MA232 Linear Algebra

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Eigenvectors and eigenvalues.

- Suppose we can *diagonalize* A, i.e. for some diagonal matrix D

$$A = SDS^{-1}$$

- Algebraic operations can be performed very efficiently

$$\begin{aligned} A^m &= (SDS^{-1})^m = (SDS^{-1})(SDS^{-1})\cdots(SDS^{-1}) = SD^m S^{-1} \\ A + B &= SD_1 S^{-1} + SD_2 S^{-1} = S(D_1 + D_2)S^{-1} \\ cA &= c(SDS^{-1}) = ScDS^{-1} \end{aligned}$$

- In fact, for any polynomial function $f(t)$

$$f(A) = Sf(D)S^{-1} = S\ [diag(f(s_{11})\ f(s_{22})\ \cdots\ f(s_{nn}))]\ S^{-1}$$

Eigenvectors and eigenvalues.

A nonzero vector $\bar{\mathbf{x}} \neq \bar{\mathbf{0}}$ is called an eigenvector of matrix A if

$$A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$$

for some scalar λ, called eigenvalue of A corresponding to $\bar{\mathbf{x}}$

– Multiplication by A does not change the direction of $\bar{\mathbf{x}}$.

Eigenvectors and eigenvalues.

- $A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}} \quad \Rightarrow \quad (A - \lambda I)\bar{\mathbf{x}} = \bar{\mathbf{0}}$
- $\bar{\mathbf{x}} \in N(A - \lambda I)$ null space of $A - \lambda I$
- If we know λ, then we can solve for $\bar{\mathbf{x}}$

Eigenvectors and eigenvalues.

- Eigenvector $\bar{\mathbf{x}} \neq \bar{\mathbf{0}}$
- $(A - \lambda I)\bar{\mathbf{x}} = \bar{\mathbf{0}}$ has a non-zero solution if $A - \lambda I$ is singular
- Therefore if λ is eigenvalue then

$$\det(A - \lambda I) = 0$$

Eigenvectors and eigenvalues.

- $\Delta(\lambda) = \det(A - \lambda I)$ is a *characteristic polynomial* of matrix A
- For $n \times n$ matrix A characteristic polynomial has degree n
- Characteristic polynomial of triangular matrix U

$$\Delta(\lambda) = \det(U - \lambda I) = (u_{11} - \lambda)(u_{22} - \lambda) \cdots (u_{nn} - \lambda)$$

Warning! If U is triangular form of A, then $\Delta(A)$ and $\Delta(U)$ may be different

Eigenvectors and eigenvalues.

- Characteristic polynomial of degree 2:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{Then}$$

$$\begin{aligned} \Delta(A) &= \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) \\ &= \lambda^2 - \mathrm{trace}(A)\lambda + \det(A) \end{aligned}$$

Eigenvectors and eigenvalues.

- Characteristic polynomial of degree 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{Then}$$

$$\Delta(A) \quad = \quad \lambda^3 - \mathrm{trace}(A)\lambda^2 + (C_{11} + C_{22} + C_{33})\lambda - \det(A)$$

- C_{ii} are cofactors of A

Eigenvectors and eigenvalues.

- Characteristic polynomial of degree n. Let A is $n \times n$ then

$$\Delta(A) = \lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + \cdots + (-1)^n S_n,$$

where S_k is the sum of principal minors of order k.

Eigenvectors and eigenvalues.

Computing eigenvalues and eigenvectors

1. Find characteristic polynomial $\Delta(A)$ of A
2. Obtain eigenvalues of A by finding roots of $\Delta(A)$
3. For each eigenvalue λ of A
   a. Form matrix $(A - \lambda I)$
   b. Find a basis of the nullspace $N(A - \lambda I)$
   c. This basis are eigenvectors of A belonging to λ

Eigenvectors and eigenvalues.

- Identity Matrix: $I\bar{\mathbf{x}} = 1 \cdot \bar{\mathbf{x}},\ \lambda = 1,\ \bar{\mathbf{x}}$ - any
- Rotation matrix:

$$\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} : \text{ no real eigenvallues unless } \varphi = k\pi$$

- Reflexion matrix:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \lambda = \pm 1$$

- Projection matrix:

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} : \lambda_1 = 0,\ \lambda_2 = 1$$

- Singular matrix always has $\lambda = 0$

Eigenvectors and eigenvalues.

- Computing eigenvalues cannot be reduced to triangular matrix. If $A = LU$, then eigenvalues of A may be different from eigenvalues of U:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}, \lambda_1 = \frac{7+\sqrt{41}}{2},\ \lambda_2 = \frac{7-\sqrt{41}}{2}; \\ U &= \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix},\ \lambda_1 = 1, \lambda_2 = 2. \end{aligned}$$

Eigenvectors and eigenvalues.

- $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A)$
- $\lambda_1 + \lambda_2 + \cdots + \lambda_n = A_{11} + A_{22} + \cdots + A_{nn}$
- Eigenvalues of real matrix may be not real numbers

Diagonalization

- Let $\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_n$ be eigenvectors and $\lambda_1, \ldots, \lambda_n$ be corresponding eigenvalues of A
- $S = [\bar{\mathbf{x}}_1 \ \bar{\mathbf{x}}_2 \ \cdots \ \bar{\mathbf{x}}_n]$ is the *eigenvector* matrix of A
- $D = \mathrm{diag}(\lambda_1 \ \lambda_2 \ \cdots \ \lambda_n)$ is the *eigenvalue* matrix of A

A is diagonizable if and only if A has n independent eigenvectors. Then

$$A = SDS^{-1} \text{ and } D = S^{-1}AS$$

Diagonalization

Proof:

$$
\begin{aligned}
AS &= A\begin{bmatrix} | & | & & | \\ \bar{\mathbf{x}}_1 & \bar{\mathbf{x}}_2 & \cdots & \bar{\mathbf{x}}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\bar{\mathbf{x}}_1 & \lambda_2\bar{\mathbf{x}}_2 & \cdots & \lambda_n\bar{\mathbf{x}}_n \\ | & | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & & | \\ \bar{\mathbf{x}}_1 & \bar{\mathbf{x}}_2 & \cdots & \bar{\mathbf{x}}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = SD
\end{aligned}
$$

– Since S has n independent columns it is invertible and

$$AS = SD \;\Rightarrow\; A = SDS^{-1} \text{ or } D = S^{-1}AS$$

Diagonalization.

Suppose $\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_k$ are nonzero eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then $\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_k$ are linearly independent.

Proof:

- Prove by contradiction: Suppose $\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_n$ are linearly dependent and $\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_s$ is the minimal dependent set, i.e. $\bar{\mathbf{x}}_2, \ldots, \bar{\mathbf{x}}_s$ are independent but

$$\bar{\mathbf{x}}_1 = a_2\bar{\mathbf{x}}_2 + a_3\bar{\mathbf{x}}_3 + \cdots + a_s\bar{\mathbf{x}}_s \qquad (\star)$$

- Note order can be taken arbitrary

Diagonalization.

Proof (continue):

- Multiply $(\star)$ by A. Using eigenvector property:

$$A\bar{\mathbf{x}}_1 = A(a_2\bar{\mathbf{x}}_2 + \cdots + a_s\bar{\mathbf{x}}_s) \Rightarrow \lambda_1\bar{\mathbf{x}}_1 = a_2\lambda_2\bar{\mathbf{x}}_2 + \cdots + a_s\lambda_s\bar{\mathbf{x}}_s$$

- Multiply $(\star)$ by λ_1

$$\lambda_1\bar{\mathbf{x}}_1 = a_2\lambda_1\bar{\mathbf{x}}_2 + \cdots + a_s\lambda_1\bar{\mathbf{x}}_s$$

- Subtract one from another we get

$$a_2(\lambda_1 - \lambda_2)\bar{\mathbf{x}}_2 + a_3(\lambda_1 - \lambda_3)\bar{\mathbf{x}}_3 + \cdots + a_s(\lambda_1 - \lambda_s)\bar{\mathbf{x}}_s = 0$$

- Since all λ_i are distinct, then all a_j, $j = 2, \ldots, s$ must be zero which contradicts assumption that $\bar{\mathbf{x}}_2, \ldots, \bar{\mathbf{x}}_s$ are linearly independent.

Diagonalization.

Matrix Diagonalization

- Let A be $n \times n$ matrix
- Find eigenvectors $\bar{\mathbf{x}}_1, \ldots \bar{\mathbf{x}}_n$ and corresponding eigenvalues of A
- Consider the collection of *distinct* vectors $\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_m$
- If $m \neq n$ then matrix is not diagonizable
- If $m = n$ then

$$A = SDS^{-1},$$

  where S is the eigenvector matrix and D is the diagonal eigenvalue matrix