To see the connection to logic, let us correspond the bit value 1 with the logical value T (True) and 0 with F (False). So we can consider a_0, b_0, c_0, s_1, s_0 to be logical variables.

a. Construct the truth tables for s_1 and s_0 in terms of a_0, b_0, c_0. For example, when $a_0 = T$, $b_0 = T$, and $c_0 = F$ F, the truth values of $s_1 = T$ and $s_0 = F$.

a_0	b_0	c_0	s_1	s_0
F	F	F	F	F
F	F	T	F	T
F	T	F	F	T
F	T	T	T	F
T	F	F	F	T
T	F	T	T	F
T	T	F	T	F
T	T	T	T	T

b. Express s_0 as a proposition using the variables a_0, b_0, c_0 and the logical connectives ¬,∧,∨. Write a similar expression for s_1.

$s_0 = (a \wedge \neg(b \wedge c)) \vee (b \wedge \neg(a \wedge c)) \vee (c \wedge \neg(a \wedge b)) \vee (a \wedge (b \wedge c))$

$s_1 = \big((a \wedge b) \wedge \neg c\big) \vee \big((a \wedge c) \wedge \neg b\big) \vee ((b \wedge c) \wedge \neg a) \vee (a \wedge (b \wedge c))$

c. Express s_0 and s_1 using only the NAND connective discussed in lecture.

$$\begin{aligned} s_0 = {} & ([((([(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))] \uparrow [(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))]) \\ & \uparrow ([(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))] \uparrow [(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))])) \\ & \uparrow (([(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))] \uparrow [(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))]) \\ & \uparrow ([(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))] \uparrow [(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))]))) \\ & \uparrow ([(c \uparrow (a \uparrow b)) \uparrow (c \uparrow (a \uparrow b))] \uparrow [(c \uparrow (a \uparrow b)) \uparrow (c \uparrow (a \uparrow b))])] \\ & \uparrow [((([(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))] \uparrow [(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))]) \\ & \uparrow ([(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))] \uparrow [(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))])) \\ & \uparrow (([(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))] \uparrow [(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))]) \\ & \uparrow ([(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))] \uparrow [(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))]))) \\ & \uparrow ([(c \uparrow (a \uparrow b)) \uparrow (c \uparrow (a \uparrow b))] \uparrow [(c \uparrow (a \uparrow b)) \uparrow (c \uparrow (a \uparrow b))])]) \\ & \uparrow ([(a \uparrow ((b \uparrow c) \uparrow (b \uparrow c))) \uparrow (a \uparrow ((b \uparrow c) \uparrow (b \uparrow c)))] \uparrow [(a \\ & \uparrow ((b \uparrow c) \uparrow (b \uparrow c))) \uparrow (a \uparrow ((b \uparrow c) \uparrow (b \uparrow c)))]) \end{aligned}$$

alternatively:

$x = [(a \uparrow (b \uparrow c)) \uparrow (a \uparrow (b \uparrow c))]$ $\quad y = [(b \uparrow (a \uparrow c)) \uparrow (b \uparrow (a \uparrow c))]$

$z = [(c \uparrow (a \uparrow b)) \uparrow (c \uparrow (a \uparrow b))]$

$w = [\big(((x \uparrow x) \uparrow (y \uparrow y)) \uparrow ((x \uparrow x) \uparrow (y \uparrow y))\big) \uparrow (z \uparrow z)]$

$v = [\big(a \uparrow ((b \uparrow c) \uparrow (b \uparrow c))\big) \uparrow \big(a \uparrow ((b \uparrow c) \uparrow (b \uparrow c))\big)]$

$s_0 = [(w \uparrow w) \uparrow (v \uparrow v)]$

$$
\begin{aligned}
s_1 = [\Big(\big[\Big(\Big(\big(\big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big] \uparrow \big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big]\big) \\
\uparrow ([(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)] \uparrow [(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)])\Big) \\
\uparrow \Big(\big(\big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big] \uparrow \big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big]\big) \\
\uparrow ([(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)] \uparrow [(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)])\Big)\Big) \\
\uparrow \big(\big[\big((b \uparrow c) \uparrow (b \uparrow c)\big) \uparrow (a \uparrow a)\big] \uparrow \big[\big((b \uparrow c) \uparrow (b \uparrow c)\big) \uparrow (a \uparrow a)\big]\big)\big] \\
\uparrow [\Big(\Big(\big(\big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big] \uparrow \big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big]\big) \\
\uparrow ([(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)] \uparrow [(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)])\Big) \\
\uparrow \Big(\big(\big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big] \uparrow \big[\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)\big]\big) \\
\uparrow ([(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)] \uparrow [(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)])\Big)\Big) \\
\uparrow \big(\big[\big((b \uparrow c) \uparrow (b \uparrow c)\big) \uparrow (a \uparrow a)\big] \uparrow \big[\big((b \uparrow c) \uparrow (b \uparrow c)\big) \uparrow (a \uparrow a)\big]\big)\big]\Big) \\
\uparrow \Big(\Big[\Big(a \uparrow \big((b \uparrow c) \uparrow (b \uparrow c)\big)\Big) \uparrow \Big(a \uparrow \big((b \uparrow c) \uparrow (b \uparrow c)\big)\Big)\Big] \\
\uparrow \Big[\Big(a \uparrow \big((b \uparrow c) \uparrow (b \uparrow c)\big)\Big) \uparrow \Big(a \uparrow \big((b \uparrow c) \uparrow (b \uparrow c)\big)\Big)\Big]\Big)]
\end{aligned}
$$

alternatively:

$x = [\big((a \uparrow b) \uparrow (a \uparrow b)\big) \uparrow (c \uparrow c)]$ $\quad$ $y = [(a \uparrow c) \uparrow (a \uparrow c)) \uparrow (b \uparrow b)]$

$z = [\big((b \uparrow c) \uparrow (b \uparrow c)\big) \uparrow (a \uparrow a)]$

$w = [\Big(\big((x \uparrow x) \uparrow (y \uparrow y)\big) \uparrow \big((x \uparrow x) \uparrow (y \uparrow y)\big)\Big) \uparrow (z \uparrow z)]$

$v = [\Big(a \uparrow \big((b \uparrow c) \uparrow (b \uparrow c)\big)\Big) \uparrow \Big(a \uparrow \big((b \uparrow c) \uparrow (b \uparrow c)\big)\Big)]$

$s_1 = [(w \uparrow w) \uparrow (v \uparrow v)]$

d. Next, let's see how to add two 2-bit numbers a_1a_0, b_1b_0 to produce the 3-bit result $s_2s_1s_0$. Recall how we usually add numbers – we first add the lowest order bits (a_0 and b_0) to get the value s_0 as well as the "carry bit" which when added with a_1 and b_1 produces s_1 and a carry bit (which is s_2 for 2-bit numbers). Express each of s_2, s_1, s_0 in terms of a_1, a_0, b_1, b_0 using the NAND connective.

$s_0 = \Big[\big[\big((a_0 \uparrow a_0) \uparrow (b_0 \uparrow b_0)\big) \uparrow (a_0 \uparrow b_0)\big] \uparrow \big[\big((a_0 \uparrow a_0) \uparrow (b_0 \uparrow b_0)\big) \uparrow (a_0 \uparrow b_0)\big]\Big]$

$c_1 = [(a_0 \uparrow b_0) \uparrow (a_0 \uparrow b_0)]$ **Carry bit

$x_1 = \big[\big(a_1 \uparrow (b_1 \uparrow s_0)\big) \uparrow \big(a_1 \uparrow (b_1 \uparrow s_0)\big)\big]$ $\quad$ $y_1 = \big[\big(b_1 \uparrow (a_1 \uparrow s_0)\big) \uparrow \big(b_1 \uparrow (a_1 \uparrow s_0)\big)\big]$

$z_1 = \big[\big(s_0 \uparrow (a_1 \uparrow b_1)\big) \uparrow \big(s_0 \uparrow (a_1 \uparrow b_1)\big)\big]$

$w_1 = [\Big(\big((x_1 \uparrow x_1) \uparrow (y_1 \uparrow y_1)\big) \uparrow \big((x_1 \uparrow x_1) \uparrow (y_1 \uparrow y_1)\big)\Big) \uparrow (z_1 \uparrow z_1)]$

$v_1 = [\Big(a_1 \uparrow \big((b_1 \uparrow s_0) \uparrow (b_1 \uparrow s_0)\big)\Big) \uparrow \Big(a_1 \uparrow \big((b_1 \uparrow s_0) \uparrow (b_1 \uparrow s_0)\big)\Big)]$

$s_1 = [(w_1 \uparrow w_1) \uparrow (v_1 \uparrow v_1)]$ ** Similar to s_0 from part a/b/c (with a_1, b_1, s_0 instead of a, b, c)

$x_2 = [\big((a_1 \uparrow b_1) \uparrow (a_1 \uparrow b_1)\big) \uparrow (c_1 \uparrow c_1)]$ $\quad y_2 = [(a_1 \uparrow c_1) \uparrow (a_1 \uparrow c_1)) \uparrow (b_1 \uparrow b_1)]$
$z_2 = [\big((b_1 \uparrow c_1) \uparrow (b_1 \uparrow c_1)\big) \uparrow (a_1 \uparrow a_1)]$
$w_2 = [\Big(\big((x_2 \uparrow x_2) \uparrow (y_2 \uparrow y_2)\big) \uparrow \big((x_2 \uparrow x_2) \uparrow (y_2 \uparrow y_2)\big)\Big) \uparrow (z_2 \uparrow z_2)]$
$v_2 = [\Big(a_1 \uparrow \big((b_1 \uparrow c_1) \uparrow (b_1 \uparrow c_1)\big)\Big) \uparrow \Big(a_1 \uparrow \big((b_1 \uparrow c_1) \uparrow (b_1 \uparrow c_1)\big)\Big)]$
$s_2 = [(w_2 \uparrow w_2) \uparrow (v_2 \uparrow v_2)]$ ** Similar to s_1 from part a/b/c (with a_1, b_1, c_1 instead of a, b, c)