Recall: We defined localization of a ring given $S$

$S^{-1}R$

Defn: Let $f: A \to B$ be a ring hom. The extension of an ideal $\mathfrak{a} \subset A$ is the ideal of $B$ that is generated by $f(\mathfrak{a})$, denoted $\mathfrak{a}^e$.
The contraction of an ideal $\mathfrak{b} \subset B$ is $f^{-1}(\mathfrak{b})$, denoted $\mathfrak{b}^c$.

Prop. Let $f: A \to B$ ring hom. $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$ nonzero ideal.

1. $\mathfrak{a} \subset \mathfrak{a}^{ec}$, $\mathfrak{b}^{ce} \subset \mathfrak{b}$.
2. $\mathfrak{a}^e = \mathfrak{a}^{ece}$, $\mathfrak{b}^{cec} = \mathfrak{b}^c$.

Fact. Contraction of a prime ideal is always a prime ideal.

Reason. Say $F: A \to B$. $\mathfrak{b} \subset B$ prime. $\mathfrak{b}^c \subset A$

Fix $ab \in \mathfrak{b}^c$. $F(ab) = F(a)F(b) \in \mathfrak{b}$.

$\mathfrak{b}$ prime $\Rightarrow F(a) \in \mathfrak{b}$ or $F(b) \in \mathfrak{b}$

$\Rightarrow a \in \mathfrak{b}^c$ or $b \in \mathfrak{b}^c$, $\mathfrak{b}^c$ is prime.

Proposition: Let $A$ be a ring and $S$ a mult. set. $x \longmapsto \frac{x}{1}$

Extension and contraction on the natural homomorphism $A \to S^{-1}A$ induce mutually inverse bijection between the set prime ideals of $S^{-1}A$ and the set of prime ideals in $A$ that do not meet $S$.

pf. Let $p \subset A$ s.t. $p \cap S = \phi$.

Let $\frac{x}{s_1} \cdot \frac{y}{s_2} \in p^e$. Note that every elt in $p^e$ can be written as $\frac{x}{s}$ with $x \in p$ and $s \in S$, because

$$\frac{x}{s_1} \frac{y}{s_2} = \frac{z}{s_3}$$

$\{\frac{x}{1} \mid x \in p\}$ generate $p^e$, and $\frac{x}{1} \cdot \frac{a}{s} = \frac{xa}{s}$ $\begin{matrix} \in p \\ \in S \end{matrix}$

$$\frac{x}{s_1} \cdot \frac{y}{s_2} = \frac{z}{s_3}$$

$$s'(xys_3 - s_1 s_2 z) = 0$$

$$xys_4 - zs_5 = 0 \,.,\; s_4, s_5 \in S.$$

$$xys_4 = zs_5 \,,\; s_4 xy \in P.$$

$s_4 \notin p$, it follows that $x \in p$ or $y \in p$.

$\frac{x}{s_1} \in p^e$ or $\frac{y}{s^2} \in p^e \implies p^e$ is prime.

WTS $p = p^{ec}$. Suppose $x \in p^{ec}$, $\frac{x}{1} \in p^e$.

Thus $\frac{x}{1} = \frac{y}{s}$ for $y \in p$, $s \in S$.

$s'(sx - y) = 0 \quad s''x = s'y^{\in p}$, and $s'' \notin p \implies x \in p$.

$p^{ec} \subset p$ and by 1st slide, $p = p^{ec}$.

$\mathfrak{q}$ prime in $S^{-1}A$. $\mathfrak{q}^c$ is a prime in $A$.

WTS $\mathfrak{q}^{ce} = \mathfrak{q}$.

Suppose $\frac{x}{s} \in \mathfrak{q}$ with $x \in A$, $s \in S$.

Then $\frac{x}{1} = s \cdot \frac{x}{s} \in \mathfrak{q}$

Since $x \mapsto \frac{x}{1}$, $x \in \mathfrak{q}^c$, $\frac{x}{1} \in \mathfrak{q}^{ce}$, $\frac{x}{1} \cdot \frac{1}{s} = \frac{x}{s} \in \mathfrak{q}^{ce}$

$\Longrightarrow \mathfrak{q} \subset \mathfrak{q}^{ce}$.

With first slide, we have $\mathfrak{q} = \mathfrak{q}^{ce}$. $\square$

Defn. A ring is called <u>local</u> if it has a unique maximal ideal.

Prop. Suppose $A$ is a local ring, then $x \in A$ is a unit iff $x \notin \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal.

Prop. Let $A$ be a ring with prime ideal $\mathfrak{p}$.

$A_{\mathfrak{p}} = S^{-1}A$, $S = \{A \setminus \mathfrak{p}\}$ is a local ring.

Pf. Let $\mathfrak{m}$ be a max ideal of $A_{\mathfrak{p}}$, in particular it is prime

$\mathfrak{m} = (\mathfrak{p}')^e$ for some $\mathfrak{p}' \subset \mathfrak{p}$. $\mathfrak{p}'^e \subset \mathfrak{p}^e$. $\mathfrak{p}^e$ proper $\Rightarrow \mathfrak{m} = \mathfrak{p}^e$.

$\mathfrak{p}^e$ is the unique max ideal. $\square$

$q \subset A_p$ Max (prime), then $q^c \subset p$, $q^{ce} \subset (f(p))$ ← proper.

Since $q^{ce} = q$, $q^{ce} = (f(p))$.

Defn. Let $A$ be a comm. ring, $p$ prime ideal, then <u>the residue field at $p$</u> is the field of fractions of the int. domain $A/p$.

Remark $A_\mathfrak{p}/\mathfrak{m} = \mathrm{Frac}(A/\mathfrak{p})$.

local ring

unique max ideal