Thus far …

1. Propositions, truth tables, laws of propositional logic, rules of inference
2. Checking validity of logical arguments
   a) Build a truth table.
   b) Use the rules of inference and laws of propositional logic to derive the conclusion, starting with the hypotheses.
   c) Apply the tree method to search for counterexamples.

Getting beyond propositions

John is taller than Dan

<u>Dan is taller than Bill</u>

∴ John is taller than Bill

Is this argument valid or invalid?

We need to specify the logical meaning of “is taller than”

Propositions only deal with nouns

Predicates

Dictionary definition: *" …the part of a sentence or clause containing a verb and stating something about the subject …"*

$isTaller(x,y)$: "x is $taller$ $than$ y"

$isTaller$ is a $predicate$, not a $proposition$

x,y : $arguments$, $variables$ of the $predicate$

Predicates

John is taller than Dan $isTaller(John, Dan)$

Dan is taller than Bill $isTaller(Dan, Bill)$

Each of these statements is a proposition.

But we still cannot conclude that John is taller than Bill.

So far we've only defined the syntax of the predicate.

Next we have to define its property.

Predicates

$(isTaller(John, Dan) \wedge isTaller(Dan, Bill)) \Rightarrow isTaller(John, Bill)$

$isTaller(John, Dan)$

$isTaller(Dan, Bill)$

With this we can conclude $isTaller(John, Bill)$

What about: John is taller than David

David is taller than Bill

$\therefore$ John is taller than Bill

Will we have to create a separate rule for every set of three persons?

Quantification

For any three persons $x,y,z\ \ (isTaller(x,y) \wedge isTaller(y,z)) \Rightarrow isTaller(x,z)$

Formally we must define the domain of discourse (set of all persons, buildings, …)

Often the domain is implicit but understood.

$$\forall x \forall y \forall z\!: \ (isTaller(x,y) \wedge isTaller(y,z)) \Rightarrow isTaller(x,z)$$

$\forall x$: "for all x"   the universal quantifier

The quantifier *binds* each variable to the domain of discourse. A quantified statement in which every variable is bound is a *quantified proposition*.

Predicates over numbers

$1+2=3$

$1+2+3=6$

$1+2+3+4 = 10$

$P(n)$: $1+\cdots+n=n(n+1)/2$ is a predicate (not a proposition)

$P(1)$, $P(2)$, $P(3)$ are all propositions that are true.

The proposition $\forall n\in\mathbb{N}:P(n)$ is true!

Quantification

John is taller than everyone. $\forall x$: $isTaller(John,x)$

Everyone is taller than Bill. $\forall x$: $isTaller(x,Bill)$

No one is taller than John. $\forall x$: $\neg isTaller(x,John)$

John is taller than someone. ???

There exists a person x : $isTaller(John, x)$

$\exists x$:$isTaller(John, x)$

$\exists$ "*there exists*" is the *existential quantifier*

Examples

H(x) = “x is a horse”

A(y) = “y is an animal”

Every horse is an animal.

$$\boxed{?}\, x\colon H(x) \Rightarrow A(x)$$

Some animals are horses.

$$\boxed{?}\, x\colon H(x) \wedge A(x)$$

More examples

$L(x,y)$ = “x loves y”

$$L(Romeo, Juliet) = True$$

Everyone loves someone.

$$\boxed{?}x\ \boxed{?}y\ : L(x,y)$$

Someone loves everyone.

$$\boxed{?}x\ \boxed{?}y : L(x,y)$$

Someone is loved by everyone.

$$\boxed{?}x\ \boxed{?}y : L(y,x)$$

Nested quantifiers

No one loves everyone.

$$\neg(\boxed{?}x\,\boxed{?}y\colon L(x,y))$$

Everyone does not love everyone.

$$\boxed{?}x\,\neg(\boxed{?}y\colon L(x,y))$$

For every person there is someone that person does not love.

$$\boxed{?}x\,\boxed{?}y\colon \neg L(x,y)$$

Pushing negation through a layer changes the quantifier!

De Morgan's Law for nested quantifiers

Inferences with quantified propositions

All the world loves a lover
Romeo loves Juliet

∴ Archie loves Betty

1. $\boxed{?}x\,\boxed{?}y\,(\boxed{?}z\!: L(y,z) \rightarrow L(x,y))$
2. $L(Romeo, Juliet)$
3. $\boxed{?}x\, L(x, Romeo)$ 1,2
4. $L(Betty, Romeo)$ 1,3 Universal Instantiation
5. $\boxed{?}x\, L(x, Betty)$ 1,4
6. $L(Archie, Betty)$ 1,5 Universal Instantiation

Logical Deductions

Everybody loves my baby
<u>My baby loves nobody but me</u>

∴ I am my baby

1. $\boxed{?}\, x\, L(x,baby)$ Hypothesis
2. $\boxed{?}\, x\,(L(baby,x) \Rightarrow E(x, me))$ $E(x,y)$: $x=y$
3. $\neg E(baby, me)$ negate the conclusion
4. $L(baby, baby)$ Universal instantiation, 1
5. $L(baby, baby) \Rightarrow E(baby, me)$ Universal Instantiation, 2

$\neg L(baby, baby)$ $E(baby, me)$

Another Example

Everyone has a parent
___

∴ Everyone has a grandparent

$\boxed{?}x\ \boxed{?}y: P(x,y)$ (parent of x is y)

$\boxed{?}x\ \boxed{?}y\ \boxed{?}z: P(x,y) \land P(y,z)$ (every x has some grandparent z)

$\neg\boxed{?}x\ \boxed{?}y\ \boxed{?}z: P(x,y) \land P(y,z)$ (negate the conclusion)

$\boxed{?}x\ \neg\boxed{?}y\ \boxed{?}z: P(x,y) \land P(y,z)$ (1)

$\neg\boxed{?}y\ \boxed{?}z: P(a,y) \land P(y,z)$ (2, Instantiation)

$\boxed{?}y: P(a,y)$ (hypothesis)

$P(a,b)$ (Instantiation)

$\boxed{?}y: P(b,y)$ (hypothesis)

$P(b,c)$ (Instantiation)

$\boxed{?}y\ \neg\boxed{?}z: P(\cancel{a},y) \land P(\cancel{y},z)$ (3, from 2)

$\neg\boxed{?}z: P(a,b) \land P(b,z)$ (Instantiation)