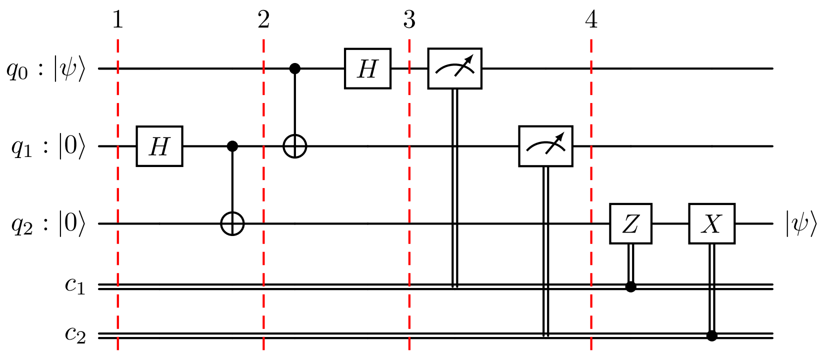
1 Quantum Teleportation Source Code

```
OPENQASM 2.0;

gate x a { U(pi,0,pi) a; }
gate z a { U(0,0,pi) a; }
gate h a { U(pi/2,0,pi) a; }

qreg q[3];
creg c0[1];
creg c1[1];
creg c2[1];

// we are trying to send q[0]

// step 0
// prepare an arbitrary qubit
U(0.3,0.2,0.1) q[0];

// step 1
// make a Bell pair
h q[1];
CX q[1],q[2];

// step 2
CX q[0],q[1];
h q[0];

// step 3
measure q[0] -> c0[0];
measure q[1] -> c1[0];

// step 4
if(c0==1) z q[2];
if(c1==1) x q[2];

// now q[0] teleported to q[2]
```

2 Quantum Teleportation Circuit



3 Formal Presentation

3.1 Preliminaries

$$
\begin{aligned}
H\left|0\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right) \\
H\left|1\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right) \\
X\left|0\right\rangle &= \left|1\right\rangle \\
X\left|1\right\rangle &= \left|0\right\rangle \\
Z\left|0\right\rangle &= \left|0\right\rangle \\
Z\left|1\right\rangle &= -\left|1\right\rangle \\
C_X\left|00\right\rangle &= \left|00\right\rangle \\
C_X\left|01\right\rangle &= \left|01\right\rangle \\
C_X\left|10\right\rangle &= \left|11\right\rangle \\
C_X\left|11\right\rangle &= \left|10\right\rangle
\end{aligned}
$$

3.2 Step 0

Alice is the one who owns $q_0 = \left|\psi\right\rangle$ and q_1 and she wants to teleport $\left|psi\right\rangle$ to Bob, who owns q_2. The teleportation protocol begins with a quantum state or qubit $\left|\psi\right\rangle$, in Alice's possession, that she wants to convey to Bob. This qubit can be written generally as:

$$\left|\psi\right\rangle_{q_0} = \alpha\left|0\right\rangle_{q_0} + \beta\left|1\right\rangle_{q_0}$$

The subscript 0 above is used only to distinguish this qubit states in the first, second and third horizontal rows of the circuit.

3.3 Step 1

Next, the protocol requires that Alice and Bob share a maximally entangled state. This state is fixed in advance, by mutual agreement between Alice and Bob, and can be any one of the four Bell states. In this circuit, Alice and Bob will share following Bell state:

$$\left|\Phi^+\right\rangle_{q_1 q_2} = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_{q_1} \otimes \left|0\right\rangle_{q_2} + \left|1\right\rangle_{q_1} \otimes \left|1\right\rangle_{q_2}\right)$$

by the process we are about to discuss. Given q_1 and q_2 both initialized to $\left|0\right\rangle$:

$$\text{Inital } q_1 \text{ and } q_2 = \left|0\right\rangle_{q_1} \otimes \left|0\right\rangle_{q_2} \ ,$$

the Hadamard gate is applied to q_1:

$$
\begin{aligned}
(H \otimes I) \cdot \left(\left|0\right\rangle_{q_1} \otimes \left|0\right\rangle_{q_2}\right) &= \left(H\left|0\right\rangle_{q_1}\right) \otimes \left(I\left|0\right\rangle_{q_2}\right) \\
&= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_{q_1} + \left|1\right\rangle_{q_1}\right) \otimes \left|0\right\rangle_{q_2} \\
&= \frac{1}{\sqrt{2}}\left(\left|00\right\rangle_{q_1 q_2} + \left|10\right\rangle_{q_1 q_2}\right)
\end{aligned}
$$

then C_X (CNOT) gate, $q1$ as a control qubit and $q2$ as a target qubit:

$$\frac{1}{\sqrt{2}}\left(\left|00\right\rangle_{q_1 q_2} + \left|10\right\rangle_{q_1 q_2}\right) \cdot C_X = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle_{q_1 q_2} + \left|11\right\rangle_{q_1 q_2}\right)$$

At this point, Alice has two particles (q_0, the one she wants to teleport, and q_1, one of the entangled pair), and Bob has one particle, q_2. In the total system, the state of these three particles is given by

$$
\begin{aligned}
|\psi\rangle_{q_0} \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{q_1 q_2} + |11\rangle_{q_1 q_2}\right) &= \left(\alpha\,|0\rangle_{q_0} + \beta\,|1\rangle_{q_0}\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{q_1 q_2} + |11\rangle_{q_1 q_2}\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right).
\end{aligned}
$$

3.4 Step 2

In step 2, we first apply C_X (CNOT) gate with q_0 as a control qubit and q_1 as a target qubit:

$$
\begin{aligned}
&(C_X \otimes I) \cdot \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle\right)
\end{aligned}
$$

then H gate to q_0:

$$
\begin{aligned}
&(H \otimes I) \cdot \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right) \\
=&\frac{1}{2}\left(\alpha\,|000\rangle + \alpha\,|100\rangle + \alpha\,|011\rangle + \alpha\,|111\rangle + \beta\,|010\rangle - \beta\,|110\rangle + \beta\,|001\rangle - \beta\,|101\rangle\right) \\
=&\frac{1}{2}\left(\alpha\,|000\rangle + \beta\,|001\rangle + \beta\,|010\rangle + \alpha\,|011\rangle + \alpha\,|100\rangle - \beta\,|101\rangle - \beta\,|110\rangle + \alpha\,|111\rangle\right).
\end{aligned}
$$

3.5 Step 3

Alice will then make two local measurement of q_0 and q_1. The measurement causes each qubit to collapse to non-superposed state. There are a total of four possible results of step 3, depending on the measurement outcomes. (Omit details about normalization.)

3.5.1 $q_0 \to c_0 = 0$, $q_1 \to c_1 = 0$

$$
\begin{aligned}
\text{resulting state} &= \frac{1}{2}\left(\alpha\,|000\rangle + \beta\,|001\rangle + \cancel{\beta\,|010\rangle} + \cancel{\alpha\,|011\rangle} + \cancel{\alpha\,|100\rangle} - \cancel{\beta\,|101\rangle} - \cancel{\beta\,|110\rangle} + \cancel{\alpha\,|111\rangle}\right) \\
&= \alpha\,|000\rangle + \beta\,|001\rangle
\end{aligned}
$$

3.5.2 $q_0 \to c_0 = 0$, $q_1 \to c_1 = 1$

$$
\begin{aligned}
\text{resulting state} &= \frac{1}{2}\left(\cancel{\alpha\,|000\rangle} + \cancel{\beta\,|001\rangle} + \beta\,|010\rangle + \alpha\,|011\rangle + \cancel{\alpha\,|100\rangle} - \cancel{\beta\,|101\rangle} - \cancel{\beta\,|110\rangle} + \cancel{\alpha\,|111\rangle}\right) \\
&= \beta\,|010\rangle + \alpha\,|011\rangle
\end{aligned}
$$

3.5.3 $q_0 \to c_0 = 1$, $q_1 \to c_1 = 0$

$$
\begin{aligned}
\text{resulting state} &= \frac{1}{2}\left(\cancel{\alpha\,|000\rangle} + \cancel{\beta\,|001\rangle} + \cancel{\beta\,|010\rangle} + \cancel{\alpha\,|011\rangle} + \alpha\,|100\rangle - \beta\,|101\rangle - \cancel{\beta\,|110\rangle} + \cancel{\alpha\,|111\rangle}\right) \\
&=\alpha\,|100\rangle - \beta\,|101\rangle
\end{aligned}
$$

3.5.4 $q_0 \to c_0 = 1$, $q_1 \to c_1 = 1$

$$
\begin{aligned}
\text{resulting state} &= \frac{1}{2}\left(\cancel{\alpha\,|000\rangle} + \cancel{\beta\,|001\rangle} + \cancel{\beta\,|010\rangle} + \cancel{\alpha\,|011\rangle} + \cancel{\alpha\,|100\rangle} - \cancel{\beta\,|101\rangle} - \beta\,|110\rangle + \alpha\,|111\rangle\right) \\
&= -\,\beta\,|110\rangle + \alpha\,|111\rangle
\end{aligned}
$$

3.6 Step 4

The result of Alice's measurement tells her which of the above four states the system is in. She can now send her result to Bob through a classical channel, c_0 and c_1. These two classical bits can communicate which of the four results she obtained.
After Bob receives the message from Alice, he will know which of the four states his particle is in. using this information, he performs a unitary operation on his particle to transform it to the desired state $\alpha \left|0\right\rangle_{q_2} + \beta \left|1\right\rangle_{q_2}$.
From now on, we only care about Bob's qubit q_2 hence omit the states for q_0 and q_1.

3.6.1 $q_0 \to c_0 = 0$, $q_1 \to c_1 = 0$

$$\begin{aligned} \text{state after step 3} &= \alpha \left|0\right\rangle_{q_2} + \beta \left|1\right\rangle_{q_2} \\ &= \left|\psi\right\rangle_{q_2} \end{aligned}$$

This is exactly the state we want. As with the circuit, Bob does not have to do anything.

3.6.2 $q_0 \to c_0 = 0$, $q_1 \to c_1 = 1$

$$\text{state after step 3} = \beta \left|0\right\rangle_{q_2} + \alpha \left|1\right\rangle_{q_2}$$

Since c_1 is 1, Bob just needs to do the X operation as represented in the circuit to get the state we want:

$$\begin{aligned} X \cdot (\text{state after step 3}) &= X \cdot \left(\beta \left|0\right\rangle_{q_2} + \alpha \left|1\right\rangle_{q_2}\right) \\ &= \alpha \left|0\right\rangle_{q_2} + \beta \left|1\right\rangle_{q_2} \\ &= \left|\psi\right\rangle_{q_2} \end{aligned}$$

3.6.3 $q_0 \to c_0 = 1$, $q_1 \to c_1 = 0$

$$\text{state after step 3} = \alpha \left|0\right\rangle_{q_2} - \beta \left|1\right\rangle_{q_2}$$

Since c_0 is 1, Bob just needs to do the Z operation as represented in the circuit to get the state we want:

$$\begin{aligned} Z \cdot (\text{state after step 3}) &= Z \cdot \left(\alpha \left|0\right\rangle_{q_2} - \beta \left|1\right\rangle_{q_2}\right) \\ &= \alpha \left|0\right\rangle_{q_2} + \beta \left|1\right\rangle_{q_2} \\ &= \left|\psi\right\rangle_{q_2} \end{aligned}$$

3.6.4 $q_0 \to c_0 = 1$, $q_1 \to c_1 = 1$

$$\text{state after step 3} = -\beta \left|0\right\rangle_{q_2} + \alpha \left|1\right\rangle_{q_2}$$

Since both c_0 and c_1 are 1, Bob needs to do the Z then also X operation as represented in the circuit to get the state we want:

$$\begin{aligned} X \cdot Z \cdot (\text{state after step 3}) &= X \cdot Z \cdot \left(-\beta \left|0\right\rangle_{q_2} + \alpha \left|1\right\rangle_{q_2}\right) \\ &= X \cdot \left(-\beta \left|0\right\rangle_{q_2} - \alpha \left|1\right\rangle_{q_2}\right) \\ &= \left(-\alpha \left|0\right\rangle_{q_2} - \beta \left|1\right\rangle_{q_2}\right) \\ &= -\left(\alpha \left|0\right\rangle_{q_2} + \beta \left|1\right\rangle_{q_2}\right) \\ &= -\left|\psi\right\rangle_{q_2} \end{aligned}$$

The global phase in quantum mechanics does not matter because it does not affect observable properties of the system due to the cancellation effect in the computation of expectation values. Therefore, we can say that $\left|\psi\right\rangle$ and $-\left|\psi\right\rangle$ are same.