MA331 Intermediate Statistics

Lecture 08 Analysis of Variance [1]

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[1] Based on Chapters 12 and 13.

0. Topics to be covered

As a continuation of two way table analysis and two sample t test, one way ANalyis Of VAriance (ANOVA) conducts statistical inference on the difference among population means of more than two groups of observations (usually due to a categorical random variable).

- Factors and responses
- One-way classification
- One-way ANOVA inference for multiple means
- Two-way classification
- Two-way ANOVA inference

1. Review and outlook

☞ Two-way table analysis checks the association between two categorical r.v.'s. In other word, whether the row r.v. (e.g., gender) has a significant effect on the column (e.g., admission)? or whether it make sense to group admission by gender?

☞ Also, it is of interest to check the significant difference in a numerical r.v. (e.g., weight) due to a categorical r.v. (e.g., gender). Two sample t-test checks the difference in the mean of a r.v. due to a categorical r.v. with two outcomes. In other word, whether a categorical r.v. with multiple outcomes (gender) has a significant effect on the quantitative r.v. (weight).

☞ ANOVA studies the difference among population means of more than two groups of observations. Equivalently, whether one categorical r.v. with more than 2 outcomes (e.g., race) has a significant effect on the quantitative r.v. (annual income) or whether it make sense to group annual income by race.

2. Data for one-way ANOVA – example 01

✍ When there is only one way (i.e., a categorical variable) to classify the population of interest, we use one-way ANOVA to analyze the data.

✍ Choosing the best magazine layout

- There are 3 magazine layouts at supermarket checkout lines;
- The sales manager is interested in whether one layout is better to catch shoppers attention and thus results in more sales;
- She randomly assigns each of 60 stores to one of the three layouts and records the number of magazines that are sold in a one-week period.
- Two variables of interest are
  - the response variable – the number of magazines sold in one week, and
  - the factor – way to layout magazines (three layouts).

3. Data for one-way ANOVA – example 02

✍ Average age of bookstore customers

- How do 5 bookstores in a city differ in the demographics of customers?
- Are certain bookstores more popular among teenagers?
- Do upper-income shoppers tend to go to some specific store?
- 50 customers of each store are asked to respond to a questionnaire.
- Two responses of interest are the customers age and income level, and factor is store.

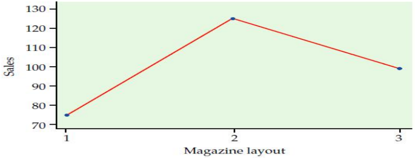
✍ These two examples are similar in the sense that

- a quantitative response variable (the number of magazines sold in one week and customer's age), and
- the goal is to compare several groups of the population (3 magazine layouts and 5 bookstores).

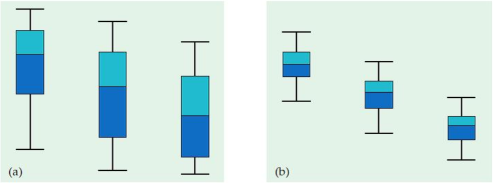
✍ The question to be answered by ANOVA is

Do all groups have the same population mean?

✍ The purpose of ANOVA is to assess whether the observed difference among sample means are statistically significant.



✍ Could a variation among the 3 sample means in the above plot be plausibly due to chance, or is it a good evidence for a difference among population means?



✍ This question can't be answered from the sample means alone. Owing to $\mathrm{Var}[\bar{X}] = \sigma^2/n$, both variations within groups and sizes of groups of observations play a role in coming up with a reasonable answer.

5. An overview of ANOVA – elementary facts

✍ ANOVA tests the null hypothesis that the population means are all equal, and the alternative is that they are not all equal.

✍ This alternative could be true because all of the means are different or simply one of them differs from the others. It is a more complex situation than comparing just two populations.

✍ If we reject the null hypothesis, we need to perform some further study to see which population mean differ from others and by how much.

✍ The computations needed for an ANOVA are a bit tedious, and the software frees us from the burden of arithmetic and helps concentrate on interpreting the output.

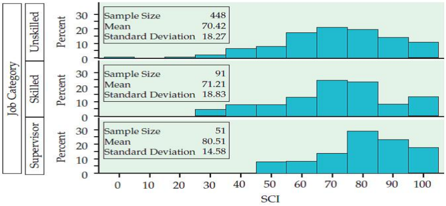
✍ Complicated computations do not guarantee a valid statistical analysis. We should always start our ANOVA with a careful examination of the data using graphical and numerical summaries.

6. An overview of ANOVA – a simple example

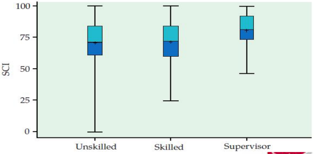
✍ Safety Climate Index (from 0 to 100) was calculated based on the ratings from three category of workers.

Job category	n	$\bar{x}$	s
Unskilled workers	448	70.42	18.27
Skilled workers	91	71.21	18.83
Supervisors	51	80.51	14.58

✍ **Histograms and descriptive statistics**: distributions are somewhat skewed to lower values. Our sample sizes, however, are sufficiently large such that we are confident that sample means are approximately normal.



✍ **Boxplot**: the unskilled and skilled groups are of similar means, while supervisors get a larger mean.



To apply ANOVA, 3 independent samples are viewed as from distinct populations with a common variance and we compare their means.

7. One way ANOVA model

✍ **Population**

The r.v. X_i is normal with mean μ_i and a common variance σ^2, $i = 1, \cdots, k$.

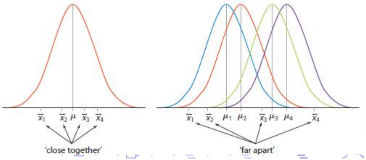
✍ **Data set**

- one column of quantitative measurements/responses (e.g., SCI) along with
- the other corresponding observations of a categorical variable (e.g., skilled, unskilled and supervisors).

✍ **Samples**

- random variable X_i has a SRS $X_{i,1}, \cdots, X_{i,n_i}$, $i = 1, \cdots, k$, and
- all k SRS's are mutually independent.

✍ **Hypotheses to be tested** $H_0 : \mu_1 = \cdots = \mu_k$ versus H_a: not all $\mu_1, \cdots, \mu_k$ are equal.



8. One way ANOVA – sum of squares

✍ Sum of Squares in Total (overall variation in the data)

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{i,j} - \bar{X}_{\cdot,\cdot})^2, \quad \bar{X}_{\cdot,\cdot} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{i,j}, \quad n = \sum_{i=1}^{k} n_i.$$

✍ Sum of Squares between Groups (variation b/w groups, also SSB, SSTR)

$$SSB = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{X}_{i,\cdot} - \bar{X}_{\cdot,\cdot})^2 = \sum_{i=1}^{k} n_i(\bar{X}_{i,\cdot} - \bar{X}_{\cdot,\cdot})^2, \quad \bar{X}_{i,\cdot} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{i,j},\ i = 1, \cdots, k.$$

✍ Sum of Squares in Error (variation due to randomness, within groups also SSW)

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{i,j} - \bar{X}_{i,\cdot})^2 = \sum_{i=1}^{k}(n_i - 1)S_i^2, \quad S_i^2 = \frac{\sum_{j=1}^{n_i}(X_{i,j} - \bar{X}_{i,\cdot})^2}{n_i - 1},\ i = 1, \cdots, k.$$

✌ Decomposition of SST

$$SST \text{ (overall)} = SSB \text{ (between groups)} + SSE \text{ (within groups)}.$$

9. One way ANOVA – logics

✍ Due to $SST = SSB + SSE$, a larger SSB compared to SSE serves as an evidence against H_0: no difference among $\mu_1, \cdots, \mu_k$.

✍ Under the null H_0, it can be proved that

- $\frac{SSB}{\sigma^2} \sim \chi^2_{k-1}$ and $\frac{SSE}{\sigma^2} \sim \chi^2_{n-k}$.
- As a result of the dependence between SSB and SSE, we have the ratio

$$F = \frac{MSB}{MSE} = \frac{SSB/(k-1)}{SSE/(n-k)} \sim \mathcal{F}_{k-1,n-k}.$$

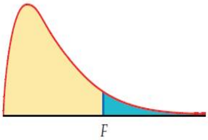
✍ Testing rule: For F observed as f, reject H_0 if

- f is larger than the critical value, i.e,

  $f > F_{1-\alpha}(k-1, n-k)$,
- or equivalently, the p-value

  $\mathrm{P}(F > f) = 1 - \mathrm{pf}(f, k-1, n-k) < \alpha.$

$F = \frac{\text{MSG}}{\text{MSE}}$



✍ Safety Climate Index was calculated based on the ratings from 3 category of workers. At significance level $\alpha = 0.05$, we test

Job category	n	$\bar{x}$	s
Unskilled workers	448	70.42	18.27
Skilled workers	91	71.21	18.83
Supervisors	51	80.51	14.58

$$H_0 : \mu_1 = \mu_2 = \mu_3 \quad \text{versus} \quad H_a\text{: they are not all equal.}$$

- Check the assumption. Assume three group populations have normal distributions sharing a common variance σ^2.
- Statistics $\bar{X}_{\cdot,\cdot} = (n_1\bar{X}_{1,\cdot} + n_2\bar{X}_{2,\cdot} + n_3\bar{X}_{3,\cdot})/(n_1 + n_2 + n_3) = 71.414$,

$$SSB = \sum_{i=1}^{3} n_i(\bar{X}_{i,\cdot} - \bar{X}_{\cdot,\cdot})^2 = 4666.025, \qquad SSE = \sum_{i=1}^{3}(n_i - 1)S_i^2 = 191745.4$$

- Testing statistic F observed as

$$f = \frac{SSB/(k-1)}{SSE/(n-k)} = \frac{4666.025/(3-1)}{191745.4/(590-3)} = 7.14217.$$

- Since the p-value $\mathrm{P}(F > 7.14) = 1 - \mathrm{pf}(7.14, 2, 587) = 0.00086 << 0.05 = \alpha$, we reject H_0, i.e., μ_1, μ_2, μ_3 are not all equal.

11. One way ANOVA – ANOVA table

✍ **ANOVA table** Computation of the observed F statistic is summarized as

Source	Degree of freedom	SS	MS	F
Group	$k-1$	SSB	$MSB = \frac{SSB}{k-1}$	$F = \frac{MSB}{MSE}$
Error	$n-k$	SSE	$MSE = \frac{SSE}{n-k}$	
Total	$n-1$	SST		

✍ **Coefficient of determination**

- Due to the decomposition $SST = SSB + SSE$, the coefficient of determination

$$R^2 = SSB/SST$$

  tells the fraction of variation in the response explained by the factor and hence is also employed to evaluate the performance of the classification.

- A larger R^2 provides much confidence of significant difference among group populations means $\mu_1, \cdots, \mu_k$.

12. One way ANOVA: R example

```
> data(InsectSprays)                          ## Get the data.
> str(InsectSprays)
'data.frame':   72 obs. of  2 variables:
 $ count: num  10 7 20 14 14 12 10 23 17 20 ...
 $ spray: Factor w/ 6 levels "A","B","C","D",..: 1 1 1 1 1 1 1 1 1 1 ...

## 6 insect sprays tested in many fields, and a count of insects found in each field after spraying.
> attach(InsectSprays)
> tapply(count, spray, mean)
        A         B         C         D         E         F
14.500000 15.333333  2.083333  4.916667  3.500000 16.666667
> tapply(count, spray, var)
        A         B         C         D         E         F
22.272727 18.242424  3.901515  6.265152  3.000000 38.606061
> tapply(count, spray, length)
 A  B  C  D  E  F
12 12 12 12 12 12

> oneway.test(count ~ spray)                  ## Do ANOVA
        One-way analysis of means (not assuming equal variances)
data:  count and spray
F = 36.0654, num df = 5.000, denom df = 30.043, p-value = 8e-12
```

13. Comparing means after ANOVA test rejects H_0

✍ A small p-value of the ANOVA F test simply tells that the group population means are not all the same. Which means differ from the others?

✍ Plotting and inspecting the means give some indication of where the difference is; However, to confirm the observed difference a formal inference has to be done.

✍ Usually, specific questions comparing population means are of interest before the data is collected, and the answers with a level of confidence are desired. E.g.,

- $\mu_3 > \mu_1$ or $\mu_3 > \mu_2$? i.e., supervisors differ from skilled or unskilled workers?
- $\mu_3 > \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$? i.e., supervisors differ from workers in average?

✍ A contrast expresses an effect in population as a combination of group means

$$\psi = a_1\mu_1 + \cdots + a_k\mu_k, \qquad a_1 + \cdots + a_k = 0.$$

- $a_1 = 1, a_2 = 0, a_3 = -1$: test $H_0 : \psi = 0 \Leftrightarrow \mu_1 = \mu_3$ versus $H_a : \mu_1 \neq \mu_3$;
- $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}, a_3 = -1$: test $H_0 : \psi = 0 \Leftrightarrow \frac{\mu_1+\mu_2}{2} = \mu_3$ versus $H_a : \frac{\mu_1+\mu_2}{2} \neq \mu_3$.

14. Statistics on a contrast

✍ To test a contrast ψ, we consider $H_0 : \psi = 0$, i.e.,

$$a_1\mu_1 + \cdots + a_k\mu_k = 0,$$

for specific contrast coefficients $(a_1, \cdots, a_k)$ such that $a_1 + \cdots + a_k = 0$.

✍ The sample version of the contrast ψ is

$$C_\psi = a_1\bar{X}_{1,\cdot} + \cdots + a_k\bar{X}_{k,\cdot} = \sum_{i=1}^{k} a_i\bar{X}_{i,\cdot}.$$

✍ Due to the independence, $C_\psi \sim \mathcal{N}(\psi, \sigma_\psi^2)$ has the variance

$$\begin{aligned}\sigma_\psi^2 = \mathrm{Var}[C_\psi] &= \mathrm{Var}\left[\sum_{i=1}^{k} a_i\bar{X}_{i,\cdot}\right] \\ &= \sum_{i=1}^{k}\mathrm{Var}[a_i\bar{X}_{i,\cdot}] = \sum_{i=1}^{k} a_i^2\mathrm{Var}[\bar{X}_{i,\cdot}] = \sum_{i=1}^{k}\frac{a_i^2\sigma^2}{n_i} = \sigma^2\sum_{i=1}^{k}\frac{a_i^2}{n_i}.\end{aligned}$$

15. Student's t-test and confidence interval on a contrast

✍ Due to the independence among samples, the common variance σ^2 may be estimated by the pooled sample variance

$$S_p^2 = \frac{\sum_{i=1}^k (n_i - 1)S_i^2}{\sum_{i=1}^k (n_i - 1)} = \frac{SSE}{n-k} = MSE.$$

✍ As a result, the variance σ_ψ^2 is estimated as $S_p^2 \sum_{i=1}^k \frac{a_i^2}{n_i}$.

✍ Under the null H_0, it proves that $T = \dfrac{C_\psi}{\sqrt{S_p^2 \sum_{i=1}^k a_i^2/n_i}} \sim t_{n-k}$.

✍ Now, the testing rule: Reject $H_0 : \psi = 0$ (versus $H_a : \psi \neq 0$) if T observed as $t = \frac{c_\psi}{\sqrt{s_p^2 \sum_{i=1}^k a_i^2/n_i}}$ having the p-value $\mathrm{P}(|T| > |t|) < \alpha$.

✍ ψ gets the level $1-\alpha$ confidence interval $c_\psi \pm t_{1-\alpha/2}(n-k)\sqrt{s_p^2 \sum_{i=1}^k a_i^2/n_i}$.

16. Student's t-test on a contrast – example

✍ Based on SCI from 3 category of workers, we test $H_0 : \mu_1 = \mu_2 = \mu_3$.

Job category	n	$\overline{x}$	s
Unskilled workers	448	70.42	18.27
Skilled workers	91	71.21	18.83
Supervisors	51	80.51	14.58

✍ ANOVA: $F = 7.14$, p-value 0.00086. Reject H_0 and μ_1, μ_2, μ_3 are different.

✍ Consider two contrasts

$$\psi_1 : a_1 = a_2 = 1/2, a_3 = -1 \quad \text{and} \quad \psi_2 : a_1 = 1, a_2 = -1, a_3 = 0.$$

✍ Testing for $H_0 : \psi_1 = 0$: T is evaluated as

$$\begin{aligned} t &= \frac{c_{\psi_1}}{\sqrt{s_p^2 \sum_{i=1}^3 a_i^2/n_i}} = \frac{a_1\bar{x}_{1,\cdot} + a_2\bar{x}_{2,\cdot} + a_3\bar{x}_{3,\cdot}}{\sqrt{MSE \cdot \left[\frac{a_1^2}{n_1} + \frac{a_2^2}{n_2} + \frac{a_3^2}{n_3}\right]}} \\ &= \frac{70.42 \cdot 0.5 + 71.21 \cdot 0.5 + 80.51 \cdot (-1)}{\sqrt{18.07 \cdot \left[\frac{(1/2)^2}{448} + \frac{(1/2)^2}{91} + \frac{(-1)^2}{51}\right]}} = \frac{9.69}{2.74} = 3.54. \end{aligned}$$

✍ In view of the degree of freedom 587, we evaluate the p-value < 0.0001, which is a strong evidence against $H_0 : \psi_1 = 0$.

17. Confidence interval on a contrast – example

✍ Testing for $H_0 : \psi_2 = 0$: T is evaluated as

$$t = \frac{c_{\psi_2}}{\sqrt{s_p^2 \sum_{i=1}^{3} a_i^2/n_i}} = \frac{\bar{x}_{1,\cdot} - \bar{x}_{2,\cdot}}{\sqrt{MSE \cdot \left[\frac{(1)^2}{n_1} + \frac{(-1)^2}{n_2}\right]}} = \frac{-0.79}{2.08} = -0.38.$$

With degree of freedom 587, the two-sided test gets p-value 0.76. So, not reject $H_0 : \psi_2 = 0$.

✍ The level 0.95 confidence intervals

- for contrast ψ_1,

$$c_{\psi_1} \pm t_{1-\alpha/2}(n-k)\sqrt{s_p^2 \sum_{i=1}^{3} a_i^2/n_i} = 9.69 \pm 1.984 \cdot 2.74 = 9.69 \pm 5.44.$$

- for contrast ψ_2,

$$c_{\psi_2} \pm t_{1-\alpha/2}(n-k)\sqrt{s_p^2 \sum_{i=1}^{3} a_i^2/n_i} = -0.79 \pm 1.984 \cdot 2.08 = -0.79 \pm 4.13.$$

18. Multiple comparison after ANOVA test rejects the null

✍ Sometimes, specific questions cannot be formulated in advance. If we reject $H_0 : \mu_1 = \cdots = \mu_k$, it is of interest to further identify which pairs of means differ.

✍ To perform a multiple-comparisons procedure, we compute Student's t statistics

$$T_{i,j} = \frac{\bar{X}_{i,\cdot} - \bar{X}_{j,\cdot}}{\sqrt{S_p^2(n_i^{-1} + n_j^{-1})}}, \quad \text{for all pair } \{i, j\} \text{ with } 1 \le i \ne j \le k.$$

✍ LSD (least significant difference) method: Reject $H_0 : \mu_i = \mu_j$ if $T_{i,j}$ is observed as $t_{i,j}$ with p-value

$$\mathrm{P}(|T_{i,j}| > |t_{i,j}|) = 2[1 - \mathrm{pt}(|t_{i,j}|, n - k)] < \alpha.$$

✍ LSD has some undesirable properties, particularly when the number of groups is large. The software also considers other alternatives such as Bonferroni and Tukey methods. Users must be careful in understanding the output of ANOVA.

19. Two-way classification

☞ Sometimes a population can be classified in two ways or the response is associated with two factors in experimental studies.

☞ It is of interest to check the significant difference in response variable (e.g., body length) due to two categorical variables. (e.g., Gender and Race). Also, one may wonder whether the categorical variables are independent with each other.

☞ Two-way ANOVA addresses comparisons in the following:

- Can an increase in the consumption of red palm oil reduce the occurrence and severity of malaria in both male and female children in Nigeria?
- Whether the sale amount of journals has something to do with the layout in the check line and the cover style?
- Do calcium supplements prevent bone loss in elderly people with and without adequate vitamin D?

20. Data for two-way ANOVA – example

✍ When there are two ways (i.e., two categorical variables) to classify the population of interest, we employ two-way ANOVA.

✍ Choosing the best magazine layout and cover

- There are 3 magazine layouts at checkout lines and 4 cover styles, and the manager is interested in the best one of $3 \times 4 = 12$ designs (combinations);
- Assign each design to 5 stores and record the number of magazines sold.
- Three variables are involved:
  A response – magazines sold, and
  Two factors – layout and cover style.

	Cover				
Layout	1	2	3	4	Total
1	5	5	5	5	20
2	5	5	5	5	20
3	5	5	5	5	20
Total	15	15	15	15	60

✍ Advantages of two-way ANOVA

- More efficient to study two factors simultaneously rather than separately.
- Reduce the error in a model by including one more influential factor.
- Investigate interactions between factors.

✍ Assumptions

- The population $X_{i,j} \sim \mathcal{N}(\mu_{i,j}, \sigma^2)$ has a SRS
  $X_{i,j,1}, \cdots, X_{i,j,n_{i,j}}, \qquad i = 1, \cdots, r, j = 1, \cdots, c.$
- The population means $\mu_{i,j}$'s and σ^2 are both unknown.
- All $r \times c$ SRS's are mutually independent.

✍ Data includes observations of the response along with two categorical variables.

✍ Two-way classification produces a 3-dimensional table.

Factor B

Factor A	1	$\cdots\cdots$	c	
1	$(X_{1,1,1}, \cdots, X_{1,1,n_{1,1}})$	$\cdots\cdots$	$(X_{1,c,1}, \cdots, X_{1,c,n_{1,c}})$	$\bar{X}_{1,\cdot,\cdot}$
$\vdots$	$\vdots$	$\cdots\cdots$	$\vdots$	$\vdots$
r	$(X_{r,1,1}, \cdots, X_{r,1,n_{r,1}})$	$\cdots\cdots$	$(X_{r,c,1}, \cdots, X_{r,c,n_{r,c}})$	$\bar{X}_{r,\cdot,\cdot}$
	$\bar{X}_{\cdot,1,\cdot}$	$\cdots\cdots$	$\bar{X}_{\cdot,c,\cdot}$	$\bar{X}_{\cdot,\cdot,\cdot}$

22. Two-way ANOVA – averages

✍ Overall average

$$\bar{X}_{\cdot,\cdot,\cdot} = \frac{1}{n} \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n_{i,j}} X_{i,j,k}, \qquad n = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{i,j}.$$

✍ Row averages

$$\bar{X}_{i,\cdot,\cdot} = \frac{1}{n_{i,+}} \sum_{j=1}^{c} \sum_{k=1}^{n_{i,j}} X_{i,j,k}, \qquad n_{i,+} = \sum_{j=1}^{c} n_{i,j}, \quad i = 1, \cdots, r.$$

✍ Column averages

$$\bar{X}_{\cdot,j,\cdot} = \frac{1}{n_{+,j}} \sum_{i=1}^{r} \sum_{k=1}^{n_{i,j}} X_{i,j,k}, \qquad n_{+,j} = \sum_{i=1}^{r} n_{i,j}, \quad j = 1, \cdots, c.$$

✍ Cell averages

$$\bar{X}_{i,j,\cdot} = \frac{1}{n_{i,j}} \sum_{k=1}^{n_{i,j}} X_{i,j,k}, \qquad i = 1, \cdots, r, \ j = 1, \cdots, c.$$

23. Two-way ANOVA – decomposition of sum of squares

✍ Sum of Squares in Total

$$SST = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_{i,j}} (X_{i,j,k} - \bar{X}_{\cdot,\cdot,\cdot})^2.$$

✍ Sum of Squares due to factor A

$$SSA = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_{i,j}} (\bar{X}_{i,\cdot,\cdot} - \bar{X}_{\cdot,\cdot,\cdot})^2 = c\sum_{i=1}^{r} n_{i,j}(\bar{X}_{i,\cdot,\cdot} - \bar{X}_{\cdot,\cdot,\cdot})^2$$

with degree of freedom $r-1$.

✍ Sum of Squares due to factor B

$$SSB = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_{i,j}} (\bar{X}_{\cdot,j,\cdot} - \bar{X}_{\cdot,\cdot,\cdot})^2 = r\sum_{j=1}^{c} n_{i,j}(\bar{X}_{\cdot,j,\cdot} - \bar{X}_{\cdot,\cdot,\cdot})^2$$

with degree of freedom $c-1$.

24. Two-way ANOVA – decomposition of sum of squares

✍ Sum of Squares due to interaction $A \times B$

$$\begin{aligned} SSAB &= \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_{i,j}}(\bar{X}_{i,j,\cdot} - \bar{X}_{\cdot,\cdot,\cdot})^2 \\ &= \sum_{i=1}^{r}\sum_{j=1}^{c} n_{i,j}(\bar{X}_{i,j,\cdot} - \bar{X}_{i,\cdot,\cdot} - X_{\cdot,j,\cdot} + \bar{X}_{\cdot,\cdot,\cdot})^2 \end{aligned}$$

with degree of freedom $(c-1)(r-1)$.

✍ Sum of Squares in Error

$$SSE = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_{i,j}}(X_{i,j,k} - \bar{X}_{i,j,\cdot})^2$$

with degree of freedom $n - cr$.

✌ Decomposition of SST

$$SST = SSA + SSB + SSAB + SSE.$$

25. Two-way ANOVA – F tests

✍ For H_{01}: groups in all rows have a common population mean, we reject if

$$F_1 = \frac{SSA/(r-1)}{SSE/(n-cr)} = \frac{MSA}{MSE},$$

is observed as f_1 having p-value $\quad \mathrm{P}(F_1 > f_1) = 1 - \mathrm{pf}(f_1, r-1, n-cr) < \alpha.$

✍ For H_{02}: groups in all columns have a common population mean, we reject if

$$F_2 = \frac{SSB/(c-1)}{SSE/(n-cr)} = \frac{MSB}{MSE}$$

is observed as f_2 and the p-value $\quad \mathrm{P}(F_2 > f_2) = 1 - \mathrm{pf}(f_2, c-1, n-cr) < \alpha.$

✍ For H_{03}: all cells have a common population mean, we reject if

$$F_3 = \frac{SSAB/[(r-1)(c-1)]}{SSE/(n-cr)} = \frac{MSAB}{MSE}$$

is observed as f_3 having p-value $\quad \mathrm{P}(F_3 > f_3) = 1 - \mathrm{pf}(f_3, (r-1)(c-1), n-cr) < \alpha.$

26. Two-way ANOVA table

✍ To compute F testing statistics, we first get the following ready.

- Overall average $\bar{X}_{\cdot,\cdot,\cdot}$.
- Row averages $\bar{X}_{i,\cdot,\cdot}$, $i = 1, \cdots, r$.
- Column averages $\bar{X}_{\cdot,j,\cdot}$, $j = 1, \cdots, c$.
- Cell averages $\bar{X}_{i,j,\cdot}$, $i = 1, \cdots, r$, $j = 1, \cdots, c$.

✍ The computation procedure is summarized as the ANOVA table

Source	Degree of freedom	SS	MS	F
A	$r-1$	SSA	$MSA = \frac{SSA}{r-1}$	$F_1 = \frac{MSA}{MSE}$
B	$c-1$	SSB	$MSB = \frac{SSB}{c-1}$	$F_2 = \frac{MSB}{MSE}$
A×B	$(r-1)(c-1)$	$SSAB$	$MSAB = \frac{SSAB}{(r-1)(c_1)}$	$F_3 = \frac{MSAB}{MSE}$
Error	$n-cr$	SSE	MSE	
Total	$n-1$	SST		

27. Two-way ANOVA – an example

✍ The experiment has two factors (Detergent and Temperature) at $r = 2$ (Super, Best) and $c = 3$ (cold, warm, hot) levels. For each combination of detergent and temperature we wash 4 loads and record the amount of dirt removed.

	Cold	Warm	Hot
Super	4,5,6,5	7,9,8,12	10,12,11,9
Best	6,6,4,4	13,15,12,12	12,13,10,13

✍ The computation procedure is summarized as a table

Source	Degree of freedom	SS	MS	F	p-value
A	2-1=1	24	$\frac{24}{1} = 24$	9.81	$1 - \text{pf}(9.81, 1, 18)$
B	3-1=2	192	$\frac{192}{2} = 96$	48.73	$1 - \text{pf}(48.73, 2, 18)$
A×B	(2-1)(3-1)=2	12	$\frac{12}{2} = 6$	3.97	$1 - \text{pf}(3.97, 2, 18)$
Error	18	38	$\frac{38}{18} = 2.11$		
Total	23	266			

28. Two-way ANOVA: R example

```
# The amount of stain removed due to a certain amount of detergent.
> wash=c(4, 5, 6, 5, 7, 9, 8, 12, 10, 12, 11, 9, 6, 6, 4, 4, 13, 15, 12, 12, 12, 13, 10, 13)
> deter=factor(c(rep(1,12),rep(2,12)))    ## Two sorts of detergent.
> deter
[1] 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2 2 2      Levels: 1 2
> water=factor(rep(gl(3,4),2))           ## Three kinds of washing styles.
> water
[1] 1 1 1 1 2 2 2 2 3 3 3 3 1 1 1 1 2 2 2 2 3 3 3 3       Levels: 1 2 3
> tapply(wash,water,mean)
   1     2     3
5.00 11.00 11.25
> tapply(wash,deter,mean)
       1         2
8.166667 10.000000
> tapply(wash,deter:water,mean)
1:1 1:2 1:3 2:1 2:2 2:3
5.0 9.0 10.5 5.0 13.0 12.0
> ss=aov(wash ~ deter*water)
> summary(ss)
            Df Sum Sq  Mean Sq Fvalue    Pr(>F)
deter        1 20.167  20.167  9.8108   0.005758 **
water        2 200.333 100.167 48.7297  5.44e-08 ***
deter:water  2 16.333  8.167   3.9730   0.037224 *
Residuals   18 37.000  2.056
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```