有、$f(x_1\cdots x_n) = x_1^{\frac{w_1}{w}} x_2^{\frac{w_2}{w}} \cdots x_n^{\frac{w_n}{w}}$

设 $\lambda_i = \frac{w_i}{w}$ 则 $\lambda_i > 0$ 且 $\lambda_1 + \cdots + \lambda_n = 1$

$\therefore f(x_1\cdots x_n) = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$

则有：$\frac{\partial f(x_1 \cdots x_n)}{\partial x_i} = \lambda_i x_i^{\lambda_i - 1} \cdot (x_1^{\lambda_1} \cdots x_{i-1}^{\lambda_{i-1}} x_{i+1}^{\lambda_{i+1}} \cdots)$

$\therefore \frac{\partial^2 f(x_1 \cdots x_n)}{\partial x_i^2} = \lambda_i(\lambda_i - 1) x_i^{\lambda_i - 2} (x_1^{\lambda_1} \cdots x_{i-1}^{\lambda_{i-1}} \cdots)$

$= \frac{f(x_1 \cdots x_n)}{x_i^2} \lambda_i(\lambda_i - 1)$

而 $\frac{\partial^2 f(x_1 \cdots x_n)}{\partial x_i \partial x_j} = \lambda_i \lambda_j x_i^{\lambda_i - 1} x_j^{\lambda_j - 1} \cdots$

$= \frac{f(x_1 \cdots x_n)}{x_i x_j} \cdot \lambda_i \lambda_j$

$\therefore$ 有 Hessian 矩阵为

$$f(x_1 \cdots x_n) \begin{pmatrix} \frac{\lambda_i(\lambda_i - 1)}{x_i^2} & & \frac{\lambda_i \lambda_j}{x_i x_j} \\ & \ddots & \\ & & \ddots \end{pmatrix}$$ 设后面的矩阵为 H

而 $(z_1 \cdots z_n) \begin{pmatrix} \frac{\lambda_i(\lambda_i - 1)}{x_i^2} & & \\ & \frac{\lambda_i \lambda_j}{x_i x_j} & \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

$= \sum_{i=1}^{n} \frac{z_i^2 \lambda_i(\lambda_i - 1)}{x_i^2} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{\lambda_i \lambda_j}{x_i x_j} \cdot z_i z_j$

$$= \sum_{i=1}^{n} \frac{\lambda_i^2 z_i^2}{x_i^2} - \sum_{i=1}^{n} \frac{\lambda_i z_i^2}{x_i^2} + \sum_{i=1}^{n} \sum_{j\neq i}^{n} \left(\frac{z_i}{x_i}\right)\left(\frac{z_j}{x_j}\right)\lambda_i\lambda_j$$

$$= \left(\sum_{i=1}^{n} \frac{z_i\lambda_i}{x_i}\right)^2 - \sum_{i=1}^{n} \left(\frac{\lambda_i^2 z_i^2}{x_i^2}\right)\cdot\frac{1}{\lambda_i}$$

记 $s_i = \frac{z_i\lambda_i}{x_i}$ 则 $s_i > 0$

且要证明 H. 半负定 仅需证明

$\left(\sum_{i=1}^{n} s_i\right)^2 - \sum_{i=1}^{n} \frac{1}{\lambda_i}\cdot s_i^2 \leq 0$ 即可

而 $\left(\sum_{i=1}^{n} \frac{1}{\lambda_i} s_i^2\right)\left(\sum_{i=1}^{n} \lambda_i\right) \geq \left(\sum_{i=1}^{n} \sqrt{\frac{1}{\lambda_i} s_i^2\cdot\lambda_i}\right)^2 = \left(\sum_{i=1}^{n} s_i\right)^2$

∴由 $\sum_{i=1}^{n} \lambda_i = 1 \Rightarrow$

$\sum_{i=1}^{n} \frac{1}{\lambda_i} s_i^2 \geq \left(\sum_{i=1}^{n} s_i\right)^2$

∴有 $\forall z = (z_1, \dots, z_n)$ 有

$z^T H z \leq 0$ ∴ H 半负定

∴有: $f(x_1, \dots, x_n)$ 为 concave