Eigenvectors and eigenvalues: symmetric matrices.

- Symmetric Matrix A: $A = A^T$
- A is is always diagonalizable
- $A = QDQ^T$, where Q is orthonormal and D always real

Eigenvectors and eigenvalues: symmetric matrices.

Let A be real symmetric matrix then each root λ of of its characteristic polynomial is real.

Eigenvectors and eigenvalues: symmetric matrix.

Proof eigenvalues are real

- Suppose $\lambda = h + ik$ is a complex eigenvalue and let $\bar{\lambda} = h - ik$ be its complex conjugate
- Then $A - \lambda I$ is singular and $\det(A - \lambda I) = 0$
- Let $B = (A - \lambda I)(A - \bar{\lambda} I)$ then $\det(B) = \det(A - \lambda I)\det(A - \bar{\lambda} I) = 0$ Therefore, B is singular
- $B = (A - \lambda I)(A - \bar{\lambda} I) = A^2 - 2hA + h^2 I + k^2 I = (A - hI)^2 + k^2 I$
- Since B is singular, then there exists $\bar{\mathbf{x}}$ s.t. $B\bar{\mathbf{x}} = \bar{\mathbf{0}}$.
- Multiply by $\bar{\mathbf{x}}^T$: $\bar{\mathbf{x}}^T B \bar{\mathbf{x}} = 0$
- $\bar{\mathbf{x}}^T B \bar{\mathbf{x}} = \bar{\mathbf{x}}^T (A - hI)^2 \bar{\mathbf{x}} + h^2 \bar{\mathbf{x}}^T \bar{\mathbf{x}} =$
  $\bar{\mathbf{x}}^T (A - hI)^T (A - hI)\bar{\mathbf{x}} + h^2 \bar{\mathbf{x}}^T \bar{\mathbf{x}} = ||(A - hI)\bar{\mathbf{x}}|| + h^2 \bar{\mathbf{x}}^T \bar{\mathbf{x}} = 0$
  Note since A is symmetric then $(A - hI) = (A - hI)^T$
- Now $||(A - hI)\bar{\mathbf{x}}|| \geq 0$ and $\bar{\mathbf{x}}^T \bar{\mathbf{x}} \neq 0$ therefore $||(A - hI)\bar{\mathbf{x}}|| + h^2 \bar{\mathbf{x}}^T \bar{\mathbf{x}} = 0$ iff $k = 0$
- Hence, $\lambda = h$ is a real number.

Eigenvectors and eigenvalues: symmetric matrices.

Let A be real symmetric matrix and $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$ be eigenvectors corresponding to distinct eigenvalues λ_1, λ_2 then $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$ are orthogonal, i.e. $\bar{\mathbf{x}}_1^T \bar{\mathbf{x}}_2 = 0$.

Eigenvectors and eigenvalues: symmetric matrix.

Proof: $\bar{\mathbf{x}}_1^T \bar{\mathbf{x}}_2 = 0$

- $A\bar{\mathbf{x}}_1 = \lambda_1 \bar{\mathbf{x}}_1$ and $A\bar{\mathbf{x}}_2 = \lambda_2 \bar{\mathbf{x}}_2$
- Multiply by $\bar{\mathbf{x}}_2^T$ and $\bar{\mathbf{x}}_1^T$ respectively:

$$
\begin{aligned}
\bar{\mathbf{x}}_2^T A\bar{\mathbf{x}}_1 &= \lambda_1 \bar{\mathbf{x}}_2^T \bar{\mathbf{x}}_1 \qquad (1) \\
\bar{\mathbf{x}}_1^T A\bar{\mathbf{x}}_2 &= \lambda_2 \bar{\mathbf{x}}_1^T \bar{\mathbf{x}}_2 \qquad (2)
\end{aligned}
$$

- Transpose (2): $\bar{\mathbf{x}}_2^T A\bar{\mathbf{x}}_1 = \lambda_2 \bar{\mathbf{x}}_2^T \bar{\mathbf{x}}_1 \qquad (3)$
- Substract (3) from (2): $0 = (\lambda_1 - \lambda_2)\bar{\mathbf{x}}_2^T \bar{\mathbf{x}}_2$
- Since $\lambda_1 \neq \lambda_2$, we have $\bar{\mathbf{x}}_2^T \bar{\mathbf{x}}_1 = 0$ i.e. $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_1$ are orthogonal.

Eigenvectors and eigenvalues: symmetric matrices.

If $A = A^T$ is a real symmetric $n \times n$ matrix, Then

1. All the eigenvalues of A are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. There is an orthonormal basis of $\mathbb{R}^n$ consisting of n eigenvectors of A.

Eigenvectors and eigenvalues: symmetric matrices.

Spectral Theorem

Every symmetric matrix A has factorization

$$A = QDQ^{-1} = QDQ^T$$

where Q is an orthonormal matrix ($QQ^T = I$) and D is a real diagonal matrix.

- Q is an eigenvector matrix
- D is an eigenvalue matrix

Special matrices: symmetric positive definite

A matrix is symmetric positive definite if it is symmetric and for any $\mathbf{x} \neq \mathbf{0}$

$$\mathbf{x}^T A \mathbf{x} > 0$$

- $\mathbf{x}^T A \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$
- Showing that inequality holds for all $\mathbf{x}$ often is hard

Special matrices: symmetric positive definite

A symmetric matrix $A = A^T$ is positive definite if and only if all of its eigenvalues are strictly positive.

Proof

- If A is positive definite then $\bar{\mathbf{x}}^T A \bar{\mathbf{x}} > 0$ for any $\bar{\mathbf{x}}$.
- Let $\bar{\mathbf{u}}$ be an eigenvector with eigenvalue λ then
$$0 < \bar{\mathbf{u}}^T A \bar{\mathbf{u}} = \bar{\mathbf{u}}^T (\lambda \bar{\mathbf{u}}) = \lambda ||\bar{\mathbf{u}}||$$
Since $||\bar{\mathbf{u}}|| > 0$ then λ must be positive

Special matrices: symmetric positive definite

- Now suppose that A has all strictly positive eigenvalues.
- Let $\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_n$ be the orthonormal eigenvector basis of $\mathbb{R}^n$
- And $A\bar{\mathbf{u}}_i = \lambda_i \bar{\mathbf{u}}_i$
- Then $\bar{\mathbf{x}} = c_1 \bar{\mathbf{u}}_1 + \cdots + c_n \bar{\mathbf{u}})_n$ and we have

$$K\bar{\mathbf{x}} = c_1 \lambda_1 \bar{\mathbf{u}}_1 + \cdots + c_n \lambda_n \bar{\mathbf{u}}_n$$

- Using the fact that $||\bar{\mathbf{u}}_i|| = 1$ and $\bar{\mathbf{u}}_i^T \bar{\mathbf{u}}_j = 0$, $i \neq j$:

$$\begin{aligned} \bar{\mathbf{x}}^T A \bar{\mathbf{x}} &= (c_1 \bar{\mathbf{u}}_1 + \cdots + c_n \bar{\mathbf{u}}_n) \cdot (c_1 \lambda_1 \bar{\mathbf{u}}_1 + \cdots + c_n \lambda_n \bar{\mathbf{u}}_n) \\ &= \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2 > 0 \end{aligned}$$

strictly positive since not all $c_1, \cdots, c_n$ can be zero.

Special matrices: symmetric positive definite

If rectangular matrix R has n independent columns then $A = R^T R$ is symmetric positive definite

- We already know that $R^T R$ is symmetric
- $\bar{\mathbf{x}}^T A \bar{\mathbf{x}} = \bar{\mathbf{x}}^T R^T R \bar{\mathbf{x}} = (R\bar{\mathbf{x}})^T (R\bar{\mathbf{x}}) = ||R\bar{\mathbf{x}}|| > 0$
- Note $||R\bar{\mathbf{x}}|| \neq 0$ because R has independent columns

Special matrices: symmetric positive definite

- All n pivots are positive
- All n eigenvalues are positive
- Each leading principal submatrix has positive determinant.
- $\bar{\mathbf{x}}^T A\bar{\mathbf{x}} > 0$ for all $\bar{\mathbf{x}} \neq \bar{\mathbf{0}}$
- $A = R^T R$, where R is an arbitrary matrix with independent columns

Special matrices: symmetric positive definite

Let A be a symmetric positive definite, then

- A is nonsingular
- Gaussian elimination can be performed without row interchanges
- Stable - no pivoting is needed

Special matrices: symmetric positive definite

- Let A be symmetric positive definite
- LU decomposition can be arranged so that $U = L^T$:

$$A = LL^T$$

- Setting $A = LL^T$ and solving for entries of L we get

$$\begin{aligned}
l_{11} &= \sqrt{a_{11}}, l_{i1} = a_{i1}/l_{11} \\
l_{kk} &= \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2} \\
l_{ik} &= \left(a_{ik} - \sum_{j=1}^{k-1} l_{ij} l_{kj} \right) / l_{kk}
\end{aligned}$$

Special types of linear systems: Cholesky factorization

PROCEDURE Cholesky factorization

INPUT: n, matrix A
OUTPUT: L s.t. $A = LL^T$

1: $l_{11} = a_{11}$

2: **FOR** $2 \leq i \leq n$ **DO** $l_{i1} = a_{i1}/l_{11}$

3: **FOR** $2 \leq i \leq n-1$ **DO**

4: $l_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2\right)^{1/2}$

5: **FOR** $i+1 \leq j \leq n$ **DO**

6: $l_{ji} = \left(a_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik}\right)/l_{ii}$

7: $l_{nn} = \left(a_{nn} - \sum_{k=1}^{n-1} l_{nk}^2\right)^{1/2}$

8: **RETURN** $L = [l_{ij}], 1 \leq i, j \leq n$

Special matrices: Cholesky factorization

A is symmetric positive definite if and only if it can be factored in the form LL^T, where L is lower triangular with nonzero diagonal elements.

If A is symmetric positive definite

- All n square roots are of positive numbers so algorithm is well defined
- No pivoting required
- Only lower triangle of A is needed
- Only $O(n^3/6)$ numerical operations (half of the Gaussian elimination)

Similar matrices

Let M be an invertible matrix. Matrix B is called *similar* to A if

$$B = M^{-1}AB$$

- If B similar to A then A is similar to B ($A = MBM^{-1}$)
- Diagonalizable matrix is similar to diagonal D ($A = S^{-1}DS$)

Similar matrices

Let A and B be similar and $B = M^{-1}AM$. Then

- they have exactly the same eigenvalues
- if $\bar{\mathbf{x}}$ is an eigenvector of A then M^{-1} is an eigenvector of B

- $B = M^{-1}AM \Rightarrow A = MBM^{-1}$
- Let $A\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$
- $A\bar{\mathbf{x}} = (MBM^{-1})\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}} \Rightarrow B(M^{-1}\bar{\mathbf{x}}) = \lambda](M^{-1}\bar{\mathbf{x}})$
- I.e. λ is an eigenvalue of B and $M^{-1}\bar{\mathbf{x}}$ is the corresponding eigenvector

– Note that non-similar matrices could have the same eigenvalues

Similar matrices

- Similar matrices share important properties.
- One way to think about it is similar matrices represent same objects but in different bases

Similar matrices

If A and B are similar then they have the same

- Eigenvalues
- Determinant
- Trace
- Rank
- Number of independent eigenvectors
- Jordan form

Similar matrices

If A and B are similar then they have different

- Eigenvectors
- Nullspace
- Columns space
- Row space
- Left nullspace

Similar matrices

Jordan normal form

Let A has s independent eigenvectors the it is similar to matrix J:

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J,$$

where J_i is a Jordan block

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \dots & \\ & & & \lambda_i \end{bmatrix}$$

- Jordan matrices are unique for similar matrices
- A and B are similar if and only if they have the same Jordan form.

Singular value decomposition

- Let A be arbitrary $m \times n$ matrix with rank r
- Singular Value Decomposition is a diagonalization in general case

Singular Value Decomposition

$$A = U\Sigma V^T$$

Singular value decomposition

Proof of SVD

- AA^T and A^TA are symmetric and they have r orthonormal eigenvectors
- Let $\lambda_i = \sigma_i^2$ be the eigenvalues of AA^T (A^TA)
- $A^TA\bar{\mathbf{v}}_i = \sigma_i^2\bar{\mathbf{v}}_i$
- Multiply by $\bar{\mathbf{v}}_i^T$: $\bar{\mathbf{x}}_i^TA^TA\bar{\mathbf{v}}_i = \sigma_i^2\bar{\mathbf{x}}_i^T\bar{\mathbf{v}}_i \Rightarrow ||A\bar{\mathbf{v}}_i||^2 = \sigma_i^2 \Rightarrow ||A\bar{\mathbf{v}}_i|| = \sigma_i$
- Multiply by A: $AA^TA\bar{\mathbf{v}}_i = \sigma_i^2A\bar{\mathbf{v}}_i \Rightarrow A\bar{\mathbf{v}}_i$ is an eigenvector of AA^T
- Dividing by $||A\bar{\mathbf{v}}_i|| = \sigma_i$ gives $\bar{\mathbf{u}}_i = A\bar{\mathbf{v}}_i/\sigma_i$ unit eigenvector of AA^T

Singular value decomposition

Proof of SVD

- So we have $A\bar{\mathbf{v}}_i = \sigma_i \bar{\mathbf{u}}_i$, $i = 1, \ldots, r$
- We can put it in matrix form:

$$AV = U\Sigma,$$

  where V is $n \times r$ eigenvector matrix of $A^T A$, U is $m \times r$ eigenvector matrix of AA^T and Σ is $r \times r$ diagonal matrix of σ_i
- Now $VV^T = I_n$, multiplying on both sides we get

$$AVV^T = A = U\Sigma V^T$$

Singular value decomposition

Singular value decomposition

$$A = U\Sigma V^T$$

- σ_i - singular values
- $\bar{\mathbf{v}}_i$ - singular vectors