University of Toronto

Department of Statistical Sciences

STA 2503 / MMF 1928 – Applied Probability for Mathematical Finance

Project 2: Dynamic Hedging

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Course: MMF-1928 Pricing Theory

Date: December 3, 2023

Abstract

This paper presents a comprehensive analysis of dynamic hedging strategies, focusing on delta and delta-gamma hedges within the Black-Scholes modeling framework. Employing Monte Carlo simulations for Geometric Brownian Motion, we analyze stock prices influenced by constant drift and volatility, emphasizing the practical aspects like transaction costs and market realities.

Our study investigates the sensitivity of Profit & Loss (P&L) to underlying stock drift, explores the impact of real-world volatility estimation errors on hedge positions, and compares outcomes for in-the-money and out-of-the-money scenarios. We demonstrate that while delta hedging results in more symmetric P&L distributions, i.e. a balanced risk profile, delta-gamma hedging is more effective at minimizing risk, albeit at a higher complexity and cost. The findings offer significant insights for option pricing and risk management, highlighting the need for cost-efficient strategies in dynamic hedging practices.

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1 Introduction

Since the beginning of financial trading activity, hedging unwanted risk has long been a desire of participants, some might even say it's the main purpose of financial markets. The earliest hedging strategy can be dated back to 1949 when Alfred Winslow Jones established the first hedge fund in the world, where he employed an early version of the "long-short equity" strategy (Dutta et al., 2017). Then comes the all-famous Black-Scholes formula (Black & Scholes, 1973), where the academia realized "portfolio with the same payoff should be priced the same", thus made famous for multiple forms of dynamic hedging with customised risk exposure.

In this paper, we take a closer look at the delta hedge and delta-gamma hedge when we sold At-the-Money call option under the Black-Scholes dynamics, where stocks have constant drift and volatility, but with a twist: We assume crossing the spread when executing hedging position, thus bringing our simulation closer to the reality. We utilise Monte Carlo simulation for Geometric Brownian Motion price process of the underlying, i.e. stock prices driven by process drift and volatility, then calculate Delta and Delta-Gamma hedging respectively. We then simulate behaviours under different stock drift, different execution results (in-the-money or out-of-money), and different volatility estimation error.

The following of this paper is divided into four parts: In Section 2, we describe our Black-Scholes pricing framework, trade and delta/delta-gamma hedging setups. In Section 3, we analysis the P&L sensitivity to the change in underlying drift term. In Section 4, we compare the hedging position difference when the sold option ends up In-the-Money and the sold option ends up Out-of-Money. In section 5, we compare the impact of different real-world volatility estimation errors to delta and delta-gamma hedges. In section 6, we conclude and link our empirical experiment back to the real-world, where we evaluate the effect of delta and delta-gamma hedge and how practical it is to deploy them in reality.

2 Methodology and Model Set-Up

2.1 Black-Scholes Model

We investigate delta and delta-gamma hedge under the Black-Scholes model setting introduced by Black & Scholes (1973). Suppose a probability space with natural filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Under such a setting, we have a traded financial instrument $S = (S_t)_{t\geq 0}$ whose pricing dynamics can be described as:

$$dS_t = \mu^S S_t dt + \sigma^S S_t dW_t$$

where $W = (W_t)_{t\geq 0}$ is the $\mathbb{P}$ standard Brownian motion, μ^S and σ^S are both constants. We also have a bank account $B = (B_t)_{t\geq 0}$ that one can borrow and save with risk-free interest rate r, the dynamics of a bank account is denoted as

$$dB_t = rB_t dt$$

Where the risk-free interest rate r is a constant. Under our setting, suppose there exist a claim written on $S = (S_t)_{t\geq 0}$ that pays $G(S_T) = (S_T - K)_+$, i.e. a European Call option with strike price K, where T is the terminal time for the option. This claim has a pricing process $g = (g_t)_{0\leq t\leq T}$ which is Markovian in S, i.e. there exists a function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that $g_t = g(t, S_t)$. Then, adopted from the generalized Black-Scholes PDE:

$$\begin{cases} (\partial_t + \mathcal{L}_t^S)g(t,s) = rg(t,x) \\ g(T,x) = G(x) \end{cases}$$

where $\mathcal{L}_t^S = rx \cdot \partial_x + \frac{1}{2}(\sigma^S x)^2 \partial_{xx}$

To solve this PDE, we need to invoke the Feynman-Kac Theorem. By Feynman-Kac theorem, a stochastic representation of the solution to this PDE is given by:

$$g(t,x) = e^{r\tau} E^{\mathbb{Q}}[G(S_T)|S_t = s]$$

where ($\tau := T - t$) and S satisfies SDE

$$dS_t = rS_t dt + \sigma^S dW_t^{\mathbb{Q}}$$

$W^{\mathbb{Q}} = (W_t^{\mathbb{Q}})_{t \geq 0}$ is the standard Brownian motion under probability measure $\mathbb{Q}$, where $\mathbb{P}$ and $\mathbb{Q}$ are probability measures under the same probability space and natural filtration. $\mathbb{Q}$ is also called the risk-neutral probability measure invoked by using the bank account as a numeraire.

To obtain an explicit expression for the option pricing, we need to solve for the expectation under $\mathbb{Q}$. By utilizing the Girsanov's Theorem (Girsanov, 1960), there exists a measure $\mathbb{Q}$ by which $W^{\mathbb{Q}}$ is a standard Brownian motion. That measure, generated by Radon-Nikodym derivative, can be expressed as

$$\eta_t = \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = exp\left(-\int_0^t \lambda dW_u - \frac{1}{2}\int_0^t \lambda^2 du\right)$$

where $\lambda = \frac{\mu^S - r}{\sigma^S}$ is the market price of risk, and under BS model, this is a constant as well. Therefore we have

$$dW^{\mathbb{P}} = dW^{\mathbb{Q}} - \lambda dt$$

Now we can rewrite the F-K representation as:

$$\begin{aligned} g(t,s) &= e^{r\tau} E^{\mathbb{Q}}[G(S_T)|S_t = s] \\ &= e^{r\tau} E^{\mathbb{Q}}[S_T \mathbb{1}_{S_T > K} - K\mathbb{1}_{S_T > K}|S_t = s] \end{aligned}$$

Then the call option becomes a combination of 1. asset or nothing option payoff and 2. a digital call option payoff. Skipping the detailed numerical calculation, we arrived at the famous BS European Call pricing:

$$g(t,S) = S\Phi(d_+) - Ke^{-r\tau}\Phi(d_-)$$

Where $d_{\pm} = \frac{log(S/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$

2.2 Trade Setup

To investigate delta hedge and delta-gamma hedge, we set up the following scenario where at $t = 0$, the underlying is trading at $S_0 = 10$. A sell-side trader just sold 10,000 numbers of the At-the-Money (ATM) European call option with a 1/4 year of expiration, denoted as option g. ATM means the option has strike $K = 10$, which is the same as S_0. The trader have the following three instruments to trade with: (S, B, h) with corresponding position of (α, β, γ). S is the underlying asset, B is the bank account and h is another ATM European Call option with expiration 0.3 year. Both position in the underlying and position in the option accept only rounded numbers i.e. $\alpha_t, \gamma_t \in \mathbb{Z}, \forall t$. The bank account can accept both positive and negative real numbers, i.e. $\beta_t \in \mathbb{R}$. The underlying follows a geometric Brownian motion, i.e.

$$dS_t = \mu^S S_t dt + \sigma^S S_t dW_t$$

The bank account accumulates at a annual rate of r, i.e.

$$dB_t = rB_t dt$$

And finally, the hedge option follows BS model pricing.

$$h(t, s) = s\Phi(d_+) - Ke^{-r\tau}\Phi(d_-)$$

Where $d_\pm = \frac{log(s/K)+(r+\frac{1}{2}(\sigma^S)^2)\tau}{\sigma^S\sqrt{\tau}}$. We assume submitting market orders when trading in the underlying and trading in the hedge option, which means we cross the spread every time. The transaction cost is assumed fixed of 0.005 per unit. We also assume at t=0 when we sell the European call option, as a sell-side we did not cross the spread, translate to reality, we were phoned by the counter-party to sell 10,000 at agreed Black-Scholes price, or our limit order at Black-Scholes price got filled completely. This is an assumption that is closer to the reality of a sell-side trader.

We also assume that the constant parameters

$$\mu^S = 0.1, \sigma^S = 0.25, r = 0.05$$

and we perform delta or delta-gamma hedge daily. At terminal of our sold call option, we would clear our position in the following order: 1. We accumulate the bank account from last period, i.e. period T-1 2. We sold our position in underlying and hedging option (if any) 3. We pay our obligation for the option sold on t=0. i.e. the cash outflow would be $(S_T - K)_+ \cdot 10,000$. we assume this to be a cash transfer so no transaction cost incurred. What remains in our bank account is our Profit & Loss. In order to see P&L distribution, we simulate 5000 path of Geometric Brownian Motion of the underlying, do hedge simulation on each of them, then record the terminal P&L to get the distribution of P&L.

2.3 Delta Hedge

At t=0, we sold option g. implied by the BS pricing model, we can compute the first order partial differential of the option price w.r.t the underlying:

$$\Delta_0^g = \partial_s g(0, s) = \Phi(d_+)$$

where $d_+ = \frac{log(S/K)+(r+\frac{1}{2}(\sigma^S)^2)\tau}{\sigma^S\sqrt{\tau}}$. The value of our portfolio is:

$$V_t = \alpha_t S_t + \beta_t B_t - 10,000 g_t$$

The portfolio delta is:

$$\Delta_t^V = \alpha_t \Delta^S + \beta_t \Delta^B - 10,000 \Delta_t^g$$
$$\Delta_t^V = \alpha_t \cdot 1 + \beta_t \cdot 0 - 10,000 \Delta_t^g$$

where the underlying has a delta of 1 because movement of underlying has a one to one effect on the movement of underlying; bank account has a delta of 0 because movement of underlying is unrelated with bank account. To bring our portfolio delta to 0, we can hedge by purchasing the position in the underlying, since the underlying has delta $\Delta_t^S = 1, \forall t$, i.e. by having $\alpha_0 = 10,000\Delta_0^g$, we completely offset the delta risk in our portfolio. Due to the position limit, we use np.ceil function to round up our position. The funding source will be our bank account, so

$$\beta_0 = -\alpha_0 \cdot S_0 = -10,000\Delta_0^g \cdot S_0$$

At every point of time, we would recalculate the option delta, then rebalance the portfolio. We assume every time we rebalance portfolio will incur transaction cost, as detailed in Trade Setup section. At terminal time, we would first accumulate bank account from last period, then sell all of our hedging position in the underlying, then pay out our obligation to the option sold at t=0, what remain in our bank account is our Profit & Loss.

2.4 Delta-gamma Hedge

In addition to delta hedge, the option we sold also have gamma, i.e. the second order partial differential. Under the Delta-gamma hedging setting, we have two things we want to hedge:

$$\Delta_0^g = \partial_s g(0, s) = \Phi(d_+)$$
$$\Gamma_0^g = \partial_{ss} g(0, s) = \phi(d_+)/(s \cdot \sigma^S \cdot \sqrt{\tau})$$

where $d_+ = \frac{log(s/K)+(r+\frac{1}{2}(\sigma^S)^2)\tau}{\sigma^S\sqrt{\tau}}$, $\tau = T - t$

Portfolio value:

$$V_t = \alpha_t S_t + \beta_t B_t + \gamma_t h_t - 10,000 g_t$$

Portfolio delta and gamma:

$$\Delta_t^V = \alpha_t + \gamma_t \Delta_t^h - 10,000\Delta_t^g$$
$$\Gamma_t^V = \gamma_t \Gamma_t^h - 10,000\Gamma_t^g$$

We can use option h to hedge the Γ^g since the underlying have a gamma of 0. The position in hedging option will also incur delta, and we want to hedge the overall portfolio delta. Our initial position will be:

$$\gamma_0 = 10,000\left(\Gamma_0^g/\Gamma_0^h\right)$$
$$\alpha_0 = 10,000\Delta_0^g - \gamma_0\Delta_0^h$$

We then source funding from our bank account, where:

$$\beta_0 = -\alpha_0 S_0 - \gamma_0 h_0$$

Same as in Delta hedge, we apply np.ceil to round up position h and position α, in that order. Again, the funding source will be from our bank account. At every time step, we would re-calculate option delta and option gamma, rebalance our portfolio accordingly, and source funding from the bank account. At terminal time, we would accumulate previous period bank account, clear our hedge positions (both underlying and hedge option position), and pay our obligation of t=0 sold option. What remains of our bank account is our Profit & Loss.

3 μ^S Sensitivity to Profit and Loss (P&L) Analysis

We generate 5000 simulated paths for the price of the underlying asset $S = (S_t)_{t\geq 0}$ whose pricing dynamics can be described as

$$dS_t = \mu^S S_t dt + \sigma^S S_t dW_t$$

under the assumption that the constant parameters are given as

$$\mu^S = 0.1, \sigma^S = 0.25, r = 0.05$$

The initial price of the asset is S_0 =\$10.

We then calculate the Profit and Loss (P&L) for each path under both Delta and Delta-Gamma hedging strategies, with the base case of $\mu^S = 10\%$, and obtain distributions of the P&L, respectively. We represent the resulting P&L distributions in Figure 1 and can observe that the P&L distributions exhibit very different shapes in the two cases.

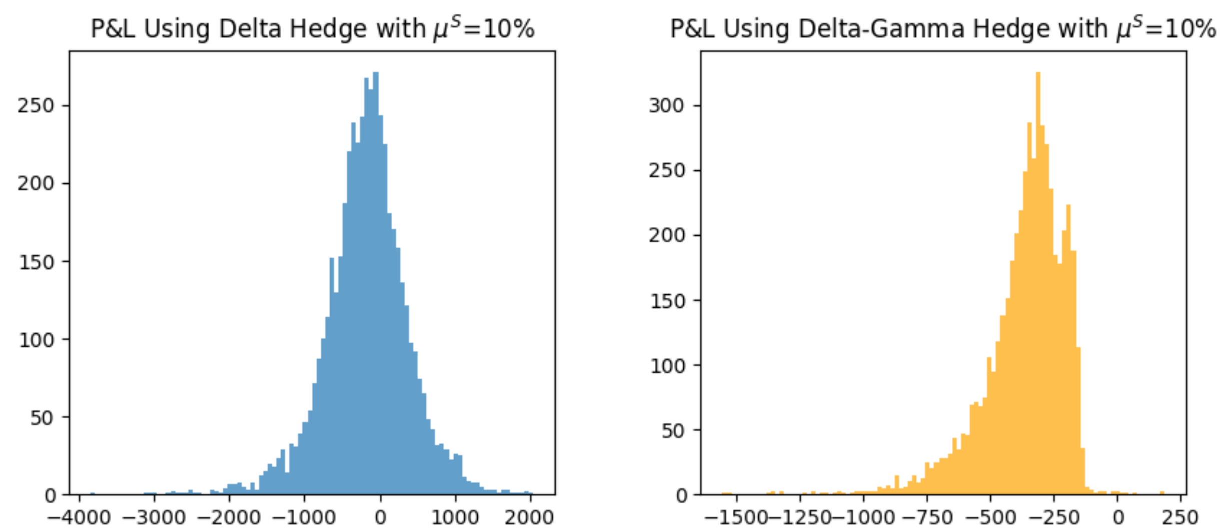


Figure 1: P&L Distributions with μ^S=10%

We repeat the process for different values of μ^S in order to discover more patterns of the P&L distributions. We set the range of μ^S as [0, 2] to reflect most of the real-world situations. We represent the resulting P&L distributions with four sample values of μ^S: $\{0.05, 0.7, 1.35, 2.0\} \in [0, 2]$ in Figure 2.

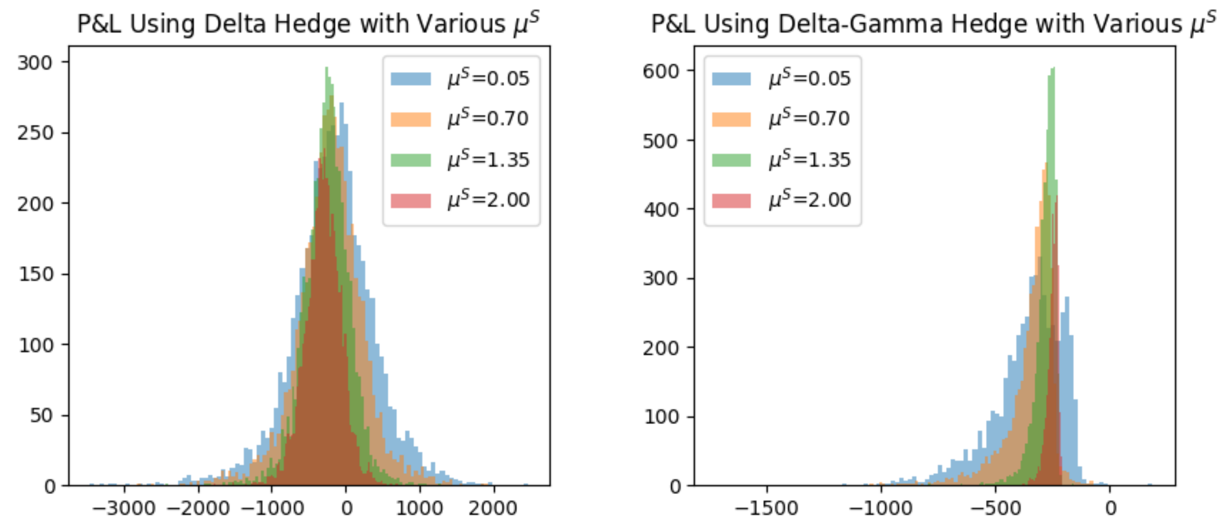


Figure 2: P&L Distributions with Sample Values of μ^S

We then collect the end P&L values for all 5000 paths to obtain the mean as expected P&L values for each μ^S value case, respectively. Then, we calculate the standard deviation and skewness of the P&L distribution for further analysis. We represent the parameters of the resulting P&L distributions with four sample values of μ^S in Table 1.

	Delta Hedge			**Delta-Gamma Hedge**		
μ^S	**Mean**	**S.D.**	**Skewness**	**Mean**	**S.D.**	**Skewness**
$\mu^S = 0.05$	-170.08	560.12	-0.34	-358.62	159.14	-1.41
$\mu^S = 0.70$	-211.62	446.81	-0.25	-343.89	110.79	-2.17
$\mu^S = 1.35$	-256.27	309.17	-0.35	-281.30	54.45	-4.24
$\mu^S = 2.00$	-304.71	245.12	-0.26	-249.30	25.83	-2.18

Table 1: P&L Distribution Parameters

We also represent some of the resulting P&L distributions with standard deviation and skewness of expected P&L in Figure 3, 4, 5 to clearly reflect the trend.

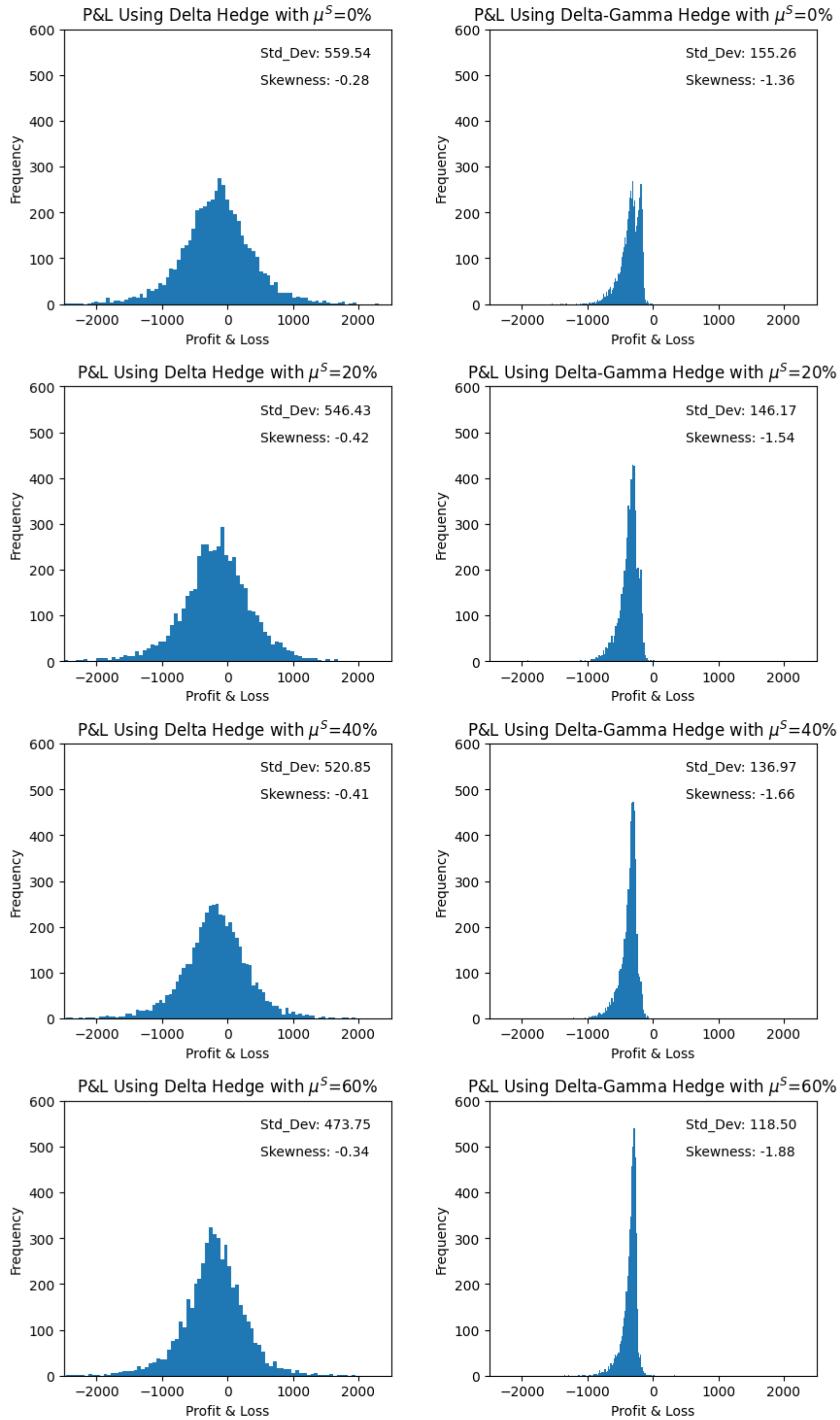


Figure 3: P&L Distributions with Standard Deviation and Skewness, $\mu^S \in [0, 0.6]$

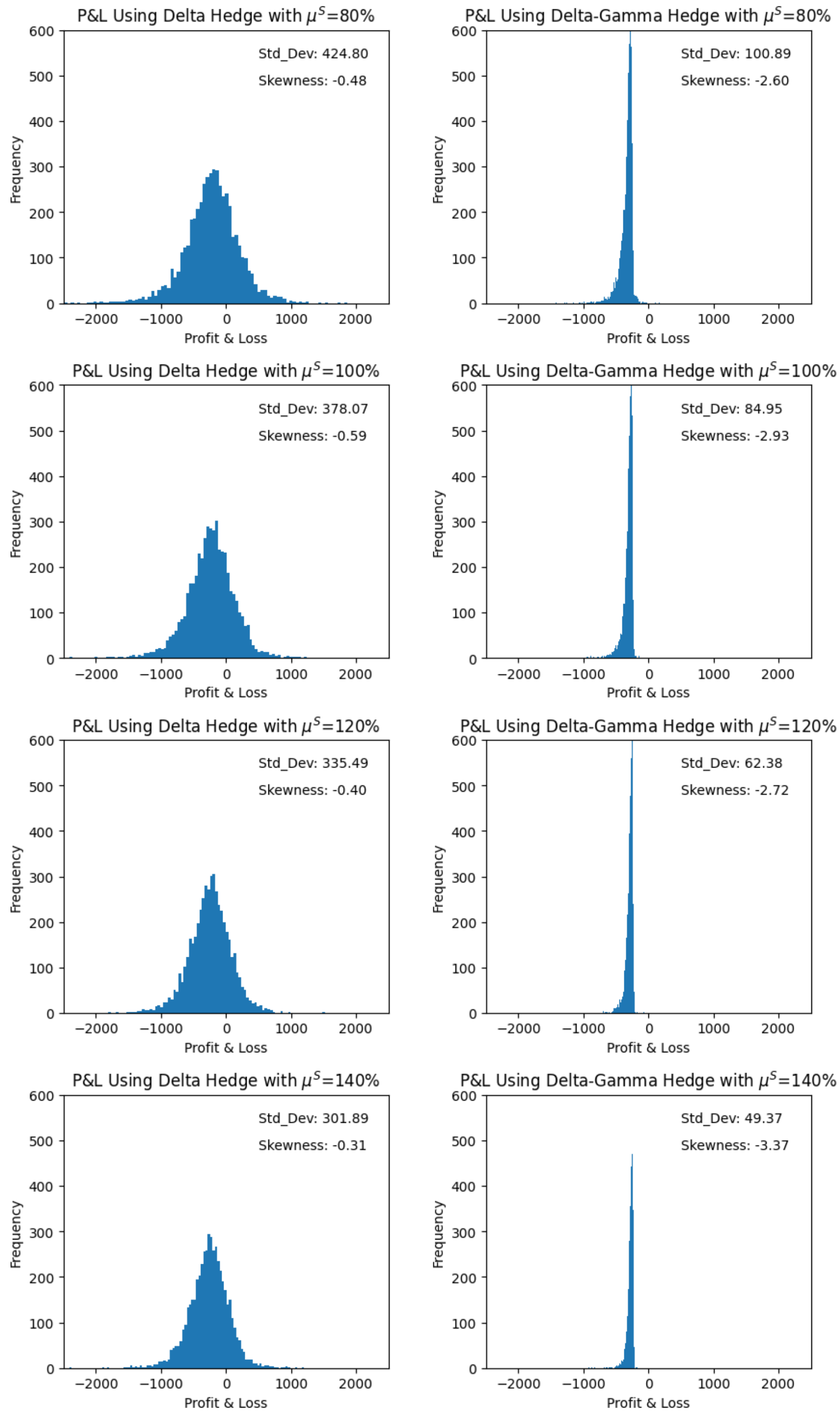


Figure 4: P&L Distributions with Skewness and Standard Deviation, $\mu^S \in [0.8, 1.4]$

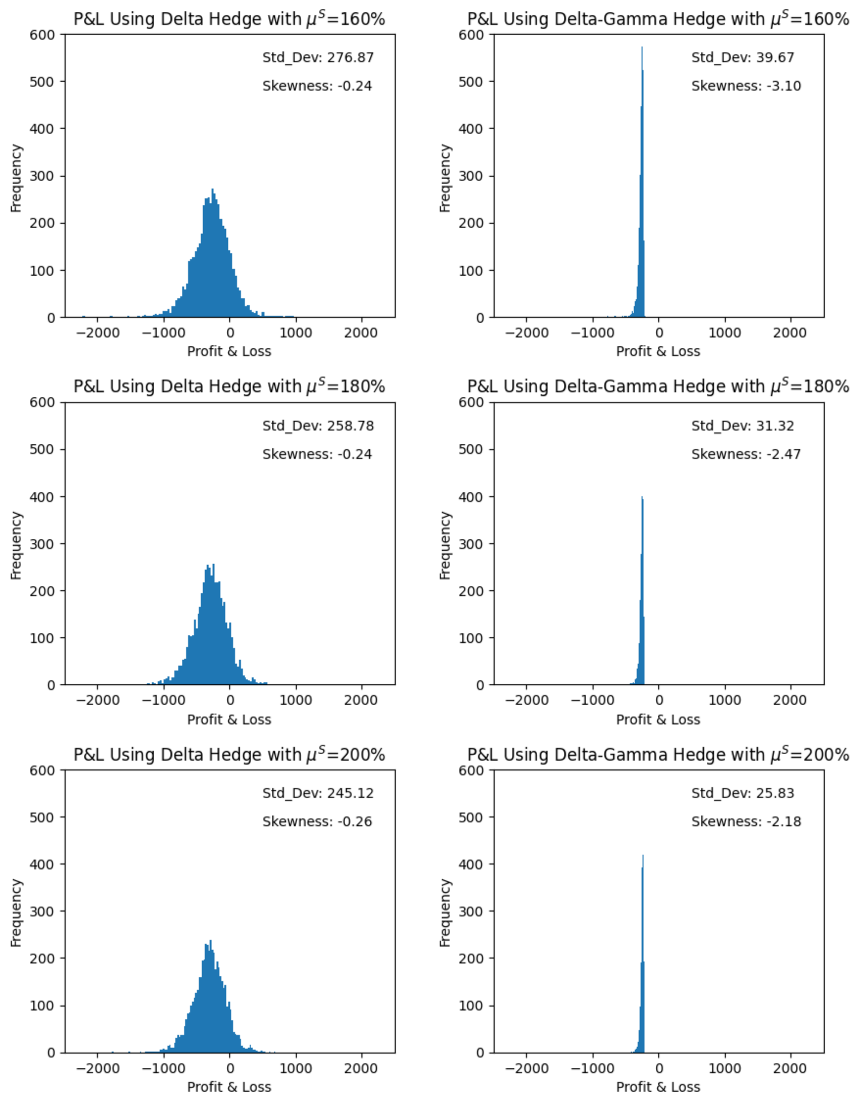


Figure 5: P&L Distributions with Skewness and Standard Deviation, $\mu^S \in [1.4, 2.0]$

We apply linear regression for all of the moments of the P&L distributions calculated and polynomial regression with degree=3 onto some of the cases. We represent the results in Figure 6.

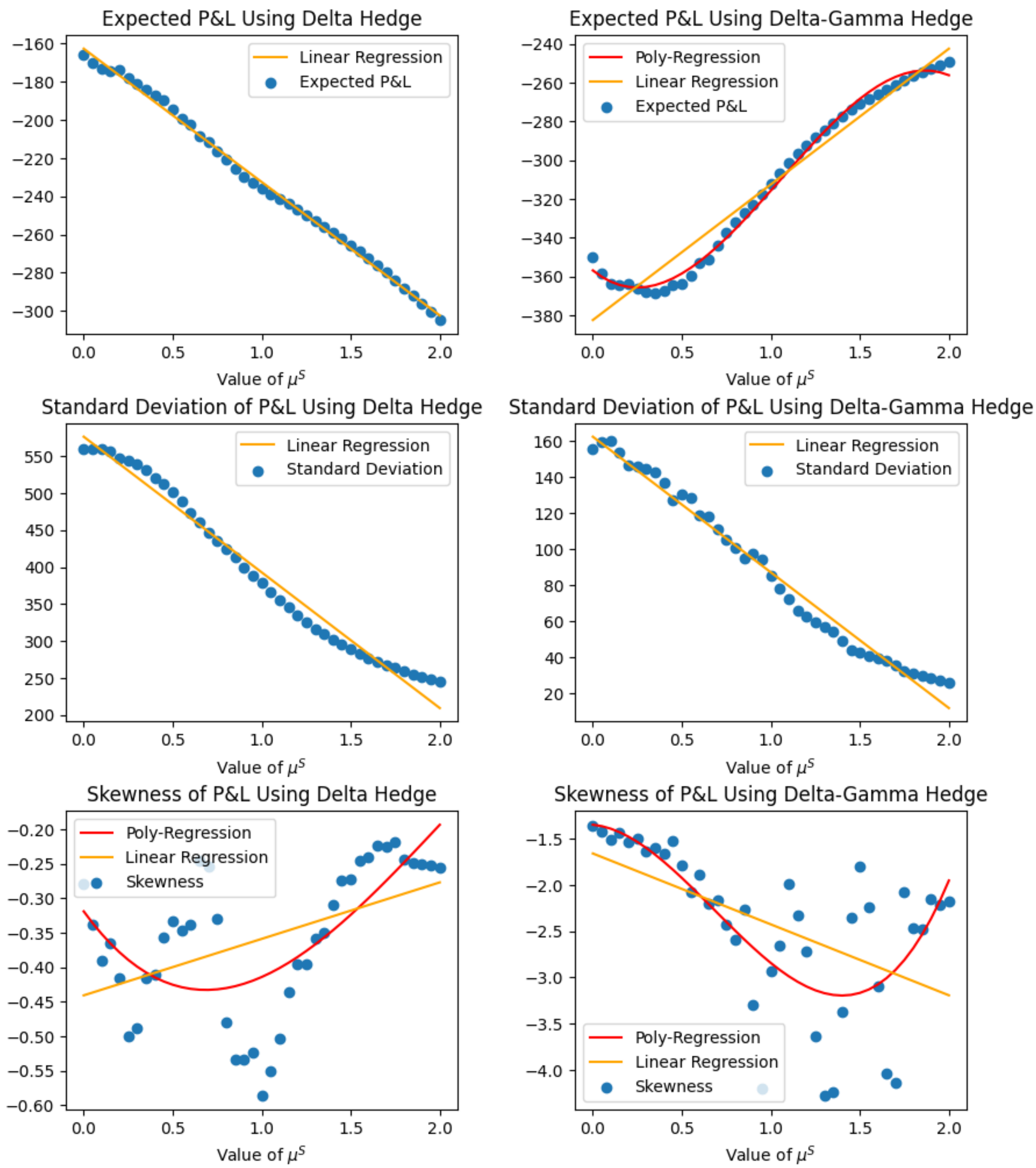


Figure 6: Regression Analysis of Expected P&L

Based on the 5000 simulated paths, we can observe several patterns of variation of the P&L distributions as μ^S varies.

3.1 Effect of μ^S on Expected P&L

The expected P&L plots and calculations (Figure 3, 4, 5, 6, Table 1) show that the Delta-Gamma hedge results in higher losses compared to the Delta hedge overall at most values of μ^S.

The regression plots (Figure 6) indicate that as μ^S increases, the expected P&L for the Delta hedge decreases. For the Delta-Gamma hedge, however, the expected P&L initially decreases, reaches the trough around $\mu^S = 0.4$, then bounces back to surpass the expected P&L at $\mu^S = 0$ and continues to increase.

These may be explained by the different sensitivity to drift (μ^S) that Delta and Delta-Gamma hedge bear.

The Delta hedge is primarily concerned with the hedging of directional risk, meaning it aims to offset changes in the option's value due to small price movements in the underlying asset. Part of the reason may be that when μ^S is low, i.e. the expected return of the underlying asset is lower, the value of the call options we sold is likely to decrease, leading to a profit on our initial short ATM call options position. However, as μ^S increases, the expected return of the underlying asset increases, leading to higher losses on the short call position due to the increase in the option's value.

The Delta-Gamma hedge not only hedges against the directional risk but also hedges against the curvature risk of the option's value (Gamma). The trough in the expected P&L plot around $\mu^S = 0.4$ likely arose as the trade-off between extra cost buying position in hedge option and the hedge benefit reached an equilibrium.

The initial decrease in expected P&L with increasing μ^S is likely consistent with the Delta hedge, as the expected increase in the underlying asset price leads to increased option prices and thus potential losses on the short position. However, the rebound and increase in expected P&L past $\mu^S = 0.4$ could be partially due to the fact that the Delta-Gamma hedge becomes more effective as the underlying asset's price moves further away from the strike price. As μ^S increases, our underlying asset moves more aggressively, our Delta-Gamma hedging approach utilizes second order Taylor expansion approximation, therefore it is likely to better capture the change in the underlying, i.e. benefit from larger μ^S.

3.2 Effect of μ^S on Standard Deviation of Expected P&L

Overall, the Delta-Gamma hedge shows a lower standard deviation across all values of μ^S compared to the Delta hedge. (Figure 3, 4, 5, 6, Table 1) This suggests that incorporating Gamma into the hedge effectively reduces the uncertainty of the P&L.

The change in standard deviation reveals that the variability in P&L decreases with an increase in μ^S for both hedging strategies. (Figure 2) This is explained as both hedging strategies aim to reduce the risk of the option position by offsetting changes in the option's value due to movements in the underlying asset's price. As μ^S increases, the expected return of the underlying asset increases, which typically leads to a rise in its price. If the underlying asset's price consistently moves in one direction (upwards, in this case), the option's value becomes more stable, reducing the variability (standard deviation) of the hedging strategy's P&L.

However, we can also observe that the decrease is more pronounced with the Delta-Gamma hedge. This suggests that it is more effective at managing the non-linear aspects of the option's price sensitivity to the underlying asset's price movements (i.e., Gamma risk). As μ^S increases, the Delta-Gamma hedge may be more adept at adapting to changes in the option's Delta and Gamma, resulting in a tighter P&L distribution.

3.3 Effect of μ^S on Skewness of Expected P&L Distributions

The varying skewness (Figure 3, 4, 5, 6, Table 1) suggests that as μ^S increases, skewness in the Delta hedge case fluctuates with an overall trend of becoming slightly more negatively skewed, while the Delta-Gamma hedge shows an increase in negative skewness.

Skewness measures the asymmetry of the distribution around its mean. A higher negative skewness for the Delta hedge with increasing μ^S implies that the distribution of P&L is becoming more left-tailed. This could mean that as the underlying asset's price tends to increase, more in-the-money scenarios for call options may occur. Given our short position, this could potentially result in larger losses, which would be reflected as a heavier left tail in the P&L distribution.

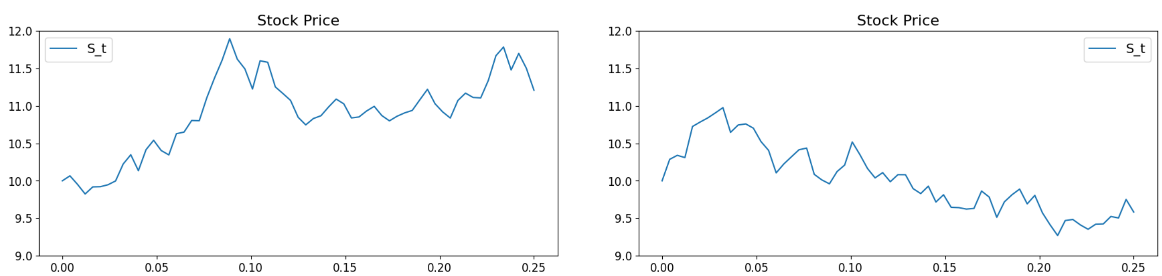
4 Comparing Execution and Non-Execution Sample Paths

We generate two sample paths, where one is ITM (In-the-Money, i.e. $S_T > K = 10$) and one is OTM (Out-of-the-Money, i.e. $S_T < K = 10$). See Figure 7.

The pricing model is based on the assumption that the constant parameters are given as

$$\mu^S = 0.1, \sigma^S = 0.25, r = 0.05$$

The initial price of the asset S_0 =\$10.



(a) Generated Path: In-the-Money (b) Generated Path: Out-of-the-Money

Figure 7: Generated Paths

We then construct the corresponding delta hedging strategy and delta-gamma hedging strategy respectively, and conduct analysis as below:

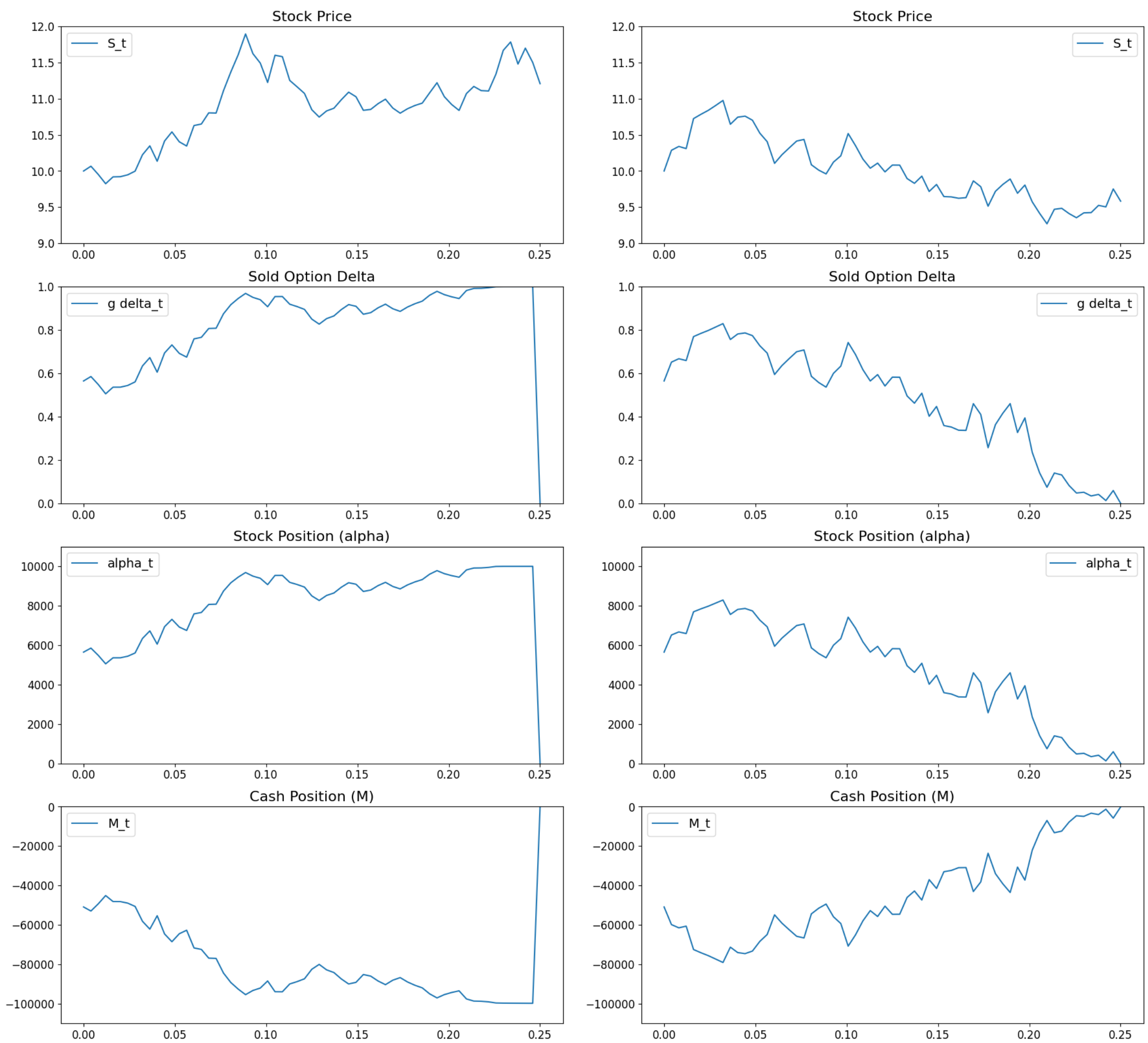
4.1 Delta Hedge for Generated Paths

For the delta hedge, we use the methodology explained in Section 2.3. The holding portfolio is (S, B, g) with the corresponding positions of (α, β, -10,000).

For each time-step, we rebalance by:

$$\alpha_t = 10,000\Delta_t^g$$
$$M_t = M_{t-1}e^{r\Delta t} - (\alpha_t - \alpha_{t-1})(S_t + 0.005)$$

Below are the positions figures of delta hedging strategy for two paths:



(a) Delta Hedge: In-the-Money

(b) Delta Hedge: Out-of-the-Money

Figure 8: Delta Hedge

We have the following observations:

1. $\Delta_0^g = 0.5$ since the option is At-the-Money when t=0. Further, Δ_t^g moves in the same direction as stock price. This is because:

   $$\Delta_t^g = \partial_x g(t,s) = \Phi(d_+(S_t))$$

   Where $d_+(S_t) = \frac{log(S_t/K)+(r+\frac{1}{2}(\sigma^S)^2)\tau}{\sigma^S\sqrt{\tau}}$, $\tau = T - t$

   As the stock price rises, the increase in d_+ causes δ to go up since Φ (the c.d.f. of standard normal distribution) is an increasing function.

2. α demonstrates a positive correlation with the stock price, following our findings in #1 and given that $\alpha_t = 10,000\Delta_t^g$. Hence, to hedge the option, investors are required to increase their holdings of underlying stocks when the stock price rises and decrease them when the stock price falls. This practice constitutes the rebalancing process.

3. M exhibits inverse fluctuations relative to the stock price. Since M serves as a self-financing account as a result of balancing the position of stock holdings, the relationship is directly attributed to the positive correlation between α and stock price. This can be verified from the figures, where stock position and cash position exhibit a symmetrical pattern.

4. As $t \to T$, in the In-the-Money scenario, $\alpha_t \to 10000$, given that $\Delta_t^g \to 1$. Conversely, in the Out-of-the-Money scenario, $\alpha_t \to 0$, as $\Delta_t^g \to 0$. This relationship arises from the definition $\alpha_t = 10000\Delta_t^g$ and the following considerations:

   $$\Delta_{T_-}^g \approx \partial_s g(T,s) = \partial_s G(s) = \partial_s(S_T - K)_+$$

   $$\Delta_{T_-}^g \approx \partial_s(S_T - K) = 1, \qquad S_T > K(\text{In-the-Money})$$

   $$\Delta_{T_-}^g \approx \partial_s(0) = 0, \qquad S_T < K(\text{Out-of-the-Money})$$

   This corresponds to the financial intuition that when options are deep In-the-money near maturity, the fluctuation on option prices are nearly equivalent those of stocks.

Therefore, the portfolio needs to predominantly consist of stocks to achieve full hedging. On the contrary, when options are far Out-of-the-money near maturity, the portfolio would be mostly consist of cash. This is a consequence of rebalancing process, align with our findings in #2.

5. Finally, α and M both reach 0 at t=T. This is because the option has reached maturity, and there is no need to keep the hedging portfolio.

In summary, in the In-the-Money scenario, our hedging strategy involves maintaining a portfolio with a higher proportion of stocks, leading to lower allocation of cash. Conversely, in the Out-of-the-Money situation, our approach entails holding fewer stocks and thus greater amount of cash. Notably, all alpha, beta and gamma positions close at 0 as the option matures.

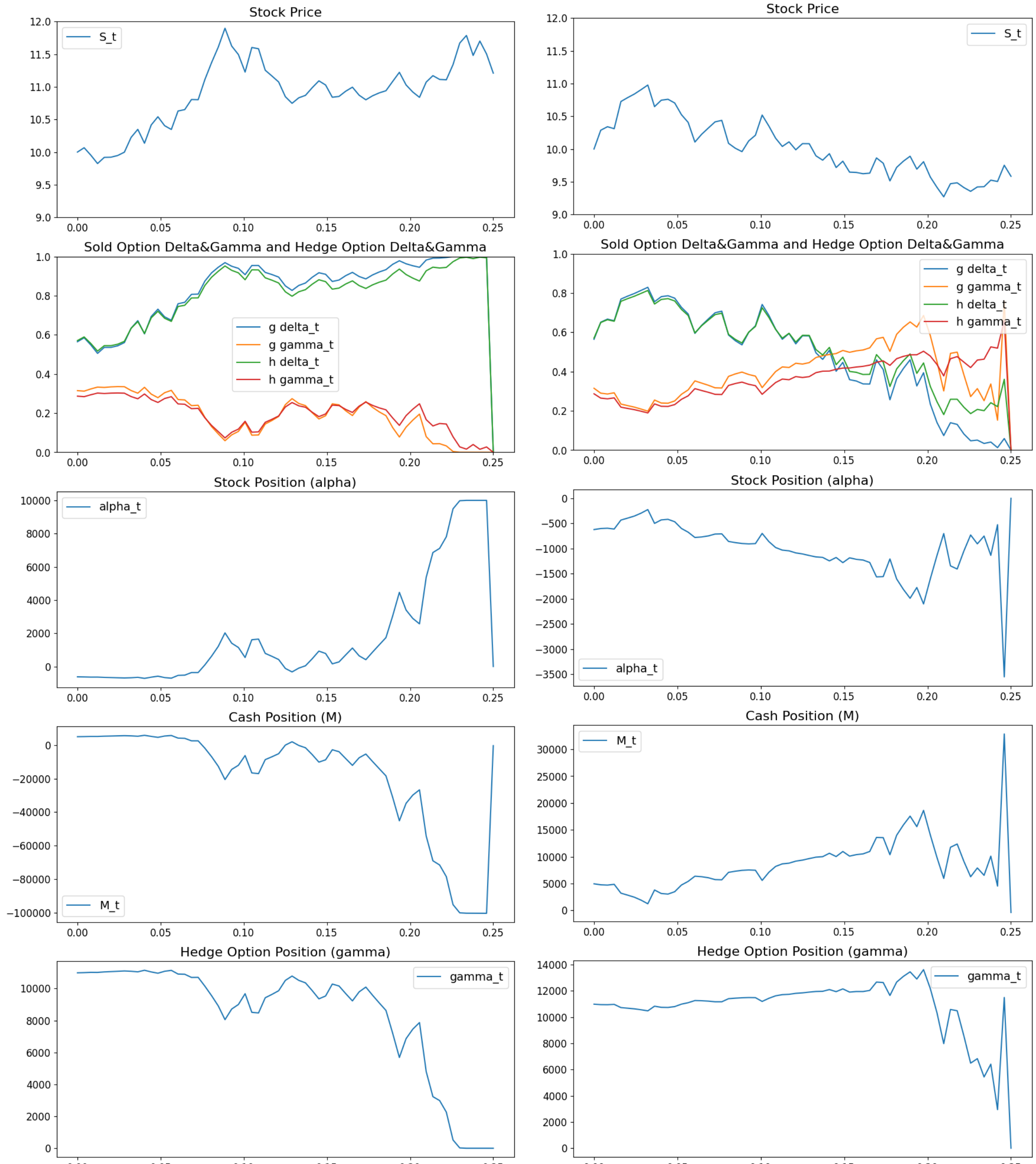
4.2 Delta-gamma Hedge for Generated Paths

For the delta-gamma hedge, we use the methodology explained in Section 2.4. The holding portfolio is (S, B, h, g) with the corresponding positions of (α, β, γ, -10,000).

For each time-step, we rebalance by:

$$\begin{aligned}
\gamma_t &= 10,000\left(\Gamma_t^g/\Gamma_t^h\right) \\
\alpha_t &= 10,000\Delta_t^g - \gamma_t\Delta_t^h \\
M_t =& M_{t-1}e^{r\Delta t} - (\alpha_t - \alpha_{t-1})(S_t + 0.005) - (\gamma_t - \gamma_{t-1})(h_t + 0.005)
\end{aligned}$$

Below are the position figures of the delta-gamma hedging strategy for two paths:



(a) Delta-gamma Hedge: In-the-Money

(b) Delta-gamma Hedge: Out-of-the-Money

Figure 9: Delta-gamma Hedge

Based on the figures, We have the following observations:

1. Γ^g (and Γ^h) exhibits a reverse correlation with the stock price. We have established that d_+ rises with an increase in stock price. Considering $\Gamma_t^g = \phi(d_+)/(S_t \cdot \sigma^S \cdot \sqrt{\tau})$, having $d_+ > 0$, and ϕ (the p.d.f. of standard normal distribution) decreasing on the positive axis, with a smaller numerator and a larger denominator, it follows that the value of Γ_t^g diminishes as S_t experiences an upward trend.

2. Γ^g (and Γ^h) tends to rise as options close to being At-the-Money (as observed around $t = 0.20$ for the Out-of-the-Money scenario); and becomes lower in value otherwise. This is because $\Delta_t^g \approx \partial_s(S_t - K) = 1$ when deep In-the-Money and $\Delta_t^g \approx \partial_s(0) = 0$ when far Out-of-the-Money, causing $\Gamma_t^g = \partial s \Delta_t^g \approx 0$.

3. $\gamma_0 \approx 0$, given that $\gamma_t = 10,000 \left(\Gamma_t^g/\Gamma_t^h\right)$, and $\Gamma_0^g \approx \Gamma_0^h$ since the difference in their time to maturity is not significant proportionally. Further, γ undergoes minor variations when t is small and became more volatile as $t \to 0.25$. This is because g is close to maturity while h is not, and the difference in term τ can lead to greater deviation in their Γ.

4. We observe that in the given context, under the interaction of parameters in delta-gamma hedging, α moves along with price changes, while γ and M demonstrate an inverse relationship with price fluctuations. From the formula $\alpha_t = 10,000\Delta_t^g - \gamma_t\Delta_t^h$, α exhibits a direct negative correlation with γ (though α and γ involve complex interaction with change in S), which can be verified by the figures.

   The patterns for α and M mirror those encountered in delta hedging, displaying a symmetrical pattern. This alignment suggests that positions in delta-gamma hedging adhere to the established principles of delta hedging, with gamma neutrality serving as a supplementary factor. This coherence is logical, given that delta-gamma hedging can be viewed as a specialized instance of delta hedging.

5. In the deep In-the-Money scenario, as illustrated by the figure on the left, similar to delta hedging, α approaches 10000 near maturity, accompanied by γ approaches 0. This is because $\Gamma \approx 0$, and no further efforts are required to maintain gamma-neutrality; hence, the portfolio can be fully hedged by stocks. Consequently, in the cases deep In-the-Money and far Out-of-the-Money, delta-gamma hedging is equivalent to delta hedging.

6. Unlike delta hedging, α, M and γ experience significant volatility as $t \to 0.25$. This is directly attributed to our finding in #3 and the fact that α and M are both calculated from γ.

7. Lastly, α, M and γ all reach 0 as time(t) progresses to maturity(T). This occurs because, at the point of maturity, it is no longer necessary to maintain the hedging portfolio. We sell all hedging positions and convert them into cash, then pay our obligation for option sold at t=0.

In conclusion, the major difference to delta hedging is the high volatility in positions near maturity for delta-gamma hedging.

Besides that, similar to delta-hedging, in the In-the-Money scenario, our hedging strategy involves progressively increasing our equity holdings and reducing cash reserves over time. Conversely, in the Out-of-the-Money scenario, our approach would be decreasing equity exposure and increasing cash holdings. γ_t is determined by the interaction of the relevant parameters. Importantly, all positions reach zero as the option reaches maturity.

5 Comparing Different Real-World $\mathbb{P}$ Volatility Estimation Error

In financial modelling, real-world volatility is crucial for generating stock paths to ensure a realistic simulation of stock price movements. This approach is grounded in the principle of accurately reflecting market dynamics and uncertainties. However, under Black-Scholes modelling, one of the restrictions is that we can only model volatility as a constant. In this section, we aim to show our analysis of how different real-world volatility impact P&L and positions for delta and delta-gamma hedging cases. The table below shows what values we use.

Real-World $\mathbb{P}$ Volatility	**Model Volatility**
20%	25%
22%	25%
25%	25%
28%	25%
30%	25%

Table 2: *Real-World and Model Volatility*

5.1 Effect of Real-World Volatility on P&L

We firstly repeatedly plot the P&L distributions using different real-world volatility for both delta and delta-gamma hedging cases to reflect most of the real-world situations. The results are shown below in Figure 10.

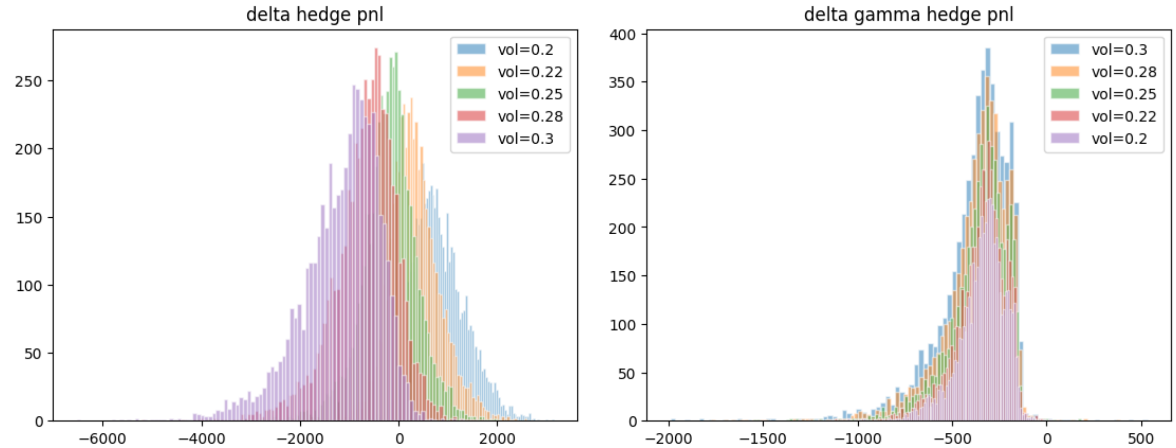


Figure 10: P&L Distributions with Different Real-World Volatility

Based on the plots, we can see that the impact of real-world volatility on hedging strategies is evident from the P&L distributions. For delta hedging, the peak of the P&L distribution shifts more noticeably with changes in volatility. The distribution is not as centred around zero as in delta-gamma hedging, suggesting a less effective neutralization of price movements. There is a more significant shift in the spread of the P&L distribution with changing volatility levels. Higher volatility broadens the distribution substantially, indicating a higher risk of experiencing large losses or gains. The distribution shows potential skewness, especially at higher volatility, where the likelihood of extreme negative outcomes appears to increase.

On the other hand, the delta-gamma hedging strategy shows the distribution remains relatively centred around zero across different volatility levels, indicating that the strategy is consistently mitigating the first and second-order risks. As the volatility decreases, the spread of the distribution narrows, signifying a reduction in the extremities of P&L. This shows that lower volatility environments tend to produce fewer large losses or gains when using delta-gamma hedging. With higher volatility the tails of the distribution are thicker, implying a greater likelihood of observing extreme P&L values due to increased market uncertainty.

In conclusion, the shift in the distribution of P&L for both hedging strategies indicates that delta-gamma hedging is more robust to changes in volatility, maintaining a stable risk profile. In contrast, delta hedging is more vulnerable to volatility changes, which can lead to a wider and potentially skewed distribution of P&L, signifying a higher risk of adverse outcomes. These shifts are critical for risk management decisions, particularly in choosing the appropriate hedging strategy based on the volatility environment and risk appetite.

5.2 Effect of Real-World Volatility on Positions

Furthermore, we compare how the alpha and gamma positions are affected by different levels of real-world volatility in both delta and delta-gamma hedging scenarios. Understanding these differences is crucial as they can significantly impact the overall risk and return of a portfolio. Through this comparison, we aim to gain insights into which hedging approach may be more effective in various market conditions. The results are shown in Figure 11 below.

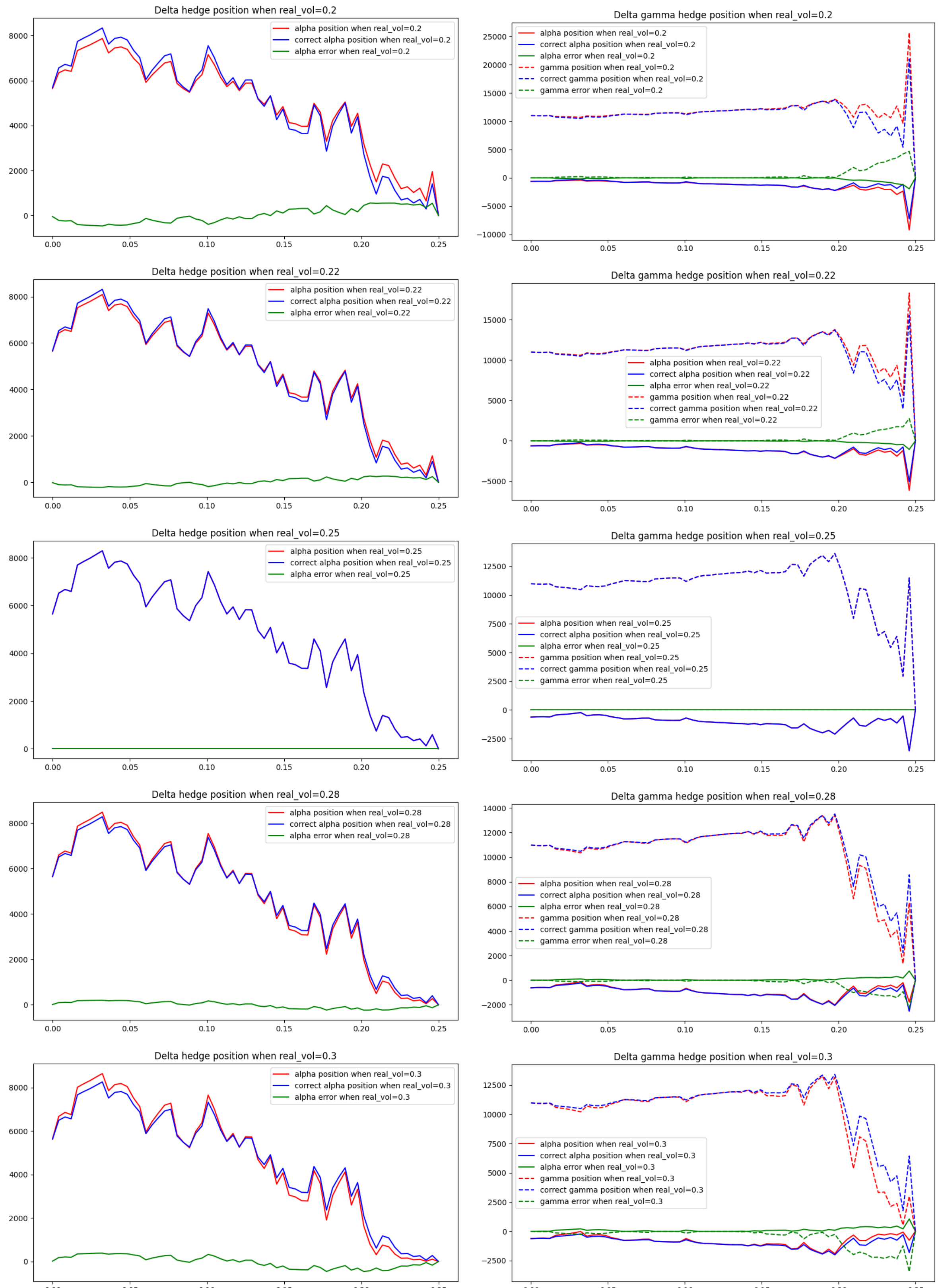


Figure 11: Asset and Hedging Position with Different Real-World Volatility

Based on the plots, for the delta hedging scenario, as real-world volatility deviates from the model's assumption, the alpha error increases. we can see where the red line diverges from the blue. The alpha error tends to remain relatively small and stable across different volatility levels, suggesting that the delta hedging strategy is somewhat robust to volatility mis-estimations. The alpha position does not significantly drift from the correct position, indicating that the delta hedge adjustments are still tracking the changes in the underlying asset's price reasonably well.

For the delta-gamma hedging scenario, the gamma error tends to be more pronounced than the alpha error, particularly at higher volatility levels, suggesting that different real-world volatility assumptions affect the gamma position more significantly. However, the position of alpha and gamma have opposite directions, so they somewhat cancel out with each other. Moreover, the difference of the magnitude of the errors can be explained by the nature of stocks and options (the delta of the stock is 1 but the option is range from 0 to 1 for the call option). As a result, the monetary value of the portfolio is not that fluctuated compared to delta hedging case, and delta-gamma hedging is less sensitive to volatility mismatches than delta hedging alone.

6 Conclusion

In this project, we performed an in-depth analysis of dynamic hedging strategies, particularly the delta and delta-gamma hedging based on the Black-Scholes model assumptions. We have arrived at several key findings and insights with significant implications for option pricing and risk management.

We have discovered that transaction cost has non-negligible downward effects on the P&L under both hedging strategies. The costs of trading the underlying asset and options can reduce the profitability of dynamic hedging and it is expensive to hedge perfectly. In real-world practice, we may need a careful analysis to determine the most cost-efficient strategy.

After carefully examining the outcomes of the delta and delta-gamma hedge, we can observe that delta hedge typically results in more symmetrical P&L distributions, indicating a more balanced risk profile. delta-gamma hedge, on the other hand, minimizes risk more effectively, despite being more complex and cost-consuming. In this project, we are forced to hedge near perfectly as reflected in the skewness of the P&L distributions. However, in real-world practice, we may require a strategic difference, such as lowering the hedging frequency, when exercising the two hedging methods to incorporate the cost concerns.

Generally, we observe that the delta-gamma hedge is more stable to variation of the drift (i.e. μ^S) compared to delta hedge, provided that the skewness is more stable. This stability can be beneficial for traders when pricing options, as it reduces the impact of mis-evaluating μ^S over the P&L outcomes.

Through the analysis on portfolio position movements along two generated sample paths (for scenarios In-the-Money and Out-of-the-Money), we gained valuable insights into the behavior of Greek letters and positions concerning changes in stock price and time decay. In the context of a call option, according to mathematical computation, Δ shows a positive correlation to stock price, whereas Γ exhibits a negative correlation. The Δ of an options tends to approach 1 when it is deep In-the-Money and approaches 0 when it is far Out-of-the-Money; consequently, Γ approaches 0 in both cases, and increases in value when it is near At-the-Money.

Moreover, for both delta hedging and delta-gamma hedging, due to the re-balancing process, α increases along with the increase in price, while β decreases as a result to finance the buying and selling of stock (and hedge options). Therefore, the proportion of stocks will rise in the scenario of In-the-Money, and fall in the scenario of Out-of-the-Money.

Comparing delta and delta-gamma hedging, the major difference lies on the high volatility in positions near maturity for delta-gamma hedging due to the deviation in Γ^g and Γ^h, especially when the option is near At-the-Money. This could lead to significant transaction costs near maturity and warrants careful attention. In contrast, when the option is deep In-the-Money or far Out-of-the-Money, delta-gamma hedging demonstrates similar behaviour as delta hedging. Based on that and the comparable patterns of α and M, we conclude that delta-gamma hedging can be considered as a specialized case of delta hedging, with gamma neutrality as an supplementary factor.

Lastly, accurately simulating stock price movements necessitates incorporating real-world volatility, a critical factor for realistic market dynamics representation. Under Black-Scholes modeling's constraint of constant volatility, this analysis explores the impact of varying real-world volatility on P&L and positions in both delta and delta-gamma hedging scenarios.

For delta hedging, increased real-world volatility results in P&L distribution shift left and more left-skewed. Larger volatility estimation error, could lead to more larger losses and the alpha error increases as well due to market uncertainties. For delta-gamma hedging scenario, it is more resilient against volatility estimation error and compared to delta hedging, the tail losses are much less and hedges more risks. Although the position error is more significant at higher volatility estimation error, the actual P&L does not deviate too much which is because the difference in price of stocks and options and some of the gamma position errors could be cancel out by the alpha position errors.

As a result, delta-gamma approach comparatively keeps loss at a higher but more certain level, which is practically a more desirable property.

7 Citation

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8 Appendix

8.1 Contribution Attestation

Daiwen Yang: Coding Q3, Q3 + Conclusion write-up

Qizi Luo: Coding Q2, Q2 + Conclusion write-up

Shihan Fang: Coding Q1, Q1 + Conclusion write-up

Kaiwen Shen: Coding the hedging object, Introduction + Methodology write-up.

Signatures:

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