Math 494 Discussion | Tensor Product | February 19th

Classification of mathematical problems as linear and nonlinear is like classification of the Universe as bananas and non-bananas.[1]

Fix a field K.

Definition. Given vector spaces $V_1, \dots, V_n$ and W, a function $\varphi : V_1 \times \cdots \times V_n \to W$ is *multilinear* if it is linear in each argument separately, e.g., fixing $v_1 \in V_1, \dots, v_n \in V_n$, then the function $V_1 \to W$ defined by $v_1 \mapsto \varphi(v_1, \dots, v_n)$ is linear.

Example. For $n = 2$, $W = K$, a multilinear map $V_1 \times V_2 \to K$ is called a *bilinear form*. Say $V_1 = V_2 = \mathbb{R}^n$, the dot product $(v_1, v_2) \mapsto v_1 \cdots v_2$ is a bilinear form.

Example. Say $K = \mathbb{C}$, $V = \mathbb{C}^n$, then we have a multilinear map $V \times \cdots \times V \to \mathbb{C}$ defined by $(v_1, \dots, v_n) \mapsto \det((v_1 \ \cdots \ v_n))$, seeing v_i as column vectors of an $n \times n$ matrix over $\mathbb{C}$.

Definition. Let U, V be vector spaces. A *tensor product* of U and V is a vector space T equipped with a bilinear map $\varphi : U \times V \to T$ that is universal, i.e., given any bilinear map $\psi : U \times V \to W$, there exists a unique linear map $\rho : T \to W$ such that $\psi = \rho \circ \varphi$.

$$\begin{tikzcd} U \times V \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & T \arrow[dl, dotted, "\exists!\rho"] \\ & W & \end{tikzcd}$$

Proposition. *Let V, W be vector spaces. Suppose (T, φ) and (T', φ') are two tensor products of V and W. Then there exists a unique isomorphism $i : T \to T'$ such that $\varphi' = i \circ \varphi$.*

Proof. There exists a unique linear map $i : T \to T'$ such that $\varphi' = i \circ \varphi$ by definition. There also exists a unique linear map $j : T' \to T$ such that $\varphi = j \circ \varphi'$ by definition. Then $i \circ j : T' \to T'$ satisfies $i \circ j \circ \varphi' = i \circ \varphi = \varphi'$, but we also know $\mathrm{id}_{T'} \circ \varphi' = \varphi'$, so by uniqueness, $i \circ j = \mathrm{id}_{T'}$. Similarly, $j \circ i = \mathrm{id}_T$.

Remark. Because of uniqueness, we abuse notation and speak of "the" tensor product.

Proposition. *The tensor product of any two vector spaces exists.*

Remark. The proof is kind of long, so see it in discussion or Jamboard notes, or look it up.

[1] I didn't say this, and I'm not sure who did.