Physics-Mathematics Dictionary

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physical language	mathematical language
physics	applied mathematics
mathematics	applied mathematics
abstract nonsense	mathematics
conjecture	vague idea
theorem	conjecture
rigorous theorem	theorem
proof	sketch of the proof
basic operations $+, -, \cdot, :$	commutative field theory
classical field theory	multi-variable calculus
mathematical analysis	calculus
linear algebra	matrix calculus
group theory	representation theory
abstract group theory	group theory
quantum mechanics	mambo jumbo Dirac notation
fancy schmancy functional analysis	quantum mechanics
Hilbert space	rigged Hilbert space
Riemannian geometry	pseudo-Riemannian geometry
tensor	components of a tensor
contravariant vector	components of a vector
covariant vector	components of a one-form
fancy schmancy index-free notation	differential geometry

$\otimes$ (of vector spaces)

part 2.

Notation. V, W vector spaces, then we write their <u>tensor product</u> as $V \otimes W$. $\pi([v|w]) = v \otimes w$.

Remark. $v \otimes w$ called pure tensors. Pure tensors span the tensor product but not every element is a pure tensor

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & V \otimes W \arrow[d, dashed, "\exists ! \rho"] \\ & U \end{tikzcd}$$

Prop. $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$.

pf. Define $\tilde{\alpha} : (V_1 \oplus V_2) \times W \longrightarrow (V_1 \otimes W) \oplus (V_2 \otimes W)$

$$(v_1 + v_2, w) \longmapsto v_1 \otimes w + v_2 \otimes w.$$

Claim $\tilde{\alpha}$ is bilinear.

$$\tilde{\alpha}(\underline{a}(v_1+v_2) + \underline{b}(v_1'+v_2'), w)$$
$$= \tilde{\alpha}(av_1 + av_2 + bv_1' + bv_2', w)$$
$$= \tilde{\alpha}((av_1 + bv_1') + (av_2 + bv_2'), w)$$
$$= (av_1 + bv_1') \otimes w + (av_2 + bv_2') \otimes w$$
$$= av_1 \otimes w + bv_1' \otimes w + av_2 \otimes w + bv_2' \otimes w$$
$$= a(v_1 + v_2) \otimes w + b(v_1' + v_2') \otimes w$$
$$= a\tilde{\alpha}(v_1 + v_2, w) + b\tilde{\alpha}(v_1' + v_2', w)$$

Now apply universal property, get a linear map

$$\alpha : (V_1 \oplus V_2) \otimes W \longrightarrow (V_1 \otimes W) \oplus (V_2 \otimes W).$$

$$\tilde{\alpha} : (V_1 \oplus V_2) \times W \longrightarrow (V_1 \otimes W) \oplus (V_2 \otimes W) \text{ bilin.}$$

Next define $\tilde{\beta}_1 : V_1 \times W \longrightarrow (V_1 \oplus V_2) \otimes W$

$$(v_1, w) \longmapsto v_1 \otimes w$$

this is bilin $\Longrightarrow$ $\beta_1 : V_1 \otimes W \to (V_1 \oplus V_2) \otimes W.$

Similarly, we can get $\beta_2 : V_2 \otimes W \longrightarrow (V_1 \oplus V_2) \otimes W.$

Combine β_1 and β_2 : $\beta : (V_1 \otimes W) \oplus (V_2 \otimes W) \longmapsto (V_1 \oplus V_2) \otimes W$

Since pure tensors span, it suffices to check on pure tensors.

$$\alpha(\beta(v_1 \otimes w_1 + v_2 \otimes w_2)) = \alpha(v_1 \otimes w_1 + v_2 \otimes w_2)$$

$$= v_1 \otimes w_1 + v_2 \otimes w_2$$

$$\beta(\alpha((v_1 + v_2) \otimes w)) = \beta(v_1 \otimes w + v_2 \otimes w) = v_1 \otimes w + v_2 \otimes w$$

$$= (v_1 + v_2) \otimes w.$$

$\alpha \circ \beta = id$, $\beta \circ \alpha = id$ $\Longrightarrow$ they are isom. $\square$.

Rmk: This actually works for infinite $\oplus$.

$$\left(\bigoplus_{i \in I} V_i\right) \otimes W \cong \bigoplus_{i \in I} (V_i \otimes W).$$

Prop. $K \otimes K$ is one dimensional.

pf. a general elt of $K \otimes K$ looks like $\sum_{i=1}^{n} a_i \otimes b_i$

$$\sum_{i=1}^{n} a_i b_i \, 1 \otimes 1 = \left(\sum_{i=1}^{n} a_i b_i\right) 1 \otimes 1.$$

$\Rightarrow$ $1 \otimes 1$ spans $K \otimes K$

$\Rightarrow$ $K \otimes K$ has dimension 1 or 0.

Suppose $K \otimes K = 0$. Consider a bilin map $\psi : K \times K \longrightarrow W$

$$\begin{tikzcd} K \times K \arrow[r] \arrow[dr, "\psi"'] & K \otimes K \arrow[d, "\rho"] \arrow[r] & 0 \\ & W & \end{tikzcd}$$

$\rho \to 0$ map

ψ must be the 0 map!

If we can construct any nonzero bilin map $\psi : K \times K \to W$, then $K \otimes K \neq 0$.

$$\begin{aligned} \psi : K \times K &\longrightarrow K \quad \text{is a bilin map and nonzero!} \\ (a,b) &\longmapsto ab \end{aligned}$$

So $K \otimes K \neq 0 \implies \dim K \otimes K = 1$. $\square$.

Rmk: the mult map induces an isom

$$\begin{aligned} K \otimes K &\xrightarrow{\sim} K \\ a \otimes b &\longmapsto ab. \end{aligned}$$

Prop. Let $e_1, \dots, e_n$ be a basis for V, $f_1, \dots, f_m$ be a basis for W. Then $e_i \otimes f_j$ for $1 \le i \le n$, $1 \le j \le m$ forms a basis for $V \otimes W$.

Pf. $V = \bigoplus_{i=1}^{n} K e_i \quad W = \bigoplus_{j=1}^{m} K f_j$.

$$V \otimes W = \left(\bigoplus_{i=1}^{n} K e_i\right) \otimes \left(\bigoplus_{j=1}^{n} K f_j\right).$$

$$= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \underbrace{(K e_i \otimes K f_j)}$$ this is a $\otimes$ of 1-d v.s., so it's 1-d

$$= \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} K\, e_i \otimes f_j. \qquad \square$$

Cor. $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

$$K^n \otimes K^m \cong K^{nm}$$

Let $f: V_1 \to V_2$, and $g: W_1 \to W_2$ linear maps.

Prop. $\exists!$ linear map $V_1 \otimes W_1 \to V_2 \otimes W_2$, denoted $f \otimes g$ with the property that $v \otimes w \mapsto f(v) \otimes g(w)$.

Pf. Have a map
$$V_1 \times W_1 \to V_2 \otimes W_2$$
$$(v_1, w_1) \mapsto f(v_1) \otimes g(w_1).$$

this is bilin map, by universal prop, $\exists!$ linear map $V_1 \otimes W_1 \to V_2 \otimes W_2$ s.t. $v_1 \otimes w_1 \mapsto f(v_1) \otimes g(w_2)$.

Rmk: This says that $\otimes$ is functorial.