1.	2.	3.	4.	5.	Total:

MA 232. **Exam 2.** **November 11, 2016.**

Print name: ____________________

Instructor: A.Myasnikov

Closed book and closed notes. Show all of your work. Answers without supporting work will not receive credit.

Pledge and sign: ____________________

Problem 1. (10pts) Prove or disprove:

(a) Let A be an $m \times n$ matrix and $rank(A) = m$, then there is a vector $\bar{\mathbf{b}}$ for which $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has unique solution.

(b) Let $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3$ be a basis for subspace W of the euclidean space $\mathbb{R}^5$. Then the basis for the orthogonal complement $W^\perp$ consists of vectors $\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_3$ such that $\bar{\mathbf{w}}_i \cdot \bar{\mathbf{v}}_j = 0$.

(c) Let Q be orthonormal matrix, then $Q^{-1} = Q^T$.

Solutions.

(a) Number of rows is greater then number of columns and all rows are linearly independent. Therefore there are free variables and infinitely many solutions.

(b) Note $dim(\mathbb{R}^5) = 5$, $dim(W) = 3$, hence $dim(W^\perp) = 5 - 3 = 2$. Basis of $W^\perp$ must contain at most two vectors, therefore the statement is false.

(c) even though $QQ^t = I$ this statement is true only if Q is square.

Problem 2. (10pts) Let W be the subspace of $\mathbb{R}^4$ spanned by vectors

$$\bar{\mathbf{u}}_1 = (1, -2, 5, -3), \quad \bar{\mathbf{u}}_2 = (2, 3, 1, -4), \quad u_3 = (3, 8, -3, -5)$$

(a) Find a basis and dimension of W

(b) Extend the basis of W to a basis of $\mathbb{R}^4$

Solution:

(a) Let A be the matrix with rows equal to $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3$. Compute its reduced form:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows are linearly independent and form a basis; $dim(W) = 2$.

(b) Any basis of $\mathbb{R}^4$ should have four vectors. We need two more vectors to form a basis for $\mathbb{R}^4$. any vectors that are linearly independent will do. The easiest way is to take the reduced form with basis vectors and add rows so that the resulting matrix is upper triangular and square:

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The resulting basis:

$$[1\ -2\ 5\ -3]^T, [0\ 7\ -9\ 2]^T, [0\ 0\ 1\ 0]^T, [0\ 0\ 0\ 1]^T$$

Problem 3. (10pts) Find a basis for the subspace $U^\perp$ of $\mathbb{R}^5$, where U is spanned by two vectors

$$u_1 = (1, 2, 3, -1, 2),\ u_2 = (2, 4, 7, 2, -1).$$

Solution:

$U^\perp = N(A)$ where $N(A)$ is a nullspace of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix}$$

Triangular form is

$$U = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -5 \end{bmatrix} \Rightarrow \begin{aligned} x + 2y + 3z - s + 2t = 0 \\ z + 4s - 5t = 0 \end{aligned}$$

There are three free variables: y, s, t. Obtain special solutions:

(a) $y = 1, s = 0, t = 0 \Rightarrow \bar{\mathbf{w}}_1 = (-2, 1, 0, 0, 0)$

(b) $y = 0, s = 1, t = 0 \Rightarrow \bar{\mathbf{w}}_2 = (13, 0, -4, 1, 0)$

(c) $y = 0, s = 0, t = 1 \Rightarrow \bar{\mathbf{w}}_3 = (-17, 0, 5, 0, 1)$

$\{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_3\}$ is a basis for $U^\perp$

Problem 4. (10pts) Let $\bar{\mathbf{v}} = (1, -1, 1)$ and W be a subspace of $\mathbb{R}^3$ spanned by

$$\bar{\mathbf{w}}_1 = (0, 1, 0),\ \bar{\mathbf{w}}_2 = (1, 0, -1)$$

Find the vector $\bar{\mathbf{w}}$ in W closest to $\bar{\mathbf{v}}$.

Solution.

The closes vector in W will be the projection of $\bar{\mathbf{v}}$ onto W. Note that vectors $\bar{\mathbf{w}}_1$ and $\bar{\mathbf{w}}_2$ are orthogonal to each other and therefore linearly independent which means they form a basis of W. Se matrix A which has vectors $\bar{\mathbf{w}}_1$ and $\bar{\mathbf{w}}_2$ as columns.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The projection matrix $P = A(A^TA)^{-1}A^T$.

$$A^TA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } (A^TA)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Then

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

The projection

$$\bar{\mathbf{w}} = P\bar{\mathbf{v}} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Problem 5. (10pts) The following is the US census data:

Year	Population mln
1800	5.3
1850	23
1900	75
1950	151
1990	249

(a) Set up a system of equations whose solution gives the best fitted line through the data above;

(b) Set up a system of equation to find a parabola closest to the data.

Do not need to solve the equations.

Solution:

Line is given by equation $y = \alpha x + \beta$ to obtain α, β solve system

$$\begin{bmatrix} 1 & 1800 \\ 1 & 1850 \\ 1 & 1900 \\ 1 & 1950 \\ 1 & 1990 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 5.3 \\ 23 \\ 75 \\ 151 \\ 249 \end{bmatrix}$$

Parabola is given by equation $y = \alpha x^2 + \beta x + \gamma$ to obtain α, β, γ solve system

$$\begin{bmatrix} 1 & 1800 & 1800^2 \\ 1 & 1850 & 1850^2 \\ 1 & 1900 & 1900^2 \\ 1 & 1950 & 1950^2 \\ 1 & 1990 & 1990^2 \end{bmatrix} \begin{bmatrix} \gamma \\ \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 5.3 \\ 23 \\ 75 \\ 151 \\ 249 \end{bmatrix}$$