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Assignment 10 - Solutions

1. We know that $\frac{1}{z^4-1} = \frac{1}{4}(\frac{1}{z-1} - \frac{1}{z+1} + \frac{i}{z-i} - \frac{i}{z+i})$, which allows us to write $\int_C \frac{dz}{z^4-1} = \int_C \frac{dz}{z-1} - \int_C \frac{dz}{z+1} + i\int_C \frac{dz}{z-i} - i\int_C \frac{dz}{z+i}$.

   For each of these four integrals, the function has only one point of non-analyticity, namely 1, -1, i and $-i$, so we can use our contour deformation theorem to get:

   $$\int_C \frac{dz}{z^4-1} = \int_C \frac{dz}{z-1} - \int_C \frac{dz}{z+1} + i\int_C \frac{dz}{z-i} - i\int_C \frac{dz}{z+i}$$
   $$= \int_{C_1} \frac{dz}{z-1} - \int_{C_2} \frac{dz}{z+1} + i\int_{C_3} \frac{dz}{z-i} - i\int_{C_4} \frac{dz}{z+i}$$

   where C_1 is the positively oriented circle of radius 1 around 1, C_2 the positively oriented circle of radius 1 around -1, C_3 the positively oriented circle of radius 1 around i and C_4 the positively oriented circle of radius 1 around $-i$.

   Defining $z_1(t) = e^{it}+1$, $z_2(t) = e^{it}-1$, $z_3(t) = e^{it}+i$ and $z_4(t) = e^{it}-i$, with $t \in [0, 2\pi]$, gives us parametrizations of C_1, C_2, C_3 and C_4 respectively, with $z_i'(t) = ie^{it}$ for all i.

   Using these parametrizations we get:

   $$\int_{C_1} \frac{dz}{z-1} = \int_{C_2} \frac{dz}{z+1} = \int_{C_3} \frac{dz}{z-i} = \int_{C_4} \frac{dz}{z+i} = \int_0^{2\pi} i\,dt = 2\pi i$$

   which allows us to conclude $\int_C \frac{dz}{z^4-1} = 2\pi i - 2\pi i - 2\pi + 2\pi = 0$.

2. a) C has as initial point 0 and as final point -3π, and our function is entire and has the anti-derivative $\sin z$. The fundamental theorem of calculus allows us to compute that $\int_C \cos z\,dz = \sin(-3\pi) - \sin(0) = 0$

   b) C has as initial point $1+i$ and as final point $-i$, and our function is entire and has the anti-derivative e^z. The fundamental theorem of calculus allows us to compute that $\int_C e^z\,dz = e^{-i} - e^{1+i} = (\cos -1 + i\sin -1) - e(\cos 1 + i\sin 1) = (1-e)\cos 1 - i(1+e)\sin 1$

3. Let $f(z) = \sin^6 z$, we know f is entire and C contains the point $z = \frac{\pi}{6}$ in its interior, so the Cauchy integral formula tells us that $f(\frac{\pi}{6}) = \frac{1}{2\pi i}\int_C \frac{\sin^6 z}{z-\frac{\pi}{6}}\,dz$.

   $f(\frac{\pi}{6}) = (\frac{1}{2})^6 = \frac{1}{64}$ so using our previous formula we have $\int_C \frac{\sin^6 z}{z-\frac{\pi}{6}}\,dz = \frac{2\pi i}{64} = \frac{\pi i}{32}$.