ARITHMETIC

MA 236 - INTRODUCTION TO MATHEMATICAL REASONING

1. The language of arithmetic

Constant: 0.

Variables: x, y, x_0, x_1, etc.

Function symbols: $s, +, \times$. Instead of $+(x, y)$, we write $(x + y)$ etc.

Predicate: $=$.

2. The axioms of Robinson arithmetic.

See Sections 6.9 and 6.10 of the text.

Axiom Q1: $\forall x \forall y (x \neq y \rightarrow sx \neq sy)$.
Axiom Q2: $\forall x\, 0 \neq sx$.
Axiom Q3: $\forall x (x \neq 0 \rightarrow \exists y\, x = sy)$.
Axiom Q4: $\forall x (x + 0) = x$.
Axiom Q5: $\forall x \forall y (x + sy) = s(x + y)$.
Axiom Q6: $\forall x (x \times 0) = 0$.
Axiom Q7: $\forall x \forall y (x \times sy) = [(x \times y) + x)]$.

Until further notice "Axioms" means these axioms.

3. The theory of Robinson arithmetic

The theory of Robinson arithmetic is the collection of all sentences which are provable from the axioms.

Example 1. $s0 \neq ss0$ (in other words $1 \neq 2$) can be proved from Axioms Q1 and Q2. The following argument is valid.

$$\begin{array}{l} \forall x \forall y (x \neq y \rightarrow sx \neq sy) \\ \forall x\, 0 \neq sx \\ \hline s0 \neq ss0 \end{array}$$

As usual, we construct a refutation tree.

Date: Spring 2017.

$$
\begin{array}{c}
1.\ \forall x\, \forall y\, (x \neq y \to sx \neq sy) \\
| \\
2.\ \forall x\, 0 \neq sx \\
| \\
3.\ s0 = ss0 \\
| \\
4.\ 0 \neq s0 \qquad (2) \\
| \\
5.\ \forall y\, (0 \neq y \to s0 \neq sy) \qquad (1) \\
| \\
6.\ 0 \neq s0 \to s0 \neq ss0 \qquad (5) \\
/ \quad \backslash \\
7.\ 0 = s0 \qquad s0 \neq ss0 \qquad (\ 6\ ) \\
| \qquad\quad | \\
8.\ \times \qquad \times \quad (3,\ 4,\ 7)
\end{array}
$$

Exercise 2. Check that the refutation tree in Example 1 is correct.

4. Pruned refutation trees

Page 94 (last pragraph) and page 95 in the text.

From now on we allow shortcuts in refutation trees. Axioms do not need to be listed explicitly as premises. Also steps like 5 and 6 in the tree of the previous section can be combined. Thus that tree becomes

$$
\begin{array}{c}
1.\ s0 = ss0 \\
| \\
2.\ 0 \neq s0 \qquad (\mathrm{Q2}) \\
| \\
3.\ 0 \neq s0 \to s0 \neq ss0 \qquad (\mathrm{Q1}) \\
/ \quad \backslash \\
4.\ 0 = s0 \qquad s0 \neq ss0 \qquad (\ 3\ ) \\
| \qquad\quad | \\
5.\ \times \qquad \times \quad (1{,}2{,}4)
\end{array}
$$

Exercise 3. Prove that $ss0 \neq sss0$.

5. Models of Robinson arithmetic

Section 6.10, Problem 2.

An interpretation in which the Axioms are true is called a model of Robinson arithmetic. Every sentence in the theory of Robinson arithmetic is true in every model of it because the conclusion of a valid argument is true whenever its premises are.

The standard model of Robinson arithmetic is the non-negative integers $N = \{0, 1, 2, \ldots\}$ with the usual extensions of $0, +, \times$. The extension of s is the successor function $sx = x + 1$. (Recall that in every interpretation the extension of $=$ is always real equality.)

Problem 4. Verify that the following interpretation of Robinson arithmetic is a model.

> **Domain:** $N \cup \{i\}$
> **Extensions:** The extension of 0 is that same as in the standard model, and the extensions of $s, +, \times$ are also same when only standard integers are involved. When i is involved,
> (1) $si = i$;
> (2) $x + i = i = i + x$ for all x in the domain;
> (3) $x \times i = i = i \times x$ for all x in the domain.

6. Incompleteness of Robinson arithmetic

Theorem 5. *Robinson arithmetic is incomplete. In other words there is a sentence which is true in the standard model N but not provable from the Axioms.*

Proof. $\forall x\, x \neq sx$ is true in the standard model because $x \neq x + 1$ for all x in N. On the other hand $\forall x\, x \neq sx$ is false in the nonstandard model of Problem 4 because $si = i$. If follows that $\forall x\, x \neq sx$ cannot be proved from the Axioms (or else it would be true in all models of Robinson arithmetic). □

Problem 2 in Section 6.10 describes a more complicated model in which some other sentences true in the standard model are false.

7. Induction

Section 6.10 Problem 3.

Robinson arithmetic is incomplete; we cannot deduce the true (in the standard model) sentence $\forall x\, x \neq sx$ from the Axioms. Let's try to add more axioms (true in the standard model) so that we can at least prove $\forall x\, x \neq sx$.

Call a subset $S \subset N$ *inductive* if

(1) 0 is in S.
(2) For any integer x, if x is in S, then so is sx.

In fact the only inductive subset of N is N itself. We would like to add an axiom which says that every inductive subset of N is equal to all of N. Such an axiom would begin "for all subsets … ", but we can only say "for all elements …".

So instead we add a new tree rule *which works only in the standard model.*

$$\begin{array}{c} \ldots 0 \ldots \\ \underline{\neg \ldots a \ldots} \\ \ldots b \ldots \\ \neg \ldots sb \ldots \end{array}$$

Problem 6. Use the new tree rule to prove $\forall x\, x \neq sx$.

8. The Compactness Theorem

Recall that the refutation tree procedure tests whether or not a set of sentences is consistent.

(1) If the procedure halts with all paths closed, the set of sentences in inconsistent. They cannot all be true in any interpretation.
(2) If the procedure halts with one or more open paths, then the sentences are consistent. Each open path specifies extensions of constants, predicates, and functions so that all the sentences are true. (If the interpretation is not completely specified by these extensions, the missing extensions may be chosen arbitrarily.)
(3) If the procedure does not halt, then there must be an infinite open path. This path specifies an interpretation as above, even though in most cases we will not know what that that interpretation is.

With more argument along the lines above one can prove the following fundamental theorem.

Theorem 7 (Compactness Theorem.)**.** *Let S be a set of sentences over a language. Either there is an interpretation which makes all the sentences true (in other words a model of S) or some finite subset of S is inconsistent.*

9. A nonstandard model

In Section 5 we constructed a nonstandard model of Robinson arithmetic. Now we use the Compactness Theorem to construct a non-standard model of the theory of N.

The theory of N is the collection of all sentences (in the language of arithmetic) which are true in the standard model. The existence of a nonstandard model shows a limitation of logic. Everything we can say about N with logic is not enought to determine N uniquely. There is another interpretation which is different from N but has the same theory.

(1) Add a new constant a to the languages of Robinson arithmetic.
(2) Define S to be the set of sentences consisting of all the Axioms together with the infinite set of new sentences

$$a \neq 0 \quad a \neq 1 \quad a \neq (1+1) \quad a \neq ((1+1)+1) \quad \cdots$$

Informally the sentences above say $a \neq 0$, $a \neq 1$, $a \neq 2$, $a \neq 3$, etc.
(3) Let T be any finite subset of S. Since T is finite, it includes only a finite number of our new sentences. Suppose T includes the first 1000 of the new sentences.
(4) The Axioms are true in N, and the first 1000 new sentences will be true as long as the extension of a is 1001 or greater.
(5) So every finite subset of S is consistent. By the Compactness Theorem S has a model.

The model we have constructed is nonstandard because it contains an element not equal to $0, 1, 2, 3, \cdots$. There is no such element in the standard model.