Name: Kai Chuen Tan

Course: CSE 276C

HW#: Homework 2

Date: 17th October 2021

Professor: Dr. Christensen

1.) Prove that the first derivative $p_2'(x)$ of the parabola interpolating $f(x)$ at $x_0 < x_1 < x_2$ is equal to the straight line which takes on the value $f[x_{i-1}, x_i]$ at the point $\frac{(x_{i-1} + x_i)}{2}$, for $i = 1, 2$.

Apply Lagrange Polynomial Interpolation:

Consider the 2th degree polynomial factored:

$$p(x) = a_0 L_0(x) + a_1 L_1(x) + a_2 L_2(x)$$

$$= a_0\left[\frac{x-x_1}{x_0-x_1}\left(\frac{x-x_2}{x_0-x_2}\right)\right] + a_1\left[\frac{x-x_0}{x_1-x_0}\left(\frac{x-x_2}{x_1-x_2}\right)\right] + a_2\left[\frac{x-x_0}{x_2-x_0}\left(\frac{x-x_1}{x_2-x_1}\right)\right]$$

$$= \frac{a_0}{(x_0-x_1)(x_0-x_2)}(x-x_1)(x-x_2) + \frac{a_1}{(x_1-x_0)(x_1-x_2)}(x-x_0)(x-x_2) + \frac{a_2}{(x_2-x_0)(x_2-x_1)}(x-x_0)(x-x_1)$$

$$= \frac{a_0}{(x_0-x_1)(x_0-x_2)}\left(x^2-(x_1+x_2)x + x_1x_2\right) + \frac{a_1}{(x_1-x_0)(x_1-x_2)}\left(x^2-(x_0+x_2)x + x_0x_2\right)$$

$$+ \frac{a_2}{(x_2-x_0)(x_2-x_1)}\left(x^2-(x_1+x_0)x + x_1x_0\right)$$

$$= \left[\frac{a_0}{(x_0-x_1)(x_0-x_2)} + \frac{a_1}{(x_1-x_0)(x_1-x_2)} + \frac{a_2}{(x_2-x_0)(x_2-x_1)}\right]x^2$$

$$- \left[\frac{a_0(x_1+x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{a_1(x_0+x_2)}{(x_1-x_0)(x_1-x_2)} + \frac{a_2(x_1+x_0)}{(x_2-x_0)(x_2-x_1)}\right]x$$

$$+ \left[\frac{a_0(x_1x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{a_1(x_0x_2)}{(x_1-x_0)(x_1-x_2)} + \frac{a_2(x_1x_0)}{(x_2-x_0)(x_2-x_1)}\right]$$

Let $A_0 = \left[\frac{a_0}{(x_0-x_1)(x_0-x_2)} + \frac{a_1}{(x_1-x_0)(x_1-x_2)} + \frac{a_2}{(x_2-x_0)(x_2-x_1)}\right]$, $A_1 = [$

Apply Newton Polynomial Interpolation:

$$p_k(x) = \sum_{j=0}^{k} a_j n_j(x) \;;\; n_j(x) = \prod_{i=0}^{j-1}(x-x_i) \text{ for } j>0 \;;\; a_j = f[x_0, x_1, \dots, x_j] \;;\; n_0(x) = 1$$

Where $f[x_0, x_1, \dots, x_j]$ is the divided differences notation

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1.) Continue (Part 2)

Consider the 2th degree polynomial factored with Newton Polynomial Interpolation, k=2

$$p_2(x) = \cancel{\sum_{j=0}^{2} a_j \prod_{i}^{j-1}} \quad a_0 n_0(x) + \sum_{j=1}^{2} a_j \prod_{i=0}^{j-1} (x - x_i)$$

$$= a_0(1) + a_1(x - x_0) + a_2(x-x_0)(x-x_1)$$

$$= a_0 + a_1 x - a_1 x_0 + a_2 x^2 - (a_2(x_0+x_1))x + a_2 x_0 x_1$$

$$p_2(x) = (a_0 - a_1 x_0 + a_2 x_0 x_1) + (a_1 - a_2(x_0+x_1))x + a_2 x^2$$

1st Derivative of $p(x)$ is the following:

$$p_2'(x) = [a_1 - a_2(x_0+x_1)] + 2a_2 x^2$$

Let $i=1$, $x = \frac{x_0+x_1}{2}$

Sub x to $p_2'(x)$

$$p_2'\left(\frac{x_0+x_1}{2}\right) = a_1 - \cancel{a_2(x_0+x_1)} + \cancel{2} a_2\left(\frac{\cancel{x_0+x_1}}{\cancel{2}}\right)$$

$$= a_1 \quad ; a_1 = f[x_0, x_1]$$

$$\boxed{p_2'\left(\frac{x_0+x_1}{2}\right) = f[x_0, x_1]}$$

Let $i=2$, $x = \frac{x_1+x_2}{2}$

Sub x to $p_2'(x)$

$$p_2'\left(\frac{x_1+x_2}{2}\right) = a_1 - a_2(x_0+x_1) + \cancel{2} a_2\left(\frac{x_1+x_2}{\cancel{2}}\right)$$

$$= a_1 + a_2(\cancel{x_1} + x_2 - x_0 - \cancel{x_1})$$

$$= a_1 + a_2(x_2 - x_0)$$

$$= f[x_0, x_1] + f[x_0, x_1, x_2](x_2 - x_0)$$

$$= f[x_0, x_1] + \frac{f[x_1, x_2] - f[x_0, x_1]}{\cancel{x_2 - x_0}} \cancel{(x_2 - x_0)}$$

$$\boxed{p_2'\left(\frac{x_1+x_2}{2}\right) = f[x_1, x_2]}$$

~~∴ The 1st derivative $p_2'(x)$ of the pro~~

∴ The 1st derivative $p_2'(x)$ of the parabola interpolating $f(x)$ at $x_0 < x_1 < x_2$ is equal to the straight line which takes on the value $f[x_{i-1}, x_i]$ at the point $\frac{x_{i-1}+x_i}{2}$, for $i \in \{1, 2\}$

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Problem 2

$f(x) = \cos x$; $0 \le x \le \pi$

The errors function in polynomial interpolation, $e(x)$ is defined in the following:

$$e_n(x) = f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i) \quad \text{———} \quad ①$$

where, n is the degree of the polynomial

$\xi_x \in [a, b]$, which means some ξ_x is between the minimum and maximum of a, b, and x.

$p_n(x)$ is the n^{th} degree polynomial interpolating $f(x)$ at a and b.

If we use linear interpolation between adjacent points, let $p_1(x)$ be the linear polynomial interpolating $f(x)$ at x_i and x_{i+1}, $\forall\ 0 \le i \le n$.

Then, the errors function from equation ① above can be written as follows:

$$e_1(x) = f(x) - p_1(x) = \frac{1}{2!} f''(\xi_x)\ (x - x_0)(x - x_1)$$

Since $p_1(x)$ is just an approximation of $f(x)$, $x \in [x_0, x_1]$, the error bound can be written as follows:

$$|e_1(x)| = |f(x) - p_1(x)| \le \frac{1}{2} \max_{x_0 \le x \le x_1} |f''(\xi_x)|\ (x - x_0)(x - x_1) \quad , \quad x \in [x_0, x_1] \ \text{———} \ ②$$

First, maximize the following expression:

$$\max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)| = \max_{x_0 \le x \le x_1} |x^2 - (x_0 + x_1)x + x_0 x_1|$$

; $d(x) = 0$ and find x to get $\max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)|$

$$d(x) = 2x - (x_0 + x_1) = 0$$

$$x = \frac{x_0 + x_1}{2}$$

$$(x - x_0)(x - x_1) = \left(\frac{x_0 + x_1}{2} - x_0\right)\left(\frac{x_0 + x_1}{2} - x_1\right)$$

$$= -\left(\frac{x_1 - x_0}{2}\right)^2$$

Let $h = x_1 - x_0$, then $(x - x_0)(x - x_1) = -\frac{h^2}{4}$

$$\max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)| = \left|-\frac{h^2}{4}\right| = \frac{h^2}{4} \quad \text{———} \quad ③$$

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Problem 2 (Part 2)

Second, maximize the following expression:

$$\max_{x_0 \le x \le x_1} \left| f''(x) \right| \quad ; \quad f(x) = \cos(x) \; ; \; f'(x) = -\sin(x) \; ; \; f''(x) = -\cos(x)$$

$$\max_{x_0 \le x \le x_1} \left| f''(x) \right| = \max_{x_0 \le x \le x_1} \left| -\cos(x) \right| = 1 \quad \text{— } 4$$

Sub 3 & 4 to 2

$$\left| e_1(x) \right| \le \frac{1}{2}(1)\left(\frac{h^2}{4}\right)$$

$$\left| e_1(x) \right| \le \frac{h^2}{8} \quad ; \quad h = x_1 - x_0$$

To obtain 6 decimal digit accuracy,

$$\left| e_1(x) \right| \le 5 \times 10^{-7}$$

$$\frac{h^2}{8} \le 5 \times 10^{-7}$$

$$h \le 0.002$$

To obtain the number of interval, denoted as k, needed to get 6 decimal digit accuracy,

$$k = \frac{\pi - 0}{h} = \frac{\pi - 0}{0.002}$$

$$k = 1570.80 \approx 1571$$

Hence, 1572 entries are needed to obtain 6 decimal digit accuracy for the cheap robot.

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Problem 2 (Part 3)

If we use quadratic interpolation between adjacent points, let $p_2(x)$ be the 2^{nd} degree polynomial interpolating $f(x)$ at x, x_0, x_1, and x_2.

Then, the errors function from equation ① from Part 1 of Problem 2 can be written as follows:

$$e_2(x) = f(x) - p_2(x) = \frac{1}{3!} f''(\xi_x)(x-x_0)(x-x_1)(x-x_2)$$

Since $p_2(x)$ is just an approximation of $f(x)$, $x \in [x_0, x_2]$, the error bound can be written as follows:

$$|e_2(x)| \leq \frac{1}{6} \max_{x_0 \leq x \leq x_2} |f''(\xi_x)| (x-x_0)(x-x_1)(x-x_2), \quad x \in [x_0, x_2] \quad \text{—— ⑤}$$

First, maximize the following expression:

$$\max_{x_0 \leq x \leq x_2} |(x-x_0)(x-x_1)(x-x_2)| \quad ;$$

Let $h = x_1 - x_0$ and $h = x_2 - x_1$

then $x_1 = x_0 + h$ and $x_2 = x_1 + h$

Consider $x = x_1 + th$,

then $x - x_0 = (x_1 + th) - (x_1 - h)$

$x - x_0 = h(t+1)$,

$x - x_1 = (x_1 + th) - x_1$

$x - x_1 = th$,

$x - x_2 = (x_1 + th) - (h + x_1)$

$x - x_2 = h(t-1)$

Lower Bound: $x_0 = x_1 + th$

$x_0 - x_1 = th$

$t = -1$

Upper Bound: $x_2 = x_1 + th$

$t = 1$

$d(x) = 0$ and find x to get $\max_{x_0 \leq x \leq x_2} |(x-x_0)(x-x_1)(x-x_2)|$

$$(x-x_0)(x-x_1)(x-x_2) = (x^2 - (x_0+x_1)x + x_0x_1)(x-x_2)$$
$$= x^3 - (x_0+x_1)x^2 + x_0x_1x - x_2x^2 + x_2(x_0+x_1)x - x_0x_1x_2$$
$$= x^3 - (x_0+x_1+x_2)x^2 + (x_0x_1 + x_0x_2 + x_1x_2)x - x_0x_1x_2$$

$$d(x) = 3x^2 - 2(x_0+x_1+x_2)x + (x_0x_1 + x_0x_2 + x_1x_2) = 0$$

$$x = \frac{2(x_0+x_1+x_2) \pm \sqrt{4(x_0+x_1+x_2)^2 - 12(x_0x_1 + x_0x_2 + x_1x_2)}}{6}$$

$$x = \frac{2(x_0+x_1+x_2) \pm \sqrt{4(x_0^2+x_1^2+x_2^2+2x_0x_1+2x_0x_2+2x_1x_2) - 12(x_0x_1+x_0x_2+x_1x_2)}}{6}$$

$$x = \frac{2(x_0+x_1+x_2) \pm \sqrt{4(x_0^2+x_1^2+x_2^2) - 4(x_0x_1+x_0x_2+x_1x_2)}}{6}$$

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Problem 2 (Part 4)

$$x = \frac{(x_0+x_1+x_2) \pm \sqrt{x_0(x_0-x_1)+x_1(x_1-x_2)+x_2(x_2-x_1)}}{3}$$; $h = x_1 - x_0$, $h = x_2 - x_1$

$$= \frac{(x_0+x_1+x_2) \pm \sqrt{-x_0h - x_1h + x_2h}}{3}$$

$$= \frac{(x_0+x_1+x_2) \pm \sqrt{h(x_2-x_1-x_0)}}{3}$$

$$= \frac{(x_0+x_1+x_2) \pm \sqrt{h(h-x_0)}}{3}$$

$$(x-x_0)(x-x_1)(x-x_2) = (x_0$$

$$\max_{x_0 \le x \le x_2} \left| (x-x_0)(x-x_1)(x-x_2) \right| = \max_{-1 \le t \le 1} \left| [h(t+1)]\, ht\, [h(t-1)] \right|$$

$$= h^3 \max_{-1 \le t \le 1} \left| t^3 - t \right|$$; $d(t) = 0$ find t; $d(t) = 3t^2 - 1 = 0$; $t = \pm\sqrt{\frac{1}{3}}$; $t = \pm\frac{1}{\sqrt{3}}$

$$= h^3 \max_{-1 \le t \le 1} \left| \left(-\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}} \right|$$

$$\max_{x_0 \le x \le x_2} \left| (x-x_0)(x-x_1)(x-x_2) \right| = \frac{2\sqrt{3}}{9} h^3 \text{ --- } 6$$

Second, maximize the following expression :

$$\max_{x_0 \le x \le x_2} \left| f'''(x) \right| \; ; \; f''(x) = -\cos(x) \; ; \; f'''(x) = \sin(x)$$

$$\max_{x_0 \le x \le x_2} \left| \sin(x) \right| = 1 \text{ --- } 7$$

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Problem 2 (Part 5)

Sub ⑥ & ⑦ to ⑤,

$$|e_2(x)| \le \frac{1}{\cancel{6}_3}(1)\left(\frac{\cancel{2}^1\sqrt{3}}{9}\right)h^3$$

$$|e_2(x)| \le \frac{\sqrt{3}}{27}h^3$$

To obtain 6 decimal digit accuracy,

$$|e_2(x)| \le 5\times10^{-7}$$

$$\frac{\sqrt{3}}{27}h^3 \le 5\times10^{-7}$$

$$h \le 0.019827$$

To obtain the number of interval, denoted as k, needed to get 6 decimal accuracy,

$$k = \frac{\pi - 0}{h} = \frac{\pi - 0}{0.019827}$$

$$k = 158.45 \approx 159$$

Hence, 160 entries are needed to obtain 6 decimal digit accuracy for the cheap robot.

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Problem 4

Planets	Mercury	Venus	Earth	Mars	Jupiter
Distance From the Sun (10^6 km), x	58	108	149.5	227	778
Days In a Planet Year , y	88	224.7	365.3	687	4331.5

Find the function with the Lagrange Method.

$$f(x) = \sum_{i=0}^{n} y_i L_i \quad ; \quad L_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x-x_j)}{(x_i - x_j)}$$

$$f(x) = \frac{(x-108)(x-149.5)(x-227)(x-778)}{(58-108)(58-149.5)(58-227)(58-778)}(88) + \frac{(x-58)(x-149.5)(x-227)(x-778)}{(108-58)(108-149.5)(108-227)(108-778)}(224.7)$$

$$+ \frac{(x-58)(x-108)(x-227)(x-778)}{(149.5-58)(149.5-108)(149.5-227)(149.5-778)}(365.3) + \frac{(x-58)(x-108)(x-149.5)(x-778)}{(227-58)(227-108)(227-149.5)(227-778)}(687)$$

$$+ \frac{(x-58)(x-108)(x-149.5)(x-227)}{(778-58)(778-108)(778-149.5)(778-227)}(4331.5)$$

$$f(x) = \frac{11(x-108)(x-149.5)(x-227)(x-778)}{69585750} - \frac{321(x-58)(x-149.5)(x-227)(x-778)}{236342500}$$

$$+ \frac{3653(x-58)(x-108)(x-227)(x-778)}{1849593009} - \frac{1374(x-58)(x-108)(x-149.5)(x-778)}{1717579955}$$

$$+ \frac{8663(x-58)(x-108)(x-149.5)(x-227)}{334113616800}$$

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Problem 4 (Part 2)

$$f(x) = \frac{11(x^2-257.5x+16146)(x^2-1005x+176606)}{69585750} - \frac{321(x^2-207.5x+8671)(x^2-1005x+176606)}{236342500}$$

$$+ \frac{3653(x^2-166x+6264)(x^2-1005x+176606)}{1849593009} - \frac{1374(x^2-166x+6264)(x^2-927.5x+116311)}{1717579955}$$

$$+ \frac{8663(x^2-166x+6264)(x^2-376.5x+33936.5)}{334113616800}$$

$$f(x) = \frac{11(x^4-1262.5x^3+451539.5x^2-61702775x+2851480476)}{69585750}$$

$$- \frac{321(x^4-1212.5x^3+393814.5x^2-45360100x+1531350626)}{236342500}$$

$$+ \frac{3653(x^4-1171x^3+349700x^2-35611916x+1106259984)}{1849593009}$$

$$- \frac{1374(x^4-1093.5x^3+276540x^2-25117486x+728572104)}{1717579955}$$

$$+ \frac{8663(x^4-542.5x^3+102699.5x^2-7991855x+212578236)}{334113616800}$$

$$\boxed{f(x) = \frac{7686468199051}{87885295406786896200000}x^4 - \frac{3391858202199959}{7030823632542951696 00}x^3 + \frac{1513243542455773777229}{17577059081357379240000}x^2 + \frac{98811891118626464159}{703082363254295169 60}x - \frac{1395061566376816 4967}{6475485956880850 00}}$$

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Problem 5

$f'(x) = e^{x} - 4$

$f(0) = 2$

Alice's internet speed function, $f(x)$

Use the bisection method to help Alice figure out when her internet will stop working.

$$f(x) = \int f'(x)\,dx$$

$$= \int e^{x} - 4\,dx$$

$f(x) = e^{x} - 4x + c$; $f(0) = 2$

$f(0) = 1 + c$

$2 = 1 + c$

$c = 1$

$f(x) = e^{x} - 4x + 1$