Math 494 Discussion | Midterm Review | March 5th

Q: What does an analytic number theorist say when he is drowning?

Problem 1. Let I be the ideal $(2+i)$ in $\mathbf{Z}[i]$. Show that $\mathbf{Z}[i]/I$ is a field.

Problem 2. Let $A = \mathbf{C}[x, y]$.

(a) Find two prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of A such that $\mathfrak{p}_1 + \mathfrak{p}_2$ is prime.

(b) Find two prime ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ of A such that $\mathfrak{q}_1 + \mathfrak{q}_2$ is not prime.

(c) Find two non-prime ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of A such that $\mathfrak{a}_1 + \mathfrak{a}_2$ is prime.

Problem 3.

(a) Suppose I is an ideal in a PID R such that $I^2 = I$. Show that $I = (0)$ or $I = R$.

(b) Give an example of an ideal I in a commutative ring R such that $I^2 = I$ but I is not (0) or R.

Problem 4. Let R be a commutative ring such that $IJ = I \cap J$ for all ideals I and J. Show that every prime ideal of R is maximal.

Problem 5. Let $R = \mathbf{F}_5[x]/(x^3)$, where $\mathbf{F}_5 = \mathbf{Z}/5\mathbf{Z}$ denotes the field with five elements. Determine the order of the unit group $R^\times$.

Problem 6. Let A be a ring in which $x^3 = x$ holds for all x. Prove that A is a subring of a (perhaps infinite) product of $\mathbf{F}_2$'s and $\mathbf{F}_3$'s.

A: Log-log, log-log, log-log, ...