Lecture 2: Logical equivalence; predicate logic

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CS 135 Discrete Structures
Spring 2015

Outline of lecture

Review of propositional logic

Logical equivalence

Predicate logic

The *boolean scalars* or *propositional values*: T and F.

"or" means "and"?

Some boolean operators: $\wedge$, $\vee$, $\neg$, $\rightarrow$, ...,

Can define by exhaustive enumeration, e.g.,

p	q	p∧q
F	F	F
F	T	F
T	F	F
T	T	T

Can also evaluate expressions that way, e.g, $(p \wedge q) \vee (\neg p \wedge q)$.
(BTW for now, do not omit parentheses here: the precedence table (p11) says $\neg$ tighter than $\wedge, \vee$ tighter than $\rightarrow, \leftrightarrow$).

Given some operators, others can be defined:
Define $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$
Define $p \oplus q = (p \vee q) \wedge \neg(p \wedge q)$

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Applications

p: sleep late

q: miss class

r: get rested

"I sleep late and get rested."

"If I sleep late I miss class."

"I really like to sleep late."

We're going to focus on boolean operators as mathematical structure.

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Functionally complete sets of operators

All of the propositional functions can be defined in terms of $\neg$ and $\wedge$, which means hardware can be built from just two kinds of "gates" that implement these functions.

Just "nand" by itself is functionally complete.

Tautology and equivalence defined

A propositional formula is a *tautology* if it is true no matter what values its variables have. It is a *contradiction* if it is false no matter what.

Some tautologies: $p \to p$, $\quad true \vee p$, $\quad p \vee true$. (Do the truth tables.)

Some contradictions: $p \wedge \neg p$, $F \vee F$

Alert: we will start using p, q etc to range over formulas, not just as variables. They're "metavariables"!

Say p and q are *logically equivalent* iff $p \leftrightarrow q$ is a tautology. Write $p \equiv q$ for short.

Examples: $(p \wedge T) \equiv p$, $\quad p \wedge q \equiv q \wedge p$, $\quad p \to p \equiv T$

Quiz: what's a tautology, in terms of equivalence

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Some equivalences

$p \vee T \equiv T$ "domination law", or T is the zero element for $\vee$

$p \vee (p \wedge q) \equiv p$ "absorption law"

These are general laws; we can *instantiate* p and q with any formula.

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Proving equivalences: four methods

How to prove an equivalence $p \equiv q$ for some given p, q? For example, $p \to q \equiv \neg q \to \neg p$.

First method: prove $p \equiv q$ by filling out truth table for p and q; they should have same value in every row.

Second: prove that $p \leftrightarrow q$ is a tautology, using a truth table.

Third: calculate algebraically: $p \equiv \ldots \equiv q$ using known laws. Why is this sound? (What does "sound" mean?)

Fourth: calculate algebraically: $p \leftrightarrow q \equiv \ldots \equiv T$ using known laws. Why is this sound?

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Exercise

Prove: $(p \to r) \vee (q \to r)$ is equivalent to $(p \wedge q) \to r$

Predicates

Conditions that pertain to things, i.e., boolean-valued functions or "propositional functions". (We should say what kind of things.)

Some named examples we'll use later.

$P(x)$: $x > 0$ (the property of being greater than zero)

$Q(x)$: x is boring (a property of games)

$R(x)$: x is infected with a virus (a property of computers)

$S(x, y)$: x is infected with virus y

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Predicates in program assertions

Alert: I'm going to use the symbol = for equality in math, as well as for assignment in code. (Better to use := for assignment.)

$//assert \quad x = a \wedge y = b$
$t = \; x;$
$//assert \quad x = a \wedge y = b \wedge t = a$
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Quiz: can we conclude $x = b \wedge y = a$ afterwords? Why?

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Logical quantifiers

$\forall x\ P(x)$ means "all numbers are positive" (it's false)
$\exists x\ P(x)$ means "some number is positive"
Domain of discourse: the values we consider possible for x.

Other notations:
$\forall x : \mathbf{N}\ .\ P(x)$ $\qquad (\forall x \in \mathbf{N} :: P(x))$
`forall int x; P(x)` in the Java Modeling Language

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Using predicate logic

Express these as predicate logic formulas:
"Every user has access to exactly one mailbox"

"There is a process that continues to run during all error conditions only if the kernel is working properly"

"All users on campus can access all websites whose url ends in .edu"

What are the nouns? (domain of discourse?) The verb phrases? (predicates) Logical connectives and quantifiers?
$A(u, m)$ means "user u has access to mailbox m"
$H(e)$ means "error condition e is in effect"
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Predicate logic tautologies

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$\forall x\ (P(x) \vee \neg P(x))$ no matter what P is

$(\forall x\ (\forall y\ R(x,y)) \leftrightarrow (\forall y\ (\forall x\ R(x,y))$ no matter what R is

$(\exists x\ P(x)) \rightarrow (\forall x\ P(x))$???

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