Tensor Product

of vector spaces

A universal object in this category is called a **tensor product** of $E_1, \ldots, E_n$ (over R).

We shall now prove that a tensor product exists, and in fact construct one in a natural way. By abstract nonsense we know of course that a tensor product is uniquely determined, up to a unique isomorphism.

Fix a field K.

Defn. Given vector spaces $V_1, \dots, V_n$, and W, a function $\varphi: V_1 \times \cdots \times V_n \to W$

is *multilinear* if it is linear in each argument.

example. For $n=2$, $W=K$. a multilinear map $V_1 \times V_2 \to K$ is called a bilinear form. Say $V_1 = V_2 = \mathbb{R}^n$, the dot product $(v_1, v_2) \mapsto v_1 \cdot v_2$ is a bilinear form.

example $W=K$. $V=\mathbb{C}^n$.

We have a multilinear map $\underbrace{V \times \cdots \times V}_{n \text{ copies}} \longrightarrow K$ by

$$(v_1, \dots, v_n) \longmapsto \det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}.$$

Point of tensor product: convert multilinear maps into linear maps.

Defn Let U and V be vector spaces. A <u>tensor product</u> of U and V is a vector space T equipped with a bilinear map $\varphi : U \times V \to T$ s.t. it is universal.

bilin. part of defn.

$$\begin{tikzcd} U \times V \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & T \arrow[d, dashed, "\exists ! \rho"] \\ & W \end{tikzcd}$$

given $\psi : U \times V \to W$ bilinear, $\exists ! \rho : T \to W$ s.t. $\psi = \rho \circ \varphi$.

Prop Suppose (T, φ), (T', φ') are tensor products of V and W. Then $\exists !$ isom $i : T \to T'$ s.t. $\varphi' = i \circ \varphi$

Pf. (Alex's favorite proof :-))

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & T \arrow[d, shift left, "i"] \\ & T' \arrow[u, shift left, dashed, "j"] \end{tikzcd}$$

bilin

bilin

<u>By defn</u>, $\exists ! \, i$ s.t. $\varphi' = i \circ \varphi$

, $\exists ! \, j$ s.t. $\varphi = j \circ \varphi'$

$i \circ j : T' \to T'$, $\left[\begin{array}{l} i \circ j \circ \varphi' = i \circ \varphi = \varphi' \\ id_{T'} \circ \varphi' = \varphi' \end{array} \right.$

uniq $\Rightarrow i \circ j = id_{T'}$

by similar argument, $j \circ i = id_T$.

Voila, they are isom $\square$.

Rmk: because of uniqueness, we speak of "the" tensor product.

Prop. The tensor product of any two v.s. exists.

pf. Let U, V be two v.s.

If (T, φ) were a tensor prod,

Then we have a bilin map $U \times V \longrightarrow T$ and so given $u \in U$, $v \in V$, get elt $\varphi(u, v) \in T$.

Define $\tilde{T}$ to be the vector space having for a basis $[u|v]$ for $u \in U$, $v \in V$.

We get a map $\tilde{\varphi}: U \times V \longrightarrow \tilde{T}$

$$(u, v) \longmapsto [u|v].$$

Note: $\tilde{\varphi}$ is <u>NOT</u> bilin.

reason: $\tilde{\varphi}(u_1 + u_2, v) = [u_1 + u_2 | v]$

$$\neq$$

$$\tilde{\varphi}(u_1, v) + \tilde{\varphi}(u_2, v) = [u_1|v] + [u_2|v]$$

Now, define T to be

$$T = \frac{\tilde{T}}{\operatorname{span}\left\{\begin{array}{ll} [\alpha u_1 + \beta u_2 | v] - \alpha[u_1|v] - \beta[u_2|v] & \forall \alpha, \beta \in K,\ u_1, u_2 \in U,\ v \in V \\ {[u|\alpha v_1 + \beta v_2] - \alpha[u|v_1] - \beta[u|v_2]} & \forall \alpha, \beta \in K,\ u \in U,\ v_1, v_2 \in V \end{array}\right.}$$

Have a quotient map $\pi : \tilde{T} \to T$

Define $\varphi := \pi \circ \tilde{\varphi}$.

$$\varphi : U \times V \to T.$$

Claim: φ is bilinear.

$$\varphi(\alpha u_1 + \beta u_2, v) - \alpha \varphi(u_1, v) - \beta \varphi(u_2, v)$$

$$= \pi([\alpha u_1 + \beta u_2 | v]) - \pi(\alpha [u_1 | v]) - \pi(\beta [u_2 | v])$$

$$= \pi([\alpha u_1 + \beta u_2 | v] - \alpha [u_1 | v] - \beta [u_2 | v])$$

$$= 0$$

(universality)

Suppose we have a bilin. map

$$\psi : U \times V \to W.$$

Want: a unique linear map $\rho : T \to W$ s.t.

$$\begin{tikzcd}
U \times V \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & T \arrow[dl, "!\rho"] \\
& W &
\end{tikzcd}$$

Want

To start, define $\tilde{\rho} : \tilde{T} \to W$ by

$$[u | v] \to \psi(u, v)$$

$$\tilde{\rho}([\alpha u_1 + \beta u_2 | v] - \alpha [u_1 | v] - \beta [u_2 | v])$$

$$= \psi(\alpha u_1 + \beta u_2, v) - \alpha \psi(u_1, v) - \beta \psi(u_2, v)$$

$= 0$ (ψ is bilin).

by mapping prop of quot.

$\exists !$ linear map $\rho : T \to W$ s.t.

$$\tilde{\rho} = \rho \circ \pi$$

$$\begin{tikzcd} U \times V \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & T \arrow[d, "\rho"] \\ & W \end{tikzcd} \qquad \begin{tikzcd} (u,v) \arrow[r, mapsto, "\varphi"] \arrow[dr, mapsto, "\psi"'] & \pi([u|v]) \arrow[d, mapsto, "\rho"] \\ & \psi(u,v) = \tilde{\rho}([u|v]) \end{tikzcd}$$

Claim: ρ is the unique lin. map making the diagram commute

reason: Must have $\rho(\varphi(u,v)) = \psi(u,v)$

$$\rho(\pi([u|v]))$$

the set $\pi(\{[u|v] \mid u \in U, v \in V\})$ span T, and

ρ is determined on a spanning set, and therefore unique. $\square$