JOINTLY DISTRIBUTED RANDOM VARIABLES

In some random experiments, we observe two numerical random characteristics at the same time.

ex/ measure (height, weight) for a population of people

ex/ measure (rainfall, temperature) over time or location

I. DISCRETE RANDOM VARIABLES

For 2 discrete r.v.'s X and Y, the JOINT PROBABILITY that $X = x_i$ and $Y = y_j$ is written as $P(X=x_i, Y=y_j)$ or $p(x_i, y_j)$ or $p(x,y)$

Properties:

1) $P(X=x_i, Y=y_j) \geq 0$ for all i and j

and

2) $\sum_{\text{all } j} \sum_{\text{all } i} P(X=x_i, Y=y_j) = 1$

ex/ Let: X = # of jobs a college graduate holds in 5 yrs.
Y = # of promotions a college graduate has in 5 yrs. (after graduation)

		Y=1	2	3	4
X	1	.10	.15	.12	.06
	2	.05	.07	.10	.05
	3	.04	.02	.14	.10

← JOINT PROBABILITY DISTRIBUTION OF X and Y

for example,

$P(2 \text{ jobs}, 3 \text{ promotions}) = P(X=2, Y=3) = .10$

<u>Marginal probability</u> : probabilities about X alone or Y alone

$P(X=1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4) = .43$

similarly

$P(X=2) = .27$

$P(X=3) = .30$

MARGINAL PROBABILITY DISTRIBUTION OF X

MARGINAL PROBABILITY DISTRIBUTION OF Y:

$P(Y=1) = P(X=1, Y=1) + P(X=2, Y=1) + P(X=3, Y=1) = .19$

$P(Y=2) = .24$

$P(Y=3) = .36$

$P(Y=4) = .21$

In general:

$$P(X=x_i) = \sum_j P(X=x_i, Y=y_j) \quad \text{for each } i$$

and

$$P(Y=y_j) = \sum_i P(X=x_i, Y=y_j) \quad \text{for each } j$$

		Y				
		1	2	3	4	marginal ~~totals (x)~~
X	1	.10	.15	.12	.06	.43
	2	.05	.07	.10	.05	.27
	3	.04	.02	.14	.10	.30
marginal totals (Y)		.19	.24	.36	.21	$\Sigma\Sigma = 1$ ✓

Note: In the context of a JOINT pmf, $P_{X,Y}(x_i, y_j)$ the pmf's of X and Y individually, $P_X(x_i)$ and $P_Y(y_j)$, are often referred to as marginal prob. distn's. The use of the word "marginal", though, is used solely for emphasis and clarity — there is absolutely no difference between a pmf and a marginal pmf.

CONDITIONAL PROBABILITY DISTRIBUTION

for ease of notation here, we will use: $P(X=x_i, Y=y_j) = P_{X,Y}(x_i, y_j)$

and $P(X=x_i) = P_X(x_i)$; $P(Y=y_j) = P_Y(y_j)$

for all i, j

$$P(X=x_i \mid Y=y_j) = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)} \quad \text{if } P(Y=y_j) \neq 0$$

or, using our new notation

(1) $$P_{X|Y}(x_i \mid y_j) = \frac{P_{X,Y}(x_i, y_j)}{P_Y(y_j)} \quad \text{if } P_Y(y_j) \neq 0$$

and

(2) $$P_{Y|X}(y_j \mid x_i) = \frac{P_{X,Y}(x_i, y_j)}{P_X(x_i)} \quad \text{if } P_X(x_i) \neq 0$$

ex/ (previous ex.)

$$P(X=2 \mid Y=3) = \frac{P_{X,Y}(2,3)}{P_Y(3)} = \frac{.10}{.36} = .278$$

ex/ Find the conditional probability distribution of X, given Y=2

$$P_X(x)^{Y=2} \begin{cases} P(X=1 \mid Y=2) = \frac{.15}{.24} = .625 \\ P(X=2 \mid Y=2) = \frac{.07}{.24} = .292 \\ P(X=3 \mid Y=2) = \frac{.02}{.24} = .083 \end{cases}$$

$$\overline{\quad 1 \quad}$$

From (1) and (2) on the previous page we have:

$$P_{X,Y}(x_i, y_j) = P_Y(y_j)\, P_{X|Y}(x_i | y_j)$$

and $P_{X,Y}(x_i, y_j) = P_X(x_i)\, P_{Y|X}(y_j | x_i)$

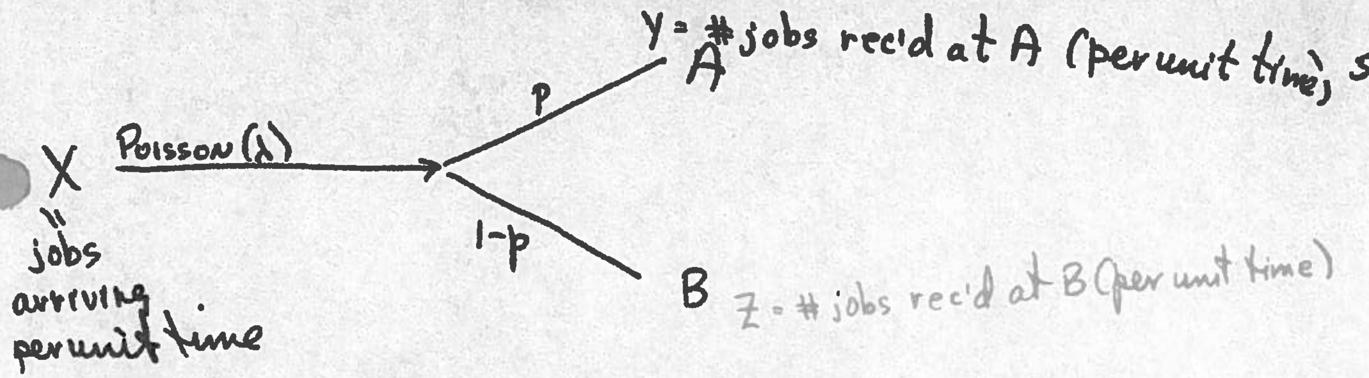
and recalling that

$$P_Y(y_j) = \sum_i P_{X,Y}(x_i, y_j) \qquad \text{for each } j$$

we have

*** $P_Y(y_j) = \sum_i P_X(x_i)\, P_{Y|X}(y_j | x_i)$ for each j

which is another form of the Theorem of Total Probability; an example using this powerful result is on the next page:



We seek Prob. Dist'n. of Y : $P(Y=k)$

$$P(Y=k) = \sum_n P(Y=k \mid X=n)\, P(X=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!}$$

$$= \frac{(\lambda p)^k e^{-\lambda}}{k!} e^{\lambda(1-p)}$$

$$= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \quad \sim \text{Poisson}(\lambda p) \,!$$

This result is sometimes stated as: the Poisson distribution is preserved under random selection.

INDEPENDENT RANDOM VARIABLES

Df:

X and Y are independent r.v.'s iff

** $P_{X,Y}(x_i, y_j) = P_X(x_i) \cdot P_Y(y_j)$ for all i and j

or, equivalently

** $P(X=x_i, Y=y_j) = P(X=x_i)\, P(Y=y_j)$ for all i and j

That is, X and Y are independent iff
$P_{X|Y}(x_i|y_j) = P_X(x_i)$ and $P_{Y|X}(y_j|x_i) = P_Y(y_j)$
for all i and j

note: One (i,j) pair such that the above is not true is enough to make X and Y dependent. That's what the "all" means.

ex/

		Y: 0	1	2	$P_X(x)$
X	0	.1	.2	.2	.5
	1	.04	.08	.08	.2
	2	.06	.12	.12	.3
$P_Y(y)$		.2	.4	.4	

$\Rightarrow$ X and Y are indep.

ex/ from page 2, bottom : X and Y are NOT indep.

JOINTLY DISTRIBUTED RANDOM VARIABLES

II CONTINUOUS RANDOM VARIABLES

We will be looking at the joint density function of several random variables, and we will begin by looking at the joint density function of two random variables: bivariate density fcn.

ex/ observations on adult males
measure: X: height and Y: weight

ex/ observations in U.S. cities
measure X: temperature and Y: humidity

where the X and Y take on a continuum of values over a range R

$f_{X,Y}(x,y)$ = joint pdf of X and Y

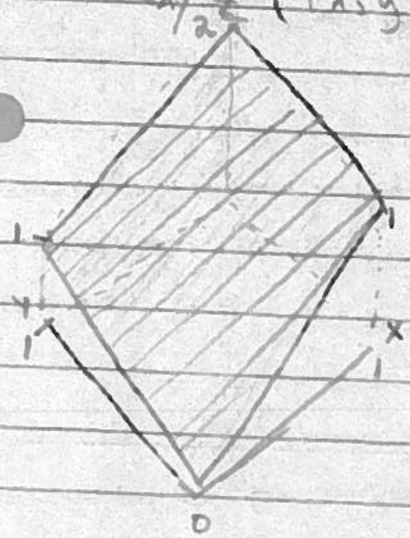
Properties of $f(x,y)$: → $\boxed{\text{a surface: } z = f(x,y)}$

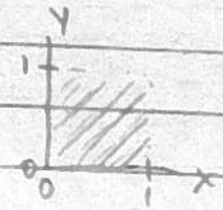
1. $f(x,y) \geq 0$ for all (x,y) in R

2. $\iint_R f(x,y)\,dx\,dy = 1$ (total volume under the surface equals 1

3. $P((x,y)\in A) = \iint_{(x,y)\in A} f(x,y)\,dx\,dy$

ex/2 $f(x,y) = x+y$





$$P(0 \le x \le 1, 0 \le y \le 1) = \int_0^1 \int_0^1 (x+y)\,dx\,dy$$

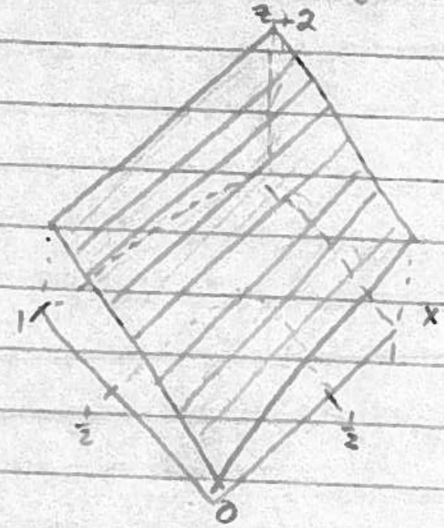
$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_{x=0}^{x=1} dy$$

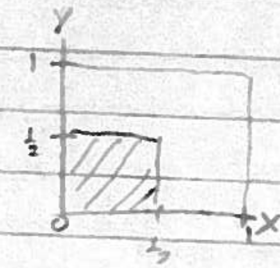
$$= \int_0^1 \left(y + \frac{1}{2}\right) dy$$

$$= \left[\frac{y^2}{2} + \frac{y}{2}\right]_{y=0}^{1}$$

$$= 1$$

$0 \le x \le 1,\ 0 \le y \le 1$





$$P\left(0 \le x \le \tfrac{1}{2}, 0 \le y \le \tfrac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (x+y)\,dx\,dy$$

$$= \frac{1}{8}$$

The 2-dimensional analog to the Uniform pdf is a joint pdf whose surface is a horizontal plane.

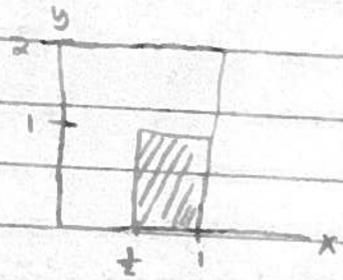
ex/ $f(x,y) = c \qquad 0 \le x \le 1,\ 0 \le y \le 2$

For the total volume under the surface to be 1, c must be $\frac{1}{2}$.

So
$$f(x,y) = \tfrac{1}{2} \qquad 0 \le x \le 1,\ 0 \le y \le 2$$

a) total volume
$$P(0 \le x \le 1, 0 \le y \le 2) = \int_0^1 \int_0^2 \tfrac{1}{2}\, dy\, dx = 1 \checkmark$$

b) $P(\frac{1}{2} \le x \le 1, 0 \le y \le 1) = \int_{\frac{1}{2}}^1 \int_0^1 \frac{1}{2}\, dy\, dx = \frac{1}{4}$



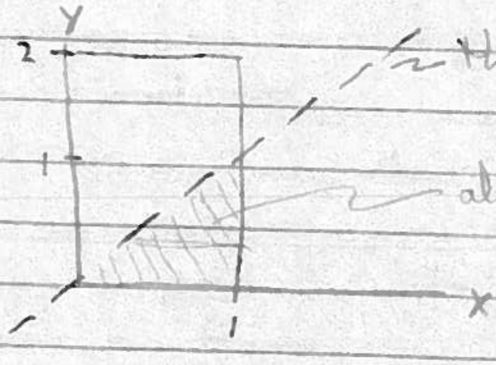
Notes:

1. As with the 1-dimensional case, $f(x,y)$ does not represent the probability of anything. However, for $\Delta x + \Delta y$ small,
$$f(x,y)\Delta x \Delta y \simeq P(x \le X \le x+\Delta x,\ y \le Y \le y+\Delta y)$$

2. As in the 1-dimensional case, we shall adopt the convention that $f(x,y) = 0$ for $(x,y) \notin R$. Therefore $f(x,y)$ is defined for all (x,y) in the plane, and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

ex| If $f(x,y) = \frac{1}{2}$ $0 \le x \le 1$, $0 \le y \le 2$

Find $P(X > Y)$.



all (x,y) points in this shaded region have $x > y$

So we integrate the surface over this region.

$$P(X>Y) = \iint_{(x,y):x>y} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_y^1 \frac{1}{2}\,dx\,dy$$

$$= \int_0^1 \left.\frac{x}{2}\right|_{x=y}^{x=1} dy = \int_0^1 \left(\frac{1}{2} - \frac{y}{2}\right) dy = \left[\frac{y}{2} - \frac{y^2}{4}\right]_0^1 = \frac{1}{4}$$

One could also write this integral in the other order:

$$P(X>Y) = \int_0^1 \int_0^x \frac{1}{2}\,dy\,dx$$

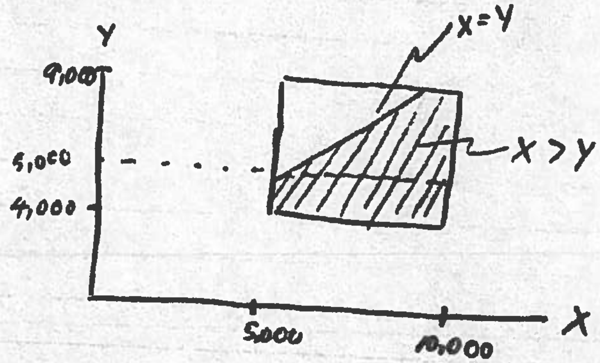
$$= \int_0^1 \left.\frac{y}{2}\right|_{y=0}^{y=x} dx = \int_0^1 \frac{x}{2}\,dx = \left.\frac{x^2}{4}\right|_0^1 = \frac{1}{4}$$

as before

ex/ X, Y uniform over $5{,}000 < X < 10{,}000$ & $4{,}000 < Y < 9{,}000$

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{5{,}000^2} \quad \text{for } X \text{ and } Y \text{ as above}$$

FIND $P(X>Y)$

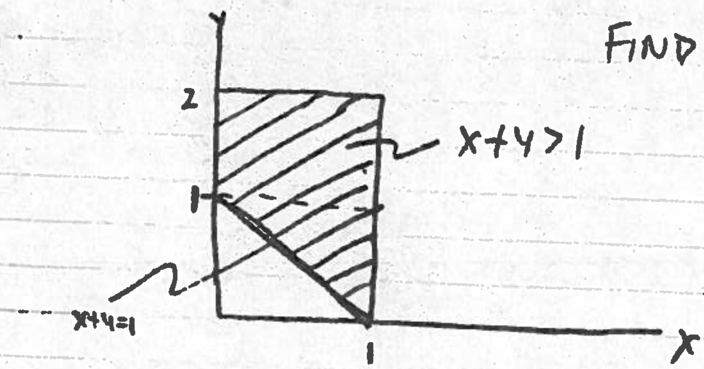


$$P(X>Y) = \int_{4000}^{5000}\int_{5000}^{10{,}000} \frac{1}{5{,}000^2}\,dx\,dy + \int_{5{,}000}^{9{,}000}\int_{y}^{10{,}000} \frac{1}{5{,}000^2}\,dx\,dy$$

$$= .68$$

ex/ $f_{X,Y}(x,y) = x^2 + \frac{xy}{3}$; $0 \le x \le 1$, $0 \le y \le 2$

FIND $P(X+Y>1)$



$$P(X+Y>1) = \int_0^1\int_{1-y}^{1}\left(x^2+\frac{xy}{3}\right)dx\,dy + \int_1^2\int_0^1\left(x^2+\frac{xy}{3}\right)dx\,dy$$

$$= \frac{65}{72}$$

MARGINAL PDF'S

If $f_{X,Y}(x,y)$ is our joint pdf

Then

$f_X(x)$ is the marginal dist. of X

and

$f_Y(y)$ is the marginal dist. of Y

where

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

ex/ $f_{X,Y}(x,y) = xy \qquad 0 \le x \le 2 \; ; \; 0 \le y \le 1$

check if density: $\int_0^1 \int_0^2 xy\,dx\,dy = \int_0^1 \frac{x^2 y}{2}\Big|_{x=0}^{x=2} dy = \int_0^1 2y\,dy = \underline{1}$ ✓

$$f_X(x) = \int_0^1 xy\,dy = \frac{xy^2}{2}\Big|_{y=0}^{y=1} = \frac{x}{2} \qquad 0 \le x \le 2$$

$$f_Y(y) = \int_0^2 xy\,dx = \frac{x^2 y}{2}\Big|_{x=0}^{x=2} = 2y \qquad 0 \le y \le 1$$

and we can answer questions about X and Y alone.

INDEPENDENT R.V.'s

X and Y are INDEPENDENT R.V.'S. IFF

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

Note:
Factoring $f_{X,Y}(x,y)$ into $g(x)\, h(y)$ is NOT enough.
They must factor into the marginals

The def'n. implies

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\, f_Y(y)}{f_Y(y)} = f_X(x)$$

and similarly

$$f_{Y|X}(y|x) = f_Y(y)$$

Note: Knowledge of the joint p.d.f is always enough for us to find the marginals, but the CONVERSE is NOT TRUE.

ex/ In a previous example, we had, for

$$f_{X,Y}(x,y) = xy \qquad 0 \le x \le 2 \quad 0 \le y \le 1$$

that

$$f_X(x) = \frac{x}{2} \qquad 0 \le x \le 2$$

and

$$f_Y(y) = 2y \qquad 0 \le y \le 1$$

which implies that X and Y are independent