MA232 Linear Algebra

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Linear equations: matrix form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots & \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
\text{ROW:} \quad \begin{bmatrix} \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{x}} \\ \bar{\mathbf{a}}_2 \cdot \bar{\mathbf{x}} \\ \vdots \\ \bar{\mathbf{a}}_n \cdot \bar{\mathbf{x}} \end{bmatrix} = A\bar{\mathbf{x}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

$$
\text{COLUMN:} \; x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

Linear equations: matrix form

Two different views on solving system of linear equations

$$A\bar{\mathbf{x}} = \bar{\mathbf{b}}$$

– ROW: find point of intersection of planes given by row equations

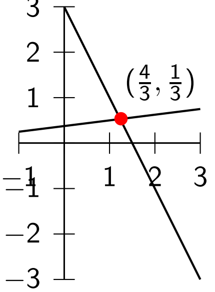
– COLUMN: find a combination of columns of A equal to $\bar{\mathbf{b}}$

Linear equations: matrix form

$$\begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

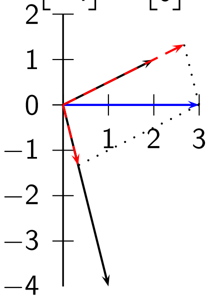
Linear equations

$$2x_1 + x_2 = 3$$
$$x_1 - 4x_2 = 0$$



Linear combination

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



Solving linear equations: elimination

If system of linear equations has upper triangular form then we can easily solve it by back substitution

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
 & & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & & & & & & \cdots & & & & \\
 & & & & & & & & a_{nn}x_n & = & b_n
\end{array}$$

GOAL: Given arbitrary system obtain equivalent upper triangular system

Solving linear equations: elimination

Triangular form is obtained by application of the following rules:

1. Any two equations can be interchanged: $E_i \leftrightarrow E_j$
2. Any equation can be multiplied by a nonzero constant: $E_i = mE_i$
3. Any multiple of one equation can be added to another equation: $E_i = E_i + mE_j$

Original system of equations and the result of a sequence of rule applications are equivalent.

Linear systems of Equations.

$$\text{Augmented matrix}: \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{11} & a_{12} & \dots & a_{1n} & b_2 \\ . & . & . & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$$

Transformations:

equations $\Leftrightarrow$ rows of augmented matrix

$$\begin{aligned} E_i \leftrightarrow E_j \quad &\Leftrightarrow \quad R_i \leftrightarrow R_j \\ E_i = mE_i \quad &\Leftrightarrow \quad R_i = mR_i \\ E_i = E_i + mE_j \quad &\Leftrightarrow \quad R_i = R_i + mR_j \end{aligned}$$

Linear systems of Equations.

- *Gaussian elimination* process transforms coefficient part of augmented matrix into upper triangular form
- Pivot - first nonzero in the row that does the elimination
- Goal is to replace all elements below pivot with zeros using only rules 1-3

Linear systems of Equations.

- Let $a_{ii} \neq 0$ be the pivot then to each row R_k, $k > i$ apply rule $R_k = R_k + mR_i$, where

$$m = -\frac{a_{ki}}{a_{ii}}$$

- If $a_{ii} = 0$, replace R_i with R_k, $k > i$, such that $a_{ki} \neq 0$
- If $a_{ii} = 0$ and all $a_{ki} = 0, k > i$ then no unique solution.

Linear systems of Equations.

– The pivots are on the diagonal after elimination stops

If, after elimination, at least one of the diagonal elements is zero then no unique solution:

- No solution: two or more equations correspond to parallel planes
- Infinitely many solutions: two or more equations correspond to the same plane

Inverse matrices

The inverse of a square matrix $A^{n \times n}$ is a matrix $B^{n \times n}$ (if it exists) such that

$$BA = AB = I_n$$

Denote inverse by A^{-1}.

- Inverse is unique: Suppose $AB = I$ and $CA = I$ then

$$C = C(AB) = (CA)B = B$$

Inverse matrices

– If A and B are invertible then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

– In general if $A_1, \dots, A_n$ are invertible then

$$(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} a_1^{-1}$$

Inverse matrices

If A is invertible then $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has unique solution:

$$A\bar{\mathbf{x}} = \bar{\mathbf{b}} \Rightarrow A^{-1}A\bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}} \Rightarrow \bar{\mathbf{x}} = A^{-1}\bar{\mathbf{b}}$$

In particular $A\bar{\mathbf{x}} = \bar{\mathbf{0}}$ must have $\bar{\mathbf{x}} = A^{-1}\bar{\mathbf{0}} = \bar{\mathbf{0}}$

Computing Matrix inverse.

By the definition of the matrix product we can write $AB = I_n$:

$$AB = \begin{bmatrix} \sum_{k=1}^{k} a_{1k} b_{k1} & \dots & \sum_{k=1}^{k} a_{1k} b_{kn} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{k} a_{nk} b_{k1} & \dots & \sum_{k=1}^{k} a_{nk} b_{kn} \end{bmatrix} = I_n$$

Computing Matrix inverse.

To obtain ith column of B solve system:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ 1 \\ . \\ 0 \end{bmatrix} = \bar{\mathbf{e}}_i$$

where $\mathbf{e}_i$ is the ith column of I_n:

Computing Matrix inverse.

- To find elements b_{ij} we need to solve n systems of linear equations (n^2 equations with total of n^2 unknowns)
- Note entries in each row have the same coefficients a_{ij}s
- Elimination process is the same for equations in each column

Computing Matrix inverse.

We can combine all n systems into one by constructing an augmented matrix

$$A' = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array}\right]$$

Computing Matrix inverse: Gauss method.

To compute coefficients of A^{-1}

- Run Gaussian elimination process on augmented matrix A'
- Compute coefficients b_{ij}, $1 \leq i \leq n$ by performing back substitution on

  $$[\ U \mid \mathbf{e}_j\ ],$$

  where U is the upper triangular matrix resulted after elimination process and $\mathbf{e}_j$ is the jth column of the transformed identity matrix.

Computing Matrix inverse: Gauss-Jordan method.

– Main Idea: instead of using back substitution continue elimination process to eliminate elements of A' above the diagonal using the same row transformation $R_i - R_i + mR_j$.

– In matrix form Gauss-Jordan process:

$$[AI] \Rightarrow [IA^{-1}]$$

Computing Matrix inverse: Gauss-Jordan method.

To compute coefficients of A^{-1}

- Run Gaussian elimination process on augmented matrix A' (Same as in Gauss)
- Eliminate entries of A' above the diagonal (Obtain *reduced echelon form*
- Divide each row y its pivot.

Elimination using matrices

– Steps of elimination process can be performed using matrix operations.

$$\underbrace{\begin{bmatrix} 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & e_{ij} & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{E_{ij}} \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i + e_{ij} b_j \\ \vdots \\ b_n \end{bmatrix}$$

E_{ij} takes ith element of $\bar{\mathbf{b}}$ and adds $e_{ij} b_j$.

Elimination using matrices

In general for a matrix A:

$$E_{ij}A = \begin{bmatrix} A_{row}(1) \\ \cdots \\ A_{row}(i) + e_{ij}A_{row}(j) \\ \cdots \\ A_{row}(n) \end{bmatrix}$$

Elimination using matrices

– To compute inverse just negate the non-zero element off diagonal.

$$E_{ij} = \begin{bmatrix} 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & e_{ij} & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \Rightarrow E_{ij}^{-1} = \begin{bmatrix} 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & -e_{ij} & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

– Product:

$$E_{ij}E_{kl} = E_{ij} + E_{kl} - I$$

Elimination using matrices

Elimination rule $R_k = R_k + mR_i$ for rows of A is equivalent to multiplication

$$E_{ij}A, \text{ where } e_{ij} = m$$

LU factorization

- If A is triangular then $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ can be solved by performing Cn^2 arithmetic operations, where C s some constant.
- Suppose we need to solve repeatedly for different values of $\bar{\mathbf{b}}$
- Suppose $A = LU$, where L and U are lower and upper triangular then $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ can be solved by two substitutions (forward and backward) $2Cn^2$:
  1. Solve $L\bar{\mathbf{y}} = \bar{\mathbf{b}}$ for $\bar{\mathbf{y}}$
  2. Solve $U\bar{\mathbf{x}} = \bar{\mathbf{y}}$ for $\bar{\mathbf{x}}$
- LU factorization requires Kn^3 operations but needs to be done once.

LU factorization

- Suppose we do not permute rows, then Gaussian elimination is a sequence of multiplications by some elimination matrices and the result is upper triangular

$$U = A_n = E_{n-1}E_{n-2}\cdots E_1 A$$

- To obtain A we multiply both sides by the inverses of E_i

$$E_1^{-1}E_2^{-1}\cdots E_{n-1}^{-1}U = A$$

  Note: that E_i^{-1} is lower triangular and the product of lower triangular matrices is lower triangular

- The product $L = E_1^{-1}E_2^{-1}\cdots E_{n-1}^{-1}$ is a lower triangular matrix and we have factorization

$$LU = A$$

Permutation matrices

Need matrix operation to perform row exchange $R_i \leftrightarrow R_j$

Permutation matrix P_{ij} is obtained from I by exchanging rows i and j

– The product $B = P_{ij}A$ is the matrix B such that

$$\begin{aligned} B_{row}(i) &= A_{row}(j) \\ B_{row}(j) &= A_{row}(i) \\ B_{row}(k) &= A_{row}(k), k \neq i, j \end{aligned}$$

Row exchange rule $R_i \leftrightarrow R_j$ for rows of A is equivalent to $P_{ij}A$.

Elimination using matrices: $A = LU$ in general

- Let U be the upper triangular matrix obtained by elimination process on a matrix A.
- Then

$$U = E_n P_n \cdots E_2 P_2 E_1 P_1 A$$

  where E_i is an elimination matrix and P_i is a permutation matrix.
- Solving for A we obtain:

$$(E_1^{-1} P_1^{-1} \cdots E_n^{-1} P_n^{-1}) U = A$$

Elimination using matrices: $A = LU$ in general

- The problem is that product

$$E_1^{-1} P_1^{-1} \cdots E_n^{-1} P_n^{-1}$$

  is not necessarily lower triangular and we do not obtain $LU = A$ form.

- The practical solution is to exchange rows in *advance*:

$$\text{Set } P = P_n \cdots P_2 P_1$$

  Compute factorization $PA = LU$

  – NOTE: P is a permutation matrix.

Elimination using matrices: $A = LU$ in general

- Now to solve system $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$

$$A\bar{\mathbf{x}} = \bar{\mathbf{b}} \Rightarrow PA\bar{\mathbf{x}} = P\bar{\mathbf{b}} \Rightarrow LU\bar{\mathbf{x}} = P\bar{\mathbf{b}}$$

We solve system using two substitutions.

Elimination using matrices

Let U be the upper triangular matrix obtained by elimination process on a matrix A.

Then

$$U = E_n P_n \cdots E_2 P_2 E_1 P_1 A$$

where E_i is an elimination matrix and P_i is a permutation matrix.

LDU factorization

- Another way to represent triangular factorization is to write:

$$A = LDU$$

where

L is lower triangular with 1s on the diagonal
U is upper triangular with 1s on the diagonal
D is a diagonal matrix

LDU factorization

If $A = LU'$, then factor $U' = DU$

$$\begin{bmatrix} d_1 & u_{12} & u_{13} & \cdot \\ & d_2 & u_{23} & \cdot \\ & & \ddots & \\ & & & d_n \end{bmatrix} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

We have $A = LDU$

LU factorization vs Matrix inverse

- Solving by matrix inverse is 3 times more expensive
- Explicit inversion gives less accurate results
- Rarely used in practice

Transpose

Transpose of $n \times m$ matrix A is the $m \times n$ matrix A^T such that columns of A^T are the rows of A:

$$A^T(i,j) = A(j,i)$$

Rules of transpose:

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$

Symmetric matrix

A matrix A is symmetric if an only if $A = A^T$, i.e. $A(i,j) = A(j,i)$

Symmetric matrix

- The inverse of symmetric matrix is also symmetric:

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

- For arbitrary matrix A

$A^T A$ and AA^T are square symmetrix matrices

- If A is symmetric then its factorization

$$A = LDL^T$$