Primary Decomposition

Prop. The set of nilpotent elements in a ring A is an ideal, $\mathfrak{R}$, and $A/\mathfrak{R}$ has no nonzero nilpotent elements

pf. Let $x \in \mathfrak{R}$, then ax for any $a \in A$ is also nilpotent ($x^n = 0$ for some n, $(ax)^n = a^n x^n = a \cdot 0 = 0$).

Let $y \in \mathfrak{R}$, $y^m = 0$ for some $m > 0$. $(x+y)^{m+n-1}$ is $\sum_{k=1}^{m+n-1} \binom{m+n-1}{k} x^k y^{m+n-1-k}$ $k < n$ and $m+n-1-k < m$ cannot happen at the same time, x^k or $y^{m+n-1-k}$ must be 0, than $(x+y)^{m+n-1}$ is 0, $x+y$ is nilpotent.

Let $\bar{x} \in A/\mathfrak{R}$, $x \in A$ be a representative. Then $\bar{x}^n$ is represented by x^n, and $\bar{x}^n = 0 \Rightarrow x^n \in \mathfrak{R}$, $(x^n)^k = 0, k > 0$, x is nilpotent and $x \in \mathfrak{R}$. $\bar{x} = 0$.

$\square$

Defn. $\mathfrak{R}$ is the <u>nilradical</u> of A.

Prop. The nilradical of A is the intersection of all prime ideals in A.

Defn. Let $\mathfrak{a} \subset A$ be an ideal, then the <u>radical</u> of $\mathfrak{a}$, denoted by $\sqrt{\mathfrak{a}}$, is $\{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n > 0\}$.

Prop. The radical of $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$.

Pf. $S: A \to A/\mathfrak{a}$ quotient map, this is surjective.

$\{\text{prime ideals in } A \text{ containing } \mathfrak{a}\} \longleftrightarrow \{\text{prime ideals in } A/\mathfrak{a}\}$.

$\square$

Defn. An ideal q in a ring A is <u>primary</u> if $q \neq A$ and if $xy \in q$, then $x \in q$ or $y^n \in q$ for some $n > 0$.

Warning: Does $\boxed{\text{not}}$ mean $xy \in q \Rightarrow x^n \in q$ or y^n in q.

Rmk. We can also say a is primary, if $xy \in a$, we have $x \in a$, or $y \in a$, or $x, y \in \sqrt{a}$.

Another equivalent definition: q is primary iff $A/q \neq 0$ and every zero-divisor of A/q that is nonzero has to be nilpotent.

Lemma. The contraction of a primary ideal is primary

pf. Let $f: A \to B$ be a ring hom, and $q \subset B$ to be a primary ideal. $f(A)$ is a subring of B, and A/q^c is isom to the subring of B/q that is the image of $f(A)$. $\square$.

Prop. Let $q \subset A$ be a primary ideal. Then $\sqrt{q}$ is the smallest prime ideal containing q.

pf. We need only show $\sqrt{q}$ is prime.

Let $xy \in \sqrt{q}$, then $(xy)^m \in q$ for some $m > 0$.

$x^m y^m \in q$, $x^m \in q$ or $(y^m)^n \in q$. for some $n > 0$. $\square$.

$x \in \sqrt{q}$ or $y \in \sqrt{q}$

Defn. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, then $\mathfrak{q}$ is said to be <u>$\mathfrak{p}$-primary</u>.

example. 1. Inside of $\mathbb{Z}$, the only primary ideals are (0) and (p^n) for p prime and $n>0$.

2. Let $A = K[x,y]$, and $\mathfrak{q} = (x, y^2)$. $A/\mathfrak{q} \cong K[y]/(y^2)$. the zero-divisors must have a factor of y, and is therefore nilpotent. $\mathfrak{q}$ is primary, $\sqrt{\mathfrak{q}} = (x,y) = \mathfrak{p}$

$x \notin \mathfrak{p}^2$. $\mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$.

3. Let $A = K[x,y,z]/(xy - z^2)$. Let $\bar{x}, \bar{y}, \bar{z}$ to be image of x,y,z. $\mathfrak{p} = (\bar{x}, \bar{z})$, $A/\mathfrak{p} \cong K[y]$ an integral domain $\Rightarrow \mathfrak{p}$ prime

$\bar{x}\bar{y} = \bar{z}^2 \in \mathfrak{p}^2$, but $\bar{x} \notin \mathfrak{p}^2$, $\bar{y} \in \sqrt{\mathfrak{p}^2} = \mathfrak{p}$, $\mathfrak{p}^2$ is not primary.