MA232 Linear Algebra

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Determinant of a matrix.

Each square $n \times n$ matrix A is assigned a special scalar called the determinant of A, denoted:

$$\det(A) \text{ or } |A|$$

Think of a determinant function

$$\det : A^{n \times n} \to \mathbb{R}$$

Determinant of a matrix.

- Determinants was discovered during investigation of systems of linear equations
- They carry a lot of information about matrices

Determinant of a matrix.

- $A = [a]$ then $\det(A) = a$
- For 2×2 matrix

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a matrix.

- For 3×3 matrix

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{c} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{array}$$

- NOTE: each summand has only one entry from each row and column!

Determinant of a matrix.

Determinant of an arbitrary $n \times n$ matrix can be computed in different ways

- Using permutations: long but convenient for proofs
- Using Gaussian elimination: efficient in practice
- Cofactors: useful when row or column has many zeros

Determinant of a matrix.

- Let $N = \{1, 2, \ldots, n\}$ be a set
- A permutation σ of the set N is one-to-one mapping $\sigma : N \to N$

$$\sigma = \begin{pmatrix} 1 \to j_1 \\ 2 \to j_2 \\ \vdots \\ n \to j_n \end{pmatrix} \text{ or } \sigma(i) = j_i \text{ or } \sigma = (j_1 \; j_2 \ldots j_n)$$

- σ^{-1} is a permutation such that $\sigma^{-1}(\sigma(i)) = i$
- There are total $n!$ possible permutations of N

Determinant of a matrix.

- An odd permutation is a permutation obtainable from an odd number of two-element swaps.
- An evenpermutation is a permutation obtainable from an even number of two-element swaps
- Sign of permutation:

$$sgn\ \sigma = \begin{cases} 1 & if\ \sigma\ is\ even \\ -1 & if\ \sigma\ is\ odd \end{cases}$$

Determinant of a matrix.

How many swaps are needed to obtain permutation from $(1\ 2\ 3\ 4)$

- $(1\ 2\ 4\ 3) : 3 \leftrightarrow 4$ is odd
- $(2\ 4\ 1\ 3) : 1 \leftrightarrow 2 = (2\ 1\ 3\ 4),$
  $1 \leftrightarrow 3 = (2\ 3\ 1\ 4),$
  $3 \leftrightarrow 4 = (2\ 4\ 1\ 3)$ is odd
- $(3\ 4\ 1\ 2) : 1 \leftrightarrow 3 = (3\ 2\ 1\ 4),$
  $2 \leftrightarrow 4 = (3\ 4\ 1\ 2)$ is even

Determinant of a matrix.

Determinant of $n \times n$ matrix A is the sum of $n!$ products

$$\det(A) = \sum_{\text{all } \sigma} (sgn\ \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Each product of n elements has

- one and only one element that comes from each row
- one and only one element that comes from each column

Properties of Determinants.

- $\det(A^T) = \det(A)$

FACT:

1. $sgn\ \sigma = sgn\ \sigma^{-1}$
2. $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} \cdots a_{\sigma(n)^{-1}n}$

- Let $B = A^T$ then $B_{ij} = A_{ji}$ and

$$\begin{aligned}
\det(A^T) &= \sum_{\sigma} (sgn\ \sigma) B_{1\sigma(1)} B_{2\sigma(2)} \cdots B_{n\sigma(n)} \\
&= \sum_{\sigma} (sgn\ \sigma) A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n} \\
&= \sum_{\sigma} (sgn\ \sigma^{-1}) A_{1\sigma^{-1}(1)} A_{2\sigma^{-1}(2)} \cdots A_{n\sigma^{-1}(n)}
\end{aligned}$$

- For each σ there is unique σ^{-1}. Hence $\det(A) = \det(A^T)$

Properties of Determinants.

B is obtained from A by one of the elementary operations:

- $R_i \Leftrightarrow R_j$, then $\det(B) = -\det(A)$
- $R_i = kR_i$, then $\det(B) = k\det(A)$
- $R_i = R_i + mR_j$, then $\det(B) = \det(A)$

The same is true for operations on columns of A. It follows from $\det(A^T) = \det(A)$

Properties of Determinants.

Prove that if $R_i \Leftrightarrow R_j$, then $\det(B) = -\det(A)$

- Row exchange is some odd permutation of indices τ $(i \leftrightarrow j)$
- Then $B_{ij} = A_{i\tau(j)}$
- For any permutation σ

$$B_{1\sigma(1)}B_{2\sigma(2)}\cdots B_{n\sigma(n)} = A_{1\tau\circ\sigma(1)}A_{2\tau\circ\sigma(2)}\cdots A_{n\tau\circ\sigma(n)}$$

- $\det(B) = \sum_\sigma (sgn\ \sigma) A_{1\tau\circ\sigma(1)}A_{2\tau\circ\sigma(2)}\cdots A_{n\tau\circ\sigma(n)}$
- Since τ is odd then $sgn(\tau\circ\sigma) = -sgn\ sigma$
- $\det(B) = -\sum_\sigma (sgn\ tau \circ \sigma) A_{1\tau\circ\sigma(1)}A_{2\tau\circ\sigma(2)}\cdots A_{n\tau\circ\sigma(n)}$
- If σ runs through all permutations then $\tau\circ\sigma$ also runs through all permutations. Hence

$$\det(B) = -\det(A)$$

Properties of Determinants.

- Prove that if $R_i = kR_i$, then $\det(B) = k\det(A)$

$$\begin{aligned}
\det(B) &= \sum_{\sigma} (sgn\ \sigma) a_{1\sigma(1)} \cdots k a_{i\sigma(i)} \cdots a_{n\sigma(n)} \\
&= k \sum_{\sigma} (sgn\ \sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} \\
&= k \det(A)
\end{aligned}$$

Corollary: $\det(kA) = k^n \det(A)$

Properties of Determinants.

- Prove that if $R_i = R_i + mR_j$, then $\det(B) = \det(A)$

$$
\begin{aligned}
\det(B) &= \sum_{\sigma} (sgn\ \sigma) a_{1\sigma(1)} \cdots a_{j\sigma(j)} \cdots (m a_{j\sigma(j)} + a_{i\sigma(i)}) \cdots a_{n\sigma(n)} \\
&= m \sum_{\sigma} (sgn\ \sigma) a_{1\sigma(1)} \cdots a_{j\sigma(j)} \cdots a_{j\sigma(j)} \cdots a_{n\sigma(n)} \\
&+ \sum_{\sigma} (sgn\ \sigma) a_{1\sigma(1)} \cdots a_{j\sigma(j)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)} \\
&= m \cdot 0 + \det(A)
\end{aligned}
$$

Properties of Determinants.

- If A has a row of zeros then $\det(A) = 0$
- If A has two identical rows $\det(A) = 0$

The same is true for operations on columns of A.

- By definition each term in $\det(A)$ contains a factor from zero row, hence each term is zero and $det(A) = 0$
- Interchange two identical rows in A, matrix A does not change, however row interchange negates determinant so we have

$$\det(A) = -\det(A) \Rightarrow \det(A) = 0$$

Properties of Determinants.

- If A triangular then $\det(A) = a_{11}a_{22}\cdots a_{nn}$
- $\det(I) = 1$

- Suppose A is lower triangular, then $A_{ij} = 0, i < j$
- Let $t = (sgn\ \sigma)A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)}$ be a term in $\det(A)$
- Suppose $\sigma(1) \neq 1$, then $1 < \sigma(1)$ and $A_{1\sigma(1)} = 0$ hence $t = 0$
- Now suppose $\sigma(1) = 1$, but $\sigma(2) \neq 2$, then $2 < \sigma(2)$ and $A_{2\sigma(2)} = 0$ hence $t = 0$
- Repeating the argument we obtain each term for which

  $$\sigma(1) \neq 1 \text{ or } \sigma(2) \neq 2 \text{ or } \ldots \text{ or } \sigma(n) \neq n \text{ is zero}$$
- The only nonzero term is $a_{11}a_{22}\cdots a_{nn} = \det(A)$

Properties of Determinants.

- If A is singular then $\det(A) = 0$
- If A invertible then $\det(A) \neq 0$

- If A is singular then
  1. it can be reduced matrix B with a zero row
  2. $\det(A) = m\det(B) = 0$
- If A invertible then
  1. it can be reduced to I using row operations
  2. Hence, $\det(A) = k\det(I) \neq 0$

Properties of Determinants.

- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = 1/\det(A)$

Properties of Determinants.

Prove $\det(AB) = \det(A)\det(B)$

FACT: if E is an elementary matrix then $\det(EA) = \det(E)\det(A)$

- If A is singular then AB is singular and $\det(A) = 0 = \det(AB)$
- IF A is nonsingular then $A = E_n \cdots E_1 I$ and

$$\begin{aligned}
\det(AB) &= \det(E_n \cdots E_1 B) \\
&= \det(E_n) \cdots \det(E_2)\det(E_1)\det(B) \\
&= \det(E_n) \cdots E_2 E_1)\det(B) \\
&= \det(A)\det(B)
\end{aligned}$$

Properties of Determinants.

Prove $\det(A^{-1}) = 1/\det(A)$

- $AA^{-1} = I$
- $\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1$
- Hence
$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Properties of Determinants.

Let P be permutation matrix then $\det(P) = \pm 1$

- Permutation matrix is obtained by performing K row exchanges $R_i \Leftrightarrow R_j$ of I
- If K is even then $\det(P) = (-1)^K \det(I) = 1$
- If K is odd then $\det(P) = (-1)^K \det(I) = -1$

Properties of Determinants.

1 $\det(A^T) = \det(A)$
2 $R_i \Leftrightarrow R_j$, then $\det(B) = -\det(A)$
3 $R_i = kR_i$, then $\det(B) = k\det(A)$
4 $R_i = R_i + mR_j$, then $\det(B) = \det(A)$
5 If A has a row of zeros then $\det(A) = 0$
6 If A has two identical rows $\det(A) = 0$
7 If A triangular then $\det(A) = a_{11}a_{22}\cdots a_{nn}$
8 $\det(I) = 1$
9 If A is singular then $\det(A) = 0$
10 If A invertible then $\det(A) \neq 0$
11 $\det(AB) = \det(A)\det(B)$
12 $\det(A^{-1}) = 1/\det(A)$
13 $\det(P) = \pm 1$
14 $\det(kA) = k^n\det(A)$

Determinant of a matrix

Theorem

For any matrix $A^{n\times n}$ the following are equivalent:

- A is nonsingular
- $\det(A) \neq 0$
- The equation $A\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$
- The equation $A\mathbf{x} = \mathbf{b}$ has unique solution for any vector $\mathbf{b}$

Determinant of a matrix

- Computing determinant from definition: $O(n!)$
- Need a practical way to compute determinant

- We know how determinant changes with elimination steps
- We know how to compute determinant of a triangular matrix
- Try to reduce to triangular form

Determinant by elimination

To compute the $\det(A)$:

1. Transform A into triangular form U using Gaussian elimination
2. Compute

$$\det(A) = (-1)^K \prod_{i=1}^{n} u_{ii}$$

   where

   - u_{ii} are diagonal elements of the triangular form U
   - K is the number of row interchanges

- If zeros occur on the diagonal, then $\det(A) = 0$, i.e. matrix is singular

Determinant by elimination

- Let $A = LU$, where L lower triangular with ones on diagonal; U is upper triangular then

$$\det(A) = \det(L)\det(U) = 1 \cdot \det(U) = \prod_i^n u_{ii}$$

- In general $PA = LU$, where P is permutation matrix:

$$\begin{aligned} \det(PA) = \det(LU) &\Rightarrow \det(P)\det(A) = \det(L)\det(U) \\ &\Rightarrow \pm 1 \det(A) = \det(U) \\ &\Rightarrow \det(A) = \pm \prod_i^n u_{ii} \end{aligned}$$

Determinant: minors and cofactors

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{c} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{array} =$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix};\ M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{13} & a_{33} \end{bmatrix};\ M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{bmatrix};$$

$$\det(A) = a_{11}\det(M_{11}) + a_{12}(-\det(M_{12})) + a_{13}\det(M_{13})$$

Determinant: minors and cofactors

- Let A be $n \times n$ matrix
- M_{ij} is $(n-1) \times (n-1)$ matrix obtained from A by deleting ith row and jth column

$$M_{ij} = \begin{bmatrix} x & x & | & x \\ x & x & | & x \\ - & - & a_{ij} & - \\ x & x & | & x \end{bmatrix}$$

- $\det(M_{ij})$ is called the minor of a_{ij}
- $C_{ij} = (-1)^{i+j} \det(M_{ij})$ is the cofactor of a_{ij}

Determinant: minors and cofactors

- Signs of cofactors form a *chessboard pattern*:

$$(-1)^{i+j} = \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- In particular to compute signs of cofactors of the first row elements start with "$+$" and then alternate

$$+ \quad - \quad + \quad - \quad + \quad \cdots$$

Determinant: Laplace Expansion

Laplace Formula

The determinant of a square matrix A is equal to the sum of products obtained by multiplying elements of any row (column) by their cofactors:

- For any choice of row i:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

- for any choice of column j

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

- Each of $2n$ ways computing the determinant lead to the same value

Determinant: Laplace Expansion

- Show for row i. Same proof works for columns since $\det(A) = \det(A^T)$
- Since every term of $\det(A)$ contains one and only one entry from row i:

$$\det(A) = a_{i1}C_{i1}^* + a_{i2}C_{i2}^* + \cdots + a_{in}C_{in}^*$$

- Need to show that $C_{ij}^* = (-1)^{i+j}\det(M_{ij}) = C_{ij}$

Determinant: Laplace Expansion

- Let $i = n, j = n$ then

$$a_{nn} C^*_{nn} = a_{nn} \sum_{\sigma} (sgn\, \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n-1\sigma(n-1)} = a_{nn} \det(M_{nn}) =$$

- Now take arbitrary i, j. We can move a_{ij} to position n, n by swapping rows $n - i$ times and columns $n - j$ times
- Note that swapping rows, columns does not change the value $\det(M_{ij})$ only the sign of C_{ij}
- Therefore

$$C^*_{ij} = (-1)^{(n-i)+(n-j)} \det(M_{ij}) = (-)^{i+j} \det(M_{ij}) = C_{ij}$$

Determinants and systems of linear equations

Cramer's Rule solves $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$

- Consider equation $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ for $n \times n$ matrix
- Let X_i, B_i be $n \times n$ matrices

$$X_i = [\bar{\mathbf{e}}_1 \ \ldots \ \underbrace{\bar{\mathbf{x}}}_{i} \ \ldots \ \bar{\mathbf{e}}_n] \text{ and } B_i = [\bar{\mathbf{a}}_1 \ \ldots \ \underbrace{\bar{\mathbf{b}}}_{i} \ \ldots \ \bar{\mathbf{a}}_n]$$

- Then
  1. $AX_i = B_i$
  2. $\det(X_i) = x_i$
  3. $\det(A)\det(X_i) = \det(B_i)$ and we have

$$x_i = \frac{\det(B_i)}{\det(A)}$$

Determinants and matrix inverse

- Solve $AA^{-1} = I$ for A^{-1}
- From matrix equality we can solve for each column of A^{-1}

$$A \cdot Col_1(A^{-1}) = \bar{\mathbf{e}}_1,\ A \cdot Col_2(A^{-1}) = \bar{\mathbf{e}}_2,\ \ldots,\ A \cdot Col_n(A^{-1}) = \bar{\mathbf{e}}_n,$$

- Using Cramer's rule

$$(A^{-1})_{ij} = \frac{\det(B_{ij})}{\det(A)},$$

where $B_{ij} = [\bar{\mathbf{a}}_1 \ \ldots \ \underbrace{\bar{\mathbf{e}}_j}_{i} \ \ldots \ \bar{\mathbf{a}}_n] = C_{ji}$

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}$$

Determinants and matrix inverse

- Define cofactor matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

- Then the inverse

$$A^{-1} = \frac{C^T}{\det(A)}$$

Determinants and volumes