Sets

A set is an unordered collection of objects, called members or elements of the set.

$x \in S$ represents the proposition "x is a member of S."

$x \notin S \equiv \neg(x \in S)$ (x is not a member of S).

Sets can contain numbers, letters, people, strings, trees, birds, … as members.

$\{1, 2, Jack, Jill, elm, sparrow, USA\}$

Can a set contain no members?

Sure, the *empty set* contains no members.

There is a unique empty set, denoted Φ

Is the proposition $\forall x \in \Phi : x = x$ true?

Yes

Is the proposition $\forall x \in \Phi : x \neq x$ true?

Yes!

Is the proposition $\exists x \in \Phi : x = x$ true?

No

Can a set contain sets as members?

Sure!

$$X=\{1, 2, \{Jack, Jill\}, \{elm, beech\}\}$$

$$Y=\{\Phi,1,2\}$$

Is $\{\Phi\}$ different from Φ?

Yes, $\{\Phi\}$ contains one member (the set Φ), but Φ contains nothing!

How many members does $\{\{\Phi\}\}$ contain?

One, its only member is the set $\{\Phi\}$.

The set $\{\Phi, \{\Phi\}, \{Jack, Jill\}, \{a,\{b,c\}\}\}$ contains 4 elements.

Can a set contain itself as a member?

Let’s see what happens if we allow that.

Now consider all the sets that don’t contain themselves:

$$S = \{X : X \notin X\}$$

Is $S \in S$? Or is $S \notin S$?

$$(\boldsymbol{S} \in \boldsymbol{S}) \Leftrightarrow (\boldsymbol{S} \notin \boldsymbol{S})\ !$$

Defining sets precisely is extremely tricky!

We’ll just agree that sets cannot contain themselves.

If A contains B then B cannot contain A.

Subsets

$A \subseteq B$ means that every member of A is also a member of B

or, $\boxed{?}x\colon (x \in A \Rightarrow x \in B)$

$A \subset B$ means that every member of A is a member of B, and B has members that are not members of A

or, $\boxed{?}x\colon (x \in A \Rightarrow x \in B) \wedge (\boxed{?}x\colon x \in B \wedge x \notin A)$

Set Notation

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$: sets of natural numbers, integers, rationals, real numbers

Sets can be represented by:

- Listing elements in the set {1, 2, 3}
- By a predicate that describes properties of elements (Set builder notation)

$\{x: P(x)\}$

$\{x \in \mathbb{N} : \boxed{?}\, y \in \mathbb{N}, (y > 1 \land x > y) \rightarrow y \nmid x\}$

This is the set of prime numbers.

Operations on Sets

Set Union: $A \cup B = \{x: (x \in A) \vee (x \in B)\}$

Intersection: $A \cap B = \{x: (x \in A) \wedge (x \in B)\}$

Difference: $A - B = \{x: (x \in A) \wedge (x \notin B)\}$

Complement (with respect to a universe S of elements):

$$\bar{A} = S - A = \{x: (x \in S) \wedge (x \notin A)\}$$

Cartesian Product: $A \times B = \{(a,b) : (a \in A) \wedge (b \in B)\}$

Example: $\{1,2\} \times \{a,b,c\} = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$

Note: $(1,a) \neq (a,1)$!

Power Sets

The power set $P(S)$ of a set S is defined as:

$$P(S)=\{X : X \subseteq S\}$$

"The set of all subsets of S"

$$P(\{1,2\})=\{\emptyset, \{1\},\{2\},\{1,2\}\}$$

$$P(\{a,b,c\})=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$$

If a finite set S has m elements, then $P(S)$ has $2\uparrow m > m$ elements.

Proving set identities

Prove that $A \cup (A \cap B) = A$

$A \cup (A \cap B) \quad = \quad \{x\text{: } (x \in A) \vee (x \in A \cap B)\}$

$= \quad \{x\text{: } (x \in A) \vee (x \in A \ \wedge x \in B)\}$

$= \quad \{x\text{: } (x \in A)\}$, because $(p \vee (p \wedge q)) = p$

$= \quad A$

Anything look familiar?

Table 3.5.1: Set identities.

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double Complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \emptyset$ $\overline{U} = \emptyset$	$A \cup \overline{A} = U$ $\overline{\emptyset} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Table 1.5.1: Laws of propositional logic.

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

The well-ordering principle

Every non-empty subset of $\mathbb{N}$ *has a least element.*

Theorem. $\forall n \in \boldsymbol{Z}\uparrow + : n > 1 \; and \; n \; is \; not \; prime \rightarrow n \; can \; be \; factored \; as \; a \; product \; of \; primes.$