## Week 17: Brownian Motion and Stability

Corrections:

**Stationary States:**

$$dX = \mu X\, dt + \lambda X\, dW$$ Wiener process: $N(0, dt)$

- $\mu = 0$ ✓
- $\sigma^2 = dt$

- $\mu, \lambda > 0 \to$ exponential growth.
- $\mu < 0 \to$ " decay.

(Absorbing state at $x=0$).

(a). $dX = \mu X\, dt + \lambda X\, dW \Rightarrow \times \frac{1}{dt}$

$$\Rightarrow \frac{dX}{dt} = \mu X + \lambda X\, \xi(t)$$ $N(0, \frac{1}{dt})$.

We are just re-writing the equation (dividing by $dt$), thus, they are equivalent.

(b). If we start at $x \geq 0$, then we will always have $x \geq 0$, since we have an absorbing state at $x=0$.

**Mean:**

$$\frac{d\langle x\rangle}{dt} = \mu\langle x\rangle + \lambda\langle x\,\xi(t)\rangle \overset{\text{Itô's prescription.}}{=}$$
$$= \mu\langle x\rangle + \lambda\langle x\rangle\underbrace{\langle \xi(t)\rangle}_{=0}$$

$$\Rightarrow \frac{d\langle x\rangle}{dt} = \mu\langle x\rangle \Rightarrow \langle x\rangle = x_0 e^{\mu t}$$

exponential growth of expected (for $\mu>0$).

**Variance:**

$$x(t+dt) = \left(x(t) + \mu x(t)\,dt\right) + \lambda x(t)\,dW(t) \Rightarrow$$
$$\Rightarrow x^2(t+dt) = x^2(t) + 2\mu x^2(t)\,dt + \lambda^2 x^2(t)\,dW^2(t) + 2\lambda x(t)\,dW(t)\left(x(t)+\mu x(t)\,dt\right)$$

neglecting terms $O(dt^2)$. $\langle dW(t)\rangle = 0$.

$\langle dW^2\rangle = dt$

$$\Rightarrow \langle x^2(t+dt)\rangle = \langle x^2(t)\rangle + 2\mu\langle x^2(t)\rangle dt + \lambda^2\langle x^2(t)\rangle dt$$
$$\Rightarrow \frac{\langle x^2(t+dt)\rangle - \langle x^2(t)\rangle}{dt} = 2\mu\langle x^2(t)\rangle + \lambda^2\langle x^2(t)\rangle \Rightarrow$$
$$\Rightarrow \frac{d}{dt}\langle x^2(t)\rangle = (2\mu+\lambda^2)\langle x^2(t)\rangle \Rightarrow$$
$$\Rightarrow \langle x^2(t)\rangle = \underbrace{k}_{\text{constant.}} e^{(2\mu+\lambda^2)t}$$

$$\sigma_x^2 = \langle x^2(t)\rangle - \langle x(t)\rangle^2 =$$
$$= k e^{(2\mu+\lambda^2)t} - x_0^2 e^{2\mu t}$$

- Assuming $\sigma_x^2(t=0) = 0 \Rightarrow k - x_0^2 = 0 \Rightarrow k = x_0^2$.
- Finally: $\sigma_x^2(t) = x_0^2 e^{(2\mu+\lambda^2)t} - x_0^2 e^{2\mu t} =$ $= x_0^2 e^{2\mu t}\left(e^{\lambda^2 t} - 1\right)$.

**Results:**

- $\mu_x = \langle x(t)\rangle = x_0 e^{\mu t}$;
- $\sigma_x^2 = \langle x^2(t)\rangle - \langle x(t)\rangle^2 = x_0^2 e^{2\mu t}\left(e^{\lambda^2 t} - 1\right)$.

From this expression, it is clear that the distribution will broaden with time.

## (C). Derivation of the Fokker-Planck eq.:

General function $\Phi$ that depends on r.v. $x$:

$\Phi = \Phi(x)$.

From Itô's lemma, we can write:

$$d\Phi(x) = \underbrace{\cancel{\frac{\partial \Phi}{\partial t}}}_{=0} dt + \frac{\partial \Phi}{\partial x} dx + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2} dx^2 .$$

Remembering that: $dx = \mu x dt + \lambda x dW$.

$$\Rightarrow d\Phi = \frac{\partial \Phi}{\partial x} \mu x dt + \frac{\partial \Phi}{\partial x} \lambda x \underbrace{dW}_{\langle dW \rangle = 0} + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2}\Big( \underbrace{\mu^2 x^2 \cancel{dt^2}}_{=0} + \lambda^2 x^2 \underbrace{dW^2}_{\langle dW^2 \rangle = dt} + 2\mu\lambda x^2 dt \underbrace{dW}_{\langle dW \rangle = 0} \Big) \Rightarrow$$

$$\Rightarrow \langle d\Phi \rangle = \mu \left\langle \frac{\partial \Phi}{\partial x} \cdot x \right\rangle dt + \frac{1}{2}\lambda^2 \left\langle \frac{\partial^2 \Phi}{\partial x^2} \cdot x^2 \right\rangle dt \Rightarrow$$

$$\Rightarrow \frac{d\langle \Phi \rangle}{dt} = \mu \left\langle \frac{\partial \Phi}{\partial x} \cdot x \right\rangle + \frac{1}{2}\lambda^2 \left\langle \frac{\partial^2 \Phi}{\partial x^2} \cdot x^2 \right\rangle \quad (1).$$

**But:**

$$\frac{d\langle \Phi \rangle}{dt} = \frac{d}{dt}\left( \int_{-\infty}^{+\infty} dx\, \Phi(x)\, p(x,t) \right) = \int_{-\infty}^{+\infty} dx\, \Phi(x) \frac{\partial p(x,t)}{\partial t} \quad (2).$$

**Setting:** $(1) = (2)$:

$$\int_{-\infty}^{+\infty} dx\, \Phi(x) \frac{\partial p(x,t)}{\partial t} = \underbrace{\int_{-\infty}^{+\infty} dx\, \mu \frac{\partial \Phi}{\partial x} \cdot x\, p(x,t)}_{A} + \underbrace{\int_{-\infty}^{+\infty} \frac{1}{2}\lambda^2 x^2 \frac{\partial^2 \Phi}{\partial x^2} \cdot p\, dx}_{B}.$$

Integration by parts: (DI method)

**A:**

| | D | I |
|---|---|---|
| + | $xp$ | $\frac{\partial \Phi}{\partial x}$ |
| − | $\frac{\partial}{\partial x}(xp)$ | $\Phi$ |

$$A = \underbrace{\cancel{\Big[ x p \Phi \Big]_{-\infty}^{+\infty}}}_{=0} - \int_{-\infty}^{+\infty} dx\, \Phi \cdot \frac{\partial}{\partial x}(xp)$$

$=0$ / $p$ goes to $0$ at $\pm\infty$

**B:**

| | D | I |
|---|---|---|
| + | $x^2 p$ | $\frac{\partial^2 \Phi}{\partial x^2}$ |
| − | $\frac{\partial}{\partial x}(x^2 p)$ | $\frac{\partial \Phi}{\partial x}$ |
| + | $\frac{\partial^2}{\partial x^2}(x^2 p)$ | $\Phi$ |

$$B = \underbrace{\cancel{\Big[ x^2 p \frac{\partial \Phi}{\partial x} \Big]_{-\infty}^{+\infty}}}_{=0} - \underbrace{\cancel{\Big[ \Phi \cdot \frac{\partial}{\partial x}(x^2 p) \Big]_{-\infty}^{+\infty}}}_{=0} + \int_{-\infty}^{+\infty} dx\, \Phi \cdot \frac{\partial^2}{\partial x^2}(x^2 p).$$

Putting all of this together, we obtain:

$$\int_{-\infty}^{+\infty} dx\, \Phi \cdot \frac{\partial p(x,t)}{\partial t} = -\mu \int_{-\infty}^{+\infty} dx\, \Phi \cdot \partial_x(xp) + \frac{1}{2}\sigma^2 \int_{-\infty}^{+\infty} dx\, \Phi \cdot \partial_{xx}(x^2 p) = 0$$

$$\Rightarrow \frac{\partial p(x,t)}{\partial t} = -\mu \cdot \frac{\partial (x\, p(x,t))}{\partial x} + \frac{1}{2}\sigma^2 \cdot \frac{\partial^2 (x^2 p(x,t))}{\partial x^2}.$$

↳ Fokker-Planck equation.

$$\Leftrightarrow \frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} \underbrace{\left\{ \mu\, x\, p(x,t) - \frac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(x^2 p(x,t)\right)\right\}}_{J(x,t)}.$$

(probability current density).

Stationary state:

$\Rightarrow J = 0. \Rightarrow$

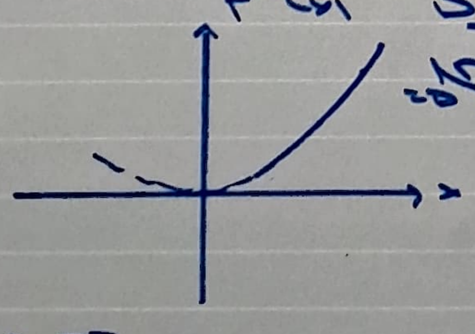
$\Rightarrow \mu\, x\, p^{st}(x) - \frac{1}{2}\sigma^2 \left(2x\, p^{st}(x) + x^2 \partial_x p^{st}(x)\right) = 0$

$\Rightarrow \mu\, x\, p^{st}(x) - \sigma^2 x\, p^{st}(x) = \frac{1}{2}\sigma^2 x^2 \partial_x p^{st}(x) \Rightarrow$

$\Rightarrow \partial_x p^{st}(x) = \frac{2(\mu - \sigma^2)}{\sigma^2} \cdot \frac{p^{st}(x)}{x} \Rightarrow$

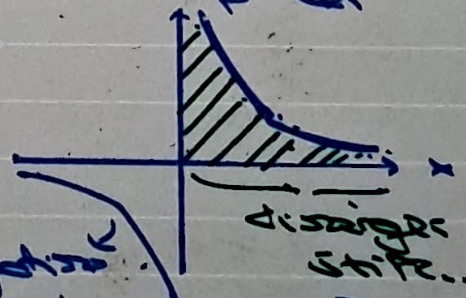
$\Rightarrow p^{st}(x) \sim x^{\frac{2(\mu-\sigma^2)}{\sigma^2}}$ . Q.E.D.

$\mu > \sigma^2$:




Diverges!
⇒ Not normalizable!

$\mu < \sigma^2$:




negative $p(x)$ (not a problem: $x>0$!).

→ If we remove the x in $x\,dW$: $\mu\, x\, p^{st}(x) - \frac{1}{2}\sigma^2 \partial_x p^{st} = 0$ (just discard this part.)

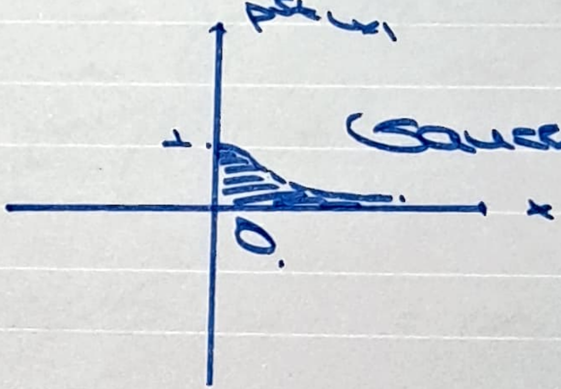
$\Rightarrow \partial_x p^{st}(x) = \frac{2\mu}{\sigma^2} \cdot x \cdot p^{st}(x)$

$\Rightarrow p^{st}(x) \sim e^{\frac{\mu}{\sigma^2} x^2}.$

$p^{st}(x) \sim e^{-\frac{\gamma}{\lambda^2} x^2}$ → if, additionally <u>$\gamma > 0$</u>.

doesn't diverge + is normalizable!

<u>good!</u>




Gaussian-like.

→ which means that it actually corresponds to a physically meaningful stationary state.