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Assignment 4 - Solutions

1. We know $u_x = 4x+1 = v_y \Rightarrow v = 4xy+y+C(x)$ and $u_y = -4y = -v_x \Rightarrow v = 4xy+C(y)$. Putting both together we have $v(x,y) = 4xy+y+C$ and $v(0,0) = 0+0+C = 0$, giving us that $v(x,y) = 4xy+y$.

2. We know $u_r = \sin\theta = \frac{1}{r}v_\theta \Rightarrow v = -r\cos\theta + C(r)$ and $u_\theta = r\cos\theta = -rv_r \Rightarrow v = -r\cos\theta + C(\theta)$. Putting both together we have $v(r,\theta) = -r\cos\theta + C$ and $v(0,0) = 0 + C = 0$, giving us that $v(r,\theta) = -r\cos\theta$.

3. a) $u_{xx} = 0$, $u_y y = 0$, $v_{xx} = 2$ and $v_{yy} = 2$, so u is harmonic but v isn't so they can't be the real and imaginary parts of an analytic function.

   b) $u_{xx} = 0$, $u_y y = 0$, $v_{xx} = 2$ and $v_{yy} = -2$, so both u and v are harmonic. However, $u_x = y$ and $v_y = -2y$ so the function could only be differentiable if $y = -2y$ so $y = 0$ and no disk is contained in that so the function couldn't be analytic anywhere.

4. a) Let $h(x,y) = f(x,y) + g(x,y)$. By basic calculus we have $h_{xx} = f_{xx} + g_{xx}$ and $h_{yy} = f_{yy} + g_{yy}$ so $h_{xx} + h_{yy} = f_{xx} + g_{xx} + f_{yy} + g_{yy} = (f_{xx} + f_{yy}) + (g_{xx} + g_{yy}) = 0 + 0 = 0$ so h is harmonic.

   b) It is enough to find f and g harmonic such that fg isn't. Let $f = x$, $g = xy$. It is straightforward to check that $f_{xx} = f_{yy} = g_{xx} = g_{yy} = 0$ so f and g are harmonic but $fg = x^2y$ so $fg_{xx} = 2y$ and $fg_{yy} = 0$ so fg is not harmonic.