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**Assignment 2 - Solutions**

1. a) $f(a+i) = (a+i)^2 + 1 = (a^2 - 1 + 1) + i(2a) = a^2 + i(2a)$. Going to the cartesian representation, with $y = 2a$ we have $x = a^2 = \frac{y^2}{4}$ so the image of the line is a parabola whose vertex is at the origin opening towards the positive $x$-axis

   b) $f(1+ib) = (1+ib)^2+1 = (1-b^2+1)+i(2b) = (2-b^2)+i(2a)$. Going to the cartesian representation, with $y = 2b$ we have $x = 2-b^2 = 2-\frac{y^2}{4}$ so the image of the line is a parabola whose vertex is at $(0,2)$ opening towards the negative $x$-axis

   c) The two parabolas divide the plane into five regions. Note that since $a, b \geq 1$ and $f(a+ib) = (a^2-b^2+1)+i(2ab)$, we have that $\Im(f(z)) > 0$ for any $z$ in our region and the only region bounded by our parabolas is the one lying above both.

2. We know from class that $(re^{i\theta})^{\frac{1}{4}} = \{+\sqrt[4]{r}e^{i\frac{\theta}{4}}, +\sqrt[4]{r}e^{i\frac{\theta+2\pi}{4}}, +\sqrt[4]{r}e^{i\frac{\theta+4\pi}{4}}, +\sqrt[4]{r}e^{i\frac{\theta+6\pi}{4}}\}$ where $+\sqrt[4]{r}$ is the positive real quartic root of a real number, giving us that the branches of the function are:

$$f_1(re^{i\theta}) = +\sqrt[4]{r}e^{i\frac{\theta}{4}},\ f_2(re^{i\theta}) = +\sqrt[4]{r}e^{i\frac{\theta+2\pi}{4}},\ f_3(re^{i\theta}) = +\sqrt[4]{r}e^{i\frac{\theta+4\pi}{4}},\ f_4(re^{i\theta}) = +\sqrt[4]{r}e^{i\frac{\theta+6\pi}{4}}$$

3. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ defined by $f(a+ib) = |a|^{|b|} + i|b|^{|a|}$

   a) Notice first that, with $f(a+ib) = |a|^{|b|} + i|b|^{|a|}$, we have that $f(a+ib) = f(-a+ib) = f(a-ib) = f(-a-ib)$ so it is enough to show that the function is continuous in the first quadrant.

   Restricting ourselves to $a, b \geq 0$, then $|a|^{|b|} = a^b$ and $|b|^{|a|} = b^a$ and we know from real calculus that both of these are continuous except at the origin, so we have that the real and imaginary parts of the function are continuous everywhere except at the origin where it isn't defined.

   b) Let $x_n = \frac{1}{n}$ and $y_n = \frac{i}{n}$, we have that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} = 0$ but $f(x_n) = (\frac{1}{n})^0 + i0^{\frac{1}{n}} = 1$ whereas $f(y_n) = 0^{\frac{1}{n}} + i(\frac{1}{n})^0 = i$ so, regardless of what value we pick for $f(0)$ we still have that either $f(0) \neq 1$ or $f(0) \neq i$ so it would still be discontinuous.