MA 234

5/5/18

Homework 12

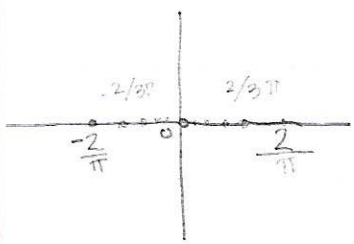
1. a. Let $f(z) = \cos\left(\frac{1}{z}\right)$. Find Laurent series of f around $z_0 = 0$.

$$\sum_{n=0}^{\infty} c_n z^{-n} \qquad c_n = \begin{cases} 0 & k = 2n+1 \\ \frac{(-1)^n}{k!} & k = 2n \end{cases}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n (2n)!} = 1 - \frac{1}{2!\, z^2} + \frac{1}{4!\, z^4} + \cdots}$$

b. Let $g(z) = \dfrac{1}{\cos\left(\frac{1}{z}\right)}$. Find all singularities of g & state if they are isolated or not.

$z = 0, \quad \frac{1}{z} = \frac{(2n+1)\pi}{2} \Rightarrow z = \frac{2}{(2n+1)\pi}$



$$\boxed{z = 0, \; z = \pm\frac{2}{\pi}, \pm\frac{2}{3\pi}, \pm\frac{2}{5\pi}, \ldots}$$

Not isolated singularities

c. Find 2 sequences $\{z_n\}$ & $\{w_n\}$ such that $\lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0$, but $\lim_{n\to\infty} |f(z_n)| = 0$, $\lim_{n\to\infty} |f(w_n)| = \infty$ for all $n \in \mathbb{N}$.

2. a. Let $f(z) = z + \frac{1}{z}$. Show that f has a pole of order 1 at 0 and zeroes of order 1 at i and -i.

- at $z=0$: f behaves like $\frac{1}{z}$

  $g(z) = z \Rightarrow$ zero of order 1 $\Rightarrow 1/z$ is pole of order 1 at 0

- at $z = i$: $i + \frac{1}{i} = i - i = 0$

  $f'(z) = 1 - 1/z^2 \neq 0$

  $f(z)$ has zero of order 1 at i

- at $z = -i$: $-i + \frac{1}{-i} = -i + i = 0$

  $f'(z) = 1 - 1/z^2 \neq 0$

  $f(z)$ has zero of order 1 at $-i$

b. Let $g(z) = \dfrac{1}{z + 1/z}$. Find all singularities of g & state if they are removable, poles (w/ order), or essential.

$* \; z=0, \; z=i, \; z=-i$

$$g(z) = \frac{z}{z^2+1} = \frac{1}{z + 1/z}$$

$z = i$: $\lim_{z \to i} \left| \frac{z}{z^2+1} \right| = \lim_{z \to i} \frac{|z|}{|z^2+1|} = \frac{i}{0}$

$\exists \delta$ s.t. $|z| > 1/2$

$$\left| \frac{z}{z^2+1} \right| > \frac{1/2}{|z^2+1|} \quad \forall z \in D_\delta(i)$$

$$\lim_{z \to i} \left| \frac{z}{z^2+1} \right| \geq \frac{1}{2|z^2+1|} = \infty + \rightarrow \text{pole order 1}$$

$z = -i$: same logic as $z = i \rightarrow$ pole order 1

$z = 0$: $\lim_{z \to 0} \left| \frac{z}{z^2+1} \right| = \lim_{z \to 0} \frac{|z|}{|z^2+1|} = \frac{0}{1} = 0$

$z=0$ is a removable singularity

c. Let $h(z) = \begin{cases} g(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$ Show h is analytic in $D_1(0)$

$$\lim_{z \to 0} \frac{h(z) - h(0)}{z - z} = \lim_{z \to 0} z \cdot g(z)$$

$$\lim_{z \to 0} |z \cdot g(z)| = \lim_{z \to 0} |z| \underbrace{|g(z)|}_{=0} = 0$$

$\Rightarrow g'$ exists,

$g'(z) = 0$ $\therefore$ Analytic

I pledge my honor that I have abided by the Stevens honor system.