Lecture 19: Equivalence relations; more relations in Scheme

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Outline of lecture

Partial orders

Equivalence relations

Partitions

Scheme practice with relations in general

Partial order (partially ordered set)

Review:

Let R be a relation on T, i.e., $R \subseteq T \times T$.

"R is *reflexive*" means: $\forall x \in T \ (x, x) \in R$.

R is *anti-symmetric*: $\forall x, y \ ((x, y) \in R \land (y, x) \in R \to x = y)$

R is *transitive*: $\forall x, y, z \ ((x, y) \in R \land (y, z) \in R \to (x, z) \in R)$

Define: R is a partial order (or: the set T is partially ordered by R) iff R is reflexive, anti-symmetric, and transitive.

Examples?

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Equality

equality — foundation of the mathematical universe

Pick your favorite set and call it S.

Define relation E on S by $(s, t) \in E$ iff $s = t$. That is, $E = \{(s, s) \mid s \in S\}$.

Is E reflexive? transitive? empty? antisymmetric? ...?

Overview and history of equivalence

Flashback to gradeschool: three apples in a pile is “the same as” three pennies in a pocket.

Definition: 3 is the equivalence class of the pennies in my pocket, where the equivalence relation is that there’s a bijection. (Really: the symbol 3 stands for the equivalence class.

Althought $sqrt(2)$ isn’t rational, there is a sequence of rationals that get’s closer and closer: 1, 1.4, 1.41, 1.414, 1.4142, 1.41421,. . .

So does 1, 1, 1.4, 1.4, 1.41, 1.42, 1.414, 1.414, . . .

Definition: a Cauchy sequence is an infinite sequence $r_0, r_1, \ldots$ of rationals such that $r_i - r_j$ converges to 0 as i, j get large.

Definition: a *real number* is an equivalence class of Cauchy sequences. For sequence r and s to be equivalent means that as i gets large, $r_i - s_i$ converges to 0.

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Equivalence relation

Let S be a set. Definition: A relation $R \subseteq S \times S$ is an *equivalence relation* iff R is reflexive, transitive, and symmetric.

Being in the same room as (a relation on people).

Having the same elements (a relation on lists without duplicates).

Having the same elements (a relation on sorted lists with duplicates).

Having the same number of credits (a relation on study plans).

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Equivalence of set?s

The Scheme predicate set? applies to lists; it says the list has no two elements that are equal?. (What's equal??)

Exercise: define (eq-set? set0 set1) to say that set0 and set1 have the same elements.

Hints: think about subset?

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Hints: think about subset?

Hashtables

Popular and often efficient data structure.

Let S be a set of data and suppose our program works with subsets of S. E.g.: check whether x is in subset T.

Choose some function $h : S \to \{0, \ldots, N\}$ where, e.g., $N = 2^{10}$. Use $b : array[0..N]$ of lists of S.

Now b represents a subset T of S, i.e., T is the union of the contents of the lists. To find whether x is in T, look for x in the list $b[h(x)]$.

Define a relation $E \subseteq R \times R$ by: $\boxed{xEy \text{ iff } h(x) = h(y)}$
("x and y have the same hash value")

The elements of $b[i]$, where $i = h(x)$, are all the elements of T that are equivalent to x.

For $i \neq j$, the lists $b[i]$ and $b[j]$ are disjoint (i.e, no elements in common)

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Back and forth: Partitions/equivalences

Suppose $E \subseteq S \times S$ is an equivalence relation.
The corresponding *partition* P is a set of subsets of S. The subsets are called *equivalence classes*, and are written $[x]_E$.
Define $[x]_E = \{y \mid xEy\}$.

Suppose P is a set of subsets of S, such that

- every element of S is in some X in P (call X a cell of P)
- if X and Y are different elements of P then $X \cap Y = \emptyset$

Define xEy iff x and y are in the same cell. (Exercise: formula)

Exercises: reflexive, transitive closure using compose

Review: Given any relation R, the reflexive transitive closure R^* of R satisfies $\boxed{(x, y) \in R^* \equiv \exists n \in \mathbf{N}.\ (x, y) \in R^n}$

Where R^n is defined by $\boxed{R^0 = id \text{ and } R^{n+1} = R \circ R^n}$

In other words, $R^* = (\bigcup_{i \in \mathbf{N}} R^i)$

If $R \subseteq S \times S$ and $size(S) \leqslant n$ then $R^* = (\bigcup_{i=0}^{n} R^i)$

Compute using the fact that
$(\bigcup_{i=0}^{k+1} R^i) = (R \circ (\bigcup_{i=0}^{k} R^i)) \cup (\bigcup_{i=0}^{k} R^i)$.

Alternatively, use the fact that
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(define (compose-rel rel0 rel1)
  ; Assume rel0 and rel1 are set?s of pairs.
  ; Return the composition rel1 o rel0 of rel1 after rel0.
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Alert: Rosen uses R^* for R^+

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