1.	2.	3.	4.	5.	6.	Total:

MA 232. **Exam 1. Solutions** **October 10, 2017.**

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Problem 1. (10pts) Prove or disprove. Show all of your work to support the argument.

(a) If $\bar{\mathbf{v}} = (1,1)$ and $\bar{\mathbf{w}} = (1,5)$ then $\bar{\mathbf{w}} - 3\bar{\mathbf{v}}$ is orthogonal to $\bar{\mathbf{v}}$

(b) The set of all $n \times n$ diagonal matrices is a vector subspace of the space of all $n \times n$ square matrices.

(c) Let A be invertible matrix then $N(A)$ contains more then one vector.

Solution:

(a) True: $\bar{\mathbf{v}} \cdot (\bar{\mathbf{w}} - 3\bar{\mathbf{v}}) = \bar{\mathbf{v}}\bar{\mathbf{w}} - 3\bar{\mathbf{v}}\bar{\mathbf{v}} = 1 \cdot 1 + 1 \cdot 5 - 3(1 \cdot 1 + 1 \cdot 1) = 6 - 6 = 0$

(b) True: To prove that diagonal matrices is a subspace we need to show that a linear combination of diagonal matrices is a subspace:

$$c \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & a_n \end{bmatrix} + d \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} ca_1 + db_1 & 0 & \cdots & 0 \\ 0 & ca_2 + db_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & ca_n + db_n \end{bmatrix}$$

(c) False: Since A is invertible then $N(A) = \{\bar{\mathbf{0}}\}$.

Problem 2. (10pts) Find the inverse of the following matrix if it exists.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Problem 3. (10pts) Find vectors that span the space of all points on the plane

$$x + 2y - 3z - t = 0.$$

Solution:

(a) Plane defined by equation $x + 2y - 3z - t = 0$ are all points (x, y, z, t) s.t. the equation is true. So we need to find all possible solutions to this equation which means we need to find the null space of matrix $[1\ 2\ \ -3\ \ -1]$. There is one row, therefore one pivot and three free variables: y, z, t. Hence there are three vectors in the basis of this plane.

Find special solutions:

- Let $y = 1, z = 0, t = 0$ then $x = -2$ and $\bar{\mathbf{v}}_1 = [-2\ 1\ 0\ 0]^T$

- Let $y = 0, z = 1, t = 0$ then $x = 3$ and $\bar{\mathbf{v}}_2 = [3\ 0\ 1\ 0]^T$
- Let $y = 0, z = 0, t = 1$ then $x = 1$ and $\bar{\mathbf{v}}_3 = [1\ 0\ 0\ 1]^T$

Vectors $\bar{\mathbf{v}}_1$, $\bar{\mathbf{v}}_2$ and $\bar{\mathbf{v}}_3$ are span the given plane.

Problem 4. (10pts) Express the polynomial $v = 4t^2 - 8t + 9$ as a linear combination of the polynomials

$$p_1 = t^2 - 2t + 5, \quad p_2 = 2t^2 - 3t, \quad p_3 = t + 1$$

[*Hint:* A polynomial is completely determined by its coefficients. The two polynomials are equal iff the corresponding coefficients are equal.]

Solution

Need to find c_1, c_2, c_3 such that $v = c_1 p1 + c_2 p_2 + c_3 p_3$. Equalizing coefficients of corresponding monomials we obtain 3 linear equations:

$$\begin{aligned} c_1 + 2c_2 &= 4 \\ -2c_1 - 3c_2 + c_3 &= -8 \\ 5c_1 + c_3 &= 9 \end{aligned}$$

with the corresponding augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ -2 & -3 & 1 & -8 \\ 5 & 0 & 1 & 9 \end{array}\right]$$

Which has the solution $c_1 = 2, c_2 = 1, c_3 = -1$, therefore

$$v = 2p_1 + p_2 - p_3$$

Problem 5. (10pts) Write a general solution to the following system of linear equations

$$\begin{aligned} x + 2y + z - 2t &= 2 \\ 2x + 4y + 4z - 3t &= 7 \\ 3x + 6y + 7z - 4t &= 12 \end{aligned}$$

Solution:

The matrix of coefficients

$$A = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 4 & 4 & -3 \\ 3 & 6 & 7 & -4 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Variables y and t are free variables. Find special solutions:

- $y = 1, t = 0$ then $z = 0$ and $x = -2$
- $y = 0, t = 1$ then $z = -\frac{1}{2}$ and $x = 2\frac{1}{2}$

The null space

$$N(A) = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2\frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

A particular solution to the system above is (could be different)

$$\bar{\mathbf{x}}^* = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

so the general solution to the system is

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}^* + \bar{\mathbf{x}}_{null\ space} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2\frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Problem 6. (10pts) Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \bar{\mathbf{b}} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix},$$

find the solution to the system of linear equations $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ without computing the matrix A explicitly.

Solution:

We are given LU decomposition of A so the most efficient way is to solve by two substitutions. First we find $\bar{\mathbf{y}}$ such that

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \bar{\mathbf{y}} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix},$$

This can be done by forward substitution and $\bar{\mathbf{y}} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

Then solve for $\bar{\mathbf{x}}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{\mathbf{x}} = \bar{\mathbf{y}}$$

The answer is $\bar{\mathbf{x}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$