CS 334 - Homework 0 (Mathematical Prerequisites)
Due 1/28/2016

Sets

1. Let f(x) = 2x, where f:$\mathbb{N} \rightarrow \mathbb{Z}_{10}$. Use set-notation to express the collection of values for which f(x) = 0.

Note: Given f:D→R, the set {x∈D | f(x) = 0}, is sometimes called the "kernel" of the function. Kernels play a central role in constructions used throughout many fields of mathematics (group theory, linear algebra, algebraic topology, etc…) This set can also be expressed using the notation $f^{-1}(0)$. More generally, we call $f^{-1}(x)$ the "preimage of x under f".

2. Let A := {x∈$\mathbb{N}$ | 2x mod 10 = 0}, and B := {x∈$\mathbb{N}$ | x mod 2 = 0}. Use set-notation to express:

   i. A $\cup$ B
   ii. A ∩ B
   iii. A ∖ B

Functions

For each of the following functions, identify *whether the function is injective, surjective, bijective, or neither injective nor surjective* for the following domain-range pairs:

i. $\mathbb{N} \rightarrow \mathbb{N}$
ii. $\mathbb{N} \rightarrow \mathbb{Z}_5$
iii. $\mathbb{N} \rightarrow \mathbb{Z}_{10}$
iv. $\mathbb{Z} \rightarrow \mathbb{Z}$

If the function is not injective, provide an example of two objects in the domain which map to the same value. If the function is not surjective, provide an example of a value in the range which is not mapped to by any object in the domain.

1. f(x) = x + 2
2. f(x) = 2x
3. f(x) = 3x

Boolean Logic

Write truth-tables for the following boolean expressions:

1. A ⇔ (B ∧ C)
2. (A ∨ B) ⇒ C
3. A ⇒ (B ⊕ C)

Strings and Languages

1. Provide an alphabet ∑ which would be sufficient to write all arithmetical expressions which include additions and subtractions of integers, as well as parentheses.
   Ex: "(3 - 5) / 2"
2. Give three examples of strings over this alphabet which would belong to ∑*, but would not be a syntactically valid arithmetical expression. Try to have each of the three strings

violate a different syntactic rule.
Ex: “((12-8(”

3. Try to give a complete list of conditions that would need to be checked in order to ensure that a given string is a valid arithmetical expression.
Ex: “No number’s first digit is 0.”

Proofs

1. *“Proof by Contrapositive”*
    a. Use truth-tables to prove that: “$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$”.
    The statement “$(\neg B \Rightarrow \neg A)$” is known as the “contrapositive” of the statement “$(A \Rightarrow B)$”.
    b. State the contrapositive of the following:
    “If $n^2 \in \mathbb{N}$ is odd, then n is also odd.”
    c. Use arithmetic to prove the contrapositive statement, and conclude that the statement from (b) is true.
2. *“Proof by Induction”*
    Given $n \in \mathbb{N}$, prove that $2^n \geq n^2$, for all $n \geq 4$.

    Hint: Since this is only for numbers greater than or equal to 4, use n = 4 as the base case of your proof - then substitute “n+1” into the inequality above, and arithmetically manipulate it, under the assumption that the inequality holds for the (n-1)-case.