MA232 Linear Algebra

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Linear combination.

Linear combination of two vectors is a vector:

$$\mathbf{u} = a\mathbf{v} + b\mathbf{w}$$

In general:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

- $a\mathbf{v}$ defines a line
- $a\mathbf{v} + b\mathbf{w}$ defines a plane if $\mathbf{v}$ and $\mathbf{w}$ are independent

Vector multiplication.

Let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

Dot product $\mathbf{v} \cdot \mathbf{w}$ is a number:

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i}^{n} v_i w_i.$$

NOTE: dot product is commutative: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

Vector multiplication.

Dot product gives information about the angle θ between two vectors:

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\ ||\mathbf{w}||}$$

In particular if $\mathbf{v}\cdot\mathbf{w} = 0$ then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal

The length of a vector.

– The length (norm) $||\mathbf{v}||$ of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ is the distance from origin to the point $(v_1, \ldots, v_n)$

It can be computed as a dot product of a vector with itself

$$||\mathbf{v}|| = \mathbf{v} \cdot \mathbf{v}$$

The length of a vector: unit vector

- Unit vector is a vector with length $||u|| = 1$
- Given arbitrary vector $\mathbf{w}$, we can obtain unit vector $u = \frac{\mathbf{w}}{||\mathbf{w}||}$
- Dot product of two unit vectors $\mathbf{u}$ and $\mathbf{u}'$:

$$\mathbf{u} \cdot \mathbf{u}' = \cos\theta$$

Matrix operations.

- Matrix is a rectangular table of numbers
- If A has m rows and n columns we say it is m by n matrix $A^{m \times n}$
- The entry in row i and column j of A is $A(i,j)$ or A_{ij}
- $A_{row}(i)$ is the ith row of A
- $A_{col}(j)$ is the jth column of A

Matrix operations.

- $A^{n\times n}$ is a square matrix and $A_{11}, A_{22}, A_{33}, \ldots, A_{nn}$ are diagonal elements
- $A^{n\times n}$ is a diagonal matrix if $A_{ij} = 0$, $\forall i \neq j$
- A diagonal matrix with all diagonal elements equal to 1 is called the identity matrix I_n
- A matrix U is called upper triangular if $U_{ij} = 0$ whenever $i > j$

Matrix operations.

- Equality:

$$A^{n\times m} = B^{n\times m} \text{ iff } A_{ij} = B_{ij} \text{ for all } i, j$$

- Scalar multiplication: $cA = B$, where $B_{ij} = cA_{ij}$
- Addition: matrices of the same shape can be added:

$$A^{m\times n} + B^{m\times n} = C^{m\times n},$$

where $C_{ij} = A_{ij} + B_{ij}$

Matrix-vector multiplication

– Row view (linear equations)

$$\begin{bmatrix} -\mathbf{r}_1- \\ -\mathbf{r}_2- \\ \vdots \\ -\mathbf{r}_n- \end{bmatrix} \cdot \mathbf{v} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{v} \\ \mathbf{r}_2 \cdot \mathbf{v} \\ \vdots \\ \mathbf{r}_n \cdot \mathbf{v} \end{bmatrix}$$

– Column view (linear combination)

$$\begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \vdots & \mathbf{c}_n \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{c}_1 + v_2\mathbf{c}_2 + \cdots v_n\mathbf{c}_n$$

Matrix multiplication.

Product:

$$A^{n\times m}B^{m\times p} = C^{n\times p}, \text{ where } C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$$

Another way to look:

$$AB = C, \text{ where } C_{ij} = A_{row(i)} \cdot B_{col}(j)$$

Important: If A has n columns, B must have n rows, otherwise product is not defined!

The Laws for addition.

$$\begin{array}{rcll} A + B & = & B + A & \text{(commutative)} \\ c(A + B) & = & cA + cB & \text{(distributive)} \\ A + (B + C) & = & (A + B) + C & \text{(associative)} \end{array}$$

The Laws for multiplication.

$$
\begin{array}{rcll}
AB & \neq & BA & \text{(NOT commutative)} \\
C(A+B) & = & CA+CB & \text{(distributive from the left)} \\
(A+B)C & = & AC+BC & \text{(distributive from the right)} \\
A(BC) & = & (AB)C & \text{(associative)}
\end{array}
$$

Matrix powers.

If A is square then

$$A^p = AA\ldots A \; p \text{ factors}$$

$$\begin{aligned} (A^p)(A^q) &= A^{p+q} \\ (A^p)^q &= A^{pq} \end{aligned}$$