Don’t forget!

Quiz 1 on Friday, February 23.

- Practice problems will be posted on Canvas this week
- Syllabus includes everything up to todays lecture

Next Monday (Feb 19) is a holiday

- Monday schedule next Wednesday
- CAs will lead a review session

No Problem Set next week

Relations

A relation R with domain A and range B is a subset of $A \times B$

A relation R over a set A is a subset of $A \times A$.

$A = \{$EWR, BOS, DCA, LAX, SFO,ORD, DEN, MIA$\}$
$FLIGHTS$=$\{($EWR,ORD$)$, $($BOS,DCA$)$,$($LAX,SFO$)$, $($DEN,LAX$)$, $($DCA,MIA$)$,$($SFO,EWR$)$,
$($ORD,DEN$)$, $($LAX,BOS$)$,$($MIA,SFO$)\}$

Properties of Relations

A relation R over a set A is:

- ***Reflexive*** if $\forall x \in A : (x,x) \in R$

$$DIVIDES = (a,b) : a,b \in \mathbb{N} \uparrow + \quad \wedge a \square b$$

- ***Anti-Reflexive*** if $\forall x \in A : (x,x) \notin R$

$$GREATER = \{(a,b) : a,b \in \mathbb{N} \quad \wedge a > b\}$$

Properties of Relations

A relation R over a set A is:

- ***Symmetric*** if $\forall x,y\in A$: $(x,y)\in R \Leftrightarrow (y,x)\in R$

$$CLOSEBY=\{(a,b):a,b\in\mathbb{N}\wedge|a-b|\leq 2\}$$

- ***Anti-Symmetric*** if $\forall x,y\in A$: $((x,y)\in R\wedge(y,x)\in R)\Rightarrow(x=y)$

$$DIVIDES=(a,b):a,b\in\mathbb{N}\uparrow+\quad\wedge a\square b$$

Properties of Relations

A relation R over a set A is:

- ***Transitive*** if $\forall x,y,z \in A: ((x,y) \in R \land (y,z) \in R) \Rightarrow (x,z) \in R$

$$DIVIDES = (a,b) : a,b \in \mathbb{N} \ \land a \square b$$

$$IMPLIES = \{(P,Q) : P \Rightarrow Q\}$$

Equivalence Relations

A relation R over a set A that is reflexive, symmetric and transitive is called an ***equivalence*** relation.

Examples:

$$\{(P,Q):P \Leftrightarrow Q\}$$

$$\{(a,b):rem(a,3)=rem(b,3)\}$$

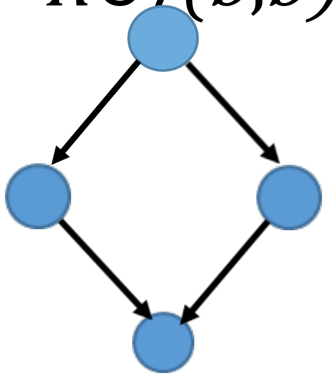
Reflexive Closure

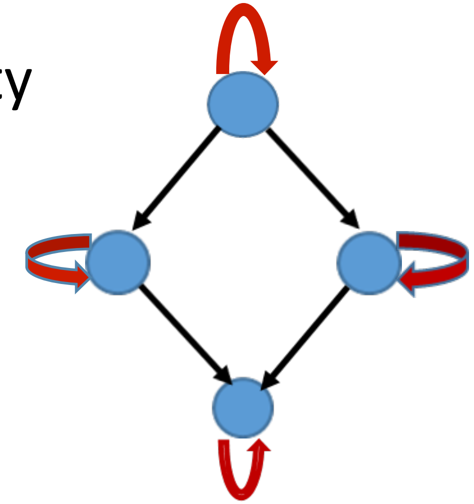
The ***reflexive closure*** of relation R is the smallest reflexive relation $r(R)$: $r(R) \supseteq R$.

Example:

$$R = \{(a,a),(a,b), (b,c)\}$$

$r(R) = R \cup \{(b,b),(c,c)\} = R \cup I$, where I is the identity





Symmetric Closure

The ***symmetric closure*** of relation R is the smallest symmetric relation $s(R)$: $s(R) \supseteq R$.

Example:

$$R = \{(a,a),(a,b), (b,c)\}$$

$$s(R) = R \cup \{(b,a),(c,b)\} \quad = \quad R \cup R\uparrow- \quad \text{where } R\uparrow- \text{ is the inverse of } R$$

Transitive Closure

The ***transitive closure*** of relation R is the smallest transitive relation $R^{+} \supseteq R$.

Example:

$$R = \{(a,a),(a,b),\ (b,c)\}$$

$$R^{+} = R \cup \{(a,c)\}$$

Composing Relations

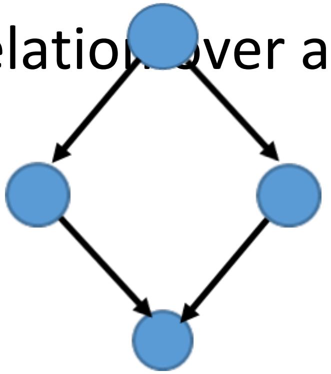
Given two relations $R{:}A\to B$, $\quad S{:}B\to C$

we define the composition

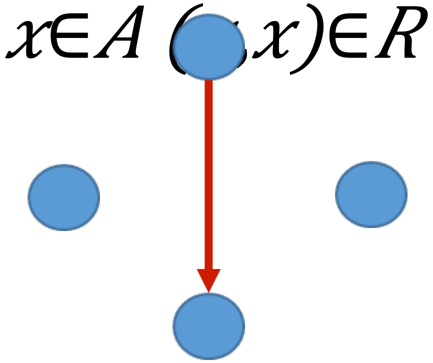
$S\circ R{:}A\to C$ as

$$\{(a,c): a\in A \wedge c\in C \wedge \exists\, b\in B: (a,b)\in R \wedge (b,c)\in S\}$$

If R is a relation over a set A then $R\circ R=\{(a,b): \exists\, x\in A\ (a,x)\in R \wedge (x,b)\in R\}$



R: direct flights



$R\circ R$: one-stop flights

Composing Relations

If R is a relation over a set A then $R \circ R = \{(a,b) : \exists x \in A \ (a,x) \in R \wedge (x,b) \in R\}$

$$R \circ (R \circ R) = \{(a,b) : \exists x, y \in A \ (a,x) \in R \wedge (x,y) \in R \wedge (y,b) \in R\}$$

R: direct flights

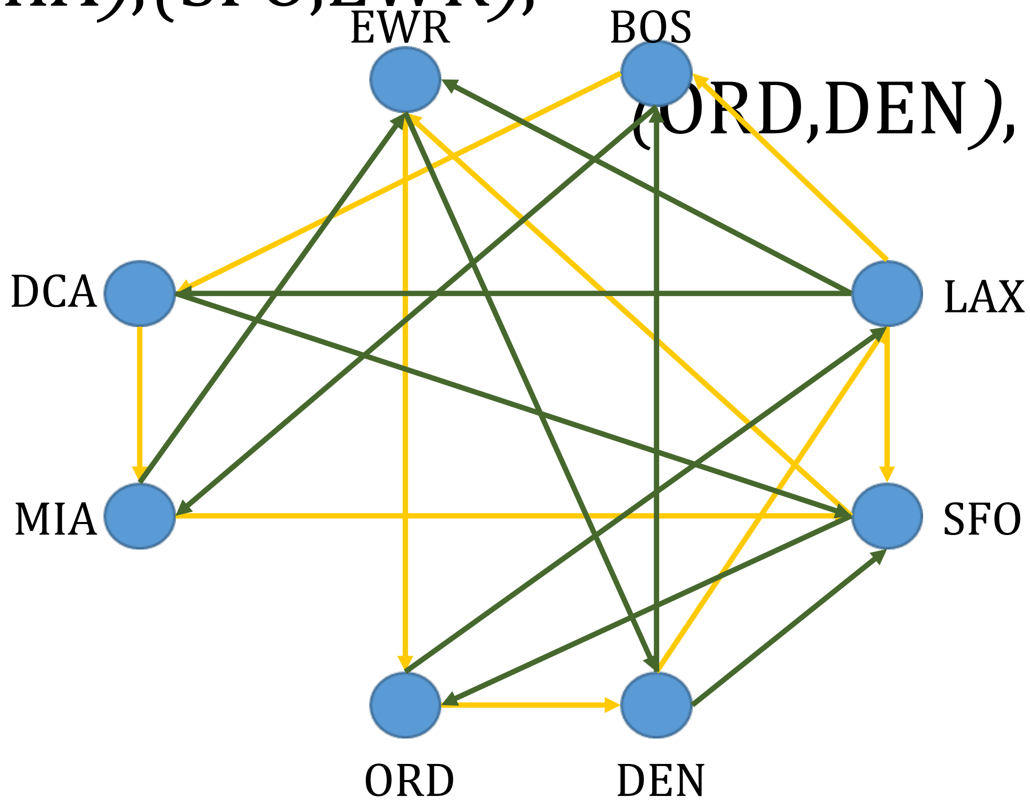
$R \circ R = R \uparrow 2$: one-stop flights

$R \circ R \circ R = R \uparrow 3$: two-stop flights

In general: $(a,b) \in R \uparrow k$ iff there is a sequence of k flights from a to b.

Our little airline

$A = \{$EWR, BOS, DCA, LAX, SFO,ORD, DEN, MIA$\}$
$FLIGHTS = \{$(EWR,ORD), (BOS,DCA),(LAX,SFO), (DEN,LAX), (DCA,MIA),(SFO,EWR), (ORD,DEN), (LAX,BOS),(MIA,SFO)$\}$



Composing Relations

Suppose A consists of n cities and that one can fly (directly or indirectly) from a to b

Then there is a sequence of k flights where $1 \leq k \leq n$. (Why not $n-1$?)

In other words, $(a,b) \in R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

Theorem: For any relation R over a set A, $|A| = n$,

$$R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

Corollary: If R is reflexive then $R^+ = R^n$