THE PROBABILITY DISTRIBUTION OF THE SAMPLE MEAN

If X is a r.v. (say, IQ score of a randomly selected person), then $\bar{X}_n$ (the average of the IQ's of n randomly selected people) is **ALSO** a r.v..
We will discuss the properties and characteristi of this new r.v., $\bar{X}_n$.

Define the r.v.:
X = the IQ score of a random person
and say that we know that the mean of X (the average value of X) is

$$E(X) = \mu = 100$$

What if we now repeatedly take samples of groups of 50 people and record the average IQ's of the groups:

Sample 1 $\bar{X}_{50} = 104.3$
Sample 2 $\bar{X}_{50} = 101.9$
Sample 3 $\bar{X}_{50} = 98.9$
Sample 4 $\bar{X}_{50} = 99.8$
$\vdots$ $\vdots$

What do you think the average of all these group averages would be? Some reflection should lead you to conclude that the average of the average IQ scores would still be 100.

That is: if $E(X) = \mu$, then $E(\bar{X}_n) = \mu$

Pf: $E(\bar{X}) = E\left(\sum_{i=1}^{n} X_i/n\right) = \frac{1}{n} E(\Sigma X_i) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n}(n\mu) = \mu$

Now, what about the variability (standard deviation) of these group averages? The group averages would tend not to vary as much as the individual IQ scores because of the tendency of group averages to have large values balance out small values. For example, an individual IQ score of 135 would not be surprising to get, but to have the average IQ of 50 randomly chosen people be 135 would be very surprising. So we have that $\sigma_{\bar{x}_n}$ will be less than σ, and the question is: by how much. The answer is:

If σ = standard deviation of the X's

and $\sigma_{\bar{x}_n}$ = standard deviation of the averages of sample of size n

Then $\sigma_{\bar{x}_n} = \dfrac{\sigma}{\sqrt{n}}$

where n = sample size

So, summarizing:

If X is a r.v. with

$$\text{mean} = E(X) = \mu$$

$$\text{and st.dev.} = \sigma$$

and if we now consider the new r.v. $\overline{X}_n$, where $\overline{X}_n$ is the average of a sample of n items, then for this r.v. we have

$$\text{mean} = E(\overline{X}_n) = \mu$$

$$\text{and st dev} = \sigma_{\overline{X}_n} = \frac{\sigma}{\sqrt{n}}$$

***

and it is also true (by the Central Limit theorem) that $\overline{X}_n$ is approximately normally distributed if either of the following hold:

- X is normally distributed
- n is sufficiently large (usually ≥ 30)

  ↑ sample size

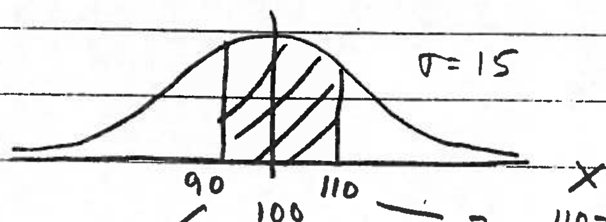
2) Assume

X = IQ score of a random person

and assume $X \sim N(100, 15)$.

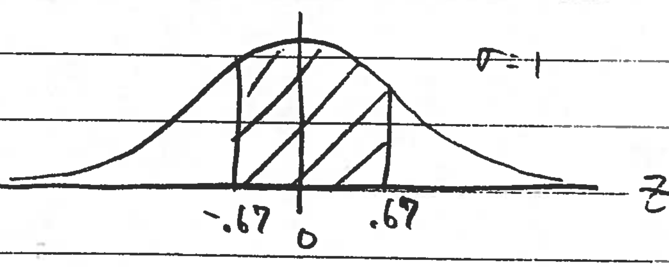
a) Find the probability that a randomly selected person will have an IQ between 90 and 110.

We seek $P(90 < X < 110)$



$z = \frac{110-100}{15} = .67$

$z = \frac{90-100}{15} = -.67$



$P(90 < X < 110) = P(-.67 < Z < .67) = .2486 + .2486 = \underline{\underline{.4972}}$

b) Now, let's choose a random sample of $n = 30$ people and compute the average IQ score ($\bar{X}_{30}$) of the group. What is the probability that the average IQ score of this group is between 90 and 110?

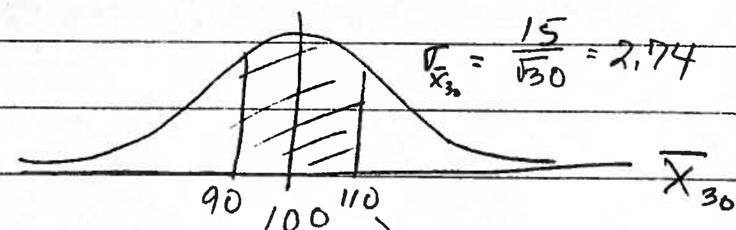
We seek: $P(90 < \bar{X}_{30} < 110)$

NOTE: OUR R.V. HERE IS $\bar{X}_{30}$!

FIRST: $\bar{X}_{30} \sim N\left(100, \frac{15}{\sqrt{30}}\right) = N(100, 2.74)$

Now we draw the $\bar{X}_{30}$ distribution, since that's the r.v. about which we're asking the question:

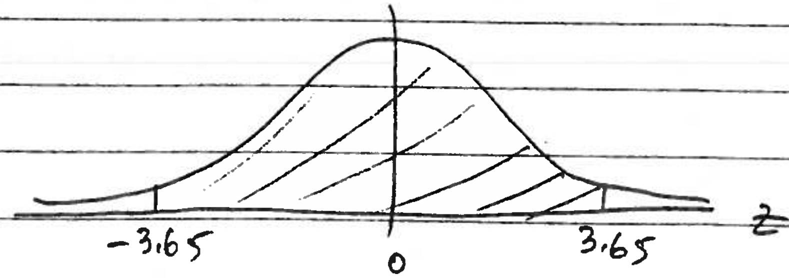
$\sigma_{\bar{X}_{30}} = \frac{15}{\sqrt{30}} = 2.74$



Now: to convert to z-scores, we use $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$z = \frac{90-100}{2.74} = -3.65$ $\qquad$ $z = \frac{110-100}{2.74} = +3.65$

So:



$P(90 < \bar{X}_{30} < 110) = P(-3.65 < z < 3.65) \doteq .9996$

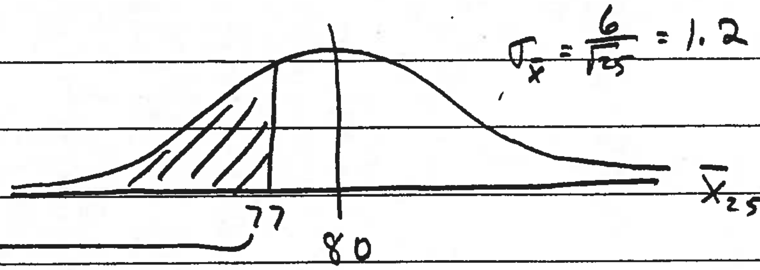
(nearly 1)

ex/ Assume that final grades in Statistics 101 are normally distributed with mean 80 and standard deviation 6. What is the probability that the class <u>average</u> (class size = 25) of Mr. Adams' Statistics 101 class is less than 77?

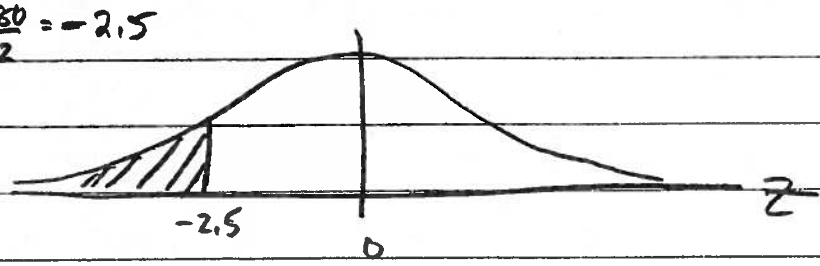
$$\bar{X}_{25} \sim N\left(80, \frac{6}{\sqrt{25}}\right) = N(80, 1.2)$$

and we seek

$$P(\bar{X}_{25} \leq 77)$$



$z = \frac{77-80}{1.2} = -2.5$



So

$$P(\bar{X}_{25} \leq 77) = P(Z < -2.5) = .5 - .4938 = \underline{\underline{.0062}}$$

POINT TO PONDER: GIVEN THIS ANSWER, WHAT MIGHT YOU CONCLUDE IF MR. ADAMS' CLASS HAD A CLASS AVERAGE OF 76?