1.	2.	3.	4.	5.	6.	Total:

MA 232. **Exam 1.** **October 7, 2016.**

Print name: ______________________________

Instructor: A.Myasnikov

Closed book and closed notes. Show all of your work. Answers without supporting work will not receive credit.

Pledge and sign: ______________________________

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Problem 1. (10pts) Prove or disprove:

(a) If $\bar{\mathbf{v}} = (1,1)$ and $\bar{\mathbf{w}} = (1,5)$ then $\bar{\mathbf{w}} - 3\bar{\mathbf{v}}$ is perpendicular to $\bar{\mathbf{v}}$

(b) Set of 2×2 invertible matrices is closed under addition and scalar multiplication.

(c) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ then $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has only 1 solution.

Solution:

(a) True: $\bar{\mathbf{v}} \cdot (\bar{\mathbf{w}} - 3\bar{\mathbf{v}}) = \bar{\mathbf{v}}\bar{\mathbf{w}} - 3\bar{\mathbf{v}}\bar{\mathbf{v}} = 1 \cdot 1 + 1 \cdot 5 - 3(1 \cdot 1 + 1 \cdot 1) = 6 - 6 = 0$

(b) False: $0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ - not invertible.

(c) False: There is one free variable so it is possible to have infinitely many solutions depending on $\bar{\mathbf{b}}$.

Problem 2. (10pts) Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

Solution:

Augment matrix with I and perform Gaussian-Jordan procedure:

$$A = \left[\begin{array}{rrr|rrr} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right]$$

The inverse is the right side of the augmented matrix;

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

Problem 3. (10pts) Compute LU-decomposition of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}$$

Solution:

Obtain rows below diagonal in the first column of A:

1. $R_2 = R_2 - 2R_1$
2. $R_3 = R_3 + 3R_1$

The corresponding elimination matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

and

$$E_2E_1A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

Obtain zero below diagonal in the second column:

$$R_3 = R_3 + R_2 \Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

And we have

$$E_3E_2E_1A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Move $E_3E_2E_1$ to the right:

$$A = E_1^{-1}E_2^{-1}E_3^{-1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Note that

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}; \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

We obtain LU-decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Problem 4. (10pts) Express the polynomial $q = t + 2$ as a linear combination of the polynomials

$$p_1 = t^2 + 2t - 1, \quad p_2 = -t^2 - t + 2, \quad p_3 = 3t^2 + 4t - 4$$

[*Hint:* the two polynomials are equal iff the corresponding coefficients are equal]

Solution

Need to find c_1, c_2, c_3 such that $v = c_1 p1 + c_2 p_2 + c_3 p_3$. Equalizing coefficients of corresponding monomials we obtain 3 linear equations:

$$\begin{aligned} c_1 + 2c_2 &= 1 \\ -2c_1 - 3c_2 + c_3 &= 4 \\ 5c_1 + c_3 &= -3 \end{aligned}$$

Which has the solution $c_1 = -\frac{17}{11}, c_2 = \frac{14}{11}, c_3 = \frac{52}{11}$, therefore

$$v = -\frac{17}{11}p_1 + \frac{14}{11}p_2 + \frac{52}{11}p_3$$

Problem 5. (10pts) Find vectors that span the space of all points on the plane

$$x + 2y + z + 2t = 0$$

Solution.

The corresponding vector space W contains all vectors $\bar{\mathbf{x}}$ that are solutions to the equation

$$\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix} \bar{\mathbf{x}} = 0$$

Let $\bar{\mathbf{x}} = (x, y, z, t)$ there are 3 free variables in the equation above, therefore we need at least 3 vectors to span W. Use special solutions to find vectors. Note they will be automatically linearly independent which guarantees that you do not need more vectors to span the subspace W.

(a) Let $y = 1, z = 0, t = 0$ then $\bar{\mathbf{x}}_1 = (-2, 1, 0, 0)$

(b) $y = 0, z = 1, t = 0$ then $\bar{\mathbf{x}}_2 = (-1, 0, 1, 0)$

(c) $y = 0, z = 0, t = 1$ then $\bar{\mathbf{x}}_3 = (-2, 0, 0, 1)$

Then

$$W = span(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3)$$

Problem 6. (10pts) Write the complete solution to the following system of linear equations

$$\begin{aligned} x + 2y - z + 3t &= 4 \\ 2x + 4y - 2z + 7t &= 10 \\ -x - 2y + z - 4t &= -6 \end{aligned}$$

Solution

In matrix form the equation is

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 7 \\ -1 & -2 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

The general solution is of the form

$$\bar{\mathbf{x}}^* = \bar{\mathbf{x}}_p + \bar{\mathbf{x}}_n,$$

where $\bar{\mathbf{x}}_p$ is any one solution to the equation and $\bar{\mathbf{x}}_n$ a vector from the nullspace of the matrix. To describe any all vectors $\bar{\mathbf{x}}_n$ we need to find the basis of the nullspace.

First perform gaussian on the matrix augmented with the right-habd side vector. This will allow us to find a particular solution and we will also have the reduced matrix which can be used to find the nullspace:

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 7 & 10 \\ -1 & -2 & 1 & -4 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There are two free variables: y and z

(a) To find $\bar{\mathbf{x}}_p$ set $y = z = 0$ and solve equation above. We have

$$\begin{aligned} x + 3t &= 4 \\ t &= 2 \end{aligned}$$

hence $x = -2$ and $\bar{\mathbf{x}}_p = (-2, 0, 0, 2)$

(b) Find the null space. Use reduced matrix on the left side:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with free variables y, z

Let $y = 1, z = 0$ then the equation is

$$\begin{aligned} x + 2 + 3t &= 0 \\ t &= 0 \end{aligned}$$

therefore $x = -2$ and the first vector in a basis of $N(A)$ is $(-2, 1, 0, 0)$

Set $y = 0, z = 1$ then

$$\begin{aligned} x - 1 + 3t &= 0 \\ t &= 0 \end{aligned}$$

therefore $x = 1$ and the first vector in a basis of $N(A)$ is $(1, 0, 1, 0)$
The complete solution:

$$\bar{\mathbf{x}}^* = \bar{\mathbf{x}}_p + \bar{\mathbf{x}}_n = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$