MA232 Linear Algebra

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General solution to $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$

$$A\bar{\mathbf{x}} = \bar{\mathbf{b}} \Rightarrow R\bar{\mathbf{x}} = \bar{\mathbf{d}}$$

- R is an echelon matrix
- Has a solution if zero rows in R correspond to zero entries in d

General solution to $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$

- $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ inhomogeneous linear system
- $A\bar{\mathbf{x}} = \bar{\mathbf{0}}$ *associated* homogeneous linear system

General solution to $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$

Theorem

Let $\bar{\mathbf{x}}^*$ be a particular solution: $A\bar{\mathbf{x}}^* = \bar{\mathbf{b}}$. The general solution to the linear system $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ is

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}^* + \bar{\mathbf{z}}$$

where $z \in N(A)$, i.e. $A\bar{\mathbf{z}} = \bar{\mathbf{0}}$, is arbitrary from the nullspace of A.

Proof: if $A\bar{\mathbf{x}}_1 = \bar{\mathbf{b}} = A\bar{\mathbf{x}}_2$ are any two solutions then

$$\bar{\mathbf{z}} = \bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 \Rightarrow A\bar{\mathbf{z}} = A(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) = A\bar{\mathbf{x}}_1 - A\bar{\mathbf{x}}_2 = \bar{\mathbf{b}} - \bar{\mathbf{b}} = \bar{\mathbf{0}}$$

- $\bar{\mathbf{z}} \in N(A)$
- Given $\bar{\mathbf{x}}_1$ any other solution $\bar{\mathbf{x}}_2 = \bar{\mathbf{x}}_1 + \bar{\mathbf{z}}$ for some $\bar{\mathbf{z}}$

Linear Dependence and Independence

Vectors $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_k$ are linearly dependent if there exist $a_1, \ldots, a_k$ with at least one $a_i \neq 0$ such that

$$a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + a_k\bar{\mathbf{v}}_k = 0$$

Otherwise vectors are linearly independent

Vectors $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_k$ are linearly independent if the only combination equal to 0 is

$$0\bar{\mathbf{v}}_1 + 0\bar{\mathbf{v}}_2 + \cdots + 0\bar{\mathbf{v}}_k = 0$$

Linear Dependence and Independence

- Suppose in $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_k$ one non-zero vector is a multiple of another, say, $\bar{\mathbf{v}}_1 = a\bar{\mathbf{v}}_2$ then the vectors are linearly *dependent*

$$\bar{\mathbf{v}}_1 - k\bar{\mathbf{v}}_2 + 0\bar{\mathbf{v}}_3 + \cdots + 0\bar{\mathbf{v}}_k$$

- $\bar{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$ are linearly *dependent* iff $\bar{\mathbf{v}}_1 = a\bar{\mathbf{v}}_2$
- If $\bar{\mathbf{0}}$ is one of the vectors $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_k$ then they must be linearly dependent. Let $\bar{\mathbf{v}}_1 = \bar{\mathbf{0}}$:

$$1\bar{\mathbf{v}}_1 + 0\bar{\mathbf{v}}_2 + \cdots + 0\bar{\mathbf{v}}_k$$

- If a set S of vectors is linearly independent then any subset of S is linearly independent
- If a set S of vectors contains a linearly dependent subset the S is linearly dependent

Linear Dependence and Independence

- To show that vectors are *dependent* provide $a_1, \ldots, a_k$ not all equal to zero such that

$$a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + 0\bar{\mathbf{v}}_k = 0$$

- Example: $\bar{\mathbf{u}} = (1, 1, 0), \bar{\mathbf{v}} = (1, 3, 2), \bar{\mathbf{w}} = (4, 9, 5)$

$$3\bar{\mathbf{u}} + 5\bar{\mathbf{v}} - 2\bar{\mathbf{w}} = (0, 0, 0) = \bar{\mathbf{0}}$$

Linear Dependence and Independence

- To show that vectors are *independent* solve

$$x_1\bar{\mathbf{v}}_1 + x_2\bar{\mathbf{v}}_2 + \cdots + x_k\bar{\mathbf{v}}_k = 0$$

- If the only solution is $x_1 = x_2 = \cdots = x_k = 0$ then independent
- If there are solutions other then $\bar{\mathbf{0}}$ then dependent

Reduced row echelon matrix

Reduce Echelon matrix using Jordan's method:

$$\begin{bmatrix} p_1 & x & x & x & x & x \\ 0 & 0 & p_2 & x & x & x \\ 0 & 0 & 0 & p_3 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & x/p_1 & x/p_1 & x/p_1 & x/p_1 & x/p_1 \\ 0 & 0 & 1 & x/p_2 & x/p_2 & x/p_2 \\ 0 & 0 & 0 & 1 & x/p_3 & x/p_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & x/p_1 & 0 & 0 & x' & x' \\ 0 & 0 & 1 & 0 & x' & x' \\ 0 & 0 & 0 & 1 & x/p_3 & x/p_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon matrix has zeros above the pivots as well as below

Reduced row echelon matrix

$$\begin{bmatrix} 1 & x & 0 & 0 & x & x \\ 0 & 0 & 1 & 0 & x & x \\ 0 & 0 & 0 & 1 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Nonzero rows cannot be expressed as a linear combination of other nonzero rows
- Nonzero rows are linearly independent: must multiply each row by zero to eliminate pivots

Reduced row echelon matrix

$$\begin{bmatrix} \color{red}{1} & x & 0 & 0 & x & x \\ 0 & 0 & \color{red}{1} & 0 & x & x \\ 0 & 0 & 0 & \color{red}{1} & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Columns with pivots cannot be expressed as a linear combination of columns with pivots
- Columns with pivots are linearly independent
- Moreover it is easy to see that columns which do not have pivots can be obtained from the ones with pivots

Rank of a matrix

The rank of a matrix A is equal to

- The number of pivots in echelon form R
- The number of non-zero rows in R
- The maximal number of linearly independent rows
- The maximal number of linearly independent columns

Rank of a matrix

To compute $rank(A)$ run elimination on A

$$A = \begin{bmatrix} p_1 & x & x & x & x & x \\ 0 & 0 & p_2 & x & x & x \\ 0 & 0 & 0 & p_3 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$rank(A) = 3$$

Rank of a matrix

Let A be $m \times n$ matrix

A is full column rank if $rank(A) = n$

- All columns of A have pivots
- There are no free variables
- $N(A) = \{\bar{\mathbf{0}}\}$
- $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has either unique solution or no solutions

Rank of a matrix

Let A be $m \times n$ matrix

A is full row rank if $rank(A) = m$

- All rows have pivots and no zero rows in reduced form
- $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has solution for every right side $\bar{\mathbf{b}}$
- $C(A) = \mathbb{R}^m$

$rank(A)$ and $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$

Let $r = rank(A)$ and A is $m \times n$

1	$r = m$ and $r = n$	A is invertible	$A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has 1 solution
2	$r = m$ and $r < n$	A is full row rank	$A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has ∞ solutions
3	$r < m$ and $r = n$	A is full column rank	$A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has 0 or 1 solution
4	$r < m$ and $r < n$	A is NOT full rank	$A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has 0 or ∞ solutions

Rank and linear independence

Let A be a $m \times n$ matrix columns of A are independent when

- $rank(A) = n$
- There are n pivots and no free variables
- $N(A) = \{\bar{\mathbf{0}}\}$

Rank and linear independence

Let $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n$ be vectors in $\mathbb{R}^m$ and $n > m$ then $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n$ must be dependent.

Basis for a Vector Space

Vectors $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n$ span V if for all $\bar{\mathbf{w}} \in V$

$$\bar{\mathbf{w}} = a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots a_n\bar{\mathbf{v}}_n$$

- Different sets may span the same vector space
- A spanning set needs a sufficient number of distinct elements.
- Having too many elements in the spanning set will violate linear independence, and cause redundancies.
- The optimal spanning sets are those that are linearly independent.

Basis for a Vector Space

A basis of a vector space V is a finite collection of elements $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n$ such that

1. $span(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n) = V$
2. $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n$ are linearly independent

Basis for a Vector Space

The elements $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n$ form a basis of V if and only if every $\bar{\mathbf{w}} \in V$ can be written uniquely as a linear combination :

$$\bar{\mathbf{w}} = a_1\bar{\mathbf{v}}_1 + \ldots a_n\bar{\mathbf{v}}_n$$

Basis for a Vector Space

Every basis of $\mathbb{R}^n$ contains exactly n vectors.
A set of n vectors $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n \in \mathbb{R}^n$ is a basis if and only if the $n \times n$ matrix $\mathrm{A} = (\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n)$ is nonsingular (invertible).

- Linear independent - the only solution to $A\bar{\mathbf{x}} = \bar{\mathbf{0}}$ is $\bar{\mathbf{x}} = \bar{\mathbf{0}}$.
- A vector $\bar{\mathbf{b}} \in \mathbb{R}^n$ is in $span(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n)$ iff $\bar{\mathbf{b}} \in C(A)$ or iff $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has a solution.
- For both results A must be nonsingular, i.e., have maximal rank n.

Basis for a Vector Space

If $span(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n) = V$ and $span(\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_n) = V$ then $n = m$.

Basis for a Vector Space

Proof:

- Suppose $m > n$
- Every $\bar{\mathbf{w}}_j = a_{1j}\bar{\mathbf{v}}_1 + \cdots + anj\bar{\mathbf{v}}_n$
- Then

$$c_1\bar{\mathbf{w}} + \cdots + c_m\bar{\mathbf{w}}_m = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}c_j\bar{\mathbf{v}}_i$$

- This linear combination is zero if $(c_1, \ldots, c_m)$ is a solution to

$$\sum_{j=1}^{m} a_{ij}c_j = 0, i = 1, \ldots, n$$

- There are n equations with m unknowns and $m > n$ therefore there is a nontrivial solution $\bar{\mathbf{c}} \neq \bar{\mathbf{0}}$
- This makes vectors $\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_m$ linearly *dependent* - contradiction
- We conclude that $m = n$

Basis for a Vector Space

The dimension $dim(V)$ of a space V is the number of vectors in the basis.

- Let $V = span(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n)$
- $A = [\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n]$
- Then

$$dim(V) = rank(A)$$

Basis for a Vector Space

Let V be a n-dimensional vector space. Then

- Every set of more than n elements of V is linearly dependent.
- No set of less than n elements spans V
- A set of n elements forms a basis if and only if it spans V if and only if it is linearly independent.

Basis for a Vector Space

Let $B = \{\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_n\}$ be a basis of V and for $\bar{\mathbf{w}} \in V$

$$\bar{\mathbf{w}} = c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{2}} + \cdots + c_n\bar{\mathbf{v}}_n$$

The coefficients $c_1, \ldots, c_n$ are called coordinates of $\bar{\mathbf{w}}$ with respect to basis B.

If $\bar{\mathbf{w}} = \bar{\mathbf{c}}$ then the basis is called *standard*

Basis for a Vector Space

$$\bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \bar{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \bar{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \bar{\mathbf{v}}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

- $\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \bar{\mathbf{v}}_4\}$ is a basis of $\mathbb{R}^4$ called *wavelet basis*

Basis for a Vector Space

To show that $\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \bar{\mathbf{v}}_4\}$ is a basis of $\mathbb{R}^4$:

1. Show that $span(\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \bar{\mathbf{v}}_4) = \mathbb{R}^4$, i.e. for any $\bar{\mathbf{b}} \in \mathbb{R}^4$

   $$\bar{\mathbf{b}} = c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + c_3\bar{\mathbf{v}}_3 + c_4\bar{\mathbf{v}}_4$$

   or show that if $A = [\bar{\mathbf{v}}_1 \ \bar{\mathbf{v}}_2 \ \bar{\mathbf{v}}_3 \ \bar{\mathbf{v}}_4]$ then

   $$A\bar{\mathbf{x}} = \bar{\mathbf{b}}$$

   has a solution for any $\bar{\mathbf{b}} \in \mathbb{R}^4$
2. Show that independent: prove that $A\bar{\mathbf{x}} - \bar{\mathbf{0}}$ has unique solution $\bar{\mathbf{x}} = \bar{\mathbf{0}}$

Basis for a Vector Space

Let $\bar{\mathbf{w}} = [4, -2, 1, 5]^T$ then

$$\bar{\mathbf{w}} = 2\bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_2 + 3\bar{\mathbf{v}}_3 - 2\bar{\mathbf{v}}_4$$

i.e. $(2, -1, 3, -2)$ are coordinates of $\bar{\mathbf{w}}$ with respect to wavelet basis $\{\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_4\}$

Basis for a Vector Space

Let $\mathcal{P}^{(n)}$ be the vector space of polynomials of degree $\leq n$ then

- $1, x, x^2, \ldots, x^n$ is the standard basis of $\mathcal{P}^{(n)}$
- $dim(\mathcal{P}^{(n)}) = n + 1$
- Every other basis of $\mathcal{P}^{(n)}$ has $n + 1$ polynomials but not every collection of $n + 1$ polynomials forms a basis

The four fundamental subspaces

Let A be $m \times n$ matrix

- The columns space $C(A)$ is the space of all vectors $\bar{\mathbf{b}}$ for which $A\bar{\mathbf{x}} - \bar{\mathbf{b}}$ has a solution.
- The null space $N(A)$ is the space of all solutions to $A\bar{\mathbf{x}} = \bar{\mathbf{0}}$

The four fundamental subspaces

- The adjoint to a linear system $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ of m equations in n unknowns is the linear system $A^T\bar{\mathbf{y}} = \bar{\mathbf{f}}$ of n equations in m unknowns. Here $\bar{\mathbf{x}} \in \mathbb{R}^n$ and $\bar{\mathbf{y}} \in \mathbb{R}^m$
- The two a closely related

The four fundamental subspaces

Introduce new subspaces
Let A be $m \times n$ matrix

- The row space of A is $C(A^T)$
- The left null space of A is $N(A^T)$

The Fundamental Theorem of Linear Algebra

- The four fundamental subspaces associated with the matrix A, are the row space, column space, nullspace and left nullspace.
- *The Fundamental Theorem of Linear Algebra* states that their dimensions are entirely prescribed by the rank (and size) of the matrix.

Let A be an $m \times n$ matrix of rank r. Then

- $\dim(\text{row space}) = \dim(\text{column space}) = r$
- $\dim(\text{nullspace}) = n - r$ and $\dim(\text{ left nullspace }) = m - r$

The Fundamental Theorem of Linear Algebra

- Row space: subspace spanned by rows of A.
  - Elementary row operations do not change the subspace.
  - row space of A = row space of its echelon form U
  - there are r nonzero rows of U and they form the basis of row space of A
- Column space:
  - column space of A = row space of A^T
  - dim(column space of A) = dim(row space of A^T) = $rank(A^T)$ = r

Dimension of the space spanned by columns of A is the same as the dimension of the space spanned by rows of A

The Fundamental Theorem of Linear Algebra

- Null space:
  - General solution to $A\bar{\mathbf{x}} = \bar{\mathbf{0}}$ is

    $$\bar{\mathbf{x}} = y_1\bar{\mathbf{s}}_1 + y_2\bar{\mathbf{s}}_2 + \cdots + y_{n-r}\bar{\mathbf{s}}_{n-r}$$

    where y_i are free variables and $\bar{\mathbf{s}}_i$ are corresponding special solutions
  - The *i*th entry of $\bar{\mathbf{x}}$ is a free variable y_i
  - $\bar{\mathbf{x}} = 0$ if and only if each $y_i = 0$
  - Therefore, $\bar{\mathbf{s}}_1, \ldots, \bar{\mathbf{s}}_{n-r}$ are linearly independent and form a basis for $N(A)$

The Fundamental Theorem of Linear Algebra

- Left nullspace:
  - left null space of $A =$ null space of A^T
  - dim(left null space of A) $=$ dim(null space of A^T) $= m - r$