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Assignment 6 - Solutions

1. a) Writing the series as $\sum_{n=0}^{\infty} c_n(z-a)^n$ we have $c_n = \cos n$ and $a = 0$, and note that $\limsup_{n\to\infty} |c_n| = 1$. Using the root test, we have $R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}} = \frac{1}{1} = 1$. If z is on the boundary, then $1 = |z-0| = |z|$ and, using the divergence test, $\lim_{n\to\infty} |c_n(z-a)^n| = \lim_{n\to\infty} |\cos n||z|^n = \lim_{n\to\infty} |\cos n|$ which does not exist, so the power series diverges on the boundary.

   b) Writing the series as $\sum_{n=0}^{\infty} c_n(z-a)^n$ we have $c_n = \frac{1}{n}$ and $a = 0$. Using the limit ratio test, we have $R = \frac{1}{\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}} = \frac{1}{\lim_{n\to\infty} \frac{n}{n+1}} = \frac{1}{1} = 1$. If z is on the boundary, then $1 = |z-0| = |z|$ so $z = 1$ and $z = -1$ are on the boundary. However, $\sum_{n=0}^{\infty} \frac{1^n}{n}$ diverges by the integral test whereas $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test, so the power series is erratic on the boundary.

   c) Writing the series as $\sum_{n=0}^{\infty} c_n(z-a)^n$ we have $c_n = \frac{1}{n^2}$ and $a = 0$. Using the limit ratio test, we have $R = \frac{1}{\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}} = \frac{1}{\lim_{n\to\infty} \frac{n^2}{(n+1)^2}} = \frac{1}{1} = 1$. If z is on the boundary, then $1 = |z-0| = |z|$ and, using the absolute convergence test, $\sum_{n=0}^{\infty} |c_n z^n| = \sum_{n=0}^{\infty} \frac{|z|^n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$ which converges by the integral test, so the power series converges on the boundary.

2. a) In power series form we have $f(z) = e^{(z^2)} = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$. Setting $z = x + iy$ we have $f(z) = e^{(x+iy)^2} = e^{(x^2-y^2)+i(2xy)} = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$.

   b) In power series form we have $f(z) = (e^z)^2 = e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n z^n}{n!}$. Setting $z = x+iy$ we have $f(z) = e^{2(x+iy)} = e^{(2x)+i(2y)} = e^{2x}\cos 2y + ie^{2x}\sin 2y$.