Lecture 18: Transitive Closure

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CS 135 Discrete Structures Spring 2015

Path in a digraph

Vertices: user IDs in Facebook
Edges: $u \to v$ if user v is friend of u
A *path* is a chain $u \to u' \to \ldots \to v$

Suppose FRN is the set of edges, i.e., $(u, v) \in FRN$ means that v is a friend of u.

Then $(u, v) \in FRN^+$ means either $(u, v) \in FRN$
or there's w such that $(u, w) \in FRN$ and $(w, v) \in FRN^+$

FRN^+ is the transitive closure of FRN

Alert: Rosen uses R^* for R^+

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Subways

Let $Stops$ be the set of subway stations in NYC.

Let $adj(s, t)$ mean that station t is one stop away from s. So $adj \subseteq Stops \times Stops$. What is adj^n?

Let $R \subseteq T \times T$. Define
$R^0 = id$ (the identity relation on T)
$R^{n+1} = R \circ R^n$ (what about $R^{n+1} = R^n \circ R$?)

Note that $id \circ R = R$ and $R \circ id = R$ and $R^1 = R$ (for any R).

Define R^+ by $(s, t) \in R^+$ iff there is $n \geqslant 1$ such that $(s, t) \in R^n$.

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Properties of relations, algebraically

What does $id \subseteq R$ say about R?
What does $R^{-1} \subseteq R$ say?

What does $R \circ R \subseteq R$ say about R?

What does $R \cap R^{-1} \subseteq id$ say about R?

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Heaps and UML models

Define R^* by $(s, t) \in R^*$ iff there is $n \geqslant 0$ such that $(s, t) \in R^n$. So $R^* = R^+ \cup id$.

Let $f(x, y)$ mean that the f field of object x has value y. What is f^*? What is $f \cup g \cup h$, where f, g, h are field names. What is $(f \cup g \cup h)^*$?

For example, $(root, x) \in (left \cup right)^*$ means that x is somewhere in the tree rooted at $root$. What about $(root, x) \in (val \circ (left \cup right)^*)$?

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Algorithms for transitive closure

Suppose $R \subseteq T \times T$ and T is a finite set.

$R^0 = id$ and $R^{n+1} = R \circ R^n$ Exercise: (ncomp rel n)

$(s, t) \in R^+$ iff $\exists i\ (i \geqslant 1 \wedge (s, t) \in R^i)$
$(s, t) \in R^*$ iff $\exists i\ (i \geqslant 0 \wedge (s, t) \in R^i)$

Algorithm to check whether (s, t) is in R^+
For i from 1 up to $size(T)$, check whether (s, t) is in (ncomp R i)
Time complexity?

Using matrix: initialize M to represent R and k to 1. Invariant: M represents $\cup_{i=1}^{k} R^i$.
That is, M represents R_i defined by $R_{i+1} = R_i \cup (R \circ R_i)$

(See textbook for Warshall's algorithm.)

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