1.	2.	3.	4.	5.	6.	Total:

MA 232. **Exam 2.** **November 16, 2017.**

Print name: ______________________________

Instructor: A.Myasnikov

Closed book and closed notes. Show all of your work. Answers without supporting work will not receive credit.

Pledge and sign: ______________________________

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Problem 1. (10pts) Prove or disprove:

(a) Let $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4$ be a basis of W, $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2$ be a basis of $W^\perp$ an d $S = span(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2)$. Then $dim(S) = 4 - 2 = 2$.

(b) $(1,0), (0,1), (0,0)$ is a basis of $\mathbb{R}^2$

(c) For any orthoormal matrix Q we have $Q^{-1} = Q^T$.

Solution:

(a) Vectors in each basis are linearly independent. In addition vectors in the basis of W are linearly independent from the vectors in the basis of $W^\perp$. So the set of vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2$ is linearly independent and, therefore, $dim(S) = 6$.

(b) Zero vector cannot be in a basis. Also dimension of $\mathbb{R}^2$ is 2 so it maust be at most two vctors in a basis.

(c) In general Q may not be a square matrix and will not have an inverse. The statement is false in general.

Problem 2. (10pts) Let $\bar{\mathbf{v}} = (2, -2, 2)$ and W be a subspace of $\mathbb{R}^3$ spanned by

$$\bar{\mathbf{w}}_1 = (0, 1, 0),\ \bar{\mathbf{w}}_2 = (1, 0, -1)$$

Find the vector $\bar{\mathbf{w}}$ in W closest to $\bar{\mathbf{v}}$.

Solution. Closest vector will be the projection of $\bar{\mathbf{v}}$ onto W. The projection

$$\bar{\mathbf{w}} = A(A^TA)^{-1}A^T\bar{\mathbf{v}}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that $\bar{\mathbf{w}}_1$ and $\bar{\mathbf{w}}_2$ are independent so the inverse $(A^TA)^{-1}$ exists.

Compute

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ Hence } (A^TA)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Then

$$\bar{\mathbf{w}} = A(A^TA)^{-1}A^T\bar{\mathbf{v}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

Problem 3. (10pts) Find a basis for the subspace $W^{\perp}$ of $\mathbb{R}^5$, where W is spanned by two vectors

$$\bar{\mathbf{u}}_1 = (1, 2, 3, -1, 2),\ \ \bar{\mathbf{u}}_2 = (2, 4, 7, 2, -1).$$

Solution:

$U^{\perp} = N(A)$ where $N(A)$ is a nullspace of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix}$$

Triangular form is

$$U = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -5 \end{bmatrix} \Rightarrow \quad \begin{aligned} x + 2y + 3z - s + 2t &= 0 \\ z + 4s - 5t &= 0 \end{aligned}$$

There are three free variables: y, s, t. Obtain special solutions:

(a) $y = 1, s = 0, t = 0 \Rightarrow \bar{\mathbf{w}}_1 = (-2, 1, 0, 0, 0)$

(b) $y = 0, s = 1, t = 0 \Rightarrow \bar{\mathbf{w}}_2 = (13, 0, -4, 1, 0)$

(c) $y = 0, s = 0, t = 1 \Rightarrow \bar{\mathbf{w}}_3 = (-17, 0, 5, 0, 1)$

$\{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_3\}$ is a basis for $U^{\perp}$

Problem 4. (10pts) Find an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by

$$\bar{\mathbf{v}}_1 = (1, 1, 1, 1),\ \ \bar{\mathbf{v}}_2 = (1, 2, 4, 5),\ \ \bar{\mathbf{v}}_3 = (1, -3, -4, -2)$$

Solution:

Apply Gram-Schmidt:

(a) $\bar{\mathbf{w}}_1 = (1, 1, 1, 1)$

(b) $\bar{\mathbf{w}}_2 = \bar{\mathbf{v}}_2 - \frac{\bar{\mathbf{v}}_2^T \bar{\mathbf{w}}_1}{\bar{\mathbf{w}}_1^T \bar{\mathbf{w}}_1} \bar{\mathbf{w}}_1 = (-2, -1, 1, 2)$

(c) $\bar{\mathbf{w}}_3 = \bar{\mathbf{v}}_3 - \frac{\bar{\mathbf{v}}_3^T \bar{\mathbf{w}}_1}{\bar{\mathbf{w}}_1^T \bar{\mathbf{w}}_1} \bar{\mathbf{w}}_1 - \frac{\bar{\mathbf{v}}_3^T \bar{\mathbf{w}}_2}{\bar{\mathbf{w}}_2^T \bar{\mathbf{w}}_2} \bar{\mathbf{w}}_2 = (8/5, -17/10, -13/10, 7/5)$

Scale by the length to obtain unit vectors:

$$\bar{\mathbf{u}}_1 = \frac{1}{2}(1, 1, 1, 1),\ \bar{\mathbf{u}}_2 = \frac{1}{\sqrt{10}}(-2, -1, 1, 2),\ \bar{\mathbf{u}}_3 = \frac{10}{\sqrt{910}}(8/5, -17/10, -13/10, 7/5) = \frac{1}{\sqrt{910}}(16, -17, -13, 14)$$

Problem 5. (10pts) Let S consists of the following vectors in $\mathbb{R}^4$

$$\bar{\mathbf{u}}_1 = (1, 1, 0, -1),\ \ \bar{\mathbf{u}}_2 = (1, 2, 1, 3),\ \ \bar{\mathbf{u}}_3 = (1, 1, -9, 2),\ \ \bar{\mathbf{u}}_4 = (16, -13, 1, 3)$$

Show that S is a basis of $\mathbb{R}^4$

Solution:

Taking dot products, we can show that $\bar{\mathbf{u}}_i \cdot \bar{\mathbf{u}}_j = 0$ when $i \neq j$. Therefore they are mutually orthogonal and S is orthogonal. Orthogonal vectors are linearly independent, hence S is linearly independent. We need four independent vectors to span $\mathbb{R}^4$ which means S is a basis for $\mathbb{R}^4$. A more standard way to solve this problem is to setup matrix with rows (or columns) being the vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4$ and show that they are linearly independent.

Problem 6. (10pts) The following is the US census data:

Year	Population mln
1800	5.3
1850	23
1900	75
1950	151
1990	249

(a) Set up a system of equations whose solution gives the best fitted line through the data above;

(b) Set up a system of equation to find a parabola closest to the data.

Note: you just need to set up the appropriate system, do not solve it.

Solution:

Line is given by equation $y = \alpha x + \beta$ to obtain α, β solve system

$$\begin{bmatrix} 1 & 1800 \\ 1 & 1850 \\ 1 & 1900 \\ 1 & 1950 \\ 1 & 1990 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 5.3 \\ 23 \\ 75 \\ 151 \\ 249 \end{bmatrix}$$

Parabola is given by equation $y = \alpha x^2 + \beta x + \gamma$ to obtain α, β, γ solve system

$$\begin{bmatrix} 1 & 1800 & 1800^2 \\ 1 & 1850 & 1850^2 \\ 1 & 1900 & 1900^2 \\ 1 & 1950 & 1950^2 \\ 1 & 1990 & 1990^2 \end{bmatrix} \begin{bmatrix} \gamma \\ \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 5.3 \\ 23 \\ 75 \\ 151 \\ 249 \end{bmatrix}$$