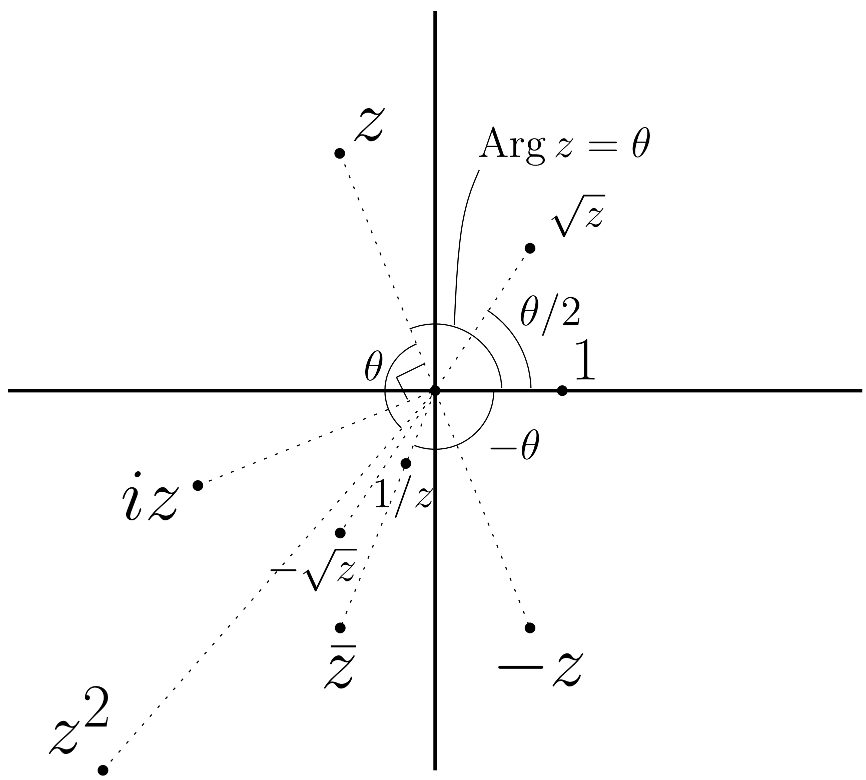
Part A. [6 points] For the complex number z shown in the figure below, depict the following in the figure:
$\bar{z}$, $-z$, iz, $\frac{1}{z}$, z^2, both square roots of z.

▷ The things to note are:

- $\bar{z}$ is mirror reflection of z in the real axis, i.e. the number with the argument $-\operatorname{Arg}(z)$;
- $-z$ is exactly the opposite of z;
- since $\operatorname{Arg} i = \pi/2$, the vector iz is *at the right angle* to z;
- $1/z$ has the same argument as $\bar{z}$, but the inverse absolute value;
- z^2 has double the argument of z;
- one of two square roots of z, say z_1, has half the argumnt of z, and the other is the exact opposite of the first one, $z_2 = -z_1$.



Part B. In this part, only provide answers. Each question is worth **2 points**.

(B1) [2pt] Find $\arg\left((\cos 2 + i\sin 2)(\cos 8 - i\sin 8)\right)$. (Answer only.)

$\triangleright$ $(\cos 2+i\sin 2)(\cos 8-i\sin 8) = \cos(2-8)+i\sin(2-8) = \cos(-6)+i\sin(-6)$, so $\arg\left((\cos 2 + i\sin 2)(\cos 8 - i\sin 8)\right) = -6 + 2\pi n$. (Remember that arg means all values of argument.)

(B2) [2pt] Find center and radius of the circle $|z-2+3i| = 16$. (Answer only.)

$\triangleright$ $|z-2+3i| = |z-(2-3i)|$, so the center is $2-3i$ and radius is 16. (Not 4. Remember that $|\cdot|$ is distance, not square of distance.)

(B3) [2pt] Which of the following five complex functions are one-to-one? (Answer only.)

$f(z) = z^{-2}, \qquad g(z) = z^{-1}, \qquad h(z) = 1, \qquad j(z) = z, \qquad k(z) = z^2.$

$\triangleright$ $z^{\pm 2}$ "glues" z and $-z$ together. $h(z) = 1$ glues *everything* together. So the only functions that are one-to-one are $g(z) = z^{-1}$ and $j(z) = z$.

(B4) [2pt] Which of the following five subsets of $\mathbb{C}$ are open? (Answer only.)

$A = \{3\}, \qquad B = \{z \in \mathbb{C} : |z-3| < 3\}, \qquad C = \{z \in \mathbb{C} : |z-3| \leq 3\},$

$D = \{z \in \mathbb{C} : |z-3| = 3\}, \qquad E = \{z \in \mathbb{C} : |z-3| > 3\}.$

$\triangleright$ A is not open since no open disks fit in it at all (in particular, there is no open disk inside centered at 3). For the rest of the sets, they all have the same boundary, the circle $|z-3| = 3$, so the only open ones are B and E.

(B5) [2pt] Let $\sqrt{}$ be the principal square root function. Give an example of a complex number z such that $\sqrt{z^2} \neq z$. (Answer only.)

$\triangleright$ Any number from left open half-plane or from negative part of the imaginary axis will do (because principal sqaure root function *only takes values* in the right open half-plane and non-negative part of the imaginary axis). For example, for $z = -1$, $z^2 = 1$ and $\sqrt{z^2} = \sqrt{1} = 1 \neq z$.

(B6) [2pt] Find the derivative $\left((z^2 + z^{-1} + i)^{2015}\right)'$. (Answer only.)

$\triangleright$ By chain rule, $\left((z^2+z^{-1}+i)^{2015}\right)' = 2015(z^2+z^{-1}+i)^{2014} \cdot (z^2+z^{-1}+i)' = 2015(z^2+z^{-1}+i)^{2014}(2z - z^{-2})$.

Part C. In this part, show your work and provide explanations. Each question is worth **5 points**.

(C1) [5pt] Find and sketch image of the region $\{x+iy : 0 < x < 2, -1 < y < 0\}$ under the mapping $f(z) = z^2$.

$\triangleright$ First we find where the boundary of this region goes under $f(z) = z^2$. After that, we figure out the original question.

The boundary is made up of segments of lines $x = 0$, $x = 2$, $y = -1$, and $y = 0$.

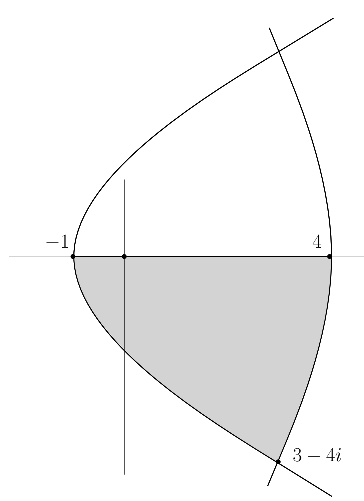
$x = 0$. These are the points of the form iy, $-1 \le y \le 0$. When squared, they become $(iy)^2 = -y^2$, i.e. the segment $-1 \le t \le 0$ of the negative part of real axis.

$x = 2$. These are the points of the form $2 + iy$, $-1 \le y \le 0$. When squared, they become $(2+iy)^2 = 4 - y^2 + 4iy = u + iv$, where $v = 4y$, so $y = v/4$ and $u = 4 - y^2 = 4 - \frac{v^2}{16}$. This is an equation of parabola open to the left on the complex (u, v)-plane. Since $-1 \le y \le 0$, we get an arc of this parabola enclosed between points with $v = 4 \cdot (-1) = -4$ and $v = 0$, i.e. between points 4 and $3 - 4i$.

$y = 0$. These are the points of the form x, $0 \le x \le 2$. When squared, they become x^2, i.e. the segment $0 \le t \le 4$ of the real line.

$y = -1$. These are the points of the form $x - i$, $0 \le x \le 2$. When squared, they become $(x-i)^2 = x^2 - 1 - 2xi = u + iv$, where $v = -2x$ and $u = x^2 - 1 = \frac{v^2}{4} - 1$. This is an equation of a parabola open to the right on the (u, v)-plane. Since $0 \le x \le 2$, we get an arc of this parabola enclosed between points with $v = -2 \cdot 2 = -4$ and $v = 0$, i.e. between points $3 - 4i$ and -1.

After we draw the segments and arc found above, we see that the original rectangular region is sent to on of the two regions with the depicted border (the inside and the outside). Since image of a bounded set under z^2 must be bounded, it must be the inside.



(C2) [5pt] Find and sketch image of the region $\{x+iy : 0 < x < 2, -1 < y < 0\}$ under the mapping $f(z) = z^{-1}$.

$\triangleright$ We can go by the same approach as we did in the preceding question, that is figure out where the boundary goes and conclude the answer from there (exercise: do that). Instead, let's do something slightly different. Observe that $1/z$ is a one-to-one function, so we can recover x, y from $1/z = u + iv$, and then just plug the obtained expressions into the inequalities that we have for x, y.

If $z = x + iy$, and $w = u + iv = \frac{1}{z}$, observe that $\frac{1}{w} = z$, or $\frac{1}{u+iv} = x + iy$, so $x + iy = \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2}$, so $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$.

Now, go through the four conditions $0 < x < 2, -1 < y < 0$ and express them in terms of u, v.

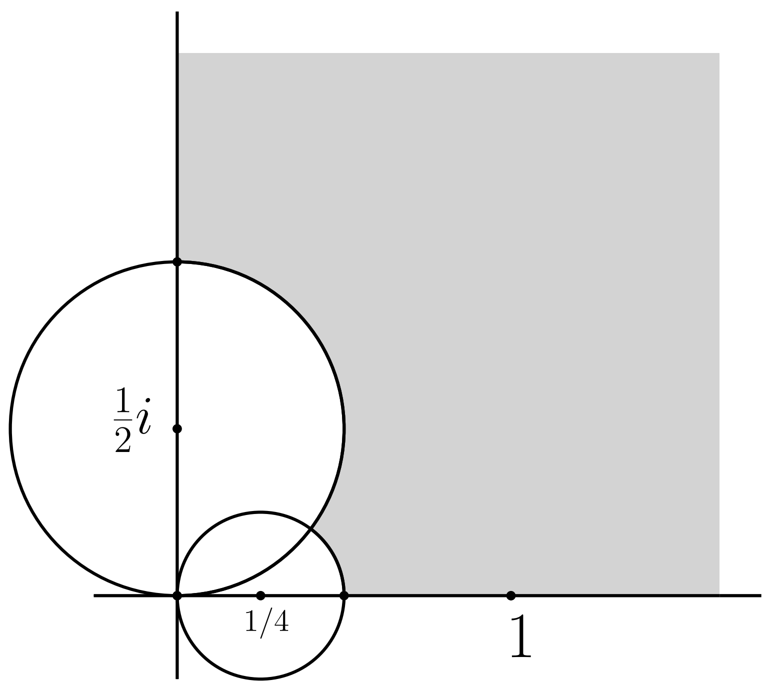
$x > 0$. We get $\frac{u}{u^2+v^2} > 0$, or simply $u > 0$.

$x < 2$. We get $\frac{u}{u^2+v^2} < 2$, or $u/2 < u^2 + v^2$, or $(u - \frac{1}{4})^2 + v^2 > \frac{1}{16}$, which is the outside of the circle of radius $1/4$ centered at $(1/4, 0)$.

$y < 0$. We get $\frac{-v}{u^2+v^2} < 0$, or simply $v > 0$.

$y > -1$. We get $\frac{-v}{u^2+v^2} > -1$, or $-v > -u^2 - v^2$, or $u^2 + (v - \frac{1}{2})^2 + > \frac{1}{4}$, which is the outside of the circle of radius $1/2$ centered at $(0, 1/2)$.

Now, the first and the third conditions simply define First Quadrant, so the answer is the region that is the part of Quadrant I that lies outside the two circles specified above.



(C3) [5pt] Find a real number A such that the function

$$f(z) = f(x+iy) = -y^3 + Ax^2y + i(Axy^2 - x^3)$$

is complex differentiable everywhere on $\mathbb{C}$.
(Question for 1 extra point: express f as a function of z.)

$\triangleright$ $u(x,y) = -y^3 + Ax^2y$, $v(x,y) = Axy^2 - x^3$. Since these functions have continuous partial derivatives, for complex differentiability it is necessary and sufficient that the Cauchy–Riemann equations are satisfied.

Compute partial derivatives: $u_x = 2Axy$, $u_y = -3y^2 + Ax^2$, $v_x = Ay^2 - 3x^2$, $v_y = 2Axy$. Then the Cauchy–Riemann equations

$$u_x = v_y, u_y = -v_x$$

take form

$$2Axy = 2Axy,$$

and

$$-3y^2 + Ax^2 = -(Ay^2 - 3x^2).$$

First equation is satisfied regardless of A. In the second equation, comparing coefficients in front of x^2, y^2 we get that $A = 3$.

Therefore, our function is $f(x+iy) = -y^3+3x^2y+i(3xy^2-x^3)$. One can observe that this is precisely $f(z) = -iz^3$.