Previous Lectures

Logical Proposition: statement that is either True or False (but not both!).
Building compound propositions using logical connectives

Negation	$\neg G$	"not G"
Conjunction	$(G \wedge H)$	"G and H"
Disjunction	$(G \vee H)$	"G or H"
Implication	$G \Rightarrow H$	"G implies H"

Truth Tables
Tautologies
Contradictions
Logical Equivalence

Logical Equivalence

Two propositions P and Q that have the same truth table are *logically equivalent*

$$P \equiv Q$$

How do we prove that two propositions are equivalent?

$\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \equiv (\neg \boldsymbol{p} \vee \neg \boldsymbol{q})$ **?**

Example:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Laws of Propositional Logic

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Proving Logical Equivalence

Method 1: Construct truth tables

Method 2: Use the laws to transform one proposition into the other

Example: Are the following propositions equivalent?

P: If I attend lectures then I will do well in CS135
OR
If I do my homework then I will do well in CS135

$$(L \Rightarrow W) \vee (H \Rightarrow W)$$

Q: If I attend lectures AND do my homework then I will do well in CS 135 $(L \wedge H) \Rightarrow W$

Let L be the proposition “I attend lectures”

W be the proposition “I will do well in CS 135”

H be the proposition “I do my homework”

Applying the Laws of Propositional Logic

$P \equiv (L \Rightarrow W) \vee (H \Rightarrow W)$

$\equiv (\neg L \vee W) \vee (\neg H \vee W)$ Conditional Identity Law (applied twice)

$\equiv (\neg L \vee W) \vee W \vee \neg H$ Commutative Law

$\equiv \neg L \vee (W \vee W) \vee \neg H$ Associative Law

$\equiv \neg L \vee W \vee \neg H$ Idempotent Law

$\equiv \neg L \vee \neg H \vee W$ Commutative Law

$\equiv \neg (L \wedge H) \vee W$ De Morgan's Law

$\equiv (L \wedge H) \Rightarrow W$ Conditional Identity

The Contrapositive

$$P \Rightarrow Q$$

$\equiv \neg P \vee Q$ Conditional Identity Law

$\equiv \quad Q \vee \neg P$ Commutative Law

$\equiv \neg\neg Q \vee \neg P$ Double Negation

$\equiv \ \neg Q \Rightarrow \neg P$ Conditional Identity

Logical Reasoning

A logical argument has the form

$$\begin{array}{l} H_1 \\ \underline{H_2} \quad \text{hypotheses} \\ \therefore C \quad \text{conclusion} \end{array}$$

Example: Socrates is a man
If Socrates is a man, he is mortal
∴ Socrates is mortal

$$\begin{array}{l} S \\ \underline{S \Rightarrow M} \\ \therefore M \end{array}$$

An argument is valid if the conclusion is true when every hypothesis is true

Logical Reasoning

The form

$$\begin{array}{ll} H_1 & \\ \underline{H_2} & \text{hypotheses} \\ \therefore C & \text{conclusion} \end{array}$$

is short-hand for the proposition

$$(H_1 \wedge H_2) \Rightarrow C$$

An argument is valid if and only if its corresponding proposition is a tautology.

Rules of Inference

p $p \to q$ $\therefore q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	Modus tollens
p $\therefore p \vee q$	Addition
$p \wedge q$ $\therefore p$	Simplification

p q $\therefore p \wedge q$	Conjunction
$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	Resolution

We can build truth tables to prove that each of these rules is a tautology.

Proving Validity

If Alfred studies he will get good grades
If Alfred does not study then he will enjoy college
<u>If Alfred does not get good grades he will not enjoy college</u>
Alfred will get good grades

$$S \Rightarrow G$$
$$\neg S \Rightarrow E$$
$$\neg G \Rightarrow \neg E$$
$$\therefore\ G$$

We will use the laws of propositional logic and the rules of inference to prove that this inference is valid

Proof of Validity

A. $S \Rightarrow G$

B. $\neg S \Rightarrow E$

C. $\neg G \Rightarrow \neg E$

$\therefore G$

1. $\neg G \Rightarrow \neg E$ hypothesis, C
2. $E \Rightarrow G$ contrapositive, 1
3. $\neg S \Rightarrow E$ hypothesis, B
4. $\neg S \Rightarrow G$ hypothetical syllogism, 3,2
5. $S \Rightarrow G$ hypothesis, A
6. $G \vee \neg S$ conditional identity, 5
7. $G \vee S$ conditional identity, 4
8. G resolution, 7

Another Example

Moriarty will escape unless Holmes acts
<u>We shall rely on Watson only if Holmes does not act</u>
If Holmes does not act, Moriarty will escape unless we rely on Watson

$$\neg H \Rightarrow M$$
$$\underline{W \Rightarrow \neg H \qquad\qquad}$$
$$\neg H \Rightarrow (\neg W \Rightarrow M)$$

Proof of Validity

A. $\neg H \Rightarrow M$

B. $W \Rightarrow \neg H$

C. $\neg H \Rightarrow (\neg W \Rightarrow M)$

1. $\neg H \Rightarrow M$ hypothesis, A
2. $H \vee M$ conditional identity, 1
3. $H \vee M \vee W$ addition, 2
4. $\neg H \Rightarrow (W \vee M)$ conditional identity, commutative law, 3
5. $\neg H \Rightarrow (\neg W \Rightarrow M)$ conditional identity, 4

Hypothesis B was never used in the proof!

Is there a more straightforward method to prove validity?