Asymptotic Theory

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Asymptotic theory deals with large number of random variables and their convergence

- Two basic convergence notions are
  - Convergence in probability
  - Convergence in distribution

- These give rise to two fundamental results in statistics and probability theory:

  - The Law of Large Numbers
  - The Central Limit Theorem

Some preliminary concepts

- Basics of statistics and probability theory
  - Random variable and its probability distribution

- Asymptotic theory
  - Sequence of random variables
  - Convergence of a sequence

Sequences

- In Mathematics, a sequence is an infinite list of numbers indexed by a natural number, denoted by, say $\{x_n\}$ where n is a natural number.

- In a sequence, each number x_n has a fixed position in the list, called the n^{th} term. The terms are determined by some function of n.

- A sequence is convergent if its terms approaches some constant **c** as the sequence expands. Such a constant, if exists, is known as the limit of the convergent sequence.

Sequences of random variables

- A sequence of random variables $\{X_n\}$, is an infinite list random variables indexed by a natural number, n.

- In the sequence, the n^{th} random variable has a probability distribution that is characterized by some function of n.
- Eg.,
  1. $\{X_n\}$ where $X_n \sim \exp(n)$
  2. $\{X_n\}$ where $X_n \sim \text{Bernoulli}(1/n)$
  3. $\{X_n\}$ where $X_n=(Y/n)$; $Y \sim \text{Uniform}(-1,1)$
  4. $\{X_n\}$ where $E(X_n) = \mu$ and $Var(X_n)= \sigma^2/n$

Convergence in probability

Sequence $\{X_n\}$ is said to converge to a constant μ *in probability* if for arbitrarily small $\varepsilon > 0$,

$$\mathbf{Pr}\left\{\left|X_n - \mu\right| \geq \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

i.e.

$$\mathbf{Pr}\left\{\left|X_n - \mu\right| \leq \varepsilon\right\} \to 1 \text{ as } n \to \infty$$

μ is known as the plim of X_n; this is denoted by

$$X_n \xrightarrow{p} \mu \quad \text{or} \quad \text{plim}\, X_n = \mu$$

Note: Convergence in probability is about convergence of the values of a sequence of r.v. to a constant.

Convergence in probability: examples

Are the following sequences convergent in probability?

1. $\{X_n\}$ where $X_n \sim \exp(n)$

2. $\{X_n\}$ where $X_n \sim \text{Bernoulli}(1/n)$

3. $\{X_n\}$ where $X_n \sim \text{Uniform}(-1,1)$

4. $\{X_n\}$ where $X_n = (Y/n)$; $Y \sim \text{Uniform}(-1,1)$

5. $\{X_n\}$ where $E(X_n) = \mu$ and $\text{Var}(X_n) = \sigma^2/n$

The Weak Law of Large Numbers

Let $\{X_n\}$ be a sequence of i.i.d. random variables with common mean μ and variance $\sigma^2 < \infty$. Then,

$$\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{P} \mu$$

Thus, the average of n random variables with common distribution converges in probability to the common underlying mean.

Proof: Skipping, but I will show the validity of the result using simulation

The Weak Law of Large Numbers: an example

- I draw 1000 random numbers from U(0,1) distribution
- I construct the sum

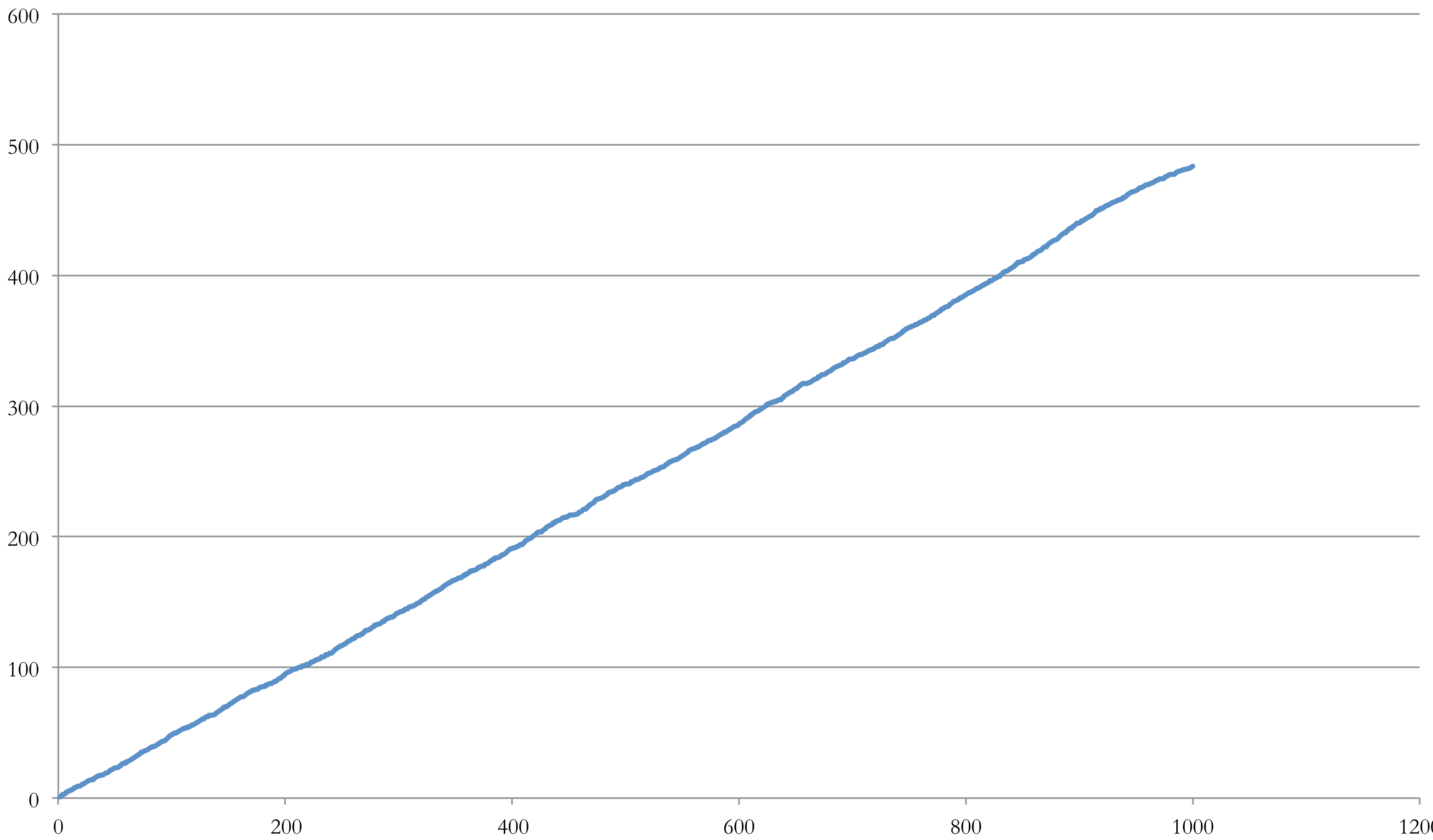
$$S_n = \sum_{i=1}^{n} u_i$$

- Then I construct the average

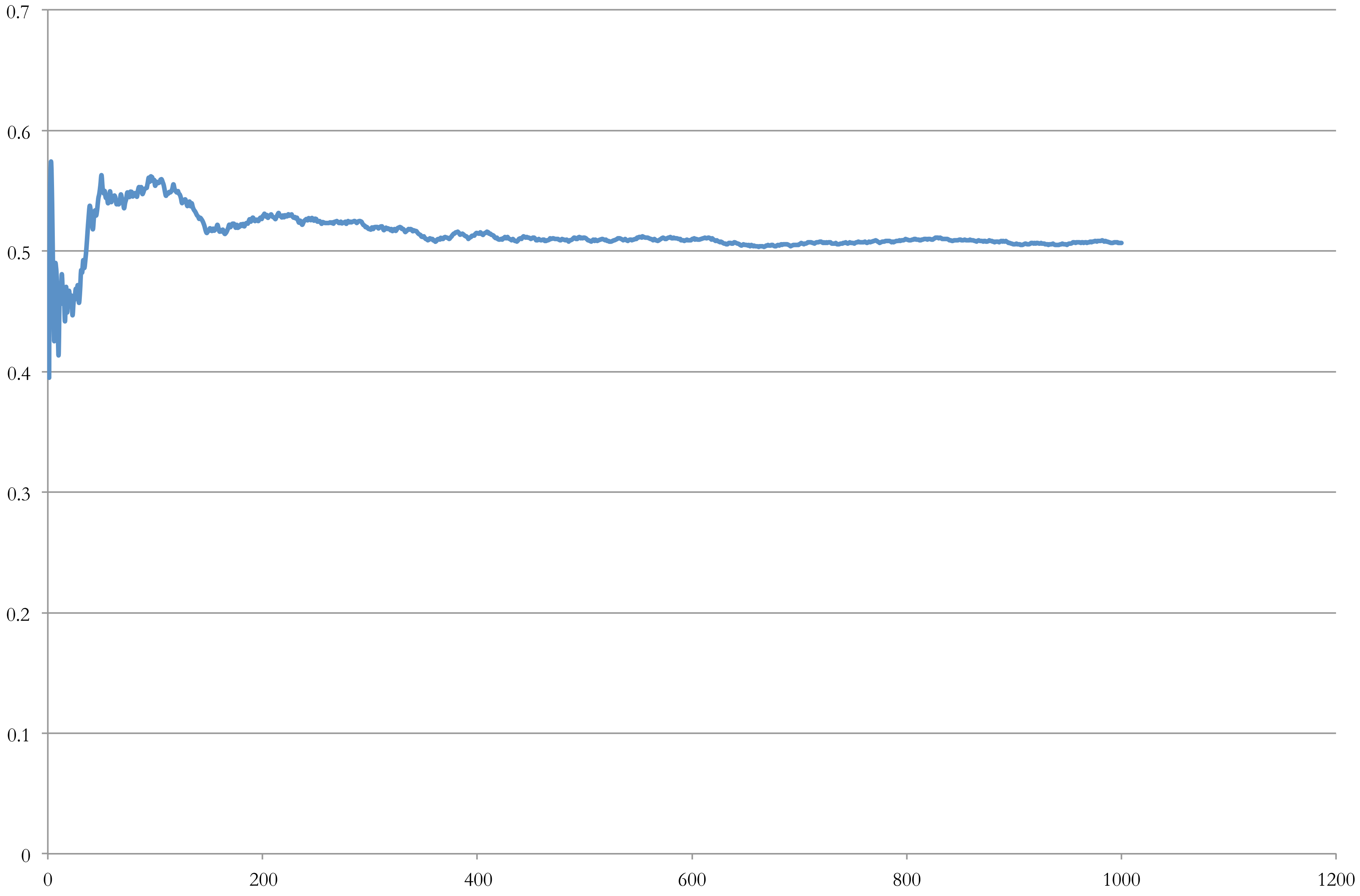
$$\bar{\mathbf{x}}_{\mathbf{n}} = \frac{\mathbf{S_n}}{\mathbf{n}}$$

- Then I plot S_n and $\bar{x}_n$ along y-axis against n along x-axis to observe their behaviour.

Plot of the sum of n U(0,1) random numbers



Plot of means of n U(1,0) random numbers against n



Convergence in distribution

Let $\{X_n\}$ be a sequence of random variables with cumulative distribution function F_n. Let X be another random variable with cdf F.

The sequence $\{X_n\}$ is said to converge in distribution to X if

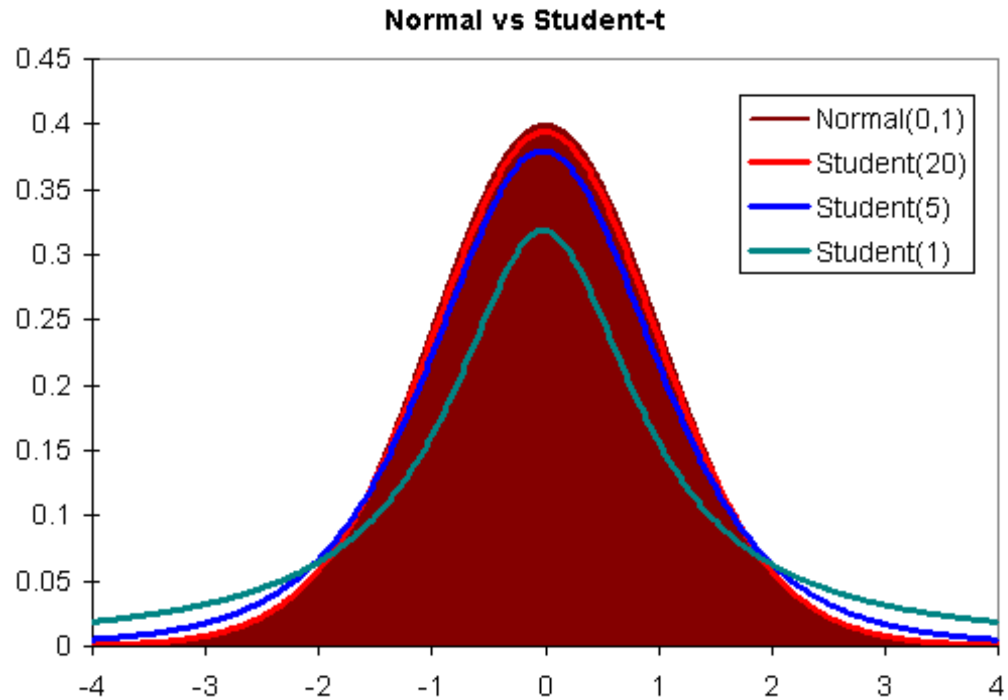
$$\mathbf{F_n \rightarrow F \text{ as } n \rightarrow \infty}$$

F is called the asymptotic or limiting distribution of the sequence. This is denoted as

$$\mathbf{X_n \xrightarrow{D} X \quad \text{or} \quad F_n(x) \xrightarrow{D} F(x)}$$

Convergence in distribution: example

$$t_n \xrightarrow{D} N(0,1) \text{ as } n \to \infty$$



Note:

1. *Convergence in distribution is about convergence of the distributions of a sequence of r.v. to a limiting distribution.*
2. *The normal distribution serves as an asymptotic distribution in many cases.*

Central Limit Theorem (CLT)

Let $\{X_n\}$ be a sequence of i.i.d. random variables with common mean μ and variance $\sigma^2 < \infty$. Then,

$$\mathbf{Y_n} = \frac{\sum_{i=1}^{n} \mathbf{X_i} - \mathbf{n}\mu}{\sqrt{\mathbf{n}}\,\sigma} \xrightarrow{\mathbf{D}} \mathbf{N(0,1)}$$

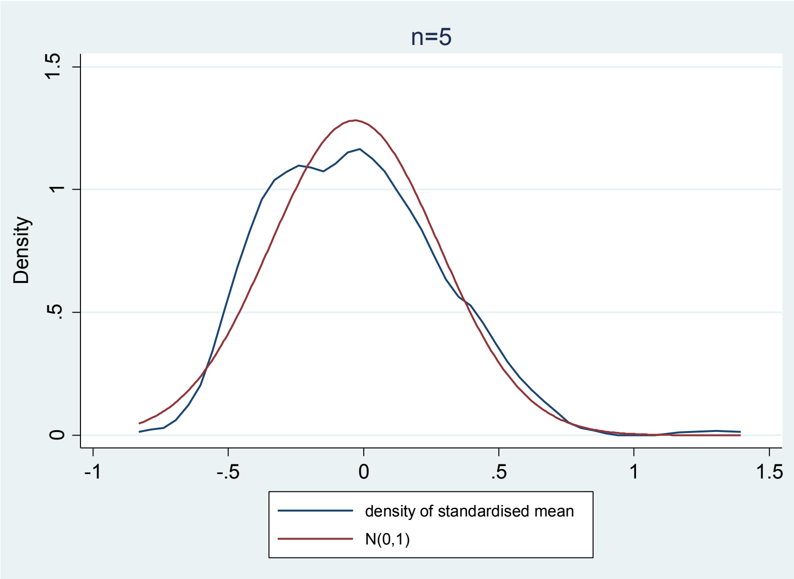
or

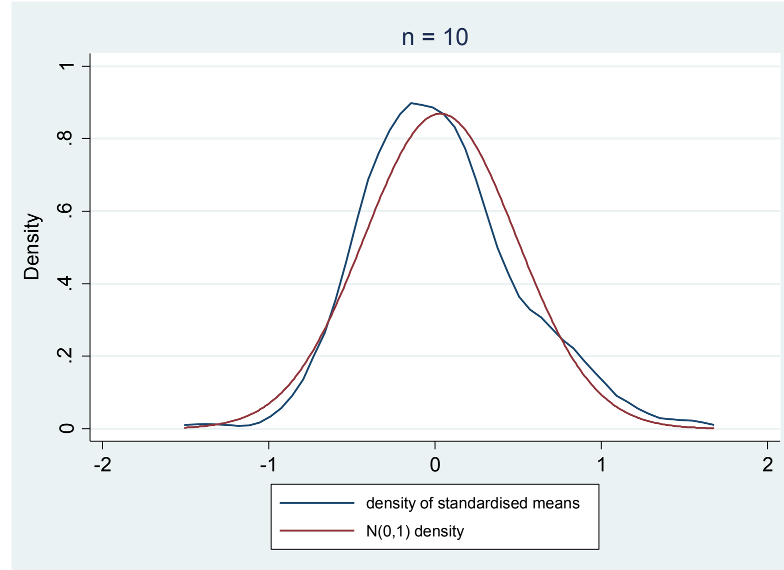
$$\mathbf{Y_n} = \frac{\bar{\mathbf{X}}_n - \mu}{\sigma / \sqrt{\mathbf{n}}} \xrightarrow{\mathbf{D}} \mathbf{N(0,1)}$$

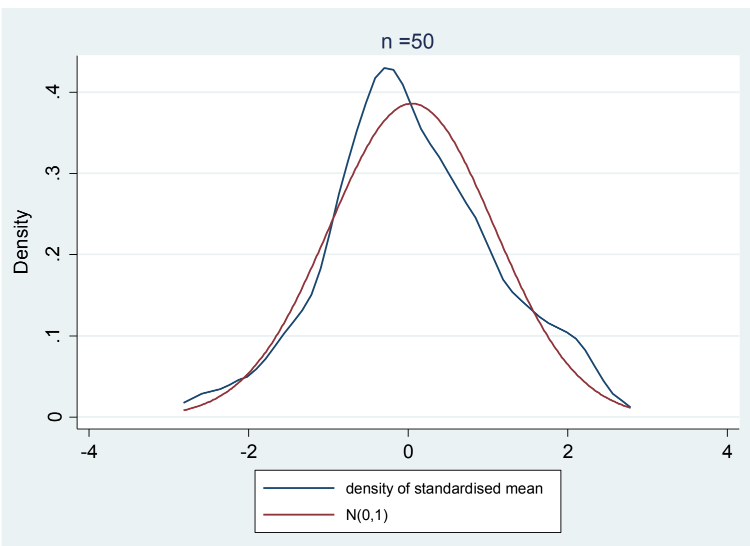
Note: This is the simplest version of CLT; the result is generalised for sequences that do not have identical distributions and those that do not have independent distributions.

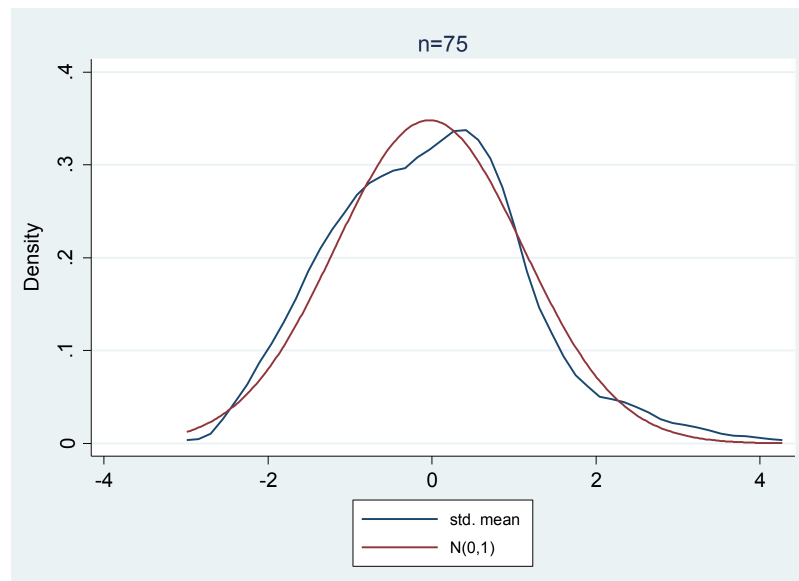
Central Limit Theorem: An Example

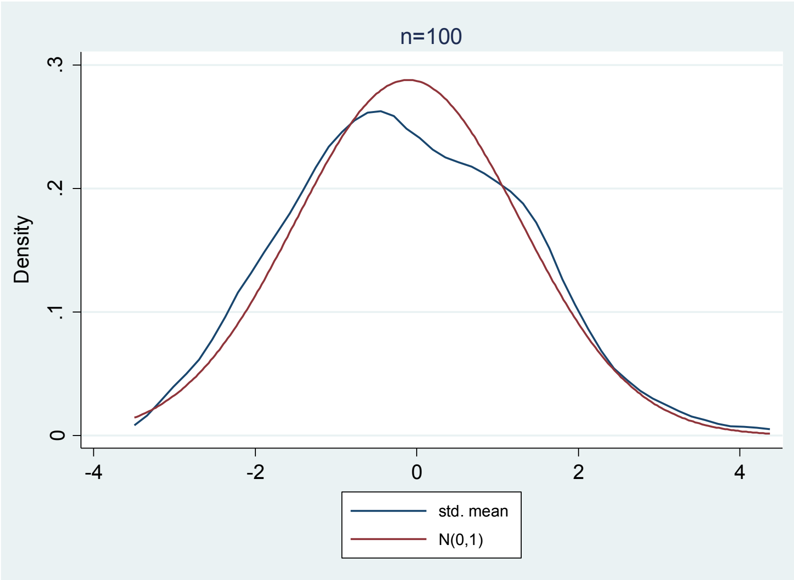
- I draw n random numbers from exp(1) distribution (for n = 5, 10, 50, 75, 100, 500, 1000, 3000)
- For each n, I compute the standardised sample mean.
- I repeat this 200 times; i.e., I simulate 200 standardised means of a sample of n exp(1) variables.
- I draw the empirical pdf of the standardised means and compare it vis-à-vis the standard normal distribution.

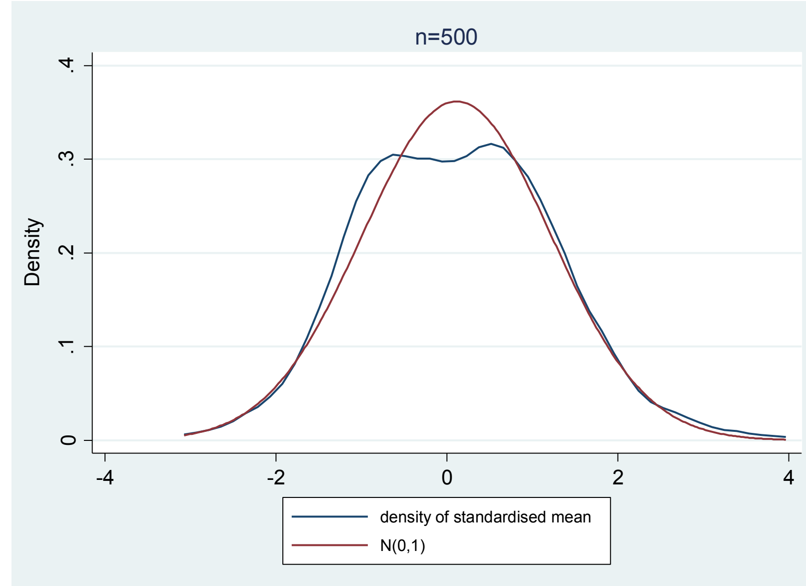

n=5
Density
0
.5
1
1.5
-1
-.5
0
.5
1
1.5
density of standardised mean
N(0,1)

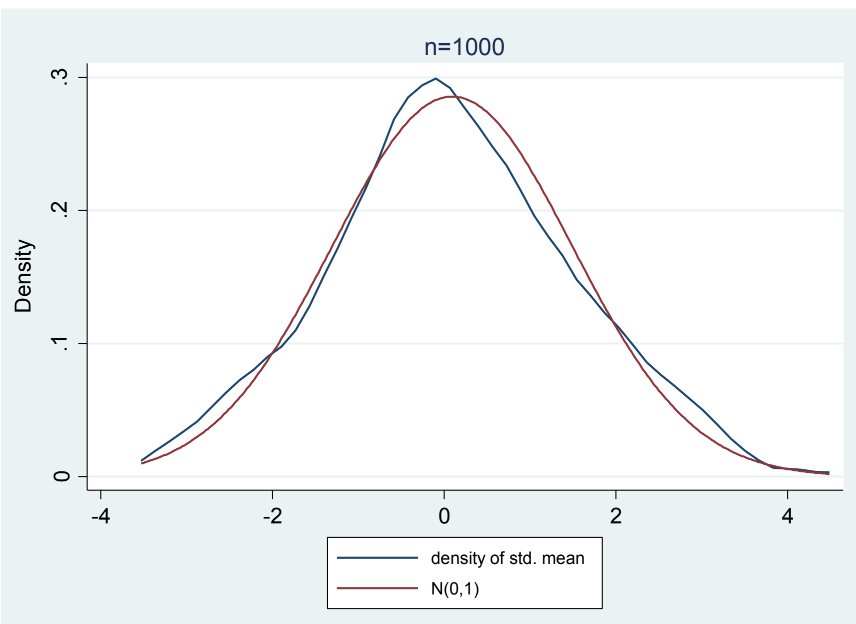

n = 10
Density
0
.2
.4
.6
.8
1
-2
-1
0
1
2
density of standardised means
N(0,1) density

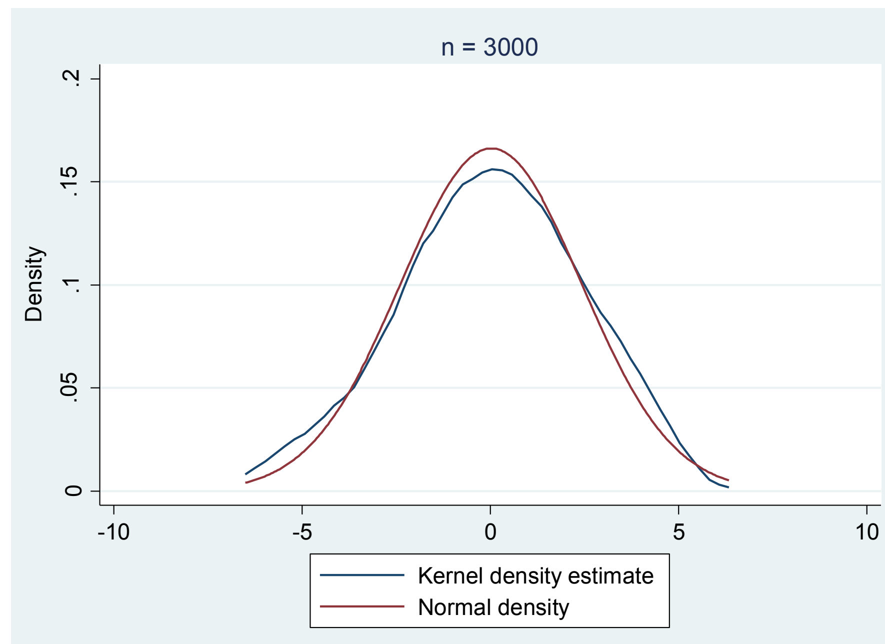

n =50
Density
0
.1
.2
.3
.4
-4
-2
0
2
4
density of standardised mean
N(0,1)


n=75
Density
0
.1
.2
.3
.4
-4
-2
0
2
4
std. mean
N(0,1)


n=100
Density
0
.1
.2
.3
-4
-2
0
2
4
std. mean
N(0,1)


n=500
Density
0
.1
.2
.3
.4
-4
-2
0
2
4
density of standardised mean
N(0,1)


n=1000
Density
0
.1
.2
.3
-4
-2
0
2
4
density of std. mean
N(0,1)


n = 3000
Density
0
.05
.1
.15
.2
-10
-5
0
5
10
Kernel density estimate
Normal density

Such convergence results are appealing when we have large data from random samples

- A sample observation from a random sample can be viewed as a realization of an underlying random variable

- A sample of n observations can be then treated as realization of n i.i.d. random variables.

- Therefore, the convergence results can be applied effectively while we use the sample to estimate population parameters (like mean, total etc.)

Basics of Estimation Theory

- Statistic: A statistic is a function of the sample observations.

- Sampling distribution: Because the sample is random, the statistic is also random variable with a probability distribution. This is called the sampling distribution of the statistic.

- Estimator: A statistic is called an estimator if we can use it to estimate a parameter of the underlying population distribution.

When is a statistic good estimator

- Let T_n is an estimator for a parameter μ, based on a sample of size n.
- Then looking at the sampling distribution of T_n, we can define some desirable properties for it to be a **"good"** estimator:

1. **Unbiased**: if $E(T_n) = \mu$
2. **Best** or **Most Efficient**: if $SE(T_n)$ is the least in a class of estimators. *(BTW, Standard Deviation of the sampling distribution is called standard error or SE.)*
3. **Consistent**: if $\mathbf{T_n} \xrightarrow{\text{Plim}} \mu$, i.e., if T_n converges to μ in probability.

Summary

- Due to WLLN, sample mean is consistent estimator of the population mean. Thus, for very large sample, sample mean collapses to population mean. This will happen for simple mean as well as weighted mean.

- Regardless of the underlying distribution, the sampling distribution of sample mean converges in distribution to the normal distribution.

Assignments for afternoon session

1. Use simulation technique to validate
   i. WLLN
   ii. CLT
2. Estimate the following using NSS 68^{th} round on Employment and Unemployment. For every estimate, provide its standard error.
   i. mean household landholding for India, for rural India and for urban India. Also for different states of India.
   ii. average age of workers in coffee plantation, by state, in India
   iii. average age of coffee curing workers for India and also state wise.
   iv. average household size of paddy workers in India