MMF1941H: Stochastic Analysis
Revision Questions Set # 8

30 Oct 2023

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1 Infinitesimal Generator of Brownian Motion

Let $(B(t))_{t\geq 0}$ be a Brownian Motion with start in x. We know that the infinitesimal generator of this process takes the form $Af = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$, ie the Laplace operator. Furthermore, we define the transition operator P_t for a function f through $P_t f(x) = \mathbb{E}\left(f(B(t)) \mid B(0) = x\right)$ for $t \geq 0$ and $x \in \mathbb{R}$.

1. Show that for $f(x) = x^k$ for $k = 1, 2, 3$ the relation $P_t f(x) = f(x) + tAf(x)$ holds exactly.
2. Show that for $f(x) = \exp(x)$ the relation $P_t f(x) = f(x) + tAf(x)$ only holds as a first order approximation for small t.

2 Heat Equation

Show that the following functions $u = u(t, x)$ solve the heat equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x).$$

1. $u(t,x) = u_1(t,x) + u_2(t,x)$ where u_1 and u_2 satisfy the heat equation individually.
2. $u(t,x) = g(t)h(x)$ where the functions g, h satisfy $\frac{1}{2}\frac{h''(x)}{h(x)} = \frac{g'(t)}{g(t)}$.
3. $u(t,x) = \exp(x + \frac{t}{2})$. (Hint: Either do this directly, or validate that the condition from the previous example is satisfied.)
4. $u(t,x) = x^2 + t$.
5. $u(t,x) = \Phi\left(\frac{y-x}{\sqrt{t}}\right)$ for $y \in \mathbb{R}$ where Φ is the cumulative distribution function of the standard normal distribution. Note that we already know that any function $u(t,x) = \mathbb{E}(f(B(t)) \mid B(0) = x)$ for f a real-valued function and B Brownian Motion solves the heat equation, so easiest is to determine that u is of this particular form for a suitable f where we also know that $f(x) = \lim_{t\to 0} u(t,x)$ holds.

3 Stochastic Integral of Brownian Motion

Let $(B(t))_{t\geq 0}$ be a standard Brownian Motion. For $\alpha \in [0,1]$ and a partition $0 = t_0 < \ldots < t_n = T$ define

$$I(T;\alpha) = \sum_{i=0}^{n-1} (\alpha B(t_i) + (1-\alpha)B(t_{i+1}))(B(t_{i+1}) - B(t_i))$$

Calculate $\mathbb{E}(I(T;\alpha))$.

4 Moments of Stochastic Integral

For an Ito Process of the form $X(t) = X(0) + \int_0^t f(u)dB(u)$ with deterministic coefficients f we know that $X(t)$ at any time t follows a normal distribution. We also know that from the Ito isometry $\mathbb{E}(X(t)^2) = \int_0^t b^2(u)du$ holds.

1. For $X(t) = \int_0^t dB(s)$ calculate $\mathrm{Var}(X(t))$.
2. For $X(t) = \int_0^t s^\alpha dB(s)$ for $\alpha > 0$ calculate $\mathrm{Var}(X(t))$.
3. For $X(t) = \int_0^t \exp(s)dB(s)$ calculate $\mathrm{Var}(X(t))$.

5 Direct Solution of Stochastic Integral

Let $(B(t))_{t\geq 0}$ be a standard Brownian Motion. Show directly from the definition of the stochastic integral that

$$\int_0^t s dB(s) = tB(t) - \int_0^t B(s)ds$$

holds. (Hint: For any partition $0 = t_0 < \ldots < t_n = t$ you can see that

$$t_{i+1}B(t_{i+1}) - t_i B(t_i) = t_i(B(t_{i+1}) - B(t_i)) + (t_{i+1} - t_i)B(t_{i+1})$$

holds for $i = 0, \ldots, n$.)

6 Ito Formula and Martingale Property

1. Use Ito's Formula to prove that

   $$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds$$

   holds.

2. Use Ito's Formula to prove that

   $$\int_0^t s dB(s) = tB(t) - \int_0^t B(s)ds$$

   holds.

7 Martingale Property of Product Process

Use Ito's Formula to write the following stochastic processes $(X(t))_{t\geq 0}$ in the form

$$dX(t) = a(t)dt + b(t)dB(t)$$

where $(B(t))_{t\geq 0}$ denotes Brownian Motion.

Define a process $(M(t))_{t\geq 0}$ by setting

$$M(t) = \exp(-\int_0^t a(u)dB(u) - \frac{1}{2}\int_0^t a^2(s)ds).$$

1. Use Ito's formula to calculate the differential form $dM(t)$.
2. Use Ito's formula to calculate the differential form $d(M(t)X(t))$.
3. Conclude that the process $(M(t)X(t))_{t\geq 0}$ is a martingale. (Hint: You simply need to validate that in the form of $M(t)X(t)$ all non-stochastic integrals cancel out).

8 Martingale Representation Theorem

Let $(B(t))_{t\geq 0}$ be a standard Brownian Motion. We know from the Martingale Representation Theorem that each martingale $(M(t))_{t\geq 0}$ can be written in the form

$$M(t) = \mathbb{E}(M(0)) + \int_0^t g(s)dB(s)$$

for a suitable integrand g. Use Ito's Formula to derive the explicit form of the integrand for the following martingales $(M(t))_{t\geq 0}$.

1. $M(t) = B(t)$.
2. $M(t) = B(t)^2 - t$.
3. $M(t) = \exp(B(t) - \frac{1}{2}t)$.

9 Assignment: Digital Option Pricing with Default Risk

- Let τ be the default time of an option seller and let $S = s\exp(\sigma Z - \frac{1}{2}\sigma^2)$ where Z follows a standard normal distribution and parameters $s > 0$, $\sigma > 0$.
- We are looking again at the *risk-adjusted digital option value*
$$v = \mathbb{E}\left(1_{\tau > T} 1_{S \geq K}\right).$$
- We are modelling $\{\tau > T\} = \{Y > a\}$ for a standard normal random variable Y and set $a = \Phi^{-1}(p)$ for some $p \in (0, 1)$.
- We express $Y = \rho Z + \sqrt{1-\rho^2}W$ for W a standard normal random variable independent of Z.
- For the context of all questions, we set $s = 1$, $\sigma = 0.2$ and $K = 1.25$.
- You are asked to submit a written assignment asking the following questions where the respective points are indicated with each question. Assignments are due on Monday Nov 6th, 5:30pm EST. Submissions are accepted late for 1 day with a 10% penalty, submissions after Nov 7th will not be accepted.
- Please have your final report typeset using LaTeX and submit your report individually. Provide code separate in a Python script file that you attach with your submission.
- You may discuss these questions with your fellow students, however the write-up must be yours and yours alone, sharing of the write-up before the deadline is not allowed.

1. (3 pts) Show that for all ρ
$$\mathbb{P}\{\tau \leq T\} = p$$
holds.
2. (2 pts) In order to make the dependency on p and ρ explicit, we write $v = v(\rho; p)$. For $\rho = 0$, state the form of $v(0; p)$ explicitly and plot $v(0; p)$ as a function of p for $p \in (0, 1)$.
3. (5 pts) With the same notation state the explicit form of $v(\rho, p)$ for $\rho = -1$ and plot $v(-1; p)$ as a function of p.
4. (5 pts) For $p = 0.05$ write Python Code to determine
$$v(\rho) = \mathbb{E}\left(1_{Y > a} 1_{S \geq K}\right)$$
through a MC simulation of both Z and W with sample size $N = 10,000$. Use this code to plot $v(\rho)$ for $\rho \in (-1, 1)$, include the code and the plot.
5. (5 pts) In the same simulation include the correlation risk $\frac{\partial}{\partial \rho} v(\rho) \approx \frac{1}{\varepsilon}(v(\rho + \varepsilon) - v(\rho))$ for small values of ε and again include the plot for $\rho \in (-1, 1)$.
6. (3 pts) We know that
$$\mathbb{E}\left(1_{\rho Z + \sqrt{1-\rho^2}W > a} 1_{S \geq K} \,|\, Z\right) = h(Z; \rho)$$

holds for a function h – which will again depend on ρ. State the explicit form of h.

7. (7 pts) Calculate the derivative of h with respect to ρ.

8. (5 pts) For $p = 0.05$ write Python Code to determine

$$v(\rho) = \mathbb{E}\left(h(Z;\rho)\right)$$

through a MC simulation of just simulating Z with sample size $N = 10,000$. Use this code to plot $v(\rho)$ for $\rho \in (-1,1)$, include the code and the plot.

9. (5 pts) Use the same simulation approach to calculate

$$\frac{\partial}{\partial \rho} v(\rho) = \mathbb{E}\left(\frac{\partial}{\partial \rho} h(Z;\rho)\right)$$

and plot the results.

10. (5 pts) Finally, we can approximate an integral over a normal distribution

$$y = \int_{\mathbb{R}} f(x)\varphi(x)dx$$

where f is an integrable function and φ the density of the standard normal distribution through

$$y \approx \sum_{i=1}^{M} f(x_i)w_i$$

where we choose $x_i = -b + \frac{2bi}{M}$ for a lower bound $-b$ and $w_i = \Phi(x_i) - \Phi(x_{i-1})$ for $i = 1, \ldots, M$ (and $w_0 = \Phi(x_0)$). Again, for the same parameters and $M = 200$ and $b = 5$, calculate

$$v(\rho) = \mathbb{E}\left(h(Z;\rho)\right)$$

through this numerical integration, therefore avoiding all simulations. Plot the results of your calculation for $\rho \in (-1,1)$.

11. (5 pts) Finally, use the same integration to calculate

$$\frac{\partial}{\partial \rho} v(\rho) = \mathbb{E}\left(\frac{\partial}{\partial \rho} h(Z;\rho)\right)$$

and plot the result for $\rho \in (-1,1)$.