Lecture 16: Relations, binary relations, and their properties (Rosen 9.1-9.3)

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Outline of lecture

Binary relations

Recall from lecture 7: a *relation* from T to U is a subset of $T \times U$. (Here T, U are sets.)

Let $users = \{\text{wabeel}, \text{camaba}, \text{kbodzak}, \text{mbrady}, \text{acoppola}\}$
$printers = \{\text{wombat}, \text{badger}, \text{ferret}\}$
$modes = \{\text{rd}, \text{wr}, \text{ex}\}$
$files = \{/bin/diff, /Users/Nau/test1, /Home/Nau/lab8.ss\}$

$CanUse$ is a binary relation from $users$ to $printers$, where
$CanUse = \{(\text{wabeel}, \text{wombat}), (\text{camaba}, \text{wombat}), (\text{wabeel}, \text{ferret})\}$.

$Access$ is a *ternary relation* on $users \times modes \times files$ where

$$\begin{aligned} Access = \; & \{(\text{wabeel}, \text{rd}, /Home/Nau/lab8.ss), \\ & (\text{mbrady}, \text{wr}, /Home/Nau/test1), \\ & (\text{mbrady}, \text{ex}, /bin/diff), \quad \ldots \quad \} \end{aligned}$$

Some relations on big sets

The $\leqslant$ relation on $\mathbf{N}$. The $\leqslant$ relation on $\mathbf{R}$.

$R0 = \{(i, j) \mid (i, j) \in \mathbf{N} \times \mathbf{N} \wedge (i \mid j) \wedge i \neq j\}$ (alert: i divides j)

$Child = \{(p, c) \mid p \text{ is the parent of } c\}$, so
$Child \subseteq People \times People$

$\{(u, \mathsf{ex}, f) \mid u \in users \wedge f \text{ is a path with prefix /bin}\}$
$\cup \{(u, \mathsf{rd}, f \mid u \in users \wedge f \text{ is a path with prefix /pub}\}$
$\cup \{(u, \mathsf{wr}, f \mid u \in users \wedge f \text{ is a path with prefix /Home/u}\}$
$\cup \{(\mathsf{root}, m, f) \mid m \in modes \wedge f \in files\}$

(Look at lect16.ss)

Relations

Recall that a function f from $\mathbf{Z}$ to $\mathbf{R}$ is a relation, i.e., subset of $\mathbf{Z} \times \mathbf{R}$, such that $\forall i \in \mathbf{Z}.\ \exists! r \in \mathbf{R}.\ (i, r) \in f$.

Which of the following relations are functions?

- $(i, r) \in R0$ iff $i < r$ and $r < i + 1$
- $(i, r) \in R1$ iff $i \geqslant 0$ and $i = r$
- $(i, r) \in R2$ iff $r^2 = i$

Relation as matrix

$CanUse \subseteq users \times printers$

$CanUse = \{(\mathsf{wabeel}, \mathsf{wombat}), (\mathsf{camaba}, \mathsf{wombat}), (\mathsf{wabeel}, \mathsf{ferret})\}$

	wombat	*badger*	*ferret*
wabeel	✓		✓
camaba	✓		
kbodzak			
mbrady			
acoppola			

Relations and predicates

$CanUse = \{(u, p) \mid CanUse?(u, p)\}$ where we define

$$CanUse?(x, y) \equiv \; (x = wabeel \wedge (y = wombat \vee y = ferret)) \vee (x = camaba \wedge y = wombat)$$

$$R0 = \{(i, j) \mid i \neq j \wedge \exists k.\ i * k = j\}$$

Suppose P_0 is a two-argument predicate and $R_0 = \{(x, y) \mid P_0(x, y)\}$, and *mutatis mutandis* for P_1 and R_1. What is $R_0 \cap R_1$? What is $R_0 \cup R_1$?

Defining sets the modern way:

$$(x, y) \in CanUse \equiv \; (x = wabeel \wedge (y = wombat \vee y = ferret)) \vee (x = camaba \wedge y = wombat)$$

Matrices

A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. Two matrices are equal if they have the same dimensions and same entries.

Suppose R is a relation from the set $\{1, 2, \ldots, m\}$ to the set $\{1, 2, \ldots, n\}$. We can represent R by an $m \times n$ matrix M of 0s and 1s like this:

$$(i, j) \in R \;\equiv\; M[i, j] = 1$$

$users = \{\text{wabeel, camaba, kbodzak, mbrady, acoppola}\}$
$printers = \{\text{wombat, badger, ferret}\}$
$CanUse = \{(\text{wabeel, wombat}), (\text{camaba, wombat}), (\text{wabeel, ferret})\}$

Number the elements of $users$ and $printers$ in the order given.

Represent $CanUse$ by this 5×3 matrix: $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Relations as boolean functions

Let S be a finite set $\{a_0, a_1, \ldots, a_m\}$ and T be $\{b_0, b_1, \ldots, b_n\}$.
Let $R \subseteq S \times T$.

Represent R by a matrix M like this:
$(x, y) \in R$ iff $M[i, j] = 1$ where $x = a_i$ and $y = b_j$

Represent R by a predicate P like this:
$(x, y) \in R$ iff $P(x, y)$ is true

In case S is $\{0, 1, \ldots, m\}$, i.e., $a_i = i$, and same for T, then
$P(i, j) \equiv (M[i, j] = 1)$.

Essentially the same concept. Matrix works well as data structure, when the sets aren't too big. But we won't be studying arrays in Scheme.

Relations as digraphs

Rosen: "A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*)."

What does "or" mean?

Draw digraphs with labelled dots as nodes and arrows for edges. Nice for humans, especially finite graphs and finite humans.

Essentially, a digraph is nothing more than a set and a relation on that set.

Some properties of relations

Let R be a relation on T, i.e., $R \subseteq T \times T$.

"R is *reflexive*" means: $\forall x \in T\ (x, x) \in R$.

R is *symmetric*: $\forall x, y\ ((x, y) \in R \to (y, x) \in R)$

R is *anti-symmetric*: $\forall x, y\ ((x, y) \in R \wedge (y, x) \in R \to x = y)$

R is *transitive*: $\forall x, y, z\ ((x, y) \in R \wedge (y, z) \in R \to (x, z) \in R)$

Properties, in terms of predicates

Suppose $R = \{(x, y) \mid P(x, y)\}$.

Then "R is *reflexive*" iff $\forall x \in S \; P(x, x)$.

R is *symmetric*: $\forall x, y \; (P(x, y) \to P(y, x))$

R is *anti-symmetric*: $\forall x, y \; (P(x, y) \wedge P(y, x) \to x = y)$

R is *transitive*: $\forall x, y, z \; (P(x, y) \wedge P(y, z) \to P(x, z))$

Operations on relations

Let $R \subseteq T \times U$.

Define R^{-1}, the *inverse*, to be a subset of $U \times T$ as follows:
$R^{-1} = \{(x, y) \mid (y, x) \in R\}$.
That is: $\boxed{(x, y) \in R^{-1} \text{ iff } (y, x) \in R}$ (for all x, y)

What if $R^{-1} \subseteq R$? What if $id \subseteq R$?

Let $S \subseteq U \times V$.

Define the *composition* $S \circ R$ to relate T to V as follows:
$(x, z) \in S \circ R$ iff $\boxed{\exists y \ ((x, y) \in R \wedge (y, z) \in S)}$.

What is $parent \circ parent$? What is $CanUse^{-1} \circ CanUse$?

a note on matrix multiplication

Sum of matrices of the same size —like disjunction of relations on the same set. Works for conjunction. What R^{-1}? What about composition of relations?

Let M be $m \times n$ and N be $n \times p$.
Define MN by $(MN)[i,j] = (\Sigma_{k=1}^{n} M[i,k] * N[k,j])$

For boolean matrix (Rosen sect.9.3), replace $*$ by $\wedge$ and Σ by $\exists$.