Breathing modes:

Collisionless Boltzmann equation:

$$\left(\frac{\partial}{\partial t} + \frac{\vec p}{m}\cdot\frac{\partial}{\partial \vec r} + \vec F(\vec r)\cdot\frac{\partial}{\partial \vec p}\right) f(t,\vec r,\vec p) = 0$$

collision integral is equal to 0 (collisionless).

Ex. $U(\vec r) = \frac{m\omega^2}{2} r^2$.

In equilibrium: $\frac{\partial f}{\partial t}=0$. Such that the Boltzmann equation becomes: (Also. $\vec F(\vec r) = -\nabla U(\vec r) = -m\omega^2 \vec r$.)

$$\frac{\vec p}{m}\cdot\nabla f - m\omega^2 \vec r\cdot\nabla_p f = 0.$$

We'll solve this through separation of variables:

$$f(\vec r,\vec p) = g(\vec r)\cdot j(\vec p).$$

$$\Rightarrow \frac{\vec p}{m}\, j(\vec p)\cdot\nabla g(\vec r) - m\omega^2\vec r\cdot\nabla_p j(\vec p)\, g(\vec r) = 0 \Rightarrow$$

$$\Rightarrow \frac{\vec p}{m}\cdot\frac{\nabla g(\vec r)}{g(\vec r)} = m\omega^2 \vec r\cdot\frac{\nabla_p j(\vec p)}{j(\vec p)} \Rightarrow$$

$$\Rightarrow \underbrace{\frac{1}{\vec r}\cdot\frac{\nabla g(\vec r)}{g(\vec r)}}_{\text{function of } \vec r} = \underbrace{\frac{m^2\omega^2}{\vec p}\cdot\frac{\nabla_p j(\vec p)}{j(\vec p)}}_{\text{function of } \vec p} = \text{const} = -\lambda$$

For LHS:

$$\int d\vec r\, \frac{\nabla g(\vec r)}{g(\vec r)} = -\int d\vec r\, \vec r\lambda \Rightarrow \ln g(\vec r) = -\frac{\lambda r^2}{2} + b \Rightarrow \boxed{g(\vec r) = b'\cdot e^{-\frac{\lambda r^2}{2}}}.$$

For RHS:

$$\int d\vec p\, \frac{\nabla_p j(\vec p)}{j(\vec p)} = -\frac{\lambda}{m^2\omega^2}\int d\vec p\, \vec p \Rightarrow \boxed{j(\vec p) = b''\cdot e^{-\frac{\lambda p^2}{2m^2\omega^2}}}.$$

$$\Rightarrow f(\vec r,\vec p) = b\cdot e^{-\frac{\lambda r^2}{2}}\cdot e^{-\frac{\lambda p^2}{2m^2\omega^2}}.$$

The constants λ and b are fixed by the following relations.

1. $\int d\vec p\int d\vec r\, f(\vec r,\vec p) = N \rightarrow$ total number of particles (constant).
2. $\langle \frac{p^2}{2m}\rangle = \frac{3}{2}Nk_BT \rightarrow$ from Equipartition theorem.

So, starting with 1:

$$N = b\cdot\int d^3r\, e^{-\frac{\lambda r^2}{2}}\int d^3p\, e^{-\frac{\lambda p^2}{2m^2\omega^2}} =$$

$$= 16\pi^2 b\int_0^{+\infty} dr\, e^{-\frac{\lambda r^2}{2}}\cdot r^2\cdot\int_0^{+\infty} dp\, e^{-\frac{\lambda p^2}{2m^2\omega^2}} p^2 =$$

$$= \cancel{16}\pi^2 b\,\frac{\sqrt{\pi}\, 2\sqrt2}{\cancel{4}\lambda^{3/2}}\,\frac{\sqrt{\pi}\, 2\sqrt2}{\cancel{4}\lambda^{3/2}}\cdot m^3\omega^3 = b\cdot\frac{8\pi^3 m^3\omega^3}{\lambda^3} = b\left(\frac{2\pi m\omega}{\lambda}\right)^3$$

$$\Rightarrow \boxed{b = \left(\frac{\lambda}{2\pi m\omega}\right)^3 N}$$

And from 2.:

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{3}{2} N k_B T \Rightarrow$$

$$\Rightarrow \frac{b}{\cancel{2}m} \cdot 16\pi^2 \int_0^{+\infty} dr\, e^{-\frac{\lambda r^2}{2}} r^2 \int_0^{+\infty} dp\, e^{-\frac{\lambda p^2}{2m^2\omega^2}} p^4 = \frac{3}{\cancel{2}} N k_B T \Rightarrow$$

$$\Rightarrow b \cdot 16\pi^2 \cdot \frac{\sqrt{\pi}\cdot 2\sqrt{2}}{\cancel{4}\lambda^{3/2}} \cdot \frac{3\sqrt{\pi}}{8\lambda^{5/2}} \cdot \cancel{4}\sqrt{2} \cdot \frac{m^5\omega^5}{m} = 3 N k_B T \Rightarrow$$

$$\Rightarrow 8b\pi^3 \frac{m^4\omega^5}{\lambda^4} = N k_B T \Rightarrow \boxed{b = \frac{N k_B T \lambda^4}{8\pi^3 m^4 \omega^5}}$$

This implies that:

$$\frac{\cancel{\lambda^3} \cancel{N}}{\cancel{8\pi^3 m^3} \cancel{\omega^3}} = \frac{\cancel{N} k_B T \cancel{\lambda^4}^{\lambda}}{\cancel{8\pi^3 m^4}^{m} \omega^{\cancel{5}2}} \Rightarrow 1 = \frac{k_B T \lambda}{m\omega^2} \Rightarrow \boxed{\lambda = \frac{m\omega^2}{k_B T}}$$

↳ this gives the right exponents!

$$f(\vec r,\vec p) = b\, e^{-\frac{\lambda r^2}{2}} e^{-\frac{\lambda p^2}{2m^2\omega^2}}$$

$$= b\, e^{-\frac{m\omega^2 r^2}{2k_BT}} e^{-\frac{p^2}{2mk_BT}}$$

finally: $b = \left(\frac{\lambda}{2\pi m\omega}\right)^3 N = \boxed{\left(\frac{\omega}{2\pi k_B T}\right)^3 N = b}$

The density of the gas in configuration space is:

$$n(\vec r) = \int d^3p\, f(\vec r,\vec p)$$

$$= b\, e^{-\frac{\lambda r^2}{2}} \cdot \frac{\cancel{4}\pi\sqrt{\pi}}{\cancel{4}\lambda^{3/2}} 2\sqrt{2}\, m^3\omega^3 = b\, e^{-\frac{\lambda r^2}{2}} \left(\frac{2\pi m^2\omega^2}{\lambda}\right)^{3/2} =$$

using $\boxed{\lambda = \frac{m\omega^2}{k_BT}}$: $b(2\pi m k_B T)^{3/2} e^{-\frac{\lambda r^2}{2}}$

At $r=0$: $n(\vec 0) = b(2\pi m k_B T)^{3/2} \Rightarrow$

$$\Rightarrow \boxed{b = n(\vec 0)\left(\frac{1}{2\pi m k_B T}\right)^{3/2}}$$

Thus, $n(\vec 0)$ represents exactly the density of the gas at the trap's centre!

Such that:

$$\Rightarrow f(\vec p,\vec r) = b\, e^{-\frac{\lambda r^2}{2}} e^{-\frac{\lambda p^2}{2m^2\omega^2}}$$

$$= n(\vec 0)\left(\frac{1}{2\pi m k_BT}\right)^{3/2} \exp\left(-\frac{p^2}{2mk_BT} - \frac{m\omega^2 r^2}{2k_BT}\right).$$

↳ This we wanted to show.

Additionally, if we want to express $n(\vec 0)$ in terms of the total particle number N, we have:

$$n(\vec 0) = b(2\pi m k_BT)^{3/2} = \left[\frac{\omega^2}{(2\pi k_BT)^2}\right]^{3/2} N (2\pi m k_BT)^{3/2}$$

$$\boxed{= \left(\frac{m\omega^2}{2\pi k_BT}\right)^{3/2} N = n(\vec 0)}$$

Done.

b). $\langle g(\vec r,\vec p)\rangle = \int d^3r \int d^3p \; f(t,\vec r,\vec p)\, g(\vec r,\vec p) \Rightarrow$

$$\Rightarrow \frac{d}{dt}\langle g(\vec r,\vec p)\rangle = \int d^3r\int d^3p \left(\frac{\vec p}{m}\cdot\frac{\partial f}{\partial \vec r} + \vec F(\vec r)\cdot\frac{\partial f}{\partial \vec p} + \frac{\partial f}{\partial t}\right) g(\vec r,\vec p) +$$

$$+ \underbrace{\int d^3r\int d^3p\; f(t,\vec r,\vec p)\left(\frac{\partial g}{\partial \vec r}\cdot\frac{\vec p}{m} + \frac{\partial g}{\partial \vec p}\cdot \vec F(\vec r)\right)}_{\left\langle \frac{\vec p}{m}\cdot\frac{\partial g(\vec r,\vec p)}{\partial \vec r}\right\rangle + \left\langle \vec F(\vec r)\cdot \frac{\partial g(\vec r,\vec p)}{\partial \vec p}\right\rangle}$$

Now, just need to show that: (which is pretty straightforward):

$$\int d^3r\int d^3p \underbrace{\left(\frac{\vec p}{m}\cdot\frac{\partial f}{\partial\vec r} + \vec F(\vec r)\cdot\frac{\partial f}{\partial\vec p} + \frac{\partial f}{\partial t}\right)}_{=0 \text{ from the collisionless Boltzmann equation!}} \cdot g(\vec r,\vec p) = 0.$$

Finally, we have:

$$\frac{d}{dt}\langle g(\vec r,\vec p)\rangle = \left\langle \frac{\vec p}{m}\cdot \frac{\partial g(\vec r,\vec p)}{\partial\vec r}\right\rangle + \left\langle \vec F(\vec r)\cdot\frac{\partial g(\vec r,\vec p)}{\partial \vec p}\right\rangle,$$

just like we wanted to show!

c). $E_{pot} = \frac{m\omega^2}{2} r^2$;

$\frac{d}{dt}\langle E_{pot}(\vec r)\rangle = \left\langle \frac{\vec p}{m}\cdot\frac{m\omega^2}{2} 2\vec r\right\rangle = \omega^2\langle \vec r\cdot\vec p\rangle \Rightarrow$ using the definition of $E_{pot} = \frac{m\omega^2}{2} r^2$.

$\Rightarrow \frac{d}{dt}\langle r^2\rangle = \frac{2}{m}\langle \vec r\cdot\vec p\rangle$

$E_{kin} = \frac{p^2}{2m}$;

$\frac{d}{dt}\left\langle \frac{p^2}{2m}\right\rangle = -\left\langle m\omega^2 \vec r\cdot\frac{\vec p}{m}\right\rangle \Rightarrow \frac{d}{dt}\langle p^2\rangle = -2m\omega^2\langle\vec r\cdot\vec p\rangle.$

$E_{corr} = \frac{\omega}{2}\vec r\cdot\vec p$;

$\frac{d}{dt}\left\langle \cancel{\frac{\omega}{2}}\vec r\cdot\vec p\right\rangle = \left\langle \frac{\vec p}{m}\cdot\cancel{\frac{\omega}{2}}\vec p\right\rangle - \left\langle m\omega^2\, \vec r\cdot\vec r\,\cancel{\frac{\omega}{2}}\right\rangle \Rightarrow$

$\Rightarrow \frac{d}{dt}\langle\vec r\cdot\vec p\rangle = \frac{1}{m}\langle p^2\rangle - m\omega^2\langle r^2\rangle.$

(Full) closed system of equations:

$$\begin{cases} \frac{d}{dt}\left(\langle \vec r\cdot\vec p\rangle\right) = \frac{1}{m}\langle p^2\rangle - m\omega^2\langle r^2\rangle & (1) \\ \frac{d}{dt}\left(\langle p^2\rangle\right) = -2m\omega^2\langle \vec r\cdot\vec p\rangle & (2) \\ \frac{d}{dt}\left(\langle r^2\rangle\right) = \frac{2}{m}\langle \vec r\cdot\vec p\rangle & (3) \end{cases}$$

(d). Differentiating (1) w.r.t. time and using (2) and (3), we get:

$$\frac{d^2}{dt^2}\left(\langle \vec{r}\cdot\vec{p}\rangle\right) = \frac{1}{m}\frac{d}{dt}\langle p^2\rangle - m\omega^2\frac{d}{dt}\langle r^2\rangle \Rightarrow$$

· using (2) and (3)

$$\Rightarrow \frac{d^2}{dt^2}\langle \vec{r}\cdot\vec{p}\rangle = \frac{1}{\cancel{m}}\left(-2\cancel{m}\omega^2\langle \vec{r}\cdot\vec{p}\rangle\right) - \cancel{m}\omega^2\left(\frac{2}{\cancel{m}}\langle \vec{r}\cdot\vec{p}\rangle\right) \Rightarrow$$

$$\Rightarrow \frac{d^2}{dt^2}\langle \vec{r}\cdot\vec{p}\rangle = -2\omega^2\langle \vec{r}\cdot\vec{p}\rangle - 2\omega^2\langle \vec{r}\cdot\vec{p}\rangle \Rightarrow$$

$$\Rightarrow \frac{d^2}{dt^2}\langle \vec{r}\cdot\vec{p}\rangle = \underline{-4\omega^2}\langle \vec{r}\cdot\vec{p}\rangle$$

↳ Oscillations with frequency: $\Omega = \sqrt{4\omega^2} = 2\omega$.

· Collisionless Boltzmann equation ⇒ no way to relax back to equilibrium (⇔ no loss/dissipative term).

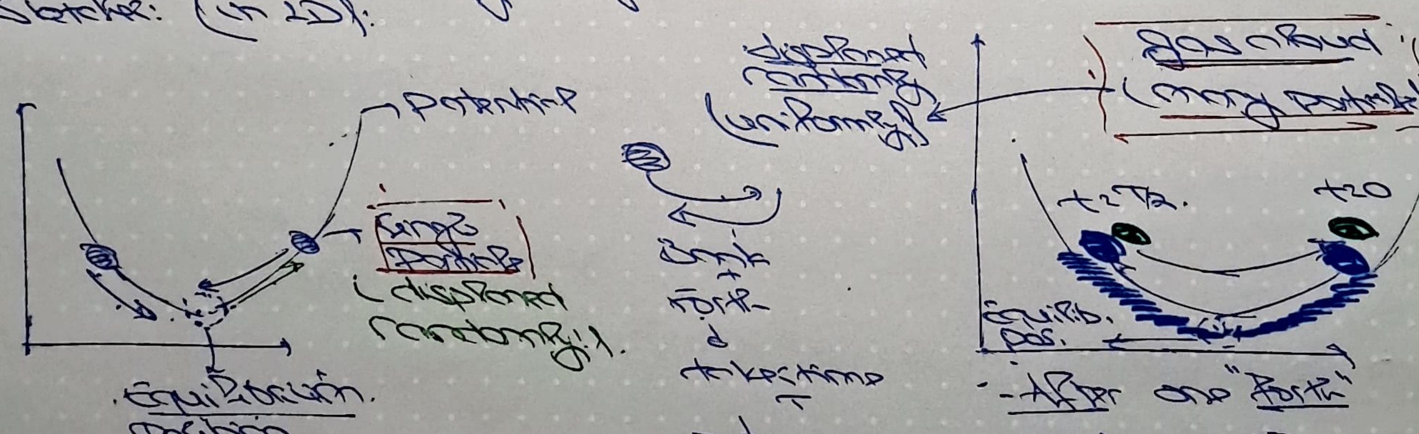
↳ particle(s) will just follow # streamline(s) in phase-space.

· The frequency is twice the trap's frequency because of the "collective motion" of the whole gas cloud, as it expands and contracts.

↳ This expansion and contraction of the whole gas cloud (as it moves) is exactly what gives the name "breathing mode" to this phenomena, since it's similar to a breathing motion.

· In order to illustrate why $\Omega = 2\omega$, consider the following rough sketch (in 1D):

· Inhale: expand. Exhale: contract.



· will move back and forth with frequency: ω.

— · —

· After one "forth" we're back to the initial position (since the particles are indistinguishable) ⇒ $\Omega = 2\omega$.

- Bogah