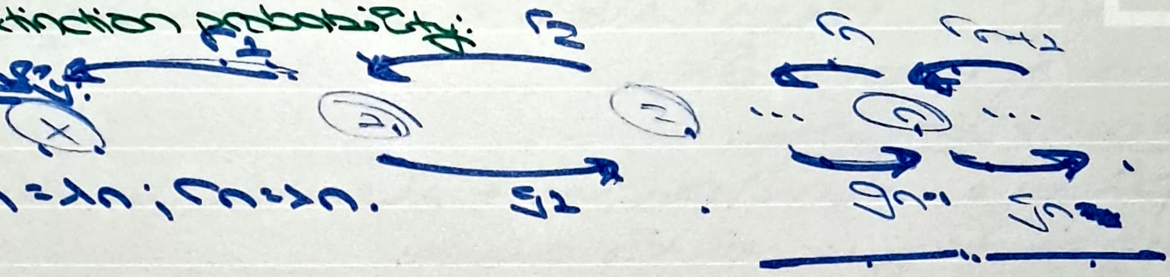
Extinction probability:

Schematically:



$g_n = \lambda n$; $r_n = \lambda n$.

(a). Master equation:

$$\dot{p}_n = g_{n-1} p_{n-1} + r_{n+1} p_{n+1} - (r_n + g_n) p_n$$
$$= \lambda(n-1) p_{n-1} + \lambda(n+1) p_{n+1} - 2\lambda n p_n.$$

(b). Defining $\mathbb{E} f(n) = f(n+1)$ and $\mathbb{E}^{-1} f(n) = f(n-1)$ (like in the lecture):

$$\dot{p}_n = (\mathbb{E}-1) r_n p_n + (\mathbb{E}^{-1}-1) g_n p_n$$

Mean-field behaviour:

$$\langle n \rangle(t) = \sum_{n=0}^{+\infty} n p_n(t) \Rightarrow$$

$$\Rightarrow \frac{d\langle n \rangle}{dt} = \sum_{n=0}^{+\infty} n \dot{p}_n(t) \Rightarrow$$

$$\Rightarrow \frac{d\langle n \rangle}{dt} = \sum_{n=0}^{+\infty} n \left\{ (\mathbb{E}-1) r_n p_n + (\mathbb{E}^{-1}-1) g_n p_n \right\} \Rightarrow$$

$$\Rightarrow \frac{d\langle n \rangle}{dt} = \sum_{n=0}^{+\infty} \left[r_n p_n \underbrace{(\mathbb{E}^{-1}-1) n}_{\cancel{n}-1-\cancel{n}} + g_n p_n \underbrace{(\mathbb{E}-1) n}_{\cancel{n}+1-\cancel{n}} \right].$$

$$\Rightarrow \frac{d\langle n \rangle}{dt} = \sum_{n=0}^{+\infty} g_n p_n - \sum_{n=0}^{+\infty} r_n p_n \Rightarrow$$

$$\Rightarrow \frac{d\langle n \rangle}{dt} = \underbrace{\langle g_n \rangle}_{\text{birth.}} - \underbrace{\langle r_n \rangle}_{\text{death.}}$$

In this case, $r_n = g_n = \lambda n$. So:

$$\frac{d\langle n \rangle}{dt} = \langle \lambda n \rangle - \langle \lambda n \rangle = 0 \Rightarrow \frac{d\langle n \rangle}{dt} = 0 \Rightarrow$$
$$\Rightarrow \langle n \rangle = n_0 \rightarrow \text{const.}$$

Having the same birth and death rates, it doesn't surprise anyone that $\langle n \rangle$ remains constant and equal to its initial value.

Q1. $P_n(t) = A(t)\cdot\alpha^{n-1}(t)\,,\ n \geq 1$ (ansatz).
And $P_n(t=0) = \delta_{n,1}$.

Master equation:

$$\dot{P}_n(t) = \lambda(n-1)P_{n-1}(t) + \lambda(n+1)P_{n+1}(t) - 2\lambda n P_n(t).$$

Substituting with the ansatz:

$$\dot{A}(t)\cdot\alpha^{n-1}(t) + A(t)(n-1)\cdot\alpha^{n-2}\cdot\dot{\alpha}(t) =$$

$$= \lambda(n-1)A(t)\cdot\alpha^{n-2} + \lambda(n+1)A(t)\cdot\alpha^{n} - 2\lambda n A(t)\alpha^{n-1} \Rightarrow$$

$$\Rightarrow \frac{\dot{A}(t)}{\alpha(t)} + \frac{A(t)(n-1)}{\alpha^2(t)}\cdot\dot{\alpha}(t) = \lambda(n-1)\frac{A(t)}{\alpha^2(t)} + \lambda(n+1)A(t) - 2\lambda n\frac{A(t)}{\alpha(t)} = 0$$

$$\Rightarrow \underbrace{\left(\frac{\dot{A}(t)}{\alpha(t)} - \frac{A(t)\cdot\dot{\alpha}(t)}{\alpha^2(t)} + \frac{\lambda A(t)}{\alpha^2(t)} - \lambda A(t)\right)}_{A} + \underbrace{\left(\frac{A(t)\cdot\dot{\alpha}(t)}{\alpha^2(t)} - \frac{\lambda A(t)}{\alpha^2(t)} - \lambda A(t) + \frac{2\lambda A(t)}{\alpha(t)}\right)}_{B} n = 0$$

$\Rightarrow$ now we require that $\underline{A = B = 0}$:

$$\begin{cases} \dfrac{\dot{A}(t)}{\alpha(t)} - \dfrac{A(t)\dot{\alpha}(t)}{\alpha^2(t)} + \dfrac{\lambda A(t)}{\alpha^2(t)} - \lambda A(t) = 0 & (1) \\ \underbrace{\cancel{A(t)}}_{A(t)\neq 0}\left(\dfrac{\dot{\alpha}(t)}{\alpha^2(t)} - \dfrac{\lambda}{\alpha^2(t)} - \lambda + \dfrac{2\lambda}{\alpha(t)}\right) = 0 & (2). \end{cases}$$

We solve (2) first, as it's independent of $A(t)$:

$$\Rightarrow \dot{\alpha}(t) - \lambda - \lambda\alpha^2(t) + 2\lambda\alpha(t) = 0 \Rightarrow$$

$$\Rightarrow \dot{\alpha} = +\lambda(\alpha^2 - 2\alpha + 1) \Rightarrow$$

$$\Rightarrow \dot{\alpha} = \lambda(\alpha - 1)^2 \Rightarrow \dot{\beta} = \lambda\beta^2,\ \text{if } \underline{\beta = \alpha - 1}.$$

$$\Rightarrow \beta = \frac{1}{k - \lambda t},\ k := \text{const.}$$

$$\Rightarrow \boxed{\alpha(t) = 1 + \frac{1}{k - \lambda t}}.$$

· In detail:

$\dot{\beta} = \lambda \beta^2 \Rightarrow \frac{d\beta/dt}{\beta^2} = \lambda \Rightarrow \frac{d\beta}{\beta^2} = \lambda dt \Rightarrow$

$\Rightarrow -\frac{1}{\beta} = \lambda t + c \Rightarrow \underbrace{\beta = \frac{1}{-c-\lambda t}}_{} \Rightarrow \beta = \frac{1}{k-\lambda t}$

Now, we add (1) and (2): $\qquad \underline{k=-c}$.

1: $\frac{\dot{A}(t)}{\alpha(t)} - 2\lambda A(t) + 2\lambda\frac{A(t)}{\alpha(t)} = 0 \Rightarrow$

$\Rightarrow \dot{A}(t) - 2\lambda A(t)\alpha(t) + 2\lambda A(t) = 0 \Rightarrow$

$\Rightarrow \dot{A}(t) = -2\lambda\left(1-\underbrace{\alpha(t)}\right)\cdot A(t)$.

Substituting α(t)

$\Rightarrow \dot{A}(t) = -2\lambda\cdot\left(\cancel{1} - \left(\cancel{1} + \frac{1}{k-\lambda t}\right)\right)A(t) \Rightarrow$

$\Rightarrow \dot{A}(t) = \frac{2\lambda}{k-\lambda t}\cdot A(t) \Rightarrow$

$\Rightarrow \int \frac{\dot{A}(t)}{A(t)}dt = 2\lambda\int\frac{1}{k-\lambda t}dt \Rightarrow$

$\Rightarrow \int \frac{dA}{A} = 2\lambda\int\frac{1}{k-\lambda t}dt \Rightarrow \ln[A(t)] = \frac{\cancel{2\lambda}\ln(k-\lambda t)}{\cancel{-\lambda}}$

$\Rightarrow \ln[A(t)] = \ln[(k-\lambda t)^{-2}] + c \Rightarrow$

$\Rightarrow A(t) = \frac{1}{(k-\lambda t)^2}\cdot e^c \Rightarrow \boxed{A(t) = \frac{C}{(k-\lambda t)^2}} \quad \underline{C = e^c}$.

· Total solution:

$P_n(t) = A(t)\cdot\alpha^{n-1}(t)$

$= \frac{C}{(k-\lambda t)^2}\left(1 + \frac{1}{k-\lambda t}\right)^{n-1}, \; n \geq 1.$

$P_n(t=0) = \delta_{n,1} \Rightarrow \frac{C}{k^2}\left(1+\frac{1}{k}\right)^{n-1} = \delta_{n,1} \Rightarrow$

$\Rightarrow \begin{cases} \frac{C}{k^2} = 1, & n=1 \\ \frac{C}{k^2}\left(1+\frac{1}{k}\right) = 0, & n \geq 2 \end{cases} \Rightarrow \begin{cases} C = 1 \\ k = -1 \end{cases}$

$$\Rightarrow \boxed{P_n(t) = \frac{1}{(1+\lambda t)^2}\left(1 - \frac{1}{1+\lambda t}\right)^{n-1},\ n \geq 1.}$$

Q1: For convenience, I define: $y = \frac{1}{1+\lambda t}$.

Mean: $\boxed{\langle n(t)\rangle} = \sum_{n=1}^{\infty} n P_n(t) = \sum_{n=1}^{\infty} n y^2 (1-y)^{n-1}$

$n=0$ for $n=0$ no contribution!

$$= -y^2 \cdot \frac{d}{dy}\left[\sum_{n=1}^{\infty} (1-y)^n\right]$$

$$= -y^2 \frac{d}{dy}\left[\frac{1-y}{\cancel{1}-(\cancel{1}-y)}\right] = -y^2 \cdot \frac{d}{dy}\left(\frac{1}{y} - 1\right)$$

$$= -y^2 \cdot \left(-\frac{1}{y^2}\right) = \boxed{1}$$ → const. and equal to $n_0 = 1$. Same as before! (Checks out.)

Variance: $\langle n^2(t)\rangle = \sum_{n=1}^{\infty} n^2 y^2 (1-y)^{n-1}$

$$= -y^2 \cdot \frac{d}{dy}\left[\underbrace{\sum_{n=1}^{\infty} n(1-y)^n}_{*}\right] = -y^2 \cdot \frac{d}{dy}\left(\frac{1-y}{(\cancel{1}-(\cancel{1}-y))^2}\right)$$

$$= -y^2 \cdot \frac{d}{dy}\left(\frac{1-y}{y^2}\right) = -y^2\left(\frac{-1\cdot y^2 - (1-y)\cdot 2y}{y^4}\right)$$

$$= \frac{-y^2 - 2y + 2y^2}{-y^2} = \frac{y^2 - 2y}{-y^2} = \frac{2}{y} - 1$$

Finally: $\boxed{\sigma^2} = \langle n^2\rangle - \langle n\rangle^2$

$$= \frac{2}{y} - 1 - 1^2 = \frac{2}{y} - 2$$

$$= 2(\cancel{1} + \lambda t) - \cancel{2} = \boxed{2\lambda t} \xrightarrow{t\to\infty} \infty\ !$$

Fluctuation will be infinite, even though $\langle n\rangle$ = const.

Also: *

$$(1-y)\cdot \underbrace{\sum_{n=1}^{\infty} n(1-y)^{n-1}}_{\text{from before...}} = (1-y)\cdot\frac{1}{y^2} = \frac{1-y}{y^2}$$

8). $P_n(t) = \frac{1}{(1+\lambda t)^2}\left(1-\frac{1}{1+\lambda t}\right)^{n-1}$, $n \geq 1$.

$P_0(t) = 1 - \sum_{n=1}^{+\infty} P_n(t)$ (normalization).

$$= 1 - \sum_{n=1}^{+\infty} \frac{1}{(1+\lambda t)^2}\left(\frac{\lambda t}{1+\lambda t}\right)^{n-1}$$

$$= 1 - \frac{1}{(1+\lambda t)^2}\left(\frac{1}{1-\frac{\lambda t}{1+\lambda t}}\right) = 1 - \frac{1}{1+\lambda t} = \boxed{\frac{\lambda t}{1+\lambda t} \xrightarrow{t\to\infty} 1}.$$

extinction is certain, as predicted before...

9). Using directly the expression derived in the lecture (for general g_n and r_n solved):

$$\boxed{T_n = \sum_{j=1}^{n} R_j \sum_{i=j}^{+\infty} \frac{1}{g_i R_i}}, \text{ with } R_j = \prod_{i=1}^{j} \frac{r_i}{g_i}$$

In our case: $g_n = r_n = \lambda n \Rightarrow R_j = \prod_{i=1}^{j} 1 = 1$.

$$\Rightarrow T_1 = \sum_{j=1}^{1} 1 \cdot \sum_{i=j}^{+\infty} \frac{1}{g_i} = \frac{1}{\lambda}\underbrace{\sum_{i=1}^{+\infty}\frac{1}{i}} \to +\infty.$$

So, we have an infinite time to extinction, but we know that it is certain.

Harmonic series diverge.

This extinction will be due to fluctuations, which get arbitrarily large, as $t\to+\infty$. This also explains why $T_1 = +\infty$.