MA232 Linear Algebra

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Vector spaces.

- Line is a space of all points in $\mathbb{R}^1$
- Plane is a space of all points in $\mathbb{R}^2$
- 3D space consistes of all points in $\mathbb{R}^3$

A vector space is the abstract formulation motivated by the most basic properties of n-dimensional Euclidean space $\mathbb{R}^n$: addition and scalar multiplication.

Vector spaces.

A vector space is a set V closed under two operations:

1. Addition: $\bar{\mathbf{v}}, \bar{\mathbf{w}} \in V \Rightarrow \bar{\mathbf{v}} + \bar{\mathbf{w}} \in V$;
2. Scalar Multiplication: $\bar{\mathbf{v}} \in V$ and $c \in \mathbb{R} \Rightarrow c\bar{\mathbf{v}} \in V$.

– NOTE: we refer to the elements of V as *"vectors"*, even though, they might be functions or matrices or some other objects.

Vector spaces.

Axioms:

1. Commutativity of Addition: $\bar{\mathbf{v}} + \bar{\mathbf{w}} = \bar{\mathbf{w}} + \bar{\mathbf{v}}$;
2. Associativity of Addition: $\bar{\mathbf{u}} + (\bar{\mathbf{v}} + \bar{\mathbf{w}}) = (\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}}$;
3. Additive Identity: There is a zero element $\bar{\mathbf{0}} \in V$ s.t. $\bar{\mathbf{v}} + \bar{\mathbf{0}} = \bar{\mathbf{0}} + \bar{\mathbf{v}} = \bar{\mathbf{0}}$;
4. Additive Inverse: For each $\bar{\mathbf{v}} \in V$ there exists $-\bar{\mathbf{v}} \in V$ s.t. $\bar{\mathbf{v}} + (-\bar{\mathbf{v}}) = \bar{\mathbf{0}}$;
5. Distributivity: $(c + d)\bar{\mathbf{v}} = (c\bar{\mathbf{v}}) + (d\bar{\mathbf{v}})$, and $c(\bar{\mathbf{v}} + \bar{\mathbf{w}}) = (c\bar{\mathbf{v}}) + (c\bar{\mathbf{w}})$;
6. Assoc. of Scalar Mult.: $c(d\bar{\mathbf{v}}) = (cd)\bar{\mathbf{v}}$;
7. Unit for Scalar Mult: there exists $1 \in R$ s.t. $1\bar{\mathbf{v}} = \bar{\mathbf{v}}$.

Vector spaces.

Immediate consequences of axioms:

1. $0\bar{\mathbf{v}} = \bar{\mathbf{0}}$;
2. $(-1)\bar{\mathbf{v}} = -\bar{\mathbf{v}}$;
3. $c\bar{\mathbf{0}} = \bar{\mathbf{0}}$;
4. If $c\bar{\mathbf{v}} = \bar{\mathbf{0}}$, then either $c = 0$ or $\bar{\mathbf{v}} = \bar{\mathbf{0}}$.

Vector spaces.

Vector spaces:

- Euclidean space $\mathbb{R}^n$;
- Set $M(n)$ of all $n \times m$ matrices;
- Set $P(n)$ of polynomials of degree $\leq n$
- Set of all real functions $f(x)$ defined on interval I

Vector spaces.

NOT vector spaces:

- Natural numbers
- Line $x = 2$ on a plane
- Euclidean space $\mathbb{R}^n$;
- Set of polynomials of degree n

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Subspace

A subspace of a vector space V is a subset $W \subseteq V$ which is also a vector space.

A subset $W \subseteq V$ of a vector space V is a subspace iff it is closed under addition and scalar multiplication:

1. $\bar{\mathbf{v}}, \bar{\mathbf{w}} \in W \Rightarrow \bar{\mathbf{v}} + \bar{\mathbf{w}} \in W$;
2. $\bar{\mathbf{v}} \in W$ and $c \in \mathbb{R} \Rightarrow c\bar{\mathbf{v}} \in W$.

Or combined together for every $c, d \in \mathbb{R}$ and $\bar{\mathbf{v}}, \bar{\mathbf{w}} \in W$:

$$c\bar{\mathbf{v}} + d\bar{\mathbf{w}} \in W$$

– NOTE: Every subspace contains the zero vector

Subspace

Examples:

- The trivial subspace $W = \{\bar{\mathbf{0}}\}$
- The entire space $W = \mathbb{R}^3$
- All vectors of the form $(x, y, 0)$
- Any plane through $(0, 0, 0)$
- The set of solutions (x; y; z) to the linear equation $3x + 2y - z = 0$
- Set of diagonal matrices.
- Set of polynomials $P(n)$ is a subset of all real functions.

Subspace

NOT: subspaces

- Set of vectors of the form $(x, y, 1)$
- Identity matrix $W = \{I\}$
- The unit sphere $S = \left\{x^2 + y^2 + z^2 = 1\right\}$

Span

- If $\bar{\mathbf{v}}, \bar{\mathbf{w}} \in V$ then their *linear combination* $c\bar{\mathbf{v}} + d\bar{\mathbf{w}} \in V$
- In general if $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n \in V$ then for any $c_1, \ldots, c_n \in \mathbb{R}$

$$c_1\bar{\mathbf{v}}_1 + \cdots + c_n\bar{\mathbf{v}}_n \in V$$

- Can we use this to define spaces?

Span

Let $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n \in V$. The span is the subset $W = span(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n) \in V$ consisting of all possible linear combinations of vectors $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n \in V$:

$$span(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n) = \{c_1\bar{\mathbf{v}}_1 + \cdots + c_n\bar{\mathbf{v}}_n \mid c_1, \ldots, c_n \in \mathbb{R}\}$$

Moreover, a span always forms a subspace

- $span((0,0,1),(0,1,0),(1,0,0)) = \mathbb{R}^3$

Subspaces and linear equations

- Consider $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$
- If A is invertible then there is a *unique* solution.
- If A does not have inverse then $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ may still have solutions for some vectors $\bar{\mathbf{b}}$
- In fact it may have infinitely many solutions.
- How do we describe solutions to such systems?

Subspaces and linear equations: Column space

- Recall *COLUMN* view of $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has a solution $x_1, \ldots, x_n$ if $\bar{\mathbf{b}}$ is a linear combination:

$$\bar{\mathbf{b}} = x_1 A_{col}(1) + x_2 A_{col}(2) + \cdots + x_n A_{col}(n)$$

Subspaces and linear equations: Column space

The column space of $m \times n$ matrix A is a subspace spanned by columns of A:

$$C(A) = span(A_{col}(1), \dots, A_{col}(n)) \subseteq \mathbb{R}^m.$$

Equation $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has a solution if and only if $\bar{\mathbf{b}} \in C(A)$

NOTE: $\bar{\mathbf{0}} \in C(A)$

Subspaces and linear equations: Null Space

- Homogenous linear system

$$A\bar{\mathbf{z}} = \bar{\mathbf{0}}$$

- *Trivial* solution $\bar{\mathbf{z}} = \bar{\mathbf{0}}$ always exists
- Our goal is to describe all solutions for arbitrary matrix A.

Subspaces and linear equations: Null Space

The null space of $m \times n$ matrix A is a subspace which contains all solutions to the system $A\bar{\mathbf{z}} = \bar{\mathbf{0}}$:

$$N(A) = \left\{\bar{\mathbf{z}} \in \mathbb{R}^n \mid A\bar{\mathbf{z}} = \bar{\mathbf{0}}\right\} \subseteq \mathbb{R}^n.$$

$N(A)$ is indeed a subspace: if $\bar{\mathbf{z}}_1, \ldots, \bar{\mathbf{z}}_k$ are solutions to $A\bar{\mathbf{z}} = \bar{\mathbf{0}}$ then so is any linear combination: $c_1\bar{\mathbf{z}}_1 + \cdots + c_k\bar{\mathbf{z}}_k$

- Set of solutions to $B\bar{\mathbf{x}} = \bar{\mathbf{b}}, \bar{\mathbf{b}} \neq \bar{\mathbf{0}}$ is not a subspace since $\bar{\mathbf{0}} \notin N(B)$
- If A is invertible then $N(A) = \left\{\bar{\mathbf{0}}\right\}$

Subspaces and linear equations: Null Space

For $m \times n$ matrix A:

- Column space $C(A)$:
  Column space $C(A)$ is a subspace spanned by columns of A and contains all vectors $\bar{\mathbf{b}}$ for which $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has at least one solution. Columns of A have m entries and therefore

$$C(A) \subseteq \mathbb{R}^m$$

- Null Space $N(A)$:
  Null space of A is a subspace which contains all solutions to $A\bar{\mathbf{z}} = \bar{\mathbf{0}}$. Each solution assigns n variables and

$$N(A) \subseteq \mathbb{R}^n$$

Subspaces and linear equations: Null Space

Compute $N(A)$:

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponds to:

$x - 2y + 3w = 0$ and $y - z - 10w = 0$

Subspaces and linear equations: Null Space

Solve $x - 2y + 3w = 0$, $y - z - 10w = 0$

- We have 2 equations and 4 variables
- 2 variables are free i.e. can take any values
- Values of the other 2 variables are obtained using the equations

Let z, w be free. Obtain some solutions:

- $z = 0, w = 1 \Rightarrow x = 17, y = 10 \Rightarrow [17, 10, 0, 1]^T$
- $z = 1, w = 0 \Rightarrow x = 2, y = 1 \Rightarrow [2, 1, 1, 0]^T$

Subspaces and linear equations: Null Space

- The general solution is a linear combination:

$$z \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 17 \\ 10 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2z + 17w \\ z + 10w \\ z \\ w \end{bmatrix}$$

- To obtain a particular solution we assign values to z, w and compute corresponding values of x, y
- The null space of A:

$$N(A) = \left\{ [2z + 17w, z + 10w, z, w]^T \mid z, w \in \mathbb{R} \right\}$$

Computing Null Space of A

- Perform elimination steps for every column and obtain Echelon Matrix:

$$\begin{bmatrix} p & x & x & x & x & x \\ 0 & 0 & p & x & x & x \\ 0 & 0 & 0 & p & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- In Echelon Matrix: pivots are the leftmost nonzero entries in the rows
- Columns which have pivots correspond to pivot variables
- Columns which have NO pivots correspond to free variables

Computing Null Space of A

- Suppose there are k *free* variables $y_1, \ldots, y_k$
- Obtain k special solutions $\bar{\mathbf{s}}_1, \ldots, \bar{\mathbf{s}}_k$. To get $\bar{\mathbf{s}}_i$:
  1. Assign values to free variables: $y_i = 1, y_j = 0, i \neq j, i = 1, \ldots, k$
  2. Solve for the pivot variables
- The general solution is:

$$y_1\bar{\mathbf{s}}_1 + y_2\bar{\mathbf{s}}_2 + \cdots + y_k\bar{\mathbf{s}}_k$$

- Null space:

$$N(A) = \{y_1\bar{\mathbf{s}}_1 + \cdots + y_k\bar{\mathbf{s}}_k \mid y_1, \ldots, y_k \in \mathbb{R}\}$$

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