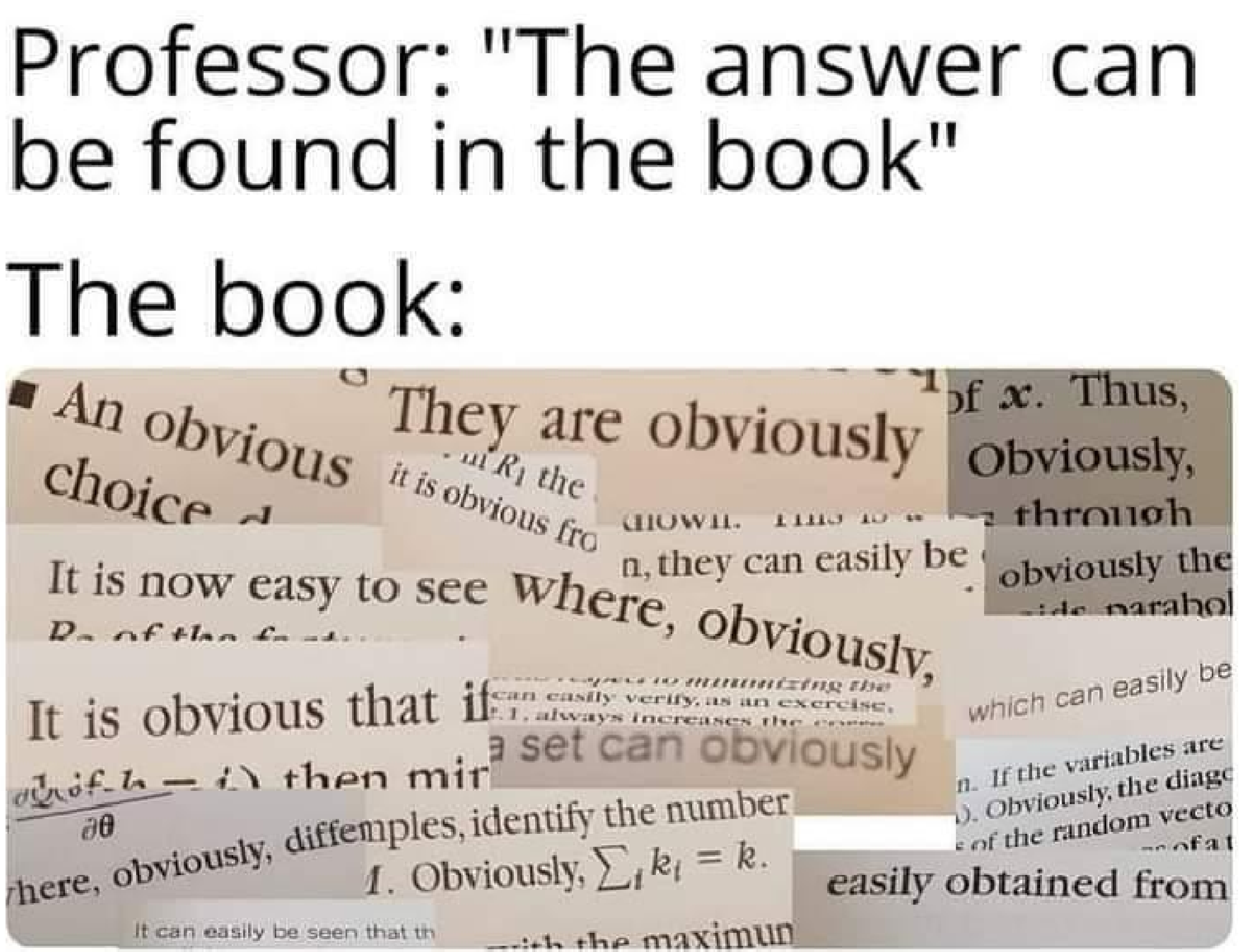
Primary Decomposition pt 2



Recall An ideal $q \subset A$ is <u>primary</u> if for $xy \in q$, we have $x \in q$, or $y \in r(q)$, i.e., $y^n \in q$.

If $r(q) = p$, then we call q <u>p-primary</u>.

Defn. If $a, b \subset A$ are ideals, then their <u>ideal quotient</u> $(a:b) = \{x \in A \mid xb \subseteq a\}$. This is an ideal.

Rmk. A special case, $(0:b)$ is the <u>annihilator</u> of b, also denoted $\mathrm{Ann}(b)$.

Rmk. The set of all zero divisors in A is $D = \bigcup_{x \neq 0} \mathrm{Ann}(x)$.

Rmk. If $b = (x)$, we write $(a:x)$.

Lemma $(\mathfrak{a}_1 \cap \mathfrak{a}_2 : \mathfrak{b}) = (\mathfrak{a}_1 : \mathfrak{b}) \cap (\mathfrak{a}_2 : \mathfrak{b})$

Pf. Let $x \in A$. $x\mathfrak{b} \in \mathfrak{a}_1 \cap \mathfrak{a}_2 \iff x\mathfrak{b} \in \mathfrak{a}_1$ and $x\mathfrak{b} \in \mathfrak{a}_2$ $\square$

Lemma. $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal, $x \in A$ an elt, then.

1) if $x \in \mathfrak{q}$, $(\mathfrak{q} : x) = (1)$
2) if $x \notin \mathfrak{q}$, then $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary, and $r(\mathfrak{q} : x) = \mathfrak{p}$.
3) if $x \notin \mathfrak{p}$, then $(\mathfrak{q} : x)$ is $\mathfrak{q}$.

Pf. 1) by defn.

2) Let $y \in (\mathfrak{q} : x)$, $xy \in \mathfrak{q}$, $x \notin \mathfrak{q} \Rightarrow y \in r(\mathfrak{q}) = \mathfrak{p}$.

$\mathfrak{q} \subseteq (\mathfrak{q} : x) \subseteq \mathfrak{p}$. $r(\mathfrak{q}) = \mathfrak{p} \Rightarrow r(\mathfrak{q} : x) = \mathfrak{p}$.

Let $yz \in (\mathfrak{q} : x)$. $xyz \in \mathfrak{q}$, $y \notin r(\mathfrak{q}) = \mathfrak{p}$,

$xz \in \mathfrak{q} \Longrightarrow z \in (\mathfrak{q} : x)$.

3). $yx \in \mathfrak{q} \subset \mathfrak{p}$, $x \notin \mathfrak{p} = r(\mathfrak{q}) \Rightarrow y \in \mathfrak{q}$. $\square$

Lemma $r(\mathfrak{a}\cap\mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.

Pf. $x \in r(\mathfrak{a}\cap\mathfrak{b})$, $\exists n$ s.t. $x^n \in \mathfrak{a}\cap\mathfrak{b}$, $x \in r(\mathfrak{a}) \cap r(\mathfrak{b})$.

$x \in r(\mathfrak{a}) \cap r(\mathfrak{b})$, $\exists n, m$ s.t. $x^n \in \mathfrak{a}$, $x^m \in \mathfrak{b}$,

$$x^n x^m = x^{n+m} \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}\cap\mathfrak{b}.$$ $\square$

Lemma If $\mathfrak{q}_i$ for $1 \le i \le n$ are $\mathfrak{p}$-primary, then $\mathfrak{q} = \bigcap_{i=1}^n \mathfrak{q}_i$ is $\mathfrak{p}$-primary.

Pf. $r(\mathfrak{q}) = r(\bigcap_{i=1}^n \mathfrak{q}_i) = \bigcap_{i=1}^n r(\mathfrak{q}_i) = \mathfrak{p}$.

Let $xy \in \mathfrak{q}$, $y \notin \mathfrak{q}$, then for some i, we have $xy \in \mathfrak{q}_i$, $y \notin \mathfrak{q}_i$, hence $x^n \in \mathfrak{q}_i$, $x \in r(\mathfrak{q}_i) = \mathfrak{p}$. $\square$

Defn. A <u>primary decomposition</u> of an ideal $\mathfrak{a} \subset A$ is an expression of $\mathfrak{a}$ as an intersection of finite primary ideals, say

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i .$$

Rmk. In general, such a primary decomp need not exist, but it does for Noetherian rings.

Rmk If the $r(\mathfrak{q}_i)$ are all distinct, and $\bigcap_{j \neq i} \mathfrak{q}_j \not\subseteq \mathfrak{q}_i$, then we say the decomp is <u>minimal</u>.

By the $\cap$ lemma, we can get the first condition, then we just omit any redundant factor ($\bigcap_{j \neq i} \mathfrak{q}_j \subseteq \mathfrak{q}_i$), we get a minimal decomposition.

Defn. $\mathfrak{a} \subset A$ is <u>decomposable</u> if it has a primary decomposition.

Theorem (1st uniqueness theorem)

Let $\mathfrak{a}$ be a decomposable ideal, $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ a minimal decomposition. Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$, then $\mathfrak{p}_i$'s are precisely the prime ideals that occur in the set of ideals $\{r(\mathfrak{a}:x) \mid x \in A\}$, and hence are independent of the particular decomposition.

Pf. For any $x \in A$, we have $(\mathfrak{a}:x) = (\bigcap \mathfrak{q}_i : x) = \bigcap (\mathfrak{q}_i : x)$,

$r(\mathfrak{a}:x) = \bigcap_{i=1}^n r(\mathfrak{q}_i : x) = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$.

Suppose $r(\mathfrak{a}:x)$ is prime, then $r(\mathfrak{a}:x) = \mathfrak{p}_j$ for some j.

general fact: if $\mathfrak{a}_i$ ideals and $\mathfrak{p} = \bigcap \mathfrak{a}_i$ is prime, then $\mathfrak{p} = \mathfrak{a}_i$ for some i.

pf. suppose $p \not\supseteq \mathfrak{a}_i \ \forall i$, then $\forall i \ \exists x_i \in \mathfrak{a}_i$ s.t. $x_i \notin p$. and $\prod x_i \in \prod \mathfrak{a}_i \subset \bigcap \mathfrak{a}_i = p$, but $\prod x_i \notin p$, $\lightning \Rightarrow \mathfrak{a}_i \subseteq p$ for some $\mathfrak{a}_i$. $\Rightarrow p = \mathfrak{a}_i$ b/c $p \subseteq \bigcap \mathfrak{a}_i \subset \mathfrak{a}_i$ $\square$

Every prime ideal of the form $r(\mathfrak{a}:x)$ is one of the p_i.

Conversely, for each i, $\exists x_i \notin q_i$, $x_i \in \bigcap_{j\neq i} q_j$ since the decomp is minimal. Then $r(\mathfrak{a}:x_i) = \bigcap_{x_i \notin q_j} p_j = p_i$ b/c $x_i \in \bigcap_{j \neq i} q_j$.

$\square$

Rmk. The proof shows that $\forall i$, $\exists x_i \in A$ s.t. $(\mathfrak{a}:x_i)$ is p_i-primary $/\!\!/$

$$(\mathfrak{a}:x_i) = \bigcap (q_j : x_i) = \begin{cases} (1) & j \neq i \\ (q_i : x_i) & p_i\text{-primary.} \end{cases}$$

example. Let $\mathfrak{a} = (x^2, xy)$ in $A = K[x,y]$.

$\mathfrak{p}_1 = (x)$, $\mathfrak{p}_2 = (x,y)$. $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2^2$.

$\mathfrak{p}_1$ is prime, $\mathfrak{p}_2$ is primary b/c $r(\mathfrak{p}_2) = (x,y)$ is max.

$\mathfrak{p}_1 \subset \mathfrak{p}_2$, $r(\mathfrak{a}) = \mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_1$, but $\mathfrak{a}$ is not primary.

Rmk: The prime ideals that show up in them are said to <u>belong to</u> $\mathfrak{a}$, or <u>associated</u> w/ $\mathfrak{a}$.

$\mathfrak{a}$ is primary iff it has only 1 assoc. prime.