The central limit theorem:

- I will consider 3 (continuous) distributions.
  - uniform;
  - gaussian;
  - Lorentzian (Cauchy).

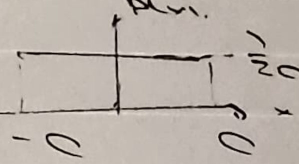
- $x = \sum_{i=1}^{N} x_i$ where
- $\langle x \rangle = N\mu = N \langle x_i \rangle$.
- $\sigma_x = \sqrt{N}\,\sigma_i$

- The x_i's are
- independently
- identically
- distributed
- i.i.d. random
- Setup:

Theoretical considerations:

(Central limit th.)

Uniform distribution:



$p(x) = \frac{1}{2a}\,\Theta(a-|x|)$

$\mu = 0$, $\sigma = \frac{a^2}{3}$

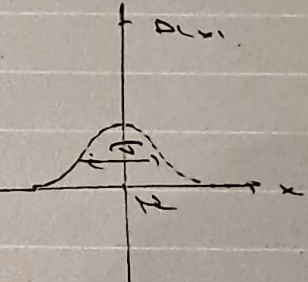
$\mu = \langle x \rangle = \int_{-a}^{a} dx\; x\, p(x)$

$= \int_{-a}^{a} dx \frac{x}{2a} = \frac{1}{2a}\left(\frac{x^2}{2}\right)\Big|_{-a}^{a} = 0$

$\langle x^2 \rangle = \frac{1}{2a}\int_{-a}^{a} dx\, x^2 = \frac{1}{2a}\left(\frac{x^3}{3}\right)\Big|_{-a}^{a} = \cancel{\frac{1}{2a}\cdot\frac{2a^3}{3}} = \frac{a^2}{3}$

$\sigma = \langle x^2 \rangle - \langle x \rangle^2 = \frac{a^2}{3} - 0 = \frac{a^2}{3}$

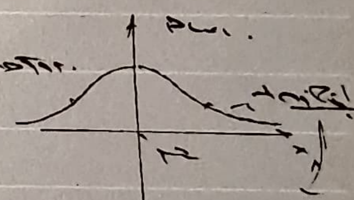
Gaussian distribution:



$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$, mean: μ

variance: σ^2

Lorentzian (cauchy) distribution:



$p(x) = \frac{1}{\pi}\frac{\gamma}{(x-\mu)^2+\gamma^2}$, ~~mean: μ~~ μ is simply a position parameter.

variance $\sigma^2 \to \infty$.

$X = \sum_i^N x_i$:

x_i	μ_X	σ^2_X
uniform	$N\cdot\mu = N\cdot 0 = 0$	$\frac{Na^2}{3}$
gaussian	$N\mu$	$N\sigma^2$
cauchy ?	not defined	$\to\infty$ (not defined)

problems? (?)

- Should also feasible one
- probably more 'behaved' than the Lorentzian. (binomial, e.g.)

even though it is discrete.

It should work to demonstrate (normalizing the CLT).

Also, I use the Poisson distribution:

$P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$

$\mu = \sigma^2 = \lambda$

(poisson dist. parameter)

Intuition:

$X = \sum_{i=1}^{N} x_i$ satisfies:

$\mu_X = N\mu = N\lambda$

$\sigma^2_X = N\sigma^2 = N\lambda$

· In addition to this explanation, I include the obtained plots (and essentially the code?) in the submission.pdf.

In [ ]:

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats as stats
plt.style.use('ggplot')
```

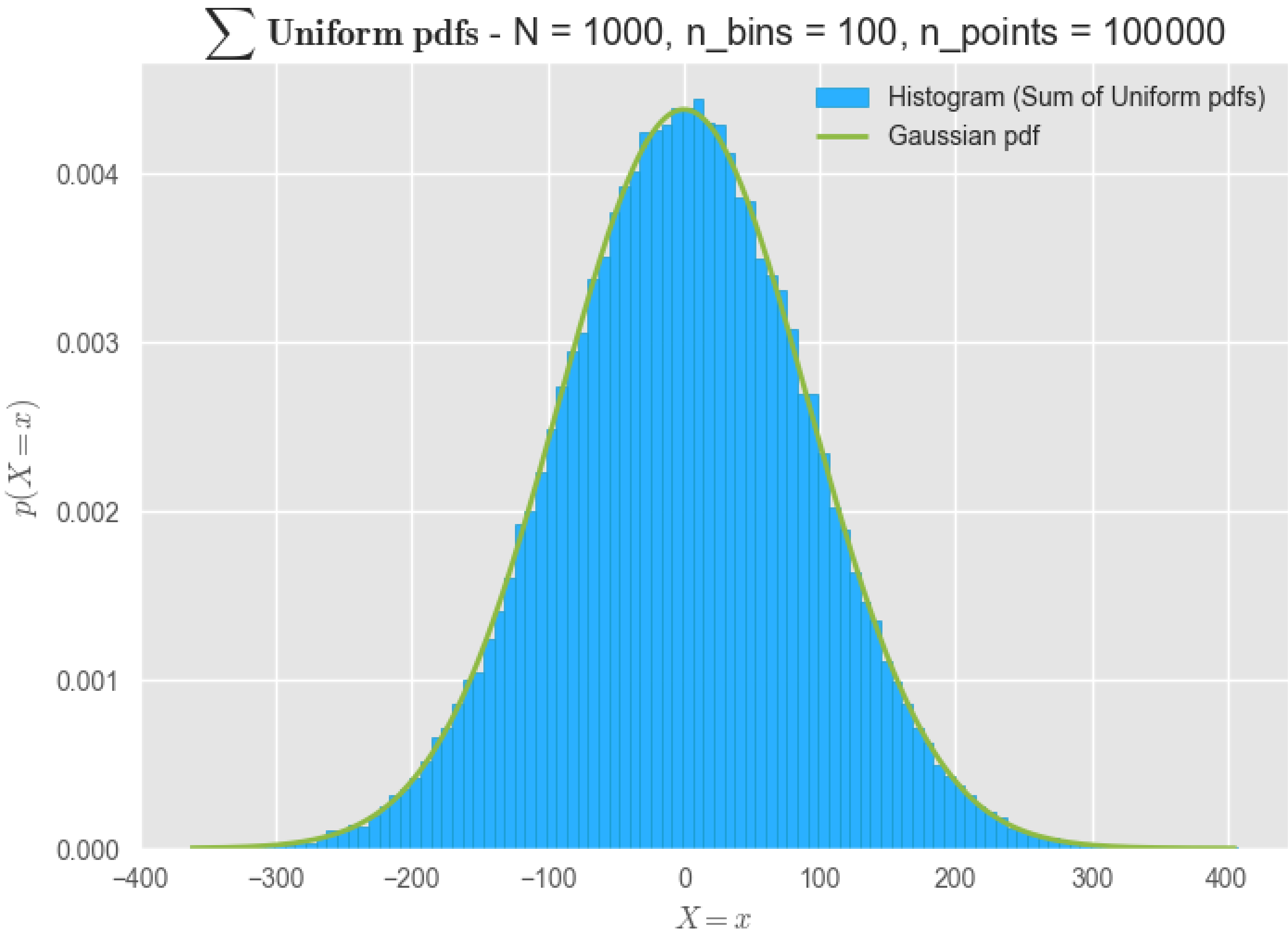
Uniform pdf:

In [ ]:

```
# Definitions
N = 1_000; c = 5; n_points = 100_000; sum_X_vals = []

for n in range(n_points):
    X = np.random.uniform(-c, c, N)
    sum_X_vals.append(sum(X))
```

In [ ]:

```
# Creating histogram
n_bins = 100
hist, bins, _ = plt.hist(sum_X_vals, density = True, bins = n_bins, label = 'Histogram (Sum of Uniform pdfs)',
                         facecolor = '#2ab0ff', edgecolor='#169acf', linewidth=0.5) # dimgray/maroon?

# Analytic parameters (computed by hand)
mu    = 0
sigma = np.sqrt(N*c**2/3)

x = np.linspace(bins.min(), bins.max(), 1_000)
plt.title(r'$\sum$ $\bf{Uniform}$ $\bf{pdfs}$ - ' + f'N = {N}, n_bins = {n_bins}, n_points = {n_points}')
plt.plot(x, stats.norm.pdf(x, mu, sigma), label = 'Gaussian pdf', color = 'C5')
plt.xlabel(r'$X = x$')
plt.ylabel(r'$p(X = x)$')
plt.legend()
plt.show()
```



Gaussian pdf:

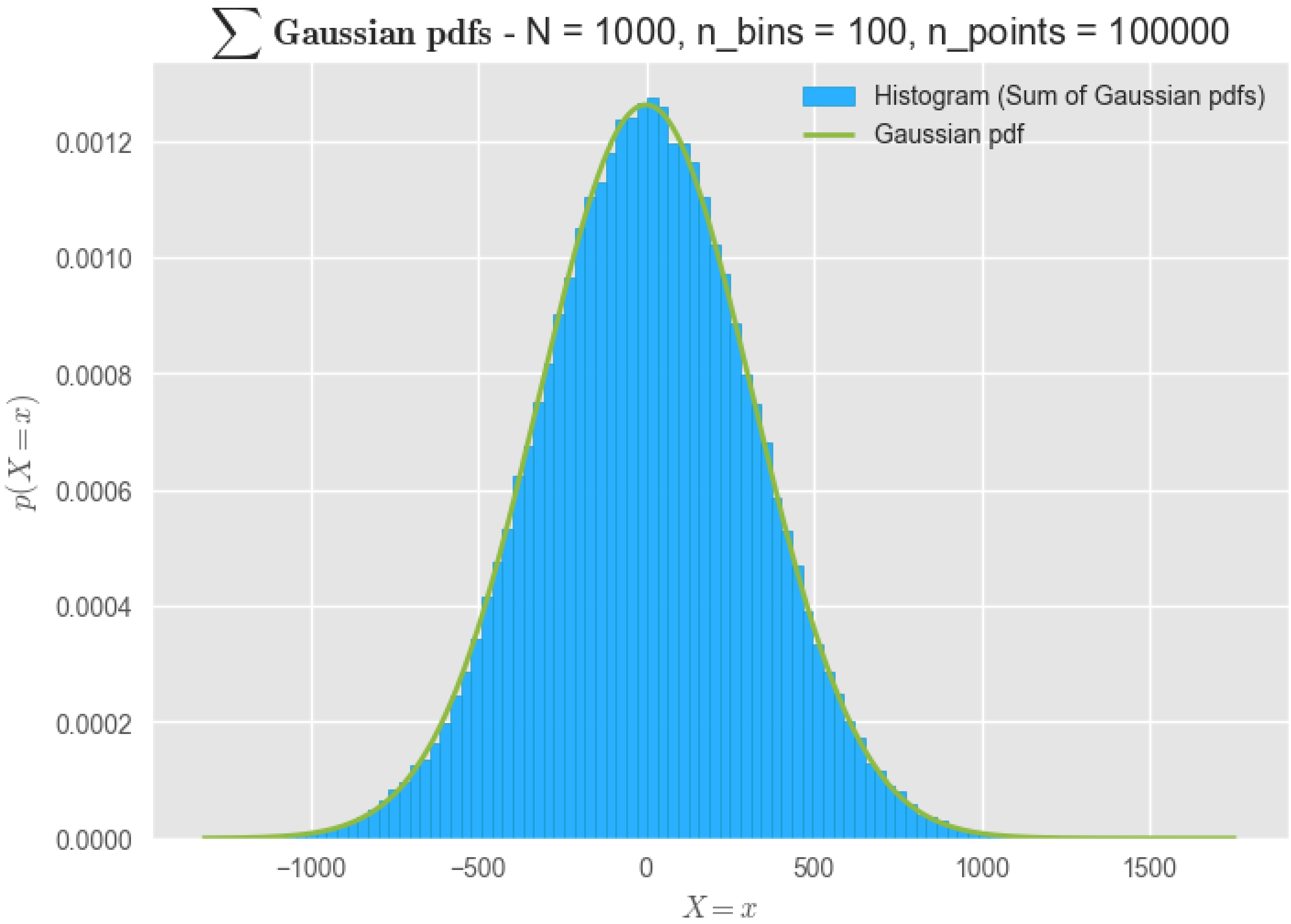
In [ ]:

```
# Definitions
N = 1_000; n_points = 100_000; sum_X_vals = []

# Initial Gaussian's parameters
mu    = 0
sigma = 10
```

```python
for n in range(n_points):
    X = np.random.normal(mu, sigma, N)
    sum_X_vals.append(sum(X))
```

In [ ]:

```python
# Creating histogram
n_bins = 100
hist, bins, _ = plt.hist(sum_X_vals, density = True, bins = n_bins, label = 'Histogram (Sum of Gaussian pdfs)',
                         facecolor = '#2ab0ff', edgecolor='#169acf', linewidth=0.5) # dimgray/maroon?

# Analytic parameters (computed by hand)
mu_X    = 0
sigma_X = np.sqrt(N) * 10

x = np.linspace(bins.min(), bins.max(), 1_000)
plt.title(r'$\sum$ $\bf{Gaussian}$ $\bf{pdfs}$ - ' + f'N = {N}, n_bins = {n_bins}, n_points = {n_points}')
plt.plot(x, stats.norm.pdf(x, mu_X, sigma_X), label = 'Gaussian pdf', color = 'C5')
plt.xlabel(r'$X = x$')
plt.ylabel(r'$p(X = x)$')
plt.legend()
plt.show()
```



Lorentzian (Cauchy) pdf:

- Here, the CLT is NOT valid!

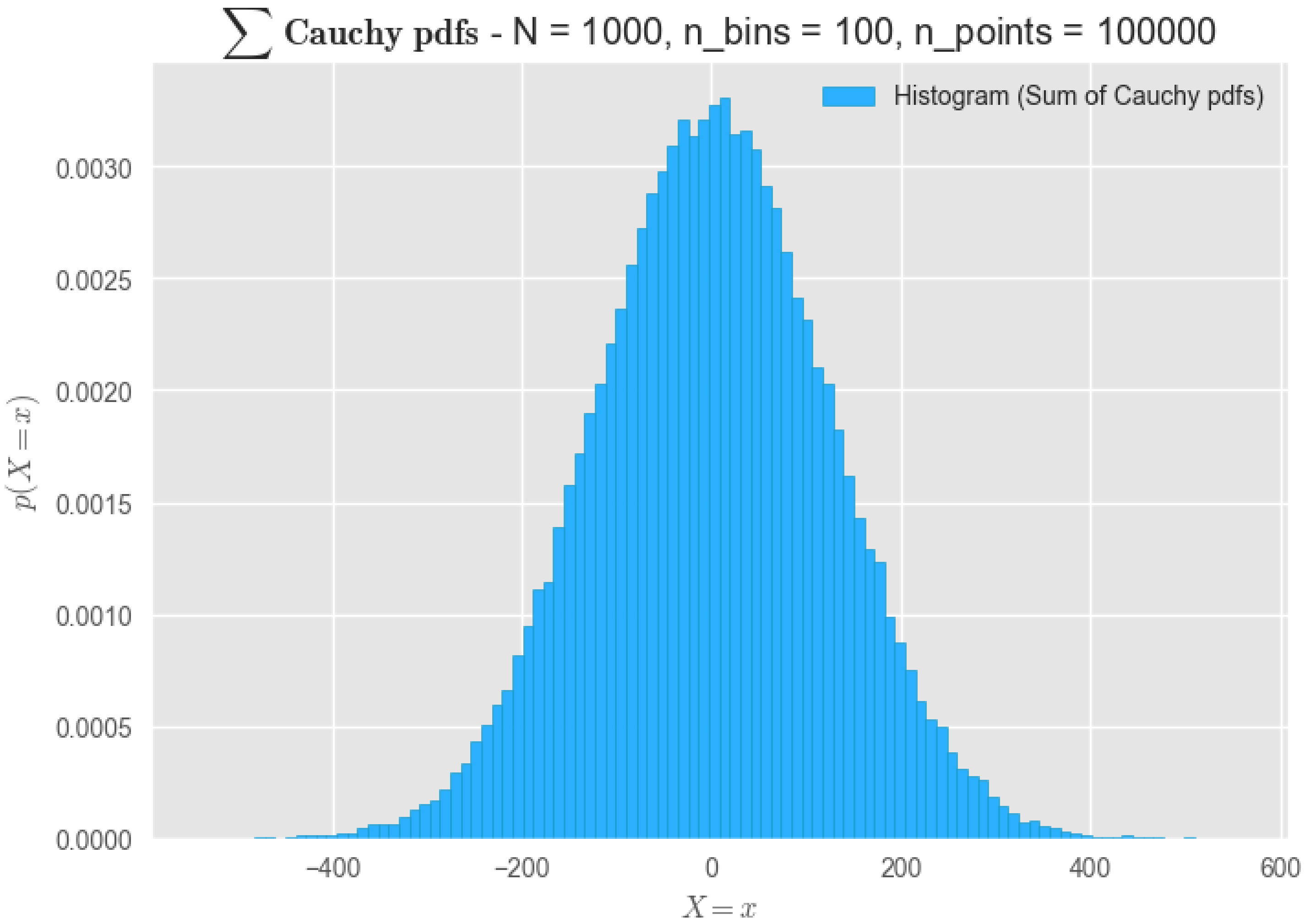
In [ ]:

```
# Definitions
N = 1_000; n_points = 100_000; sum_X_vals = []

# Standard Cauchy distribution's parameters. Not used in the code. Just here for reference/completeness.
```

```python
x0    = 0
gamma = 1

for n in range(n_points):
    X = np.random.standard_cauchy(N)
    X = X[(X>-25) & (X<25)] # Truncate distribution so it plots well
    sum_X_vals.append(sum(X))
```

In [ ]:

```python
# Creating histogram
n_bins = 100
hist, bins, _ = plt.hist(sum_X_vals, density = True, bins = n_bins, label = 'Histogram (Sum of Cauchy pdfs)',
                         facecolor = '#2ab0ff', edgecolor='#169acf', linewidth=0.5) # dimgray/maroon?

x = np.linspace(bins.min(), bins.max(), 1_000)
plt.title(r'$\sum$ $\bf{Cauchy}$ $\bf{pdfs}$ - ' + f'N = {N}, n_bins = {n_bins}, n_points = {n_points}')
plt.xlabel(r'$X = x$')
plt.ylabel(r'$p(X = x)$')
plt.legend()
plt.show()
```



Poisson distribution:

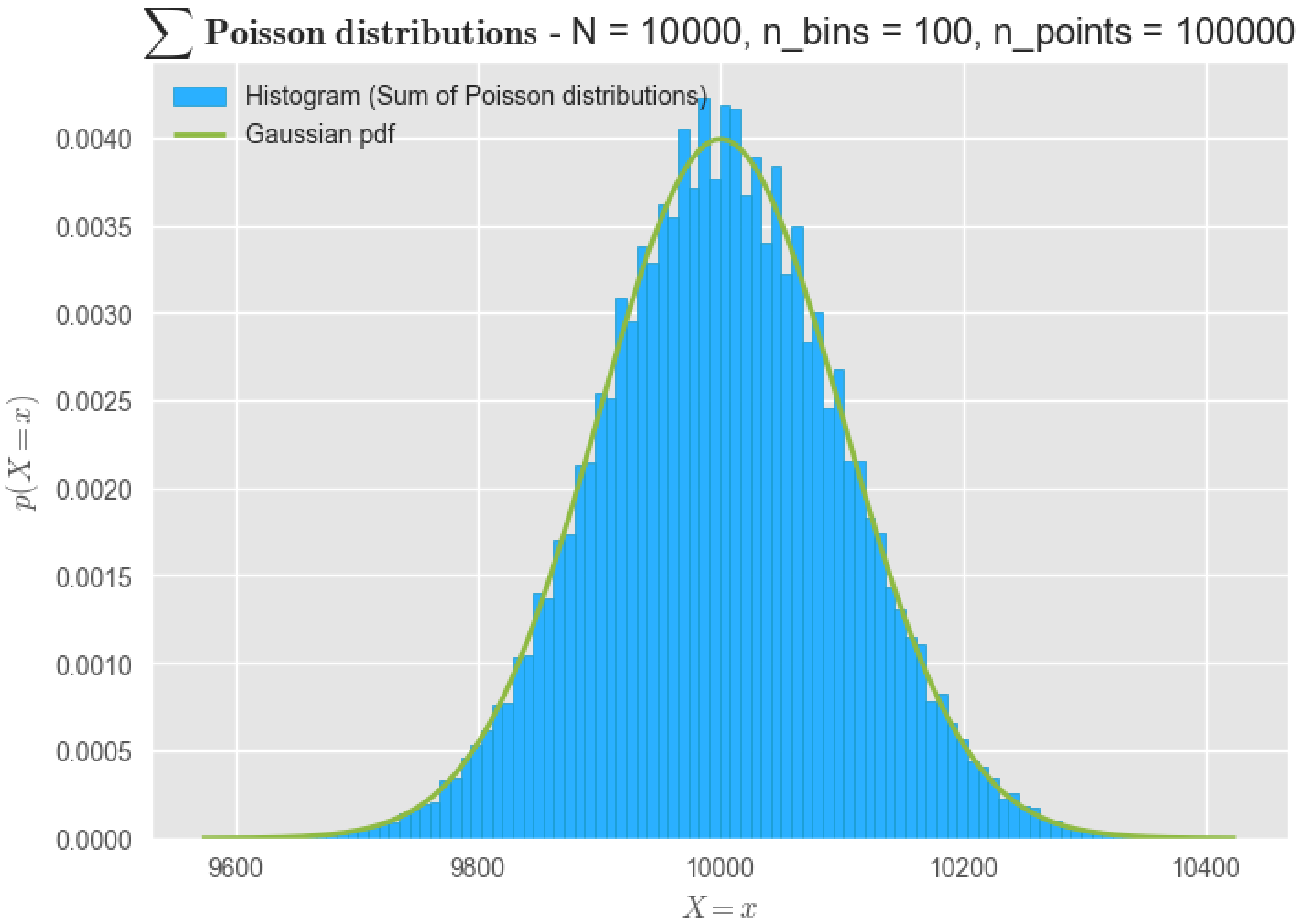
In [ ]:

```
# Definitions
N = 10_000; n_points = 100_000; sum_X_vals = []

# Poisson's lambda parameter
lambda_par = 1

for n in range(n_points):
```

```python
    X = np.random.poisson(lambda_par, N)
    sum_X_vals.append(sum(X))
```

In [ ]:

```python
# Creating histogram
n_bins = 100
hist, bins, _ = plt.hist(sum_X_vals, density = True, bins = n_bins, label = 'Histogram (Sum of Poisson distributions)',
                         facecolor = '#2ab0ff', edgecolor='#169acf', linewidth=0.5) # dimgray/maroon?

# Analytic parameters (computed by hand)
mu    = N * lambda_par
sigma = np.sqrt(N * lambda_par)

x = np.linspace(bins.min(), bins.max(), 1_000)
plt.title(r'$\sum$ $\bf{Poisson}$ $\bf{distributions}$ - ' + f'N = {N}, n_bins = {n_bins}, n_points = {n_points}')
plt.plot(x, stats.norm.pdf(x, mu, sigma), label = 'Gaussian pdf', color = 'C5')
plt.xlabel(r'$X = x$')
plt.ylabel(r'$p(X = x)$')
plt.legend()
plt.show()
```



As can be seen from the plots, the Central Limit Theorem (CLT) is verified, except for the Cauchy distribution. In fact, this distribution's mean and variance are not defined. As such, we cannot expect the CLT to hold.