STA 2503 / MMF 1928 Project - 1 American Options

Suppose that an asset price process $S = (S_{t_k})_{k\in\{0,1,\dots,N\}}$ (with $t_k = k\,\Delta t$ and $\Delta t = \frac{T}{N}$, for a fixed N) are given by the stochastic dynamics

$$S_{t_k} = S_{t_{k-1}}\, e^{r\,\Delta t + \sigma\sqrt{\Delta t}\,\epsilon_k},$$

where ϵ_k are iid rv with $\epsilon_k \in \{+1, -1\}$ and

$$\mathbb{P}(\epsilon_k = \pm 1) = \tfrac{1}{2}\left(1 \pm \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right).$$

Here, $r \geq 0$ and $\sigma > 0$ are constants. This is a variation of the Cox, Ross, Rubenstein (CRR) model.

Moreover, let $B = (B_{t_k})_{k\in\{0,1,\dots,N\}}$ denote the bank account with $B_t = e^{r\,t}$.

1. Derive the branching probabilities $\mathbb{Q}(\epsilon_k = \pm 1)$ and $\mathbb{Q}^S(\epsilon_k = \pm 1)$, as well as the $\mathbb{Q}$ and $\mathbb{Q}^S$.
   [Here, $\mathbb{Q}$ refers to the martingale measure induced by using the bank account B as a numeraire, and $\mathbb{Q}^S$ refers to the martingale measure induced by using the asset S as a numeraire.]

   For those that are keen, derive the distribution of $\log \frac{S_T}{S_0}$ as $N \to \infty$. You cannot use the central limit theorem, as the probabilities vary as N increases, rather you have to compute the mgf and prove that it converges to something that then allows you to determine the distribution.

2. In this part, you will evaluate an American put option. Assume that $T = 1$, $S_0 = 10$, $\mu = 5\%$, $\sigma = 20\%$, and the risk-free rate $r = 2\%$. Use $N = 5000$ and a strike $K = 10$.

   (a) Implement the valuation and exercise boundary of the American put option using two methods: (i) with B as the numeraire, and (ii) with S as the numerarie.

      i. Plot the exercise boundary as a function of t for both approaches.

      ii. Generate two sample paths where in sample path 1) the option is exercised early (say around $t = \frac{1}{2}$), 2) the option is not exercised

      iii. Along the two sample paths above, plot the hedging strategy that a trader would use to hedge the option as a function of time.

      iv. Illustrate how the results in i,ii, and iii vary as volatility and risk-free rate change (pari-wise). *[For example, $\sigma = 10\%, 20\%, 30\%$ and $r = 0\%, 2\%, 4\%$]*

   (b) Assume you have purchased the American option using the parameters $T = 1$, $S_0 = 10$, $\mu = 5\%$, $\sigma = 20\%$, and the risk-free rate $r = 2\%$. Use $N = 5000$ and a strike $K = 10$.

      i. Simulate 10,000 sample paths of the asset and generate distributions for the P&L and distribution and the stopping time for a trader who purchased the option. Conditioned on those paths that are exercised (otherwise the distribution will have point masses), but record the probability of exercise to account for the point mass.

      ii. Repeat the above for various values of r and σ.

      iii. Suppose that the realized volatility is $\sigma = 10\%, 15\%, 20\%, 25\%, 30\%$, but you were able to purchase the option with a volatility of $\sigma = 20\%$ and you use the $\sigma = 20\%$ exercise boundary in your trading strategy. Explore how the distributions of profit and loss and exercise time vary in this case.