equations as a system and solve the system. Since $y=0$ is one of the equations, we can substitute this into the equation of the circle:

$$x^2+0^2=9$$
$$x=\pm\sqrt{9}=\pm 3$$

The points of intersection are then $(-3,0)$ and $(3,0)$. Only $(-3,0)$ is found in the answer choices.

8. $\boxed{C}$ Use the area to find the radius:

$$\pi r^2 = 36\pi$$
$$r=\sqrt{36}=6$$

The circumference of the circle is then $2\pi r = 2\pi(6) = 12\pi$. Because the equilateral triangle divides the circumference of the circle into 3 equal parts, arc AB must be one-third of the circumference. Therefore, its length is $\frac{1}{3}\times 12\pi = 4\pi$.

9. $\boxed{B}$ The circle has a circumference of $2\pi r = 2\pi(10) = 20\pi$ feet. Arc CD makes up $\frac{72^\circ}{360^\circ} = \frac{1}{5}$ of the circle. Its length is then $\frac{1}{5}\times 20\pi = 4\pi$ feet.

10. $\boxed{C}$ Circle A has center $(1,-2)$. Circle B is up 7 units from circle A, so circle B must have center $(1,-2+7)=(1,5)$ and the same radius as circle A. The equation of circle B is then $(x-1)^2+(y-5)^2=64$.

11. $\boxed{18}$ Since arc QR measures 144°, central angle QPR also measures 144°. Since PQ and PR are both radii, $PQ=PR$ and triangle PQR is an isosceles triangle. Therefore, angle PQR measures $(180-144)\div 2 = 18^\circ$.

12. $\boxed{36}$ The circle has center $(-4,-6)$, which is 4 units away from the y-axis and 6 units away from the x-axis. Since the circle intersects the x-axis at exactly one point (i.e. is tangent to the x-axis), the circle must have a radius of 6. Therefore, $k=6^2=36$.

13. $\boxed{D}$ The center of the circle is at the midpoint of the diameter's endpoints. Thus, the center is at $\left(\frac{-8+(-8)}{2}, \frac{-5+5}{2}\right) = (-8,0)$. Since the endpoints of the given diameter have the same x-coordinates, the length of the diameter is just the distance between the y-coordinates: $5-(-5)=10$. The radius must then be 5, and the equation of the circle is $(x+8)^2+y^2=25$.

14. $\boxed{9}$ Complete the square to get the equation of the circle in standard form:

$$(x+1)^2-1+(y-4)^2-16=64$$
$$(x+1)^2+(y-4)^2=81$$

Now it's easy to see that the radius is $\sqrt{81}=9$.

EXERCISE 2:

1. $\boxed{D}$ Use the area to find the radius:

$$\pi r^2 = 49\pi$$
$$r^2=49$$
$$r=\sqrt{49}=7$$

The standard form of a circle with center (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$. The equation of the given circle must then be $(x+2)^2+y^2=49$.

2. $\boxed{A}$ Based on the equation, circle A has center $(-3,6)$. Circle B is 5 units to the right of circle A, so circle B must have center $(-3+5,6)=(2,6)$.

3. $\boxed{40}$ Since angle APB intercepts arc AB, its measure is half the measure of arc AB. Since angle AQB also intercepts arc AB, its measure is also half the measure of arc AB. Thus, angles APB and AQB have the same measure because they intercept the same arc. Both angles measure 40°.

4. $\boxed{B}$ The diameter of the circle is equal in length to the side of the square. Therefore, the circumference of the circle is $\pi d = 14\pi$. Because the square divides the circle into 4 equal parts at their points of intersection, arc AB makes up one-quarter of the circle. Its length is then $\frac{1}{4}\times 14\pi = \frac{7}{2}\pi$.

5. $\boxed{D}$ The standard form equation of a circle with center $(1,-1)$ is $(x-1)^2+(y+1)^2=r^2$, where r is the radius. Since the point $(-5,7)$ lies on the circle, it must satisfy this equation. Plugging it in yields

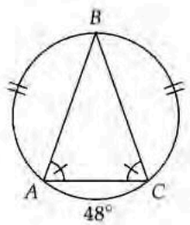
$$(-5-1)^2+(7+1)^2=r^2$$
$$100=r^2$$

Therefore, the standard form equation of the circle is $(x-1)^2+(y+1)^2=100$. Alternatively, we could've plugged $(-5,7)$ into answers B and D to see which equation is satisfied. Yet a third approach is to solve directly for the radius r by calculating the distance between $(-5,7)$ and center $(1,-1)$:
$\sqrt{(y_2-y_1)^2+(x_2-x_1)^2} =$
$\sqrt{(7-(-1))^2+(-5-1)^2} = \sqrt{64+36} = 10$.

6. $\boxed{C}$ Since the measure of arc MN is $24°$, its length makes up $\frac{24}{360} = \frac{1}{15}$ of the circle. The circumference of the entire circle is then $15 \times 8\pi = 120\pi$. With this information, we can solve for the radius:

$$2\pi r = 120\pi$$
$$r = 60$$

7. $\boxed{156}$



Angles A, B, and C are all inscribed angles, which have half the measure of the arc they intercept. Thus, the measure of arc AB is twice the measure of angle C, and the measure of arc BC is twice the measure of angle A. Since the measures of angles A and C are equal, the measures of arc AB and arc BC must also be equal. Since arc AB and arc BC sum to $360° - \text{arc } AC = 360° - 48° = 312°$, arc AB must measure $312° \div 2 = 156°$.

8. $\boxed{784}$ Given that the standard form of a circle with center (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$, the radius of circle A is $\sqrt{16} = 4$ and its diameter is $4 \times 2 = 8$. The diameter of circle B is then $8 + 48 = 56$ and its radius is $56 \div 2 = 28$. Thus, $n = 28^2 = 784$.

9. $\boxed{126}$ Draw radius $\overline{OB}$. Because $\overline{OA}$ and $\overline{OB}$ are radii, triangle OAB is isosceles. In isosceles triangles, angles opposite equal sides are equal in measure, so the measure of angle BAO ($38°$) is equal to the measure of angle ABO ($38°$). Because $\overline{OB}$ and $\overline{OC}$ are radii, triangle OBC is also isosceles and the measure of angle BCO ($25°$) is equal to the measure of angle OBC ($25°$). Since angle ABC is the sum of angles ABO and OBC, the measure of angle ABC is $38° + 25° = 63°$. And since angle ABC is an inscribed angle that intercepts arc AC, arc AC has twice the measure of angle ABC: $63 \times 2 = 126°$.

10. $\boxed{C}$ The circle given by answer choice C has center $(7, 5)$ and radius $\sqrt{4} = 2$. The x-coordinate of the center is $7 - 5 = 2$ units away from the line $x = 5$. Since this distance is equal to the radius of the circle, this circle must intersect the line $x = 5$ at exactly one point (i.e. must be tangent to the line $x = 5$).

11. $\boxed{15}$ A radius that bisects a chord is perpendicular to that chord. Therefore, angle OED is a right angle. Now, drawing radius $\overline{OD}$ yields right triangle OED, with $DE = 28 \div 2 = 14$ and $OD = 16$. Applying the pythagorean theorem to this triangle, we get

$$OE^2 + 14^2 = 16^2$$
$$OE = \sqrt{256 - 196}$$
$$OE = \sqrt{60} = \sqrt{2 \cdot 2 \cdot 15} = 2\sqrt{15}$$

Therefore, $k = 15$.

12. $\boxed{230}$ The standard form equation of the circle is

$$(x-9)^2 + (y+12)^2 = 1^2$$
$$(x^2 - 18x + 81) + (y^2 + 24y + 144) = 1$$
$$x^2 + y^2 - 18x + 24y + 81 + 144 - 1 = 0$$
$$x^2 + y^2 - 18x + 24y + 224 = 0$$

Therefore, $a = -18$, $b = 24$, and $c = 224$. Finally, $a + b + c = -18 + 24 + 224 = 230$.

13. $\boxed{C}$ Plugging the point in, we get

$$(8-h)^2 + (27-k)^2 = 144$$

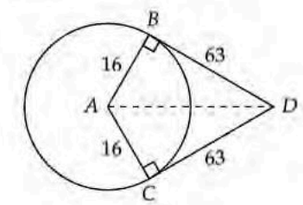
Now let's test answer choice A. Plugging in $k = -8$ yields

$$(8-h)^2 + (27-(-8))^2 = 144$$
$$(8-h)^2 + (35)^2 = 144$$
$$(8-h)^2 = -1081$$

Now compare the left and right-hand sides. The square of something can never be negative, so this equation is not "solvable" for h (i.e. h is undefined). Therefore, -8 is not a possible value for k. Repeating these steps for answer choices B and D, we see that the same thing happens. Only answer choice C, $k = 39$, yields an equation in which h has a defined value.

14. $\boxed{11}$ The center of the circle is at the midpoint of the diameter's endpoints. Thus, the center is at $\left(\frac{-4+8}{2}, \frac{3+3}{2}\right) = (2, 3)$. Since the endpoints of the given diameter have the same y-coordinates, the length of the diameter is just the distance between the x-coordinates: $8 - (-4) = 12$. The radius must then be 6, and the equation of the circle is $(x-2)^2 + (y-3)^2 = 6^2$. As a result, $a = 2$, $b = 3$, $c = 6$, and $a + b + c = 2 + 3 + 6 = 11$.

15. $\boxed{1008}$ Tangent segments to a circle from the same external point are congruent. Therefore, $CD = BD = 63$. Furthermore, any radius that intersects a tangent line segment at its point of tangency is perpendicular to that line segment. Thus, angles ABD and ACD are right angles.



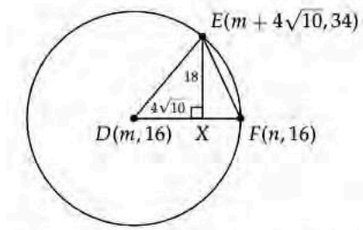
To find the area of quadrilateral $ABCD$, we can draw line segment AD to split the quadrilateral into two congruent triangles. Each triangle has an area of $\frac{1}{2}(63)(16) = 504$ square units, so the quadrilateral must have an area of $2 \times 504 = 1008$ square units.

16. $\boxed{3}$ Complete the square to get the equation of the circle in standard form:

$$\left(x + \frac{7}{2}\right)^2 - \frac{49}{4} + \left(y + \frac{1}{2}\right)^2 - \frac{1}{4} = -\frac{7}{2}$$
$$\left(x + \frac{7}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = -\frac{7}{2} + \frac{25}{2} = 9$$

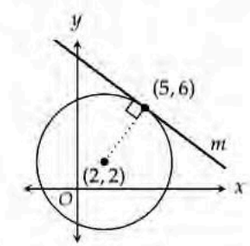
The radius is then $\sqrt{9} = 3$.

17. $\boxed{A}$ Since D and F have the same y-coordinate value, DF is a horizontal line segment parallel to the x-axis. This allows us to draw altitude $\overline{EX}$, which has a length of $34 - 16 = 18$. Since the horizontal distance between D and E is $(m + 4\sqrt{10}) - m = 4\sqrt{10}$, $DX = 4\sqrt{10}$.



Now we can use the pythagorean theorem to find the length of radius DE: $DE = \sqrt{(4\sqrt{10})^2 + 18^2} = 22$. Since DF is a radius of the circle, its length is also 22. Therefore, the area of triangle DEF is $\frac{1}{2}bh = \frac{1}{2}(22)(18) = 198$ square units.

18. $\boxed{\frac{39}{4} \text{ or } 9.75}$ Any radius that's drawn to a tangent line at its point of tangency is perpendicular to that line.



The endpoints of this radius are the center of the circle, $(2, 2)$, and the point of tangency, $(5, 6)$. The slope of the radius is then $\frac{6-2}{5-2} = \frac{4}{3}$. Since line m is perpendicular to the radius, the slope of line m must be $-\frac{3}{4}$, the negative reciprocal of the slope of the radius. Line m passes through the point of tangency, $(5, 6)$, so we can use point-slope form to find the equation of line m:

$$y - 6 = -\frac{3}{4}(x - 5)$$
$$y = -\frac{3}{4}x + \frac{15}{4} + 6$$
$$y = -\frac{3}{4}x + 9\frac{3}{4}$$

From this equation, we can see that the y-intercept of line m is 9.75.

Chapter 20: Radians

EXERCISE:

1. $\boxed{255}$ $\frac{17\pi}{12}$ radians $\times \frac{180^\circ}{\pi \text{ radians}} = 255^\circ$

2. $\boxed{C}$ $225^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{5\pi}{4}$ radians

3. $\boxed{B}$ The measure of angle A is $\frac{5\pi}{9} \times \frac{180^\circ}{\pi} = 100^\circ$. The measure of angle B is $\frac{7\pi}{36} \times \frac{180^\circ}{\pi} = 35^\circ$ less than the measure of angle A. Thus, the measure of angle B is $100^\circ - 35^\circ = 65^\circ$.

4. $\boxed{B}$ $162^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{9}{10}\pi$ radians

5. $\boxed{195}$ 3.4 radians $\times \frac{180^\circ}{\pi \text{ radians}} \approx 194.8 \approx 195^\circ$

6. $\boxed{C}$ A full circle has a measure of 2π radians. Since arc AB is $\frac{1}{5}$ of the circumference, central angle AOB must be $\frac{1}{5}$ of a full circle. Thus, angle AOB measures $\frac{1}{5} \times 2\pi = \frac{2\pi}{5}$ radians.

7. $\boxed{5}$ Converting to radians yields $900^\circ \times \frac{\pi \text{ radians}}{180^\circ} = 5\pi$ radians. Therefore, $k = 5$.

8. $\boxed{A}$ The radii extending to the corners of the triangle divide the circle into three equal parts, so the measure of angle ADB must be $360 \div 3 = 120^\circ$, which is equivalent to $120^\circ \times \frac{\pi}{180^\circ} = \frac{2\pi}{3}$ radians.

9. $\boxed{\frac{2}{3} \text{ or } .6666 \text{ or } .6667}$ The measure of a full circle is 2π radians. Since arc MN measures $\frac{4\pi}{3}$ radians, its length must constitute $\frac{\frac{4\pi}{3}}{2\pi} = \frac{4\pi}{3(2\pi)} = \frac{4\pi}{6\pi} = \frac{2}{3}$ of the circle's circumference.

10. $\boxed{D}$ Since OX and OY are radii, triangle OXY is an isosceles triangle, in which angles opposite equal sides have equal measure. Thus, the measure of angle OYX is also $\frac{\pi}{9}$ radians. Since the angles of a triangle sum to π radians (180°), the measure of angle XOY is $\pi - \frac{\pi}{9} - \frac{\pi}{9} = \frac{7\pi}{9}$ radians.

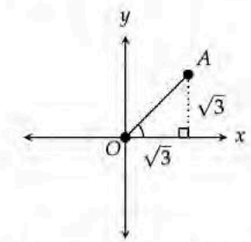
11. $\boxed{C}$ Since the area of the circle is $\pi r^2 = \pi(6)^2 = 36\pi$ square inches, the shaded sector constitutes $\frac{10\pi}{36\pi} = \frac{5}{18}$ of the circle. The measure of central angle AOB must then be $\frac{5}{18}$ of 2π radians (a full circle = 2π radians): $\frac{5}{18} \times 2\pi = \frac{5\pi}{9}$ radians.

12. $\boxed{D}$ If the length of arc AD is three times the length of arc BD, then the measure of central angle AOD must be three times the measure of central angle BOD. Let the measure of angle BOD be x. Then the measure of angle AOD is $3x$. Since angles AOD and BOD form a line (are supplementary), their measures sum to π radians (180°):

$$x + 3x = \pi$$
$$4x = \pi$$
$$x = \frac{\pi}{4}$$

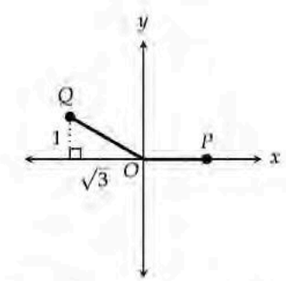
Therefore, the measure of angle BOD is $\frac{\pi}{4}$ radians, which means the measure of angle AOD is $3x = \frac{3\pi}{4}$ radians.

13. $\boxed{B}$ Draw a picture. The line $y = 0$ is the x-axis.



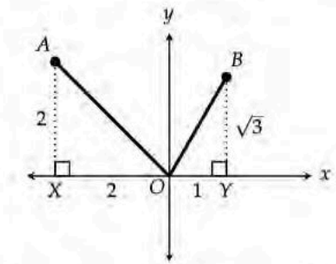
Add a line segment from point A to the x-axis to form a right triangle. Since this right triangle has congruent legs of length $\sqrt{3}$, this triangle must be a 45–45–90 isosceles right triangle. Thus, the measure of the acute angle formed by $\overline{OA}$ and the line $y = 0$ is 45°, which is equivalent to $45^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4}$ radians.

14. $\boxed{C}$



Draw a line segment from point Q to the x-axis to form a right triangle as shown above. Since the legs of this right triangle are 1 and $\sqrt{3}$, this triangle must be a 30–60–90 triangle, in which the angle opposite the 1-unit side is 30°. Angle POQ is then $180 - 30 = 150^\circ$, which is equivalent to $150^\circ \times \frac{\pi}{180^\circ} = \frac{5\pi}{6}$ radians.

15. $\boxed{B}$ Draw line segments from points A and B to the x-axis to form right triangles.



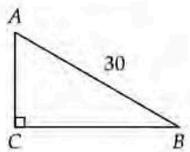
Triangle AOX on the left has two congruent legs of length 2, so it must be a 45–45–90 isosceles right triangle. Triangle BOY on the right has legs of length 1 and length $\sqrt{3}$, so it must be a 30–60–90 triangle. Thus, the measure of angle AOX is 45° and the measure of angle BOY is 60°. The measure of angle AOB is then $180^\circ - 45^\circ - 60^\circ = 75^\circ$, which is equivalent to $75^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{5}{12}\pi$ radians.

Chapter 21: Trigonometry

EXERCISE 1:

1. $\boxed{C}$ Draw the triangle. Then $\tan(A) = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{4}{3}$.

2. $\boxed{25}$ Let's draw the triangle.

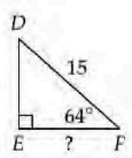


$$\cos(A) = \frac{5}{6}$$
$$\frac{AC}{30} = \frac{5}{6}$$
$$AC = \frac{5}{6} \times 30 = 25$$

3. $\boxed{B}$ **Solution 1:** Angles A and B sum to $90°$, which means $\cos(B) = \sin(A)$. Therefore, $\cos(B)$ also has a value of $\frac{12}{13}$.

Solution 2: Since $\sin(A) = \frac{12}{13}$, we can let BC, the opposite side, be 12 and AB, the hypotenuse, be 13. The value of $\cos(B)$ is then $\frac{\text{adj}}{\text{hyp}} = \frac{BC}{AB} = \frac{12}{13}$.

4. $\boxed{B}$ Draw the triangle.



$$\cos(F) = \frac{\text{adj}}{\text{hyp}}$$
$$\cos 64° = \frac{EF}{15}$$
$$15 \cos 64° = EF$$

5. $\boxed{.352}$ Since angle B corresponds to angle E, they must have equal measures. Therefore, $\sin(E)$ is also equal to 0.352.

6. $\boxed{A}$ Since BC is the longest side of triangle ABC, it must be the hypotenuse. Angle A is then the right angle. Since Q corresponds to B, $\cos(Q) = \cos(B) = \frac{AB}{BC} = \frac{198}{390} = \frac{33}{65}$.

7. $\boxed{\frac{24}{25} \text{ or } .96}$ **Solution 1:** Angles A and C sum to $90°$, which means $\sin(C) = \cos(A)$. Therefore, $\sin(C)$ also has a value of $\frac{24}{25}$.

Solution 2: Angle B must be the right angle. Since $\cos(A) = \frac{24}{25}$, we can let AB, the adjacent side, be 24 and AC, the hypotenuse, be 25. The value of $\sin(C)$ is then $\frac{\text{opp}}{\text{hyp}} = \frac{AB}{AC} = \frac{24}{25}$.

8. $\boxed{\frac{35}{37} \text{ or } .9459}$ Since $\sin(E) = \frac{24}{74}$, we can let DF, the opposite side, be 24 and EF, the hypotenuse, be 74. By the pythagorean theorem, $DE = \sqrt{74^2 - 24^2} = 70$. Now we can determine that $\sin(F) = \frac{\text{opp}}{\text{hyp}} = \frac{DE}{EF} = \frac{70}{74} = \frac{35}{37}$.

9. $\boxed{\frac{1}{4} \text{ or } .25}$ Since angle B corresponds to angle E, $\tan(E) = \tan(B) = 4$. In triangle DEF, we can then let DF, the side opposite angle E, be 4 and EF, the side adjacent to angle E, be 1. The value of $\tan(D)$ is then $\frac{\text{opp}}{\text{adj}} = \frac{EF}{DF} = \frac{1}{4}$.

10. $\boxed{\frac{65}{72} \text{ or } .9027 \text{ or } .9028}$ Given that $\cos J = \frac{65}{97}$, we can let adjacent side JK be 65 and hypotenuse JL be 97. And since $\cos L = \frac{72}{97}$, we can let KL, the side adjacent to angle L, be 72. Now, angle W corresponds to angle L, so $\tan W = \tan L = \frac{\text{opp}}{\text{adj}} = \frac{JK}{KL} = \frac{65}{72}$.

11. $\boxed{\frac{28}{53} \text{ or } .5283}$ Since angle C corresponds to angle T, $\tan(C) = \tan(T) = \frac{28}{45}$. In triangle RST, we can then let RS, the side opposite angle T, be 28 and ST, the side adjacent to angle T, be 45. By the pythagorean theorem, hypotenuse $RT = \sqrt{28^2 + 45^2} = 53$. The value of $\sin(T)$ is then $\frac{\text{opp}}{\text{hyp}} = \frac{RS}{RT} = \frac{28}{53}$.

12. $\boxed{D}$ **Solution 1:** Since $\cos(90° - x°) = \sin x°$, $\sin x° = \frac{8}{17}$. So we can let the side opposite angle $x°$ be 8 and the hypotenuse be 17. Using the pythagorean theorem, the side adjacent to angle x has length $\sqrt{17^2 - 8^2} = 15$. Finally, $\cos x° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$.

Solution 2: Since $x°$ is the measure of the bottom-right angle, $90° - x°$ must be the measure of the top angle, which we'll label angle Z. Since $\cos Z = \frac{8}{17}$, we can let the side adjacent to angle Z be 8 and the hypotenuse be 17. Now we can use the pythagorean theorem and continue on as we did in Solution 1.

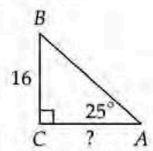
13. $\boxed{B}$ Since angle F corresponds to angle L, $\cos(F) = \cos(L) = \frac{2}{3}$. Now we can set up the following equation from triangle LMN:

$$\cos(L) = \frac{2}{3}$$
$$\frac{LN}{LM} = \frac{2}{3}$$
$$\frac{LN}{18} = \frac{2}{3}$$
$$LN = \frac{2}{3}(18) = 12$$

By the pythagorean theorem, $MN = \sqrt{LM^2 - LN^2} = \sqrt{18^2 - 12^2} = \sqrt{180} = \sqrt{6 \cdot 6 \cdot 5} = 6\sqrt{5}$.

EXERCISE 2:

1. $\boxed{D}$ Draw the triangle.



$$\tan(A) = \frac{\text{opp}}{\text{adj}}$$
$$\tan 25^\circ = \frac{16}{AC}$$
$$(AC)\tan 25^\circ = 16$$
$$AC = \frac{16}{\tan 25^\circ}$$

2. $\boxed{C}$ We can determine the length of $\overline{AB}$ from the length of $\overline{AC}$ by using the cosine of angle A. Alternatively, we can determine the length of $\overline{AB}$ from the length of $\overline{BC}$ by using the tangent of angle A. I is sufficient and II is sufficient.

3. $\boxed{A}$ Draw a triangle. Since the sine of one of the acute angles is $\frac{\sqrt{3}}{2}$, we can let the side opposite this angle be $\sqrt{3}$ and the hypotenuse be 2. By the pythagorean theorem, the side adjacent to the angle has length $\sqrt{2^2 - (\sqrt{3})^2} = 1$. Notice that this is a 30–60–90 triangle. With the triangle complete, the sine of the other acute angle must then be $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}$.

4. $\boxed{D}$ Use Desmos to compare the decimal value of $\sin\frac{76\pi}{3}$ (≈ -0.866) with the decimal value of each answer choice. Answer choice D is the only one that gives the same decimal value. Alternatively, use the unit circle with a reference angle of $\frac{\pi}{3}$ in the third quadrant. With the help of a 30–60–90 triangle, it's easy to see that the y-coordinate of the intercepted point is $-\frac{\sqrt{3}}{2}$. Therefore, the value of $\sin\frac{76\pi}{3}$ is also $-\frac{\sqrt{3}}{2}$.

5. $\boxed{38}$ Angle T must be acute because triangle RST is a right triangle (no angles greater than 90° in a right triangle). The cosine of an acute angle is always equal to the sine of the complementary angle (complementary angles sum to 90°). Therefore,

$$\cos T = \sin(90 - T)$$
$$\sin 52^\circ = \sin(90 - T)$$
$$52 = 90 - T$$
$$T = 38^\circ$$

6. $\boxed{B}$ Since $90^\circ - x^\circ$ is the measure of the triangle's top angle,
$\tan(90^\circ - x^\circ) = \tan(\text{top angle}) = \frac{\text{opp}}{\text{adj}} = \frac{20}{21}$.

7. $\boxed{A}$ Use Desmos to compare the decimal value of $\tan\frac{49\pi}{6}$ (≈ 0.577) with the decimal value of each answer choice. Just plug each value in for k. Only answer choice A gives the same decimal value ($\frac{1}{3}\cdot\sqrt{3} \approx 0.577$). Alternatively, use the unit circle with a reference angle of $\frac{\pi}{6}$ in the first quadrant. With the help of a 30–60–90 triangle, it's easy to see that the y-coordinate of the intercepted point is $\frac{1}{\sqrt{3}}$, or $\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, when the denominator is rationalized. Therefore, the value of $\tan\frac{49\pi}{6}$ is also $\frac{\sqrt{3}}{3}$. Since $\frac{\sqrt{3}}{3}$ can be written as $\frac{1}{3}\cdot\sqrt{3}$, $k = \frac{1}{3}$.

8. $\boxed{\frac{36}{5} \text{ or } 7.2}$ Since angle S corresponds to angle N, $\tan(S) = \tan(N) = \frac{5}{9}$. Given that tangent is opposite over adjacent, we can set up the following equation for triangle QRS:

$$\tan(S) = \frac{5}{9}$$
$$\frac{QR}{RS} = \frac{5}{9}$$

Cross multiplying yields

$$5RS = 9QR$$
$$RS = \frac{9}{5}QR = \frac{9}{5}(4) = \frac{36}{5}$$

9. $\boxed{C}$ Since angle T corresponds to angle L, $\cos(T) = \cos(L) = \frac{119}{169}$. In triangle RST, we can then let ST, the side adjacent to angle T, be 119 and RT, the hypotenuse, be 169. By the pythagorean theorem,
$RS = \sqrt{RT^2 - ST^2} = \sqrt{169^2 - 119^2} = 120$.
The value of $\tan(T)$ is then $\frac{\text{opp}}{\text{adj}} = \frac{RS}{ST} = \frac{120}{119}$.

10. $\boxed{\frac{25}{2} \text{ or } 12.5}$ Notice that triangles ABC and DBE are similar, which means angle A is equivalent to angle BDE. Since $\tan(A)$ is 1.25, $\tan(\angle BDE)$ is also 1.25.

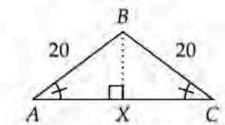
$$\tan(\angle BDE) = 1.25$$
$$\frac{\text{opp}}{\text{adj}} = 1.25$$
$$\frac{BE}{DE} = 1.25$$
$$BE = (1.25)(DE) = (1.25)(10) = 12.5$$

11. $\boxed{6}$ Since $\tan A = \sqrt{7}$ and $AB = 2$,

$$\frac{BC}{AB} = \sqrt{7}$$
$$BC = AB\cdot\sqrt{7} = 2\sqrt{7}$$

By the pythagorean theorem, hypotenuse
$AC = \sqrt{AB^2 + BC^2} = \sqrt{2^2 + (2\sqrt{7})^2} = \sqrt{4 + 28} = \sqrt{32} = 4\sqrt{2}$. Since DF, which corresponds to AC, is $12\sqrt{2} \div 4\sqrt{2} = 3$ times as long as AC, we can infer that the sides of triangle DEF are 3 times as long as the corresponding sides of similar triangle ABC. Therefore, $DE = 3AB = 3(2) = 6$.

12. $\boxed{15}$ Since angles A and C have the same measure, the sides opposite them have the same length: $AB = BC = 20$. Triangle ABC is isosceles. Now we can draw a line segment from B straight down to $\overline{AC}$ to form two congruent triangles (by angle-angle-side). Let the intersection point be X.



Given that $\cos A = \frac{3}{8}$, we can use triangle ABX on the left to construct the following equation:

$$\frac{AX}{AB} = \frac{3}{8}$$
$$AX = \frac{3}{8}(AB) = \frac{3}{8}(20) = 7.5$$

Finally,
$AC = AX + XC = 2AX = 2(7.5) = 15$.

13. $\boxed{C}$ Since triangle ABC is inscribed in a semicircle, angle B measures 90°. Thus, triangle ABC is a right triangle. We can consider the height to be AB and the base to be BC. Since hypotenuse $AC = 1$,

$$\sin x = AB/1 = AB$$
$$\cos x = BC/1 = BC$$

The area of triangle ABC is then
$\frac{1}{2}(BC)(AB) = \frac{1}{2}(\cos x)(\sin x) = \frac{\sin x\cos x}{2}$.

Chapter 22: Area, Perimeter, & Volume

EXERCISE 1:

1. $\boxed{A}$ Density $= \frac{8 \text{ kg}}{20 \text{ m}^3} = 0.4$ kg per m^3

2. $\boxed{7350}$ The cube must have an edge length of $\sqrt[3]{42{,}875} = 35$ inches. The area of each face is then $35^2 = 1{,}225$ square inches. Since a cube has 6 faces, the surface area is $6 \times 1{,}225 = 7{,}350$ square inches.

3. $\boxed{2156}$ Volume of a pyramid $= \frac{1}{3} \times$ Area of base $\times$ height $= \frac{1}{3}(14^2)(33) = 2{,}156$ cm^3

4. $\boxed{28}$

$$\pi r^2 h = 980\pi$$
$$\pi r^2(5) = 980\pi$$
$$r^2 = 196$$
$$r = \sqrt{196} = 14 \text{ feet}$$

The base diameter is then $2r = 2(14) = 28$ ft.

5. $\boxed{B}$ Square A has a side length of $\sqrt{64} = 8$ cm and a perimeter of $8 \times 4 = 32$ cm. The perimeter of square B is then $4 \times 32 = 128$ cm. Its side length is then $128 \div 4 = 32$ cm. Note that this question could have been done a little faster—since the perimeter of square B is 4 times the perimeter of square A, the side length of square B must be 4 times the side length of square A.

6. $\boxed{B}$ For similar polygons, the ratio of the areas is equal to the square of the ratio of the side lengths. The longest side of quadrilateral Y is 3 times the longest side of quadrilateral X. Therefore, the area of quadrilateral Y is $3^2 = 9$ times the area of quadrilateral X.

7. $\boxed{1936}$ $2(16 \times 18) + 2(16 \times 20) + 2(18 \times 20) = 1{,}936$ square inches

8. $\boxed{B}$ Each of the 4 sides of square A contributes equally to the difference in perimeter. Therefore, the side length of square A is $144 \div 4 = 36$ inches greater than the side length of square B.

9. $\boxed{B}$ The easiest solution is to test each answer choice. The mathematical solution is to solve the following equation:

$$(x-8)(x+5) = 68$$
$$x^2 - 3x - 40 = 68$$
$$x^2 - 3x - 108 = 0$$
$$(x-12)(x+9) = 0$$
$$x = -9 \text{ or } 12$$

Since any rectangle must have a positive width and a positive length, the answer is 12.

10. $\boxed{C}$ For similar polygons, the ratio of their areas is equal to the square of the ratio of their corresponding side lengths. Since side AB corresponds to side PQ, the ratio of the area of $ABCD$ to the area of $PQRS$ must be $3^2{:}2^2 = 9{:}4$. Therefore, the area of $ABCD$ is $\frac{9}{4} \times 72 = 162$ square units.

11. $\boxed{2430}$ If we let l and w be the length and width of rectangle B, then the length and width of rectangle A can be expressed as $54l$ and $45w$, respectively. The area of rectangle A is then $(54l)(45w) = 2430lw$, which is 2430 times the area of rectangle B.

12. $\boxed{30}$ The base of the parallelogram is 5 units, and its height is 6 units (don't use the slant height!). The area of the parallelogram is then $bh = (5)(6) = 30$ square units.

13. $\boxed{C}$ The volume of the object is $225 \text{ grams} \times \frac{1 \text{ cubic cm}^3}{390 \text{ grams}} \approx 0.58$ cm^3. Its edge length is then $\sqrt[3]{0.58} \approx 0.83$ cm.

14. $\boxed{B}$

$$\frac{4}{3}\pi r^3 = 15^3$$
$$r^3 = 15^3 \times \frac{3}{4\pi}$$
$$r = \sqrt[3]{15^3 \times \frac{3}{4\pi}} \approx 9.3 \text{ inches}$$

15. $\boxed{100}$ Draw the height of the cone from the center of the base to the top. This height forms a right triangle. By the pythagorean theorem, the height is $\sqrt{13^2 - 5^2} = 12$ inches. The volume of the cone is then $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5)^2(12) = 100\pi$ cubic inches. Therefore, $k = 100$.

16. $\boxed{98}$ Let r_a and h_a be the radius and height of cylinder A, and let r_b and h_b be the radius and height of cylinder B. The volume of cylinder A is then $\pi r_a^2 h_a = \pi(14r_b)^2(\frac{1}{2}h_b) = 14^2 \cdot \frac{1}{2}\pi r_b^2 h_b = 98\pi r_b^2 h_b$. Therefore, the volume of cylinder A is 98 times the volume of cylinder B.

EXERCISE 2:

1. $\boxed{C}$ For similar polygons, the ratio of the areas is equal to the square of the ratio of the side lengths. Since all squares are similar, square A must have a side length that is $\sqrt{576} = 24$ times the side length of square B. Thus, the perimeter of square A is 24 times the perimeter of square B. As proof, if we let the side length of square B be x, then the side length of square A is $24x$. Their perimeters are then $4x$ and $4(24x) = 96x$, respectively, and $96x \div 4x = 24$.

2. $\boxed{A}$ Since there are 6 faces on a cube, each face has an area of $24a^2 \div 6 = 4a^2$. The side length of each face is then $\sqrt{4a^2} = 2a$. Finally, the volume of the cube is $(2a)^3 = 8a^3$.

3. $\boxed{D}$ The volume of the cylinder is Area of base $\times$ height $= 28 \times 4 = 112$ cubic centimeters. To find the mass of the cylinder, we can use the density as a conversion factor:

$$112 \text{ cm}^3 \times \frac{2.4 \text{ grams}}{1 \text{ cm}^3} \approx 269 \text{ grams}$$

4. $\boxed{\frac{175}{9} \text{ or } 19.44}$ For similar solids, the ratio of the volumes is equal to the cube of the ratio of the corresponding edge lengths. Therefore, the volume of rectangular prism Y is $3^3 = 27$ times the volume of rectangular prism X. Since the volume of rectangular prism Y is 525 cubic meters, the volume of rectangular prism X must be $\frac{525}{27} = \frac{175}{9}$ cubic meters.

5. $\boxed{A}$ The general formula for the volume of a cylinder is $\pi r^2 h$. Since $f(x) = \pi x^3 + \pi x^2 = \pi x^2(x+1)$, where x is the radius, $(x+1)$ must be the height of the cylinder.

6. $\boxed{17}$ For similar polygons, the ratio of the areas is equal to the square of the ratio of the side lengths. Therefore, the side lengths of triangle ABC are $\sqrt{16} = 4$ times the corresponding side lengths of triangle XYZ. Since side AB corresponds to side XY, side XY must have a length of $68 \div 4 = 17$ inches.

7. $\boxed{C}$ The surface area of a rectangular prism is $2lw + 2lh + 2wh$, where l, w, and h represent the length, width, and height, respectively. For the prism in question, x and y represent the length and the height, so

$$2xy + 2xh + 2yh = 2xy + 100x + 100y$$

Comparing the coefficients, we can see that $2h = 100$, which yields $h = 50$.

8. $\boxed{D}$ The area of a rectangle is length $\times$ width. Now look at the given equation, $A = x\left(\frac{75-2x}{2}\right)$. Since x is the length, $\left(\frac{75-2x}{2}\right)$ must be the width. If we let y represent the width of the rectangle, then

$$\frac{75-2x}{2} = y$$
$$75 - 2x = 2y$$
$$75 = 2x + 2y$$

Since $2x + 2y$ represents the perimeter of the

rectangle, the perimeter must be 75 inches. Note that this is not a solution you would come up with directly. This question is best tackled using process of elimination. Once you get to answer choice D and assume that the perimeter is 75 inches, you only need to prove that the width is $\frac{75-2x}{2}$ for this answer to be correct (not hard to do).

9. $\boxed{522}$ Let l and w represent the original length and width of the image. Since the area of the original image is $lw = 464$ square centimeters, the area of the edited image is $(1.25l)(0.90w) = (1.25)(0.90)lw = (1.25)(0.90)(464) = 522$ square centimeters.

10. $\boxed{\frac{41}{5} \text{ or } 8.2}$ If we let x be the length of the rectangle, then its width is $5x - 6$. Since the area of the rectangle, lw, is 287 square inches,

$$x(5x-6) = 287$$
$$5x^2 - 6x - 287 = 0$$
$$(x+7)(5x-41) = 0$$
$$x = -7 \text{ or } \frac{41}{5}$$

Therefore, the rectangle has a length of $\frac{41}{5}$ inches (the length must be positive).

11. $\boxed{C}$ Cone M has a volume of $\frac{1}{3}\pi r^2 h$. Now let's test each of the answer choices to see which one results in a volume of cone N that is 144 times the volume of cone M:

Answer A yields a volume of $\frac{1}{3}\pi(4r)^2(6h) = 4^2 \cdot 6\left(\frac{1}{3}\pi r^2 h\right) = 96(\text{volume of cone M})$.

Answer B yields a volume of $\frac{1}{3}\pi(4r)^2(36h) = 4^2 \cdot 36\left(\frac{1}{3}\pi r^2 h\right) = 576(\text{volume of cone M})$.

Answer C yields a volume of $\frac{1}{3}\pi(6r)^2(4h) = 6^2 \cdot 4\left(\frac{1}{3}\pi r^2 h\right) = 144(\text{volume of cone M})$.

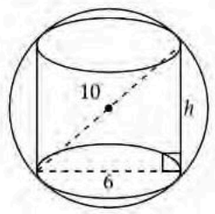
Answer D yields a volume of $\frac{1}{3}\pi(6r)^2(12h) = 6^2 \cdot 12\left(\frac{1}{3}\pi r^2 h\right) = 432(\text{volume of cone M})$.

Therefore, the answer is C.

12. $\boxed{C}$ What makes this problem difficult is that we don't know what x represents. If x represents the width, then the length is $x - 6$ and the height is $x - 12$. The volume is then $x(x-6)(x-12)$, which is not equivalent to the expression we're given, so x cannot represent the width. If x represents the length, then the width is $x + 6$ and the height is $x + 6 - 12 = x - 6$. The volume, in this case, can be represented by $x(x+6)(x-6)$, which is equivalent to the expression given. Therefore, x represents the length and $x - 6$ represents the height.

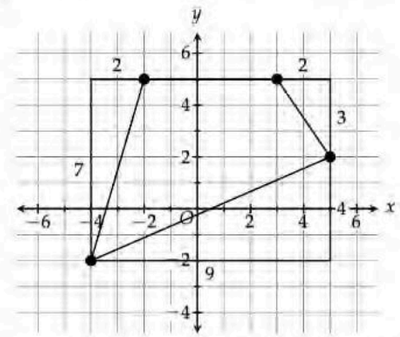
13. $\boxed{378}$ Let r and h represent the radius and height of the original artifact, and let R and H represent the radius and height of the replica. Since the volume of the original artifact is $\pi r^2 h = 112$ cubic inches, the volume of the replica is $\pi R^2 H = \pi(1.5r)^2(1.5h) = (1.5)^2(1.5)(\pi r^2 h) = (1.5)^3(112) = 378$ cubic inches.

14. $\boxed{C}$ To find the height of the cylinder, we first draw the two dashed lines shown below. The first line is a diameter of the lower base of the cylinder. The second line is a diameter of the sphere from the lower base of the cylinder to the upper base. These two lines help form a right triangle that allows us to solve for the cylinder's height.



By the pythagorean theorem, the height of the cylinder is $\sqrt{10^2 - 6^2} = 8$ cm. Since its base radius is $6 \div 2 = 3$ cm, the volume of the cylinder is $\pi r^2 h = \pi(3)^2(8) = 72\pi$ cubic cm. Since the radius of the sphere is $10 \div 2 = 5$ cm, its volume is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(5)^3 = \frac{500}{3}\pi$ cubic cm. Therefore, the volume of the space in the sphere not taken up by the cylinder is $\frac{500}{3}\pi - 72\pi \approx 297$ cubic cm.

15. $\boxed{35}$ Draw the following rectangle around the quadrilateral:

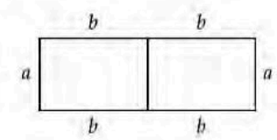


The area of the quadrilateral is the area of the rectangle minus the area of the three triangles that should be excluded:
$(7)(9) - \frac{1}{2}(2)(7) - \frac{1}{2}(2)(3) - \frac{1}{2}(9)(4) = 63 - 7 - 3 - 18 = 35$ square units.

16. $\boxed{\frac{63}{5} \text{ or } 12.6}$ Note that rectangles X and Y are identical. Let their shorter sides have length a and their longer sides have length b. Since they each have an area of $\frac{49}{5}$ square meters and a perimeter of P meters,

$$ab = \frac{49}{5} \quad \text{and} \quad 2a + 2b = P$$

Now let's draw the rectangles when they are glued together:



Since the perimeter of this resulting rectangle is $\frac{14}{9}P$,

$$2a + 4b = \frac{14}{9}P$$

Subtracting the earlier equation $2a + 2b = P$ from this equation yields

$$2b = \frac{14}{9}P - P$$
$$2b = \frac{5}{9}P$$
$$b = \frac{5}{18}P$$

Plugging this result back into $2a + 2b = P$ to solve for a, we get

$$2a + 2\left(\frac{5}{18}P\right) = P$$
$$2a + \frac{5}{9}P = P$$
$$2a = \frac{4}{9}P$$
$$a = \frac{2}{9}P$$

Since $ab = \frac{49}{5}$,

$$\left(\frac{2}{9}P\right)\left(\frac{5}{18}P\right) = \frac{49}{5}$$
$$\frac{5}{81}P^2 = \frac{49}{5}$$
$$P^2 = \frac{49}{5} \cdot \frac{81}{5}$$
$$P = \sqrt{\frac{49 \cdot 81}{25}} = \frac{7 \cdot 9}{5} = \frac{63}{5}$$

Chapter 23: Reading Data

EXERCISE:

1. $\boxed{38}$ princess + superhero + bear + bunny = $11 + 14 + 8 + 5 = 38$ children.

2. $\boxed{C}$ The graph reaches 45% at 8 days after the drive began.

3. $\boxed{144}$ $(N_2 + N_4) - (N_1 + N_3 + N_5) = (447 + 388) - (254 + 315 + 122) = 144$ cookies.

4. $\boxed{C}$ Only answer choice C gives an interval for which the graph only goes downward from the start of the interval to the end of the interval (strictly decreasing).

5. $\boxed{B}$ The candidate who slept the least is represented by the lowest point on the plot, $(39, 5.5)$. These coordinates represent a 39-year-old who slept for 5.5 hours. The closest answer choice is 40.

6. $\boxed{\frac{2}{3} \text{ or } .6666 \text{ or } .6667}$ $\frac{\text{Wolf's Brain}}{\text{Pig's Brain}} = \frac{120}{180} = \frac{2}{3}$

7. $\boxed{D}$ Since x represents the number of minutes *after* the end of the 3-mile run, $x = 0$ must represent the time at which the 3-mile run ended. At $x = 0$, the graph gives a y-value (body temperature) of 40.5 degrees Celsius.

8. $\boxed{C}$ The points corresponding to July through September are the highest in both 2013 and 2014.

9. $\boxed{D}$ Let's estimate the total commute time for each day:

Day	Commute Time
Day 1	$25 + 60 = 85$
Day 2	$38 + 40 = 78$
Day 3	$80 + 20 = 100$
Day 4	$45 + 80 = 125$

Even though these are estimates, it's clear that the longest commute was on Day 4.

10. $\boxed{C}$ The greater the average rate of change, the greater the slope. The portion of the graph between year 7 and year 8 has the greatest slope, so the average rate of change was greatest between year 7 and year 8. Note that the slope between year 7 and year 8 is above 200 whereas the slope between year 1 and year 2 is clearly below 200.

11. $\boxed{C}$ $\frac{\text{Sept}}{\text{June}} = \frac{150}{250} = \frac{3}{5} = 60\%$

12. $\boxed{C}$ March 2017 corresponds to zero along the x-axis. The point at $x = 0$ has a y-value slightly above 3.75, which represents roughly 3.75 million passenger arrivals. The closest answer choice is 3.8 million.

13. $\boxed{B}$ Console A generated $250{,}000 \times 100 = \$25{,}000{,}000$. Console B generated $225{,}000 \times 150 = \$33{,}750{,}000$. Console C generated $125{,}000 \times 250 = \$31{,}250{,}000$. Console D generated $50{,}000 \times 300 = \$15{,}000{,}000$. Console B generated the highest revenue.

14. $\boxed{16200}$ Average Rate of Change = $\frac{\text{2010 Total Amount} - \text{2005 Total Amount}}{2010 - 2005} = \frac{477{,}000 - 396{,}000}{5} = \frac{81{,}000}{5} = 16{,}200$ metric tons per year.

15. $\boxed{D}$ We can choose any point and see how long it takes to reach a point that is double. At the start (0 months), the population is 1,250. At 18 months, the population is double at 2,500. Therefore, it takes 18 months for the population to double.

16. $\boxed{\frac{7}{2} \text{ or } 3.5}$ The smaller the average rate of change between two points, the smaller the slope. The portion of the graph between 60 and 80 seconds has the smallest slope, so the smallest average rate of change given by the graph is $\frac{640 - 570}{80 - 60} = \frac{70}{20} = 3.5$ beats per minute per second.

17. $\boxed{A}$ From the graph, we can see that it took Greg's glucose level 2.5 hours to return to its initial value (140 mg/dL) after breakfast and $8 - 4 = 4$ hours to return to its initial value (also 140 mg/dL) after lunch. Thus, his glucose recovery time after lunch was $4 - 2.5 = 1.5$ hours greater than his glucose recovery time after breakfast.

18. $\boxed{C}$ From 2010 to 2011, the percent decrease was $\frac{30 - 40}{40} = -\frac{1}{4} = -25\%$ (percent decreases are negative). From 2013 to 2014, the percent increase was $\frac{25 - 20}{20} = \frac{1}{4} = 25\%$, which is equal in absolute value to the percent decrease from 2010 to 2011.

19. $\boxed{6}$ The graph indicates that at 30 miles per hour, the car travels 25 miles per gallon of gas. Driving for 5 hours at 30 miles per hour means that a distance of $5 \times 30 = 150$ miles will be traveled. The amount of gas needed to travel this distance is

$$150 \text{ miles} \times \frac{1 \text{ gallon}}{25 \text{ miles}} = 6 \text{ gallons}$$

Chapter 24: Probability

EXERCISE 1:

1. $\boxed{36}$ $80 \times 0.45 = 36$

2. $\boxed{\frac{11}{30} \text{ or } .3666 \text{ or } .3667}$ $\frac{\text{Gray}}{\text{Total}} = \frac{165}{450} = \frac{11}{30}$

3. $\boxed{B}$ $\frac{\text{Gold \& Platinum rings}}{\text{Total rings}} = \frac{11+17}{11+30+17+6} = \frac{28}{64} = \frac{7}{16}$

4. $\boxed{D}$ $\frac{\text{Round photochromic lenses}}{\text{Total lenses}} = \frac{5}{9+7+5+6} = \frac{5}{27}$

5. $\boxed{D}$ The percentage of silver cars is $100 - 20 - 33 - 10 - 14 = 23\%$. So, red and silver make up $20 + 23 = 43\%$ of the cars.

6. $\boxed{C}$ The second roll is independent of the first roll. The probability of rolling a '3' is $\frac{1}{12}$, since there are 12 sides and '3' is just one of the sides.

7. $\boxed{B}$ $\frac{\text{Red ballpoint pens}}{\text{Ballpoint pens}} = \frac{41}{67+112+41+93} = \frac{41}{313}$

8. $\boxed{B}$ There are 9 numbers from 11 to 19 $(19 - 11 + 1 = 9)$. There are 4 even numbers from 11 to 19: 12, 14, 16, and 18. Therefore, the probability of selecting an even number is $\frac{4}{9}$.

9. $\boxed{C}$ $\frac{\text{Stop sign violations by truck drivers}}{\text{Stop sign violations}} = \frac{39}{90}$

10. $\boxed{C}$ $\frac{\text{Games won as underdogs}}{\text{Games played as underdogs}} = \frac{10}{45} = \frac{2}{9}$

11. $\boxed{D}$ The probability of selecting a guppy is $\frac{g}{n}$. The probability of selecting a fish that is not a guppy is then $1 - \frac{g}{n}$.

12. $\boxed{B}$ $\frac{\text{Defective units from assembly line A}}{\text{Defective units}} = \frac{300}{800} = \frac{3}{8}$

13. $\boxed{B}$ In data set A, which has a total of 6 values, there are 2 values (3 and 15) that are not found in data set B. Therefore, the probability is $\frac{2}{6} = \frac{1}{3}$.

14. $\boxed{\frac{19}{25} \text{ or } .76}$ $\frac{\text{Wool scarves at least \$21}}{\text{Wool scarves}} = \frac{13+44}{75} = \frac{19}{25} = .76$

15. $\boxed{C}$ $\frac{\text{Juices w/ no more than 30 grams sugar}}{\text{Beverages w/ no more than 30 grams sugar}} = \frac{22+56}{39+91} = \frac{78}{130} = \frac{3}{5}$

16. $\boxed{\frac{5}{9} \text{ or } .5555 \text{ or } .5556}$ The outer circle has an area of $\pi(6)^2 = 36\pi$. The inner circle has an area of $\pi(4)^2 = 16\pi$. The area of the shaded region is then $36\pi - 16\pi = 20\pi$. Since the total area of the figure is 36π (the outer circle), the probability of selecting a point within the shaded region is $\frac{20\pi}{36\pi} = \frac{5}{9}$.

17. $\boxed{B}$ The percentage of respondents whose favorite flavor is neither vanilla nor coffee is $100\% - 28\% - 12\% = 60\%$. If we let x be the total number of respondents, then $\frac{\text{Cookies and cream}}{\text{Neither vanilla nor coffee}} = \frac{0.18x}{0.60x} = \frac{18}{60} = \frac{3}{10}$.

EXERCISE 2:

1. $\boxed{B}$ $\frac{\text{Buildings constructed in 2015}}{\text{Total number of buildings}} = \frac{112}{364} = \frac{4}{13}$

2. $\boxed{C}$ $\frac{\text{Brand A items in stock}}{\text{Brand A items}} = \frac{12}{12+21} = \frac{12}{33} = \frac{4}{11}$

3. $\boxed{75}$ The proportion of the group that specializes in pediatrics is $1 - \frac{2}{7} - \frac{15}{28} = \frac{28}{28} - \frac{8}{28} - \frac{15}{28} = \frac{5}{28}$. Therefore, $\frac{5}{28} \times 420 = 75$ nurses specialize in pediatrics.

4. $\boxed{C}$ $\frac{\text{jar of honey or peanut butter}}{\text{total number of jars}} = \frac{h+p}{h+m+p}$

5. $\boxed{B}$ The total number of samples contaminated with glyphosate is $(450 \times 0.08) + (550 \times 0.06) = 69$. The probability of selecting a soil sample contaminated with glyphosate is then $\frac{\text{Contaminated samples}}{\text{All samples}} = \frac{69}{450+550} = 0.069$.

6. $\boxed{A}$ Evaluate each of the answer choices.

   A) $\frac{\text{Off campus \& College C}}{\text{College C}} = \frac{596}{870} \approx 0.69$

   B) $\frac{\text{On campus \& College C}}{\text{College C}} = \frac{274}{870} \approx 0.31$

   C) $\frac{\text{Off campus \& College C}}{\text{Off campus}} = \frac{596}{1{,}055} \approx 0.56$

   D) $\frac{\text{On campus \& College C}}{\text{On campus}} = \frac{274}{1{,}445} \approx 0.19$

   Answer choice A gives the greatest value.

7. $\boxed{A}$ $\frac{\text{Company A scientists \& spoken before}}{\text{Company A scientists}} - \frac{\text{Company B scientists \& spoken before}}{\text{Company B scientists}} = \frac{40}{40+80} - \frac{30}{30+120} = \frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$

8. $\boxed{\frac{2}{3} \text{ or } .6666 \text{ or } .6667}$ The bakery produced a total of $700 \times 3 + 300 \times 5 = 3{,}600$ bagels on Friday. During the first 4 hours of operation, the bakery produced $700 \times 3 + 300 = 2{,}400$ bagels. Thus, the probability of selecting a bagel that was produced during the first 4 hours is $\frac{2{,}400}{3{,}600} = \frac{2}{3}$.

9. $\boxed{B}$ The indicator is wrong when the test gives positive indicators to patients who don't have the virus and negative indicators to patients who do, a total of $30 + 50 = 80$ occurrences. Thus, the probability that the test is wrong is $\frac{80}{1000} = \frac{8}{100} = 8\%$.

10. $\boxed{\frac{2}{5} \text{ or } .4}$ $\frac{\text{Members at least 45 \& tennis is favorite}}{\text{Members at least 45}} = \frac{14+26}{50+50} = \frac{40}{100} = \frac{2}{5}$

11. $\boxed{\frac{2}{5} \text{ or } .4}$ To find the number of integers from a to b, we subtract a from b and add 1. For example, from 1 to 10, there are 10 numbers $(10 - 1 + 1 = 10)$. From 101 to 300, there are $300 - 101 + 1 = 200$ integers. Therefore, the probability of selecting a number less than 301 is $\frac{200}{500} = \frac{2}{5}$.

12. $\boxed{240}$ **Solution 1:** Let the number of students who answered "gym equipment" be x. The total number of students is then $x + 160$. Since $\frac{1}{3}$ of them answered "gym equipment,"

$$x = \frac{1}{3}(x + 160)$$
$$3x = x + 160$$
$$2x = 160$$
$$x = 80$$

Thus, there are a total of $80 + 160 = 240$ students at this school.

Solution 2: If $\frac{1}{3}$ of the students answered "gym equipment," then $1 - \frac{1}{3} = \frac{2}{3}$ of the students answered "computers." If we let the total number of students be x, then

$$\frac{2}{3}x = 160$$

$$x = 160 \times \frac{3}{2} = 240$$

13. $\boxed{B}$ There are $7 \times 6 = 42$ paperback books published in 2011, and $9 \times 8 = 72$ hardcover books published in 2011. Therefore, the probability of selecting a paperback book, given that the book was published in 2011, is $\frac{\text{Number of 2011 paperback books}}{\text{Total number of 2011 books}} = \frac{42}{42+72} = \frac{42}{114} = \frac{7}{19}$.

14. $\boxed{A}$ The area of the rectangle is $10 \times 7 = 70$. The unshaded triangle has a base of 3 cm and a height of 10 cm, so its area is $\frac{1}{2}(3)(10) = 15$. The area of the shaded region is then $70 - 15 = 55$. The difference between the shaded and unshaded probabilities is then $\frac{55}{70} - \frac{15}{70} = \frac{40}{70} = \frac{4}{7}$.

15. $\boxed{\frac{6}{25} \text{ or } .24}$ **Solution 1:** To find the number of integers from a to b, we subtract a from b and add 1. For example, from 1 to 10, there are 10 numbers ($10 - 1 + 1 = 10$). From 1 to 125, there are 125 positive integers ($125 - 1 + 1 = 125$). The smallest two-digit multiple of 3 is 12 and the largest two-digit multiple of 3 is 99:

$$12, 15, 18, \ldots, 99$$

How many multiples of 3 are there from 12 to 99? Well, if we divide each number in this list by 3, we get

$$4, 5, 6, \ldots, 33$$

So the question becomes: how many integers are there from 4 to 33? There are $33 - 4 + 1 = 30$ integers. So there must be 30 multiples of 3 from 12 to 99. The probability of selecting a two-digit multiple of 3 is then $\frac{30}{125} = \frac{6}{25}$.

Solution 2: From 1 to 125, there are 125 positive integers, as explained in Solution 1. The number of multiples up to (and including) a certain number can be found through integer division. So there are $99 \div 3 = 33$ multiples of 3 up to 99, and there are $9 \div 3 = 3$ multiples of 3 up to 9. Subtracting the number of single-digit multiples from the total number of multiples up to 99 gives us the number of two-digit multiples: $33 - 3 = 30$. The desired probability is then $\frac{30}{125} = \frac{6}{25}$.

16. $\boxed{\frac{13}{18} \text{ or } .7222}$ If we let x be the total number of participants, then $\frac{\text{At least 150 pounds}}{\text{At least 110 pounds}} = \frac{0.47x + 0.18x}{0.25x + 0.47x + 0.18x} = \frac{0.65x}{0.90x} = \frac{65}{90} = \frac{13}{18}$.

17. $\boxed{\frac{3}{5} \text{ or } .6}$ Let x be the number of plants that inhabit the side with poor soil. Then $2x$ is the number of plants that inhabit the side with rich soil. The probability that a selected laurel plant inhabits the side with rich soil is then $\frac{\text{Laurel plants on rich soil}}{\text{Total number of laurel plants}} = \frac{0.15(2x)}{0.20x + 0.15(2x)} = \frac{0.30x}{0.50x} = \frac{3}{5} = .6$.

Chapter 25: Statistics I

EXERCISE 1:

1. $\boxed{9}$ Data set P has 9 values and data set Q has 8 values. The median of data set P is its $9 \div 2 = 4.5 \rightarrow$ 5th value, which is 47. The median of data set Q is the average of its $8 \div 2 = 4$th and 5th values, which are both 38. The average of 38 and 38 is, of course, 38. The difference is then $47 - 38 = 9$.

2. $\boxed{11}$ Max value − Min value $= 28 - 17 = 11$. The frequencies don't matter when finding the range.

3. $\boxed{6}$ The median is represented by the $27 \div 2 = 13.5 \rightarrow$ 14th dot from the start, which has a value of 6.

4. $\boxed{D}$ The median score, designated by the line segment in the middle of the "box", is approximately 83. The lowest individual score, designated by the line segment at the left end of the plot, is approximately 72. The difference between the two values is $83 - 72 = 11$.

5. $\boxed{C}$ The sum of the students' heights in the first class is $14 \times 63 = 882$ inches. The sum of the students' heights in the second class is $21 \times 68 = 1{,}428$ inches. The sum of the heights in the combined class is then $882 + 1428 = 2{,}310$ inches, which results in a mean height of $\frac{\text{Sum of the heights}}{\text{Total number of students}} = \frac{2{,}310}{14 + 21} = 66$ inches.

6. $\boxed{B}$ A range of 3 days means the difference between the longest shelf life and the shortest shelf life among the units is 3. This could be 10 days vs. 13 days or 25 days vs. 28 days. The range says nothing about the mean or median.

7. $\boxed{D}$ Box plots don't tell us anything about the number of data points being represented, so answer choice A is incorrect. The median of set A is 25 while the median of set B is 35, so answer choice B is incorrect. The range of set A is $45 - 10 = 35$ while the range of set B is $45 - 15 = 30$, so answer choice C is incorrect. The maximum of set A is 45 and the maximum of set B is also 45, so answer choice D is correct.

8. $\boxed{50}$ Since the mean of data set A is $\frac{30 + 24 + 61 + 37}{4} = 38$, the mean of data set B must be $2 \times 38 = 76$. The sum of the values in data set B is then $4 \times 76 = 304$. Finally, $x = 304 - 77 - 63 - 114 = 50$.

9. $\boxed{67}$ The median low temperature is the average of the low temperatures for the 14th and 15th days, both of which were 67°F. The average of 67 and 67 is, of course, 67.

10. $\boxed{B}$ One of the cases in which the mean is equal to the median is when each value on the left of the median has a counterpart on the right that is the same distance away from the median. This rule is especially helpful for this question because the answer choices have values that are spaced at certain distances from the medians. For example, in answer choice A, the median is 13, and the three 3's and the three 23's are spaced 10 away from this median. The two 13's are also counterparts since both are spaced "zero" away from the median. Since each value on the left has a counterpart on the right that is equidistant from the median, the mean must be equal to the median. The same holds true for answer choices C and D. In answer choice B, the 4's are 4 away from the median of 8, but the 16's are 8 away. Therefore, the mean is not equal to the median for this data set.

11. $\boxed{B}$ The mean is greater when there are relatively more of the higher values than of the lower values. The answer is B because there are less of the lower values. In other words, the data set in answer choice B is weighted towards the higher values, resulting in a greater mean than if all the values were weighted equally.

12. $\boxed{B}$ Because there are 20 editors, the median is the average of the 10th and 11th editors' number of books read. From the histogram, the 10th and 11th editors each read between 11 and 15 books last year, which means the average for the two of them must also be between 11 and 15. The only answer choice between 11 and 15 is 12.

13. $\boxed{B}$ The mean $= \dfrac{(5\cdot 2)+(6\cdot 1)+(8\cdot 4)+(9\cdot 2)+(10\cdot 1)}{2+1+4+2+1} = \dfrac{76}{10} = 7.6$, and the range $= 10-5=5$.

14. $\boxed{C}$ From lowest to highest, the values are 84, 88, 91, 91, 96. The median is currently the 3rd value, 91. If 100 is added to the data set, the median becomes the average of the 3rd and 4th values, which are both 91. Therefore, the median will remain the same. As for the mean, the value of 100 is greater than any of the numbers in the data set, so the mean will certainly increase if 100 is added to the data set.

15. $\boxed{C}$ The frequencies are the same in both tables, but the values are not. Since there are higher values in data set B, the mean of data set B is greater than the mean of data set A. The standard deviations of the two data sets are the same because each data set is spread out in exactly the same way. The values in each data set are consecutive integers (spaced one apart), and corresponding values across the two data sets have equal frequencies. The actual values themselves don't matter when it comes to comparing standard deviation, only their spacing relative to one another.

16. $\boxed{A}$ The frequency table summarizes $4+6+2+4+2=18$ values. The median of the data set is the average of the $18\div 2=9$th and 10th values, which are both 27. The average of 27 and 27 is, of course, 27, so the median is 27. The mean of the data set is $\dfrac{24\cdot 4+27\cdot 6+30\cdot 2+33\cdot 4+36\cdot 2}{18} = \dfrac{522}{18} = 29$. Therefore, the mean is $29-27=2$ greater than the median.

17. $\boxed{B}$ From lowest to highest, the scores are 75, 83, 87, 87, 90, 91, 98. The average is $\dfrac{75+83+87+87+90+91+98}{7} \approx 87.3$. The mode is 87. The median is also 87. The range is $98-75=23$. Thus, statement I is false, statement II is false, and statement III is true.

EXERCISE 2:

1. $\boxed{D}$ By definition, at least half the values are greater than or equal to the median and at least half the values are less than or equal to the median.

2. $\boxed{C}$ The numbers of bicycles rented each day are 810, 540, 180, 720, and 450. From least to greatest, the values are 180, 450, 540, 720, 810. The median is the 3rd value, 540. The mean is $\dfrac{180+450+540+720+810}{5} = 540$. Therefore, the mean is equal to the median.

3. $\boxed{44}$ The sum of the values of the 19 pins is $19\times 98 = 1{,}862$ dollars. The sum of the values of the remaining 18 pins is $18\times 101 = 1{,}818$ dollars. The difference between these two sums must be the value of the chosen pin: $1{,}862-1{,}818=44$ dollars.

4. $\boxed{C}$ The medians are represented by the $19\div 2 = 9.5 \rightarrow$ 10th value in each data set. The 10th value in data set A is 47, and the 10th value in data set B is also 47. Therefore, the medians are equal.

5. $\boxed{C}$ There are $6+4=10$ data values in the combined data set. The sum of the data values in the combined data set is $10\times 94 = 940$. The sum of the data values in data set A is $6\times 92 = 552$. Subtracting this sum from the combined sum yields the sum of the data values in data set B: $940-552=388$. The mean of data set B is then $\dfrac{388}{4} = 97$.

6. $\boxed{D}$ The mean does not imply anything about the median, so it's not possible to compare the medians of two data sets based off of their means. Therefore, statement I is not necessarily true. It's also not possible to compare the maximum values of two data sets based off of their means. A lower mean does not always correspond to a lower maximum value, and a higher mean does not always correspond to a higher maximum value. One data set could have one very high maximum value but a low mean if the rest of the values in the data set are low. Therefore, statement II is not necessarily true either.

7. $\boxed{A}$ A box plot does not convey any information about the mean, so statement I is not necessarily true. The range of data set 1 is $23-12=11$, and the range of data set 2 is $21-13=8$. Statement II is definitely not true.

8. $\boxed{C}$ We can compare the means without calculating them. To go from data set A to data set B, 2 dots at 22 move down by 2, and 3 dots at 24 move up by 1. The net change in the sum of the values is then $2\times -2+3\times 1=-1$. In other words, the sum of the values in data set B is 1 less than the sum of the values in data set A. Since data sets A and B have the same number of values, the mean of data set B must be less than the mean of data set A. Statement I is false. Visually, data set B is more spread out than data set A, so the standard deviation of data set B must be greater than the standard deviation of data set A. Statement II is also false.

9. $\boxed{D}$ Currently, the median is the average of the prices for the $30\div 2=15$th and 16th items, which are both 55 dollars. The average of 55 and 55 is, of course, 55, so the median is 55 dollars. If a 25 dollar item is removed, the frequency of the \$25 items decreases by 1, and the median becomes the price of the $29\div 2 = 14.5 \rightarrow$ 15th item of the new data set, which is also 55 dollars. Therefore, the median will remain the same. As for the mean, 25 dollars is the lowest value in the data set, pulling down the mean, so removing a 25 dollar item will undoubtedly increase the mean.

10. $\boxed{D}$ When each value in a data set is multiplied by 2, each resulting corresponding value remains in the same relative position within the data set, and so the resulting median also remains in the same position. Thus, the median of data set B is twice the median of data set A. As evidence, there are 17 values in data set A. Its median is represented by the $17\div 2 = 8.5 \rightarrow$ 9th dot, which has a value of 21. Once data set B is created, it also has 17 values and a median represented by the 9th dot, which has a value of 42. Data set A has a maximum value of 23 and a minimum value of 18, yielding a range of $23-18=5$. Data set B has a maximum value of $23\times 2=46$ and a minimum value of $18\times 2=36$, yielding a range of $46-36=10$. Therefore, the range of data set B is twice the range of data set A.

11. $\boxed{B}$ Standard deviation is a measure of how spread out the values in a data set are. The more spread out they are, the greater the standard deviation. The actual values themselves don't matter, only their spacing in relation to one another. In this question, the data sets are plotted on the same scale with values spaced 1 unit apart, so it's easy to visually compare their standard deviations. Because the values in data set B are closely clustered around 66, it's the least spread out and thus has the lowest standard deviation. The values in data set A are loosely clustered around 64, with 61 and 67 being values that are further out, so its standard deviation is higher than data set B's. Finally, data set C is the most spread out because none of the values are clustered together. Instead, they are evenly spaced across the entire axis, resulting in the highest standard deviation among the three data sets. Putting everything together, $b<a<c$.

12. $\boxed{A}$ The two integers between 210 and 250 are less than any of the other integers, so they bring the mean down. Thus, the mean of data set A is greater than the mean of data set B. However, it's not necessarily true that the median of data set A is greater than the median of data set B. For example, if all the values in data set A were the same, then the medians of data sets A and B would be the same. Lastly, data set A must have a smaller range than data set B because 1) their maximums are the same, and 2) data set B's minimum is smaller because it must be one of the two integers between 210 and 250. This smaller minimum value results in a greater range for data set B. The answer to this question is I only.

13. $\boxed{B}$ Before the car is removed, the median is represented by the 8th car (23 mpg). After the car is removed, the median is represented by the average of the 7th and 8th cars (still 23 mpg). So the median stays the same. However, the mean and the standard deviation both decrease. We're removing a data point higher than all the others so the mean decreases. We're also reducing the spread in the data so the standard deviation decreases.

14. $\boxed{62}$ The mean of the 7 given integers is 78, so their sum is $7 \times 78 = 546$. Since the mean of data set Q is an integer less than 78, the 8th integer must be less than 78 to bring the mean down. Given that the range is greater than 20 but less than 30, the 8th integer must be between $85 - 30 = 55$ and $85 - 20 = 65$, not including 55 and 65. So let's do a little trial and error. If the mean of data set Q were 77, the 8th integer would then be $8 \times 77 - 546 = 70$, which is not between 55 and 65. If the mean of data set Q were 76, the 8th integer would be $8 \times 76 - 546 = 62$, which is between 55 and 65. Therefore, the 8th integer must be 62. For the sake of completeness, if the mean of data set Q were 75, the 8th integer would be $8 \times 75 - 546 = 54$, which is not between 55 and 65. Thus, 62 is the only possible value.

15. $\boxed{A}$ Standard deviation is a measure of how spread out the values in a data set are. The more spread out they are, the greater the standard deviation. The actual values themselves don't matter, only their spacing in relation to one another. Here, the frequencies in both data sets are the same, but the travel times are spaced further apart for bus B. The travel times for bus A are spaced much closer together (44, 45, 47, 48 vs. 25, 30, 35, 40). Therefore, the standard deviation of travel times for bus A is less than the standard deviation of travel times for bus B.

16. $\boxed{18}$ The median of data set Y is its $25 \div 2 = 12.5 \rightarrow$ 13th value, which, according to the histogram, is greater than or equal to 90 but less than 95. Since all the values in data set X are less than 90, the median of data set Y must be greater than any possible mean of data set X. Therefore, the largest possible difference between the median of data set Y and the mean of data set X is found by making the median of data set Y as large as possible and the mean of data set X as small as possible. Since both data sets consist of whole numbers, the largest possible median of data set Y is 94. The smallest possible mean of data set X is $\frac{70 \cdot 9 + 75 \cdot 6 + 80 \cdot 6 + 85 \cdot 4}{25} = 76$. The largest possible difference is then $94 - 76 = 18$.

17. $\boxed{D}$ The lowest possible mean monthly fee charged to the existing clients is $\frac{8 \cdot 170 + 11 \cdot 200 + 21 \cdot 240}{8 + 11 + 21} = \frac{8{,}600}{40} = \$215$. The highest possible mean monthly fee is $\frac{8 \cdot 190 + 11 \cdot 230 + 21 \cdot 300}{8 + 11 + 21} = \frac{10{,}350}{40} = \$258.75$. Since the new client's \$250 monthly fee falls between \$215 and \$258.75, we don't have enough information to determine how the mean will be affected. If the current mean is less than \$250 (e.g. \$215), then the new \$250 fee will increase the mean. If the current mean is greater than \$250 (e.g. \$258.75), then the new \$250 fee will decrease the mean.

Chapter 26: Statistics II

EXERCISE 1:

1. $\boxed{C}$ $0.12 \times 8{,}400 = 1{,}008$ users

2. $\boxed{C}$ At $x = 8.5$, the line of best fit gives a y-value between 44 and 45. The only answer choice between 44 and 45 is 44.8.

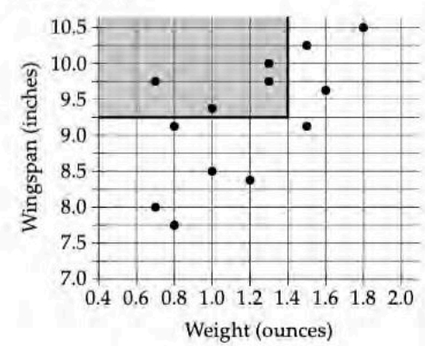
3. $\boxed{4}$ There are 4 points below the line of best fit.

4. $\boxed{B}$ Answer choice A is incorrect because an estimate based on a sample is just an estimate. The population percentage could very well be different, which is why there is a margin of error associated with the sample estimate. Since we have a percentage of 58% and the margin of error is 6%, it is plausible (reasonable) that the percentage for the population is between 52% and 64%. Based on the margin of error, answer choices C and D are not plausible conclusions.

5. $\boxed{B}$ At $x = 21$, the y-value predicted by the line of best fit is 70, which exceeds the actual y-value of 67 by $70 - 67 = 3$.

6. $\boxed{C}$ The poll was based on a random sample of 21 professors from a certain liberal arts program at a university. So the results of the poll can only be generalized to professors in that program.

7. $\boxed{4}$ The sparrows that weigh less than 1.4 ounces and have a wingspan greater than 9.25 inches are represented by the points within the upper-left rectangle shown. There are 4 of them.



8. $\boxed{A}$ If you extend the line of best fit, the y-intercept is approximately 11.5. The slope can be estimated from any two points on the line. The points $(30, 10)$ and $(50, 9)$ yield a slope of $\frac{9 - 10}{50 - 30} = \frac{-1}{20} = -0.05$. The equation of the line of best fit is then $y = 11.5 - 0.05x$.

9. $\boxed{C}$ We can use the approximate coordinates of any two points on the line to estimate its slope. The points $(3, 4)$ and $(6, 6)$ yield a slope of $\frac{6 - 4}{6 - 3} = \frac{2}{3} \approx 0.66$. The closest answer choice to 0.66 is 0.64.

10. $\boxed{B}$ The most we can conclude is that there is a negative association between the price of food and population density in U.S. cities (as one goes up, the other goes down). We CANNOT conclude that there is a cause and effect relationship between the two (i.e. that one causes the other).

11. $\boxed{D}$ Draw a reasonable line of best fit. For any reasonable line of best fit, the y-intercept is roughly 24. The approximate coordinates of any two points on the line can be used to estimate the slope. The points $(0, 24)$ and $(7, 1)$ yield a slope of $\frac{1 - 24}{7 - 0} = \frac{-23}{7} \approx -3.29$. Of the answer choices, the linear model $y = -3.4x + 24.2$ comes closest to having the values we found.

12. $\boxed{C}$ Of the 500 people chosen, $50 + 75 + 100 = 225$ used public transportation at least twice in the last month. Therefore, out of a population of 36,000, we would expect a total of $\frac{225}{500} \times 36{,}000 = 16{,}200$ people to have used public transportation at least twice in the last month.

13. $\boxed{C}$ An estimated mean wait time based on a sample is an estimate for the mean wait time, not the actual wait times of the customers. For example, one customer could wait 1 minute and another customer could wait 13 minutes and the mean wait time would be 7 minutes, even though neither of the customers waited for 7 minutes. Therefore, answer choices A and B are incorrect. An associated margin of error of 2 minutes means that it's plausible the mean wait time is between 5 and 9 minutes. Even though this is the plausible range of values for the mean, it's possible that the mean falls outside these values. Therefore, answer choice D is incorrect.

14. $\boxed{D}$ The study was done using a random sample of adults who are competing in a chess tournament. Therefore, the results of the study can only be generalized to all the adults who are competing in that tournament.

15. $\boxed{C}$ The second test had a larger sample size than the first, so we would expect the margin of error from the second test to be smaller than the margin of error from the first test.

16. $\boxed{D}$ To draw a reliable conclusion about the effectiveness of the new vaccine, the patients must be randomly assigned to their treatment. Only answer D leads to random assignment. Note that answer C does not because the patients are allowed to group themselves as they desire. For example, three friends might want to be in the same group, leading to assignment that isn't random.

EXERCISE 2:

1. $\boxed{C}$ $\frac{660}{900} \times 75{,}300 = 55{,}220$ voters

2. $\boxed{A}$ The survey was done using a random sample of math teachers from one of the school districts in the state. Therefore, the results of the survey can only be generalized to all math teachers from that one school district.

3. $\boxed{D}$ Variability (i.e. standard deviation) refers to how spread out a data set is. The higher the variability in a population data set, the higher the margin of error when it comes to estimating the mean. A population that eats 1, 3, 5, or 7 servings of vegetables per day results in a more variable data set (and thus a higher margin of error) than a population that eats only 4 servings of vegetables per day. Therefore, surveying students who are following the same daily diet plan would most likely result in the smallest margin of error because they are presumably eating the same number of servings of vegetables each day. The other answer choices are likely to yield data sets that are much more variable.

4. $\boxed{C}$ The mean for a sample is not necessarily the mean for the entire population, so statement I is not necessarily true. The mean for a sample is also not necessarily the mean for another sample, so statement II is not necessarily true either.

5. $\boxed{C}$ The data points at $x = 2, 4$, and 6 have y-values that are at least 2 greater than the y-value predicted by the line of best fit. That's 3 data points in total.

6. $\boxed{C}$ A 65% estimate with an associated margin of error of 4% means that it's plausible for the population percentage to be between 61% and 69%. Therefore, it is unlikely that the population percentage is 60%. Answer A is incorrect because the range of values is too narrow. Answer B is wrong because it misinterprets what margin of error means. And answer D is wrong because it *is* possible that the population percentage is 70%—it's just unlikely.

7. $\boxed{D}$ Extend the line of best fit just a little and it's clear that it must have a negative y-intercept. Answer choices A and B are thus eliminated. Since the line has a positive slope, we can also eliminate answer choice C, leaving answer choice D as the correct answer.

8. $\boxed{A}$ Based on the sample, candidate A is expected to receive $\frac{110}{250} \times 500{,}000 = 220{,}000$ votes. Candidate B is expected to receive $\frac{140}{250} \times 500{,}000 = 280{,}000$ votes. So candidate B is expected to receive $280{,}000 - 220{,}000 = 60{,}000$ more votes.

9. $\boxed{B}$ We can use the approximate coordinates of any two points on the line to estimate its slope. The points $(1, 17)$ and $(4, 13)$ yield a slope of $\frac{13-17}{4-1} = \frac{-4}{3} \approx -1.33$. The closest answer choice to -1.33 is -1.38.

10. $\boxed{C}$ The survey results are unlikely to represent the opinions of all the town's residents because only those who mailed their responses back were represented. It's very possible, for instance, that the residents who bothered to mail a response back are the ones who care the most about the issue. In any case, the survey was not based on a random sample of the town's residents so its results are likely biased.

11. $\boxed{B}$ An 82% estimate with an associated margin of error of 6% means that it's plausible for the true percentage to be between 76% and 88%. Therefore, it's plausible that between $0.76 \times 900 = 684$ employees and $0.88 \times 900 = 792$ employees are satisfied with the company's training materials.

12. $\boxed{C}$ The y-intercept of the given line of best fit is roughly 2.5. The approximate coordinates of any two points on the line can be used to estimate the slope. The points $(0, 2.5)$ and $(3, 6)$ yield a slope of $\frac{6-2.5}{3-0} = \frac{3.5}{3} \approx 1.2$. A rough equation of the line of best fit shown is then $y = 1.2x + 2.5$. Because every point in data set B is 1.6 units up from its corresponding point in data set A, we can obtain data set B's line of best fit by shifting data set A's line of best fit up by 1.6 units. To do so, we just add 1.6 to the equation we found: $y = 1.2x + 2.5 + 1.6 = 1.2x + 4.1$ (see the chapter on Function Transformations). This equation best matches answer choice C.

13. $\boxed{B}$ For a valid conclusion, the students must be randomly assigned to their "treatment." In this case, it's studying either at a library or at home. Asking the students who live closest to a library to study at a library may result in a faulty conclusion because students who live close to a library may have something in common that skews the result. Perhaps they are more likely to prefer studying at a library, for example. Answer A is incorrect because one particular library may have unique features that make it easier/harder to study than others. We would not be able to generalize the results of the study if only one library were used.

14. $\boxed{B}$ The survey was based on a random sample of the company's employees. Therefore, the results of the survey can be generalized to all employees at the company. It does not matter that the sample included only 1 full-time employee. Perhaps there are 59 part-time employees for every 1 full-time employee at the company. In any case, it's expected that random samples do not always reflect the population. That's why we have margins of error.

15. $\boxed{D}$ Sample B resulted in a greater margin of error. Therefore, its sample size was likely smaller (i.e. fewer customers). Answer choices A and B may be true but don't explain why the margins of error are different.

16. $\boxed{D}$ To obtain an unbiased estimate of the mean age, a random sample of the residents must be used. Randomly selecting 3 floors seems like it would result in a random sample of residents, but it doesn't. For any individual that's chosen, it's guaranteed that all the neighbors on the same floor are also chosen. That's not random. In a random sample, you would expect a mix of residents from many different floors, not just 3. Given that a flawed sampling method was used, the mean age of all the residents should not be estimated.