Localization (all rings commutative)

Motivating: obtain $\mathbb{Q}$ from $\mathbb{Z}$.

Defn. A multiplicative set (system) in a ring R is a subset S of R that contains 1 and is closed under mult. ($x, y \in S \Rightarrow xy \in S$).

example: $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$.

Goal: Given a multiplicative set $S \subset R$, define a ring $S^{-1}R$ whose elements are fractions, $\frac{a}{s}$ $a \in R$, $s \in S$.

Formally, we'll start with ordered pairs (a, s), $a \in R$, $s \in S$.

Attempt to define an equivalence relation:

Problem: $a_1/s_1 = a_2/s_2 \longleftrightarrow s_1 a_2 = s_2 a_1$

Suppose $(a_1, s_1) \sim (a_2, s_2)$, $(a_2 s_2) \sim (a_3, s_3)$.

$$s_1 a_2 = s_2 a$$

$$s_2 a_3 = s_3 a_2$$

$$\Rightarrow \underline{s_3 s_2 a_1} = s_1 s_3 a_2 = s_2 \underline{s_1 a_3}$$

$$\Rightarrow s_2 a_1 s_3 = s_2 a_3 s_1 \not\Rightarrow a_1 s_3 = a_3 s_1$$

Solution: $(a_1, s_1) \sim (a_2, s_2)$ if $\exists s' \in S$ s.t. $s'(a_1 s_2 - a_2 s_1) = 0$.

exercise : this is an equivalence relation.

Defn. As a set, $S^{-1}R$ is the set of equivalence classes of pairs (a,s) with $a \in R$, $s \in S$.

notation $\frac{a}{s}$ for the elt rep. by (a,s).

addition & mult. defined using usual formulae.

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$$

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2 a_1 + s_1 a_2}{s_1 s_2}$$

Can check this defines ring structure on $S^{-1}R$

Example:

- $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$, $S^{-1}R \cong \mathbb{Q}$.
- $R = \mathbb{C}[x]$, $S = R \setminus \{0\}$, $S^{-1}R \cong \mathbb{C}(x)$
- R a ring, $0 \in S$, $S^{-1}R = 0$. (trivial ring).

Reason: WTS $(a_1, s_1) \sim (a_2, s_2)$ $\forall$ a_1, a_2, s_1, s_2.

need $s' \in S$ s.t. $s'(s_2 a_1 - s_1 a_2) = 0$.

Yes! take $s' = 0$.

$\exists$ natural homomorphism $f : R \longrightarrow S^{-1}R$

$$a \longmapsto \frac{a}{1}.$$

Observation: If $s \in S$, then $f(s)$ is a unit in $S^{-1}R$.

Reason: $f(s) = \frac{s}{1} = (s,1)$. Inverse is $\frac{1}{s}$.

$$\frac{s}{1} \cdot \frac{1}{s} = \frac{s}{s} = \frac{1}{1} = 1.$$

Proposition (universal property)

Giving a hom. $g: S^{-1}R \longrightarrow T$ is equivalent to giving a hom. $g_0 : R \longrightarrow T$ s.t. g_0 take elts of S to units of T.

natural hom.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "g_0"'] & S^{-1}R \arrow[d, dashed, "\exists ! g"] \\ & T \end{tikzcd}$$

For every $g_0 : R \to T$ that s.t. $g_0(S) \subseteq T^\times$, $\exists ! g$ s.t. $g_0 = g \circ f$.

$g_0(s^{-1}) = g_0(s)^{-1}$

pf. Given g, define $g_0 = g \circ f$. ← takes $S \to$ units of $S^{-1}R$

g ring hom, takes units to units.

Conversely, let g_0 be given.

Define $g\left(\frac{a}{s}\right) = \frac{g_0(a)}{g_0(s)}$. ($g: S^{-1}R \to T$; $a \in R$)

$g_0(s) =$ unit in T

$g\left(\frac{a}{s}\right) = g\left(\frac{a}{1}\right) g\left(\frac{1}{s}\right)$

$g_0(a) g_0(s)^{-1}$

$\frac{a}{s}$ might be represented by $\frac{a'}{s'}$!

exercise: checking g is well-defined. $\square$

$g \circ f(s) = g\left(\frac{s}{1}\right) = g_0(s)$

$g\left(\frac{1}{s}\right) = g_0(s)^{-1}$

Q: When is $f: R \to S^{-1}R$ injective?

example: $\underline{R = \mathbb{Z}/6\mathbb{Z}}$, $S = \{1, \underset{\triangle}{3}, 5\}$. $S^{-1}R \cong \mathbb{Z}/3\mathbb{Z}$, a field

$\underline{2 \cdot 3 = 0}$

$$\frac{2}{5} = \frac{0}{3} \quad 1(2 \cdot 3 - 5 \cdot 0) = 6 \equiv 0 \pmod 6.$$

Answer Think about $\ker(f)$. Suppose $a \in \ker(f)$

$$0 = \frac{0}{1} = f(a) = \frac{a}{1}, \quad \text{so } (a, 1) \sim (0, 1)$$

So $\exists$ s' s.t. $s'(a \cdot 1 - 1 \cdot 0) = 0$, $s'a = 0$.

Corollary: f is injective $\Longleftrightarrow$ S doesn't contain zero divisors.

Special cases: ◦ R = int. domain, $S = R \setminus \{0\}$, $S^{-1}R$ = field of fraction.

◦ Let $\mathfrak{p} \subset R$ be a prime ideal, $S = R \setminus \mathfrak{p}$.

(because if $a, b \in R \setminus \mathfrak{p}$, $ab \in R \setminus \mathfrak{p}$).

$S^{-1}R = R_{\mathfrak{p}}$ called the localization of R at $\mathfrak{p}$.

◦ Let $f \in R$, $S = \{ f^n \mid n \in \mathbb{Z}_{\geq 0} \}$, $S^{-1}R = R[\frac{1}{f}]$