Math 494 Discussion | Localization | January 29th

All rings in this note are commutative.

Definition. A *multiplicative set* in a ring R is a subset S of R that contains 1 and is closed under multiplication, i.e., $x, y \in S \implies xy \in S$.

Example.

- $-R = \mathbb{Z}, S = \mathbb{Z} \setminus \{0\}.$
- R an integral domain, $S = R \setminus \{0\}$.
- R a ring, $a \in R \setminus \{0\}, S = \{a^n \mid n \in \mathbb{Z}_{>0}\}.$
- R a ring, $\mathfrak{p} \subset R$ a prime ideal, $S = R \setminus \mathfrak{p}$.

Definition. As a set, $S^{-1}R$ is the set of equivalence classes of pairs (a, s) with $a \in R, s \in S$, under the equivalence relation \sim defined by $(a_1, s_1) \sim (a_2, s_2)$ if $\exists s' \in S$ s.t. $s'(s_2a_1 - s_1a_2) = 0$.

Exercise. Check the above definition of \sim indeed gives an equivalence relation.

Notation. We write $\frac{a}{s}$ for the element represented by (a, s). Addition and multiplication are defined using the usual formulas:

$$\begin{split} \frac{a_1}{s_1} + \frac{a_2}{s_2} &= \frac{s_2 a_1 + s_1 a_2}{s_1 s_2} \\ \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} &= \frac{a_1 a_2}{s_1 s_2} \end{split}$$

Exercise. Check this gives $S^{-1}R$ a well-defined ring structure.

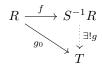
Remark. Checking that the choice of representative is well-defined is common when working with quotients and equivalence classes.

Example.

- If $R = \mathbb{Z}, S = \mathbb{Z} \setminus \{0\}$, then $S^{-1}R \cong \mathbb{Q}$. This construction works for any integral domains, the localization is called the *field of fractions*.
- If $R = \mathbb{C}[x], S = R \setminus \{0\}$, then $S^{-1}R \cong \mathbb{C}(x)$.
- If $0 \in S$, then $S^{-1}R = 0$, the trivial ring.
- If $R = \mathbb{Z}/6\mathbb{Z}, S = \{1, 3, 5\}$, then $S^{-1}R \cong \mathbb{Z}/3\mathbb{Z}$, a field.
- If R is a ring, $a \in R \setminus \{0\}$, $S = \{a^n \mid n \in \mathbb{Z}_{\geq 0}\}$, then $S^{-1}R$ is denoted $R[\frac{1}{a}]$.
- If R is a ring, $\mathfrak{p} \subset R$ is a prime ideal, $S = R \setminus \mathfrak{p}$, then $S^{-1}R$ is denoted $R_{\mathfrak{p}}$ and called *localization of* R *at* \mathfrak{p} .

Observation. There is a natural ring homomorphism $f: R \to S^{-1}R$, where $f(a) = \frac{a}{1}$. If $s \in S$, then f(s) is a unit of $S^{-1}R$.

Proposition. (universal property) Let T be an arbitrary ring. Giving a ring homomorphism $g: S^{-1}R \to T$ is equivalent to giving a ring homomorphism $g_0: R \to T$ s.t. g_0 takes elements of S to units of T.



Proof. Given g, we define $g_0 = g \circ f$. Conversely, given g_0 , we define $g(\frac{a}{s}) = \frac{g_0(a)}{g_0(s)}$. Checking g is well-defined is left as exercise. It's clear that $g_0 = g \circ f$.

Observation. $f: R \to S^{-1}R$ is injective $\iff S$ does not contain zero divisors.

4

Problem 1. Let R = C[x,y]. Give an example of a multiplicative set $S \subset R$ such that the localization $S^{-1}R$ is a PID, and not a field. (Note: the zero ring is not a PID).

Problem 2. A commutative ring is called *local* if it has a unique maximal ideal. Suppose A is local and \mathfrak{m} is its unique maximal ideal. Show that $x \in A$ is a unit if and only if $x \notin \mathfrak{m}$.

Problem 3. Let A be a ring and let S be a multiplicative subset. Prove that extension and contraction induce mutually inverse bijections between the set of prime ideals of $S^{-1}A$ and the set of prime ideals of A that do not meet S.

[Some definitions: let $f: A \to S^{-1}A$ be the canonical ring homomorphism sending a to a/1. The *extension* of an ideal \mathfrak{a} of A is the ideal of $S^{-1}A$ generated by $f(\mathfrak{a})$. The *contraction* of an ideal \mathfrak{b} of $S^{-1}A$ is $f^{-1}(\mathfrak{b})$.]

Problem 4. Let A be a commutative ring with a prime ideal $\mathfrak{p} \subset A$. Show that $A_{\mathfrak{p}}$ is a local ring.