1.	2.	3.	4.	5.	6.	Total:

MA 232. **Exam 3.** **December 7, 2017.**

Print name: ______________________________

Instructor: A.Myasnikov

Closed book and closed notes. Show all of your work. Answers without supporting work will not receive credit.

Pledge and sign: ______________________________

____

Problem 1. (10pts) Prove or disprove. Show work supporting your argument:

(a) If P_1 and P_2 are two permutation matrices then absolute value $det(P_1 P_2^{-1}) = 1$

(b) If A is 2×2 matrix with eigenvalues $\lambda_1 = 0.5$ and $\lambda_2 = 0.3$ then $A^\infty \to I$.

(c) Let A be a nonsingular matrix then it is possible to have just one distinct eigenvalue.

Solution:

(a) True: For any permutation matrix P: $det(P) = \pm 1$. $det(P^{-1}) = \frac{1}{det(P)} = \pm 1$ Therefore, $det(P_1 P_2^{-1}) = \pm 1$ and absolute value is 1.

(b) False: A is 2×2 and has 2 distinct eigenvalues - diagonalisable:

$$A = S \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix} S^{-1}$$

Then

$$A^\infty = S \begin{bmatrix} 0.5^\infty & 0 \\ 0 & 0.3^\infty \end{bmatrix} S^{-1} = S \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) True: Identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is non-singular and has characteristic polynomial $(1-\lambda)^2 = 0$, i.e. there is one distinct eigenvalue $\lambda = 1$ of multiplicity 2.

Problem 2. (5pts) Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Compute the determinant of the the matrix

$$B = 3A^{-1}A^T A^5 (A^{-1})^T$$

Solution: Determinant of A is -2. Using properties of the determinant:

$$|B| = |3A^{-1}A^T A^5 (A^{-1})^T| = 3^2_1 \frac{1}{det(A)} det(A)^5 \frac{1}{det(A)} = 9 \cdot 16 = 144$$

Problem 3. (5pts) Find b, if it exists, which makes the following matrix symmetric positive definite. Explain your answers.

$$\begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

Solution: Use the fact that matrix is symmetric positive definite iff determinants of all principle submatrices are strictly positive. In this case we see that $|2| > 0$ and $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} > 0$ So need to find b s.t. determinant of the whole matrix is positive. Computing determinant gives $(b-2)(b+1) < 0$ this is possible if

(a) $b - 2 > 0$ and $b + 1 < 0$, or

(b) $b - 2 < 0$ and $b + 1 > 0$

The first case gives $b > 2$ and $b < -1$ which is not possible. The second is $b < 2$ and $b > -1$ or $-1 < b < 2$.

Problem 4. (10pts) Let

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Find all *real* eigenvalues of A.

Solution: To compute the characteristic polynomial note that the last column of the matrix $(A - \lambda I)$ has all zeros except the last element. Using the last column we compute $\Delta(A)$:

$$(1-\lambda) \cdot \begin{vmatrix} -\lambda & -3 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)\left(-\lambda(-1-\lambda)+3\right) = (1-\lambda)\left(\lambda^2+\lambda+3\right)$$

$\lambda^2 + \lambda + 3 = 0$ does not have real solutions so the only real eigenvalue is

$$\lambda = 1$$

Problem 5. (10pts) Let

$$A = \begin{bmatrix} -8 & 9 \\ 0 & 1 \end{bmatrix}$$

Compute $\sqrt[3]{A}$.

Solution: A is an upper triangular matrix. The determinant of a tringular matrix is the product of diagonal elements. Eigenvalues of A: $(-8-\lambda)(1-\lambda) = 0$ and $\lambda_1 = -8$, $\lambda_2 = 1$. Two distinct eigenvalues, therefore A is diagonalisable and

$$D = \begin{bmatrix} -8 & 0 \\ 0 & 1 \end{bmatrix}$$

Find eigenvectors.

$\lambda_1 = -8$: $\bar{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\lambda_2 = 1$: $\bar{\mathbf{x}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Hence

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A = SDS^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -8 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

To compute the root, we compute roots of the diagonal in D:

$$\sqrt[3]{A} = SDS^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt[3]{-8} & 0 \\ 0 & \sqrt[3]{1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$$

Problem 6. (10pts) Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

Find an ***orthonormal*** basis of $\mathbb{R}^3$ consisting of eigenvectors of A or ***prove*** that it is not possible.

Solution: The characteristic polynomial of the A is

$$\Delta(A) = (1-\lambda)(5-\lambda)(4-\lambda)-(-2)(-2)(5-\lambda) = (5-\lambda)\left[(1-\lambda)(4-\lambda)-4\right] = (5-\lambda)(\lambda^2-5\lambda) = -\lambda(5-\lambda)^2 = 0$$

so we have two eigenvalues. One is 0 and another is 5 multiplicity 2. Find eigenvectors.

$\lambda = 0$ and the corresponding eigenvector is in the basis of the nullspace of the matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \end{bmatrix}$$

There is one free variable so there is one eigenvector

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 5$: the corresponding eigenvectors are the basis of the nullspace of the matrix

$$\begin{bmatrix} -4 & 0 & -2 \end{bmatrix}$$

There are two free variables, the corresponding eigenvectors:

$$\bar{\mathbf{x}}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \bar{\mathbf{x}}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Easy to see that $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3$ are orthogonal as was expected since A is symmetric. To obtain normal basis, compute corresponding unit vectors:

$$\bar{\mathbf{u}}_1 = \frac{\bar{\mathbf{x}}_1}{||\bar{\mathbf{x}}_1||} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{\mathbf{u}}_2 = \frac{\bar{\mathbf{x}}_2}{||\bar{\mathbf{x}}_2||} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\bar{\mathbf{u}}_3 = \frac{\bar{\mathbf{x}}_3}{||\bar{\mathbf{x}}_3||} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$