Where we are

1. Logical reasoning
    - Propositions and Predicates
        - Laws of logic, Rules of Inference
        - Valid arguments
2. Sets
    - Set Operations: union, intersection, cross-product, difference, complement
    - Set identities
    - Set cardinality: finite, countably infinite, well-ordering principle, uncountable
3. Relations
    - Properties: reflexive, symmetric, transitive, equivalence, closures
4. Functions
    - Properties: injective, surjective, bijective

Where we're headed

Use what we've learned to build stronger proof techniques

- Principle of Induction

Use our proof techniques to solve problems

- Counting
  - How many instructions will my program execute?
- Number theory
  - Design cryptographic systems that are hard to crack
- Graph theory
  - Exploit structure in real world applications to create efficient solutions

Predicates over $\mathbb{N}$

Let $P(x)$ be a predicate over the natural numbers.

Example:

Predicate: $P(n)$: $0+1+2+\cdots+n = n(n+1)/2$

Propositions: $P(0)$: $0 = 0\cdot(0+1)/2$

$P(1)$: $0+1 = 1\cdot(1+1)/2$

$P(2), P(3), P(4), \ldots$

Is each of these (infinitely many) propositions TRUE?

How do we prove the truth of infinitely many propositions?

Modus Ponens Revisited

Let $P(x)$ be a predicate over the natural numbers.

Suppose we are told that each of the following is true:

$$P(0)$$

$$P(0) \Rightarrow P(1)$$

We can conclude that $P(1)$ is true.

What if we're also told that $P(1) \Rightarrow P(2)$ is true?

We can conclude that $P(2)$ is true.

Modus Ponens Ad Infinitum

Suppose we are told that

$P(0)\ is\ true$

$P(0) \Rightarrow P(1)$

$P(1) \Rightarrow P(2)$

$P(2) \Rightarrow P(3)$

$P(3) \Rightarrow P(4)$

.

.

.

The Principle of Induction (PI)

$$P(0)$$
$$\forall k \geq 0:\ P(k) \Rightarrow P(k+1)$$
$$\therefore\ \forall n \in \mathbb{N}:\ P(n)$$

To prove that $P(x)$ is TRUE for all natural numbers:

Step 1. **(BASIS)** Show that $P(0)$ is TRUE

Step 2. **(INDUCTIVE HYPOTHESIS)** $P(k)$

Step 3. **(INDUCTIVE STEP)** Show that $P(k) \Rightarrow P(k+1)$ is TRUE for all k

PI in Action

Theorem. $\forall n \in \mathbb{N}\ P(n)$, where $P(n)$: $0+1+2+\cdots+n = n(n+1)/2$

PROOF:

Step 1. (Basis) $P(0)$: $0 = 0 \cdot (0+1)/2$ is TRUE.

Step 2. (Inductive Hypothesis) $P(k): 0+1+\cdots+k = k(k+1)/2$

Step 3. (Inductive Step) $0+1+\cdots+(k+1) = (0+1+\cdots+k)+(k+1)$

Apply the inductive hypothesis to the first summand on the right:

So the LHS $= k(k+1)/2 + (k+1) = (k+1)(k/2+1) = (k+1)(k+2)/2$

In other words, $P(k+1)$ is TRUE.

So we have shown that $P(k) \Rightarrow P(k+1)$, and the theorem follows from PI.

Another Predicate over $\mathbb{N}$

Consider the sequence created by summing the odd numbers

$$1,\ 1+3,\ 1+3+5,\ 1+3+5+7,\ 1+3+5+7+9,\ \ldots$$

The sequence is $\quad 1,\ 4,\ 9,\ 16,\ 25,\ \ldots$

But this is the same as $1 \uparrow 2\ , 2 \uparrow 2\ , 3 \uparrow 2\ , 4 \uparrow 2\ , 5 \uparrow 2\ , \ldots$

The first n odd numbers are $\quad 1,\ 3,\ \cdots,\ (2n-1)$

Let $P(n)$: $\quad 1+3+\cdots+(2n-1)=n \uparrow 2$

$P(1)$: $\quad 1=1 \uparrow 2$

$P(2)$: $\quad 1+(2\cdot 2-1)=2 \uparrow 2$

Are $P(1), P(2), P(3), P(4), P(5), P(6), \ldots$ *all TRUE?*

PI to the rescue!

Theorem. $\forall n>0 \;\; P(n)$, where $P(n)$: $\;1+3+\cdots+(2n-1)=n\uparrow 2$

PROOF:

Step 1. (Basis) $P(1)$: $\;1=1\uparrow 2$ is TRUE.

Step 2. (Inductive Hypothesis) $P(k)$:$1+\cdots+(2k-1)=k\uparrow 2$

Step 3. (Inductive Step) $1+\cdots+(2(k+1)-1)=(1+\cdots+(2k-1))+(2k+1)$

Apply the inductive hypothesis to the first summand on the right:

So the LHS $=k\uparrow 2+(2k+1)=(k+1)\uparrow 2$

In other words, $P(k+1)$ is TRUE.

since $P(k)\Rightarrow P(k+1)$, the theorem follows from PI.

Proof of PI

$$P(0) \wedge \ \forall k \geq 0 : (P(k) \rightarrow P(k+1))$$

$$\forall n \in \boldsymbol{N} : P(n)$$

Proof. (by contradiction):

Let $C = \{n \in \boldsymbol{N} : P(n) = False\}$

Suppose that C is non-empty.

Then, by the well-ordering principle, it has a least element k.

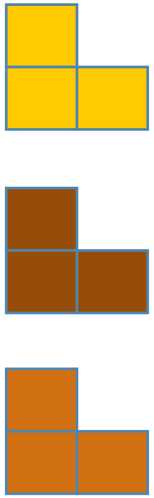
Now, $k \neq 0$, because $P(0)$ is true. Therefore $k > 0$.

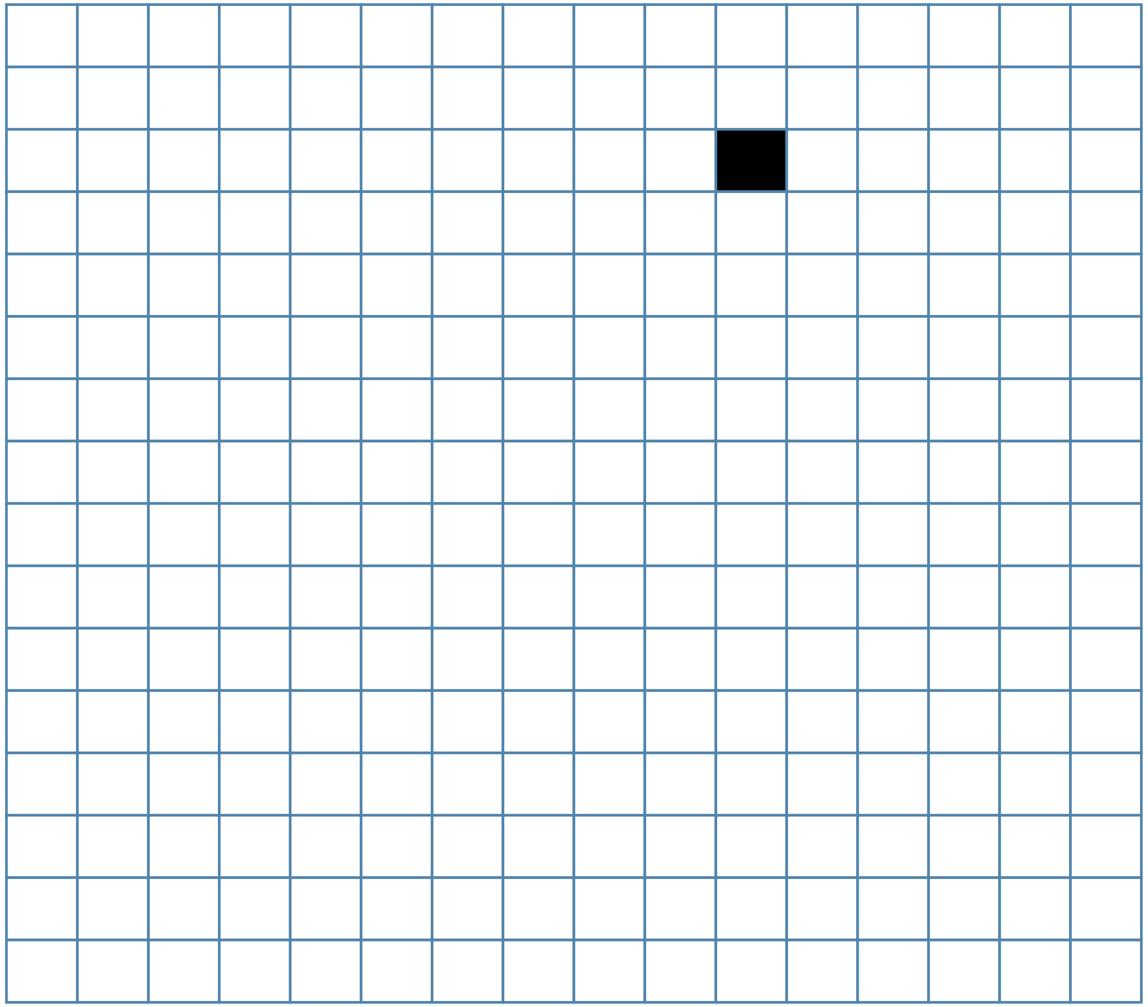
Since $k-1 \notin C$, $P(k-1) = True$

By the inductive step, $P(k-1) \Rightarrow P(k)$

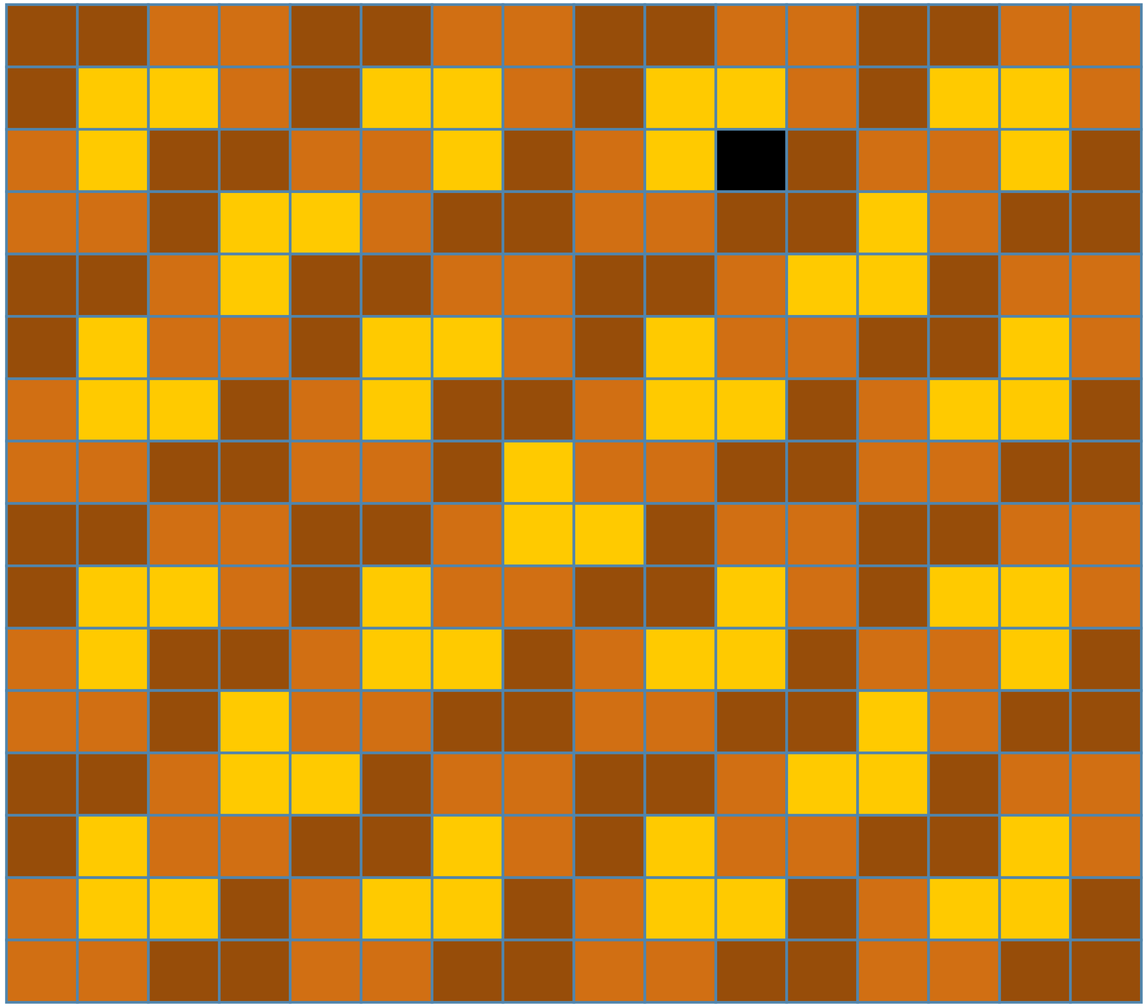
Therefore, $P(k)$ is true. Contradiction!

Induction: Tiling a Courtyard





Tile the 16 x 16 courtyard using multiple tiles of the given shape.



Tiling the Courtyard

Define the predicate $P(n)$: Every $2\uparrow n \times 2\uparrow n$ courtyard with one hole can be tiled using triominoes.

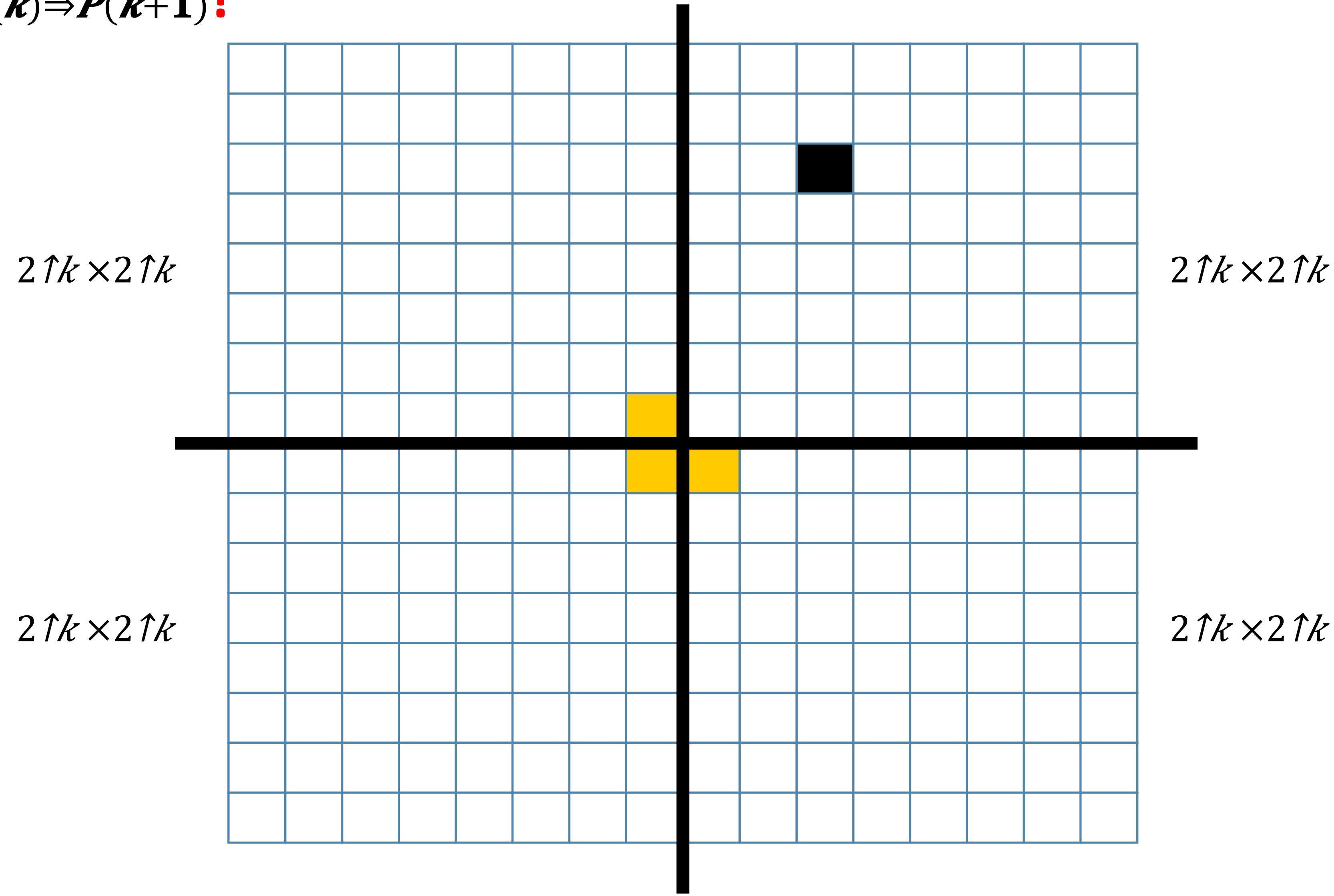
Theorem: $\forall n \in \mathbb{N} \;\; P(n)$.

Basis: $P(0)$: 1×1 courtyard with a hole (!). Vacuously TRUE.

Inductive hypothesis: $P(k)$: Claim is true for any $k \geq 0$.

Inductive step: Show that $P(k) \Rightarrow P(k+1)$.

$\boldsymbol{P(k) \Rightarrow P(k+1)}$!



Recursively tiling the courtyard