Economics 205 Final Exam

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Fall 2019

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Instructions:

- You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets. Calculators are not allowed.
- All logarithms are base $\mathrm{e} = 2.718281828\dots$, so $\ln x$ and $\log x$ are the same.
- Questions are not necessarily ordered according to difficulty, and some parts are (far) easier than others. Make sure to look at all questions and parts before deciding the order to solve.

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	120
Score:							

1. (a) (5 points) Let A be a square matrix. Show that

$$(I-A)(I+A+\cdots+A^{k-1}) = I - A^k.$$

Solution:

$$(I-A)(I+A+\cdots+A^{k-1}) = (I+A+\cdots+A^{k-1}) - A(I+A+\cdots+A^{k-1}) = I - A^k.$$

(b) (15 points) Let A be a nonnegative square matrix. Show that if $z > \rho(A)$, then the matrix $zI - A$ is regular and $(zI - A)^{-1}$ is nonnegative. (Here $\rho(A)$ is the spectral radius of A, which is the largest absolute value of all eigenvalues.)

Solution: Let $z > \rho(A)$ and $B = \frac{1}{z}A$. Then B is nonnegative and $\rho(B) = \rho(A)/z < 1$. Since

$$\left\|I + B + \cdots + B^{k-1}\right\| \leq \frac{1 - \|B\|^k}{1 - \|B\|} \leq \frac{1}{1 - \|B\|},$$

the geometric series $C = \sum_{k=0}^{\infty} B^k$ is convergent. Furthermore,

$$(I-B)C = \sum_{k=0}^{\infty} B^k - B\sum_{k=0}^{\infty} B^k = \sum_{k=0}^{\infty} B^k - \sum_{k=1}^{\infty} B^k = I.$$

Therefore $I - B$ is regular and $(I-B)^{-1} = C \geq 0$. Since $A = zB$, it follows that $zI - A = z(I - B)$ is regular and $(zI - A)^{-1} = \frac{1}{z}C$ is nonnegative.

2. Let $C \subset \mathbb{R}^N$ be a convex set. A nonnegative function $f : C \to \mathbb{R}$ is called *log-convex* if either (i) $f(x) = 0$ for all $x \in C$ or (ii) $f(x) > 0$ for all $x \in C$ and $\log f$ is convex. Prove the followings.

(a) (5 points) If f is log-convex, then f^λ is log-convex for all $\lambda \geq 0$.

Solution: If f is log-convex, then $\log f^\lambda = \lambda \log f$ is convex, so f^λ is log-convex.

(b) (5 points) If f, g are log-convex, then fg is log-convex.

Solution: If f, g are log-convex, $\log(fg) = \log f + \log g$ is convex, so fg is log-convex.

(c) (10 points) If f, g are log-convex, then $f + g$ is log-convex. (Hint: you may use Hölder's inequality

$$\sum_{n=1}^{N} u_n v_n \leq \left(\sum_{n=1}^{N} u_n^p\right)^{1/p} \left(\sum_{n=1}^{N} v_n^q\right)^{1/q},$$

where $u_n \geq 0$, $v_n \geq 0$, and $p, q > 1$ are numbers such that $1/p + 1/q = 1$.)

Solution: Take any $x_1, x_2 \in C$ and $\alpha \in (0,1)$. Since f is log-convex, we have

$$\begin{aligned}
&\log f((1-\alpha)x_1 + \alpha x_2) \le (1-\alpha)\log f(x_1) + \alpha \log f(x_2) \\
&\implies f((1-\alpha)x_1 + \alpha x_2) \le f(x_1)^{1-\alpha} f(x_2)^{\alpha}.
\end{aligned}$$

The same inequality holds for g. Define the vectors $u = (f(x_1)^{1/p}, g(x_1)^{1/q})' \in \mathbb{R}^2$ and $v = (f(x_2)^{1/p}, g(x_2)^{1/q})' \in \mathbb{R}^2$, where $1/p = 1-\alpha$ and $1/q = \alpha$. Then by Hölder's inequality we obtain

$$\begin{aligned}
f((1-\alpha)x_1 + \alpha x_2) + g((1-\alpha)x_1 + \alpha x_2) &\le f(x_1)^{1-\alpha} f(x_2)^{\alpha} + g(x_1)^{1-\alpha} g(x_2)^{\alpha} \\
&= u_1 v_1 + u_2 v_2 \le \|u\|_p \|v\|_q \\
&= (f(x_1) + g(x_1))^{1-\alpha} (f(x_2) + g(x_2))^{\alpha}.
\end{aligned}$$

Taking the logarithm, we obtain

$$\log(f+g)((1-\alpha)x_1 + \alpha x_2) \le (1-\alpha)\log(f+g)(x_1) + \alpha \log(f+g)(x_2),$$

so $f+g$ is log-convex.

3. (a) (5 points) Let $M_N(\mathbb{R})$ be the set of real $N \times N$ matrices. What is the definition of a matrix norm on $M_N(\mathbb{R})$?

Solution: A matrix norm $\|\cdot\| : M_N(\mathbb{R}) \to \mathbb{R}$ is a function such that (i) $\|A\| \ge 0$ for all A, with $\|A\| = 0$ if and only if $A = 0$, (ii) $\|\alpha A\| = |\alpha| \|A\|$ for all $\alpha \in \mathbb{R}$, (iii) $\|A + B\| \le \|A\| + \|B\|$, and (iv) $\|AB\| \le \|A\| \|B\|$.

(b) (5 points) For $A = (a_{mn}) \in M_N(\mathbb{R})$, define $\|A\| = \sqrt{\sum_{m=1}^N \sum_{n=1}^N a_{mn}^2}$. Prove that $\|\cdot\|$ is a matrix norm.

Solution: Properties (i) and (ii) are trivial. Property (iii) is immediate from the Cauchy-Schwarz inequality. To show (iv), let

$$A = \begin{bmatrix} a_1' \\ \vdots \\ a_N' \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & \cdots b_N \end{bmatrix},$$

so a_m is the m-th row vector of A and b_n is the n-th column vector of B. Then the (m,n)-th entry of AB is $a_m' b_n = \langle a_m, b_n \rangle$. By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|AB\|^2 &= \sum_{m,n} \langle a_m, b_n \rangle^2 \le \sum_{m,n} \|a_m\|^2 \|b_n\|^2 \\
&= \left(\sum_{m=1}^N \|a_m\|^2 \right) \left(\sum_{n=1}^N \|b_n\|^2 \right) = \|A\|^2 \|B\|^2.
\end{aligned}$$

(c) (10 points) Let C be a convex set. Let $A(x) = (a_{mn}(x))$ be a nonnegative matrix, where each $a_{mn} : C \to \mathbb{R}$ is log-convex. (See the previous problem for the definition of a log-convex function.) Let $f(x) = \rho(A(x))$, where ρ is the spectral radius. Show that f is log-convex. (Hint: you may use the property $\rho(A) = \lim_{k\to\infty} \left\|A^k\right\|^{1/k}$.)

Solution: By the results from the previous problem, the set of log-convex functions is closed under addition, multiplication, and nonnegative powers. Since $a_{mn} \geq 0$ and a_{mn} is log-convex, it follows that all elements of A^k are log-convex. Consider the matrix norm $\|A\| = \sqrt{\sum_{m,n} a_{mn}^2}$. Since this matrix norm is the composition of addition and positive powers, it follows that $\left\|A^k(x)\right\|$ is log-convex. Therefore $\left\|A^k(x)\right\|^{1/k}$ is also log-convex, and so is the limit

$$f(x) = \rho(A(x)) = \lim_{k\to\infty} \left\|A^k(x)\right\|^{1/k}.$$

4. Consider the optimization problem

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g_i(x) \leq 0 \quad (i = 1, \dots, I), \end{aligned}$$

where $f, g_i : \mathbb{R}^N \to \mathbb{R}$ are differentiable. Let $C = \left\{x \in \mathbb{R}^N \,\middle|\, (\forall i) g_i(x) \leq 0\right\}$ be the constraint set and $\bar{x} \in C$.

(a) (5 points) Define the tangent cone of C at $\bar{x}$, denoted by $T_C(\bar{x})$.

Solution: According to the lecture note,

$$T_C(\bar{x}) = \left\{y \in \mathbb{R}^N \,\middle|\, (\exists)\, \{\alpha_k\} \geq 0, \{x_k\} \subset C, \lim_{k\to\infty} x_k = \bar{x}, y = \lim_{k\to\infty} \alpha_k (x_k - \bar{x})\right\}.$$

(b) (5 points) Define the linearizing cone of C at $\bar{x}$, denoted by $L_C(\bar{x})$.

Solution: According to the lecture note,

$$L_C(\bar{x}) = \left\{y \in \mathbb{R}^N \,\middle|\, (\forall i \in I(\bar{x}))\, \langle \nabla g_i(\bar{x}), y \rangle \leq 0\right\},$$

where $I(\bar{x}) = \{i \,|\, g_i(\bar{x}) = 0\}$ is the active set (the index of binding constraints).

(c) (10 points) Define the Guignard constraint qualification. Prove that when all constraints are linear, so $g_i(x) = \langle a_i, x \rangle - c_i$ for some $0 \neq a_i \in \mathbb{R}^N$ and $c_i \in \mathbb{R}$, then the Guignard constraint qualification holds.

Solution: According to the lecture note, the Guignard constraint qualification is the condition $L_C(\bar{x}) \subset \operatorname{co} T_C(\bar{x})$, where co denotes the convex hull. Suppose $g_i(x) = \langle a_i, x\rangle - c_i$ for all i. Since $\nabla g_i = a_i$, the linearizing cone is

$$L_C(\bar{x}) = \left\{ y \in \mathbb{R}^N \,\middle|\, (\forall i \in I(\bar{x}))\, \langle a_i, y\rangle \le 0 \right\}.$$

Take any $y \in L_C(\bar{x})$ and consider the point $x(t) = \bar{x} + ty$ for $t \ge 0$. Let us show that $x(t) \in C$ if $t \ge 0$ is small enough. If $i \in I(\bar{x})$, then $\langle a_i, \bar{x}\rangle - c_i = 0$. Therefore

$$\langle a_i, x(t)\rangle - c_i = \langle a_i, \bar{x}\rangle - c_i + t\,\langle a_i, y\rangle \le 0.$$

If $i \notin I(\bar{x})$, then $\langle a_i, \bar{x}\rangle - c_i < 0$. Therefore by continuity $\langle a_i, x(t)\rangle - c_i < 0$ if $t \ge 0$ is small enough. Now for $t > 0$ we have

$$y = \frac{\bar{x} + ty - \bar{x}}{t} = \frac{x(t) - \bar{x}}{t},$$

so letting $t \to 0$ we obtain $y = \lim_{t \downarrow 0} \frac{x(t)-\bar{x}}{t} \in T_C(\bar{x})$. Therefore $L_C(\bar{x}) \subset T_C(\bar{x}) \subset \operatorname{co} T_C(\bar{x})$.

5. Consider the problem

$$\begin{array}{ll} \text{maximize} & \frac{4}{3}x_1^3 + \frac{1}{3}x_2^3 \\ \text{subject to} & x_1 + x_2 \le 1, \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

(a) (5 points) Are the Karush-Kuhn-Tucker conditions necessary for a solution? Answer yes or no, then explain why.

Solution: Yes (2 points). Since the objective function is continuous and the constraint set is compact, there is a solution. Since the constraints are linear, the constraint qualification holds automatically. Therefore the KKT conditions hold (3 points).

(b) (5 points) Are the Karush-Kuhn-Tucker conditions sufficient for a solution? Answer yes or no, then explain why.

Solution: No (2 points). Since the objective function is convex but the problem is a maximization problem, the first-order conditions are not sufficient (3 points). Note that KKT conditions are sufficient for *convex minimization* or *concave maximization* problems.

(c) (5 points) Write down the Lagrangian.

Solution:

$$L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = \frac{4}{3}x_1^3 + \frac{1}{3}x_2^3 + \lambda_1(1 - x_1 - x_2) + \lambda_2 x_1 + \lambda_3 x_2.$$

(d) (5 points) Compute the solution.

Solution: Since the objective function is increasing in both x_1 and x_2, clearly the constraint $x_1 + x_2 \leq 1$ binds. Therefore $x_1 + x_2 = 1$. The first-order condition is

$$0 = \frac{\partial L}{\partial x_1} = 4x_1^2 - \lambda_1 + \lambda_2,$$
$$0 = \frac{\partial L}{\partial x_2} = x_2^2 - \lambda_1 + \lambda_3.$$

If $x_1 = 0$, then $x_2 = 1$, and the function value is $\frac{1}{3}$. If $x_2 = 0$, then $x_1 = 1$, and the function value is $\frac{4}{3}$. If $x_1, x_2 > 0$, then by complementary slackness $\lambda_2 = \lambda_3 = 0$. Solving the first-order conditions we get $x_1 = \frac{1}{2\sqrt{\lambda_1}}$ and $x_2 = \frac{1}{\sqrt{\lambda_1}}$. Since $x_1 + x_2 = 1$, we get $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$. Then the function value is

$$\frac{4}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{3}\left(\frac{2}{3}\right)^3 = \frac{4+8}{81} = \frac{4}{27}.$$

Therefore the solution is $(x_1, x_2) = (1, 0)$.

6. (20 points) Choose one of the following theorems and state it as precisely as possible:

- separating hyperplane theorem
- contraction mapping theorem
- implicit function theorem

Solution: See the lecture note.

You can detach this sheet and use it as scratch paper.