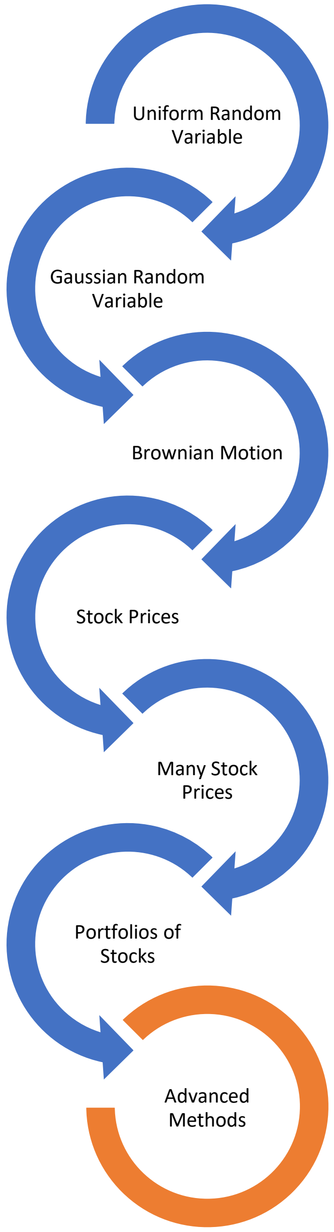
Introduction to Numerical Methods in Finance

MMF Course No.2021H

September 2023 – November 2023

Course Agenda

- Module 1: Uncertainty and Simulating the Future
    1. Random Variables
    2. Random Walks
    3. Portfolios of Random Walks
- Module 2: Risk and Reward in the Future
    1. Value at Risk
    2. Pricing Derivatives
    3. PDE's and Copulas
- Module 3: Predicting the Future
    1. **Credit Risk**
    2. Econometrics/Optimization/Factor Analysis
    3. Statistical Pattern Recognition
- What you should be able to do after this course:
    - Create a macro-economic forecast
    - ✓ Price a tailor-made derivative instrument
    - ✓ Calculate the risk of a portfolio
    - Optimize the performance of a portfolio of risks



Admin

- Assignments
  - 3 Projects – 30% + 30% + 30%
  - Participation/Homework – 10%
  - No final exam

- References that might be helpful
- Chatfield. The Analysis of Time Series: An Introduction, 1997.
- Martinez and Martinez. Computational Statistics Handbook with MATLAB, 2002
- Alexander. Market Models, 2001
- Neftci. Principles of Financial Engineering, 2004.
- Brandimarte. Numerical Methods in Finance and Economics, 2006.
- Lewinson. Python for Finance Cookbook, 2022.

- By the end of this course you should have 3 projects that you can show to potential employers to highlight your understanding
  - Pairs Trading
  - Credit Risk

- Structure of Lecture
  - 3 hours 3 topics
  - Describe a focus problem for the lecture using all 3 topics to solve
  - Introduction
  - Theory, methods, and tools
  - Solve the focus problem together
  - 1-2 Breaks and questions

- Send homework to taha.jaffer@gmail.com

Project 2 – Credit Risk

- Consider a portfolio of two BBB bonds with two different counterparties, with marginal default probabilities given by the table below, and a dependence structure given by:
  - A. A multivariate Gaussian distribution with a correlation of 0.5
  - B. A Gaussian copula with correlation of 0.5

	Bond Rating	Probability(%)	Value	$\sum_i p_i V_i$	$\sum_i p_i (V_i - m)^2$
	AAA	0.02	**$109.37**	0.02	0.00
	AA	0.33	**$109.19**	0.36	0.01
	A	5.95	**$108.66**	6.47	0.15
BBB · ⟶	BBB	86.93	**$107.55**	93.49	0.19
	BB	5.30	**$102.02**	5.41	1.36
	B	1.17	**$98.10**	1.15	0.95
	CCC	0.12	**$83.64**	0.10	0.66
	Default	0.18	**$51.13**	0.09	5.64

- Simulate one million future portfolio values to calculate the expected credit loss and the 99% quantile of the loss distribution
- Compare A and B above. Which one produces the highest loss prediction? Can you tell why?
- Due: November TBD, 2023 (last class or a little thereafter)

Introduction

- Topic #1. Introduction to Credit Risk
- Topic #2. Merton and KMV Models
- Topic #3. Estimating Credit Risk with Copulas

- Today's Problem:
  - Simple Merton model
  - 2 bond portfolio
- Today's Homework
  - None. Focus on Project #2

Module 2
Risk and Reward in the Future

Class 7. Credit Risk

Topic 1

Credit Risk

Credit Risk

- Credit risk is the probability of a financial loss resulting from a borrower's failure to repay a loan. Essentially, credit risk refers to the risk that a lender may not receive the owed principal and interest, which results in an interruption of cash flows and increased costs for collection.

- In bond trading, a credit spread, also known as a yield spread, is the difference in yield between two debt securities of the same maturity but different credit quality. Credit spreads are measured in basis points, with a 1% difference in yield equal to a spread of 100 basis points.

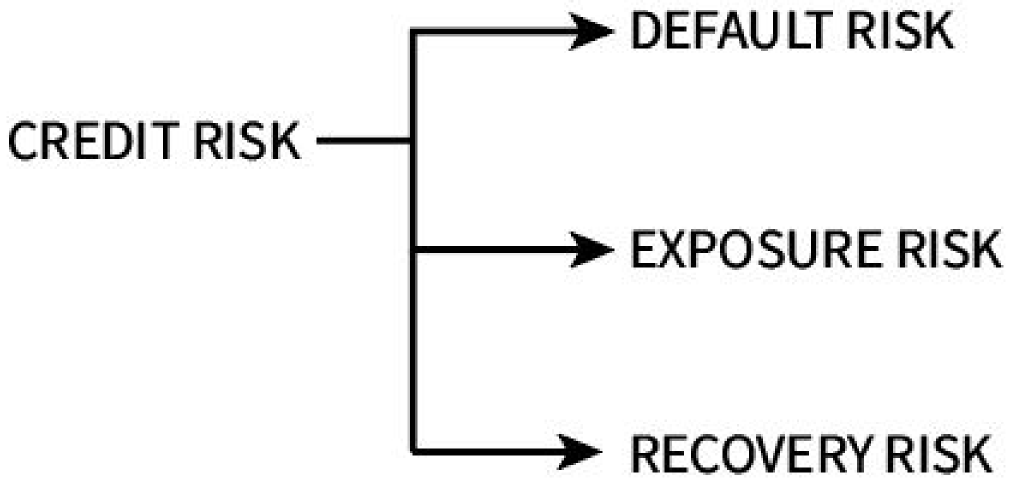
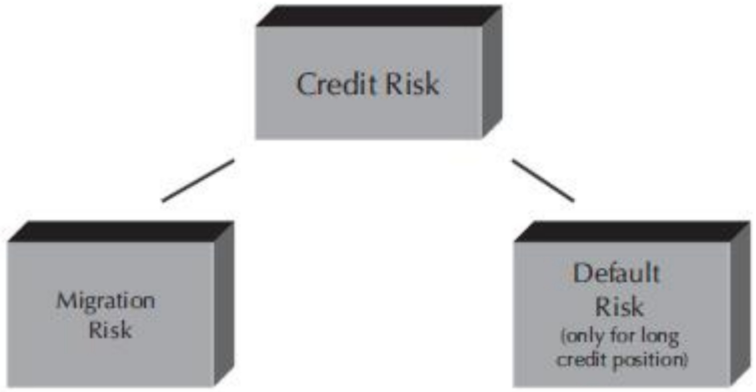


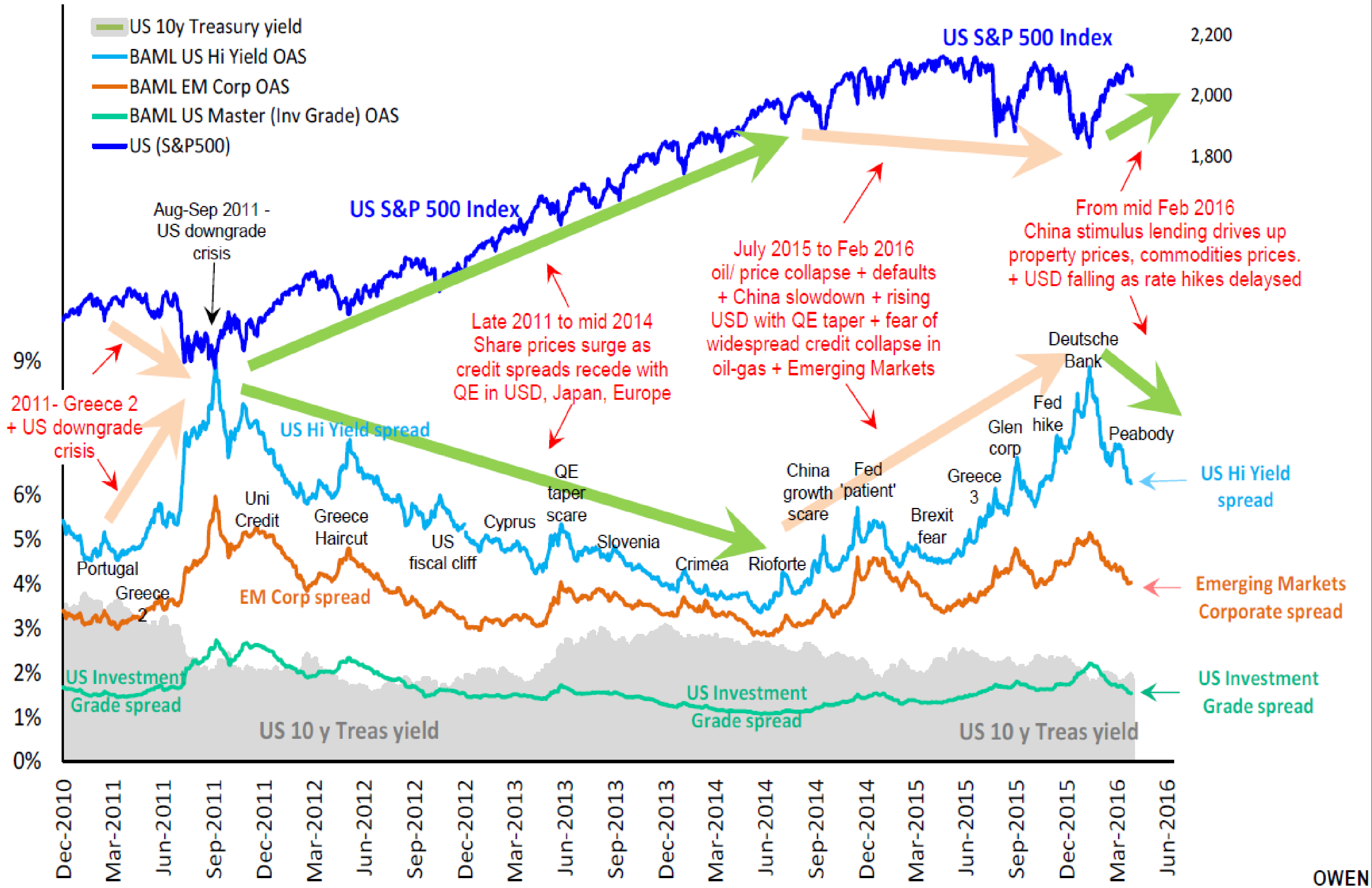
Figure 11.1 Types of credit risk



Credit spreads

- The higher the level of credit risk, the greater the cost of borrowing.
- Once companies lose investment grade status, the cost of borrowing increases significantly (high yield / junk bonds).

Credit Spreads & Share Prices



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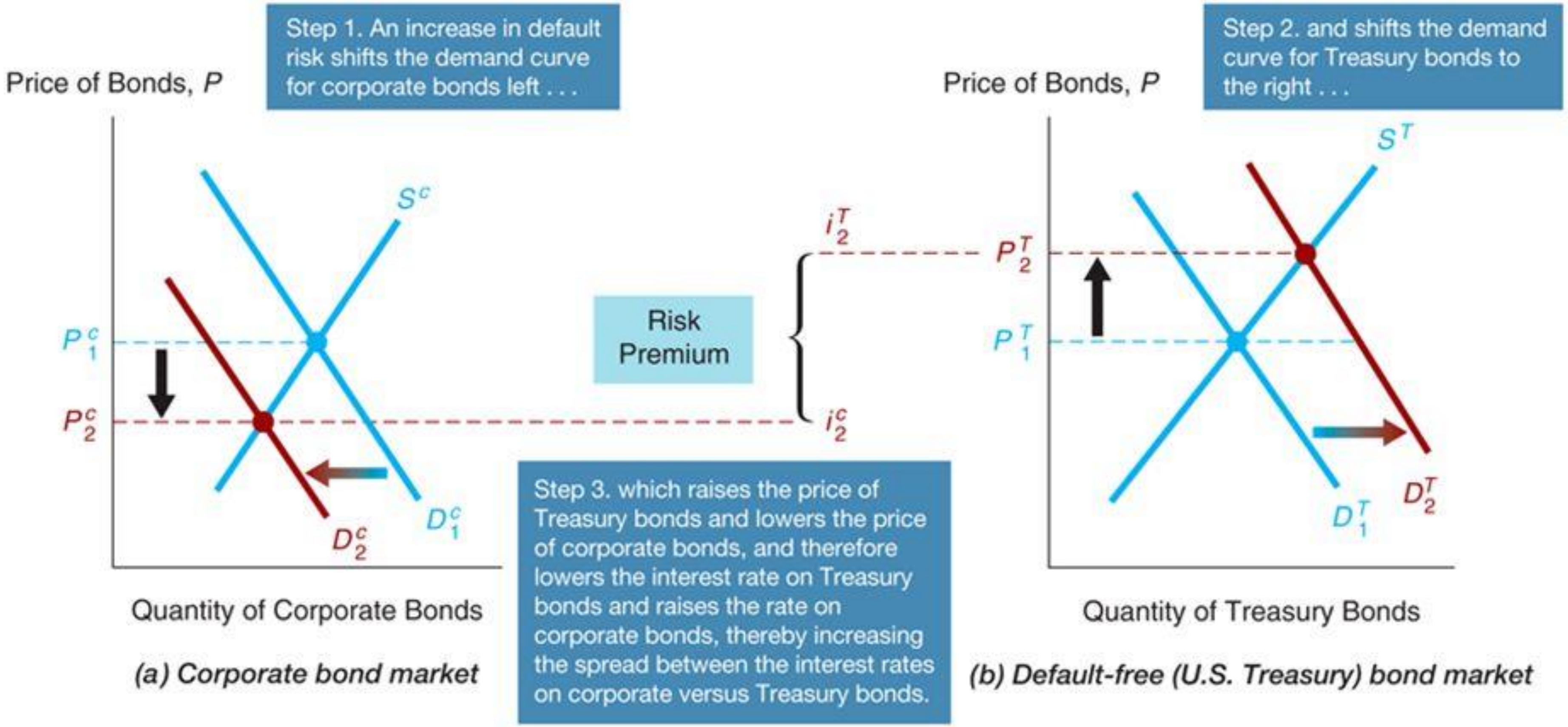
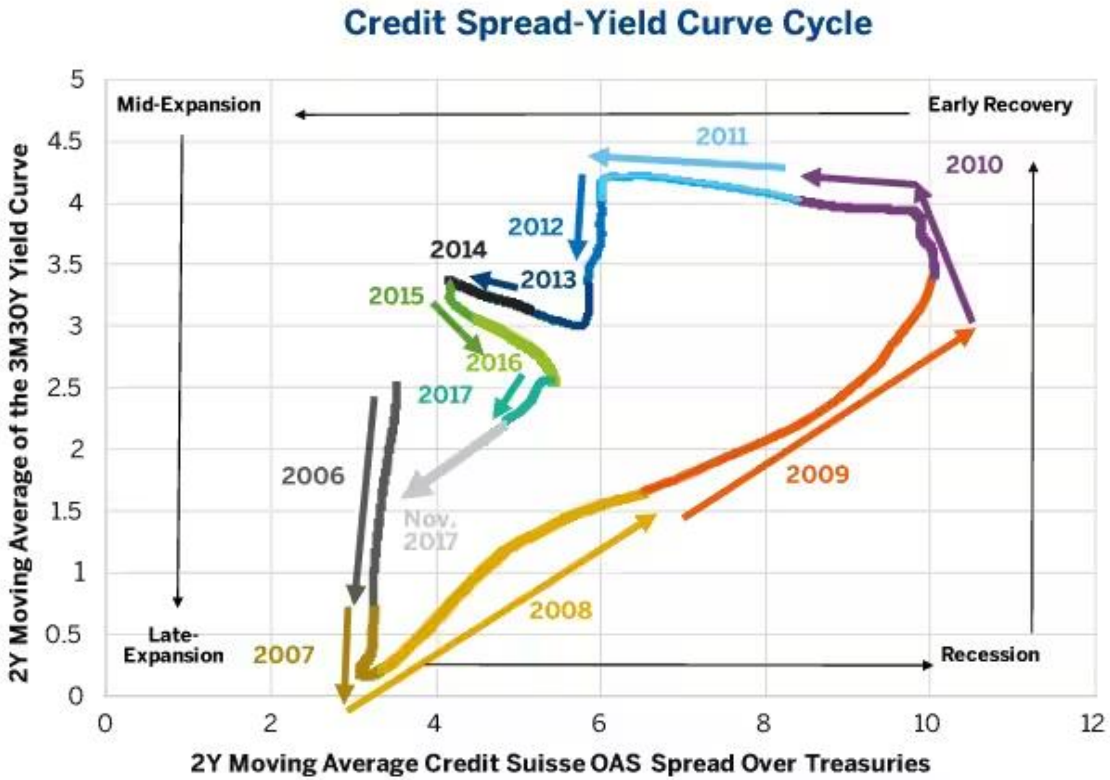
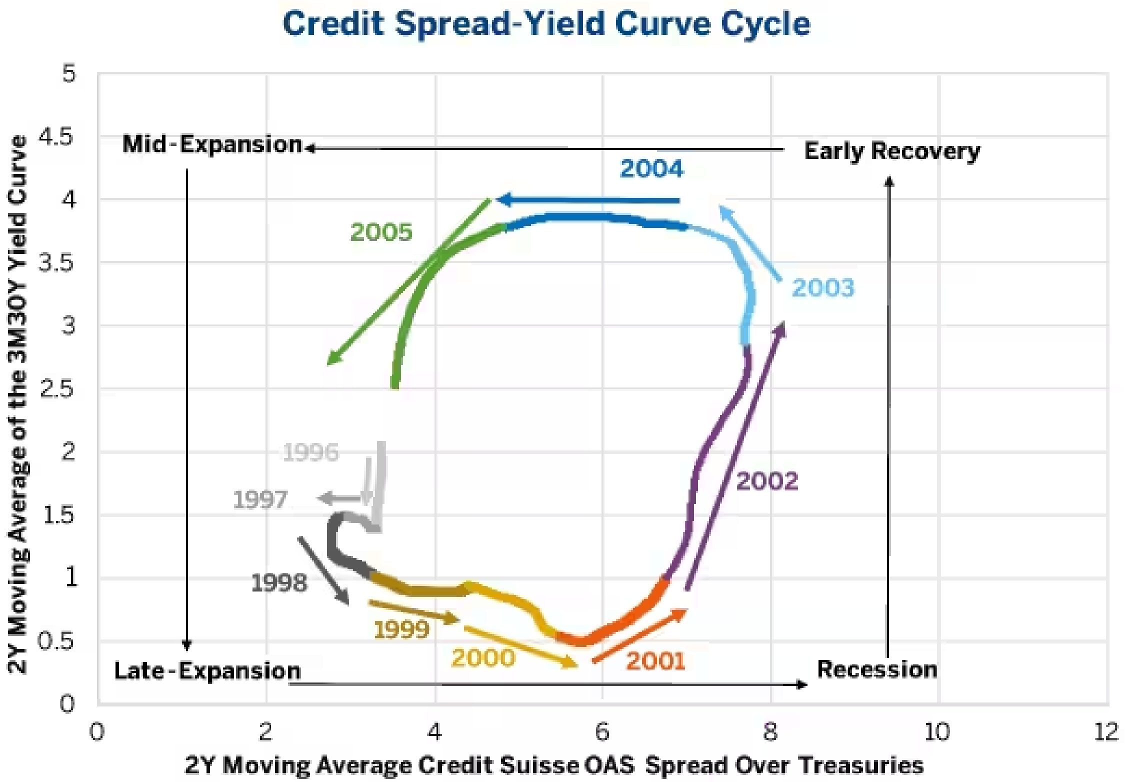
Response to an Increase in Default Risk on Corporate Bonds



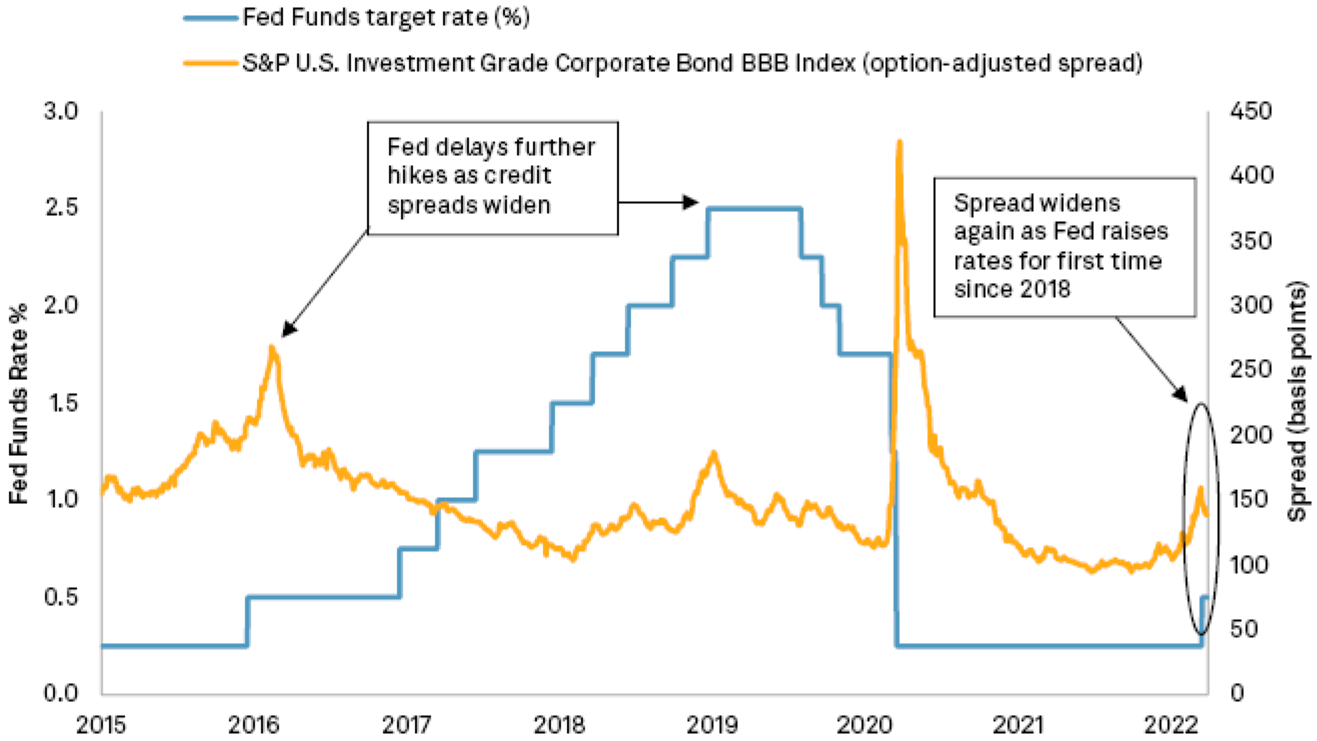


Source: Bloomberg Professional (GB3, USGG30YR, LF98OAS), CME Economic Research Calculations



Source: Bloomberg Professional (GB3, USGG30YR, LF98OAS), CME Economics Research Calculations

Previous pivots in Fed policy have led to spikes in credit spreads



Data as of March 30, 2022.
Sources: S&P Global Market Intelligence; S&P Dow Jones Indices

Chart 1

Comparison of distribution of market returns and credit returns

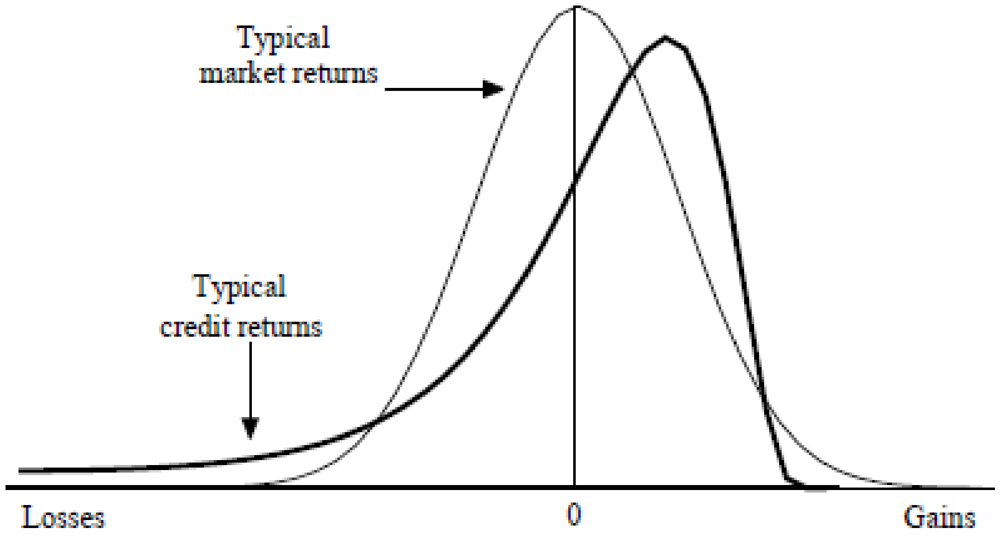
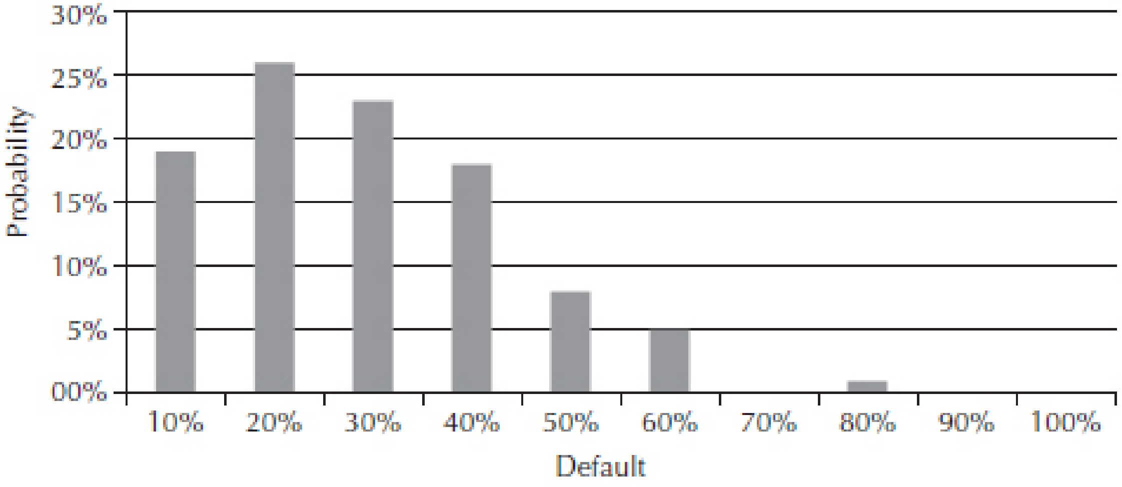


Figure 6.8 A loss distribution of the Gaussian copula model. Inputs are: 10 assets, default probability of every asset 5%, recovery rate 5%, correlations as in Table 6.1. See www.dersoft.com/CDOGausseducational.xlsm.



One-Year Rating Transition Matrix (% probability, Moody's 1970-2010)

Table 18.1 page 401

Initial Rating	Rating at year end Aaa	Aa	A	Baa	Ba	B	Caa	Ca-C	Default
Aaa	90.42	8.92	0.62	0.01	0.03	0.00	0.00	0.00	0.00
Aa	1.02	90.12	8.38	0.38	0.05	0.02	0.01	0.00	0.02
A	0.06	2.82	90.88	5.52	0.51	0.11	0.03	0.01	0.06
Baa	0.05	0.19	4.79	89.41	4.35	0.82	0.18	0.02	0.19
Ba	0.01	0.06	0.41	6.22	83.43	7.97	0.59	0.09	1.22
B	0.01	0.04	0.14	0.38	5.32	82.19	6.45	0.74	4.73
Caa	0.00	0.02	0.02	0.16	0.53	9.41	68.43	4.67	16.76
Ca-C	0.00	0.00	0.00	0.00	0.39	2.85	10.66	43.54	42.56
Default	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Cumulative default rates and transition matrix

Table 24

Global Corporate Average Cumulative Default Rates (1981 - 2016) (%)

	--Time horizon (years)--														
Rating	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
AAA	0.00	0.03	0.13	0.24	0.35	0.46	0.52	0.60	0.66	0.72	0.75	0.78	0.81	0.88	0.94
AA	0.02	0.06	0.13	0.23	0.33	0.44	0.54	0.62	0.69	0.77	0.85	0.91	0.98	1.05	1.11
A	0.06	0.15	0.25	0.38	0.53	0.69	0.88	1.05	1.23	1.41	1.57	1.73	1.89	2.03	2.20
BBB	0.18	0.51	0.88	1.33	1.78	2.24	2.63	3.01	3.39	3.76	4.16	4.48	4.79	5.10	5.43
BB	0.72	2.24	4.02	5.80	7.45	8.97	10.26	11.41	12.42	13.33	14.06	14.71	15.29	15.80	16.34
B	3.76	8.56	12.66	15.87	18.32	20.32	21.96	23.23	24.37	25.43	26.34	27.03	27.64	28.21	28.80
CCC/C	26.78	35.88	40.96	44.06	46.42	47.38	48.56	49.52	50.38	51.03	51.55	52.10	52.81	53.37	53.37
Investment grade	0.10	0.27	0.46	0.71	0.96	1.21	1.45	1.67	1.89	2.11	2.33	2.51	2.69	2.86	3.05
Speculative grade	3.83	7.48	10.63	13.20	15.29	17.01	18.45	19.65	20.71	21.67	22.47	23.13	23.73	24.27	24.80
All rated	1.52	2.99	4.27	5.35	6.25	7.02	7.67	8.22	8.72	9.18	9.58	9.91	10.22	10.50	10.78

Sources: S&P Global Fixed Income Research and S&P CreditPro®.

Figure 2. Graphical illustration of thresholds for representing transitions through a standard normal variable.

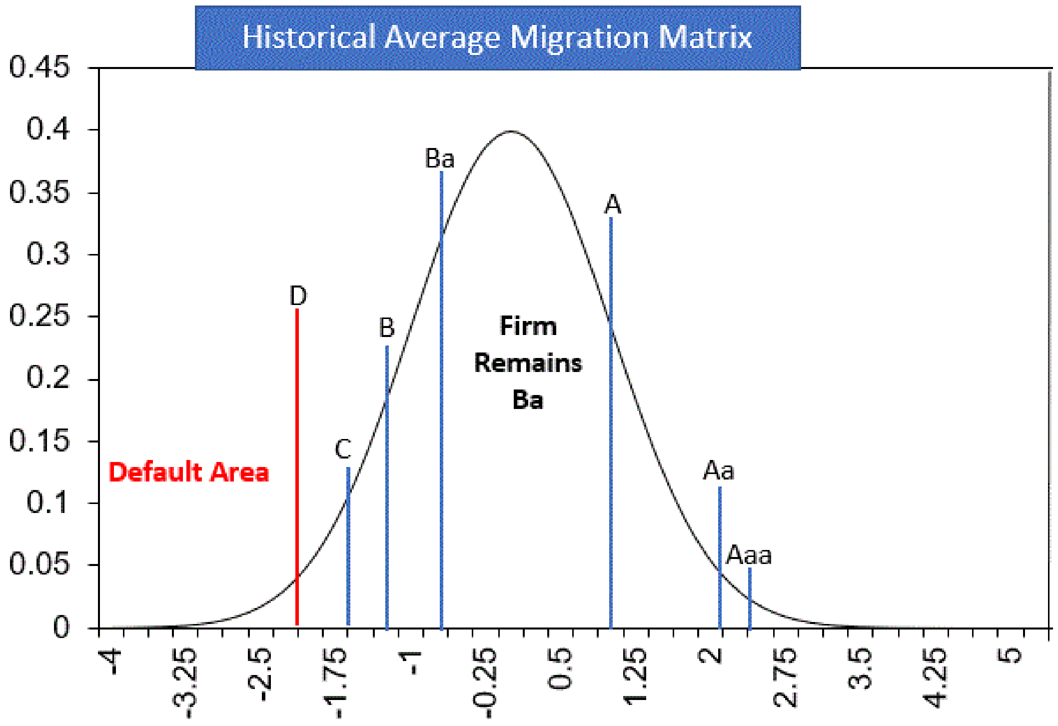
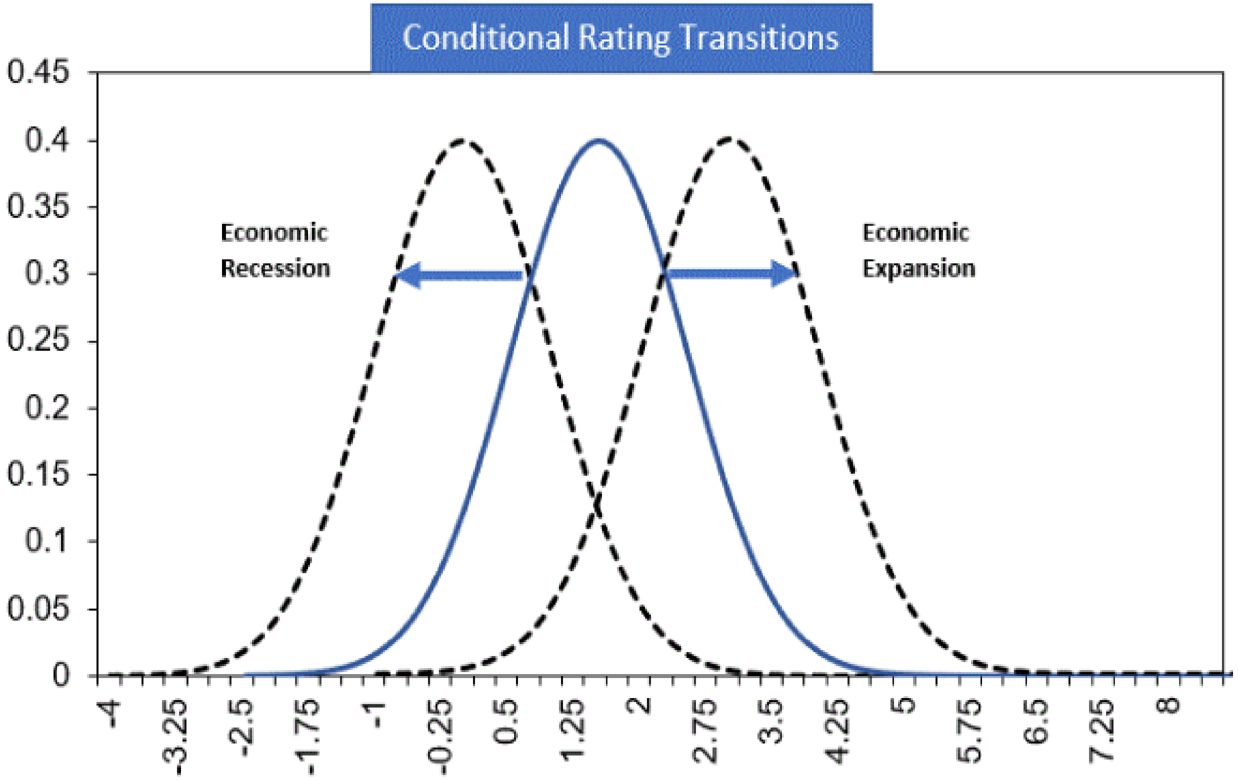


Figure 3. Graphical illustration of shifting the distribution function to change transition probabilities.

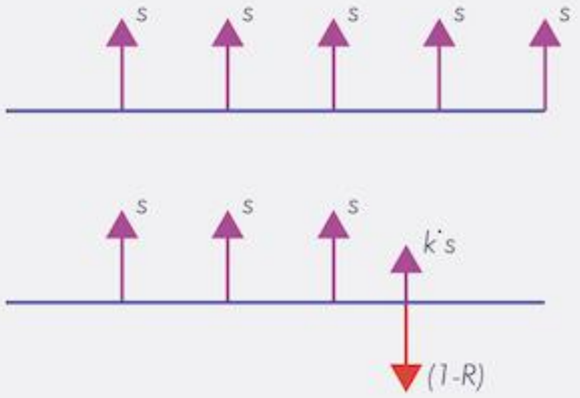


Pricing a Credit Default Swap

- Given a set of default probabilities, we can calculate the fair premium for a CDS
- To do this, consider a CDS as a series of contingent cash flows...
- ...the cash flows depending upon whether a credit event occurs:

No Default

Default

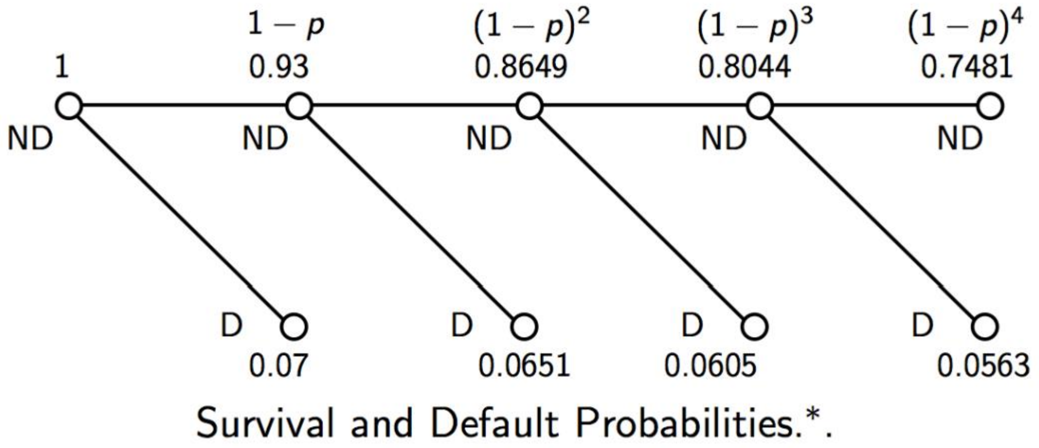


where:

s is the CDS premium

k is the day count fraction when default occurred

R is the recovery rate



Survival and Default Probabilities.*.

Single Period

$1 = (1-p) x ($1 + s) + p x R x $1

$$s = \frac{p(1-R)}{1-p} \approx p(1-R)$$

Approximating Default and Survival Probabilities

There are some approximations to the formulas given earlier:

- One-Period Default Probability

$$PD \approx \frac{s}{1-R} \quad e.g.\ PD_{6mth} = \frac{10.46\%/2}{1-0.3} = 7.5\%$$

  - This compares to the exact answer of 7.0%

- Term Survival Probability

$$PS_N \approx \left(1-\frac{s_N}{1-R}\right)^N \quad e.g.\ PS_{5yr} = \left(1-\frac{6.14\%/2}{1-0.3}\right)^{10} = 63.9\%$$

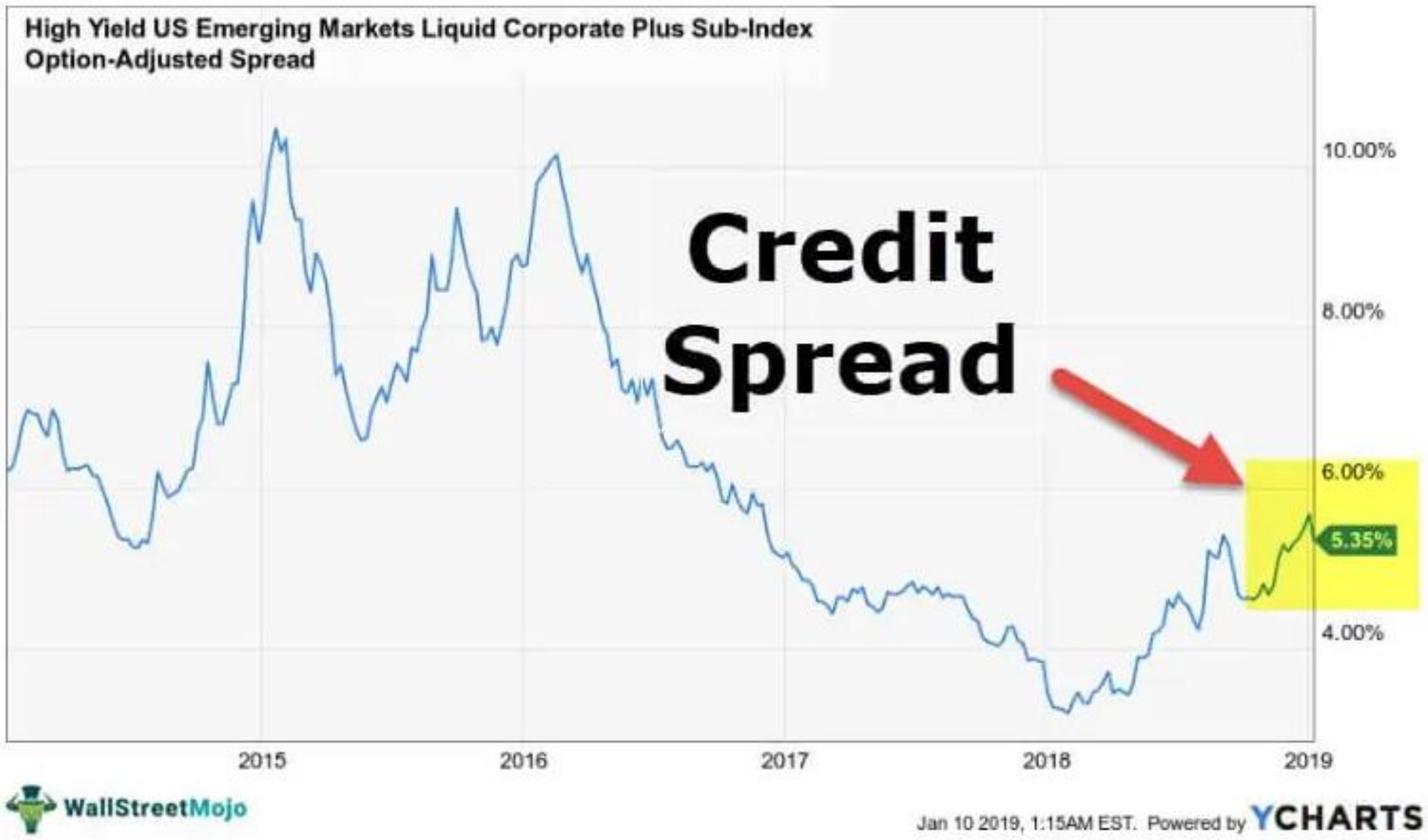
  - This compares to the exact answer of 67.8%

Formula

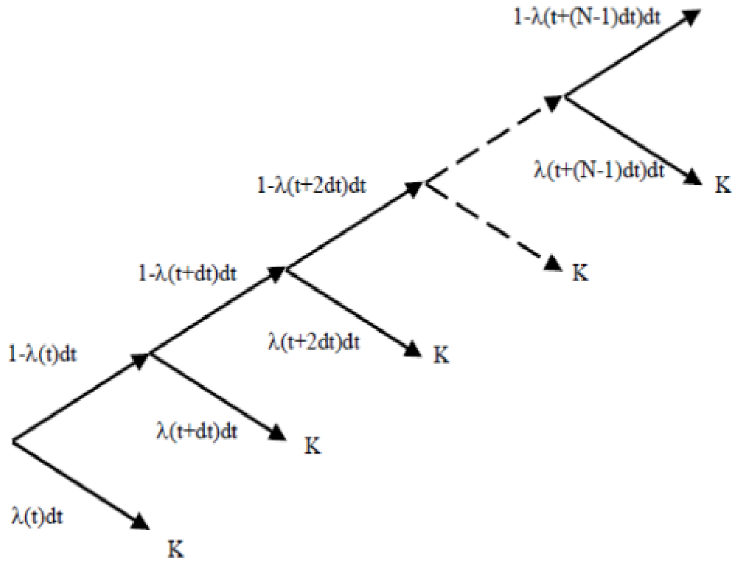
Following is the Credit Spread Formula-

Credit Spread = (1 – Recovery Rate) (Default Probability)

The formula simply states that credit spread on a bond is simply the product of the issuer's probability of default times 1 minus possibility of recovery on the respective transaction.

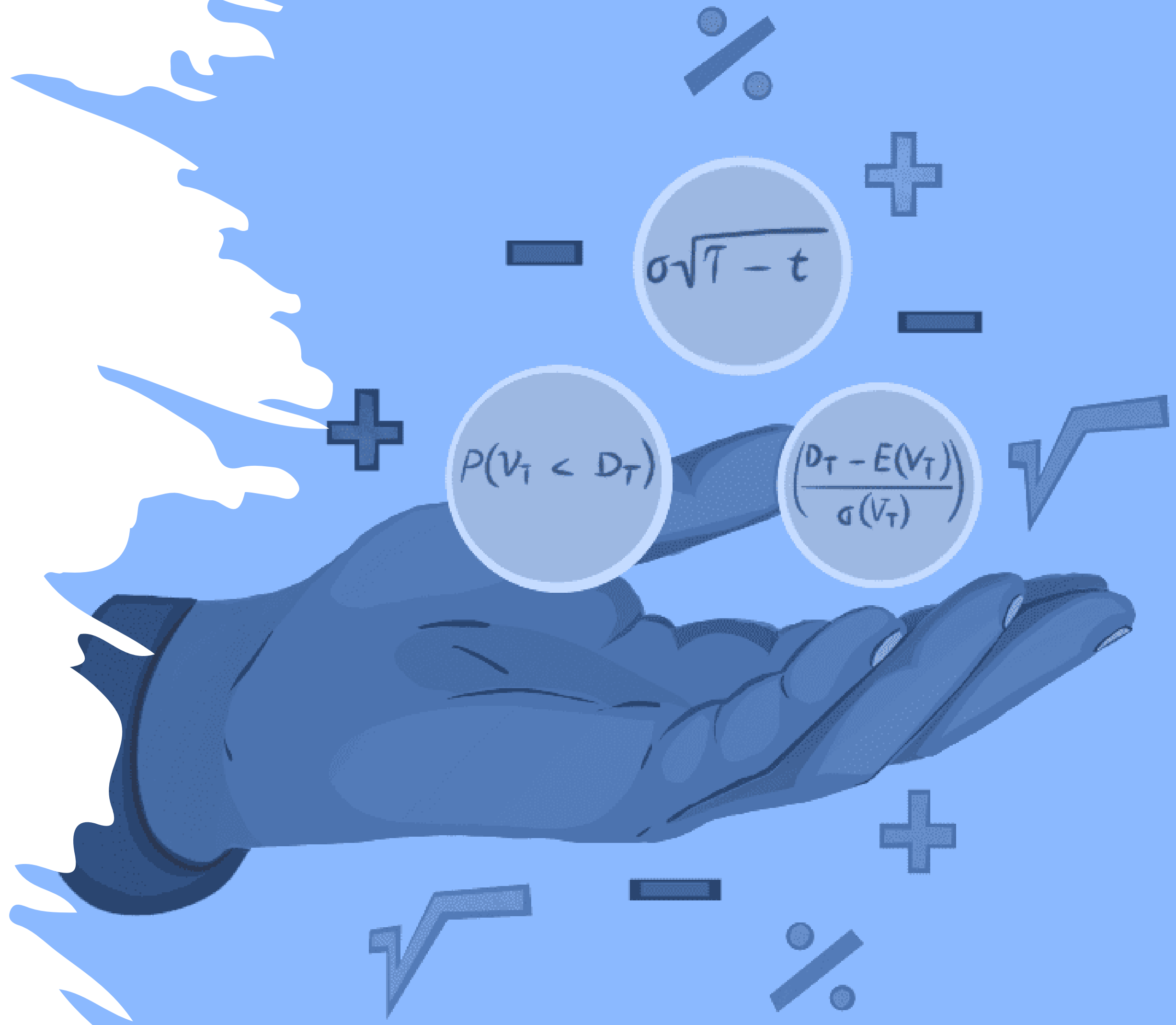


Jarrow and Turnbull model default using a simple two period model. Extending this model to multiple time periods it can easily be seen how they modelled default using a binomial tree, as shown below:



Topic #2

Merton and KMV Models of Credit Risk



Merton Model

['mər-t^{ə}n 'mä-d^{ə}l]

A mathematical formula that stock analysts and commercial loan officers, among others, can use to judge a corporation's risk of credit default.

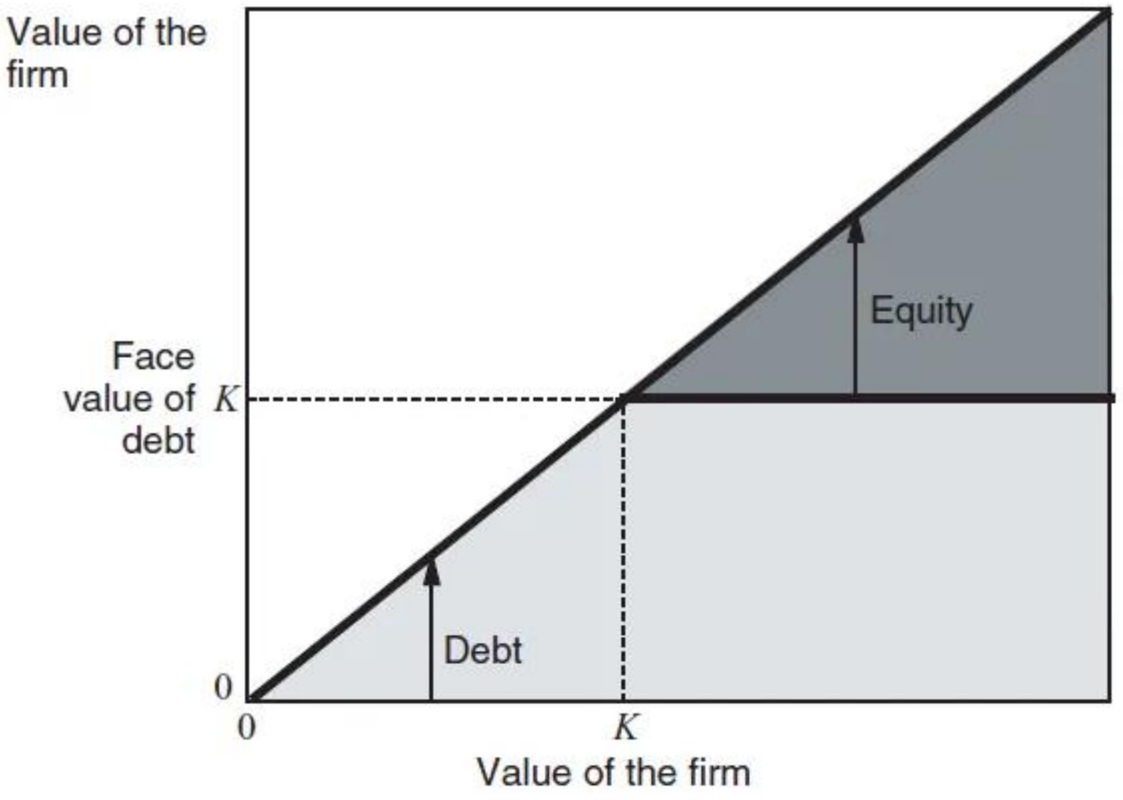
What Is the Merton Model?

The Merton model is a mathematical formula that stock analysts and commercial loan officers, among others, can use to judge a corporation's risk of credit default. Named for economist Robert C. Merton, who proposed it in 1974, the Merton model assesses the structural credit risk of a company by modeling its equity as a call option on its assets.

KEY TAKEAWAYS

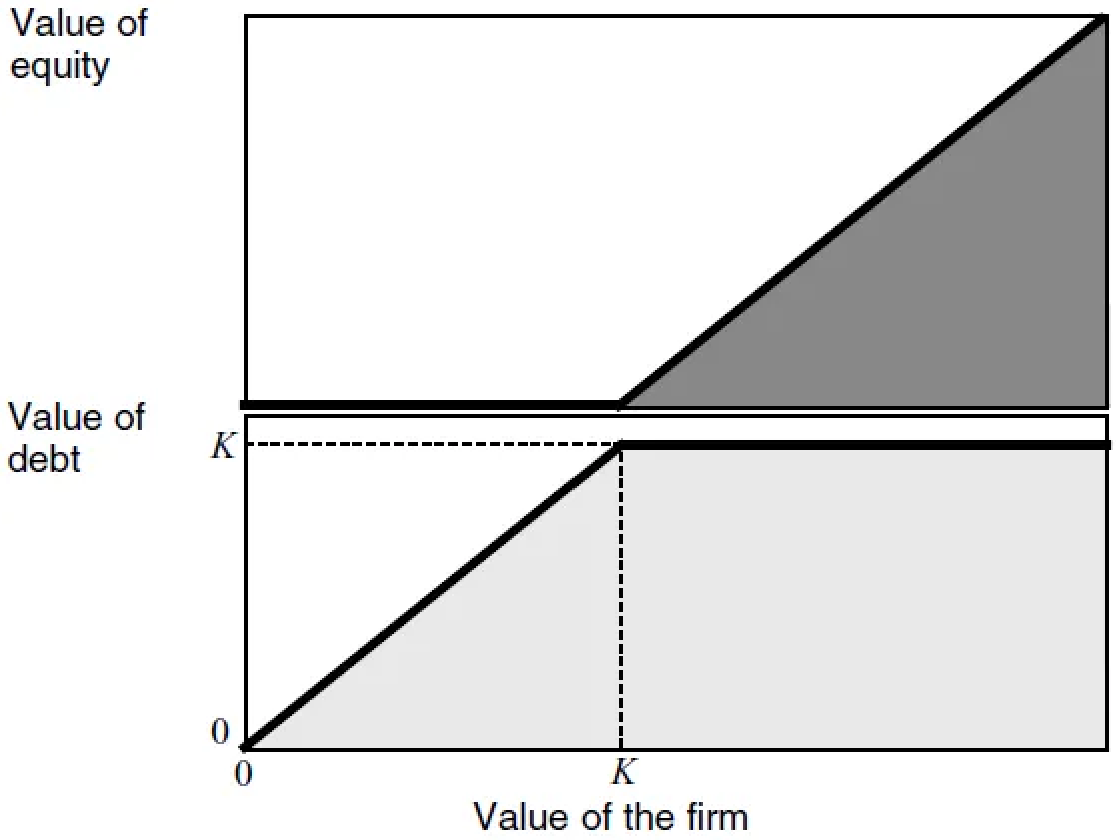
- In 1974, economist Robert C. Merton proposed a model for assessing the credit risk of a company by modeling its equity as a call option on its assets.
- The Merton model is used today by stock analysts, commercial loan officers, and others.
- Merton's work, and that of fellow economist Myron S. Scholes, earned the Nobel Prize for economics in 1997.

Company Value = Equity + Debt



$$B_T = V_T - S_T = V_T - \mathrm{Max}(V_T - K, 0) = \mathrm{Min}(V_T, K)$$

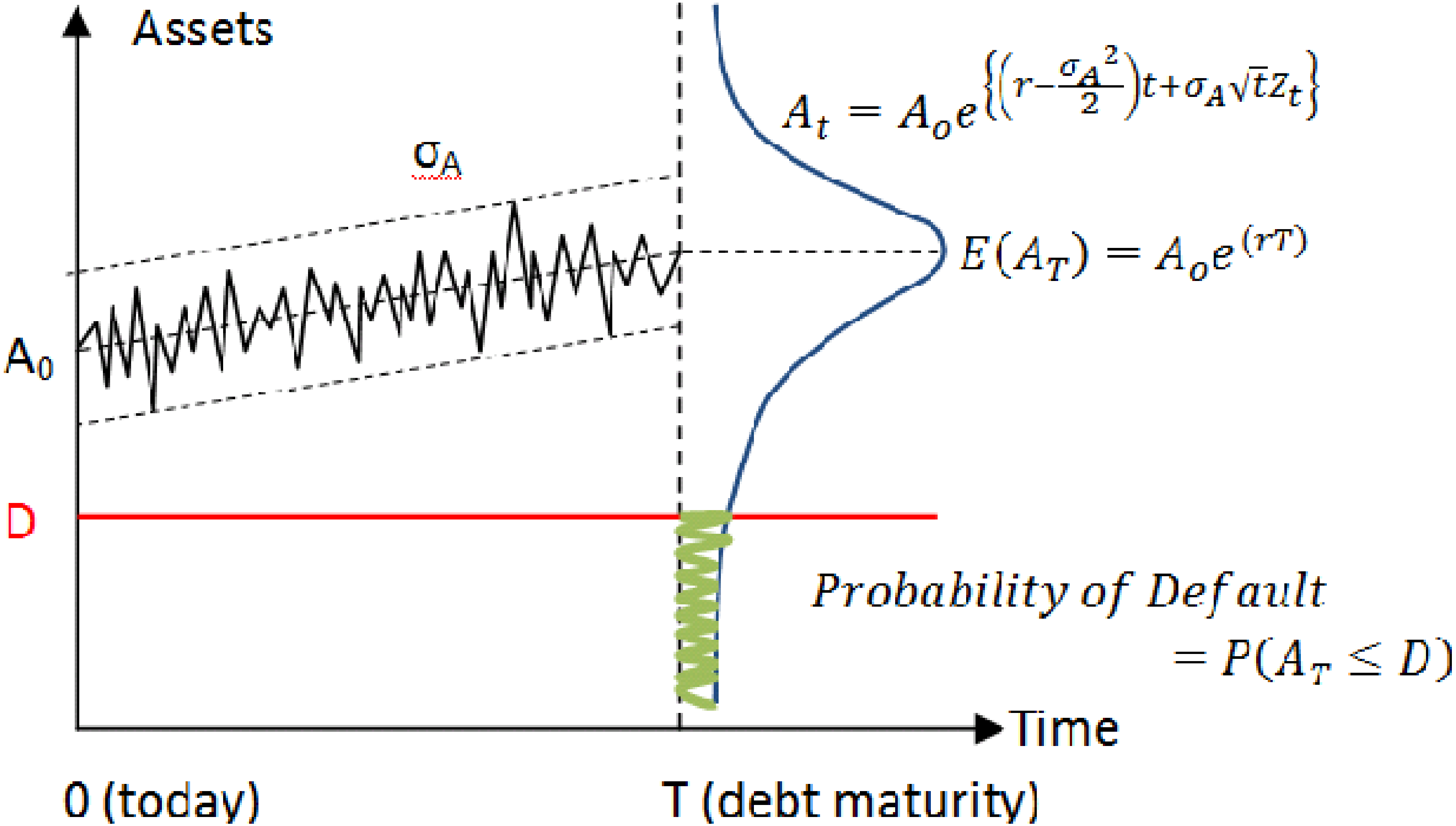
$$S_T = \mathrm{Max}(V_T - K, 0)$$



$$B_T = K - \mathrm{Max}(K - V_T, 0)$$

- A long position in a risky bond is equivalent to a long position in a risk-free bond plus a short put option.
- The short put option is really a credit derivative, like the risky bond.
- This explains the left skewness in credit losses for risky debt (similar to a short option position.)
- It also shows that equity is equivalent to an option on the value's assets; due to the limited liability of the firm, investors can lose no more than their original investment.

Stock Price = max(Assets – Liabilities, 0)



The Formula for the Merton Model

$$E = V_t N(d_1) - Ke^{-r\Delta T} N(d_2)$$

where:

$$d_1 = \frac{\ln \frac{V_t}{K} + \left(r + \frac{\sigma_v^2}{2}\right)\Delta T}{\sigma_v \sqrt{\Delta T}}$$

and

$$d_2 = d_1 - \sigma_v \sqrt{\Delta t}$$

E = Theoretical value of a company's equity

V_t = Value of the company's assets in period t

K = Value of the company's debt

t = Current time period

T = Future time period

r = Risk-free interest rate

N = Cumulative standard normal distribution

e = Exponential term ($i.e.$ 2.7183...)

σ = Standard deviation of stock returns

What Does the Merton Model Tell You?

The Merton model allows for easier valuation of a company and helps analysts determine if it will be able to retain solvency, by analyzing the maturity dates of its debt and its debt totals.

The Merton model calculates the theoretical pricing of European put and call options without considering dividends paid out during the life of the option. The model can, however, be adapted to consider dividends by calculating the ex-dividend date value of underlying stocks.

The Merton model makes the following basic assumptions:

- All options are European options and are exercised only at the time of expiration.
- No dividends are paid out.
- Market movements are unpredictable (efficient markets).
- No commissions are included.
- Underlying stocks' volatility and risk-free rates are constant.
- Returns on underlying stocks are regularly distributed.

1.4 Merton's model strengths

- It is easy to compute and provide intuitive results.
- It is able to depict the potential conflict of interest between shareholders and debt holders. The first ones have an interest in the company investing in risky asset project, that increases the volatility of the underlying asset by the way guarantee higher returns, while the second one prefers a less volatile and less risky asset's company value.

1.5 Merton's model flaws

- It does not take into account extrem or rare events by assuming that we are in the Gaussian world.
- The default occurs only at maturity and this is not a realistic since default can occurs before maturity.
- The default corresponds to the liquidation of the company from the market but this is not always the case in most legislation. In US for example, the so-called Chapter 11 allows companies under bankruptcy to try a reorganization, in order to try to become profitable again.

Topic 3

Credit Risk with Copulas

The Portfolio Loss Distribution

- As the number of loans goes to infinity we can derive the limiting CDF of the loss rate L to be

$$F_L(x; PD, \rho) = \Pr[L < x] = \Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)$$

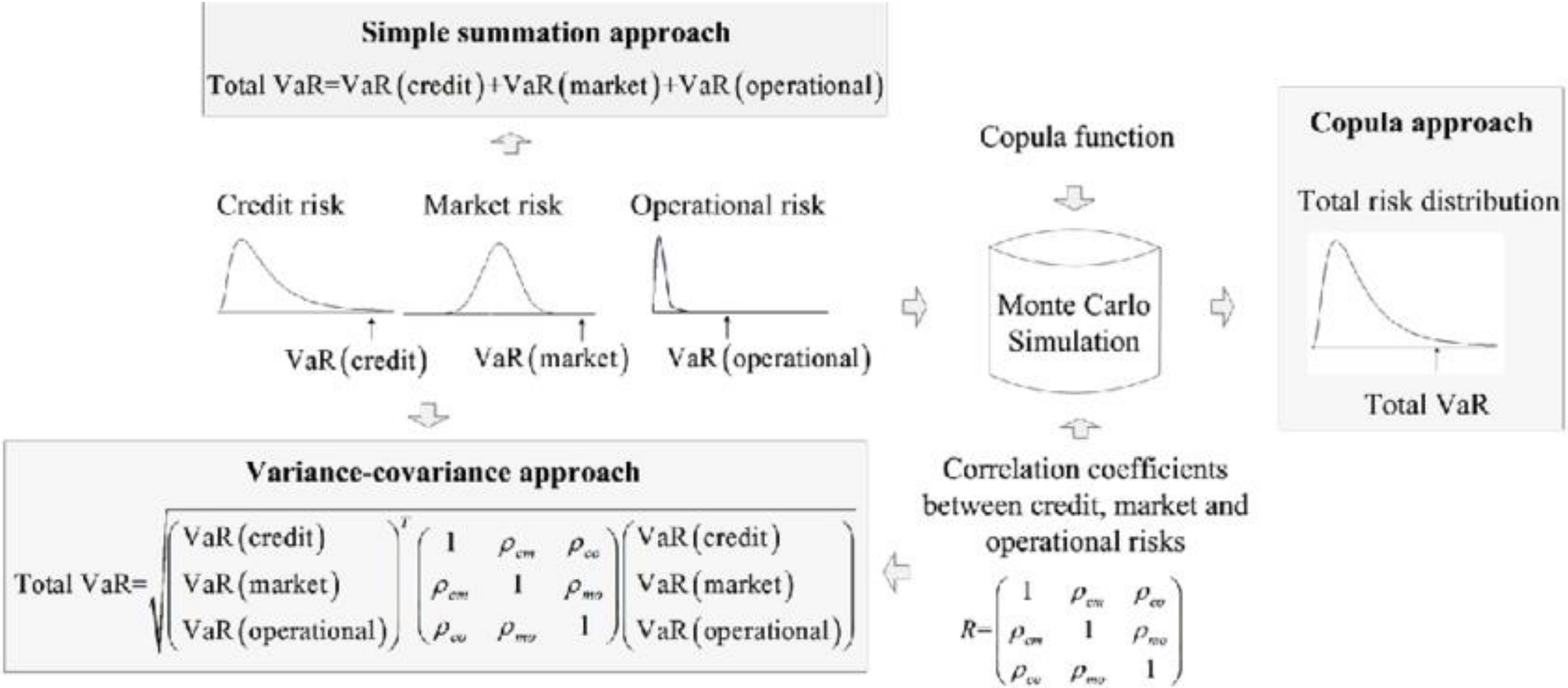
- where $\Phi^{-1}(\bullet)$ is the standard normal inverse CDF
- The portfolio loss rate distribution thus appears to have similarities with the normal distribution but the presence of the $\Phi^{-1}(x)$ term makes the distribution highly nonnormal
- This distribution is sometimes known as the Vasicek distribution, from Oldrich Vasicek who derived it

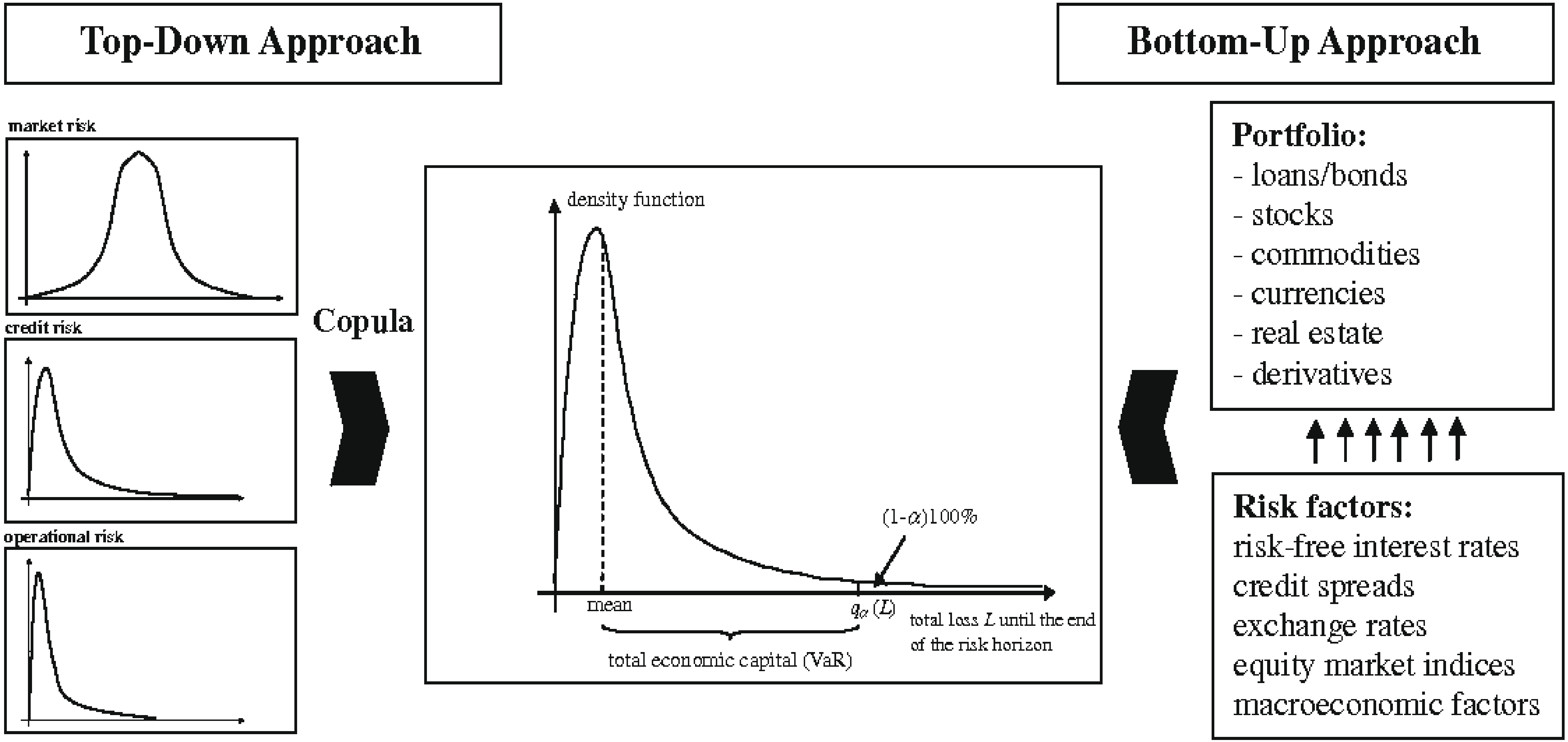
Example 24.7

Suppose that a bank has a total of \$100 million of retail exposures. The 1-year probability of default averages 2% and the recovery rate averages 60%. The copula correlation parameter is estimated as 0.1. In this case,

$$V(0.999, 1) = N\left(\frac{N^{-1}(0.02) + \sqrt{0.1}\,N^{-1}(0.999)}{\sqrt{1-0.1}}\right) = 0.128$$

showing that the 99.9% worst case default rate is 12.8%. The 1-year 99.9% credit VaR is therefore $100 \times 0.128 \times (1 - 0.6)$ or \$5.13 million.

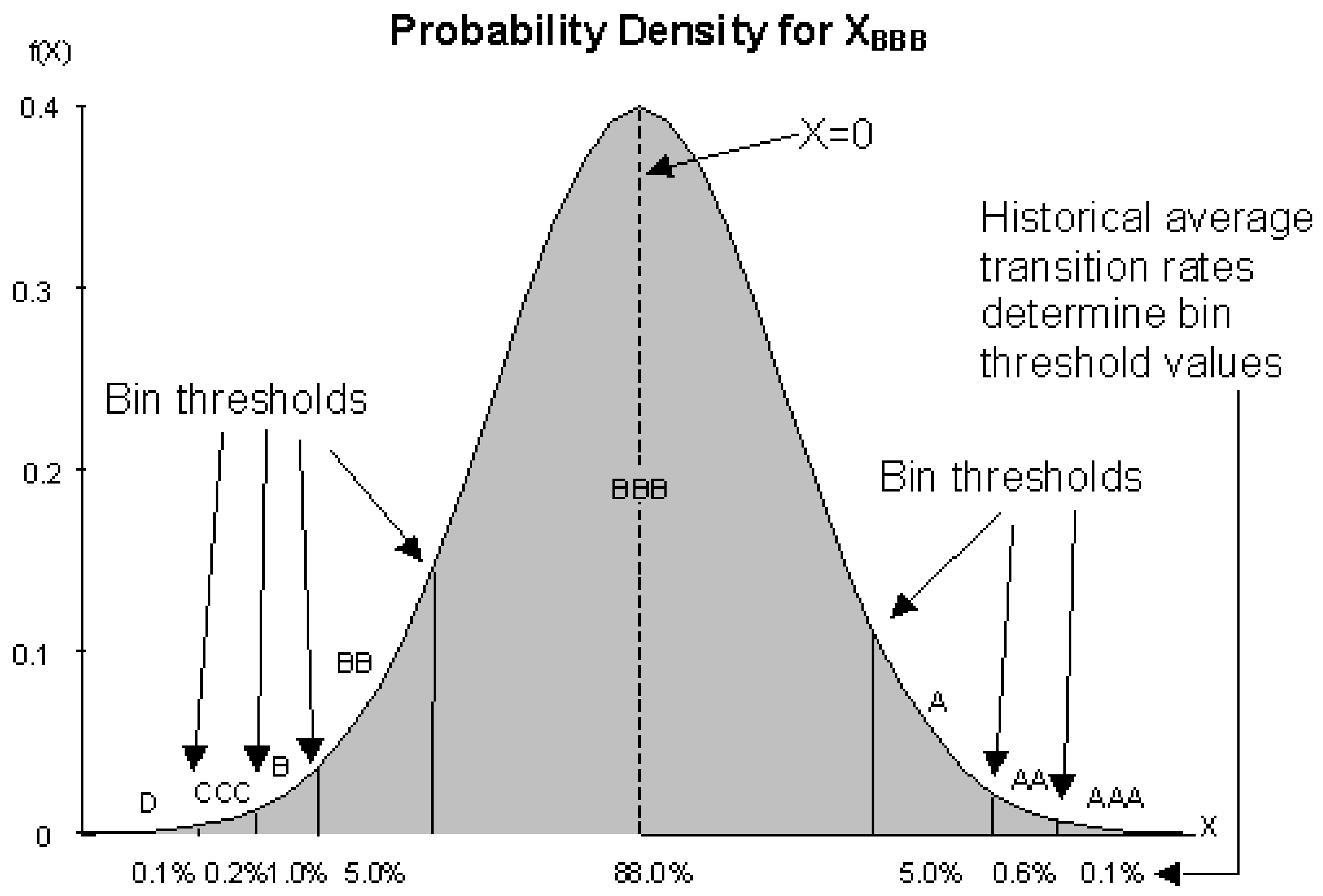

Simple summation approach
Total VaR=VaR (credit)+VaR (market)+VaR (operational)
Credit risk
Market risk
Operational risk
VaR (credit)
VaR (market)
VaR (operational)
Copula function
Monte Carlo Simulation
Copula approach
Total risk distribution
Total VaR
Variance-covariance approach
Total VaR=
VaR (credit)
VaR (market)
VaR (operational)
Correlation coefficients between credit, market and operational risks
R=

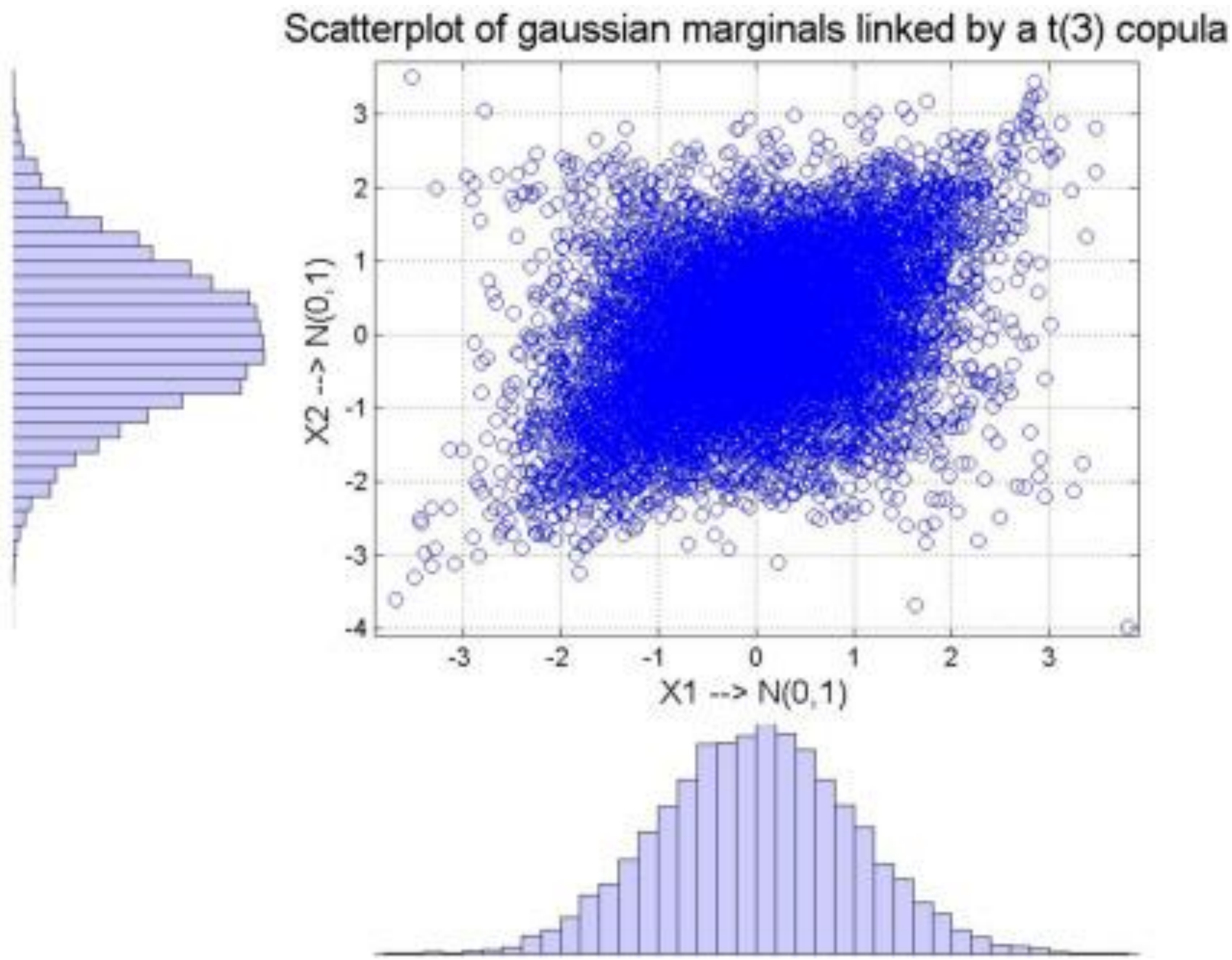


Top-Down Approach
Bottom-Up Approach
market risk
credit risk
operational risk
Copula
density function
mean
$q_\alpha(L)$
(1-α)100%
total loss L until the end of the risk horizon
total economic capital (VaR)
Portfolio:
- loans/bonds
- stocks
- commodities
- currencies
- real estate
- derivatives
Risk factors:
risk-free interest rates
credit spreads
exchange rates
equity market indices
macroeconomic factors

Current Rating	Possible Future Rating	Probability	Resulting Value
	AAA	0.02%	$ 101.69
	AA	0.33%	$ 101.47
	A	5.95%	$ 101.03
BBB	BBB	86.9%	$ 100.00
	BB	5.30%	$ 94.86
	B	1.17%	$ 91.21
	C	0.12%	$ 77.77
	D	0.18%	$ 47.54

Exhibit 1: Relationship Between Continuous Credit Index X and Rating Transitions




Scatterplot of gaussian marginals linked by a t(3) copula
X1 --> N(0,1)
X2 --> N(0,1)
-3
-2
-1
0
1
2
3
-4

Appendix

A common way to model the default probability is by the hazard rate. As @Bob correctly mentions, a traditional requirement is for it to satisfy (see Option Futures and Other Derivatives section 23.4 in which the author discusses also other more exact approximations):

$$\lambda(t) = \frac{S(t)}{1-R}.$$

This is associated with the default probability by (see Poisson Process):

$$P(t, t+h) = \lambda(t)h + o(h)\,,$$

with $P(t, t+h)$ the probability of a default occurring between t and $t+h$. Therefore:

$$P(0,T) = \int_0^T (1-P(0,t))P(t,t+dt) = \int_0^T \lambda(t)(1-P(0,t))dt\,,$$

where the first term of the integral is "default has not occurred so far" and the second is "default occurs on the next time step". This means that P satisfies:

$$\frac{dP(0,t)}{dt} = \lambda(t)(1-P(0,t)).$$

If the CDS is assumed to be constant then λ is constant and a solution would be:

$$P(0,t) = 1 - \exp\left(\frac{-St}{1-R}\right).$$

Equivalently solution for the CDS is:

$$S = \frac{R-1}{t}\log(1-P(0,t)).$$

Jarrow and Turnbull model default using a simple two period model. Extending this model to multiple time periods it can easily be seen how they modelled default using a binomial tree, as shown below: