Orthonormal bases.

Recall that a basis is called *orthogonal* if its vectors are mutually orthogonal.

A basis $\{\bar{\mathbf{q}}_1, \ldots, \bar{\mathbf{q}}_n\}$ is called orthonormal if

1. it is orthogonal: $\bar{\mathbf{q}}_i^T \bar{\mathbf{q}}_j = 0, i \neq j$
2. each vector has unit length: $||\bar{\mathbf{q}}_i|| = 1$, $i = 1, \ldots, n$

Given orthogonal basis $\{\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_n\}$ we always construct orthonormal basis $\{\bar{\mathbf{q}}_1, \ldots, \bar{\mathbf{q}}_n\}$, where $\bar{\mathbf{q}}_i = \bar{\mathbf{w}}_i / ||\bar{\mathbf{w}}_i||$

Orthonormal bases.

A matrix is called orthonormal id its columns form an arthonormal basis.
If matrix Q is orthonormal then $Q^TQ = I$

$$
\begin{aligned}
Q^TQ &= \begin{bmatrix} -\bar{\mathbf{q}}_1^T- \\ -\bar{\mathbf{q}}_2^T- \\ \dots \\ -\bar{\mathbf{q}}_n^T- \end{bmatrix} \begin{bmatrix} | & | & & | \\ \bar{\mathbf{q}}_1 & \bar{\mathbf{q}}_2 & \dots & \bar{\mathbf{q}}_n \\ | & | & & | \end{bmatrix} \\
&= \begin{bmatrix} \bar{\mathbf{q}}_1^T\bar{\mathbf{q}}_1 & \bar{\mathbf{q}}_1^T\bar{\mathbf{q}}_2 & \cdots & \bar{\mathbf{q}}_1^T\bar{\mathbf{q}}_n \\ \bar{\mathbf{q}}_2^T\bar{\mathbf{q}}_1 & \bar{\mathbf{q}}_2^T\bar{\mathbf{q}}_2 & \cdots & \bar{\mathbf{q}}_2^T\bar{\mathbf{q}}_n \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{q}}_n^T\bar{\mathbf{q}}_1 & \bar{\mathbf{q}}_n^T\bar{\mathbf{q}}_2 & \cdots & \bar{\mathbf{q}}_1^n\bar{\mathbf{q}}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\end{aligned}
$$

Orthonormal bases.

Let Q be orthonormal

- If Q square then $Q^{-1} = Q^T$
- Q preserves dot products: $(Q\bar{\mathbf{x}})^T)Q\bar{\mathbf{y}}) = \bar{\mathbf{x}}^T\bar{\mathbf{y}}$
- Q preserves length $||Q\bar{\mathbf{x}}|| = ||\bar{\mathbf{x}}||$
- Product of two orthonormal matrices is orthonormal

$$Q^TQ = I = S^TS \Rightarrow (QS)^T(QS) = S^TQ^TQS = I$$

- Projection of $\bar{\mathbf{b}}$ onto space spanned by columns of Q is

$$\bar{\mathbf{p}} = Q(Q^TQ)^{-1}Q^T\bar{\mathbf{b}} = QQ^T\bar{\mathbf{b}}$$

Orthonormal bases.

Let Q be orthonormal

Solution to least squares problem $Q\bar{\mathbf{x}} = \bar{\mathbf{b}}$ is

$$\bar{\mathbf{x}} = (Q^T Q)^{-1} Q^T \bar{\mathbf{b}} = Q^T \bar{\mathbf{b}}$$

Orthonormal bases.

Orthogonal bases (matrices) are easier to work with.
Questions:

- How we obtain an orthonormal basis of a vector space
- How we can use it to improve the solution to the least squares problem

Gram-Schmidt Orthogonalization process.

Let $\{\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n\}$ be basis for a vector space V.
Our goal is to find an orthogonal basis $\{\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_n\}$ of V.

Gram-Schmidt Orthogonalization process.

We construct orthogonal basis as follows:

$$
\begin{aligned}
\bar{\mathbf{w}}_1 &= \bar{\mathbf{v}}_1 \\
\bar{\mathbf{w}}_2 &= \bar{\mathbf{v}}_2 - \frac{\bar{\mathbf{v}}_2 \cdot \bar{\mathbf{w}}_1}{\bar{\mathbf{w}}_1 \cdot \bar{\mathbf{w}}_1}\bar{\mathbf{w}}_1 \\
\bar{\mathbf{w}}_3 &= \bar{\mathbf{v}}_3 - \frac{\bar{\mathbf{v}}_3 \cdot \bar{\mathbf{w}}_1}{\bar{\mathbf{w}}_1 \cdot \bar{\mathbf{w}}_1}\bar{\mathbf{w}}_1 - \frac{\bar{\mathbf{v}}_3 \cdot \bar{\mathbf{w}}_2}{\bar{\mathbf{w}}_2 \cdot \bar{\mathbf{w}}_2}\bar{\mathbf{w}}_2 \\
& \cdots \\
\bar{\mathbf{w}}_n &= \bar{\mathbf{v}}_n - \frac{\bar{\mathbf{v}}_n \cdot \bar{\mathbf{w}}_1}{\bar{\mathbf{w}}_1 \cdot \bar{\mathbf{w}}_1}\bar{\mathbf{w}}_1 - \frac{\bar{\mathbf{v}}_n \cdot \bar{\mathbf{w}}_2}{\bar{\mathbf{w}}_2 \cdot \bar{\mathbf{w}}_2}\bar{\mathbf{w}}_2 - \cdots - \frac{\bar{\mathbf{v}}_n \cdot \bar{\mathbf{w}}_{n-1}}{\bar{\mathbf{w}}_{n-1} \cdot \bar{\mathbf{w}}_{n-1}}\bar{\mathbf{w}}_{n-1}
\end{aligned}
$$

Orthonormal basis:

$$\{\bar{\mathbf{q}}_i \mid \bar{\mathbf{q}}_i = \bar{\mathbf{w}}_i / ||\bar{\mathbf{w}}_i||, i = 1, \ldots, n\}$$

Gram-Schmidt Orthogonalization process.

- Each $\bar{\mathbf{w}}_k$ is a linear combination of $\bar{\mathbf{v}}_k$ and $\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_{k-1}$. Therefore $\bar{\mathbf{w}}_k$ is a linear combination of $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n$, i.e. $\bar{\mathbf{w}}_k \in V$
- Set $\{\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_n\}$ is orthogonal and, hence, vectors are linearly independent
- Since $dim(V) = n$ and $\{\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_n\}$ are linearly independent vectors which span V then they form a basis for V.

Any finite-dimensional vectors space has an orthogonal basis.

QR factorization.

- Let $\{\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n\}$ and $\{\bar{\mathbf{q}}_1, \ldots, \bar{\mathbf{q}}_n\}$ be orthonormal basis that spans the same space.
- Set $A = [\bar{\mathbf{v}}_1 \ \ldots \ \bar{\mathbf{v}}_n]$ and $Q = [\bar{\mathbf{q}}_1 \ \ldots \ \bar{\mathbf{q}}_n]$

$$A = QR, \text{ where } R \text{ is upper triangular}$$

QR factorization.

Find R:

$$A = QR \Rightarrow Q^T A = Q^T Q R = R$$

$$R = \begin{bmatrix} -\bar{\mathbf{q}}_1^T- \\ -\bar{\mathbf{q}}_2^T- \\ \dots \\ -\bar{\mathbf{q}}_n^T- \end{bmatrix} \cdot \begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{q}}_1^T\bar{\mathbf{v}}_1 & \bar{\mathbf{q}}_1^T\bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{q}}_1^T\bar{\mathbf{v}}_n \\ 0 & \bar{\mathbf{q}}_2^T\bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{q}}_1^T\bar{\mathbf{v}}_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\mathbf{q}}_n^T\bar{\mathbf{v}}_n \end{bmatrix}$$

QR factorization.

The QR factorization is $A = QR$

$$\begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{q}}_1 & \bar{\mathbf{q}}_2 & \cdots & \bar{\mathbf{q}}_n \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathbf{q}}_1^T \bar{\mathbf{v}}_1 & \bar{\mathbf{q}}_1^T \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{q}}_1^T \bar{\mathbf{v}}_n \\ 0 & \bar{\mathbf{q}}_2^T \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{q}}_1^T \bar{\mathbf{v}}_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\mathbf{q}}_n^T \bar{\mathbf{v}}_n \end{bmatrix}$$

QR factorization.

- Given leat squares problem $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ we would like reduce it to $Q\bar{\mathbf{x}} = \bar{\mathbf{b}}'$ for some orthonormal matrix Q
- QR factorization gives $A = QR$
- Solution to $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$:

$$\begin{aligned} A^T A\bar{\mathbf{x}} &= A^T\bar{\mathbf{b}} \Rightarrow ((QR)^T(QR))\bar{\mathbf{x}} = (QR)^T\bar{\mathbf{b}} \\ &\Rightarrow (R^T Q^T QR)\bar{\mathbf{x}} = R^T Q^T\bar{\mathbf{b}} \Rightarrow R^T R\bar{\mathbf{x}} = R^T Q^T\bar{\mathbf{b}} \\ &\Rightarrow R\bar{\mathbf{x}} = Q^T\bar{\mathbf{b}} \end{aligned}$$

- Recall that R is upper triangular. We can solve for $\bar{\mathbf{x}}$ using back substitution!