

2 Truth Trees

Testing an argument for validity amounts to testing the premises together with the negation of the conclusion for consistency. The argument will be *valid* if and only if the premises together with the negation of the conclusion are *inconsistent*.

Truth tables are one way to do this test, and truth trees are another.

2.1 A closed tree

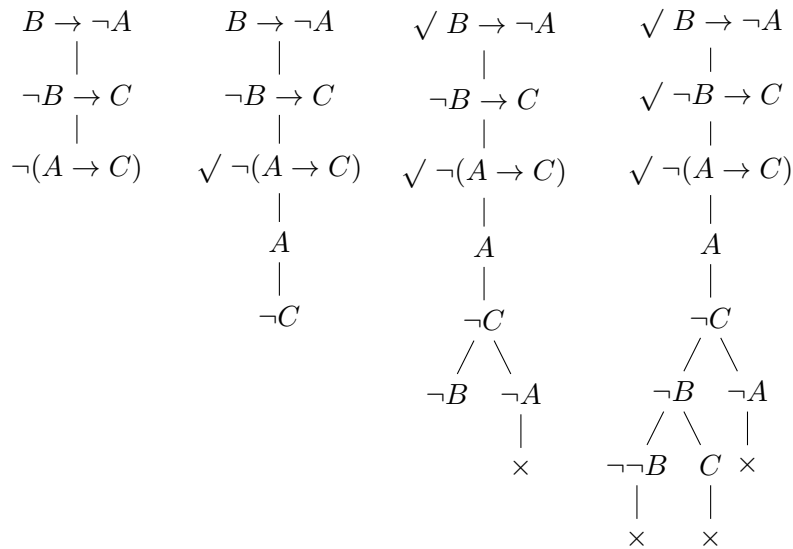
Use the tree method to test the following argument for validity.

$$\frac{B \rightarrow \neg A \quad \neg B \rightarrow C}{A \rightarrow C}$$

In other words use the tree method to test the following collection for consistency.

$$\{B \rightarrow \neg A, \quad \neg B \rightarrow C, \quad \neg(A \rightarrow C)\}$$

There are three steps. Each step moves from a tree (starting with the tree at the left) to the adjacent tree on the right.



The first step uses the rule

$$\frac{\sqrt{\neg(\bigcirc \rightarrow \triangle)}}{\begin{array}{c} \bigcirc \\ \triangle \end{array}}$$

applied to $\neg(A \rightarrow C)$. The second step uses

$$\frac{\sqrt{\bigcirc \rightarrow \triangle}}{\neg \bigcirc \quad \triangle}$$

applied to $B \rightarrow \neg A$, and the last step uses the same rule applied to $\neg B \rightarrow C$.

After the second step the path ending with $\neg A$ contains A too. There is no way to make all the sentences in that path true, since then $\neg A$ and A would have to both be true, which is impossible. Thus that path is closed with an \times . In the same way the two paths remaining after the third step are both closed. All paths are closed, so the collection

$$\{B \rightarrow \neg A, \quad \neg B \rightarrow C, \quad \neg(A \rightarrow C)\}$$

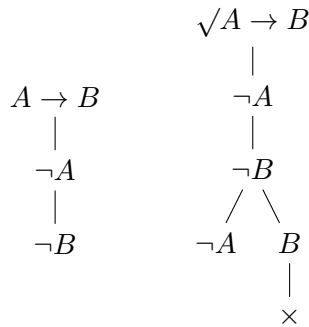
is inconsistent. The argument is valid.

2.2 A Tree Which Does Not Close

If an argument is invalid, the tree does not close, and each unclosed branch gives a counterexample to the argument.

$$\frac{\begin{array}{c} A \rightarrow B \\ \neg A \end{array}}{B}$$

The tree test works as follows.



After one move using the rule

$$\frac{\sqrt{\quad} \quad \bigcirc \rightarrow \Delta}{\neg \bigcirc \quad \Delta}$$

There is nothing more to do. On the open branch, A and B are both false. It is easy to check that these truth values are a counterexample to the argument given at the beginning of this section.

2.3 Double Denial

$$\frac{\sqrt{\quad} \quad \neg\neg\bigcirc}{\bigcirc}$$

2.4 Flowchart for “ \neg ” and “ \rightarrow ”

2.5 Rules of Inference

2.6 Problems

1. By the tree method test the following set of three sentences for consistency.

If Paul lives in Paris, he lives in France. It’s false that if Paul lives in London, he lives in France. It’s false that if Paul lives in Paris he lives in England.

2. Write a paragraph comparing and contrasting the concepts of validity, tautology, and consistency. Assume the reader knows about sentences and truth tables.