Previous Lecture

Sets

Set-builder description

Set Union, Intersection, Difference, Complement

Proving Set identities

Power Sets

The Well Ordering Principle

The well-ordering principle

Every non-empty subset of $\mathbb{N}$ *has a least element.*

Theorem. $\forall n \in \mathbb{Z}^{+} : n>1 \ and\ n\ is\ not\ prime \rightarrow$
$n\ can\ be\ factored\ as\ a\ product\ of\ primes$.

Proof. (By contradiction.) Let C be the non-empty set of counterexamples.

Then, by the WOP, C has a least element. Let's call it m.

m is not prime and $m>1$ and m cannot be factored as a product of primes.

Since m is not prime, $m = a \cdot b$ where $1 < a,b < m$.

a,b are not in C: *(because* m *is the smallest element in* C*)*

$$a=p_1 .p_2 \ldots p_k \ \ and\ b=q_1 .q_2 \ldots q_l ,$$ where

$\boxed{?}\, i,j\ p_i \ \ and\ q_j \ \ are\ primes$.

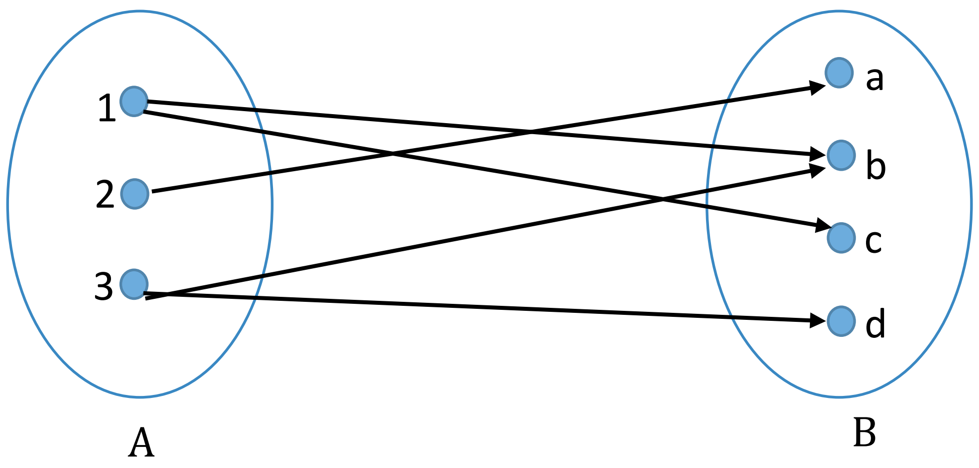
Relations

A relation R from a domain A to a range B is a subset of A x B.

Example:

R: {1,2,3} →{a,b,c,d}

R = {(1,b), (1,c), (2,a), (3,b), (3,d)}



Functions

A (total) function f from a domain A to a range B is a relation such that:

$$\forall x \in A\ \exists b \in B : f(x) = b$$

$$\forall x \in A, \forall b_1 \in B, \forall b_2 \in B : (f(x) = b_1 \wedge f(x) = b_2) \rightarrow b_1 = b_2$$

"Every domain element is mapped to exactly one element in the range."

Example: $Domain = \boldsymbol{N}, Range = \boldsymbol{N}$

$$f(x) = x^2$$

Example: $Domain = \boldsymbol{N}, Range = \boldsymbol{R}$

$$f(x) = x^2$$

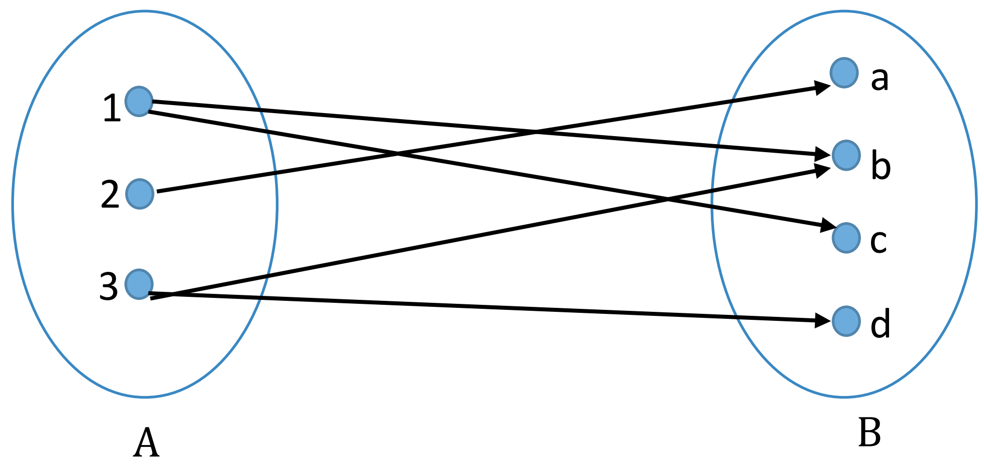
$$f(x) = \sqrt{x}$$

Functions

R: $\{1,2,3\} \rightarrow \{a,b,c,d\}$

$R = \{(1,b), (1,c), (2,a), (3,b), (3,d)\}$

Is R a function?



Types of Functions

Definiton 1: A function $f{:}A \rightarrow B$ *is* *one-to-one* (also called *injective*) if

$$\forall x_1, x_2 \in A: (x_1 \neq x_2) \rightarrow f(x_1) \neq f(x_2)$$

"every domain element is mapped to a unique element in the range."

Definition 2: A function $f{:}A \rightarrow B$ *is* *onto* (also called *surjective*) if

$$\forall y \in B\ \exists x \in A: f(x) = y$$

"every element in the range is the target of at least one domain element."

Definition 3: A function $f{:}A \rightarrow B$ is a one-to-one correspondence (also called bijective) if f is both injective and surjective.

"every domain element is matched with exactly one element in the range, and vice versa."

Examples

Let $f\colon \boldsymbol{N} \to \boldsymbol{N}$

$$f(x) = x\uparrow 2$$

one-to-one but not onto

$$f(x) = x\uparrow 2 - 1$$

not a function!

$$f(x) = (x-1)\uparrow 2$$

not one-to-one, not onto!

$$f(x) = x$$

one-to-one and onto

$$f(0) = 0,\ \forall x > 0\colon f(x) = x - 1$$

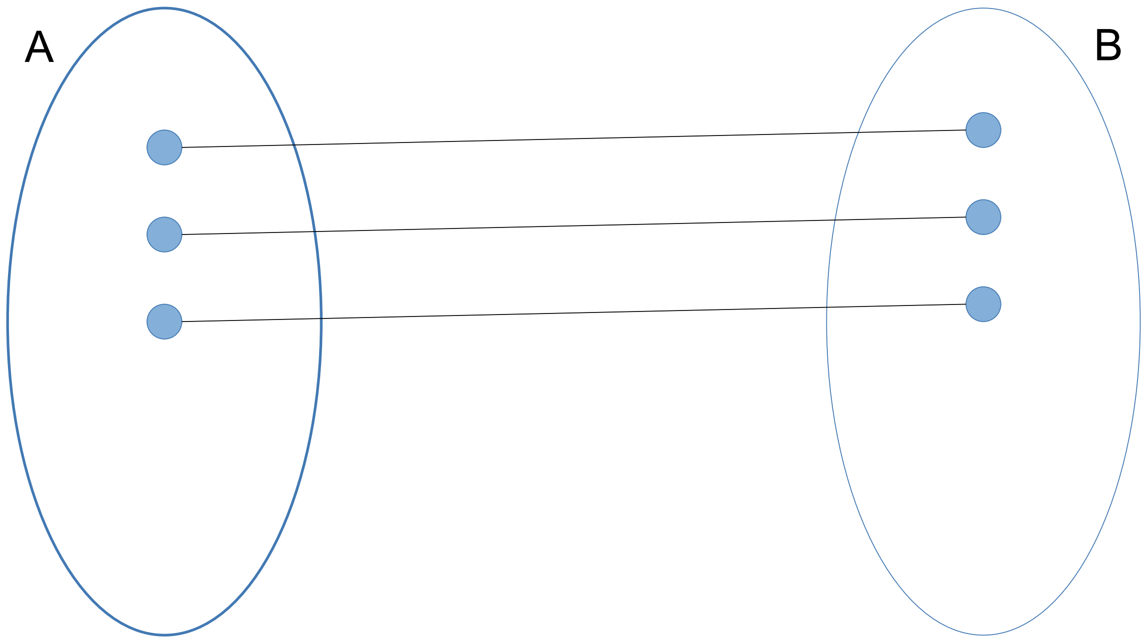
not one-to-one, but onto

Which is bigger?

1. {1, 2, 3} or {Alice, Bob, Charlie}
2. {10, 20, 25} or {234, 567}
3. {10, 20, 25} or {10, 20, 250}
4. {0, 1, 2, …} or {1, 2, 3, …}

What does it even mean for two sets to be equal in size?

Sets of equal size



Associate ***every*** member of A with a ***unique*** member of B.
If every member of B is associated with a unique member of A then $|A| = |B|$.

Or, $|A|=|B|$ if and only if there is a bijection from A to B.

This is the definition of equality of set sizes, even for infinite sets!

Cardinality of Sets

Sets A and B have the same cardinality (denoted |A|=|B|) if

$\exists f{:}A \to B$ and f is bijective (one-to-one correspondence).

Example: $f{:}\boldsymbol{N} \to \boldsymbol{Z}$ where $f(x)= \begin{cases} x/2, & if\ x\ is\ even \\ -(x+1)/2, & if\ x\ is\ odd \end{cases}$

f is bijective. Therefore, $|\boldsymbol{N}| = |\boldsymbol{Z}|$