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Assignment 12 - Solutions

1. a) We know $\cos(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n}$, convergent everywhere. It follows that, as long as $\frac{1}{z}$ exists, we can find the Laurent series centered at $z_0 = 0$ by replacing w with $\frac{1}{z}$, giving us $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n} = \sum_{n=0}^{-} \infty \frac{(-1)^n}{(-2n)!} z^{2n}$, convergent everywhere except at $z = 0$.

   Since this gives us a power series centered at 0 and convergent in the annulus $A(0, 0, \infty)$, this is the Laurent series centered at 0.

   b) Since cos is entire and $\frac{1}{z}$ is analytic everywhere except at $z = 0$, we have that g is analytic unless $z = 0$ or $\cos(\frac{1}{z}) = 0$. By a computation seen in class, we know $\cos(\frac{1}{z}) = 0 \iff \frac{1}{z} = (n + \frac{1}{2})\pi \iff z = \frac{1}{(n+\frac{1}{2})\pi}$ for some $n \in \mathbb{Z}$, so the set of singularities is $\{0\} \bigcup \{\frac{1}{(n+\frac{1}{2})\pi}, n \in \mathbb{Z}\}$

   If $z = \frac{1}{(n+\frac{1}{2})\pi}$, then taking $r = \min\{|z - \frac{1}{(n+1+\frac{1}{2})\pi}|, |z - \frac{1}{(n-1+\frac{1}{2})\pi}|\}$ we have that there are no singularities in $D_r(z)$ so z is an isolated singularity. However, since $\lim_{n\to\infty} \frac{1}{(n+\frac{1}{2})\pi} = 0$, we have that 0 is not an isolated singularity.

   c) Picking as before $z_n = \frac{1}{(n+\frac{1}{2})\pi}$, we have that $f(z_n) = 0 \forall n$ and $\lim_{n\to\infty} z_n = 0$. Taking now $w_n = \frac{-i}{n}$, $f(w_n) = \cos(\frac{n}{-i}) = \cos(in) = \cosh(n)$ so $\lim_{n\to\infty} w_n = 0$ but $\lim_{n\to\infty} |f(w_n)| = \infty$.

   d) Since $\lim_{n\to\infty} |f(z_n)| = 0$, the theorem of classification of singularities tells us that 0 is not a pole of f since otherwise the modulus of f would go to infinity as we approach 0. Since $\lim_{n\to\infty} |f(w_n) = \infty|$ we have that $|f|$ is not bounded in any neighbourhood of 0 and Riemann's theorem tells us 0 can't be a removable singularity.

   By elimination, 0 must be an essential singularity. a) verifies this as the Laurent series has an infinite number of $c_{-n} \neq 0$.

2. a) $f(z) = (z^2 + 1)\frac{1}{z} = (z + i)(z - i)\frac{1}{z}$. Defining $g_0(z) = (z + i)(z - i)$, $g_i(z) = (z + i)\frac{1}{z}$ and $g_{-i}(z) = (z - i)\frac{1}{z}$ we have that $g_0(0) \neq 0$, $g_i(i) \neq i$ and $g_{-i}(-i) \neq 0$ with all three being analytic at their respective points.

   Since $f(z) = (z - i)g_i(z) = (z + i)g_{-i}(z)$ then, by a theorem seen in class and the properties above, we have that i and $-i$ are zeroes of order 1; likewise since $f(z) = \frac{1}{(z-0)} g_0(z)$ then 0 must be a pole of order 1.

   b) By a theorem seen in class, since $g(z) = \frac{1}{f(z)}$, zeroes become poles of equal order and poles become removable singularities. It follows that i and $-i$ are poles of order 1 of g and 0 is a removable singularity.

c) Let $H(z) = \frac{z}{z^2+1}$. If $z \neq 0$, $H(z) = \frac{z}{z}g(z) = g(z)$, if $z = 0$, $H(z) = 0$. It follows that $H(z) = h(z)$ and, since $H = z\frac{1}{z+i}\frac{1}{z-i}$, using the same argument as in a) H has only two points of non-analiticity at i and $-i$, neither of which are in $D_1(0)$.