Do not be seduced by the lotus-eaters into infatuation with untethered abstraction. ¹

Fix a field K.

**Notation.** Given vector spaces V and W, we write  $V \otimes W$  as the tensor product of V and W. For  $v \in V$ ,  $w \in W$ , we write  $v \otimes w = \pi([v \mid w])^2$ , this is called a *pure tensor*.

**Remark.** Pure tensors span the tensor product, but not every element of the tensor product is a pure tensor.

**Recall.** Let U, V be vector spaces. We defined  $U \otimes V$  equipped with a bilinear map  $\varphi : U \times V \to U \otimes V$  that is universal, i.e., given any bilinear map  $\psi : U \times V \to W$ , there exists a unique linear map  $\rho : U \otimes V \to W$  such that  $\psi = \rho \circ \varphi$ , and it maps (u, v) to  $u \otimes v$ .

$$U\times V \xrightarrow{\varphi} U\otimes V$$

$$\downarrow^{\psi}$$

$$W$$

**Proposition.**  $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$ .

*Proof.* Define  $\tilde{\alpha}: (V_1 \oplus V_2) \times W \to (V_1 \otimes W) \oplus (V_2 \otimes W)$  by  $(v_1 + v_2, w) \mapsto v_1 \otimes w + v_2 \otimes w$ .

 $\tilde{\alpha}$  is bilinear, so we apply the universal property of tensor product and obtain linear map  $\alpha: (V_1 \oplus V_2) \otimes W \to (V_1 \otimes W) \oplus (V_2 \otimes W)$  such that  $\alpha((v_1 + v_2) \otimes w) = v_1 \otimes w + v_2 \otimes w$ .

Next we define  $\beta_1: V_1 \times W \to (V_1 \oplus V_2) \otimes W$  by  $(v_1, w) \mapsto v_1 \otimes w$ .

Again by the universal property of tensor product we get a linear map  $\beta_1$ :  $V_1 \otimes W \to (V_1 \oplus V_2) \otimes W$  such that  $\beta_1(v_1 \otimes w) = v_1 \otimes w$ .

Similarly we have  $\beta_2: V_2 \otimes W \to (V_1 \oplus V_2) \otimes W$  such that  $\beta_2(v_2 \otimes w) = v_2 \otimes w$ . Together  $\beta_1, \beta_2$  gives  $\beta: (V_1 \otimes W) \oplus (V_2 \otimes W) \to (V_1 \oplus V_2) \otimes W$ , where  $\beta(v_1 \otimes w + v_2 \otimes w') = v_1 \otimes w + v_2 \otimes w'$ .

Since the pure tensors span, we can check that  $\alpha$  and  $\beta$  are inverse to each other by checking on the pure tensors.

Remark. This works for infinite direct sums:

$$(\bigoplus_{i\in I} V_i) \otimes W \cong \bigoplus_{i\in I} (V_i \otimes W).$$

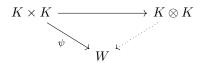
<sup>&</sup>lt;sup>1</sup>Ravi Vakil, The Rising Sea: Foundations of Algebraic Geometry.

<sup>&</sup>lt;sup>2</sup>This notation is from the construction presented in discussion on Feb. 19th.

**Proposition.**  $K \otimes K$  is one dimensional.

**Proof.** An element of  $K \otimes K$  has the form  $\sum_{i=1}^{n} a_i \otimes b_i$  for  $a_i, b_i \in K$ .  $\sum_{i=1}^{n} a_i \otimes b_i = \sum_{i=1}^{n} a_i b_i (1 \otimes 1) = (\sum_{i=1}^{n} a_i b_i) (1 \otimes 1)$ , so  $1 \otimes 1$  spans  $K \otimes K$ , and  $K \otimes K$  has dimension at most 1.

Suppose  $K \otimes K = 0$ . Then consider a bilinear map  $\psi : K \times K \to W$ .



 $K \otimes K = 0$  would imply that  $\psi = 0$ . Therefore, if we can construct any nonzero bilinear map out of  $K \times K$ , then  $K \otimes K \neq 0$ .

The multiplication map  $K \times K \to K$  given by  $(a,b) \mapsto ab$  is a nonzero bilinear map, so  $K \otimes K \neq 0$  and  $K \otimes K$  has dimension 1.

**Remark.** The multiplication map induces isomorphism  $K \otimes K \xrightarrow{\sim} K$ .

**Proposition.** Let  $e_1, \ldots, e_n$  be a basis for  $V, f_1, \ldots, f_m$  be a basis for W. Then  $e_i \otimes f_j$  for  $1 \leq i \leq n$ ,  $1 \leq j \leq m$  forms a basis for  $V \otimes W$ .

Proof.

$$V \otimes W = (\bigoplus_{i=1}^{n} Ke_i) \otimes (\bigoplus_{j=1}^{m} Kf_j)$$
$$= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (Ke_i \otimes Kf_j)$$
$$= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} Ke_i \otimes f_j$$

Corollary.  $\dim(V \otimes W) = \dim(V) \times \dim(W)$ .

**Proposition.** (Tensor product is functorial) Let  $f: V_1 \to V_2$  and  $g: W_1 \to W_2$  be linear maps. There exists a unique linear map  $V_1 \otimes W_1 \to V_2 \otimes W_2$  denoted  $f \otimes g$  with the property that  $v \otimes w \mapsto f(v) \otimes g(w)$ .

*Proof.* We have a map  $V_1 \times W_1 \to V_2 \otimes W_2$  that takes  $(v, w) \mapsto f(v) \otimes g(w)$  that is bilinear, so universal property once again gives us what we want.