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Assignment 3 - Solutions

1. a) $f(z)-f(z_0)=z^n-z_0^n=(z-z_0)(z^{n-1}+z^{n-2}z_0+...+z_0^{n-1})$ so using the limit definition of derivative we have:

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}z^{n-1}+z^{n-2}z_0+...+z_0^{n-1}=z_0^{n-1}+...+z_0^{n-1}=nz_0^{n-1}$$

   b) $h(z)-h(z_0)=(af(z)+bg(z))-(af(z_0)-bg(z_0))=a(f(z)-f(z_0))+b(g(z)-g(z_0))$, so using the limit definition of derivative we have:

$$\lim_{z\to z_0}\frac{h(z)-h(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{a(f(z)-f(z_0))+b(g(z)-g(z_0))}{z-z_0}$$

$$=a\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}+b\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}=af'(z_0)+bg'(z_0)$$

2. By problem 1a). we know $f_k(z)=z^k$ is entire for any $k\in\mathbb{N}$ as the derivative exists everywhere so, by a result seen in class, $p(z)=a_0+a_1z+...+a_nz^n=a_0+a_1f_1(z)+...+f_n(z)$ must be entire.

   By problem 1b) we know that $p'=(a_0+a_1f_1+...+a_nf_n)'=a_0(1)'+a_1f_1'+...+a_nf_n'$. By problem 1b) we know $f_k'(z)=kz^{k-1}$ for any $k\in\mathbb{N}$ and by example seen in class we know $(1)'=0$, completing the proof.

3. $\frac{i^3-i^2+i-1}{i^3+i^2+i+1}=\frac{0}{0}$ so we can use L'Hopital's rule. Using the formula from problem 2 we have

$$\lim_{z\to i}\frac{z^3-z^2+z-1}{z^3+z^2+z+1}=\lim_{z\to i}\frac{3z^2-2z+1}{3z^2+2z+1}=\frac{3(-1)-2i+1}{3(-1)+2i+1}=\frac{-2-2i}{-2+2i}=i$$

4. We have $u(x,y)=x^2$ and $v(x,y)=y^2$, so $u_x=2x$, $u_y=0$, $v_x=0$ and $v_y=2y$. $u_y=0=-v_x$ everywhere, so to be differentiable we must have $u_x=v_y\Rightarrow 2x=2y\Rightarrow x=y$. It follows that f is only differentiable only on the straight line $x=y$, analytic nowhere since the straight line contains no disks and where it is differentiable $f'(x+iy)=u_x+iv_x=2x$.

5. Let $f(re^{i\theta})=re^{i2\theta}$. Use the polar Cauchy-Riemann equations to determine where f is differentiable, where it is analytic and what the derivative is where it exists. We have $U(r,\theta)=r\cos 2\theta$ and $V(r,\theta)=r\sin 2\theta$ so $U_r=\cos 2\theta$, $U_\theta=-2r\sin 2\theta$, $V_r=\sin 2\theta$ and $V_\theta=2r\cos 2\theta$. To be differentiable we must have that $U_r=\frac{1}{r}V_\theta\Rightarrow\cos 2\theta=2\cos 2\theta\Rightarrow\cos 2\theta=0$ and $V_r=\frac{-1}{r}U_\theta\Rightarrow\sin 2\theta=2\sin 2\theta\Rightarrow\sin 2\theta=0$. Since we can never satisfy both equations simultaneously, f is nowhere differentiable or analytic and the derivative doesn't exist.