Homework 5 continued

Problem 2) a)

$T_0 = 2$ Derive a formula of form $T_n = A_n + B_n$

$T_1 = 3$ A_n, B_n must be well known sequences

$T_2 = 6$

$n \geq 3 \rightarrow T_n = (n+4)T_{n-1} - 4n\,T_{n-2} + (4n-8)\,T_{n-3}$

$T_3 = 14$

$T_4 = 40$

$T_5 = 152$

$\cancel{A_n = T_{n-1} + T_{n-2}}$

$$\boxed{\begin{array}{l} A_n = n! \quad , \quad B_n = 2^n \\ T_n = n! + 2^n \end{array}}$$

Problem 2) b)

Base case(s): $T_0 = 0! + 2^0 = 1 + 1 = 2$ ✓

$T_1 = 1! + 2^1 = 1 + 2 = 3$ ✓

$T_2 = 2! + 2^2 = 2 + 4 = 6$ ✓

Assume for $n = k-1, k-2, k-3$, we have

$$T_{k-1} = (k-1)! + 2^{k-1} \quad , \quad T_{k-2} = (k-2)! + 2^{k-2} \quad , \quad T_{k-3} = (k-3)! + 2^{k-3}$$

Want to prove: $T_k = k! + 2^k$

Plug in to original formula

$$T_k = (k+4)\left[(k-1)! + 2^{k-1}\right] - 4k\left[(k-2)! + 2^{k-2}\right] + (4k-8)\left[(k-3)! + 2^{k-3}\right]$$

$$T_k = \underbrace{(k+4)(k-1)! - 4k(k-2)! + (4k-8)(k-3)!}_{\text{factorial part}} + \underbrace{(k+4)2^{k-1} - 4k \cdot 2^{k-2} + (4k-8) \cdot 2^{k-3}}_{\text{Powers of 2 part}}$$

Homework 5 continued

Problem 2) b) continued

Solve factorial part first:

$$(k+4)(k-1)! - 4k(k-2)! + (4k-8)(k-3)!$$

$$(k+4)(k-1)(k-2)(k-3)! - 4k(k-2)(k-3)! + (4k-8)(k-3)!$$

$$\left[(k+4)(k-1)(k-2) - 4k(k-2) + (4k-8)\right](k-3)!$$

$$\left[(k+4)(k^2-3k+2) - 4k^2 + 8k + 4k - 8\right](k-3)!$$

$$\left[k^3 + k^2 - 10k + 8 - 4k^2 + 12k - 8\right](k-3)!$$

$$\left[k^3 - 3k^2 + 2k\right](k-3)!$$

$$\left[k(k^2 - 3k + 2)\right](k-3)!$$

$$\left[k(k-1)(k-2)\right](k-3)! = \underline{k!} \checkmark$$

Now Powers of 2 part:

$$(k+4)2^{k-1} - 4k \cdot 2^{k-2} + (4k-8)2^{k-3}$$

$$(k+4)\cdot 2 \cdot 2 \cdot 2^{k-3} - 4k \cdot 2 \cdot 2^{k-3} + (4k-8)\cdot 2^{k-3}$$

$$\left[4(k+4) - 8k + 4k - 8\right]\cdot 2^{k-3}$$

$$\left[4k + 16 - 4k - 8\right]\cdot 2^{k-3}$$

$$8 \cdot 2^{k-3} = 2^3 \cdot 2^{k-3} = \underline{2^k} \checkmark$$

Thus, by induction

$$\underline{T_k = k! + 2^k}$$