Math 494 Discussion | Localization | January 29th

All rings in this note are commutative.

Definition. A *multipicative set* in a ring R is a subset S of R that contains 1 and is closed under multiplication, i.e., $x, y \in S \implies xy \in S$.

Example.

- $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$.
- R an integral domain, $S = R \setminus \{0\}$.
- R a ring, $a \in R \setminus \{0\}$, $S = \{a^n \mid n \in \mathbb{Z}_{\geq 0}\}$.
- R a ring, $\mathfrak{p} \subset R$ a prime ideal, $S = R \setminus \mathfrak{p}$.

Definition. As a set, $S^{-1}R$ is the set of equivalence classes of pairs (a, s) with $a \in R, s \in S$, under the equivalence relation $\sim$ defined by $(a_1, s_1) \sim (a_2, s_2)$ if $\exists\ s' \in S$ s.t. $s'(s_2a_1 - s_1a_2) = 0$.

Exercise. Check the above definition of $\sim$ indeed gives an equivalence relation.

Notation. We write $\frac{a}{s}$ for the element represented by (a, s). Addition and multiplication are defined using the usual formulas:

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2a_1 + s_1a_2}{s_1s_2}$$
$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2}$$

Exercise. Check this gives $S^{-1}R$ a well-defined ring structure.

Remark. Checking that the choice of representative is well-defined is common when working with quotients and equivalence classes.

Example.

- If $R = \mathbb{Z}, S = \mathbb{Z} \setminus \{0\}$, then $S^{-1}R \cong \mathbb{Q}$. This construction works for any integral domains, the localization is called the *field of fractions*.
- If $R = \mathbb{C}[x], S = R \setminus \{0\}$, then $S^{-1}R \cong \mathbb{C}(x)$.
- If $0 \in S$, then $S^{-1}R = 0$, the trivial ring.
- If $R = \mathbb{Z}/6\mathbb{Z}, S = \{1, 3, 5\}$, then $S^{-1}R \cong \mathbb{Z}/3\mathbb{Z}$, a field.
- If R is a ring, $a \in R \setminus \{0\}$, $S = \{a^n \mid n \in \mathbb{Z}_{\geq 0}\}$, then $S^{-1}R$ is denoted $R[\frac{1}{a}]$.
- If R is a ring, $\mathfrak{p} \subset R$ is a prime ideal, $S = R \setminus \mathfrak{p}$, then $S^{-1}R$ is denoted $R_{\mathfrak{p}}$ and called *localization of R at $\mathfrak{p}$*.

Observation. There is a natural ring homomorphism $f : R \to S^{-1}R$, where $f(a) = \frac{a}{1}$. If $s \in S$, then $f(s)$ is a unit of $S^{-1}R$.

Proposition. *(universal property) Let T be an arbitrary ring. Giving a ring homomorphism $g : S^{-1}R \to T$ is equivalent to giving a ring homomorphism $g_0 : R \to T$ s.t. g_0 takes elements of S to units of T.*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "g_0"'] & S^{-1}R \arrow[d, dotted, "\exists! g"] \\ & T \end{tikzcd}$$

Proof. Given g, we define $g_0 = g \circ f$.

Conversely, given g_0, we define $g(\frac{a}{s}) = \frac{g_0(a)}{g_0(s)}$.

Checking g is well-defined is left as exercise.

It's clear that $g_0 = g \circ f$.

Observation. $f : R \to S^{-1}R$ is injective $\iff$ S does not contain zero divisors.

Problem 1. Let $R = C[x, y]$. Give an example of a multiplicative set $S \subset R$ such that the localization $S^{-1}R$ is a PID, and not a field. (Note: the zero ring is not a PID).

Problem 2. A commutative ring is called *local* if it has a unique maximal ideal. Suppose A is local and $\mathfrak{m}$ is its unique maximal ideal. Show that $x \in A$ is a unit if and only if $x \notin \mathfrak{m}$.

Problem 3. Let A be a ring and let S be a multiplicative subset. Prove that extension and contraction induce mutually inverse bijections between the set of prime ideals of $S^{-1}A$ and the set of prime ideals of A that do not meet S.

[Some definitions: let $f\colon A \to S^{-1}A$ be the canonical ring homomorphism sending a to $a/1$. The *extension* of an ideal $\mathfrak{a}$ of A is the ideal of $S^{-1}A$ generated by $f(\mathfrak{a})$. The *contraction* of an ideal $\mathfrak{b}$ of $S^{-1}A$ is $f^{-1}(\mathfrak{b})$.]

Problem 4. Let A be a commutative ring with a prime ideal $\mathfrak{p} \subset A$. Show that $A_{\mathfrak{p}}$ is a local ring.