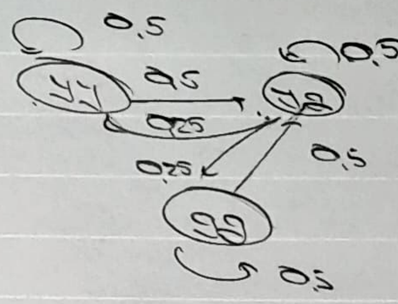
Open Ed: Week 1:

Population Inheritance:

Ordering:
- $YY = 1$
- $Yg = 2$
- $gg = 3$
- Stochastic matrix:

(a).



$$S = \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix}.$$

$YY \to Yg$ (1/2)
$\quad\to YY$ (1/2)

crossed with Yg

$Yg \to YY$ (1/4)
$\quad\to Yg$ (1/2)
$\quad\to gg$ (1/4)

$gg \to Yg$ (1/2)
$\quad\to gg$ (1/2)

This can be taken to be a Markov model since it has no memory: the next state only depends on the current state (genetic composition). The previous states are not relevant for the future evolution.

(b). $S = \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix}$ $\det(S - \lambda I) = 0 \Rightarrow$
$\Rightarrow (0.5-\lambda)^3 - \frac{1}{2}\cdot\frac{1}{4}\left(\frac{1}{2}-\lambda\right) - \frac{1}{2}\frac{1}{4}\left(\frac{1}{2}-\lambda\right) = 0 \Rightarrow$

$\Rightarrow (0.5-\lambda)\left((0.5-\lambda)^2 - 0.25\right) = 0 \Rightarrow$
$\Rightarrow \lambda = 0.5 \lor 0.5-\lambda = \pm 0.5 \Rightarrow \lambda = 0.5 \lor \lambda = 0 \lor \lambda = 1$

$\lambda_1 = 0$:

$(S - \lambda_1 I) = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \bar{s}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. $\sum_j s_j = 0$.

$\lambda_2 = 0.5$:

$(S - \lambda_2 I) = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \bar{s}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
$\sum_j s_j = 0$.

$\lambda_3 = 1$:

$(S - \lambda_3 I) = \frac{1}{4}\begin{pmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{pmatrix} \Rightarrow \bar{s}_3 = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.
normalisation.

Properties:
- $\sum_i S_{ij} = 1, \forall j$.
- $\bar{s}_3$ has $\lambda = 1$: steady state.
- $|\lambda_j| \leq 1, \forall j$.
- Other $\lambda \neq 1$, $\sum_j s_j = 0$.
(the ...).

General time evolution: eigenvector expansion:

$$p(t) = \sum_j c_j(t)\,\vec{s}_j = W^t \sum_j c_j(0)\,\vec{s}_j = \sum_j \left(c_j(0)\,\lambda_j^t\right)\vec{s}_j$$

$$\Rightarrow c_j(t) = \underbrace{c_j(0)}_{\text{initial condition}}\,\lambda_j^t = c_j(0)\cdot e^{t\ln\lambda_j} = e^{-t/\tau_j}$$

relaxation time ... $\tau_j = -\frac{1}{\ln\lambda_j} > 0$. $\tau \to 0$, if $|\lambda_j| < 1 \Rightarrow$ no decay. steady state

$c_1(0)\,e^{-t}\,\bar{s}_1 +$

$$p(t) = \underbrace{c_1(0)}_{e^{-t}}\,\bar{s}_1 + c_2(0)\,\bar{s}_2 + c_3(0)\,\bar{s}_3 \xRightarrow{t\to\infty} p(t) \to c_3(0)\,\bar{s}_3 = \bar{s}_3$$ at eigen.

e.g. if $p(0)$: $\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} = c_2(0)\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_3(0)\begin{pmatrix}0.25\\0.5\\0.25\end{pmatrix} + c_1(0)\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$\vec{c} = (c_1(0), c_2(0), c_3(0)) = \frac{1}{4}(1, 2, 4)$ [illegible]

6. Long-term limit = steady state: $\bar{s}_3 = \frac{1}{4}\begin{pmatrix}1\\2\\1\end{pmatrix}$

$$p_i(t+1) - p_i(t) = \sum_{j\neq i}\left(W_{ij}\,p_j - W_{ji}\,p_i\right)$$

Detailed balance: gain = loss

$W_{ij}\,p_j = W_{ji}\,p_i \;\forall i,j$ (from the steady-st. vector)

$$\begin{cases} W_{12}\,p_2 = W_{21}\,p_1 \\ W_{13}\,p_3 = W_{31}\,p_1 \\ W_{23}\,p_3 = W_{32}\,p_2 \end{cases} \Rightarrow \begin{cases} 0.25\,(0.5) = 0.5\,(0.25) \\ 0\,(0.25) = 0\,(0.25) \\ 0.5\,(0.25) = 0.25\,(0.5) \end{cases} \Rightarrow \begin{cases} 0.125 = 0.125 \\ 0 = 0 \\ 0.125 = 0.125 \end{cases}$$

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3\cdot\cancel{2}}{\cancel{2}} = 3.$$

detailed balance: checked for the steady-state.

If the experiment is repeated many times, we expect to have a ratio of $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ respectively for dominant, hybrid and recessive.