Homework 5 continued

Problem 4) a)

$$S = \{1,\ 1-x,\ (1-x)^2\}$$

$$S = \{1,\ 1-x,\ x^2 - 2x + 1\}$$

can be expressed as vectors

$$V = \{(1,0,0),\ (1,-1,0),\ (1,-2,1)\}$$

Put them in a matrix (each vector is a row)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{multiply } R_2 \text{ by } -1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

thus linearly independent.
and it spans the space
~~by~~ because the space for polynomials up to degree 2 is 3-dimensional, and we have 3 linearly indepent vectors.

$\boxed{\text{Thus, the set } S \text{ forms a basis for the space of polynomials up to degree 2!}}$

Homework 5 continued

Problem 4) b)

$S = \{x_1(t), x_2(t), x_3(t)\} \ni x_1(t) = t^2, \quad x_2(t) = t, \quad x_3(t) = 1,$

$t \in [-1, 1] \in \mathbb{R}$, construct an orthonormal basis for S using the inner product $(x, y) = \int_{-1}^{1} x(t)y(t)\,dt$

Set $\cancel{x_1}\, v_1 = x_1(t) = t^2$

↳ normalize to get e_1: $\|v_1\|^2 = (t^2, t^2) = \int_{-1}^{1} t^4\,dt$

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{t^2}{\sqrt{2/5}} = \sqrt{\frac{5}{2}}\, t^2$$

$$= \left[\frac{t^5}{5}\right]_{-1}^{1} = \frac{1}{5} - \left[-\frac{1}{5}\right] = \frac{2}{5}$$

$\|v_1\| = \sqrt{2/5}$

~~Orthogonalize $x_2 = t$ against~~

get e_2:

$$v_2 = x_2 - (x_2, e_1)e_1 = t - (t, e_1)e_1$$

$$= t - \left[\int_{-1}^{1} t \cdot \sqrt{\tfrac{5}{2}}\, t^2\,dt\right] \cdot \sqrt{\tfrac{5}{2}}\, t^2$$

$$= t - \left[\sqrt{\tfrac{5}{2}} \cdot \left[\frac{t^4}{4}\right]_{-1}^{1}\right] \cdot \sqrt{\tfrac{5}{2}}\, t^2$$

$$= t - \sqrt{\tfrac{5}{2}} \cdot (0) \cdot \sqrt{\tfrac{5}{2}}\, t^2$$

$$= t$$

$v_2 = t$

$$\|v_2\| = \sqrt{\int_{-1}^{1} t^2\,dt} = \sqrt{\frac{2}{3}}$$

$$e_2 = \sqrt{\frac{3}{2}}\, t$$

Homework 5 continued

Problem 4) b) continued

get e_3: $v_3 = x_3 - (x_3, e_2)e_2 - (x_3, e_1)e_1$

$$v_3 = 1 - \left[\int_{-1}^{1} 1 \cdot \sqrt{\tfrac{3}{2}}\, t \, dt\right] \cdot \sqrt{\tfrac{3}{2}}\, t - \left[\int_{-1}^{1} 1 \cdot \sqrt{\tfrac{5}{2}}\, t^2 \, dt\right] \cdot \sqrt{\tfrac{5}{2}}\, t^2$$

$$v_3 = 1 - 0 - \sqrt{\tfrac{5}{2}} \cdot \tfrac{2}{3} \cdot \sqrt{\tfrac{5}{2}}\, t^2$$

$$\underline{v_3 = -\tfrac{5}{3} t^2 + 1}$$

$$\|v_3\|^2 = \int_{-1}^{1} \left(-\tfrac{5}{3}t^2 + 1\right)^2 dt = \int_{-1}^{1} \left(\tfrac{25}{9} t^4 - \tfrac{10}{3} t^2 + 1\right) dt$$

$$= \frac{10}{9} - \frac{20}{9} + 2$$

$$= \frac{-10}{9} + \frac{18}{9} = \frac{8}{9}$$

$$\|v_3\| = \sqrt{\tfrac{8}{9}}$$

$$e_3 = \sqrt{\tfrac{9}{8}}\left(1 - \tfrac{5}{3} t^2\right)$$

Orthonormal Basis:

$$\boxed{\left\{\sqrt{\tfrac{5}{2}}\, t^2,\ \sqrt{\tfrac{3}{2}}\, t,\ \sqrt{\tfrac{9}{8}}\left(1 - \tfrac{5}{3} t^2\right)\right\}}$$