LOG(M) PROJECT REPORT

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1. Introduction

A translation surface is a collection of polygons with pairs of parallel edges identified by translation, up to cut and paste equivalence, meaning when we cut along a line, identify the two edges formed by the cut, and perhaps glue together two edges that are identified, we say the new surface is equivalent to the original one.

A motivation for studying such geometric objects is that they help us understand the illumination problems, which is a class of mathematical problems that study the illumination of rooms with mirrored walls by point light sources. For example, it has been proven that rational polygons, which are polygons with all angles rational multiples of 2π, can have at most finitely many dark points[LMW16].

One problem that arises naturally as we explore in this direction is as follows: when are we able to cut and paste the translation surface to obtain a convex polygon? This is due to the fact that a room of convex polygon shape can always be illuminated. We will study this problem in some special cases with the help of SageMath in this LoG(M) project.

2. Background

Let us introduce some definitions that are necessary to make sense of the problem. They are simplified for the purpose of this project. We refer the readers to [MT02] and [Zor06] for a more detailed and technical introduction to the subject matter. It requires substantial knowledge in differential geometry, complex analysis, and other related fields.

Definition 2.1 ([LW15] Translation surface)**.** A *translation surface* is a collection of polygons with pairs of parallel edges identified by translation, up to cut and paste equivalence.

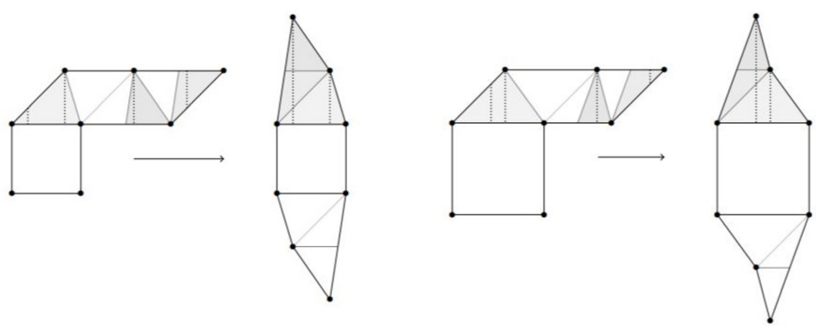

Figure 1. Example of convex and non-convex presentation of translation surfaces from [LW15].

It is implied in this definition that the identified edges have identical length and orientation.

We refer to the vertices of the polygons as the vertices of the translation surface. If two vertices of the polygons are identified by an edge, we treat them as the same vertex in the translation surface.

Definition 2.2 (Cone angle)**.** The *cone angle* of a vertex in a translation surface is the angle around the vertex one has to travel in order to return to where they started.

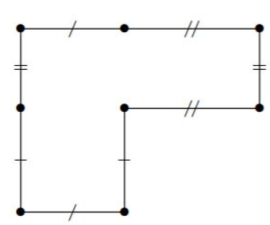

Figure 2. Translation surface with cone angle 6π. [LW15].

Observe that the cone angles of a translation surface will always be a multiple of 2π due to the nature of the translation surfaces.

Definition 2.3 (Singular point)**.** A vertex of a translation surface is a *singular point* if its cone angle is not 2π.

The cone angle of a singular point could be represented as $2\pi(\kappa+1)$ for some positive integer κ.[1] With this quantity, the translation surfaces can then be classified into *strata*.

Definition 2.4 (Stratum)**.** A *stratum* $\mathcal{H}(\kappa_1, \dots, \kappa_n)$ is the set of translation surfaces with n singular points, and the ith singular point has cone angle $2\pi(\kappa_i + 1)$.

Note that the classification is up to ordering of the singular points, and it's conventional to put the κ_i in decreasing order.

Upon identifying the edges, we obtain a surface from the translation surface.

Definition 2.5 (Genus)**.** The *genus* of a connected, orientable surface is the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. The most common concept, the genus of an (orientable) surface, is the number of "holes" it has.

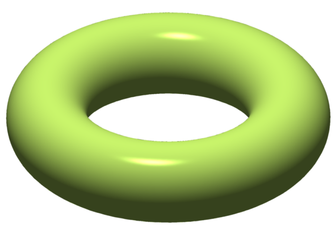

FIGURE 3 Surface of genus 1

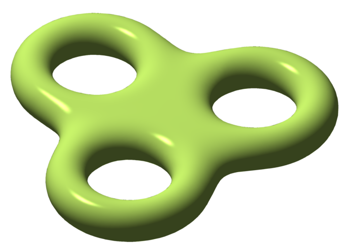

FIGURE 4 Surface of genus 3

The genus of the surface we obtained is given by the following formula, a special case of the Gauss-Bonnet theorem.

Theorem 2.6 (Gauss-Bonnet)**.** *For a translation surface in the stratum $\mathcal{H}(\kappa_1, \dots, \kappa_n)$, its genus g is given by*

$$\sum_{i=1}^{n} \kappa_i = 2g - 2.$$

Definition 2.7 (Saddle connection)**.** A *saddle connection* is a line segments joining singular points that do not contain additional singular points.

[1]The quantity κ corresponds to the degrees of zero of a form associated to the singular point.

Every point in a stratum, which is a translation surface, can be distinguished from others by a finite number of saddle connections.[2]

Identifying points in a stratum by saddle connections gives coordinates to each point. This gives us a way of identifying a neighborhood of any point in a stratum with an open subset of $\mathbb{R}^d$, where d is the dimension of the stratum, given by the following proposition. It also determines the topology on the stratum.

Proposition 2.8. *The dimension of a stratum $\mathcal{H}(\kappa_1, \ldots, \kappa_n)$ is always $2(2g+n-1)$, where g is the genus of the surface.*

Now consider $\mathrm{GL}(2,\mathbb{R})$, the group of invertible 2×2 matrices over $\mathbb{R}$. It acts on $\mathbb{R}^2$, in particular polygons in $\mathbb{R}^2$, and translation surfaces are comprised of such polygons. We are interested in the orbit of points in a stratum under this action, i.e., all the possible translation surfaces we could obtain by applying elements of $\mathrm{GL}(2,\mathbb{R})$ to a point. The following is a result in this direction.

Theorem 2.9 ([Mas82], [Vee82]). *The* $\mathrm{GL}(2,\mathbb{R})$ *orbit of almost every translation surface is dense in a connected component of a stratum.*

An approach to finding points without dense orbits is to produce closed proper subsets of a stratum that are invariant under the action of $\mathrm{GL}(2,\mathbb{R})$. Such a subset is called an *invariant subvariety.*

One such example is the *translation covers.*

Definition 2.10 (translation cover). A translation surface is called a *translation cover* if it admits a map to another translation surface that is locally distance-preserving with respect to the flat metric and respects cardinal directions.

It turns out that the collection of all translation covers forms a closed proper invariant subvariety of a stratum.

Finally, surfaces with a convex presentation only exist in strata where every surface has a 180° rotational symmetry, these are called *hyperelliptic components of strata.* A recent theorem describes the $\mathrm{GL}(2,\mathbb{R})$ orbit closures in hyperelliptic components of strata.

Theorem 2.11 ([Api18]). *For hyperelliptic components of strata, the* $\mathrm{GL}(2,\mathbb{R})$ *orbit closures are either closed or collections of translation covers.*

There have also been studies of translation surfaces that admit no convex presentation, [LW15] gives such examples.

[2]This is sensible because if you were to triangulate any flat surface of genus g with n singular points, you will have to specify $2g+n-1$ edges to determine the length of every edge in the triangulation.

3. Our Work

3.1. Preliminaries.

Definition 3.1 (Origami)**.** An *origami* is a translation surface made of unit squares.

Origamis can be easily specified by the gluing between edges. For an n-orgami, it is just a pair of permutations of $[n]$ specifying the gluing between horizonal edges and gluing between vertical edges.

We focus our attention to origamis because any translation surfaces on the plane with rational coordinates could be scaled appropriately to have integer coordinates, which could then be square-tiled up to cut and paste and represented by an origami. In other words, up to scaling, we can approximate any translation surface arbitrarily well. We also note that origamis are translation covers of the square torus (the unit squares).

In determining the convexity of a translation surface, the *cylinders* are important.

Definition 3.2 (Cylinder)**.** A *cylinder* is a rectangle with one pair of opposite edges glued together.

Definition 3.3 (Simple cylinder)**.** A *simple cylinder* is a cylinder for which each boundary is a single saddle connection.

One useful observation is that, for an origami, the 1-cylinder directions must be rational, and therefore can be brought to the horizontal direction after applying the action of an appropriate element of $\mathrm{SL}(2,\mathbb{Z})$.

It is well-known that $\mathrm{SL}(2,\mathbb{Z})$ is generated by two elements, one representing shear and one representing rotation, so the action of $\mathrm{SL}(2,\mathbb{Z})$ can be implemented on SageMath. Furthermore, the *Veech group* $\mathrm{SL}(x) := \{g \in \mathrm{SL}(2,\mathbb{Z}) \mid gx = x\}$ of an origami is known to have finite index in $\mathrm{SL}(2,\mathbb{Z})$, so we can actually enumerate the $\mathrm{SL}(2,\mathbb{Z})$ orbit of each orgami, as it will be finite.

With these tools, we can implement the Lelièvre-Weiss strict convexity test on SageMath. It can be described as follows:

(1) Find all simple cylinders of the translation surface, which are horizontal up to rotation.
(2) Apply shearing to obtain vertical saddle connection on the simple cylinder.
(3) For a surface in $\mathcal{H}(2g-2)$ with strictly convex presentation, for some horizontal simple cylinder with vertical saddle connection, every southward pointing arrow from singularities will intersect the top of the simple cylinder. Find the intersections.
(4) The intersection divides the top of the cylinder into $2g-1$ intervals. For a surface with convex presentation, traveling upward from those intervals will result in intervals on the bottom of the cylinder in reverse direction.

(5) For a surface with strictly convex presentation, when we plot the points above the simple cylinder, the resulting region is strictly convex in the plane.

All of our work are in SageMath.

3.2. **Outcomes.** The following are our progress on the project.

(1) We implemented a function that gives the resulting origami of an origami under the action by any given matrix in $\mathrm{SL}(2, \mathbb{Z})$.
(2) We implemented a function to enumerate all n-origamis in $\mathcal{H}(2)$ for any input n.
(3) We finished implementing the Lelièvre-Weiss test for convexity.

3.3. **Future Directions.**

(1) Enumerate origamis in $\mathcal{H}(4)$.
(2) Use the test to study surfaces in $\mathcal{H}(2g - g)$ for larger genus g.
(3) Explore non-strict convexity tests for translation surfaces.
(4) Explore tests for surfaces that are not origamis up to cut and paste.

References

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