Math 494 Discussion | Primary Decomposition[1] | March 19th

Mathematics is the art of giving the same name to different things.[2]

All rings in this note are commutative.

Proposition. *The set $\mathfrak{N}$ of all nilpotent elements in a ring A is an ideal, and $A/\mathfrak{N}$ has no nonzero nilpotent elements*

Proof. If $x \in \mathfrak{N}$, clearly $ax \in \mathfrak{N}$ for all $a \in A$. For $x, y \in \mathfrak{N}$ where $x^m = 0$, $y^n = 0$, $(x+y)^{m+n-1} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} x^{m+n-1-k} y^k$. Observe that $m+n-1-k < m$ and $k < n$ cannot happen at the same time, so every term in the expansion is in fact 0. Therefore $x + y \in \mathfrak{N}$ and $\mathfrak{N}$ is an ideal.

Let $\bar{x} \in A/\mathfrak{N}$ be represented by x. Then $\bar{x}^n$ is represented by x^n, and $\bar{x}^n = 0$ implies $x^n \in \mathfrak{N}$, $(x^n)^k = 0$ for some $k > 0$. x is nilpotent and $x \in \mathfrak{N}$ means $\bar{x} = 0$.

Definition. $\mathfrak{N}$ is the *nilradical* of A.

Proposition. *The nilradical of A is the intersection of all the prime ideals of A.*

Definition. Let $\mathfrak{a}$ be an ideal of A. The *radical* of $\mathfrak{a}$, denoted $\sqrt{\mathfrak{a}}$, is $\sqrt{\mathfrak{a}} = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n > 0\}$.

Proposition. *The radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals containing $\mathfrak{a}$.*

Proof. The quotient map is surjective, so the prime ideals of $A/\mathfrak{a}$ are in bijection with prime ideals containing $\mathfrak{a}$.

Definition. An ideal $\mathfrak{q}$ in a ring A is *primary* if $\mathfrak{q} \neq A$ and $xy \in \mathfrak{q}$ implies either $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some $n > 0$.

Remark. Rephrasing the definition in a more symmetric way, we can say an ideal $\mathfrak{a}$ is primary if, for every $xy \in \mathfrak{a}$, we have $x \in \mathfrak{a}$ or $y \in \mathfrak{a}$, or $x, y \in \sqrt{\mathfrak{a}}$.

Another equivalent definition: $\mathfrak{q}$ is primary iff $A/\mathfrak{q} \neq 0$ and every zero-divisor in $A/\mathfrak{q} \neq 0$ is nilpotent.

Lemma. *The contraction of a primary ideal is primary.*

Proof. Let $f : A \to B$ be a ring homomorphism, and $\mathfrak{q} \subset B$ a primary ideal. $f(A)$ is a subring of B, so $A/\mathfrak{q}^c$ is isomorphic to a subring of $B/\mathfrak{q}$.

[1] Reference: Atiyah, MacDonald, *Introduction to Commutative Algebra*

[2] Henri Poincaré

Proposition. *Let $\mathfrak{q}$ be a primary ideal in a ring A. Then $\sqrt{\mathfrak{q}}$ is the smallest prime ideal containing $\mathfrak{q}$.*

Proof. Since $\sqrt{\mathfrak{q}}$ is the intersection of all prime ideals containing $\mathfrak{q}$, we need only prove that it is prime. Let $xy \in \mathfrak{q}$, then $(xy)^m \in \mathfrak{q}$ for some $m > 0$, and therefore $x^m \in \mathfrak{q}$ or $y^{mn} \in \mathfrak{q}$ for some $n > 0$, i.e., $x \in \sqrt{\mathfrak{q}}$ or $y \in \sqrt{\mathfrak{q}}$.

Definition. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, then $\mathfrak{q}$ is said to be $\mathfrak{p}$-*primary*.

Example.

1. The primary ideals in $\mathbf{Z}$ are (0) and (p^n), where p is prime. For these are the only ideals in $\mathbf{Z}$ with prime radical *(what are the radical ideals in* $\mathbf{Z}$*?)*, and they are indeed primary.

2. Let $A = K[x, y]$, and $\mathfrak{q} = (x, y^2)$. Then $A/\mathfrak{q} \cong K[y]/(y^2)$, in which the zero-divisors are precisely all the multiples of y, which are nilpotent. Hence $\mathfrak{q}$ is primary, and its radical $\mathfrak{p}$ is (x, y). $\mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$. This example shows that a primary ideal need not be a prime power.

3. Let $A = K[x, y, z]/(xy - z^2)$ and let $\bar{x}, \bar{y}, \bar{z}$ denote the image of x, y, z in A respectively. $\mathfrak{p} = (\bar{x}, \bar{z})$ is prime since $A/\mathfrak{p} \cong K[y]$ is an integral domain. We have $\bar{x}\bar{y} = \bar{z}^2 \in \mathfrak{p}^2$, but $\bar{x} \notin \mathfrak{p}^2$ and $\bar{y} \notin \sqrt{\mathfrak{p}^2} = \mathfrak{p}$, hence $\mathfrak{p}^2$ is not primary. This example shows that a prime power need not be primary.