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Assignment 11 - Solutions

1. It is easy to see that $f(z) = 2\pi z$ if $\Im(z) \in (\frac{-1}{2}, \frac{1}{2}]$ and $f(z) = 2\pi z - 2\pi i)$ if $\Im(z) \in (\frac{1}{2}, \frac{3}{2}]$.

   Let us then take C to be the positively oriented square of side $\frac{1}{2}$ centered at $\frac{i}{2}$. We cannot compute this integral directly as f is not continuous on C. However, we can define two half-squares A_ϵ and B_ϵ where $A_\epsilon = [\frac{1}{4} + i(\frac{1}{2} + \epsilon), \frac{1}{4} + \frac{3i}{4}] \bigcup [\frac{1}{4} + \frac{3i}{4}, \frac{-1}{4} + \frac{3i}{4}] \bigcup [\frac{-1}{4} + \frac{3i}{4}, \frac{-1}{4} + i(\frac{1}{2} + \epsilon)]$ and $B_\epsilon = [\frac{-1}{4} + i(\frac{1}{2} - \epsilon), \frac{-1}{4} + \frac{i}{4}] \bigcup [\frac{-1}{4} + \frac{i}{4}, \frac{1}{4} + \frac{i}{4}] \bigcup [\frac{1}{4} + \frac{i}{4}, \frac{1}{4} + i(\frac{1}{2} - \epsilon)]$, where $[,]$ denote straight lines and the order gives us orientation. It is plain to see that $C = \lim_{\epsilon \to 0} A_\epsilon \bigcup B_\epsilon$.

   However, f is analytic on A_ϵ and B_ϵ for any $\epsilon > 0$, so we can use our fundamental theorem of calculus.

   A_ϵ begins at $\frac{1}{4} + i(\frac{1}{2} + \epsilon)$ and ends at $\frac{-1}{4} + i(\frac{1}{2} + \epsilon)$ with the anti-derivative being $\pi z^2 - 2\pi i z$, so we have

   $$\int_{A_\epsilon} f(z)\, dz = \pi(\frac{-1}{4} + i(\frac{1}{2} + \epsilon))^2 - 2\pi i(\frac{-1}{4} + i(\frac{1}{2} + \epsilon)) - \pi(\frac{1}{4} + i(\frac{1}{2} + \epsilon))^2 + 2\pi i(\frac{1}{4} + i(\frac{1}{2} + \epsilon))$$

   $$\Rightarrow \lim_{\epsilon \to 0} \int_{A_\epsilon} f(z)\, dz = \pi(\frac{-1}{4} + \frac{i}{2})^2 - \pi(\frac{1}{4} + \frac{i}{2})^2 + \pi i$$

   B_ϵ begins at $\frac{-1}{4} + i(\frac{1}{2} - \epsilon)$ and ends at $\frac{1}{4} + i(\frac{1}{2} - \epsilon)$ with the anti-derivative being πz^2, so we have

   $$\int_{B_\epsilon} f(z)\, dz = \pi(\frac{1}{4} + i(\frac{1}{2} - \epsilon))^2 - \pi(\frac{-1}{4} + i(\frac{1}{2} - \epsilon))^2$$

   $$\Rightarrow \lim_{\epsilon \to 0} \int_{B_\epsilon} f(z)\, dz = \pi(\frac{1}{4} + \frac{i}{2})^2 - \pi(\frac{-1}{4} + \frac{i}{2})^2 +$$

   Putting them together gives us that $\int_C f(z)\, dz = \lim_{\epsilon \to 0} \int_{A_\epsilon} f(z)\, dz + \lim_{\epsilon \to 0} \int_{B_\epsilon} f(z)\, dz = \pi i$.

2. Suppose there exists an entire function f such that $|f(x+iy)| = x^2 + y^2 + 1$. We know that $x^2 + y^2 \geq 0$ for all $x, y \in \mathbb{R}$. It follows $|f(x + iy)| = x^2 + y^2 + 1 \geq 0 + 1 = 1$ for all $x + iy \in \mathbb{C}$.

   By our corollary to Liouville's theorem, since f is entire and its modulus is bounded, we know f is constant. However, that would mean $|f|$ is constant, but $x^2 + y^2 + 1$ isn't. Contradiction.

3. a) $f(z) = (z + i)^{-1}$, $f'(z) = -(z + i)^{-2}$,
   $f''(z) = 2(z + i)^{-3}$, $f'''(z) = -6(z + i)^{-4}$
   $\Rightarrow f(0) = -i$, $f'(0) = 1$, $f''(0) = 2i$, $f''(0) = -6$
   $\Rightarrow T_4(z) = -i + z + iz^2 - z^3$

b) By looking at our pattern above, we can see that for arbitrary n we have $f^{(n)}(z) = (-1)^n n!(z+i)^{-n-1} = i^{2n} n!(z+i)^{-n-1}$.

This means that $f^{(n)}(0) = i^{2n} n! i^{-n-1} = n! i^{n-1}$. Using this into our Taylor series formula gives us:

$T(z) = \sum_{n=0}^{\infty} \frac{f^{(n)} z^n}{n!} = \sum_{n=0}^{\infty} i^{n-1} z^n$

c) f has a single point of non-analyticity at $z = -i$, which is at distance 1 from our basepoint 0. By the theorem of convergence of Taylor series, this means the radius of convergence of the Taylor series of f centered at 0 is 1.