University of Illinois at Urbana-Champaign

ECE 311: Digital Signal Processing Lab

LAB 3: SOLUTIONS

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Problem 1

(a) Using $T = \frac{2\pi}{\Omega_s} = \frac{1}{10^4}$

$$x[n] = x_a(nT) = cos\left(2\pi \times 10^3 \times \frac{1}{10^4}n\right) + cos\left(2\pi \times 1.4 \times 10^3 \times \frac{1}{10^4}n\right)$$
$$= cos(\frac{\pi}{5}n) + cos(\frac{7\pi}{25}n)$$

(b) Figure 1 shows the plot of magnitude of the sampled version of $X_a(\Omega)$. It can be seen from the figure that we can distinguish between the frequencies.

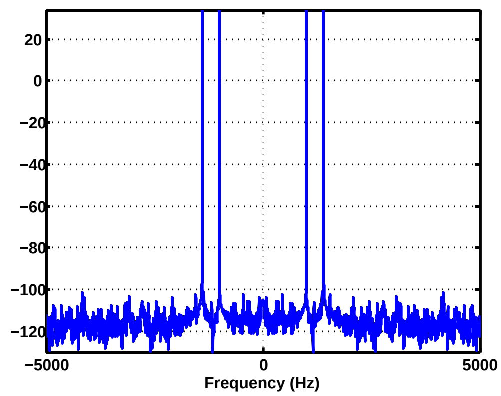

Figure 1: Problem 1b: Magnitude of the sampled version of $X_a(\Omega)$ for $N = 5000$

(c) Figures 2 and 3 show the magnitude of the sampled version of $X_a(\Omega)$ for $N = 100$ and $N = 20$ respectively. It can be seen that for we can distinguish between f_1 and f_2 for $N = 100$.

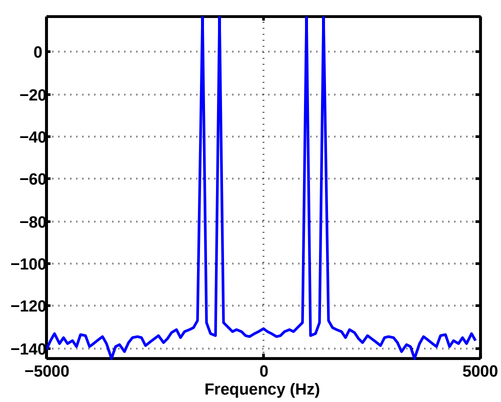

Figure 2: Problem 1c: Magnitude of the sampled version of $X_a(\Omega)$ for N=100

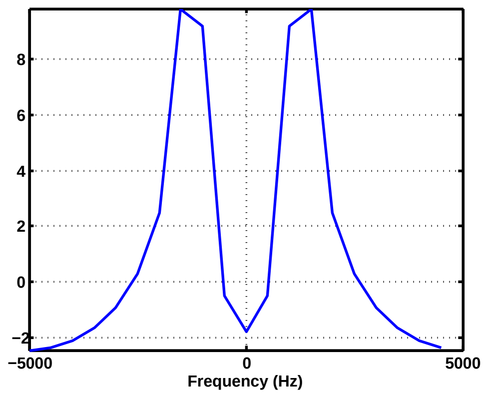

Figure 3: Problem 1c: Magnitude of the sampled version of $X_a(\Omega)$ for $N = 20$

(d) The condition that main lobes of the two functions do not overlap is that,

$$\begin{aligned} N &> \frac{1}{T} \frac{4\pi}{\Omega_1 - \Omega_0} \\ &= 10^4 \frac{4\pi}{2\pi \times 0.4 \times 10^3} \\ &= \frac{200}{4} = 50 \end{aligned}$$

The above result agrees with our observation in Parts (b) and (c).

Problem 2

(a) Plot is given in Fig. 4 below.

(b) The relation between analog frequency Ω and DFT (length-M) index k is $\Omega = \omega/T = (2\pi k/M)/T$ (need to adjust by 2π for $k = 173$).

(i)	$(2\pi 73/256)/(1/240) = 136.9\pi$	$(2\pi 173/256 - 2\pi)/(1/240) = -155.6\pi$
(ii)	$(2\pi 73/256)/(1/30) = 17.1\pi$	$(2\pi 173/256 - 2\pi)/(1/30) = -19.5\pi$
(iii)	$(2\pi 73/256)/(1/5) = 2.85\pi$	$(2\pi 173/256 - 2\pi)/(1/5) = -3.24\pi$
(iv)	$(2\pi 73/256)/(1/120) = 68.4\pi$	$(2\pi 173/256 - 2\pi)/(1/120) = -77.8\pi$
(v)	$(2\pi 73/256)/(1/30) = 17.1\pi$	$(2\pi 173/256 - 2\pi)/(1/30) = -19.5\pi$
(vi)	$(2\pi 73/256)/(1/15) = 8.55\pi$	$(2\pi 173/256 - 2\pi)/(1/15) = -9.73\pi$

(c) Consider a single sinusoidal component that is sampled with a sampling period T:

$$x[n] = A\cos(\Omega_0 n T) \ , \ \ 0 \le n \le N-1$$

The DTFT of $\{x[n]_{n=0}^{N-1}\}$ is:

$$\begin{aligned} X_d(\omega) &= \sum_{n=0}^{N-1} A\cos(\Omega_0 n T) e^{-j\omega n} \\ &= \frac{A}{2} \sum_{n=0}^{N-1} [e^{-j(\omega-\Omega_0 T)n} + e^{-j(\omega+\Omega_0 T)n}] \\ &= e^{-j(\omega-\Omega_0 T)\frac{N-1}{2}} \frac{AN}{2} \frac{\text{sinc}[(\omega-\Omega_0 T)\frac{N}{2}]}{\text{sinc}[(\omega-\Omega_0 T)\frac{1}{2}]} + e^{-j(\omega+\Omega_0 T)\frac{N-1}{2}} \frac{AN}{2} \frac{\text{sinc}[(\omega+\Omega_0 T)\frac{N}{2}]}{\text{sinc}[(\omega+\Omega_0 T)\frac{1}{2}]} \\ &= T_1(\omega) + T_2(\omega) \end{aligned}$$

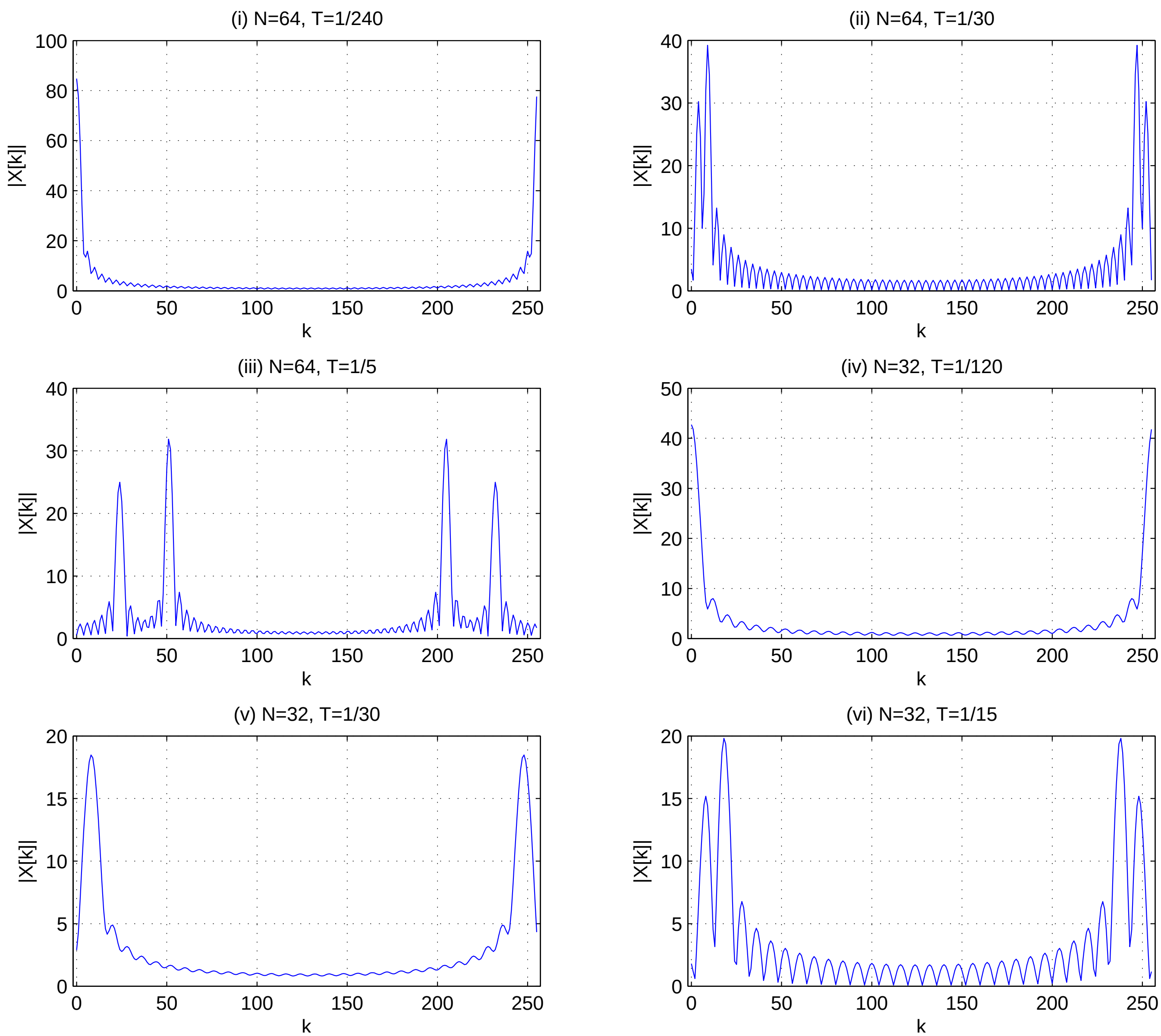

Figure 4: Problem 2a: The DFT of the given sequences.

Therefore, the DTFT of a single sinusoidal signal with a finite length consists of two periodic sincs, $T_1(\omega)$ and $T_2(\omega)$, with peaks located at $\omega = \pm\Omega_0 T$. The M-point DFT of $\{x[n]_{n=0}^{N-1}\}$ is a set of samples of the DTFT:

$$X[k] = X_d(\omega = \frac{2\pi}{M}k) \ , \ \ k = 0, \ldots, M-1$$

Effect of the number of samples N and the sampling interval T:

i) Number of samples N:
From the $X_d(\omega)$ expression obtained at the beginning of part (c), it is seen that N determines the amplitude of the two periodic sincs $(= \frac{AN}{2})$ and the width of the two mainlobes $(= \frac{4\pi}{N}$ (in radians) or $\frac{2.256}{N}$ (in number of indices)). The larger N gives the higher magnitude peaks. The smaller N gives the wider mainlobes. This hypothesis can be verified in plots of (ii) and (v) above.

ii) Sampling interval T:
The Fourier transform of $A\cos(\Omega_0 nT)$ has two deltas at $\Omega = \pm\Omega_0$ and the DTFT of the finite duration $A\cos(\Omega_0 nT)$ has two periodic sincs at $\omega = \pm(\Omega_0 T \bmod 2\pi)$ (the two periodic sincs will converge to deltas as the number of samples N increases). By taking a set of M samples of $X_d(\omega)$ on $[0, 2\pi)$, the DFT $\{X[k]_{k=0}^{M-1}\}$ is obtained (M is fixed at 256 in this problem). If $|\Omega_0 T| > \pi$, $\cos(\Omega_0 Tn) = \cos((\Omega_0 T \bmod 2\pi)n)$, where $(\Omega_0 T \bmod 2\pi) \in [-\pi, \pi]$. Then, the analog frequency higher than the Nyquist frequency ($|\Omega_0| > \pi/T$) is **wrapped** or **aliased** to a lower digital frequency and appears to correspond to a lower analog frequency. This effect can be seen in the plot of case (iii). Note: If an analog signal is bandlimited to Ω_{max}, the requirement for no aliasing ($\Omega_{max}T < \pi$) in A/D is equivalent to the Nyquist criterion ($T < \frac{1}{2f_{max}}$).

(d) There are two points to consider:

i) Using the criterion that there be no more than 50% overlap of the two peaks, then to distinguish the two frequencies it is required that

$$NT \geq \frac{2\pi}{|\Omega_1 - \Omega_2|} = \frac{2\pi}{\left|2\pi - \frac{14\pi}{15}\right|} = 1.875$$

Since $NT = 2$, $NT \geq 1.875$, this condition is satisfied.

ii) T must also satisfy the Nyquist criterion

$$T < \frac{1}{2f_{max}} = \frac{1}{2 \cdot 1} = 0.5 \text{ sec}$$

Note: $f_{max} = \frac{1}{2\pi}\max(2\pi, \frac{14\pi}{15}) = 1$. Therefore, choosing $T < 0.5$ sec would resolve the sinusoidal components in $x_a(t)$.

(e) The frequency separation condition used is $NT \geq \frac{2\pi}{|\Omega_1 - \Omega_2|}$. Therefore, for $N = 128$,

$$T \geq \frac{2\pi}{N|\Omega_1 - \Omega_2|} = \frac{2\pi}{128\left|2\pi - \frac{14\pi}{15}\right|} = \frac{15}{1024} \approx \frac{1}{64}$$

The Nyquist criterion is also required to avoid aliasing: $T < \frac{1}{2}$ since $f_{max} = 1$ Hz. Therefore, the best choice would be approximately $\frac{1}{64} \leq T \leq \frac{1}{2}$. Small T in this interval means fast sampling of the given analog signal. However, the implementation of fast sampling is usually expensive, and fast sampling is not necessary as long as no aliasing occurs. Therefore, it is desired that the sampling interval be as long as possible. So, $T = 0.5$ sec.

(f) With $T = 1/30$,

$$NT \geq \frac{2\pi}{\left|2\pi - \frac{14\pi}{15}\right|} \Rightarrow N_{min} = 57$$

Thus, $N = 57$ is needed for the separation of the two peaks. Experimenting in MATLAB, it is found that the mainlobes corresponding to the two sinusoids do not overlap for $N = 56$ and experience increase overlap for smaller N. In fact, with MATLAB, $N_{min} = 40$. Note that, as mentioned in the notes, the two peaks may be resolvable even when the mainlobes have minor overlap. Therefore, the $N = 57$ estimate for the minimum number of samples is conservative. Zoomed-in plots are shown in Fig. 5 to highlight the resolution of the sinusoids.

Problem 3

(a) The two sequences are shown below,

$$x[n] = [1, 1, 1, 1, 1, 1]$$
$$h[n] = [0, 1, 2, 3, 4, 5]$$

Flipping $h[n]$ and performing the convolution yields the following result,

$$y[n] = [0, 1, 3, 6, 10, 15, 15, 14, 12, 9, 5]$$

(b) The convolution by `conv` command can be done as follows,

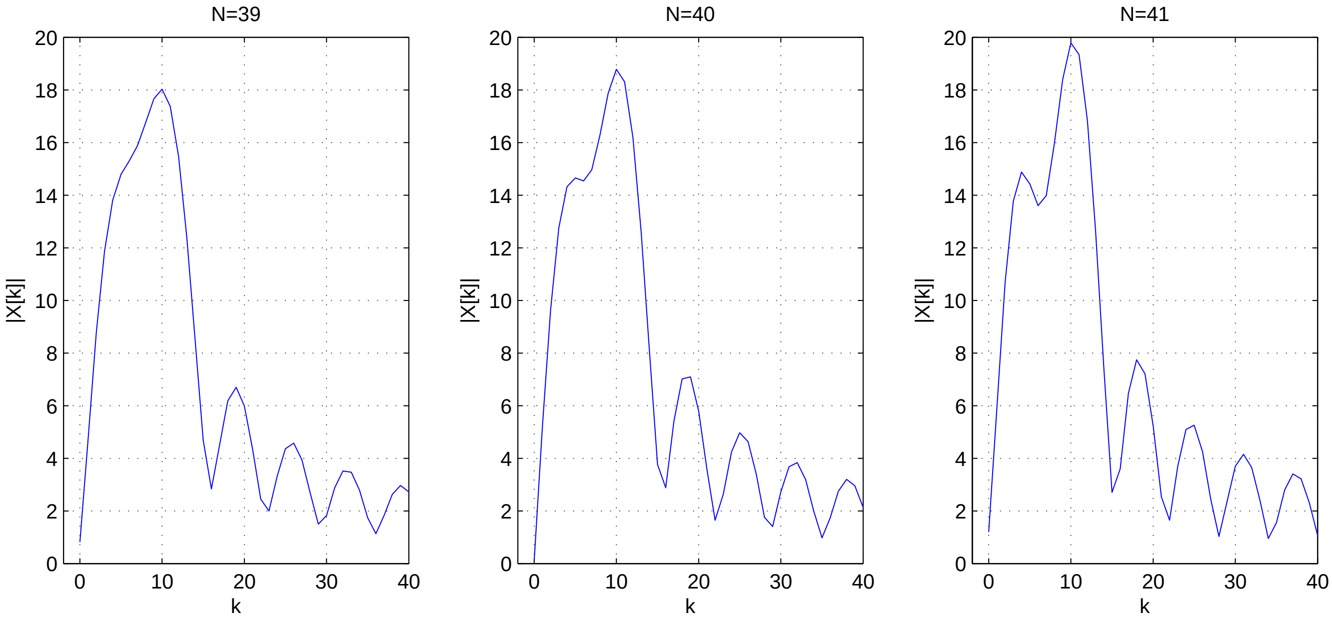

Figure 5: Problem 2f: Plot for Problem 2f

```
x = [1 1 1 1 1 1];
h = [0 1 2 3 4 5];
y = conv(x,h);
```

In this case the sequences start at $n = 0$, hence the output `y` is between $0 \leq n \leq 8$. The result can be plotted using `stem` command.

```
idx = (0:8);
stem(idx,y);
title('Convolved result for problem 3b'),xlabel('time index, n'), ylabel('convolved output, y[n]=h[n]*x[n]')
```

The plot of the convolved result is shown in Fig. 6.

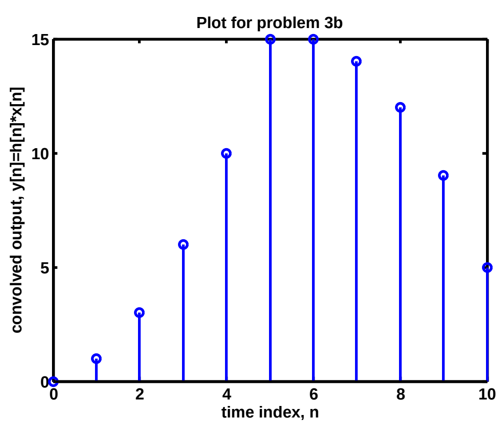

Figure 6: Problem 3: Plot for Problem 3b

(c) Note that the sequence $h[n+5]$ is advanced in time. The result will be time shifted with the same sample values. The result is shown in Fig. 7

```
idx = (0:8)-4;
stem(idx,y);
title('Convolved result for problem 3b'),xlabel('time index, n'), ylabel('convolved output, y[n]=h[n+5]*x[n]
```

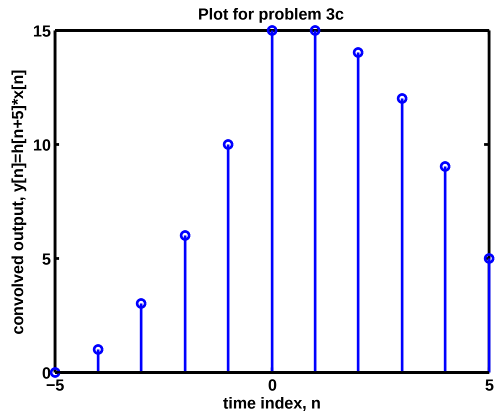

Figure 7: Problem 3: Plot for Problem 3c

Problem 4

(a) The plots for $h_1[n]$ and $h_2[n]$ are shown in Figs. 8 and 9 respectively. Note that the `filter` command can be used to compute $h_1[n]$ as shown below,

```
x = [1 zeros(1,19)];
h1 =filter([1],[1 -3/5],[1 zeros(1,19)]);
```

Please refer to the MATLAB code `func_prob4a` for computing $h_2[n]$. The function can be called by using the input `inp = [1 zeros(1,19)]` as shown below,

```
h2 = func_prob4a(inp);
```

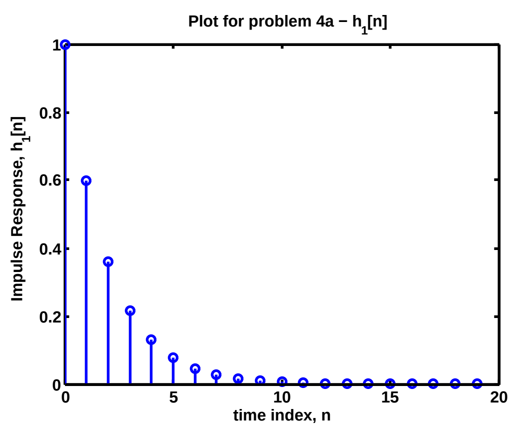

Figure 8: Problem 4a: Impulse Response $h_1[n]$

(b) The step response for System I can be calculated as follows,

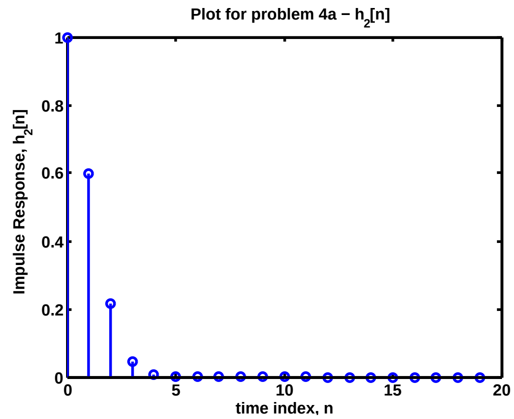

Figure 9: Problem 4a: Impulse response $h_2[n]$

```
x = [1 zeros(1,19)];
s1 =filter([1],[1 -3/5],[1 ones(1,19)]);
```

The step response for System II can be computed using the same function `func_prob4a`. The input vector is `inp = ones(1,20)`,

```
h2 = func_prob4a(inp);
```

(c) The convolution for the impulse responses computed in part (a) can be computed using the `conv` command.

```
z1 = conv(h1,ones(1,20));
z2 = conv(h2,ones(1,20));
```

(d) Figures 10 and 11 show the plots of `s1,z1` and `s2,z2`. Note that only the first 20 samples of `z1` and `z2` are shown.It can be seen that `s1,z1` are the same while `s2,z2` are different. This is because System II is a time-varying system. The `conv` and `filter` commands assume that the system is LSI. The commands for generating the plots are shown below,

```
>> x = (0:19);
>> subplot(2,1,1)
>> stem(x,s1)
>> subplot(2,1,2)
>> stem(x,z1(1:20))

>> subplot(2,1,1)
>> stem(x,s2)
>> subplot(2,1,2)
>> stem(x,z2(1:20))
```

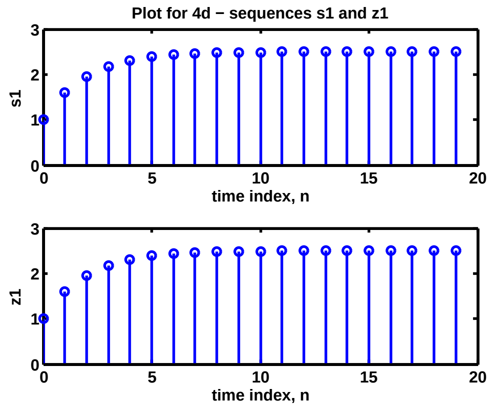

Figure 10: Problem 4a: Impulse Response $h_1[n]$

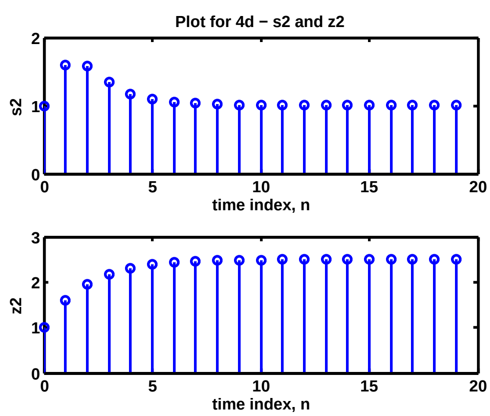

Figure 11: Problem 4a: Impulse response $h_2[n]$