Report - Assignment # 2:

· Bottom node:

want to compute: $\dfrac{J_{g,1} - J_g(0)}{\Delta z}$

· $J_{g,1} = -\dfrac{D_{g,1} \times D_{g,2}}{\Delta z/2} \cdot \dfrac{\hat{\phi}_{g,2} - \hat{\phi}_{g,1}}{D_{g,1} + D_{g,2}}$ from equation (A14).

· But $J_g(0)$ requires special attention (since there's no node below this):

(Assume Marshak-boundary-conditions):

$$\begin{cases} J_g(0) = -D_{g,1} \cdot \dfrac{\phi_{g,1} - \phi_g(0)}{\Delta z/2} \\ J_g(0) = -\dfrac{\phi_g(0)}{2} \end{cases} \Rightarrow \begin{cases} \phi_g(0) = \dfrac{\phi_{g,1}}{1 + \dfrac{\Delta z}{4 D_{g,1}}} \\ J_g(0) = -\dfrac{1}{2} \cdot \dfrac{\phi_{g,1}}{1 + \dfrac{\Delta z}{4 D_{g,1}}} \end{cases}$$

So that:

$$\frac{J_{g,1} - J_g(0)}{\Delta z} = -\frac{2}{\Delta z^2} \cdot \frac{D_{g,1} D_{g,2}}{D_{g,1} + D_{g,2}} \left(\hat{\phi}_{g,2} - \hat{\phi}_{g,1} \right) + \frac{1}{2\Delta z} \cdot \frac{1}{1 + \dfrac{\Delta z}{4 D_{g,1}}} \cdot \hat{\phi}_{g,1}$$

$$= \underbrace{\left(+ \frac{2}{\Delta z^2} \cdot \frac{D_{g,1} \cdot D_{g,2}}{D_{g,1} + D_{g,2}} + \frac{1}{2\Delta z} \cdot \frac{1}{1 + \dfrac{\Delta z}{4 D_{g,1}}} \right)}_{a_{g,1}} \hat{\phi}_{g,1} + \underbrace{-\frac{2}{\Delta z^2} \cdot \frac{D_{g,1} \cdot D_{g,2}}{D_{g,1} + D_{g,2}}}_{b_{g,1}} \hat{\phi}_{g,2}$$

$$\Rightarrow \begin{cases} a_{g,1} = \dfrac{2}{\Delta z^2} \cdot \dfrac{D_{g,1} \cdot D_{g,2}}{D_{g,1} + D_{g,2}} + \dfrac{1}{\Delta z} \cdot \dfrac{1/2}{1 + \dfrac{\Delta z}{4 D_{g,1}}} \\ b_{g,1} = -\dfrac{2}{\Delta z^2} \cdot \dfrac{D_{g,1} \cdot D_{g,2}}{D_{g,1} + D_{g,2}} \rightarrow \text{Same that we got in the lecture.} \\ c_{g,1} = 0 \end{cases}$$

· Now, for the top node:

want to compute: $\dfrac{J_g(H) - J_{g,Nz}}{\Delta z}$.

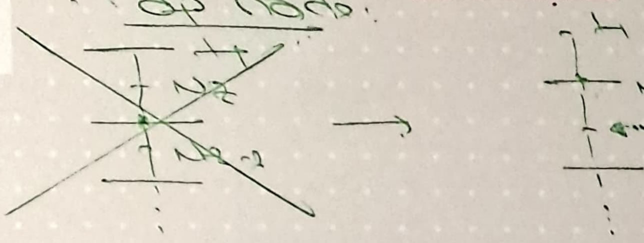
$J_{g,Nz} = -\dfrac{D_{g,Nz} \cdot D_{g,Nz-1}}{D_{g,Nz} + D_{g,Nz-1}} \cdot \dfrac{\hat{\phi}_{g,Nz} - \hat{\phi}_{g,Nz-1}}{\Delta z/2}$

Boundary conditions:

$$\begin{cases} J_g(H) = -D_{g,Nz} \cdot \dfrac{\phi_g(H) - \phi_{g,Nz}}{\Delta z/2} \\ J_g(H) = +\dfrac{\phi_g(H)}{2} \end{cases}$$

Check note in the next page.

1D mesh: **Preliminaries:** **Darcy:**



$$J_{g,Nz}^{-} = -D_{g,Nz}\,\frac{\Phi_{g,Nz}^{-}-\Phi_{g,Nz}}{\Delta z/2}$$

$$J_{g,Nz}^{-} = -D_{g,Nz-1}\,\frac{\Phi_{g,Nz-1}-\Phi_{g,Nz}^{-}}{\Delta z/2}$$

$$\Phi_{g,Nz}^{-} = \Phi_{g,Nz-1}^{+}$$

$$-D_{g,Nz}\cdot\frac{\Phi_{g,Nz}-\Phi_{g,Nz}^{-}}{\cancel{\Delta z/2}} = -D_{g,Nz-1}\cdot\frac{\Phi_{g,Nz}^{-}-\Phi_{g,Nz-1}}{\cancel{\Delta z/2}}$$

$$\Rightarrow +D_{g,Nz}\cdot\Phi_{g,Nz}^{-}+D_{g,Nz-1}\cdot\Phi_{g,Nz}^{-} = +D_{g,Nz-1}\cdot\Phi_{g,Nz-1} + D_{g,Nz}\cdot\Phi_{g,Nz} \Rightarrow$$

$$\Rightarrow \Phi_{g,Nz}^{-} = \underbrace{\frac{D_{g,Nz-1}\cdot\Phi_{g,Nz-1}+D_{g,Nz}\cdot\Phi_{g,Nz}}{D_{g,Nz}+D_{g,Nz-1}}}.$$

$$\Rightarrow J_{g,Nz}^{-} = -\frac{D_{g,Nz}}{\Delta z/2}\left(\Phi_{g,Nz}-\frac{D_{g,Nz-1}\cdot\Phi_{g,Nz-1}+D_{g,Nz}\cdot\Phi_{g,Nz}}{D_{g,Nz}+D_{g,Nz-1}}\right) =$$

$$= -\frac{D_{g,Nz}}{\Delta z/2}\cdot\frac{1}{D_{g,Nz}+D_{g,Nz-1}}\left(\cancel{D_{g,Nz}\Phi_{g,Nz}}+D_{g,Nz-1}\Phi_{g,Nz} - D_{g,Nz-1}\Phi_{g,Nz-1} - \cancel{D_{g,Nz}\Phi_{g,Nz}}\right) =$$

$$= -\frac{D_{g,Nz}}{\Delta z/2}\cdot\frac{D_{g,Nz-1}}{D_{g,Nz}+D_{g,Nz-1}}\left(\Phi_{g,Nz}-\Phi_{g,Nz-1}\right).$$

Want to compute: $\dfrac{J_g(t)-J_{g,Nz}^{-}}{\Delta z}$.

$$J_{g,Nz}^{-} = -\frac{D_{g,Nz}\cdot D_{g,Nz-1}}{D_{g,Nz}+D_{g,Nz-1}}\cdot\frac{\Phi_{g,Nz}-\Phi_{g,Nz-1}}{\Delta z/2}.$$

Boundary conditions:

$$\begin{cases} J_g(t) = -D_{g,Nz}\cdot\dfrac{\Phi_g(t)-\Phi_{g,Nz}}{\Delta z/2} \\ J_g(t) = +\dfrac{\Phi_g(t)}{2} \end{cases}$$

$$\Rightarrow -D_{g,Nz}\cdot\frac{\Phi_g(t)-\Phi_{g,Nz}}{\Delta z/2} = \frac{\Phi_g(t)}{2} \Rightarrow$$

$$\Rightarrow -\frac{2}{\Delta z}\cdot D_{g,Nz}\cdot\Phi_g(t) - \frac{1}{2}\cdot\Phi_g(t) = -\frac{2}{\Delta z}D_{g,Nz}\Phi_{g,Nz} \Rightarrow$$

$$\Rightarrow \left(-\frac{2}{\Delta z}\cdot D_{g,Nz}-\frac{1}{2}\right)\Phi_g(t) = -\frac{2}{\Delta z}D_{g,Nz}\cdot\Phi_{g,Nz} \Rightarrow$$

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$$\Rightarrow \boxed{\bar{\Phi}_g(H) = + \frac{\Phi_{g,Nz}}{1 + \frac{\Delta z}{4 D_{g,Nz}}}}$$

$$\Rightarrow \boxed{\bar{\Phi}_g(H) = \frac{\Phi_g(H)}{2} = + \frac{1/2}{1 + \frac{\Delta z}{4 D_{g,Nz}}} \cdot \Phi_{g,Nz}}$$

Final:

$$\bar{\Phi}_g(H) - \frac{J_{g,Nz}}{\Delta z} = \qquad (\Phi_{g,Nz} - \Phi_{g,Nz-1})$$

$$= + \frac{1}{2\Delta z} \cdot \frac{1}{1 + \frac{\Delta z}{4 D_{g,Nz}}} \cdot \Phi_{g,Nz} + \frac{D_{g,Nz} \cdot D_{g,Nz-1}}{D_{g,Nz} + D_{g,Nz-1}} \cdot \frac{2}{\Delta z^2}$$

$$= \underbrace{\left(+ \frac{1}{2\Delta z} \cdot \frac{1}{1 + \frac{\Delta z}{4 D_{g,Nz}}} + \frac{2}{\Delta z^2} \cdot \frac{D_{g,Nz} \cdot D_{g,Nz-1}}{D_{g,Nz} + D_{g,Nz-1}} \right)}_{a_{g,Nz}} \Phi_{g,Nz}$$

$$+ \underbrace{\left(- \frac{2}{\Delta z^2} \cdot \frac{D_{g,Nz} \cdot D_{g,Nz-1}}{D_{g,Nz} + D_{g,Nz-1}} \right)}_{c_{g,Nz}} \Phi_{g,Nz-1} + \underbrace{(0)}_{b_{g,Nz}} \cdot \Phi_{g,Nz+1}$$

$$\Rightarrow \begin{cases} c_{g,Nz} = - \dfrac{2}{\Delta z^2} \cdot \dfrac{D_{g,Nz} \cdot D_{g,Nz-1}}{D_{g,Nz} + D_{g,Nz-1}} \\[2ex] b_{g,Nz} = 0 \\[2ex] a_{g,Nz} = \dfrac{1}{2\Delta z} \cdot \dfrac{1}{1 + \frac{\Delta z}{4 D_{g,Nz}}} + \dfrac{2}{\Delta z^2} \cdot \dfrac{D_{g,Nz} \cdot D_{g,Nz-1}}{D_{g,Nz} + D_{g,Nz-1}} \end{cases}$$

Proof.

Wielandt's shift technique: modification to power iteration method.

$$A_\omega \cdot \Phi = \lambda \cdot \Phi \Rightarrow$$

$$\Rightarrow \left(A^{-1} - \frac{1}{k_{est}}\right)\Phi = \left(\frac{1}{k} - \frac{1}{k_{est}}\right)\Phi \quad \text{(definitions)}.$$

In our case: $\left(M \Phi = \frac{1}{k} F \Phi\right)$.

$A = M^{-1}F \qquad \Rightarrow k\Phi = M^{-1}F\Phi$

$\Rightarrow A^{-1} = F^{-1} M \qquad \Rightarrow k\Phi = A\Phi$

- But, F^{-1} doesn't exist (F has blank of zeros $\Rightarrow$ not invertible!)

So,

$$A_\omega \cdot \Phi = \lambda\Phi = \left(\frac{1}{k} - \frac{1}{k_{est}}\right)\Phi \Rightarrow$$

$$\Rightarrow \left(A^{-1} - \frac{1}{k_{est}}\right)\Phi = \left(\frac{1}{k} - \frac{1}{k_{est}}\right)\Phi \Rightarrow$$

$$\Rightarrow \left(F^{-1}M - \frac{1}{k_{est}}\right)\Phi = \left(\frac{1}{k} - \frac{1}{k_{est}}\right)\Phi \Rightarrow$$

simplify $\times F$

$$\Rightarrow \left(M - \frac{F}{k_{est}}\right)\Phi = \left(\frac{1}{k} - \frac{1}{k_{est}}\right)F\Phi$$

So, we have arrived at a "modified" Wielandt shift method:

- $\boxed{\Phi^{(p)} = \frac{1}{K^{(p-1)}}\left(M - \frac{F}{k_{est}}\right)^{-1} F \cdot \Phi^{(p-1)}}$; $A\Phi = S$.
- $\boxed{K^{(p)} = K^{(p-1)} \frac{\Phi^{(p-1)} \cdot \Phi^{(p)}}{\Phi^{(p-1)} \cdot \Phi^{(p-1)}}}$; $\to$ by k.
- $\boxed{\frac{1}{K^{(p)}} = \frac{1}{k^{(p)}} - \frac{1}{k_{est}}}$; same $k \neq$ by k (K).

Wielandt's shift comes in the definition of k_{est}:

Fundamental mode: $\boxed{k_{est} = k^{(p-1)} - k_{shift}}$

Higher order modes: $k_{est} = k_{est\_mod} - k_{shift}$

Same (Wielandt's) shift.

"estimation of the "desired" higher order mode."

Step 11: pdf export.