Random walk with pauses: step = right, left, step = still.

- prob. of a particular path: $p^{n_R} \cdot q^{n_L} \cdot s^{n_S}$, $s = 1 - p - q$.
- the combination: $\dfrac{N!}{n_R!\, n_L!\, n_S!} = \dbinom{N}{n_R, n_L, n_S}$.

So, $P_N(n_R, n_L) = \dbinom{N}{n_R, n_L, n_S} p^{n_R} q^{n_L} s^{n_S}$.

Additionally, we know that: $\begin{cases} n_R - n_L = m \\ n_R + n_L + n_S = N \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} n_R = \dfrac{N+m-n_S}{2} \\ n_L = \dfrac{N-m-n_S}{2} \end{cases} \Rightarrow P_N(m, n_S) = \dbinom{N}{\frac{N+m-n_S}{2}, \frac{N-m-n_S}{2}, n_S} \cdot p^{\frac{N+m-n_S}{2}} \cdot q^{\frac{N-m-n_S}{2}} \cdot s^{n_S}.$$

· $P_N(m, n_s)$ – probability of ending up in position m, having stood still for n_s steps.

$$\langle m\rangle = \sum_{m=-N}^{N} m P_N(m, n_s).$$

Aux: Multinomial theorem:

$$\sum_{\substack{n_R+n_L+n_S=N \\ n_R,n_L,n_S\geq 0}} \binom{N}{n_R, n_L, n_S} p^{n_R} q^{n_L} s^{n_S} = (p+q+s)^N.$$

$$\langle m\rangle = \langle n_R - n_L\rangle = \langle n_R\rangle - \langle n_L\rangle.$$

$$= u\frac{d}{du}u^{n_R}\Big|_{u=1}.$$

$$\langle n_R\rangle = \sum_{\substack{n_R+n_L+n_S=N \\ n_R,n_L,n_S\geq 0}} \boxed{n_R} \binom{N}{n_R, n_L, n_S} p^{n_R} q^{n_L} s^{n_S} =$$

$$= u\frac{d}{du}\left[\sum \binom{N}{n_R, n_L, n_S} (pu)^{n_R} q^{n_L} s^{n_S}\right]_{u=1}$$

$$= u\frac{d}{du}(pu+q+s)^N\Big|_{u=1} = uN(pu+q+s)^{N-1}\cdot p\Big|_{u=1}$$

$$= 1\cdot N\underbrace{(p+q+s)}_{=1}^{N-1}\cdot p = Np.$$

Similarly: $\langle n_L\rangle = Nq$.

Then finally: $\langle m\rangle = \langle n_R\rangle - \langle n_L\rangle = Np - Nq = N(p-q)$.

$\sigma_m^2 = \langle m^2\rangle - \langle m\rangle^2$.

$\langle m\rangle^2 = N^2(p-q)^2$.

$\langle m^2\rangle = \langle n_R^2\rangle - 2\langle n_R n_L\rangle + \langle n_L^2\rangle$.

$$u\frac{d}{du}\left(u\frac{d}{du}u^{n_R}\right)\Big|_{u=1} = n_R^2.$$

$$\langle n_R^2\rangle = \sum \boxed{n_R^2} P_N(n_R, n_L)^2$$

$$= u\frac{d}{du}\left(u\frac{d}{du}\left[\sum \binom{N}{n_R, n_L, n_S}(pu)^{n_R}\cdot q^{n_L}\cdot s^{n_S}\right]\right)_{u=1}$$

$$= u\frac{d}{du}\left(u\frac{d}{du}\left[(pu+q+s)^N\right]\right)_{u=1}$$

$$= u\frac{d}{du}\left(u\cdot N(pu+q+s)^{N-1}\cdot p\right)_{u=1} = Npu\frac{d}{du}\left(u(pu+q+s)^{N-1}\right)_{u=1}$$

$$= Npu\left(\overbrace{(pu+q+s)^{N-1}}^{=1\,(u=1)} + u(N-1)\overbrace{(pu+q+s)^{N-2}}^{=1\,(u=1)}\cdot p\right)_{u=1}$$

$= Np(1+(N-1)p).$

- Similarly: $\langle n_L^2\rangle = Nq(1+(N-1)q)$. ✓✓✓.

- Now we're still missing $\langle n_R n_L\rangle$:

$\langle n_R n_L\rangle = \sum \boxed{n_R}\boxed{n_L} P_N(n_R, n_L).$

- $n_R = u\frac{d}{du}(\ldots)|_{u=1}$

- $n_L = s\frac{d}{ds}(\ldots)|_{s=1}$

$$= \left(u\frac{d}{du}\right)\left(s\frac{d}{ds}\right)\left(\sum \binom{N}{n_R, n_L, n_s} (pu)^{n_R}(qs)^{n_L} s^{n_s}\right)\Big|_{u,s=1} =$$

$$= \left(u\frac{d}{du}\right)\left(s\frac{d}{ds}\right)(pu+qs+s)^N\Big|_{u,s=1} =$$

$$= \left(u\frac{d}{du}\right) s\,N(pu+qs+s)^{N-1}.q\,\Big|_{u,s=1} =$$

$$= us(N-1)N(pu+qs+s)^{N-2}pq\,\Big|_{u,s=1} =$$

$$= Npq(N-1).$$

- Finally: $\sigma^2 = \langle m^2\rangle - \langle m\rangle^2 =$

$$= \langle n_R^2\rangle - 2\langle n_R n_L\rangle + \langle n_L^2\rangle - N^2(p-q)^2 =$$

$$= Np(1+(N-1)p) - 2Npq(N-1) + Nq(1+(N-1)q) - N^2(p-q)^2 =$$

$$= N\left[p + (N-1)p^2 - 2pq(N-1) + q + (N-1)q^2 - N(p-q)^2\right] =$$

$$= N\left[(N-1)(\underline{p^2-2pq+q^2}) + p + q - N(p-q)^2\right].$$

$$= N\left[(\cancel{N}-1)(p-q)^2 + p + q - \cancel{N(p-q)^2}\right] =$$

$$= N\left[p+q-(p-q)^2\right].$$

- **Note**: this gives the correct results, but I don't agree with the solution too much. We're asked to compute the distribution for the ~~position~~ ~~m~~ after N steps, regardless of n_s! In my opinion, the distribution asked should be:

$$\sum_{\substack{n_R+n_L+n_s=N \\ n_R-n_L=m \\ n_R, n_L, n_s \geq 0}} \binom{N}{n_R, n_L, n_s} p^{n_R} q^{n_L} s^{n_s}.$$

- Or, in terms of m and N:

$$P_N(m) = \sum_{\substack{n_s=0 \\ n_R, n_L, n_s \geq 0}}^{N} \binom{N}{\underbrace{\tfrac{N+m-n_s}{2}}_{=n_R}, \underbrace{\tfrac{N-m-n_s}{2}}_{=n_L}, n_s} p^{\frac{N+m-n_s}{2}} \cdot q^{\frac{N-m-n_s}{2}} \cdot s^{n_s}$$

· Intuitively: ($N=20$, e.g.)

$\xrightarrow{5 \text{ steps}}$ + $\xleftarrow{5 \text{ steps}}$ + 10 steps still $\Rightarrow m=0$, but $\xrightarrow{10 \text{ steps}}$ + $\xleftarrow{10 \text{ steps}}$ $\Rightarrow m=0$, too.

· The expression $P_N(n_R, n_L) = \binom{N}{n_R, n_L, n_S} p^{n_R} q^{n_L} s^{n_S}$ assumes a certain number of still steps, n_S, to get to m. But, there's other ways to get to m, which are left unaccounted for...

However, if we try to use the other expression, we must simplify it further (the sum...).

$$\sum_{\substack{n_R+n_L+n_S=N \\ n_R-n_L=m \\ n_R, n_L, n_S \geq 0}} \binom{N}{n_R, n_L, n_S} p^{n_R} q^{n_L} s^{n_S}.$$