Taylor Series Problems

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a. $f(x) = sin(2x),\ f(0) = 0$
$f'(x) = 2cos(2x),\ f'(0) = 2$
$f''(x) = -4sin(2x),\ f''(0) = 0$
$f'''(x) = -8cos(2x),\ f'''(0) = -8$
$sin(2x) \approx 2x - \frac{4}{3}x^3$
$sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} x^{2n+1}$

Radius of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} x^{2n+1}$

$lim_{n\to\infty} \frac{2^{2n+3} \cdot x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{2n+1} x^{2n+1}} = 4x^2 lim_{n\to\infty} 1/((2n+3)(2n+2)) = 0$
By the ratio test, the radius of convergence is $(-\infty, \infty)$

b. $f(x) = ln(2x) = ln(2) + ln(x)\ f(1) = ln(2)$
$f'(x) = x^{-1},\ f'(1) = 1$
$f''(x) = -x^{-2},\ f''(1) = -1$
$f'''(x) = -2x^{-3},\ f'''(1) = 2$
$ln(2x) = ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x-1)^n$

Radius of convergence:

$lim_{n\to\infty} \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} = lim_{n\to\infty} (x-1) \cdot \frac{n}{n+1} = (x-1)$
In this case the radius of convergence is 1.

c. $f(x) = e^{2x},\ f(1) = e^2$
$f'(x) = 2e^{2x},\ f'(1) = 2e^2$
$e^{2x} = e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot (x-1)^n$

Radius of convergence:

$lim_{n\to\infty} \frac{2^{n+1}(x-1)^{n+1} n!}{(n+1)! 2^n (x-1)^n} = lim_{n\to\infty} 2(x-1)/(n+1) = 0$
The radius of convergence is $(-\infty, \infty)$

d. $f(x) = 3x^2 - 2x + 5$
Written as the taylor series this would be:
$5 - 2x + 3x^2$
The radius of convergence is $(-\infty, \infty)$

e. $f(x) = 3x^2 - 2x + 5$
The taylor series centered at x=1 is:
$6 + 4(x-1) + 3(x-1)^2$
Which can be shown to equal the given equation, so the interval of convergence is $(-\infty, \infty)$

f. $f(x) = (3x^2 - 2x + 5)^{-1},\ f(1) = 1/6$
$f'(x) = (2 - 6x)(3x^2 - 2x + 5)^{-2},\ f'(1) = -1/9$
$f''(x) = 2(2-6x)^2(3x^2 - 2x + 5)^{-3} - 6(3x^2 - 2x + 5)^{-2},\ f''(1) = -1/54$
I would do more terms, but wolfram alpha shows the coefficients in front of each additional term getting smaller, so adding more will have miniscule effect. I also don't see a way to find a closed form infinite series representation.
$\frac{1}{6} - \frac{x-1}{9} - \frac{(x-1)^2}{108}$
Since this is an approximation of a polynomial, the series will converge, and has an infinite radius of convergence.

g. $f(x) = cosh(3 - x),\ f(1) = cosh(2)$
$f'(x) = -sinh(3 - x),\ f'(1) = -sinh(2)$
$f''(x) = cosh(3 - x),\ f''(1) = cosh(2)$
$cosh(3 - x) \approx cosh2 - sinh(2)(x - 1) + \frac{cosh(2)}{2}(x - 1)^2$
$\sum_{if\ n\ mod\ 2=1 \wedge n \geq 0}^{\infty} \frac{sinh(2)(x-1)^n}{n!} + \sum_{if\ n\ mod\ 2=0 \wedge n \geq 0}^{\infty} \frac{cosh(2)(x-1)^n}{n!}$
We can check both series convergences at the same time because of their constant multiple, so:
$lim_{n \to \infty} \frac{(x-1)^{n+1} n!}{(n+1)!(x-1)^n} = lim_{n \to \infty} \frac{x-1}{n+1} = 0$
The radius of convergence is infinite.

h. $f(x) = a,\ f(a) = a$
$f'(x) = 0,\ f'(0) = 0$
$f(x) = a$ is the taylor series expansion of the function $f(x) = a$, which has an infinite ratio of convergence.

i.

j.