Classification of mathematical problems as linear and nonlinear is like classification of the Universe as bananas and non-bananas.

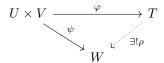
Fix a field K.

**Definition.** Given vector spaces  $V_1, \ldots, V_n$  and W, a function  $\varphi : V_1 \times \cdots \times V_n \to W$  is *multilinear* if it is linear in each argument separately, e.g., fixing  $v_1 \in V_1, \ldots, v_n \in V_n$ , then the function  $V_1 \to W$  defined by  $v_1 \mapsto \varphi(v_1, \ldots, v_n)$  is linear.

**Example.** For n=2, W=K, a multilinear map  $V_1 \times V_2 \to K$  is called a *bilinear form*. Say  $V_1=V_2=\mathbb{R}^n$ , the dot product  $(v_1,v_2)\mapsto v_1\cdots v_2$  is a bilinear form.

**Example.** Say  $K = \mathbb{C}$ ,  $V = \mathbb{C}^n$ , then we have a multilinear map  $V \times \cdots \times V \to \mathbb{C}$  defined by  $(v_1, \ldots, v_n) \mapsto \det((v_1 \cdots v_n))$ , seeing  $v_i$  as column vectors of an  $n \times n$  matrix over  $\mathbb{C}$ .

**Definition.** Let U, V be vector spaces. A *tensor product* of U and V is a vector space T equipped with a bilinear map  $\varphi: U \times V \to T$  that is universal, i.e., given any bilinear map  $\psi: U \times V \to W$ , there exists a unique linear map  $\rho: T \to W$  such that  $\psi = \rho \circ \varphi$ .



**Proposition.** Let V, W be vector spaces. Suppose  $(T, \varphi)$  and  $(T', \varphi')$  are two tensor products of V and W. Then there exists a unique isomorphism  $i: T \to T'$  such that  $\varphi' = i \circ \varphi$ .

**Proof.** There exists a unique linear map  $i: T \to T'$  such that  $\varphi' = i \circ \varphi$  by definition. There also exists a unique linear map  $j: T' \to T$  such that  $\varphi = j \circ \varphi'$  by definition. Then  $i \circ j: T' \to T'$  satisfies  $i \circ j \circ \varphi' = i \circ \varphi = \varphi'$ , but we also know  $\mathrm{id}_{T'} \circ \varphi' = \varphi'$ , so by uniqueness,  $i \circ j = \mathrm{id}_{T'}$ . Similarly,  $j \circ i = \mathrm{id}_{T}$ .

**Remark.** Because of uniqueness, we abuse notation and speak of "the" tensor product.

**Proposition.** The tensor product of any two vector spaces exists.

**Remark.** The proof is kind of long, so see it in discussion or Jamboard notes, or look it up.

<sup>&</sup>lt;sup>1</sup>I didn't say this, and I'm not sure who did.