有: (a): feasible set:

$2 \le x \le 4$:

optimal value: $2^2+1 = 5$

optimal solution: 2

(b): $L(x, \lambda) = x^2+1 + \lambda(x-2)(x-4)$

Because: $p^* = \min_x (x^2+1)$ (x subject to ---)

$\therefore \exists\ x_*$ s.t $(x_* - 2)(x_* - 4) \le 0$

且 $p^* = x_*^2 + 1$

$\therefore$ 有: $\lambda(x_*-2)(x_*-4) \le 0 \quad \forall \lambda \ge 0$

则有: $p^* = x_*^2+1 \ge x_*^2+1+\lambda(x_*-2)(x_*-4) = L(x_*, \lambda)$

$\ge \inf_x L(x, \lambda)$

而:且:有: $\frac{\partial L}{\partial x} = 0$ 有: $2x + \lambda(2x-6) = 0$

则有: $x + \lambda(x-3) = 0$ 则 $(1+\lambda)x = 3\lambda$

$\therefore$ $x = \frac{3\lambda}{1+\lambda}$ 则 $L(x, \lambda) = x^2 + \lambda(x-2)(x-4)$

$= \left(\frac{3\lambda}{1+\lambda}\right)^2 + 1 + \lambda$ ---- -

(3): 有: $d = \max_{\lambda \ge 0}\left(\min_x L(x, \lambda)\right) = \max_{\lambda \ge 0}\left(\frac{(3\lambda)^2}{(1+\lambda)^2} + 1 + \right.$

$\left.\lambda\left(\frac{3\lambda}{1+\lambda} - 2\right)\left(\frac{3\lambda}{1+\lambda} - 4\right)\right)$

有: $g(\lambda) =$ 括号内 $= \frac{9\lambda^2}{(1+\lambda)^2} + 1 + \lambda \cdot \frac{\lambda-2}{1+\lambda} \cdot \frac{-(4+\lambda)}{1+\lambda}$

$$= \frac{9\lambda^2}{(1+\lambda)^2} + 1 - \frac{\lambda^2+2\lambda-8}{(1+\lambda)^2} = \frac{9\lambda^2+\lambda^2+2\lambda+1-\lambda^2-2\lambda+8}{(1+\lambda)^2}$$

$$= \frac{9(\lambda^2+1)}{(1+\lambda)^2}$$ 证明：$\frac{\lambda^2+1}{(1+\lambda)^2}$ 在 $[0,+\infty)$ 内 concave 即可。

而有：$f_2(\lambda) = \frac{\lambda^2+1}{(1+\lambda)^2} = \frac{\lambda^2+2\lambda+1-2\lambda}{(1+\lambda)^2} = 1 - \frac{2\lambda}{(1+\lambda)^2}$

∴有 $\frac{\partial f_2}{\partial \lambda} = -2\,\frac{(1+\lambda)^2 - \lambda\cdot(2\times(1+\lambda))}{(1+\lambda)^4} = -2\cdot\frac{1-\lambda}{(1+\lambda)^3}$

∴ $\underline{\underline{\lambda^*=1}}$

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(4)：有：$x^2+1 = f_0(x)$

$h_1(x) = (x-2)(x-4) - u \leq 0$

∴有：$L(x,\lambda) = x^2+1 + \lambda(x-2)(x-4) - \lambda u$

∴有：$\frac{\partial L}{\partial x} = 0 \Rightarrow$ 则有！ $2x + \lambda(2x-6) = 0$

∴，$x = \frac{3\lambda}{1+\lambda}$

而有：$g(\lambda) = \frac{9(\lambda^2+1)}{(\lambda+1)^2} - \lambda\cdot u = 9\left(1 - \frac{2\lambda}{(\lambda+1)^2}\right) - \lambda u$

∴有 $g'(\lambda) = 9\times(-2)\frac{1-\lambda}{(1+\lambda)^3} - u = 18\,\frac{\lambda-1}{(1+\lambda)^3} - u$ 令 $g'(\lambda)=0$

则 $\frac{\lambda-1}{(1+\lambda)^3} = \frac{u}{18}$ 设解为 λ_0

而：有：$\max g(\lambda) = g(\lambda_0) = 9\left(1 - \frac{2\lambda_0}{(\lambda_0+1)^2}\right) - \lambda_0 u$

而 $(\lambda_0+1)^3 = \frac{18(\lambda_0-1)}{u}$ ∴有 $(\lambda_0+1)^2 = \frac{18(\lambda_0-1)}{u(\lambda_0+1)}$

∴ $g(\lambda_0) = 9 - 18\,\frac{\lambda_0\cdot u(\lambda_0+1)}{18(\lambda_0-1)} - \lambda_0 u$

$= 9 - \frac{\lambda_0 u(\lambda_0+1)}{\lambda_0-1} - \lambda_0 u$

$$= \rho - \lambda_0 u\left(\frac{\lambda_0+1}{\lambda_0-1}+1\right) = \rho - \lambda_0 u \cdot \frac{2\lambda_0}{\lambda_0-1}$$

$$= \rho - 2u\frac{\lambda_0^2}{\lambda_0-1} \Rightarrow \left(\text{而 } u = \frac{18(\lambda_0-1)}{(\lambda_0+1)^3}\right)$$

$$\therefore = \rho - 36\frac{\lambda_0^2}{(\lambda_0+1)^3} \quad \left(\lambda_0 \text{ 为 } \frac{u}{18} = \frac{\lambda-1}{(\lambda+1)^3} \text{ 的解}\right)$$

而 $$\frac{dp^*(0)}{du} \Rightarrow \lim_{\lambda_0\to\lambda^*}\frac{\rho-36\frac{\lambda_0^2}{(\lambda_0+1)^3}-\left(\rho-36\frac{\lambda_*^2}{(\lambda_*+1)^3}\right)}{u-0}$$

$$= \lim_{\lambda\to\lambda^*}\frac{36\left(\frac{\lambda_*^2}{(\lambda_*+1)^3}-\frac{\lambda_0^2}{(\lambda_0+1)^3}\right)}{18\cdot\left(\frac{\lambda_0-1}{(\lambda_0+1)^3}-\frac{\lambda_*-1}{(\lambda_*+1)^3}\right)}$$

$$= \lim_{\lambda_0\to\lambda^*} 2\cdot\frac{(\lambda_*^2-\lambda_0^2)}{(\lambda_0-\lambda_*)} = \lim_{\lambda_0\to\lambda^*} 2\cdot\times(-1)(\lambda_*+\lambda_0)$$

?