Previous Lecture

Relations

Functions

- Injective (one-to-one)
- Surjective (onto)
- Bijective

Sets of equal cardinality

- $|A|=|B|$ if there is a *bijective* function from A to B.

- $|A|<|B|$ if there is an injective function from A to B but
  no surjective function from A to B

Infinite Sets

A set S is *countable* if it is finite or if $|A|=|\mathbb{N}|$.

Examples of countable sets:

$|\mathbb{N}|=|\mathbb{Z}|$

$|\{0,2,4,6,\ldots\}|=|\{1,3,5,7,\ldots\}|=|\mathbb{N}|$

$|\mathbb{N}|=|\mathbb{Z}\times\mathbb{Z}|$

$|PRIMES|=|\mathbb{N}|$

$|\mathbb{Q}|=|N|$

Is every set countable?

Power Sets

The power set $P(S)$ of a set S is defined as:

$$P(S)=\{X\colon X\subseteq S\}$$

"The set of all subsets of S"

$$P(\{1,2\})=\{\emptyset, \{1\},\{2\},\{1,2\}\}$$

$$P(\{a,b,c\})=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$$

If a finite set S has m elements, then $P(S)$ has $2\uparrow m > m$ elements.

The Power Set Hierarchy

Theorem. For every set S, $|S| < |P(S)|$ where $P(S)$ is the power set of S.

Proof Outline:

Case 1. If S is finite and has m elements then $P(S)$ has $2\uparrow m > m$ elements.

Therefore the proposition $|S| < |P(S)|$ is true.

Case 2. If S is infinite then we will show that no function from S to $P(S)$ is surjective (i.e., no bijection exists), and therefore the proposition is true.

Since, there are no other cases to consider, the proposition is always true and therefore the theorem is valid.

(But we need to prove Case 2!)

Proof of Case 2.

Case 2. If S is infinite then we will show that no function from S to $\mathrm{P}(S)$ is surjective (i.e., no bijection exists).

Proof: Let $f\colon S \to P(S)$ be surjective.

Define $X = \{x \in S : x \notin f(x)\}$

X is a well-defined subset of S, and therefore is an element of $P(S)$.

Let $a \in S$ be any element in S.

$a \in X \Rightarrow \;\; a \notin f(a)$ Therefore, $f(a) \neq X$

$a \notin X \Rightarrow \;\; a \in f(a)$ Therefore, $f(a) \neq X$

Since, $\forall a \in S : f(a) \neq X$, it follows that f is not surjective!

Some uncountable sets

Consider the set $\mathrm{P}(\mathbb{N})$, the power set of $\mathbb{N}$.

1. $\mathrm{P}(\mathbb{N})$ is infinite.

2. By the theorem, there is no bijection from $\mathbb{N}$ to $\mathrm{P}(\mathbb{N})$.

Therefore, $\mathrm{P}(\mathbb{N})$ is not countable.

In other words, $/\mathbb{N}/ < /\mathrm{P}(\mathbb{N})/$

Also, by the theorem, there is no bijection from $\mathrm{P}(\mathbb{N})$ to $\mathrm{P}(\mathrm{P}(\mathbb{N}))$

So $|\mathrm{P}(\mathbb{N})| < |\mathrm{P}(\mathrm{P}(\mathbb{N}))|$

We can keep going!

The Infinite Hierarchy of Infinite Sets

$\boldsymbol{N}$	$\boldsymbol{P(N)}$	$\boldsymbol{P(P(N))}$	$\boldsymbol{P(P(P(N)))}$	**…**
$\aleph\downarrow 0$	$\aleph\downarrow 1$	$\aleph\downarrow 2$	$\aleph\downarrow 3$	…

$$|\boldsymbol{P(N)}| = |\mathrm{R}|$$

Cantor's Continuum Hypothesis

https://www.ias.edu/ideas/2011/kennedy-continuum-hypothesis

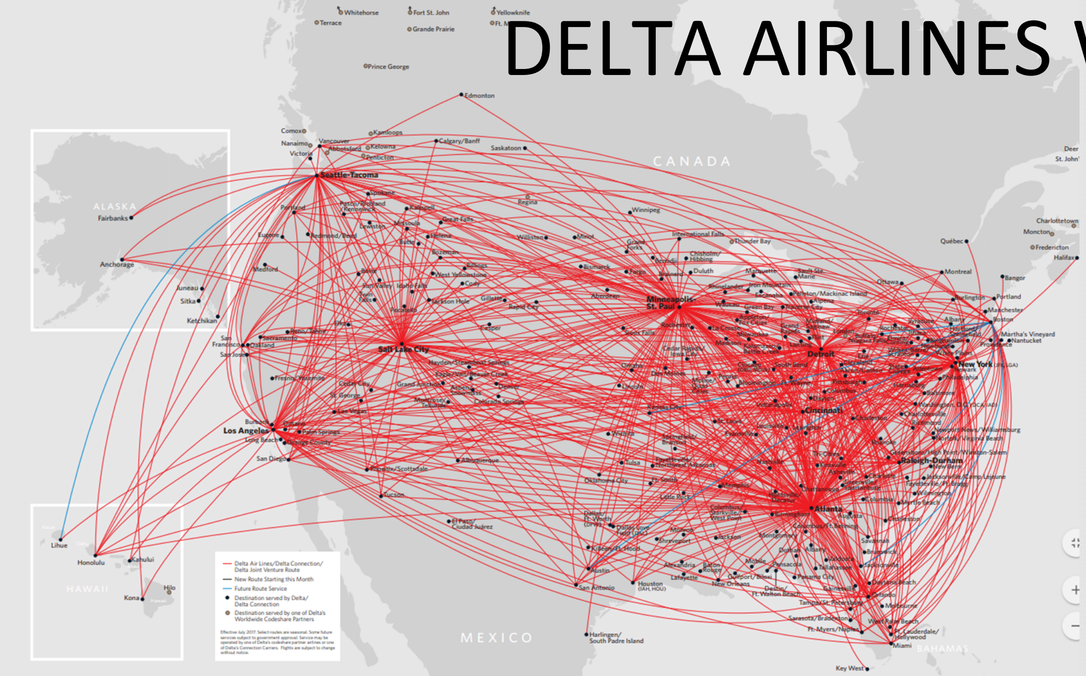
Relations

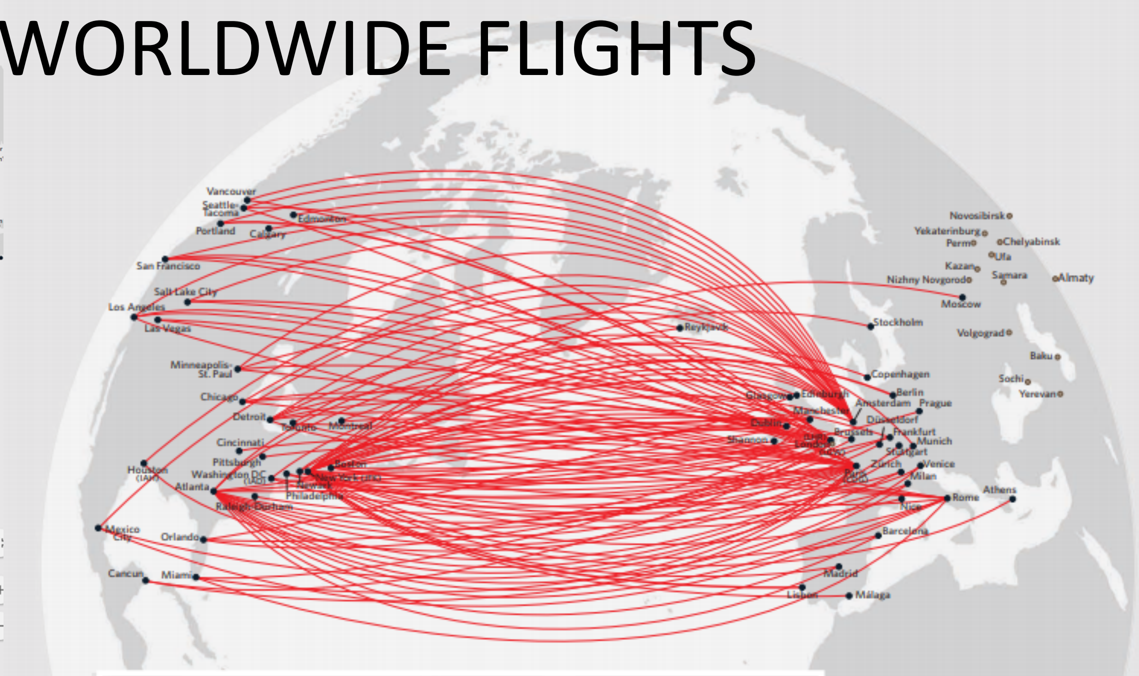
A relation R with domain A and range B is a subset of $A \times B$

A relation R over a set A is a subset of $A \times A$.

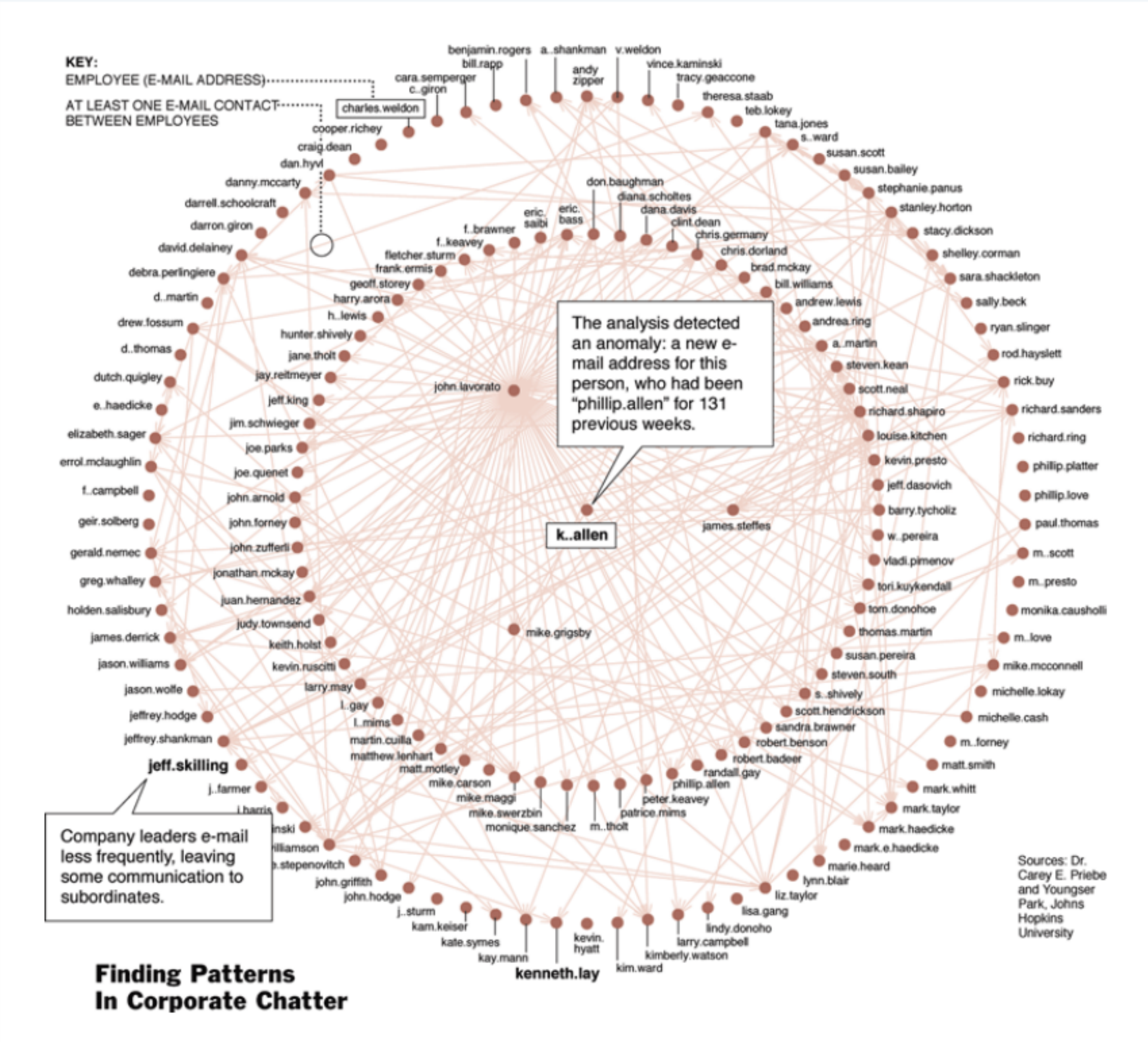
$A = \{$EWR, BOS, DCA, LAX, SFO,ORD, DEN, MIA$\}$
$FLIGHTS = \{($EWR,ORD$)$, $($BOS,DCA$)$,$($LAX,SFO$)$, $($DEN,LAX$)$, $($DCA,MIA$)$,$($SFO,EWR$)$,
$($ORD,DEN$)$, $($LAX,BOS$)$,$($MIA,SFO$)\}$

DELTA AIRLINES WORLDWIDE FLIGHTS

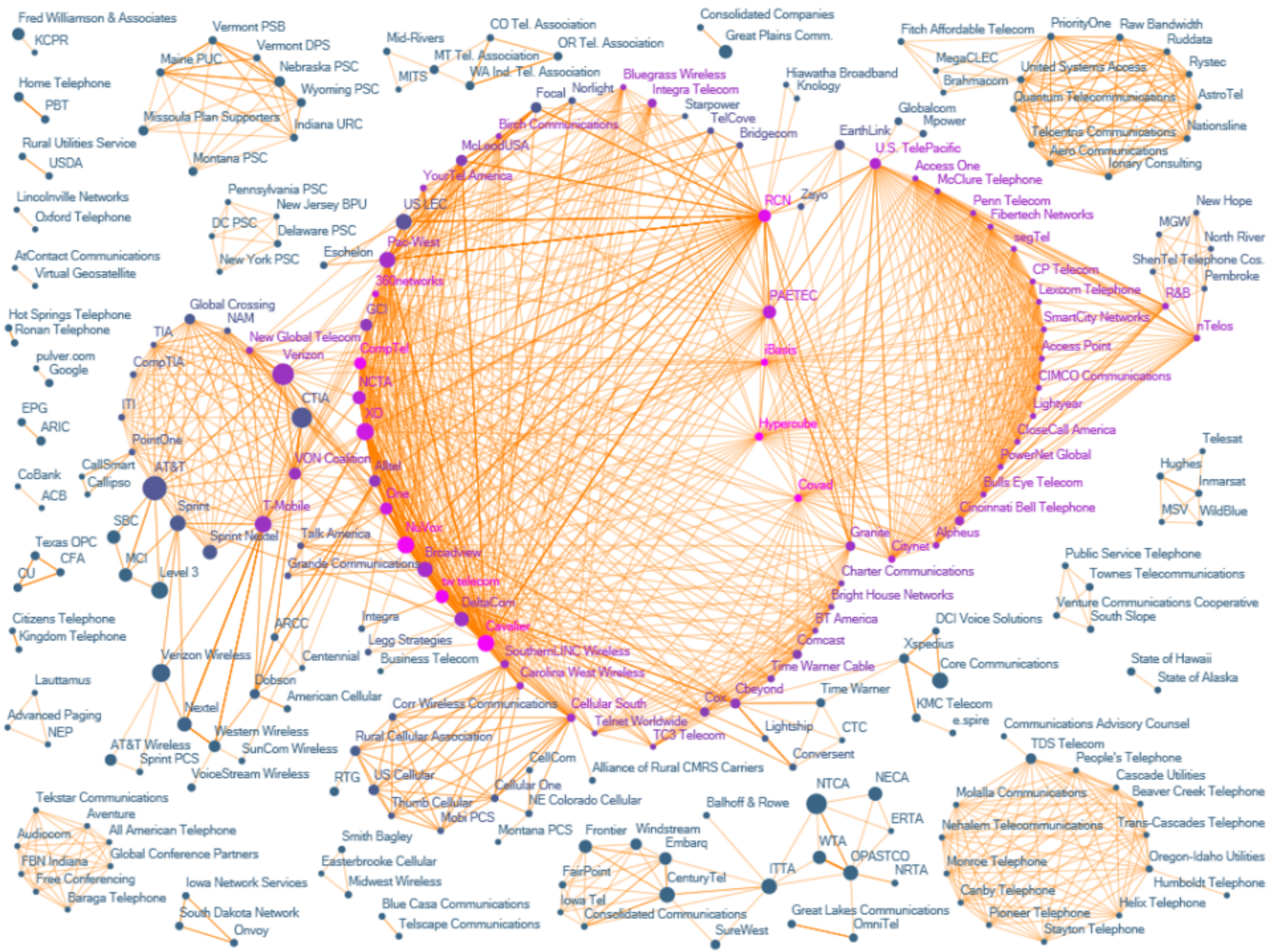




One week of Enron emails



The evolution of FCC lobbying coalitions



Properties of Relations

A relation R over a set A is:

- ***Reflexive*** if $\forall x \in A: (x,x) \in R$
- ***Anti-Reflexive*** if $\forall x \in A: (x,x) \notin R$
- ***Symmetric*** if $\forall x,y \in A: \ (x,y) \in R \Leftrightarrow (y,x) \in R$
- ***Anti-Symmetric*** if $\forall x,y \in A: \ ((x,y) \in R \wedge (y,x) \in R) \Rightarrow (x = y)$
- ***Transitive*** if $\forall x,y,z \in A: ((x,y) \in R \wedge (y,z) \in R) \Rightarrow (x,z) \in R$

Properties of Relations

A relation R over a set A is:

- ***Reflexive*** if $\forall x \in A: (x,x) \in R$

$$DIVIDES = (a,b): a,b \in \mathbb{N} \ \wedge a \square b$$

- ***Anti-Reflexive*** if $\forall x \in A: (x,x) \notin R$

$$GREATER = \{(a,b): a,b \in \mathbb{N} \ \wedge a > b\}$$

Properties of Relations

A relation R over a set A is:

- ***Symmetric*** if $\forall x,y \in A$: $(x,y) \in R \Leftrightarrow (y,x) \in R$

$$CLOSEBY = \{(a,b) : a,b \in \mathbb{N} \land |a-b| \leq 2\}$$

- ***Anti-Symmetric*** if $\forall x,y \in A$: $((x,y) \in R \land (y,x) \in R) \Rightarrow (x = y)$

$$DIVIDES = (a,b) : a,b \in \mathbb{N} \ \land a \square b$$

Properties of Relations

A relation R over a set A is:

- ***Transitive*** if $\forall x,y,z\in A:((x,y)\in R\wedge(y,z)\in R)\Rightarrow(x,z)\in R$

$$DIVIDES=(a,b):a,b\in\mathbb{N}\ \ \wedge a\square b$$