Category Theory

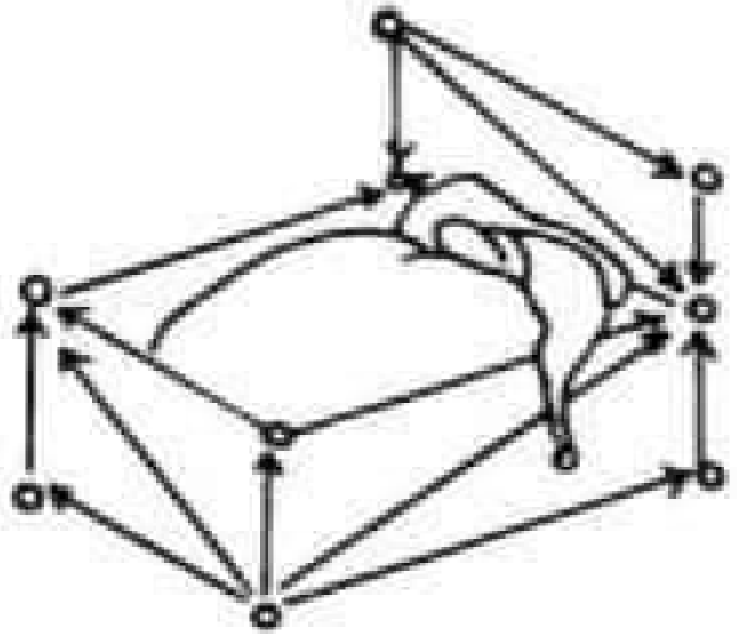
4.4 REMARK

For every subcategory **A** of a category **B** there is a naturally associated **inclusion functor** $E : \mathbf{A} \hookrightarrow \mathbf{B}$. Moreover, each such inclusion is

(1) an embedding;

(2) a full functor if and only if **A** is a full subcategory of **B**.

As the next proposition shows, inclusions of subcategories are (up to isomorphism) precisely the embedding functors and (up to equivalence) precisely the faithful functors.



A full embedding

Defn. A <u>Category</u> $\mathcal{C}$ consists of the following:

- A collection $Ob(\mathcal{C})$ of things called <u>objects</u>.
- For any $X, Y \in Ob(\mathcal{C})$, a set $Hom_{\mathcal{C}}(X,Y)$ of <u>morphisms</u>.
- Given objects X, Y, Z, a function

$$Hom_{\mathcal{C}}(Y,Z) \times Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{C}}(X,Z)$$

$$(f,g) \longmapsto f \circ g \text{ (or } fg)$$

called <u>composition</u>

- composition is associative
- for every object $X \in Ob(\mathcal{C})$, $\exists\ id_X \in Hom_{\mathcal{C}}(X,X)$ s.t. $f \circ id_X = f$, $id_X \circ g = g$, for all f and g that makes sense.

examples.

1. the category of sets.
   - objects are sets
   - morphisms are functions.

   $Hom(X,Y)$ = all fns from X to Y
2. the category of groups.
   - objects are groups
   - morphisms are gp homs
3. the category of rings
   - ______ ring
   - ______ ring hom

One nice feature of Cat Thy:
it provides the right language to talk about universal properties.

- If R is comm ring, T also a comm ring.

$$\mathrm{Hom}(R[x], T) \cong \mathrm{Hom}(R, T) \times T$$
$$f \longleftrightarrow (f_0, t)$$

cartesian product

Functor (Morphism of categories)

Defn. Let $\mathcal{C}, \mathcal{D}$ be two categories.
A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a "function" $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
$$x \longmapsto F(x)$$
- a rule associate to every morphism $\varphi : X \to Y$ in $\mathcal{C}$ to a morphism $F(\varphi) : F(X) \to F(Y)$ in $\mathcal{D}$.

s.t.
- $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ $\forall x \in \mathrm{Ob}(\mathcal{C})$.
- F is compatible w/ composition

given $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ in $\mathcal{C}$

then $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ in $\mathcal{D}$

example

There is a functor

$Gp \longrightarrow Set$, taking a gp to the underlying set

This is an example of <u>forgetful functor</u>.

· <u>Abelianization</u> defines a functor from $Gp \to Ab$ ← abelian gps.

G a gp, its commutator subgroup is denote $[G,G]$, is the subgp gen'd by $\{aba^{-1}b^{-1} \mid \ldots\}$

example (not functor)

Let G a gp, H an ab gp.

Given $\varphi : G \to H$, we have

$$
\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi"', "\text{quotient}"] & H \\
G_{ab} \arrow[ur, dashed] &
\end{tikzcd}
$$

$\exists !$ hom ψ s.t. $\varphi = \psi \circ \pi$.

G_{ab} is the abelianization, $G/[G,G]$

example (not functor)

$$\mathrm{Hom}(T, R\times S) \cong \mathrm{Hom}(T,R)\times \mathrm{Hom}(T,S)$$

$$h : T \to R\times S \longleftrightarrow \begin{matrix} f : T \to R \\ g : T \to S \end{matrix}$$

$h = (f,g)$

$\mathcal{C}$ = any category, $X \in Ob(\mathcal{C})$, $\exists$ functor $h_X : \mathcal{C} \to Set$.

$h_X(Y) = Hom_{\mathcal{C}}(X, Y)$

given $\varphi : Y \to Z$

$h_X(\varphi) : h_X(Y) \to h_X(Z)$

$$\underset{\underset{Hom_{\mathcal{C}}(X,Y)}{\cap}}{(\psi : X \to Y)} \mapsto \underset{\underset{Hom_{\mathcal{C}}(X,Z)}{\cap}}{\varphi \circ \psi : X \to Z}$$

Rmk. $h_X'(Y) = Hom_{\mathcal{C}}(Y, X)$ is a <u>contravariant functor</u>:

i.e., given $Y_1 \xrightarrow{\varphi} Y_2 \xrightarrow{\psi} Y_3$

$h_X'(\psi \circ \varphi) = h_X'(\varphi) \circ h_X'(\psi)$.