(A1) [2pt] Which of the following limits converge to a finite complex number? (Answer only.)

$$\lim_{n\to\infty} \tfrac{n+(1+i)^n}{n^2}, \qquad \lim_{n\to\infty} \tfrac{n+i^n}{n^2}, \qquad \lim_{n\to\infty} \tfrac{n^2+i^n}{n^2}.$$

Answer: The second and the third limits.

▷ Note that $|1+i|>1$, so the first limit goes to ∞. The other are finite:

$$\lim_{n\to\infty} \frac{n+i^n}{n^2} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{i^n}{n^2}}{1} = 0,$$

$$\lim_{n\to\infty} \frac{n^2+i^n}{n^2} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{i^n}{n^2}}{1} = 1+0=1.$$

(A2) [2pt] Find radius of convergence of the following power series. (Answer only.)

$$1+3+5^2z^2+3^3z^3+5^4z^4+\ldots=\sum_{n=0}^{\infty}(4+(-1)^n)^n z^n.$$

Answer: $\frac{1}{5}$.

▷ We have $\Lambda = \limsup_{n\to\infty} \sqrt[n]{(4+(-1)^n)^n} = \limsup_{n\to\infty}(4+(-1)^n) = 5$.
So $R = 1/\Lambda = 1/5$.

(A3) [2pt] Which of the following complex power functions have only finitely many values for a given $z \neq 0$? (Answer only.)

$$z^{-2017}, \qquad z^{\sqrt{2}}, \qquad z^{i}, \qquad z^{\frac{3}{4}i}, \qquad z^{\frac{3}{4}}.$$

Answer: Only z^{-2017} and $z^{\frac{3}{4}}$.

▷ The power function has finitely many values if and only if the exponent is rational. Therefore, of the listed functions, z^{-2017} and $z^{\frac{3}{4}}$ have finitely many values for a given z. The rest of the listed functions have infinitely many values.

(A4) [3pt] Arrange the following numbers in the order of increasing absolute value. (Answer only.)

$$(2+i)^8, \qquad \sinh(2017i), \qquad e^{4-20i}, \qquad \mathrm{Log}(5e^{2017i}).$$

Answer: $\sinh(2017i)$, $\text{Log}(5e^{2017i})$, e^{4-20i}, $(2+i)^8$.

$\triangleright$ Note that $\sinh(2017i) = i\sin 2017$, so absolute value does not exceed 1.

Next, $\text{Log}(5e^{2017i}) = \ln 5 + i\theta$, where θ is some number which is between $-\pi$ and π, so the absolute value is at lest $\ln 5 > 1$, and is not more than $\ln 5 + \pi < 5+4 < 10$.

Next, $|e^{4-20i}| = e^4$, which is more than $2^4 > 10$.

Next, $|(2+i)^8| = \sqrt{5}^8 = 5^4$, which is more than e^4 since $5 > e$.

(A5) [3pt] Suppose C is a contour with endpoints z_0 and z_1 which does not pass through 0. For which of the following functions $f(z)$ is the integral $\int_C f(z)dz$ path independent? (Answer only.)

$$\text{Log}\, z, \qquad \cos z^3, \qquad \tfrac{1}{e^z}, \qquad \tfrac{1}{z}, \qquad \bar{z}.$$

Answer: Only $\cos z^3$ and $1/e^z$.

$\triangleright$ $\cos z^3$ and $1/e^z = e^{-z}$ are entire, so the integral is path independent by Cauchy Integral Theorem.

For $1/z$, the integral along a circle is equal to $2\pi i \neq 0$ (which everybody has seen about 100 times by now), so it's not path independent.

$\bar{z}$ also has a nonzero integral along a circle, as seen in HW10. (And there is no reason so even suspect that it's path independent since it's not even analytic at any point.)

$\text{Log}\, z$ is not even continuous so its integral has no chance of being path independent.
[If you are curious, there are two ways to explain it rigorously. (1) Compare integral of $\text{Log}\, z$ along a real line segment, say $[-2,-1]$ to the integral along a path that with the same endpoints that goes a little below real axis. Im(Log) changes its sign to opposite, therefore the integral changes. (2) By Morera's theorem, if integrals are path independent in a simply connected domain, then function is analytic. Pick domain $\text{Re}\, z < 0$. If the integrals are path independent, then $\text{Log}\, z$ is analytic in that domain, but that function is not even continuous.]

(A6) [3pt] Find
$$\frac{1}{2\pi i}\int_C \frac{\text{Log}\, z}{(z-2i)^2}dz,$$
where $\text{Log}\, z$ is the principal value of the logarithm, and C is a circle of radius 1 centered at $2i$ traversed in the positive direction. (Answer only. Simplify the answer.)

Answer: $-\frac{1}{2}i$.

$\triangleright$ Recall that by Cauchy integral formula for derivatives,

$$\frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^2}dz = f'(\alpha),$$

where C is any appropriate contour (see the statement of the corresponding theorem for precise requirements).

With $f = \operatorname{Log} z$ we get

$$\frac{1}{2\pi i}\int_C \frac{\operatorname{Log} z}{(z-2i)^2}dz = (\operatorname{Log} z)'|_{z=2i} = \frac{1}{2i} = -\frac{1}{2}i.$$

Part B. In this part, show your work and provide explanations.

(B1) [4pt] Find all solutions of the equation $\cos z = 2ie^{-iz}$. (Give the answer in the form $x+iy$.)

$\triangleright$ Recall that $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, so the equation is

$$\frac{e^{iz}+e^{-iz}}{2} = 2ie^{-iz},$$

which is, after algebraic manipulations,

$$e^{iz} = (4i-1)e^{-iz},$$

or

$$e^{2iz} = 4i-1.$$

From this we conclude that

$$2iz = \log(4i-1),$$

so

$$z = -i\frac{\log(4i-1)}{2}.$$

We further simplify this as (remember that $-1+4i$ in QII)

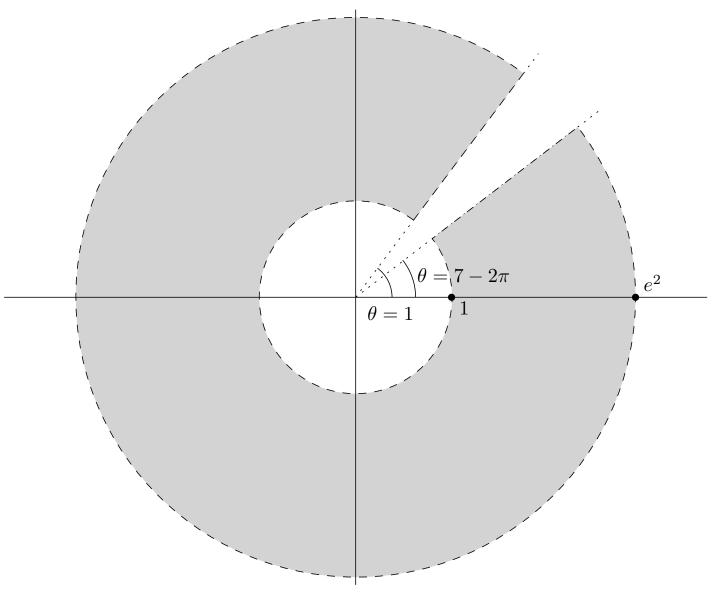
$$\begin{aligned} z = -i\frac{\log(4i-1)}{2} &= -\frac{i}{2}\left(\ln\sqrt{17} + i\arctan\left(-\frac{1}{4}\right) + i\pi + i2\pi n\right) = \\ &= -\frac{i\ln\sqrt{17}}{2} - \frac{\arctan(-\frac{1}{4})}{2} + \frac{\pi}{2} + \pi n. \end{aligned}$$

(B2) [4pt] Find and sketch image of the region $\{x + iy : 0 < x < 2,\ 1 < y < 7\}$ under the mapping $f(z) = e^z$.

▷ Recall that

$$e^{x+iy} = e^x(\cos y + i \sin y),$$

so in our case absolute value of e^z ranges from $e^0 = 1$ to e^2, and, independently, argument of e^z ranges from 1 to 7 (in both cases not including endpoints). Taking into account that $7 > 2\pi$ but $7 < 1 + 2\pi$, we see that this defines an annular sector, as shown in the figure.



(B3) [7pt] Use Cauchy Integral Theorem and Cauchy Integral Formula (and its version for derivative) to evaluate the following integral:

$$\int_{C_R(0)} \left(\frac{e^z}{2z - \pi i} + \frac{e^{3z}}{(z-4)^3} \right) dz$$

for $R = 1$, for $R = 2$, and for $R = 10$.

(Reminder: $C_R(0)$ is a circle of radius R centered at 0 traversed in the positive direction.)

▷ The integrand is analytic except at points $z = \frac{\pi}{2}i$ and $z = 4$. Therefore:

$R = 1$. The integrand is analytic inside this contour, so by Cauchy Integral Theorem the integral is 0.

$R=2$. The point $z=\frac{\pi i}{2}$ is inside the contour, while the point $z=4$ is outside. Therefore, the second term $\frac{e^{3z}}{(z-4)^3}$ defines an analytic function inside $C_2(0)$, and the corresponding integral is 0. We have

$$\int\limits_{C_2(0)}\left(\frac{e^z}{2z-\pi i}+\frac{e^{3z}}{(z-4)^3}\right)dz=\int\limits_{C_2(0)}\frac{e^z}{2z-\pi i}dz=\frac{1}{2}\int\limits_{C_2(0)}\frac{e^z}{z-\frac{\pi}{2}i}dz=$$
$$=\frac{1}{2}\int\limits_{C_1}\frac{e^z}{z-\frac{\pi}{2}i}dz=2\pi i\cdot\frac{1}{2}e^z\Big|_{z=\frac{\pi}{2}i}=\pi i\cdot i=-\pi,$$

by Cauchy integral formula (here C_1 denotes a small circle centered at $\frac{\pi}{2}i$; we can replace $C_2(0)$ with C_1 by Extended Cauchy integral theorem).

$R=10$. Both points $z=\frac{\pi i}{2}$ and $z=4$ is inside the contour. We have by Extended Cauchy integral theorem

$$\begin{aligned}\int\limits_{C_{10}(0)}\left(\frac{e^z}{2z-\pi i}+\frac{e^{3z}}{(z-4)^3}\right)dz &= \int\limits_{C_{10}(0)}\frac{e^z}{2z-\pi i}dz+\int\limits_{C_{10}(0)}\frac{e^{3z}}{(z-4)^3}dz= \\ &= \int\limits_{C_1}\frac{e^z}{2z-\pi i}dz+\int\limits_{C_2}\frac{e^{3z}}{(z-4)^3}dz,\end{aligned}$$

where C_1 denotes a small circle centered at $\frac{\pi}{2}i$, and C_2 a small circle centered at 4.

The former integral is computed above and equals $-\pi$. To compute the latter integral, recall that by Cauchy integral formula for derivatives,

$$\frac{2!}{2\pi i}\int_{C_r(z_0)}\frac{f(z)}{(z-z_0)^3}dz=f''(z_0),$$

so we have

$$\int\limits_{C_2}\frac{e^{3z}}{(z-4)^3}dz=\frac{2\pi i}{2!}(e^{3z})''\Big|_{z=4}=9\pi i e^{12}.$$

Putting the two summands together, we get

$$\int\limits_{C_{10}(0)}\left(\frac{e^z}{2z-\pi i}+\frac{e^{5z}}{(z-3)^3}\right)dz=-\pi+9\pi i e^{12}.$$