Math 494 Discussion | Tensor Product | March 12th

Do not be seduced by the lotus-eaters into infatuation with untethered abstraction.[1]

Fix a field K.

Notation. Given vector spaces V and W, we write $V \otimes W$ as the tensor product of V and W. For $v \in V$, $w \in W$, we write $v \otimes w = \pi([v \mid w])$[2], this is called a *pure tensor*.

Remark. Pure tensors span the tensor product, but not every element of the tensor product is a pure tensor.

Recall. Let U, V be vector spaces. We defined $U \otimes V$ equipped with a bilinear map $\varphi : U \times V \to U \otimes V$ that is universal, i.e., given any bilinear map $\psi : U \times V \to W$, there exists a unique linear map $\rho : U \otimes V \to W$ such that $\psi = \rho \circ \varphi$, and it maps (u, v) to $u \otimes v$.

$$\begin{tikzcd} U \times V \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & U \otimes V \arrow[dl, dotted, "\exists! \rho"] \\ & W & \end{tikzcd}$$

Proposition. $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$.

Proof. Define $\tilde{\alpha} : (V_1 \oplus V_2) \times W \to (V_1 \otimes W) \oplus (V_2 \otimes W)$ by $(v_1 + v_2, w) \mapsto v_1 \otimes w + v_2 \otimes w$.

$\tilde{\alpha}$ is bilinear, so we apply the universal property of tensor product and obtain linear map $\alpha : (V_1 \oplus V_2) \otimes W \to (V_1 \otimes W) \oplus (V_2 \otimes W)$ such that $\alpha((v_1 + v_2) \otimes w) = v_1 \otimes w + v_2 \otimes w$.

Next we define $\tilde{\beta_1} : V_1 \times W \to (V_1 \oplus V_2) \otimes W$ by $(v_1, w) \mapsto v_1 \otimes w$.

Again by the universal property of tensor product we get a linear map $\beta_1 : V_1 \otimes W \to (V_1 \oplus V_2) \otimes W$ such that $\beta_1(v_1 \otimes w) = v_1 \otimes w$.

Similarly we have $\beta_2 : V_2 \otimes W \to (V_1 \oplus V_2) \otimes W$ such that $\beta_2(v_2 \otimes w) = v_2 \otimes w$.

Together β_1, β_2 gives $\beta : (V_1 \otimes W) \oplus (V_2 \otimes W) \to (V_1 \oplus V_2) \otimes W$, where $\beta(v_1 \otimes w + v_2 \otimes w') = v_1 \otimes w + v_2 \otimes w'$.

Since the pure tensors span, we can check that α and β are inverse to each other by checking on the pure tensors.

Remark. This works for infinite direct sums:

$(\bigoplus_{i \in I} V_i) \otimes W \cong \bigoplus_{i \in I} (V_i \otimes W)$.

[1] Ravi Vakil, *The Rising Sea: Foundations of Algebraic Geometry*.

[2] This notation is from the construction presented in discussion on Feb. 19th.

Proposition. *$K \otimes K$ is one dimensional.*

Proof. An element of $K \otimes K$ has the form $\sum_{i=1}^n a_i \otimes b_i$ for $a_i, b_i \in K$. $\sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^n a_i b_i (1 \otimes 1) = (\sum_{i=1}^n a_i b_i)(1 \otimes 1)$, so $1 \otimes 1$ spans $K \otimes K$, and $K \otimes K$ has dimension at most 1.

Suppose $K \otimes K = 0$. Then consider a bilinear map $\psi : K \times K \to W$.

$$\begin{tikzcd} K \times K \arrow[rr] \arrow[dr, "\psi"'] & & K \otimes K \arrow[dl, dotted] \\ & W & \end{tikzcd}$$

$K \otimes K = 0$ would imply that $\psi = 0$. Therefore, if we can construct any nonzero bilinear map out of $K \times K$, then $K \otimes K \neq 0$.

The multiplication map $K \times K \to K$ given by $(a, b) \mapsto ab$ is a nonzero bilinear map, so $K \otimes K \neq 0$ and $K \otimes K$ has dimension 1.

Remark. The multiplication map induces isomorphism $K \otimes K \xrightarrow{\sim} K$.

Proposition. *Let $e_1, \ldots, e_n$ be a basis for V, $f_1, \ldots, f_m$ be a basis for W. Then $e_i \otimes f_j$ for $1 \leq i \leq n$, $1 \leq j \leq m$ forms a basis for $V \otimes W$.*

Proof.

$$\begin{aligned} V \otimes W &= (\bigoplus_{i=1}^n K e_i) \otimes (\bigoplus_{j=1}^m K f_j) \\ &= \bigoplus_{i=1}^n \bigoplus_{j=1}^m (K e_i \otimes K f_j) \\ &= \bigoplus_{i=1}^n \bigoplus_{j=1}^m K e_i \otimes f_j \end{aligned}$$

Corollary. $\dim(V \otimes W) = \dim(V) \times \dim(W)$.

Proposition. *(Tensor product is functorial) Let $f : V_1 \to V_2$ and $g : W_1 \to W_2$ be linear maps. There exists a unique linear map $V_1 \otimes W_1 \to V_2 \otimes W_2$ denoted $f \otimes g$ with the property that $v \otimes w \mapsto f(v) \otimes g(w)$.*

Proof. We have a map $V_1 \times W_1 \to V_2 \otimes W_2$ that takes $(v, w) \mapsto f(v) \otimes g(w)$ that is bilinear, so universal property once again gives us what we want.