Active swimmer

(a). Langevin equation

$dx = s\,dt + \sqrt{D}\cdot dW \Rightarrow$

$\Rightarrow \frac{dx}{dt} = \underset{\text{constant velocity}}{s} + \underset{\text{additive noise}}{\sqrt{D}\cdot \xi_t}$

Wiener process: mean $=0$, $\sigma^2 = dt$.

white noise: $\langle \xi_t \rangle = 0$, $\sigma^2 = \frac{1}{dt}$. $\langle \xi(t_1)\,\xi(t_2)\rangle = \delta(t_1 - t_2)$.

This Langevin equation describes an active swimmer that moves with velocity s (deterministic term), with random pushes from the surrounding fluid particles.

(b). $x' = x - st \overset{s \text{ is constant!}}{\Rightarrow} dx' = dx - s\,dt \Rightarrow$

$\Rightarrow dx = dx' + s\,dt.$

- Substituting in the previously obtained Langevin eq.:

$(dx' + \cancel{s\,dt}) = \cancel{s\,dt} + \sqrt{D}\cdot dW \Rightarrow$

$\Rightarrow dx' = \sqrt{D}\,dW.$

This is precisely the diffusion equation for x'.

"Forward Fokker-Planck" equation (general form):

$\partial_t P_t(x) = -\frac{\partial}{\partial x}\left(A(x,t)\,P_t(x)\right) + \frac{\partial^2}{\partial x^2}\left(\frac{B(x,t)}{2}\cdot P_t(x)\right).$

In this case, $A(x,t) = 0$, $B(x,t) = D$.

$\Rightarrow \boxed{\frac{\partial P_t(x')}{\partial t} = \frac{D}{2}\frac{\partial^2}{\partial x'^2}\left(P_t(x')\right)} \rightarrow$ Diffusion equation.

$\Rightarrow P_t(x') = \frac{1}{\sqrt{2\pi\cdot Dt}}\cdot e^{-\frac{x'^2}{2Dt}} \Rightarrow$

$\Rightarrow P_t(x) = \frac{1}{\sqrt{2\pi Dt}}\cdot e^{-\frac{(x-st)^2}{2Dt}} \xrightarrow{t\to 0} \delta(x).$

(Initial condition satisfied)

to see this explicitly, use:

$$\lim_{D\to 0} \frac{1}{\sqrt{2\pi D}} \cdot e^{-\frac{x^2}{2D}} = \delta(x).$$

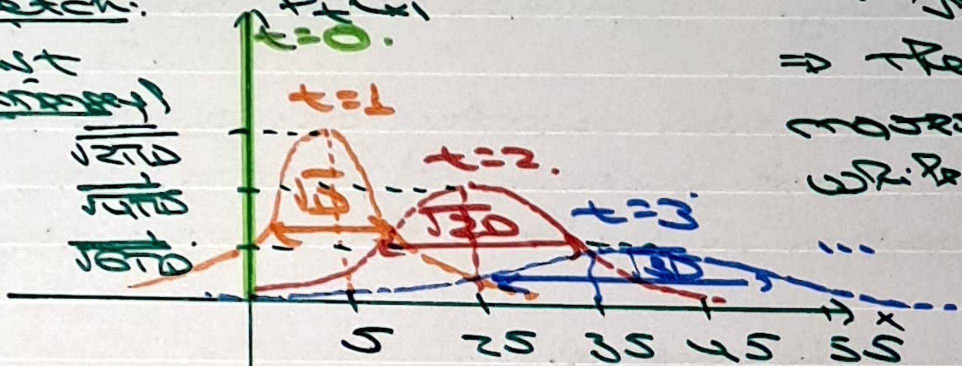
Also: $\delta(x-st)|_{t=0} = \delta(x)$.

$$P_t(x) = \frac{1}{\sqrt{2\pi Dt}} \cdot e^{-\frac{(x-st)^2}{2Dt}} \rightarrow$$

· Gaussian with:
· $\mu = st$;
· $\sigma^2 = Dt$.

⇒ the distribution moves to the right, while flattening out. (as t increases).

Sketch: (as t increases)



$\frac{\sigma_t}{\mu_t} = \frac{\sqrt{Dt}}{st} \sim \frac{1}{\sqrt{t}}$ "self-averaging" for $t\to\infty$

Diffusion + movement with velocity s.

(c). Rewriting the Fokker-Planck equation:

$$\partial_t P_t(x) = -\partial_x \underbrace{\left(s P_t(x) - \frac{D}{2}\partial_x P_t(x)\right)}_{J}.$$

$$\partial_t P_t(x) = -\partial_x J$$

$J \rightarrow$ probability current density.

· If we now require that J vanishes, we get:

$$J = 0 \Rightarrow s P^{st}_t(x) - \frac{D}{2}\partial_x P^{st}_t(x) = 0 \Rightarrow$$
$$\Rightarrow \partial_x P^{st}_t(x) = \frac{2s}{D} P^{st}_t(x) \Rightarrow$$
$$\Rightarrow P^{st}_t(x) = P_0\, e^{\frac{2sx}{D}}.$$

stationary case: doesn't depend on t.

· But this expression explodes for $x\to\infty$, so it cannot be normalised! If we allow x to go to infinity, we will never achieve a stationary distr., since the swimmer will, on average, just keep swimming to the right, indefinitely.

If we really want to have a stationary state, we can introduce another wall at $\underline{x=a}$.

Then:

normalization $\Rightarrow$ $f_0 = \dfrac{1}{\int_0^a e^{\frac{2sx}{D}}\,dx} = \dfrac{2s}{D}\cdot\dfrac{1}{e^{\frac{2sa}{D}}-1}$.

And now, the average position of the swimmer will be:

$$\int_0^a x\cdot f_0\cdot e^{\frac{2sx}{D}}\,dx =$$

$$= f_0\left(x-\frac{D}{2s}\right)\cdot\frac{D}{2s}\,e^{\frac{2sx}{D}}\Bigg|_{x=0}^{x=a} =$$

$$= f_0\left[\left(a-\frac{D}{2s}\right)\frac{D}{2s}\,e^{\frac{2sa}{D}} + \left(\frac{D}{2s}\right)^2\right] =$$

D	I
$+x$	$e^{\frac{2sx}{D}}$
-1	$\frac{D}{2s}e^{\frac{2sx}{D}}$
$+0$	$\left(\frac{D}{2s}\right)^2 e^{\frac{2sx}{D}}$

$$= \cancel{\frac{2s}{D}}\cdot\frac{1}{e^{\frac{2sa}{D}}-1}\cdot\cancel{\frac{D}{2s}}\left[a\,e^{\frac{2sa}{D}} - \frac{D}{2s}\,e^{\frac{2sa}{D}} + \frac{D}{2s}\right] =$$

$$= -\frac{D}{2s}\cdot\frac{1}{e^{\frac{2sa}{D}}-1}\cdot\left[e^{\frac{2sa}{D}} - 1 - \frac{a\,2s}{D}\,e^{\frac{2sa}{D}}\right] =$$

$$= -\frac{D}{2s}\left[\frac{e^{\frac{2sa}{D}}-1}{e^{\frac{2sa}{D}}-1} - \frac{2as}{D}\cdot\frac{e^{\frac{2sa}{D}}}{e^{\frac{2sa}{D}}-1}\right] =$$

(good, that's correct. $a\to\infty$.)

$$= -\frac{D}{2s} + \cancel{\frac{2as}{D}}\cdot\frac{1}{1-e^{-\frac{2sa}{D}}} = \frac{a}{1-e^{-\frac{2sa}{D}}} - \frac{D}{2s} \to +\infty$$

Final result:

$$\langle x\rangle = \frac{a}{1-e^{-\frac{2sa}{D}}} - \frac{D}{2s}.$$

· If we increase s, $\langle x\rangle$ also strictly increases, as expected.

· On the other hand, an increase in D leads to the strict decrease of $\langle x\rangle$, which intuitively makes sense, since we allow for greater deviations from the constant velocity case.