Name: Kai Chuen Tan
PID: A59011493
Course: ECE 276A
Assignment #: HW1
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Professor: Dr. Atanasov

Problem 4

$X \sim \mathcal{N}(\underbrace{\mu}_{\text{unknown}}, \overbrace{\sigma^2}^{\text{known}})$

$\{x_i\}_{i=1}^{n}$, where n is an independent samples obtained from X.

a.) Formulate a maximum likelihood estimation (MLE) problem to determine the unknown mean of X.

Apply Likelihood Function: $L(x_1,\ldots,x_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} f_x(x_i \mid \mu, \sigma^2)$; $f_x(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$L(x_1,\ldots,x_n \mid \mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Take the natural logarithm of the likelihood function,

$$\ell(x_1,\ldots,x_n \mid \mu, \sigma^2) = \ln(L(x_1,\ldots,x_n \mid \mu, \sigma^2))$$

$$= \ln\left((2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}\right)$$

$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

To $\max_{\mu,\sigma^2} \ell(x_1,\ldots,x_n \mid \mu, \sigma^2)$, $\frac{\partial}{\partial\sigma^2}\ell(x_1,\ldots,x_n \mid \mu, \sigma^2) = 0$

$$\frac{\partial}{\partial\sigma^2}\left(-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$= -\frac{n}{2\sigma^2} + \left(\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right)\left(\frac{1}{\sigma^4}\right)$$

$$\boxed{\frac{\partial \ell}{\partial\sigma^2} = \frac{1}{2\sigma^2}\left[\frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu)^2\right] - n\right]} = 0$$

$$\boxed{\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\frac{\partial}{\partial\mu}\ell(\mu,\sigma^2 \mid x_1,\ldots,x_n) = 0 \Rightarrow \frac{\partial}{\partial\mu}\left(-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$\boxed{\frac{\partial \ell}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)} = 0$$

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b.) Solve the problem in part (a) to obtain maximum likelihood estimate $\hat{u}_{MLE}$

$$\frac{\partial l}{\partial u} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - u) = 0$$

$$\left(\sum_{i=1}^{n} x_i - nu\right) = 0$$

$$\boxed{\hat{u}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

c.) Formulate a maximum a posteriori (MAP) problem to determine the unknown mean of X. Suppose that a prior Gaussian distribution $\mathcal{N}(u_o, \sigma_o^2)$ with known u_o and σ_o^2 is available.

Given that $f(u) = \frac{1}{\sqrt{2\pi\sigma_o^2}} e^{-\frac{(u-u_o)^2}{2\sigma_o^2}}$ → prior probability function, apply Bayes' Rule:

$$f(u \mid x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n \mid u)\, f(u)}{h(x_1, \ldots, x_n)} \quad ; \quad H = h(x_1, \ldots, x_n)$$

$$= \frac{\left[(2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - u)^2}\right](2\pi\sigma_o^2)^{-\frac{1}{2}} e^{-\frac{(u-u_o)^2}{2\sigma_o^2}}}{H}$$

To $\max_u \ln(f(u \mid x_1, \ldots, x_n)) = \left(\sum_{i=1}^{n} -\ln(\sqrt{2\pi\sigma^2}) - \frac{(x_i - u)^2}{2\sigma^2}\right) - \ln(\sqrt{2\pi\sigma_o^2}) - \frac{(u-u_o)^2}{2\sigma_o^2}$

$$\boxed{\frac{\partial \ln(f(u \mid x_1, \ldots, x_n))}{\partial u} = \left(\sum_{i=1}^{n} \frac{x_i - u}{\sigma^2}\right) - \frac{u - u_o}{\sigma_o^2}} = 0$$

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d.) Solve the problem in part (c) to obtain the MAP estimate $\hat{u}_{MAP}$.

$$\frac{\partial \ln(f(u \mid x_1, \ldots, x_n))}{\partial u} = 0$$

$$\left(\sum_{i=1}^{n} \frac{x_i - u}{\sigma^2}\right) - \frac{u - u_0}{\sigma_0^2} = 0$$

$$\frac{u - u_0}{\sigma_0^2} = \sum_{i=1}^{n} \frac{x_i - u}{\sigma^2}$$

$$= \frac{1}{\sigma^2}\left[\left(\sum_{i=1}^{n} x_i\right) - nu\right]$$

$$\frac{(\sigma^2 + n\sigma_0^2)u}{\cancel{\sigma^2\sigma_0^2}} = \frac{\left(\sigma_0^2 \sum_{i=1}^{n} x_i\right) + \sigma^2 u}{\cancel{\sigma^2\sigma_0^2}}$$

$$\boxed{\hat{u}_{MAP} = \frac{\left(\sigma_0^2 \sum_{i=1}^{n} x_i\right) + \sigma^2 u}{\sigma^2 + n\sigma_0^2}}$$